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Separable and tree-like asymptotic cones of groups

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Abstract. Using methods from nonstandard analysis, we will discuss which metric spaces can be realized as ultralimits (in particular, asymptotic cones). For example, we will show that any separable ultralimit is proper. Applying the results we will find in the context of groups, we will classify the real trees appearing as asymptotic cones of (not necessarily hyperbolic) finitely generated groups. Also, we show that all proper metric spaces can be realized as asymptotic cones.

1. Introduction

The asymptotic cones of a metric space are obtained "rescaling the metric by an infinitesimal factor", in such a way that "infinitely far away" points become close, while points which are not far enough are identified.

They have been introduced by Gromov in [6] in the proof that a finitely generated group of polynomial growth is virtually nilpotent. Van den Dries and Wilkie gave a different and more general definition in [22], where they slightly generalize and simplify the proof of the aforementioned result of Gromov.

Since then, asymptotic cones have been used in several contexts, such as the proof of quasi-isometric rigidity results for cocompact lattices in higher rank semisimple groups ([11]), fundamental groups of Haken manifolds ([9, 10]), relatively hyperbolic groups ([2]) and others. Also, there is a close connection between many quasi-isometric invariants for groups (e.g. growth and order of Dehn functions) and the topology and geometry of the asymptotic cones, see [1] for a survey.

To define the asymptotic cones formally we will use nonstandard methods, which are powerful tools to formally deal with concepts such as "infinitesimals", "infinite numbers", "points infinitely far away", etc. More generally, we will deal with ultralimits of metric spaces, asymptotic cones being a special case of these.

We will then use those methods to investigate the following question: which metric spaces can be obtained as an asymptotic cone (resp. ultralimit) of another metric space or of a group? Our results rely on the well-known fact that an internal set (i.e. an ultraproduct of a sequence of sets) is either finite or has cardinality at least $2^{\aleph_0}$ (Lemma 5.1), and they often reflect this dichotomy. Perhaps this lemma inspired Gromov (see [7]) to ask if it is true that an asymptotic cone of a group (endowed with a word metric) has either trivial or uncountable fundamental group (however, this is not true, see [16]).

Our first main result is the following, (3) being just a reformulation of (2) in the language of ultrafilters. Recall that the metric space Z is *doubling* if there exists N so that each ball of radius $r > 0$ in Z can be covered by at most N balls of radius $r/2$.

Theorem 1.1.

(1) Suppose that the metric space X is an ultralimit. Then if X is separable, it is proper. More precisely, if X is not proper, then it contains $2^{\aleph_0}$ disjoint balls.

(2) Fix nonstandard extensions of $\mathbb{R}$ and the metric space Y. Suppose that for some $0 \ll \mu \ll \nu$ and some $p \in {}^*Y$ each asymptotic cone of Y with scaling factor ν' such that $\mu \leq \nu' \leq \nu$ and basepoint p is separable. If the asymptotic cone X of Y with scaling factor ν and basepoint p is homogeneous, then every closed ball in X is doubling.

(3) Fix a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Suppose that for some sequences (r_n), (s_n) satisfying $\lim_{\mathcal{U}} r_n = \lim_{\mathcal{U}} s_n/r_n = +\infty$ and some sequence of basepoints (p_n) of the metric space Y each asymptotic cone of Y with scaling factor (r'_n) satisfying $\{n \mid r_n \leq r'_n \leq s_n\} \in \mathcal{U}$ is separable. If the asymptotic cone X with respect to the ultrafilter $\mathcal{U}$, scaling factor (s_n) and basepoint (p_n) is homogeneous, then every closed ball in X is doubling.

For example, a corollary of point (1) is that the separable Hilbert space cannot appear as an asymptotic cone.

It follows from a recent result of Hrushovski [8, Thm. 7.1] that a finitely generated group with one proper (or, in view of point (1), separable) asymptotic cone is virtually nilpotent. We point out that as a corollary of Theorem 1.1 and the main result of [18] one can get the following somewhat weaker result: if G is a finitely generated group endowed with a word metric and there exist $\nu_1 \ll \nu_2$ such that all the asymptotic cones of G with scaling factor $\nu \in [\nu_1, \nu_2]$ are separable, then G is virtually nilpotent. These issues are discussed in Remark 5.11.

We will also prove two results about real trees appearing as asymptotic cones (Corollary 5.5 and Proposition 5.7). These are interesting objects in view of the fact that a geodesic metric space is hyperbolic if and only if each of its asymptotic cones is a real tree [7] (see also [1], [4]). In particular, we show that there are three possible real trees appearing as the asymptotic cone of a finitely generated group, up to isometry. More precisely:

Theorem 1.2 (Corollary 5.9). *If the real tree X is the asymptotic cone of a finitely generated group (endowed with a word metric), then it is a point, a line or a real tree with valency $2^{\aleph_0}$ at each point. The same holds true for asymptotic cones of a separable topological Hausdorff group endowed with a left invariant metric.*

Recall from [15] that if T_1, T_2 are homogeneous real trees such that the valency at a point in T_1 is the same as the valency at a point in T_2, then T_1 and T_2 are isometric, so that the theorem actually gives us three real trees up to isometry.

If a group is finitely presented and has one asymptotic cone which is a real tree, then it is hyperbolic (see the appendix of [16] by Kapovich and Kleiner). Therefore, in the case of finitely presented groups, this classification follows from the results in [3]. However, there are examples of finitely generated groups such that just some of their asymptotic cones are real trees (see [21, 16]) and our results can be applied to metric spaces in general. Finitely generated groups such that at least one of their asymptotic cones is a real tree are called lacunary hyperbolic and they are studied in [16].

Theorem 1.2 is used in [20] to study tree-graded spaces arising as asymptotic cones of finitely generated groups, see also [17].

Finally, we will prove that all proper metric spaces can be realized as asymptotic cones:

Theorem 1.3 (Theorem 5.12). *If the metric space X is proper, then it is an asymptotic cone of some metric space Y. If X is also geodesic and unbounded, we can choose Y to be geodesic as well.*

The definition of asymptotic cone we will present is not stated in the way it is usually found in the literature, with some exceptions [12, 13]. In fact, use of nonstandard methods tends to be avoided and a definition based on ultrafilters is usually given, even though the ultrafilters based definition is just a restatement of the nonstandard definition. The author thinks that the nonstandard definition is far more convenient because, besides providing a lighter formalism, it allows to directly apply basic results about nonstandard extensions, particular cases of which ought to be proved in most arguments if the other definition is used. Also, the nonstandard definition is "philosophically" closer to the idea of looking at a metric space from infinitely far away, while the other one is closer to the idea of Gromov of convergence of rescaled metric spaces, which is more complicated to "visualize".

2. Basic notation and definitions

If X is a metric space, $x \in X$ and $r \in \mathbb{R}^+$, we will denote by $B(x,r)$ (resp. $\overline{B}(x,r)$) the open (resp. closed) ball with center x and radius r.

Recall that a metric space X is proper if closed balls in X are compact.

A geodesic (parametrized by arc length) in the metric space X is a curve $\gamma : [0, l] \to X$ such that $d(\gamma(t), \gamma(s)) = |t - s|$ for each $t, s \in [0, l]$. The metric space X is geodesic if for each $x, y \in X$ there exists a geodesic from x to y.

Definition 2.1. A *tripod* is a geodesic triangle such that each side is contained in the union of the other two sides.

A *real tree* is a geodesic metric space such that all its geodesic triangles are tripods.

Convention 2.2. From now on all real trees are implied to be complete metric spaces.

If T is a real tree and $p \in T$, the valency of T at p is the number of connected components of $T\backslash\{p\}$.

3. Nonstandard extensions

For the following sections we will need basic results about the theory of nonstandard extensions. The treatment will be rather informal, for a more formal one see for example [5]. Let us start with a motivating example. It is quite evident that being allowed to use nonzero infinitesimals (i.e. numbers x different from 0 such that $|x| < 1/n$ for each $n \in \mathbb{N}^+$) would be very helpful in analysis. Unfortunately, $\mathbb{R}$ does not contain infinitesimals. The idea is therefore to find an extension of $\mathbb{R}$, denoted by ${}^*\mathbb{R}$, which contains infinitesimals. Let us construct such an extension.

Definition 3.1. Let I be any infinite set. A *filter* $\mathcal{U} \subseteq \mathcal{P}(I)$ on I is a collection of subsets of I such that for each $A, B \subseteq I$:

(1) If A is finite, $A \notin \mathcal{U}$ (in particular $\varnothing \notin \mathcal{U}$).
(2) $A, B \in \mathcal{U} \Rightarrow A \cap B \in \mathcal{U}$.
(3) $A \in \mathcal{U}, B \supseteq A \Rightarrow B \in \mathcal{U}$.

An *ultrafilter* is a filter satisfying the further property:

(4) $A \notin \mathcal{U} \Rightarrow A^c \in \mathcal{U}$.

This is not the standard definition of ultrafilter: the usual one requires only that $\varnothing \notin \mathcal{U}$ instead of property (1), and the ultrafilters not containing finite sets are usually called nonprincipal ultrafilters. However, we will only need nonprincipal ultrafilters.

Fix any infinite set I. An example of filter on I is the collection of complements of finite sets. An easy application of Zorn's Lemma shows that there actually exists an ultrafilter $\mathcal{U}$, which extends the mentioned filter. Fix such an ultrafilter. We are ready to define ${}^*\mathbb{R}$.

Definition 3.2. Define the following equivalence relation $\sim$ on $\mathbb{R}^I = \{f \mid I \to \mathbb{R}\}$:

$$f \sim g \iff \{i \in I \mid f(i) = g(i)\} \in \mathcal{U}.$$

Let ${}^*\mathbb{R}$ be the quotient set of $\mathbb{R}^I$ modulo this relation.

It is easily seen using the properties of an ultrafilter (in fact, of a filter) that $\sim$ is indeed an equivalence relation. We can define the sum and the product on ${}^*\mathbb{R}$ componentwise, as this is easily seen to be well defined. Using also property (4), we obtain that ${}^*\mathbb{R}$, equipped with this operations, is a field. We can also define an order ${}^*\!\leq$ on ${}^*\mathbb{R}$ in the following way:

$$[f]\ {}^*\!\leq [g] \iff \{i \in I \mid f(i) \leq g(i)\} \in \mathcal{U}.$$

Using the properties of ultrafilters it is easily seen that this is a total order on ${}^*\mathbb{R}$ (property (4) is required only to show that it is total), and that ${}^*\mathbb{R}$ is an ordered field. An embedding of ordered fields $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ can be defined simply by $r \mapsto f_r$, where f_r is the function with constant value r. We can identify $\mathbb{R}$ with its image in ${}^*\mathbb{R}$.

Notice that in the definition we gave of ${}^*\mathbb{R}$ we can substitute $\mathbb{R}$ with any set X. Doing so, we obtain the definition of ${}^*X$, which can be considered as an extension of X, just as we considered ${}^*\mathbb{R}$ as an extension of $\mathbb{R}$. In the case of ${}^*\mathbb{R}$, we showed that this extension preserves the basic properties of $\mathbb{R}$, i.e. being an ordered field. The idea is that this is true in general, as we will see.

Before proceeding, notice that if $f : X \to Y$ is any function, we can define componentwise a function ${}^*f : {}^*X \to {}^*Y$ (which is well defined), called the nonstandard extension of f. This function coincides with f on (the subset of ${}^*X$ identified with) X. Also relations have nonstandard extensions (see the definition of ${}^*\!\leq$). Let us give another definition (in a quite informal way), and then we will see which properties are preserved by nonstandard extensions.

Definition 3.3. A formula ϕ is *bounded* if all quantifiers appear in expressions like $\forall x \in X$, $\exists x \in X$ (bounded quantifiers).

The *nonstandard interpretation* of ϕ, denoted ${}^*\phi$, is obtained by adding * before any set, relation or function (not before quantified variables).

An example will make these concepts clear: consider

$$\forall X \subseteq \mathbb{N},\ X \neq \varnothing\ \exists x \in X\ \forall y \in X \quad x \leq y,$$

which expresses the fact that any nonempty subset of $\mathbb{N}$ has a minimum. This formula is not bounded, because it contains "$\forall X \subseteq \mathbb{N}$". However, it can be turned into a bounded formula by substituting "$\forall X \subseteq \mathbb{N}$" with "$\forall X \in \mathcal{P}(\mathbb{N})$". The nonstandard interpretation of the modified formula reads

$$\forall X \in {}^*\mathcal{P}(\mathbb{N}),\ X \neq {}^*\varnothing\ \exists x \in X\ \forall y \in X \quad x\ {}^*\!\leq y. \tag{3.1}$$

These definitions are fundamental for the theory of nonstandard extensions in view of the following theorem, which will be referred to as the transfer principle.

Theorem 3.4. *(Łoš Theorem) Let ϕ be a bounded formula. Then $\phi \iff {}^*\phi$.*

This theorem roughly tells us that the nonstandard extensions have the same properties, up to paying attention to state these properties correctly (for example, replacing "$\forall X \subseteq \mathbb{N}$" with "$\forall X \in \mathcal{P}(\mathbb{N})$"). Easy consequences of this theorem are, for example, that the nonstandard extension $({}^*G, {}^*\cdot)$ of a group $(G, \cdot)$ is a group, or that the nonstandard extension $({}^*X, {}^*d)$ of a metric space (X, d) is a ${}^*\mathbb{R}$-metric space (that is ${}^*d : {}^*X \times {}^*X \to {}^*\mathbb{R}$ satisfies the axioms of distance, which make sense as ${}^*\mathbb{R}$ is in particular an ordered abelian group). To avoid too many *'s, we will often drop them before functions or relations, for example we will denote the "distance" on ${}^*X$ as above simply by "d". In view of the transfer principle, the following definition is very useful:

Definition 3.5. $A \subseteq {}^*X$ will be called *internal* subset of X if $A \in {}^*\mathcal{P}(X)$. An internal set is an internal subset of some ${}^*X$.

$f : {}^*X \to {}^*Y$ will be called *internal* function if $f \in {}^*(Y^X) = {}^*\{f : X \to Y\}$.

One may think that "living inside the nonstandard world" one only sees internal sets and functions, and therefore, by the transfer principle, one cannot distinguish the standard world from the nonstandard world.

Notice that ${}^*\mathcal{P}(X) \subseteq \mathcal{P}({}^*X)$ by the transfer principle applied to the formula

$$\forall A \in \mathcal{P}(X)\ \forall a \in A \quad a \in X.$$

Also, $\{{}^*A \mid A \in \mathcal{P}(X)\} \subseteq {}^*\mathcal{P}(X)$, by the transfer principle applied to $(\forall a \in A,\ a \in X) \Rightarrow A \in \mathcal{P}(X)$. To sum up

$$\{{}^*A \mid A \in \mathcal{P}(X)\} \subseteq {}^*\mathcal{P}(X) \subseteq \mathcal{P}({}^*X).$$

Analogously, in the case of maps we have

$$\{{}^*f \mid f \in Y^X\} \subseteq {}^*(Y^X) \subseteq ({}^*Y)^{{}^*X}.$$

However, the equalities are in general not true, as we will see.

Another example: the transfer principle applied to formula (3.1), which tells that each nonempty subset of $\mathbb{N}$ has a minimum, gives that each *internal* nonempty subset of ${}^*X$ has a minimum (${}^*\varnothing = \varnothing$ as, for each set A, $\exists a \in A \iff \exists a \in {}^*A$).

Łoš Theorem alone is not enough to prove anything new. In fact, it holds for the trivial extension, that is, if we set ${}^*X = X$, ${}^*f = f$ and ${}^*R = R$ for each set X, function f and relation R. However, the nonstandard extensions we defined enjoy another property, referred to as saturation. First, a definition, and then the statement.

Definition 3.6. A collection of sets $\{A_j\}_{j \in J}$ has the finite intersection property (FIP) if for each $n \in \mathbb{N}$ and $j_0, \dots, j_n \in J$, we have $A_{j_0} \cap \dots \cap A_{j_n} \neq \varnothing$.

Theorem 3.7. *Suppose that the collection of internal sets $\{A_n\}_{n \in \mathbb{N}}$ has the FIP. Then $\bigcap_{n \in \mathbb{N}} A_n \neq \varnothing$.*

Let us use this theorem to prove that ${}^*\mathbb{R}$ contains infinitesimals. It is enough to consider the collection of sets $\{{}^*(0, 1/n)\}_{n \in \mathbb{N}^+}$ and apply the theorem to it. Notice that for $n \in \mathbb{N}^+$, ${}^*(0, 1/n) \in {}^*\mathcal{P}(\mathbb{R})$ as it is of the form ${}^*A$ for $A \in \mathcal{P}(\mathbb{R})$. More in general, however, for each $x, y \in {}^*\mathbb{R}$, $(x, y) \in {}^*\mathbb{R}$ (we should use a different notation for intervals in $\mathbb{R}$ and intervals in ${}^*\mathbb{R}$, but hopefully it will be clear from the context which kind of interval is under consideration). In fact, we can apply the transfer principle to the formula $\forall x, y \in \mathbb{R}\ (x, y) \in \mathcal{P}(\mathbb{R})$. To be more formal, "$(x, y) \in \mathcal{P}(\mathbb{R})$" should be substituted by

$$\exists A \in \mathcal{P}(\mathbb{R})\ \forall z \in \mathbb{R}\ (z \in A \iff x < z \text{ and } z < y).$$

Notice that it can be proved similarly that ${}^*\mathbb{N}$ and ${}^*\mathbb{R}$ contain infinite numbers. We will need the following refinement of this:

Lemma 3.8.

(1) Let $\{\xi_n\}_{n\in\mathbb{N}}$ be a sequence of infinitesimals. There exists an infinitesimal ξ greater than any ξ_n.

(2) Let $\{\rho_n\}_{n\in\mathbb{N}}$ be a sequence of positive infinite numbers (in ${}^*\mathbb{R}$ or ${}^*\mathbb{N}$). There exists an infinite number ρ smaller than any ρ_n.

Proof. Let us prove (1), the proof of (2) being very similar.

The collection $\{(\xi_n, 1/(n+1))\}_{n\in\mathbb{N}}$ of internal subsets of ${}^*\mathbb{R}$ has the FIP. An element $\xi \in \bigcap(\xi_n, 1/(n+1))$ has the required properties. □

Convention 3.9. The definition of the nonstandard extensions depends on the infinite set I and the ultrafilter $\mathcal{U}$. From now on we set $I = \mathbb{N}$ and we fix an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, and we will consider the nonstandard extensions constructed from this data.

The reader is suggested to forget the definition of nonstandard extensions, as Theorem 3.4, Theorem 3.7 and the remark below are all we need, and the definition will never be used again.

Remark 3.10. The nonstandard extension of a set of cardinality at most $2^{\aleph_0}$ has cardinality at most $2^{\aleph_0}$ (this is a consequence of the fact that we set $I = \mathbb{N}$).

Now, a lemma which is frequently used when working with nonstandard extensions, usually referred to as overspill.

Lemma 3.11. *Suppose that the internal subset $A \subseteq {}^*\mathbb{R}^+$ (or $A \subseteq {}^*\mathbb{N}$) contains, for each $n \in \mathbb{N}$, an element greater than n. Then A contains an infinite number.*

Proof. The collection of internal sets $\{A\} \cup \{(n, +\infty)\}_{n\in\mathbb{N}}$ has the FIP, therefore $\bigcap_{n\in\mathbb{N}}(n, \infty) \cap A \neq \varnothing$. (For clarity, here by (n, ∞) we mean $\{x \in {}^*\mathbb{R} \mid x > n\}$.) An element in the intersection is what we were looking for. □

We will also need:

Lemma 3.12. *Suppose that $A \subseteq X \subseteq {}^*X$ is internal. Then it is finite.*

Let us introduce some (quite intuitive) notation, which we are going to use from now on.

Definition 3.13. Consider $\xi, \eta \in {}^*\mathbb{R}$, with $\eta \neq 0$. We will write:

- $\xi \in o(\eta)$ (or $\xi \ll \eta$ if ξ, η are nonnegative) if ξ/η is infinitesimal,
- $\xi \in O(\eta)$ if ξ/η is finite,
- $\xi \gg \eta$ if ξ, η are nonnegative and ξ/η is infinite.

For example, $o(1)$ is the set of infinitesimals, and $O(1) = \{\xi \in {}^*\mathbb{R} \mid |\xi| < r \text{ for some } r \in \mathbb{R}^+\}$.

The map we given by the following lemma plays a fundamental role in nonstandard analysis, and will be used in the definition of asymptotic cone:

Proposition 3.14. *There exists a unique map* $\mathrm{st} : O(1) \to \mathbb{R}$ *such that, for each* $\xi \in {}^*\mathbb{R}$*,* $\xi - \mathrm{st}(\xi)$ *is infinitesimal.*

We will call $\mathrm{st}(\xi)$ the standard part of ξ. Notice that $\mathrm{st}(\xi) = 0 \iff \xi$ is infinitesimal.

Many common definitions have interesting nonstandard counterparts. Here is an example which will be used later.

Proposition 3.15. *The metric space* X *is compact if and only if for each* $\xi \in {}^*X$ *there exists* $x \in X$ *such that* $d(x, \xi) \in o(1)$*.*

4. Asymptotic cones and ultralimits

Let (Y, δ) be an internal metric space. This means that Y is an internal set, and that $\delta : Y \times Y \to {}^*\mathbb{R}$ is an internal map satisfying the usual axioms of a metric. If A is a set and $\sim$ is an equivalence relation on A, the equivalence class of $a \in A$ will be denoted by $[a]$ and the quotient set of A will be denoted by $A/_\sim$.

Definition 4.1. Define on Y the equivalence relation $x \sim y \iff \delta(x, y) \in o(1)$. The ultralimit $U(Y, p)$ of Y with basepoint $p \in Y$ is defined by:

$$\{[x] \in Y/_\sim \mid \delta(x, p) \in O(1)\}.$$

The distance on $U(Y, p)$ is defined as $d([x], [y]) = \mathrm{st}\,(\delta(x, y))$.

An interesting example of ultralimits are asymptotic cones, where (Y, δ) is chosen to be $({}^*X, {}^*d/\nu)$, where (X, d) is a metric space and $\nu \gg 1$. Let us spell this out.

Definition 4.2. Consider a metric space (X, d) and $\nu \in {}^*\mathbb{R}$, $\nu \gg 1$. Define on ${}^*X$ the equivalence relation $x \sim y \iff d(x, y) \in o(\nu)$. The asymptotic cone $C(X, p, \nu)$ of X with basepoint $p \in {}^*X$ and scaling factor ν is defined by

$$\{[x] \in {}^*X/_\sim \mid d(x, p) \in O(\nu)\}.$$

The distance on $C(X, p, \nu)$ is defined as $d([x], [y]) = \mathrm{st}\,({}^*d(x, y)/\nu)$.

This definition of asymptotic cone is basically due to van den Dries and Wilkie [22].

We now present the definition of ultralimits in terms of ultrafilters, for comparison. Recall that if $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ and (r_n) is a sequence of nonnegative real numbers one can define $\lim_{\mathcal{U}} r_n \in [0, +\infty]$ as the only $l \in [0, +\infty]$ so that for each neighborhood U of l we have $\{n \mid r_n \in U\} \in \mathcal{U}$.

Definition 4.3. Fix an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ and let (X_n, d_n) be a sequence of metric spaces. Define on ΠX_n the equivalence relation $(x_n) \sim (y_n) \iff \lim_{\mathcal{U}} d_n(x_n, y_n) = 0$. The ultralimit $U_{\mathcal{U}}((X_n),(p_n))$ with basepoint $(p_n) \in \Pi X_n$ is defined as

$$\{[(x_n)] \in \Pi X_n/_{\sim} \mid \lim_{\mathcal{U}} d_n(x_n, p_n) < +\infty\}.$$

The distance on $U_{\mathcal{U}}((X_n),(p_n))$ is defined by $d([(x_n)],[(y_n)]) = \lim_{\mathcal{U}} d_n(x_n, y_n)$.

The two definitions are actually equivalent. In fact, let (r_n) be any sequence, which represents the element $\xi \in O(1) \subseteq {}^*\mathbb{R}$. Then $\mathrm{st}(\xi) = \lim_{\mathcal{U}} r_n$, as by definition of $\lim_{\mathcal{U}}$ for each $\epsilon \in \mathbb{R}^+$ we have $\{n \in \mathbb{N} \mid |r_n - \lim_{\mathcal{U}} r_n| \leq \epsilon\} \in \mathcal{U}$ so that $|\xi - \lim_{\mathcal{U}} r_n| \leq \epsilon$. So $|\xi - \lim_{\mathcal{U}} r_n|$ is infinitesimal. Using this description of st and expanding the nonstandard definition of ultralimit with $Y = \Pi X_n/\mathcal{U}$ (i.e. $Y = \Pi X_n/_{\sim}$ where $(x_n) \sim (y_n) \iff \{n \mid x_n = y_n\} \in \mathcal{U}$) one obtains the standard one.

Before proceeding, a few definitions. If Y is an internal metric space, $q \in Y$ and $d(p,q) \in O(1)$, so that $[q] \in U(Y,p)$, then $[q]$ will be called the projection of q on $U(Y,p)$. Similarly, if $A \subseteq \{x \in Y \mid d(x,p) \in O(1)\}$, the projection of A on $U(Y,p)$ is $\{[a] \mid a \in A\}$.

The following properties of asymptotic cones are well-known:

Lemma 4.4.

(1) Any ultralimit is a complete metric space.
(2) Any ultralimit of an internal geodesic metric space is a geodesic metric space.

5. Use of nonstandard methods for asymptotic cones

The main aim of this section is to show how nonstandard methods can be used to prove results about asymptotic cones.

We will see that the following well-known lemma gives several obstructions for a space to be realized as an asymptotic cone.

Lemma 5.1. *An internal set is finite or has cardinality at least $2^{\aleph_0}$.*

Proof. Any set is finite or admits an injective function from $\mathbb{N}$. By the transfer of this property, we have that every internal set admits a bijective (internal) function from $\{0,\dots,\nu\}$ for some $\nu \in {}^*\mathbb{N}$ or an injective (internal) function from ${}^*\mathbb{N}$.

So it is enough to prove that the set $\{0,\dots,\nu\}$ is uncountable for every infinite ν. The fact that the map

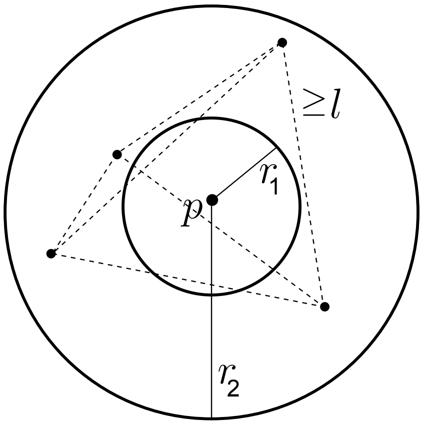
$$\alpha \in \{0,\dots,\nu\} \mapsto \mathrm{st}(\alpha/\nu) \in [0,1]$$

is surjective implies the claim. □

Let X be a metric space. For $p \in X$ and $r_1, r_2, l \geq 0$ denote by $F_X(p, r_1, r_2, l)$ the supremum of the cardinalities of sets M satisfying

(1) $\forall x \in M,\ r_1 \leq d(x,p) \leq r_2$,
(2) $\forall x, y \in M,\ x \neq y \Rightarrow d(x,y) \geq l$.

A set M satisfying the above properties will be called, for $\alpha \leq |M|$, a test for $F_X(p, r_1, r_2, l) \geq \alpha$.



Remark 5.2.

- If $F_X(p, r_1, r_2, l)$ is finite, then it is a maximum,
- for each $\alpha < F_X(p, r_1, r_2, l)$ we can find a test for $F_X(p, r_1, r_2, l) \geq \alpha$.

For convenience, if $\rho \in {}^*\mathbb{N}$ is infinite, we set $\mathrm{st}(\rho) = 2^{\aleph_0}$. We also set $\mathrm{st}(\alpha) = 2^{\aleph_0}$ for each *cardinality $\alpha \geq {}^*\aleph_0$.

We will often use the following easy properties:

Lemma 5.3.

(1) $F_X(p, r_1, r_2, l) \leq F_X(p', r_1', r_2', l')$ *if and only if for each test* M *for* $F_X(p, r_1, r_2, l) \geq \alpha$ *there is a test* M' *for* $F_X(p', r_1', r_2', l') \geq \alpha$*,*

(2) if $r_1 \geq r_1'$*,* $r_2 \leq r_2'$ *and* $l \geq l'$ *then* $F_X(p, r_1, r_2, l) \leq F_X(p, r_1', r_2', l')$*,*

(3) if X *is the ultralimit of* Y *with basepoint* p*, then there exists an infinitesimal* ξ *so that*

(a) $\mathrm{st}\left(F_Y(p, \rho_1, \rho_2, \lambda)\right) \leq F_X([p], \mathrm{st}(\rho_1), \mathrm{st}(\rho_2), \mathrm{st}(\lambda))$,

(b) $$\min\left\{F_X([p], \mathrm{st}(\rho_1), \mathrm{st}(\rho_2), \mathrm{st}(\lambda)), 2^{\aleph_0}\right\} \leq \mathrm{st}\left(F_Y(p, \rho_1 - \xi, \rho_2 + \xi, \lambda - \xi)\right),$$

where ρ_1, ρ_2 *and* λ *are finite and* $\mathrm{st}(\lambda) > 0$.

Proof. (1) is straightforward from the definitions.

(2) If M is a test for $F_X(p, r_1, r_2, l) \geq \alpha$, then it is also a test for $F_X(p, r_1', r_2', l') \geq \alpha$.

(3) By our convention on the standard part and the fact that $|{}^*\mathbb{N}| = 2^{\aleph_0}$, $\mathrm{st}(\nu)$ is the cardinality of $\nu = \{0, \ldots, \nu - 1\}$. In fact, if ν is finite, $\mathrm{st}(\nu) = \nu$, otherwise $\mathrm{st}(\nu) = 2^{\aleph_0} \leq |\nu| \leq |{}^*\mathbb{N}| = 2^{\aleph_0}$ (we used Lemma 5.1). If M is a test for $F_Y(p, \rho_1, \rho_2, \lambda) \geq \alpha$, its projection on X is a test for $F_X([p], \mathrm{st}(\rho_1), \mathrm{st}(\rho_2), \mathrm{st}(\lambda)) \geq \mathrm{st}(\alpha)$ (projections of distinct elements of M are distinct because their distance is at least $\mathrm{st}(\lambda) > 0$). This proves the first inequality. On the other hand, if M is finite and a test for $F_X([p], \mathrm{st}(\rho_1), \mathrm{st}(\rho_2),$

$\mathrm{st}(\lambda)) \geq n$, then it can be lifted to a test for $F_Y(p, \rho_1 - \xi_n, \rho_2 + \xi_n, \lambda - \xi_n) \geq n$ for an appropriate infinitesimal ξ_n. This settles the second inequality if $F_X([p], \mathrm{st}(\rho_1), \mathrm{st}(\rho_2), \mathrm{st}(\lambda))$ is finite. If it is infinite, by Lemma 3.8 we can choose a positive infinitesimal ξ with $\xi \geq |\xi_n|$ for each n and we have $F_Y(p, \rho_1 - \xi, \rho_2 + \xi, \lambda - \xi) \geq n$ for each natural n, hence the same holds true for some infinite n. □

Proposition 5.4. *Let X be a metric space. If X is an ultralimit then for each p, r_1, r_2, l, with $l > 0$, if $F_X(p, r_1, r_2, l)$ is infinite then it is at least $2^{\aleph_0}$.*

Proof. Assume that X is the ultralimit of Y, and fix p, r_1, r_2, l as above and such that $F_X(p, r_1, r_2, l) \geq \aleph_0$. Fix a representative $\pi \in Y$ of p. By point (3b) of Lemma 5.3 for some infinitesimal ξ we have that $F_Y(\pi, r_1 - \xi, r_2 + \xi, l - \xi)$ is infinite, and hence it is at least $2^{\aleph_0}$ by Lemma 5.1. The conclusion follows from point (3a) of Lemma 5.3 □

We will now study the consequences of this proposition for real trees appearing as asymptotic cones.

Corollary 5.5. *If the ultralimit X is a real tree such that every geodesic can be extended to a geodesic ray (e.g.: a complete homogeneous real tree) and the valency at a point p is infinite, then this valency is at least $2^{\aleph_0}$.*

Notice that in a complete real tree T where every point has valency at least 2 all geodesics can be extended to geodesic rays.

Proof. Our assumption on geodesics implies that, for each $r > 0$, $F_X(p, r, r, 2r)$ equals the valency at p. □

Definition 5.6. In a real tree, a point of valency greater than 2 will be called a *branching point*.

Proposition 5.7. *Let X be a real tree such that each nontrivial geodesic can be extended to a geodesic ray and the valency at a point p is finite. If X is an ultralimit then p is isolated from the other branching points.*

Proof. Let n be the valency of X at p. For each $r > 0$, $F_X(p, r, r, 2r) = n$. Assume that p is not isolated from the other branching points. Then for each $k \in \mathbb{N}$ (and $k > 1/2r$) we have that $F_X(p, r, r, 2r - 1/k)$ is infinite. If X is an ultralimit of Y and $\pi \in {}^*Y$ is a representative for p, by Lemma 5.3(3b) for each k we can find a positive infinitesimal ξ_k and a positive infinite μ_k such that $F_Y(\pi, r - \xi_k, r + \xi_k, 2r - 1/k - \xi_k) \geq \mu_k$. Let us fix a positive infinitesimal ξ greater than any ξ_k and a positive infinite μ smaller than any μ_k. We have that $\{\alpha \mid F_Y(\pi, r - \xi, r + \xi, 2r - \alpha) \geq \mu\}$ contains, for each k, elements of ${}^*\mathbb{R}$ smaller than $1/k$ (for example $1/(k+1) + \xi_{k+1}$), hence it contains an infinitesimal η. This implies that $F_X(p, r, r, 2r)$ is infinite (using point (3a) of Lemma 5.3), a contradiction. □

Putting together Corollary 5.5 and Proposition 5.7 in the case of homogeneous real trees, we have:

Corollary 5.8. *If X is a homogeneous real tree and an ultralimit, then it is a point, a line or a complete real tree with valency at least $2^{\aleph_0}$ at each point.*

Asymptotic cones of finitely generated groups (as well as of separable metric spaces and more generally of metric spaces of cardinality at most $2^{\aleph_0}$) have cardinality at most $2^{\aleph_0}$, and hence:

Corollary 5.9. *If the real tree X is the asymptotic cone of a finitely generated group endowed with a word metric, then it is a point, a line or a complete real tree with valency $2^{\aleph_0}$ at each point. The same holds true for asymptotic cones of separable topological Hausdorff groups endowed with a left invariant metric.*

Now, let us analyze the consequences of Proposition 5.4 in the special case $r_1 = 0$, proving Theorem 1.1 which we restate for the convenience of the reader.

Theorem 5.10.

(1) Suppose that the metric space X is an ultralimit. Then if X is separable, it is proper. More precisely, if X is not proper, then it contains $2^{\aleph_0}$ disjoint balls.

(2) Suppose that for some $0 \ll \mu \ll \nu$ and some $p \in {}^*Y$ each asymptotic cone of the metric space Y with scaling factor ν' such that $\mu \leq \nu' \leq \nu$ and basepoint p is separable. If $X = C(Y, p, \nu)$ is homogeneous then every closed ball in X is doubling.

Proof. (1) Let X be the ultralimit of Y with basepoint p. Suppose that $B = \overline{B}([p], r) \subseteq X$ is not compact. Then, as X and therefore B is complete (by Lemma 4.4(1)), there exists some $0 < \epsilon < 1$ such that B cannot be covered by finitely many balls of radius ϵr. Then it is readily checked that $F_X([p], 0, r, \epsilon r)$ is infinite, for otherwise B would be covered by balls of radius ϵr around points in a maximal test. By Lemma 5.3(3b), $F_Y(p, 0, r+\xi, \epsilon r - \xi)$ is infinite as well, for some infinitesimal ξ. Projecting a test M for $F_Y(p, 0, r+\xi, \epsilon r - \xi) \geq \nu$ to X, for some infinite ν, and considering balls of radius $\epsilon r/2$ around the points obtained in this way we find $2^{\aleph_0}$ disjoint balls in B, as required. In fact if $m_1 \neq m_2$ and $m_i \in M$, then by definition of test $d([m_1], [m_2]) \geq \epsilon r$ so that $B([m_1], \epsilon r/2) \cap B([m_2], \epsilon r/2) = \varnothing$.

(2) By (1), we know that fixing $r \in \mathbb{R}^+$ we can choose $n(r)$ such that $B([p], r)$ can be covered by at most $n(r)$ balls of radius $r/2$. Suppose by contradiction that $n(r)$ is not bounded when $r \to 0$. By Lemma 5.3(3b), $F_{{}^*Y}(p, 0, (2r + \xi_r)\nu, (r - \xi_r)\nu)$ is not bounded by any finite number for some appropriately chosen infinitesimals ξ_r. In particular, we can find an infinitesimal ξ such that $F_{{}^*Y}(p, 0, 3\xi\nu, \xi\nu/2) \geq \rho$ for some infinite ρ (notice that $2r + \xi_r \leq 3r$ and $r - \xi_r \geq r/2$), and we can also choose it so that $\xi\nu \geq \mu$. Projecting a test as in (1) we get that $C(Y, p, \xi\nu)$ is not separable, a contradiction. □

Remark 5.11. In this remark groups are finitely generated and endowed with a word metric unless otherwise stated. Point proved in [18] that if a group has one proper asymptotic cone of finite Minkowski dimension then it is virtually nilpotent. It is easily checked that doubling metric spaces have finite

Minkowski (as well as Hausdorff) dimension, so that a corollary of Theorem 1.1 is the following: if G is a group and there exist $\nu_1 \ll \nu_2$ such that all the asymptotic cones of G with scaling factor $\nu \in [\nu_1, \nu_2]$ are separable, then G is virtually nilpotent.

However, a stronger result holds true in view of [8, Thm. 7.1]: if the group G has one separable asymptotic cone then it is virtually nilpotent. A version of this probably holds for locally compact compactly generated groups endowed with word metric with respect to a compact generating set, with "virtually nilpotent" replaced by "polynomial growth" (locally compact compactly generated groups of polynomial growth are characterized in [14]). In order to state the mentioned theorem, recall that a k-approximate subgroup A is a finite subset of a group G so that $1 \in A, A = A^{-1}$ and $A \cdot A$ is contained in at most k left cosets of A (in [8] this notion is defined in terms of right cosets, the given definition is equivalent as $A = A^{-1}$ and $A \cdot A = (A \cdot A)^{-1}$). The content of [8, Thm. 7.1] is that a group such that, for a given k, all finite subsets of it are contained in a k-approximate subgroup is virtually nilpotent. Notice that, given a finite subset $F \subseteq G$ and $k \in \mathbb{N}$, F is contained in a k-approximate subgroup of G if and only if it is contained in an *internal* k-approximate subgroup of ${}^*G$. So, suppose that G has a proper asymptotic cone. Then it is readily seen that there exists $k \in \mathbb{N}$ and $\nu \gg 1$ so that $\overline{B}(1, \nu) \subseteq {}^*G$ can be covered by at most k balls of radius $\nu/2$, as in $C(G, 1, \nu)$ balls of radius 1 can be covered by at most k balls of radius, say, 1/3. In particular, $\overline{B}(1, \nu/2) \subseteq {}^*G$ is an internal k-approximate subgroup, and it contains all finite subsets of G, as it contains G. Thus, G is virtually nilpotent.

Notice that a slight variation of the argument above involving a ball of radius not necessarily equal to 1 shows that if there is a ball of radius $R > 0$ in an asymptotic cone of the group G that can be covered by finitely many balls of radius $R/3$, then G is virtually nilpotent. In particular, by an argument based on Proposition 5.4, if the group G is not virtually nilpotent then any asymptotic cone of G has the property that any ball of radius $R > 0$ contains $2^{\aleph_0}$ disjoint balls of radius $R/10$. This implies, for example, that such asymptotic cone is not quasi-isometric to any proper metric space.

Theorem 1.1 provides many examples of metric spaces which do not appear as asymptotic cones, for example the separable Hilbert space. In contrast, we will prove below a "positive" result on spaces which are realized as asymptotic cones.

Theorem 5.12. *If the metric space X is proper, then it is an asymptotic cone of some metric space Y. If X is also geodesic and unbounded, we can choose Y to be geodesic as well.*

The first part of the statement above has been proven independently by Scheele in [19], using a different construction. Our construction is a slight variation of the one which appears in [4, Sec. 5], translated in the nonstandard setting.

Remark 5.13. Notice that if X is not geodesic, then Y cannot be geodesic by Lemma 4.4. Also, if Y is bounded and not a point, then it cannot be the asymptotic cone of a geodesic metric space X. Indeed, if the bounded metric space Y containing at least two points is the asymptotic cone of the metric space X with basepoint p and scaling factor ν, then X needs to be unbounded. Hence, we can choose $q \in {}^*Y$ with $d(q,p) \gg \nu$. If X was geodesic, we could choose an internal geodesic in ${}^*X$ connecting p to q (i.e. an internal map $\gamma : ([0, d(p,q)] \subseteq {}^*\mathbb{R}) \to {}^*X$ so that $\gamma(0) = p, \gamma(d(p,q)) = q$ and $d(\gamma(x), \gamma(y)) = |x-y|$ for each $x, y \in [0, d(p,q)]$), which induces in a natural way a geodesic ray in Y. But Y cannot contain geodesic rays, so that X cannot be geodesic.

Proof of Theorem 5.12. Let us first assume that X is unbounded. Let $\{p_n\}$ be a sequence of points of X such that $d(p_0, p_n) \to \infty$. Set $Y = (X \times \mathbb{N}) \cup \bigcup_{n\in\mathbb{N}}(\{p_n\} \times [n, n+1])$. Define a distance $\widetilde{d}$ on Y in the following way:

$$\widetilde{d}((x,t),(x',t')) = \begin{cases} t \cdot d(x, p_{\lfloor t \rfloor}) + t' \cdot d(p_{\lfloor t' \rfloor}, x') + |t-t'|, & \text{if } t \neq t', \\ t \cdot d(x,x'), & \text{if } t = t'. \end{cases}$$

It is quite clear that Y is a metric space, and that it is geodesic if X is geodesic. Consider now ${}^*Y = ({}^*X \times {}^*\mathbb{N}) \cup \bigcup_{\mu \in {}^*\mathbb{N}}(\{p_\mu\} \times [\mu, \mu+1])$, and an infinite $\nu \in {}^*\mathbb{N}$. We want to show that the asymptotic cone Z of Y with basepoint (p_0, ν) and scaling factor ν is isometric to X. The isometry $i : X \to Z$ can be defined simply by $x \mapsto [(x, \nu)]$. It is readily checked that it is an isometric embedding. So far we did not use properness or that $d(p_0, p_n) \to \infty$, so we obtained the following.

Remark 5.14. Any metric space X can be isometrically embedded in an asymptotic cone of a metric space Y. If X is geodesic, we can require Y to be geodesic.

Section 5 of [4] already contains a proof of this fact.

We are left to prove that i is surjective. First of all, notice that the distance of any element of ${}^*Y \backslash ({}^*X \times \{\nu\})$ from (p_0, ν) is at least

$$\min\{\nu d(p_0, p_{\nu-1}), \nu d(p_0, p_{\nu+1})\} \gg \nu,$$

as $d(p_0, p_n) \to \infty$ and so $d(p_0, p_\mu) \gg 1$ for each infinite $\mu \in {}^*\mathbb{N}$. Therefore no element of ${}^*Y \backslash ({}^*X \times \{\nu\})$ projects onto an element of Z. What remains to prove is that for each $y \in {}^*X$ with $\widetilde{d}((p_0, \nu), (y, \nu))/\nu = d(p_0, y) \in O(1)$ there exists $x \in X$ such that $\widetilde{d}((x, \nu), (y, \nu))/\nu = d(x, y) \ll 1$. Consider y as above and some $r > d(p_0, y)$, $r \in \mathbb{R}$. We have that $B = \overline{B}_X(p_0, r)$ is compact and $y \in {}^*B$. By the nonstandard characterization of compact metric spaces (Proposition 3.15), there exists $x \in B$ such that $d(x, y) \ll 1$, and we are done.

The case that X is bounded can be handled similarly. Fix $p \in X$ and set $Y = X \times \mathbb{N}$. Define

$$\widetilde{d}((x,n),(x',n')) = \begin{cases} n \cdot d(x,p) + n' \cdot d(p, x') + |n^2 - (n')^2|, & \text{if } n \neq n', \\ n \cdot d(x,x'), & \text{if } n = n'. \end{cases}$$

Modifying the previous proof, it is easily shown that, for any infinite $\nu \in {}^*\mathbb{N}$, the asymptotic cone of Y with basepoint (p, ν) and scaling factor ν is isometric to X. □

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