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Doubloons and q-secant numbers

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Abstract. Based on the evaluation at $t=-1$ of the generating polynomial for the hyperoctahedral group by the number of descents, an observation recently made by Hirzebruch, a new q-secant number is derived by working with the Chow-Gessel q-polynomial involving the flag major index. Using the doubloon combinatorial model we show that this new q-secant number is a polynomial with positive integral coefficients, a property apparently hard to prove by analytical methods.

1. Introduction

This paper, in harmony with our previous two papers on doubloons [11, 12], is motivated by our intention of finding a combinatorial connection between the Eulerian polynomials, on the one hand, and the trigonometric functions, tangent and secant, on the other hand, when the connection is further carried over to a q-analog environment.

Let $(t;q)_n := (1-t)(1-tq)\cdots(1-tq^{n-1})$ if $n\ge 1$ and $(t;q)_0 := 1$ be the traditional q-ascending factorial and $[\,j\,]_q := 1+q+\cdots+q^{j-1}$ be the q-analog of the positive integer j. The *q-analogs* $A_n(t,q)$, introduced by Carlitz [3, 4], of the *Eulerian polynomials*, may be defined by the identity

$$\frac{A_n(t,q)}{(t;q)_{n+1}} = \sum_{j\ge 0} t^j([j+1]_q)^n \quad (n\ge 0). \tag{1}$$

For each $n\ge 0$ the q-analog $A_n(t,q)$ is a polynomial with positive integral coefficients [in short, a PIC *polynomial*], such that $A_n(t,1)$ is equal to the traditional *Eulerian polynomial* $A_n(t)$ introduced by Euler himself [7], who also derived the exponential generating function:

$$\sum_{n\ge 0}\frac{u^n}{n!}A_n(t) = \frac{1-t}{-t+\exp(u(t-1))}. \tag{2}$$

As $A_n(1,1)=A_n(1)=n!$, each PIC polynomial $A_n(t)$ (resp. $A_n(t,q)$) has been regarded as a generating function for the symmetric group $\mathfrak{S}_n$ by several integral-valued statistics (resp. pairs of such statistics) [20, 13, 4]. Note that (2) is easily derived from (1).

In the same manner, the next two identities

$$\frac{B_n(t,q)}{(t;q^2)_{n+1}} = \sum_{j\ge 0} t^j([2j+1]_q)^n \quad (n\ge 0); \tag{3}$$

$$\sum_{n\ge 0} \frac{u^n}{n!} B_n(t) = \frac{(1-t)\exp(u(t-1))}{-t+\exp(2u(t-1))}; \tag{4}$$

may serve to define two families of polynomials $(B_n(t))$, $(B_n(t,q))$ $(n\ge 0)$.

Again, both $B_n(t)$ and $B_n(t,q)$ are PIC polynomials and $B_n(t)=B_n(t,1)$. Moreover, (4) is easily derived from (3). The interpretation of $B_n(t)$ as a generating polynomial for the hyperoctahedral group B_n, together with the derivations of (3) for $q=1$ and (4), was first obtained by Reiner [19], also by Cohen [6] in the general context of the Coxeter groups of spherical type. Formula (3) was derived and fully interpreted by Chow and Gessel [5].

While studying the signatures of the toric varieties, Hirzebruch [16] is led to calculate the values of both polynomials $A_n(t)$ and $B_n(t)$ at $t=-1$. He first quotes Euler's identities [7]

$$A_{2n}(-1)=0 \ (n\ge 1); \quad (-1)^n A_{2n+1}(-1) = T_{2n+1} \ (n\ge 0), \tag{5}$$

where the coefficients T_{2n+1} $(n\ge 0)$ are the *tangent numbers* occurring in the Taylor expansion of $\tan u$:

$$\begin{aligned} \tan u &= \sum_{n\ge 0} \frac{u^{2n+1}}{(2n+1)!} T_{2n+1} \\ &= \frac{u}{1!}1 + \frac{u^3}{3!}2 + \frac{u^5}{5!}16 + \frac{u^7}{7!}272 + \frac{u^9}{9!}7936 + \frac{u^{11}}{11!}353792 + \cdots \end{aligned} \tag{6}$$

Then, he notes that

$$B_{2n+1}(-1)=0 \ (n\ge 0); \quad (-1)^n B_{2n}(-1) = 2^{2n} E_{2n} \ (n\ge 0), \tag{7}$$

where the coefficients E_{2n} $(n\ge 0)$ are the *secant numbers* occurring in the Taylor expansion of $\sec u$

$$\begin{aligned} \sec u = \frac{1}{\cos u} &= \sum_{n\ge 0} \frac{u^{2n}}{(2n)!} E_{2n} \\ &= 1 + \frac{u^2}{2!}1 + \frac{u^4}{4!}5 + \frac{u^6}{6!}61 + \frac{u^8}{8!}1385 + \frac{u^{10}}{10!}50521 + \cdots \end{aligned} \tag{8}$$

since, by (4),

$$\sum_{n\ge 0} \frac{(iu)^n}{2^n n!} B_n(-1) = \frac{2}{e^{iu}+e^{-iu}} = \sec u = \sum_{n\ge 0} \frac{u^{2n}}{(2n)!} E_{2n}.$$

It so happens that (7) is just the relation needed to construct a new q-analog of the secant number, in parallel with what has been done already for the tangent number.

Theorem 1.1. *Let $(B_n(t,q))$ $(n \ge 0)$ be the sequence of polynomials defined by (3) and let*

$$E_{2n}(q) := (-1)^n q^{n^2} B_{2n}(-q^{-2n}, q) \quad (n \ge 1). \tag{9}$$

Then,
(*a*) *each $E_{2n}(q)$ is a* PIC *polynomial;*
(*b*) *it admits the factorization*

$$E_{2n}(q) = (1+q^2)(1+q^4)\cdots(1+q^{2n})F_{2n}(q), \tag{10}$$

where $F_{2n}(q)$ is a PIC *polynomial;*
(*c*) $E_{2n}(1) = 2^n F_{2n}(1) = 2^{2n} E_{2n}$ (E_{2n} *the secant number*);
(*d*) $B_{2n+1}(-q^{-(2n+1)}, q) = 0$ $(n \ge 0)$.

Property (*c*) follows from (7) and (9). Property (*d*) is proved in Section 5. As is often the case, it is much harder to derive the factorization shown in (*b*) and prove that the coefficients of $E_{2n}(q)$ are *positive.* It requires a long *combinatorial* development, given in the next three Sections. We reproduce the first values of the polynomials $B_n(t,q)$ and $E_{2n}(q)$ in Tables 1.1 and 1.2.

$$\begin{aligned}
&B_1(t,q) = 1 + qt; \qquad B_2(t,q) = 1 + (2q + 2q^2 + 2q^3)t + q^4t^2;\\
&B_3(t,q) = 1 + (3q + 5q^2 + 7q^3 + 5q^4 + 3q^5)t\\
&\qquad\qquad + (3q^4 + 5q^5 + 7q^6 + 5q^7 + 3q^8)t^2 + q^9t^3;\\
&B_4(t,q) = 1 + (4q + 9q^2 + 16q^3 + 18q^4 + 16q^5 + 9q^6 + 4q^7)t\\
&\quad + (6q^4 + 16q^5 + 30q^6 + 40q^7 + 46q^8 + 40q^9 + 30q^{10} + 16q^{11} + 6q^{12})t^2\\
&\quad + (4q^9 + 9q^{10} + 16q^{11} + 18q^{12} + 16q^{13} + 9q^{14} + 4q^{15})t^3 + q^{16}t^4.
\end{aligned}$$

Table 1.1. The polynomials $B_n(t,q)$.

$$\begin{aligned}
&E_2(q) = (1+q^2)2; \qquad E_4(q) = (1+q^2)(1+q^4)(6+8q+6q^2);\\
&E_6(q) = (1+q^2)(1+q^4)(1+q^6)(20 + 60q + 104q^2 + 120q^3 + 104q^4\\
&\qquad\qquad + 60q^5 + 20q^6);\\
&E_8(q) = (1+q^2)(1+q^4)(1+q^6)(1+q^8)(70 + 336q + 910q^2 + 1760q^3\\
&\qquad\qquad + 2702q^4 + 3440q^5 + 3724q^6 + 3440q^7 + 2702q^8 + 1760q^9 + 910q^{10}\\
&\qquad\qquad + 336q^{11} + 70q^{12}).
\end{aligned}$$

Table 1.2. The polynomials $E_{2n}(q)$.

Following the method developed in [11] and [12], the proof of Theorem 1.1 (*a*) and (*b*) will consist of making the polynomial $E_{2n+2}(q)$, defined in (9), appear as a generating function by an appropriate statistic "smaj," combined with a sign "sgn"

$$E_{2n+2}(q) = \sum_{w \in B_{2n+2}} \operatorname{sgn} w \, q^{\operatorname{smaj} w} \qquad (n \ge 1)$$

and constructing a sign-reversing involution on B_{2n+2}, in such a way that after its application the remaining terms in the sum have positive signs. We leave out the banal case: $E_2(q) = 2(1+q^2)$.

The final step is then to prove the identity

$$E_{2n+2}(q) = (1+q^2)(1+q^4)\cdots(1+q^{2n+2}) \sum_{w\in\mathcal{SN}_{2n+2}} q^{\operatorname{smaj} w}, \tag{11}$$

where the sum is over a specific class $\mathcal{SN}_{2n+2}$ of signed permutations, called *normalized signed doubloons* (see Section 4).

More importantly, the generating polynomial for $\mathcal{SN}_{2n+2}$ occurring in (11) will be explicitly calculated by means of the *doubloon polynomials* $(d_{n,j}(q))$ $(n\ge 1,\ 2\le j\le 2n)$, which are defined by the recurrence

$(D1)$ $d_{0,j}(q)=\delta_{1,j}$ (Kronecker symbol);
$(D2)$ $d_{n,j}(q)=0$ for $n\ge 1$ and $j\le 1$ or $j\ge 2n+1$;
$(D3)$ $d_{n,2}(q)=\sum_j q^{j-1}\, d_{n-1,j}(q)$ for $n\ge 1$;
$(D4)$ $d_{n,j}(q)-2\,d_{n,j-1}(q)+d_{n,j-2}(q)$

$$= -(1-q)\sum_{i=1}^{j-3} q^{n+i+1-j}\, d_{n-1,i}(q)$$
$$-(1+q^{n-1})\,d_{n-1,j-2}(q)+(1-q)\sum_{i=j-1}^{2n-1} q^{i-j+1}\, d_{n-1,i}(q)$$

for $n\ge 2$ and $3\le j\le 2n$;

the first values being:

$d_{1,2}(q)=1$; $d_{2,2}(q)=q$; $d_{2,3}(q)=q+1$; $d_{2,4}(q)=1$;
$d_{3,2}(q)=2q^3+2q^2$; $d_{3,3}(q)=2q^3+4q^2+2q$; $d_{3,4}(q)=q^3+4q^2+4q+1$;
$d_{3,5}(q)=2q^2+4q+2$; $d_{3,6}(q)=2q+2$;
$d_{4,2}(q)=5q^6+12q^5+12q^4+5q^3$; $d_{4,3}(q)=5q^6+17q^5+24q^4+17q^3+5q^2$;
$d_{4,4}(q)=3q^6+15q^5+29q^4+29q^3+15q^2+3q$;
$d_{4,5}(q)=q^6+9q^5+25q^4+34q^3+25q^2+9q+1$;
$d_{4,6}(q)=3q^5+15q^4+29q^3+29q^2+15q+3$;
$d_{4,7}(q)=5q^4+17q^3+24q^2+17q+5$; $d_{4,8}(q)=5q^3+12q^2+12q+5$.

Those polynomials were introduced and used in [12] to evaluate a new q-analog

$$T_{2n+1}(q) := (-1)^n q^{\binom{n}{2}} A_{2n+1}(-q^{-n},q) \tag{12}$$

of the *tangent number* based on the Carlitz q-Eulerian polynomial $A_n(t,q)$ defined in (1). It was shown that $T_{2n+1}(q)$ was a PIC polynomial equal to

$$T_{2n+1}(q) = (1+q)(1+q^2)\cdots(1+q^n)\sum_{k=2}^{2n+2} d_{n,k}(q). \tag{13}$$

The parallel expression for the PIC polynomials $E_{2n+2}(q)$ is next stated.

Theorem 1.2. *For each $n\ge 1$ the polynomial $E_{2n+2}(q)$ has the following expression:*

$$E_{2n+2}(q) = (1+q^2)(1+q^4)\cdots(1+q^{2n+2})\sum_{k=2}^{2n} d_{n,k}(q^2)P_{n,k}(q), \tag{14}$$

where the coefficients $P_{n,k}(q)$ $(n \geq 1,\ 2 \leq j \leq 2n)$ *are defined by*

$$P_{n,k}(q) := \sum_{i=0}^{2n+1-k} q^{n-1-2i} \sum_{l=i+1}^{i+k} \binom{2n+2}{l} q^l. \tag{15}$$

The quantities $Q_{n,k}(q) := q^{n+1-k} P_{n,k}(q)$ are PIC polynomials. Their first values are listed in Table 1.3.

$Q_{1,2}(q) = 6 + 8q + 6q^2;$
$Q_{2,2}(q) = 15 + 26q + 30q^2 + 26q^3 + 15q^4;$
$Q_{2,3}(q) = 20 + 30q + 32q^2 + 30q^3 + 20q^4;\quad Q_{2,4}(q) = Q_{2,2}(q)$
$Q_{3,2}(q) = 28 + 64q + 98q^2 + 112q^3 + 98q^4 + 64q^5 + 28q^6;$
$Q_{3,3}(q) = 56 + 98q + 120q^2 + 126q^3 + 120q^4 + 98q^5 + 56q^6;$
$Q_{3,4}(q) = 70 + 112q + 126q^2 + 128q^3 + 126q^4 + 112q^5 + 70q^6;$
$Q_{3,5}(q) = Q_{3,3}(q);\quad Q_{3,6}(q) = Q_{3,2}(q);$
$Q_{4,2}(q) = 45 + 130q + 255q^2 + 372q^3 + 420q^4 + 372q^5 + 255q^6 + 130q^7 + 45q^8;$
$Q_{4,3}(q) = 120 + 255q + 382q^2 + 465q^3 + 492q^4 + 465q^5 + 382q^6 + 255q^7 + 120q^8;$
$Q_{4,4}(q) = 210 + 372q + 465q^2 + 502q^3 + 510q^4 + 502q^5 + 465q^6 + 372q^7 + 210q^8;$
$Q_{4,5}(q) = 252 + 420q + 492q^2 + 510q^3 + 512q^4 + 510q^5 + 492q^6 + 420q^7 + 252q^8;$
$Q_{4,6}(q) = Q_{4,4}(q);\quad Q_{4,7}(q) = Q_{4,3}(q);\quad Q_{4,8}(q) = Q_{4,2}(q).$

Table 1.3. The polynomials $Q_{n,k}(q)$.

The proofs of Theorems 1.1 and 1.2 are given in Sections 3 and 4. In the last Section we obtain a global expression for the generating polynomial for the group B_n by a five-variable statistic, which takes the two classical *descent* definitions into account.

To end this introduction we point out that the identity

$$T_{2n+1} = 2^n \sum_{k=2}^{2n} d_{n,k}, \tag{16}$$

which is the $q = 1$ version of (13), is originally due to Christiane Poupard [18], who worked out the recurrence for the now called *Poupard triangle* $d_{n,k} := d_{n,k}(1)$ $(n \geq 1,\ 2 \leq k \leq 2n)$, obtainable from $(D1)$–$(D4)$ for $q = 1$.

We reproduce the first values of the Poupard triangle $(d_{n,k})$, together with the first values of

$$Q_{n,k} := Q_{n,k}(1) = P_{n,k}(1) = \sum_{i=0}^{2n+1-k} \sum_{l=i+1}^{i+k} \binom{2n+2}{l}. \tag{17}$$

Both $d_{n,k}$ and $Q_{n,k}$ are displayed in triangles $(2 \leq k \leq 2n,\ 1 \leq n \leq 4)$, as shown in Fig. 1.4.

The $q = 1$ version of identity (14) reads:

$$2^{n+1} E_{2n+2} = \sum_{k=2}^{2n} d_{n,k} Q_{n,k}. \tag{18}$$

For instance, (18) for $n = 2$ yields: $2^3E_6 = 8 \times 61 = 488 = 1 \times 112 + 2 \times 132 + 1 \times 112$. There exists a rich formulary of relations for tangent and secant numbers (see, e.g., the old monograph by Nielsen [17]). Identities (16) and (18) provide a new parametrization of those coefficients by means of the Poupard triangle $(d_{n,k})$.

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & 2 & 1 & & \\ & 4 & 8 & 10 & 8 & 4 & \\ 34 & 68 & 94 & 104 & 94 & 68 & 34 \end{array}$$

The Poupard triangle $(d_{n,k})$

$$\begin{array}{ccccccc} & & & 20 & & & \\ & & 112 & 132 & 112 & & \\ & 492 & 674 & 744 & 674 & 492 & \\ 2024 & 2936 & 3608 & 3860 & 3608 & 2936 & 2024 \end{array}$$

The coefficients $(Q_{n,k})$

Fig. 1.4.

2. STATISTICS ON THE HYPEROCTAHEDRAL GROUP

The elements of the hyperoctahedral group B_n, usually called *signed permutations*, may be viewed as *words* $w = x_1x_2\cdots x_n$, where each x_i belongs to the set $\{-n, \ldots, -1, 1, \ldots, n\}$ and $|x_1||x_2|\cdots|x_n|$ is a permutation of $12\ldots n$. The *set* (resp. the *number*) of *negative* letters among the x_i's is denoted by $\operatorname{Neg} w$ (resp. $\operatorname{neg} w$). In the same manner, let $\operatorname{Pos} w$ (resp. $\operatorname{pos} w$) be the set (resp. the number) of all positive letters in w. It is convenient to write $\bar{i} := -i$ for each integer i. There are $2^n n!$ signed permutations of order n. The symmetric group $\mathfrak{S}_n$ may be considered as the subset of all w from B_n such that $\operatorname{Neg} w = \varnothing$.

For each statement A let $\chi(A) = 1$ or 0 depending on whether A is true or not. The usual *number of descents* and *major index* of each word $w = x_1x_2\cdots x_n$ are defined by

$$\operatorname{des} w := \sum_{i=1}^{n-1} \chi(x_i > x_{i+1}); \tag{19}$$

$$\operatorname{maj} w := \sum_{i=1}^{n-1} i\,\chi(x_i > x_{i+1}). \tag{20}$$

When B_n is regarded as a Coxeter group, an extra descent is counted, when the first letter x_1 of the signed permutation $w = x_1x_2\cdots x_n$ is *negative.* In the literature two definitions are then used:

$$\operatorname{des}_B w := \chi(x_1 < 0) + \operatorname{des} w; \tag{21}$$

$$\operatorname{fdes} w := \chi(x_1 < 0) + 2\operatorname{des} w. \tag{22}$$

Furthermore, a *flag major index* "fmaj" defined by

$$\operatorname{fmaj} w := 2\operatorname{maj} w + \operatorname{neg} w, \tag{23}$$

has been adopted for B_n, because it is equidistributed with the *Coxeter length* "ℓ" for B_n (see, e.g., [1, 8]), a property that extends the corresponding property for the symmetric group $\mathfrak{S}_n$, which says that the major index "maj" and the number of inversions "inv" (the Coxeter length for $\mathfrak{S}_n$) are equidistributed.

Proposition 2.1. *The polynomial $B_n(t,q)$ defined by (3) has the following combinatorial interpretation:*

$$B_n(t,q)=\sum_{w\in B_n} t^{\operatorname{des}_B w} q^{\operatorname{fmaj} w}. \tag{24}$$

In other words, $B_n(t,q)$ is the generating polynomial for the hyperoctahedral group B_n by the pair $(\operatorname{des}_B, \operatorname{fmaj})$.

The proof of the proposition can be found in [5]. This is also a consequence of Theorem 6.2, that takes both "des_B" and "fdes" into account (see (75) and (76)).

From the definition of the polynomials $E_{2n+2}(q)$ given in (9) and (24) it follows that

$$E_{2n+2}(q)=(-1)^{n+1}q^{(n+1)^2}B_{2n+2}(-q^{-2n+2},q)$$

may be expressed as

$$E_{2n+2}(q)=(-1)^{n+1}\sum_{w=x_1\cdots x_{2n+2}\in B_{2n+2}}(-1)^{\chi(x_1<0)+\operatorname{des} w}q^{\operatorname{smaj} w}, \tag{25}$$

where "smaj" is a new statistic — call it *signed major index* — defined for each signed permutation $w=x_1x_2\cdots x_{2n+2}\in B_{2n+2}$ by

$$\operatorname{smaj} w:=(n+1)^2-2(n+1)\big(\chi(x_1<0)+\operatorname{des} w\big)+2\operatorname{maj} w+\operatorname{neg} w. \tag{26}$$

A *compressed major index* "cmaj" was defined in [11, 12] on the symmetric group $\mathfrak{S}_n$. Extend its definition to each $w\in B_{2n+2}$, as follows

$$\operatorname{cmaj} w:=\operatorname{maj} w-(n+1)\operatorname{des} w+(n-1)n/2. \tag{27}$$

The next lemma only needs a straightforward calculation.

Lemma 2.2. *For each $w=x_1x_2\cdots x_{2n+2}\in B_{2n+2}$ we have:*

$$\operatorname{smaj} w-2\operatorname{cmaj} w=3n+1+\operatorname{neg} w-2(n+1)\chi(x_1<0); \tag{28}$$

so that

$$\operatorname{smaj} w-2\operatorname{cmaj} w=n+\operatorname{neg} w-1, \qquad \textit{if } x_1<0. \tag{29}$$

The *mirror image* of a signed permutation $w=x_1x_2\cdots x_{2n+2}$ is defined by $\mathbf{r}\,w:=x_{2n+2}\cdots x_2x_1$. It is easily verified that

$$\operatorname{des}\mathbf{r}\,w=(2n+1)-\operatorname{des} w; \tag{30}$$

$$\operatorname{maj}\mathbf{r}\,w=(2n+2)(2n+1)/2-(2n+2)\operatorname{des} w+\operatorname{maj} w. \tag{31}$$

Those two relations suffice to prove the next lemma.

Lemma 2.3. *For each $w=x_1x_2\cdots x_{2n+2}\in B_{2n+2}$ we have:*

$$\operatorname{smaj}\mathbf{r}\,w-\operatorname{smaj} w=2(n+1)\big(\chi(x_1<0)-\chi(x_{2n+2}<0)\big); \tag{32}$$

$$(-1)^{\operatorname{des}\mathbf{r}\,w+\chi(x_{2n+2}<0)}\times(-1)^{\operatorname{des} w+\chi(x_1<0)}=-(-1)^{\chi(x_1<0)+\chi(x_{2n+2}<0)}. \tag{33}$$

The sum displayed in (25) may be decomposed into four subsums:

$$\begin{aligned}\sum_{w=x_1\cdots x_{2n+2}\in B_{2n+2}} &= \sum_{\substack{x_1x_{2n+2}>0,\\ x_1<x_{2n+2}}} + \sum_{\substack{x_1x_{2n+2}>0,\\ x_1>x_{2n+2}}} + \sum_{\substack{x_1<0,\\ x_{2n+2}>0}} + \sum_{\substack{x_1>0,\\ x_{2n+2}<0}}\\ &= \sum_{\substack{x_1x_{2n+2}>0,\\ x_1<x_{2n+2}}} + \mathbf{r}\sum_{\substack{x_1x_{2n+2}>0,\\ x_1<x_{2n+2}}} + \sum_{\substack{x_1<0,\\ x_{2n+2}>0}} + \mathbf{r}\sum_{\substack{x_1<0,\\ x_{2n+2}>0}}.\end{aligned}$$

It follows from Lemma 2.3 that the sum of the first two subsums vanishes, and the fourth one is equal to the product of the third one by q^{2n+2}. Thus,

$$E_{2n+2}(q) = (-1)^{n+1}(1+q^{2n+2}) \sum_{\substack{w=x_1\cdots x_{2n+2}\in B_{2n+2},\\ x_1<0<x_{2n+2}}} (-1)^{\operatorname{des} w+1} q^{\operatorname{smaj} w} \tag{34}$$

since $\chi(x_1 < 0) = 1$ for every w occurring in the sum. To pursue the calculation of $E_{2n+2}(q)$ we use the doubloon calculus, as developed in our previous two papers.

3. Doubloons

A *doubloon* of order $(2n+1)$ is defined to be a permutation of the word $012\cdots(2n+1)$, represented as a $2\times(n+1)$-matrix $\delta = \binom{a_0\,\cdots\,a_n}{b_0\,\cdots\,b_n}$. The word $a_0\cdots a_n b_n\cdots b_0$ is called the *reading* $\rho(\delta)$ of δ. Define $\operatorname{stat}\delta := \operatorname{stat}\rho(\delta)$, whenever "stat" is equal to "des," "maj," "fmaj," "cmaj," or "smaj." Let $F\,\delta := a_0$, $L\,\delta := b_0$. The set of all doubloons of order $(2n+1)$ is denoted by $\mathcal{D}_{2n+1}$. The subset of all doubloons δ such that $L\,\delta = j$ (resp. $F\,\delta = i$ and $L\,\delta = j$) is denoted by $\mathcal{D}_{2n+1,j}$ (resp. $\mathcal{D}^i_{2n+1,j}$).

Each doubloon $\delta = \binom{a_0\,\cdots\,a_n}{b_0\,\cdots\,b_n}$ from $\mathcal{D}_{2n+1}$ is said to be *interlaced* (resp. *normalized*), if for every $k = 1, 2, \ldots, n$ the sequence $(a_{k-1}, a_k, b_{k-1}, b_k)$ or one of its three *cyclic rearrangements* is monotonic increasing or decreasing (resp. decreasing). Let $\mathcal{I}^i_{2n+1}$ (resp. $\mathcal{I}^i_{2n+1,j}$, resp. $\mathcal{N}^i_{2n+1}$, resp. $\mathcal{N}^i_{2n+1,j}$) denote the set of all doubloons δ from $\mathcal{D}^i_{2n+1}$, which are interlaced (resp. interlaced with $L\,\delta = j$, resp. normalized, resp. normalized with $L\,\delta = j$).

For instance, the doubloon $\delta = \binom{0\,4\,3}{2\,1\,5}$ is normalized, since both sequences $(4,2,1,0)$ and $(5,4,3,1)$, which are cyclic rearrangements of $(0,4,2,1)$ and $(4,3,1,5)$, respectively, are decreasing.

The geometry of interlaced and normalized doubloons has been studied in [11]. The connection between interlaced doubloons and *split-pair arrangements*, introduced by Graham and Zang [14], is explicitly made in [12].

We now recall several properties on doubloons already proved in [11, 12]. For each doubloon $\delta = \binom{a_0\,a_1\,\cdots\,a_n}{b_0\,b_1\,\cdots\,b_n}$ from $\mathcal{D}_{2n+1}$ and each integer h let $\delta + h$ be the doubloon

$$\delta + h := \begin{pmatrix} a_0+h & a_1+h & \cdots & a_n+h\\ b_0+h & b_1+h & \cdots & b_n+h\end{pmatrix}, \tag{35}$$

where each entry is expressed as a residue $\bmod(2n+2)$.

Property 3.1. *The mapping $\delta \mapsto \delta + h$ is a bijection of $\mathcal{I}^i_{2n+1,j}$ (resp. $\mathcal{N}^i_{2n+1,j}$) onto $\mathcal{I}^{i+h}_{2n+1,j+h}$ (resp. $\mathcal{N}^{i+h}_{2n+1,j+h}$) (superscript and subscript being taken* $\bmod(2n+2)$*).*

See [12], Proposition 2.1.

Property 3.2. *Let $0 \le i < j$ and $\delta = \binom{a_0\, a_1 \cdots a_n}{b_0\, b_1 \cdots b_n}$ be a doubloon from $\mathcal{D}^i_{2n+1,j}$, so that $\delta - i = \binom{0 \quad a_1 - i \cdots a_n - i}{j-i\, b_1 - i \cdots b_n - i}$ belongs to $\mathcal{D}^0_{2n+1,j-i}$. Then,*

$$\operatorname{des}(\delta - i) = \operatorname{des}\delta, \qquad \operatorname{cmaj}(\delta - i) = \operatorname{cmaj}\delta + i. \tag{36}$$

See [12], Lemma 3.2.

Property 3.3. *For each integer k there is a sign-reversing involution on $\mathcal{D}^0_{2n+1,k} \setminus \mathcal{I}^0_{2n+1,k}$ having the property that*

$$\sum_{\delta \in \mathcal{D}^0_{2n+1,k}} (-1)^{n+\operatorname{des}\delta} q^{\operatorname{cmaj}\delta} = \sum_{\delta \in \mathcal{I}^0_{2n+1,k}} q^{\operatorname{cmaj}\delta}. \tag{37}$$

Moreover,

$$\sum_{\delta \in \mathcal{I}^0_{2n+1,k}} q^{\operatorname{cmaj}\delta} = (1+q)(1+q^2)\cdots(1+q^n) \sum_{\delta' \in \mathcal{N}^0_{2n+1,k}} q^{\operatorname{cmaj}\delta}. \tag{38}$$

Proof. Refer to the proofs of Theorems 4.2 and 1.6 in [11], and observe that the first column $\binom{0}{k}$ is left invariant under each macro flip. $\square$

4. Signed doubloons

Now, we extend the notion of doubloon to the group of signed permutations and speak of *signed doubloons*, but only for those signed permutations $w = x_1 x_2 \cdots x_{2n+2} \in B_{2n+2}$ occurring in the summation displayed in (34). They have the property that $F\,w := x_1 < 0 < x_{2n+2} =: L\,w$. We represent them as $2 \times (n+1)$-matrices $w = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n+1} \\ x_{2n+2} & x_{2n+1} & \cdots & x_{n+2} \end{pmatrix}$. The set of all those signed doubloons will be denoted by $\mathcal{SD}_{2n+2}$.

For each $w = \begin{pmatrix} x_1 & x_2 & \cdots & x_{n+1} \\ x_{2n+2} & x_{2n+1} & \cdots & x_{n+2} \end{pmatrix}$ from $\mathcal{SD}_{2n+2}$ let ϕ_w be the increasing bijection of $\{x_1, x_2, \ldots, x_{2n+2}\}$ onto $\{0, 1, 2, \ldots, 2n+1\}$ and form the (unsigned) doubloon $\delta_w := \begin{pmatrix} \phi_w(x_1) & \phi_w(x_2) & \cdots & \phi_w(x_{n+1}) \\ \phi_w(x_{2n+2}) & \phi_w(x_{2n+1}) & \cdots & \phi_w(x_{n+2}) \end{pmatrix}$. The signed doubloon w is characterized by the pair $(\delta_w, -\operatorname{Neg} w)$. Moreover, $\operatorname{stat} w = \operatorname{stat}\delta_w$ whenever "stat" is equal to "des," "maj," "fmaj," "cmaj," or "smaj." The signed doubloon w is said to be *interlaced* (resp. *normalized*), if δ_w is interlaced (resp. normalized).

As $F\,w < 0 < L\,w$ when w belongs to $\mathcal{SD}_{2n+2}$, the mapping

$$w \mapsto (\delta_w, -\operatorname{Neg} w) \tag{39}$$

is a bijection of the set $\mathcal{SD}_{2n+2}$ onto the set of pairs (δ, J) such that $\delta \in \mathcal{D}_{2n+1}$, and J a subset of $\{1, 2, \ldots, 2n+2\}$ such that $F\,\delta + 1 \le \#J \le L\,\delta$.

For instance, if $\delta = \binom{042}{315} \in \mathcal{D}_5$, then $F\delta + 1 = 1 \le \#J \le 3 = L\delta$. Take $J = \{3\}, \{1,3\}, \{2,3,5\}$ for example, the three signed doubloons $w \in \mathcal{SD}_6$ associated with those three subsets J are the following:

$$\begin{aligned}
\binom{\overline{3}52}{416} &\mapsto \left(\binom{042}{315}, \{3\}\right),\\
\binom{\overline{3}52}{4\overline{1}6} &\mapsto \left(\binom{042}{315}, \{1,3\}\right),\\
\binom{\overline{5}4\overline{2}}{1\overline{3}6} &\mapsto \left(\binom{042}{315}, \{2,3,5\}\right).
\end{aligned}$$

If $(\delta_w, -\operatorname{Neg} w) = (\delta, J)$, then (see (29))

$$\operatorname{des} w = \operatorname{des} \delta; \quad \operatorname{smaj} w = 2\operatorname{cmaj} \delta + \#J + n - 1. \tag{40}$$

We next make the composition product of the two mappings described in (35) and (39).

Theorem 4.1. *For each pair (i,k) of integers such that $1 \le k \le 2n$ and $0 \le i \le 2n+1-k$ the mapping*

$$w \mapsto (\delta_w - i,\ -\operatorname{Neg} w) \tag{41}$$

is a bijection of the set $\mathcal{SD}^i_{2n+2,i+k}$ of the signed doubloons w satisfying $F\,\delta_w = i$, $L\,\delta_w = i+k$ onto the set of pairs (δ, J) such that $\delta \in \mathcal{D}^0_{2n+1,k}$ and $J \subset [1, 2n+2]$ with $i+1 \le \#J \le i+k$. Moreover, if w is interlaced (resp. normalized), so is $\delta_w - i$, and conversely. Finally, if $\delta = \delta_w - i$, then

$$\operatorname{des} w = \operatorname{des} \delta; \quad \operatorname{smaj} w = 2\operatorname{cmaj} \delta - 2i + \#J + n - 1. \tag{42}$$

Proof. The theorem is a consequence of Properties 3.1 and 3.2 and the properties of the bijection $w \mapsto (\delta_w, -\operatorname{Neg} w)$ given in (40). $\square$

Identity (34) may be rewritten as

$$\begin{aligned}
E_{2n+2}(q) &= (-1)^{n+1}(1+q^{2n+2}) \sum_{w \in \mathcal{SD}_{2n+2}} (-1)^{\operatorname{des} w + 1} q^{\operatorname{smaj} w}\\
&= (1+q^{2n+2}) \sum_{k=1}^{2n} \sum_{i=0}^{2n+1-k} \sum_{w \in \mathcal{SD}^i_{2n+2,i+k}} (-1)^{n + \operatorname{des} w} q^{\operatorname{smaj} w}.
\end{aligned}$$

Let

$$P_{n,i,k}(q) := q^{n-1-2i} \sum_{l=i+1}^{i+k} \binom{2n+2}{l} q^l. \tag{43}$$

Using the preceding theorem and Property 3.3 we evaluate the third sum as follows.

$$\begin{aligned}
\sum_{w\in\mathcal{SD}^i_{2n+2,i+k}} & (-1)^{n+\operatorname{des} w} q^{\operatorname{smaj} w} \\
&= \sum_{\delta\in\mathcal{D}^0_{2n+1,k}} \sum_{i+1\le \#J\le i+k} (-1)^{n+\operatorname{des}\delta} q^{2\operatorname{cmaj}\delta-2i+\#J+n-1} \\
&= q^{n-1-2i} \sum_{\delta\in\mathcal{D}^0_{2n+1,k}} (-1)^{n+\operatorname{des}\delta} q^{2\operatorname{cmaj}\delta} \sum_{l=i+1}^{i+k} \binom{2n+2}{l} q^l \\
&= P_{n,i,k}(q) \sum_{\delta\in\mathcal{D}^0_{2n+1,k}} (-1)^{n+\operatorname{des}\delta} q^{2\operatorname{cmaj}\delta} \\
&= P_{n,i,k}(q) \sum_{\delta\in\mathcal{I}^0_{2n+1,k}} q^{2\operatorname{cmaj}\delta} \\
&= (1+q^2)\cdots(1+q^{2n}) P_{n,i,k}(q) \sum_{\delta\in\mathcal{N}^0_{2n+1,k}} q^{2\operatorname{cmaj}\delta} \\
&= (1+q^2)\cdots(1+q^{2n}) P_{n,i,k}(q)\, d_{n,k}(q^2), \qquad (44)
\end{aligned}$$

where the last equality follows from [12], Theorem 1.2. By multiplying (44) by $(1+q^{2n+2})$ and summing over all pairs (k,i) such that $1\le k\le 2n$ and $0\le i\le 2n+1-k$ we derive identity (14), keeping in mind that $P_{n,k}(q) = \sum_{0\le i\le 2n+1-k} P_{n,i,k}$.

This achieves the proofs of both Theorems 1.1 and 1.2, except part (d).

Let $\mathcal{SN}^i_{2n+2,i+k}$ be the set of the normalized signed doubloons w satisfying $F\,\delta_w = i$, $L\,\delta_w = i+k$. It also follows from Theorem 4.1 that

$$\sum_{w\in\mathcal{SN}^i_{2n+2,i+k}} q^{\operatorname{smaj} w} = P_{n,i,k}(q) \sum_{\delta\in\mathcal{N}^0_{2n+1,k}} q^{2\operatorname{cmaj}\delta}. \qquad (45)$$

From (44) it follows that

$$\sum_{w\in\mathcal{SD}^i_{2n+2,i+k}} (-1)^{n+\operatorname{des} w} q^{\operatorname{smaj} w} = (1+q^2)\cdots(1+q^{2n}) \sum_{w\in\mathcal{SN}^i_{2n+2,i+k}} q^{\operatorname{smaj} w}. \qquad (46)$$

By multiplying (46) by $(1+q^{2n+2})$ and summing over all pairs (k,i) such that $1\le k\le 2n$ and $0\le i\le 2n+1-k$ we derive identity (11).

5. Proof of theorem 1.1 (d)

Recall that for each $w = x_1x_2\cdots x_{2n+1}\in B_{2n+1}$ we have used the notations $F\,w := x_1$ and $L\,w := x_{2n+1}$. As $B_{2n+1}(t,q) = \sum_{w\in B_{2n+1}} t^{\operatorname{des}_B w} q^{\operatorname{fmaj} w}$, we may write

$$B_{2n+1}(-q^{-(2n+1)}, q) = \sum_{w\in B_{2n+1}} (-1)^{\operatorname{sgn} w} q^{\operatorname{smaj} w}, \qquad (47)$$

where

$$\text{(48)}\qquad \operatorname{sgn} w := (-1)^{\operatorname{des} w + \chi(F\,w<0)};$$

$$\text{(49)}\qquad \operatorname{smaj} w := 2\operatorname{maj} w + \operatorname{neg} w - (2n+1)(\operatorname{des} w + \chi(F\,w<0)),$$

as there is no ambiguity to adopt this definition of "smaj" for signed permutations from B_{2n+1}.

For proving the identity $A_{2n}(-q^{-n},q)=0$ in [11] we had recourse to the classical properties of the dihedral group acting on $\mathfrak{S}_{2n}$. Actually, the mirror image $\mathbf{r}$ provided the sign-reversing involution that was needed. With the group B_{2n+1} the supplementary descent to be counted, when the first letter is negative, makes it necessary to include another dihedral group involution, as well as a sign change operation.

In this section the elements of B_{2n+1} will be regarded as two-row matrices $w=\binom{|w|}{\epsilon}:=\binom{|x_1|\,|x_2|\,\cdots\,|x_{2n+1}|}{\epsilon_1\ \epsilon_2\ \cdots\ \epsilon_{2n+1}}$, where $|w|:=|x_1||x_2|\cdots|x_{2n+1}|$ becomes an *ordinary* permutation and $\epsilon:=\epsilon_1\epsilon_2\cdots\epsilon_{2n+1}$ is the sign word defined by $\epsilon_i:=1$ or -1, depending on whether x_i is positive or negative $(1\le i\le 2n+1)$.

Three operations $\mathbf{r}$, $\mathbf{c}$, $\mathbf{s}$ are now introduced and further extended to all of B_{2n+1}: first, the *mirror image*

$$\mathbf{r}: y_1y_2\cdots y_{2n+1}\mapsto y_{2n+1}\cdots y_2y_1,$$

defined for every *arbitrary word*; second, the *complement to* $(2n+2)$, defined for each *permutation* from $\mathfrak{S}_{2n+1}$, by

$$\mathbf{c}: y_1y_2\cdots y_{2n+1}\mapsto (2n+2-y_1)(2n+2-y_2)\cdots(2n+2-y_{2n+1});$$

third, the *sign change* $\mathbf{s}$, defined for each *binary word*, such as $\epsilon=\epsilon_1\epsilon_2\cdots\epsilon_{2n+1}$, whose letters are equal to $+1$ or -1, by

$$\mathbf{s}: \epsilon_1\epsilon_2\cdots\epsilon_{2n+1}\mapsto \overline{\epsilon}_1\overline{\epsilon}_2\cdots\overline{\epsilon}_{2n+1}.$$

We use the same symbols for their extensions to B_{2n+1}:

$$\text{(50)}\qquad \mathbf{r}:\binom{|w|}{\epsilon}=\binom{|x_1|\,|x_2|\cdots|x_{2n+1}|}{\epsilon_1\ \epsilon_2\ \cdots\ \epsilon_{2n+1}}\mapsto\binom{\mathbf{r}\,|w|}{\mathbf{r}\,\epsilon}=\binom{|x_{2n+1}|\cdots|x_2||x_1|}{\epsilon_{2n+1}\ \cdots\ \epsilon_2\ \epsilon_1};$$

$$\text{(51)}\qquad \mathbf{c}:\binom{|w|}{\epsilon}=\binom{|x_1|\cdots|x_{2n+1}|}{\epsilon_1\ \cdots\ \epsilon_{2n+1}}\mapsto\binom{\mathbf{c}\,|w|}{\epsilon}=\binom{(2n+2-|x_1|)\cdots(2n+2-|x_{2n+1}|)}{\epsilon_1\qquad\cdots\qquad\epsilon_{2n+1}};$$

$$\text{(52)}\qquad \mathbf{s}:\binom{|w|}{\epsilon}=\binom{|x_1|\,|x_2|\cdots|x_{2n+1}|}{\epsilon_1\ \epsilon_2\ \cdots\ \epsilon_{2n+1}}\mapsto\binom{|w|}{\mathbf{s}\,\epsilon}=\binom{|x_1|\,|x_2|\cdots|x_{2n+1}|}{\overline{\epsilon}_1\ \overline{\epsilon}_2\ \cdots\ \overline{\epsilon}_{2n+1}}.$$

Note that the three involutions $\mathbf{r}$, $\mathbf{c}$, $\mathbf{s}$, defined on B_{2n+1} by (50), (51) and (52) *commute*. The composition product $\mathbf{b}:=\mathbf{c}\,\mathbf{s}\,\mathbf{r}$ can also be written as

$$\text{(53)}\qquad \mathbf{b}:\binom{|w|}{\epsilon}=\binom{|x_1|\cdots|x_{2n+1}|}{\epsilon_1\ \cdots\ \epsilon_{2n+1}}\mapsto\binom{\mathbf{c}\,\mathbf{r}\,|w|}{\mathbf{r}\,\mathbf{s}\,\epsilon}=\binom{(n+2-|x_{2n+1}|)\cdots(n+2-|x_1|)}{\overline{\epsilon}_{2n+1}\qquad\cdots\qquad\overline{\epsilon}_1}.$$

Theorem 5.1. *The composition product* $\mathbf{b}$ *defined in (53) is a sign-reversing involution of* B_{2n+1}*, i.e.,*

$$\text{(54)}\qquad (\operatorname{sgn},\operatorname{smaj})\binom{|w|}{\epsilon}=(-\operatorname{sgn},\operatorname{smaj})\mathbf{b}\binom{|w|}{\epsilon}.$$

The proof of the theorem is based on the next three lemmas. The first two ones being easy to verify are given without proofs.

Lemma 5.2. *For each* $w = \binom{|w|}{\epsilon} \in B_{2n+1}$ *we have*

$$\operatorname{sgn} \mathbf{r}\, w = \operatorname{sgn} w \cdot (-1)^{\chi(L\, w<0)+\chi(F\, w<0)}; \tag{55}$$

$$\operatorname{smaj} \mathbf{r}\, w = \operatorname{smaj} w + (2n+1)\big(\chi(F\, w < 0) - \chi(L\, w < 0)\big). \tag{56}$$

Lemma 5.3. *For each* $w = \binom{|w|}{\epsilon} \in B_{2n+1}$ *we have:*

$$\operatorname{sgn} \mathbf{s}\, w = -\operatorname{sgn} w; \tag{57}$$

$$\operatorname{smaj} \mathbf{s}\, w = -\operatorname{smaj} w. \tag{58}$$

The third lemma requires a careful analysis.

Lemma 5.4. *For each* $w = \binom{|w|}{\epsilon} = \binom{|x_1|\,|x_2|\,\cdots\,|x_{2n+1}|}{\epsilon_1\ \epsilon_2\ \cdots\ \epsilon_{2n+1}} \in B_{2n+1}$ *we have:*

$$\operatorname{sgn} \mathbf{c}\, w = (-1)^{\chi(F\, w<0)-\chi(L\, w<0)} \operatorname{sgn} w; \tag{59}$$

$$\operatorname{smaj} \mathbf{c}\, w = -\operatorname{smaj} w - (2n+1)\big(\chi(F\, w < 0) - \chi(L\, w < 0)\big). \tag{60}$$

Proof. If $\binom{|x_i|}{\epsilon_i} > \binom{|x_{i+1}|}{\epsilon_{i+1}}$ (resp. $\binom{|x_i|}{\epsilon_i} < \binom{|x_{i+1}|}{\epsilon_{i+1}}$) say that i is an *interior* descent (resp. rise), if $|x_i| > |x_{i+1}|$ (resp. $|x_i| < |x_{i+1}|$) *and* $\epsilon_i = \epsilon_{i+1}$. Denote the set of all descents (resp. rises) of w by $\operatorname{DES} w$ (resp. $\operatorname{RISE} w$), the set of all *interior* descents (resp. rises) being designated by $\operatorname{DES}^i w$ (resp. $\operatorname{RISE}^i w$), so that $\operatorname{DES} w = \operatorname{DES}^i w + \operatorname{DES} \epsilon$ and $\operatorname{RISE} w = \operatorname{RISE}^i w + \operatorname{RISE} \epsilon$.

First, $\operatorname{DES}^i w = \operatorname{RISE}^i \mathbf{c}\, w$ and $\operatorname{RISE}^i w = \operatorname{DES}^i \mathbf{c}\, w$. Hence,

$$\begin{aligned}
\operatorname{des} w + \operatorname{des} \mathbf{c}\, w &= (\#\operatorname{DES}\epsilon + \#\operatorname{DES}^i w) + (\#\operatorname{DES}\epsilon + \#\operatorname{DES}^i \mathbf{c}\, w)\\
&= (\#\operatorname{DES}^i w + \#\operatorname{DES}\epsilon) + (\#\operatorname{RISE}^i w + \#\operatorname{RISE}\epsilon)\\
&\qquad\qquad + (\#\operatorname{DES}\epsilon - \#\operatorname{RISE}\epsilon)\\
&= 2n + (\#\operatorname{DES}\epsilon - \#\operatorname{RISE}\epsilon)\\
&:= 2n + \operatorname{drise}\epsilon.
\end{aligned}$$

In the same way, let $\operatorname{DRISE}\epsilon := \sum_i i\,\big(\chi(i \in \operatorname{DES}\epsilon) - \chi(i \in \operatorname{RISE}\epsilon)\big)$. Then,

$$\begin{aligned}
\operatorname{maj} w + \operatorname{maj} \mathbf{c}\, w &= \sum_i \big(i\,\chi(i \in \operatorname{DES}\epsilon) + i\,\chi(i \in \operatorname{DES}^i w)\big)\\
&\qquad + \sum_i \big(i\,\chi(i \in \operatorname{DES}\epsilon) + i\,\chi(i \in \operatorname{DES}^i \mathbf{c}\, w)\big)\\
&= \sum_i \big(i\,\chi(i \in \operatorname{DES} w) + i\,\chi(i \in \operatorname{RISE} w)\big)\\
&\qquad + \sum_i i\,\big(\chi(i \in \operatorname{DES}\epsilon) - \chi(i \in \operatorname{RISE}\epsilon)\big)\\
&= (1 + 2 + \cdots + 2n) + \operatorname{DRISE}\epsilon\\
&= n(2n+1) + \operatorname{DRISE}\epsilon.
\end{aligned}$$

Let $d_1 < d_2 < \cdots$ (resp. $r_1 < r_2 < \cdots$) denote the sequence of the descents (resp. rises) of ϵ, when reading the word ϵ from left to right. Four cases are now considered.

(a) $\epsilon_1=\epsilon_{2n+1}=-1$; the rises and descents alternate in such a way that $1\le r_1<d_1<r_2<d_2<\cdots<r_k<d_k\le 2n$ and $k\ge 0$. Hence, $\operatorname{drise}\epsilon=0$ and $\operatorname{DRISE}\epsilon=\sum_{i=1}^{k}(d_i-r_i)=\operatorname{pos}w$.

(b) $\epsilon_1=+1$, $\epsilon_{2n+1}=-1$; the alternation becomes: $1\le d_1<r_1<d_2<r_2<\cdots<d_k<r_k<d_{k+1}\le 2n$ ($k\ge 0$). In this case, $\operatorname{drise}\epsilon=1$ and $\operatorname{DRISE}\epsilon=\operatorname{pos}w$.

(c) $\epsilon_1=\epsilon_{2n+1}=1$; the sequence is then: $1\le d_1<r_1<d_2<r_2<\cdots<d_k<r_k\le 2n$ ($k\ge 0$). Hence, $\operatorname{drise}\epsilon=0$ and $\operatorname{DRISE}\epsilon=-\operatorname{neg}w$.

(d) $\epsilon_1=-1$, $\epsilon_{2n+1}=1$; then $1\le r_1<d_2<r_2<\cdots<r_k<d_k<r_{r+1}\le 2n$ ($k\ge 0$). Hence, $\operatorname{drise}\epsilon=-1$ and $\operatorname{DRISE}\epsilon=-\operatorname{neg}w$.

Thus, $\operatorname{sgn}w+\operatorname{sgn}\mathbf{c}\,w=\operatorname{des}w+\chi(\epsilon_1<0)+\operatorname{des}\mathbf{c}\,w+\chi(\epsilon_1<0)\equiv\operatorname{drise}\epsilon \pmod 2$, which is 0 when ϵ_1 and ϵ_{2n+1} are of the same sign (cases (a) and (c)), equal to 1 when $\epsilon_1=1$, $\epsilon_{2n+1}=-1$ (case (b)) and -1 when $\epsilon_1=-1$ and $\epsilon_{2n+1}=+1$ (case (d)). Gathering in a common formula: $\operatorname{sgn}w+\operatorname{sgn}\mathbf{c}\,w\equiv\chi(\epsilon_1=1)-\chi(\epsilon_{2n+1}=1)$. This implies (59).

Finally,

$$\begin{aligned}
\operatorname{smaj}\mathbf{c}\,w+\operatorname{smaj}w&=2(\operatorname{maj}\mathbf{c}\,w+\operatorname{maj}w)+\operatorname{neg}\mathbf{c}\,w+\operatorname{neg}w\\
&\qquad-(2n+1)(\operatorname{des}\mathbf{c}\,w+\operatorname{des}w+2\chi(\epsilon_1<0))\\
&=2n(2n+1)+2\operatorname{DRISE}\epsilon+2\operatorname{neg}w\\
&\qquad-(2n+1)(2n+\operatorname{drise}\epsilon+2\chi(\epsilon_1<0))\\
&=2\operatorname{DRISE}\epsilon+2\operatorname{neg}w\\
&\qquad-(2n+1)(\operatorname{drise}\epsilon+2\chi(\epsilon_1<0))
\end{aligned}$$

$$=\begin{cases}2\operatorname{pos}w+2\operatorname{neg}w-(2n+1)2=0, & \text{in case } (a);\\ 2\operatorname{pos}w+2\operatorname{neg}w-(2n+1)=2n+1, & \text{in case } (b);\\ -2\operatorname{neg}w+2\operatorname{neg}w-(2n+1)0=0, & \text{in case } (c);\\ -2\operatorname{neg}w+2\operatorname{neg}w-(2n+1)=-(2n+1), & \text{in case } (d).\end{cases}$$

Altogether, $\operatorname{smaj}\mathbf{c}\,w=-\operatorname{smaj}w-(2n+1)\big(\chi(\epsilon_1=-1)-\chi(\epsilon_{2n+1}=-1)\big)$. This proves (60) and also Lemma 5.4. □

Proof of Theorem 5.1. Let $w\in B_{2n+1}$. By the previous three lemmas

$$\begin{aligned}
\operatorname{sgn}\mathbf{r}\,\mathbf{c}\,w&=\operatorname{sgn}\mathbf{c}\,w\cdot(-1)^{\chi(L\ \mathbf{c}\,w<0)+\chi(F\ \mathbf{c}\ w<0)}\\
&=\operatorname{sgn}w(-1)^{\chi(F\,w<0)-\chi(L\,w<0)}(-1)^{\chi(L\,w<0)+\chi(F\,w<0)}\\
&=\operatorname{sgn}w;\\
\operatorname{smaj}\mathbf{r}\,\mathbf{c}\,w&=\operatorname{smaj}\mathbf{c}\,w+(2n+1)(\chi(F\,\mathbf{c}\,w<0)-\chi(L\,\mathbf{c}\,w<0))\\
&=\operatorname{smaj}\mathbf{c}\,w+(2n+1)(\chi(F\,w<0)-\chi(L\,w<0))\\
&=-\operatorname{smaj}w;\\
\operatorname{sgn}\mathbf{s}\,\mathbf{r}\,\mathbf{c}\,w&=-\operatorname{sgn}\mathbf{r}\,\mathbf{c}\,w=-\operatorname{sgn}w;\\
\operatorname{smaj}\mathbf{s}\,\mathbf{r}\,\mathbf{c}\,w&=-\operatorname{smaj}\mathbf{r}\,\mathbf{c}\,w=\operatorname{smaj}w.
\end{aligned}$$

□

6. Which descent for the hyperoctahedral group?

The purpose of this Section is to work out a *global* expression for the generating polynomial for B_n by the five-term statistic $(\mathrm{neg}, \mathrm{pos}, \Xi, \mathrm{des}, \mathrm{fmaj})$, where $\Xi\, w$ is equal to 1 or 0, depending on whether the first letter of w is negative or positive, and to derive the specializations when the pair (Ξ, des) is replaced either by "des_B," or by "fdes," defined in (21) and (22). Our main result is the following.

Theorem 6.1. *Let*

$$B_n(X,Y,Z;t,q) = \sum_{w\in B_n} X^{\mathrm{neg}\, w} Y^{\mathrm{pos}\, w} Z^{\chi(x_1<0)} t^{\mathrm{des}\, w} q^{\mathrm{fmaj}\, w}. \tag{61}$$

Then,

$$\begin{aligned}\frac{B_n(X,Y,Z;t,q)}{(t;q^2)_{n+1}} &= \frac{t-Z}{t-1}\sum_{s\geq 0} t^s\big((qX+Y)[s+1]_{q^2}\big)^n\\ &\quad + \frac{Z-1}{t-1}\sum_{s\geq 0} t^s\big((qX+Y)[s+1]_{q^2} - Xq^{2s+1}\big)^n.\end{aligned} \tag{62}$$

When $q=1$, write $B_n(X,Y,Z;t) := B_n(X,Y,Z;t,1)$. The exponential generating function for the latter polynomials can be derived in the following form.

Theorem 6.2. *The following identity holds:*

$$\sum_{n\geq 0}\frac{u^n}{n!}\, B_n(X,Y,Z;t) = \frac{Z-t+(1-Z)\exp(uX(t-1))}{-t+\exp(u(X+Y)(t-1))}. \tag{63}$$

Proof of Theorem 6.1. Let $w = x_1x_2\cdots x_n$ be a signed permutation from B_n and ϕ be the unique increasing bijection of the set $\{x_1,x_2,\ldots,x_n\}$ onto the interval $[n]{:=}\{1,2,\ldots,n\}$. The word

$$\sigma = \sigma(1)\sigma(2)\cdots\sigma(n) := \phi(x_1)\phi(x_2)\cdots\phi(x_n)$$

is then an (ordinary) permutation from $\mathfrak{S}_n$ and the map $w\mapsto(\mathrm{Neg}\, w,\sigma)$ a bijection of B_n onto the Cartesian product $2^{[n]}\times\mathfrak{S}_n$ having the following properties:

$$\chi(x_1<0) = \chi(\sigma(1)\leq \mathrm{neg}\, w);\quad \mathrm{des}\, w = \mathrm{des}\,\sigma;\quad \mathrm{fmaj}\, w = \mathrm{fmaj}\,\sigma.$$

For convenience, introduce the polynomial

$$A_n^k(Z;t,q) := \sum_{\sigma} Z^{\chi(\sigma(1)\leq k)} t^{\mathrm{des}\,\sigma} q^{\mathrm{maj}\,\sigma}\quad (\sigma=\sigma(1)\cdots\sigma(n)\in\mathfrak{S}_n)$$

and express $B_n(X,Y,Z;t,q)$ in terms of the latter polynomials, to get:

$$\begin{aligned} B_n(X,Y,Z;t,q) &= \sum_{k=0}^{n} \sum_{|E|=k} \sum_{\mathrm{Neg}\, w=E} (qX)^{\mathrm{neg}\, w} Y^{\mathrm{pos}\, w} Z^{\chi(x_1<0)} t^{\mathrm{des}\, w} q^{2\,\mathrm{maj}\, w} \\ &= \sum_{k=0}^{n} (qX)^k Y^{n-k} \sum_{|E|=k} \sum_{(E,\sigma)} Z^{\chi(\sigma(1)\le k)} t^{\mathrm{des}\,\sigma} q^{2\,\mathrm{maj}\,\sigma} \\ &= \sum_{k=0}^{n} \binom{n}{k} (qX)^k Y^{n-k} A_n^k(Z;t,q^2). \end{aligned}$$

Next, with each permutation $\sigma = k\,\sigma(2)\cdots\sigma(n)$ starting with k associate the permutation $\sigma' = \sigma'(1)\cdots\sigma'(n-1) := \psi(\sigma(2))\cdots\psi(\sigma(n))$, where ψ is the unique increasing bijection of $[n]\setminus\{k\}$ onto $[n-1]$. If $\sigma(2)\le k-1$, then $\mathrm{des}\,\sigma = \mathrm{des}\,\sigma' + 1$, while $\mathrm{maj}\,\sigma = \mathrm{maj}\,\sigma' + \mathrm{des}\,\sigma' + 1$ and $\sigma'(1)\le k-1$. If $\sigma(2)\ge k+1$, then $\mathrm{des}\,\sigma = \mathrm{des}\,\sigma'$, while $\mathrm{maj}\,\sigma = \mathrm{maj}\,\sigma' + \mathrm{des}\,\sigma'$ and $\sigma'(1)\ge k$. Hence,

$$\begin{aligned} \sum_{\sigma(1)=k,\sigma(2)\le k-1} Z^{\chi(\sigma(1)\le k)} t^{\mathrm{des}\,\sigma} q^{\mathrm{maj}\,\sigma} &= Z \sum_{\sigma'(1)\le k-1} t^{\mathrm{des}\,\sigma'+1} q^{\mathrm{maj}\,\sigma'+\mathrm{des}\,\sigma'+1} \\ &= Z \sum_{\sigma'(1)\le k-1} (tq)^{\chi(\sigma(1)\le k-1)} (tq)^{\mathrm{des}\,\sigma'} q^{\mathrm{maj}\,\sigma'}; \end{aligned}$$

while

$$\begin{aligned} \sum_{\substack{\sigma(1)=k,\\ \sigma(2)\ge k+1}} Z^{\chi(\sigma(1)\le k)} t^{\mathrm{des}\,\sigma} q^{\mathrm{maj}\,\sigma} &= Z \sum_{\sigma'(1)\ge k} t^{\mathrm{des}\,\sigma'} q^{\mathrm{maj}\,\sigma'+\mathrm{des}\,\sigma'} \\ &= Z \sum_{\sigma'(1)\ge k} (tq)^{\chi(\sigma(1)\le k-1)} (tq)^{\mathrm{des}\,\sigma'} q^{\mathrm{maj}\,\sigma'}. \end{aligned}$$

Altogether

$$\sum_{\sigma(1)=k} Z^{\chi(\sigma(1)\le k)} t^{\mathrm{des}\,\sigma} q^{\mathrm{maj}\,\sigma} = Z\, A_{n-1}^{k-1}(tq;tq,q).$$

In the same manner,

$$\sum_{\sigma(1)=k} Z^{\chi(\sigma(1)\le k-1)} t^{\mathrm{des}\,\sigma} q^{\mathrm{maj}\,\sigma} = A_{n-1}^{k-1}(tq;tq,q).$$

Consequently, we have the relation:

$$A_n^k(Z;t,q) = A_n^{k-1}(Z;t,q) + (Z-1)A_{n-1}^{k-1}(tq;tq,q). \tag{64}$$

By iteration we are led to:

$$\begin{aligned} A_n^k(Z;t,q) &= A_n^0(Z;t,q) \\ &\quad + \frac{Z-1}{t-1} \sum_{j=1}^{k} \binom{k}{j} (t-1)(tq-1)\cdots(tq^{j-1}) A_{n-j}^0(tq^j, tq^j, q). \end{aligned} \tag{65}$$

But, the variable Z vanishes from $A_n^k(Z;t,q)$ when $k=0$ and then $A_n^0(Z;t,q)=A_n(t,q)$, which is the Carlitz q-analog of the Eulerian polynomial ([3, 4]) appearing in (1). Hence,

$$\begin{aligned}A_n^k(Z;t,q) &= A_n(t,q)+\frac{Z-1}{t-1}\sum_{j=1}^{k}\binom{k}{j}(-1)^j\,(t;q)_j\,A_{n-j}(tq^j,q)\\ &= \frac{t-Z}{t-1}A_n(t,q)+\frac{Z-1}{t-1}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\,(t;q)_j\,A_{n-j}(tq^j,q).\end{aligned}$$

The next step is to report this new expression of $A_n^k(Z;t,q)$ into the polynomial $B_n(X,Y,Z;t,q)$. We get:

$$\begin{aligned}B_n(X,Y,Z;t,q) &= \sum_{k=0}^{n}\binom{n}{k}(qX)^kY^{n-k}A_n^k(Z;t,q^2)\\ &= \sum_{k=0}^{n}\binom{n}{k}(qX)^kY^{n-k}\Big(\frac{t-Z}{t-1}A_n(t,q^2)\\ &\quad+\frac{Z-1}{t-1}\sum_{j=0}^{k}\binom{k}{j}(-1)^j\,(t;q^2)_j\,A_{n-j}(tq^{2j},q^2)\Big)\\ &= \frac{t-Z}{t-1}(qX+Y)^nA_n(t,q^2)\\ &\quad+\frac{Z-1}{t-1}\sum_{\substack{j,l,m\geq 0\\ j+l+m=n}}\frac{n!}{j!\,l!\,m!}(qX)^{j+l}Y^m(-1)^j\,(t;q^2)_j\,A_{l+m}(tq^{2j},q^2),\end{aligned}$$

where $k=j+l$.

Next, with $r=l+m$ we get

$$\begin{aligned}\frac{B_n(X,Y,Z;t,q)}{(t;q^2)_{n+1}} &= \frac{t-Z}{t-1}(qX+Y)^n\frac{A_n(t,q^2)}{(t;q^2)_{n+1}}\\ &\quad+\frac{Z-1}{t-1}\sum_{j+r=n}\frac{n!}{r!\,j!}(-qX)^j\frac{A_r(tq^{2j},q^2)}{(tq^{2j};q^2)_{r+1}}\sum_{l+m=r}\frac{r!}{l!\,m!}(qX)^lY^m\\ &= \frac{t-Z}{t-1}(qX+Y)^n\frac{A_n(t,q^2)}{(t;q^2)_{n+1}}\\ &\quad+\frac{Z-1}{t-1}\sum_{j+r=n}\frac{n!}{r!\,j!}(-qX)^j\frac{A_r(tq^{2j},q^2)}{(tq^{2j};q^2)_{r+1}}(qX+Y)^r.\end{aligned}$$

Furthermore,

$$\begin{aligned}\sum_{n\geq 0}\frac{B_n(X,Y,Z;t,q)}{(t;q^2)_{n+1}}\frac{u^n}{n!} &= \frac{t-Z}{t-1}\sum_{n\geq 0}\frac{A_n(t,q^2)}{(t;q^2)_{n+1}}\frac{((qX+Y)u)^n}{n!}\\ &\quad+\frac{Z-1}{t-1}\sum_{j\geq 0}\frac{(-qXu)^j}{j!}\sum_{r\geq 0}\frac{A_r(tq^{2j},q^2)}{(tq^{2j},q^2)_{r+1}}\frac{((qX+Y)u)^r}{r!}.\end{aligned}$$

Now, make use of the classical identity on the Carlitz q-Eulerian polynomials

$$\sum_{n\ge 0}\frac{u^n}{n!}\frac{A_n(t,q)}{(t;q)_{n+1}}=\sum_{s\ge 0}t^s\ \exp(u[s+1]_q),$$

to obtain

$$\sum_{n\ge 0}\frac{B_n(X,Y,Z;t,q)}{(t;q^2)_{n+1}}\frac{u^n}{n!}=\frac{t-Z}{t-1}\sum_{s\ge 0}t^s\ \exp((qX+Y)u\,[s+1]_{q^2})$$
$$+\frac{Z-1}{t-1}\sum_{j\ge 0}\frac{(-qXu)^j}{j!}\sum_{s\ge 0}(tq^{2j})^s\exp((qX+Y)u\,[s+1]_{q^2}). \tag{66}$$

There remains to extract the coefficient of u^n on both sides. This leads to:

$$\frac{B_n(X,Y,Z;t,q)}{n!\,(t;q^2)_{n+1}}=\frac{t-Z}{t-1}\sum_{s\ge 0}t^s\,\bigl((qX+Y)\,[s+1]_{q^2}\bigr)^n+\frac{Z-1}{t-1}C, \tag{67}$$

where C is the coefficient of u^n in

$$\sum_{j\ge 0}\frac{(-qXu)^j}{j!}\sum_{s\ge 0}(tq^{2j})^s\sum_{m\ge 0}\frac{((qX+Y)u\,[s+1]_{q^2})^m}{m!},$$

that is,

$$\begin{aligned}C&=\sum_{s\ge 0}t^s\sum_{j\ge 0}\frac{(-qX)^j}{j!}q^{2js}\frac{((qX+Y)\,[s+1]_{q^2})^{n-j}}{(n-j)!}\\
&=\frac{1}{n!}\sum_{s\ge 0}t^s\sum_{j\ge 0}\binom{n}{j}(-qXq^{2s})^j((qX+Y)\,[s+1]_{q^2})^{n-j}\\
&=\frac{1}{n!}\sum_{s\ge 0}t^s\bigl((qX+Y)[s+1]_{q^2}-Xq^{2s+1}\bigr)^n\end{aligned}$$

Reporting the last expression in (67) yields identity (62). □

Proof of Theorem 6.2. When $q=1$ in (62), we obtain

$$\frac{B_n(X,Y,Z;t)}{(1-t)^{n+1}}=\frac{t-Z}{t-1}\sum_{s\ge 0}t^s\bigl((X+Y)(s+1)\bigr)^n$$
$$+\frac{Z-1}{t-1}\sum_{s\ge 0}t^s\bigl((X+Y)(s+1)-X\bigr)^n. \tag{68}$$

Hence,

$$\begin{aligned}
&\sum_{n\geq 0}\frac{u^n}{(1-t)^n}B_n(X,Y,Z;t)\\
&=(Z-t)\sum_{s\geq 0}t^s\sum_{n\geq 0}\frac{\big(u(X+Y)(s+1)\big)^n}{n!}\\
&\quad+(1-Z)\sum_{s\geq 0}t^s\sum_{n\geq 0}\frac{\big(u(X+Y)(s+1)-X\big)^n}{n!}\\
&=(Z-t)\sum_{s\geq 0}t^s\exp(u(X+Y)(s+1))\\
&\quad+(1-Z)\sum_{s\geq 0}t^s\sum_{n\geq 0}\exp(u(X+Y)(s+1)-X)\\
&=\big((Z-t)\exp(u(X+Y))+(1-Z)\exp(uY)\big)\sum_{s\geq 0}t^s\exp(u(X+Y)s)\\
&=\frac{(Z-t)\exp(u(X+Y)(1-Z))\exp(uY)}{1-t\exp(u(X+Y))},
\end{aligned}$$

which is identity (63) by replacing u by $u(1-t)$. □

Next, we derive specializations of Theorems 6.1 and 6.2 when the pair (Ξ,des) is replaced by "des_B" and "fdes" (see (21) and (22)). We get:

$$\sum_{w\in B_n}X^{\mathrm{neg}\,w}Y^{\mathrm{pos}\,w}t^{\mathrm{des}_B\,w}q^{\mathrm{fmaj}\,w}=B_n(X,Y,t;t,q);\tag{69}$$

$$\sum_{w\in B_n}X^{\mathrm{neg}\,w}Y^{\mathrm{pos}\,w}t^{\mathrm{fdes}\,w}q^{\mathrm{fmaj}\,w}=B_n(X,Y,t;t^2,q).\tag{70}$$

Also, note that $B_n(0,1,1;t,q)$ is the Carlitz q-Eulerian polynomial $A_n(t,q)$.

First,

$$\frac{B_n(X,Y,t;t,q)}{(t;q^2)_{n+1}}=\sum_{s\geq 0}t^s\big((qX+Y)[s+1]_{q^2}-Xq^{2s+1}\big)^n;\tag{71}$$

$$\frac{B_n(1,1,t;t,q)}{(t;q^2)_{n+1}}=\sum_{w\in B_n}t^{\mathrm{des}_B\,w}q^{\mathrm{fmaj}\,w}=\sum_{s\geq 0}t^s\big([2s+1]_q\big)^n.\tag{72}$$

Second,

$$\begin{aligned}
\frac{B_n(X,Y,t;t^2,q)}{(t^2;q^2)_{n+1}}&=\frac{t^2-t}{t-1}\sum_{s\geq 0}t^{2s}\big((qX+Y)\,[s+1]_{q^2}\big)^n\\
&\quad+\frac{t-1}{t^2-1}\sum_{s\geq 0}t^{2s}\big((qX+Y)[s+1]_{q^2}-Xq^{2s+1}\big)^n,
\end{aligned}$$

so that

$$\frac{(1+t)B_n(X,Y,t;t^2,q)}{(t^2;q^2)_{n+1}} = \sum_{s\ge 0} t^{2s+1}\big((qX+Y)\,[s+1]_{q^2}\big)^n + \sum_{s\ge 0} t^{2s}\big((qX+Y)[s]_{q^2}+Yq^{2s}\big)^n. \tag{73}$$

In particular,

$$\frac{(1+t)B_n(1,1,t;t,q)}{(t^2;q^2)_{n+1}} = \sum_{w\in B_n} t^{\mathrm{fdes}\,w} q^{\mathrm{fmaj}\,w} = \sum_{s\ge 0} t^s\big([s+1]_q\big)^n. \tag{74}$$

The specializations of (72) and (74) for $q=1$ are banal and not reproduced. However, it is worth writing the exponential generating functions for the polynomials $B_n(1,1,t;t)$ and $B_n(1,1,t;t^2)$ directly obtained from (63):

$$\sum_{n\ge 0}\frac{u^n}{n!}\,B_n(1,1,t;t) = \sum_{n\ge 0}\frac{u^n}{n!}\sum_{w\in B_n} t^{\mathrm{des}_B\,w} = \frac{(1-t)\exp(u(t-1))}{-t+\exp(2u(t-1))}; \tag{75}$$

$$\sum_{n\ge 0}\frac{u^n}{n!}\,B_n(1,1,t;t^2) = \frac{(1-t)(t+\exp(u(t^2-1)))}{-t^2+\exp(2u(t^2-1))};$$

so that

$$\sum_{n\ge 0}\frac{u^n}{n!}\,B_n(1,1,t;t^2) = \sum_{n\ge 0}\frac{u^n}{n!}\sum_{w\in B_n} t^{\mathrm{fdes}\,w} = \frac{1-t}{-t+\exp(u(t^2-1))}. \tag{76}$$

The statistics "fdes" and "fmaj" were introduced by Adin and Roichman [2]. Identity (74) with their equivalent adaptations were derived by Brenti et al. [1], Haglund et al. [15] and reproved by the authors ([9, 10]) as specializations of identities involving several-variable statistics. Note that (76) implies that $\sum_{w\in B_n}(-1)^{\mathrm{fdes}\,w}$ is null for every $n\ge 1$. Accordingly, the statistic "fdes" would have been a wrong choice for obtaining a q-extension!

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