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Survey on geometric group theory

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Abstract. This article is a survey article on geometric group theory from the point of view of a non-expert who likes geometric group theory and uses it in his own research.

Introduction

This survey article on geometric group theory is written by a non-expert who likes geometric group theory and uses it in his own research. It is meant as a service for people who want to receive an impression and read an introduction about the topic and possibly will later pass to more elaborate and specialized survey articles or to actual research articles. There will be no proofs. Except for Theorem 7.4 all results have already appeared in the literature.

There is to the author's knowledge no obvious definition what geometric group theory really is. At any rate the basic idea is to pass from a finitely generated group to the geometry underlying its Cayley graph with the word metric. It turns out that only the large scale geometry is really an invariant of the group itself but that this large scale or coarse geometry carries a lot of information. This leads also to a surprising and intriguing variety of new results and structural insights about groups.

A possible explanation for this may be that humans have a better intuition when they think in geometric terms. Moreover, it is helpful to understand groups in the way as they have appeared naturally in mathematics, namely, as groups of symmetries. In other words, basic information about a group can be obtained by studying its actions on nice spaces.

The personal interest of the author comes from questions of the type whether a group satisfies the conjectures due to Baum-Connes, Borel, Farrell-Jones, Kaplansky, Novikov, Hopf, Singer or yields a positive answer to Atiyah's question on L^2-Betti numbers. They are all of the kind that one wants to know whether for a given group G its group ring RG, its reduced group C^*-algebra $C_r^*(G)$, or an aspherical closed manifold with G as fundamental group satisfy certain algebraic or geometric properties concerning their structure as rings or C^*-algebras, their K- or L-theory, rigidity properties or the spectrum of the

Laplace operator of the universal covering. A priori these problems do not seem to be related to questions about the geometry of the group. However, most of the proofs for certain classes of groups contain an important part, where one uses certain geometric properties of the groups, very often properties such as being negatively or non-positively curved in some metric sense. For instance, there is the, on first sight purely ring theoretic, conjecture that for a torsionfree group G and an integral domain R the group ring RG contains no idempotents except 0 and 1. It is surprising that a proof of it can be given for certain rather large classes of groups by exploiting their geometry, and no algebraic proof is known in these cases.

The author has done his best to sort out interesting problems and results and to include the relevant references, and apologizes if an important aspect or reference is missing, it was left out because of ignorance, not on purpose.

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The paper is organized as follows:

1. Classical examples

A classical example of geometric methods used in group theory is the topological proof of Schreier's theorem.

Theorem 1.1 (Schreier's Theorem)**.** *Let G be a free group and $H \subseteq G$ be a subgroup. Then H is free. If the rank* $\mathrm{rk}(G)$ *and the index* $[G : H]$ *are finite, then the rank of H is finite and satisfies*

$$\mathrm{rk}(H) = [G : H] \cdot \big(\mathrm{rk}(G) - 1\big) + 1.$$

Proof. Let G be a free group on the set S. Take the wedge $X = \bigvee_S S^1$ of circles, one copy for each element in S. This is a 1-dimensional CW-complex with $\pi_1(X) \cong G$ by the Seifert-van Kampen Theorem. Let $p\colon \overline{X} \to X$ be the

covering associated to $H \subseteq G = \pi_1(X)$. We have $\pi_1(\overline{X}) \cong H$. Since X is a 1-dimensional CW-complex, $\overline{X}$ is a 1-dimensional CW-complex. If $T \subseteq \overline{X}$ is a maximal tree, then $\overline{X}$ is homotopy equivalent to $\overline{X}/T = \bigvee_{\overline{S}} S^1$ for some set $\overline{S}$. By the Seifert-van Kampen Theorem $H \cong \pi_1(\overline{X})$ is the free group generated by the set $\overline{S}$.

Suppose that $\mathrm{rk}(G)$ and $[G : H]$ are finite. Since $|S| = \mathrm{rk}(G)$, the CW-complex X is compact. Since $[G : H]$ is finite, the CW-complex $\overline{X}$ and hence $\overline{X}/T$ are compact. Hence $\mathrm{rk}(H) = |\overline{S}|$ is finite. We obtain for the Euler characteristics

$$1 - |\overline{S}| = \chi(\overline{X}) = [G : H] \cdot \chi(X) = [G : H] \cdot (1 - |S|)\,.$$

Since $|S| = \mathrm{rk}(G)$ and $|\overline{S}| = \mathrm{rk}(H)$, the claim follows. □

Another example of this type is the topological proof of Kurosh's Theorem, which can be found for instance in [130, Theorem 14 in I.5 on page 56]. The interpretation of amalgamated products and HNN-extensions in terms of topological spaces by the Seifert-van Kampen Theorem or actions of groups on trees are in the same spirit (see for instance [8],[28], [35], [91], [130]).

2. Basics about quasiisometry

A very important notion is the one of quasiisometry since it yields a bridge between group theory and geometry by assigning to a finitely generated group a metric space (unique up to quasiisometry), namely, its Cayley graph with the word metric. There are many good reasons for this passage, see for instance the discussion in [63, Item 0.3 on page 7 ff.]). At any rate this concept has led to an interesting and overwhelming variety of new amazing results and applications and to intriguing and stimulating activities.

Definition 2.1. Let $X_1 = (X_1, d_1)$ and $X_2 = (X_2, d_2)$ be two metric spaces. A map $f\colon X_1 \to X_2$ is called a *quasiisometry* if there exist real numbers $\lambda, C > 0$ satisfying:

(i) The inequality

$$\lambda^{-1} \cdot d_1(x, y) - C \leq d_2\big(f(x), f(y)\big) \leq \lambda \cdot d_1(x, y) + C$$

holds for all $x, y \in X_1$;

(ii) For every x_2 in X_2 there exists $x_1 \in X_1$ with $d_2(f(x_1), x_2) < C$.

We call X_1 and X_2 *quasiisometric* if there is a quasiisometry $X_1 \to X_2$.

Remark 2.2 (Quasiisometry is an equivalence relation)**.** If $f\colon X_1 \to X_2$ is a quasiisometry, then there exists a quasiisometry $g\colon X_2 \to X_1$ such that both composites $g \circ f$ and $f \circ g$ have bounded distance from the identity map. The composite of two quasiisometries is again a quasiisometry. Hence the notion of quasiisometry is an equivalence relation on the class of metric spaces.

Definition 2.3 (Word-metric)**.** Let G be a finitely generated group. Let S be a finite set of generators. The *word metric*

$$d_S \colon G \times G \to \mathbb{R}$$

assigns to (g,h) the minimum over all integers $n \geq 0$ such that $g^{-1}h$ can be written as a product $s_1^{\epsilon_1} s_2^{\epsilon_2} \dots s_n^{\epsilon_n}$ for elements $s_i \in S$ and $\epsilon_i \in \{\pm 1\}$.

The metric d_S depends on S. The main motivation for the notion of quasi-isometry is that the quasiisometry class of (G, d_S) is independent of the choice of S by the following elementary lemma.

Lemma 2.4. *Let G be a finitely generated group. Let S_1 and S_2 be two finite sets of generators. Then the identity* $\mathrm{id} \colon (G, d_{S_1}) \to (G, d_{S_2})$ *is a quasiisometry.*

Proof. Choose λ such that for all $s_1 \in S_1$ we have $d_{S_2}(s_1, e), d_{S_2}(s_1^{-1}, e) \leq \lambda$ and for $s_2 \in S_2$ we have $d_{S_1}(s_2, e), d_{S_1}(s_2^{-1}, e) \leq \lambda$. Take $C = 0$. □

Definition 2.5 (Cayley graph)**.** Let G be a finitely generated group. Consider a finite set S of generators. The *Cayley graph* $\mathrm{Cay}_S(G)$ is the graph whose set of vertices is G and there is an edge joining g_1 and g_2 if and only if $g_1 = g_2 s$ for some $s \in S$.

A *geodesic* in a metric space (X, d) is an isometric embedding $I \to X$, where $I \subset \mathbb{R}$ is an interval equipped with the metric induced from the standard metric on $\mathbb{R}$.

Definition 2.6 (Geodesic space)**.** A metric space (X, d) is called a *geodesic space* if for two points $x, y \in X$ there is a geodesic $c \colon [0, d(x,y)] \to X$ with $c(0) = x$ and $c(d(x,y)) = y$.

Notice that we do not require the unique existence of a geodesic joining two given points.

Remark 2.7 (Metric on the Cayley graph)**.** There is an obvious procedure to define a metric on $\mathrm{Cay}_S(G)$ such that each edge is isometric to $[0,1]$ and such that the distance of two points in $\mathrm{Cay}_S(G)$ is the infimum over the length over all piecewise linear paths joining these two points. This metric restricted to G is just the word metric d_S. Obviously the inclusion $(G, d_S) \to \mathrm{Cay}_S(G)$ is a quasiisometry. In particular, the quasiisometry class of the metric space $\mathrm{Cay}_S(G)$ is independent of S.

The Cayley graph allows to translate properties of a finitely generated group to properties of a geodesic metric space.

Lemma 2.8 (Švarc-Milnor Lemma)**.** *Let X be a geodesic space. Suppose that G acts properly, cocompactly and isometrically on X. Choose a base point $x \in X$. Then the map*

$$f \colon G \to X, \quad g \mapsto gx$$

is a quasiisometry.

Proof. See [20, Proposition 8.19 in Chapter I.8 on page 140]. □

Example 2.9. Let $M = (M, g)$ be a closed connected Riemannian manifold. Let $\widetilde{M}$ be its universal covering. The fundamental group $\pi = \pi_1(M)$ acts freely on $\widetilde{M}$. Equip $\widetilde{M}$ with the unique π-invariant Riemannian metric for which the projection $\widetilde{M} \to M$ becomes a local isometry. The fundamental group π is finitely generated. Equip it with the word metric with respect to any finite set of generators.

Then π and $\widetilde{M}$ are quasiisometric by the Švarc-Milnor Lemma 2.8.

Definition 2.10. Two groups G_1 and G_2 are *commensurable* if there are subgroups $H_1 \subseteq G_1$ and $H_2 \subseteq G_2$ such that the indices $[G_1 : H_1]$ and $[G_2 : H_2]$ are finite and H_1 and H_2 are isomorphic.

Lemma 2.11. *Let G_1 and G_2 be finitely generated groups. Then:*

(i) A group homomorphism $G_1 \to G_2$ is a quasiisometry if and only if its kernel is finite and its image has finite index in G_2;
(ii) If G_1 and G_2 are commensurable, then they are quasiisometric.

There are quasiisometric groups that are not commensurable as the following example shows.

Example 2.12. Consider a semi-direct product $G_\phi = \mathbb{Z}^2 \rtimes_\phi \mathbb{Z}$ for an isomorphism $\phi\colon \mathbb{Z}^2 \to \mathbb{Z}^2$. For these groups a classification up to commensurability and quasiisometry has been given in [19] as explained next.

These groups act properly and cocompactly by isometries on precisely one of the 3-dimensional simply connected geometries $\mathbb{R}^3$, Nil or Sol. (A *geometry* is a complete locally homogeneous Riemannian manifold.) If ϕ has finite order, then the geometry is $\mathbb{R}^3$. If ϕ has infinite order and the eigenvalues of the induced $\mathbb{C}$-linear map $\mathbb{C}^2 \to \mathbb{C}^2$ have absolute value 1, then the geometry is Nil. If ϕ has infinite order and one of the eigenvalues of the induced $\mathbb{C}$-linear map $\mathbb{C}^2 \to \mathbb{C}^2$ has absolute value > 1, then the geometry is Sol.

These metric spaces given by the geometries $\mathbb{R}^3$, Nil or Sol are mutually distinct under quasiisometry. By Example 2.9 two groups of the shape G_ϕ are quasiisometric if and only if they belong to the same geometry.

Two groups G_ϕ and $G_{\phi'}$ belonging to the same geometry $\mathbb{R}^3$ or Nil respectively contain a common subgroup of finite index and hence are commensurable. However, suppose that G_ϕ and $G_{\phi'}$ belong to Sol. Then they are commensurable if and only if the eigenvalues Λ and Λ' with absolute value > 1 of ϕ and ϕ', respectively, have a common power (see [19]). This obviously yields examples of groups G_ϕ and $G_{\phi'}$ that belong to the geometry Sol and are quasiisometric but are not commensurable.

The classification up to quasiisometry of finitely presented non-poly-cyclic abelian-by-cyclic groups is presented in [44, Theorem 1.1].

3. Properties and invariants of groups invariant under quasiisometry

Recall that, given a property (P) of groups, we call a group virtually-(P) if it contains a subgroup of finite index having property (P). In particular a group is virtually trivial if and only it is finite. It is virtually finitely generated abelian if and only if it contains a normal subgroup of finite index which is isomorphic to $\mathbb{Z}^n$ for some integer $n \geq 0$.

A finitely generated group G is *nilpotent* if G possesses a finite *lower central series*

$$G = G_1 \supset G_2 \supset \ldots \supset G_s = \{1\} \qquad G_{k+1} = [G, G_k].$$

A group G is called *amenable* if there is a (left) G-invariant linear operator $\mu\colon l^\infty(G, \mathbb{R}) \to \mathbb{R}$ with $\mu(1) = 1$ that satisfies for all $f \in l^\infty(G, \mathbb{R})$

$$\inf\{f(g) \mid g \in G\} \leq \mu(f) \leq \sup\{f(g) \mid g \in G\}.$$

Abelian groups and more generally solvable groups are amenable. The class of amenable groups is closed under extensions and directed unions. A group which contains a non-abelian free group as subgroup is not amenable. A brief survey on amenable groups and the definition and a brief survey on accessible groups can be found for instance in [82, Section 6.4.1 on page 256 ff.] and [34, III.15 on page 52]. The book [109] is devoted to amenability. The notion of a hyperbolic space and a hyperbolic group will be explained in Definition 5.2 and Definition 7.1.

Theorem 3.1 (Group properties invariant under quasiisometry)**.** *The following properties of groups are* geometric properties*, i.e., if the finitely generated group G has the property, then every finitely generated group that is quasiisometric to G also has this property:*

(i) Finite;
(ii) Infinite virtually cyclic;
(iii) Finitely presented;
(iv) Virtually abelian;
(v) Virtually nilpotent;
(vi) Virtually free;
(vii) Amenable;
(viii) Hyperbolic;
(ix) Accessible;
(x) The existence of a model for the classifying space BG with finite n-skeleton for given $n \geq 2$;
(xi) The existence of a model for BG of finite type, i.e., all skeletons are finite.

Proof. (i) Having bounded diameter is a quasiisometry invariant of metric spaces.

(ii) This follows from Theorem 3.4 (iii) and Theorem 3.5 (i).

(iii) See [34, Proposition 4 In Chapter V.A on page 119].

(iv) See [58, Chapter I].

(v) This follows from Theorem 3.5 (iv) and Theorem 3.8.

(vi) See [58, Theorem 19 in Chapter I] and Theorem 3.5(i).

(vii) This follows from [48]. See also [61, Chapter 6].

(viii) Quasiisometric groups have quasiisometric Cayley graphs and it is not difficult to see that the property being hyperbolic is a quasiisometry invariant of geodesic spaces.

(ix) See [133].

(x) This follows from [63, Item $1.C_2'$ on page 25]. See also [3].

(xi) This follows from assertion (x). □

If S is a finite set of generators for the group G, let $b_S(n)$ be the number of elements in G which can be written as a word in n letters of $S \cup S^{-1} \cup \{1\}$, i.e., the number of elements in the closed ball of radius n around 1 with respect to d_S.

The following definition is indeed independent of the choice of the finite set S of generators.

Definition 3.2 (Growth)**.** The group G has *polynomial growth of degree not greater than* d if there is C with $b_S(n) \leq Cn^d$ for all $n \geq 1$.

We say that G has *polynomial growth* if it has polynomial growth of degree not greater than d for some $d > 0$.

It has *exponential growth* if there exist $C > 0$ and $\alpha > 0$ such that for $n \geq 1$ we have

$$b_S(n) \geq C \cdot \alpha^n.$$

It has *subexponential growth* if it has neither polynomial growth nor exponential growth.

The free abelian group $\mathbb{Z}^n$ of rank n has polynomial growth rate of precisely degree n. A finitely generated non-abelian free group has exponential growth rate.

Recall that the *Hirsch rank* of a solvable group G is defined to be

$$h(G) = \sum_{i \geq 0} \dim_{\mathbb{Q}}(G_{i+1}/G_i \otimes_{\mathbb{Z}} \mathbb{Q}),$$

where G_i is the i-th term in the derived series of G.

A metric is called *proper* if every closed ball is compact. Let X be a proper geodesic space. A *proper ray* is a map $[0, \infty) \to X$ such that the preimage of a compact set is compact again. Two proper rays $c_0, c_1 \colon [0, \infty) \to X$ *converge to the same end* if for every compact subset $C \subset X$ there is $R > 0$ such that $c_0([R, \infty))$ and $c_1([R, \infty))$ lie in the same component of $X \setminus C$. This defines an equivalence relation on the set of proper rays. The set of equivalence classes is the *set of ends*.

The *number of ends* of X is the cardinality of this set. It is a quasiisometry invariant (see [20, Proposition 8.29 on page 128]). Hence the following definition makes sense.

Definition 3.3 (Number of ends)**.** The *number of ends* of a finitely generated group G is defined to be the number of ends of the Cayley graph $\mathrm{Cay}_S(G)$ for any choice of a finite set S of generators.

Theorem 3.4 (Ends of groups)**.**

(i) A finitely generated group has 0, 1, 2 *or infinitely many ends;*
(ii) It has 0 *ends precisely if it is finite;*
(iii) It has two ends precisely if it is infinite and virtually cyclic.
(iv) It has infinitely many ends if and only if G can be expressed as an amalgamated product $A*_C B$ or as an HNN*-extension $A*_C$ with finite C and $|A/C| \geq 3$ and $|B/C| \geq 2$.*

Proof. See [20, Theorem 8.32 in Chapter I.8 on page 146]. □

Theorem 3.5 (Invariants under quasiisometry)**.** *Let G_1 and G_2 be two finitely generated groups which are quasiisometric. Then:*

(i) They have the same number of ends;
(ii) Let R be a commutative ring. Then we get

$$\mathrm{cd}_R(G_1) = \mathrm{cd}_R(G_2)$$

if one of the following assumptions is satisfied:
(a) The cohomological dimensions $\mathrm{cd}_R(G_1)$ and $\mathrm{cd}_R(G_2)$ are both finite;
(b) There exist finite models for BG_1 and BG_2;
(c) One of the groups G_1 and G_2 is amenable and $\mathbb{Q} \subseteq R$;
(iii) If they are solvable, then they have the same Hirsch length;
(iv) Suppose that G_1 has polynomial growth of degree not greater than d, intermediate growth,or exponential growth, respectively. Then the same is true for G_2;
(v) Let G_1 and G_2 be nilpotent. Then their real cohomology rings $H^*(G_1;\mathbb{R})$ and $H^*(G_2;\mathbb{R})$ are isomorphic as graded rings. In particular the Betti numbers of G_1 and G_2 agree.

Proof. (i) See [18, Corollary 2.3] or [57, Corollary 1].

(ii) See [123, Theorem 1.2]. The case $R = \mathbb{Z}$ under condition ((ii)b) has already been treated in [57, Corollary 2].

(iii) This follows from assertion (ii) since $\mathrm{cd}_{\mathbb{Q}}(G)$ is the Hirsch rank for a virtually poly-cyclic group G (see [123, Corollary 1.3]).

(iv) See [34, Proposition 27 in VI.B on page 170].

(v) See [123, Theorem 1.5]. The statement about the Betti numbers was already proved by Shalom [131, Theorem 1.2].

□

We mention that there is an extension of the notion of quasiisometry to groups which are not necessarily finitely generated and that some of the the results of Theorem 3.5 are still true in this more general setting (see [123], [131]).

Conjecture 3.6 (Folk)**.** *Let G_1 and G_2 be two finitely generated torsionfree nilpotent groups. Let L_1 and L_2 be the simply connected nilpotent Lie groups given by their Mal'cev completion. (These are uniquely determined by the fact that G_i is cocompactly embedded in L_i.)*

If G_1 and G_2 are quasi-isometric, then L_1 and L_2 are isomorphic as Lie groups.

Remark 3.7. Evidence for Conjecture 3.6 comes from the following facts. The graded Lie algebra associated to the Mal'cev completion of a finitely generated torsionfree nilpotent group G is a quasiisometry invariant of G by a result of Pansu [104]. The result of Sauer mentioned in Theorem 3.5 (v) follows from Conjecture 3.6 since the cohomology algebras of the Lie algebra of the Mal'cev completion and the cohomology algebra of G itself are isomorphic (see [96, Theorem 1]).

The following celebrated theorem due to Gromov [60] is one of the milestones in geometric group theory. A new proof can be found in [72].

Theorem 3.8 (Virtually nilpotent groups and growth)**.** *A finitely generated group is virtually nilpotent if and only if it has polynomial growth.*

Remark 3.9 (Virtually solvable groups)**.** This raises the question whether solvability is a geometric property. However, there exists a finitely generated solvable group which is quasiisometric to a finitely generated group which is not virtually solvable (see [36]). This counterexample is not finitely presented. It is still not known whether two finitely presented quasiisometric groups both have to be virtually solvable if one of them is.

Remark 3.10 (Free products)**.** Let G_1, G_1', G_2 and G_2' be finitely generated groups. Suppose that G_i and G_i' are quasiisometric for $i = 1, 2$. Assume that none of the groups G_1, G_1', G_2 and G_2' is trivial or $\mathbb{Z}/2$. Then the free products $G_1 * G_2$ and $G_1' * G_2'$ are quasiisometric. (They are actually Lipschitz equivalent). See [34, 46. (ii) in IV.B on page 105] and [108, Theorem 0.1]. The corresponding statement is false if one replaces quasiisometric by commensurable (see [34, 46. (iii) in IV.B on page 106]).

Remark 3.11 (Property (T))**.** Kazhdan's Property (T) is not a quasiisometry invariant. (This is due to Furman and Monod and stated in [54, page 173]).

Remark 3.12 (The sign of the Euler characteristic)**.** The sign of the Euler characteristic of a group with a finite model for BG is not a quasiisometry invariant. See [34, 46. (iii) in IV.B on page 105].

Remark 3.13 (Minimal dimension of EG and $\underline{E}G$)**.** We have already mentioned in Theorem 3.5 (ii) that the cohomological dimension $\mathrm{cd}_{\mathbb{Z}}(G)$ is a quasiisometry invariant under the assumption that there exists a G-CW-model for EG which is finite or, equivalently, cocompact.

There always exists a $\max\{3, \mathrm{cd}_{\mathbb{Z}}(G)\}$-dimensional model for BG (see [21, Theorem 7.1 in Chapter VII on page 295]). Notice that the existence of a d-dimensional CW-model for BG is equivalent to the existence of a d-dimensional G-CW-model for EG since EG is the universal covering of BG. Hence $\mathrm{cd}_{\mathbb{Z}}(G)$ is equal to the minimal dimension of a model for EG if $\mathrm{cd}_{\mathbb{Z}}(G) \geq 3$.

If $H \subset G$ is a subgroup of finite index of the torsionfree group G and there is a finite dimensional model for EG, then the cohomological dimensions of G and H agree by a result of Serre (see [21, Theorem 3.2 in Chapter VIII.3 on page 190], [129]) and hence also the minimal dimension for EH and EG agree if the cohomological dimension of G is greater or equal to 3.

The corresponding statement is false if one replaces EG by the universal space $\underline{E}G$ for proper group G-actions (see Definition 9.1). Namely, there exists a group G with a torsionfree subgroup H of finite index such that there exists a d-dimensional model for $EH = \underline{E}H$ but no d-dimensional model for $\underline{E}G$ (see [78, Theorem 6]).

Hence the minimal dimension of a model for $\underline{E}G$ is not at all a quasiisometry invariant in general.

Remark 3.14 (L^2-invariants)**.** If the finitely generated groups G_1 and G_2 are quasiisometric and there exist finite models for BG_1 and BG_2 then

$$b_p^{(2)}(G_1) = 0 \Leftrightarrow b_p^{(2)}(G_2) = 0$$

holds (see [63, page 224], [105]). But it is general not true that in the situation above there exists a constant $C > 0$ such that $b_p^{(2)}(G_1) = C \cdot b_p^{(2)}(G_2)$ holds for all $p \geq 0$ (see [82, page 313], [106]).

It is unknown whether the vanishing of the L^2-torsion of appropriate groups or the Novikov-Shubin invariants of appropriate groups are quasiisometry invariants. (see [82, Question 7.35 and Question 7.36 on page 313]). Partial results in this direction have been obtained in [123, Theorem 1.6] and in [138] for amenable respectively elementary amenable groups.

Remark 3.15 (Asymptotic cone)**.** The notion of an asymptotic cone using ultralimits was introduced by Van den Dries and Wilkie [136]. It assigns to a metric space a new space after the choice of a non-principal ultrafilter on the set of natural numbers, a scaling sequence and a sequence of observation points. The asymptotic cone does in general depend on these extra choices. Roughly speaking, an asymptotic cone of a metric space is what one sees when one looks at the space from infinitely far away.

Applied to the Cayley graph of a finitely generated group an asymptotic cone yields a complete geodesic homogeneous metric space, which captures the coarse properties. It depends on the ultrafilter and the scaling sequence but not on the sequence of observation points. A quasiisometry induces a bi-Lipschitz

homeomorphism between the asymptotic cones (for the same ultrafilters and scaling constants). So as in the case of the boundary of a hyperbolic group (see Section 6) we can assign to a group a metric space such that a quasiisometry induces a "nice" map between the associated structures.

Further information and a discussion of some applications to quasiisometry can be found for instance in [20, Chapter I.5 on page 77 ff.] and [37]. Asymptotic cones play a significant role in the proof of certain rigidity results, for instance in the proof of the rigidity of quasiisometries for symmetric spaces and Euclidean buildings due to Kleiner-Leeb [73] or in the proof the rigidity under quasiisometry of the mapping class group (see Theorem 4.5) due to Behrstock-Kleiner-Minsky-Mosher [9] and Hamenstädt [65]. Asymptotic cones and quasiisometry classes of fundamental groups of 3-manifolds are investigated in [71].

Remark 3.16 (Group splittings)**.** A lot of activity in geometric group theory has been focused on extending the Jaco-Johannson-Shalen decomposition for 3-manifolds to finitely presented groups (see for instance [17], [38], [119], [126], [128]). Its quasiisometry invariance has been proved in [107].

Further information about quasiisometry invariants can be found for instance in [13], [20], [34], [59], [63].

4. Rigidity

An explanation of the following two theorems and a list of papers that have made significant contributions to their proof can be found in [42]. It includes Eskin [40], Eskin-Farb [41], Farb-Schwarz [45], Kleiner-Leeb [73], Pansu [104], and Schwartz [124] and [125].

In the sequel *semisimple Lie group* means non-compact, connected semisimple Lie group with finite center. *Lattice* means a discrete subgroup of finite covolume. A lattice is called *uniform* if it is cocompact.

Theorem 4.1 (Rigidity of the class of lattices)**.** *Let Γ be a finitely generated group. If Γ is quasiisometric to an irreducible lattice in a semisimple Lie group G, then Γ is almost a lattice in G, i.e., there is a lattice Λ' in G and a finite group F such that there exists an exact sequence*

$$1 \to F \to \Gamma \to \Lambda' \to 1.$$

Theorem 4.2 (Classification among lattices)**.** *The quasiisometry classes of irreducible lattices in semisimple Lie groups are precisely:*

- *(i) One quasiisometry class for each semisimple Lie group, consisting of the uniform lattices in G;*
- *(ii) One quasiisometry class for each commensurability class of irreducible non-uniform lattices, except in $G = \mathrm{SL}_2(\mathbb{R})$, where there is precisely one quasiisometry class of non-uniform lattices.*

The following result is the main result of [43]. Recall that for $n \geq 2$ the solvable *Baumslag-Solitar-group* is defined by

$$BS(1,n) = \langle a, b \mid bab^{-1} = a^n \rangle.$$

Theorem 4.3 (Rigidity of Baumslag-Solitar groups)**.** *Let G be a finitely generated group. Suppose that G is quasiisometric to* $\mathrm{BS}(1,n)$ *for some $n \geq 2$. Then there is an exact sequence*

$$1 \to F \to G \to \Gamma \to 1,$$

where F is finite and Γ is commensurable to $BS(1,n)$.

Remark 4.4 (Abelian by cyclic groups)**.** The quasiisometry rigidity of finitely presented abelian-by-cyclic groups is investigated in [44, Theorem 1.2].

The following result is due Behrstock-Kleiner-Minsky-Mosher [9] and Hamenstädt [65].

Theorem 4.5 (Rigidity of mapping class groups)**.** *Let S be an oriented closed surface. Let $M(S)$ be the associated mapping class group. Let G be a finitely generated group that is quasiisometric to $M(S)$. Let* $\mathrm{cent}(M(S))$ *be the center of $M(S)$ which is a finite group.*

Then there is a finite index subgroup G' in G and a homomorphism $G' \to M(S)/\mathrm{cent}(M(S))$ with finite kernel and finite index image.

5. Hyperbolic spaces and CAT(κ)-spaces

Recall that we have introduced the notion of a geodesic space in Definition 2.6.

Example 5.1 (Geodesic spaces)**.** A complete Riemannian manifold inherits the structure of a geodesic metric space from the Riemannian metric by defining the distance of two points to be the infimum over the length of any curve joining them.

A graph inherits the structure of a metric space by defining the distance of two points to be the infimum over the length of any piecewise linear path joining them, where each edge is isometrically identified with the unit interval $[0,1]$. A graph is connected if and only if it is a geodesic space with respect to this metric.

A *geodesic triangle* in a geodesic space X is a configuration of three points x_1, x_2 and x_3 in X together with a choice of three geodesics g_1, g_2 and g_3 such that g_1 joins x_2 to x_3, g_2 joins x_1 to x_3 and g_3 joins x_1 to x_2. For $\delta > 0$ a geodesic triangle is called *δ-thin* if each edge is contained in the closed δ-neighborhood of the union of the other two edges.

Definition 5.2 (Hyperbolic space)**.** Consider $\delta \geq 0$. A *δ-hyperbolic space* is a geodesic space whose geodesic triangles are all δ-thin.

A geodesic space is called *hyperbolic* it is δ-hyperbolic for some $\delta > 0$.

Remark 5.3 (Equivalent definitions of hyperbolic space)**.** There are many equivalent definitions of hyperbolic spaces, which are useful and can be found under the key words "fine triangles", "minsize", "insize", "Gromov's inner product and the 4-point-condition", "geodesic divergence" and "linear isoperimetric inequality". (see for instance [16], [20], [58], [62]).

Remark 5.4 (Examples and non-examples for hyperbolic spaces)**.** Every geodesic space with bounded diameter is hyperbolic. Every complete Riemannian manifold whose sectional curvature is bounded from above by a negative constant is a hyperbolic space. In particular the hyperbolic n-space $\mathbb{H}^n$ and every closed Riemannian manifold with negative sectional curvature are hyperbolic spaces. The Euclidean space $\mathbb{R}^n$ is not hyperbolic. A tree is δ-hyperbolic for every $\delta \geq 0$.

For $\kappa \leq 0$ let M_κ be the up to isometry unique simply connected complete Riemannian manifold whose sectional curvature is constant with value κ. Consider a metric space X. For every geodesic triangle Δ with edges x_1, x_2 and x_3 in X there exists a geodesic triangle $\overline{\Delta}$ in M_κ with edges $\overline{x_1}$, $\overline{x_2}$, and $\overline{x_3}$ which is a geodesic triangle and satisfies $d_X(x_i, x_j) = d_{M_\kappa}(\overline{x_i}, \overline{x_j})$ for $i, j \in \{1, 2, 3\}$. We call such a triangle $\overline{\Delta}$ a *comparison triangle*. It is unique up to isometry. For every point x in Δ there is unique comparison point $\overline{x}$ determined by the property that $\overline{x}$ lies on the edge from $\overline{x_i}$ to $\overline{x_j}$ if x lies on the edge joining x_i and x_j and the distance of $\overline{x}$ and $\overline{x_i}$ agrees with the distance of x and x_i.

Definition 5.5 (CAT(κ)-space)**.** Let X be a geodesic space and let $\kappa \leq 0$. Then X satisfies the CAT(κ)*-condition* if for every geodesic triangle Δ and points $x, y \in \Delta$ and any comparison triangle $\overline{\Delta}$ in M_κ and comparison points $\overline{x}$ and $\overline{y}$ we have

$$d_X(x, y) \leq d_{M_\kappa}(\overline{x}, \overline{y}).$$

A CAT(κ)*-space* is a geodesic space which satisfies the CAT(κ)-condition.

A geodesic space is *of curvature* $\leq \kappa$ for some $\kappa \leq 0$ if it satisfies the CAT(κ)-condition locally. It is called *negatively curved* or *non-positively curved* respectively if it is of curvature $\leq \kappa$ for some $\kappa < 0$ or $\kappa \leq 0$ respectively.

A space Y is called *aspherical* if it is path connected and $\pi_n(Y, y)$ vanishes for one (and hence all) $y \in Y$. Provided that Y is a CW-complex, Y is aspherical if and only if it is connected and its universal covering is contractible.

Theorem 5.6 (CAT(κ)-spaces)**.** *Fix $\kappa \leq 0$. Then:*

(i) A simply connected Riemannian manifold has sectional curvature $\leq \kappa$ if and only if it is a CAT(κ)*-space with respect to the metric induced by the Riemannian metric;*

(ii) A CAT(κ)*-space is contractible;*

(iii) A simply connected complete geodesic space of curvature $\leq \kappa$ is a CAT(κ)*-space;*

(iv) A complete geodesic space of curvature $\leq \kappa$ has a CAT(κ)*-space as universal covering and is aspherical;*

(v) *Consider $\kappa \leq \kappa' \leq 0$. If X is a $\mathrm{CAT}(\kappa)$-space of curvature $\leq \kappa$, then X is a $\mathrm{CAT}(\kappa')$-space of curvature $\leq \kappa'$;*

(vi) *A proper $\mathrm{CAT}(0)$-space is hyperbolic if and only if it contains no subspace isometric to $\mathbb{R}^2$;*

(vii) *For $\kappa < 0$ a $\mathrm{CAT}(\kappa)$-space is hyperbolic;*

(viii) *A tree is a $\mathrm{CAT}(\kappa)$-space for all $\kappa \leq 0$.*

Proof. (i) See [20, Corollary 1A.6 in Chapter II.1 on page 173].

(ii) See [20, Corollary 1.5 in Chapter II.1 on page 161].

(iii) See [20, Theorem 4.1 (2) in Chapter II.4 on page 194].

(iv) This follows from assertions (ii) and (iii).

(v) See [20, Theorem 1.12 in Chapter II.1 on page 165].

(vi) See [20, Theorem 1.5 in Chapter III.H on page 400].

(vii) See [20, Proposition 1.2 in Chapter III.H on page 399].

(viii) See [20, Example 1.15 (4) in Chapter II.1 on page 167]. □

Remark 5.7. The condition of being hyperbolic is a condition in the large. For instance, a local change of the metric on a compact subset does not destroy this property. This is not true for the condition being $\mathrm{CAT}(\kappa)$. For example, any compact metric space is hyperbolic, whereas it is not $\mathrm{CAT}(\kappa)$ for some $\kappa \leq 0$ in general.

In general it makes a significant difference whether a space is negatively curved or non-positively curved.

There is no version of the CAT(0)-condition known that is like the condition hyperbolic defined in the large.

6. The boundary of a hyperbolic space

Let X be a hyperbolic space. A *geodesic ray* is a geodesic $c\colon [0,\infty) \to X$ with $[0,\infty)$ as source. We call two geodesic rays $c, c'\colon [0,\infty) \to X$ *asymptotic* if there exists $C \geq 0$ such that $d_X(c(t), c'(t)) \leq C$ holds for all $t \in [0,\infty)$.

Definition 6.1 (Boundary of a hyperbolic space)**.** Let ∂X be the set of equivalence classes of geodesic rays. Put

$$\overline{X} = X \amalg \partial X.$$

The description of the topology on $\overline{X}$ and the proof of the following two results can be found in [20, Chapter III.H on pages 429-430 and Exercise 3.18 (4) in Chapter III.H on page 433].

Lemma 6.2. *There is a topology on $\overline{X}$ such that $\overline{X}$ is compact and metrizable, the subspace topology of $X \subseteq \overline{X}$ agrees with the topology coming from the metric, $X \subseteq \overline{X}$ is open and dense, and $\partial X \subseteq \overline{X}$ is closed.*

Lemma 6.3. *Let X and Y be hyperbolic spaces. Let $f\colon X \to Y$ be a quasi-isometry. It induces a map*

$$\overline{f}\colon \overline{X} \to \overline{Y},$$

which restricts on the boundary to a homeomorphism

$$\partial f\colon \partial X \xrightarrow{\cong} \partial Y.$$

In particular, the boundary is a quasiisometry invariant of a hyperbolic space.

Remark 6.4 (Mostow rigidity)**.** Let $f\colon M \to N$ be a homotopy equivalence of hyperbolic closed manifolds of dimension $n \geq 3$. *Mostow rigidity* says that f is homotopic to an isometric diffeomorphism. Lemma 6.3 plays a role in its proof as we briefly explain next. More details can be found for instance in [11].

Notice that the universal coverings $\widetilde{M}$ and $\widetilde{N}$ are isometrically diffeomorphic to the n-dimensional hyperbolic space $\mathbb{H}^n$. The boundary of $\mathbb{H}^n$ can be identified with S^{n-1} and $\overline{\mathbb{H}^n}$ with D^n. Since M and N are compact, the map $\widetilde{f}\colon \widetilde{M} \to \widetilde{N}$ is a quasiisometry. Hence it induces a homeomorphism $\partial\widetilde{f}\colon \partial\widetilde{M} \xrightarrow{\cong} \partial\widetilde{N}$. Next one shows that the volume of a closed hyperbolic manifold is a homotopy invariant, for instance using the notion of the simplicial volume due to Gromov and Thurston. This is used to prove that an ideal simplex in $\overline{\mathbb{H}^n}$ with vertices $x_0, x_1, \ldots, x_n$ on $\partial\mathbb{H}^n$ has the same volume as the ideal triangle with vertices $\partial\widetilde{f}(x_0), \partial\widetilde{f}(x_1), \ldots, \partial\widetilde{f}(x_n)$. This implies that there is an isometric diffeomorphism $\widetilde{g}\colon \widetilde{M} \to \widetilde{N}$ with $\partial\widetilde{g} = \partial\widetilde{f}$ such that $\widetilde{g}$ is compatible with the actions of the fundamental groups and passes to an isometric diffeomorphism $g\colon M \to N$ which induces on the fundamental groups the same map as f and hence is homotopic to f.

In the last step the condition $n \geq 3$ enters. Indeed, Mostow rigidity does not hold in dimension $n = 2$.

7. Hyperbolic groups

Definition 7.1 (Hyperbolic group)**.** A finitely generated group is called *hyperbolic* if its Cayley graph is hyperbolic.

Recall that the quasiisometry type of the Cayley graph of a finitely generated group G depends only on G as a group but not on the choice of a finite set of generators and the notion hyperbolic is a quasiisometry invariant for geodesic spaces. Hence the definition above makes sense, i.e., being hyperbolic is a property of the finitely generated group G itself and does not depend on the choice of a finite set of generators.

Let G be hyperbolic. Its *boundary* ∂G is the boundary of the Cayley graph. This is well-defined up to homeomorphism, i.e., independent of a choice of a finite set of generators because of Lemma 6.3.

The notion of the *classifying space for proper G-actions* $\underline{E}G$ will be explained in Definition 9.1.

A *Dehn presentation* of a group G with a finite set of generators S is a finite list of words $u_1, v_1, \ldots, u_n, v_n$ such that $u_1 = v_1, \ldots, u_n = v_n$ holds in G, and $d_S(e, v_i) \leq d_S(e, u_i)$ is true for $i = 1, 2, \ldots, n$ and any word represents the identity element e only if it contains one of the words u_i as a subword. Now there is an obvious algorithm to decide whether a word w represents the unit

element e in G: Look whether it contains one of the words u_i. If the answer is no, the process ends, if the answer is yes, replace u_i by v_i. By induction over $d_S(e, w)$ one sees that this process stops after at most $d_S(e, w)$ steps. The word w represents e if and only if the process ends with the trivial word.

A survey article about *Poincaré duality groups* is [31].

The property of being hyperbolic has a lot of consequences:

Theorem 7.2 (Properties of hyperbolic groups)**.**

(i) Geometric:
The property "hyperbolic" is geometric;

(ii) Characterization by actions:
A group G is a hyperbolic group if and only if it acts isometrically, properly and cocompactly on a proper hyperbolic space X. In this case ∂G is homeomorphic to ∂X;

(iii) Characterization by asymptotic cones:
A finitely generated group is hyperbolic if and only if all its asymptotic cones are $\mathbb{R}$-trees. A finitely presented group is hyperbolic if and only if one (and hence all) asymptotic cones are $\mathbb{R}$-trees;

(iv) Presentations:

- *(a) A finitely generated group is hyperbolic if and only if it possesses a Dehn presentation;*
- *(b) Suppose that the finitely presented group G is a small cancellation group in the sense that it admits a presentation which satisfies the condition $C'(1/6)$ or which satisfies both the conditions $C'(1/4)$ and $T(4)$ (see* [58, Definition 3 in Chapter 8 on page 228]*). Then G is hyperbolic;*

(v) Classifying spaces and finiteness properties:

- *(a) If G is hyperbolic, then there exists a finite model for the universal space for proper G-actions $\underline{E}G$;*
- *(b) If G is hyperbolic, then there is a model for BG of finite type, $H_n(G;\mathbb{Z})$ is finitely generated as $\mathbb{Z}$-module for $n \geq 0$ and $H_n(G;\mathbb{Q})$ is trivial for almost all $n \geq 0$;*
- *(c) If G is hyperbolic, then G is finitely presented;*
- *(d) Suppose that G is hyperbolic. Then there are only finitely many conjugacy classes of finite subgroups;*
- *(e) If G is hyperbolic and torsionfree, then there is a finite model for BG, the abelian group $H_n(G;\mathbb{Z})$ is finitely generated for $n \geq 0$ and $H_n(G;\mathbb{Z})$ is trivial for almost all $n \geq 0$;*

(vi) Subgroups:

- *(a) Let $C \subseteq G$ be an infinite cyclic subgroup of G. Suppose that G is hyperbolic. Then C has finite index in both its centralizer C_GC and its normalizer N_GC. In particular, G does not contain a subgroup isomorphic to $\mathbb{Z}^n$ for $n \geq 2$;*

(b) *Any subgroup of a hyperbolic group is either virtually cyclic or contains a free group of rank two as subgroup. In particular, an amenable subgroup of a hyperbolic group is virtually cyclic;*

(c) *Given r elements $g_1, g_2, \ldots, g_r$ in a hyperbolic group, then there exists an integer $n \geq 1$, such that $\{g_1^n, g_2^n, \ldots, g_r^n\}$ generates a free subgroup of rank at most r;*

(vii) Torsion groups:

Let G be a torsion group, i.e., each element in G has finite order. Then G is hyperbolic if and only if G is finite;

(viii) Inheritance properties:

(a) *The product $G_1 \times G_2$ of two hyperbolic groups is again hyperbolic if and only if one of the two groups G_1 and G_2 is finite;*

(b) *The free product of two hyperbolic groups is again hyperbolic;*

(ix) Decision problems:

(a) *The word-problem and the conjugation-problem is solvable for a hyperbolic group;*

(b) *The isomorphism-problem is solvable for torsionfree hyperbolic groups;*

(x) The boundary:

(a) *Let G be a hyperbolic group that is virtually torsionfree. Then*

$$\mathrm{vcd}(G) - 1 = \dim(\partial G),$$

where $\mathrm{vcd}(G)$ is the virtual cohomological dimension of G and $\dim(\partial G)$ is the topological dimension of ∂G;

(b) *Let G be hyperbolic and infinite and let $n \geq 2$ be an integer. Suppose that ∂G contains an open subset which is homeomorphic to $\mathbb{R}^n$. Then ∂G is homeomorphic to S^n;*

(c) *Let G be hyperbolic. Then ∂G is homeomorphic to S^1 if and only if G is a Fuchsian group;*

(d) *A torsionfree hyperbolic group G is a Poincaré duality group of dimension n if and only if ∂G has the integral Čech cohomology of S^{n-1};*

(e) *A torsionfree hyperbolic group G is a Poincaré duality group of dimension 3 if and only if ∂G is homeomorphic to S^2;*

(xi) Rationality:

Let G be a hyperbolic group. Let S be a finite set of generators. For the integer $n \geq 0$ let $\sigma(n)$ be the number of elements $g \in G$ with $d_S(g, e) = n$;

Then the formal power series $\sum_{n=0}^{\infty} \sigma(n) \cdot t^n$ is a rational function.

The same is true if one replaces $\sigma(n)$ by the number $\beta(n)$ of elements $g \in G$ with $d_S(g, e) \leq n$;

(xii) Further group theoretic properties:

(a) *A hyperbolic group is weakly amenable in the sense of Cowling-Haagerup* [29]*;*

(b) *A hyperbolic group has finite asymptotic dimension;*

(c) A finitely generated subgroup H of a torsionfree hyperbolic group is Hopfian, i.e., every epimorphism $H \to H$ is an isomorphism;

(xiii) Being hyperbolic is generic:

In a precise statistical sense almost all finitely presented groups are hyperbolic.

Proof. (i) Quasiisometric groups have quasiisometric Cayley graphs and it is not difficult to see that the property being hyperbolic is a quasiisometry invariant of geodesic spaces.

(ii) See [69, Theorem 2.24], [62].

(iii) See [97, Section 1.1].

(iv)a See [20, Theorem 2.6 in Chapter III.Γ on page 450].

(iv)b See [58, Theorem 36 in Chapter 8 on page 254].

(v)a One can assign to a hyperbolic group its Rips complex for a certain parameter. If this parameter is chosen large enough, then the Rips complex is a model for $\underline{E}G$ (see [92]). The Rips complex is known to be a G-CW-complex which is finite or, equivalently, cocompact.

(v)b This follows from assertion (v)a (see [80, Theorem 4.2]).

(v)c This follows from assertion (v)b.

(v)d This follows from assertion (v)a (see [80, Theorem 4.2]).

(v)e This follows from assertion (v)a.

(vi)a See [20, Corollary 3.10 in Chapter III.Γ on page 462].

(vi)b This follows from [58, Theorem 37 in Chapter 8 on page 154] and the fact that an amenable group cannot contain a free group of rank 2 as subgroup.

(vi)c See [20, Proposition 3.20 in Chapter III.Γ on page 467].

(vii) See [20, Proposition 2.22 in Chapter III.Γ on page 458].

(viii)a This follows from assertions (vi)b and (vii).

(viii)b See [58, Exercise 34 in Chapter 1 on page 19].

(ix)a See [20, Theorem 2.8 in Chapter III.Γ on page 451].

(ix)b See [127].

(x)a See [12, Corollary 1.4 (e)].

(x)b See [69, Theorem 4.4].

(x)c See [26], [49], [52].

(x)d See [12, Corollary 1.3].

(x)e This follows from assertion (x)d. See [31, Corollary 6.3].

(xi) See [20, Theorem 2.21 in Chapter III.Γ on page 457].

(xii)a See [102].

(xii)b See [120].

(xii)c See [22].

(xiii) See [98]. □

Remark 7.3 (The boundary of a hyperbolic group)**.** The boundary ∂X of a hyperbolic space and in particular the boundary ∂G of a hyperbolic group G are metrizable. Any compact metric space can be realized as the boundary of a hyperbolic space. However, not every compact metrizable space can occur as the boundary of a hyperbolic group. Namely, exactly one of the following three cases occurs:

(i) G is finite and ∂G is empty;
(ii) G is infinite virtually cyclic and ∂G consists of two points;
(iii) G contains a free group of rank two as subgroup and ∂G is an infinite perfect, (i.e., without isolated points) compact metric space.

The metric structure on ∂X for a hyperbolic space X is not canonical. One can actually equip ∂X with the structure of a *visual metric* (see [20, Definition 3.20 on page 343]). Again the structure of a space with a visual metric is not canonical, not even for ∂G of a hyperbolic group G. However, the induced *quasiconformal structure* and the induced *quasi-Möbius structure* associated to some visual metric on ∂G of a hyperbolic group G are canonical, i.e., independent of the choice of a visual metric.

These structures are quasiisometry invariants. Namely, a quasiisometry of finitely generated hyperbolic groups $G_1 \to G_2$ (with respect to some choice of finite sets of generators) induces a homeomorphism $\partial G_1 \to \partial G_2$ which is quasiconformal and quasi-Möbius homeomorphism with respect to any visual metric. The converse is also true in the sense that a homeomorphism $\partial G_1 \to \partial G_2$, which is a quasi-Möbius equivalence or a quasiconformal homeomorphism, comes from a quasiisometry $G_1 \to G_2$. (see [15], [69, Section 3], [110]).

The induced action of G on the boundary ∂G is also an important invariant of G.

For more information about the boundary of a hyperbolic group we refer for instance to [69].

We mention the following result whose proof will appear in a forthcoming paper by Bartels, Lück and Weinberger [6].

Theorem 7.4 (High-dimensional spheres as boundary)**.** *Let G be a torsionfree hyperbolic group and let n be an integer ≥ 6. Then:*

(i) The following statements are equivalent:
  (a) The boundary ∂G is homeomorphic to S^{n-1};
  (b) There is a closed aspherical topological manifold M such that $G \cong \pi_1(M)$, its universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and the compactification of $\widetilde{M}$ by ∂G is homeomorphic to D^n.

(ii) The following statements are equivalent:
  (a) The boundary ∂G has the integral Čech cohomology of S^{n-1};
  (b) There is a closed aspherical ANR-homology manifold M with $G \cong \pi_1(M)$.

(iii) Let M and N be two aspherical closed n-dimensional manifolds together with isomorphisms $\phi_M \colon \pi_1(M) \xrightarrow{\cong} G$ and $\phi_N \colon \pi_1(N) \xrightarrow{\cong} G$. Then there exists a homeomorphism $f \colon M \to N$ such that $\pi_1(f)$ agrees with $\phi_N^{-1} \circ \phi_M$ (up to inner automorphisms).

Remark 7.5 (Algorithm for the homeomorphism problem)**.** By unpublished work of Bartels and Lück [5] on the Borel Conjecture for hyperbolic groups two closed aspherical manifolds with hyperbolic fundamental groups and dimension $n \geq 5$ are homeomorphic if and only if their fundamental groups are isomorphic. Combining this with the result of Sela [127] stated in Theorem 7.2 (ix)b shows for any integer $n \geq 5$: There exists an algorithm which takes as input two closed aspherical n-dimensional manifolds with hyperbolic fundamental groups and which (after a finite amount of time) will stop and answers *yes* or *no* according to whether or not the manifolds are homeomorphic.

The following is already pointed out in [20, page 459]: There is a technical problem here with how the closed aspherical manifolds are given. They must be given by a finite amount of information (from which one can read off a presentation of the fundamental group).

Remark 7.6 (Lacunary groups)**.** Olshanskii-Osin-Sapir [100] introduced the notion of a *lacunary* group as a finitely generated group one of whose asymptotic cones is an $\mathbb{R}$-tree. They show that such a group can always be obtained as a colimit of a directed system of hyperbolic groups $G_1 \to G_2 \to G_3 \to \cdots$, where the structure maps are epimorphisms of hyperbolic groups with certain additional properties. A finitely presented lacunary group is hyperbolic. The class of lacunary groups is very large and contains some examples with unusual properties, e.g., certain infinite torsionfree groups whose proper subgroups are all cyclic and infinite torsion-groups whose proper subgroups are all of order p for some fixed prime number p.

Remark 7.7. Colimits of directed systems of hyperbolic groups which come from adding more and more relations have been used to construct exotic groups. Other constructions come from random groups (see [64]). Here are some examples:

(i) Let G be a torsionfree hyperbolic group which is not virtually cyclic. Then there exists a quotient of G which is an infinite torsiongroup whose proper subgroups are all finite (or cyclic) (See [99]);
(ii) There are hyperbolic groups which do have Kazhdan's property (T) (see Zuk [140]);
(iii) There exist groups with expanders. They play a role in the construction of counterexamples to the Baum-Connes Conjecture with coefficients due to Higson, Lafforgue and Skandalis [67].

Remark 7.8 (Exotic aspherical manifolds)**.** For every $n \geq 5$ there exists an example of a closed aspherical topological manifold M of dimension n that is a piecewise flat, non-positively curved polyhedron such that the universal covering $\widetilde{M}$ is not homeomorphic to $\mathbb{R}^n$ (see [33, Theorem 5b.1 on page 383]).

This manifold is not homeomorphic to a closed smooth manifold with Riemannian metric of non-positive sectional curvature by Hadamard's Theorem. There is a variation of this construction that uses the strict hyperbolization of Charney-Davis [27] and produces closed aspherical manifolds whose universal cover is not homeomorphic to Euclidean space and whose fundamental group is hyperbolic.

There exists a strictly negatively curved polyhedron N of dimension 5 whose fundamental group is hyperbolic, which is homeomorphic to a closed aspherical smooth manifold and whose universal covering is homeomorphic to $\mathbb{R}^n$, but the ideal boundary of its universal covering, which is homeomorphic to ∂G, is not homeomorphic to S^{n-1} (see [33, Theorem 5c.1 on page 384]). Notice N is not homeomorphic to a closed smooth Riemannian manifold with negative sectional curvature.

Remark 7.9 (Cohomological characterization of hyperbolic groups)**.** There exist also characterizations of the property hyperbolic in terms of cohomology. A finitely presented group G is hyperbolic if and only if $H_1^{(1)}(G,\mathbb{R}) = \overline{H}_1^{(1)}(G,\mathbb{R}) = 0$ holds for the first l^1-homology and the first reduced l^1-homology (see [2]). For a characterization in terms of bounded cohomology we refer to [93].

8. CAT(0)-groups

Definition 8.1 (CAT(0)-group)**.** A group is called CAT(0)-*group* if it admits an isometric proper cocompact action on some CAT(0)-space.

Theorem 8.2 (Properties of CAT(0)-groups)**.**

(i) Classifying spaces and finiteness properties:
- *(a) If G is a* CAT(0)*-group, then there exists a finite model for the universal space of proper G-actions $\underline{E}G$ (see Definition 9.1);*
- *(b) If G is a* CAT(0)*-group, then there is a model for BG of finite type, $H_n(G;\mathbb{Z})$ is finitely generated as $\mathbb{Z}$-module for $n \geq 0$ and $H_n(G;\mathbb{Q})$ is trivial for almost all $n \geq 0$;*
- *(c) If G is a* CAT(0)*-group, then G is finitely presented;*
- *(d) Suppose that G is a* CAT(0)*-group. Then there are only finitely many conjugacy classes of finite subgroups of G;*
- *(e) If G is a torsionfree* CAT(0)*-group, then there is a finite model for BG, the abelian group $H_n(G;\mathbb{Z})$ is finitely generated for $n \geq 0$ and $H_n(G;\mathbb{Z})$ is trivial for almost all $n \geq 0$;*

(ii) Solvable subgroups:
Every solvable subgroup of a CAT(0)*-group is virtually $\mathbb{Z}^n$;*

(iii) Inheritance properties:
- *(a) The direct product of two* CAT(0)*-groups is again a* CAT(0)*-group;*
- *(b) The free product with amalgamation along a virtually cyclic subgroup of two* CAT(0)*-groups is again a* CAT(0)*-group;*

(c) *The* HNN*-extension of a* CAT(0)*-group along a finite group is again a* CAT(0)*-group;*

(iv) Examples:
   (a) *Limit groups in the sense of Sela are* CAT(0)*-groups;*
   (b) *Coxeter groups are* CAT(0)*-groups;*
   (c) *Three-dimensional FC Artin groups are* CAT(0)*-groups;*

(v) Decision problems:
The word-problem and the conjugation-problem are solvable for a CAT(0)*-group;*

(vi) Hyperbolic:
Let G act isometrically, properly and cocompactly on the CAT(0)*-space X. Then G is hyperbolic if and only if X does not contain an isometrically embedded copy of a Euclidean plane;*

(vii) Weak Hyperbolization Theorem:
Let G be a three-dimensional Poincaré duality group. Suppose that in addition that G is a CAT(0)*-group. Then G satisfies the Weak Hyperbolization Conjecture, i.e., either G contains $\mathbb{Z}^2$ or G is hyperbolic.*

Proof. (i)a Let X be a CAT(0)-space on which G acts properly, isometrically and cocompactly. Then it is easy to show that X is a model for $\underline{J}G$ for the numerable version of the classifying space for proper G-actions. (Notice that X is not necessarily a CW-complex. But this implies that there is a cocompact model for $\underline{E}G$. Details will appear in [87].

(i)b This follows from assertion (i)a (see [80, Theorem 4.2]).

(i)c This follows from assertion (i)b.

(i)d This follows from assertion (i)a (see [80, Theorem 4.2]).

(i)e This follows from assertion (i)a.

(ii) See [20, Theorem 1.1 in Chapter III.Γ on page 439].

(iii)a See [20, Theorem 1.1 in Chapter III.Γ on page 439].

(iii)b See [20, Theorem 1.1 in Chapter III.Γ on page 439].

(iii)c See [20, Theorem 1.1 in Chapter III.Γ on page 439].

(iv)a [1].

(iv)b This is a result due to Moussong. See [32, Theorem 12.3.3 on page 235], [95].

(iv)c See [10].

(vi) See [20, Theorem 3.1 in Chapter III.Γ on page 459].

(vii) See [70, Theorem 2]. □

Interesting results about CAT(0)-groups and CAT(0)-lattices including rigidity statements have been proved by Caprace and Monod [24].

9. Classifying spaces for proper actions

Very often information or basic properties of groups are reflected in interesting actions of the group. In this context the notion of a classifying space for proper G-actions is important. This notion and the more general notion of a classifying space for a family of subgroups was introduced by tom Dieck (see [134], [135, I.6]).

A *G-CW-complex* X is a CW-complex with a G-action such that for every open cell e and every $g \in G$ with $g \cdot e = e$ we have $gx = x$ for every $g \in G$ and $x \in e$. The barycentric subdivision of a simplicial complex with simplicial G-action is a G-CW-complex. A G-CW-complex X is *proper* if and only if all its isotropy groups are finite (see [79, Theorem 1.23]).

Definition 9.1 (Classifying space for proper actions)**.** Let G be a group. A model for the *classifying space of proper G-actions* is a proper G-CW-complex $\underline{E}G$ such that $\underline{E}G^H$ is contractible for all finite subgroups $H \subseteq G$.

Theorem 9.2 (Homotopy characterization of $\underline{E}G$)**.**

(i) There exists a model for $\underline{E}G$;

(ii) A G-CW-complex Y is a model for $\underline{E}G$ if and only if for every proper G-CW-complex X there is up to G-homotopy precisely one G-map $X \to Y$. In particular any two models for the classifying space for proper G-actions are G-homotopy equivalent.

Proof. See for instance [85, Theorem 1.9 on page 275]. □

If G is torsionfree, then a model for $\underline{E}G$ is a model for EG, i.e., the total space of the universal G-principal bundle $G \to EG \to BG$. A group G is finite if and only if G/G is a model for $\underline{E}G$.

Some prominent groups come with prominent actions on prominent spaces. Often it turns out that these are models for the classifying space for proper G-actions. Here we give a list of examples. More explanations and references can be found in the survey article [85].

- Discrete subgroups of almost connected Lie groups
  Let L be a Lie group with finitely many path components. Let $K \subseteq L$ be any maximal compact subgroup, which is unique up to conjugation. Let $G \subseteq L$ be a discrete subgroup. Then L/K is diffeomorphic to $\mathbb{R}^n$ and becomes with the obvious left G-action a model for $\underline{E}G$.
- Hyperbolic groups and the Rips complex
  Let G be a hyperbolic group. Let $P_d(G)$ be the Rips complex. Then $P_d(G)$ is a model for $\underline{E}G$ if d is chosen large enough.
- Proper isometric actions on simply connected complete Riemannian manifolds with non-positive sectional curvature
  Suppose that G acts isometrically and properly on a simply connected complete Riemannian manifold M with non-positive sectional curvature. Then M is a model for $\underline{E}G$;

- Proper actions on trees
  Let T be a tree. Suppose that G acts on T by tree automorphisms without inversion such that all isotropy groups are finite. Then T is a model for $\underline{E}G$;
- Arithmetic groups and the Borel-Serre compactification
  Let $G(\mathbb{R})$ be the $\mathbb{R}$-points of a semisimple $\mathbb{Q}$-group $G(\mathbb{Q})$ and let $K \subseteq G(\mathbb{R})$ be a maximal compact subgroup. If $A \subseteq G(\mathbb{Q})$ is an arithmetic group, then $G(\mathbb{R})/K$ with the left A-action is a model for $\underline{E}A$. The A-space $G(\mathbb{R})/K$ is not necessarily cocompact. However, the Borel-Serre completion of $G(\mathbb{R})/K$ is a finite A-CW-model for $\underline{E}A$;
- Mapping class groups and Teichmüller space
  Let $\Gamma_{g,r}^s$be the *mapping class group* of an orientable compact surface $F_{g,r}^s$ of genus g with s punctures and r boundary components. This is the group of isotopy classes of orientation preserving self-diffeomorphisms $F_{g,r}^s \to F_{g,r}^s$ that preserve the punctures individually and restrict to the identity on the boundary. We require that the isotopies leave the boundary pointwise fixed. We will always assume that $2g+s+r > 2$, or, equivalently, that the Euler characteristic of the punctured surface $F_{g,r}^s$ is negative. Then the associated *Teichmüller space* $\mathcal{T}_{g,r}^s$ is a model for $\underline{E}\Gamma_{g,r}^s$;
- $\mathrm{Out}(F_n)$ and outer space
  Let F_n be the free group of rank n. Denote by $\mathrm{Out}(F_n)$ the group of outer automorphisms of F_n. Culler and Vogtmann [30], [137] have constructed a space X_n called *outer space*, on which $\mathrm{Out}(F_n)$ acts with finite isotropy groups. It is a model for $\underline{E}\,\mathrm{Out}(F_n)$.

  The space X_n contains a *spine* K_n which is an $\mathrm{Out}(F_n)$-equivariant deformation retract. This space K_n is a simplicial complex of dimension $(2n-3)$ on which the $\mathrm{Out}(F_n)$-action is by simplicial automorphisms and cocompact. Hence the barycentric subdivision of K_n is a finite $(2n-3)$-dimensional model of $\underline{E}\,\mathrm{Out}(F_n)$;
- One-relator groups
  Let G be a one-relator group. Let $G = \langle (q_i)_{i\in I} \mid r\rangle$ be a presentation with one relation. There is up to conjugacy one maximal finite subgroup C which turns out to be cyclic. Let $p\colon *_{i\in I}\mathbb{Z} \to G$ be the epimorphism from the free group generated by the set I to G that sends the generator $i \in I$ to q_i. Let $Y \to \bigvee_{i\in I} S^1$ be the G-covering associated to the epimorphism p. There is a 1-dimensional unitary C-representation V and a C-map $f\colon SV \to \mathrm{res}_G^C Y$ such that the following is true: The induced action on the unit sphere SV is free. If we equip SV and DV with the obvious C-CW-complex structures, the C-map f can be chosen to be cellular and we obtain a G-CW-model

for $\underline{E}G$ by the G-pushout

$$\begin{array}{ccc} G \times_C SV & \xrightarrow{\overline{f}} & Y \\ \downarrow & & \downarrow \\ G \times_C DV & \longrightarrow & \underline{E}G \end{array}$$

where $\overline{f}$ sends (g, x) to $gf(x)$. Thus we get a 2-dimensional G-CW-model for $\underline{E}G$ such that $\underline{E}G$ is obtained from G/C for a maximal finite cyclic subgroup $C \subseteq G$ by attaching free cells of dimensions ≤ 2 and the CW-complex structure on the quotient $G\backslash\underline{E}G$ has precisely one 0-cell, precisely one 2-cell and as many 1-cells as there are elements in I.

Remark 9.3 (Isomorphism Conjectures)**.** The space $\underline{E}G$ and its version for the family of virtually cyclic subgroups play an important role in the formulation of the Isomorphism Conjectures for K- and L-theory of group rings and reduced group C^*-algebras or Banach algebras due to Farrell-Jones (see [47, 1.6 on page 257]), Baum-Connes (see [7, Conjecture 3.15 on page 254]) and Bost. Methods and results from geometric group theory enter the proofs of these conjectures for certain classes of groups. A survey on these conjectures, their status and the methods of proof can be found for instance in [89].

Remark 9.4 (Small models)**.** As one can ask whether there are small models for BG (or, equivalently, for the G-CW-complex EG) such as finite models, models of finite type or finite-dimensional models, the same question is interesting for the G-CW-complex $\underline{E}G$ and has been studied for instance in [74], [80], [88].

Although there are often nice small models for $\underline{E}G$, these spaces can be arbitrarily complicated. Namely, for any CW-complex X there exists a group G such that $G\backslash\underline{E}G$ and X are homotopy equivalent (see [77]). There can also be dramatic changes in the complexity and size of $\underline{E}G$ if one passes from $\underline{E}H$ to $\underline{E}G$ for a subgroup $H \subseteq G$ of finite index (see [78]).

Remark 9.5 (Compactifications of $\underline{E}G$)**.** It is very important to find appropriate compactifications of $\underline{E}G$. Finding the right one which is "small at infinity" leads to injectivity results concerning the Isomorphism Conjectures (see for instance [25], [121], [122]). We have seen for a hyperbolic group that its boundary yields a powerful compactification of the associated Rips complex. A CAT(0)-space comes with a natural compactification by adding its boundary. There is a whole theory of compactifications of the Teichmüller space. For arithmetic groups the Borel-Serre compactification is crucial.

Remark 9.6 (Computations)**.** A good understanding of the spaces $\underline{E}G$ can be used to make explicit computations of the homology or topological K-theory $H_*(BG)$ and $K_*(BG)$ or various K- and L-groups such as $K_*(RG)$, $L_*(RG)$ and $K_*(C_r^*(G))$. See for instance [81], [83], [84], [86], [90].

10. Measurable group theory

Gromov [63, $0.2.C_2'$ on page 6] (see also [34, Exercise 35 in IV.B on page 98] or [131, Theorem 2.1.2]) observed that the notion of quasiisometry can be reformulated as follows.

Lemma 10.1. *Two finitely generated groups G_1 and G_2 are quasiisometric if and only if there exists a locally compact space on which G_1 and G_2 act properly and cocompactly and the actions commute.*

This led Gromov to the following measure theoretic version (see [63, 0.5E], [50] and [51]). A *Polish space* is a separable topological space which is metrizable by a complete metric. A measurable space is called a *standard Borel space* if it is isomorphic to a Polish space with its standard Borel σ-algebra. Let Ω be a standard Borel space with a Borel measure μ. Let G act on Ω by Borel automorphisms. A *measure theoretic fundamental domain* for the G-action is a Borel subset $X \subseteq \Omega$ such that $\mu(g \cdot X \cap X) = 0$ for every $g \in G, g \neq 1$ and $\mu(\Omega - G \cdot X) = 0$ hold.

Definition 10.2 (Measure equivalence)**.** Two countable groups G and H are called *measure equivalent* if there is a standard Borel space Ω with a non-zero Borel measure on which G and H act by measure-preserving Borel automorphisms such that the actions commute and the actions of both G and H admit finite measure fundamental domains.

The actions appearing in Definition 10.2 are automatically *essentially free*, i.e., the stabilizer of almost every point is trivial, because of the existence of the measure fundamental domains. Measure equivalence defines an equivalence relation on countable groups (see [50, Section 2]).

Remark 10.3 (Lattices)**.** Let Γ and Λ be two lattices in the locally compact second countable topological group G, i.e., discrete subgroups with finite covolume with respect to a Haar measure on G. Then Λ and Γ are quasiisometric provided that they are cocompact. An important feature of measure equivalence is that Λ and Γ are measure equivalent without the hypothesis of being cocompact (see [63, $0.5.E_2$]).

An action $G \curvearrowright X$ of a countable group G is called *standard* if X is a standard Borel space with a probability measure μ, the group G acts by μ-preserving Borel automorphisms and the action is essentially free.

Definition 10.4 ((Weak) orbit equivalence)**.** Two standard actions $G \curvearrowright X$ and $H \curvearrowright Y$ are called *weakly orbit equivalent* if there exist Borel subsets $A \subseteq X$ and $B \subseteq Y$ meeting almost every orbit and a Borel isomorphism $f\colon A \to B$ which preserves the normalized measures on A and B, respectively, and satisfies for almost all $x \in A$

$$f(G \cdot x \cap A) = H \cdot f(x) \cap B.$$

If A and B have full measure in X and Y, the two actions are called *orbit equivalent*.

The following result is formulated and proved in [51, Theorem 3.3], where credit is also given to Gromov and Zimmer.

Theorem 10.5 (Measure equivalence versus weak orbit equivalence)**.** *Two countable groups G and H are measure equivalent if and only if there exist standard actions of G and H that are weakly orbit equivalent.*

The next result is due to Ornstein-Weiss [101].

Theorem 10.6.

(i) Let G_1 and G_2 be two infinite countable amenable groups. Then any two standard actions of G_1 and G_2 are orbit equivalent;
(ii) Any infinite amenable group G is measure equivalent to $\mathbb{Z}$.

On the other hand we have the following result due to Epstein [39, Corollary 1.2], the case of a group with property (T) has been treated by Hjorth [68] before.

Theorem 10.7. *A countable non-amenable group admits a continuum of standard actions which are not pairwise orbit equivalent.*

The following result is due to Gaboriau-Popa [56].

Theorem 10.8. *Let G be a non-abelian free group. Then there exists a continuum of standard actions $G \curvearrowright X$ which are pairwise not orbit equivalent and whose associated von Neumann algebras $L^\infty(X) \rtimes G$ are pairwise not isomorphic.*

Remark 10.9 (Quasiisometry versus measure equivalence)**.** In general two finitely presented measure equivalent groups need not be quasiisometric. For example $\mathbb{Z}^n$ and $\mathbb{Z}^m$ for $n, m \geq 1$ are quasiisometric if and only if $n = m$ (see Theorem 3.5) and they are always measure equivalent (see Theorem 10.5).

We mention that property (T) is invariant under measure equivalence (see [50, Theorem 8.2]) but is not a quasiisometry invariant (see Remark 3.11).

In general two finitely presented quasiisometric groups need not be measure equivalent as the following example shows. If F_g denotes the free group on g generators, then define $G_n := (F_3 \times F_3) * F_n$ for $n \geq 2$. The groups G_m and G_n are quasiisometric for $m, n \geq 2$ (see [34, page 105 in IV-B.46], [139, Theorem 1.5]) and have finite models for their classifying spaces. One easily checks that $b_1^{(2)}(G_n) = n$ and $b_2^{(2)}(G_n) = 4$ (see [82, Example 1.38 on page 41]). By the following result of Gaboriau [53, Theorem 6.3] the groups G_n and G_m are measure equivalent if and only if $m = n$ holds.

Theorem 10.10 (Measure equivalence and L^2-Betti numbers)**.** *Let G_1 and G_2 be two countable groups that are measure equivalent. Then there is a constant $C > 0$ such that for all $p \geq 0$*

$$b_p^{(2)}(G_1) = C \cdot b_p^{(2)}(G_2).$$

Remark 10.11 (Measure equivalence rigidity)**.** In view of Theorem 10.6 one realizes that measurable equivalence cannot capture any group theoretic property which can be separated within the class of amenable groups and is highly non-rigid for amenable groups. Nevertheless, there is a deep and interesting rigidity theory underlying the notion of orbit equivalence. For information about this topic we refer for instance to the survey article of Shalom [132]. We give as an illustration some examples below.

The next result follows from Furman [51, Corollary B] and is stated in the present sharpened form in [132, Theorem 3.1].

Theorem 10.12. *Fix an odd natural number $n \geq 3$. Consider the obvious standard action of $\mathrm{SL}_n(\mathbb{Z})$ on the n-torus T^n equipped with the Lebesgue measure. Suppose that it is orbit equivalent to a standard action of the group Λ. Then $\Lambda \cong \mathrm{SL}_n(\mathbb{Z})$ and the orbit equivalence is induced by an isomorphism of actions.*

The next result is due to Monod-Shalom [94, Theorem 1.18] and may be viewed as the measure theoretic definition of a negatively curved group.

Theorem 10.13. *The condition that the second bounded cohomology $H_b^2(G, l^2(G))$ with coefficients in $l^2(G)$ does not vanish is an invariant under measure equivalence. Hyperbolic groups have this property.*

The next result is taken from [94, Corollary 1.11 and Theorem 1.16]. For a countable group G and any probability distribution μ (different from Dirac) on the interval $[0, 1]$, the natural shift action on $\prod_G([0, 1], \mu)$ is called a *Bernoulli G-action*.

Theorem 10.14.

(i) Let G be the direct product of two torsionfree groups G_1 and G_2 with non-trivial $H_b^2(G_1, l^2(G_1))$ and $H_b^2(G_2, l^2(G_2))$. If a Bernoulli G-action is orbit equivalent to a Bernoulli H-action for some group H, then $G \cong H$ and the actions are isomorphic by a Borel isomorphism which induces the given orbit equivalence;

(ii) Let $G_1, \ldots, G_m$ and $H_1, \ldots, H_n$ be torsionfree groups with non-vanishing $H_b^2(G_i, l^2(G_i))$ and $H_b^2(H_j, l^2(H_j))$. Suppose that $\prod_{i=1}^m G_i$ and $\prod_{j=1}^n H_j$ are measure equivalent. Then $m = n$ and for an appropriate permutation σ the groups G_i and $H_{\sigma(i)}$ are measure equivalent for $i = 1, 2, \ldots n$.

There are many more interesting results in the spirit that the orbit structure of an action remembers the group and the action, and relations between orbit equivalence and questions about von Neumann algebras and bounded cohomology. In particular, Popa has proved spectacular results on fundamental groups of II_1-factors and on rigidity. See for instance [23], [55], [56], [94], [103], [111], [112], [113], [114], [115], [117], [116], [118].

11. Some open problems

Here is a list of interesting open problems. It reflects some of the interests (and limited knowledge) of the author:

11.1. **Hyperbolic groups.**

(i) Is every hyperbolic group virtually torsionfree?

(ii) Is every hyperbolic group residually finite?

(iii) Suppose that the space at infinity of a hyperbolic group is homeomorphic to S^2. Does this imply that it acts properly isometrically and cocompactly on the 3-dimensional hyperbolic space?

Partial results in this direction have been proved in [14].

(iv) Has the boundary of a hyperbolic group the integral Čech cohomology of a sphere if and only if it occurs as the fundamental group of an aspherical closed manifold M?

(v) Is the boundary ∂G of a hyperbolic group G homeomorphic to S^n if and only if it occurs as the fundamental group of an aspherical closed manifold M whose universal covering $\widetilde{M}$ is homeomorphic to $\mathbb{R}^n$ and its compactification $\widetilde{M} \cup \partial G$ by ∂G is homeomorphic to D^n?

The answer for the last problems is yes for $n \geq 6$ (see Theorem 7.4).

(vi) Which topological spaces occur as boundary of a hyperbolic group?

(vii) Is every hyperbolic group a CAT(0)-group?

11.2. **Isomorphism Conjectures.**

(i) Are the Conjectures due to Baum-Connes, Farrell-Jones and Borel true for the following groups?

- $\mathrm{SL}_n(\mathbb{Z})$ for $n \geq 3$;
- Mapping class groups;
- $\mathrm{Out}(F_n)$.

The fact that these conjectures are not known for these groups indicates that we do not understand enough about the geometry of these groups. Probably any successful proof will include new interesting information about these groups.

(ii) Are the Conjectures due to Farrell-Jones and Borel true for amenable groups?

The Baum-Connes Conjecture is known for groups with the Haagerup property and hence in particular for amenable groups (see [66], [46].) The Farrell-Jones Conjecture and the Borel Conjecture for these groups are harder since there one has to take into account all virtually cyclic subgroups and not only all finite subgroups as in the Baum-Connes setting and one encounters Nil-phenomena which do not occur in the Baum-Connes setting.

(iii) Is there a property for groups known such that Isomorphism Conjectures mentioned above are not known for any group having this property. If yes, can one use this property to produce counterexamples?

For some time property (T) was thought to be such a property for the Baum-Connes Conjecture until Lafforgue (see [75], [76]) proved the Baum-Connes Conjecture for certain groups having property (T). The counterexamples to the Baum-Connes Conjecture by Higson-Lafforgue-Skandalis [67] given by groups with expanders have indicated another source of possible counterexamples. Such groups can be constructed by directed colimits of hyperbolic groups. However, for directed colimits of hyperbolic groups the Farrell-Jones Conjecture and the Borel Conjecture in dimension ≥ 5 are known to be true by unpublished work of Bartels and Lück [5] and the Bost Conjecture with C^*-coefficients has been proved by Bartels-Echterhoff-Lück [4].

11.3. **Quasiisometry.**

(i) Are there finitely presented groups that are quasiisometric such that one is solvable but the other is not? See Remark 3.9.
(ii) Is the property of being poly-cyclic invariant under quasiisometry?
(iii) Is the Mal'cev completion of a finitely generated torsionfree nilpotent group an invariant under quasiisometry? See Conjecture 3.6.
(iv) Are the Novikov-Shubin invariants or the vanishing of the L^2-torsion invariants under quasiisometry? See Remark 3.14.

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