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Zeta functions and topological entropy of the Markov-Dyck shifts

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Abstract. The Markov-Dyck shifts arise from finite directed graphs. An expression for the zeta function of a Markov-Dyck shift is given. The derivation of this expression is based on a formula in Keller [12]. For a class of examples that includes the Fibonacci-Dyck shift the zeta functions and topological entropy are determined.

1. Introduction

Let Σ be a finite alphabet, and let S_Σ be the left shift on $\Sigma^{\mathbb{Z}}$,

$$S_\Sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}, \qquad (x_i)_{i\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}.$$

The closed shift-invariant subsystems of the shifts S_Σ are called subshifts. For an introduction to their theory, which belongs to symbolic dynamics, we refer to [13] and [20]. A finite word in the symbols of S_Σ is called admissible for the subshift $X \subset \Sigma^{\mathbb{Z}}$ if it appears somewhere in a point of X. A subshift is uniquely determined by its language of admissible words that we denote by $\mathcal{L}(X)$. $\mathcal{L}_n(X)$ will denote the set of words in $\mathcal{L}(X)$ of length $n \in \mathbb{N}$. The topological entropy of the subshift $X \subset \Sigma^{\mathbb{Z}}$ is given by

$$h(X) = \lim_{n\to\infty} \frac{1}{n} \log \operatorname{card} \mathcal{L}_n(X).$$

Denoting by $\Pi_n(X)$ the number of points of period n of a shift-invariant set $X \subset \Sigma^{\mathbb{Z}}$, the zeta function of X is given by

$$\zeta_X(z) = e^{\sum_{n\in\mathbb{N}} \frac{\Pi_n(X) z^n}{n}}.$$

In this paper we are concerned with a class of subshifts that arise from finite directed graphs as a special case of constructions that were described in [14], [15], [10]. Following the line of terminology of [24], we call these subshifts Markov-Dyck shifts. Let G be a finite directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$. We denote the initial vertex of $e \in \mathcal{E}$ by $s(e)$ and the final vertex by $r(e)$. Let G^- be the graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}^-$ a copy of

$\mathcal{E}$. Reverse the directions of the edges in $\mathcal{E}^-$ to obtain the reversed graph G^+ of G^- with vertex set $\mathcal{V}$ and edge set $\mathcal{E}^+$. Denote by $\mathcal{P}^-$ (resp. $\mathcal{P}^+$) the set of finite paths in G^- (resp. G^+). The mapping $e^- \to e^+$ ($e^- \in \mathcal{E}^-$) extends to the bijection $w^- \to w^+$ ($w^- \in \mathcal{P}^-$) of $\mathcal{P}^-$ onto $\mathcal{P}^+$ that reverses direction. With idempotents P_v, $v \in \mathcal{V}$, the set $\mathcal{E}^- \cup \{P_v \mid v \in \mathcal{V}\} \cup \mathcal{E}^+$ is the generating set of the graph inverse semigroup of G, where, besides $P_u^2 = P_u$, $v \in \mathcal{V}$, the relations are (see for instance [27])

$$P_u P_w = 0, \qquad u, w \in \mathcal{V},\ u \neq w,$$

$$f^- g^+ = \begin{cases} P_{s(f)}, & (f = g), \\ 0 & (f \neq g),\ f, g \in \mathcal{E}, \end{cases} \tag{1.1}$$

$$P_{s(f)} f^- = f^- P_{r(f)}, \qquad P_{r(f)} f^+ = f^+ P_{s(f)}, \qquad f \in \mathcal{E}.$$

The alphabet of the Markov-Dyck shift D_G of G is $\mathcal{E}^- \cup \mathcal{E}^+$ and a word $(e_k)_{1 \le k \le K}$ is admissible for D_G precisely if

$$\prod_{1 \le k \le K} e_k \neq 0.$$

For the directed graph with one vertex and loops e_n, $1 \le n \le N$, $N > 1$, the relations take the form

$$e_n^- e_n^+ = \mathbf{1}, \quad 1 \le n \le N, \qquad e_l^- e_m^+ = 0, \quad 1 \le l, m \le N,\ l \neq m. \tag{1.2}$$

and one sees the Dyck inverse monoid [26], together with the Dyck shifts that were first described in [14].

The relations (1.2) can be viewed as the multiplicative relations among the relations that are satisfied by the generators of a Leavitt algebra [19] or a Cuntz algebra [5], and the relations (1.1) can be viewed as the multiplicative relations among the relations that are satisfied by generators of a Leavitt path algebra of the directed graph G [1], or the generators of the graph C^*-algebra of G [6], [8].

The zeta functions of the Dyck shifts were determined in [12], and K-theoretic invariants were computed in [22] and [17]. For related systems, the Motzkin shifts that add a symbol $\mathbf{1}$ to the alphabets of the Dyck shifts, the zeta functions were determined in [11] and K-theoretic invariants were computed in [21]. In Section 2 we will obtain an expression for the zeta function of a Markov-Dyck shift by applying a formula of Keller [12]. In Section 3 we consider the subsystems of the Markov-Dyck shifts that are obtained by allowing the paths in the Markov-Dyck shift to go from $\mathcal{E}^-$ to $\mathcal{E}^-$ or vice-versa only when entering a given vertex, giving estimates of topological entropies. Our approach to zeta functions and topological entropy is via the generating functions that are associated to circular codes and Markov codes. Compare here [23, Sec. 6]. For related material see [18]. In Section 4 we determine the zeta functions and topological entropy of the Markov-Dyck shifts that arise from directed graphs with adjacency matrix $F(a,b,c) = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, $a, b, c \in \mathbb{N}$. K-theoretic invariants of the $D_{F(1,1,1)}$ shift were computed in [25].

The length of a word w we denote by $\ell(w)$ and we denote the generating function of a formal language $\mathcal{L}$ by $g_{\mathcal{L}}$,

$$g_{\mathcal{L}}(z) = \sum_{n=0}^{\infty} \operatorname{card}\{w \in \mathcal{L} \mid \ell(w) = n\} z^n.$$

2. Zeta functions

Keller [12] has introduced the notion of a circular Markov code. Here we find ourselves in a situation where we will want a Markov code to be given by a set $\mathcal{C}$ of non-empty words in the symbols of a finite alphabet Σ together with a finite set $\mathcal{V}$ and mappings $r : \mathcal{C} \to \mathcal{V}$, $s : \mathcal{C} \to \mathcal{V}$. To $(\mathcal{C}, r, s)$ there is associated the shift invariant set $X_{\mathcal{C}} \subset \Sigma^{\mathbb{Z}}$ of points $x \in \Sigma^{\mathbb{Z}}$ such that there are indices I_k, $k \in \mathbb{Z}$, such that

$$I_0 \leq 0 < I_1, \quad I_k < I_{k+1}, \quad k \in \mathbb{Z}, \tag{2.1}$$

and such that

$$x_{[I_k, I_{k+1})} \in \mathcal{C}, \qquad k \in \mathbb{Z}, \tag{2.2}$$

and

$$r(x_{[I_k, I_{k+1})}) = s(x_{[I_{k+1}, I_{k+2})}), \qquad k \in \mathbb{Z}. \tag{2.3}$$

$(\mathcal{C}, r, s)$ is said to be a circular Markov code if for every periodic point x in $X_{\mathcal{C}}$ the indices I_k, $k \in \mathbb{Z}$, such that (2.1), (2.2), and (2.3) hold, are uniquely determined by x and can then be denoted by $I_k(x)$, $k \in \mathbb{Z}$. If $\mathcal{V}$ contains one element then one has a circular code (see e.g. [2]).

Generalizing the formula for the zeta function of $X_{\mathcal{C}}$, where $\mathcal{C}$ is a circular code, Keller [12] has proven a formula for the zeta function of $X_{\mathcal{C}}$, where $\mathcal{C}$ is a circular Markov code. For completeness we reproduce here Keller's proof for the special case that we have in mind.

Given a circular Markov code $(\mathcal{C}, s, r)$ denote by $\mathcal{C}(u, w)$ the set of words $c \in \mathcal{C}$ such that $s(c) = u$, $r(c) = w$ for $u, w \in \mathcal{V}$. Set

$$g_{\mathcal{C}(u,v),n} = \operatorname{card}\{c \in \mathcal{C} \mid s(c) = u,\ r(c) = v,\ \ell(c) = n\},$$
$$g_{\mathcal{C}(u,v)}(z) = \sum_{n=0}^{\infty} g_{\mathcal{C}(u,v),n} z^n$$

where $g_{\mathcal{C}(u,v),0} = 0$, and introduce the matrix

$$H^{(\mathcal{C})}(z) = (g_{\mathcal{C}(u,v)}(z))_{u,v \in \mathcal{V}}.$$

Theorem 2.1 (Keller). *For a circular Markov code* $(\mathcal{C}, s, r)$,

$$\zeta_{X_{\mathcal{C}}}(z) = \det(I - H^{(\mathcal{C})}(z))^{-1}.$$

Proof. Let $n \in \mathbb{N}$. Consider triples of the form $(j, c_1, c_1 \cdots c_k)$, where $k \in \mathbb{N}$, and where

$$c_l \in \mathcal{C}, \quad 1 \le l \le k, \quad s(c_1) = r(c_k), \quad r(c_l) = s(c_{l+1}), \quad 1 \le l \le k,$$
$$\ell(c_1 \cdots c_k) = n, \quad j = \ell(c_1)$$

To every point $x \in X_{\mathcal{C}}$ of period n one assigns a triple of this kind, where $k \in \mathbb{N}$ is determined by

$$n = I_k(x) - I_0(x), \qquad j = I_1(x) - I_0(x),$$

and

$$c_l = x_{[I_{l-1}(x), I_l(x))}, \quad 1 \le l < k.$$

Due to the circularity of $(\mathcal{C}, r, s)$ this assignment is bijective.

Denote by $\gamma_i(f)$ the i-th coefficient of the power series of a function f. From

$$\gamma_j((H^k)_{u,w}) = \mathrm{card}\{c_1 \cdots c_k \mid c_l \in \mathcal{C},\ 1 \le l \le k,\ s(c_1) = u,\ r(c_k) = w,$$
$$r(c_l) = s(c_{l+1}),\ 1 \le l \le k,\ \ell(c_1 \cdots c_k) = j\}, \quad j, k \in \mathbb{Z}_+,$$

one has

$$\begin{aligned}
&(\gamma_j(H)\gamma_{n-j}(H^k))_{u,w} \\
&\quad = \mathrm{card}\{c_1 \cdots c_k \mid c_l \in C,\ 1 \le l \le k,\ s(c_1) = u,\ r(c_k) = w, \\
&\qquad\qquad r(c_l) = s(c_{l+1}),\ 1 \le l \le k,\ \ell(c_1) = j,\ \ell(c_1 \cdots c_k) = n - j\}, \\
&\qquad\qquad\qquad 1 \le j \le n,\ k \in \mathbb{Z}_+.
\end{aligned}$$

It follows that

$$\begin{aligned}
\log \zeta_{X_{\mathcal{C}}}(z) &= \sum_{n \in \mathbb{N}} \frac{z^n}{n} \sum_{1 \le j \le n} j \ \mathrm{card}\{c_1 \cdots c_k \mid c_l \in C,\ 1 \le l \le k,\ r(c_l) = s(c_{l+1}), \\
&\qquad\qquad s(c_1) = r(c_k),\ \ell(c_1) = j,\ \ell(c_1 \cdots c_k) = n\} \\
&= \sum_{n \in \mathbb{N}} \frac{z^n}{n} \sum_{1 \le j \le n} j \ \mathrm{trace}\Big(\sum_{k \in \mathbb{Z}_+} \gamma_j(H)\gamma_{n-j}(H^k) \Big) \\
&= \sum_{n \in \mathbb{N}} \frac{z^n}{n} \sum_{1 \le j \le n} j \ \mathrm{trace}(\gamma_j(H)\gamma_{n-j}((I - H)^{-1})) \\
&= \sum_{n \in \mathbb{N}} \frac{z^n}{n} \sum_{0 \le j < n} (j+1)\mathrm{trace}(\gamma_{j+1}(H)\gamma_{n-1-j}((I - H)^{-1})) \\
&= \sum_{n \in \mathbb{N}} \frac{z^n}{n} \sum_{0 \le j < n} \mathrm{trace}(\gamma_j(H')\gamma_{n-1-j}(I - H)^{-1})) \\
&= \sum_{n \in \mathbb{N}} \frac{1}{n} \ \mathrm{trace}(\gamma_{n-1}(H'(I - H)^{-1})z^n) \\
&= -\sum_{n \in \mathbb{N}} \mathrm{trace}(\gamma_n(\log(I - H))z^n) \\
&= -\mathrm{trace}(\log(I - H)).
\end{aligned}$$

By the formula (see [9, Sec. 1.1.10])

$$\operatorname{trace}(\log(I-H)) = \log\det(I-H),$$

the theorem follows. □

We state Keller's formula for the case of a circular code (see [29], [28], and references given in [4]) as a corollary.

Corollary 2.2. *For a circular code $\mathcal{C}$*

$$\zeta_{X_{\mathcal{C}}}(z) = \frac{1}{1-g_{\mathcal{C}}(z)}.$$

Note that also the formula for the zeta function of a subshift of finite type in terms of a presenting polynomial matrix [3] is a special case of Keller's formula.

Let G be a finite directed graph with adjacency matrix A_G. We introduce the Markov-Dyck codes $\mathcal{C}_v$, $v \in \mathcal{V}$, of words $c = (c_k)_{1\le k\le K}$ such that

$$\prod_{1\le k\le K} c_k = P_v, \qquad \prod_{1\le j\le J} c_k \neq P_v, \quad 1 \le J < K.$$

The codes $\mathcal{C}_v$, $v \in \mathcal{V}$, are circular codes. Standard methods of combinatorics (as for instance described in [7]) give

$$g_{\mathcal{C}_u}(z) = z^2 \sum_{v\in V} \frac{A_G(u,v)}{1-g_{\mathcal{C}_v}(z)}, \qquad u \in \mathcal{V}, \tag{2.4}$$

and by the implicit function theorem (2.4) has a unique solution. Set

$$\mathcal{C} = \bigcup_{v\in\mathcal{V}} \mathcal{C}_v.$$

Also denote by $\mathcal{C}^-$ the set of admissible words that are concatenations of an element (possibly empty) of $\mathcal{P}^-$ with a word in $\mathcal{C}$ and denote by $\mathcal{C}^+$ the set of admissible words that are concatenations of a word in $\mathcal{C}$ and an element (possibly empty) of $\mathcal{P}^+$. $(\mathcal{C}^-, s, r)$ and $(\mathcal{C}^+, s, r)$ are circular Markov codes. Denote by $D(A_G, z)$ the diagonal matrix with entries $g_{\mathcal{C}_v}(z)$, $v \in \mathcal{V}$, and denote by $D^*(A_G, z)$ the diagonal matrix with entries $g_{\mathcal{C}_v^*}(z) = \frac{1}{1-g_{\mathcal{C}_v}(z)}$, $v \in \mathcal{V}$.

Theorem 2.3. *The zeta function of the Markov Dyck shift D_G is*

$$\begin{aligned}\zeta_{D_G}(z) &= \frac{1}{\det((I - D(A_G,z) - A_G z)(I - D^*(A_G,z)A_G z))} \\ &= \frac{\det(D^*(A_G,z))}{\det(I - D^*(A_G,z)A_G z)^2}.\end{aligned}$$

Proof. Since $\sum_{k\in\mathbb{Z}_+} A_G^k z^k = (I - A_G z)^{-1}$ one has

$$H^{(\mathcal{C}^+)}(z) = D(A_G, z)(I - A_G z)^{-1},$$

and $H^{(\mathcal{C}^-)}(z)$ is the adjoint of $H^{(\mathcal{C}^+)}(z)$. Applying Proposition 2.1 and collecting all contributions to the zeta function, one has

$$\begin{aligned}\zeta_{D_G}(z) &= \Big(\prod_{u\in\mathcal{V}} g_{\mathcal{C}^*_u}(z)^{-1}\Big)\det(I-A_Gz)^{-2}\det(I-D(A_G,z)(I-A_Gz)^{-1})^{-2}\\ &= \Big(\prod_{u\in\mathcal{V}} g_{\mathcal{C}^*_u}(z)^{-1}\Big)\det(D^*(A_G,z)^{-1}-A_Gz)^{-2}\\ &= \det((D^*(A_G,z)^{-1}-A_Gz)(I-D^*(A_G,z)A_Gz))^{-1}\\ &= \frac{1}{\det((I-D(A_G,z)-A_Gz)(I-D^*(A_G,z)A_Gz))}\\ &= \frac{\det(D^*(A_G,z))}{\det(I-D^*(A_G,z)A_Gz)^2}.\end{aligned}$$

□

Inserting into the formula for the case of the graph with one vertex and N-loops the generating function

$$g_{\mathcal{C}_v}(z) = \frac{1-\sqrt{1-4Nz^2}}{2},$$

one obtains again the zeta function of the Dyck shift D_N as

$$\zeta_{D_N}(z) = \frac{2(1+\sqrt{1-4Nz^2})}{(1-2Nz+\sqrt{1-4Nz^2})^2}$$

(see [12]).

3. Topological entropy

Proposition 3.1. *For the Markov-Dyck shift D_G*

$$h(D_G) = \lim_{n\to\infty}\frac{1}{n}\log\Pi_n(D_G).$$

Proof. A word $b=(b_m)_{1\le m\le n}\in\mathcal{L}_n(D_G)$ determines words $a^+(b)\in\mathcal{C}^+$, $a^-(b)\in\mathcal{C}^-$ by

$$\prod_{1\le m\le n} b_m = a^+(b)a^-(b),$$

as well as an index $I(b)$, $1\le I(b)\le n$, by

$$I(b) = \min\Big\{ i \;\Big|\; \prod_{1\le j\le i} b_j = a^+(b)\Big\}.$$

Denote by $\mathcal{K}_n$ the set of $b\in\mathcal{L}(D_G)$ of length $n\in\mathbb{N}$ such that $I(b)=1$. Choose for $u,w\in\mathcal{V}$ a path $c(u,w)$ in G from u to w of shortest length $\lambda(u,w)$ and set

$$L = \max_{u,w\in\mathcal{V}}\lambda(u,w).$$

We define a mapping Ψ_n of $\mathcal{L}_n(D_G)$ into $\cup_{n\le m\le n+2L}\mathcal{K}_m$ by

$$\Psi_n(b) = b_{[I(b),n]}*c(b_n,b_{I(b)})*a^-*c(b_1,b_{I(b)}),\; a^+(b)=a^+,\; b\in\mathcal{L}_n(D_G),\; n\in\mathbb{N}.$$

A word in $\mathcal{K}_m$, $n \leq m \leq n+2L$, has at most $n \cdot \mathrm{card}\mathcal{V}$ inverse images under the mapping Ψ_n, therefore

$$\text{card } \mathcal{L}_n(D_G) \leq n \cdot \mathrm{card}\mathcal{V} \sum_{n \leq m \leq n+2L} |\mathcal{K}_m|. \tag{3.1}$$

Every word in $\mathcal{K}_m$, $n \leq m \leq n+2L$, determines a periodic point in D_G and (3.1) implies that

$$\lim_{n\to\infty} \frac{1}{n} \log |\mathcal{L}_n(D_G)| \leq \liminf_{n\to\infty} \frac{1}{n} \log \Pi_n(D_G).$$

By [20, Prop. 4.1.5], the formula of the proposition follows. □

Corollary 3.2. *For the Markov-Dyck shift D_G the topological entropy $h(D_G)$ is the negative logarithm of the smallest positive solution of the equation:*

$$\det(I - D^*(A_G, z)A_G z) = 0.$$

For $v \in \mathcal{V}$ let X_v denote the subsystem of the Markov-Dyck shift D_G that is obtained by excluding the words

$$e(-)e(+), \qquad e \in \mathcal{E}, \quad r(e) \in \mathcal{V} \setminus \{v\}$$

and the words

$$f(+)g(-), \qquad g, f \in \mathcal{E}, \quad s(f) \in \mathcal{V} \setminus \{v\}.$$

We will estimate the asymptotic growth rate of the periodic points of X_v which, by a proof that is similar to the proof of Proposition 3.1, is actually equal to the topological entropy of X_v. In this way, we will also obtain estimates of the topological entropy of the Markov-Dyck shifts.

For $v \in \mathcal{V}$ denote by $\mathcal{D}_v$ the circular code of elementary Markov-Dyck words that start and end at v, and that are admissible for X_v, and denote by $\mathcal{C}_v$ the circular code of paths in G that start at v and end at v when they return for the first time to v. ρ denotes the inverse of the Perron eigenvalue of A_G.

Proposition 3.3.

$$g_{\mathcal{D}_v^* * \mathcal{C}_v^*}(z) = \frac{1 - \sqrt{1 - 4g_{\mathcal{C}_v}(z^2)}}{2(1 - g_{\mathcal{C}_v}(z))}, \qquad v \in \mathcal{V}.$$

Proof. One has $g_{\mathcal{D}_v * \mathcal{C}_v^*} = g_{\mathcal{D}_v} g_{\mathcal{C}_v^*}$ and $g_{\mathcal{D}_v}$ satisfies the equation

$$g_{\mathcal{D}_v}(z) = \frac{g_{\mathcal{C}_v}(z^2)}{1 - g_{\mathcal{D}_v}(z)}.$$

It follows that

$$g_{\mathcal{D}_v}(z) = \frac{1}{2}\Big(1 - \sqrt{1 - 4g_{\mathcal{C}_v}(z^2)}\Big),$$

which yields the proposition. □

Denoting by $p(z)$ the determinant of the matrix $I - A_G z$ and by $p_v(z)$ the determinant of the matrix $I - A_G z$ with the v-th row and the v-th column deleted, $v \in \mathcal{V}$, we set

$$q_v = \frac{p}{p_v}, \qquad v \in \mathcal{V}.$$

Theorem 3.4. *Let $v \in \mathcal{V}$ be such that*

$$q_v(\rho^2) > \frac{3}{4} \tag{3.2}$$

and let μ_v denote the minimum of the derivative or q_v on $0 \leq z \leq \rho$. Then

$$h(X_v) > -\log\rho + \frac{\sqrt{q_v(\rho^2) - \frac{3}{4}}\left(\frac{1}{2} - q_v(\rho) - \sqrt{q_v(\rho^2) - \frac{3}{4}}\right)}{|\mu_v|\left(\rho + \sqrt{q_v(\rho^2) - \frac{3}{4}}\right)}. \tag{3.3}$$

Proof. Corollary 2.2 and Proposition 3.3 imply that $h(X_v)$ is equal to $-\log\kappa$, where κ is the solution of the equation

$$1 = 2g_{\mathcal{C}_v}(z) - \sqrt{1 - 4g_{\mathcal{C}_v}(z^2)}, \qquad 0 < z < \rho,$$

or, equivalently, of the equation

$$\frac{1}{2} = q_v(z) + \sqrt{q_v(z^2) - \frac{3}{4}}, \qquad 0 < z < \rho.$$

By (3.2), one has the estimate

$$\rho - \kappa > \frac{\sqrt{q_v(\rho^2) - \frac{3}{4}}\left(\frac{1}{2} - q_v(\rho) - \sqrt{q_v(\rho^2) - \frac{3}{4}}\right)}{|\mu_v|\left(\rho + \sqrt{q_v(\rho^2) - \frac{3}{4}}\right)}$$

and (3.3) follows. □

Proposition 3.5. *Let*

$$\rho < \frac{1}{4}. \tag{3.4}$$

Then there is a $v \in \mathcal{V}$ such that

$$q_v(\rho^2) > \frac{3}{4}.$$

Proof. Assume the contrary. Then

$$\frac{4}{3}\operatorname{card}\mathcal{V} \leq \sum_{v\in\mathcal{V}} \frac{p_v(\rho^2)}{p(\rho^2)} = \operatorname{trace}(I - A_G\rho^2)^{-1} \leq \frac{1}{1-\rho}\operatorname{card}\mathcal{V},$$

contradicting (3.4). □

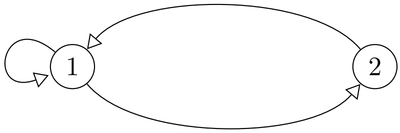

FIGURE 1

4. A class of examples

We consider first the Fibonacci-Dyck shift D_F that is produced by the directed graph (Figure 1) with adjacency matrix $F = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

The generating functions $g_{\mathcal{C}_1}$ and $g_{\mathcal{C}_2}$ of the two Fibonacci-Dyck codes $\mathcal{C}_1$ and $\mathcal{C}_2$ satisfy the equations

$$g_{\mathcal{C}_1}(z) = (g_{\mathcal{C}_1^*}(z) + g_{\mathcal{C}_2^*}(z))z^2, \tag{4.1}$$

$$g_{\mathcal{C}_2}(z) = g_{\mathcal{C}_1^*}(z)z^2, \tag{4.2}$$

where

$$g_{\mathcal{C}_1^*} = \frac{1}{1 - g_{\mathcal{C}_1}}, \qquad g_{\mathcal{C}_2^*} = \frac{1}{1 - g_{\mathcal{C}_2}}, \tag{4.3}$$

or

$$g_{\mathcal{C}_1} = 1 - \frac{1}{g_{\mathcal{C}_1^*}}, \qquad g_{\mathcal{C}_2} = 1 - \frac{1}{g_{\mathcal{C}_2^*}}. \tag{4.4}$$

By (4.1) and (4.4)

$$g_{\mathcal{C}_1^*}(z) = 1 + g_{\mathcal{C}_1^*}(z)(g_{\mathcal{C}_1^*}(z) + g_{\mathcal{C}_2^*}(z))z^2, \tag{4.5}$$

and from (4.2) and (4.3)

$$g_{\mathcal{C}_2^*}(z) = 1 + g_{\mathcal{C}_1^*}(z)g_{\mathcal{C}_2^*}(z)z^2. \tag{4.6}$$

From (4.2), (4.4) and (4.5)

$$g_{\mathcal{C}_2^*}(z)^3 z^2 - g_{\mathcal{C}_2^*}(z) + 1 = 0, \tag{4.7}$$

and from (4.6) and (4.7)

$$g_{\mathcal{C}_1^*} = g^2_{\mathcal{C}_2^*}. \tag{4.8}$$

From (4.2) and (4.3)

$$\det(I - Fz - D(F,z)) = \frac{z}{g_{\mathcal{C}_2}(z)}(g_{\mathcal{C}_2}(z)^2 - (2z+1)g_{\mathcal{C}_2}(z) + z). \tag{4.9}$$

Setting $\xi(z) = g_{\mathcal{C}_2^*}(z)z$ one has from (4.7)

$$\xi(z)^3 - \xi(z) + z = 0 \tag{4.10}$$

and from (4.9) and (4.10)

$$\det(I - Fz - D(F,z)) = -\frac{z^2}{\xi(z)^2}(2\xi(z)^2 + \xi(z) - 1). \tag{4.11}$$

By Theorem 2.3 and by (4.8) and (4.11)

$$\zeta_{D_F}(z) = \frac{\xi(z)}{z(2\xi(z)^2 + \xi(z) - 1)^2}, \tag{4.12}$$

where one identifies ξ as the solution of equation (4.10) vanishing at the origin that is given by

$$\xi(z) = \frac{2}{\sqrt{3}} \sin\left(\frac{1}{3} \arcsin \frac{3\sqrt{3}}{2} z\right), \qquad 0 \leq z \leq \frac{2}{3\sqrt{3}}. \tag{4.13}$$

By Theorem 3.1 and by (4.12) the topological entropy of the Fibonacci-Dyck shift is equal to the negative logarithm of the solution of

$$2\xi(z)^2 + \xi(z) - 1 = 0.$$

By (4.10) (or by (4.13)),

$$h(D_F) = 3\log 2 - \log 3. \tag{4.14}$$

We turn to the Markov-Dyck shift that is produced by the directed graph with adjacency matrix $F(a,b,c) = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, $a, b, c \in \mathbb{N}$. Here

$$g_{\mathcal{C}_1}(z) = (a g_{\mathcal{C}_1^*}(z) + b g_{\mathcal{C}_2^*}(z)) z^2, \tag{4.15}$$

$$g_{\mathcal{C}_2}(z) = c g_{\mathcal{C}_1^*}(z) z^2 \tag{4.16}$$

and one has from (4.16) that

$$g_{\mathcal{C}_1}(z) = 1 - \frac{cz^2}{g_{\mathcal{C}_2}(z)}. \tag{4.17}$$

From (4.15) and (4.17)

$$a g_{\mathcal{C}_2}(z)^3 - (a+c) g_{\mathcal{C}_2}(z)^2 + c(1 + (c-b)z^2) g_{\mathcal{C}_2}(z) - c^2 z^2 = 0. \tag{4.18}$$

From (4.17)

$$\begin{aligned} &\det(I - F(a,b,c)z - D(F(a,b,c),z)) \\ &\qquad = \frac{z}{g_{\mathcal{C}_2}} (a g_{\mathcal{C}_2}(z)^2 - (a + c(1+b)z) g_{\mathcal{C}_2}(z) + cz). \end{aligned} \tag{4.19}$$

Theorem 2.3 and (4.17) and (4.19) give

$$\zeta_{D_{F(a,b,c)}}(z) = \frac{c g_{\mathcal{C}_2}(z)(1 - g_{\mathcal{C}_2}(z))}{(a g_{\mathcal{C}_2}(z)^2 - (a + c(1+b)z) g_{\mathcal{C}_2}(z) + cz)^2}. \tag{4.20}$$

Setting

$$\begin{aligned} \mu(z) &= (c-a)^2 + ac - 3ac(c-b)z^2, \\ \nu(z) &= 2(a+c)^3 - 9ac(a+c) - (27a^2c^2 - 9ac(a+c)(c-b))z^2, \end{aligned}$$

one identifies $g_{\mathcal{C}_2}(z)$ as the solution of (4.18) that vanishes at the origin that is given by

$$g_{\mathcal{C}_2}(z) = \frac{a+c}{3a} + \frac{2}{3a}\sqrt{\mu(z)}\cos\left(\frac{1}{3}\left(2\pi + \arccos\frac{\nu(z)}{\mu(z)\sqrt{\mu(z)}}\right)\right).$$

We determine the topological entropy of $D_{F(a,b,c)}$, $a,b,c,\in\mathbb{N}$. Set

$$\begin{aligned}P_{a,b,c}(z) = &(1+c)[a(b-c) - c(1+b)^2]z^3\\ &+ (c[(1+b)(1+c) - 2ab] + a(1+a-b))z^2\\ &+ (bc - a - (1+a)(a-c))z + a - c.\end{aligned}$$

Theorem 4.1.

(a) $h(D_{F(a,b,c)})$ *is equal to the negative logarithm of the smallest positive solution of* $P_{a,b,c}(z) = 0$, $a,b,c\in\mathbb{N}$.

(b) $h(D_{F(a,b,a+b)}) = \log(1+a+b)$, $a,b\in\mathbb{N}$.

Proof. Let $z>0$ be such that the equations

$$ay^2 - (a + c(1+b)z)y + cz = 0 \tag{4.21}$$

and

$$ay^3 - (a+c)y^2 + c(1+(c-b)z^2)y - c^2z^2 = 0 \tag{4.22}$$

have a common solution y. Then y also solves the equation

$$(1-(1+b)z)y^2 - (1 - z + (c-b)z^2)y + cz^2 = 0 \tag{4.23}$$

and, as is seen from (4.21) and (4.23), it also solves the equation

$$(1-(1+a+b)z)y = 1 - (1+a)z - b(1+c)z^2. \tag{4.24}$$

From (4.21) and (4.24)

(4.25)

$$\begin{aligned}bzP_{a,b,c}(z) =&(1-(1+a)z - b(1+c)z^2)\\ &\{a(1-(1+a)z - b(1+c)z^2) - (1-(1+a+b)z)(a + c(1+b)z)\}\\ &+ cz(1-(1+a+b)z)^2 = 0.\end{aligned}$$

This shows that for every $z>0$ such that equations (4.21) and (4.22) have a common solution, $P_{a,b,c}(z) = 0$.

Equation (4.23) is a multiple of equation (4.21) precisely if $c = a+b$ and $z = \frac{1}{1+a+b}$, and from this one sees, consulting (4.24), that both solutions of equation (4.21) are also solutions of equation (4.22) precisely if $c = a+b$ and $z = \frac{1}{1+a+b}$. Moreover, as is seen from (4.25), $P_{a,b,c}(z) = 0$ has the root $\frac{1}{1+a+b}$ if and only if $c = a+b$.

For the case that $c \neq a+b$, let $z>0$,

$$P_{a,b,c}(z) = 0, \tag{4.26}$$

and reverse the argument, setting

$$y = \frac{1-(1+a)z-b(1+c)z^2}{1-(1+a+b)z}. \tag{4.27}$$

Consult then (4.24) and find from (4.26) that y as given by (4.27) solves equation (4.21) and therefore also equations (4.23) and (4.22). Apply now Corollary 3.2 together with (4.18) and (4.20) to prove part (a) of the theorem for the case $c \neq a+b$.

Consider the case that $c = a+b$. One checks that $\frac{1}{1+a+b}$ is the unique positive root of $P_{a,b,a+b}(z) = 0$. It has already been shown that $P_{a,b,a+b}(z) = 0$ for every $z > 0$ such that equations (4.21) and (4.22), or, in this case, the equations

$$ay^2 - (a+(a+b)(1+b)z)y + (a+b)z = 0 \tag{4.28}$$

and

$$ay^3 - (2a+b)y^2 + (a+b)(1+az^2)y - (a+b)^2z^2 = 0$$

have a common solution. One checks that for $z = \frac{1}{1+a+b}$ a root of (4.28), in fact the smaller one, is equal to $g_{\mathcal{C}_2}(\frac{1}{1+a+b})$. Apply now again Corollary 3.2 together with (4.18) and (4.20) to prove part (a) of the theorem for the case that $c = a+b$ and also part (b). □

The corollary reconfirms (4.14).

Corollary 4.2.

$$h(D_{F(a,1,a)}) = \log(a+1) - \log(a+2) + \log(a+3), \qquad a \in \mathbb{N}.$$

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