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Euler characteristics of categories and barycentric subdivision

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Abstract. We show that three Euler characteristics of categories, Leinster's Euler characteristic, the series Euler characteristic and the Euler characteristic of $\mathbb{N}$-filtered acyclic categories, are invariant under barycentric subdivision for finite acyclic categories. An acyclic category is a small category in which all endomorphisms and isomorphisms are identities, that is, an acyclic category is a skeletal scwol. We show for any small category $\mathcal{I}$, the opposite subdivision category $\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}$ is of type (L^2) if and only if $\mathcal{I}$ is finite acyclic. We also extend the definition of the L^2-Euler characteristic and prove our extended L^2-Euler characteristic is invariant under barycentric subdivision for a wider class of finite categories.

Contents

1. Introduction

Euler characteristics are defined for many mathematical objects, for example, cell complexes, manifolds, varieties, graphs and so on. But the most basic one is the Euler characteristic for simplicial complexes which is defined by the alternating sum of the number of faces. Rota defined the Euler characteristic for finite posets [14]. The relation between the Euler characteristic of simplicial complexes and the one of posets is described by the following diagram

$$\begin{array}{ccc} \textbf{Finite posets} & \xrightarrow{\text{order complex}} & \textbf{Finite simplicial complexes} \\ & {\scriptstyle \chi_{\mathrm{Rota}}} \searrow \quad \swarrow {\scriptstyle \chi} & \\ & \mathbb{Z}. & \end{array}$$

Here, the order complex of a finite poset P is an abstract simplicial complex having totally ordered $(n+1)$-subsets of P as its n-simplices.

Leinster extended Rota's theory. He defined the Euler characteristic χ_L for finite categories which satisfy certain conditions on the underlying directed graph, including finite posets, finite groups, orbifolds, directed graphs and so on [8]. At present, we have various invariants of categories, *the series Euler characteristic* $\chi_{\sum}$ [2], *the* L^2*-Euler characteristic* $\chi^{(2)}$ [5], *the* L^2*-Betti numbers of discrete measured groupoids* [15], *the Euler characteristic of an* $\mathbb{N}$*-filtered acyclic category* χ_{fil} [13], *the cardinality of categories* [1] and so on.

In this paper, we show three Euler characteristics of categories, Leinster's Euler characteristic, the series Euler characteristic and the Euler characteristic of $\mathbb{N}$-filtered acyclic categories, are invariant under barycentric subdivision for finite acyclic categories. An *acyclic category* is a small category in which all endomorphisms and isomorphisms are identities, that is, an acyclic category is a skeletal scwol [3]. We show for any small category $\mathcal{I}$, the opposite subdivision category $\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}$ is of type (L^2) if and only if $\mathcal{I}$ is finite acyclic. We also extend the definition of the L^2-Euler characteristic and prove our extended L^2-Euler characteristic is invariant under barycentric subdivision for a wider class of finite categories.

First of all, let us review four Euler characteristics of categories.

Leinster's Euler characteristic χ_L and the series Euler characteristic $\chi_{\sum}$ are defined for finite categories satisfying certain conditions on the underlying directed graph. When a finite category $\mathcal{I}$ has a Möbius inversion, they coincide $\chi_L(\mathcal{I}) = \chi_{\sum}(\mathcal{I})$ ([2, Thm. 3.2]). Here, a finite category $\mathcal{I}$ with the set of objects $\mathrm{Ob}(\mathcal{I}) = \{x_1, \ldots, x_n\}$ has a *Möbius inversion* if the matrix $Z_{\mathcal{I}} = (\#\mathrm{Hom}_{\mathcal{I}}(x_i, x_j))_{i,j}$ is invertible. But if $\mathcal{I}$ does not have a Möbius inversion, there are various situations, that is, these two Euler characteristics take the same values, they take different values, one is defined but the other is not, both of them are not defined.

The L^2-Euler characteristic is defined not only for finite categories but also infinite categories satisfying a certain homological condition. For a finite, free, skeletal EI-category $\mathcal{I}$, Leinster's Euler characteristic and the L^2-Euler characteristic coincide, $\chi_L(\mathcal{I}) = \chi^{(2)}(\mathcal{I})$ ([5, Lemma 7.3]). Here, an *EI-category* is a small category whose endomorphisms are isomorphisms. A small category $\mathcal{J}$ is *free* if the left $\mathrm{Aut}(y)$-action on $\mathrm{Hom}_{\mathcal{J}}(x, y)$ is free for any objects x, y of $\mathcal{J}$. Free here is different than being a freely generated category. The L^2-Euler characteristic sometimes takes different values from $\chi_L(\mathcal{I})$ and $\chi_{\sum}(\mathcal{I})$. The reason that it is different from χ_L and $\chi_{\sum}$ is that those only depend on the underlying graph and ignore the composition of $\mathcal{I}$, while the L^2-Euler characteristic detects composition also. For instance, let $M = \{0, 1\}$ be the commutative monoid where 0 is the unit element and $1 + 1 = 1$. A monoid can be regarded as a category with one object. Then, as pointed out in [5, Rem. 7.2], $\chi^{(2)}$ distinguishes between M and $\mathbb{Z}_2$, that is,

$$\chi^{(2)}(M) \neq \chi^{(2)}(\mathbb{Z}_2) = \frac{1}{2},$$

but χ_L and $\chi_{\sum}$ do not, that is,

$$\chi_L(M) = \chi_L(\mathbb{Z}_2) = \frac{1}{2}$$

and

$$\chi_{\sum}(M) = \chi_{\sum}(\mathbb{Z}_2) = \frac{1}{2}.$$

The next invariant χ_{fil} is the Euler characteristic for $\mathbb{N}$-filtered acyclic categories. An $\mathbb{N}$-*filtered acyclic category* is a pair $(\mathcal{A}, \mu)$ of an acyclic category $\mathcal{A}$ and a filtration μ, called an $\mathbb{N}$-*filtration*, on the set of objects in $\mathcal{A}$. For a finite acyclic category $\mathcal{A}$, these four Euler characteristics coincide

$$\chi_L(\mathcal{A}) = \chi_{\sum}(\mathcal{A}) = \chi^{(2)}(\mathcal{A}) = \chi_{\mathrm{fil}}(\mathcal{A}, \mu)$$

for any $\mathbb{N}$-filtration μ of $\mathcal{A}$. The first equality is implied by the fact that $\mathcal{A}$ has a Möbius inversion and [2, Thm. 3.2]. The second equality is implied by the fact $\mathcal{A}$ is a finite, free, skeletal EI-category and [5, Lemma 7.3]. The definition of $\chi_{\mathrm{fil}}(\mathcal{A}, \mu)$ and [8, Cor. 1.5] directly imply the third equality.

Moreover, χ_{fil} is suitable for barycentric subdivision of small categories. The *barycentric subdivision* of small categories is a functor from the category of small categories to itself

$$\mathrm{Sd} : \textbf{Small categories} \longrightarrow \textbf{Small categories}$$

(see [4] and [6]). For a small category $\mathcal{J}$, $\mathrm{Sd}(\mathcal{J})$ is an acyclic category and its objects are the nondegenerate chains of morphisms of $\mathcal{J}$. In addition, $\mathrm{Sd}(\mathcal{J})$ has naturally an $\mathbb{N}$-filtration. Since the Euler characteristic of simplicial complexes is invariant under barycentric subdivision, we expect a categorical analogue of this fact would hold for a certain class of small categories. But we have to note that $\mathrm{Sd}(\mathcal{J})$ is often infinite even if $\mathcal{J}$ is finite. The category $\mathrm{Sd}(\mathcal{J})$ is finite if and only if $\mathcal{J}$ is a finite acyclic category (see Lemma 3.6). So we can not always use Leinster's Euler characteristic and the series Euler characteristic for this purpose. In [13], the following theorem was proved.

Theorem 1.1 ([13, Thm. 4.9]). *Let $\mathcal{I}$ be a finite category for which the series Euler characteristic can be defined. Then, $\chi_{\mathrm{fil}}(\mathrm{Sd}(\mathcal{I}), L)$ is also defined and they coincide*

$$\chi_{\sum}(\mathcal{I}) = \chi_{\mathrm{fil}}(\mathrm{Sd}(\mathcal{I}), L),$$

that is, we have the following commutative diagram

$$\begin{array}{ccc} \chi_{\sum}\text{-}\mathbf{categories} & \xrightarrow{\mathrm{Sd}} & \chi_{\mathrm{fil}}\text{-}\mathbf{categories} \\ & {\scriptstyle \chi_{\sum}} \searrow \quad \swarrow {\scriptstyle \chi_{\mathrm{fil}}} & \\ & \mathbb{Q}, & \end{array}$$

where $\chi_{\sum}$-**categories** *denotes the category of finite categories for which the series Euler characteristic can be defined and* χ_{fil}-**categories** *denotes the category of* $\mathbb{N}$*-filtered acyclic categories for which its Euler characteristic can be defined (see Section 3.8, Section 3.12). Here,* L *is the* $\mathbb{N}$*-filtration of* $\mathrm{Sd}(\mathcal{I})$ *which is defined by taking the length of chains.*

Since the L^2-Euler characteristic is defined for a certain class of infinite categories, we can consider a similar problem; is the L^2-Euler characteristic invariant under barycentric subdivision? In this paper, the following theorem is obtained.

Theorem 1.2 (Theorem 3.25)**.** *For any small category* $\mathcal{I}$*, the opposite of the subdivision* $\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}$ *is of type* (L^2) *if and only if* $\mathcal{I}$ *is finite acyclic. In this case,*

$$\chi^{(2)}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}) = \chi^{(2)}(\mathcal{I}) = \chi^{(2)}(\mathrm{Sd}(\mathcal{I})).$$

Thus, the L^2-Euler characteristic is invariant under barycentric subdivision for finite acyclic categories. But $\mathrm{Sd}(\mathcal{A})$ is a finite category for a finite acyclic category $\mathcal{A}$ and $\chi_L(\mathrm{Sd}(\mathcal{A}))$ and $\chi_{\sum}(\mathrm{Sd}(\mathcal{A}))$ exist. Furthermore, we obtain

$$\chi_L(\mathcal{A}) = \chi_L(\mathrm{Sd}(\mathcal{A})) \text{ (Section 3.1)}, \ \chi_{\sum}(\mathcal{A}) = \chi_{\sum}(\mathrm{Sd}(\mathcal{A})) \text{ (Section 3.8)}$$

for a finite acyclic category $\mathcal{A}$. For any $\mathbb{N}$-filtration μ of $\mathcal{A}$ we obtain

$$\chi_{\mathrm{fil}}(\mathcal{A}, \mu) = \chi_{\mathrm{fil}}(\mathrm{Sd}(\mathcal{A}), L) \text{ (Section 3.12)}.$$

Theorem 1.2 suggests that extensions of the notion of "type (L^2)" and the L^2-Euler characteristic are desirable in order to have a wider class of categories $\mathcal{I}$ such that both $\chi^{(2)}_{\mathrm{ex}}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}})$ and $\chi^{(2)}_{\mathrm{ex}}(\mathcal{I})$ exist and are equal. We do this in Section 4 and obtain the following theorem.

Main Theorem (Theorem 4.5)**.** *Let* $\mathcal{I}$ *be a finite category. Then* $\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}$ *is of type extended* (L^2) *if and only if the power series* $f_{\mathcal{I}}(t) = \sum_{n=0}^{\infty} \#\overline{N_n}(\mathcal{I})t^n$ *is rational with a nonvanishing denominator at* $t = -1$*. In this case, we have*

$$\chi_{\sum}(\mathcal{I}) = \chi^{(2)}_{\mathrm{ex}}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}).$$

If $\mathcal{I}$ *is additionally acyclic, these are equal to* $\chi^{(2)}(\mathcal{I})$*,* $\chi^{(2)}(\mathrm{Sd}(\mathcal{I}))$*, and* $\chi^{(2)}_{\mathrm{ex}}$ $(\mathrm{Sd}(\mathcal{I}))$*.*

Here, $\overline{N_n}(\mathcal{I})$ is the set of nondegenerate chains of morphisms of $\mathcal{I}$ of length n.

We note that the $\mathbb{N}$-filtration L appears on the way to prove our main theorem, although the L^2-Euler characteristic and the Euler characteristic of

$\mathbb{N}$-filtered acyclic categories were independently found. When we compute $\chi^{(2)}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}})$ and $\chi^{(2)}_{\mathrm{ex}}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}})$, the definition of the L^2-Euler characteristic requires us to have a projective resolution of the constant functor $\underline{\mathbb{C}}$ in the functor category $\mathrm{Func}(\mathrm{Sd}(\mathcal{I}), \mathbb{C}\text{-vect})$. The following is a projective resolution of $\underline{\mathbb{C}}$ we will construct

$$\cdots \xrightarrow{\partial_2} \bigoplus_{\mathbf{f_1} \in \overline{N_1}(\mathcal{I})} \mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{I})}(\mathbf{f_1}, -)] \xrightarrow{\partial_1}$$

$$\bigoplus_{\mathbf{f_0} \in \overline{N_0}(\mathcal{I})} \mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{I})}(\mathbf{f_0}, -)] \xrightarrow{\partial_0} \underline{\mathbb{C}} \longrightarrow 0$$

where $\mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{I})}(\mathbf{f_n}, -)]$ is a projective object corresponding to each $\mathbf{f_n}$ of $\overline{N_n}(\mathcal{I})$ (Note that $\mathbf{f_n}$ is an object in $\mathrm{Sd}(\mathcal{I})$). The functor $\mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{I})}(\mathbf{f_n}, -)]$ means to compose the free vector space functor with the functor $\mathrm{Hom}_{\mathrm{Sd}(\mathcal{I})}(\mathbf{f_n}, -)$. Thus, this projective resolution gives the $\mathbb{N}$-filtration L on $\mathrm{Sd}(\mathcal{I})$ and conversely L gives the projective resolution. Furthermore, on the way to compute

$$\chi_{\sum}(\mathcal{I}),\ \chi^{(2)}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}),\ \chi^{(2)}_{\mathrm{ex}}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}),\ \chi_{\mathrm{fil}}(\mathrm{Sd}(\mathcal{I}), L),$$

the power series

$$\sum_{n=0}^{\infty} \# \overline{N_n}(\mathcal{I}) z^n$$

always appears. The Euler characteristics $\chi^{(2)}_{\mathrm{ex}}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}})$ and $\chi_{\mathrm{fil}}(\mathrm{Sd}(\mathcal{I}), L)$ are just the series Euler characteristic $\chi_{\sum}(\mathcal{I})$ and it can be indicated that the series is very important to consider the Euler characteristic of categories.

This paper is organized as follows.

In Section 2, we give some notation and basic definitions. We recall the homological algebra of a functor category, which is used in the definition of the L^2-Euler characteristic.

In Section 3, we prove the four Euler characteristics of categories mentioned above are invariant under barycentric subdivision for finite acyclic categories. To prove the exactness of the sequence above

$$\cdots \xrightarrow{\partial_2} \bigoplus_{\mathbf{f_1} \in \overline{N_1}(\mathcal{I})} \mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{I})}(\mathbf{f_1}, -)] \xrightarrow{\partial_1}$$

$$\bigoplus_{\mathbf{f_0} \in \overline{N_0}(\mathcal{I})} \mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{I})}(\mathbf{f_0}, -)] \xrightarrow{\partial_0} \underline{\mathbb{C}} \longrightarrow 0$$

we introduce the notion of an equivalence n-simplex and we prove that it forms an acyclic chain complex.

In Section 4, we extend the domain of the definition of the L^2-Euler characteristic and give a proof of our main theorem.

2. Preliminaries

2.1. **Notation.**

(1) Natural numbers mean nonnegative integers. So
$$\mathbb{N} = \{0, 1, 2, \dots\}.$$
(2) For a natural number n, let $[n] = \{0, 1, \dots, n\}$ equipped with usual ordering.
(3) Let X be a set. Then, $\mathbb{C}[X]$ denotes the free $\mathbb{C}$-vector space generated by X.
(4) Let X be a finite set. Then, we denote the number of elements of X by $\#X$.
(5) Let $\varphi : \mathcal{J} \to \mathcal{I}$ be a functor between small categories and let i be an object of $\mathcal{I}$. Then, the category *φ-over i* is denoted by $(\varphi \downarrow i)$ and the category *φ-under i* is denoted by $(i \downarrow \varphi)$.
(6) A *discrete category* X is a category that consists of only objects and identity morphisms. In particular, if a discrete category has exactly one object, it is called *one-point category*, denoted by $*$.
(7) Suppose $\mathcal{J}$ is a small category and $\mathscr{C}$ is a category. The *functor category* $\mathrm{Func}(\mathcal{J}, \mathscr{C})$ consists of functors from $\mathcal{J}$ to $\mathscr{C}$ as its objects and natural transformations between them as its morphisms. Sometimes we simply write it $\mathscr{C}^{\mathcal{J}}$.

2.2. **Basic definitions.** In this subsection, we recall basic definitions.

Definition 2.3. A small category $\mathcal{A}$ is *acyclic* if every endomorphism and every isomorphism is an identity morphism.

Remark 2.4. This is the same as a skeletal scwol [3].

Define an order on the set $\mathrm{Ob}(\mathcal{A})$ of objects of $\mathcal{A}$ by $x \le y$ if there exists a morphism $x \to y$. Then, $\mathrm{Ob}(\mathcal{A})$ is a poset.

Definition 2.5. Let $\mathcal{J}$ be a small category. The *nerve* $N_*(\mathcal{J})$ of $\mathcal{J}$ is the simplicial set whose set of n-simplices $N_n(\mathcal{J})$ is defined as follows:
$$N_n(\mathcal{J}) = \{(f_1, f_2, \dots, f_n) \mid t(f_i) = s(f_{i+1}) \text{ for all } 0 \le i < n\}$$
where $t(f)$ is the target of a morphism f and $s(f)$ is the source of a morphism f.

The *nondegenerate nerve* of $\mathcal{J}$, denoted by $\overline{N_*}(\mathcal{J})$, is the $\mathbb{N}$-graded subset of $N_*(\mathcal{J})$ defined by the following:
$$\overline{N_n}(\mathcal{J}) = \{(f_1, f_2, \dots, f_n) \in N_n(\mathcal{J}) \mid \text{none of } f_i \text{ is the identity morphism}\}$$
where $\overline{N_0}(\mathcal{J})$ is defined by $\overline{N_0}(\mathcal{J}) = N_0(\mathcal{J})$. We remark that when $\mathcal{J}$ is acyclic, $\overline{N_*}(\mathcal{J})$ is a Δ-set, that is, a simplicial set without degeneracy operators, a semisimplicial set in today's terminology.

For any objects x and y of $\mathcal{J}$, define
$$\overline{N_n}(\mathcal{J})^x_y = \left\{ (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} x_n) \in \overline{N_n}(\mathcal{J}) \;\middle|\; x_0 = x, x_n = y \right\}$$

and

$$\overline{N_n}(\mathcal{J})_y = \left\{ (x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} x_n) \in \overline{N_n}(\mathcal{J}) \;\middle|\; x_n = y \right\}.$$

Definition 2.6. [4, 6] Let $\mathcal{J}$ be a small category. Then, *barycentric subdivision* $\mathrm{Sd}(\mathcal{J})$ of $\mathcal{J}$ is the small category whose objects are the nondegenerate chains of morphisms in $\mathcal{J}$ and the set of morphisms between $\mathbf{f}$ and $\mathbf{g}$ is the quotient set of order-preserving maps $\varphi : [n] \to [m]$ satisfying $\mathbf{g} \circ \varphi = \mathbf{f}$ under the relation defined below where n and m are the length of $\mathbf{f}$ and $\mathbf{g}$, respectively. Here, $\mathbf{f}$ and $\mathbf{g}$ are regarded as functors from posets $[n]$ and $[m]$ to $\mathcal{J}$, respectively. So the condition $\mathbf{g} \circ \varphi = \mathbf{f}$ implies the commutativity of the diagram

$$\begin{array}{ccc} & \mathcal{J} & \\ {}^{\mathbf{f}}\nearrow & & \nwarrow^{\mathbf{g}} \\ [n] & \xrightarrow{\varphi} & [m] \end{array}$$

in the category of small categories.

The equivalence relation is generated by the following relation: Given order-preserving maps $\varphi, \psi : [n] \to [m]$ satisfying $\mathbf{g} \circ \varphi = \mathbf{f}, \mathbf{g} \circ \psi = \mathbf{f}$, respectively, define $\varphi \sim \psi$ if for any $0 \leq i \leq n$, $\mathbf{g}(\min\{\varphi(i), \psi(i)\} \to \max\{\varphi(i), \psi(i)\})$ is an identity morphism. Here,

$$\min\{\varphi(i), \psi(i)\} \to \max\{\varphi(i), \psi(i)\}$$

is a morphism in $[m]$. The composition in $\mathrm{Sd}(\mathcal{J})$ is defined by the composition of order-preserving maps.

This is equivalent to the definition of [6], as the author of that paper told me. The functor Sd preserves homotopy type (see [6, Thm. 32]) and opposite, that is, for any small category $\mathcal{J}$, $B\mathcal{J}$ is homotopy equivalent to $B\mathrm{Sd}(\mathcal{J})$ and $\mathrm{Sd}(\mathcal{J})$ is isomorphic to $\mathrm{Sd}(\mathcal{J}^{\mathrm{op}})$. Here, we briefly describe an isomorphic functor between $\mathrm{Sd}(\mathcal{J})$ and $\mathrm{Sd}(\mathcal{J}^{\mathrm{op}})$. Define a functor $F_{\mathcal{J}} : \mathrm{Sd}(\mathcal{J}) \to \mathrm{Sd}(\mathcal{J}^{\mathrm{op}})$ by the following. Define $F_{\mathcal{J}}(\mathbf{f}) = \mathbf{f}^{\mathrm{op}}$ for any object $\mathbf{f}$ of $\mathrm{Sd}(\mathcal{J})$. For any $[\varphi] : \mathbf{f} \to \mathbf{g}$, define $F_{\mathcal{J}}([\varphi]) : \mathbf{f}^{\mathrm{op}} \to \mathbf{g}^{\mathrm{op}}$ by $F_{\mathcal{J}}([\varphi])(i) = m - \varphi(n-i)$ for any i of $[n]$ where n and m are the length of $\mathbf{f}$ and $\mathbf{g}$, respectively. It is easy to show that this functor is well-defined and $F_{\mathcal{J}} \circ F_{\mathcal{J}^{\mathrm{op}}} = \mathrm{id}$ and $F_{\mathcal{J}^{\mathrm{op}}} \circ F_{\mathcal{J}} = \mathrm{id}$.

On the other hands, $\mathrm{Sd}(\mathcal{J})$ is not isomorphic to $\mathrm{Sd}(\mathcal{J})^{\mathrm{op}}$. The category $\mathcal{J} = x \longrightarrow y$ is an example. This subdivision and the subdivision for simplicial sets form the following commutative diagram

$$\begin{array}{ccc} \textbf{Small categories} & \xrightarrow{N_*} & \textbf{Simplicial sets} \\ \downarrow{\scriptstyle \mathrm{Sd}} & & \downarrow{\scriptstyle \mathrm{Sd}} \\ \textbf{Small categories} & \xleftarrow[C]{} & \textbf{Simplicial sets} \end{array}$$

where C is the categorization functor ([6, Rem. 13]).

Remark 2.7. We summarize important properties we will often use. For proofs see [13].

(1) For a small category $\mathcal{J}$, $\mathrm{Sd}(\mathcal{J})$ is an acyclic category (see [13, Prop. 3.4]).
(2) For a morphism $[\varphi] : \mathbf{f} \to \mathbf{g}$ in $\mathrm{Sd}(\mathcal{J})$ and for any representative ψ of $[\varphi]$, $\psi : [n] \to [m]$ is an order-preserving injection (see [13, Lemma 3.3]).

2.8. **Homological algebra of a functor category.** In this subsection, let us recall the definition and basic properties of the Kan extensions. See [7] and [12] for more details.

Suppose $\varphi : \mathcal{J} \to \mathcal{I}$ is a functor between small categories and $\mathscr{C}$ is a category. Then, φ induces a functor φ^* by precomposition

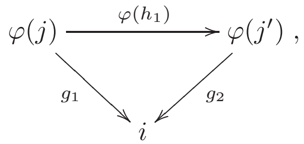
$$\mathrm{Func}(\mathcal{I},\mathscr{C}) \underset{\varphi^\dagger}{\overset{\varphi^\ddagger}{\leftleftarrows}} \!\!\xrightarrow{\ \varphi^*\ }\! \mathrm{Func}(\mathcal{J},\mathscr{C}).$$

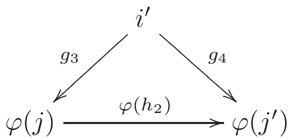
If $\mathscr{C}$ is closed under all small limits and colimits, φ^* has a left and a right adjoint $\varphi^\dagger$ and $\varphi^\ddagger$, respectively. These functors can be described as follows. For any $\beta : \mathcal{J} \to \mathscr{C}$,

$$\varphi^\dagger(\beta) : \mathcal{I} \to \mathscr{C},\ \varphi^\dagger(\beta)(i) = \operatorname*{colim}_{(\varphi\downarrow i)} \beta \circ P_i$$

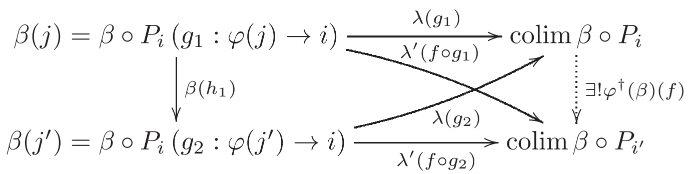
$$\varphi^\ddagger(\beta) : \mathcal{I} \to \mathscr{C},\ \varphi^\ddagger(\beta)(i) = \lim_{(i\downarrow\varphi)} \beta \circ Q^i$$

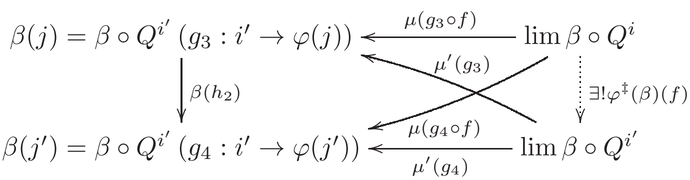
where $P_i : (\varphi \downarrow i) \to \mathcal{J}$ and $Q^i : (i \downarrow \varphi) \to \mathcal{J}$ are the projections. For a morphism $f : i \to i'$ in $\mathcal{I}$, $\varphi^\dagger(\beta)(f)$ and $\varphi^\ddagger(\beta)(f)$ are determined by the universal properties. For morphisms





in $(\varphi \downarrow i)$ and $(i' \downarrow \varphi)$, respectively, we obtain the following diagrams





where λ, λ' are the colimiting cones of $\operatorname{colim} \beta \circ P_i$ and $\operatorname{colim} \beta \circ P_{i'}$ respectively and μ, μ' are the limiting cones of $\lim \beta \circ Q^i$ and $\lim \beta \circ Q^{i'}$ respectively.

Since $\varphi^\dagger$ and $\varphi^\ddagger$ are a left and a right adjoint of φ^*, respectively, we have the following bijections

$$\operatorname{Hom}_{\operatorname{Func}(\mathcal{I},\mathscr{C})}(\varphi^\dagger(\beta), \alpha) \cong \operatorname{Hom}_{\operatorname{Func}(\mathcal{J},\mathscr{C})}(\beta, \varphi^*(\alpha))$$
$$\operatorname{Hom}_{\operatorname{Func}(\mathcal{J},\mathscr{C})}(\varphi^*(\alpha), \beta) \cong \operatorname{Hom}_{\operatorname{Func}(\mathcal{I},\mathscr{C})}(\alpha, \varphi^\ddagger(\beta)).$$

Recall that for an abelian category $\mathscr{A}$, the functor category $\operatorname{Func}(\mathcal{I}, \mathscr{A})$ is an abelian category.

Lemma 2.9. *Suppose $\varphi : \mathcal{J} \to \mathcal{I}$ is a functor between small categories and $\mathscr{A}$ is an abelian category closed under all small colimits. If P is a projective object in* $\operatorname{Func}(\mathcal{J}, \mathscr{A})$*, then $\varphi^\dagger(P)$ is projective in* $\operatorname{Func}(\mathcal{I}, \mathscr{A})$.

Proof. It is equivalent that $\varphi^\dagger(P)$ is projective and $\operatorname{Hom}_{\mathcal{A}^{\mathcal{I}}}(\varphi^\dagger(P), -)$ is an exact functor. Given a short exact sequence

$$0 \longrightarrow F_1 \xrightarrow{\alpha} F_2 \xrightarrow{\beta} F_3 \longrightarrow 0$$

in $\operatorname{Func}(\mathcal{I}, \mathcal{A})$, we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Hom}_{\mathcal{A}^{\mathcal{I}}}(\varphi^\dagger(P), F_1) & \xrightarrow{\alpha_*} & \operatorname{Hom}_{\mathcal{A}^{\mathcal{I}}}(\varphi^\dagger(P), F_2) & \xrightarrow{\beta_*} & \operatorname{Hom}_{\mathcal{A}^{\mathcal{I}}}(\varphi^\dagger(P), F_3) & \longrightarrow & 0 \\ & & \downarrow\cong & & \downarrow\cong & & \downarrow\cong & & \\ 0 & \longrightarrow & \operatorname{Hom}_{\mathcal{A}^{\mathcal{J}}}(P, \varphi^*(F_1)) & \xrightarrow{\varphi^*(\alpha)_*} & \operatorname{Hom}_{\mathcal{A}^{\mathcal{J}}}(P, \varphi^*(F_2)) & \xrightarrow{\varphi^*(\beta)_*} & \operatorname{Hom}_{\mathcal{A}^{\mathcal{J}}}(P, \varphi^*(F_3)) & \longrightarrow & 0. \end{array}$$

Note that these vertical maps are just bijections, not isomorphisms and this diagram is commutative since the definition of an adjoint functor is required to be natural. Since

$$0 \longrightarrow F_1 \xrightarrow{\alpha} F_2 \xrightarrow{\beta} F_3 \longrightarrow 0$$

is exact in $\operatorname{Func}(\mathcal{I}, \mathcal{A})$,

$$0 \longrightarrow F_1(i) \xrightarrow{\alpha(i)} F_2(i) \xrightarrow{\beta(i)} F_3(i) \longrightarrow 0$$

is exact in $\mathcal{A}$ for any object i of $\mathcal{I}$. So for any object j of $\mathcal{J}$ and $\varphi(j)$, we have

$$0 \longrightarrow F_1(\varphi(j)) \xrightarrow{\alpha(\varphi(j))} F_2(\varphi(j)) \xrightarrow{\beta(\varphi(j))} F_3(\varphi(j)) \longrightarrow 0$$

is exact. Hence,

$$0 \longrightarrow \varphi^*(F_1) \xrightarrow{\varphi^*(\alpha)} \varphi^*(F_2) \xrightarrow{\varphi^*(\beta)} \varphi^*(F_3) \longrightarrow 0$$

is also exact. Since $\operatorname{Hom}_{\mathcal{A}^{\mathcal{J}}}(P, -)$ is an exact functor,

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{A}^{\mathcal{J}}}(P, \varphi^*(F_1)) \xrightarrow{\varphi^*(\alpha)_*} \operatorname{Hom}_{\mathcal{A}^{\mathcal{J}}}(P, \varphi^*(F_2)) \xrightarrow{\varphi^*(\beta)_*} \operatorname{Hom}_{\mathcal{A}^{\mathcal{J}}}(P, \varphi^*(F_3)) \longrightarrow 0$$

is exact. This follows that $\varphi^*(\beta)_*$ is a surjection and $\varphi^*(\alpha)_*$ is an injection. Since the ladder diagram is commutative and the vertical maps in the diagram are bijections, we obtain α_* is an injection and β_* is an surjection. Hence, $\mathrm{Hom}_{\mathcal{A}^{\mathcal{I}}}(\varphi^\dagger(P), -)$ is an exact functor. $\square$

Let $\mathcal{I}$ be a small category and let i be an object of $\mathcal{I}$. Define $I_i : * \to \mathcal{I}$ to be the inclusion functor into i. Then we have

$$\mathrm{Func}(\mathcal{I}, \mathbb{C}\text{-vect}) \underset{I_i^\dagger}{\overset{I_i^\ddagger}{\leftleftarrows}} \xrightarrow{I_i^*} \mathbb{C}\text{-vect}$$

where $\mathbb{C}$-vect is the category of $\mathbb{C}$-vector spaces. The comma category $(I_i \downarrow j)$ can be determined easily for any object j of $\mathcal{I}$.

Lemma 2.10. *Suppose $\mathcal{I}$ is a small category and i is an object of $\mathcal{I}$. For the inclusion functor $I_i : * \to \mathcal{I}$ into i, $(I_i \downarrow j)$ is the discrete category* $\mathrm{Hom}_{\mathcal{I}}(i, j)$ *for any object j of $\mathcal{I}$.*

Proposition 2.11. *Let $\mathcal{I}$ be a small category and i be an object of $\mathcal{I}$. Then, for the functor*

$$I_i^\dagger(\mathbb{C}) : \mathcal{I} \longrightarrow \mathbb{C}\text{-vect},$$

we have

$$I_i^\dagger(\mathbb{C})(j) = \mathbb{C}[\mathrm{Hom}_{\mathcal{I}}(i, j)]$$

and

$$I_i^\dagger(\mathbb{C})(f) = f_* : \mathbb{C}[\mathrm{Hom}_{\mathcal{I}}(i, j)] \longrightarrow \mathbb{C}[\mathrm{Hom}_{\mathcal{I}}(i, j')]$$

for any object j of $\mathcal{I}$ and for any morphism $f : j \to j'$ of $\mathcal{I}$.

Proof. By Lemma 2.10, $(I_i \downarrow j)$ is the discrete category $\mathrm{Hom}_{\mathcal{I}}(i, j)$. Hence, we obtain

$$\begin{aligned} I_i^\dagger(\mathbb{C})(j) &= \operatorname*{colim}_{(I_i \downarrow j)} \mathbb{C} \\ &= \mathbb{C}[\mathrm{Hom}_{\mathcal{I}}(i, j)]. \end{aligned}$$

The universal property of the colimit implies $I_i^\dagger(\mathbb{C})(f) = f_*$. $\square$

Corollary 2.12. *Let $\mathcal{I}$ be a small category and i be an object of $\mathcal{I}$. Then, $I_i^\dagger(\mathbb{C})$ is projective in* $\mathrm{Func}(\mathcal{I}, \mathbb{C}\text{-vect})$.

Proof. This is a special case of Lemma 2.9. $\square$

3. Invariance of the Euler characteristics under barycentric subdivision for finite acyclic categories

In this section, we prove that four Euler characteristics of categories (Leinster's Euler characteristic [8], the series Euler characteristic [2], the L^2-Euler characteristic [5] and the Euler characteristic of $\mathbb{N}$-filtered acyclic categories [13]) are invariant under barycentric subdivision for finite acyclic categories.

3.1. **Leinster's Euler characteristics of categories.** Let us recall the definition of Leinster's Euler characteristic [8]. Suppose $\mathcal{I}$ is a finite category and the set of objects $\mathrm{Ob}(\mathcal{I})$ is labeled by natural numbers as follows:

$$\mathrm{Ob}(\mathcal{I}) = \{x_1, x_2, \dots, x_n\}$$

Let $Z_{\mathcal{I}}$ be the $n \times n$-matrix whose (i, j)-entry is the number of morphisms of $\mathcal{I}$ from x_i to x_j.

Definition 3.2 (Leinster [8])**.** Let $\mathbf{w}, \mathbf{c}$ be row vectors in $\mathbb{Q}^n$. Then, we say $\mathbf{w}$ is a *weighting* on $\mathcal{I}$ if

$$Z_{\mathcal{I}}{}^{\mathrm{t}}\mathbf{w} = Z_{\mathcal{I}} \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}.$$

We say $\mathbf{c}$ is a *coweighting* on $\mathcal{I}$ if

$$\mathbf{c}Z_{\mathcal{I}} = (c_1, c_2, \dots, c_n)Z_{\mathcal{I}} = (1, \dots, 1).$$

Definition 3.3 (Leinster [8])**.** A finite category $\mathcal{I}$ has *Euler characteristic* if it admits both a weighting $\mathbf{w}$ and a coweighting $\mathbf{c}$. Its *Euler characteristic* is then

$$\chi_L(\mathcal{I}) = \sum_i w_i = \sum_i c_i \in \mathbb{Q}$$

for any weighting $\mathbf{w}$ and coweighting $\mathbf{c}$.

Definition 3.4 (Leinster [8])**.** We say $\mathcal{I}$ has a *Möbius inversion* if $Z_{\mathcal{I}}$ has an inverse matrix. Then, the Möbius inversion μ is the map

$$\mu : \mathrm{Ob}(\mathcal{I}) \times \mathrm{Ob}(\mathcal{I}) \longrightarrow \mathbb{Q}$$

defined by $\mu(x_i, x_j) = (i, j)$-entry of $Z_{\mathcal{I}}^{-1}$.

A finite category $\mathcal{I}$ has a Möbius inversion if and only if there uniquely exist a weighting and a coweighting on $\mathcal{I}$. Then, we have

$$\sum_{i,j} \mu(x_i, x_j) = \sum_i w_i = \sum_i c_i$$

and $\chi_L(\mathcal{I}) = \sum_{i,j} \mu(x_i, x_j)$.

Example 3.5. Let $\mathcal{A}$ be a finite acyclic category. Then, [8, Cor. 1.5] implies that $\mathcal{A}$ has a Möbius inversion given by

$$\mu(x, y) = \sum_{n \geq 0} (-1)^n \# \overline{N_n}(\mathcal{A})_y^x$$

(see Definition 2.5). Note that this sum is a finite sum. Hence, Leinster's Euler characteristic of $\mathcal{A}$ is

$$\chi_L(\mathcal{A}) = \sum_{n \geq 0} (-1)^n \# \overline{N_n}(\mathcal{A}).$$

We will use this fact many times later.

Lemma 3.6. *Let $\mathcal{J}$ be a small category. Then, the following are equivalent*

(1) $\mathcal{J}$ is finite acyclic.
(2) $\mathrm{Sd}(\mathcal{J})$ is a finite category.
(3) $\overline{N_k}(\mathcal{J})$ is finite for any k and there exists a sufficiently large integer M such that $\overline{N_n}(\mathcal{J}) = \varnothing$ for $n > M$.

Proof. It is clear that (1) and (3) are equivalent. We only give a proof which shows (2) and (3) are equivalent.

Suppose $\mathrm{Sd}(\mathcal{J})$ is a finite category. Then, the set of objects of $\mathrm{Sd}(\mathcal{J})$ is a finite set. Here, since the set of objects of $\mathrm{Sd}(\mathcal{J})$ is $\coprod_{n\geq 0} \overline{N_n}(\mathcal{J})$, so each $\overline{N_k}(\mathcal{J})$ is finite and there exists such an M.

Conversely, if we suppose (3), then the set of objects of $\mathrm{Sd}(\mathcal{J})$ is a finite set since it is equal to $\coprod_{n\geq 0} \overline{N_n}(\mathcal{J})$. The category $\mathrm{Sd}(\mathcal{J})$ is locally finite, that is, each Hom-set is a finite set. So $\mathrm{Sd}(\mathcal{J})$ is a finite category. □

Proposition 3.7. *Let $\mathcal{I}$ be a finite category. Then $\mathrm{Sd}(\mathcal{I})$ has Euler characteristic in the sense of Leinster if and only if $\mathcal{I}$ is acyclic. In this case, we have*

$$\chi_L(\mathcal{I}) = \chi_L(\mathrm{Sd}(\mathcal{I})).$$

Proof. Suppose $\mathrm{Sd}(\mathcal{I})$ has Euler characteristic. Then, $\mathrm{Sd}(\mathcal{I})$ must be a finite category since χ_L is defined for only finite categories. Lemma 3.6 implies $\mathcal{I}$ is finite acyclic.

Conversely, if $\mathcal{I}$ is acyclic, then $\mathrm{Sd}(\mathcal{I})$ is finite by Lemma 3.6. Example 3.5 implies $\mathrm{Sd}(\mathcal{I})$ has Euler characteristic. So the first claim is proven.

Suppose $\mathcal{I}$ is finite acyclic. Then, since $\mathrm{Sd}(\mathcal{I})$ is finite acyclic, we can apply [8, Cor. 1.5] and we obtain a Möbius inversion μ which is given by

$$\mu(\mathbf{f}, \mathbf{g}) = \sum_{n\geq 0} (-1)^n \# \overline{N_n}(\mathrm{Sd}(\mathcal{I}))^{\mathbf{f}}_{\mathbf{g}}.$$

Here, we note Remark 2.7 part 2. For an element

$$\left(\mathbf{f} \xrightarrow{\varphi_1} \mathbf{f_1} \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_n} \mathbf{g} \right)$$

of $\overline{N_n}(\mathrm{Sd}(\mathcal{I}))^{\mathbf{f}}_{\mathbf{g}}$, we have

$$L(\mathbf{f}) < L(\mathbf{f_1}) < \cdots < L(\mathbf{f_{n-1}}) < L(\mathbf{g})$$

where L is the length function since each φ_k is an order-preserving injection and φ_k is not the identity morphism. Hence, the alternating sum is

$$\mu(\mathbf{f}, \mathbf{g}) = \sum_{n\geq 0}^{L(\mathbf{g})} (-1)^n \# \overline{N_n}(\mathrm{Sd}(\mathcal{I}))^{\mathbf{f}}_{\mathbf{g}}.$$

So we have

$$\begin{aligned}
\chi_L(\mathrm{Sd}(\mathcal{I})) &= \sum_{\mathbf{f},\mathbf{g}\in \mathrm{Ob}(\mathrm{Sd}(\mathcal{I}))} \mu(\mathbf{f},\mathbf{g}) \\
&= \sum_{\mathbf{g}\in \mathrm{Ob}(\mathrm{Sd}(\mathcal{I}))} \left(\sum_{\mathbf{f}\in \mathrm{Ob}(\mathrm{Sd}(\mathcal{I}))} \mu(\mathbf{f},\mathbf{g}) \right) \\
&= \sum_{\mathbf{g}\in \coprod_{n=0}^{M} \overline{N_n}(\mathcal{I})} \left(\sum_{\mathbf{f}\in \coprod_{n=0}^{L(\mathbf{g})} \overline{N_n}(\mathcal{I})} \mu(\mathbf{f},\mathbf{g}) \right) \\
&= \sum_{\mathbf{g}\in \coprod_{n=0}^{M} \overline{N_n}(\mathcal{I})} \left(\sum_{n=0}^{L(\mathbf{g})} (-1)^n \# \overline{N_n}(\mathrm{Sd}(\mathcal{I}))_{\mathbf{g}} \right). \qquad (1)
\end{aligned}$$

By [13, Thm. 4.7] and substituting -1 for s, we obtain

$$\sum_{n=0}^{L(\mathbf{g})} (-1)^n \# \overline{N_n}(\mathrm{Sd}(\mathcal{I}))_{\mathbf{g}} = (-1)^{L(\mathbf{g})}.$$

Thus, equation (1) is

$$\begin{aligned}
\chi_L(\mathrm{Sd}(\mathcal{I})) &= \sum_{\mathbf{g}\in \coprod_{n=0}^{M} \overline{N_n}(\mathcal{I})} (-1)^{L(\mathbf{g})} \\
&= \sum_{n=0}^{M} (-1)^n \# \overline{N_n}(\mathcal{I}) \\
&= \chi_L(\mathcal{I}).
\end{aligned}$$

□

3.8. **The series Euler characteristic.** We recall the *series Euler characteristic* [2] and show that it is invariant under barycentric subdivision for finite acyclic categories.

We have the following commutative diagram of rings:

$$\begin{array}{ccc}
\mathbb{Z}[t] & \hookrightarrow & \mathbb{Z}[[t]] \\
\downarrow & & \downarrow \\
\mathbb{Q}(t) & \hookrightarrow & \mathbb{Q}((t)).
\end{array}$$

Here, $\mathbb{Z}[t]$ is the polynomial ring with the coefficients in $\mathbb{Z}$ and $\mathbb{Z}[[t]]$ is the ring of formal power series over $\mathbb{Z}$. The rings $\mathbb{Q}(t)$ and $\mathbb{Q}((t))$ are the respective quotient fields.

Let $\mathcal{I}$ be a finite category. Then, the formal power series

$$f_{\mathcal{I}}(t) := \sum_{n=0}^{\infty} \# \overline{N_n}(\mathcal{I}) t^n$$

is rational over $\mathbb{Q}$ ([2, Thm. 2.2]).

Definition 3.9. Let $f(t)$ be a formal power series over $\mathbb{Z}$. If there exists a rational function $g(t)/h(t)$ in $\mathbb{Q}(t)$ such that $f(t) = g(t)/h(t)$ in $\mathbb{Q}((t))$, then define

$$f|_{t=-1} = \frac{g(-1)}{h(-1)} \in \mathbb{Q}$$

if $h(-1) \neq 0$.

Definition 3.10. A finite category $\mathcal{I}$ has *series Euler characteristic* if $f_{\mathcal{I}}|_{t=-1}$ can be defined as in Definition 3.9. In this case, its *series Euler characteristic* is

$$\chi_{\sum}(\mathcal{I}) = f_{\mathcal{I}}|_{t=-1}.$$

When $\mathcal{A}$ is finite acyclic, the power series $f_{\mathcal{A}}(t)$ is a polynomial by Lemma 3.6. Hence, the series Euler characteristic $\chi_{\sum}(\mathcal{A})$ is

$$\chi_{\sum}(\mathcal{A}) = \sum_{n=0}^{M} (-1)^n \# \overline{N_n}(\mathcal{A})$$

where M is a sufficiently large integer.

Proposition 3.11. *Let $\mathcal{I}$ be a finite category. Then,* $\mathrm{Sd}(\mathcal{I})$ *has series Euler characteristic if and only if $\mathcal{I}$ is acyclic. In this case, we have*

$$\chi_{\sum}(\mathcal{I}) = \chi_{\sum}(\mathrm{Sd}(\mathcal{I})).$$

Proof. Suppose $\mathrm{Sd}(\mathcal{I})$ has series Euler characteristic. Then, $\mathrm{Sd}(\mathcal{I})$ must be a finite category since $\chi_{\sum}$ is defined for only finite categories. Lemma 3.6 implies $\mathcal{I}$ is finite acyclic.

Conversely, if $\mathcal{I}$ is acyclic, then $\mathrm{Sd}(\mathcal{I})$ is finite by Lemma 3.6, so that $\mathrm{Sd}(\mathcal{I})$ has series Euler characteristic since $\mathrm{Sd}(\mathcal{I})$ is acyclic. [2, Thm. 3.2] and Proposition 3.7 complete this proof. □

3.12. **The Euler characteristic of $\mathbb{N}$-filtered acyclic categories.** We recall the *Euler characteristic of an $\mathbb{N}$-filtered acyclic category* [13] and show that it is invariant under barycentric subdivision for finite acyclic categories.

Definition 3.13. Let $\mathcal{A}$ be an acyclic category. A functor $\mu : \mathcal{A} \to \mathbb{N}$ satisfying $\mu(x) < \mu(y)$ for $x < y$ in $\mathrm{Ob}(\mathcal{A})$ is called *an $\mathbb{N}$-filtration of $\mathcal{A}$*. A pair $(\mathcal{A}, \mu)$ is called *an $\mathbb{N}$-filtered acyclic category.*

Example 3.14. Let $\mathcal{J}$ be a small category. Then, $\mathrm{Sd}(\mathcal{J})$ is an acyclic category. The length functor L gives a natural $\mathbb{N}$-filtration on $\mathrm{Sd}(\mathcal{J})$ where the functor L is defined by $L(\mathbf{f}) = n$ for $\mathbf{f}$ of $\overline{N_n}(\mathcal{J})$. Thus, we obtain an $\mathbb{N}$-filtered acyclic category $(\mathrm{Sd}(\mathcal{J}), L)$.

Definition 3.15. Let $(\mathcal{A}, \mu)$ be an $\mathbb{N}$-filtered acyclic category. Then, define $\chi_{\mathrm{fil}}(\mathcal{A}, \mu)$ as follows.

For natural numbers i and n, let

$$\overline{N_n}(\mathcal{A})_i = \{\mathbf{f} \in \overline{N_n}(\mathcal{A}) \mid \mu(t(\mathbf{f})) = i\}$$

where $t(\mathbf{f}) = x_n$ if

$$\mathbf{f} = \left(x_0 \xrightarrow{f_1} x_1 \xrightarrow{f_2} \dots \xrightarrow{f_n} x_n \right).$$

Suppose each $\overline{N_n}(\mathcal{A})_i$ is finite. Define the formal power series $f_\chi(\mathcal{A}, \mu)(t)$ over $\mathbb{Z}$ by

$$f_\chi(\mathcal{A}, \mu)(t) = \sum_{i=0}^{\infty} (-1)^i \left(\sum_{n=0}^{i} (-1)^n \# \overline{N_n}(\mathcal{A})_i \right) t^i.$$

Then, define

$$\chi_{\text{fil}}(\mathcal{A}, \mu) = f_\chi(\mathcal{A}, \mu)|_{t=-1}$$

if $f_\chi(\mathcal{A}, \mu)(t)$ is rational and has a nonvanishing denominator at $t = -1$.

Lemma 3.16. *Let $\mathcal{A}$ be a finite acyclic category. Then $\mathcal{A}$ has an $\mathbb{N}$-filtration.*

Proof. We can label the set of objects of $\mathcal{A}$ such that if $x_i < x_j$ in the ordering of Remark 2.4, then $i < j$ in the usual ordering on integers

$$\text{Ob}(\mathcal{A}) = \{x_1, \dots, x_n\}.$$

Indeed, take a maximal element x of $\text{Ob}(\mathcal{A})$ and label it as x_n. Inductively, we obtain such labeling. This labeling gives an $\mathbb{N}$-filtration to $\mathcal{A}$. □

Proposition 3.17. *Let $\mathcal{A}$ be a finite acyclic category. Then, we have*

$$\chi_{\text{fil}}(\mathcal{A}, \mu) = \chi_{\text{fil}}(\text{Sd}(\mathcal{A}), L)$$

where μ is any $\mathbb{N}$-filtration of $\mathcal{A}$ and L is the length $\mathbb{N}$-filtration (see Example 3.14).

Proof. We have

$$\begin{aligned} f_\chi(\mathcal{A}, \mu)(t) &= \sum_{i=0}^{\infty} (-1)^i \left(\sum_{n=0}^{i} (-1)^n \# \overline{N_n}(\mathcal{A})_i \right) t^i \\ &= \sum_{i=0}^{M} (-1)^i \left(\sum_{n=0}^{i} (-1)^n \# \overline{N_n}(\mathcal{A})_i \right) t^i \end{aligned}$$

for a sufficiently large integer M. Hence, $f_\chi(\mathcal{A},\mu)(t)$ is a polynomial. Thus, we obtain

$$\begin{aligned}\chi_{\mathrm{fil}}(\mathcal{A},\mu) &= f_\chi(\mathcal{A},\mu)|_{t=-1}\\ &= f_\chi(\mathcal{A},\mu)(-1)\\ &= \sum_{i=0}^{M}(-1)^i\left(\sum_{n=0}^{i}(-1)^n\#\overline{N_n}(\mathcal{A})_i\right)(-1)^i\\ &= \sum_{i=0}^{M}\left(\sum_{n=0}^{i}(-1)^n\#\overline{N_n}(\mathcal{A})_i\right)\\ &= \sum_{n=0}^{M}(-1)^n\#\overline{N_n}(\mathcal{A})\\ &= \chi_{\sum}(\mathcal{A}).\end{aligned}$$

Since $\mathcal{A}$ has series Euler characteristic, [13, Thm. 4.9] implies

$$\chi_{\sum}(\mathcal{A}) = \chi_{\mathrm{fil}}(\mathrm{Sd}(\mathcal{A}),L).$$

Hence, we obtain

$$\chi_{\mathrm{fil}}(\mathcal{A},\mu) = \chi_{\sum}(\mathcal{A}) = \chi_{\mathrm{fil}}(\mathrm{Sd}(\mathcal{A}),L).$$ □

3.18. **The $\mathbf{L^2}$-Euler characteristic.** In this subsection, we show the invariance of the L^2-Euler characteristic under barycentric subdivision for finite acyclic categories.

First, we recall the *L^2-Euler characteristic* of [5]. Let k be a commutative ring and let $\mathcal{J}$ be a small category. We denote the category of left k-modules by k-Mod.

Definition 3.19. If $M : \mathcal{J}^{\mathrm{op}} \to k\text{-Mod}$ and $N : \mathcal{J} \to k\text{-Mod}$ are functors, then the *tensor product* $M \otimes_{k\mathcal{J}} N$ is the quotient of the k-module

$$\bigoplus_{x\in\mathrm{Ob}(\mathcal{J})} M(x)\otimes_k N(x)$$

by the k-submodule generated by elements of the form

$$(M(f^{\mathrm{op}})m)\otimes n - m\otimes(N(f)n)$$

where $f : x \to y$ is a morphism in $\mathcal{J}$, m is an element of $M(y)$ and n is an element of $N(x)$.

For a discrete group G, we denote the *group von Neumann algebra* by $\mathcal{N}(G)$. It is a von Neumann algebra and when G is a finite group $\mathcal{N}(G)$ is just the group ring $\mathbb{C}[G]$. We briefly recall its dimension theory, see [5], [10] and [11] for more details. The *von Neumann dimension* $\dim_{\mathcal{N}(G)}$ is a map which assigns real numbers or $+\infty$ to right $\mathcal{N}(G)$-modules

$$\dim_{\mathcal{N}(G)} : \text{Mod-}\mathcal{N}(G) \longrightarrow [0,+\infty].$$

Here, we ignore the functional analytic aspects of $\mathcal{N}(G)$, so we regard it purely algebraically. An $\mathcal{N}(G)$-*chain complex* is a chain complex of $\mathcal{N}(G)$-modules and its homology is also the usual module homology. We often use the fact that when G is a finite group, $\dim_{\mathcal{N}(G)} = \frac{1}{\# G} \dim_{\mathbb{C}}$. For an object x of $\mathcal{J}$, the group von Neumann algebra $\mathcal{N}(\mathrm{Aut}(x))$ is simply denoted by $\mathcal{N}(x)$.

Definition 3.20. Let C_* be an $\mathcal{N}(G)$-chain complex. The *p-th L^2-Betti number of C_** is the von Neumann dimension of the $\mathcal{N}(G)$-module given by its p-th homology, namely

$$b_p^{(2)}(C_*) = \dim_{\mathcal{N}(G)}(H_p(C_*)) \in [0, \infty].$$

Definition 3.21. Let C_* be an $\mathcal{N}(G)$-chain complex. Define

$$h^{(2)}(C_*) = \sum_{p \geq 0} b_p^{(2)}(C_*) \in [0, \infty].$$

If $h^{(2)}(C_*) < \infty$, the *L^2-Euler characteristic of C_** is defined by

$$\chi^{(2)}(C_*) = \sum_{p \geq 0} (-1)^p b_p^{(2)}(C_*) \in \mathbb{R}.$$

The following definition actually comes from [9].

Definition 3.22. Let $\mathcal{J}$ be a small category and let x be an object of $\mathcal{J}$. Define *the splitting functor* at x

$$S_x : \mathrm{Func}(\mathcal{J}^{\mathrm{op}}, \mathbb{C}\text{-vect}) \longrightarrow \mathrm{Func}(\mathrm{Aut}(x)^{\mathrm{op}}, \mathbb{C}\text{-vect})$$

as follows.

For a functor $F : \mathcal{J}^{\mathrm{op}} \to \mathbb{C}$-vect,

$$S_x F : \mathrm{Aut}(x)^{\mathrm{op}} \longrightarrow \mathbb{C}\text{-vect}$$

is defined by

$$S_x F(*) = \mathrm{Coker}\left(\bigoplus_{u : x \to y \text{ in } \mathcal{J}, \nexists u^{-1}} F(u^{\mathrm{op}}) : \bigoplus_{u : x \to y \text{ in } \mathcal{J}, \nexists u^{-1}} F(y) \longrightarrow F(x) \right)$$

where this direct sum runs over all the morphisms $u : x \to y$ in $\mathcal{J}$ which are not invertible. For g^{op} of $\mathrm{Aut}(x)^{\mathrm{op}}$,

$$S_x(g^{\mathrm{op}}) : S_x F(*) \longrightarrow S_x F(*)$$

is defined by $S_x(g^{\mathrm{op}})[m] = [F(g^{\mathrm{op}})(m)]$ for any $[m]$ of $S_x F(*)$.

For a natural transformation $\alpha : F \Rightarrow G$, $S_x \alpha$ is defined by the universal property of the cokernels.

$$\begin{array}{ccccc} \bigoplus F(y) & \xrightarrow{\bigoplus F(u^{\mathrm{op}})} & F(x) & \longrightarrow & \mathrm{Coker} = S_x F \\ {\scriptstyle \bigoplus \alpha(y)} \downarrow & & \downarrow {\scriptstyle \alpha(x)} & & \vdots {\scriptstyle \exists ! S_x \alpha} \\ \bigoplus G(y) & \xrightarrow{\bigoplus G(u^{\mathrm{op}})} & G(x) & \longrightarrow & \mathrm{Coker} = S_x G \end{array}$$

Definition 3.23. A small category $\mathcal{J}$ *of type* (L^2) if for some projective resolution P_* in $\mathrm{Func}(\mathcal{J}^{\mathrm{op}}, \mathbb{C}\text{-vect})$ of the constant functor $\underline{\mathbb{C}}$ we have

$$h^{(2)}(\mathcal{J}) := \sum_{[x]\in \mathrm{iso}(\mathcal{J})} h^{(2)}(S_x P_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)) < \infty.$$

It is equivalent to require this for some (single) projective resolution or for any resolution.

Definition 3.24. Suppose that $\mathcal{J}$ is of type (L^2). The L^2*-Euler characteristic* of $\mathcal{J}$ is the real number

$$\chi^{(2)}(\mathcal{J}) := \sum_{[x]\in \mathrm{iso}(\mathcal{J})} \chi^{(2)}(S_x P_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)) \in \mathbb{R},$$

where P_* is a projective resolution of the constant functor $\underline{\mathbb{C}}$ in $\mathrm{Func}(\mathcal{J}^{\mathrm{op}}, \mathbb{C}\text{-vect})$.

As remarked in [5], this definition makes sense since the condition (L^2) ensures that the sum $\sum_{[x]\in \mathrm{iso}(\mathcal{J})} \chi^{(2)}(S_x P_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x))$ is absolutely convergent.

The following is our main theorem of this section and the proof is given later.

Theorem 3.25. *For any small category $\mathcal{I}$, the opposite of the subdivision* $\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}$ *is of type (L^2) if and only if $\mathcal{I}$ is finite acyclic. In this case,*

$$\chi^{(2)}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}) = \chi^{(2)}(\mathcal{I}) = \chi^{(2)}(\mathrm{Sd}(\mathcal{I})).$$

To prove this theorem we need Lemma 3.26 and Proposition 3.35. In Lemma 3.26, we characterize the splitting functor for an acyclic category. In Proposition 3.35, we construct a projective resolution of $\underline{\mathbb{C}}$ in $\mathrm{Func}(\mathrm{Sd}(\mathcal{J}), \mathbb{C}\text{-vect})$.

Lemma 3.26. *Let $\mathcal{A}$ be an acyclic category and x and y be objects of $\mathcal{A}$. For the functor*

$$S_x : \mathrm{Func}(\mathcal{A}^{\mathrm{op}}, \mathbb{C}\text{-vect}) \longrightarrow \mathbb{C}\text{-vect},$$

we have

$$S_x \mathbb{C}[\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(y, -)] = \begin{cases} \mathbb{C}, & \text{if } x = y, \\ 0, & \text{if } x \neq y. \end{cases}$$

Proof. For the functor

$$\mathbb{C}[\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(y, -)] : \mathcal{A}^{\mathrm{op}} \to \mathbb{C}\text{-vect},$$

we have

$$\begin{aligned} \mathbb{C}[\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(y, -)](z) &= \mathbb{C}[\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(y, z)] \\ &= \mathbb{C}[\mathrm{Hom}_{\mathcal{A}}(z, y)] \end{aligned}$$

for an object z of $\mathcal{A}^{\mathrm{op}}$. We have

$$S_x\mathbb{C}[\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(y,-)] = \mathrm{Coker}\left(\bigoplus_{\substack{u:x\to z\\ u\neq 1}} u^* : \bigoplus_{\substack{u:x\to z\\ u\neq 1}} \mathbb{C}[\mathrm{Hom}_{\mathcal{A}}(z,y)] \to \mathbb{C}[\mathrm{Hom}_{\mathcal{A}}(x,y)] \right).$$

If $x = y$, then

$$S_x\mathbb{C}[\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(x,-)] = \mathrm{Coker}\left(\bigoplus_{\substack{u:x\to z\\ u\neq 1}} u^* : \bigoplus_{\substack{u:x\to z\\ u\neq 1}} \mathbb{C}[\mathrm{Hom}_{\mathcal{A}}(z,x)] \to \mathbb{C} \right).$$

Here, all of the running $u : x \to z$ are not 1_x, so $x \neq z$. Since $\mathcal{A}$ is acyclic, $\mathrm{Hom}_{\mathcal{A}}(z,x)$ are empty sets if there exists a morphism $u : x \to z$. Hence,

$$\begin{aligned} S_x\mathbb{C}[\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(x,-)] &= \mathrm{Coker}\,(0 : 0 \to \mathbb{C}) \\ &= \mathbb{C}. \end{aligned}$$

Suppose $x \neq y$. If $\mathrm{Hom}_{\mathcal{A}}(x,y) = \varnothing$, then we obtain

$$S_x\mathbb{C}[\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(y,-)] = \mathrm{Coker}\left(\bigoplus_{\substack{u:x\to z\\ u\neq 1}} u^* : \bigoplus_{\substack{u:x\to z\\ u\neq 1}} \mathbb{C}[\mathrm{Hom}_{\mathcal{A}}(z,x)] \to 0 \right) = 0.$$

If $\mathrm{Hom}_{\mathcal{A}}(x,y) \neq \varnothing$, then such $u : x \to z$ runs over $\mathrm{Hom}_{\mathcal{A}}(x,y)$ and

$$\coprod_{\substack{u:x\to y\\ u\neq 1}} \mathrm{Hom}_{\mathcal{A}}(y,y) \subseteq \coprod_{\substack{u:x\to z\\ u\neq 1}} \mathrm{Hom}_{\mathcal{A}}(z,y).$$

This maps onto $\mathrm{Hom}_{\mathcal{A}}(x,y)$ since 1_y is in each copy of $\mathrm{Hom}_{\mathcal{A}}(y,y)$. Hence we obtain $S_x\mathbb{C}[\mathrm{Hom}_{\mathcal{A}^{\mathrm{op}}}(y,-)] = 0$. □

For a small category $\mathcal{J}$ we construct a projective resolution of the constant functor $\underline{\mathbb{C}}$ in $\mathrm{Func}(\mathrm{Sd}(\mathcal{J}), \mathbb{C}\text{-vect})$. Let $P(\mathrm{Sd}(\mathcal{J}))_*$ be the sequence

$$\cdots \xrightarrow{\partial_2} \bigoplus_{\mathbf{f_1}\in\overline{N_1}(\mathcal{J})} \mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_1},-)] \xrightarrow{\partial_1}$$

$$\bigoplus_{\mathbf{f_0}\in\overline{N_0}(\mathcal{J})} \mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_0},-)] \xrightarrow{\partial_0} \underline{\mathbb{C}} \longrightarrow 0$$

where each ∂_k is defined as follows. The map

$$\partial_k(\mathbf{g}) : \bigoplus_{\mathbf{f_k}\in\overline{N_k}(\mathcal{J})} \mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_k},\mathbf{g})] \longrightarrow \bigoplus_{\mathbf{f_{k-1}}\in\overline{N_{k-1}}(\mathcal{J})} \mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_{k-1}},\mathbf{g})]$$

is defined by

$$\partial_k(\mathbf{g})([\varphi]) = \sum_{\ell\in F(\mathbf{f_k})} (-1)^\ell [\varphi \circ d^\ell]$$

for any $[\varphi]$ of $\mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_k},\mathbf{g})$ where

$$F(\mathbf{f_k}) = \{\ell \in [k] \mid d_\ell(\mathbf{f_k}) \in \overline{N_{k-1}}(\mathcal{J})\}.$$

Note that if $\mathcal{J}$ is acyclic, then $F(\mathbf{f_k}) = [k]$. For a morphism $[\psi] : \mathbf{g} \to \mathbf{g'}$ in $\mathrm{Sd}(\mathcal{J})$, the following diagrams are commutative

$$\begin{array}{ccc}
\bigoplus_{\mathbf{f_k}\in\overline{N_k}(\mathcal{J})} \mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_k},\mathbf{g})] & \xrightarrow{\partial_k(\mathbf{g})} & \bigoplus_{\mathbf{f_{k-1}}\in\overline{N_{k-1}}(\mathcal{J})} \mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_{k-1}},\mathbf{g})] \\
\downarrow [\psi]_* & & \downarrow [\psi]_* \\
\bigoplus_{\mathbf{f_k}\in\overline{N_k}(\mathcal{J})} \mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_k},\mathbf{g'})] & \xrightarrow{\partial_k(\mathbf{g'})} & \bigoplus_{\mathbf{f_{k-1}}\in\overline{N_{k-1}}(\mathcal{J})} \mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_{k-1}},\mathbf{g'})]
\end{array}$$

$$\begin{array}{ccc}
[\varphi] & \xmapsto{\partial_k(\mathbf{g})} & \sum_{\ell\in F(\mathbf{f_k})}(-1)^\ell[\varphi\circ d^\ell] \\
\downarrow [\psi]_* & & \downarrow [\psi]_* \\
[\psi\circ\varphi] & \xmapsto{\partial_k(\mathbf{g'})} & \sum_{\ell\in F(\mathbf{f_k})}(-1)^\ell[\psi\circ\varphi\circ d^\ell]
\end{array}$$

for $[\varphi] : \mathbf{f_k} \to \mathbf{g}$ of $\mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_k},\mathbf{g})$. Therefore, ∂_k is a natural transformation. At $k = 0$, ∂_0 is the augmentation, that is, for $\mathbf{g}$ of $\overline{N_k}(\mathcal{J})$,

$$\partial_0(\mathbf{g}) : \bigoplus_{\mathbf{f_0}\in\overline{N_0}(\mathcal{J})} \mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_0},\mathbf{g})] \longrightarrow \mathbb{C}$$

$\partial_0(\mathbf{g})([\varphi]) = 1$ for any $[\varphi]$ of $\mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_0},\mathbf{g})$.

To prove $P(\mathrm{Sd}(\mathcal{J}))_*$ is exact we introduce the notion of *equivalence n-simplex*. It is a generalization of a combinatorial n-simplex and it is obtained by exclusion and identification of some faces of an n-simplex. We prove that an equivalence n-simplex satisfying a certain condition generates an acyclic chain complex and this fact implies $P(\mathrm{Sd}(\mathcal{J}))_*$ is exact.

Definition 3.27. Let n be a natural number and let $\sim$ be an equivalence relation on $[n]$. For $0 \le k \le n-1$, let

$$\Delta_k^{(n)} = \{(i_0, i_1, \ldots, i_k) \in [n]^{k+1} \mid i_0 < \cdots < i_k\}$$

and

$$E_k^{(n)} = \{(i_0, i_1, \ldots, i_k) \in \Delta_k^{(n)} \mid \text{for all } 0 \le \ell \le k-1,\ i_\ell \not\sim i_{\ell+1}\}$$

and

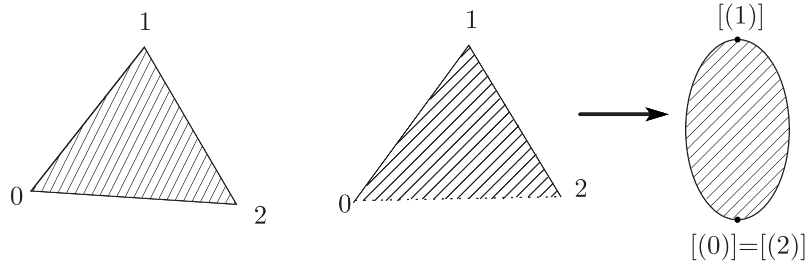
$$C_k^{(n)} = E_k^{(n)}/\approx$$

where $(i_0, i_1, \ldots, i_k) \approx (j_0, j_1, \ldots, j_k)$ if and only if $i_\ell \sim j_\ell$ for every ℓ. For $k = -1$, let $\Delta_{-1}^{(n)} = C_{-1}^{(n)} = *$. We also define $C_n^{(n)} = \{[(1, 2, \ldots, n)]\}$. We call the family $\{C_k^{(n)}\}_{k\ge -1}$ an *equivalence n-simplex*.

Example 3.28. Suppose $n = 2$ and $0 \sim 2$. Then, we have

$$C_0^{(2)} = \{[(0)] = [(2)], [(1)]\}$$
$$C_1^{(2)} = \{[(0,1)], [(1,2)]\}$$
$$C_2^{(2)} = \{[(0,1,2)]\}.$$

The families $\{\Delta_k^{(2)}\}_{k \geq -1}$ and $\{C_k^{(2)}\}_{k \geq -1}$ are visualized as follows.



The left hand side is $\{\Delta_k^{(2)}\}_{k \geq -1}$ and the right hand side is $\{C_k^{(2)}\}_{k \geq -1}$.

The *face operator* $d_\ell : \Delta_k^{(n)} \to \Delta_{k-1}^{(n)}$ is the map which eliminates the ℓ-th coordinate,

$$d_\ell(i_0, i_1, \ldots, i_k) = (i_0, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_k).$$

It is partially defined on $C_k^{(n)}$. We give the definition in the following.

Lemma 3.29. *Let $\{C_k^{(n)}\}_{k \geq -1}$ be an equivalence n-simplex. For $[(i_0, \ldots, i_k)]$ of $C_k^{(n)}$, define*

$$F([(i_0, \ldots, i_k)]) = \{\ell \in [k] \mid d_\ell(i_0, \ldots, i_k) \in E_{k-1}^{(n)}\}.$$

Then, $F([(i_0, \ldots, i_k)])$ does not depend on the choice of the representation of $[(i_0, \ldots, i_k)]$.

Proof. Suppose $(i_0, \ldots, i_k) \approx (j_0, \ldots, j_k)$. For ℓ of $F([(i_0, \ldots, i_k)])$, we have $i_{\ell-1} \not\sim i_{\ell+1}$. Then we also have $j_{\ell-1} \not\sim j_{\ell+1}$, since $j_{\ell-1} \sim i_{\ell-1} \not\sim i_{\ell+1} \sim j_{\ell+1}$. Hence, $F([(i_0, \ldots, i_k)])$ contains ℓ if and only if $F([(j_0, \ldots, j_k)])$ contains ℓ,

$$F([(i_0, \ldots, i_k)]) = F([(j_0, \ldots, j_k)]).$$

□

Definition 3.30. Let $\{C_k^{(n)}\}$ be an equivalence n-simplex. For $[(i_0, \ldots, i_k)]$ of $C_k^{(n)}$ and ℓ of $F([(i_0, \ldots, i_k)])$, define the *face operator*

$$d_\ell([(i_0, \ldots, i_k)]) = [d_\ell(i_0, \ldots, i_k)].$$

If $(i_0, \ldots, i_k) \approx (j_0, \ldots, j_k)$, then $i_m \sim j_m$ for every m. So

$$(i_0, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_k) \approx (j_0, \ldots, j_{\ell-1}, j_{\ell+1}, \ldots, j_k).$$

Hence, this map is well-defined.

Definition 3.31. Let $\{C_k^{(n)}\}_{k\geq -1}$ be an equivalence n-simplex. For $0 < k$ define $D_k : \mathbb{C}[C_k^{(n)}] \to \mathbb{C}[C_{k-1}^{(n)}]$ by

$$D_k([(i_0,\dots,i_k)]) = \sum_{\ell\in F([(i_0,\dots,i_k)])} (-1)^\ell d_\ell([(i_0,\dots,i_k)])$$

for any $[(i_0,\dots,i_k)]$ of $C_k^{(n)}$. For $k=0$ define $D_0 : \mathbb{C}[C_0^{(n)}] \to \mathbb{C}$ to be the augmentation, that is,

$$D_0\Big(\sum_{x_i\in C_0^{(n)}} \alpha_i x_i\Big) = \sum_{x_i\in C_0^{(n)}} \alpha_i .$$

Proposition 3.32. *Let $\{C_k^{(n)}\}_{k\geq -1}$ be an equivalence n-simplex. Then,*

$$D_k \circ D_{k-1} = 0.$$

Hence,

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{C} \xrightarrow{D_n} \mathbb{C}[C_{n-1}^{(n)}] \xrightarrow{D_{n-1}} \cdots$$

$$\cdots \xrightarrow{D_2} \mathbb{C}[C_1^{(n)}] \xrightarrow{D_1} \mathbb{C}[C_0^{(n)}] \xrightarrow{D_0} \mathbb{C} \longrightarrow 0$$

is a chain complex.

Proof. We prove this claim by comparing this complex with the familiar chain complex $\{\mathbb{C}[\Delta_k^{(n)}], \partial_k\}_{k\geq -1}$ where $\partial_k : \mathbb{C}[\Delta_k^{(n)}] \to \mathbb{C}[\Delta_{k-1}^{(n)}]$ is defined by the alternating sum of the face operators,

$$\partial_k = \sum_{\ell=0}^{k} (-1)^\ell d_\ell .$$

This chain complex is isomorphic to the augmented chain complex of usual n-simplex with coefficients in $\mathbb{C}$.

Define a map $p_k : \mathbb{C}[\Delta_k^{(n)}] \to \mathbb{C}[C_k^{(n)}]$ by

$$p_k((i_0,\dots,i_k)) = \begin{cases} [(i_0,\dots,i_k)], & \text{if } (i_0,\dots,i_k) \in E_k^{(n)}, \\ 0, & \text{if } (i_0,\dots,i_k) \notin E_k^{(n)}. \end{cases}$$

In particular, define $p_{-1} = 1_{\mathbb{C}}$. We show $\{p_k\} : \{\mathbb{C}[\Delta_k^{(n)}], \partial_k\} \to \{\mathbb{C}[C_k^{(n)}], D_k\}$ is a chain map. It suffices to show that the following two types of diagrams are commutative

$$\begin{array}{ccc} \mathbb{C}[\Delta_k^{(n)}] & \xrightarrow{\partial_k} & \mathbb{C}[\Delta_{k-1}^{(n)}] \\ \downarrow{\scriptstyle p_k} & & \downarrow{\scriptstyle p_{k-1}} \\ \mathbb{C}[C_k^{(n)}] & \xrightarrow{D_k} & \mathbb{C}[C_{k-1}^{(n)}] \end{array} \qquad \begin{array}{ccc} \mathbb{C}[\Delta_0^{(n)}] & \xrightarrow{\partial_0} & \mathbb{C} \\ \downarrow{\scriptstyle p_0} & & \downarrow{\scriptstyle 1_{\mathbb{C}}} \\ \mathbb{C}[C_0^{(n)}] & \xrightarrow{D_0} & \mathbb{C} \end{array}$$

where ∂_0 is the augmentation and $1 \leq k \leq n$.

Since p_0 is a natural projection, the diagram of the right hand side is commutative.

Next we show the commutativity of the left diagram. Take $(i_0, \ldots, i_k)$ of $\Delta_k^{(n)}$. Suppose $E_k^{(n)}$ does not contain it. We have

$$\begin{aligned} D_k \circ p_k((i_0, \ldots, i_k)) &= D_k(0) \\ &= 0. \end{aligned}$$

Since $E_k^{(n)}$ does not contain $(i_0, \ldots, i_k)$, there exists ℓ such that $0 \leq \ell < k$ and $i_\ell \sim i_{\ell+1}$. Here, we have to consider two cases,

(1) the existence of such ℓ is unique,
(2) there is another such ℓ'.

In the first case,

$$p_{k-1} \circ \partial_k((i_0, \ldots, i_k)) = p_{k-1}\left(\sum_{j=0}^{k} (-1)^j d_j(i_0, \ldots, i_k)\right).$$

For $0 \leq j \leq \ell - 1$ or $\ell + 2 \leq j \leq k$, $d_j(i_0, \ldots, i_k)$ contains i_ℓ and $i_{\ell+1}$ which are consecutive, hence $p_{k-1}(d_j(i_0 \ldots, i_k)) = 0$. Since $i_\ell \sim i_{\ell+1}$, $d_\ell(i_0, \ldots, i_k) \approx d_{\ell+1}(i_0, \ldots, i_k)$. This fact implies

$$\begin{aligned} p_{k-1} \circ \partial_k((i_0, \ldots, i_k)) &= p_{k-1}\left((-1)^\ell d_\ell(i_0, \ldots, i_k) + (-1)^{\ell+1} d_{\ell+1}(i_0, \ldots, i_k)\right) \\ &= (-1)^\ell [d_\ell(i_0, \ldots, i_k)] + (-1)^{\ell+1} [d_{\ell+1}(i_0, \ldots, i_k)] \\ &= 0. \end{aligned}$$

The second case can be done by induction, with some case distinctions if three or more consecutive entries are equivalent.

If $E_k^{(n)}$ contains $(i_0, \ldots, i_k)$, it is easy to see

$$p_{k-1} \circ \partial_k((i_0, \ldots, i_k)) = D_k \circ p_k((i_0, \ldots, i_k)).$$

Hence, $\{p_k\}$ is a chain map. Since each p_k is a surjection and $\partial_k \circ \partial_{k-1} = 0$, we obtain $D_k \circ D_{k-1} = 0$. □

Proposition 3.33. *Let $\{C_k^{(n)}\}_{k \geq -1}$ be an equivalence n-simplex. Suppose its equivalence relation satisfies the property that $i \not\sim i + 1$ for all $0 \leq i \leq n - 1$. Then,*

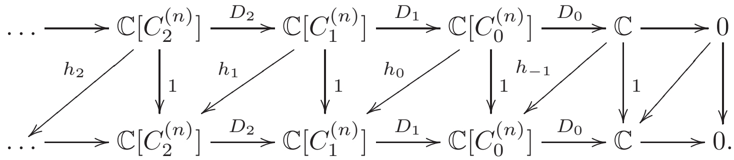
$$H_m(\{C_k^{(n)}, D_k\}_{k \geq -1}) = 0$$

for any m.

Proof. Define a chain homotopy between 0 and identity $h_k : \mathbb{C}[C_k^{(n)}] \to \mathbb{C}[C_{k+1}^{(n)}]$ by

$$h_k([(i_0, \ldots, i_k)]) = \begin{cases} [(0, i_0, \ldots, i_k)], & \text{if } 0 \not\sim i_0, \\ 0, & \text{if } 0 \sim i_0. \end{cases}$$

In particular, for $k = -1$ define $h_{-1} : \mathbb{C} \to \mathbb{C}[C_0^{(n)}]$ by $h_{-1}(*) = [(0)]$. Then we have the following diagram



We have

$$\begin{aligned} D_0 \circ h_{-1}(1) &= D_0[(0)] \\ &= 1. \end{aligned}$$

For $0 \leq k < n$, we show $h_{k-1} \circ D_k + D_{k+1} \circ h_k = 1$. Take an element $[(i_0, \ldots, i_k)]$ of $C_k^{(n)}$. If $i_0 \sim 0$, then

$$(2) \quad \begin{aligned} (h_{k-1} \circ D_k + D_{k+1} \circ h_k)([(i_0, \ldots, i_k)]) &= h_{k-1} \circ D_k([(i_0, \ldots, i_k)]) \\ &= h_{k-1}\Bigg(\sum_{\ell \in F([(i_0,\ldots,i_k)])} (-1)^\ell d_\ell[(i_0, \ldots, i_k)] \Bigg). \end{aligned}$$

Here, for ℓ of $F([(i_0, \ldots, i_k)])$ such that $\ell > 0$, we have

$$\begin{aligned} h_{k-1}((-1)^\ell d_\ell[(i_0, \ldots, i_k)]) &= h_{k-1}((-1)^\ell[(i_0, \ldots, i_{\ell-1}, i_{\ell+1}, \ldots, i_k)]) \\ &= 0. \end{aligned}$$

Since $i_0 \not\sim i_1$, we have $i_1 \not\sim 0$. Thus, equation (2) is

$$\begin{aligned} h_{k-1}((-1)^0 d_0[(i_0, \ldots, i_k)]) &= h_{k-1}([(i_1, \ldots, i_k)]) \\ &= [(0, i_1, \ldots, i_k)] \\ &= [(i_0, i_1, \ldots, i_k)]. \end{aligned}$$

If $i_0 \not\sim 0$, then we have

$$\begin{aligned} D_{k+1} \circ h_k([(i_0, \ldots, i_k)]) &= D_{k+1}([(0, i_0, \ldots, i_k)]) \\ &= \sum_{\ell \in F([(0,i_0,\ldots,i_k)])} (-1)^\ell d_\ell[(0, i_0, \ldots, i_k)] \end{aligned}$$

and

$$h_{k-1} \circ D_k([(i_0, \ldots, i_k)]) = h_{k-1}\Bigg(\sum_{\ell \in F([(i_0,\ldots,i_k)])} (-1)^\ell d_\ell[(i_0, \ldots, i_k)] \Bigg).$$

For $\ell > 0$, $F([(i_0, \ldots, i_k)])$ contains ℓ if and only if $F([(0, i_0, \ldots, i_k)])$ contains $\ell + 1$. Thus, we obtain

$$\begin{aligned} (h_{k-1} \circ D_k + D_{k+1} \circ h_k)([(i_0, \ldots, i_k)]) &= (-1)^0 d_0[(0, i_0, \ldots, i_k)] \\ &= [(i_0, \ldots, i_k)]. \end{aligned}$$

When $k = n$, h_n is the 0-map. Hence, we have

$$(3)\quad (h_{n-1} \circ D_n + D_{n+1} \circ h_n)([(0,1,\dots,n)]) = h_{n-1} \circ D_n([(0,1,\dots,n)])$$
$$= h_{n-1}\left(\sum_{\ell \in F([(0,1,\dots,n)])} (-1)^\ell d_\ell[(0,1,\dots,n)] \right).$$

If there exists ℓ such that $\ell > 0$ and ℓ of $F([(0,1,\dots,n)])$, we have

$$h_{n-1}((-1)^\ell d_\ell[(0,1,\dots,n)]) = h_{n-1}((-1)^\ell[(0,1,\dots,\ell-1,\ell+1,\dots,n)])$$
$$= 0.$$

Here, we use the asymmetric condition that $i \not\sim i+1$ for all $0 \leq i \leq n-1$. Since $i \not\sim i+1$ for all $0 \leq i \leq n-1$, $F([(0,1,\dots,n)])$ contains 0. Thus the equation (3) is

$$(4)\quad h_{n-1}((-1)^0 d_0[(0,1,\dots,n)]) = h_{n-1}([(1,\dots,n)]).$$

Since 0 and 1 is consecutive, we have $0 \not\sim 1$. Hence, equation (4) is

$$h_{n-1}([(1,\dots,n)]) = [(0,1,\dots,n)].$$

We conclude $\{C_k^{(n)}, D_k\}_{k \geq -1}$ is an exact sequence. □

Remark 3.34. We used the asymmetric condition that $i \not\sim i+1$ for all $0 \leq i \leq n-1$ at the last part of the proof above. If we drop this assumption, we do not expect this result.

If there exists i such that $i \sim i+1$, then $F([(0,1,\dots,n)])$ does not contain ℓ which satisfies $0 \leq \ell < i$ or $i+1 < \ell \leq n$. If $0 \leq \ell < i$, then

$$d_\ell(0,1,\dots,n) = (0,1,\dots,\ell-1,\ell+1,\dots,i,i+1,\dots,n),$$

but $E_{n-1}^{(n)}$ does not contain it. Hence, $F([(0,1,\dots,n)])$ does not contain ℓ. In the same way, we can show $F([(0,1,\dots,n)])$ does not contain ℓ if $i+1 < \ell \leq n$. Therefore, if there is another such i', the set $F([(0,1,\dots,n)])$ is an empty set. So the differential $D_n : \mathbb{C} \to \mathbb{C}[C_{n-1}^{(n)}]$ is the 0-map. If the existence of such i is unique, we have $F([(0,1,\dots,n)]) = \{i, i+1\}$. Then, we have

$$D_n([(0,1,\dots,n)]) = (-1)^i d_i([(0,1,\dots,n)]) + (-1)^{i+1} d_{i+1}([(0,1,\dots,n)])$$
$$= (-1)^i([(0,1,\dots,i-1,i+1,\dots,n)])$$
$$+ (-1)^{i+1}([(0,1,\dots,i,i+2,\dots,n)])$$
$$= 0.$$

In any case, if the equivalence relation does not satisfy the property that $i \not\sim i+1$ for all $0 \leq i \leq n-1$, the differential D_n is the 0-map. Hence, we obtain $H_n(\{C_k^{(n)}, D_k\}_{k \geq -1}) = \mathbb{C}$ and it is not what we expect.

This result is a homological interpretation of [13, Prop. 4.6] which proved the *reduced Euler characteristic* of an equivalence n-simplex $\{C_k^{(n)}\}_{k\geq -1}$ is zero, that is,

$$\widetilde{\chi}(\{C_k^{(n)}\}) = \sum_{k=-1}^{n} (-1)^k \# C_k^{(n)} = 0$$

when its equivalence relation satisfies the property that $i \not\sim i+1$ for all $0 \leq i \leq n-1$.

Proposition 3.35. *For a small category $\mathcal{J}$, $P(\mathrm{Sd}(\mathcal{J}))_*$ is a projective resolution of $\underline{\mathbb{C}}$ in* $\mathrm{Func}(\mathrm{Sd}(\mathcal{J}), \mathbb{C}\text{-vect})$.

Proof. Since each $\mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f}, -)]$ is projective for any object $\mathbf{f}$ of $\mathrm{Sd}(\mathcal{J})$ by Proposition 2.11 and Corollary 2.12,

$$\bigoplus_{\mathbf{f_k} \in \overline{N_k}(\mathcal{J})} \mathbb{C}[\mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_k}, -)]$$

is also projective for any k. Next we show exactness of $P(\mathrm{Sd}(\mathcal{J}))_*$. Note that $P(\mathrm{Sd}(\mathcal{J}))_*$ is exact if and only if each $P(\mathrm{Sd}(\mathcal{J}))_*(\mathbf{g})$ is exact for any $\mathbf{g}$ of $\overline{N_n}(\mathcal{J})$. Take $\mathbf{g}$ of $\overline{N_n}(\mathcal{J})$ and define an equivalence relation $\sim_{\mathbf{g}}$ on $[n]$ by $i \sim_{\mathbf{g}} j$ if

$$\mathbf{g}(\min\{i,j\} \to \max\{i,j\}) = \mathrm{id}.$$

Then, $\sim_{\mathbf{g}}$ is an equivalence relation and it satisfies $i \not\sim_{\mathbf{g}} i+1$ for all $0 \leq i \leq n-1$. For this equivalence relation, we obtain an equivalence n-simplex and its chain complex $\{C_k^{(n)}, D_k\}_{k\geq -1}$.

Then, the chain complex is isomorphic to $P(\mathrm{Sd}(\mathcal{J}))_*(\mathbf{g})$ as we now show. Define two maps

$$\varphi_k : C_k^{(n)} \longrightarrow \coprod_{\mathbf{f_k} \in \overline{N_k}(\mathcal{J})} \mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_k}, \mathbf{g})$$

$$\psi_k : \coprod_{\mathbf{f_k} \in \overline{N_k}(\mathcal{J})} \mathrm{Hom}_{\mathrm{Sd}(\mathcal{J})}(\mathbf{f_k}, \mathbf{g}) \longrightarrow C_k^{(n)}$$

by

$$\varphi_k([(i_0, \ldots, i_k)]) : [k] \longrightarrow [n]$$
$$\varphi_k([(i_0, \ldots, i_k)])(\ell) = i_\ell$$

and

$$\psi_k([\alpha]) = [(\alpha(0), \ldots, \alpha(k))]$$

for any $[(i_0, \ldots, i_k)]$ of $C_k^{(n)}$ and any $[\alpha] : \mathbf{f_k} \to \mathbf{g}$. In general, a morphism $[\varphi] : \mathbf{f} \to \mathbf{g}$ in $\mathrm{Sd}(\mathcal{J})$ satisfies $\mathbf{f} = \mathbf{g} \circ \varphi$, so $\mathbf{g}$ and φ determine $\mathbf{f}$. Thus, the order-preserving injection $\varphi_k([(i_0, \ldots, i_k)])$ and $\mathbf{g}$ determine the domain of the map $\varphi_k([(i_0, \ldots, i_k)]) :? \to \mathbf{g}$. Then, φ_k and ψ_k are well-defined. Indeed, if $\alpha_1 \sim \alpha_2 : \mathbf{f_k} \to \mathbf{g}$, then

$$\mathbf{g}(\min\{\alpha_1(i), \alpha_2(i)\} \to \min\{\alpha_1(i), \alpha_2(i)\}) = \mathrm{id}$$

for any i, that is, $\alpha_1(i) \sim_{\mathbf{g}} \alpha_2(i)$. Hence,

$$\begin{aligned}\psi_k([\alpha_1]) &= [(\alpha_1(0), \ldots, \alpha_1(k))] \\ &= [(\alpha_2(0), \ldots, \alpha_2(k))] \\ &= \psi_k([\alpha_2]).\end{aligned}$$

If $[(i_0, \ldots, i_k)] = [(j_0, \ldots, j_k)]$, then $i_\ell \sim_{\mathbf{g}} j_\ell$ for any ℓ. So we have

$$\mathbf{g}\left(\min\{i_\ell, j_\ell\} \to \max\{i_\ell, j_\ell\}\right) = \mathrm{id}$$

and we have $\varphi_k([(i_0, \ldots, i_k)]) \sim \varphi_k([(j_0, \ldots, j_k)])$. It is clear that $\varphi_k \circ \psi_k = 1$ and $\psi_k \circ \varphi_k = 1$ for any k. Moreover, $\{\varphi_k\}$ is compatible with the differentials, so $\{\varphi_k\}$ is a chain map. Hence, $P(\mathrm{Sd}(\mathcal{J}))_*(\mathbf{g})$ is isomorphic to $\{C_k^{(n)}, D_k\}_{k \geq -1}$. Proposition 3.33 implies $\{C_k^{(n)}, D_k\}_{k \geq -1}$ is exact, so $P(\mathrm{Sd}(\mathcal{J}))_*(\mathbf{g})$ is also. □

Finally, we give a proof of Theorem 3.25 which states the opposite of the subdivision $\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}$ is of type (L^2) if and only if $\mathcal{I}$ is finite acyclic for any small category $\mathcal{I}$ and in this case

$$\chi^{(2)}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}) = \chi^{(2)}(\mathcal{I}) = \chi^{(2)}(\mathrm{Sd}(\mathcal{I})).$$

Proof of Theorem 3.25. To compute $\chi^{(2)}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}})$ we work on the category

$$\mathrm{Func}((\mathrm{Sd}(\mathcal{I})^{\mathrm{op}})^{\mathrm{op}}, \mathbb{C}\text{-vect}) = \mathrm{Func}(\mathrm{Sd}(\mathcal{I}), \mathbb{C}\text{-vect}).$$

We have the projective resolution $P(\mathrm{Sd}(\mathcal{I}))_*$ of the constant functor $\underline{\mathbb{C}}$. Since $\mathrm{Sd}(\mathcal{I})$ is acyclic, so is $\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}$. Hence, we can apply Lemma 3.26. Since the splitting functor preserves direct sums, for any object $\mathbf{f}$ of $\mathrm{Sd}(\mathcal{I})$ we obtain

$$S_{\mathbf{f}} P(\mathrm{Sd}(\mathcal{I}))_* = \ \cdots \longrightarrow 0 \longrightarrow \mathbb{C} \longrightarrow 0 \longrightarrow \cdots$$

where $\mathbb{C}$ is only in the dimension $L(\mathbf{f})$. Since $\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}$ is acyclic, $\mathrm{Aut}(\mathbf{f})$ is trivial, hence the tensor operation $- \bigotimes_{\mathbb{C}[\mathbf{f}]} \mathcal{N}(\mathbf{f})$ is trivial. Thus, we have

$$\begin{aligned} h^{(2)}\left(S_{\mathbf{f}} P(\mathrm{Sd}(\mathcal{I}))_* \bigotimes_{\mathbb{C}[\mathbf{f}]} \mathcal{N}(\mathbf{f})\right) &= h^{(2)}\left(S_{\mathbf{f}} P(\mathrm{Sd}(\mathcal{I}))_*\right) \\ &= \sum_{n \geq 0} \dim_{\mathcal{N}(\mathbf{f})}\left(H_n(S_{\mathbf{f}} P(\mathrm{Sd}(\mathcal{I}))_*)\right) \\ &= 1. \end{aligned}$$

Note that $\dim_{\mathcal{N}(\mathbf{f})}$ is just the dimension as $\mathbb{C}$-vector spaces. We obtain

$$\begin{aligned}
h^{(2)}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}) &= \sum_{\mathbf{f}\in \mathrm{Ob}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}})} h^{(2)}\left(S_{\mathbf{f}} P(\mathrm{Sd}(\mathcal{I}))_* \bigotimes_{\mathbb{C}[\mathbf{f}]} \mathcal{N}(\mathbf{f})\right) \\
&= \sum_{\mathbf{f}\in \mathrm{Ob}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}})} 1 \\
&= \sum_{\mathbf{f}\in \mathrm{Ob}(\mathrm{Sd}(\mathcal{I}))} 1 \\
&= \sum_{n=0}^{\infty} \# \overline{N_n}(\mathcal{I}). \qquad (5)
\end{aligned}$$

The series (5) converges if and only if each $\overline{N_n}(\mathcal{I})$ is finite and there exists a sufficiently large integer M such that $\overline{N_n}(\mathcal{I}) = \varnothing$ for $n > M$. In other words, $\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}$ is of type (L^2) if and only if $P(\mathrm{Sd}(\mathcal{I}))_*$ is an L^2-resolution if and only if (5) converges if and only if $\mathcal{I}$ is finite acyclic by Lemma 3.6.

If $\mathcal{I}$ is finite acyclic, the series (5) converges, hence $\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}$ is of type (L^2). We have

$$\begin{aligned}
\chi^{(2)}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}) &= \sum_{\mathbf{f}\in \mathrm{Ob}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}})} \chi^{(2)}\left(S_{\mathbf{f}} P(\mathrm{Sd}(\mathcal{I}))_* \bigotimes_{\mathbb{C}[\mathbf{f}]} \mathcal{N}(\mathbf{f})\right) \\
&= \sum_{\mathbf{f}\in \mathrm{Ob}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}})} (-1)^{L(\mathbf{f})} \\
&= \sum_{n=0}^{M} (-1)^n \# \overline{N_n}(\mathcal{I}) \\
&= \chi_L(\mathcal{I})
\end{aligned}$$

for a sufficiently large integer M. [5, Lemma 7.3] implies $\chi_L(\mathcal{I}) = \chi^{(2)}(\mathcal{I})$. Hence, we obtain

$$\chi^{(2)}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}) = \chi^{(2)}(\mathcal{I}).$$

Since $\mathcal{I}$ is finite acyclic, $\mathrm{Sd}(\mathcal{I})$ is finite acyclic by Lemma 3.6. [5, Lemma 7.3] implies $\chi_L(\mathrm{Sd}(\mathcal{I})) = \chi^{(2)}(\mathrm{Sd}(\mathcal{I}))$. Proposition 3.7 implies

$$\chi^{(2)}(\mathcal{I}) = \chi_L(\mathcal{I}) = \chi_L(\mathrm{Sd}(\mathcal{I})) = \chi^{(2)}(\mathrm{Sd}(\mathcal{I})).$$

Hence, we obtain the result. □

4. The extended L^2-Euler characteristic

In this section, we extend the definition of the L^2-Euler characteristic. As we have seen, the equation $\chi^{(2)}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}) = \chi^{(2)}(\mathcal{I})$ only holds when $\mathcal{I}$ is finite acyclic, because $\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}$ is of type (L^2) if and only if $\mathcal{I}$ is finite acyclic. We show the extended L^2-Euler characteristic is invariant under barycentric subdivision for a wider class of finite categories, that is, the class for which the series Euler characteristic can be defined (Section 3.8).

Definition 4.1. A small category $\mathcal{J}$ is called *of type extended* (L^2) if for some projective resolution P_* of the constant functor $\underline{\mathbb{C}}$ in $\mathrm{Func}(\mathcal{J}^{\mathrm{op}}, \mathbb{C}\text{-vect})$,

$$h_n^{(2)}(\mathcal{J}) := \sum_{[x]\in \mathrm{iso}(\mathcal{J})} \dim_{\mathcal{N}(x)} \Big(H_n(S_x P_* \otimes_{\mathbb{C}[x]} \mathcal{N}(x)) \Big)$$

converges, the radius of convergence ρ of the power series with complex variable

$$f_{\mathcal{J}}^{(2)}(z) := \sum_{n=0}^{\infty} h_n^{(2)}(\mathcal{J}) z^n$$

is not zero, there exist a real number ε and a complex function g such that

(1) $\varepsilon \in [1, \infty]$
(2) g is holomorphic on the open ε-disk around 0, except possibly at finitely many poles
(3) $-1 \in \mathrm{dom}\, g$
(4) $g = f_{\mathcal{J}}^{(2)}$ on the open ρ-disk around 0
(5) If $\varepsilon = 1$, then we additionally require g is continuous along the real line segment $[-1, 0]$.

Then we define *the extended L^2-Euler characteristic* $\chi_{\mathrm{ex}}^{(2)}(\mathcal{J})$ of $\mathcal{J}$ by

$$\chi_{\mathrm{ex}}^{(2)}(\mathcal{J}) = g(-1) = \lim_{\substack{z\to -1 \\ z\in[-1,0]}} g(z).$$

If there exist another δ and h, then the uniqueness of the analytic continuity assures $g = h$ in $U(0; 1)$ since $\varepsilon, \delta \geq 1$. Hence, $g(z) = h(z)$ for any z of $(-1, 0]$. Therefore,

$$\lim_{\substack{z\to -1 \\ z\in[-1,0]}} g(z) = \lim_{\substack{z\to -1 \\ z\in[-1,0]}} h(z),$$

so that $g(-1) = h(-1)$. Hence, this definition is well-defined.

Remark 4.2. The definition does not depend on the choice of projective resolution. This follows from the fundamental lemma of homological algebra for $R\Gamma$-modules, as in [9]. See also [5, § 5.3].

Proposition 4.3. *If a small category $\mathcal{J}$ is of type (L^2), then $\mathcal{J}$ is of type extended (L^2) and $\chi_{\mathrm{ex}}^{(2)}(\mathcal{J}) = \chi^{(2)}(\mathcal{J})$.*

Proof. Suppose $\mathcal{J}$ is of type (L^2), so that $h^{(2)}(\mathcal{J})$ converges absolutely, and hence also each $h_n^{(2)}(\mathcal{J})$ does. By comparison, we see that $f_{\mathcal{J}}^{(2)}(z)$ converges on the closed disk of radius 1. Namely if $|z| \leq 1$ we have

$$\begin{aligned}\sum_{n=0}^{\infty} |h_n^{(2)}(\mathcal{J}) z^n| &= \sum_{n=0}^{\infty} |h_n^{(2)}(\mathcal{J})||z|^n \\ &\leq \sum_{n=0}^{\infty} h_n^{(2)}(\mathcal{J}) < \infty.\end{aligned}$$

Thus, the radius of convergence ρ of the power series $f^{(2)}_{\mathcal{J}}$ is at least 1. If $\rho > 1$, then we may take $g = f^{(2)}_{\mathcal{J}}$ and $\varepsilon := \rho$ so that Definition 4.1 is satisfied.

If $\rho = 1$, take $\varepsilon = 1$, then an application of Abel's Theorem for Power Series to the real series $\sum(-1)^n h^{(2)}_n(\mathcal{J})$ shows that $g := f^{(2)}_{\mathcal{J}}$ is continuous on the interval $[-1, 0]$. Namely, since $\sum(-1)^n h^{(2)}_n(\mathcal{J})$ converges, Abel's Theorem implies

$$g(-1) = \sum_{n\geq 0}^{\infty}(-1)^n h^{(2)}_n(\mathcal{J}) = \lim_{\substack{z\to -1\\ z\in[-1,0]}} g(z),$$

so that Definition 4.1 is satisfied.

Moreover, we have

$$\begin{aligned}\chi^{(2)}_{\mathrm{ex}}(\mathcal{J}) &= g(-1)\\ &= f^{(2)}_{\mathcal{J}}(-1)\\ &= \chi^{(2)}(\mathcal{J}).\end{aligned}$$

□

Example 4.4. Let $\mathcal{G}$ be the skeleton of the category of finite sets and bijections. Then, $\mathcal{G}$ is a groupoid. By [5, Ex. 5.12], we have

$$H_p\Big(S_x P_* \bigotimes_{\mathbb{C}[x]} \mathcal{N}(x)\Big) = \begin{cases}\mathbb{C}\bigotimes_{\mathbb{C}[x]}\mathcal{N}(x), & \text{if } p = 0,\\ 0, & \text{if } p > 0.\end{cases}$$

Since each $\mathrm{Aut}(x)$ is finite, $\mathcal{N}(x) = \mathbb{C}[x]$. Hence, we have

$$H_p\Big(S_x P_* \bigotimes_{\mathbb{C}[x]} \mathcal{N}(x)\Big) = \begin{cases}\mathbb{C}, & \text{if } p = 0,\\ 0, & \text{if } p > 0.\end{cases}$$

The von Neumann dimension $\dim_{\mathcal{N}(x)}$ is also easy, that is,

$$\dim_{\mathcal{N}(x)} = \frac{1}{\#\mathrm{Aut}(x)}\dim_{\mathbb{C}}.$$

Hence, $h^{(2)}_n(\mathcal{J}) = 0$ if $n > 0$ and

$$h^{(2)}_0(\mathcal{J}) = \sum_{n=0}^{\infty}\frac{1}{n!} = e,$$

so that the series $f^{(2)}_{\mathcal{G}}(z) = e$. The constant polynomial clearly satisfies all of Definition 4.1 and $f^{(2)}_{\mathcal{G}}(-1) = e$, so that $\chi^{(2)}_{\mathrm{ex}}(\mathcal{G}) = e$.

This example was studied by Baez–Dolan on [1, p. 15] from a different point of view.

Theorem 4.5. *Let $\mathcal{I}$ be a finite category. Then $\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}$ is of type extended (L^2) if and only if the power series $f_{\mathcal{I}}(t) = \sum_{n=0}^{\infty}\#\overline{N_n}(\mathcal{I})t^n$ is rational with a nonvanishing denominator at $t = -1$. In this case, we have*

$$\chi_{\sum}(\mathcal{I}) = \chi^{(2)}_{\mathrm{ex}}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}).$$

If $\mathcal{I}$ is additionally acyclic, these are equal to $\chi^{(2)}(\mathcal{I})$, $\chi^{(2)}(\mathrm{Sd}(\mathcal{I}))$, and $\chi^{(2)}_{\mathrm{ex}}$ $(\mathrm{Sd}(\mathcal{I}))$.

Proof. In equation (5) of proof of Theorem 3.25, we obtain

$$f^{(2)}_{\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}}(z) = \sum_{n=0}^{\infty} \#\overline{N_n}(\mathcal{I})z^n = f_{\mathcal{I}}(z). \tag{6}$$

As pointed out by [2] in the proof of Theorem 2.2, the number $\#\overline{N_n}(\mathcal{I})$ can be expressed by using matrices, that is, $\#\overline{N_n}(\mathcal{I}) = \mathrm{sum}\{(Z_{\mathcal{I}} - E)^n\}$. Since entries of $(Z_{\mathcal{I}} - E)$ are natural numbers, we obtain

$$\#\overline{N_n}(\mathcal{I}) = \mathrm{sum}\{(Z_{\mathcal{I}} - E)^n\} \le \{\mathrm{sum}(Z_{\mathcal{I}} - E)\}^n.$$

Hence, we have

$$\begin{aligned} \sum_{n=0}^{\infty} |\#\overline{N_n}(\mathcal{I})z^n| &= \sum_{n=0}^{\infty} \#\overline{N_n}(\mathcal{I})|z^n| \\ &\le \sum_{n=0}^{\infty} \{\mathrm{sum}(Z_{\mathcal{I}} - E)\}^n |z^n|. \end{aligned} \tag{7}$$

For $0 \le |z_0| < \frac{1}{\mathrm{sum}(Z_{\mathcal{I}} - E)}$, the series (7) converges, hence $f^{(2)}_{\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}}(z_0)$ also converges. So the radius of convergence of $f^{(2)}_{\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}}(z)$ is not zero.

By [2, Thm. 2.2] and equation (6), it follows that $f^{(2)}_{\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}}$ has the rational expression

$$f^{(2)}_{\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}}(z) = \frac{\mathrm{sum}(\mathrm{adj}(E - (Z_{\mathcal{I}} - E)z))}{\det(E - (Z_{\mathcal{I}} - E)z)}.$$

The rational function $f^{(2)}_{\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}}$ has finitely many poles on $U(0;\infty)$. Hence, $\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}$ is of type extended (L^2) if and only if (6) does not have a pole at -1 and this is equivalent to the existence of $\chi_{\sum}(\mathcal{I})$. So we obtain

$$\begin{aligned} \chi^{(2)}_{\mathrm{ex}}(\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}) &= \frac{\mathrm{sum}(\mathrm{adj}(E - (Z_{\mathcal{I}} - E)(-1)))}{\det(E - (Z_{\mathcal{I}} - E)(-1))} \\ &= \chi_{\sum}(\mathcal{I}). \end{aligned}$$

Furthermore, since $\mathrm{Sd}(\mathcal{I})^{\mathrm{op}}$ is acyclic, Theorem 3.25 and Proposition 4.3 imply the last part of the claim. □

Remark 4.6. We defined an extension of the L^2-Euler characteristic which turns out to be not invariant under equivalence of categories, since the series Euler characteristic is not. In [5, Lemma 5.15], it was proven that the L^2-Euler characteristic is invariant under equivalence of categories for directly finite categories.

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