Mathematik

The spectral flow theorem for families of twisted Dirac operators

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Abstract

In this thesis we generalize the construction of the two, a priori different, versions of the index difference to the case of families of twisted Spin Dirac operators and give a proof that they are mapped to each other under the Bott isomorphism in KK-theory. The classical versions of the index difference assign to a pair of positive scalar metric on a closed spin manifold an index in the Real K-theory of a point using Spin Dirac operators. By replacing the Spin Dirac operators by twisted Spin Dirac operators one obtains two new versions taking the fundamental group of the manifold into account. This is done not only for pair of positive scalar curvature metric but also for compact families of positive scalar curvature metrics. Therefore the proof, called the spectral flow index theorem, that the two definitions of the classical index difference agree does not work anymore. We give a proof of this by calculating a Kasparov product. As an application a result of Ebert and Randal-Williams concerning the homotopy of the space of positive scalar metrics on even dimensional manifolds can be generalize to odd dimensional manifolds.

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Introduction

Foreword. The local geometry of two homeomorphic Riemannian manifolds can be very different. This suggests a fundamental research interest in the theory of smooth Riemannian manifolds, namely the relation between the local geometry and the global topology of a given smooth manifold M. It turns out that the *scalar curvature* has a rich interaction with topology. The scalar curvature $\mathrm{scal}(g)$ is the simplest curvature invariant of a Riemannian manifold (M, g). Its value at a point $x \in M$ is given by the double trace of the Riemannian curvature tensor at x. In dimension two the scalar curvature is twice the Gaussian curvature. Using the classical Gauss Bonnet Theorem, this entails that the only orientable 2-dimensional closed manifold with non-negative scalar curvature which is not flat is $\mathbb{S}^2$. Of course this does not continue to hold for $d \geqslant 3$. For that reason the space $R^+(M)$ of all Riemannian metrics on M with *positive scalar curvature* (hereafter psc) is well worth studying. There are two pivotal questions concerning $R^+(M)$:

Question 1. *Which closed Riemannian manifolds M admit a metric with positive scalar curvature, i. e. for which manifolds M is the space $R^+(M)$ non-empty?*

Question 2. *Provided that $R^+(M) \neq \emptyset$, what is the homotopy type of $R^+(M)$, i. e. what do its homotopy groups $\pi_k(R^+(M), g)$ look like?*

This thesis helps to examine the second question. The main result of this thesis is the generalization of the *spectral flow index theorem* of [Ebe17] taking the fundamental group of M into account. As a consequence it allows to apply the results of [ERW17] about the homotopy type of $R^+(M)[B^{-1}]$ also to odd dimensional manifolds.

Literature review. We will first discuss answers to question 1 about the existence of psc metrics on a given smooth simply connected manifold M. In the presence of a *spin structure* the answer is given purely in topological terms and only depends on the spin-cobordism class of the manifold:

Let (M^d, g) be a closed connected d-dimensional Riemannian spin manifold. A spin structure is given by a lift of the Gauss map $\tau\colon M \to BO(d)$ classifying the tangent bundle along the covering map $BSpin(d) \to BO(d)$ to the 2-connected cover $BSpin(d)$. A spin manifold has a KO-fundamental class $[M]_{KO} \in KO_d(M)$ determined by the spinor bundle $\not{\mathfrak{S}}(M) \to M$ and the *Spin Dirac* operator $\not{D}$. This is a first order elliptic operator acting on the smooth sections $\Gamma(\not{\mathfrak{S}}(M))$ of the spinor bundle. Its index $\mathrm{ind}(\not{D})$ can be interpreted as an element of the real K-theory $KO^{-d}(pt)$ of a point in degree $-d$ due to the Clifford symmetries encoded in $\not{\mathfrak{S}}(M)$ and $\not{D}$. Since M is compact, the map $M \to \{pt\}$ is proper and hence induces a map $p_*\colon KO_d(M) \to KO_d(pt) \cong KO^{-d}(pt)$ under which $[M]_{KO}$ is sent to $\mathrm{ind}(\not{D})$. If g has psc then $\not{D}$ is invertible. This is a consequence of the *Schrödinger-Lichnerowicz formula* [Lic63] which states that

$$\not{D}^2 = \not{\nabla}^*\not{\nabla} + \frac{1}{4}\,\mathrm{scal}(g)$$

Therefore $\mathrm{ind}(\not{D})$ must be zero. It follows that the index of the Spin Dirac operator is the obstruction to psc on M. The Atiyah-Singer Index Theorem computes $\mathrm{ind}(\not{D})$ in terms of the $\hat{A}$-genus of M, a topological invariant independent of the metric, defined using characteristic classes. Thus $R^+(M) = \emptyset$ if $\hat{A}(M) \neq 0$, i. e. parts of the geometry of M are determined by a topological invariant.

Indeed $\alpha(M) := \mathrm{ind}(\not{D})$ is not only an invariant of the manifold M^d but also an invariant of the cobordism class $[M] \in \Omega_d^{spin}$ of M. Moreover Gromov and Lawson proved [GL80a] that psc is stable under suitable surgeries. In other words, if the closed d-dimensional spin manifold M_1 is obtained from the closed d-dimensional spin manifold M_0 by suitable surgeries then $R^+(M_0) \neq \emptyset$ if and only if $R^+(M_1) \neq \emptyset$. Stolz was able to prove for a simply connected manifold that the necessary condition of having trivial $\hat{A}$-genus for admitting a psc metric is also sufficient, i. e. he proved

Theorem ([Sto92])**.** *Let M be a simply connected closed spin manifold of dimension $d \geqslant 5$. Then $R^+(M) \neq \emptyset$ if and only if $\mathrm{ind}(\not{D}) = 0$.*

The case when M is not simply connected is merely conjectured and called the (unstable) *Gromov-Lawson-Rosenberg-Conjecture.* It predicts that the result continues to hold with $\mathrm{ind}(\not{D})$ replaced by a refinement, called the *Rosenberg-index* $\alpha_r^R(M) \in KO_d(C_r^*\pi_1(M))$ with values in the real K-theory of the reduced C^*-algebra of $\pi_1(M)$:

Conjecture. *Let* M^d *be a connected closed spin manifold of dimension* $d \geqslant 5$*. Then* $R^+(M) \neq \emptyset$ *if and only if* $\alpha_r^R(M) \neq 0$*.*

However, there are counterexamples to this conjecture, compare [DSS02] and [Sch98]. This suggests that the conjecture has to be modified: The stable version of this conjecture asserts that $\alpha_r^R(M) = 0$ if and only if there exists $k \in \mathbb{N}$ such that $R^+(M \times B^k) \neq \emptyset$. Here B is the *Bott manifold*, an 8-dimensional simply connected spin manifold with $\hat{A}(B) = \beta \in KO^{-8}(pt)$. When $\pi_1(M)$ satisfies the *strong Novikov conjecture*, the stable Gromov-Lawson-Rosenberg conjecture is true due to a theorem of Stolz [Sto02].

Next we discuss answers to question 2 about the homotopy type of the space $R^+(M)$ provided that it is non-empty. In the case not taking the fundamental group into account all previous results concerning the homotopy type of $R^+(M)$ such as [Hit74, GL80b, HSS14] or [CS13] were superseded by the results of Botvinnik, Ebert and Randal-Williams in [BERW17]. In loc. cit. the space $R^+(W)_h$ of psc metrics on a compact d-dimensional manifold W with collared boundary M which are of the form $h + dt^2$ on $M \times [-\varepsilon, 0]$ is studied. The aforementioned authors combined a parameterised version of the surgery theory of Gromov and Lawson and secondary index theory with techniques coming from the study of moduli spaces of manifolds. Using this they were able to construct maps $\pi_k(R^+(W)_h, g_0) \to KO^{-*}(pt)$ which are nontrivial whenever $k \geqslant 0$, $d \geqslant 6$ and the target is nontrivial. In a second paper the two last named authors extended the methods used in [BERW17] to closed connected even-dimensional spin manifolds with a map φ to the classifying space BG of a discrete group G. They investigated the homotopy type of the homotopy colimit $R^+(M)[B^{-1}]$ of the sequence

$$R^+(M) \to R^+(M \times B) \to R^+(M \times B \times B) \to \cdots$$

In the case that G is torsion-free and satisfies the *Baum-Connes Conjecture* and $\pi_1(M) \xrightarrow{\varphi_*} G$ is split-surjective they were able to show that $\pi_k(R^+(M)[B^{-1}], g_0)$ surjects onto $KO_{k+d+1}(C_r^* G)$ for all $k \geqslant 0$.

Methodology. One of the central methods to detect non-trivial elements in the homotopy groups of $R^+(M)$ is provided by a secondary index theoretic invariant called *the index difference*. Introduced in two a priori different ways by Hitchin in [Hit74] and Gromov-Lawson in [GL83] the index difference became an important tool in the study of the homotopy type of $R^+(M)$. The construction of Hitchin differs substantially from the construction given by Gromov and Lawson. The way, roughly speaking, Hitchin defined the index difference is as follows: Let M^d be a closed connected spin manifold such that $R^+(M)$ is non-empty. Choose a base point $g_0 \in R^+(M)$ and consider for every $g_1 \in R^+(M)$ the path of Riemannian metrics $g_t := tg_1 + (1-t)g_0$ from g_0 to g_1. One obtains a corresponding path of Spin Dirac operators $\not{D}_t$ defining a path in the space Fred^d of $\mathbf{Cl}^{d,0}$-linear odd self-adjoint Fredholm operators. The operators $\not{D}_0$ and $\not{D}_1$ are invertible and the subspace of invertible operators is contractible due to Kuipers theorem. Therefore we obtain a map $[([0,1],\{0,1\}),(\mathrm{Fred}^d,\mathrm{pt})]$ given by the path of Spin Dirac operators. Since the space Fred^d represents the functor $KO^{-d}(\_)$ we obtain an element $\mathrm{inddiff}^H_{g_0}(g_1) \in KO^{-d}([0,1],\{0,1\}) \cong KO^{-d-1}(\mathrm{pt})$. It only depends on the pair $(g_0, g_1) \in R^+(M) \times R^+(M)$ of psc metrics. This construction likewise works for compact manifolds W with boundary ∂W. Furthermore a smooth map $\mathscr{G}\colon X \to R^+(M) \times R^+(M)$ from a compact manifold X defines an element $\mathrm{inddiff}^H_{\mathscr{G}} \in KO^{-d-1}(X)$. Hence for $g_0 \in R^+(W)$ we obtain a well-defined homotopy class of maps

$$\mathrm{inddiff}^H_{g_0}\colon R^+(W)_h \longrightarrow \Omega^{\infty+d+1}\mathbf{KO}$$

into the infinite loop space representing real K-theory. The key idea in [BERW17] is the construction of a map ρ from an infinite loop space Z to $R^+(W)_h$ and the identification of the composition $\mathrm{inddiff}^H_{g_0} \circ \rho$ with another well-known infinite loop map $\mathscr{A}\colon Z \to \Omega^{\infty+d+1}\mathbf{KO}$ which is surjective on homotopy groups. Using this Botvinnik, Ebert and Randal-Williams were able to derive that the induced maps on homotopy groups

$$\pi_k(R^+(W)_h, g_0) \xrightarrow{(\mathrm{inddiff}^H_{g_0})_*} KO^{-k-d-1}(\mathrm{pt})$$

are nontrivial whenever $k \geqslant 0$, $d \geqslant 6$ and the target is nontrivial. The construction of the map ρ requires the dimension of W to be even. The odd-dimensional case of the identification of $\mathrm{inddiff}_{g_0} \circ \rho$ with an infinite loop map is deduced from the even-dimensional case using the second version of the index difference $\mathrm{inddiff}^{GL}_{g_0}$ due to Gromov and Lawson, as well as the *spectral flow index theorem* proven in [Ebe17]. The index difference due to Gromov and Lawson also takes two psc metrics $g_0, g_1 \in R^+(M)$ as input. However, instead of considering the path of metrics between g_0 and g_1, one endows $M \times \mathbb{R}$ with a complete metric $g \in R(M \times \mathbb{R})$ which is cylindrical on the ends, i. e.

$$g = \begin{cases} g_0 + dt^2, & \text{on } M \times (-\infty, 0] \\ g_1 + dt^2, & \text{on } M \times [1, \infty) \end{cases}$$

As the scalar curvature is positive on the ends, the Spin Dirac operator $\not{\mathfrak{D}}$ on the

$(d+1)$-dimensional manifold $M \times \mathbb{R}$ has an index in $KO^{-d-1}(pt)$. One defines $\mathrm{inddiff}^{GL}_{g_0}(g_1) \coloneqq \mathrm{ind}(\not{\mathfrak{D}})$. This construction generalizes also to families. Fixing $g_0 \in R^+(W)$ this also yields a well-defined homotopy class of maps

$$\mathrm{inddiff}^{GL}_{g_0} \colon R^+(W)_h \longrightarrow \Omega^{\infty+d+1}\mathbf{KO}$$

The spectral flow index theorem as in [Ebe17] states that the maps $\mathrm{inddiff}^{H}_{g_0}$ and $\mathrm{inddiff}^{GL}_{g_0}$ are weakly homotopic. Now the deduction of the odd dimensional case from the even dimensional case goes roughly as follows: The first step is the reduction to the case $W = \mathbb{D}^{2n+1}$ and the identification of the spaces $\Omega_{g_0} R^+(\mathbb{S}^{2n}) \xrightarrow{T} R^+(\mathbb{D}^{2n+1})_{g_0}$. Since $\mathbb{S}^{2n}$ is even-dimensional, $\mathscr{A}$ is weakly homotopic to $\mathrm{inddiff}^{GL}_{g_0} \circ \rho$. Therefore

$$\Omega(\mathscr{A}) \simeq \Omega(\mathrm{inddiff}^{GL}_{g_0} \circ \rho) \simeq \Omega(\mathrm{inddiff}^{GL}_{g_0}) \circ \Omega(\rho).$$

Hence one can use T and the spectral flow index theorem to deduce that

$$\Omega(\mathrm{inddiff}^{GL}_{g_0}) \circ \Omega(\rho) \simeq \mathrm{inddiff}^{H}_{g_0} \circ T \circ \Omega(\rho).$$

The crucial point here is that the spectral flow index theorem was only proven for operators with indices in $KO^{-*}(pt)$. The proof in [Ebe17] does not allow for a generalization to the case of operators with indices in $KO_*(C^*_r G)$ as the proof boils down to an index calculation for which the knowledge of $KO_*(C^*_r G)$ is needed.

In [ERW17] Ebert and Randal-Williams were able to generalize the methods of [BERW17] to the non-simply connected case. This goes as follows: Let W^d be a compact connected spin manifold of even dimension $d = 2n$ with collared boundary $M \coloneqq \partial W$ and a map $\varphi \colon W \to BG$. The index difference $\mathrm{inddiff}^H$ induces a map

$$R^+(W)_h \times R^+(W)_h \longrightarrow KK(\mathbf{Cl}^{0,d+1}, C^*_r G)$$

depending on φ by replacing $\mathrm{ind}(\not{D}_t) \in KO^{-d+1}(pt)$ by $\alpha^{\mathbb{R}}_r(M, g_t) \in KO_{d+1}(C^*_r G)$. As in the previous case the authors used the index difference to factor a map $\mathscr{A}_G$ of infinite loop spaces through the space $R^+(W)_h$. Contrary to the former case, $\mathscr{A}_G$ is itself a composition and factors through the real K-homology $KO_*(BG)$ of the classifying space of G. Indeed the map $\mathscr{A}_G$ is induced by the composition of the maps

$$\begin{aligned} \Omega^{spin}_d(BG) &\to KO_{d+1}(BG) \to KO_{d+1}(C^*_r G) \\ [W, \varphi] &\mapsto \varphi_*([W]_{KO}) \mapsto \nu(\varphi_*([W]_{KO})) \end{aligned}$$

or rather by the maps between the infinite loop spaces of the spectra representing these generalized homology theories. The second map is given by the *Novikov assembly map* $\nu : KO_*(BG) \to KO_*(C^*_r G)$. If G is torsion free then the Novikov assembly map can be identified with the *Baum-Connes assembly map* μ. If G furthermore satisfies the *Baum-Connes conjecture* then μ, hence also ν, is an isomorphism. Provided that G is torsion free and satisfies the Baum-Connes conjecture and that $\varphi_* : \pi_1(M) \to G$ is split-surjective, Ebert and Randal-Williams proved in [ERW17] that the induced maps

$$\pi_k(R^+(M)[B^{-1}], g_0) \xrightarrow{(\mathrm{inddiff}^H_{g_0}[B^{-1}])_*} KO_{d+1}(C^*_r G)$$

are surjective for all k.

Research aim and results. As mentioned earlier the restriction to even dimensional manifolds relies on the lack of a proof of the spectral flow index theorem taking the fundamental group into account. The thesis arose from necessity to provide a proof of the spectral flow index theorem in this case. This goal was achieved: Let M^d be a closed connected d-dimensional spin manifold with a map $\varphi \colon M \to BG$ to the classifying space of a countable discrete group. For example let $\pi_1(M) \cong G$. Furthermore let $R^+(M)$ be non-empty and let X be a compact smooth manifold possibly with boundary. For every smooth map $\mathscr{G} \colon X \to R^+(M) \times R^+(M)$ we construct elements

$$\mathrm{inddiff}^H_{\mathscr{G}} \in KK(\mathbf{Cl}^{0,d}, C_0(\mathbb{R} \times X, C^*_r G)) \text{ and } \mathrm{inddiff}^{GL}_{\mathscr{G}} \in KK(\mathbf{Cl}^{0,d+1}, C(X, C^*_r G)),$$

and prove the following

Theorem. *Let* $\mathrm{bott}^{-1} \colon KK(\mathbf{Cl}^{0,d}, C_0(\mathbb{R} \times X, C^*_r G)) \to KK(\mathbf{Cl}^{0,d+1}, C(X, C^*_r G))$ *be the inverse of the Bott isomorphism in* KK*-theory. Then* $\mathrm{bott}^{-1}([\mathrm{inddiff}^H_{\mathscr{G}}]) = [\mathrm{inddiff}^{GL}_{\mathscr{G}}]$ *holds.*

This is the most general form of the spectral flow index theorem. With the aid of this theorem the restriction to even dimensional manifolds in [ERW17] can be dropped.

Overview of the thesis and outline of the argument. The proof of the theorem is provided by using Kasparovs KK-theory. In fact it boils down to a calculation of an *unbounded* Kasparov product in theorem 5.2.1 using a theorem of Kucerovsky [Kuc97]. The first chapter can be viewed as a short overview of this topic starting with C^*-algebras, Hilbert modules and (unbounded) operators. In section 1.3 the reader will find the relevant definitions concerning KK-theory. The geometric input such as bundles of Hilbert modules and (families of) **A**-linear differential operators is explained in chapter 2. The thesis is organized is such a way that a reader who is familiar with these concepts can skip the first two chapters. Chapter 3 is devoted to a different description of the groups $KK(\mathbf{A}, C(X, \mathbf{B}))$ in terms of continuous fields of Hilbert modules and operator families. It is adaption of the results of [Ebe18]. This description is adequate to construct Kasparov modules from families of manifolds and families of operators. The unbounded Kasparov modules $\mathrm{inddiff}^{H}_{\mathscr{G}} \in \Psi(\mathbf{Cl}^{0,d}, C_0(\mathbb{R} \times X, C^*_r G))$ and $\mathrm{inddiff}^{GL}_{\mathscr{G}} \in \Psi(\mathbf{Cl}^{0,d+1}, C(X, C^*_r G))$ are constructed in chapter 4. Finally the product is calculated in chapter 5.

To prove theorem 5.0.1 we consider an unbounded product. Kucerovsky provided criteria under which an unbounded Kasparov module **z** represents the Kasparov product of the two unbounded modules **x** and **y**. In more detail, he gave sufficient conditions when the *bounded transform*, i. e. the image of **z** under the surjection $b\colon \Psi(\mathbf{A}, \mathbf{B}) \twoheadrightarrow KK(\mathbf{A}, \mathbf{B})$, is equal to the Kasparov product $b(\mathbf{x}) \# b(\mathbf{y})$.

The first step of the proof of theorem 5.0.1 is to represent $\mathrm{inddiff}^{H}_{\mathscr{G}}$ and $\mathrm{inddiff}^{GL}_{\mathscr{G}}$ by (unbounded) Kasparov-modules. Both versions of the index difference can be obtained as special cases from the following general construction developed in [Ebe18] and presented in chapter 3: A submersion $\pi\colon N \to X$ together with a bundle $E \to N$ of Hilbert-**B**-modules defines a continuous field $L^2_X(\pi, E)$, cf. definition 3.2.4, of Hilbert-**B**-modules over X. A **B**-linear differential operator $D\colon \Gamma^\infty_{cv}(N, E) \to \Gamma^\infty_{cv}(N, E)$ induces an (unbounded) operator family on $L^2_X(\pi, E)$. When (N, D) is fiberwise complete, compare definition 3.2.6, the closure $\overline{D}$ is self-adjoint and regular by theorem 2.2.3. Moreover by theorem 3.2.9 the resolvent of the closure is compact, if D is bounded from below by a fiberwise coercive function, cf. definition 3.2.8. If the bundle $E \to N$ is Real, graded by η and endowed with a Clifford action ρ of $\mathbf{Cl}^{0,d}$ such that $\overline{D}$ is Real, odd and anti-commutes with ρ, then the tuple $(L^2_X(\pi, E), \eta, c, \overline{D})$ defines an unbounded Kasparov-$(\mathbf{Cl}^{0,d}, C(X, \mathbf{B}))$-module. This is the content of theorem 3.2.10.

To define $\mathrm{inddiff}^{H}_{\mathscr{G}}$ let M^d be a closed connected spin manifold with a map $\varphi\colon M \to BG$ and suppose that $R^+(M)$ is non-empty. Furthermore let X be a smooth compact manifold endowed with a smooth map $\mathscr{G}\colon X \to R^+(M) \times R^+(M)$. We consider the submersion $\pi\colon M \times \mathbb{R} \times X \to \mathbb{R} \times X$ with a fiberwise Riemannian metric induced by $\mathscr{G}$. The bundle E is given by the spinor bundle $\not{\mathfrak{S}}(\pi)$ of the vertical tangent bundle $T_v\pi = TM \times \mathbb{R} \times X$ twisted with the pullback of the Miščenko-Fomenko line bundle $\mathcal{L}_G \to BG$ along φ. The operator family is induced by the twisted fiberwise Spin Dirac operators $\not{D}_{\lambda,x}$ on $\pi^{-1}(\lambda, x) \cong M$. The element $\mathrm{inddiff}^{H}_{\mathscr{G}} \in \Psi(\mathbf{Cl}^{0,d}, C_0(\mathbb{R} \times X, C^*_r G))$ is obtained from this data in the way explained above, see theorem 4.2.9. The element $\mathrm{inddiff}^{GL}_{\mathscr{G}} \in \Psi(\mathbf{Cl}^{0,d+1}, C(X, C^*_r G))$

is constructed in a similar way in theorem 4.2.18 by considering the submersion $\Pi\colon M \times \mathbb{R} \times X \to X$ and the (unbounded) operator family induced by the Spin Dirac operators $\not{\mathfrak{D}}_x$ on the non-compact manifolds $\Pi^{-1}(x) \cong M \times \mathbb{R}$. In the case that $X = \{pt\}$ and $G = \{1\}$ we get back the classical definition of the index difference of either Hitchin or Gromov and Lawson.

Let $\mathrm{bott}^{-1}\colon KK(\mathbf{Cl}^{0,d}, C_0(\mathbb{R} \times X, C_r^*G)) \xrightarrow{\cong} KK(\mathbf{Cl}^{0,d+1}, C(X, C_r^*G))$ be the inverse of the Bott isomorphism. It is given by the Kasparov product with the module $\tau_{C(X,C_r^*G)}(\alpha)$. Here $\alpha \in KK(C_0(\mathbb{R}), \mathbf{Cl}^{0,1})$ is the inverse of the Bott element $\beta \in KK(\mathbf{Cl}^{0,1}, C_0(\mathbb{R}))$ and the homomorphism $\tau_{C(X,C_r^*G)}\colon KK(C_0(\mathbb{R}), \mathbf{Cl}^{0,1}) \to KK(C_0(\mathbb{R}) \hat{\otimes} C(X, C_r^*G), \mathbf{Cl}^{0,1} \hat{\otimes} C(X, C_r^*G))$ is given by the exterior product. Hence the proof of theorem 5.0.1 would be accomplished by showing that

$$b(\mathrm{inddiff}^{H}_{\mathscr{G}})\#\tau_{C(X,C_r^*G)}(\alpha) = b(\mathrm{inddiff}^{GL}_{\mathscr{G}}).$$

However it is hard to prove this directly. Therefore the second step is to construct an unbounded Kasparov module $\mathbf{z}' \in \Psi(\mathbf{Cl}^{0,d+1}, C(X, C_r^*G))$ for which it is easier to prove that it represents the Kasparov product of $\mathrm{inddiff}^{H}_{\mathscr{G}}$ and $\tau_{C(X,C_r^*G)}(\alpha)$. This is done in lemma 5.1.4. Subsequently we show that $\mathbf{z}' \sim \mathrm{inddiff}^{GL}_{\mathscr{G}}$. To prove that this is indeed the case we make use of the following elementary fact: If the metric on $M \times \mathbb{R}$ is cylindrical, then the spinor bundle $\not{\mathfrak{S}}(M \times \mathbb{R})$ is isomorphic to $\not{\mathfrak{S}}(M) \hat{\otimes} \not{\mathfrak{S}}(\mathbb{R})$ and the Spin Dirac operator $\not{\mathfrak{D}}$ of $M \times \mathbb{R}$ is of the form $\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D_{\mathbb{R}}$. In the case of $\mathrm{inddiff}^{GL}_{\mathscr{G}}$ we endow $M \times \mathbb{R}$ with a metric which is cylindrical outside a compact subset. Hence the difference of the operators $\not{\mathfrak{D}}$ and $\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D_{\mathbb{R}}$ has compact support $M \times [0, 1]$. It follows that the induced unbounded operator families are identical up to a compact perturbation. As a consequence both modules represent the same element, cf. proposition 5.2.6. The last step is to show that

$$b(\mathbf{z}') = b(\mathrm{inddiff}^{H}_{\mathscr{G}})\#\tau_{C(X,C_r^*G)}(\alpha).$$

To this end we have to show that the "difference" of the unbounded operators $\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D_{\mathbb{R}}$ and $D_{\mathbb{R}}$ is bounded on the domain of $D_{\mathbb{R}}$ and that the "graded commutator" $\{\not{D} \hat{\otimes} 1, \not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D_{\mathbb{R}}\}$ is semi-bounded. Here we make use of the special form of the operator $\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D_{\mathbb{R}}$: The difference is given by multiplying $u \in C_c^\infty(\mathbb{R}, S_1)$ with a fixed section $\not{D}(s)$ with compact support. Therefore it defines a bounded map, see proposition 5.2.3 and eq. (5.7). The graded commutator conforms $\not{D}^2 \hat{\otimes} 1$. Using the Lichnerowicz formula for twisted bundles, see eq. (4.3), and the assumption about the scalar curvature this implies the semi-boundedness of the graded commutator, see proposition 5.2.5. This completes the proof of theorem 5.0.1 using theorem 1.3.9.

Contents

Chapter 1

C^*-algebras, Hilbert modules and KK-Theory

In this chapter we give an overview over the functional analysis needed in this thesis. Section 1.2 provides the definitions and concepts needed for KK-theory. This includes graded C^*-algebras, Hilbert modules and their morphisms. We also need to consider unbounded operators on Hilbert modules. It should be viewed as a summary and a reader who is familiar with Kasparov's (unbounded) KK-theory can skip this chapter.

1.1 C^*-algebras

In his original paper G. G. Kasparov [Kas80] defines the bivariant KK-functor for real, Real and complex $\mathbb{Z}/2$-graded C^*-algebras. For our purpose it suffices to consider only the class of Real $\mathbb{Z}/2$-graded C^*-algebras, i. e. the class of (complex) C^*-algebras $\mathbf{A}$ equipped with a *$\mathbb{Z}/2$-grading* and a *Real structure*. A $\mathbb{Z}/2$-grading is a self-adjoint unitary $\iota\colon \mathbf{A} \to \mathbf{A}$, such that $\iota^2 = \mathrm{id}$. Whereas a Real structure is a conjugate-linear and self-adjoint unitary $\tau\colon \mathbf{A} \to \mathbf{A}$, such that $\tau^2 = id$. The two structures are required to be compatible in the sense that $\iota\tau = \tau\iota$. The grading defines a decomposition $\mathbf{A}^{(0)} \oplus \mathbf{A}^{(1)}$ of $\mathbf{A}$ into the eigenspaces $\mathbf{A}^{(i)} := \{\mathbf{a} \in \mathbf{A}\colon \iota(\mathbf{a}) = (-1)^i \mathbf{a}\}$ of the unitary ι. The elements of the eigenspaces are called *homogeneous* and we define the *degree* of $\mathbf{a} \in \mathbf{A}^{(i)}$ by $\partial \mathbf{a} = i$. The *graded commutator* of two homogeneous elements is defined by $\{\mathbf{a}, \mathbf{b}\} := \mathbf{ab} - (-1)^{\partial \mathbf{a} \partial \mathbf{b}} \mathbf{ba}$. This definition can be extended by linearity to the entire algebra. Note that any C^*-algebra can be $\mathbb{Z}/2$-graded using the trivial grading id. A trivial example is the field of complex numbers $\mathbb{C}$ with the trivial grading and Real structure coming from the complex conjugation. This Real graded C^*-algebra is denoted by $\mathbf{R}$.

As tensor product we take the *spatial tensor product* within the class of Real $\mathbb{Z}/2$-graded C^*-algebras. This entails that the tensor product $\mathbf{A} \hat{\otimes} \mathbf{B}$ of two Real $\mathbb{Z}/2$-graded C^*-algebras is again a Real $\mathbb{Z}/2$-graded C^*-algebra such that:

- $\iota(\mathbf{a} \hat{\otimes} \mathbf{b}) = \iota(\mathbf{a}) \hat{\otimes} \iota(\mathbf{b})$,
- $\tau(\mathbf{a} \hat{\otimes} \mathbf{b}) = \tau(\mathbf{a}) \hat{\otimes} \tau(\mathbf{b})$,
- $(\mathbf{a}_1 \hat{\otimes} \mathbf{b}_1) \cdot (\mathbf{a}_2 \hat{\otimes} \mathbf{b}_2) = (-1)^{\partial \mathbf{b}_1 \partial \mathbf{a}_2} (\mathbf{a}_1 \mathbf{a}_2 \hat{\otimes} \mathbf{b}_1 \mathbf{b}_2)$,
- $(\mathbf{a} \hat{\otimes} \mathbf{b})^* = (-1)^{\partial \mathbf{a} \partial \mathbf{b}} (\mathbf{a}^* \hat{\otimes} \mathbf{b}^*)$.

The C^*-norm is obtained in the usual way, but with *graded states* instead of states. See [WO93, Appendix T]. Let X and Y be locally compact Hausdorff spaces. Then the spatial tensor product $C_0(X) \otimes \mathbf{A}$ is isomorphic to $C_0(X, \mathbf{A})$ and the spatial tensor product $C_0(X) \otimes C_0(Y)$ is isomorphic to $C_0(X \times Y)$. Compare [WO93, Proposition T.5.21]. For Real C^*-algebras see also [Sch93]. We will finish this section with an example.

The reduced C^*-algebra of a group: Let G be a countable discrete group and denote by $\mathbb{C}G$ the complex group ring of G. The complex Hilbert space of square summable functions $G \to \mathbb{C}$ is denoted by $\ell^2 G$. There is a representation of $\mathbb{C}G$ on $\ell^2 G$ by means of the *regular* representation ρ_r:

$$\rho_r(g) := [f \mapsto f(g^{-1} \cdot \_)] \in \mathbf{Lin}(\ell^2 G) \tag{1.1}$$

Using the regular representation we define the norm of $x \in \mathbb{C}G$ by $\|x\|_r := \|\rho_r(x)\|$. The completion of $\mathbb{C}G$ inside $\mathbf{Lin}(\ell^2 G)$ with respect to $\|\cdot\|_r$ is by definition *the reduced group C^*-algebra* $C_r^* G$. We consider $C_r^* G$ to be trivially graded and endowed with the Real structure given by complex conjugation, i. e. $\tau(\sum \lambda_g g) := \sum \bar{\lambda}_g g$.

1.2 Hilbert modules

Let $\mathbf{A}$ be a C^*-algebra and E a right $\mathbf{A}$-module. An $\mathbf{A}$*-valued inner product* on E is a sesquilinear map (linear in the second and conjugate linear in the first variable) $(\cdot,\cdot)\colon E \times E \longrightarrow \mathbf{A}$, such that for all $x, y \in E$ and $\mathbf{a} \in \mathbf{A}$ the following hold:

- $(x, x) \geqslant 0$,
- $(x, y\mathbf{a}) = (x, y)\mathbf{a}$,
- $(x, y)^* = (y, x)$.

Note that $(x,x) \geqslant 0$ means that (x,x) is a *positive* element in $\mathbf{A}$, i. e. it is a self-adjoint element with positive spectrum. Using this inner product and the C^*-norm $\|\cdot\|_{\mathbf{A}}$ on $\mathbf{A}$ we define the norm of an element $x \in E$ as follows:

$$\|x\| := \sqrt{\|(x,x)\|_{\mathbf{A}}} \tag{1.2}$$

If E is a Banach space with respect to this norm, then E is called a *Hilbert-$\mathbf{A}$-module*. As in the case of C^*-algebras we will also consider Real graded Hilbert modules over a Real graded C^*-algebra $\mathbf{A}$:

A *grading* on a Hilbert-$\mathbf{A}$-module E is a $\mathbb{C}$-linear self-adjoint involution η such that $\eta(x\mathbf{a}) = \eta(x)\iota(\mathbf{a})$ and $(\eta(x),\eta(y)) = \iota((x,y))$. A *Real structure* on E is a conjugate-linear involution κ satisfying the same compatible conditions as a grading with ι replaced by τ.

The Hilbert-$\mathbf{A}$-module E is called *finitely generated and projective* if E is isomorphic as a Hilbert-$\mathbf{A}$-module to an orthogonal direct summand of the Hilbert-$\mathbf{A}$-module $\mathbf{A}^n$ for some $n \in \mathbb{N}$. While a Hilbert-$\mathbf{A}$-module E is *countably generated* if there exists a countable set of generators, i. e. a countable set $\{x_n\} \subset E$ such that the linear span of $\{x_n\mathbf{a} \colon n \in \mathbb{N}, \mathbf{a} \in \mathbf{A}\} \subset E$ is dense in E. When $\mathbf{A}$ is unital then the *standard Hilbert-$\mathbf{A}$-module*

$$\mathcal{H}_{\mathbf{A}} := \{(x_k) \in \prod_{1}^{\infty} \mathbf{A} : \sum_{k=1}^{\infty} x_k^* x_k \text{ is norm convergent in } \mathbf{A}\}$$

is countably generated.

1.2.1 Bounded operators on Hilbert modules

The well-behaved operators between two Hilbert-$\mathbf{A}$-modules E and E' are those which admit an *adjoint*, i.e. those maps $D \colon E \to E'$ for which there is a map $D^* \colon E' \to E$ such that

$$(D(x), y) = (x, D^*(y)) \tag{1.3}$$

holds for all $x \in E$ and for all $y \in E'$. One can show that these maps, a priori neither linear nor bounded, are in fact linear module maps and that the vector space $\mathbf{Lin}_{\mathbf{A}}(E)$ of all *adjointable* maps from E to itself equipped with the supremum norm is a C^*-algebra, see [WO93, Proposition 15.2.4]. A grading ι on E induces a grading on $\mathbf{Lin}_{\mathbf{A}}(E)$ as follows: An operator $D \in \mathbf{Lin}_{\mathbf{A}}(E)$ is called odd, if $D \circ \iota = -\iota \circ D$ and even if $D \circ \iota = \iota \circ D$. When E is Real, we say that $D \in \mathbf{Lin}_{\mathbf{A}}(E)$ is *Real*, if $\kappa(Dx) = D\kappa(x)$. Compact operators on Hilbert-$\mathbf{A}$-modules are defined as follows:

Let $x, y \in E$ and consider the map

$$\theta_{x,y} \colon E \longrightarrow E\ ,\ z \mapsto x \cdot (y|z). \tag{1.4}$$

It is adjointable and hence defines a bounded operator on E, called *rank one operator*. These operators form a two-sided $*$-ideal inside $\mathbf{Lin_A}(E)$. The *compact operators* $\mathbf{Kom_A}(E) \subset \mathbf{Lin_A}(E)$ on E are defined as the norm closure of the linear span of the rank one operators.

1.2.2 Unbounded operators on Hilbert modules

We also have to consider *unbounded operators* on Hilbert-**A**-modules. We refer to [Lan95] for a general treatment. However we also recall a result from [KL17]. It is sufficient to consider only unbounded operators on a single Hilbert-**A**-module E. These are **A**-linear maps D defined on a dense **A**-submodule $\mathrm{dom}(D) \subset E$, called the *domain of* D, whose range lie in E. As in the case of unbounded operators between Hilbert spaces one defines the *graph* of D as the following submodule of $E \oplus E$:

$$G(D) \coloneqq \{(x, Dx) \colon x \in \mathrm{dom}(D)\}.$$

On the graph of D there is the following **A**-valued inner product:

$$((x, Dx), (y, Dy))_{G(D)} \coloneqq (x, y) + (Dx, Dy).$$

If $G(D)$ is complete with respect to this inner product then D is called *closed*. D is called *symmetric*, if $(Dx, y) = (x, Dy)$ holds for all $x, y \in \mathrm{dom}(D)$. Let $D \colon \mathrm{dom}(D) \to E$ be an unbounded operator then

$$\mathrm{dom}(D^*) \coloneqq \{y \in E \colon \exists z \in E\ \ \forall x \in \mathrm{dom}(D)\ (Tx, y) = (x, z)\}$$

is an **A**-submodule of E. For $y \in \mathrm{dom}(D^*)$ the map $y \mapsto z$ is well-defined and defines an **A**-linear map $D^* \colon \mathrm{dom}(D^*) \to E$ called the *adjoint* of D.

Remark. So far the treatment of unbounded operators on Hilbert modules is analogous to the theory of unbounded operators on Hilbert spaces. While a densely defined unbounded operator on a Hilbert space has a densely defined adjoint this does not need to hold in the Hilbert module case.

To account for this defect we say that an unbounded and densely defined operator D is *regular*, if D admits a densely defined adjoint D^* and $1 + D^*D$ has dense range. A closed densely defined symmetric operator D is self-adjoint and regular if and only if there exists $t \in \mathbb{R}$ such that the operators $(D \pm it)\colon \operatorname{dom}(D) \to E$ are invertible. See [KL17, Proposition 4.1]. The self-adjoint and regular operators are those operators for which there exists a *functional calculus*. Let $C(\overline{\mathbb{R}})$ be the C^*-algebra of continuous functions $f\colon \mathbb{R} \to \mathbf{R}$ such that $\lim_{\lambda \to \pm\infty} f(\lambda)$ exists. The functional calculus is a unital $*$-homomorphism

$$\Phi_D\colon C(\overline{\mathbb{R}}) \to \mathbf{Lin}_{\mathbf{A}}(E)\ ,\ f \mapsto f(D)$$

with the following properties:

1. $\|f(D)\| \leqslant \|f\|_{C_0}$
2. if D is Real and f is real-valued then $f(D)$ is Real and
3. if D and f are odd then $f(D)$ is odd.

See for example [Lan95, p. 118-120] or [Ebe18, Theorem 2.19]

1.2.3 Tensor products of Hilbert modules

There are two different types of (graded) tensor products of graded Hilbert modules. The first one is the *external tensor product* of a Hilbert-$\mathbf{A}$-module E_0 and a Hilbert-$\mathbf{B}$-module E_1, which is a graded Hilbert module over the graded tensor product $\mathbf{A} \hat{\otimes} \mathbf{B}$ constructed as follows: The algebraic tensor product $E_0 \odot E_1$ has a right $\mathbf{A} \hat{\otimes} \mathbf{B}$-action

$$(x_0 \hat{\otimes} x_1) \cdot (\mathbf{a} \hat{\otimes} \mathbf{b}) \coloneqq (-1)^{\partial x_1 \partial \mathbf{a}}(x_0\mathbf{a} \hat{\otimes} x_1\mathbf{b})$$

and a $\mathbf{A} \hat{\otimes} \mathbf{B}$-valued inner product

$$(x_0 \hat{\otimes} x_1, y_0 \hat{\otimes} y_1) \coloneqq (-1)^{\partial x_1 \cdot (\partial x_0 + \partial y_0)}(x_0, y_0) \hat{\otimes} (x_1, y_1).$$

The completion of $E_0 \odot E_1$ with respect to the norm induced by this inner product is $E_0 \hat{\otimes} E_1$. It can be graded by $\eta(x_0 \hat{\otimes} x_1) = \eta_0(x_0) \hat{\otimes} \eta_1(x_1)$. Using the exterior tensor product of E_0 and E_1 one gets an embedding

$$\mathbf{Lin}_{\mathbf{A}}(E_0) \hat{\otimes} \mathbf{Lin}_{\mathbf{B}}(E_1) \longrightarrow \mathbf{Lin}_{\mathbf{A}\hat{\otimes}\mathbf{B}}(E_0 \hat{\otimes} E_1)$$

given by $(D_0 \hat{\otimes} D_1)(x_0 \hat{\otimes} x_1) \coloneqq (-1)^{\partial D_1 \partial x_0}(D_0(x_0) \hat{\otimes} D_1(x_1))$. This induces an isomorphism between $\mathbf{Kom}_{\mathbf{A}}(E_0) \hat{\otimes} \mathbf{Kom}_{\mathbf{B}}(E_1)$ and $\mathbf{Kom}_{\mathbf{A}\hat{\otimes}\mathbf{B}}(E_0 \hat{\otimes} E_1)$. The second one is the *internal tensor product* $E_0 \hat{\otimes}_\varphi E_1$ of two Hilbert modules E_0 and E_1 over $\mathbf{A}$ and $\mathbf{B}$ respectively, over a graded $*$-homomorphism $\varphi\colon \mathbf{A} \to \mathbf{Lin}_{\mathbf{B}}(E_1)$. The internal tensor product $E_0 \hat{\otimes}_\varphi E_1$ is the Hilbert-$\mathbf{B}$-module (the right $\mathbf{B}$-module structure is induced from the right **B**-module structure on the algebraic tensor product $E_0 \hat{\odot} E_1$) obtained as the completion of $E_0 \odot E_1$ with respect to the norm induced by the **B**-valued inner product

$$(x_0 \hat{\otimes}_\varphi x_1, y_0 \hat{\otimes}_\varphi y_1) := (x_1, \varphi((x_0, y_0)_{E_0}) y_1)_{E_1}$$

and graded in the same way as $E_0 \hat{\otimes} E_1$. In the case of the internal tensor product, there is only an embedding

$$\mathbf{Lin_A}(E_0) \to \mathbf{Lin_B}(E_0 \hat{\otimes}_\varphi E_1) \text{ , } D \mapsto D \hat{\otimes}_\varphi 1$$

In general there is no way to define a map $\mathbf{Lin_B}(E_1) \to \mathbf{Lin_B}(E_0 \hat{\otimes}_\varphi E_1)$. See [Bla98, 14.] or [JK12, §1.2.].

1.3 KK-Theory

Kasparov modules: From now on **A**, **B**, **C**,... will always denote Real $\mathbb{Z}/2$-graded C^*-algebras and when we say C^*-algebra we mean a complex C^*-algebra equipped with a grading and a Real structure. Moreover we require all C^*-algebras to be *separable*. For us this is no restriction at all since with only one exception (which is separable) all C^*-algebras we consider are even unital. The C^*-algebra $C([0,1], \mathbf{B})$ of continuous functions from the unit interval $\mathbf{I} := [0,1]$ to **B** is denoted by **IB**.

Definition 1.3.1. *Let* **A** *and* **B** *be two* C^**-algebras,* E *a countably generated Hilbert-***B***-module,* $\rho\colon \mathbf{A} \to \mathbf{Lin_B}(E)$ *a Real graded* $*$*-homomorphism and* $D \in \mathbf{Lin_B}(E)$ *an odd and Real operator. The triple* (E, ρ, D) *is called a Kasparov-*$(\mathbf{A}, \mathbf{B})$*-module provided that*

1. $\rho(\mathbf{a})(D - D^*)$,
2. $\rho(\mathbf{a})(D^2 - 1)$ *and*
3. $\{D, \rho(\mathbf{a})\}$

are compact operators for every $\mathbf{a} \in \mathbf{A}$*. If the elements 1., 2. and 3. are zero, then* (E, ρ, D) *is called degenerate.*

Let $\mathcal{E}(\mathbf{A}, \mathbf{B})$ be the set of (bounded) Kasparov-$(\mathbf{A}, \mathbf{B})$-modules and $\mathcal{D}(\mathbf{A}, \mathbf{B})$ the subset of degenerate Kasparov-$(\mathbf{A}, \mathbf{B})$-modules. The addition of two Kasparov-$(\mathbf{A}, \mathbf{B})$-modules is given by the direct sum of the triples. We want to turn $\mathcal{E}(\mathbf{A}, \mathbf{B})$ into a group. To this end we introduce an equivalence relation on the set of Kasparov-$(\mathbf{A}, \mathbf{B})$-modules. Two Kasparov-$(\mathbf{A}, \mathbf{B})$-modules (E_i, ρ_i, T_i), $i = 0, 1$, are *unitary equivalent*, denoted by $(E_0, \rho_0, T_0) \sim_u (E_1, \rho_1, T_1)$, if there exists a unitary $u \in \mathbf{Lin_B}(E_0, E_1)$ of even degree such that $\rho_0 = u^* \rho_1 u$ and $T_0 = u^* T_1 u$.

Definition 1.3.2. *A homotopy between two Kasparov-*$(\mathbf{A},\mathbf{B})$*-modules* (E_0,ρ_0,T_0) *and* (E_1,ρ_1,T_1) *is a Kasparov-*$(\mathbf{A},\mathbf{IB})$*-module* (E,ρ,T)*, satisfying*

$$(\mathrm{ev}_i)_*((E,\rho,T)) \sim_u (E_i,\rho_i,T_i)\ ,\ \textit{for}\ i=0,1$$

where $\mathrm{ev}_i : \mathbf{IB}\to\mathbf{B}$ *is given by* $\varphi\mapsto\varphi(i)$ *and* $(\mathrm{ev}_i)_*((E,\rho,T))$ *is given by the Kasparov-*$(\mathbf{A},\mathbf{B})$*-module* $(E\,\hat{\otimes}_{\mathrm{ev}_i}\,\mathbf{B}, \mathrm{ev}_i\circ\rho, T\,\hat{\otimes}_{\mathrm{ev}_i}\,1)$.

The notion of homotopy defines an equivalence relation $\approx$ on $\mathcal{E}(\mathbf{A},\mathbf{B})$ under which a degenerate Kasparov module is homotopic to the 0-module $(\mathcal{H}_{\mathbf{B}},0,0)$.

Definition 1.3.3. $\mathbf{KK}(\mathbf{A},\mathbf{B}) := \mathcal{E}(\mathbf{A},\mathbf{B})/\approx$.

A priori $\mathbf{KK}(\mathbf{A},\mathbf{B})$ is only a semi group, but using the notion of homotopy one proves that

Lemma 1.3.4. $\mathrm{KK}(\mathbf{A},\mathbf{B})$ *is an abelian group.*

Proof. See for example [Kas80, Theorem 1.] or [Bla98, Proposition 17.3.3]. ☐

Remark. Let $\mathbf{R}$ denote the Real trivially graded C^*-algebra of complex numbers. Then $\mathrm{KK}(\mathbf{R},\mathbf{B})$ is isomorphic to $KO_0(\mathbf{B})$, the real K-theory of $\mathbf{B}$ in degree zero. See [Sch93, Theorem 2.3.8.].

1.3.1 Functoriality

The KK-groups are *contravariant* in the first and *covariant* in the second variable with respect to graded $*$-homomorphisms, in the sense that a graded $*$-homomorphism $\varphi\colon \mathbf{A}_0\to\mathbf{A}_1$ induces a group homomorphism $\varphi^*\colon \mathrm{KK}(\mathbf{A}_1,\mathbf{B})\to \mathrm{KK}(\mathbf{A}_0,\mathbf{B})$ and a graded $*$-homomorphism $\psi\colon \mathbf{B}_0\to\mathbf{B}_1$ induces a group homomorphism $\psi_*\colon \mathrm{KK}(\mathbf{A},\mathbf{B}_0)\to\mathrm{KK}(\mathbf{A},\mathbf{B}_1)$. Since these properties are not needed we will not go into details. Using the exterior tensor product of Hilbert modules one obtains a map

$$\mathcal{E}(\mathbf{A},\mathbf{B})\to\mathcal{E}(\mathbf{A}\,\hat{\otimes}\,\mathbf{D},\mathbf{B}\,\hat{\otimes}\,\mathbf{D})\ ,\ (E,\rho,D))\mapsto(E\,\hat{\otimes}\,\mathbf{D},\rho\,\hat{\otimes}\,1,D\,\hat{\otimes}\,\mathrm{id})$$

This map is compatible with direct sums and respects the equivalence relation and hence induces a homomorphism

$$\tau_{\mathbf{D}} : \mathrm{KK}(\mathbf{A},\mathbf{B})\to\mathrm{KK}(\mathbf{A}\,\hat{\otimes}\,\mathbf{D},\mathbf{B}\,\hat{\otimes}\,\mathbf{D})$$

which is natural in each variable.

1.3.2 The Kasparov product

One major feature of KK-theory is the *Kasparov product* originally defined in [Kas80]. Its construction was simplified through the notion of *connections* introduced by Cones and Skandalis [CS84].

Let E_i be Hilbert-$\mathbf{B}_i$-modules, $i = 0, 1$ and $\varphi_1 \colon \mathbf{B}_0 \to \mathbf{Lin}_{\mathbf{B}_1}(E_1)$ a graded $*$-homomorphism. For $x_0 \in E_0$ we define the $\mathbf{B}_1$-linear $T_{x_0} \colon E_1 \to E_0 \hat{\otimes}_{\varphi_1} E_1$ by $x_1 \mapsto x_0 \hat{\otimes} x_1$. Then T_{x_0} is adjointable with adjoint $T^*_{x_0}(y_0 \hat{\otimes} x_1) := \varphi_1((x_0, y_0)_{E_0})(x_1)$. Hence T_{x_0} defines an element in $\mathbf{Lin}_{\mathbf{B}_1}(E_1, E_0 \hat{\otimes}_{\varphi} E_1)$ called *tensor operator* or *creation operator*. Using tensor operators we can formulate the connection property of an operator $D \in \mathbf{Lin}_{\mathbf{B}_1}(E_0 \hat{\otimes}_{\varphi_1} E_1)$ for $D_1 \in \mathbf{Lin}_{\mathbf{B}_1}(E_1)$:

Definition 1.3.5. *The operator* D *is called a* D_1*-connection on* $E_0 \hat{\otimes}_{\varphi_1} E_1$*, if*

$$T_{x_0} \circ D_1 - (-1)^{\partial x_0} D \circ T_{x_0} \tag{1.5}$$

$$D_1 \circ T^*_{x_0} - (-1)^{\partial x_0} T^*_{x_0} \circ D \tag{1.6}$$

are compact operators for every $x_0 \in E_0$.

It is a consequence of the Stabilization Theorem [WO93, Theorem 15.4.6] that, if E_0, E_1, φ_1 and D_1 are as above, there always exists a D_1-connection D on $E_0 \hat{\otimes}_{\varphi_1} E_1$. Compare [JK12, §2.2.] or [Bla98, 18.3.].

Definition 1.3.6. *Suppose that* $\mathbf{x} := (E_0, \rho_0, D_0)$ *is a Kasparov-*$(\mathbf{A}, \mathbf{B}_0)$*-module and that* $\mathbf{y} := (E_1, \rho_1, D_1)$ *is a Kasparov-*$(\mathbf{B}_0, \mathbf{B})$*-module. Their Kasparov product* $\mathbf{x}\#\mathbf{y}$ *is given by a Kasparov-*$(\mathbf{A}, \mathbf{B})$*-module* $(E_0 \hat{\otimes}_{\rho_1} E_1, \rho_0 \hat{\otimes} 1, D)$ *such that*

1. *The operator* D *is a* D_1*-connection on* $E_0 \hat{\otimes}_{\rho_1} E_1$ *and*

2. *for all* $\mathbf{a} \in \mathbf{A}$ *the graded commutator* $\rho(\mathbf{a})\{D_0 \hat{\otimes} 1, D\}\rho(\mathbf{a})^*$ *is positive modulo compact operators.*

Theorem 1.3.7. *Let* $\mathbf{A}$ *be separable and* $\mathbf{B}_0$ *be* σ*-unital. Then for every* $\mathbf{x} \in \mathcal{E}(\mathbf{A}, \mathbf{B}_0)$ *and* $\mathbf{y} \in \mathcal{E}(\mathbf{B}_0, \mathbf{B})$ *there exists a Kasparov product* $\mathbf{x}\#\mathbf{y}$ *which is unique up to homotopy. Hence the Kasparov product induces a bilinear pairing*

$$\#\colon KK(\mathbf{A}, \mathbf{B}_0) \times KK(\mathbf{B}_0, \mathbf{B}) \to KK(\mathbf{A}, \mathbf{B}) \tag{1.7}$$

compatible with $\tau_{\mathbf{D}}$ *in the sense that* $\tau_{\mathbf{D}}(\mathbf{x}\#\mathbf{y}) = \tau_{\mathbf{D}}(\mathbf{x})\#\tau_{\mathbf{D}}(\mathbf{y})$.

See [Bla98, Theorem 18.4.3] and [Bla98, Theorem 18.4.4].

1.3.3 Unbounded Kasparov modules

Later we will make use of the notion of *unbounded Kasparov modules* introduced by Baaj and Julg in [BJ83]. The setting is the same as in the case of bounded Kasparov modules:

Definition 1.3.8. *Let* $\mathbf{A}, \mathbf{B}$ *be two* C^**-algebras,* E *a countably generated Hilbert-*$\mathbf{B}$*-module,* $\rho\colon \mathbf{A} \to \mathbf{Lin}_B(E)$ *a Real graded* $*$*-homomorphism and* D *an unbounded regular self-adjoint Real and odd operator on* E*. The set* $\Psi(\mathbf{A}, \mathbf{B})$ *of unbounded Kasparov modules is given by triples* (E, ρ, D) *as above satisfying*

1. $(1 + D^2)^{-1}\rho(\mathbf{a})$ *is a compact operator for every* $\mathbf{a} \in \mathbf{A}$ *and*
2. *the set of all* $\mathbf{a} \in \mathbf{A}$ *such that* $\{D, \rho(\mathbf{a})\}$ *is densely defined and extends to an element in* $\mathbf{Lin_B}(E)$ *is a dense subset of* $\mathbf{A}$.

Let $b\colon \mathbb{R} \to \mathbb{R}$ be the function $\lambda \mapsto \lambda(\lambda^2 + 1)^{-\frac{1}{2}}$. Then $b(D)$ is called the *bounded transform* of the unbounded regular and self-adjoint operator D. The operator $b(D)$ is bounded and for $(E, \rho, D) \in \Psi(\mathbf{A}, \mathbf{B})$ the triple $(E, \rho, b(D))$ is a Kasparov-$(\mathbf{A}, \mathbf{B})$-module, see [Bla98] or [BJ83]. Moreover Baaj and Julg proved in loc. cit. that the map

$$b\colon \Psi(\mathbf{A}, \mathbf{B}) \to KK(\mathbf{A}, \mathbf{B})\ ,\ (E, \rho, D) \mapsto (E, \rho, b(D)) \tag{1.8}$$

is surjective if $\mathbf{A}$ is separable. We will denote the image of an unbounded Kasparov module $\mathbf{x}$ under the map 1.8 by $b(\mathbf{x})$. The reason why we introduced unbounded Kasparov modules is that in some cases it is easier to calculate their Kasparov product. Let $\mathbf{x} := (E_0, \rho_0, D_0) \in \Psi(\mathbf{A}, \mathbf{B}_0)$ and $\mathbf{y} := (E_1, \rho_1, D_1) \in \Psi(\mathbf{B}_0, \mathbf{B})$. Furthermore let $\mathbf{z} := (E_0 \hat{\otimes}_{\rho_1} E_1, \rho_0 \hat{\otimes}_{\rho_1} 1, D) \in \Psi(\mathbf{A}, \mathbf{B})$. Suppose that $\mathrm{dom}(D) \subset \mathrm{dom}(D_0 \hat{\otimes}_{\rho_1} 1)$, then Kucerovsky proved the following:

Theorem 1.3.9 ([Kuc97] Theorem 13)**.** *Suppose that the operator*

$$\left\{ \begin{pmatrix} D & 0 \\ 0 & D_1 \end{pmatrix}, \begin{pmatrix} 0 & T_{x_0} \\ T^*_{x_0} & 0 \end{pmatrix} \right\}$$

is bounded on $\mathrm{dom}(D \oplus D_1)$ *for all* x_0 *in a dense subset of* $\rho_0(\mathbf{A})E_0$*. Furthermore assume that there exists a constant* $c \in \mathbb{R}$ *such that*

$$(D_0 \hat{\otimes}_{\rho_1} 1(x), Dx) + (Dx, D_0 \hat{\otimes}_{\rho_1} 1(x)) \geqslant c \cdot (x, x)$$

for all x *in the domain of* D*. Then* $(E_0 \hat{\otimes}_{\rho_1} E_1, \rho_0 \hat{\otimes}_{\rho_1} 1, b(D))$ *represents the Kasparov product of* $(E_0, \rho_0, b(D_0))$ *and* $(E_1, \rho_1, b(D_1))$.

Definition 1.3.10. *We say that an unbounded Kasparov module* $\mathbf{z}$ *represents the Kasparov product of the unbounded Kasparov modules* $\mathbf{x}$ *and* $\mathbf{y}$ *if* $\mathbf{z}$ *satisfies the assumptions of theorem 1.3.9.*

1.3.4 Clifford algebras and Periodicity

As we work in the category of Real $\mathbb{Z}/2$-graded C^*-algebras we will only introduce the *Real Clifford algebras* $\mathbf{Cl}^{p,q}$. The standard reference is [LML16, Chapter I]. Let $\mathbb{R}^{p,q}$ be the real Euclidean vector space with basis $\{e_1, \cdots, e_p, \epsilon_1, \cdots, \epsilon_q\}$. We define the Clifford algebra $\mathbf{Cl}^{p,q}$ to be the $\mathbb{C}$-algebra generated by the basis elements subject to the relations

$$\begin{aligned}
e_i e_j + e_j e_i &= -2\delta_{ij} \text{ for } 1 \leqslant i, j \leqslant p,\\
\epsilon_i \epsilon_j + \epsilon_j \epsilon_i &= 2\delta_{ij} \text{ for } 1 \leqslant i, j \leqslant q,\\
e_i \epsilon_j + \epsilon_j e_i &= 0 \text{ for } 1 \leqslant i \leqslant p \text{ and } 1 \leqslant j \leqslant q.
\end{aligned}$$

Defining $e_i^* = -e_i$ and $\epsilon_j^* = \epsilon_j$ and extending this to a $\mathbb{C}$-antilinear antiautomorphism turns $\mathbf{Cl}^{p,q}$ into a $*$-algebra. A $\mathbb{Z}/2$-grading ι on $\mathbf{Cl}^{p,q}$ is obtained by setting $\iota(e_i) = -e_i$ and $\iota(\epsilon_j) = -\epsilon_j$ and extending this to a $\mathbb{C}$-linear automorphism. The Real structure is defined to be the identity on the generators and then to be extended to a $\mathbb{C}$-antilinear automorphism. The norm on $\mathbf{Cl}^{p,q}$ is introduced as follows: Let $\mathcal{S}_{p,q} := \Lambda^*\mathbb{R}^{p+q} \otimes \mathbb{C}$ be complexified exterior algebra of $\mathbb{R}^{p+q}$ with its natural inner product, even/odd-grading and Real structure coming from the complex conjugation. $\mathcal{S}_{p,q}$ is called *the canonical Clifford module*, cf. [Ebe18, Definition 4.2]. We obtain an injective $*$-homomorphism $c : \mathbf{Cl}^{p,q} \hookrightarrow \mathrm{End}(\mathcal{S}_{p,q})$ and define $\|\mathbf{a}\| := \|c(\mathbf{a})\|$ for $\mathbf{a} \in \mathbf{Cl}^{p,q}$. If $p = q$ then c is an isomorphism. The graded tensor product $\mathbf{Cl}^{p,q} \hat{\otimes} \mathbf{Cl}^{p',q'}$ of two Clifford algebras is identified with the Clifford algebra $\mathbf{Cl}^{p+p',q+q'}$ by means of the isomorphism induced by $v \hat{\otimes} 1 \mapsto (v, 0)$ and $1 \hat{\otimes} v' \mapsto (0, v')$ for $v \in \mathbb{R}^{p,q}$ and $v' \in \mathbb{R}^{p',q'}$.

Using Clifford algebras one defines the *graded* KK-groups as follows:

Definition 1.3.11. $\mathrm{KK}^{p,q}_{p',q'}(\mathbf{A}, \mathbf{B}) := \mathrm{KK}(\mathbf{A} \hat{\otimes} \mathbf{Cl}^{p',q'}, \mathbf{B} \hat{\otimes} \mathbf{Cl}^{p,q})$

For a fixed $d := (q - p) - (q' - p')$ all these groups are isomorphic. See [Kas80, Theorem 5.4]. Hence we define the higher KK-groups as follows:

Definition 1.3.12. $\mathrm{KK}^{p-q}(\mathbf{A}, \mathbf{B}) := \mathrm{KK}_{p-q,0}(\mathbf{A}, \mathbf{B})$, *if* $p \geqslant q$ *and* $\mathrm{KK}^{p-q}(\mathbf{A}, \mathbf{B}) := \mathrm{KK}_{0,q-p}(\mathbf{A}, \mathbf{B})$, *if* $q \geqslant p$.

Since the Kasparov product is compatible with the homomorphism $\tau_{\mathbf{D}}$, it also induces a bilinear pairing $\# : \mathrm{KK}^p(\mathbf{A}, \mathbf{B}_0) \times \mathrm{KK}^q(\mathbf{B}_0, \mathbf{B}) \to \mathrm{KK}^{p+q}(\mathbf{A}, \mathbf{B})$. Since the KK-groups are *stable*, c.f. [Kas80, Theorem 5.1], and $\mathbf{Cl}^{8,0} \cong \mathbf{Cl}^{0,8} \cong \mathbf{Cl}^{4,4} \cong \mathrm{Mat}_{16}(\mathbb{R})$ we obtain the *formal part* of the Bott periodicity.

Proposition 1.3.13 ([Kas80], Theorem 5.5)**.** $\mathrm{KK}^p(\mathbf{A},\mathbf{B}) \cong \mathrm{KK}^{p+8}(\mathbf{A},\mathbf{B})$.

1.3.5 KK-equivalence and Bott isomorphism

Let H be an infinite dimensional Real graded separable Hilbert space with scalar multiplication λ. Let F be a Real odd Fredholm operator on H whose index is equal to 1. Then $\mathbf{e} := (H,\lambda,F)$ is a Kasparov-$(\mathbf{R},\mathbf{R})$-module and for every $\mathbf{x} \in \mathrm{KK}(\mathbf{A},\mathbf{B})$ the following holds:

$$\tau_{\mathbf{A}}(\mathbf{e})\#\mathbf{x} = \mathbf{x} = \mathbf{x}\#\tau_{\mathbf{B}}(\mathbf{e})$$

See [Kas80, Theorem 4.5].

Definition 1.3.14. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be separable. They are called* KK*-equivalent, if there exists* $\mathbf{x} \in \mathrm{KK}(\mathbf{A},\mathbf{B})$ *and* $\mathbf{y} \in \mathrm{KK}(\mathbf{B},\mathbf{A})$ *such that*

$$\mathbf{x}\#\mathbf{y} = \tau_{\mathbf{A}}(\mathbf{e}) \text{ and } \mathbf{y}\#\mathbf{x} = \tau_{\mathbf{B}}(\mathbf{e}).$$

The next theorem is called Bott periodicity in [Kas80].

Theorem 1.3.15 ([Kas80], Theorem 5.7)**.** *The* C^**-algebras* $C_0(\mathbb{R}^{1,0})$ *and* $\mathbf{Cl}^{1,0}$ *are* KK*-equivalent.*

The theorem is equivalent to: There exists $\alpha \in \mathrm{KK}(C_0(\mathbb{R}^{1,0}),\mathbf{Cl}^{1,0})$ and $\beta \in \mathrm{KK}(\mathbf{Cl}^{1,0}, C_0(\mathbb{R}^{1,0}))$ such that

$$\alpha\#\beta = \tau_{C_0(\mathbb{R}^{1,0})}(\mathbf{e}) \text{ and } \beta\#\alpha = \tau_{\mathbf{Cl}^{1,0}}(\mathbf{e}).$$

The element $\alpha \in \mathrm{KK}(C_0(\mathbb{R}),\mathbf{Cl}^{1,0})$ is given by the triple $(L^2(\mathbb{R},\mathbf{Cl}^{1,0}),\mu,b(D_{\mathbb{R}}))$. The Hilbert-$\mathbf{Cl}^{1,0}$-module $L^2(\mathbb{R},\mathbf{Cl}^{1,0})$ is the completion of $C_c^\infty(\mathbb{R},\mathcal{S}_{1,1})$ and the action μ of $C_0(\mathbb{R})$ on $L^2(\mathbb{R},\mathbf{Cl}^{1,0})$ is given by multiplication. The operator $\mathbf{D}_{\mathbb{R}}$ is given by $D_{\mathbb{R}} := e_1 \cdot \partial_\lambda$. The Kasparov module β is sometimes called the *Bott class*. It is given by the tuple $(C_0(\mathbb{R},\mathbf{Cl}^{0,1}),\lambda,\lambda(x)(1+\|x\|^2)^{-\frac{1}{2}}) \in \mathrm{KK}(\mathbf{R}, C_0(\mathbb{R},\mathbf{Cl}^{0,1})) \cong \mathrm{KK}(\mathbf{Cl}^{1,0}, C_0(\mathbb{R}))$. The isomorphism is given by $\tau_{\mathbf{Cl}^{1,0}}$. Kasparov proved that $\tau_{\mathbf{Cl}^{1,0}}(\beta)\#\alpha = 1 \in \mathrm{KK}(\mathbf{R},\mathbf{R}) \cong \mathbb{Z}$ and that $\alpha\#\tau_{\mathbf{Cl}^{1,0}}(\beta) = [\mathrm{id}] \in \mathrm{KK}(C_0(\mathbb{R}), C_0(\mathbb{R}))$. See [Kas80, §5 Theorem 7].

Chapter 2

Bundles and operators

The generalization of the theory of ordinary differential operators to the case of $\mathbf{A}$-linear differential operators based substantial on the work of Miščenko and Fomenko in [MF79]. They behave quite similar as ordinary differential operators, but have to be treated with care. They act on $\mathbf{A}$-bundle, which are introduced in section 2.1. At the end of this section we define real K-theory of $\mathbf{A}$-bundle in a similar way one defines real K-theory for vector bundles. In section 2.2 we present the basic definitions concerning $\mathbf{A}$-linear differential operators. Finally the concept of a family of operators in introduced. This will be important for the definition of the index difference.

2.1 Bundles of Hilbert Modules

Definition 2.1.1. *Let P be a Hilbert-$\mathbf{A}$-module. A bundle of Hilbert-$\mathbf{A}$-modules over a locally compact Hausdorff space M is a fiber bundle $E \to M$ with fiber P and structure group $\mathrm{Aut}_{\mathbf{A}}(P)$ the $\mathbf{A}$-linear automorphisms of P.*

When P is a finitely generated projective Hilbert-$\mathbf{A}$-module, E is called a bundle of finitely generated projective Hilbert-$\mathbf{A}$-modules. We will abbreviate the term *bundle of finitely generated projective* $\mathbf{A}$*-modules* by $\mathbf{A}$*-bundle*. A *graded* $\mathbf{A}$-bundle is a fiber bundle $E \to M$ with *graded* fiber P and structure group the *even* $\mathbf{A}$-linear automorphisms of P. One defines *Real* $\mathbf{A}$-bundles similarly.

If the base space M has the structure of a smooth manifold, a smooth structure on $E \to M$ is an atlas of local trivializations such that all transition functions $\phi_1 \circ \phi_0^{-1} \colon U_0 \cap U_1 \to \mathrm{Aut}_{\mathbf{A}}(P)$ are smooth. This makes sense since the group $\mathrm{Aut}_{\mathbf{A}}(P)$ is a Banach Lie group because it is an open subgroup of the C^*-algebra $\mathbf{Lin}_{\mathbf{A}}(P)$. For a compact smooth manifold M there always exists a unique (up to isomorphism) smooth structure on an $\mathbf{A}$-bundle, see [Sch05, Theorem 3.14(6)].

Definition 2.1.2. *The space of smooth sections of an* **A***-bundle* E *over a smooth manifold* M *is denoted by* $\Gamma^\infty(M, E)$.

If it is clear from context which base space we consider, we sometimes omit it in the notation and simply write $\Gamma^\infty(E)$ instead of $\Gamma^\infty(M, E)$. With $\Gamma_c^\infty(M, E)$ we denote the space of compactly supported smooth sections and with $\Gamma_0^\infty(M, E)$ those sections vanish at infinity. The space of smooth sections $\Gamma^\infty(E)$ has the natural structure of a (right) **A**-module. Moreover every **A**-bundle $E \to M$ over a smooth manifold admits a smooth fiberwise inner product with values in **A** using the **A**-valued inner product $(\cdot, \cdot)$ of the fiber P. Therefore for two sections $s_0, s_1 \in \Gamma^\infty(M, E)$ we obtain a smooth function

$$(s_0, s_1) \colon M \to \mathbf{A} \text{ , } p \mapsto (s_0, s_1)(p) := (s_0(p), s_1(p))$$

If M is a Riemannian manifold and $s_0, s_1 \in \Gamma_c^\infty(E)$, we can integrate the function (s_0, s_1) with respect to the volume form vol(M) on M and obtain an **A**-valued inner product on $\Gamma_c^\infty(M, E)$ as follows:

$$\langle s_0, s_1 \rangle := \int_M (s_0, s_1) d\mathrm{vol}(M). \tag{2.1}$$

Endowed with this inner product the space of compactly supported section has the structure of a pre-Hilbert-**A**-module.

Definition 2.1.3. *We define the Hilbert-***A***-module* $L^2(M, E)$ *as the completion of the space of compactly supported smooth sections* $\Gamma_c^\infty(M, E)$ *with respect to the norm induced by this inner product.*

2.1.1 Connections

The following can be found in [Scho5, Chapter 4]. Let $M \times \mathbf{A}$ be the trivial **A**-bundle over M. The differential $d\varphi \in \Gamma^\infty(T^*M \otimes (M \times \mathbf{A}))$ of $\varphi \in \Gamma^\infty(M \times \mathbf{A}) = C^\infty(M, \mathbf{A})$ is given locally by $d\varphi = \sum dx_i \otimes \frac{\partial \varphi}{\partial x_i}$.

Definition 2.1.4. *Let* $E \to M$ *be a smooth* **A***-bundle over a smooth manifold* M. *An* **A***-linear map*

$$\nabla \colon \Gamma^\infty(E) \longrightarrow \Gamma^\infty(T^*M \otimes E)$$

is called a connection, if $\nabla(s \cdot \varphi) = s \cdot d\varphi + \nabla(s) \cdot \varphi$ *holds for every* $s \in \Gamma^\infty(E)$ *and* $\varphi \in C^\infty(M, \mathbf{A})$.

Lemma 2.1.5 ([Scho5] Lemma 4.12)**.** *Assume that* V *is a smooth finite dimensional complex vector bundle over* M *endowed with a bundle metric and let* E *be a smooth* **A**-

bundle both equipped with connections ∇ *and* ∇_E*, respectively. Their tensor product (over* $\mathbb{C}$*) is again an* **A***-bundle with connection*

$$\nabla_{\otimes} := \nabla \otimes 1 + 1 \otimes \nabla_E$$

The curvature $\Omega_{\otimes}$ *of* $\nabla_{\otimes}$*, which is by definition* $\nabla_{\otimes} \circ \nabla_{\otimes}$*, is equal to*

$$\Omega_V \otimes 1 + 1 \otimes \Omega_E$$

where Ω_V *and* Ω_E *are the curvature of* V *and* E *respectively.*

Proofs and further properties of **A**-bundles can also be found in [Kar71] or [MS77].

2.1.2 KO-theory of A-bundles

Definition 2.1.6. *Let* M *be a compact Hausdorff space and* $\mathbf{A}$ *a Real graded* C^**-algebra. Then* $KO(M, \mathbf{A})$ *is defined as the Grothendieck group of isomorphism classes of Real graded* **A***-bundles over* M*.*

The group $KO(M, \mathbf{A})$ can be identified with the real K-theory $KO_0(C(M, \mathbf{A}))$ of the Real trivial graded C^*-algebra $C(M, \mathbf{A})$ in degree zero, see [Sch05, Proposition 3.17]. Using the identification $KO_0(C(M, \mathbf{A})) \cong KK(\mathbf{R}, C(M, \mathbf{A}))$ one obtains that $KO(M, \mathbf{A}) \cong KK(\mathbf{R}, C(M, \mathbf{A}))$. With the aid of this isomorphism we define the higher real K-theory groups by $KO^{-d}(M, \mathbf{A}) := KK(\mathbf{Cl}^{0,d}, C(M, \mathbf{A}))$. By Bott periodicity the latter groups are isomorphic to $KK(\mathbf{R}, C_0(M \times \mathbb{R}^d, \mathbf{A}))$, which is a more common description of the groups $KO^{-d}(M, \mathbf{A})$. See again [Sch05, Definition 3.20].

We finish this section with an important example: **The Miščenko-Fomenko line bundle**. Let G be a countable discrete group with classifying space BG. The universal cover EG $\to$ BG is a principal G-bundle with contractible total space. Since G acts on the reduced C^*-algebra C_r^*G by unitaries, see 1.1, we define the Miščenko-Fomenko line bundle

$$\mathcal{L}_G := EG \times_{\rho} C_r^*G \to BG \tag{2.2}$$

using the Borel construction. The universal Miščenko-Fomenko line bundle is a bundle of free (rank one) Real trivial graded Hilbert-C_r^*G-modules over BG. The C_r^*G-valued inner product is given by the formula $(\mathbf{a}, \mathbf{b}) := \mathbf{a}^*\mathbf{b}$. Since G acts by unitaries this inner product is invariant under left-multiplication by elements of

G. When M is a closed manifold with a map $\varphi : M \to BG$ the we can pullback $\mathcal{L}_G \to BG$ along φ.

Definition 2.1.7. *We define the Miščenko-Fomenko line bundle of the map* φ *by*

$$\mathcal{L}_\varphi \coloneqq \varphi^*(\mathcal{L}_G) \to M \tag{2.3}$$

The bundle $\mathcal{L}_\varphi$ is a C_r^*G-bundle over M defining a class in $KO(M, C_r^*G)$ given by the Kasparov module $(\Gamma^\infty(M, \mathcal{L}_\varphi), \_, 0) \in KK(\mathbf{R}, C(M, C_r^*G))$. By construction $\mathcal{L}_\varphi$ can be equipped with a flat connection.

2.2 A-linear differential operators

Next we will consider differential operators on a **A**-bundle E over a smooth Riemannian manifold M of dimension d. For us it suffices to consider only differential operators of order one. For the general theory of **A**-linear differential operators see [MF79].

Definition 2.2.1. *Let* M *be a smooth manifold and* $E \to M$ *a smooth* **A***-bundle. An* **A***-linear differential operator of order one is an* **A***-linear map* $D\colon \Gamma_c^\infty(M, E) \to \Gamma_c^\infty(M, E)$ *such that for every chart* $x = (x_1, \ldots, x_d)\colon U \to \mathbb{R}^d$ *and every local trivialization* $\phi\colon E_{|U} \to U \times P$ *there exist smooth functions* $a_1, \ldots, a_d, b\colon U \to \mathbf{Lin_A}(P)$ *such that the operator is given with respect to these coordinates by*

$$(Ds)(p) = \sum_{i=1}^{d} a_i(p)\partial_{x_i} s(p) + b(p)s(p).$$

Since for our purpose it suffices to consider only **A**-linear differential operators of order one let us make the convention that the term **A**-linear differential operator will always mean **A**-linear differential operator of order one. When we consider D as an operator on the pre-Hilbert-**A**-module $\Gamma_c^\infty(E)$ with the inner product 2.1, then we say that D is formally self-adjoint, if

$$\langle Ds_0, s_1\rangle = \langle s_0, Ds_1\rangle \tag{2.4}$$

holds for all $s_i \in \Gamma_c^\infty(E)$. A grading η of E induces a grading η of $\Gamma_c^\infty(E)$ and D is called *odd*, if D anticommutes with the grading, i.e. if $D\eta + \eta D = 0$. When there exists a Real structure on the bundle we say that the operator D is *Real* if $\kappa(Ds) = D\kappa(s)$. The *symbol* $\sigma(D)$ of D is defined as the map

$$\sigma(D)\colon T^*M \otimes E \to E\ ,\ \sigma(D)(p, \xi)(s(p)) \coloneqq i[D, f]s(p), \tag{2.5}$$

where f is a function such that $d_p f = \xi$ and $s \in \Gamma^\infty(M, E)$. This only depends on ξ and $s(p)$. In particular it does not depend on the specific choice of f. A formally self-adjoint differential operator whose symbol $\sigma(D)$ satisfies $\sigma(D)(p, \xi)^2 = -\|\xi\|^2$ for all $p \in M$ and $\xi \in T_p^*M$ is called a *Dirac operator*. A Dirac operator is a special case of the larger class of *elliptic operators*. These are differential operators whose symbols are invertible for all non-zero cotangent vectors. These are all easy generalizations of the theory of ordinary differential operators found in [HJ00], [LML16] or [Roe99].

2.2.1 Differential operators as unbounded A-linear operators

The proof that every ordinary symmetric first order differential operator on a manifold M is essentially self-adjoint as long as M is complete for D (see 10.2.10 of [HJ00]) has a generalization to the case of **A**-linear differential operators. In [Ebe18] Ebert gave sufficient conditions under which the closure of $D\colon \Gamma_c^\infty(E) \to \Gamma_c^\infty(E)$ defines an unbounded self-adjoint and regular operator $\overline{D}\colon \overline{\mathrm{dom}(D)} \to L^2(M, E)$. In the case of twisted operators Dirac operators Hanke, Pape and Schick [HPS14] gave the proof of a similar result due to Vassout. Let us recall the definitions from [Ebe18]

Definition 2.2.2. *A proper smooth function $h\colon M \to \mathbb{R}$ bounded from below is called a coercive function. The pair (M, D) is called complete if there exists a coercive function h on M such that $[D, h]$ is bounded.*

Theorem 2.2.3 ([Ebe18],Theorem 1.14)**.** *If (M, D) is complete, then the closure of $D\colon \mathrm{dom}(D) \to L^2(M, E)$ is self-adjoint and regular.*

2.2.2 Families of A-linear operators

Now we introduce the concept of families of **A**-linear differential operators. The set up is the following:

Let $\pi\colon M \to X$ be a submersion and $E \to M$ a smooth **A**-bundle on M. Since M is smooth E carries a fiberwise smooth **A**-valued inner product $(\cdot, \cdot)$. The *vertical tangent bundle* $T_v\pi \to E$ of π is by definition the kernel of the differential $d\pi\colon TM \to TX$. If $f\colon M \to \mathbb{R}$ is a smooth function with differential $df\colon TM \to \mathbb{R}$, then the restriction $d_v f$ of df to $T_v\pi$ is called the *fiberwise differential of* f.

By definition a *fiberwise Riemannian metric* on M is a smooth bundle metric on $T_v\pi$. In particular a fiberwise Riemannian metric endows each fiber $M_x := \pi^{-1}(x)$ of the submersion π with the structure of a smooth Riemannian manifold. We denote the restriction of E to the submanifold M_x by E_x and the space of smooth sections with compact support of the bundle $E_x \to M_x$ by $\Gamma_c^\infty(M_x, E_x)$. Then $\Gamma_c^\infty(M_x, E_x)$ becomes a pre-Hilbert-**A**-module. The **A**-valued inner product is given by

$$\langle s_0, s_1 \rangle_x := \int_{M_x} (s_0(p), s_1(p)) dvol(M_x),$$

cf. eq. (2.1). Let $D\colon \Gamma_c^\infty(M, E) \to \Gamma_c^\infty(M, E)$ be an **A**-linear differential operator. Then D defines a *family of* ***A****-linear differential operators* if and only if $[D, f \circ \pi] = 0$ for all smooth functions $f\colon X \to \mathbb{R}$. In this case the restriction $Ds_{|M_x}$ of $Ds \in \Gamma_c^\infty(M, E)$ to the submanifold M_x only depends on $s_{|M_x} \in \Gamma^\infty(M_x, E_x)$. Hence for every $x \in X$ we obtain differential operators D_x on $\Gamma_c^\infty(M_x, E_x)$ and the family $(D_x)_{x \in X}$ determines D uniquely.

The family of **A**-linear differential operators is called formally self-adjoint if D_x is formally self-adjoint for every $x \in X$ with respect to the inner product or equivalently if $\langle Ds_0, s_1 \rangle = \langle s_0, Ds_1 \rangle$ holds for all $s_i \in \Gamma_c^\infty(M, E)$. Using the fiberwise differential of a smooth function $f\colon M \to \mathbb{R}$ we can define the fiberwise symbol $\sigma(D)$ of the family by $\sigma(D)(d_v f) := i[D, f]$.

The family is called a Dirac family if the fiberwise symbol satisfies $\sigma(D)(d_v f)^2 = -\|d_v f\|^2$. Hence a family is a Dirac family if and only if each operator D_x is a Dirac operator.

Chapter 3

Continuous fields

In this chapter we will briefly describe a variation of K-theory based on *continuous fields of Hilbert modules* developed in [Ebe18]. The goal is to give another description of the KK-group $KK(\mathbf{Cl}^{0,d}, C(X, \mathbf{A}))$. As an application we show how to assign an unbounded Kasparov module to a submersion $\pi\colon M \to X$, an $\mathbf{A}$-bundle $E \to M$ and an $\mathbf{A}$-linear differential operator $D\colon \Gamma^{\infty}_{cv}(E) \to \Gamma^{\infty}_{cv}(E)$. We will be very brief and refer to [Ebe18] for a detailed treatment of the category of continuous fields.

3.1 Continuous Fields

We start with the definition of a continuous field of Banach spaces first considered by Dixmier and Douady in [DD63]. For the rest of the chapter let X be a compact Hausdorff space.

Definition 3.1.1. *A continuous field* $(\mathcal{E}, \Gamma)$ *of Banach spaces over* X *is a family* $\mathcal{E} := (E_x)_{x\in X}$ *of Banach spaces over* X *together with a subspace* $\Gamma \subset \prod_{x\in X} E_x$ *satisfying*

1. Γ *is a* $C(X)$*-submodule of* $\prod_{x\in X} E_x$*,*
2. $\forall \xi_x \in E_x \; \exists s \in \Gamma$ *such that* $s_x = \xi_x$*,*
3. $\forall s \in \Gamma$ *the map* $x \mapsto \|s_x\|$ *is continuous and*
4. *If* $t \in \prod_{x\in X} E_x$ *satisfies that for each* $x \in X$ *and for every* $\varepsilon > 0$ *there exists a neighborhood* $U \subset X$ *of* x *and* $s \in \Gamma$ *such that* $\sup_{y\in U} \|t_y - s_y\| \leqslant \varepsilon$*, then* $t \in \Gamma$*.*

The submodule Γ *is called the space of continuous sections of the continuous field and we can regard the elements* $s \in \Gamma$ *as a functions* $X \to \prod_{x\in X} E_x$ *,* $x \mapsto s(x) := s_x$*.*

We will exclusively consider continuous fields of *Hilbert modules* over X. These are given by a Banach field $(\mathcal{E}, \Gamma)$ over X together with a compatible Hilbert module structure on every E_x:

Definition 3.1.2. *A continuous field of Hilbert-$\mathbf{A}$-modules over X is a continuous field of Banach spaces $(\mathcal{E}, \Gamma)$ over X endowed with an $\mathbf{A}$-valued inner product $(\cdot,\cdot)_x$ and a right $\mathbf{A}$-module structure μ_x on each E_x such that:*

1. *$(\cdot,\cdot)_x$ induces the norm on E_x,*
2. *$(E_x, \mu_x, (\cdot,\cdot)_x)$ is a Hilbert-$\mathbf{A}$-module,*
3. *Γ is a right $C(X,\mathbf{A})$-module with respect to the action given by $(s \cdot f)(x) := \mu_x(s(x), f(x))$ for $f \in C(X,\mathbf{A})$ and $s \in \Gamma$ and*
4. *for all $s, t \in \Gamma$ the function $x \mapsto (s(x), t(x))_x$ is in $C(X,\mathbf{A})$.*

Morphisms between continuous fields of Hilbert modules are called *bounded operator families*. They are defined as follows:

Definition 3.1.3. *A bounded operator family $\mathcal{T}\colon (\mathcal{E}, \Gamma) \to (\mathcal{E}, \Gamma)$ is a family $\mathcal{T} := (T_x)_{x\in X}$ of adjointable maps $T_x \in \mathbf{Lin}_{\mathbf{A}}(E_x)$ such that*

1. *the map $x \mapsto \|T_x\|$ is locally bounded,*
2. *$\mathcal{T}(\Gamma) \subset \Gamma$ and*
3. *the family of adjoints $\mathcal{T}^* := (T_x^*)_{x\in X}$ satisfies also 1 and 2.*

The family is self-adjoint if $\mathcal{T} = \mathcal{T}^*$ holds, i. e. if T_x is self-adjoint for every $x \in X$.

The vector space $\mathbf{Lin}_{X,\mathbf{A}}(\mathcal{E}, \Gamma)$ of bounded operator families on $(\mathcal{E}, \Gamma)$ is a $*$-algebra and equipped with the norm $\|\mathcal{T}\| := \sup_{x\in X} \|T_x\|$ it becomes a C^*-algebra.

Lemma 3.1.4. *The space of continuous sections Γ of a continuous field $(\mathcal{E}, \Gamma)$ of Hilbert-$\mathbf{A}$-modules over a compact Hausdorff space X is a Hilbert-$C(X,\mathbf{A})$-module.*

Proof. Γ is by definition a right $C(X,\mathbf{A})$-module with a $C(X,\mathbf{A})$-valued inner product. Since X is compact we can define the norm of $s \in \Gamma$ by $\|s\| := \sup_{x\in X} \|s(x)\|$. Now it is only left to show that Γ is complete with respect to this norm. But this follows from the fourth axiom for continuous fields of Banach spaces. □

It is also proven in [Ebe18] that every Hilbert-$C(X,\mathbf{A})$-module can be realized as the space of continuous sections of a continuous field of Hilbert-$\mathbf{A}$-modules over a compact Hausdorff space X. Moreover a bounded operator family $\mathcal{T}\colon (\mathcal{E}, \Gamma) \to (\mathcal{E}, \Gamma)$ induces a bounded operator, denoted by T, on the Hilbert-$C(X,\mathbf{A})$-module Γ. In particular this defines an isomorphism [Ebe18, Lemma 3.21]

$$\mathbf{Lin}_{X,\mathbf{A}}(\mathcal{E}, \Gamma) \xrightarrow{\cong} \mathbf{Lin}_{C(X,\mathbf{A})}(\Gamma) \tag{3.1}$$

between the bounded operator families on the continuous field $(\mathcal{E}, \Gamma)$ and the adjointable operators on the Hilbert module Γ. In fact the category of continuous fields of Hilbert-$\mathbf{A}$-modules over X and bounded operator families is equivalent to the category of Hilbert-$C(X, \mathbf{A})$-modules and adjointable operators. Using this equivalence of categories we say that a bounded operator family $\mathcal{T} \in \mathbf{Lin}_{X,\mathbf{A}}(\mathcal{E}, \Gamma)$ is *compact* if and only if the induced operator $T \in \mathbf{Lin}_{C(X,\mathbf{A})}(\Gamma)$ on the Hilbert-$C(X, \mathbf{A})$-module Γ is compact. Similarly a bounded operator family $\mathcal{F} \in \mathbf{Lin}_{X,\mathbf{A}}(\mathcal{E}, \Gamma)$ is *Fredholm* precisely when $F \in \mathbf{Lin}_{C(X,\mathbf{A})}(\Gamma)$ is invertible modulo compact operators. A *grading* on $(\mathcal{E}, \Gamma)$ is given by a family $(\eta_x)_{x\in X}$ of gradings of the Hilbert modules E_x such that the direct sum decomposition $\Gamma^{(0)} \oplus \Gamma^{(1)}$ of the Hilbert-$C(X, \mathbf{A})$-module Γ defines a grading on Γ, where

$$\Gamma^{(i)} \coloneqq \{s \in \Gamma \mid \ \forall x \in X\colon \ \eta_x(s(x)) = (-1)^i s(x)\}.$$

One defines a *Real structure* on $(\mathcal{E}, \Gamma)$ analogously. When $(\mathcal{E}, \Gamma)$ is Real and graded we require that the grading and the Real structure are compatible in the sense that they commute. A family $\mathcal{T} \in \mathbf{Lin}_{X,\mathbf{A}}(\mathcal{E}, \Gamma)$ is *odd* if T is odd and *even* if T is even. Furthermore $\mathcal{T}$ is *Real* if T commutes with the Real structure.

3.2 Application

For the rest of this section let X be a smooth compact connected manifold possibly with boundary and let $\pi\colon M \to X$ be a submersion with d-dimensional fibers endowed with a fiberwise Riemannian metric g. Furthermore let $E \to M$ be a smooth $\mathbf{A}$-bundle over M with fiber P. We will use the same notation as in section 2.2.2. The space $\Gamma_c^\infty(M_x, E_x)$ of the compactly supported smooth sections of the $\mathbf{A}$-bundle $E_x \to M_x$ endowed with the inner product defined in eq. (2.1) and denoted with $\langle\cdot,\cdot\rangle_x$ is a pre-Hilbert-$\mathbf{A}$-module.

Definition 3.2.1. *A section $s \in \Gamma^\infty(M, E)$ has compact vertical support if the restriction $\pi_{|\mathrm{supp}(s)}\colon \mathrm{supp}(s) \to X$ of π to the support of s is a proper map.*

Therefore the space $\Gamma_{cv}^\infty(M, E)$ of smooth sections of the bundle $E \to M$ with compact vertical support is a subset of the direct product $\prod_{x\in X} \Gamma_c^\infty(M_x, E_x)$. If $s \in \Gamma_{cv}^\infty(M, E)$, then $s_x \coloneqq s_{|M_x} \in \Gamma_c^\infty(M_x, E_x)$ denotes its restriction to M_x. We make the following two simple observations.

Lemma 3.2.2. *For every compactly supported section $s \in \Gamma_c^\infty(M_x, E_x)$ there exists a section $t \in \Gamma_{cv}^\infty(M, E)$ such that $t_{|M_x} = s$.*

Proof. By the inverse mapping theorem we can cover M by "box neighborhoods" $\mathbb{R}^d \times \mathbb{R}^k$. On a box neighborhood, π is the projection to the second factor and E

is trivial. Suppose that s is supported in a box neighborhood. Then s is given by a function $\mathbb{R}^d \times \{x\} \subset \mathbb{R}^d \times \mathbb{R}^k \to P$. Let μ be a bump function on $\mathbb{R}^k$ such that $\mu(x) = 1$ and define $t : \mathbb{R}^d \times \mathbb{R}^k \to P$ by $(p, q) \mapsto s(p)\mu(q)$. If s is not supported in a box neighborhood we can cover its support by finitely many open sets such that these open sets are contained in box neighborhoods. As before we get sections t_i of E. Using a partition of unity we obtain a smooth section t of E with compact vertical support such that $t_{|M_x} = s$. □

Lemma 3.2.3. *For every* $s, t \in \Gamma^\infty_{cv}(M, E)$ *the map* $x \mapsto \langle s, t\rangle(x) := \langle s_x, t_x\rangle_x$ *is smooth.*

Proof. Because smoothness is a local property we can assume that both s and t have compact support in a box neighborhood $\mathbb{R}^d \times \mathbb{R}^k$. Over a box neighborhood the bundle E is trivial with fiber P and the volume measure on M_x is given by $b(x, y)dx$, with b a smooth function. Since $p \mapsto (s_y(p), t_y(p))$ is smooth and bounded in norm from above we can use the dominated convergence theorem to prove that

$$y \mapsto \langle s, t\rangle(y) = \int_{\mathbb{R}^d} (s_y, t_y) b(x, y)dx.$$

is a smooth map $\mathbb{R}^k \to P$. □

Therefore the pair $((\Gamma^\infty_c(M_x, E_x))_{x\in X}, \Gamma^\infty_{cv}(M, E))$ is called a *pre-field of Hilbert modules* over X, cf. [Ebe18]. In loc. cit. Lemma 3.7 it is proven how to complete a pre-field to obtain a continuous field of Hilbert modules over X. This completion is unique (up to isomorphism). It is given by the pair $((L^2(M_x, E_x))_{x\in X}, \Gamma)$. The Hilbert modules $L^2(M_x, E_x)$ are the completions of $\Gamma^\infty_c(M_x, E_x)$ with respect to the inner products $\langle\cdot, \cdot\rangle_x$. The space Γ of continuous sections is the subspace of all $s \in \prod_{x\in X} L^2(M_x, E_x)$ such that for each $x \in X$ and every $\varepsilon > 0$ there exists a neighborhood U of x and $t \in \Gamma^\infty_{cv}(M, E)$ such that $\sup_{y\in U} \|s_y - t_y\| \leqslant \varepsilon$. In particular $\Gamma^\infty_{cv}(M, E) \subset \Gamma$.

Definition 3.2.4. $L^2_X(M, E) := ((L^2(M_x, E_x))_{x\in X}, \Gamma)$ *denotes the continuous field of Hilbert-**A**-modules over* X *obtain as the completion of* $((\Gamma^\infty_c(M_x, E_x))_{x\in X}, \Gamma^\infty_{cv}(M, E))$.

A grading on the bundle $E \to M$ induces a gradings on $\Gamma^\infty_c(M_x, E_x)$ and on $\Gamma^\infty_{cv}(M, E)$ respectively. This is also true for a Real structure. Therefore a Real graded **A**-bundle E defines a Real graded continuous field $L^2_X(M, E)$ over X.

An **A**-linear differential operator $D\colon \Gamma^\infty_{cv}(M, E) \to \Gamma^\infty_{cv}(M, E)$ induces a family of **A**-linear differential operators over X, cf. section 2.2, i. e. a family $(D_x)_{x\in X}$ of **A**-linear differential operators D_x on $\Gamma^\infty_c(M_x, E_x)$. Therefore the family D induces, by definition, cf. [Ebe18, Definition 3.22], a *densely defined unbounded operator family*

$$\mathcal{D}\colon \operatorname{dom}(\mathcal{D})_X \longrightarrow L^2_X(M,E)$$

with domain $\operatorname{dom}(\mathcal{D})_X := ((\Gamma^\infty_c(M_x,E_x))_{x\in X}, \Gamma^\infty_{cv}(M,E))$ on the continuous field $L^2_X(M,E)$. The unbounded densely defined operator family is *symmetric* if the family of operators is formally self-adjoint, i. e. if each D_x is formally self-adjoint. We will only consider symmetric operator families. As in the case of unbounded operators on Hilbert modules one can introduce the *graph scalar product* on $\operatorname{dom}(\mathcal{D})_X$: For $s,t \in \Gamma^\infty_{cv}(M,E)$ we define

$$\langle s,t\rangle_{G(D_x)} := \langle s,t\rangle_x + \langle D_x s, D_x t\rangle_x.$$

Endowed with the graph scalar product, $((\Gamma^\infty_c(M_x,E_x))_{x\in X}, \Gamma^\infty_{cv}(M,E))$ is also field of pre-Hilbert-**A**-modules. The *closure* $\overline{\mathcal{D}}$ of the unbounded operator family $\mathcal{D}$ is by definition the unbounded operator family given by $(\overline{D}_x)_{x\in X}$ with domain $\overline{\operatorname{dom}(\mathcal{D})}_X$. The operators $\overline{D}_x\colon \overline{\Gamma^\infty_c(M_x,E_x)} \to L^2(M_x,E_x)$ are the closures of the densely defined unbounded symmetric operators D_x on the Hilbert-**A**-modules $L^2(M_x,E_x)$.

Definition 3.2.5. *The closed densely defined symmetric unbounded operator family $\overline{\mathcal{D}}$ is called self-adjoint and regular if the individual operators $\overline{D}_x$ are self-adjoint and regular.*

Remark. Note that it is not always the case that a property of an operator family $\mathcal{D} = (D_x)_{x\in X}$ is determined by the properties of the individual operators D_x. For example a family $K = (K_x)_{x\in X}$ of compact operators K_x does not have to be a compact operator family.

However to prove self-adjointness and regularity of the closed densely defined symmetric unbounded operator family $\mathcal{D} = (D_x)_{x\in X}$ it suffices to prove this for every operator D_x. This will be ensured by the existence of a *fiberwise coercive function* h, i. e. a smooth function $h\colon M \to \mathbb{R}$ bounded from below such that the map $M \to X\times\mathbb{R}$, $p \mapsto (\pi(p), h(p))$ is proper and the commutator $[\mathcal{D}, h]$ is locally bounded in norm.

Definition 3.2.6. *We say that $(M,\mathcal{D})$ is fiberwise complete if there exists a fiberwise coercive function $h\colon M \to \mathbb{R}$.*

As long as X is compact the boundedness of the commutator is no restriction at all. When h is a fiberwise coercive function then (M_x, D_x) is complete, cf. definition 2.2.2. Therefore the closure of each D_x is self-adjoint and regular by theorem 2.2.3. This proves

Lemma 3.2.7 ([Ebe18] 3.28)**.** *Suppose that* $(M, \mathcal{D})$ *is fiberwise complete. Then the closure of the unbounded operator family* $\mathcal{D}\colon \operatorname{dom}(D)_X \to L^2_X(M, E)$ *is self-adjoint and regular.*

For a self-adjoint and regular operator family $\mathcal{D} = (D_x)_{x\in X}$ there exists a *functional calculus* by applying the functional calculus for self-adjoint and regular operators on Hilbert modules fiberwise, i. e. for $f \in C(\overline{\mathbb{R}})$ we define $f(\mathcal{D}) := (f(D_x))_{x\in X}$. Since $\|f(D_x)\| \leqslant \|f\|$ the operator family $f(\mathcal{D})$ is *bounded* and induces a bounded operator on the Hilbert-$C(X, \mathbf{A})$-module Γ. We will adapt the notation from the ordinary functional calculus, e.g. $(\mathcal{D}^2 + 1)^{-1}$ denotes the bounded operator family $r(\mathcal{D}) := (r(D_x))_{x\in X}$ with $r(\lambda) := (\lambda^2 + 1)^{-1}$.

We already mentioned that compactness of a bounded operator family is not determined by the individual operators, which makes it difficult to prove that a given family is compact. Yet sometimes there are additional assumptions about the operator D inducing the operator family $\mathcal{D}$ which ensure that $f(\mathcal{D})$ is compact. This is the case when $D\colon \Gamma^\infty_{cv}(M, E) \to \Gamma^\infty_{cv}(M, E)$ is *elliptic* and the induced operator family $\mathcal{D}$ is bounded from below by a fiberwise coercive function:

Definition 3.2.8. *A fiberwise coercive function* $h\colon M \to \mathbb{R}$ *is a lower bound for the operator family* $\mathcal{D} = (D_x)_{x\in X}$ *on* $L^2_X(M, E)$*, if*

$$\langle \mathcal{D}s, t\rangle_x \geqslant \langle hs, t\rangle_x \tag{3.2}$$

holds for every $x \in X$ *and all* $s, t \in \Gamma^\infty_{cv}(M, E)$*. In that case we will write* $\mathcal{D} \geqslant h$*.*

Theorem 3.2.9 ([Ebe18], Theorem 4.40)**.** *Let* $\mathcal{D}\colon \operatorname{dom}(\mathcal{D})_X \to L^2_X(M, E)$ *be an elliptic self-adjoint and regular (unbounded) operator family and* h *a fiberwise coercive function such that* $\mathcal{D}^2 \geqslant h$*. Then the bounded operator family* $(\mathcal{D}^2 + 1)^{-1}$ *is compact.*

The next theorem summarizes what we achieved so far.

Theorem 3.2.10. *Let* $\pi\colon M \to X$ *and* $E \to M$ *be as above. Suppose that* E *is endowed with a Real structure and a grading and that the* $\mathbf{A}$*-linear differential operator* $D\colon \Gamma^\infty_{cv}(M, E) \to \Gamma^\infty_{cv}(M, E)$ *is elliptic. Furthermore let* $h\colon M \to \mathbb{R}$ *be a fiberwise coercive function such that* $\mathcal{D}^2 \geqslant h$*. Then we obtain an unbounded Kasparov-*$(\mathbf{R}, C(X, \mathbf{A}))$*-module* $(\Gamma, \lambda\cdot, \overline{D})$*.*

Proof. An unbounded Kasparov-$(\mathbf{R}, C(X, \mathbf{A}))$-module is given by definition by a Real graded Hilbert-$C(X, \mathbf{A})$-module Γ, a Real graded $*$-homomorphism $\_\colon \mathbf{R} \to \mathbf{Lin}_{C(X,\mathbf{A})}(\Gamma)$ and a Real odd self-adjoint and regular operator with compact resolvent. The Hilbert-$C(X, \mathbf{A})$-module Γ is the space of continuous sections of the continuous field $L^2_X(M, E)$ obtained from $((\Gamma^\infty_c(M_x, E_x))_{x\in X}, \Gamma^\infty_{cv}(M, E))$. The $*$-homomorphism $\lambda\cdot$ is given by scalar multiplication. The elliptic $\mathbf{A}$-linear differential operator D induces a family of operator $(D_x)_{x\in X}$ over X and hence an unbounded operator family $\mathcal{D}$ with domain $\mathrm{dom}_X(\mathcal{D})$. Since X is compact $[D, h]$ is bounded in norm and therefore the closure $\overline{\mathcal{D}}$ of $\mathcal{D}$ is self-adjoint and regular. By assumption $\mathcal{D}^2 \geqslant h$. Hence theorem 3.2.9 implies that the resolvent of $\overline{\mathcal{D}}$ is compact. Therefore the induced operator $\overline{D}$ on the Hilbert-$C(X, \mathbf{A})$-module Γ is self-adjoint and regular and has compact resolvent. Since the operator is linear it commutes with the scalar multiplication. □

3.2.1 Extension by zero

To define the index difference of Hitchin we have to consider a submersion $\pi\colon M \to \mathbb{R} \times X$ over the non-compact manifold $\mathbb{R} \times X$. Let $E \to M$ and $D : \Gamma^\infty_{cv}(M, E) \to \Gamma^\infty_{cv}(M, E)$ as above. We obtain a continuous field $L^2_{\mathbb{R}\times X}(M, E) := (\mathcal{E}, \Gamma)$ of Hilbert-$\mathbf{A}$-modules over $\mathbb{R} \times X$ together with an unbounded operator family $\mathcal{D}$ given by $(D_{\lambda,x})_{(\lambda,x)\in\mathbb{R}\times X}$. However the space Γ of continuous sections of $L^2_{\mathbb{R}\times X}(M, E)$ is not a Hilbert module since $\|s\| := \sup_{(\lambda,x)\in\mathbb{R}\times X} \|s_{\lambda,x}\|$ can be infinite. We claim that the subspace $\Gamma_0 \subset \Gamma$ of all elements vanishing at infinity is a Hilbert-$C_0(\mathbb{R} \times X)$-module. This can be proven as follows: Let $(\mathbb{R} \times X)^+ := (\mathbb{R} \times X) \cup \{\infty\}$ be the one-point compactification of $\mathbb{R} \times X$. Extending the continuous field $L^2_{\mathbb{R}\times X}(M, E)$ by zero at the point at infinity defines a continuous field $(\mathcal{E}^+, \Gamma^+)$ of Hilbert-$\mathbf{A}$-modules over $(\mathbb{R} \times X)^+$. It is given by the following family over $(\mathbb{R} \times X)^+$

$$H_{\lambda,x} := \begin{cases} L^2(M_{\lambda,x}, E_{\lambda,x}), & (\lambda, x) \in \mathbb{R} \times X, \\ 0, & \text{else.} \end{cases}$$

The space of continuous sections Γ^+ can be canonical identified with Γ_0. Therefore Γ_0 is a Hilbert-$C_0(\mathbb{R} \times X, \mathbf{A})$-module. In a similar way we can extend the unbounded operator family $\mathcal{D}$ to a unbounded operator family on $(\mathcal{E}^+, \Gamma_0)$: A fiberwise coercive function $h\colon M \to \mathbb{R}$ such that $[\mathcal{D}, h]$ is bounded and $\mathcal{D}^2 \geqslant h$ implies that the closure of $\mathcal{D}$ on $L^2_{\mathbb{R}\times X}(M, E)$ is self-adjoint and regular with compact resolvent. Therefore the operator family $\overline{\mathcal{D}}_0$ on $(\mathcal{E}^+, \Gamma_0)$ defined by

$$\overline{\mathcal{D}}_0 := \begin{cases} \overline{D}_{\lambda,x}\colon \Gamma^\infty_c(M_x, E_x) \to L^2(M_x, E_x), & (\lambda, x) \in \mathbb{R} \times X, \\ 0, & \text{else} \end{cases}$$

is self-adjoint and regular. Since $\|(\overline{D}^2_{\lambda,x} + 1)^{-1}\| \to 0$ as $(x, \lambda) \to \infty$, the operator family induced by the resolvent of $\overline{\mathcal{D}}$ on $(\mathcal{E}^+, \Gamma_0)$ is compact by an application of [Ebe18, Lemma 4.8]. Hence the triple $(\Gamma_0, \lambda\cdot, \mathcal{D}_0)$ defines an unbounded Kasparov-$(\mathbf{R}, C_0(\mathbb{R} \times X, \mathbf{A}))$-module.

Chapter 4

Two Versions of the Index Difference

In this chapter we will define both versions of the index difference. In section 4.1 we will recall some basic concepts from differential geometry and index theory. A standard reference is [LML16]. In section 4.2 the general setup is introduced. In the last two sections the unbounded Kasparov modules $\mathrm{inddiff}^{H}$ and $\mathrm{inddiff}^{GL}$ are constructed using the methods from chapter 3.

4.1 Preliminaries

Let (M, g) be a smooth connected Riemannian manifold. By the fundamental theorem of Riemannian geometry M possesses an unique symmetric connection ∇ which is compatible with the metric g. It is called *Levi-Civita connection*, [Roe99, Theorem 1.9]. There are several *curvature operators* uniquely determined by the Levi-Civita connection. Since ∇ is uniquely determined by the metric, the curvature is also uniquely determined by the metric. We will not go into detail and refer again to [Roe99, Chapter 1] for a comprehensive overview. The simplest curvature form of a Riemannian metric g is the *scalar curvature*.

Definition 4.1.1. *We define the scalar curvature* $\mathrm{scal}_g(x)$ *of* $g \in \Gamma^\infty(M, \mathrm{Sym}^2(TM))$ *at* $x \in M$ *as the double trace of the Riemannian curvature tensor* R *evaluated at* x. *We say that* g *has positive scalar curvature, if* $\mathrm{scal}_g(x) > 0$ *holds for every* $x \in M$.

If M is compact, then the space $R^+(M)$ of all Riemannian metrics on M whose scalar curvatures are positive is an open subspace of the Fréchet space $R(M)$ of all Riemannian metrics on M. When $R^+(M)$ is non-empty we say that M *has psc*.

A *spin structure* on a real orientable Riemannian vector bundle $V \to M$ of rank d is a lift of its classifying map $\tau\colon M \to BO(d)$ along the covering map $BSpin(d) \to$

$BO(d)$ to the 2-connected cover $BSpin(d)$ of $BO(d)$. A spin structure is determined by a $Spin(d)$-principal bundle $P \to M$ and an isometry $u\colon P \times_{Spin(d)} \mathbb{R}^d \xrightarrow{\cong} V$. Given a spin structure on V there exists the *spinor bundle* $\not{S}(V) \to M$ of V. The spinor bundle is a Real graded vector bundle with a bundle metric, such that the fibers $\not{S}(V)_x$ are $\mathbf{Cl}(V_x) \hat{\otimes} \mathbf{Cl}^{0,d}$-modules. The (pointwise-)action of $\mathbf{Cl}(V_x)$ on V_x is referred to as *Clifford multiplication*. It is denoted by $c(v_x)$. Clifford multiplication is *skew-adjoint* with respect to the bundle metric. The action of $\mathbf{Cl}^{0,d}$ on V_x defines a graded homomorphism $\rho\colon \mathbf{Cl}^{0,d} \to \mathrm{End}(V)$. The images $\rho(\epsilon_i)$ of the generators $\epsilon_1, \cdots, \epsilon_d$ of $\mathbf{Cl}^{0,d}$ are called *multigrading operators* and V is called a d-*multigraded* vector bundle, cf. [HJoo]. The spinor bundle $\not{S}(V)$ can be constructed by replacing the fiber of $P \times_{Spin(d)} \mathbb{R}^d$ by the Hilbert space $S_{d,d}$,compare 1.3.4, using the action of $Spin(d)$ on $S_{d,d}$ given by the identification $\mathrm{End}(S_{d,d}) \cong \mathbf{Cl}^{d,d}$.

An orientable smooth Riemannian manifold M^d is called a *spin manifold* provided that there exists a *spin structure* on its tangent bundle TM. The corresponding spinor bundle over M is denoted by $\not{S}(M)$. A standard reference is [LML16]. From now on M will always be a connected closed spin manifold.

The Levi-Civita connection ∇ on M induces a connection $\not{\nabla}$ on $\not{S}(M)$, called *spinor connection*. It is an *even* and $\mathbf{Cl}^{0,d}$-*linear* first order differential operator satisfying

$$\not{\nabla}_X(c(Y)s) = c(\nabla_X(Y))s + c(Y)\not{\nabla}_X(s) \tag{4.1}$$

for all vector fields $X, Y \in \Gamma^\infty(TM)$ and sections $s \in \Gamma^\infty(\not{S}(M))$. Using the Clifford multiplication and the spinor connection we define the following first order differential operator $\not{D}$ on $\Gamma^\infty(\not{S}(M))$:

Definition 4.1.2. *The operator* $\not{D}\colon \Gamma^\infty(\not{S}(M)) \to \Gamma^\infty(\not{S}(M))$ *on* (M, g) *is defined as the composition*

$$\Gamma^\infty(\not{S}(M)) \xrightarrow{\not{\nabla}} \Gamma^\infty(TM \otimes \not{S}(M)) \xrightarrow{c} \Gamma^\infty(\not{S}(M)).$$

$\not{D}$ is called the $\mathbf{Cl}^{0,d}$-*linear Atiyah-Singer operator* in [LML16]. As it is also a Dirac operator, we will call $\not{D}$ the *Spin Dirac operator*. If $\{e_1, \cdots, e_d\}$ is a local orthonormal frame of TM then $\not{D}$ is locally given by

$$\sum_{i=1}^{d} c(e_i) \cdot \not{\nabla}_{e_i}$$

The next lemma summarizes the properties of the operator $\not{D}$ proven in [LML16, Chapter II].

Lemma 4.1.3. *The operator $\not{D}$ is a Real odd formally self-adjoint $\mathbf{Cl}^{0,d}$-antilinear operator with symbol $\sigma(\not{D})(x,\xi) = c(i \cdot \xi_x)$. In particular $\not{D}$ is elliptic.*

Because $\not{D}$ is elliptic and M is compact, it is *Fredholm*, i. e. has finite dimensional kernel and cokernel. This is a consequence of the Sobolev embedding theorem and the Rellich lemma. Moreover $\not{D}$ is of the form

$$\not{D} = \begin{pmatrix} 0 & \not{D}_1 \\ \not{D}_0 & 0 \end{pmatrix} \tag{4.2}$$

with respect to the grading on $\not{S}(M)$. Since $\not{D}$ is formally self-adjoint $\not{D}_1^* = \not{D}_0$. The kernel of $\not{D}_0 \colon \Gamma^\infty(\not{S}(M)) \to \Gamma^\infty(\not{S}(M))$ is a finite dimensional $\mathbf{Cl}^{d-1,0}$-*module*. Therefore it defines an element $[\ker \not{D}_0]$ in the real K-theory $KO^{-d}(pt) \cong KO_d(\mathbf{R}) \cong KK(\mathbf{Cl}^{0,d}, \mathbf{R})$ of a point in degree $-d$ due to the work of Atiyah, Bott and Shapiro [ABS64].

Definition 4.1.4. *We define the index of $\not{D}$ by* $\operatorname{ind}(\not{D}) := [\ker \not{D}_0] \in KO^{-d}(pt)$.

The following theorem proven by Lichnerowicz establishes the link between positive scalar curvature and topology.

Theorem 4.1.5 ([LML16] Theorem II.8.8)**.** *Let M be a spin manifold with spinor bundle $\not{S}(M)$ and induced connection $\not{\nabla}$. Let $\not{D}$ be the Spin Dirac operator. Then*

$$\not{D}^2 = \not{\nabla}^* \not{\nabla} + \frac{1}{4} \operatorname{scal}_g .$$

When M has psc, Theorem 4.1.5 implies that $\not{D}$ is *invertible* and therefore $\operatorname{ind}(\not{D}) = 0$.

In the case M is not simply connected there exists a refinement of $\operatorname{ind}(\not{D})$. This is based on the theory of Hilbert-**A**-module bundles and **A**-linear differential operators introduced by Miščenko-Fomenko [MF79] and Rosenberg [Ros86]. See also chapter 2 for an overview. This done as follows:

Let G be a countable discrete group and suppose that M is endowed with a map $\varphi \colon M \to BG$. The Miščenko-Fomenko line bundle $\mathcal{L}_\varphi$ is a Real bundle of Hilbert-C_r^*G-modules over M. See definition 2.1.7. It is trivially graded and carries a *flat* connection ∇_φ. The tensor product $\not{S}(M) \otimes \mathcal{L}_\varphi \to M$ of the spinor bundle of M and the Miščenko-Fomenko line bundle is a Real graded bundle of Hilbert-C_r^*G-modules. It has a Clifford multiplication and is d-multigraded.

Definition 4.1.6. *The operator $\not{D}_\varphi$ on $\Gamma^\infty(\not{S}(M) \hat{\otimes} \mathcal{L}_\varphi)$ is defined as the composition*

$$\Gamma^\infty(\not{S}(M) \otimes \mathcal{L}_\varphi) \xrightarrow{\not{\nabla} \otimes 1 + 1 \otimes \nabla_\varphi} \Gamma^\infty(TM \otimes \not{S}(M) \otimes \mathcal{L}_\varphi) \xrightarrow{c} \Gamma^\infty(\not{S}(M) \otimes \mathcal{L}_\varphi)$$

of the product connection and the Clifford multiplication. The operator is called the twisted (Spin) Dirac operator.

The operator $\not{D}_\varphi$ is an elliptic C_r^*G-linear first order differential operator, which is formally self-adjoint and anticommutes with the multigrading operators. Moreover Miščenko-Fomenko proved that its closure is C_r^*G-Fredholm, compare [MF79]. Therefore one can define an index $[(M, \not{D}_\varphi)] \in KO^{-d}(M, C_r^*G) := KK(\mathbf{Cl}^{0,d}, C(M, C_r^*G)) \cong KO_d(C(M, C_r^*G))$. See again loc. cit. §1. Since M is compact the map $c\colon M \to \{pt\}$ is proper and hence induces a map $KO^{-d}(M, C_r^*G) \to KO_d(C_r^*G)$.

Definition 4.1.7. *The Rosenberg index of* $\not{D}_\varphi$ *is defined by* $\alpha_r^R(M) := c_*([(M, \not{D}_\varphi)] \in KO_d(C_r^*G)$.

Because the bundle $\mathcal{L}_\varphi$ carries a *flat* connection, i. e. has zero curvature, theorem 4.1.5 is still true:

$$\not{D}_\varphi^2 = (\not{\nabla}^*\not{\nabla} + \frac{1}{4}\operatorname{scal}_g) \otimes 1 \tag{4.3}$$

This follows from lemma 2.1.5 since $\Omega_{\mathcal{L}_\varphi} = 0$. Therefore psc also implies that $\alpha_r^R(M)$ is zero.

4.2 The Index Difference

The index difference is an invariant assigned to a *pair* or a *family* of psc metrics on a given Riemannian manifold M. It takes values in $KK(\mathbf{Cl}^{0,d}, C_0(\mathbb{R} \times X, C_r^*G)) \cong KK(\mathbf{Cl}^{0,d+1}, C(X, C_r^*G))$.

Throughout this section we will fix a closed connected d-dimensional Riemannian spin manifold M, a map φ from M to the classifying space BG of a discrete countable group G, and a compact smooth manifold X possibly with boundary, which one should think of as a parameter space. Furthermore we make the assumption that the space $R^+(M)$ of Riemannian metrics with positive scalar curvature is non-empty. Let $\mathscr{G}\colon X \longrightarrow R^+(M) \times R^+(M)$, $x \mapsto (g_0(x), g_1(x))$ be a smooth map.

Definition 4.2.1. *Then we define a family $(g_\lambda(x))_{(\lambda,x)\in\mathbb{R}\times X}$ of Riemannian metrics on M by*

$$g_\lambda(x) := \frac{\chi(\lambda)+1}{2} g_1(x) - \frac{\chi(\lambda)-1}{2} g_0(x), \tag{4.4}$$

where $\chi\colon \mathbb{R} \to [-1, 1]$ is an arbitrary normalization function, i. e. a smooth, odd function such that $\chi(\lambda) = 1$, if $\lambda \geqslant 1$ and $\chi(\lambda) = -1$, if $\lambda \leqslant -1$.

This family can be understood as a smooth map

$$\mathscr{G}^H_\chi\colon \mathbb{R} \times X \longrightarrow R(M) \ , \ (\lambda, x) \mapsto g_\lambda(x) \tag{4.5}$$

from $\mathbb{R} \times X$ to the space of all Riemannian metrics on M, such that $g_\lambda(x) \in R^+(M)$ for $|\lambda| \geqslant 1$. It is also possible to define a map

$$\mathscr{G}^{GL}_\chi\colon X \to R(M \times \mathbb{R}) \ , \ x \mapsto g_\lambda(x) + d\lambda^2 \tag{4.6}$$

and obtain a family of Riemannian metrics on $M \times \mathbb{R}$.

Remark. Neither of the definitions of the Index difference will depend on the special choice of the normalization function χ. In fact we could choose an arbitrary smooth function χ such that $\chi(\lambda) = -1$ for $\lambda \ll -1$ and $\chi(\lambda) = 1$ for $\lambda \gg 1$ to define $\mathscr{G}^H_\chi$ as well as $\mathscr{G}^{GL}_\chi$. Moreover both definitions will only depend on the *homotopy class* of the smooth map $\mathscr{G}$.

4.2.1 Hitchin's Version

Associated to the map $\mathscr{G}_X^H$, i.e. to the family $(g_\lambda(x))_{(\lambda,x)\in\mathbb{R}\times X}$ of Riemannian metrics on the closed manifold M, there is a family $\mathcal{D} := (\not{D}_{\lambda,x})_{(\lambda,x)\in\mathbb{R}\times X}$ of twisted Spin Dirac operators and a family $\mathcal{E} := (L^2(\not{\mathfrak{S}}(M)\,\hat{\otimes}\,\mathcal{L}_\varphi))_{(\lambda,x)\in\mathbb{R}\times X}$ of Hilbert modules. The Index difference of Hitchin assigns to (the homotopy class of) the smooth map $\mathscr{G}\colon X \to R^+(M)\times R^+(M)$ an unbounded Kasparov module $\mathrm{inddiff}^H_{\mathscr{G}} := (\Gamma_0, \rho, D) \in \Psi(\mathbf{Cl}^{0,d}, C_0(\mathbb{R}\times X, C^*_r G))$. The Hilbert-$C_0(\mathbb{R}\times X, C^*_r G)$-module Γ_0 will be determined by the family $\mathcal{E}$ of Hilbert-$C^*_r G$-modules over $\mathbb{R}\times X$ and the operator D will be determined by the family $\mathcal{D}$ of twisted Dirac operators.

Let $\pi : M\times\mathbb{R}\times X \to \mathbb{R}\times X$ be the trivial M-bundle. It is a submersion with d-dimensional closed fiber M. Let $T_v\pi := \ker d\pi = TM\times\mathbb{R}\times X \to M\times\mathbb{R}\times X$ be the vertical tangent bundle of π. The smooth map $\mathscr{G}_X^H$ endows $T_v\pi$ with a bundle metric, such that the induced Riemannian metric on each fiber $M(\lambda,x) := \pi^{-1}(\lambda,x)$ is given by $g_\lambda(x)$.

The bundle $T_v\pi \to M\times\mathbb{R}\times X$ has a spin structure determined by the spin structure on M. The associated spinor bundle of $T_v\pi$ is denoted by $\not{\mathfrak{S}}(\pi) := \not{\mathfrak{S}}(T_v\pi) \to M\times\mathbb{R}\times X$. Its restriction to $M(\lambda,x)$ is isomorphic to the spinor bundle $\not{\mathfrak{S}}(\lambda,x)\to M(\lambda,x)$, i.e.

$$\begin{array}{ccc} \not{\mathfrak{S}}(\pi)_{|M\times\{\lambda\}\times\{x\}} & \xrightarrow{\cong} & \not{\mathfrak{S}}(\lambda,x) \\ \downarrow & & \downarrow \\ M\times\{\lambda\}\times\{x\} & = & M(\lambda,x) \end{array}$$

In particular the Clifford-multiplication of $\mathbf{Cl}((T_v\pi)_{(p,\lambda,x)})$ on $\not{\mathfrak{S}}(\pi)_{(p,\lambda,x)}$ agrees with the Clifford multiplication of $\mathbf{Cl}(T_pM(\lambda,x))$ on $\not{\mathfrak{S}}(\lambda,x)_p$. Moreover the restriction of the multigrading operators of $\not{\mathfrak{S}}(\pi)$ to $\not{\mathfrak{S}}(\pi)_{|M(\lambda,x)}$ agree with the multigrading operators of $\not{\mathfrak{S}}(\lambda,x)$.

We can pullback the Miščenko-Fomenko line bundle $\mathcal{L}_G := EG\times_\rho C^*_r G \to BG$ along $\varphi\colon M\to BG$ and the projection $\mathrm{pr}_M\colon M\times\mathbb{R}\times X\to M$ and obtain a bundle $\mathcal{L} := \mathcal{L}_{\varphi\circ\mathrm{pr}_M} \to M\times\mathbb{R}\times X$ of finitely generated and free Hilbert-$C^*_r G$-modules over $M\times\mathbb{R}\times X$.

Definition 4.2.2. *We define the bundle* $\Lambda(\pi)\to M\times\mathbb{R}\times X$ *by* $\Lambda(\pi) := \not{\mathfrak{S}}(\pi)\,\hat{\otimes}\,\mathcal{L}$.

The bundle $\Lambda(\pi)$ is a smooth $C^*_r G$-bundle. Moreover it is endowed with a Real structure and a grading. The grading is given by the decomposition

$$\Lambda(\pi) := (\not{\mathfrak{S}}(\pi)^{(0)}\otimes\mathcal{L}) \oplus (\not{\mathfrak{S}}(\pi)^{(1)}\otimes\mathcal{L}). \tag{4.7}$$

The Real structure is given by the tensor product of the Real structures of the bundles $\mathfrak{S}(\pi)$ and $\mathcal{L}$. Furthermore the Clifford action of $\mathbf{Cl}^{0,d}$ on $\mathfrak{S}(\pi)$ induces a Clifford action of $\mathbf{Cl}^{0,d}$ on $\Lambda(\pi)$. We denote the restriction of $\Lambda(\pi)$ to $M(\lambda, x)$ by $\Lambda(\lambda, x)$. It is clear that $\Lambda(\lambda, x) \cong \mathfrak{S}(\lambda, x) \hat{\otimes} \mathcal{L}_\varphi$.

Definition 4.2.3. *The operator* $\not{D}\colon \Gamma^\infty_{cv}(M \times \mathbb{R} \times X, \Lambda(\pi)) \to \Gamma^\infty_{cv}(M \times \mathbb{R} \times X, \Lambda(\pi))$ *is defined by the family* $(\not{D}_{\lambda,x})_{(\lambda,x)\in\mathbb{R}\times X}$ *of the twisted Spin Dirac operators* $\not{D}_{\lambda,x}\colon \Gamma^\infty(\Lambda(\lambda, x)) \to \Gamma^\infty(\Lambda(\lambda, x))$ *on* $\Lambda(\lambda, x)$.

The operator $\not{D}$ is a $C^*_r G$-linear formally self-adjoint elliptic differential operator. It is Real odd and anticommutes with the multigrading operators, i. e. with the Clifford action.

Lemma 4.2.4. *If* $|\lambda| \geqslant 1$, *then* $\not{D}_{\lambda,x}$ *is invertible.*

Proof. This follows from eq. (4.3) since $\mathrm{scal}(g_\lambda(x)) > 0$, if $|\lambda| \geqslant 1$. □

Hence we assigned to $(M, \varphi, \mathscr{G})$

- a submersion $\pi\colon M \times \mathbb{R} \times X \to \mathbb{R} \times X$ with closed d-dimensional fiber M and fiberwise Riemannian metric,
- a Real graded $C^*_r G$-bundle $\Lambda(\pi) \to M \times \mathbb{R} \times X$ with an action ρ of $\mathbf{Cl}^{0,d}$ and
- a $C^*_r G$-linear Real graded formally self-adjoint elliptic differential operator $\not{D}$ acting on $\Gamma^\infty_{cv}(\Lambda(\pi))$ which anticommutes with ρ.

Using the result of section 3.2 we get a continuous field $L^2_{\mathbb{R}\times X}(M \times \mathbb{R} \times X, \Lambda(\pi))$ of Real graded Hilbert-$C^*_r G$-modules over $\mathbb{R} \times X$. The operator $\not{D}$ defines a symmetric unbounded operator family also denoted by $\not{D}$ on $L^2_{\mathbb{R}\times X}(M \times \mathbb{R} \times X, \Lambda(\pi))$ with domain

$$\mathrm{dom}(\not{D})_{\mathbb{R}\times X} := ((\Gamma^\infty(\Lambda(\lambda, x)))_{(\lambda,x)\in\mathbb{R}\times X}, \Gamma^\infty_{cv}(M \times \mathbb{R} \times X, \Lambda(\pi))).$$

Lemma 4.2.5. *The closure of the symmetric unbounded densely defined operator family* $\not{D}$ *is a self-adjoint and regular family on* $L^2_{\mathbb{R}\times X}(M \times \mathbb{R} \times X, \Lambda(\pi))$.

Proof. We have to show that the closure of every operator $\not{D}_{\lambda,x}$ is self-adjoint and regular. However this is clear, since $M(\lambda, x)$ is closed. □

By abuse of notation we will denote the closure of the family by $\not{D}$. The existence of a fiberwise coercive function $f\colon M \times \mathbb{R} \times X \to \mathbb{R}$ such that $\not{D}^2 \geqslant f$ would ensure that $\not{D}$ has compact resolvent. Moreover the resolvent of the induced operator $\not{D}_0$ on the Hilbert-$C_0(\mathbb{R} \times X, C^*_r G)$-module Γ_0 would also be compact. Therefore the triple $(\Gamma_0, \rho, \not{D}_0)$ would be an unbounded Kasparov-$(\mathbf{Cl}^{0,d}, C_0(\mathbb{R} \times X, C^*_r G))$-module.

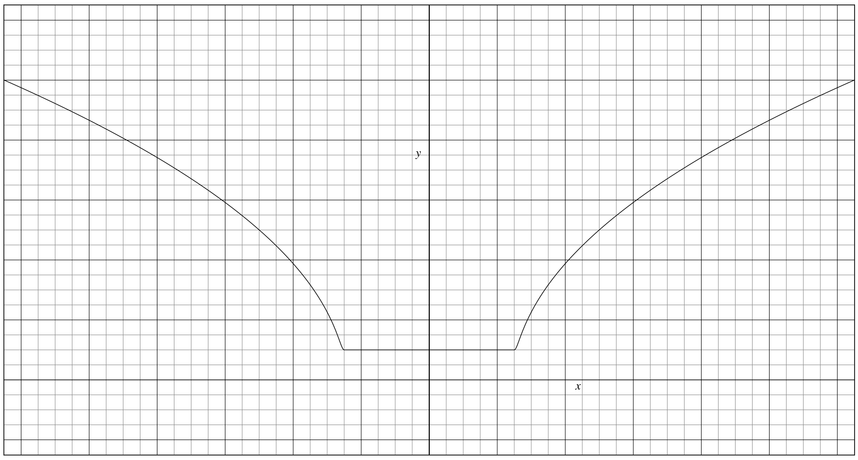

Figure 4.1: Graph of the function h

Let $h\colon M \times \mathbb{R} \times X \to \mathbb{R}$ be a smooth approximation of the function given by

$$(p, \lambda, x) \mapsto \begin{cases} 1, & \text{if } |\lambda| \leqslant 1, \\ (\lambda^2 + 1)^{\frac{1}{4}}, & \text{if } |\lambda| > 1. \end{cases}$$

Then h is proper and bounded from below. Since $\not{D}$ is a family of linear operators over $\mathbb{R} \times X$ the next lemma is rather obvious.

Lemma 4.2.6. $h\not{D} = \not{D}h$

Proof. This follows since $\not{D} = (\not{D}_{\lambda,x})_{(\lambda,x)\in\mathbb{R}\times X}$ is a family over $\mathbb{R} \times X$ and $h_{\lambda,x} := h_{|M(\lambda,x)}\colon M(\lambda, x) \to \mathbb{R}$ is constant. □

The family $h\not{D}h := (h_{\lambda,x}\not{D}_{\lambda,x}h_{\lambda,x})_{(\lambda,x)\in\mathbb{R}\times X}$ induces an unbounded family $h\not{D}h$ on $L^2_{\mathbb{R}\times X}(M \times \mathbb{R} \times X, \Lambda(\pi))$ with $\mathrm{dom}(h\not{D}h)_{\mathbb{R}\times X} = \mathrm{dom}(\not{D})_{\mathbb{R}\times X}$. Moreover this family is symmetric and its closure, also denoted by $h\not{D}h$, is self-adjoint and regular. Let $m(x)$ be the minimum of the smooth function

$$\mathrm{scal}_x\colon M \times [-1, 1] \to \mathbb{R}\ ,\ (p, \lambda) \mapsto \mathrm{scal}_{g_\lambda(x)}(p).$$

Define a smooth function $f_x\colon M \times \mathbb{R} \times \{x\}$ such that

$$f_x(p, \lambda) = \begin{cases} \frac{1}{4}m(x), & \text{if } |\lambda| \leqslant 1, \\ \ln(|\lambda|), & \text{if } |\lambda| \gg 1. \end{cases} \tag{4.8}$$

The function $f\colon M \times \mathbb{R} \times X \to \mathbb{R}$, $(p, \lambda, x) \mapsto f_x(p, \lambda)$ is smooth, fiberwise proper and bounded from below (since X is compact). Moreover f commutes with $h\not{D}h$ since

$f_{\lambda,x} \coloneqq f_{|M(\lambda,x)}$ is constant. Therefore f is a fiberwise coercive function.

Lemma 4.2.7. *The coercive function f satisfies $(h\not{D}h)^2 \geqslant f$.*

Proof. We have to show that for all $(\lambda, x) \in \mathbb{R} \times X$ and every $s \in \mathrm{dom}(\not{D}^2)_{\mathbb{R}\times X}$

$$\langle (h\not{D}h)^2 s, s\rangle_{\lambda,x} \geqslant \langle fs, s\rangle_{\lambda,x}$$

holds. Since $(h\not{D}h)^2 = h\not{D}h^2\not{D}h = h^4\not{D}^2$ we obtain that

$$\begin{aligned}\langle (h\not{D}h)^2 s, s\rangle_{\lambda,x} &\overset{\text{Def.}}{=} \langle h^4 \not{D}^2_{\lambda,x} s_{\lambda,x}, s_{\lambda,x}\rangle_{\lambda,x}\\ &\overset{4.3}{=} \langle h^4(\nabla^*\nabla + \frac{1}{4}\,\mathrm{scal}_{\lambda,x})\hat{\otimes} 1(s_{\lambda,x}), s_{\lambda,x}\rangle_{\lambda,x}\\ &\geqslant \langle h^4 \frac{1}{4}\,\mathrm{scal}_{\lambda,x}\, s_{\lambda,x}, s_{\lambda,x}\rangle_{\lambda,x}.\end{aligned}$$

If $|\lambda| > 1$ the scalar curvature of $M(\lambda, x)$ is positive and hence $(\lambda^2+1)\,\mathrm{scal}_{\lambda,x} \geqslant \ln(|\lambda|)$ for $|\lambda| \gg 1$. If $|\lambda| \leqslant 1$ then $\frac{1}{4}\,\mathrm{scal}_{\lambda,x} \geqslant \frac{1}{4} m(x)$. It follows that

$$\langle h^4 \frac{1}{4}\,\mathrm{scal}_{\lambda,x}\, s_{\lambda,x}, s_{\lambda,x}\rangle_{\lambda,x} \geqslant \langle f_{\lambda,x} s_{\lambda,x}, s_{\lambda,x}\rangle_{\lambda,x}$$

for all $(\lambda, x) \in \mathbb{R} \times X$ possibly after multiplying f_x by a constant $k_x < 1$ depending smoothly on x to ensure that $(\lambda^2+1)\mathrm{scal}_{\lambda,x} \geqslant \ln(|\lambda|)$ holds for all $|\lambda| > 1$. Note that this will only be necessary, if $0 < \mathrm{scal}_{\lambda,x} < 1$ and in this case k_x is equal to $\mathrm{scal}_{\lambda,x}$. □

Now we can apply theorem 3.2.9 to prove

Corollary 4.2.8. *The bounded operator family $((h\not{D}h)^2+1)^{-1}$ is a compact family.*

Now we can prove

Theorem 4.2.9. *The triple $(\Gamma_0, \rho, h\not{D}h)$ an unbounded Kasparov-$(\mathbf{Cl}^{0,d}, C_0(\mathbb{R}\times X, C^*_r G))$-module.*

Proof. Γ_0 is a Real graded Hilbert-$C_0(\mathbb{R}\times X, C^*_r G)$-module. The action ρ of $\mathbf{Cl}^{0,d}$ on $\Lambda(\pi)$ defines a Real graded $*$-homomorphism $\mathbf{Cl}^{0,d} \to \mathbf{Lin}_{C_0(\mathbb{R}\times X, C^*_r G)}(\Gamma_0)$. The operator $h\not{D}h$ induces a Real odd $C^*_r G$-linear self-adjoint and regular operator family on $L^2_{\mathbb{R}\times X}(M \times \mathbb{R} \times X, \Lambda(\pi))$. By the previous corollary $h\not{D}h$ has compact resolvent. The induced operator on Γ_0 is also denoted by $h\not{D}h$. It is a Real odd $C^*_r G$-linear self-adjoint and regular operator with compact resolvent by the results of section 3.2. Therefore it is only left to show that the set of all $\mathbf{a} \in \mathbf{Cl}^{0,d}$ such that $\{h\not{D}h, \rho(\mathbf{a})\}$ is densely defined and extend to a bounded operator is dense. Since $h\not{D}h$ anticommutes with the Clifford action of $\mathbf{Cl}^{0,d}$ the graded commutator $\{h\not{D}h, \rho(\mathbf{a})\}$ is zero for every $\mathbf{a} \in \mathbf{Cl}^{0,d}$. □

Definition 4.2.10. *Let* $(M, \varphi, \mathscr{G})$ *as above. Then we define* $\mathrm{inddiff}^{H}_{\mathscr{G}} \coloneqq [(\Gamma_0, \rho, h\not{D}h)]$.

Lemma 4.2.11. *The definition* $\mathrm{inddiff}^{H}_{\mathscr{G}}$ *only depends on the homotopy class of the smooth map* $\mathscr{G}$*.*

Proof. Let $\mathscr{G}' \colon X \to R^{+}(M) \times R^{+}(M)$ be a smooth map homotopic to $\mathscr{G}$. We can choose the homotopy $\mathscr{H}$ to be a smooth map $\mathscr{H} \colon X \times I \to R^{+}(M) \times R^{+}(M)$. We obtain an unbounded Kasparov module **h** in $\Psi(\mathbf{Cl}^{0,d}, C(I, C_0(\mathbb{R} \times X, C^*_r G)))$ exactly in the same way as before. It follows that $b(\mathbf{h})$ is a homotopy between $b(\mathrm{inddiff}^{H}_{\mathscr{G}})$ and $b(\mathrm{inddiff}^{H}_{\mathscr{G}'})$. □

4.2.2 Gromov and Lawson's Version

The construction of the index difference of Gromov and Lawson is similar to the construction of the index difference of Hitchin in the previous section. However instead of the submersion π we will consider the trivial $M \times \mathbb{R}$-bundle $\Pi\colon M \times \mathbb{R} \times X \to X$. Using the smooth map $\mathscr{G}_X^{GL}$ we define a bundle metric on the vertical tangent bundle $T_v\Pi = T(M \times \mathbb{R}) \times X \to M \times \mathbb{R} \times X$. The induced Riemannian metric on each fiber $W_x := \Pi^{-1}(x)$ is given by $g_x := g_\lambda(x) + d\lambda^2$. The spinor bundle of $T_v\Pi$ is denoted by $\mathfrak{S}(\Pi) \to M \times \mathbb{R} \times X$. Adapting the constructions of the previous section we make the following two definitions:

Definition 4.2.12. *The bundle* $\Lambda(\Pi) \to M \times \mathbb{R} \times X$ *is defined by* $\Lambda(\Pi) := \mathfrak{S}(\Pi) \hat{\otimes} \mathcal{L}$.

Definition 4.2.13. *The operator* $\mathfrak{D}\colon \Gamma_{cv}^\infty(M \times \mathbb{R} \times X, \Lambda(\Pi)) \to \Gamma_{cv}^\infty(M \times \mathbb{R} \times X, \Lambda(\Pi))$ *is defined by the family* $(\mathfrak{D}_x)_{x\in X}$ *of the twisted Dirac operators* $\mathfrak{D}_x\colon \Gamma_c^\infty(\Lambda(x)) \to \Gamma_c^\infty(\Lambda(x))$ *on* $\Lambda(x)$.

This can be summarized as before: We assigned to $(M, \varphi, \mathscr{G})$

- a submersion $\Pi\colon M \times \mathbb{R} \times X \to X$ with $(d+1)$-dimensional fiber $M \times \mathbb{R}$ and fiberwise Riemannian metric,
- a Real graded C_r^*G-bundle $\Lambda(\Pi) \to M \times \mathbb{R} \times X$ with an action ρ of $\mathbf{Cl}^{0,d+1}$ and
- a C_r^*G-linear Real graded formally self-adjoint elliptic differential operator $\mathfrak{D}$ acting on $\Gamma_{cv}^\infty(\Lambda(\Pi))$ which anticommutes with ρ.

Using the result of section 3.2 we obtain a continuous field $L_X^2(M \times \mathbb{R} \times X, \Lambda(\Pi))$ of Real graded Hilbert-C_r^*G-modules over the compact manifold X. The operator $\mathfrak{D}$ defines a symmetric unbounded operator family also denoted by $\mathfrak{D}$ on $L_X^2(M \times \mathbb{R} \times X, \Lambda(\Pi))$ with domain

$$\mathrm{dom}(\mathfrak{D})_X := (\Gamma_c^\infty(\Lambda(x)), \Gamma_{cv}^\infty(M \times \mathbb{R} \times X, \Lambda(\Pi))).$$

Since X is compact the space Γ of continuous sections of $L_X^2(M \times \mathbb{R} \times X, \Lambda(\Pi))$ is a Hilbert-$C(X, C_r^*G)$-module. Let $h, f\colon M \times \mathbb{R} \times X \to \mathbb{R}$ be as before. To obtain an unbounded Kasparov-$(\mathbf{Cl}^{0,d+1}, C(X, C_r^*G))$-module we follow the same strategy as we did before and consider the operator induced by $h\mathfrak{D}h$.

Lemma 4.2.14. *The commutator* $[h\mathfrak{D}h, f]$ *of the family* $h\mathfrak{D}h$ *and the coercive function* f *is bounded in* X.

Proof. Let $f\colon M \times \mathbb{R} \times X \to \mathbb{R}$ be as above. Then

$$[h\not{\mathfrak{D}}h, f] = h(hf\not{\mathfrak{D}} + [\not{\mathfrak{D}}, hf]) - fh(h\not{\mathfrak{D}} + [\not{\mathfrak{D}}, h]).$$

Since the commutator of a Dirac operator with any smooth function is given by Clifford multiplication with the gradient of the function we get

$$[h\not{\mathfrak{D}}h, f] = h^2 f\not{\mathfrak{D}} + hc(\mathrm{grad}(hf)) - fh^2\not{\mathfrak{D}} - fhc(\mathrm{grad}(h)) = h^2 c(\mathrm{grad}(f)).$$

Let h_x be the restriction of h to W_x. Similarly denote the restriction of f to W_x by f_x. Then

$$\|[h\not{\mathfrak{D}}h, f]_x\| = \|h_x^2 c(\mathrm{grad}(f_x))\| \leqslant \sup_{|\lambda|>1} \|\frac{\sqrt{\lambda^2+1}}{\lambda}\| < \infty,$$

because $h^2(p,\lambda) = (\lambda^2+1)^{\frac{1}{2}}$ and $\mathrm{grad}(f) \leqslant \frac{1}{\lambda}$. Therefore $[h\not{\mathfrak{D}}h, f]$ is bounded in norm. □

Lemma 4.2.15. *The closure of* $h\not{\mathfrak{D}}h\colon \mathrm{dom}(\not{\mathfrak{D}})_X \to L^2_X(M \times \mathbb{R} \times X, \Lambda(\Pi))$ *is self-adjoint and regular.*

Proof. Since $[h\not{\mathfrak{D}}h, f]$ is bounded it is locally bounded. It follows that $(M \times \mathbb{R} \times X, h\not{\mathfrak{D}}h)$ is fiberwise complete. Therefore the closure of $(h\not{\mathfrak{D}}h)_x$ is self-adjoint and regular for every $x \in X$ by theorem 2.2.3. □

By abuse of notation $h\not{\mathfrak{D}}h$ will denote the closure of $h\not{\mathfrak{D}}h$.

Lemma 4.2.16. *The coercive function* f *satisfies* $(h\not{\mathfrak{D}}h)^2 \geqslant f$.

Proof. We have to show that for all $x \in X$ and every $s \in \mathrm{dom}_X(\not{\mathfrak{D}}^2)$

$$\langle (h\not{\mathfrak{D}}h)^2 s, s\rangle_x \geqslant \langle fs, s\rangle_x$$

holds. We compute

$$\begin{aligned}
\langle (h\not{\mathfrak{D}}h)^2 s, s\rangle_x &= \int_{W_x} ((h\not{\mathfrak{D}}h)^2 s, s)\,\mathrm{dvol}(x) = \int_{W_x} (h\not{\mathfrak{D}}hs, h\not{\mathfrak{D}}hs)\,\mathrm{dvol}(x) \\
&= \int_{W_x} h^2(\not{\mathfrak{D}}hs, \not{\mathfrak{D}}hs)\,\mathrm{dvol}(x) = \int_{W_x} h^2(\not{\mathfrak{D}}^2 hs, hs)\,\mathrm{dvol}(x) \\
&= \int_{W_x} h^2((\not{\nabla}^*\not{\nabla} + \frac{1}{4}\mathrm{scal}(g_\lambda(x)) \hat{\otimes} 1)(hs), hs)\,\mathrm{dvol}(x) \\
&\geqslant \int_{W_x} h^2(\frac{1}{4}\mathrm{scal}(g_\lambda(x))hs, hs)\,\mathrm{dvol}(x)
\end{aligned}$$

$$= \int\limits_{W_x} \frac{1}{4} \operatorname{scal}(g_\lambda(x))(h^4 s, s)\, dvol(x) \geqslant \int\limits_{W_x} (fs, s)\, dvol(x)$$
$$= \langle fs, s\rangle_X.$$

In the first inequality we used that $(\nabla\!\!\!/^* \nabla\!\!\!/ \hat{\otimes} 1(s), s) = (\nabla\!\!\!/ \hat{\otimes} 1(s), \nabla\!\!\!/ \hat{\otimes} 1(s)) \geqslant 0$. The second inequality follows since the scalar curvature is positive outside $M \times [-1, 1] \times \{x\}$ and on $M \times [-1, 1] \times \{x\}$ bounded from below by f. □

Using theorem 3.2.9 we obtain

Corollary 4.2.17. *The bounded operator family* $((h D\!\!\!\!/ h)^2 + 1)^{-1}$ *is a compact family.*

Therefore we can prove

Theorem 4.2.18. *The triple* $(\Gamma, \rho, h D\!\!\!\!/ h)$ *is an unbounded Kasparov-*$(\mathbf{Cl}^{0,d+1}, C(X, C_r^* G))$*-module.*

Proof. The space of continuous sections Γ of the continuous field $L^2_X(M \times \mathbb{R} \times X, \Lambda(\Pi))$ over the compact space X is a Hilbert-$C(X, C_r^* G)$-module. The Real structure and the grading of $\Lambda(\Pi)$ induce a Real structure and a grading on Γ. The closure of the Real odd unbounded and densely defined operator family $h D\!\!\!\!/ h$ defines a Real odd self-adjoint and regular operator $h D\!\!\!\!/ h \in \mathbf{Lin}_{C(X, C_r^* G)}(\Gamma)$. The Clifford action ρ on $\Lambda(\Pi)$ given by the multigrading operators induces a Real graded $*$-homomorphism $\mathbf{Cl}^{0,d+1} \to \mathbf{Lin}_{C(X, C_r^* G)}(\Gamma)$ and the operator anticommutes with this action. By 4.2.17 the resolvent of $h D\!\!\!\!/ h$ is compact. Hence the triple $(\Gamma, \rho, h D\!\!\!\!/ h)$ is an unbounded Kasparov module. □

Definition 4.2.19. *Let* $(M, \varphi, \mathscr{G})$ *as above. Then we define* $\operatorname{inddiff}^{GL}_{\mathscr{G}} := [(\Gamma, \rho, h D\!\!\!\!/ h)]$.

Remark. This definition also only depends on the homotopy class of the map $\mathscr{G}$. The proof is the same as in the case of $\operatorname{inddiff}^{H}_{\mathscr{G}}$.

Chapter 5

The spectral flow theorem for families of twisted Dirac operators

We already put some effort into constructing KK-cycles representing the index difference of either Hitchin or Gromov and Lawson. Now we will show that they are mapped to each other under the Bott isomorphism in KK-theory. The next theorem is called the spectral flow theorem for families of twisted Dirac operators.

Theorem 5.0.1. *Let* X *be a compact smooth manifold and* M^d *a closed Riemannian spin manifold endowed with a map* $\varphi\colon M \to BG$. *Suppose that* $R^+(M) \neq \emptyset$ *and let* $\mathscr{G}\colon X \to R^+(M) \times R^+(M)$ *be smooth. Let* $\mathrm{inddiff}^{H}_{\mathscr{G}}$ *and* $\mathrm{inddiff}^{GL}_{\mathscr{G}}$ *be the unbounded Kasparov modules constructed from* $(M, \varphi, \mathscr{G})$. *Then*

$$\mathrm{bott}([b(\mathtt{inddiff}^{GL}_{\mathscr{G}})]) = [b(\mathrm{inddiff}^{H}_{\mathscr{G}})], \tag{5.1}$$

where $\mathrm{bott}\colon KK(\mathbf{Cl}^{0,d+1}, C(X, C^*_r G)) \xrightarrow{\cong} KK(\mathbf{Cl}^{0,d}, C_0(\mathbb{R} \times X, C^*_r G))$ *is the Bott map.*

In KK-theory the Bott isomorphism is given by the Kasparov product with the Bott element β and its inverse is given by the Kasparov product with the element α. Hence eq. (5.1) is equivalent to

$$\tau_{\mathbf{Cl}^{1,0}}(b(\mathtt{inddiff}^{GL}_{\mathscr{G}}))\#\tau_{C(X,C^*_r G)}(\beta) = b(\mathrm{inddiff}^{H}_{\mathscr{G}}). \tag{5.2}$$

Therefore to prove theorem 5.0.1 it is sufficient to show that

$$b(\mathtt{inddiff}^{GL}_{\mathscr{G}}) = \tau_{\mathbf{Cl}^{0,1}}(b(\mathtt{inddiff}^{H}_{\mathscr{G}}))\#\tau_{C(X,C^*_r G)}(\alpha). \tag{5.3}$$

This will be done in three steps: We will first construct an unbounded Kasparov module $\mathbf{z}' \in \Psi(\mathbf{Cl}^{0,d+1}, C(X, C^*_r G))$ in section 5.1. Afterwards we will prove in section 5.2 that

$$b(\mathbf{z}') = \tau_{\mathbf{Cl}^{0,1}}(b(\mathtt{inddiff}^{H}_{\mathscr{G}}))\#\tau_{C(X,C^*_rG)}(\alpha). \tag{5.4}$$

using theorem 1.3.9. Finally we will show that $[b(\mathbf{z}')] = [b(\mathrm{inddiff}^{GL}_{\mathscr{G}})]$ by construction an homotopy $\mathbf{H} \in KK(\mathbf{Cl}^{0,d+1}, IC(X, C^*_rG))$.

5.1 An approximation of inddiff^{GL}

Consider the unbounded Kasparov module $\mathbf{x} \in \Psi(\mathbf{Cl}^{0,d+1}, C_0(\mathbb{R} \times X, C^*_rG) \hat{\otimes} \mathbf{Cl}^{0,1})$ given by $\mathbf{x} := \tau_{\mathbf{Cl}^{0,1}}(\mathrm{inddiff}^{H}_{\mathscr{G}}) = (\Gamma_0 \hat{\otimes} \mathbf{Cl}^{0,1}, \rho \hat{\otimes} 1, h\not{D}h \hat{\otimes} 1)$.
Let $\mathbf{y} \in \Psi(C_0(\mathbb{R}, \mathbf{Cl}^{0,1}) \hat{\otimes} C(X, C^*_rG), C(X, C^*_rG))$ be the unbounded Kasparov module given by $\mathbf{y} := \tau_{C(X,C^*_rG)}(\alpha) = (L^2(\mathbb{R}, \mathbb{S}_{1,1}) \hat{\otimes} C(X, C^*_rG), \mu \hat{\otimes} 1, D_{\mathbb{R}} \hat{\otimes} 1)$.
The Kasparov product of $\mathbf{x}$ and $\mathbf{y}$ is given by an unbounded Kasparov module $\mathbf{z} := ((\Gamma_0 \hat{\otimes} \mathbf{Cl}^{0,1}) \hat{\otimes}_{\mu \hat{\otimes} 1} (L^2(\mathbb{R}, \mathbb{S}_{1,1}) \hat{\otimes} C(X, C^*_rG)), \rho \hat{\otimes} 1, F) \in \Psi(\mathbf{Cl}^{0,d+1}, C(X, C^*_rG))$ satisfying the assumptions of theorem 1.3.9 with respect to $\mathbf{x}$ and $\mathbf{y}$. The goal of this section is to give an "easy" description of the unbounded Kasparov module $\mathbf{z}$. With it we intend that it is relatively "easy" to check the criteria of theorem 1.3.9.

Let $\Sigma \to M \times \mathbb{R} \times X$ be the trivial Hilbert bundle with fiber $\mathbb{S}_{1,1}$ and obvious Real structure grading and $\mathbf{Cl}^{1,1}$-action. The bundle $\Sigma\Lambda(\pi) := \Lambda(\pi) \hat{\otimes} \Sigma \to M \times \mathbb{R} \times X$ has then the structure of a Real graded C^*_rG-bundle. We will regard $\Lambda(\pi) \hat{\otimes} \Sigma$ as a bundle over the submersion $\Pi\colon M \times \mathbb{R} \times X \to X$. In this way we obtain a continuous field $L^2_X(M \times \mathbb{R} \times X, \Lambda(\pi) \hat{\otimes} \Sigma)$ of Hilbert-C^*_rG-modules over X as before. The space Γ' of continuous sections of $L^2_X(M \times \mathbb{R} \times X, \Lambda(\pi) \hat{\otimes} \Sigma)$ is a Real graded Hilbert-$C(X, C^*_rG)$-module.

Lemma 5.1.1. *The Hilbert modules* $\left(\Gamma_0 \hat{\otimes} \mathbf{Cl}^{0,1}\right) \hat{\otimes}_{\mu \hat{\otimes} 1} \left(L^2(\mathbb{R}, \mathbb{S}_{1,1}) \hat{\otimes} C(X, C^*_rG)\right)$ *and* Γ' *are isomorphic as Real graded Hilbert-*$C(X, C^*_rG)$*-modules.*

Proof. It is sufficient to construct a surjective isometry

$$\Phi\colon (\Gamma_0 \hat{\otimes} \mathbf{Cl}^{0,1}) \hat{\otimes}_{\mu \hat{\otimes} 1} (L^2(\mathbb{R}, \mathbb{S}_{1,1}) \hat{\otimes} C(X, C^*_rG)) \to \Gamma'.$$

For $(s \hat{\otimes} \mathbf{a}) \hat{\otimes} (u \hat{\otimes} \gamma) \in \left(\Gamma^\infty_0(M \times \mathbb{R} \times X, \Lambda(\pi)) \hat{\otimes} \mathbf{Cl}^{0,1}\right) \hat{\otimes}_{\mu \hat{\otimes} 1} \left(L^2(\mathbb{R}, \mathbb{S}_{1,1}) \hat{\otimes} C(X, C^*_rG)\right)$ we define for every $x \in X$:

$$\Phi((s \hat{\otimes} \mathbf{a}) \hat{\otimes} (u \hat{\otimes} \gamma))_x := [(p, \lambda) \mapsto s_x(p, \lambda)\gamma(x) \hat{\otimes} \mathbf{a}u(\lambda)],$$

where s_x denotes the restriction of $s \in \Gamma^\infty_0(M \times \mathbb{R} \times X, \Lambda(\pi))$ to $W_x := M \times \mathbb{R} \times \{x\}$.

We claim that $\Phi((s \hat{\otimes} \mathbf{a}) \hat{\otimes} (u \hat{\otimes} \gamma))_x \in L^2(W_x, \Sigma\Lambda(\pi)_x)$, i. e. that

$$\begin{aligned} &\langle \Phi((s \hat{\otimes} \mathbf{a}) \hat{\otimes} (u \hat{\otimes} \gamma))_x, \Phi((s \hat{\otimes} \mathbf{a}) \hat{\otimes} (u \hat{\otimes} \gamma))_x \rangle_x \\ \overset{\text{Def.}}{=} &\int_{W_x} (s_x\gamma(x) \hat{\otimes} \mathbf{a}u(\lambda), s_x\gamma(x) \hat{\otimes} \mathbf{a}u(\lambda)) \,\mathrm{dvol}(x) \\ = &\int_{\mathbb{R}} \langle s\gamma, s\gamma \rangle_{\Gamma_0}(\_, x)\mathbf{a}^*\mathbf{a}u(\lambda)^*u(\lambda)d\lambda \end{aligned}$$

exists. However since $\langle s\gamma, s\gamma \rangle_{\Gamma_0}(\_, x) \in C_0(\mathbb{R}, C_r^*G)$ and $u \in L^2(\mathbb{R}, \mathbb{S}_{1,1})$ the last integral exists. Because $\Gamma_c^\infty(M \times \mathbb{R} \times X, \Lambda(\pi)) \subset \Gamma_0^\infty(M \times \mathbb{R} \times X, \Lambda(\pi))$ is dense, for each $x \in X$ and every $\varepsilon > 0$ there exists a neighborhood U of x and a compactly supported smooth section t of $\Sigma\Lambda(\pi)$ such that

$$\sup_{y \in U} \|\Phi((s \hat{\otimes} \mathbf{a}) \hat{\otimes} (u \hat{\otimes} \gamma))_y - t_y\| \leqslant \varepsilon.$$

Therefore $\Phi(s, \mathbf{a}, u, \gamma) := (\Phi((s \hat{\otimes} \mathbf{a}) \hat{\otimes} (u \hat{\otimes} \gamma))_x)_{x \in X}$ is indeed an element in Γ'. It is clear that Φ is graded and preserves the Real structure. Next we show that Φ is an isometry. It is sufficient to show that

$$\langle (s \hat{\otimes} \mathbf{a}) \hat{\otimes} (u \hat{\otimes} \gamma), (t \hat{\otimes} \mathbf{b}) \hat{\otimes} (\nu \hat{\otimes} \delta) \rangle(x) = \langle \Phi((s \hat{\otimes} \mathbf{a}) \hat{\otimes} (u \hat{\otimes} \gamma)), \Phi((t \hat{\otimes} \mathbf{b}) \hat{\otimes} (\nu \hat{\otimes} \delta)) \rangle(x)$$

holds for every $x \in X$. Note that the formulas for the interior tensor product in the ungraded and graded case agree. Moreover $\langle u \hat{\otimes} \gamma, \nu \hat{\otimes} \delta \rangle = \langle u \otimes \gamma, \nu \otimes \delta \rangle$ since $C(X, C_r^*G)$ is trivially graded. Therefore

$$\begin{aligned} \langle (s \hat{\otimes} \mathbf{a}) \hat{\otimes} (u \hat{\otimes} \gamma), (t \hat{\otimes} \mathbf{b}) \hat{\otimes} (\nu \hat{\otimes} \delta) \rangle(x) &\overset{\text{Def.}}{=} \langle u \hat{\otimes} \gamma, \mu \hat{\otimes} 1(\langle s \hat{\otimes} \mathbf{a}, t \hat{\otimes} \mathbf{b} \rangle)(\nu \hat{\otimes} \delta) \rangle(x) \\ &= \langle u \hat{\otimes} \gamma, \mu \hat{\otimes} 1(\langle s, t \rangle_{\Gamma_0} \hat{\otimes} \mathbf{a}^*\mathbf{b})(\nu \hat{\otimes} \delta) \rangle(x) \\ &= \int_{\mathbb{R}} \langle s, t \rangle_{\Gamma_0}(\_, x)u(\lambda)^*\mathbf{a}^*\mathbf{b}\nu(\lambda)\gamma(x)^*\delta(x)d\lambda. \end{aligned}$$

On the other hand

$$\begin{aligned} \langle \Phi(s, \mathbf{a}, u, \gamma), \Phi(t, \mathbf{b}, \nu, \delta) \rangle(x) &\overset{\text{Def.}}{=} \langle \Phi((s \hat{\otimes} \mathbf{a}) \hat{\otimes} (u \hat{\otimes} \gamma))_x, \Phi((t \hat{\otimes} \mathbf{b}) \hat{\otimes} (\nu \hat{\otimes} \delta))_x \rangle_x \\ &= \int_{W_x} (s_x\gamma(x) \hat{\otimes} \mathbf{a}u, t_x\delta(x) \hat{\otimes} \mathbf{b}\nu) \,\mathrm{dvol}(x) \\ &= \int_{W_x} (s_x \hat{\otimes} \mathbf{a}u, t_x \hat{\otimes} \mathbf{b}\nu)\gamma(x)^*\delta(x) \,\mathrm{dvol}(x) \\ &= \int_{\mathbb{R}} \langle s, t \rangle_{\Gamma_0}(\_, x)(\mathbf{a}u(\lambda))^*\mathbf{b}\nu(\lambda)\gamma(x)^*\delta(x)d\lambda. \end{aligned}$$

In particular both inner products are equal and hence Φ is an isometry, once we have shown that Φ is surjective. To show that Φ is surjective, it suffices to show that Φ has dense image. Therefore it is sufficient to prove that each $t \in \Gamma_c^\infty(M \times \mathbb{R} \times X, \Sigma\Lambda(\pi))$ can be approximated in norm by elements in the image of Φ. By the inverse mapping theorem we can cover $M \times \mathbb{R} \times X$ by relatively compact "box neighborhoods" $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^k$. Since $\mathrm{supp}(t)$ is compact, we can assume that t is supported in the unit ball of a box neighborhood, using a partition of unity. Over such a box neighborhood the bundle $\Lambda(\pi)$ is trivial. Hence t is given by $t := t_0 \hat{\otimes} t_1 \colon \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^k \to (\mathcal{S}_d \hat{\otimes} C_r^*G) \hat{\otimes} \mathcal{S}_{1,1}$, such that t_0 is a (local) section of $\Lambda(\pi)$ and t_1 is a (local) section of the trivial bundle Σ. Then we can choose $s_0 \in \Gamma_0^\infty(M \times \mathbb{R} \times X, \Lambda(\pi))$ and $u \in L^2(\mathbb{R}, \mathcal{S}_{1,1})$ approximating t on its support. □

By the previous lemma we can identify the Hilbert modules $(\Gamma_0 \hat{\otimes} \mathbf{Cl}^{0,1}) \hat{\otimes}_{\mu \hat{\otimes} 1} (L^2(\mathbb{R}, \mathcal{S}_{1,1}) \hat{\otimes} C(X, C_r^*G))$ and Γ' by means of the even isometry Φ. Therefore it is sufficient to construct an operator F on Γ' such that the triple (Γ', ρ', F) is an unbounded Kasparov module representing the unbounded product of $\mathbf{x}$ and $\mathbf{y}$.

Lemma 5.1.2. *The bundles $\Lambda(\Pi)$ and $\Lambda(\pi) \hat{\otimes} \Sigma$ over $M \times \mathbb{R} \times X$ are isomorph and the Clifford action of $\mathbf{Cl}^{0,d+1}$ on both bundles agree.*

Proof. Since all bundles are trivial in X it is sufficient to show that $\Lambda(\Pi)_x \cong \Lambda(\pi)_x \hat{\otimes} \Sigma_x$ for all $x \in X$. This follows once we have shown that $\not{\mathfrak{S}}(\Pi)_x \cong \not{\mathfrak{S}}(\pi)_x \hat{\otimes} \Sigma_x$. However $\not{\mathfrak{S}}(\Pi)_x = \not{\mathfrak{S}}(W_x) \cong \not{\mathfrak{S}}(T_v\pi)_{|W_x} \hat{\otimes} \Sigma_x = \not{\mathfrak{S}}(\pi)_x \hat{\otimes} \Sigma_x$. □

From now on we will identify both bundles by means of the above isomorphism. On the bundle $\Lambda(\pi) \hat{\otimes} \Sigma$ we have the family

$$\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D : \Gamma_{cv}^\infty(M \times \mathbb{R} \times X, \Lambda(\pi) \hat{\otimes} \Sigma) \to \Gamma_{cv}^\infty(M \times \mathbb{R} \times X, \Lambda(\pi) \hat{\otimes} \Sigma) \tag{5.5}$$

where $\not{D}$ is given by the family of twisted Dirac operators $(\not{D}_{\lambda,x})_{(\lambda,x) \in \mathbb{R} \times X}$. The operator D is given by differentiating in the $\mathbb{R}$-direction, i. e. by the operator

$$\begin{pmatrix} 0 & -\nabla_{\partial_\lambda} \\ \nabla_{\partial_\lambda} & 0 \end{pmatrix}$$

with respect to the direct sum decomposition of $\Gamma_{cv}^\infty(M \times \mathbb{R} \times X, \Lambda(\pi) \hat{\otimes} \Sigma)$ induced by the grading on $\Sigma\Lambda(\pi)$. The induced unbounded operator family

$$\mathcal{D} := \not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D \colon \ \mathrm{dom}(\mathcal{D})_X \to L_X^2(M \times \mathbb{R} \times X, \Lambda(\pi) \hat{\otimes} \Sigma) \tag{5.6}$$

on $L_X^2(M \times \mathbb{R} \times X, \Lambda(\pi) \hat{\otimes} \Sigma)$ is densely defined and symmetric. This unbounded operator family also defines an unbounded operator on the Hilbert-C_r^*G-module

Γ' denoted again by $\not{\partial} \hat{\otimes} 1 + 1 \hat{\otimes} D$ with domain $\Gamma^\infty_{cv}(M \times \mathbb{R} \times X, \Lambda(\pi) \hat{\otimes} \Sigma)$.

Lemma 5.1.3. *The operator* $Q := \not{\mathfrak{D}} - \not{\partial} \hat{\otimes} 1 + 1 \hat{\otimes} D \colon \Gamma^\infty_{cv}(M \times \mathbb{R} \times X, \Lambda(\pi) \hat{\otimes} \Sigma) \to \Gamma'$ *is a compactly supported operator of order zero. In particular* $q := \|Q\|$ *is finite.*

Proof. It is a general fact, that for a cylindrical metric on $M \times \mathbb{R}$ the Levi-Civita connection $\nabla^{M \times \mathbb{R}}$ on $T(M \times \mathbb{R})$ and the product connection $\nabla^M \otimes 1 + 1 \otimes \nabla^{\mathbb{R}}$ on $TM \otimes T\mathbb{R}$ agree. It follows that the Spin Dirac operators on $\not{\mathfrak{S}}(M \times \mathbb{R}) = \not{\mathfrak{S}}(M) \hat{\otimes} \not{\mathfrak{S}}(\mathbb{R})$ are identical. In particular the twisted Spin Dirac operators agree. Outside $K := M \times [-1, 1] \times X$ the metric $g_\lambda(x) + dt^2$ on W_x becomes cylindrical. Therefore outside K the operators $\not{\mathfrak{D}}_x$ and $(\not{\partial} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x$ are identical for every $x \in X$. Over K the difference $\not{\mathfrak{D}}_x - (\not{\partial} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x$ is a differential operator of order zero. As its support is compact, its operator norm is finite. □

Lemma 5.1.3 implies that Q and $-Q$ are relatively $\not{\mathfrak{D}}$-bounded, i. e. that

$$\|Qs\| \leqslant \|Q\| \cdot \|s\| \leqslant \epsilon \|\not{\mathfrak{D}} s\| + q \cdot \|s\|$$

for $s \in \mathrm{dom}(\not{\mathfrak{D}})$ and every $\epsilon > 0$ and with $q > 0$. Hence we can use the Kato-Rellich Theorem ([KL17, Theorem 4.5]) to conclude that $\not{\mathfrak{D}} - Q = \not{\partial} \hat{\otimes} 1 + 1 \hat{\otimes} D$ has the same domain as $\not{\mathfrak{D}}$ and is self-adjoint and regular. Using the same arguments provided in the previous chapter one also proves that the unbounded operator (family) $h(\not{\partial} \hat{\otimes} 1 + 1 \hat{\otimes} D)h$ has compact resolvent. We can summarize this as follows:

Lemma 5.1.4. *The triple* $(\Gamma', \rho', h(\not{\partial} \hat{\otimes} 1 + 1 \hat{\otimes} D)h)$ *defines an unbounded Kasparov-*$(\mathbf{Cl}^{0,d+1}, C(X, C^*_r G))$*-module.*

Proof. The space Γ' is the space of continuous sections of a continuous field of Real graded Hilbert-$C^*_r G$-modules over the compact space X. Therefore it also has the structure of an Real graded Hilbert-$C(X, C^*_r G)$-module. The Clifford action on $\Lambda(\pi) \hat{\otimes} \Sigma$ induces the Clifford action ρ' of $\mathbf{Cl}^{0,d+1}$ on Γ' and the unbounded odd Real self-adjoint and regular operator $h(\not{\partial} \hat{\otimes} 1 + 1 \hat{\otimes} D)h$ anticommutes with ρ'. Since the operator also has compact resolvent, the triple $(\Gamma', \rho', h(\not{\partial} \hat{\otimes} 1 + 1 \hat{\otimes} D)h)$ is indeed an unbounded Kasparov-$(\mathbf{Cl}^{0,d+1}, C(X, C^*_r G))$-module. □

5.2 Calculation of the Kasparov product

Our next goal is

Theorem 5.2.1. *The unbounded Kasparov module* $(\Gamma', \rho', h(\not{\partial} \hat{\otimes} 1 + 1 \hat{\otimes} D)h)$ *represents the Kasparov product of* **x** *and* **y**.

To prove theorem 5.2.1 we will show that the unbounded Kasparov module $\mathbf{z}' := (\Gamma', \rho', h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h)$ satisfies the conditions of theorem 1.3.9.

It is obvious that $\not{D} \hat{\otimes} 1 \circ \Phi = \Phi \circ (\not{D} \hat{\otimes} 1) \hat{\otimes}_{\mu \hat{\otimes} 1} (1 \hat{\otimes} 1)$. In fact as, the notation suggests, both operators are induced by the same family of operators. To ease notation we will write $\not{D}_\mu$ instead of $(\not{D} \hat{\otimes} 1) \hat{\otimes}_{\mu \hat{\otimes} 1} (1 \hat{\otimes} 1)$.

Definition 5.2.2. *For $s \hat{\otimes} \boldsymbol{a} \in \Gamma_0 \hat{\otimes} \mathbf{Cl}^{0,1}$ we define $\Phi_{s \hat{\otimes} \boldsymbol{a}} \colon L^2(\mathbb{R}, \mathcal{S}_{1,1}) \hat{\otimes} C(X, C^*_r G) \to \Gamma'$ as the composition*

$$L^2(\mathbb{R}, \mathcal{S}_{1,1}) \hat{\otimes} C(X, C^*_r G) \xrightarrow{T_{s\hat{\otimes} a}} \left(\Gamma_0 \hat{\otimes} \mathbf{Cl}^{0,1}\right) \hat{\otimes}_{\mu \hat{\otimes} 1} \left(L^2(\mathbb{R}, \mathcal{S}_{1,1}) \hat{\otimes} C(X, C^*_r G)\right) \xrightarrow{\Phi} \Gamma'.$$

This map has an adjoint $\Phi^*_{s \hat{\otimes} \mathbf{a}} = T^*_{s \hat{\otimes} \mathbf{a}} \circ \Phi^*$. In particular $\Phi_{s \hat{\otimes} \mathbf{a}}$ is bounded.

Proposition 5.2.3. *The map $h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h \circ \Phi_{s \hat{\otimes} \mathbf{a}} - (-1)^{\partial(s \hat{\otimes} \mathbf{a})} \Phi_{s \hat{\otimes} \mathbf{a}} \circ h(D_\mathbb{R} \hat{\otimes} 1)h$ from $\mathrm{dom}(D_\mathbb{R} \hat{\otimes} 1)$ to Γ' is bounded for every $s \hat{\otimes} \mathbf{a} \in \mathrm{dom}(\not{D} \hat{\otimes} 1) \subset \Gamma_0 \hat{\otimes} \mathbf{Cl}^{0,1}$.*

Proof. Let $u \hat{\otimes} \gamma \in \mathrm{dom}(D_\mathbb{R} \hat{\otimes} 1)$. The commutator $[(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D), h]$ is given by Clifford multiplication with $\mathrm{grad}(h)$. Similarly $[(D_\mathbb{R} \hat{\otimes} 1), h] = hc(h')$. Moreover $\rho'(\mathrm{grad}(h)) \circ \Phi_{s \hat{\otimes} \mathbf{a}} = (-1)^{\partial(s \hat{\otimes} \mathbf{a})} \Phi_{s \hat{\otimes} \mathbf{a}} \circ c(h')$. Using this we compute

$$\begin{aligned}
&\left(h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h \circ \Phi_{s \hat{\otimes} \mathbf{a}} - (-1)^{\partial(s \hat{\otimes} \mathbf{a})} \Phi_{s \hat{\otimes} \mathbf{a}} h(D_\mathbb{R} \hat{\otimes} 1)h\right)(u \hat{\otimes} \gamma) \\
&= h^2\left((\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D) \circ \Phi_{s \hat{\otimes} \mathbf{a}} - (-1)^{\partial(s \hat{\otimes} \mathbf{a})} \Phi_{s \hat{\otimes} \mathbf{a}} \circ h^2(D_\mathbb{R} \hat{\otimes} 1)\right)(u \hat{\otimes} \gamma) \\
&= h^2\left((\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)(s\gamma \hat{\otimes} \mathbf{a}u) - (-1)^{\partial(s \hat{\otimes} \mathbf{a})} s\gamma \hat{\otimes} \mathbf{a} D_\mathbb{R}(u)\right) \\
&= h^2\left(\not{D}(s\gamma) \hat{\otimes} \mathbf{a}u + (-1)^{\partial s} s\gamma \hat{\otimes} D(\mathbf{a}u) - (-1)^{\partial(s \hat{\otimes} \mathbf{a})} s\gamma \hat{\otimes} \mathbf{a} D_\mathbb{R}(u)\right) \\
&= h^2\left(\not{D}(s\gamma) \hat{\otimes} \mathbf{a}u + s\gamma \hat{\otimes} \mathbf{a}\big(\pm D(u) \mp D_\mathbb{R}(u)\big)\right)
\end{aligned}$$

Outside $K := M \times [-1, 1] \times X$ the metric becomes cylindrical. This implies that $\nabla_{\partial_\lambda} = \partial_\lambda$. Therefore $(1 \hat{\otimes} D) \circ \Phi_{s \hat{\otimes} \mathbf{a}}$ and $\Phi_{s \hat{\otimes} \mathbf{a}} \circ (D_\mathbb{R} \hat{\otimes} 1)$ are equal on the complement of K. It follows that $s\gamma \hat{\otimes} (D(u) - D_\mathbb{R}(u))$ has compact support. In particular the map $J := [u \hat{\otimes} \gamma \mapsto (1 \hat{\otimes} D) \circ \Phi_{s \hat{\otimes} \mathbf{a}}(u \hat{\otimes} \gamma) - \Phi_{s \hat{\otimes} \mathbf{a}} \circ (D_\mathbb{R} \hat{\otimes} 1)(u \hat{\otimes} \gamma)]$ is bounded. Hence

$$\begin{aligned}
&\|h^2\left((\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D) \circ \Phi_{s \hat{\otimes} \mathbf{a}} - (-1)^{\partial(s \hat{\otimes} \mathbf{a})} \Phi_{s \hat{\otimes} \mathbf{a}} \circ (D_\mathbb{R} \hat{\otimes} 1)\right)(u \hat{\otimes} \gamma)\| \\
&\leqslant \|h^2 \not{D}(s\gamma) \hat{\otimes} \mathbf{a}u\| + \|h^2 s\gamma \hat{\otimes} \mathbf{a}\big(D(u) - D_\mathbb{R}(u)\big)\| \\
&= \|h^2 \Phi_{\not{D}(s) \hat{\otimes} \mathbf{a}}(u \hat{\otimes} \gamma)\| + \|h^2 s\gamma \hat{\otimes} \mathbf{a}\big(D(u) - D_\mathbb{R}(u)\big)\| \\
&\leqslant (C_0 + C_1)\|u \hat{\otimes} \gamma\|
\end{aligned}$$

with constants C_0 and C_1 given by $C_0 := \sup_{(p,\lambda,x) \in \mathrm{supp}(\not{D}s)} h^2(p, \lambda, x) \|\Phi_{\not{D}(s) \hat{\otimes} \mathbf{a}}\|$ and $C_1 := \sup_{(p,\lambda,x) \in K} \|s(p, \lambda, x)\| \cdot \|J\|$. Note that since $s \in \mathrm{dom}(\not{D})$ the support of s and hence the support of $\not{D}s$ is compact. Therefore $\sup_{(p,\lambda,x) \in \mathrm{supp}(s)} h^2(p, \lambda, x)$ is finite. □

As a corollary of proposition 5.2.3 we obtain the first condition of theorem 1.3.9:

Corollary 5.2.4. *The operator*

$$\left\{ \begin{pmatrix} h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h & 0 \\ 0 & h(D_{\mathbb{R}} \hat{\otimes} 1)h \end{pmatrix}, \begin{pmatrix} 0 & \Phi_{s\hat{\otimes}\mathbf{a}} \\ \Phi^*_{s\hat{\otimes}\mathbf{a}} & 0 \end{pmatrix} \right\} \tag{5.7}$$

is bounded on its domain for all $s \hat{\otimes} \mathbf{a}$ *in the dense submodule* $\operatorname{dom}(\not{D} \hat{\otimes} 1) \subset \Gamma_0 \hat{\otimes} \mathbf{Cl}^{0,1}$.

Proof. The graded commutator 5.7 equals

$$\left\{ \begin{pmatrix} h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h & 0 \\ 0 & h(D_{\mathbb{R}} \hat{\otimes} 1)h \end{pmatrix}, \begin{pmatrix} 0 & \Phi_{s\hat{\otimes}\mathbf{a}} \\ \Phi^*_{s\hat{\otimes}\mathbf{a}} & 0 \end{pmatrix} \right\} = \begin{pmatrix} 0 & T_0 \\ T_1 & 0 \end{pmatrix}$$

with $T_0 = h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h \circ \Phi_{s\hat{\otimes}\mathbf{a}} - (-1)^{\partial(s\hat{\otimes}\mathbf{a})} \Phi_{s\hat{\otimes}\mathbf{a}} \circ h(D_{\mathbb{R}} \hat{\otimes} 1)h$ and $T_1 = h(D_{\mathbb{R}} \hat{\otimes} 1)h \circ \Phi^*_{s\hat{\otimes}\mathbf{a}} - (-1)^{\partial(s\hat{\otimes}\mathbf{a})} \Phi^*_{s\hat{\otimes}\mathbf{a}} \circ h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h$. By proposition 5.2.3 T_0 has a bounded extension and hence its adjoint T_0^* is everywhere defined and therefore bounded. Since $h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h$ and $h(D_{\mathbb{R}} \hat{\otimes} 1)h$ are self-adjoint, T_0^* equals $\Phi^*_{s\hat{\otimes}\mathbf{a}} \circ h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x h - (-1)^{\partial(s\hat{\otimes}\mathbf{a})} h(D_{\mathbb{R}} \hat{\otimes} 1)h \circ \Phi^*_{s\hat{\otimes}\mathbf{a}}$. It follows that T_0^* and T_1 agree on the domain of T_1. In particular T_1 has a bounded extension which must be T_0^*. Altogether this proves that 5.7 is bounded on its domain. □

The second condition of theorem 1.3.9 involves the graded commutator of the unbounded operators $\not{D}_{h,\mu} := (h\not{D}h \hat{\otimes} 1) \hat{\otimes}_{\mu\hat{\otimes}1} (1 \hat{\otimes} 1)$ and $h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h$. However we must be careful since the operators are not defined on the same Hilbert modules. Since $\Phi \circ \not{D}_{h,\mu} = h(\not{D} \hat{\otimes} 1)h \circ \Phi$ it is clear that $\operatorname{dom}(h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h) \subset \Phi(\operatorname{dom}(\not{D}_{h,\mu}))$. For $\Phi(\xi) \in \operatorname{dom}(h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h)$ we define

$$\begin{aligned} S(\xi)_x &:= \langle \Phi \circ (\not{D}_{h,\mu})_{\lambda,x}(\xi), h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x h \circ \Phi(\xi) \rangle_x \text{ and} \\ T(\xi)_x &:= \langle h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x h \circ \Phi(\xi), \Phi \circ (\not{D}_{h,\mu})_{\lambda,x}(\xi) \rangle_x. \end{aligned}$$

The next proposition ensures that the "graded commutator" of the operators $\not{D}_{h,\mu}$ and $h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h$ is semi-bounded.

Proposition 5.2.5. *There exists* $c > 0$ *such that*

$$S(\xi)_x + T(\xi)_x \geqslant c\langle \xi, \xi \rangle_x \tag{5.8}$$

for all $\Phi(\xi) \in \operatorname{dom}(h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h)$ *and every* $x \in X$.

Proof. Since $\Phi \circ (\not{D}_{h,\mu})_{\lambda,x} = h(\not{D} \hat{\otimes} 1)h_x \circ \Phi$ on $\operatorname{dom}(\not{D}_{h,\mu})$ we have

$$S(\xi)_x = \langle h(\not{D} \hat{\otimes} 1)_x h \circ \Phi(\xi), h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x h \circ \Phi(\xi) \rangle_x \text{ and}$$

$$T(\xi)_x = \langle h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x h \circ \Phi(\xi), h(\not{D} \hat{\otimes} 1)_x h \circ \Phi(\xi)\rangle_x.$$

Since $h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h_x$ is regular and self-adjoint it suffices to consider only elements in $\mathrm{dom}(h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h_x^2)$. Let $\zeta \in \mathrm{dom}(h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h_x^2)$. It follows that ζ is also in $\mathrm{dom}(h(\not{D} \hat{\otimes} 1)h_x^2) = \Phi(\mathrm{dom}(\not{D}_{h,\mu})^2)$ since $\mathrm{dom}(h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h_x^2) = \mathrm{dom}(h(\not{D}^2 \hat{\otimes} 1)h_x) \cap \mathrm{dom}(h(1 \hat{\otimes} D^2)h_x)$. Moreover $h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h_x(\zeta) \in \mathrm{dom}(h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)h_x) \subset \mathrm{dom} h((\not{D} \hat{\otimes} 1)h_x) = \Phi(\mathrm{dom}(\not{D}_{h,\mu}))$. Let $\zeta := \Phi(\xi) \in \mathrm{dom}((\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x^2)$. We first compute $S(\xi)_x$:

$$\begin{aligned}
S(\xi)_x &= \langle h(\not{D} \hat{\otimes} 1)_x h(\zeta), h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x(\zeta)\rangle_x \\
&= \int_{W_x} \Big(h(\not{D} \hat{\otimes} 1)h(\zeta), h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x h(\zeta)\Big)\, \mathrm{dvol}(x) \\
&= \int_{W_x} h^2\Big((\not{D} \hat{\otimes} 1)_x h(\zeta), (\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x(\zeta)\Big)\, \mathrm{dvol}(x) \\
&= \int_{W_x} h^2\Big(h(\zeta), (\not{D} \hat{\otimes} 1)_x(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x h(\zeta)\Big)\, \mathrm{dvol}(x) \\
&= \int_{W_x} h^2\Big(h(\zeta), (\not{D} \hat{\otimes} 1)_x^2 h(\zeta) + (\not{D} \hat{\otimes} 1)_x(1 \hat{\otimes} D)_x h(\zeta)\Big)\, \mathrm{dvol}(x) \\
&= \int_{W_x} h^2\Big(h(\zeta), (\not{D} \hat{\otimes} 1)_x^2 h(\zeta)\Big) + h^2\Big(h(\zeta), (\not{D} \hat{\otimes} 1)_x(1 \hat{\otimes} D)_x h(\zeta)\Big)\, \mathrm{dvol}(x) \\
&= \int_{W_x} h^2\Big((\not{D} \hat{\otimes} 1)_x^2 h(\zeta), h(\zeta)\Big) + h^2\Big((1 \hat{\otimes} D)_x(\not{D} \hat{\otimes} 1)_x h(\zeta), h(\zeta)\Big)\, \mathrm{dvol}(x) \\
&= \int_{W_x} h^2\Big((\not{D} \hat{\otimes} 1)_x^2 h(\zeta), h(\zeta)\Big) - h^2\Big((\not{D} \hat{\otimes} D)_x h(\zeta), h(\zeta)\Big)\, \mathrm{dvol}(x)
\end{aligned}$$

The minus sign in the last equation follows from the Koszul rule for multiplying graded tensors: $(1 \hat{\otimes} D)_x \circ (\not{D} \hat{\otimes} 1)_x = (-1)^{\partial D \partial \not{D}}(\not{D} \hat{\otimes} D)_x$. Now we compute $T(\xi)_x$:

$$\begin{aligned}
T(\zeta)_x &= \langle h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x h(\zeta), h(\not{D} \hat{\otimes} 1)_x h(\zeta)\rangle_x \\
&= \int_{W_x} h^2\Big(((\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x h(\zeta), (\not{D} \hat{\otimes} 1)_x h(\zeta))\Big)\, \mathrm{dvol}(x) \\
&= \int_{W_x} h^2\Big(((\not{D} \hat{\otimes} 1)_x(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D)_x h(\zeta), h(\zeta))\Big)\, \mathrm{dvol}(x) \\
&= \int_{W_x} h^2\Big(((\not{D} \hat{\otimes} 1)_x^2 h(\zeta), h(\zeta))\Big) + h^2\Big(((\not{D} \hat{\otimes} 1)_x(1 \hat{\otimes} D)h(\zeta), h(\zeta))\Big)\, \mathrm{dvol}(x)
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
S(\xi)_x + T(\xi)_x &= \int\limits_{W_x} 2h^2\Big(((\not{D}\hat{\otimes}1)_x^2 h(\zeta), h(\zeta))\Big)\,\mathrm{dvol}(x) \\
&= \int\limits_{W_x} 2h^2\Big(((\not{D}\hat{\otimes}1)_x^2 h(\zeta), h(\zeta))\Big)\,\mathrm{dvol}(x) = 2\langle h^2(\not{D}\hat{\otimes}1)_x^2 h(\zeta), h(\zeta)\rangle_x \\
&= 2\langle h^3\Phi((\not{D}_\mu)_{\lambda,x}^2(\xi)), h\Phi(\xi)\rangle_x \geqslant 2\langle h^2\Phi(\frac{1}{4}\operatorname{scal}(g_\lambda(x))\xi), h^2\Phi(\xi)\rangle_x \\
&= \langle\frac{1}{2}\operatorname{scal}(g_\lambda(x))h^2\zeta, h^2\zeta\rangle_x = \frac{1}{2}\int\limits_{W_x}\operatorname{scal}(g_\lambda(x))\Big(h^2\zeta, h^2\zeta\Big)\,\mathrm{dvol}(x) \\
&\geqslant \frac{1}{2}m(x)\int\limits_{W_x}\|h^2_{|\mathrm{supp}(\zeta)}\|(\zeta,\zeta)\,\mathrm{dvol}(x) = \frac{1}{2}C\langle\zeta,\zeta\rangle_x \\
&= \frac{1}{2}C\langle\Phi(\xi),\Phi(\xi)\rangle_x = \frac{1}{2}C\langle\xi,\xi\rangle_x
\end{aligned}$$

with $C = m(x)\|h^2_{|\mathrm{supp}(\zeta)}\|$ and $m(x) = \min_{|\lambda|\leqslant 1}(\operatorname{scal}(g_\lambda(x)))$. □

The proof of theorem 5.2.1 is now an easy corollary of proposition 5.2.4 and proposition 5.2.5 using theorem 1.3.9:

Proof of 5.2.1. By propositions 5.2.4 and 5.2.5 the unbounded Kasparov module $\mathbf{z}'$ satisfies the assumptions of theorem 1.3.9. Therefore the bounded Kasparov-$(\mathbf{Cl}^{0,d+1}, C(X, C_r^*G))$-module $b(\mathbf{z}') = (\Gamma', \rho', b(h(\not{D}\hat{\otimes}1 + 1\hat{\otimes}D)h))$ represents the Kasparov product of the bounded Kasparov modules $b(\mathbf{x})$ and $b(\mathbf{y})$. □

Proposition 5.2.6. $[b(\mathrm{inddiff}^{GL}_{\mathscr{G}})] = [b(\mathbf{z}')]$ *in* $KK(\mathbf{Cl}^{0,d+1}, C(X, C_r^*G))$.

Proof. We will construct a homotopy between $b(\mathrm{inddiff}^{GL}_{\mathscr{G}})$ and $b(\mathbf{z}')$, i. e. a Kasparov module $\mathbf{H} \in KK(\mathbf{Cl}^{0,d+1}, \mathbf{I}C(X, C_r^*G)) = KK(\mathbf{Cl}^{0,d+1}, C(X\times\mathbf{I}, C_r^*G))$, such that $(ev_0)_*(\mathbf{H}) = b(\mathbf{z}')$ and $(ev_1)_*(\mathbf{H}) = b(\mathrm{inddiff}^{GL}_{\mathscr{G}})$. To begin with we construct the following unbounded Kasparov module $\mathbf{h} := (\Gamma_\mathbf{I}, \rho_\mathbf{I}, h(\not{D}\hat{\otimes}1 + 1\hat{\otimes}D + \lambda\cdot Q)h)$. The Hilbert module $\Gamma_\mathbf{I}$ is the space of continuous sections of the pullback of the continuous field $L^2_X(M\times\mathbb{R}\times X, \Lambda(\Pi))$ via $pr : X\times\mathbf{I} \to X$. The Clifford action $\rho_\mathbf{I}$ is induced by the Clifford action ρ of $\mathbf{Cl}^{0,d+1}$ on $L^2_X(M\times\mathbb{R}\times X, \Lambda(\Pi))$. The operator $Q = \not{\mathfrak{D}} - \not{D}\hat{\otimes}1 + 1\hat{\otimes}D$ is a compactly supported bundles endomorphism of $\Lambda(\Pi)$ by lemma 5.1.3. We claim that $\mathbf{h}$ is an unbounded Kasparov-$(\mathbf{Cl}^{0,d+1}, C(X\times\mathbf{I}, C_r^*G)$-module. To prove that we only need to check that the resolvent of $h(\not{D}\hat{\otimes}1 + 1\hat{\otimes}D + \lambda\cdot Q)h = h(\not{D}\hat{\otimes}1 = 1\hat{\otimes}D)h + \lambda\cdot h^2Q$ is compact. However, since Q has order zero and compact support, $\|h^2Q\|$ is finite. Therefore by changing the coercive function f only on the compact subset $M\times[-1,1]\times X$ by a finite constant implies that $h(\not{D}\hat{\otimes}1 + 1\hat{\otimes}D)h + \lambda\cdot h^2Q$ is bounded from below by a coercive function. It follows by theorem 3.2.9 that the resolvent of $h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D + \lambda \cdot Q)h$ is compact. We define $\mathbf{H} := b(\mathbf{h}) \in KK(\mathbf{Cl}^{0,d+1}, C(X \times \mathbf{I}, C_r^* G))$. Then $(ev_0)_*(\mathbf{H}) = b(\mathbf{z}')$. Furthermore sine $h(\not{D} \hat{\otimes} 1 + 1 \hat{\otimes} D + Q)h = h\not{\mathfrak{D}}h$ we have that $(ev_1)_*(\mathbf{H}) = b(\text{inddiff}_{\mathscr{G}}^{GL})$. Therefore $\mathbf{H}$ is a homotopy between the Kasparov-$(\mathbf{Cl}^{0,d+1}, C(X, C_r^* G))$-modules $b(\mathbf{z}')$ and $b(\text{inddiff}_{\mathscr{G}}^{GL})$. □

Now we can prove theorem 5.0.1:

Proof of 5.0.1. $[b(\mathbf{z}')] = [b(\text{inddiff}_{\mathscr{G}}^{GL})]$ by proposition 5.2.6. However

$$b(\mathbf{z}') = \tau_{\mathbf{Cl}^{0,1}}(b(\text{inddiff}_{\mathscr{G}}^{H}))\# b(\tau_{C(X,C_r^* G)}(\alpha))$$

by theorem 5.2.1. Therefore

$$b(\text{inddiff}_{\mathscr{G}}^{GL}) = \tau_{\mathbf{Cl}^{0,1}}(b(\text{inddiff}_{\mathscr{G}}^{H}))\# \tau_{C(X,C_r^* G)}(\alpha).$$

This finishes the proof of theorem 5.0.1. □

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