Guzhvina Galina.

The Action of the Ricci Flow on Almost Flat Manifolds.

2007

Reine Mathematik

The Action of the Ricci Flow on Almost Flat Manifolds.

Inaugural-Dissertation zur Erlangung des Doktorgrades der
Naturwissenschaften im Fachbereich Mathematik und Informatik der
Mathematisch-Naturwissenschaftlichen Fakultät
der Westfälischen Wilhelms-Universität Mnster

vorgelegt von
Galina Guzhvina
2007

Dekan:	Prof. Dr. Joachim Cuntz
Erster Gutachter:	Prof. Dr. Burkard Wilking
Zweiter Gutachter:	Prof. Dr. Christoph Böhm
Tag der mündlichen Prüfung:	09.07.2007
Tag der Promotion:	09.07.2007

Abstract

The present work studies how the Ricci flow acts on almost flat manifolds. We show that the Ricci flow exists on any ε-flat Riemannian manifold (M, g) with ε small enough for any $t \in \mathbb{R}_{\geq 0}$, that $\lim_{t\to\infty} |K|_{(M,g_t)} \cdot diam^2(M, g_t) = 0$ along the Ricci flow and in the case when the fundamental group of (M, g_t) is (almost) Abelian we obtain the C^0-convergence of the metric to a flat limit metric. The cases of $\pi_1(M, g_t)$ Abelian and non-Abelian are handled in two different ways. Apart from that we give examples that show that the pinching constant in the Gromov's theorem necessarily depends on the dimension of the manifold.

Contents

Acknowledgments.

First of all I want to express my deepest gratitude to my supervisor Professor Burkhard Wilking for having proposed this interesting topic as well as for his constant attention to the work and his ever present readiness to help. I thank also Dr. Christoph Böhm who kindly accepted to be my second referee, for having conscientiously read the text and the valueable remarks he made. I thank equally all the other members of the Arbeitsgruppe Differentialgeometrie for interesting discussions and pleasant atmosphere in which I worked. Last but not least I want to use this opportunity to say my thanks to my family for their patience and moral support.

Introduction

A compact Riemannian manifold M^n is called ε-flat if its curvature is bounded in terms of the diameter as follows:

$$|K| \leq \varepsilon \, . \, d(M)^{-2},$$

where K denotes the sectional curvature and $d(M)$ the diameter of M. If one scales an $\varepsilon-$flat metric it remains $\varepsilon-$flat.

By almost flat we mean that the manifold carries ε-flat metrics for arbitrary $\varepsilon > 0$.

The (unnormalized) Ricci flow is the geometric evolution equation in which one starts with a smooth Riemannian manifold (M^n, g_0) and evolves its metric by the equation:

$$\frac{\partial}{\partial t} g = -2ric_g, \tag{1}$$

where ric_g denotes the Ricci tensor of the metric g.

When M^n is compact, one often considers the normalized Ricci flow

$$\frac{\partial}{\partial t} g = -2ric_g + \frac{2sc(g)}{n} g, \tag{2}$$

where $sc(g)$ is the average of the scalar curvature of M^n. Under the normalized flow the volume of the solution metric is constant in time, equations (1) and (2) differ only by a change of scale in space by a function of t and change of parametrization in time (see [4]).

The present paper studies how the Ricci flow acts on almost flat manifolds. We show that on a sufficiently flat Riemannian manifold (M, g_0) the Ricci flow exists for all $t \in [0, \infty)$, $\lim_{t\to\infty} |K|_{g(t)} \cdot d(M, g(t))^2 = 0$ as $g(t)$ evolves along (1), moreover, if $\pi_1(M, g_0)$ is abelian, $g(t)$ converges along the Ricci flow to a flat metric. More precisely, we establish the following result:

Main Theorem 1(Ricci Flow on Almost Flat Manifolds.)

In any dimension n there exists an $\varepsilon(n) > 0$ such that for any $\varepsilon \leq \varepsilon(n)$ an ε-flat Riemannian manifold (M^n, g) has the following properties:

(i) the solution $g(t)$ to the Ricci flow (1)

$$\frac{\partial g}{\partial t} = -2ric_g, \qquad g(0) = g,$$

exists for all $t \in [0, \infty)$,

(ii) along the flow (1) one has

$$\lim_{t\to\infty} |K|_{g_t} \cdot d^2(M, g_t) = 0$$

(iii) $g(t)$ converges to a flat metric along the flow (1), if and only if the fundamental group of M is (almost) abelian (= abelian up to a subgroup of finite index).

Note that in (iii) convergence is of class C^0. Actually, a little more can be said: the limit manifold (M, g_∞), where g_∞ is the limit of $g(t)$ along the Ricci flow (1), is isometric to a flat manifold.

Example 1. (cf. [1], Introduction)

Consider a compact Riemannian manifold (M^2, g) such that

$$|K| d(M^2)^2 \leq 1.73 \qquad (\star)$$

Along the normalized Ricci flow (2) with $g(0) = g$, $g(t)$ converge to a flat metric.

Indeed, normalize the curvature as $|K| \leq 1$. Applying Gauss-Bonnet theorem, we have the following estimate on the Euler characteristic of M^2 :

$$|\chi(M^2)| \leq \frac{1}{2\pi} \int_M |K| d\theta$$

From Rauch comparison argument,

$$\frac{1}{2\pi} \int_M |K| d\theta \leq \int_0^{d(M)} \sinh r dr = \cosh d(M) - 1.$$

Therefore, if $|K| d(M)^2 \leq arccosh^2(1+a)$, then $|\chi| \leq a$. Then it is easy to see that $(\star)$ implies that $\chi = 0$.

In the case $n = 2$ the normalized Ricci flow (2) looks like

$$\frac{\partial}{\partial t} g = (sc(g) - K) g. \qquad (\star\star).$$

For this flow we have the following classical result (see [3], chapter 5):

Uniformization Theorem $(\chi(M) = 0)$

On a closed Riemannian manifold (M^2, g_0) *with* $\chi(M) = 0$, *the normalized Ricci flow* $(\star\star)$ *with* $g(0) = g_0$ *has a unique solution for all time, moreover, as* $t \to \infty$, *the metrics* $g(t)$ *converge uniformly in any* C^k*-norm to a flat metric* g_∞.

Thus the pinching constant $\varepsilon(n)$ in the case of $n = 2$ can be chosen as $\varepsilon(2) = 1.73$.

□

Fundamental results concerning the algebraic structure of almost flat manifolds were obtained by Gromov at the end of 70's.
In fact, Gromov [1] showed that each nilmanifold (= compact quotient of a nilpotent Lie group) is almost flat. It means that almost flat manifolds which do not carry

flat metrics exist and occur rather naturally. Moreover, the next theorem asserts that nilmanifolds are, up to finite quotients, the only almost flat manifolds.

Theorem (Gromov)

Let M^n be an $\varepsilon(n)-$ flat manifold, where $\varepsilon(n) = \exp(-\exp(\exp n^2))$. Then M is finitely covered by a nilmanifold (compact quotient of a nilpotent Lie group). More precisely:

(i) The fundamental group $\pi_1(M)$ contains a torsion-free nilpotent normal subgroup Γ of rank n,

(ii) The quotient $G = \pi_1(M)/\Gamma$ has finite order and is isomorphic to a subgroup of $O(n)$,

(iii) the finite covering of M with fundamental group Γ and deckgroup G is diffeomorphic to a nilmanifold N/Γ,

(iv) The simply connected nilpotent group N is uniquely determined by $\pi_1(M)$.

From this theorem it is not clear whether M is diffeomorphic to the quotient of N by a uniform discrete group of isometries for a suitable left invariant metric on N. Ruh [10] proved a stronger version of this theorem. He showed that, under strict curvature assumptions, M itself, and not only the finite cover, possesses a locally homogeneous structure.

Theorem (Ruh)

For a compact Riemannian manifold M^n and a suitably small number $\varepsilon = \varepsilon(n)$ the fact that M^n is $\varepsilon-$flat implies that M is diffeomorphic to the compact quotient N/Γ, where N is a simply connected nilpotent Lie group and Γ is an extension of a lattice L in N by a finite group H.

According to this theorem it is possible to specify the manner in which the fundamental group of M acts on N. The construction of the proof provides a Riemannian metric as well as a connection D compatible with this metric. Left translation in N coincides with parallel translation with respect to D. Γ acts as a group of affine isometries of N, i.e. the elements of Γ can be viewed as diffeomorphisms of N preserving the connection D as well as the metric. The finite cover M' of M with covering group $H = L/\Gamma$ is a nilmanifold N/L, and we recover Gromov's Theorem.

When M is flat, it is finitely covered by $N = \mathbb{R}^n$, and the results of Gromov-Ruh yield the classical Bieberbach Theorem:

Theorem (Bieberbach)

Let M^n be a compact flat Riemannian manifold, $\pi_1(M)$ its fundamental group acting on $T_pM = \mathbb{R}^n$ by rigid motions (deck transformations), and Γ be the set of all translations in $\pi_1(M)$. Then Γ is a free abelian normal subgroup of rank n; the factor group $G = \pi_1(M)/\Gamma$ has finite order and is obtained as the group of rotational parts of the $\pi_1(M)-$action on T_pM; T_pM/Γ is a torus which covers M with deckgroup G. Moreover, in any dimension there are only finitely many affine equivalence classes of flat compact connected Riemannian manifolds.

Recall that two flat compact connected Riemannian manifolds are affinely equivalent if and only if they have isomorphic fundamental groups.

From the results of Bieberbach, Gromov and Ruh it follows, in particular, that an almost flat manifold carries a flat metric if and only if its fundamental group is almost abelian (abelian up to a subgroup of finite index).

In the theorems of Gromov and Ruh the pinching constant $\varepsilon(n)$ is put equal to $\exp(-\exp(\exp n^2))$. This estimation may not be optimal. Actually, the question arises whether $\varepsilon(n)$ should necessarily depend on the dimension. The second main result of this paper answers this question in the affirmative:

Main Theorem 2 (Gromov's Pinching Constant).

In every dimension n there exists a compact Riemannian manifold (M^n, g) with $|K|_M \cdot d(M^n, g)^2 < \frac{14}{n^2}$, which can not be finitely covered by a nilmanifold.

The Ricci flow (1) was introduced by Hamilton in 1982 [4]; the innovations that originated in [4] and subsequent papers have had a profound influence on geometric analysis. In particular, Perelman's recent ground-breaking work is based on Ricci flow techniques. One of the fundamental results proved by Hamilton is the short-time existence for the Ricci flow with an arbitrary smooth initial metric:

Theorem (Hamilton)

If (M^n, g_0) is a closed compact Riemannian manifold, there exists a unique solution $g(t)$ to the Ricci flow (1) defined on some time interval $[0, \varepsilon)$ such that $g(0) = g_0$. The lifetime of a maximal solution is bounded below by $\frac{C(n)}{max_{M^n}\|R\|_{g_0}}$, where $C(n)$ is *a universal constant depending only on the dimension and R is the curvature tensor of M.*

This basic result allows one to use the Ricci flow as a practical tool for improving metrics on Riemannian manifolds. In particular, a number of smoothing results in Riemannian geometry can be proved using the short-time existence of the flow combined with the derivative estimates. In their turn, the derivative estimates (Bernstein-Bando-Shi estimates) show that assuming an initial curvature bound allows one to bound all derivatives of the curvature for a short time. Bernstein-Bando-Shi estimates enable one to prove the long-time existence of the flow, which states that a unique solution of the Ricci flow exists as long as its curvature remains bounded. From the Main Theorem 1 it follows that in the case of an almost flat manifold the smoothing properties of the Ricci flow present themselves in another context, namely, we observe the drastic improvement of the Gromov's pinching constant along the Ricci flow.

When we study the Ricci flow on a nilpotent Lie group it often makes sense to consider instead of the flow (1) the normalised Ricci flow:

$$\frac{\partial g}{\partial t} = -2ric_g - 2\|ric_g\|_g^2 g, \tag{3}$$

where $\|ric_g\|_g^2 = trRic_g^2$ and we normalise the scalar curvature $sc(g_0) = -1$.
Under the flow (3) the (constant) scalar curvature of the solution metric $g(t)$ remains constant in time. Furthermore, equation (3) differs from the Ricci flow (1) only by a change of scale in space by a function of t and a change of parametrisation in time. Note that in this case our denotation differs from the standard one: usually under the normalized Ricci flow the volume-preserving Ricci flow (2) is understood. However, in this paper it should not be the cause of confusion, since in the sequel we use only flows (1) and (3).

The following example gives the intuition of how the Ricci flow acts on nilmanifolds in low dimensions:

Example 2. (cf. [3], chapter 1)
Let G^3 denote the 3-dimensional Heisenberg group, g is a left invariant metric on G^3 and let $N^3 \cong G^3/\Gamma$ be its compact quotient. Then the Ricci flow exists on N^3 for all $t \in \mathbb{R}_{\geq 0}$ and along the Ricci flow $|K(N^3)|_t \cdot diam_t^2(N^3, g) \to 0$ as $t \to \infty$.

In [8], Milnor classified all 3-dimensional unimodular Lie groups. From this classification it follows that any simply connected unimodular nilpotent Lie group in dimension 3 must be isomorphic to G^3.

Algebraically, G^3 is isomorphic to the set of upper-triangular 3×3 matrices

$$G^3 \cong \left\{ \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \right\} x, y, z \in \mathbb{R}.$$

endowed with the usual matrix multiplication. Topologically, G^3 is diffeomorphic to $\mathbb{R}^3$ under the map

$$G^3 \ni \gamma = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix} \mapsto (x, y, z) \in \mathbb{R}^3.$$

Under this identification, left multiplication by γ corresponds to the map

$$L_\gamma(a, b, c,) = (a + x, b + y, c + xb + z).$$

Introduce a Milnor frame $\{F_i\}$ for some left-invariant metric g on G^3 (cf. [8]). In this frame g may be written as

$$g = A\omega^1 \otimes \omega^1 + B\omega^2 \otimes \omega^2 + C\omega^3 \otimes \omega^3,$$

where A, B, C are positive and $\{\omega^i\}$ are 1-forms dual to $\{F_i\}$. Then the sectional curvatures of (G^3, g) are given by

$$K(F_2 \wedge F_3) = -3\frac{A}{BC}, \qquad K(F_3 \wedge F_1) = \frac{A}{BC}, \qquad K(F_1 \wedge F_2) = \frac{A}{BC}$$

and the Ricci tensor is

$$ric_g = 2\frac{A^2}{BC}\omega^1 \otimes \omega^1 - 2\frac{A}{B}\omega^2 \otimes \omega^2 - 2\frac{A}{B}\omega^3 \otimes \omega^3.$$

Hence the Ricci flow on $(G^3, g(t))$ is given by

$$\frac{d}{dt}A = -4\frac{A^2}{BC},$$

$$\frac{d}{dt}B = 4\frac{A}{C},$$

$$\frac{d}{dt}C = 4\frac{A}{B}.$$

This system of ODE can be solved explicitly. The solution for given initial data $A_0, B_0, C_0 > 0$ is of the form

$$A = A_0^{2/3}B_0^{1/3}C_0^{1/3}(12t + B_0C_0/A_0)^{-1/3},$$

$$B = A_0^{1/3}B_0^{2/3}C_0^{-1/3}(12t + B_0C_0/A_0)^{1/3},$$

$$C = A_0^{1/3}B_0^{-1/3}C_0^{2/3}(12t + B_0C_0/A_0)^{1/3}.$$

Thus we have that for any choice of the initial data $A_0, B_0, C_0 > 0$, the Ricci flow (1) has a unique solution on $(G^3, g(t))$ for all $t \in \mathbb{R}_{\geq 0}$. Moreover, from the explicit form of the solution it follows that there exist constants $0 < c_1 \leq c_2 < \infty$ depending only on the initial data such that each sectional curvature K is bounded for all $t \geq 0$ by

$$\frac{c_1}{t} \leq K \leq \frac{c_2}{t},$$

and such that the diameter of any compact quotient N^3 of G^3 is bounded for all $t \geq 0$ by

$$c_1 t^{1/6} \leq diam N^3 \leq c_2 t^{1/6}.$$

So, the pinching constant of G^3 tend to zero along the Ricci flow:

$$\frac{c_1^3}{t^{2/3}} \leq |K|_{N^3} diam^2(N^3) \leq \frac{c_2^3}{t^{2/3}}.$$

□

Chapter 1

Ricci Flow on Almost Flat Manifolds. Abelian Case.

Theorem A
In any dimension n *there exists an* $\varepsilon(n) > 0$ *such that for any* $\varepsilon \leq \varepsilon(n)$ *and for any* ε*-flat* n*-dimensional Riemannian manifold* M^n *with (almost) abelian* $\pi_1(M)$ *the solution of the Ricci flow (1)*

$$\frac{\partial g_t}{\partial t} = -2ric_g$$

exists for all $t \in [0, \infty)$ *and converges to a flat metric on* M^n*.*

So, we want to show that on any $\varepsilon-$flat Riemannian manifold with the abelian fundamental group the Ricci flow (1) converges to a flat metric for ε small enough. Of course, this condition on the fundamental group is also necessary.In fact, if (M, g_t) converges to a flat manifold (M, g_∞), then $\pi_1(M) = \pi_1(M, g_\infty)$ is almost abelian by the classical Bieberbach's theorem.
Now some notes on the proof.
In the flat case $G = \pi_1(M)/\Gamma$ is naturally isomorphic to the holonomy group of M at p, and Γ, the set of translations in $\pi_1(M)$, can also be described as the set of loops at p with rotational parts $\leq \frac{1}{2}$. In the almost flat case the group Γ is generated by those "short" loops (i. e. with lengths $\leq 4(6\pi)^{\frac{1}{2}n(n-1)}d(M)$) which have a rotational part ≤ 0.48. Again, these rotational parts are in fact much smaller than 0.48; they have an upper bound proportional to the length of the loop and decrease as ε decreases, so that Γ is almost translational (see, for example, [1]). Moreover, if one chooses shortest loops at p in the equivalence classes of $\pi_1(M)$ modulo Γ then their holonomy rotations are, after a small correction, a subgroup of $O(n)$ isomorphic to $\pi_1(M)/\Gamma$.

In the case of an almost flat manifold with almost abelian fundamental group the generators of the lattice Γ have rotational parts much smaller still, namely, proportional to ε (Section 2), which permits us to obtain nice estimations on the derivatives of the curvature tensors of such manifolds (Section 3), more precisely, we conclude that $\|\nabla R\| \leq c(n) \cdot d(M)(\|\nabla^2 R\| + \|R\|^2)$. Remark that this inequality is only valid in the abelian case.
The convergence theory developed by Cheeger and Gromov for metric spaces turns out to be a useful tool if we want to understand how the Ricci flow behaves on an almost flat manifold, because the structure of the limit space in this case is often simple enough to be understood completely.
In Section 5 we show that, given a sequence of almost flat (but not flat) manifolds, the sequence of the corresponding universal covers subconverges to a nilmanifold, if all the derivatives of the corresponding curvature tensors are uniformly bounded by norm and the pinching constant tends to zero.
This result permits us to establish that on a sufficiently flat Riemannian manifold $\|\nabla R\| \leq c(n)\|R\|^{\frac{3}{2}}$ for some $c(n)$ depending only on the dimension of the manifold. The method used in the proof consists in showing that the geometric structure of the limit space of the universal covers obtained in argument by contradiction is incompatible with the structure of a nilpotent Lie group.
Finally, we use the Bando-Bernstein-Shi estimations (BBS estimations) on the derivatives of the curvature tensor along the Ricci flow to obtain the main result. We get that in the almost abelian case the metric gets considerably flatter along the Ricci flow - the fact that simultaneously proves its long time existence.

1.1 Short Geodesic Loops. General Information.

This section does not contain any new results. We provide the necessary background material from the theory of Gromov's almost flat manifolds.
The general information on the short geodesic loops is taken from the article by Buser and Karcher ([1]).
Let M be a Riemannian manifold. Assume curvature bounds $|K| < \Lambda^2$ and fix a point $p \in M$. Then the conjugate radius can be bounded from below as

$$conj \geq \pi \cdot \Lambda^{-1}. \tag{1.1}$$

In a ball B_ρ of radius $\rho < \pi \cdot \Lambda^{-1}$ around $0 \in T_pM$ we pull back the Riemannian metric from M via $\exp^{-1}$. If $v, w \in B_\rho$, $|v| + |w| \leq \rho$, then from Rauch estimates we have the following comparison between the lifted Riemannian and the Euclidean metric of T_pM :

$$\frac{\Lambda}{sinh(\Lambda\rho)} \cdot d(v,w) \leq |v-w| \leq \frac{\Lambda}{sin(\Lambda\rho)} \cdot d(v,w). \tag{1.2}$$

Any closed curve c at p of length $\leq \rho$ can be lifted via $\exp^{-1}$ to a curve $\tilde{c}$ in T_pM. By continuously replacing longer and longer arcs of $\tilde{c}$ by geodesic segments one has "natural" length decreasing homotopies from $\tilde{c}$ to the geodesic ray $[0,1] \cdot \tilde{c}(1)$. In particular, for any $v \in T_pM$, $|v| \leq \rho - d(M)$, let c be the closed curve which is obtained from the geodesic $t \to \exp tv$, $(0 \leq t \leq 1)$ by joining the endpoint $\exp v$ by a shortest geodesic ($\leq d(M)$) to the initial point p. Then its lift $\tilde{c}$ provides a geodesic triangle ovw with $d(v,w) \leq d(M)$ and $\exp w = p$. The homotopy of $\tilde{c}$ to the geodesic ray $[0,1] \cdot \exp v$ gives a geodesic loop.

Definition 1.1.1 *A geodesic loop $c : [0,1] \to M$ on a Riemannian manifold (M,g) is a geodesic with $c(0) = c(1)$ (but not necessarily $\dot{c}(0) = \dot{c}(1)$).*

Definition 1.1.2 *A homotopy of loops is called short if each of its curves is shorter than the conjugate radius conj of* exp.

Proposition 1.1.3 *([1], 2.2.2)*
For any point $p \in M$ there is exactly one geodesic loop in each short homotopy class at p.

Definition 1.1.4 (Gromov's product of short geodesic loops) *Take any two loops α, β at p, assume $|\alpha| + |\beta| < \pi\Lambda^{-1}$. Then $\beta \star \alpha$ is the unique geodesic loop in the short homotopy class of the curve $\beta \cdot \alpha$, where $\beta \cdot \alpha$ is the product used in homotopy theory.*

Proposition 1.1.5 *([1], 2.2.5)*
The above defined operation introduces a structure on the set of short geodesic loops, namely
a) $\{p\} =: 1$,
b) if $2|\alpha| < \pi\Lambda^{-1}$, α^{-1} is the loop α with the reversed parametrisation, then

$$\alpha^{-1} \star \alpha = 1,$$

c) if $|\alpha| + |\beta| + |\gamma| < \pi\Lambda^{-1}$, $\alpha \star (\beta \star \gamma) = (\alpha \star \beta) \star \gamma$.

Remark 1.1.6 *One of the important intermediary results in the proof of Gromov's Theorem is that in case of sufficiently flat manifolds all sufficiently short geodesic loops represent different homotopy classes in $\pi_1(M)$ and all algebraic properties of the short geodesic loops remain the same if we replace the Gromov product $\star$ by the standard product in $\pi_1 M$.*

To each geodesic loop α at p we associate its holonomy motion

$$m(\alpha) : T_pM \to T_pM,$$

$$m(\alpha)(x) = r(\alpha)x + t(\alpha), \tag{1.3}$$

where $r(\alpha)$ is a parallel transport around α (rotational part of $m(\alpha)$) and

$$t(\alpha) = \dot{\alpha}(0) \tag{1.4}$$

- the translational part of α.

We use the distance on the orthogonal group to compare the holonomy motion of the Gromov product to the composition of the holonomy motions of α and β : for $A, B \in SO(n)$

$$d(A, B) := max\{|\angle(Av, Bv)|, v \in \mathbb{R}^n, |v| = 1\}, \tag{1.5}$$

where $\angle(Av, Bv)$ is the minimal ($\leq \pi$) angle between the vectors Av and Bv.

$$\|A\| := d(A, id).$$

Proposition 1.1.7 (The holonomy map is almost homomorphic) *([1], 2.3.1)*
Let (M, g) be a complete Riemannian manifold, $|K_M| < \Lambda^2$. Consider loops α and β such that $|\alpha|, |\beta| < \frac{1}{\Lambda}$. Then $\beta \star \alpha$ is defined and

$$d(r(\alpha) \cdot r(\beta), r(\beta \star \alpha)) \leq \Lambda^2 \cdot |t(\alpha)| \cdot |t(\beta)|, \tag{1.6}$$

$$|t(m(\alpha) \cdot m(\beta)) - t(\beta \star \alpha)| \leq \Lambda^2 |t(\alpha)||t(\beta)|(|t(\alpha)| + |t(\beta)|). \tag{1.7}$$

Proposition 1.1.8 (Commutator estimates) *([1], 2.4.1)*
Let (M, g) be a complete Riemannian manifold, $|K_M| < \Lambda^2$. Consider α, β at $p \in M$ such that $|\alpha| + |\beta| \leq \frac{\pi}{3\Lambda}$. Then

$$d([r(\alpha), r(\beta)], r[\beta, \alpha]) \leq \frac{5}{3}\Lambda^2 \cdot |t(\alpha)| \cdot |t(\beta)| + \frac{5}{6}\Lambda^2 |t[\alpha, \beta]|(|t(\alpha)| + |t(\beta)|),$$

(1.8)

$$|t[m(\alpha)\cdot m(\beta)] - t[\beta,\alpha]| \le \frac{10}{3}\Lambda^2 |t(\alpha)||t(\beta)|(|t(\alpha)| \tag{1.9}$$
$$+ \quad |t(\beta)|) + \frac{10}{6}|t([\beta,\alpha])|\Lambda^2|t(\alpha)||t(\beta)|(|t(\alpha)| + |t(\beta)|).$$

Propositions 1.1.7, 1.1.8 show that the Gromov product and commutator of geodesic loops are almost compatible with the easily computable product and commutator of the holonomy motions of the loops. The error is curvature controlled.
Let (M, g) be a compact Riemannian manifold, $|K_M| < \Lambda^2$. Define

$$\Gamma_\rho(M) = \{\alpha \in \pi_1(M) : |t(\alpha)| \le \rho, \|r(\alpha)\| < 0.48\}. \tag{1.10}$$

The next two propositions assert that, under strong curvature assumptions, there exists a $\rho >> d(M)$ (also bounded from above), such that for any $\alpha \in \Gamma_\rho$

$$\|r(\alpha)\| < \frac{\theta}{\rho}|t(\alpha)|, \tag{1.11}$$

where θ is an adjustable parameter.

Proposition 1.1.9 (Relative denseness of loops with small rotational parts) *([1], 3.2)*
Let (M^n, g) be a compact Riemannian manifold.
Fix $m = 10^n$, $\eta \le 0,48$ is an adjustable parameter,
$L = 3 + 2(\frac{7}{\eta})^{dimSO(n)}$,
$w \ge w(n) := 2 \cdot 14^{dimSO(n)}$,
and assume the bounds on the curvature and the diameter of M^n:

$$d(M)\Lambda \le \frac{\eta \cdot m^{-L-1}}{2w}. \tag{1.12}$$

Then there exists a number $\rho_0 = \rho_0(\eta, m, w)$,

$$2w \cdot d(M) \cdot m^4 \le \rho_0 \le 2w \cdot d(M) \cdot m^L, \tag{1.13}$$

such that
for any $v \in T_pM$ with $|v| \le \rho_0(1 - \frac{1}{m})$ there is an $\alpha \in \Gamma_{\rho_0}$ such that

$$\|r(\alpha)\| \le \eta, \qquad |t(\alpha) - v| \le \frac{1}{m-1}\rho_0. \tag{1.14}$$

Proposition 1.1.10 *([1], Key Proposition 3.3.1)*
Let (M, g) be a compact Riemannian manifold, $0 < \theta < \frac{1}{7}$ - an adjustable parameter. Choose η in 1.1.9 as $\eta = \frac{1}{3}\theta \cdot (2.1)^{-q(n)}$, where $q(n) = [3.02^{\frac{n(n+1)}{2}}]$. Then under the curvature bounds from 1.1.9 and for $\rho = \rho_0$

(i) all $q(n)-$fold commutators in Γ_ρ exist and are trivial,

(ii) for any $\alpha \in \Gamma_\rho$ $\|r(\alpha)\| \leq \theta \cdot 2.1^{q-q(n)}$, where $q \leq q(n)$ is the order of the nilpotency of α.

So, if the holonomy motion of not too long a loop γ has its rotational part $\|r(\gamma)\| < 0,48$, then it follows that $\|r(\gamma)\| \leq \theta$, where θ can be taken very small in the case of $\varepsilon-$flat manifold, where ε is very small.

1.2 Short Geodesic Loops in the Abelian Case.

In the special case of abelian fundamental group we can establish better estimates on the holonomies of loops from Γ_ρ for ρ appropriately chosen. Namely, in the next theorem we show that in this setting for any $\gamma \in \Gamma_\rho$ on an $\varepsilon-$flat M with ε sufficiently small, $\|r(\gamma)\|$ is of order ε.

Theorem 1.2.1 *In any dimension n there exists an $\varepsilon(n) > 0$ such that for any $\varepsilon \leq \varepsilon(n)$ and for any ε-flat n-dimensional manifold (M, g) with (almost) abelian fundamental group for a ρ taken as in proposition 1.1.9, i.e.*

$$\rho \leq c_2(n)d(M),$$

for $c_2(n)$ defined as in 1.1.9,
for any geodesic loop $\alpha \in \Gamma_\rho$ we have that

$$\|r(\alpha)\| < 10^n c_2(n)^2 \varepsilon. \tag{1.15}$$

Remark 1.2.2 *By passing to a finite covering of M we can suppose that the fundamental group of the manifold in question is not almost (virtually) but really abelian. In each dimension there is only a finite number of isomorphism classes of groups which can serve as fundamental groups for the flat manifolds (The Theorem of Bieberbach). Thus in each dimension the index of the normal abelian subgroup in $\pi_1(M^n)$ and, hence, the diameter of the covering manifold will remain uniformly bounded.*

Proof. Take $\alpha, \beta \in \Gamma_\rho(M)$ where ρ is from propositions 1.1.9, 1.1.10 on a n-dimensional Riemannian manifold (M, g) with the corresponding bounds on $|K|_M d(M)^2$ and abelian fundamental group. Rescale the curvature as $|K|_M \leq \Lambda^2$ for some positive Λ. Thus $[\alpha, \beta] = id$ in the sense of the short loop multiplication. Apply proposition 1.1.8 to α and β:

$$|t[m(\alpha), m(\beta)]| \leq \frac{10}{3}\Lambda^2 |t(\alpha)||t(\beta)|(|t(\alpha)| + |t(\beta)|). \tag{1.16}$$

Define the matrix norm for any matrix D' as

$$\|D'\| = max_i |\lambda_i|, \tag{1.17}$$

where λ_i are the eigenvalues of D' over $\mathbb{C}$.
Note that for convenience we have changed the definition of the matrix norm in comparison with (1.5). The new definition is valid up to the end of this section.
Let $\alpha = (A, \sigma), \beta = (B, \tau)$, where $A = id + A'$, $B = id + B'$. Then it is easy to see that

$$A^{-1} = id - A' + \Sigma_{i=2}^{\infty}(-1)^i A'^i, \qquad B^{-1} = id - B' + \Sigma_{i=2}^{\infty}(-1)^i B'^i. \tag{1.18}$$

From proposition 1.1.10, the rotational parts of $\alpha, \beta \in \Gamma_\rho$ are small, more precisely, $\|r(\alpha)\|$, $\|r(\beta)\| \leq 2^{-q(n)}$, $q(n) = [3.02^{\frac{n(n+1)}{2}}]$. So we can rewrite A'^{-1}, B'^{-1} as

$$A'^{-1} = id - A' + r(A'), \qquad B'^{-1} = id - B' + r(B'), \tag{1.19}$$

where

$$\|r(A')\| \leq 2\|A'\|^2, \qquad \|r(B')\| \leq 2\|B'\|^2. \tag{1.20}$$

Direct calculation shows that

$$\begin{aligned} |t[m(\alpha), m(\beta)]| &= |\sigma + A\tau - ABA^{-1}\sigma - ABA^{-1}B^{-1}\tau| \\ &= |\sigma + (id + A')\tau - (id + A')(id + B')(id - A' + r(A'))\sigma \\ &- (id + A')(id + B')(id - A' + r(A'))(id - B' + r(B'))\tau| \\ &\geq |A'\tau - B'\sigma| - 36(\|A'\|^2 + \|B'\|^2)(|\sigma| + |\tau|). \end{aligned} \tag{1.21}$$

Now (1.16) can be rewritten as

$$|A'\tau - B'\sigma| < \frac{10}{3}\Lambda^2|\sigma||\tau|(|\sigma| + |\tau|) + 36(\|A'\|^2 + \|B'\|^2)(|\sigma| + |\tau|). \tag{1.22}$$

Choose α so that $\|A'\| = max\{\|D'\|, \delta = (id + D', \nu), \delta \in \Gamma_\rho\}$. Since A orthogonal and $\|A'\| < 2^{-q(n)}$, A' is skew-symmetric modulo terms of the order $\|A'\|^2$. Indeed, in the appropriate basis A' can be decomposed in 2×2 blocks of the form

$$\begin{pmatrix} cos\varphi_j - 1 & sin\varphi_j \\ -sin\varphi_j & cos\varphi_j - 1 \end{pmatrix}.$$

For any φ

$$\cos\varphi - 1 = -\varphi^2 + \Sigma_{i=2}^{\infty}(-1)^i\frac{\varphi^{2i}}{(2i)!}, \qquad \sin\varphi = \varphi + \Sigma_{i=2}^{\infty}(-1)^{i-1}\frac{\varphi^{2i-1}}{(2i-1)!.} \tag{1.23}$$

Thus, since for every j $|\varphi_j| \leq \|A'\|$,

$$|\cos\varphi_j - 1| \leq 2\varphi_j^2 \leq 2\|A'\|^2, \qquad |\sin\varphi_j| \leq |\varphi_j| + |\varphi|_j^3 \leq \|A'\| + \|A'\|^3. \tag{1.24}$$

Let the mutually orthogonal $\tilde{\tau}_1$, $\tilde{\tau}_2$, $|\tilde{\tau}_1| = |\tilde{\tau}_2|$, span an invariant subspace of A', corresponding to the "maximal" (maximal by absolute value) eigenvalue $\|A'\|$, so that

$$A'\tilde{\tau}_i = (-1)^{i+1}\|A'\|\tilde{\tau}_{i+1} + \delta_1\tilde{\tau}_i + \delta_2\tilde{\tau}_{i+1}, \tag{1.25}$$

where $\delta_1 + \delta_2 \leq 3\|A'\|^2$.
Fix $|\tilde{\tau}_i| = (1 - \frac{1}{10^n})\rho$. From proposition 1.1.9 there exist the geodesic loops $\beta_1, \beta_2 \in \Gamma_\rho$, $\beta_1 = (B_1, \tau_1)$, $\beta_2 = (B_2, \tau_2)$, such that

$$|\tau_i - \tilde{\tau}_i| \leq \frac{1}{10^n - 1}\rho, \tag{1.26}$$

thus

$$(1 - \frac{3}{10^n})\rho \leq |\tau_i| \leq \rho. \tag{1.27}$$

As a consequence,

$$|A'\tau_i - A'\tilde{\tau}_i| \leq \frac{\|A'\|}{10^n - 1}\rho. \tag{1.28}$$

Then, from the triangle inequality,

$$
\begin{aligned}
|B'_i\sigma-(-1)^{i+1}\|A'\|\tau_{i+1}| &\le |B'_i\sigma-A'\tau_i|+|A'\tau_i-A'\tilde{\tau}_i| \qquad (1.29)\\
&+\quad |A'\tilde{\tau}_i-(-1)^{i+1}\|A'\|\tilde{\tau}_{i+1}|+|(-1)^{i+1}\|A'\|\tilde{\tau}_{i+1}-(-1)^{i+1}\|A'\|\tau_{i+1}|,
\end{aligned}
$$

and from (1.22),(1.25),(1.26),(1.28) and proposition 1.1.10 it follows that

$$
\begin{aligned}
&|B'_i\sigma-(-1)^{i+1}\|A'\|\tau_{i+1}| \qquad (1.30)\\
\le\ & \frac{10}{3}\Lambda^2|\sigma||\tau_i|(|\sigma|+|\tau_i|)+\|A'\|\cdot\frac{3}{10^n}\rho+72\|A'\|^2(|\sigma|+|\tau_i|)+6\|A'\|^2\rho.
\end{aligned}
$$

Recall that $\rho\le c_2\cdot d(M)$. Suppose, by contradiction, that $\|A'\|>10^n c_2(n)^2\Lambda^2\cdot d(M)^2$, then we can rewrite (1.30) as follows:

$$
|B'_i\sigma-(-1)^{i+1}\|A'\|\tau_{i+1}|\le\frac{12}{10^n}\|A'\|\rho. \qquad (1.31)
$$

And the following estimation on the norm of B' can be obtained:

$$
\|A'\||\sigma|\ge\|B'_i\||\sigma|\ge|B'_i\sigma|\ge\|A'\|\rho\Big(1-\frac{20}{10^n}\Big)\ge\|B'_i\||\sigma|\Big(1-\frac{20}{10^n}\Big). \qquad (1.32)
$$

In the sequel we will use the lemma below:

Lemma 1.2.3 *Let $B\in Skew(n,\mathbb{R})$ (the set of skew-symmetric $n\times n$ matrices with real entries) and σ is a vector in $\mathbb{R}^n$ such that $|B\sigma|\ge\ \|B\||\sigma|\Big(1-\frac{20}{10^n}\Big)$. Then for $n\ge4$*
$\Big|B^2\sigma+\|B\|^2\sigma\Big|\le\frac{20}{10^{n/2}}\frac{n}{2}\|B\|^2|\sigma|.$

We postpone the proof of Lemma 1.2.3 and continue with the proof of 1.2.1. Rewrite B'_i in such a basis that

$$
B'_i=B_i^{(a)}+B_i^{(d)}, \qquad (1.33)
$$

where $B_i^{(a)}\in Skew(n,\mathbb{R}), B_i^{(d)}\in Diag(n,\mathbb{R})$,

$$
\|B_i^{(a)}\|=\|B'_i\|,\|B_i^{(d)}\|\le2\|B'_i\|^2. \qquad (1.34)
$$

Then ${B'}_i^2\sigma={B_i^{(a)}}^2\sigma+\Delta\sigma$, $\|\Delta\|\le4\|B'_i\|^3$.
And from lemma 1.2.3 it follows that
$\Big|{B'}_i^2\sigma+\|B'_i\|^2\sigma\Big|\le\Big|{B^{(a)}}_i^2\sigma+\|B'_i\|^2\sigma\Big|+\|\Delta\||\sigma|\le\frac{20}{10^{\frac{n}{2}}}\frac{n}{2}\|B'_i\|^2|\sigma|+4\|B'_i\|^3|\sigma|.$
Then, from (1.31),

$$\Big|\|B_i'\|^2\sigma-(-1)^{i+1}\|A'\|B_i'\tau_{i+1}\Big|\le\frac{20}{10^{\frac{n}{2}}}\frac{n}{2}\|B_i'\|^2|\sigma|+\frac{20}{10^n}\|A'\|\|B_i'\|\rho+4\|B_i'\|^3|\sigma|,$$

From (1.32) we have that

$$\|B_i'\|^2\ge\Big(1-\frac{20}{10^n}\Big)^2\|A'\|^2\ge\Big(1-\frac{40}{10^n}\Big)\|A'\|^2, \tag{1.35}$$

hence,

$$\Big|\|A'\|^2\sigma-(-1)^{i+1}\|A'\|B_i'\tau_{i+1}\Big|\le\frac{20}{10^{\frac{n}{2}}}\frac{n}{2}\|A'\|^2|\sigma|+\frac{60}{10^n}\|A'\|^2\rho+4\|A'\|^3|\sigma|. \tag{1.36}$$

So, σ,τ_i, roughly speaking, "almost" span the invariant subspaces of B_{i+1}' corresponding to the maximal eigenvalue. From (1.36) for each i and for $n\ge 4$ follows:

$$\Big|\|A'\|B_1'\tau_2-\|A'\|^2\sigma\Big|\le\frac{21}{10^{\frac{n}{2}}}\frac{n}{2}\|A'\|^2\rho,$$

$$\Big|\|A'\|B_2'\tau_1+\|A'\|^2\sigma\Big|\le\frac{21}{10^{\frac{n}{2}}}\frac{n}{2}\|A'\|^2\rho,$$

or, equivalently,

$$\Big|-B_1'\tau_2+\|A'\|\sigma\Big|\le\frac{21}{10^{\frac{n}{2}}}\frac{n}{2}\|A'\|\rho, \tag{1.37}$$

$$\Big|B_2'\tau_1+\|A'\|\sigma\Big|\le\frac{21}{10^{\frac{n}{2}}}\frac{n}{2}\|A'\|\rho. \tag{1.38}$$

From the properties of the vector norm, from (1.37),(1.38), we can get

$$\Big|-B_1'\tau_2+\|A'\|\sigma\Big|+\Big|B_2'\tau_1+\|A'\|\sigma\Big|\ge\Big|B_2'\tau_1-B_1'\tau_2+2\|A'\|\sigma\Big|\ge 2\|A'\||\sigma|-|B_1'\tau_2-B_2'\tau_1|.$$

thus

$$2\|A'\||\sigma|-|B_1'\tau_2-B_2'\tau_1|\le\frac{21}{10^{\frac{n}{2}}}n\|A'\||\sigma|. \tag{1.39}$$

On the other hand, since (1.22) is applicable also to the loops β_1, β_2, we have that

$$|B'_1\tau_2 - B'_2\tau_1| \leq 10/3\Lambda^2|\tau_1||\tau_2|(|\tau_1| + |\tau_2|) + 144\|A'\|^2\rho.$$

From (1.32) we have that

$$|\sigma| \geq \rho\Big(1 - \frac{20}{10^n}\Big), \tag{1.40}$$

hence (1.39) can be rewritten as

$$\|A'\|\Big(1 - \frac{24n/2}{10^{n/2}}\Big) \leq 10/3\Lambda^2\rho^2 + 72\|A'\|^2. \tag{1.41}$$

And, since $\|A'\| \leq 2^{-q(n)}$,
finally, we get that

$$\|A'\| \leq \frac{20}{3}c_2^2\Lambda^2 d^2, \tag{1.42}$$

which is a contradiction.

Proof of lemma 1.2.3

Let

$$B = \begin{pmatrix} \begin{pmatrix} 0 & \lambda_1 \\ -\lambda_1 & 0 \end{pmatrix} & 0 & \\ & \dots & \\ & & \begin{pmatrix} 0 & \lambda_{[\frac{n}{2}]} \\ -\lambda_{[\frac{n}{2}]} & 0 \end{pmatrix} \end{pmatrix}$$

$\sigma = (\sigma_1, ..., \sigma_n)^T$.

Without loss of generality consider the case of n even.

Then

$B\sigma = (\lambda_1\sigma_2, -\lambda_1\sigma_1, ..., \lambda_{\frac{n}{2}}\sigma_n, -\lambda_{\frac{n}{2}}\sigma_{n-1})^T$.

Consider σ as $\sigma = \sigma' + \sigma''$, where without loss of generality, $\sigma' = (\sigma_1, ..., \sigma_k, 0, ..., 0)^T$, belongs to $I_1 \times ... \times I_{\frac{k}{2}}$ - the union of invariant 2-subspaces of B corresponding to λ_j, $j = 1, .., \frac{k}{2}$ such that $|\lambda_j| \geq \|B\|\Big(1 - \frac{40}{10^n}\Big)$.

$B^2\sigma' = (-\lambda_1^2\sigma_1, -\lambda_1^2\sigma_2, ..., -\lambda_{\frac{k}{2}}^2\sigma_{k-1}, -\lambda_{\frac{k}{2}}^2\sigma_k, 0, ..., 0)^T$ and

$$\begin{aligned}\Big|\|B\|^2\sigma' + B^2\sigma'\Big| &\leq max_i\Big(\|B\|^2 - \lambda_i^2\Big)|\sigma'| \\ &\leq \Big(1 - \Big(1 - \frac{40}{10^n}\Big)^2\Big)\|B\|^2|\sigma'| \leq \frac{80}{10^n}\|B\|^2|\sigma'|.\end{aligned}$$

On the other hand, $\sigma'' = \sigma''_{\frac{k}{2}+1} + ... + \sigma''_{\frac{n}{2}}$, σ''_j corresponds to an invariant 2-subspace characterized by λ_j, $|\lambda_j| = (1-l_j)\|B\|$, $l_j > \frac{40}{10^n}$.
Then, since

$$|B\sigma|^2 = |B\sigma'|^2 + \Sigma_j |B\sigma''_j|^2 \le \|B\|^2(|\sigma'|^2 + \Sigma_j (1-l_j)^2 |\sigma''_j|^2)$$

and

$$|B\sigma|^2 \ge \Big(1 - \frac{20}{10^n}\Big)^2 \|B\|^2 (|\sigma'|^2 + \Sigma_j |\sigma''_j|^2),$$

we have that

$$\begin{aligned} &\|B\|^2|\sigma'|^2 + \|B\|^2\Sigma_j(1-l_j)^2|\sigma''_j|^2 \\ \ge\ & \Big(1 - \frac{20}{10^n}\Big)^2 \|B\|^2|\sigma'|^2 + \Big(1 - \frac{20}{10^n}\Big)^2 \|B\|^2|\Sigma_j|\sigma''_j|^2. \end{aligned} \tag{1.43}$$

Substract $\Big(1 - \frac{20}{10^n}\Big)^2 \|B\|^2|\sigma'|^2 + \|B\|^2\Sigma_j(1-l_j)^2|\sigma''_j|^2$ from the both parts of (1.43).
Then for any j holds

$$\Big(1 - \Big(1 - \frac{20}{10^n}\Big)^2\Big)\|B\|^2|\sigma|^2 \ge \Big(\Big(1 - \frac{20}{10^n}\Big)^2 - \Big(1 - l_j\Big)^2\Big)\|B\|^2|\sigma''_j|^2$$

and

$$|\sigma''_j|^2 \le \frac{1 - (1 - \frac{20}{10^n})^2}{(1 - \frac{20}{10^n})^2 - (1 - l_j)^2}|\sigma|^2.$$

Accordingly,

$$\begin{aligned} &\Big|B^2\sigma''_j + \|B\|^2\sigma''_j\Big| \le \|B\|^2\Big(1 - \Big(1 - l_j\Big)^2\Big)|\sigma''_j| \\ \le\ & \|B\|^2\Big(1 - \Big(1 - l_j\Big)^2\Big)\left(\frac{1 - (1 - \frac{20}{10^n})^2}{(1 - \frac{20}{10^n})^2 - (1 - l_j)^2}\right)^{\frac{1}{2}}|\sigma| \\ \le\ & \frac{20}{10^{n/2}}\|B\|^2|\sigma|. \end{aligned}$$

then it follows that

$$\left|B^2\sigma + \|B\|^2\sigma\right| \le \left|B^2\sigma' + \|B\|^2\sigma'\right| + \Sigma_j\left|B^2\sigma''_j + \|B\|^2\sigma''_j\right|$$
$$\le \frac{20}{10^{\frac{n}{2}}}\frac{n}{2}\|B\|^2|\sigma|.$$

And the Lemma is proved. □

So, in the abelian case, all the holonomy angles of geodesic loops are of order ε. This is a strong result as concerns the metric properties of corresponding almost flat manifolds. The next section demonstrates its utility.

1.3 The First Derivative of the Curvature Tensor on Almost Flat Riemannian Manifolds in the Abelian case.

This section illustrates how the estimates of the previous section can be used. By direct calculation we get the specific estimates of the first derivative of the curvature tensor on almost flat manifolds in terms of the higher ones. This result holds only in the abelian case. It is a crucial tool for the proof of Theorem A.
Here and further on under the norm of the derivative of the tensor $T_{i_1 \dots i_k}$ we understand the following:

$$\|\nabla T\| = sup_{|v|=1}\|\nabla_v T\|_\infty, \tag{1.44}$$

where $\|\cdot\|_\infty$ is a square $sup-$norm, i. e.

$$\|T\| := \|T\|_\infty = sup_{x\in M}(\Sigma_{i_1\dots i_k}T^2_{i_1\dots i_k}(x))^{1/2}, \tag{1.45}$$

with $T_{i_1\dots i_k}(x) = T(e_{i_1}, ..., e_{i_k})$, where $e_i, ..., e_n$ is an orthonormal basis at T_xM. When we consider the tensor T at $x \in M$ as a vector in the vector space $\otimes^4 T_xM$, we use the standard square vector norm denoted by $|\cdot|$.

Theorem 1.3.1 *In any dimension n there exists an $\varepsilon(n)$ such that for any $\varepsilon \le \varepsilon(n)$ and for any ε-flat manifold (M^n, g) with (almost) abelian fundamental group, there exists a constant $c = c(n)$, depending only n such that*

$$\|\nabla R\| \le c(n)\cdot d(M)(\|\nabla^2 R\| + \|R\|^2), \tag{1.46}$$

where R is the curvature tensor of (M^n, g).

Proof

Consider $R = R(t)$ as it evolves by parallel transport along γ - a closed smooth curve on a manifold M^n, i.e. $\dot\gamma(0) = \dot\gamma(\tau)$. We may assume that γ can be parametrised by arc length. Let τ be the length of γ, $A \in SO(n^4)$ - the holonomy operator around γ. We can regard $R(t)$ at $t = 0$ as a vector in the n^4-dimensional vector space $\otimes^4 T_p M$, so along $\gamma(t)$

$$R(t) = R^1(t)e_1(t) + ... + R^{n^4}(t)e_{n^4}(t),$$

where $e_i(t)$ is the parallel transport of the i-th basis vector $e_i(0)$ of $\otimes^4 T_{\gamma(0)}(M)$. The basis in $\otimes^4 T_{\gamma(0)} M$ is taken so that the holonomy matrix A is decomposed in 2×2 blocks, j-th invariant subspace of $\mathbb{R}^{n^4}$ corresponds to the rotation by angle φ_j:

$$\begin{pmatrix} cos\varphi_j & sin\varphi_j \\ -sin\varphi_j & cos\varphi_j \end{pmatrix}$$

Take an arbitrary coordinate R^i in this basis and consider it along $\gamma(t)$ as a function of one variable, $R^i = R^i(t), t \in [0, \infty)$. We have that

$$R^i(\tau) = R^i(0)cos(\varphi_j) + R^{i+1}(0)sin(\varphi_j), \tag{1.47}$$

for $i = 2j - 1$ and

$$R^i(\tau) = -R^{i-1}(0)sin(\varphi_j) + R^i(0)cos(\varphi_j), \tag{1.48}$$

for $i = 2j$.

If φ_j is small,

$$R^i(\tau) = R^i(0) + R^{i+1}(0)\varphi_j + |R|\delta, \tag{1.49}$$

for $i = 2j - 1$ and

$$R^i(\tau) = R^i(0) - R^{i-1}(0)\varphi_j + |R|\delta, \tag{1.50}$$

for $i = 2j$.

where

$$\delta \le |sin\varphi_j - \varphi_j| + |1 - cos\varphi_j| = O(\varphi_j^2). \tag{1.51}$$

Since

$$R^i(\tau) - R^i(0) = \frac{dR^i(t_i)}{dt}\tau \tag{1.52}$$

for some $t_i \in [0, \tau)$,

$$\frac{dR^i(t_i)}{dt} \le \frac{2}{\tau}\|R\|\varphi_j. \tag{1.53}$$

The Taylor expansion of $\frac{d}{dt}R^i$ at $t = 0$ for any $t \in [0, \tau)$ (in particular, for $t = t_i$), looks like

$$\frac{dR^i(0)}{dt} = \frac{dR^i(t)}{dt} - \int_0^t \frac{d^2R^i(s)}{ds^2}ds. \tag{1.54}$$

Recall that

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma}R = \frac{d^2}{ds^2}R - \nabla_{\nabla_{\dot\gamma}\dot\gamma}R. \tag{1.55}$$

Then, obviously,

$$\left(\frac{dR^i(0)}{dt}\right)^2 \le 3\left(\frac{dR^i(t_i)}{dt}\right)^2 + 3\left(\int_0^{t_i} \nabla_{\dot\gamma}\nabla_{\dot\gamma}R^i(s)ds\right)^2 + 3\left(\int_0^{t_i} \nabla_{\nabla_{\dot\gamma}\dot\gamma}R^i(s)ds\right)^2. \tag{1.56}$$

By Cauchy-Schwartz inequality,

$$\left(\int_0^{t_i} \nabla_{\dot\gamma}\nabla_{\dot\gamma}R^i(s)ds\right)^2 \le t_i\int_0^{t_i} (\nabla_{\dot\gamma}\nabla_{\dot\gamma}R^i(s))^2ds. \tag{1.57}$$

And

$$\int_0^{t_i} \nabla_{\nabla_{\dot\gamma}\dot\gamma}R^i(s)ds \le \int_0^{t_i} |\nabla_{\nabla_{\dot\gamma}\dot\gamma}R^i(s)|ds \le n\|\nabla R\| \int_0^{\tau} |\nabla_{\dot\gamma}\dot\gamma(s)|ds \tag{1.58}$$

1.53, 1.56, 1.57 and 1.58 combined give
$(\frac{dR^i(0)}{dt})^2 \le 3t_i\int_0^{t_i}(\nabla_{\dot\gamma}\nabla_{\dot\gamma}R^i(s))^2ds + 3n^2\|\nabla R\|^2(\int_0^{\tau}|\nabla_{\dot\gamma}\dot\gamma(s)|ds)^2 + 3(\frac{2}{\tau}\|R\|\varphi_j)^2$.
Now

$$\begin{aligned}\Sigma_{i=1}^{n^4}\left(\frac{dR^i(0)}{dt}\right)^2 &\le 3\Sigma_{i=1}^{n^4}t_i\int_0^{t_i}\left(\nabla_{\dot\gamma}\nabla_{\dot\gamma}R^i(s)\right)^2ds \\ &+ 3\Sigma_{i=1}^{n^4}n^2\|\nabla R\|^2\left(\int_0^{\tau}|\nabla_{\dot\gamma}\dot\gamma(s)|ds\right)^2 + 3\Sigma_{i=1}^{n^4}\left(\frac{2}{\tau}\|R\|\varphi_j\right)^2,\end{aligned} \tag{1.59}$$

and, again using Cauchy-Schwarz inequality, we get
$|\frac{d}{dt}R(0)| \leq \sqrt{3}\sqrt{\tau}(\int_0^\tau |\nabla_{\dot\gamma}\nabla_{\dot\gamma}R(s)|^2 ds)^{1/2} + \sqrt{3}n^3\|\nabla R\| \int_0^\tau |\nabla_{\dot\gamma}\dot\gamma(s)|ds + \frac{2\sqrt{3}n^2}{\tau}\|R\|\varphi$,
where φ is the maximal angle of rotation of A. Now, supposing that $sup_x|\frac{d}{dt}R(x)|$ is realized at $x = \gamma(0)$, we get

$$\|\frac{dR}{dt}\|_\infty \leq \sqrt{3}\sqrt{\tau}\left(\int_0^\tau |\nabla_{\dot\gamma}\nabla_{\dot\gamma}R^i(s)|^2 ds\right)^{1/2} + \sqrt{3}n^3\|\nabla R\| \int_0^\tau |\nabla_{\dot\gamma}\dot\gamma(s)|ds + \frac{2\sqrt{3}n^2}{\tau}\|R\|_\infty|\varphi| \tag{1.60}$$

and, accordingly,

$$\|\nabla_{\dot\gamma}R\|_\infty \leq 2\sqrt{\tau}\left(\int_0^\tau |\nabla_{\dot\gamma}\nabla_{\dot\gamma}R(s)|^2 ds\right)^{1/2} + \frac{2\sqrt{3}n}{\tau}\|R\|_\infty|\varphi| + 2n^3\|\nabla R\| \cdot \Gamma, \tag{1.61}$$

where $\Gamma = \int_0^\tau |\nabla_{\dot\gamma}\dot\gamma(s)|ds$ is the geodesic curvature of the chosen curve γ.

Let v be a unit vector such that $\|\nabla R\| = \|\nabla_v R\|_\infty$.

Choose the pinching constant of M as in theorem 1.2.1. Then for a chosen direction v there is a geodesic loop $\gamma_1 \in \Gamma_\rho$ in the direction $\frac{\dot\gamma_1(0)}{|\dot\gamma_1(0)|}$, δ-close to τv, where $\delta = \frac{1}{10^n - 1}\rho$, ρ is defined as in 1.1.9, $\tau = \rho$. Therefore

$$\|\nabla R\| \leq \left|\nabla_{\frac{\dot\gamma_1(0)}{|\dot\gamma_1(0)|}} R\right| + \frac{\delta}{\tau}\|\nabla R\|, \tag{1.62}$$

$$\left(1 - \frac{2}{10^n}\right)\|\nabla R\| \leq \left|\nabla_{\frac{\dot\gamma_1(0)}{|\dot\gamma_1(0)|}} R\right|. \tag{1.63}$$

However, a geodesic loop is not a smooth closed curve. "Smoothen" it. On the tangent space to M at $\gamma_1(0)$ consider the vectors $\dot\gamma_1(0)$ and $-\dot\gamma_1(\rho)$. Let $\pi - \alpha$ be the angle between them. On a 2-plane defined by these two vectors construct an inner arc of the circle tangent to the rays generated by these vectors, the points of intersection lying on the rays at a distance ϵ from the origin. The exponential function maps the so obtained arc to the manifold, giving the opportunity to replace the cusp region of the geodesic loop by a smooth one. So we obtain the smooth curve γ. Parametrize γ by the arc length. Since $\gamma_1(t)$ is geodesic on $(0, \epsilon]$,

$$\left|\nabla_{\frac{\dot\gamma_1(0)}{|\dot\gamma_1(0)|}} R\right| \leq \left|\nabla_{\frac{\dot\gamma_1(\epsilon)}{|\dot\gamma_1(0)|}} R\right| + \epsilon\|\nabla^2 R\|. \tag{1.64}$$

Fix $\epsilon = \frac{\rho}{10^n}$. Then direct calculation shows that Γ- the geodesic curvature of γ - is equal to $\alpha + o(\epsilon^2)$, moreover, $\Gamma \leq \|r(\gamma)\|$ - the holonomy angle of the geodesic loop γ. It is not difficult to see also that φ - the maximal rotational angle of $A \in SO(n^4)$ - is not greater than $4\|r(\gamma)\|$ as it is defined in theorem 1.2.1 , so $\varphi \leq 4 \cdot 10^n c_2^2(n)\|R\|d^2(M)$, and hence, from (1.61), there exists a constant $c(n)$ such that

$$\|\nabla R\| \leq c(n) \cdot d(M)(\|\nabla^2 R\| + \|R\|^2). \tag{1.65}$$

□

Remark 1.3.2 *The analogous estimates hold also for an arbitrary tensor on M, in particular, for every derivative of R there exists a constant $c(n,m)$ such that*

$$\|\nabla^m R\| \leq c(n) \cdot d(M)(\|\nabla^{m+1} R\| + \|\nabla^{m-1} R\|\|R\|^2). \tag{1.66}$$

1.4 Convergence of Riemannian Manifolds.

In this section we will give an introduction to the convergence ideas of Riemannian manifolds by developing the weakest convergence concept: Gromov-Hausdorff convergence and stating the important results such as the Convergence theorem of Riemannian geometry and its generalisations. For more information on Gromov-Hausdorff convergence, we refer to Petersen, [9].

Definition 1.4.1 *Let (X,d) be a metric space, $A, B \subset X$; Then a Hausdorff distance between A and B is equal to*
$d_H(A,B) = inf\{\varepsilon > 0 : B \subset S_\varepsilon(A), A \subset S_\varepsilon(B)\}$,
where $S_\varepsilon(A) = \{z : d(z,A) \leq \varepsilon\}$.

Thus $d_H(A,B)$ is small iff every point of A is close to a point of B and vice versa. One can easily see that the Hausdorff distance defines a metric on the closed subsets of X and this collection is compact when X is compact.

If X, Y are metric spaces then an admissible metric on the disjoint union $X \amalg Y$ is a metric that extends the given metrics on X and Y.

Definition 1.4.2 *For two metric spaces X, Y the Gromov-Hausdorff distance can be defined as $d_{G-H}(X,Y) = inf d_H(X,Y)$, where inf is taken over all the admissible metrics on $X \amalg Y$.*

Thus we are trying to define distances between points in X and Y without violating the triangle inequality.

Let $(\mathfrak{M}', d_{G-H})$ define the collection of compact metric spaces. It can be regarded as a metric space on its own right. The next proposition justifies it:

Proposition 1.4.3 *Two complete locally compact metric spaces X, Y are isometric iff*

$$d_{G-H}(X,Y) = 0.$$

Both symmetry and triangle inequality are easily established for d_{G-H}. Thus the set $(\mathfrak{M}, d_{G-H})$ of isometry classes of compact metric spaces endowed with the Gromov-Hausdorff distance is a metric space.

Proposition 1.4.4 $(\mathfrak{M}, d_{G-H})$ *is separable and complete.*

So far, the Gromov-Hausdorff distance has been defined for compact metric spaces. Introduce the analogous notion in general case:

Definition 1.4.5 *The pointed Gromov-Hausdorff distance is defined as*

$$d((X,x),(Y,y)) = inf d_H(X,Y) + d(x,y).$$

Here we take as usual the infimum over all Hausdorff distances and in addition require the selected points to be close. The above results are still true for the modified distance.

We can introduce the Gromov-Hausdorff topology on the set of isometry classes of proper pointed metric spaces $\mathfrak{M}_\star = (X, x, d)$ in the following way:
We say that (X_i, x_i, d_i) converges to (X, x, d) in the pointed Gromov-Hausdorff topology if for any $r > 0$ the closed metric balls $(B(x_i, r), x_i, d_i)_{G-H} \to (B(x, r), x, d)$ with respect to the pointed Gromov-Hausdorff metric.
Suppose, now, we have $f_k : X_k \to Y_k$, $X_k \in X$, $Y_k \to Y$.

Definition 1.4.6 f_k *converges to* $f : X \to Y$ *if for every sequence* $x_k \in X_k$ *converging to* $x \in X$, $f_k(x_k) \to f(x)$.

Note that the convergence of functions preserves such properties as being distance preserving or submetries. In particular,

Proposition 1.4.7 *Suppose* $(X_k, p_k) \to (X, p)$ *in the (pointed) Gromov-Hausdorff topology and* $f_k : X_k \to X_k$ *are isometries with* $d(p_k, f_k(p_k))$ *bounded. Then* f_k *subconverges to* $f : X \to X$, f *is an isometry.*

A sequence of functions as in 1.4.6 is called equicontinuos if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $f_k(B(x_k, \delta)) \subset B(f_k(x_k), \varepsilon)$ for any k, any $x_k \in X_k$.

Lemma 1.4.8 (Ascoli-Arzela) *An equicontinous family* $f_k : X_k \to Y_k$, *where* $X_k \to X$, $Y_k \to Y$ *in the (pointed) Gromov-Hausdorff topology, has a convergent subsequence. If the spaces are not compact we also presume that* f_k *preserves the base point.*

M. Gromov provided also criteria for a collection of (pointed) spaces to be compact. Here is a corollary to this result applied to Riemannian manifolds.

Theorem 1.4.9 (Precompactness Theorem) *For any integer $n \geq 2$, $k \in \mathbb{R}$, $D > 0$ the following classes are precompact:*
(i) the collection of closed Riemannian $n-$manifolds with $Ric_g \geq (n-1)k$, and $diam \leq D$,
(ii) the collection of pointed complete Riemannian $n-$manifolds with
$Ric_g \geq (n-1)k$.

More generally, if we fix a closed manifold M (or a precompact subset $A \subset M$,) then we say that a sequence of functions converges in $C^{m,\alpha}-$topology on A, if they converge in the charts for some fixed finite covering of coordinate patches. Here $C^{m,\alpha}$ is the Hölder space. This definition is independent of the finite covering we choose.
These considerations enable us to speak about convergence of Riemannian metrics on compact subsets of a fixed manifold.

Definition 1.4.10 *A sequence of pointed Riemannian manifolds is said to converge in the pointed $C^{m,\alpha}-$topology $(M_i, p_i, g_i) \to (M, p, g)$ if for any $R > 0$ we can find a domain $\Omega \supset B(p, R)$ and embeddings $\varphi_i : \Omega \to M_i$ for large i such that $B(p_i, R) \subset \varphi_i(\Omega)$ and $\varphi_i^\star g_i \to g_i$ on Ω in the $C^{m,\alpha}-$topology.*

When all manifolds in question are closed, φ_i are diffeomorphisms. This means that for closed manifolds we can speak about unpointed convergence.

Theorem 1.4.11 (The Convergence Theorem of Riemannian Geometry) *Given $n \geq 2$, $\Lambda \in (0, \infty)$ and $i_0 > 0$, the class of Riemannian manifolds with $|Ric| \leq \Lambda$, $inj \geq i_0$ is compact in the pointed $C^{1,\alpha}-$topology for $\alpha \in (0,1)$.*

Theorem 1.4.11 is a combined result due to J. Cheeger, M. Gromov and M. Anderson. Along its lines the following interesting corollary can be proved:

Corollary 1.4.12 *Take $i_0, \Lambda > 0$ and define the class of Riemannian manifolds (M, g) such that $inj(M) \geq i_0$, $\|\nabla^k R_M\| \leq \Lambda$, for $k = 0, 1, ..., m$. This class is compact in the pointed $C^{m+1,\alpha}$ topology for all $\alpha < 1$.*

In the setting of the Ricci flow, the convergence theory permits us to improve the Gromov's compactness theorem and to obtain C^∞ convergence of a sequence of solutions to a smooth limit solution:

Theorem 1.4.13 (Compactness theorem) *([3], section 7)*
Let $(M_i^n, g_i(t), O_i, F_i : i \in N)$ *be a sequence of complete solutions to the Ricci flow (1) existing for* $t \in [0, \omega)$, *where* $\omega \leq +\infty$. *Each solution is marked by an origin* $O_i \in M_i^n$ *and a frame* $F_i = \{e_1^i, ..., e_n^i\}$ *at* O_i *which is orthonormal with respect to* $g_i(0)$. *Suppose that there exists* $K < \infty$ *such that the sectional curvatures of the sequence are uniformly bounded by* K *in the sence that*

$sup_{M_i^n \times (0,\omega)} |K|_{g_i} \leq K$

for all $i \in N$. *Suppose further that there exists* $\delta > 0$ *such that the injectivity radii of the sequence are bounded at* $O_i \in M_i^n$ *and* $t = 0$ *in the sense that*

$inj_{g_i(0)}(O_i) \geq \delta$

for all $i \in N$. *Then there exists a subsequence which converges in the pointed category to a complete solution* $(M_\infty^n, g_\infty(t), O_\infty, F_\infty)$ *of the Ricci flow existing for all* $t \in (0, \omega)$ *with the properties that*

$sup_{M_\infty^n \times (0,\omega)} |K|_{g_\infty} \leq K$

and

$inj_{g_\infty(0)}(O_\infty) \geq \delta.$

The convergence is understood in the sence of the above-defined convergence of Riemannian manifolds in C^m-*topology for every* m.

1.5 Convergence of Almost Flat Riemannian Manifolds.

The results of the previous section applied to almost flat manifolds lead to interesting conclusions on the algebraic structure of the limit space as is illustrated by the following theorem:

Theorem 1.5.1 *For any sequence of numbers* $\varepsilon_k \to 0$ *take a sequence of* ε_k-*flat Riemannian* $n-$*manifolds* (M_k^n, g_k) *such that* $max_{M_k} \|R\|_k = 1$ *and for any* $i \in \mathbb{N} \cup \{0\}$ *there exists a constant* $c(i, n)$ *such that* $\|\nabla^i R\|_k \leq c(i, n)$ *for all* k. *Consider the sequence* $(\tilde{M}_k^n, \tilde{g}_k)$ *of the universal coverings of* (M_k^n, g_k) *with the covering metrics. Then* $(\tilde{M}_k^n, \tilde{g}_k)$ *subconverges w. r. to Gromov-Hausdorff topology to a nilpotent Lie group with a left-invariant metric on it.*

Proof.
To justify the use of 1.4.12, we need the following

Proposition 1.5.2 *The injectivity radius of* $(\tilde{M}_k, \tilde{g}_k)$ *is uniformly bounded from below.*

Proof

Argue by contradiction. Suppose that $i(\tilde{M}_k) \to 0$ as $k \to \infty$. Choose for any manifold the points p_k, q_k so that $i(\tilde{M}_k) = d_k(p_k, q_k)$, i.e. so that the distance between them realizes the corresponding injectivity radius. Then, by the well-known Klingenberg's Lemma (cf, for example, [2]), either there exist minimising geodesics from p_k to q_k such that q_k are conjugate to p_k along them or there exist non-trivial geodesic loops of lengths $2i(\tilde{M}_k)$ centred at p_k and passing through q_k. The first variant would imply that the conjugate radius $conj(\tilde{M}_k) \to 0$ as $k \to \infty$, which is impossible, since the sequence $conj(\tilde{M}_k)$ is bounded from below by π. The existence of non-trivial geodesic loops centred at p_k is in contradiction to the result (cf, for example, [1]) that each sufficiently short geodesic loop represents a distinct homotopy class of M . Recall that our sequence $\tilde{M}_k$ consists of simply connected manifolds, hence each short geodesic loop will be null-homotop.

□

Now, from proposition 1.4.12, $(\tilde{M}^n_k, g_k)$ subconverges to a smooth Riemannian manifold which we will denote by $(\tilde{M}^n_\infty, g_\infty)$. Since the curvatures of the manifolds $(\tilde{M}^n_k, g_k)$ are parametrized so that $max_{M^k}\|R\|_k = 1$, $max_{\tilde{M}_\infty}\|R\| = 1$ and $d(M_k)_{k\to\infty} \to 0$.

Proposition 1.5.3 *$(\tilde{M}^n_\infty, g_\infty)$ is a homogeneous nilmanifold, i.e. there is a nilpotent group of isometries N acting transitively on $\tilde{M}_\infty$.*

Proof.
Since the manifolds in the sequence M_k are almost flat, by Gromov's theorem, for each k there exists $N_k \in \pi(M_k)$ - a nilpotent subgroup of a finite index of the corresponding fundamental group and hence a nilpotent subgroup of the isometry group $Iso(\tilde{M}_k)$ acting of the k-th universal covering. In the Gromov-Hausdorff topology isometries (sub)converge to isometries (proposition 1.4.7) and the nilpotency condition is also preserved. Now take any $p, q \in \tilde{M}$ and such sequences of points $p_k, q_k \in \tilde{M}_k$ that $p_k \to p$, $q_k \to q$ and such a sequence of isometries $i_k \in N_k$ that $i_k(p_k) \to i(p) \in \tilde{M}$ and for any k $d_k(i_k(p_k), q_k) \leq d(M_k)$. Recall that we have parameterized the curvature so that $d(M_k) = d(\tilde{M}_k/\pi(M_k)) \to 0$. Since the subgroups N_k are of finite index, we have a uniform convergence $d(\tilde{M}_k/N_k) \to 0$, and it follows that N acts transitively on $\tilde{M}$.

□

Now, by a result due to Wilson [12], any homogeneous nilmanifold is isometric to a nilpotent Lie group endowed with a left-invariant Riemannian metric.

□

The following observation about the properties of the curvature tensor of almost flat manifolds is one of the applications of the above result.

Theorem 1.5.4 *In any dimension n there exist an $\varepsilon(n)$ and a $c(n)$ such that for any $\varepsilon \leq \varepsilon(n)$ and for any ε-flat manifold M we have*

$$\|R\|_M^{3/2} \leq c(n)\|\nabla R_M\|. \tag{1.67}$$

Proof.
Argue by contradiction. Suppose that there exists an n such that for a sequence $\varepsilon_k \to 0$ there is a sequence of ε_k-flat Riemannian manifolds (M_k) such that for some sequence $c_k \to \infty$ holds:

$$\|R\|_k^{3/2} > c_k\|\nabla R\|_k \tag{1.68}$$

Scale $max_{M_k}\|R\|_k = 1$.
Consider the sequence $(\tilde{M}_k)$ of the universal coverings of (M_k).

Lemma 1.5.5 *The sequence $(\tilde{M}_k)$ subconverges to a symmetric space in the pointed Gromov - Hausdorff topology.*

Proof
By the choice of our scaling, $\|R\|_k = 1, \|\nabla R\|_k \to 0$, so, by corollary 1.4.12, $(\tilde{M}_k)$ subconverges in $C^{2,\alpha}-$topology for all $\alpha < 1$.
Therefore, the limit space $\tilde{M}_\infty$ is a C^2-Riemannian manifold, so the curvature tensor R_∞ can be defined on $\tilde{M}_\infty$. Choose a local coordinate chart of $(\tilde{M}_k, g_k)$ such that

$$x_k : U_r(p_k) \to B_r(0) \tag{1.69}$$

are such that $(B_r(0), x_k^{-1*}g_k)$ converges in the $C^{2,\alpha}$ topology. Put $\bar{g}_k := x_k^{-1*}g_k$. Let $c(t) : [0,1] \to \tilde{M}_\infty$ be a geodesic with respect to the limit metric. Notice that parallel transports Par_c^k along the corresponding c_k with respect to $\bar{g}_k$ converge to the parallel transport of c with respect to $\bar{g}_\infty$. So

$$\|(Par_c^k)^{-1}(R_{c(1)}) - R_{c(0)}\| \leq \|\nabla R\|_k \cdot L(c)_{k\to\infty} \to 0. \tag{1.70}$$

It imples that R_∞ is invariant under parallel transport with respect to $\bar{g}_\infty$.

So the first derivative of the curvature tensor of the limit manifold is defined and equal to zero and so are all the subsequent derivatives. Thus, from proposition 1.4.12, the limit manifold is of the class C^∞ and a symmetric space.

□

Lemma 1.5.6 *$\tilde{M}_\infty$ is a flat Riemannian manifold.*

Proof
According to the de Rham theorem the decomposition of the tangent bundle of $(\tilde{M}_\infty, g_{ij})$ into irreducible components w.r.t the holonomy:

$$T\tilde{M}_\infty = \eta_1 + ... + \eta_k, \tag{1.71}$$

the manifold $(\tilde{M}_\infty, g_{ij})$ itself can be decomposed globally as

$$\tilde{M}_\infty = \mathbb{R}^k \times \tilde{M}_1^c \times ... \times \tilde{M}_p^c \times \tilde{M}_1^n \times ... \times \tilde{M}_q^n, \tag{1.72}$$

with $T\tilde{M}_i = \eta_i$ and where each component of the decomposition is an Einstein space, $\tilde{M}_j^c$ stand for the spaces with positive Ricci curvature, $\tilde{M}_j^n$ - for the spaces with negative Ricci curvature. From the Bonnet-Myers theorem it follows that the first ones are compact. Now, since the manifold $\tilde{M}_\infty$ possesses a nilpotent Lie group structure, so does each of the components in the decomposition. There is a result due to Milnor [8], according to which if a Lie algebra is nilpotent but not commutative, for any left-invariant metric on it there is a direction of strictly negative Ricci curvature and a direction of strictly positive Ricci curvature. Whence follows that each component of the decomposition is a commutative Lie algebra and hence flat.

□

Since we have $C^{2,\alpha}$ convergence, this implies $\|R\|_\infty = 1$, which is a contradiction. The theorem is proved.

□

1.6 The Curvature Tensor along the Ricci Flow.

The Ricci flow can be regarded as a quasilinear parabolic PDE (see, for example, Chow-Knopf ([3])) or a nonlinear heat equation for the metric. Moreover, the intrinsically defined curvatures of a Riemannian metric evolving by the Ricci flow all obey parabolic equations with quadratic non-linearities. The classical fact from PDE's is that parabolic equations possess certain smoothing properties. Therefore, it is reasonable to expect that appropriate bounds on the geometry of the given manifold (M^n, g_0) would induce a priori bounds on the geometry of the unique solution $g(t)$ of the Ricci flow such that $g(0) = g_0$. Moreover, we would expect the geometry to improve, at least for the short time.

Theorem 1.6.1 (Shi) *Let $(M^n, g_{ij}(x))$ be a compact manifold with its curvature tensor R_{ijkl} satisfying $|R_{ijkl}|^2 \le k_0$ on M, $0 < k_0 < \infty$. Then there exists a constant $T(n, k_0) > 0$, s. t. the evolution equation (1)*

$$\frac{\partial(g_{ij}(x,t))}{\partial t} = -2ric_{ij} \tag{1.73}$$

on M,

$$g_{ij}(x, o) = g_{ij}(x) \tag{1.74}$$

for any x on M

has a smooth solution $g_{ij}(x) > 0$ on M for a short time $0 \le t \le T(n, k_0)$ and satisfies the following estimates: for any integer $m \ge 0$ there exist constants $c(m, n)$, depending only on m and n such that

$$\|\nabla^m R_{ijkl}(x,t)\|^2 \le \frac{c(m,n) \cdot k_0}{t^m} \tag{1.75}$$

for any $t \in [0, T]$

Note that the estimates in theorem 1.6.1 follow the natural parabolic scaling in which time behaves as distance squared. Note too that the estimates are stated in a form that deteriorates as $t \to 0$. This is the best one can do without making further assumptions on the initial metric. Recall also, that the lifetime of a maximal solution is bounded below by $\frac{C(n)}{\sqrt{k_0}}$, where $C(n)$ is a universal constant depending only on the dimension.

1.7 The Proof of the Main Result in the Abelian Case.

Now we are ready to prove the main result.
Let (M, g) be an $\varepsilon-$flat Riemannian manifold with abelian fundamental group. Put $max_M |R|_0 = 1$, $\varepsilon_0 := \varepsilon$, $\delta_0 = \frac{\varepsilon^{\frac{1}{8}}}{\|R_0\|}$. For ε_0 sufficiently small, the Ricci flow a priori exists for $t \le \delta_0$ and, from 1.6.1, for $t \le \delta_0$

$$max_M |R|_t \le c(0, n). \tag{1.76}$$

Hence we have the following estimation for $\|Ric\|$ on the same segment:

$$\|Ric\|_t \le n^2 \|R\|_t \le c(0, n) n^2 \|R\|_0 \tag{1.77}$$

for any $t \le \delta_0$.

Lemma 1.7.1 *Define δ_0 as above and let $g(t)$ is the metric on M evolving along the Ricci flow* (1). *Then for any $t \in [0, \delta_0]$,*

$$e^{-2\int_0^t \|ric_s\|_{g(s)} ds} g(0) \le g(t) \le e^{2\int_0^t \|ric_s\|_{g(s)} ds} g(0). \tag{1.78}$$

this lemma is proved in [11].
We have the following estimate on $\int_0^{\delta_0} \|ric_t\|$ along the segment $[0, \delta_0]$:

$$\int_0^{\delta_0} \|ric_t\|_{g(t)} \le n^2 \cdot c(0, n) \cdot \delta_0 \|R_0\| \le n^2 \cdot c(0, n) \cdot \varepsilon_0^{\frac{1}{8}}. \tag{1.79}$$

Which means that, for ε_0 sufficiently small and for any integers $i, j \le n$ and any $t \in [0, \delta_0]$,

$$\frac{1}{2} g_{ij}(0) \le g_{ij}(t) \le 2 g_{ij}(0) : \tag{1.80}$$

From Theorem 1.5.4,

$$\|R\|_{\delta_0}^{3/2} \le c_1(n) \|\nabla R\|_{\delta_0}. \tag{1.81}$$

From Theorem 1.3.1, for any point t of the Ricci flow

$$\|\nabla R\|_t \le c_2(n) \cdot d(M, g_t)(\|\nabla^2 R\|_t + \|R\|_t^2). \tag{1.82}$$

From Theorem 1.6.1,

$$\|\nabla^2 R\|_{\delta_0} \le \frac{c(2, n)}{\delta_0} \|R\|_0. \tag{1.83}$$

From these three inequalities we have finally

$$\|R\|_{\delta_0}^{3/2} \le 2c_1(n) c_2(n) \cdot d(M, g_0) \|R\|_0 \left(c(0, n) \|R\|_0 + \frac{c(2, n)}{\varepsilon_0^{1/8}} \|R\|_0 \right),$$

thus there exists a constant $c(n)$ such that

$$\|R\|_{\delta_0}^{3/2} \le \frac{c(n)^{3/2} \cdot d(M, g_0) \cdot \|R_0\|^2}{\varepsilon_0^{1/8}} \le c(n)^{3/2} \cdot \varepsilon_0^{3/8} \|R\|_0^{3/2},$$

$$\|R\|_{\delta_0} \le c(n) \cdot \varepsilon_0^{\frac{1}{4}} \|R\|_0. \tag{1.84}$$

So, after the time δ_0, the initial curvature diminishes by absolute value by $\varepsilon_0^{1/4}$. We have also the following estimation for the pinching constant ε_1 at $t = \delta_0$:

$$\varepsilon_1 = \|R\|_{\delta_0} \cdot d^2(M, d_{\delta_0}) \leq 4c(n) \cdot \varepsilon_0^{\frac{5}{4}}. \tag{1.85}$$

Define the sequence of points t_i on $\mathbb{R}_{\geq 0}$ such that $t_0 = 0$, $t_{i+1} = t_i + \delta_i$, where $\delta_i := \frac{\varepsilon_i^{\frac{1}{8}}}{\|R_{t_i}\|}$, $\varepsilon_i = d(M, g_{t_i})^2 \cdot \|R_{t_i}\|$.
Note that

$$\delta_i = \frac{(d(M, g_{t_i})^2 \cdot \|R\|_{t_i})^{\frac{1}{8}}}{\|R\|_{t_i}} \geq \frac{d(M, g_{t_{i-1}})^{\frac{1}{4}}}{2^{\frac{1}{4}} c(n)^{\frac{7}{8}} \varepsilon_{i-1}^{\frac{7}{32}} \cdot \|R\|_{t_{i-1}}^{\frac{7}{8}}}$$
$$\geq \frac{\varepsilon_{i-1}^{\frac{1}{8}}}{2^{\frac{1}{4}} c(n)^{\frac{7}{8}} \varepsilon_{i-1}^{\frac{7}{32}} \cdot \|R\|_{t_{i-1}}} = \frac{1}{2^{\frac{1}{4}} c(n)^{\frac{7}{8}} \varepsilon_{i-1}^{\frac{3}{32}} \cdot \|R\|_{t_{i-1}}}, \tag{1.86}$$

It means that

$$\delta_i \geq \frac{\delta_{i-1}}{2^{\frac{1}{4}} c(n)^{\frac{7}{8}} \varepsilon_{i-1}^{\frac{7}{32}} \cdot \|R\|_{t_{i-1}}}, \tag{1.87}$$

hence the segments $[t_i, t_{i+1}]$ cover the whole of $\mathbb{R}_{\geq 0}$.

The last inequality and (1.77) permit us to estimate $\int_0^\infty \|ric\|_{g_s} ds$:

$$\int_0^\infty \|ric\|_{g_s} ds \leq \Sigma_{i=0}^\infty \int_{t_i}^{t_{i+1}} \|ric\|_{g_s} ds$$
$$\leq \quad \Sigma_{i=0}^\infty \delta_i \cdot \|Ric\|_{t_i} \leq n^2 \Sigma_{i=0}^\infty \delta_i \cdot c(0, n) \|R\|_{t_i} \leq n^2 \Sigma_{i=0}^\infty \varepsilon_i^{\frac{1}{8}} \cdot c(0, n). \tag{1.88}$$

From the same considerations as in (1.85) we get that

$$\varepsilon_i \leq 4c(n) \cdot \varepsilon_{i-1}^{\frac{5}{4}}, \tag{1.89}$$

so, the series $\Sigma_{i=0}^\infty \varepsilon_i^{\frac{1}{8}}$ is a geometric progression, therefore, for ε_0 small enough, it converges.
Hence the integral converges on the real line and curvature along the Ricci flow tends to zero.
Note, that the convergence of the metrics along the Ricci flow (1) is of class C^0, since from (1.78) we have

$$\|g_t - g_\infty\|_{g_t} \leq -1 + e^{2\int_t^\infty \|ric_{g_\tau}\| d\tau} \to 0$$

as $t \to \infty$.

The limit manifold (M, g_∞) is a Gromov-Hausdorff limit of the family (M, g_t) of Riemannian manifolds, where g_t evolves along (1). As $t \to \infty$ we have the convergence of the corresponding sectional curvatures to zero: $|K_t|_{t\to\infty} \to 0$. We can also show that the volumes of (M, g_t) remain bounded from below ($vol_t \geq v > 0$) for any t. Indeed, from (1.78), for any $t \in \mathbb{R}_{\geq 0}$ we have

$$g_t \geq e^{-2\int_0^t \|ric_{g_\tau}\| d\tau} g_0 \tag{1.90}$$

Therefore, for ε_0 small enough, $vol(M, g_t) \geq e^{-n\int_0^t \|ric_{g_\tau}\| d\tau} vol(M, g_0)$ for any t.
Now we can use the argument of Cheeger (cf., for example, [9]):

Theorem 1.7.2 (Cheeger) *Given $n \geq 2$ and $v, k \in (0, \infty)$ and a compact n-manifold (M, g) with*
$|K| \leq k$, $volB(p, 1) \geq v$
for all $p \in M$, then $injM \geq i_0$, where i_0 depends only on n, k and v.

So, we can conclude that the injectivity radius of (M, g_t) is uniformly bounded from below and the Convergence Theorem of Riemannian Geometry (1.4.11) can be applied to this family of manifolds. We get that any sequence in this family subconverges in the Gromov-Hausdorff topology to a flat manifold.
Now, since the Gromov-Hausdorff limit is unique up to isometries, we can conclude that (M, g_∞) is isometric to a flat manifold.
So, we have shown that on any $\varepsilon-$flat Riemannian manifold with the abelian fundamental group the Ricci flow (1) converges to a flat metric for ε small enough. Of course, this condition on the fundamental group is also necessary.

Chapter 2

Almost Flat Manifolds with Non-Abelian Fundamental Group.

Theorem B.
For any $n \in \mathbb{N}$ there exists an $\varepsilon(n)$ such that for any $\varepsilon \leq \varepsilon(n)$ an ε-flat Riemannian manifold (M^n, g) has the following properties:
(i) the solution of the Ricci flow (1)

$$\frac{\partial g}{\partial t} = -2ric_g$$

exists on M for all $t \in [0, \infty)$,
(ii) along the flow (1)

$$\lim_{t \to \infty} |K|_{g_t} \cdot d^2(M, g_t) = 0$$

From Chapter 1 we know that the Gromov-Hausdorff limits of sequences of the universal covers of almost flat manifolds are nilmanifolds. This gives a motivation to understand first the Ricci flow on nilpotent Lie groups.
In this direction important results were obtained by J. Heber [6] and J. Lauret [7]. They described the behavior of so called nilsolitons - special solutions of the Ricci flow on nilpotent Lie groups - from the geometric and algebraic viewpoints.
Ricci soliton on a nilpotent group N is a left invariant metric g on N such that

$$Ric_g = c \cdot id + D, \tag{2.1}$$

where $c \in \mathbb{R}$, D is some derivation of the Lie algebra η of N and Ric_g denotes the Ricci operator of (N, g). Lauret ([7]) proved, that such a metric on the given N is unique up to isometry and scaling. The following characterization is also given: (N, g) is a Ricci soliton if and only if (N, g) admits a standard solvable extension whose corresponding standard solvmanifold $(S, \tilde{g})$ is Einstein. Note that not all the nilpotent Lie groups carry the Ricci soliton metrics.
In a certain sense, Ricci soliton metrics are the most "privileged" left invariant metrics on nilpotent Lie groups. From [8] we know that only compact Lie groups and some solvable Lie groups are known to admit Einstein left invariant metrics and other groups as nilpotent Lie groups do not admit any. The weaker condition 2.1 can be rewritten in the following way:

$$\delta(Ric_g) = c[\cdot, \cdot,] \tag{2.2}$$

where $\delta : C^1(\eta, \eta) = End(\eta) \to C^2(\eta, \eta) = \Lambda^2\eta^\star \otimes \eta$ is a coboundary operator and $[\cdot, \cdot]$ denotes the Lie bracket of η. Here we consider Chevalley cohomology, i.e. Lie algebra cohomology of η relative to the adjoint representation. Note that (2.2) can be viewed as the Einstein equation $Ric_g = c \cdot id$, but in the second cohomology group $C^2(\eta, \eta)$.
Example(see [3], vol.2)
Let N denote the 3-dimensional Heisenberg group (see Example 2, Introduction). N is diffeomorphic to $\mathbb{R}^3$. Endow $\mathbb{R}^3$ with the standard coordinates (x_1, x_2, x_3) and define the frame field

$$F_1 = 2\frac{\partial}{\partial x_1}, \qquad F_2 = 2(\frac{\partial}{\partial x_2} - x_1\frac{\partial}{\partial x_3}), \qquad F_3 = 2\frac{\partial}{\partial x_3}. \tag{2.3}$$

It is straightforward to check that (F_1, F_2, F_3) is the Milnor frame for N (see [8]). The connection 1-forms may be displayed as

$$\begin{pmatrix} \nabla_{F_1}F_1 & \nabla_{F_1}F_2 & \nabla_{F_1}F_3 \\ \nabla_{F_2}F_1 & \nabla_{F_2}F_2 & \nabla_{F_2}F_3 \\ \nabla_{F_3}F_1 & \nabla_{F_3}F_2 & \nabla_{F_3}F_3 \end{pmatrix} = \begin{pmatrix} 0 & -F_3 & F_2 \\ F_3 & 0 & -F_1 \\ F_2 & -F_1 & 0 \end{pmatrix}.$$

Using the dual field

$$\omega^1 = \frac{1}{2}dx_1, \qquad \omega^2 = \frac{1}{2}dx_2, \qquad \omega^3 = \frac{1}{2}(x_1dx_2 + dx_3),$$

define the left-invariant metric on N by

$$g = 4(\omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 + \omega^3 \otimes \omega^3).$$

Recalling the standard formula

$$< R(X,Y)Y, X >= \frac{1}{4}\|(ad_X)^*(Y) + (ad_Y)^*(X)\|^2 - < (ad_X)^*(X), (ad_Y)^*(Y) >$$
$$-\frac{3}{4}\|[X,Y]\|^2 - \frac{1}{2} < [[X,Y],Y], X > -\frac{1}{2} < [[Y,X],X], Y >,$$

it is straightforward to compute that

$$ric_g = -2\omega^1 \otimes \omega^1 - 2\omega^2 \otimes \omega^2 + 2\omega^3 \otimes \omega^3.$$

Define the vector field

$$X = -\frac{1}{2}x_1 F_1 - \frac{1}{2}x_2 F_2 - (\frac{1}{2}x_1 x_2 + x_3)F_3. \tag{2.4}$$

It is then easy to see that the coordinates $(\nabla_j X^i)$ of $= \nabla_j X^i \omega^j F_i$ correspond to the matrix

$$\begin{pmatrix} -1 & -(\frac{1}{2}x_1x_2 + x_3) & -\frac{1}{2}x_2 \\ \frac{1}{2}x_1x_2 + x_3 & -1 & \frac{1}{2}x_1 \\ \frac{1}{2}x_2 & -\frac{1}{2}x_1 & -2 \end{pmatrix}.$$

Then we have directly that

$$-2ric_g = L_X g + 3g. \tag{2.5}$$

Thus (N, g) is a Ricci soliton.

□

To obtain the necessary estimates it often makes sense to consider instead of (1) the normalized Ricci flow (3):

$$\frac{\partial g}{\partial t} = -2ric_g - 2\|ric_g\|_g^2 g,$$

where $\|ric_g\|_g^2 = trRic_g^2$ and we normalize the scalar curvature $sc(g_0) = -1$.
The normalized Ricci flow (3) differs from the unnormalized one (1) only by parametrization and scaling; moreover, the scalar curvature is preserved along (3) (cf. 2.1.4). Therefore, since on a nilpotent Lie group (N, μ), where μ denotes the algebraic structure of N, we have that

$$sc(\mu) = -\frac{1}{4}\|\mu\|^2,$$

along the flow (3) the curvature on a nilmanifold remains bounded in norm.
In section 1 we establish an important property of the Ricci soliton metrics on nilpotent Lie groups: these metrics strongly contract along the normalized Ricci flow (3), i. e. there exists a $\lambda > 0$ such that for any $t \geq 0$, $h > 0$, for any soliton metric holds: $g(t+h) < e^{-\lambda h} g(t)$, whereby g is considered as a symmetric operator. Note that the constant λ is universal for all solitons.
In section 2 we prove the equivalence between the Ricci flow of the metric and the corresponding flow of the algebraic structure in the space of all nilpotent Lie groups. We show that the Ricci flow on a nilpotent Lie group is the gradient flow of a natural functional defined on a vector space which contains all the homogeneous manifolds of a given dimension as a real algebraic set. From [7] it is known that the soliton metrics are exactly the critical points of the same functional.
Finally, we get that for t sufficiently large, on a nilmanifold holds:

$$\|R\|_t g(t) \leq \frac{1}{2}\|R\|_0 g(0)$$

along (3), and, by invariance under rescalings, the same is true also along (1).
The combination of these results leads to the conclusion that on any nilmanifold the Ricci flow (1) exists for all $t \in \mathbb{R}_{\geq 0}$ and is non-contracting only for a finite time. Since nilmanifolds are limits for the almost flat manifolds, the same holds for any (M, g) whose pinching constant is sufficiently small. The standard analytic argument similar to one applied in section 5 of Chapter 1 concludes the proof.

2.1 Ricci Solitons on Nilpotent Lie groups.

Definition 2.1.1 *A steady Ricci soliton is a special solution of the Ricci flow on a Riemannian manifold M which moves along the equation by diffeomorphisms, that is, where the metric $g(t)$ is the pull-back of the initial metric $g(0)$ by a one-parameter group of diffeomorphisms.*

In general,a Ricci soliton is a special solution of the Ricci flow which moves along the equation by a diffeomorphism and also shrinks or expands by a factor at the same time: if φ_t *is a one-parameter group of diffeomorphisms generated by some vector field* $X \in \chi(M)$, $g(t) = e^{ct}\varphi_t^\star g$, $g(0) = g_0$, *for some* $c \in \mathbb{R}$.
The Ricci soliton (M, g) *is called expanding if* $c > 0$ *and shrinking if* $c < 0$.

Proposition 2.1.2 *([12], Thm. 2,4)*
g(t) is a homothetic Ricci soliton of the flow (1) if and only if
$ric_g = cg + L_X g$,
where L_X *is the usual Lie derivative.*

Let N be a simply connected nilpotent Lie group and g a left invariant metric on N. As a differentiable manifold, N is the Euclidean space.
For N holds the following precision of 2.1.2 due to J. Lauret ([7]):

Proposition 2.1.3 *([7], 1.1)*
A left-invariant metric g on a nilpotent Lie group N is a homothetic Ricci soliton if and only if

$$Ric_g = c \cdot id + D, \tag{2.6}$$

for some $c \in \mathbb{R}$, $D \in Der(\eta)$,
where η *is the Lie algebra of* N.

The essential point of this result is that in case when the manifold N possesses the structure of a nilpotent Lie group we can consider the one-parameter group of diffeomorphisms φ_t in the definition of the homothetic Ricci soliton as a one-parameter group of automorphisms of N. If $\varphi_t = \exp(-\frac{t}{2}D)$, $D \in Der(\eta)$ then $\frac{\partial}{\partial t}|_0 \varphi_t^\star g = g(-D\cdot, \cdot)$.

Proposition 2.1.4 *(i)* g_0 *is a Ricci soliton of the Ricci flow (1) if and only if* g_0 *is a steady Ricci soliton of the flow:*

$$\frac{\partial g}{\partial t} = -2ric_g - 2\|ric_g\|_g^2 g,$$

where $\|ric_g\|_g^2 = trRic_g^2$ *and we normalize the scalar curvature* $sc(g_0) = -1$.
(ii) Under the flow (3) the (constant) scalar curvature of the solution metric $g(t)$ *remains constant in time. Furthermore, equation (3) differs from the Ricci flow (1) only by a change of scale in space by a function of t and a change of in time.*

Proof
(ii) It is well known (cf., for example, [4], 7.5) that the scalar curvature evolves along the Ricci flow defined by the equation (1) as follows:

$$\frac{\partial}{\partial t} sc(g) = \Delta sc(g) + 2\|ric_g\|_g^2. \tag{2.7}$$

Hence, along (3) we have

$$\frac{\partial}{\partial t} sc(g) = \Delta sc(g). \tag{2.8}$$

Now, on any homogeneous manifold $\Delta sc(g) = 0$, thus along the flow (3) the scalar curvature $sc(g)$ remains constant.
(i) Let g_0 be a Ricci soliton of the Ricci flow (1), $g(t) = e^{ct}\varphi_t^\star g$, for some family of diffeomorphisms φ_t. Then $Ric_{g(0)} = c \cdot id + D$, $Ric_{g(t)} = ce^{ct} \cdot id + e^{ct}D$, for any $t \in \mathbb{R}_{\geq 0}$. Since $sc(g(0)) = sc(g(t))$, $tr(Ric_{g(0)}) = tr(Ric_{g(t)})$ for any $t \in \mathbb{R}_{\geq 0}$, but it is possible only when $c = 0$. Hence, (N, g_t) is isomorphic to (N, g_0).

□

Lauret [7] gives also another, more algebraic, characterization of Ricci soliton metrics. First some preliminaries.
A *solvmanifold* is a solvable Lie group endowed with a left invariant Riemannian metric. The following proposition is due to Heber ([6], 4.4):

Proposition 2.1.5 *Let $(s = \alpha \oplus \eta)$ be a metric Lie algebra such that α is abelian, η is nilpotent, α orthogonal to η and all operators adA, $A \in \alpha - 0$ are symmetric and non-zero.$\oplus$ is understood as a semi-direct sum, i. e. $[\eta, \eta] \subset \eta$, $[\alpha, \eta] \subset \eta$, $[\alpha, \alpha] = 0$. Then the Ricci tensor $ric_{\tilde{g}}$ of the corresponding solvmanifold $(S, \tilde{g})$ is given by:*
(i) $ric_{\tilde{g}}(A, B) = -tr(adAadB)$, $A, B \in \alpha$,
(ii) $ric_{\tilde{g}}(A, X) = 0$, $A \in \alpha$, $X \in \eta$,
(iii) $ric_{\tilde{g}}(X, Y) = -\tilde{g}(adH_{\tilde{g}}X, Y) + ric_g(X, Y)$, $X, Y \in \eta$,
where ric_g denotes the Ricci tensor of $(N, g = \tilde{g}|_{\eta\times\eta})$ and $H_{\tilde{g}} \in \alpha$ is defined by $g(H_{\tilde{g}}, X) = tr(adX)$ for any $X \in s$.

Definition 2.1.6 *A metric solvable extension of (η, g) is a metric solvable Lie algebra of the form $(s = \alpha \oplus \eta)$ such that*
$[s, s]_s = \eta$, $[\cdot, \cdot]_s|_{\eta\times\eta} = [\cdot, \cdot]_\eta$, $\tilde{g}|_{\eta\times\eta} = g$,
where $[\cdot, \cdot]_s$ and $[\cdot, \cdot]_\eta$ denote the Lie brackets of s and η respectively.

Lauret provides also an algebraic condition for the metric to be a Ricci soliton ([7], 3.7):

Proposition 2.1.7 *A left invariant metric g on a nilpotent Lie group N is a Ricci soliton if and only if (η, g) admits a metric solvable extension $(s = \alpha \oplus \eta)$ with α abelian, whose corresponding solvmanifold $(S, \tilde{g})$ is Einstein.*

Let $Sym(\eta)$ be the vector space of symmetric real-valued bilinear forms on η and $P \subset Sym(\eta)-$ the open convex cone of positive definite scalar products on η, which can be identified with the left-invariant metrics on N. Every $g \in P$ induces a natural inner product $< \cdot, \cdot >_g$ on $Sym(\eta)$ given by $< a, b >_g = trA_aB_b$, where $a(X,Y) = g(A_aX, Y)$. So P can be endowed with the Riemannian metric $< \cdot, \cdot >$ given by $< \cdot, \cdot >_g$ on the tangent space $T_gP = Sym(\eta)$ for any $g \in P$.
A natural left action of $Gl(\eta)$ on P is given by pullback, that is

$$\varphi^\star \cdot g = g(\varphi \cdot, \varphi \cdot), \qquad \forall \varphi \in Gl(\eta), g \in P. \tag{2.9}$$

This action is transitive and isometric with respect to $< \cdot, \cdot >$ and for any fixed $g_0 \in P$, the isotropy group is $Gl(\eta)_{g_0} = O(\eta, g_0)$ and thus $(P, < \cdot, \cdot >)$ can be identified with the symmetric space $(Gl(\eta)/O(\eta, g_0), < \cdot, \cdot >)$. The automorphism group $Aut(\eta)$ is a closed subgroup of $Gl(\eta)$ and we have that (η, g) is isometric to (η, g') if and only if $g' = \varphi^\star \cdot g$ for some $\varphi \in Aut(\eta)$, where $g, g' \in P$ and the action is as described above.
The Ricci curvature tensor ric_g and the Ricci operator Ric_g of (N, g) are given by

$$\begin{aligned} &ric_g(X,Y) = g(Ric_gX, Y) = \\ &- \frac{1}{2}\sum_{ij} g([X, X_i], X_j)g([Y, X_i], X_j) + \frac{1}{4}\sum_{ij} g([X_i, X_j], X)g([X_i, X_j], Y) \end{aligned} \tag{2.11}$$

for all $X, Y \in \eta$, where $[\cdot, \cdot]$ denotes the Lie bracket of η and $X_1, ..., X_n$ is any orthonormal basis of (η, g). The scalar curvature $sc(g) = trRic_g$ of (N, g) equals

$$sc(g) = -\frac{1}{4}\sum_{ijk} g([X_j, X_j], X_k)^2. \tag{2.12}$$

Lemma 2.1.8 *([6], 3.3)*
The functional $sc : P \to \mathbb{R}_{\leq 0}$, $sc(g)$ is the scalar curvature of (N, g), is real analytic and satisfies the following properties:
(i) $grad(sc)_g = -Ric_g$ for any $g \in P$.
(ii) The functional sc is constant on the geodesic of the form $\alpha(t) = e^{tD}g$, where D is some symmetric derivation of (η, g).

These considerations and proposition 2.1.4 permit us to give the following definition:

Definition 2.1.9 *A left invariant metric g on a Nilpotent Lie group with $Ric_g \subset \mathbb{R} \cdot id \oplus Der(\eta)$ is said to be a Ricci nilsoliton.*

This establishes a relationship between a geometric object (the Ricci operator) and an algebraic object (the Lie algebra $Der(\eta)$). This condition is invariant under isometry and scaling. Indeed, if $b > 0$, $\varphi \in Aut(\eta)$ and $g' = b\varphi^{\star} \cdot g$ then $Ric_{g'} = b^2\varphi^{-1}Ric_g\varphi$. Thus $Ric_{g'} \subset \mathbb{R} \cdot id \oplus Der(\eta)$ if and only if $Ric_g \subset \mathbb{R} \cdot id \oplus Der(\eta)$, since $\varphi^{-1}Der(\eta)\varphi = Der(\eta)$ for any $\varphi \in Aut(\eta)$.

Theorem 2.1.10 *([6], 4.14)*
Take the normal operator

$$adH_{|\eta} : \eta \to \eta \tag{2.13}$$

from (2.1.5). For some positive multiple $H_0 = \xi H$, the normal operator $ad_{H_{0|\eta}}$ has as eigenvalues positive integers with no common divisor.

Let $\lambda_1 < ... < \lambda_m$ be the eigenvalues of ad_{H_0}, d_i, $i = 1, ..., m$ be the corresponding multiplicities. Then we call the tuple

$$(\lambda_1 < ... < \lambda_m; d_1, ..., d_m)$$

the *eigenvalue type* of the standard Einstein solvmanifold (2.1.7).

Corollary 2.1.11 *([6], 4.11)*
In every dimension, only finitely many eigenvaue types can occur.

We are now ready to establish the key property of the nilsoliton metrics which serves as a starting point for the proof of Theorem B.

Theorem 2.1.12 *Every nilsoliton strongly contracts the metric. In other words, there exists a constant $\lambda > 0$, such that, if (N, g) is a Ricci nilsoliton, then along the flow (3), for any $t \geq 0$, $h > 0$, holds $g(h + t) < e^{-\lambda h}g(t)$, where g is considered as a symmetric operator on η.*

Proof
Since (η, g) is a Ricci nilsoliton, from 2.1.7 it follows that it admits a metric solvable extension $(s = \alpha \oplus \eta)$ whose corresponding solvmanifold $(S, \tilde{g})$ is Einstein. According to [6] we can suppose that adH is symmetric for any $H \in \alpha$. Thus we can apply 2.1.5, obtaining that $Ric_{\tilde{g}}\eta \subset \eta$ and furthermore

$$Ric_{\tilde{g}}|_{\eta} = -ad(H_{\tilde{g}})|_{\eta} + Ric_g, \tag{2.14}$$

where $Ric_{\tilde{g}}$ denotes the Ricci operator of the solvamanifold $(S, \tilde{g})$. Since $(S, \tilde{g})$ is Einstein, we have $Ric_g = c \cdot id + ad(H_{\tilde{g}})|_{\eta} \in R \cdot id \oplus Der(\eta)$. So for some $D_g \in Der(\eta)$ $ad(H_{\tilde{g}}) = D_g$. Further on, Ric_g is orthogonal to $\mathfrak{P}_g := Der(\eta) \cap Sym(\eta, g)$, since, from 2.1.8, for any $A \in \mathfrak{P}_g$, $< A, grad(sc)_g >= \frac{d}{dt} sc(e^{tA}g) = 0$.
Multiply the equation $Ric_g = c \cdot id + D_g$ by D_g. We obtain $c = -trD_g^2/trD_g = trRic_g^2$, if $sc(g) = trRic_g = -1$.
Now, using propositions 2.1.3, 2.1.4 we can show that the flow (3) can be considered as a one-parameter group of automorphisms on N:

$$exp(-\frac{t}{2}D) : N \to N, \tag{2.15}$$

and for any $v, w \in \eta$

$$g_t(v, w) = g_0(exp(-\frac{t}{2}D)v, exp(-\frac{t}{2}D)w). \tag{2.16}$$

From 2.1.10 all the eigenvalues of the operator D are positive multiples of a tuple of positive integers. Moreover, from 2.1.11, in given dimension the number of such tuples is finite. We want to prove that there exists an apriori bound from below for the eigenvalues of D.
Suppose, there exists a sequence of Ricci nilsolitons (N_k, g_k) such that the minimal egenvalues $\lambda_k^{(min)}$ of the corresponding normal operators D_k tend to zero as $k \to \infty$. From 2.1.11 it follows that *all* the eigenvalues of D_k tend to zero as $k \to \infty$. Hence $\frac{trD_k^2}{trD_k} \to 0$ as $k \to \infty$. Recall that $Ric_k = c_k \cdot id + D_k$ and $c_k = -trD_k^2/trD_k = trRic_k^2$. Therefore, $Ric_{k \to \infty} \to 0$, which is a contradiction, since along the flow (3), $sc = -1$.
□

2.2 Ricci flow on Nilmanifolds.

Now let us adapt another approach. Consider an n-dimensional inner product space $(\eta, < \cdot, \cdot >)$. The set $\mathfrak{N}$ of nilpotent algebra brackets on η can be viewed as a subset of $\Lambda^2\eta^\star \otimes \eta$, the space of all bilinear anti-symmetric forms on $\eta \times \eta$ to η. An element $\mu \in \Lambda^2\eta^\star \otimes \eta$ lies in $\mathfrak{N}$ if and only if μ satisfies the Jacobi identity and the nilpotency conditions. In other words, $\mu \in \mathfrak{N}$ if and only if certain polynomials vanish at μ. Hence $\mathfrak{N}$ constitutes a real algebraic set in the Euclidean space $\Lambda^2\eta^\star \otimes \eta$.
There is a natural action of $Gl(\eta)$ on $\Lambda^2\eta^\star \otimes \eta$ given by

$$\varphi^\star \cdot \mu = \varphi^{-1}\mu(\varphi\cdot, \varphi\cdot), \qquad \forall \varphi \in Gl(\eta), \mu \in \mathfrak{N}. \tag{2.17}$$

Isomorphic Lie algebra brackets lie in the same $Gl(\eta)-$ orbit. $\mathfrak{N}$ is $Gl(\eta)-$invariant. Using the fixed inner product $< \cdot, \cdot >$ on η, we regard each element $\mu \in \mathfrak{N}$ as a homogeneous nilmanifold, denoted by $(N_\mu, < \cdot, \cdot >)$, where N_μ is a simply connected Lie group with Lie algebra (η, μ) endowed with the left-invariant metric determined by $< \cdot, \cdot >$. If $\varphi \in Gl(\eta)$, then $(N_{\varphi^\star \cdot \mu}, < \cdot, \cdot >)$ is isometric to $(N_\mu, \varphi^\star \cdot < \cdot, \cdot >)$ via φ and so $Gl(\eta)\mu$ contains all homogeneous nilmanifolds with the underlying Lie group isomorphic to N_μ. Thus $\mathfrak{N}$ is in correspondence with the set of all $n-$ dimensional homogeneous manifolds. $\mu, \lambda \in \mathfrak{N}$ are isometric if and only if $\lambda = \varphi^\star \cdot \mu$ for some $\varphi \in O(\eta)$, where $O(\eta)$ denotes the orthogonal group $O(\eta, < \cdot, \cdot >)$.
The inner product on η determines naturally an inner product on $\Lambda^2\eta^\star \otimes \eta$, which is given by

$$\langle \mu, \lambda \rangle = \Sigma_{ijk} \langle \mu(X_i, X_j), X_k \rangle \langle \lambda(X_i, X_j), X_k \rangle, \tag{2.18}$$

where $X_1, ..., X_n$ is any orthonormal basis of $(\eta, < \cdot, \cdot >)$. For any $\mu \in \mathfrak{N}$ the scalar curvature of $(N_\mu, < \cdot, \cdot >)$ is given by $sc(\mu) = -\frac{1}{4}\|\mu\|^2$. The Ricci curvature tensor and the Ricci curvature operator $Ric_\mu : \eta \to \eta$ for any $\mu \in \Lambda^2\eta^\star \otimes \eta$ is given by

$$\begin{aligned} ric_\mu(X, Y) &= \langle Ric_\mu X, Y \rangle \\ = & -\frac{1}{2}\sum_{ij} \langle \mu(X, X_i), X_j \rangle \langle \mu(Y, X_i), X_j \rangle + \frac{1}{4}\sum_{ij} \langle \mu(X_i, X_j), X \rangle \langle \mu(X_i, X_j), Y \rangle \end{aligned}$$

for any $X, Y \in \eta$.

Fix some $\mu_0 \in \mathfrak{N}$. Define the family u_t of endomorphisms on η such that $u_0 = id$ and u_t evolves by the equation

$$u_t' = Ric_{g(t)} \cdot u_t. \tag{2.19}$$

where g_{ij} is the corresponding left-invariant metric. Note that the evolution of g_{ij} along the family u_t^{-1} is the Ricci flow (1):
$\frac{\partial}{\partial t} g = -2ric_g$.
Indeed, u_t^{-1} satisfies the equation

$$(u_t^{-1})' = -u_t^{-1} \cdot Ric_g. \tag{2.20}$$

and

$$\begin{aligned} &\frac{d}{dt}[(u_t^{-1})^\star \cdot g_0(\cdot, \cdot)] = \frac{d}{dt} g_0(u_t^{-1} \cdot, u_t^{-1} \cdot) = \\ - \quad & g_0(u_t^{-1} Ric_g \cdot, u_t^{-1} \cdot) - g_0(u_t^{-1} \cdot, u_t^{-1} Ric_g \cdot) = -g_t(2Ric_g \cdot, \cdot) = -2ric_g. \end{aligned}$$

In the last equality but one we used the fact that $\frac{d}{dt}u^{-1}$ is a symmetric operator. As can be easily seen, with respect to this family of endomorphisms the pullback metric remains constant in time: $u_t^* \cdot g_t = g_0$.

Proposition 2.2.1 *Evolve an arbitrary algebraic structure μ_0 along the family of endomorphisms u_t defined above. Then*

$$\frac{d\mu}{dt} = \delta_\mu Ric_\mu, \tag{2.21}$$

where $\delta : C^1(\eta,\eta) = End(\eta) \to C^2(\eta,\eta) = \Lambda^2\eta^\star \otimes \eta$ is the coboundary operator in the sence of the Chevalley cohomology.

Proof
For any $X, X_i, X_j \in \eta$ such that X_i, X_j are orthonormal with respect to g_0

$$g_0(\mu_t(X, X_i), X_j) = g_0(u_t^{-1}\mu_0(u_tX, u_tX_i), X_j) = g_t(\mu_0(u_tX, u_tX_i), u_tX_j). \tag{2.22}$$

This implies that
$ric_{\mu_t}(X, Y) = ric_{g_t}(u_tX, u_tY)$ and the corresponding curvature operators are conjugate.
Now we can compute $\frac{d\mu_{ijk}}{dt}$:

$$\begin{aligned}
\frac{d\mu_{ijk}}{dt} &= \frac{d}{dt} < u_t^{-1}\mu_0(u_te_i, u_te_j), e_k > = < \frac{d}{dt}u_t^{-1}\mu_0(u_te_i, u_te_j), e_k > \\
&\quad + < u_t^{-1}\mu_0(\frac{d}{dt}u_te_i, u_te_j), e_k > + < u_t^{-1}\mu_0(u_te_i, \frac{d}{dt}u_te_j), e_k > \\
&= - < u_t^{-1}Ric_g\mu_0(u_te_i, u_te_j), e_k > \\
&\quad + < u_t^{-1}\mu_0(Ric_gu_t(e_i), u_t(e_j)) + < u_t^{-1}\mu_0(u_te_i, Ric_gu_t(e_j)), e_k > \\
&= - < Ric_\mu\mu(e_i, e_j), e_k > + < \mu(Ric_\mu e_i, e_j), e_k > \\
&\quad + < \mu(e_i, Ric_\mu e_j), e_k > \\
&= \delta_\mu Ric_\mu
\end{aligned}$$

□

Together with 2.21 we have the following obvious proposition:

Proposition 2.2.2 *Fix the initial data g_0 and μ_0. Let g evolve along the flow (1) and μ evolve along the flow (2.21). This two flows are equivalent in the sence that their solutions exist on the same time interval, and for any t on the interval of existence $ric(g_t, \mu_0)(\cdot,\cdot) = ric(g_0, \mu_t)(u_t^{-1}\cdot, u_t^{-1}\cdot)$, with u_t defined as above.*

Define a functional $F : \Lambda^2\eta^\star \otimes \eta \to \mathbb{R}$ homogeneous in curvature. Remark that for any $t > 0$, $(N_{t\mu}, < \cdot, \cdot >)$ and $(N_\mu, < \cdot, \cdot >)$ are homothetic and any such functional F will satisfy $F(t\mu) = c(t)F(\mu)$ for some $c(t) > 0$.

Lemma 2.2.3 ([7], 4.1) *If $F : \Lambda^2\eta^\star \otimes \eta \to \mathbb{R}$ is defined by $F(\mu) = trRic_\mu^2$, then*

$$grad(F)_\mu = -\delta_\mu(Ric_\mu). \tag{2.23}$$

From (2.2.1) and (2.2.3) immediately follows the next theorem:

Theorem 2.2.4 *On a nilpotent Lie group Ricci flow (1) is a gradient flow.*

Recall that together with the Ricci flow (1) we considered the normalized Ricci flow (3):

$$\frac{\partial g}{\partial t} = -2ric_g - 2\|ric_g\|_g^2 g,$$

where $\|ric_g\|_g^2 = trRic_g^2$ and we normalize the scalar curvature $sc(g_0) = -1$.
Applying the same procedure as in propositions 2.2.1, 2.2.3 to the flow (3) we get the following modification of theorem 2.2.4:

Theorem 2.2.5 *On a nilpotent Lie group the normalized Ricci flow (3) is a gradient flow of the functional $F(\mu) = trRic_\mu^2$ restricted to the sphere of $\Lambda^2\eta^\star \otimes \eta$,*

$$S = \{\mu \in \Lambda^2\eta^\star \otimes \eta : \|\mu\|^2 = 4\}. \tag{2.24}$$

Proof.
It is easy to see that the corresponding flow for the algebraic structure is given by

$$\frac{d\mu}{dt} = \delta_\mu Ric_\mu + F \cdot \mu, \tag{2.25}$$

which is exactly the flow (2.21) restricted to the sphere S, since in the radial direction $\mu = te$, where e is the unit vector in this direction, $F \cdot \mu = t^5 F(e)e = \frac{\partial}{\partial t}F(\mu) = 4t^3F(e)e$, if $t^2 = 4$. Moreover, the algebraic normalization to the sphere S coincides on $\mathfrak{N}$ with the following kind of geometric normalization:

$$S \cap \mathfrak{N} = \{\mu \in \mathfrak{N} : sc(\mu) = -1\}. \tag{2.26}$$

□

Theorem 2.2.6 ([7], 4.2) *For a structure μ on $S \cap \mathfrak{N} \in \Lambda^2\eta^\star \otimes \eta$ the following properties are equivalent:*
(i) $(N_\mu, < \cdot, \cdot >)$ is a Ricci soliton,
(ii) μ is a critical point of F restricted to $S = \mu \in \Lambda\eta^\star \otimes \eta : \|\mu\| = 1$,
(iii) μ is a critical point of F restricted to $S \cap Gl(\eta) \cdot \mu$, where $Gl(\eta) \cdot \mu$ can be identified with the set of all left-invariant metrics on N_μ.

Now let U be a neighbourhood of all solitons in $\mathfrak{N}' := S \cap \mathfrak{N}$ such that on any manifold in U the corresponding left-invariant metric contracts along the flow (3) in the sence of 2.1. More precisely, there exists a $\lambda > 0$ such that for any $(N_\mu, g) \in U$, as long as $g(t) \in U$, $\forall t > 0, \forall h > 0$, holds: $g(t+h) < e^{-\frac{\lambda}{2}h} g(t)$ along the flow (3). Such a neighbourhood exists, as follows from the theory of the continuous dependence of solutions of ODE's on the initial data.
Thus theorem 2.2.5 and proposition 2.2.6 permit us to obtain the following important result:

Proposition 2.2.7 *Choose a neighbourhood of the critical set as above. There exists a constant C such that for any nilmanifold $(N_\mu, g) \in \mathfrak{N}'$, along the normalized Ricci flow flow (3) the measure of the set $I := \{t : (N, g(t)) \notin U\}$ is less or equal then C.*

Proof
On the algebraic variety $S \cap \mathfrak{N} = \{\mu \in \mathfrak{N} : sc(\mu) = -1\}$ the flow (3) is a gradient flow of the functional $F(\mu) = trRic^2_\mu$, whose critical points are Ricci solitons. Thus there exists a constant $\delta > 0$ such that $|gradF| \geq \delta$ on $\mathfrak{N}' \backslash U$. Along any gradient line $\mu(t)$, by definition,

$$\frac{dF}{dt} =< gradF, \dot{\mu} > \tag{2.27}$$

and $\dot{\mu} = -gradF$. Then for any $t, \Delta t > 0$,

$$F(t) - F(t + \Delta t) \geq \delta^2 \Delta t. \tag{2.28}$$

As a continuous function, F is bounded on the compact set $S \cap \mathfrak{N}$, hence the length of each gradient line outside U is bounded below by $\frac{sup_{\mathfrak{N}\backslash U} F(\mu)}{\delta^2}$. It means that each metric along (3) remains outside U only for a finite time. Moreover, the constant C does not depend on the metric.

□

Recall that up to now we considered nilmanifolds with scalar curvature normalised to -1, or, equivalently, with the norm of the algebraic structure constants normalized to 2. The sectional curvature function K can be expressed as a quadratic function of the structure constants μ_{ijk} (cf, for example, [8]):

$$K(e_1,e_2)=\sum_k(\frac{1}{2}\mu_{12k}(-\mu_{12k}+\mu_{2k1}+\mu_{k12})$$
$$-\quad\frac{1}{4}(\mu_{12k}-\mu_{2k1}+\mu_{k12})(\mu_{12k}+\mu_{2k1}-\mu_{k12})-\mu_{k11}\mu_{k22}) \tag{2.29}$$

This means that, since along the flow (3) $sc_g\equiv -1$, there exists a constant $L(n)>1$ depending only on the dimension such that

$$\frac{1}{\sqrt{L}}\leq\|R\|\leq\sqrt{L} \tag{2.30}$$

for the curvature tensor R. Thus along (3) for any $h>0$, $t\geq 0$ holds:

$$\|R\|_{(t+h)}\leq L\|R\|_t. \tag{2.31}$$

On a compact manifold the flow (3) is equivalent to the flow (1) up to reparametrisation and scaling. More precisely,

$$g_{(1)}(\tilde{t})=\psi(t)g_{(2)}(t), \tag{2.32}$$

$$\|R_{(1)}\|_{\tilde{t}}=\frac{1}{\psi(t)}\|R_{(2)}\|_t, \tag{2.33}$$

where $g_{(1)},g_{(2)}$ and $R_{(1)},R_{(2)}$ are metric and curvature tensors corresponding, accordingly, to the flows (3) and (1), and

$$\psi(t)=e^{2\int_0^t\|ric_g\|^2ds}, \tag{2.34}$$

$$\tilde{t}=\int\psi(t)dt. \tag{2.35}$$

Remark 2.2.8 *Note, that while the norm of the curvature* $\|R\|\in[\frac{1}{\sqrt{L}},\sqrt{L}]$ *with* L *defined as above along the normalized Ricci flow (3), 2.33 and 2.34 show that* $\|R\|_t\to 0$ *along the Ricci flow (1).*
For any $\tilde{t}\in\mathbb{R}_{\geq 0}$

$$\|R_{(1)}\|_{\tilde{t}}\leq\frac{\|R_{(2)}\|_t}{e^{\frac{2t}{n^2L}}}\leq\frac{L}{e^{\frac{2t}{n^2L}}}\|R_{(1)}\|_0, \tag{2.36}$$

hence for $\tilde{t}\geq\int_0^{n^2L\cdot lnL}\psi(t)dt$, $\|R_{(1)}\|_{\tilde{t}}\leq\frac{1}{L}\|R_{(1)}\|_0$.
Thus on a nilmanifold the curvature always shrinks along the Ricci flow.

Theorem 2.2.9 *In any dimension n there exist constants $c_1(n), c_2(n) \geq 1$ such that for any n-dimensional nilmanifold (N, g) along the Ricci flow*

$$\frac{\partial g}{\partial t} = -2ric_g$$

with $g(0) = g$
holds: if $\|R\|_0 \in [\frac{1}{10c_2(n)}, 10]$, then
(i)

$$\|R\|_t g(t) < \frac{1}{2c_2(n)} \|R\|_0 g(0) \tag{2.37}$$

for any $t > c_1(n)$,
(ii)

$$\|R\|_t g(t) < c_2(n) \|R\|_0 g(0) \tag{2.38}$$

for any $t > 0$.

Proof
Take any (N, g) and consider $g(t)$ as it evolves along the flow (3). Recall, that for any $t \in \mathbb{R}_{\geq 0}$,

$$\|R\|_t \leq L\|R\|_0. \tag{2.39}$$

$g'(t)$ is bounded on $t \in \mathbb{R}_{\geq 0}$, since $g'(t) = -2ric_g$ and $\|ric_g\| \in [\frac{1}{n^2\sqrt{L}}, n^2\sqrt{L}]$ for all t with L defined as above.
Put $I = \{t : (N, g(t)) \notin U\}$, where U is a neighbourhood defined as in 2.2.7. We know that $g(t)$ decreases outside I. It means that $g(t)$ remains bounded on $t \in \mathbb{R}_{\geq 0}$, i.e. there exists a constant $c_g \geq 1$ such that for any $t \in \mathbb{R}_{\geq 0}$

$$g(t) \leq c_g g(0). \tag{2.40}$$

Estimates on the metric and on the curvature combined imply that for any $t \in \mathbb{R}_{>0}$, as $g(t)$ evolves along (3), there exists a constant c_2 depending only on the dimension, such that

$$g(t)\|R\|_t \leq \tilde{c}_2 g(0)\|R\|_0. \tag{2.41}$$

Since $f(t) = g(t)\|R\|_t$ is invariant under rescaling,

$$g_{(1)}(\tilde{t})\|R_{(1)}\|_{\tilde{t}} = g_{(2)}(t)\|R_{(2)}\|_t, \tag{2.42}$$

and we get that for any $\tilde{t} \in \mathbb{R}_{>0}$,

$$g_{(1)}(\tilde{t})\|R_{(1)}\|_{\tilde{t}} \leq c_2 g_{(1)}(0)\|R_{(1)}\|_0. \tag{2.43}$$

(ii) is proved. □

Consider I as $I = \cup_i I_i$, $I_i = [t_{2i-1}, t_{2i}]$, $\mathbb{R}\backslash I = \cup_i J_i$.
Recall that $\Sigma_{i=0}^{\infty} l(I_j) \leq C$, so $l(I_i)_{i\to\infty} \to 0$. Put the length of J_i equal to δ_i. The series $\Sigma_{i=0}^{\infty}\delta_i$ diverges.
Define for any i $\Delta_i g(t) := g(t_{2i}) - g(t_{2i-1})$.
Then for any i

$$\begin{aligned} g(t_{2i}) &\leq g(t_{2i-1}) + \Delta_i g(t) \leq e^{-\frac{\lambda}{2}\cdot\delta_i} g(t_{2(i-1)}) + \Delta_i g(t) \\ &\leq \ \dots \leq e^{-\frac{\lambda}{2}\sum_{k=1}^{i}\delta_k} g(t_0) + \sum_{k=1}^{i} e^{-\frac{\lambda}{2}\sum_{j=i-k}^{i-1}\delta_j}\Delta_{i-k} g(t) + \Delta_i g(t). \end{aligned} \tag{2.44}$$

Analogously, for any $t > t_{2i}$

$$g(t) \leq e^{-\frac{\lambda}{2}\sum_{k=1}^{i}\delta_k} g(t_0) + \sum_{k=1}^{i} e^{-\frac{\lambda}{2}\sum_{j=i-k}^{i-1}\delta_j}\Delta_{i-k} g(t) + \sum_{k=i}^{\infty}\Delta_k g(t). \tag{2.45}$$

Remark, that, since on $\mathbb{R}\backslash I$ $g(t)$ decreases, for any i

$$\sum_{j=1}^{i}\Delta_j g(t) \leq g(t_i) \leq c_g g(0), \tag{2.46}$$

where c_g is the constant from (ii).
Also, since the series $\Sigma_{i=0}^{\infty}\delta_i$ diverges, for any constant M and any i there exists a k such that

$$\sum_{j=i}^{ki}\delta_j > M. \tag{2.47}$$

Now, take the constant c_2 as in (ii).
Since $\Sigma_{i=0}^{\infty} l(I_i)$ converges and $g'(t)$ is bounded on $\mathbb{R}_{\geq 0}$, we can choose i so that

$$\sum_{k=i}^{\infty}\Delta_k g(t) < \frac{1}{8c_2 L} g(0), \tag{2.48}$$

where L is defined as in (2.39), namely, so that for any $t \in \mathbb{R}_{\geq 0}$,

$$\|R\|_t \leq L\|R\|_0.$$

For this i choose k so that

$$e^{-\frac{\lambda}{2}\sum_{j=i}^{ki}\delta_j} < \frac{1}{8c_2c_gL}. \tag{2.49}$$

Then, obviously, $e^{-\frac{\lambda}{2}\sum_{j=1}^{ki}\delta_j} < \frac{1}{8c_2c_gL}$.
Therefore, for any $t > t_{ki}$,

$$g(t) \leq e^{-\frac{\lambda}{2}\sum_{j=1}^{ki}\delta_j} c_g g(0) + e^{-\frac{\lambda}{2}\sum_{j=i}^{ki}\delta_j} c_g g(0) + \frac{g(0)}{8c_2L} < \frac{g(0)}{2c_2L} \tag{2.50}$$

Put $c_1' := t_{ki}$.
Taking into account the fact that $\|R\|_t < L\|R\|$ along the flow (3), we get that

$$\|R\|_t g(t) < \frac{1}{2c_2}\|R\|_0 g(0) \tag{2.51}$$

for any $t > c_1'$.
Since the function $g(t)\|R\|_t$ is invariant under rescaling,

$$g_{(1)}(\tilde{t})\|R_{(1)}\|_{\tilde{t}} = g_{(2)}(t)\|R_{(2)}\|_t, \tag{2.52}$$

and we get that for any $\tilde{t} > max\{\int_0^{c_1'}\psi(t)dt, \int_0^{n^2L\cdot lnL}\psi(t)dt\} =: c_1$

$$\|R\|_{\tilde{t}} g(\tilde{t}) < \frac{1}{2c_2}\|R\|_0 g(0) \tag{2.53}$$

along the flow (1). The theorem is proved.

□

Remark 2.2.10 *Using 2.2.8, we can conclude that, from the choice of constant* c_1 *(*$c_1 \geq \int_0^{n^2L\cdot lnL}\psi(t)dt$*), for* $t \geq c_1$ *along the Ricci flow (1),* $\|R\|_t \leq \frac{1}{L}\|R\|_0$.

Note also that the constants obtained in 2.2.9, are universal for all left-invariant metrics and a given algebraic structure, since $Gl(n)$ acts transitively on $Sym(\eta)$ and we can always fix the metric by passing to the algebraic model.

2.3 Ricci Flow on Almost Flat Manifolds

Now we prove the main result on a segment. In the proof we again make use of the ideas of Gromov-Hausdorff convergence. The structure of the limit space provides us with information about the behavior of the spaces "close" to it. Recall (cf. (1.45)) that under the norm of the tensor R at t (which is denoted as $\|R\|_t$) we understand the *sup*-norm of R on the manifold $(M, g(t))$.

Theorem 2.3.1 *In any dimension n and any $T > 0$ there exists an $\varepsilon(n)$ such that for any $\varepsilon \leq \varepsilon(n)$ and any ε - flat Riemannian manifold (M^n, g)*
(i) the solution of the Ricci flow (1)

$$\frac{\partial g}{\partial t} = -2ric_g$$

exists on M for all $t \in [0, T)$,
(ii) parametrize the curvature at $t = 0$ as $\|R\|_0 = 1$. Then, for c_1, c_2 defined as in 2.2.9 along the flow (1)

$$\|R\|_t g(t) < \frac{1}{2}\|R\|_0 g(0) \tag{2.54}$$

at $t = 2c_1$,

$$\|R\|_t g(t) < 2c_2\|R\|_0 g(0) \tag{2.55}$$

for any $t \in [0, 2c_1]$,
and

$$\|R\|_{2c_1} \leq \|R\|_0. \tag{2.56}$$

Proof
(i) Suppose the claim of the theorem is not true and there exists a dimension n such that for any sequence of numbers $\varepsilon_k \to 0$ there exists a sequence of ε_k-flat manifolds (M_k^n, g_k) and a sequence of numbers $0 < t_k < T$ such that on each (M_k^n, g_k) the Ricci flow exists on the maximal interval $[0, t_k)$. Parametrise the curvatures at $t = 0$ as $max_{M_k}\|R_k\| = 1$. Note, that from Hamilton's theorem, $t_k > 10^{-n} =: \delta$.
Pass to the universal covers of (M_k) with the corresponding covering metrics: $X_k = (\tilde{M}_k, g_k)$. From Gromov's Compactness theorem, $(\tilde{M}_k, g_k(t))$ subconverges on any compact subset of $(0, \delta)$ in the pointed Gromov-Hausdorff topology to a complete solution of the Ricci flow.
To get a contradiction, study the limit manifold $(\tilde{M}_\infty, g_\infty)$. From BBS estimations (1.6.1), at $t = \delta$ all the derivatives of corresponding curvature tensors R_k are uniformly bounded, hence, by the Convergence theorem of Riemannian geometry, $(\tilde{M}_\infty, g_\infty(\delta))$ is a smooth manifold.
Suppose, first, $\lim_{k\to\infty}\|R_k\|_\delta = \nu > 0$. In this case, by 1.5.1, the limit manifold $(\tilde{M}_\infty, g_\infty(t))$ is a nilmanifold. From Section 2, on a nilmanifold the Ricci flow exists for all $t \in [0, \infty)$.

If $\lim_{k\to\infty}\|R_k\|_\delta = 0$, it corresponds to a trivial solution of the Ricci flow on the limit manifold $(\tilde{M}_\infty, g_\infty)$, $\tilde{g}_\infty$ is a flat metric and the solution of (1) also exists for all t.
That means that in both cases $\|R_\infty\|_T$ is bounded, whereas, by supposition, it should explode for any k on X_k as $t \to t_k < T$. A contradiction.
So, for $T = 2c_1$, we can choose an $\varepsilon(T)$ such that for any $\varepsilon \leq \varepsilon(T)$ and on any $\varepsilon-$flat manifold the Ricci flow (1) exists on the interval $[0, 4c_1]$.
(ii) To prove (2.54), suppose, as in (i), that the claim is not true and there exists a dimension n such that for any sequence of numbers $\varepsilon_k \to 0$ there exists a sequence of ε_k-flat manifolds (M^n_k, g_k) and a sequence of numbers $t_k \in [c_1, 2c_1]$ such that on each manifold of the sequence holds: for some vectors $v_{kp_k}M_k$

$$\|R_k\|_{t_k} g_k(t_k)(v_k, v_k) \geq \frac{1}{2}\|R_k\|_0 g_k(0)(v_k, v_k), \tag{2.57}$$

where the curvatures at $t = 0$ are normalized as $max_{(M,g_k)}\|R_k\|_0 = 1$.
Passing to the universal coverings of (M_k) with the corresponding covering metrics $X_k = (\tilde{M}_k, g_k(t))$ we establish the subconvergence of X_k to a Riemannian manifold in the sence of Gromov-Hausdorff for any $t \in (0, 2c_1)$, in particular, for $t = \delta = 10^{-n}$, (cf. theorem of Hamilton). Now the following three cases can occur:
(a) at $t = \delta$ the curvature on the limit space belongs to the interval: $\|R\|_t \in [\frac{1}{10c_2}, 10]$. Then on $t \in [\delta, 2c_1]$ we have the subconvergence of $(M_k, g_k(t))$ to the Ricci flow on a nilmanifold. In this case, by 2.2.9, (i) and the hypotesis, we have that for any k and any $t \in [c_1, 2c_1]$

$$\|R_k\|_{t_k} g(t_k)(v_k, v_k) - \|R\|_t g(t)(v_k, v_k) \geq (\frac{1}{2} - \frac{1}{2c_2})\|R\|_0 g(0)(v_k, v_k), \tag{2.58}$$

which is a contradiction to the established convergence.
(b) at $t = \delta$ $\|R\|_t$ subconverges to ν, $0 < \nu < \frac{1}{10c_2}$. Then, as in (a), on $t \in [\delta, 2c_1]$ we have the subconvergence of $(M_k, g_k(t))$ to the Ricci flow on a nilmanifold. From the choice of δ we have also that $g_k(\delta) \leq 2g_k(0)$ for any k along the Ricci flow (cf. Section 7, Part 1). Further on, from 2.2.9, (ii), for any $t \in [\delta, 2c_1]$

$$\|R\|_t g(t) \leq c_2\|R\|_\delta g(\delta) \leq \frac{1}{5}\|R\|_0 g(0), \tag{2.59}$$

which is again a contradiction.
(c) at $t = \delta$ the curvatures tend to zero: $\lim_{k\to\infty}\|R_k\|_{t_o} \to 0$. From theorem 1.6.1 on the limit manifold the metric $g(t)$ remains bounded on the same segment and on the interval $[\delta, \frac{10^{-n}}{\|R\|_\delta})$, $\|R\|_t \leq C(0, n)\|R\|_\delta$. So, for any $t \in [c_1, 2c_1]$, $\|R\|_t g(t) \leq \frac{1}{2}\|R\|_0 g(0)$. A contradiction.
(2.55) and (2.56) can be proved analogously.

2.4 The Proof of the Main Result in the Non-Abelian Case.

Consider an $\varepsilon-$flat manifold (M^n, g_0) with ε small enough for M to satisfy the hypotheses of 2.3.1 and with the curvature parametrized as $\|R\|_0 = 1$. Evolve the initial metric g_0 on M under the Ricci flow (1). From 2.3.1, at $t = 2c_1$ the manifold (M, g_t) is $\frac{\varepsilon}{2}-$flat. Moreover, from 2.2.10,

$$\|R\|_{2c_1} \leq \|R\|_0. \tag{2.60}$$

From (2.56) we can put $\|R\|_{2c_1} = \nu$, $0 < \nu \leq 1$. Applying theorem 2.3.1 to the rescaled manifold $(M, \frac{1}{\nu}g)$ shows that the Ricci flow continues to exist on the interval $[2c_1, 2c_1 + \frac{2c_1}{\nu}]$. At $t_2 = 2c_1 + \frac{2c_1}{\nu}$ the manifold (M, g_{t_2}) is $\frac{\varepsilon}{4}-$flat and

$$\|R\|_{t_2} \leq \|R\|_{t_1} \leq 1. \tag{2.61}$$

So, by induction, we obtain a sequence $t_i \to \infty$ of times such that

$$d(M, g_{t_i})^2 \cdot \|R\|_{t_i} < \frac{\varepsilon}{2i}, \tag{2.62}$$

Furthermore, for any $t \in [t_i, t_{i+1}]$

$$d(M, g_t)^2 \cdot \|R\|_t < \frac{c_2\varepsilon}{2i}. \tag{2.63}$$

So, the Ricci flow exists on the whole of $\mathbb{R}_{\geq 0}$ and $\lim_{t\to\infty} |K|_{g_t} \cdot d^2(M, g_t) = 0$. Theorem B is proved.

Chapter 3

Gromov's Pinching constant.

The pinching constant in the theorem of Gromov (see Introduction) is taken equal to

$$\varepsilon(n) = \exp(-\exp(\exp n^2)), \tag{3.1}$$

where n is the dimension of the manifold. It is clear that this constant may not be optimal. In particular, there arises a question whether ε should necessarily depend on the dimension of the manifold. The answer is contained in the following

Theorem 3.0.1 *In every dimension n there exists a manifold (M^n, g) with*

$$|K|_M \cdot d(M^n, g)^2 < \frac{14}{n^2} \tag{3.2}$$

which can not be finitely covered by a nilmanifold.

Proof
Consider a Lie group $S = \mathbb{R}^n \rtimes \mathbb{R}$ with the group operation $L_{(v,t)}(w, s) = (v + h(t)w, t + s)$, where $h(t) = Exp(tA), A \in GL(n, \mathbb{R})$. As can be easily seen, S is solvable but in general not nilpotent.
Describe a lattice L' in S.

Lemma 3.0.2 *A matrix $B \in GL(n, \mathbb{R})$ preserves a lattice in $\mathbb{R}^n$ ($BL = L$) if and only if B is conjugate to a matrix in $GL(\mathbb{Z}, n)$.*

Proof
Note first, that, if a matrix B preserves a lattice, then its conjugate TBT^{-1} also preserves a lattice. Indeed, if L is a lattice, TL is also a lattice and is preserved by TBT^{-1}. Let now B preserves a canonical lattice (the one spanned to an orthonormal basis of vectors). Straightforward computation shows that in this case B and B^{-1}

have got the determinant equal to 1 or -1 and integer entrees, hence, B belongs to $GL(\mathbb{Z}, d)$. And the other way round: direct computation shows that a matrix B from $GL(n, \mathbb{Z})$ preserves a canonical lattice, therefore, its conjugate preserves a given one. $\square$

The above Lemma shows that we can choose a lattice in S in the form $L' = L \rtimes \mathbb{Z}$, provided that L is a lattice in $\mathbb{R}^n$ invariant under $ExpA$. Notice also that if $L' = L \rtimes \mathbb{Z}$ is a lattice, $L'' = \frac{1}{h} L \rtimes \mathbb{Z}$ is also a lattice for $h \neq 0$.

Lemma 3.0.3 *Suppose that $ExpA$ has eigenvalues with absolute value different from 1. A quotient manifold of S by a uniform discreet subgroup (a lattice) $L' = L \rtimes \mathbb{Z}$ can not be covered by a nilmanifold.*

Proof

Suppose, there exists a covering of $M' = S/L'$, where $L' = L \rtimes \mathbb{Z}$, by a nilmanifold N/Γ, where Γ is a lattice in N. Without loss of generality, $\Gamma \subset \pi_1(M)$ as a normal subgroup of finite index k. Since S is simply connected, $\pi_1(M')$ $\mathfrak{w}$ L'. So, according to our assumption, the nilpotent group Γ is contained in the solvable but not nilpotent group L' as a normal subgroup of finite index k. It means that $L'^k = < g^k | g \in L' > \subset \Gamma$ is nilpotent. Then the map $i : L'^k \to L'^k, i = id - Exp(kA)$ is nilpotent. Indeed, $[(v, o), (0, k)] = i(v)$, $\underbrace{[\cdots}_{l}[[(v,0),(0,k)], \underbrace{(0,k)],]\cdots}_{l}] = i^l(v)$. From the nilpotency of L'^k follows that there exists an l such that $i^l(v) = 0$. This means that all the roots of the characteristic polynomial of $Exp(kA)$ are equal to 1. Hence the eigenvalues of $ExpA$ are roots of 1, hence equal to one by absolute value. This contradicts the hypothesis and the lemma is proved. $\square$

Estimate the sectional curvature of the Lie group S.

Let A be such that it can be represented as

$$\begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} & & & & 0 \\ & \cdots & & & \\ 0 & & \begin{pmatrix} \lambda_l & 0 \\ 0 & \lambda_l \end{pmatrix} & & 0 \\ & & & \lambda_{l+1} & \\ 0 & & & & \lambda_m \end{pmatrix}$$

$$\begin{pmatrix} \begin{pmatrix} 0 & \varphi_1 \\ -\varphi_1 & 0 \end{pmatrix} & & & & 0 \\ & \cdots & & & \\ 0 & & \begin{pmatrix} 0 & \varphi_l \\ -\varphi_l & 0 \end{pmatrix} & & 0 \\ & & & 0 & \\ 0 & & & & 0 \end{pmatrix}$$

with λ_i real.
Then $Exp(tA) =$

$$\begin{pmatrix} e^{t\lambda_1}\begin{pmatrix} cos(t\varphi_1) & sin(t\varphi_1) \\ -sin(t\varphi_1) & cos(t\varphi_1) \end{pmatrix} & & & & 0 \\ & \cdots & & & \\ 0 & & e^{t\lambda_l}\begin{pmatrix} cos(t\varphi_l) & sin(t\varphi_l) \\ -sin(t\varphi_l) & cos(t\varphi_l) \end{pmatrix} & & 0 \\ & & & e^{t\lambda_{l+1}} & 0 \\ 0 & & & 0 & e^{t\lambda_m} \end{pmatrix}$$

Define a standard left-invariant metric on S:

$$< v_1, v_2 >_{(w,t)} = < (dL^{-1}_{(w,t)})_{(w,t)} v_1, (dL^{-1}_{(w,t)})_{(w,t)} v_2 >_{(0,0)} = < \tilde{h}(-t)v_1, \tilde{h}(-t)v_2 >_{(0,0)}$$

where

$$\tilde{h}(t) = \begin{pmatrix} h(t) & 0 \\ 0 & 1 \end{pmatrix}$$

and $v_1, v_2 \in s$, for s - the Lie algebra of S. From the explicit expression for this metric follows the obvious

Lemma 3.0.4 *S is isometric to $\tilde{S}$, where $\tilde{S}$ is a Lie group corresponding to the matrix A equal to*

$$\begin{pmatrix} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1 \end{pmatrix} & & & & 0 \\ & \cdots & & & \\ 0 & & \begin{pmatrix} \lambda_l & 0 \\ 0 & \lambda_l \end{pmatrix} & & 0 \\ & & & \lambda_{l+1} & \\ 0 & & & & \lambda_m \end{pmatrix}$$

Lemma 3.0.5 *Put* $\lambda_{max} = max\{|\lambda_i|, i = 1, ..., l\}$. *Then the sectional curvature of* S *is bounded from above by* $|K| \leq 3\lambda_{max}^2$

Proof
The curvature of a left-invariant metric on a Lie group is given by

$$< R(X,Y)Y, X >= \frac{1}{4}\|(ad_X)^*(Y) + (ad_Y)^*(X)\|^2 - < (ad_X)^*(X), (ad_Y)^*(Y) >$$
$$-\frac{3}{4}\|[X,Y]\|^2 - \frac{1}{2} < [[X,Y],Y], X > -\frac{1}{2} < [[Y,X],X], Y >$$

where X,Y are left-invariant vector fields,

$$X_{(v,t)} = (dL_{(v,t)})X_e = \tilde{h}(t)X_e,$$

$$Y_{(v,t)} = (dL_{(v,t)})Y_e = \tilde{h}(t)Y_e$$

(cf., for example, [2]).

To simplify the computations, for metrical estimates we can use the group $\tilde{S}$. The Lie bracket for $\tilde{S}$ is given by

$$[X,Y] = x_0 AY' - y_0 AX'$$

where

$$X = \begin{pmatrix} X', & x_0 \end{pmatrix}$$

$$Y = \begin{pmatrix} Y', & y_0 \end{pmatrix}$$

$X', Y' \in \mathbb{R}^n, x_0, y_0 \in \mathbb{R}$.
So,

$$\|ad_X Y\| = \|[X,Y]\| \leq |\max_i \lambda_i| \|X\| \|Y\|$$

and the same estimation holds for the matrix of the adjoint operator $(ad_X)^*$.
Hence, finally, $|K| \leq 3\lambda_{max}^2$. □
So we see that the sectional curvature is controlled by the eigenvalues of A.

Remark 3.0.6 *If the eigenvalues of* $Exp(A)$ *satisfy the equation*

$$x^n + 1 = 0$$

it corresponds to the case when S *is flat.*

Consider the equation

$$x^{2k} + 3x^k + 1 = 0 \tag{3.3}$$

Lemma 3.0.7 *The left-hand side of the equation (3.3) is the characteristic polynomial of a matrix $T' \in GL(\mathbb{Z}, n)$ if $n = 2k$.*

Proof
Let $T' =$

$$\left(\begin{array}{cccccc} 0 & 0 & 0 & \dots & 0 & 1 \\ -1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -1 & 0 & \dots & 0 & 0 \\ & \dots & & & & \\ 0 & 0 & 0 & \dots & 0 & a \\ & \dots & & & & \\ 0 & 0 & 0 & \dots & -1 & 0 \end{array} \right),$$

(the first rows are marked with a brace labelled k)

where $a = (-1)^{3k+1} \cdot 3$.
Direct computation shows that this matrix is indeed in $GL(\mathbb{Z}, n)$ the characteristic polynomial is exactly the polynomial on the left-hand side of the equation (3.3). Note also that the matrix T' is semisimple (each of its invariant subspaces has a complement invariant subspace), hence can be decomposed over the reals into 2×2 blocks. In particular, T' is conjugate to

$$\left(\begin{array}{ccc} e^{\lambda_1} \left(\begin{array}{cc} cos(\varphi_1) & sin(\varphi_1) \\ -sin(\varphi_1) & cos(\varphi_1) \end{array} \right) & & 0 \\ & \dots & \\ 0 & & e^{\lambda_l} \left(\begin{array}{cc} cos(\varphi_l) & sin(\varphi_l) \\ -sin(\varphi_) & cos(\varphi_l) \end{array} \right) \end{array} \right)$$

where $\lambda_i = ln|r_i|$, r_i are the roots of the characteristic equation (3.3) and corresponding φ's.
Straightforward computation shows that for any $i = 1, ..., n$, $ln|r_i| < \frac{2}{n}$.
Hence, $|\lambda_{max}| < \frac{2}{n}$.

□

Remark 3.0.8 *If $n = 2k + 1$ we consider the polynomial*

$$(x + 1)(x^{2k} + 3x^k + 1) = 0 \tag{3.4}$$

and the corresponding matrix $T'' \in GL(\mathbb{Z}, n)$

$$T' = \left(\begin{array}{ccccccc|l} 0 & 0 & 0 & \dots\dots\dots & 0 & 0 & 1 & \\ -1 & 0 & 0 & \dots\dots\dots & 0 & 0 & -1 & \\ 0 & -1 & 0 & \dots\dots\dots & 0 & 0 & 0 & \Big\} k+1 \\ & \dots & & & & & & \\ 0 & 0 & 0 & \dots\dots\dots & 0 & 0 & a_1 & \\ 0 & 0 & 0 & \dots\dots\dots & 0 & 0 & a_2 & \\ & \dots & & & & & & \\ 0 & 0 & 0 & \dots\dots\dots & -1 & 0 & -1 & \\ 0 & 0 & 0 & \dots\dots\dots & 0 & -1 & 0 & \end{array} \right),$$

where $a_1 = (-1)^{3k+3} \cdot 3, a_2 = (-1)^{3k+4} \cdot 3$.
Estimate now the diameter of the quotient manifold S/L'.

Lemma 3.0.9

$$\lim_{h\to\infty} diam(S/\frac{1}{h}L') = diam(\mathbb{R}/\mathbb{Z}) = 1. \tag{3.5}$$

Proof
First note that $S/\frac{1}{h}L' \rtimes \mathbb{Z}$ fibers over $S/\mathbb{R}^n \rtimes \mathbb{Z}$. The natural projection

$$pr : S/\frac{1}{h}L' \rtimes \mathbb{Z} \to S/\mathbb{R}^n \rtimes \mathbb{Z} = S^1$$

is a Riemannian submersion (a maximal rank surjective map, preserving the lengths of vectors orthogonal to the fiber.) It is easy to see, that the diameter of the fiber tends to zero. Thus

$$diam M^1 \to diam S^1 = 1$$

□

Thus we have shown that

$$\lim_{h\to\infty} diam(S/\frac{1}{h}L' \rtimes \mathbb{Z})^2 \cdot |K(S/\frac{1}{h}L' \rtimes \mathbb{Z})| \leq \frac{12}{n^2}.$$

The theorem 3.0.1 is proved.
We can conclude the the pinching constant in the Gromov's Theorem decreases with the dimension at least as $\frac{14}{n^2}$.

Bibliography

[1] Buser P., Karcher H.: Gromov's almost flat manifolds. In: Périodique mensuel de la Société Mathématique de France, 1981, Astérisque 81.

[2] Cheeger J., Ebin D. G.,: Comparison Theorems in Riemannian Geometry, North Holland, Amsterdam 1975.

[3] Chow B., Knopf D.,: The Ricci Flow: An Introduction. Mathematical Surveys and Monographs, vol. 110, AMS, 2004.

[4] Hamilton R., Three-manifolds with positive curvature operator. J. Differential Geom. 17 (1982), n. 2, 255–306.

[5] Hamilton R., Four-manifolds with positive curvature operator. J. Differential Geom. 24(1986), n. 2, 153–179.

[6] Heber Jens, Noncompact homogeneous Einstein spaces. Invent. Math. 133 (1998), n. 2, 279–352.

[7] Lauret, J., Ricci soliton homogeneous manifolds. Math. Ann. 319 (2001), n. 4, 715–733.

[8] Milnor, John. Curvatures of left-invariant metrics on Lie groups. Advances in Math. 21 (1976), n. 3, 293–329.

[9] Petersen P., Riemannian Geometry, Springer 171 (1962).

[10] Ruh E., Almost flat manifolds. J. Differential Geom. 17 (1982), no. 1, 1-14.

[11] Shi, Wan-Xiong Deforming the metric on complete Riemannian manifolds. J. Differential Geom. 30 (1989), no. 1, 223–301.

[12] Wilson, Edward N. Isometry groups on homogeneous manifolds. Geom. Dedicata 12 (1982), n. 3, 337–346.