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On groups and fields interpretable in torsion-free hyperbolic groups

Chloé Perin, Anand Pillay, Rizos Sklinos and Katrin Tent

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Abstract. We prove that the generic type of a noncyclic torsion-free hyperbolic group G is foreign to any interpretable abelian group, hence also to any interpretable field. This result depends, among other things, on the definable simplicity of a noncyclic torsion-free hyperbolic group, and we take the opportunity to give a proof of the latter using Sela's description of imaginaries in torsion-free hyperbolic groups. We also use the description of imaginaries to prove that if $\mathbb{F}$ is a free group of rank > 2 then no orbit of a (nontrivial) finite tuple from $\mathbb{F}$ under $Aut(\mathbb{F})$ is definable.

1. Introduction

This paper concerns the first order theories of torsion-free hyperbolic groups. There is an increasing model theoretic interest in the subject motivated by the positive solution to Tarski's problem (i.e. is the theory of nonabelian free groups complete?) by Sela and Kharlampovich–Myasnikov. Subsequently Sela proved the stability of all noncyclic torsion-free hyperbolic groups [14]. These are in fact remarkable examples of "new stable groups", given by nature.

If we fix a torsion-free hyperbolic group G, then understanding the category of definable/interpretable sets in models of $Th(G)$, informed by stability-theoretic tools and notions, is a challenge. Sela's work on imaginaries in torsion-free hyperbolic groups [13] is part of this endeavor and will be used in the current paper.

Our paper contributes to the following conjecture:

Conjecture 1. *Let G be a torsion-free hyperbolic group. Then no infinite field is interpretable in (any model of) $Th(G)$.*

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In general, also the nature and complexity of interpretable groups in a theory is important and we make another conjecture which will only indirectly be touched on in the current paper:

Conjecture 2. *Let G be a torsion-free hyperbolic group. Any group interpretable in a model $\mathcal{M}$ of $Th(G)$ is definably isomorphic to a definable subgroup of $\mathcal{M} \times \ldots \times \mathcal{M}$.*

Note that this is basically the situation in 1-based groups (see [1]).

Let G be a noncyclic torsion-free hyperbolic group. Then the first order theory of G is connected and hence has a unique generic type, which we call p_0^G (see Section 2). In the special case of nonabelian free groups a considerable amount of information has been obtained around the complexity of $p_0^{\mathbb{F}_n}$ (by all the authors). For example $p_0^{\mathbb{F}_n}$ has infinite weight ([11], [16]), and witnesses the fact that the free group is n-ample for all n ([18]). As a matter of fact, in [7] it was proved that the theory of any (noncyclic) torsion-free hyperbolic group is n-ample for all n. The ampleness result is consistent with the existence of an infinite interpretable field which interacts with p_0^G. Corollary 2 rules this out. Thus, the results of the current paper provide a partial solution to Conjecture 1, yielding additional information on p_0^G: No interpretable abelian group (hence also interpretable field) can interact with p_0^G. See Section 6 for the stability-theoretic definitions.

Theorem 1. *Let G be a noncyclic torsion-free hyperbolic group. Then p_0^G is foreign to any interpretable abelian group.*

Corollary 2. *Let G be a noncyclic torsion-free hyperbolic group. Then p_0^G is foreign to any interpretable field or skew field.*

We will need to know that noncyclic torsion-free hyperbolic groups are definably simple (no definable proper nontrivial normal subgroup). The stronger result that the only proper definable subgroups of torsion-free hyperbolic groups are cyclic, has been stated in several places, such as [2]. We take the opportunity here to give an independent proof in Section 4, using Sela's description of imaginaries [13] as a "black box":

Theorem 3. *The only definable proper subgroups of a torsion-free hyperbolic group are cyclic.*

In the same section we also prove:

Theorem 4. *Let $\mathbb{F}$ be a free group of rank at least 3. Then no orbit of a finite (nontrivial) tuple under $Aut(\mathbb{F})$ is definable.*

Let us stress that by *definable* we mean definable possibly *with parameters.*

The paper is organized as follows. In the following section we give some model theoretic background around torsion-free hyperbolic groups. We give special emphasis to the notion of imaginaries and we give a precise account of elimination of imaginaries in model theoretic terminology as this is crucial for questions regarding interpretability.

In Section 3 we record, for the benefit of the reader, some well-known material about normal forms in groups that have the structure of a free product.

In Section 4, we give certain nondefinability results. In particular we use Sela's elimination of imaginaries result to prove Theorem 3 and Theorem 4.

In Section 5 we prove a result that forbids abelian interpretable groups in nonabelian free groups to "gain" an element in higher rank free groups. This result is an elaboration on material in Chapter 8 of the third author's Ph.D. thesis [17], but now we have to consider not only real tuples but imaginaries. We also prove an analogous result for torsion-free hyperbolic groups.

Finally in Section 6, we bring everything together to prove Theorem 1.

2. Some model theory of torsion-free hyperbolic groups

We start our discussion with the free group case. Our notation is fairly standard. By $\mathbb{F}_n$ we denote the free group on n generators and we usually denote a basis of $\mathbb{F}_n$ by $e_1, \ldots, e_n$. By T_{fg} we denote the common theory of nonabelian free groups. We also note that the natural embedding of $\mathbb{F}_n$ in $\mathbb{F}_m$ for $2 \leq n < m$ is elementary (as proved by Sela and Kharlampovich–Myasnikov).

In [12] Poizat proved that $\mathbb{F}_\omega$ is connected (thus T_{fg} is connected). Moreover the following theorem has been proved by the second named author in [10].

Theorem 2.1. *Let $\mathbb{F}_\omega := \langle e_1, \ldots, e_n, \ldots \rangle$. Then $(e_i)_{i<\omega}$ is a Morley sequence in $p_0^{\mathbb{F}_n}$. In particular $tp(e_{n+1}/\mathbb{F}_n)$ is generic.*

Now, let G be a torsion-free hyperbolic group. Sela assigns to such a group its elementary core $EC(G)$, which is an elementary subgroup of G provided G is not elementarily equivalent to a free group (for a definition and further properties see [15]). Note that if G is elementarily equivalent to a free group, $EC(G)$ is the trivial group. The elementary core of $EC(G) * \mathbb{Z}$ is again $EC(G)$, so the latter is an elementary subgroup of the former. This observation led Ould Houcine to the following result [6]

Theorem 2.2. *Let G be a torsion-free hyperbolic group not elementarily equivalent to a free group. Then G is connected. Moreover if $H := EC(G) * \langle e \rangle$, then $tp^H(e/EC(G))$ is generic.*

In [7] the following useful result was proved.

Proposition 2.3. *Let G be a torsion-free hyperbolic group not elementarily equivalent to a free group. Suppose K is a free factor of a free group $\mathbb{F}$. Then $EC(G) * K$ is an elementary subgroup of $G * \mathbb{F}$. In particular since $EC(EC(G)) = EC(G)$ we have an elementary chain*

$$EC(G) \prec EC(G) * \mathbb{Z} \prec \ldots \prec EC(G) * \mathbb{F}_n \prec \ldots.$$

2.4. **Imaginaries in torsion-free hyperbolic groups.** We first give a quick overview of the model theoretic notion of imaginaries as well as various notions of elimination of imaginaries. We then specialize to torsion-free hyperbolic groups and give Sela's result.

Recall that $\mathcal{M}^{eq}$ is constructed from $\mathcal{M}$ by adding a new sort for each $\varnothing$-definable equivalence relation, $E(\bar{x}, \bar{y})$, together with a class function $f_E : M^n \to M_E$, where M_E (the domain of the new sort corresponding to E) is the set of all E-equivalence classes. The elements in these new sorts are called *imaginaries.* Note that any automorphism of $\mathcal{M}$ has a canonical extension to $\mathcal{M}^{eq}$.

We say that $\mathcal{M}$ eliminates imaginaries if it has a saturated elementary extension $\mathbb{M}$ in which for any element $\mathfrak{e}$ of $\mathbb{M}^{eq}$, there is a finite tuple $\bar{b} \in \mathbb{M}$ such that $\mathfrak{e} \in dcl^{eq}(\bar{b})$ and $\bar{b} \in dcl^{eq}(\mathfrak{e})$.

We say that $\mathcal{M}$ weakly eliminates imaginaries if it has a saturated elementary extension $\mathbb{M}$ in which for any element $\mathfrak{e}$ of $\mathbb{M}^{eq}$, there is a finite tuple $\bar{b} \in \mathbb{M}$ such that $\mathfrak{e} \in dcl^{eq}(\bar{b})$ and $\bar{b} \in acl^{eq}(\mathfrak{e})$.

We now specialize to torsion-free hyperbolic groups.

Definition 2.5. *Let G be a torsion-free hyperbolic group. The following equivalence relations in G are called basic.*

- (i) $E_1(a, b)$ *if and only if there is* $g \in G$ *such that* $a^g = b$. *(conjugation)*
- (ii)$_m$ $E_{2_m}((a_1, b_1), (a_2, b_2))$ *if and only if either* $b_1 = b_2 = 1$ *or* $b_1 \neq 1$ *and* $C_G(b_1) = C_G(b_2) = \langle b \rangle$ *and* $a_1^{-1}a_2 \in \langle b^m \rangle$. *(m-left-coset)*
- (iii)$_m$ $E_{3_m}((a_1, b_1), (a_2, b_2))$ *if and only if either* $b_1 = b_2 = 1$ *or* $b_1 \neq 1$ *and* $C_G(b_1) = C_G(b_2) = \langle b \rangle$ *and* $a_1 a_2^{-1} \in \langle b^m \rangle$. *(m-right-coset)*
- (iv)$_{m,n}$ $E_{4_{m,n}}((a_1, b_1, c_1), (a_2, b_2, c_2))$ *if and only if either* $a_1 = a_2 = 1$ *or* $c_1 = c_2 = 1$ *or* $a_1, c_1 \neq 1$ *and* $C_G(a_1) = C_G(a_2) = \langle a \rangle$ *and* $C_G(c_1) = C_G(c_2) = \langle c \rangle$ *and there is* $\gamma \in \langle a^m \rangle$ *and* $\epsilon \in \langle c^n \rangle$ *such that* $\gamma b_1 \epsilon = b_2$. *($m, n$-double-coset)*

The following is Theorem 4.4 in [13]:

Theorem 2.6. *Let G be a torsion-free hyperbolic group. Let $E(\bar{x}, \bar{y})$ be a definable equivalence relation in G, with $|\bar{x}| = m$. Then there exist $k, l < \omega$ and a definable relation*

$$R_E \subseteq G^m \times G^k \times S_1(G) \times \ldots \times S_l(G)$$

such that

- (i) *each $S_i(G)$ is one of the basic sorts;*
- (ii) *there is some $s \in \mathbb{N}$ such that for each $\bar{a} \in G^m$, $|R_E(\bar{a}, G^{eq})| \leq s$;*
- (iii) $\forall \bar{z}(R_E(\bar{a}, \bar{z}) \leftrightarrow R_E(\bar{b}, \bar{z}))$ *if and only if* $E(\bar{a}, \bar{b})$.

In the case of $\varnothing$-definable equivalence relations a slight variation of the above theorem is true (see Theorem 4.4 and Proposition 4.5 in [13]):

Theorem 2.7. *Let G be a torsion-free hyperbolic group. Let $E(\bar{x}, \bar{y})$ be a $\varnothing$-definable equivalence relation in G, with $|\bar{x}| = m$. Then there exist $k, l < \omega$, finitely many "exceptional" $\varnothing$-definable equivalence classes, defined by $\phi_i(\bar{x})$, $i \leq n$, and a $\varnothing$-definable relation*

$$R_E \subseteq G^m \times G^k \times S_1(G) \times \ldots \times S_l(G)$$

such that

(i) *each $S_i(G)$ is one of the basic sorts;*
(ii) *there is some $s \in \mathbb{N}$ such that for each $\bar{a} \in G^m$, $|R_E(\bar{a}, G^{eq})| \leq s$;*
(iii)
$$G^{eq} \models \forall \bar{x}(\phi_1(\bar{x}) \vee \phi_2(\bar{x}) \vee \ldots \vee \phi_n(\bar{x}) \rightarrow R_E(\bar{x}, 1, 1, \ldots, (1,1,1)) \wedge |R_E(\bar{x}, \bar{z})| = 1),$$
i.e. R_E assigns to each tuple in any of the "exceptional" classes a single tuple consisting of identity elements in the corresponding sorts;
(iv) *for any two tuples $\bar{a}, \bar{b}$ not in the union of the "exceptional" classes we have $G^{eq} \models \forall \bar{z}(R_E(\bar{a}, \bar{z}) \leftrightarrow R_E(\bar{b}, \bar{z}))$ if and only if $E(\bar{a}, \bar{b})$.*

Let G be a group elementarily equivalent to a torsion-free hyperbolic group, by G^{we} we denote the expansion of G by the above basic sorts, i.e. $G^{we} = (G, S_1(G), \{S_{2_m}(G)\}_{m<\omega}, \{S_{3_m}(G)\}_{m<\omega}, \{S_{4_{m,n}}(G)\}_{m,n<\omega})$. The above theorem easily implies that G^{we} weakly eliminates imaginaries.

Corollary 2.8. *Let G be a torsion-free hyperbolic group. Then G^{we} weakly eliminates imaginaries.*

Proof. We work in a saturated extension $\mathbb{G}$ of G. Let $\mathfrak{e}$ be an element of $\mathbb{G}^{eq}$. Then there is a $\varnothing$-definable equivalence relation $E(\bar{x}, \bar{y})$ in G such that $\mathfrak{e} = [\bar{a}]_E$ for some element $\bar{a} \in \mathbb{G}$. By Theorem 2.7, E is assigned a $\varnothing$-definable relation R_E such that $R_E(\bar{a}, \bar{z})$ has finitely many solutions in $\mathbb{G}^{we}$.

First assume that $\bar{a}$ belongs to an "exceptional" class, then we claim that $\mathfrak{e} \in dcl^{eq}(1)$, indeed since each exceptional class is $\varnothing$-definable any automorphism will fix $\mathfrak{e}$. Now suppose that $\bar{a}$ does not belong to an exceptional class and $(\bar{a}_1\bar{a}_2 \ldots \bar{a}_k)$ is the concatenation of the solutions of $R_E(\bar{a}, \bar{z})$ in G^{eq}. We claim that $\mathfrak{e} \in dcl^{eq}(\bar{a}_1\bar{a}_2 \ldots \bar{a}_k)$ and $\bar{a}_1\bar{a}_2 \ldots \bar{a}_k \in acl^{eq}(\mathfrak{e})$. The formula $\exists \bar{y}(f_E(\bar{y}) = x \wedge R_E(\bar{y}, \bar{a}_1) \wedge \ldots \wedge R_E(\bar{y}, \bar{a}_k) \wedge \forall \bar{w}(R_E(\bar{y}, \bar{w}) \rightarrow \bigvee_{i=1}^k \bar{w} = \bar{a}_i))$ defines $\mathfrak{e}$ in $\mathbb{G}^{eq}$. Now consider the formula $\exists \bar{y}(R_E(\bar{y}, \bar{x}_1) \wedge \ldots \wedge R_E(\bar{y}, \bar{x}_k) \wedge f_E(\bar{y}) = \mathfrak{e})$, the solution set of this formula is $\{\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_k\}^k$, which is finite. □

3. Normal forms for free products

In this section we quickly recall some basic background on normal forms of elements with respect to a free product of groups $A * B$. The material in this section can be found in [4, Chap. IV].

Definition 3.1. *Let $G := A*B$ be a free product. Then a sequence of elements in G, $g_1, g_2, \ldots, g_n$ (with $n \geq 0$), is a reduced sequence if the following hold:*
- *For each $i \leq n$, $g_i \in A \cup B$;*
- *two consecutive g_i's live in different factors.*

We say that a product of elements $g_1g_2 \ldots g_n$ in G is in normal form if $g_1, g_2, \ldots, g_n$ is a reduced sequence.

Theorem 3.2 (The normal form theorem)**.** *Let $G := A * B$ be a free product. Then each element $g \in G$ can be uniquely expressed as a product $g = g_1g_2 \ldots g_n$ in normal form.*

If $g = g_1g_2 \ldots g_n$ is in normal form (with respect to a free product $A * B$), we say that g is *cyclically reduced* if g_1, g_n live in different free factors or $n \leq 1$.

Theorem 3.3 (The conjugacy theorem). *Let $G := A * B$ be a free product. Then each element of G can be conjugated to a cyclically reduced element.*

Moreover, if $a = g_1 g_2 \dots g_m$, $b = h_1 h_2 \dots h_n$ are two cyclically reduced elements which are conjugate in $A * B$, then:

- *If $n > 1$ we have that $m = n$ and $h_1, h_2, \dots, h_n$ is a cyclic permutation of $g_1, g_2, \dots, g_m$.*
- *If $n \leq 1$ we have that $m = n$, a, b are elements of the same free factor and they are conjugates in that factor.*

Let us also remark that the above results naturally generalize to free products with more than two free factors.

4. Some (non) definability results

We first prove that the only definable proper subgroups of a torsion-free hyperbolic group are cyclic. This implies, in the case where G is noncyclic, that $Th(G)$ is definably simple.

Let G be a noncyclic torsion-free hyperbolic group and $H := G * \langle e \rangle$. If a is an element in G, we denote by f_a the automorphism of H which is the identity on G and satisfies $f_a(e) = ea$.

Lemma 4.1. *Let $\beta \in H^{eq} \setminus G^{eq}$ be an element from a basic sort. Suppose β is not contained in the following list:*

(1) *$\beta = [b]_{E_1}$, where $b = b_1 e^{i_1} b_2 \dots b_m e^{i_m}$ with $m > 1$, $b_i \in G \setminus \{1\}$ for all $i \leq m$ and $b_i \in C(a)$ for some $i \leq m$; or*
(2) *$\beta = [(c, b)]_E$ for $E = E_{2_p}$ or $E = E_{3_p}$, or $\beta = [(b, c, d)]_{E_{4_{p,q}}}$, or $\beta = [(d, c, b)]_{E_{4_{p,q}}}$, where $b = b_1 e^{i_1} b_2 e^{i_2} \dots b_m\ e^{i_m} b_{m+1}$ with $m > 1$, $b_i \in G$ (if $1 < i < m+1$, then b_i is nontrivial) and $b_2 \in C(a)$; or*
(3) *$\beta = [(c, b)]_E$ for $E = E_{2_p}$ or $E = E_{3_p}$, or $\beta = [(b, c, d)]_{E_{4_{p,q}}}$, where $b, d \in G$ and $c = c_1 e^{i_1} c_2 \dots c_m e^{i_m} c_{m+1}$ with $c_i \in G$ (if $1 < i < m+1$, then c_i is nontrivial). Moreover, either some c_i, for $1 < i < m+1$, is in $C(a)$ or b and/or d belong to the centralizer of some conjugate of a (by an element of G).*

Then β has infinite orbit under $\langle f_a^i : i \in \mathbb{Z} \rangle$.

Proof. Assume that β has finite orbit. We take cases according to the list of basic sorts.

- Let $\beta = [b]_{E_1}$. Let $b_1 e^{i_1} b_2 \dots b_m e^{i_m}$ be the (cyclically reduced) normal form of b with respect to the decomposition $G * \langle e \rangle$. Suppose for some $j \leq m$ we have that $|i_j| > 1$. Then $ea^i e$ or $e^{-1} a^{-i} e^{-1}$ will appear in the (cyclically reduced) normal form of $f^i(b)$ and the result follows easily. Suppose, for some j we have that $i_{j (\mathrm{mod}\, m)} = i_{j+1 (\mathrm{mod}\, m)}$. Then $ea^i b_{j+1 (\mathrm{mod}\, m)} e$ or $e^{-1} b_{j+1 (\mathrm{mod}\, m)} a^{-i} e^{-1}$ will appear in the (cyclically reduced) normal form of $f^i(b)$ so there must be k so that $b_k = a^i b_{j+1 (\mathrm{mod}\, m)}$ or $b_k = b_{j+1 (\mathrm{mod}\, m)} a^{-i}$ for infinitely many values of i, which is impossible. If $m = 1$, then is not hard to see that $[b_1 e^{\pm 1}]_{E_1}$ has infinite orbit under $\langle f^i | i < \omega \rangle$. Lastly, suppose for some j, $i_{j(\mathrm{mod}\, m)} = 1$ and $i_{j+1(\mathrm{mod}\, m)} = -1$ but $b_{j+1(\mathrm{mod}\, m)} \notin C(a)$. Then $ea^i b_{j+1(\mathrm{mod}\, m)} a^{-i} e^{-1}$ will appear in the (cyclically reduced) normal form of $f^i(b)$ and since $[a, b_{j+1(\mathrm{mod}\, m)}] \neq 1$ the result follows.

- Let $\beta = [(b,c,d)]_{E_{4_{p,q}}}$. We take two subcases:
  - (i) Suppose that $b \in H \setminus G$. Since H is a torsion-free hyperbolic group we have that there is a $\gamma \in H \setminus G$ such that $f^i(b) = \gamma$ for infinitely many i's. To see this recall that in a torsion-free hyperbolic group if $C(\alpha) = C(f(\alpha))$, then $f(\alpha) = \alpha$ or $f(\alpha) = \alpha^{-1}$. Now, let $b_1 e^{i_1} b_2 \dots b_m e^{i_m} b_{m+1}$ be the normal form of b with respect to the decomposition $G * \langle e \rangle$. It follows as in the case above that $|i_j| = 1$ for all $j \leq m$, and $i_j \neq i_{j+1}$ for all $j < m$. Moreover, $m > 1$ and if for some $j < m$, $i_j = 1$ and $i_{j+1} = -1$, then $b_{j+1} \in C(a)$. What remains to be shown is that $i_1 \neq -1$, which is trivial to check.
  - (ii) Suppose $b, d \in G$ and let $c_1 e^{i_1} c_2 \dots c_m\, e^{i_m} c_{m+1}$ be the normal form of c with respect to the decomposition $G * \langle e \rangle$. Suppose that $|i_j| > 1$ for some $j \leq m$, then $f^i(c)$ will have a normal form containing $ea^i e$ or $e^{-1}a^{-i}e^{-1}$, thus $\epsilon f^i(c)\delta \neq f^j(c)$ for any $\epsilon, \delta \in G$ and the result follows. Similarly one can prove that $i_j \neq i_{j+1}$, for all $j < m$ and if for some $j < m$, $i_j = 1$ and $i_{j+1} = -1$ then $c_{j+1} \in C(a)$. We now take cases that depend on the value of m. If $m > 2$, it is clear that either c_2 or c_3 will be in $C(a)$. Suppose $m = 1$ and $c = c_1 e c_2$ (or $c = c_1 e^{-1} c_2$). Then $f^i(c) = c_1 e a^i c_2$ (or $f^i(c) = c_1 a^{-i} e^{-1} c_2$) and $\epsilon f^i(c)\delta = f^j(c)$ only if $\delta \in C(c_2^{-1} a c_2)$ (only if $\epsilon \in C(c_1 a c_1^{-1})$). Lastly suppose $m = 2$, in this case we can only have $i_1 = -1$ and $i_2 = 1$ (otherwise $c_2 \in C(a)$), thus we have $c = c_1 e^{-1} c_2 e c_3$ and $\epsilon f^i(c)\delta = f^j(c)$ only if $\epsilon \in C(c_1 a c_1^{-1})$ and $\delta \in C(c_3^{-1} a c_3)$.
- Let $\beta = [(c,b)]_E$ for $E = E_{2_p}$ or $E = E_{3_p}$. The proof is similar to the above and is left to the reader. ☐

This immediately yields the following:

Corollary 4.2. *Let $a, c \in G$ with $C(a) \neq C(c)$. Then for any element $\beta \in H^{we} \setminus G^{we}$, β has infinite orbit under $\langle f_a^i, f_c^i | i < \omega \rangle$. In particular, if G is a nontrivial free product and $\beta \in H^{we}$, then $Aut(H).\beta$ is infinite.*

We note that the particular case of Corollary 4.2 fails in $\mathbb{F}_2$: The conjugacy class of $[e_1, e_2]$ has exactly two images under $Aut(\mathbb{F}_2)$ (see [4, Prop. 5.1, p. 44]).

Theorem 4.3. *Let G be a torsion-free hyperbolic group. Then any definable proper subgroup of G is cyclic.*

Proof. We may assume that G is not cyclic. Suppose the result is not true. We first consider the case where G is elementarily equivalent to a free group. Then there is a definable (over $\mathbb{F}_2$) nonabelian subgroup of $\mathbb{F}_2$, which we denote by A. Let E_A be the (definable over $\mathbb{F}_2$) equivalence relation defined by $E_A(a,b)$ if and only if $a \cdot A = b \cdot A$. Let R_A be the relation given by Theorem 2.6. Let $R_A(e_3, \mathbb{F}_3^{eq}) = \{\bar{b}_1, \dots, \bar{b}_k\}$. We claim that there is $i \leq k$ such that $\bar{b}_i$ is in

$\mathbb{F}_3^{eq} \setminus \mathbb{F}_2^{eq}$. Otherwise, since $\mathbb{F}_2$ is an elementary substructure, we find some $c \in \mathbb{F}_2$ with $R_A(c, \mathbb{F}_3^{eq}) = \{\bar{b}_1, \ldots, \bar{b}_k\}$, so c and e_3 are equivalent. But then $e_3^{-1} \cdot c$ is in A. Since A is definable over $\mathbb{F}_2$ we have that A is generic, contradicting the connectedness of T_{fg}. So, without loss of generality, $\bar{b}_1 \in \mathbb{F}_3^{eq} \setminus \mathbb{F}_2^{eq}$. Now consider the automorphisms that fix $\mathbb{F}_2$ and send e_3 to $e_3 \cdot a$ for some $a \in A$: Every such automorphism fixes R_A, and clearly $E_A(e_3, e_3 \cdot a)$. Since A is not abelian however, we can find a, c such that $[a, c] \neq 1$: By Corollary 4.2, $\bar{b}_1$ has infinitely many images under the iterates of f_a and f_c, a contradiction.

Finally, suppose G is not elementarily equivalent to a free group. Is not hard to see that the above argument is still valid if we replace $\mathbb{F}_2$ with the elementary core of G and use Theorem 2.2. □

The following corollary follows immediately from the fact that the normalizer of a cyclic subgroup in a torsion-free hyperbolic group coincides with its (cyclic) centralizer:

Corollary 4.4. *Let G be a noncyclic torsion-free hyperbolic group. Then $Th(G)$ is definably simple.*

We give another application of the weak elimination of imaginaries extending a result of Kharlampovich–Myasnikov (see [2, Cor. 2]) that proved that the set of primitive elements of $\mathbb{F}_n$ is not definable if $n > 2$.

In [3, Thm. 4.8] it was proved that no (nontrivial) type in $S(T_{fg})$ is isolated, equivalently since free groups are homogeneous structures no finite tuple has $\varnothing$-definable orbit under $Aut(\mathbb{F}_n)$. An easy application of the weak elimination of imaginaries allows us, in free groups of rank at least 3, to strengthen this result by showing that no orbit is definable even with parameters. Intuitively the reason is that every such orbit is $Aut(\mathbb{F}_n)$-invariant but every nontrivial canonical parameter can be "moved" by an automorphism of $\mathbb{F}_n$, thus the orbit can only be definable over $\varnothing$, contradicting the above mentioned theorem. Note that it is well-known that the orbit of any tuple in $\mathbb{F}_2$ (under $Aut(\mathbb{F}_2)$) is definable in $\mathbb{F}_2$ by a result of Nielsen [5].

Proposition 4.5. *Let $\bar{v}$ be a nontrivial tuple of elements in a nonabelian free group $\mathbb{F}$ of rank at least 3. Then $Aut(\mathbb{F}).\bar{v}$ is not definable.*

Proof. Suppose it is, and let $\mathfrak{e} = [\bar{a}]_E$ be the canonical parameter for the definable set $X := Aut(\mathbb{F}).\bar{v}$, in $\mathbb{F}^{eq}$. By Theorem 2.7, $|R_E(\bar{a}, \mathbb{F}^{eq})|$ is finite and by Theorem 4.8 in [3], it contains a nontrivial tuple, thus by Lemma 4.2 there is an automorphism such that $f(\mathfrak{e}) \neq \mathfrak{e}$, but then $f(X) \neq X$, a contradiction. □

5. ABELIAN INTERPRETABLE GROUPS IN TORSION-FREE HYPERBOLIC GROUPS

We show that if a formula ϕ over $\mathbb{F}_n^{eq}$ "gains" an element in $\mathbb{F}_{n+1}^{eq}$, i.e. $\phi(\mathbb{F}_n^{eq}) \neq \phi(\mathbb{F}_{n+1}^{eq})$, then it cannot be given definably the structure of an abelian group. In the case of a torsion-free hyperbolic group not elementarily equivalent to a free group, we prove an analogous result for the elementary core.

Lemma 5.1. *Let $G := F_0 * X_0 * X_1$ be a free product of groups and let h be an automorphism of G fixing F_0 pointwise and acting on $X_0 * X_1$ as an automorphism of prime order p whose fixed point set is exactly X_0.*

Suppose that h fixes the conjugacy class of a cyclically reduced element $a \in G \setminus (F_0 * X_0 \cup X_0 * X_1)$, of the form

$$a = a_1 x_1 a_2 x_2 \cdots a_m x_m \text{ with } a_i \in F_0 \setminus \{1\} \quad \text{and } x_i \in X_0 * X_1 \setminus \{1\} \text{ for } i = 1, \ldots m.$$

Then there is a permutation $\sigma \in \langle (1 \ldots m) \rangle \leq Sym(\{1, \ldots, m\})$ such that:

(i) *$a_i = a_{\sigma(i)}$ and $h(x_i) = x_{\sigma(i)}$;*
(ii) *p divides the order $o(\sigma)$ of σ.*

Proof. Since h is the identity on F_0 and leaves $X_0 * X_1$ invariant and because a is cyclically reduced and conjugate to $h(a)$, we see that

$$h(a) = a_1 h(x_1) a_2 h(x_2) \ldots a_m h(x_m)$$

differs from a by a cyclic shift only. This implies (i).

Now (ii) follows from (i): Since $h^j(x_i) = x_{\sigma^j(i)}$, we see that $h^{o(\sigma)}$ fixes each of the x_i's. The elements of $X_0 * X_1$ fixed by h (and thus by any element generating $\langle h \rangle$) are exactly those of X_0. Thus $h^{o(\sigma)}$ must be the identity so $o(h) = p$ divides $o(\sigma)$. □

We will need the following small lemma:

Lemma 5.2. *Let M be a finite cyclic group, let $s, t \in M$ and let $d > 1$. Then there are k, l with $s^k = t^l$ and such that d does not divide both k and l.*

Proof. Write $o(s) = ef$, and $o(t) = eg$ where f, g are coprime. Then s^f and t^g have order e, hence generate the same subgroup of M. So each can be written as a power of the other and the conclusion follows. □

Lemma 5.3. *Let $G := U_f * W * U_g$ be a free product of groups. Let f and g be automorphisms of G of prime order p such that f is the identity on W and on U_f and acts without nontrivial fixed point on U_g, and g is the identity on W and on U_g and acts without nontrivial fixed point on U_f. If a is an element of G whose conjugacy class is fixed both by f and by g, then a lies up to conjugacy in $W * U_f$ or in $W * U_g$.*

Proof. Suppose that $a \notin (W * U_f) \cup (W * U_g)$. We may choose $a = a_1 x_1 a_2 x_2 \cdots a_m x_m$ cyclically reduced with $a_i \in U_f * W, x_i \in U_g$, and $a_i \neq 1 \neq x_i$ for $i = 1, \ldots m$. By Lemma 5.1 applied to f and the decomposition $G = (U_f * W) * 1 * U_g$ on the one hand, and to g and the decomposition $G = U_g * W * U_f$ on the other hand we find $\sigma, \tau \in \langle (1 \ldots m) \rangle$ such that for $i = 1, \ldots m$ we have

$$a_i = a_{\sigma(i)} \text{ and } f(x_i) = x_{\sigma(i)}$$
$$g(a_i) = a_{\tau(i)} \text{ and } x_i = x_{\tau(i)}.$$

By Lemma 5.2 we can choose k, l with $\sigma^k = \tau^l$ such that p does not divide both k and l.

For $i = 1, \dots m$ we now have

$$f^k(x_i) = x_{\sigma^k(i)} = x_{\tau^l(i)} = x_i,$$
$$g^l(a_i) = a_{\tau^l(i)} = a_{\sigma^k(i)} = a_i.$$

Thus f^k is the identity (and so p divides k) and g^l fixes each a_i. Since p does not divide l, we conclude that $a_i \in W$ for $i = 1, \dots m$ showing that $a \in W * U_g$. This contradiction proves the lemma. □

Lemma 5.4. *Let $H := G * C$ be a torsion-free hyperbolic group. Let f be an automorphism of H of prime order $p > 2$ that fixes G and acts on C without nontrivial fixed point.*

Suppose $\beta \in H^{we} \setminus G^{we}$ is not a conjugacy class, i.e. β is not in $S_1(H)$. Then $|\{\beta, f(\beta), f^2(\beta), \dots, f^{p-1}(\beta)\}| = p$.

Proof. We first note that no nontrivial power of f fixes an element in $H \setminus G$. This also implies that $f^i(d) \neq d^{-1}$ for all $d \in H \setminus G$, otherwise we would have $f^{2i}(d) = d$. We now take cases according to the list of basic sorts.

We will only treat the case where $\beta = [(b_1, b_2, b_3)]_{E_{4_{k,l}}}$. Suppose for the sake of contradiction that $(f^i(b_1), f^i(b_2), f^i(b_3)) \sim_{E_{4_{k,l}}} (f^j(b_1), f^j(b_2), f^j(b_3))$ for $i \not\equiv j \mod p$. So we have $[f^i(b_1), f^j(b_1)] = 1$ and $[f^i(b_3), f^j(b_3)] = 1$. Since H is a torsion-free hyperbolic group we have $f^{i-j}(b_1) = b_1$ or $f^{i-j}(b_1) = b_1^{-1}$. But since f^{i-j} cannot invert or fix an element in $H \setminus G$, it follows that $b_1 \in G$ and similarly $b_3 \in G$. Now since $\beta \in H^{we} \setminus G^{we}$, we must have $b_2 \in H \setminus G$. But then $\gamma f^i(b_2)\epsilon = f^j(b_2)$ for some $\gamma, \epsilon \in G$, and an easy calculation shows that this is not possible. □

We will apply Lemma 5.4 to the group $H := G * \mathbb{F}_p$ where G is torsion-free hyperbolic, $p > 2$ is a prime and f is the automorphism of H that fixes G and cyclically permutes $(e_1, \dots, e_p)$.

Proposition 5.5. *Let X be a definable set in $\mathbb{F}_n^{eq}$. Suppose $X(\mathbb{F}_{n+1}^{eq}) \neq X(\mathbb{F}_n^{eq})$. Then X cannot be given definably the structure of an abelian group.*

Proof. Suppose otherwise and let $(X, \odot)$ be an abelian group. Suppose X is a subset of sort S_E (for some $\varnothing$-definable equivalence relation E). By Theorem 2.6 we can assign to E a definable equivalence relation R_E such that $R_E(\bar{a}, \bar{y})$ cannot have more than s solutions (for any $\bar{a}$) and each solution is a tuple containing l-many elements in imaginary sorts.

Let p be a prime greater than $\max\{s, 2\}$. Let $A_i := \langle e_{n+ip+1}, \dots, e_{n+(i+1)p} \rangle$ for $i \leq l \cdot s$ and f_i be the automorphism of $\mathbb{F}_n * A_i$ which is the identity on $\mathbb{F}_n$ and cyclically permutes the given basis of A_i.

Now let $\beta_0 \in X(\mathbb{F}_{n+1}^{eq}) \setminus X(\mathbb{F}_n^{eq})$ and β_i be the image of β_0 under the automorphism of $\mathbb{F}_\omega := \langle e_1, e_2, \dots, e_n, \dots \rangle$ which exchanges e_{n+1} with e_{n+ip+1} and fixes all other elements of the basis.

For each $0 \leq i \leq s \cdot l$ we consider the following product of elements of $X(\mathbb{F}_\omega^{eq})$.

$$\beta_i \odot f_i(\beta_i) \odot \dots \odot f_i^{p-1}(\beta_i) = \gamma_i$$

Is not hard to see that γ_i is an element of $(\mathbb{F}_n * A_i)^{eq} \setminus \mathbb{F}_n^{eq}$. Since $(X, \odot)$ is abelian we have $f_i(\gamma_i) = \gamma_i$.

Finally we consider the product

$$\gamma_0 \odot \gamma_1 \odot \ldots \odot \gamma_{l \cdot s} = \epsilon$$

Let $G := \mathbb{F}_n * A_0 * A_1 * \ldots * A_{ls}$ and $G_j := \mathbb{F}_n * B_j$, where B_j is a free product with strictly less than $l \cdot s + 1$ distinct free factors from the list $\{A_0, \ldots, A_{l \cdot s}\}$. Then is not hard to see that ϵ is an element in $G^{eq} \setminus \bigcup_j G_j^{eq}$. Note that since all the γ_i are fixed by each f_j (or rather by the obvious extension of f_j to $\mathbb{F}_\omega$), we have ϵ is fixed by each f_j.

Let $\epsilon = [\bar{e}]_E$, and suppose that $R_E(\bar{e}, G^{eq})$ contains an element (in a tuple) which is not a conjugacy class and lives in $G^{eq} \setminus \mathbb{F}_n^{eq}$. Then by Lemma 5.4 this element has p distinct images by the powers of some f_i, a contradiction. Thus, we may assume that all elements in the tuples of the solution set $R_E(\bar{e}, G^{eq})$ that live in $G^{eq} \setminus \mathbb{F}_n^{eq}$ are conjugacy classes. By repeatedly applying Lemma 5.3, we see that each conjugacy class is an element in $(\mathbb{F}_n * A_i)^{eq}$ for some $0 \leq i \leq l \cdot s$, but since there are less than $l \cdot s$ conjugacy classes this contradicts the fact that ϵ is an element in $G^{eq} \setminus \bigcup_j G_j^{eq}$. □

Proposition 5.6. *Let G be a torsion-free hyperbolic group not elementarily equivalent to a free group. Let X be a set definable in $EC(G)^{eq}$ such that $X(EC(G)^{eq}) \neq X((EC(G) * \mathbb{Z})^{eq})$. Then X cannot be given definably the structure of an abelian group.*

Proof. The proof is identical to the proof of Proposition 5.5 replacing $\mathbb{F}_n$ by $EC(G)$ and using Proposition 2.3. □

6. The generic type is foreign to any interpretable abelian group

In this section we bring everything together in order to prove Theorem 1. Before we start we give a brief account of the stability theoretic tools we use. For a more thorough exposition the reader is referred to [9]. Let us fix a complete stable theory T, countable if you wish, and a very saturated model $\mathbb{M}$ of T. $\mathcal{M}, \mathcal{N}, \ldots$ denote small elementary submodels, and $A, B, \ldots$ small subsets. We repeat a definition and fact from Chapter 7 of [9]. See Definition 7.4.1 and 7.4.7 there (originally due to Hrushovski).

Definition 6.1. *Let $p(x) \in S(A)$ be a stationary type and $\Sigma(y)$ a partial type over some small set B of parameters.*

(i) *We say that p is foreign to Σ if for any model $\mathcal{M}$ containing $A \cup B$, any realization a of $p|\mathcal{M}$ (the nonforking extension of p over $\mathcal{M}$) and any realization b of Σ, a is independent from b over $\mathcal{M}$.*

(ii) *We say that p is internal to Σ if for some $\mathcal{M}$ containing $A \cup B$, and some realization a of $p|\mathcal{M}$ there is a tuple c of realizations of Σ such that $a \in dcl(\mathcal{M}, c)$.*

Fact 6.2. *Suppose G is a connected definable group, defined over a set A, and let $p(x) \in S(A)$ be the generic type of G. Suppose that $\Sigma(y)$ is a partial type over some small set B of parameters and that p is not foreign to Σ. Then there is a normal $A \cup B$-definable subgroup N of G such that the generic type (over $A \cup B$) of G/N is internal to Σ.*

We now specialize to torsion-free hyperbolic groups. Let G be a noncyclic torsion-free hyperbolic group and $\mathcal{M} \models Th(G)$. We write $p_0^G|\mathcal{M}$ for the unique nonforking extension of p_0^G over $\mathcal{M}$.

Theorem 6.3. *The generic type of a noncyclic torsion-free hyperbolic group is foreign to any interpretable abelian group.*

Proof. Suppose p_0^G is not foreign to some interpretable abelian group A. By Fact 6.2 and Corollary 4.4, there is a model $\mathcal{M}$ of $Th(G)$ (over which A is defined), and a realization b of $p_0^G|\mathcal{M}$ and a tuple $(c_1, \ldots, c_n)$ of elements of A, such that $b \in dcl(\mathcal{M}, c_1, \ldots, c_n)$. We may assume that $\mathcal{M}$ is an elementary extension of the model $\mathbb{F}_2$ (respectively $EC(G)$ in the case where $G \not\models T_{fg}$). Suppose $b = f(c_1, \ldots, c_n, m)$ where $f(-)$ is a partial $\varnothing$-definable function, and m is a tuple from $\mathcal{M}$. We may assume A is defined over m too, by formula $\psi(y, m)$ (where of course y is a variable from the appropriate imaginary sort). Then the formula $\theta(x, m)$: "$\psi(y, m)$ defines an abelian group" $\wedge\ (\exists y_1, \ldots, y_n((\wedge_i \psi(y_i, m)) \wedge x = f(y_1, \ldots, y_n, m))$ is in $p_0^G|\mathcal{M}$. As $p_0^G|\mathcal{M}$ is definable over $\varnothing$, we can find $m' \in \mathbb{F}_2$ (respectively $EC(G)$) such that $\theta(x, m') \in p_0^G|\mathbb{F}_2$ (respectively $p_0^G|EC(G)$), so is satisfied by e in $\mathbb{F}_3 := \mathbb{F}_2 * \langle e \rangle$ (respectively $EC(G) * \langle e \rangle$). The formula $\psi(y, m')$ then defines an abelian group, B say, and there are elements $d_1, \ldots, d_n \in B$ such that $e = f(d_1, \ldots, d_n, m')$. As $\mathbb{F}_2 * \langle e \rangle$ (respectively $EC(G) * \langle e \rangle$) is a model containing $\mathbb{F}_2$ (respectively $EC(G)$) and e we can find such $d_1, \ldots, d_n \in B(\mathbb{F}_3^{eq})$ (respectively $(EC(G) * \mathbb{Z})^{eq}$). Hence e and $(d_1, \ldots, d_n)$ are interdefinable over $\mathbb{F}_2$ (respectively $EC(G)$). So, for some $i \leq n$ we have $d_i \in B(\mathbb{F}_3^{eq}) \setminus \mathbb{F}_2^{eq}$ (respectively $B((EC(G) * \mathbb{Z})^{eq}) \setminus EC(G)^{eq}$), contradicting Proposition 5.5 (respectively Proposition 5.6). ☐

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Chloé Perin
Einstein Institute for Mathematics, Edmond J. Safra Campus, Givat Ram
The Hebrew University of Jerusalem, Jerusalem, 9190401, Israel
E-mail: `cperin@math.unistra.fr`

Anand Pillay
University of Notre Dame, Dept. of Mathematics
Notre Dame, IN 46556, USA
E-mail: `apillay@nd.edu, pillay@maths.leeds.ac.uk`

Rizos Sklinos
Institut Camille Jordan, Universit Claude Bernard Lyon 1
43 boulevard du 11 novembre 1918, 69622, Villeurbanne cedex, France
E-mail: `rizozs@gmail.com`

Katrin Tent
Westfälische Wilhelms-Universität Münster, Mathematisches Institut
Einsteinstr. 62, 48149 Münster, Germany
E-mail: `tent@uni-muenster.de`