Hochschild cohomology and p-adic Lie groups

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Abstract. We carry out a detailed comparison of group cohomology and Lie algebra cohomology in the context of a compact p-adic Lie group G admitting a p-valuation ω, using Hochschild cohomology as an intermediary. As a result we provide a new spectral sequence for $\mathbb{F}_p[\![G]\!]$-bimodules W which computes $H^*(G, W^{\mathrm{ad}})$ from an E_1-page of Lie algebra cohomology. This generalizes the May spectral sequence for (one-sided) $\mathbb{F}_p[\![G]\!]$-modules. We believe our results are new even in the case where W is the trivial bimodule, in which case we can quantify at which stage the spectral sequence collapses in terms of the amplitude of ω. When G is equi-p-valued we recover the Lazard isomorphism with $\bigwedge \mathrm{Hom}(G, \mathbb{F}_p)$ as an edge map. We include various applications, such as the computation of the Hochschild cohomology of the mod p Iwasawa algebra $\mathbb{F}_p[\![G]\!]$ with coefficients in a discrete quotient $\mathbb{F}_p[\![G]\!]/I$. The mod p cohomology of the p-adic quaternion group $\mathcal{O}_D^\times$ is worked out in detail for $p > 3$ as an example.

1. Introduction

The cohomology of Lie groups has a long history. One of the first results is [1, Thm. 15.2] stating that $H^*(G, \mathbb{R}) \simeq H^*(\mathfrak{g}, \mathbb{R})$ for a connected compact real Lie group G with Lie algebra $\mathfrak{g}$. This can be made more precise, and more explicitly, $H^*(G, \mathbb{R})$ is an exterior algebra $\bigwedge(\xi_1, \ldots, \xi_l)$ on generators ξ_i of various odd degrees $2d_i - 1$. Here $l = \mathrm{rank}(G)$. The mod p cohomology $H^*(G, \mathbb{F}_p)$ was understood much later by Kac in the eighties. In [9, p. 73] it is shown that

$$H^*(G, \mathbb{F}_p) \simeq \mathbb{F}_p[x_1, \ldots, x_r]/(x_1^{p^{k_1}}, \ldots, x_r^{p^{k_r}}) \otimes \bigwedge(\xi_1, \ldots, \xi_l)$$

for $p > 2$. Here $\deg(\xi_i) = 2d_{i,p} - 1$ and $\deg(x_i) = 2d_{i,p}$, where the $d_{i,p}$ are defined in [9, Thm. 3], along with r and the k_i. We should emphasize that in this purely motivational paragraph, $H^*(G, \mathbb{R})$ and $H^*(G, \mathbb{F}_p)$ indicate the cohomology of G as a topological space, and not continuous group cohomology which can be thought of as the cohomology of the classifying space BG. In that

regard one can identify $H^*(BG, \mathbb{R})$ with a polynomial algebra $\mathbb{R}[x_1, \ldots, x_l]$ in variables of even degrees $\deg(x_i)$.

Our main interest here is the mod p cohomology $H^*(G, \mathbb{F}_p)$ of a compact p-adic Lie group G. We stress that from now on $H^*(G, \mathbb{F}_p)$ means the continuous group cohomology, and not the cohomology of G as a topological space which is rather unmanageable. Since p-adic Lie groups are totally disconnected, entirely new methods are called for. Lazard and Serre were the pioneers in the subject, and the monumental treatise [12] studied these questions systematically based on the notion of a p-valuation on G, and on the completed group algebras associated with G. Central to the theory is the graded Lie algebra $\mathfrak{g}$ attached to G, which shares many features with the classical theory. For instance, $H^*(\mathfrak{g}, \mathbb{F}_p)$ determines $H^*(G, \mathbb{F}_p)$ for an arbitrary p-valued group G (as we will show). When G is equi-p-valued, which means $\mathfrak{g}$ is concentrated in a single degree, Lazard showed that $H^*(G, \mathbb{F}_p) \simeq \bigwedge H^1(\mathfrak{g}, \mathbb{F}_p)$. Lazard's isomorphism is central to this paper whose goal is to "compute" $H^*(G, \mathbb{F}_p)$ for p-valued groups G which are not equi-p-valued. In fact, we can deal with nontrivial coefficients and "compute" $H^*(G, M)$ for discrete G-modules M with $pM = 0$, via a spectral sequence starting from $H^*(\mathfrak{g}, \operatorname{gr} M)$. We give the precise result later in this introduction.

It is known (due to Lazard) that any compact p-adic Lie group contains an open equi-p-valuable subgroup, cp [12, Ch. V, 2.2.7.1]. This may give the impression that the distinction between p-valued and equi-p-valued groups is somewhat nuanced, which is true for questions regarding finite generation of $H^*(G, \mathbb{F}_p)$, Poincaré duality, cohomological dimension, Euler characteristics, for example. However, there are many examples of naturally occurring p-valuable groups G which are not equi-p-valuable, where detailed information about $H^*(G, \mathbb{F}_p)$ is paramount. For example, unipotent groups (i.e., the $\mathbb{Z}_p$-points of the unipotent radical of a Borel in a split reductive group), Serre's standard groups with $e > 1$ as in [8, Lem. 2.2.2], pro-p Iwahori subgroups for large enough p, and $1 + \mathfrak{m}_D$, where D is the quaternion division algebra over $\mathbb{Q}_p$ for $p > 3$. The latter example $1 + \mathfrak{m}_D$ plays an important role both in number theory (in the Jacquet–Langlands correspondence for instance) and algebraic topology. In stable homotopy theory, $1 + \mathfrak{m}_D$ is known as the (strict) Morava stabilizer, and $H^*(1 + \mathfrak{m}_D, \mathbb{F}_p)$ somehow controls certain localization functors with respect to Morava K-theory, as explained in [6, 7]. In Section 6.6 we calculate $H^*((1 + \mathfrak{m}_D)^{\operatorname{Nrd}=1}, \mathbb{F}_p)$ explicitly to illustrate the use of our techniques, assuming $p > 3$.

We now state our main result precisely. Let (G, ω) be a p-valued group, and assume ω takes values in $\frac{1}{e}\mathbb{Z}$ for some $e \in \mathbb{N}$. Fix a perfect field k of characteristic p and consider the completed group algebra $\Omega(G) = k[\![G]\!]$. The p-valuation ω gives a filtration by two-sided ideals $\operatorname{Fil}^i \Omega(G)$ $(i = 0, 1, 2, \ldots)$, and the associated graded algebra $\operatorname{gr} \Omega(G)$ can be naturally identified with $U(\mathfrak{g})$, where $\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$ is the graded Lie k-algebra attached to G as in [12]. (Here the variable π acts by sending $g \mapsto g^p$.)

Start with an arbitrary finite filtered $\Omega(G)$-bimodule W. Thus W is finite-dimensional over k and carries a filtration $\mathrm{Fil}^i W$ indexed by $i \in \mathbb{Z}$ such that $\mathrm{Fil}^i W = W$ for $i \ll 0$ and $\mathrm{Fil}^i W = 0$ for $i \gg 0$. Moreover, the filtration is compatible with the bimodule structure in the sense that

$$\mathrm{Fil}^i \Omega(G) \times \mathrm{Fil}^j W \times \mathrm{Fil}^k \Omega(G) \longrightarrow \mathrm{Fil}^{i+j+k} W.$$

With any such W we associate a G-module W^{ad} and a $\mathfrak{g}$-module $(\mathrm{gr}\, W)^{\mathrm{ad}}$ as follows:

(i) The underlying k-vector space of W^{ad} is that of W, and the G-action is defined by the formula $g \bullet w = gwg^{-1}$ (where the right-hand side uses the bimodule structure via $G \hookrightarrow \Omega(G)^{\times}$).

(ii) As a k-vector space, $(\mathrm{gr}\, W)^{\mathrm{ad}}$ is the same as $\mathrm{gr}\, W$. The latter carries a $U(\mathfrak{g})$-bimodule structure, and the $\mathfrak{g}$-action is given by the recipe $\xi \bullet w = \xi w - w\xi$ (via the embedding $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$).

Our main theorem is the following spectral sequence computing the continuous cohomology of W^{ad} from the Lie algebra cohomology of $(\mathrm{gr}\, W)^{\mathrm{ad}}$.

Theorem 1.1. *There is a convergent spectral sequence collapsing at a finite stage,*

$$E_1^{s,t} = H^{s,t}\big(\mathfrak{g}, (\mathrm{gr}\, W)^{\mathrm{ad}}\big) \Longrightarrow H^{s+t}(G, W^{\mathrm{ad}}).$$

When W is equipped with a pairing $W \otimes_{\Omega(G)} W \to W$ of filtered $\Omega(G)$-bimodules, the spectral sequence is multiplicative.

(The bigrading $H^{s,t}$ of the Lie algebra cohomology comes from the grading of $\mathfrak{g}$. See Definition 4.4 and the discussion after Theorem 5.5 in the main text for more details.)

Although the end result (1.1) does not mention Hochschild cohomology at all, the latter plays a crucial role in its proof. In fact, Theorem 1.1 may be reformulated using only (continuous) Hochschild cohomology, cp Theorem 5.4 for this version of the main result. The overall strategy behind the proof of Theorem 1.1 is to relate group and Lie algebra cohomology using Hochschild cohomology as a go-between:

$$H^*(G, W^{\mathrm{ad}}) \overset{(1)}{\rightsquigarrow} HH^*(\Omega(G), W) \overset{(2)}{\rightsquigarrow} HH^*(\mathrm{gr}\, \Omega(G), \mathrm{gr}\, W)$$

$$\overset{(3)}{=\!=} HH^*(U(\mathfrak{g}), \mathrm{gr}\, W) \overset{(4)}{\rightsquigarrow} H^*(\mathfrak{g}, (\mathrm{gr}\, W)^{\mathrm{ad}}).$$

Here step (1) is essentially what is known as Mac Lane isomorphism. We discuss it in detail in Section 3, where we also establish an adjunction between pseudocompact $\Omega(G)$-modules and pseudocompact $\Omega(G)$-bimodules. In that section we denote W^{ad} by $R(W)$. Step (2) is the spectral sequence (cp Theorem 5.4) obtained by filtering the Hochschild complex $\mathrm{Hom}_k(\Omega(G)^{\otimes \bullet}, W)$ in a natural way. Step (3) is Lazard's identification $\mathrm{gr}\, \Omega(G) \overset{\sim}{\longrightarrow} U(\mathfrak{g})$. Finally, step (4) is a detailed study of the anti-symmetrization map relating the Hochschild cohomology of $U(\mathfrak{g})$ to the Lie algebra cohomology of $\mathfrak{g}$. In Section 4 we give a streamlined exposition of some of the arguments in [2, Ch. XIII], and extend some of their results for our needs. For instance, by analogy with the

pseudocompact case, we give an adjunction between $U(\mathfrak{g})$-modules and $U(\mathfrak{g})$-bimodules.

Taking $W = k$ to be the trivial bimodule in Theorem 1.1 gives a multiplicative convergent spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, k) \Longrightarrow H^{s+t}(G, k),$$

and we can quantify at which stage it collapses. In Proposition 6.2 we show that $E_r = E_\infty$ in the range $r \geq \mathrm{rank}(G) \cdot e \cdot \mathrm{amp}(\omega)$. Here the amplitude of ω is the largest value of $\omega(g_i)$ minus the smallest value for an ordered basis (g_i). When $\mathrm{amp}(\omega) = 0$, we have an equi-p-valued group, and the spectral sequence collapses on the first page. In this case Theorem 1.1 boils down to Lazard's famous isomorphism, cp [12, p. 183]:

$$(1) \qquad\qquad H^*(G, k) \xrightarrow{\sim} \bigwedge \mathrm{Hom}(\mathfrak{g}, k).$$

Curiously Lazard proves this without making explicit use of any spectral sequences whatsoever. We feel our proof is a bit more conceptual, and it allows us to interpret the Lazard isomorphism as an edge map of a spectral sequence.

In Section 6.4 we apply Theorem 1.1 to compute the (truncated) Hochschild cohomology of $\Omega(G)$ for equi-p-valued groups. This is an example of Theorem 1.1 for a nontrivial bimodule of coefficients; one takes W to be a finite-dimensional quotient of $\Omega(G)$. In this situation, the E_1-page is closely related to the Koszul resolution for the symmetric algebra $S(\mathfrak{g})$. We have not seen this application mentioned anywhere, and we believe the result is new. See (4) for the precise statement.

In the case of a p-valued group (G, ω) our Theorem 1.1 simplifies and generalizes what is referred to as the May spectral sequence in [22, Thm. 5.1.12]. See also May's original paper [14]. Two key differences are (1) we work with $\Omega(G)$-bimodules instead of just modules, and (2) our Lie algebras and universal enveloping algebras are not p-restricted (our filtration of $\Omega(G)$ comes from ω and is not given by powers of the augmentation ideal). On the other hand [22, Thm. 5.1.12] applies to any finitely generated pro-p group.

Let us draw a more precise comparison between Theorem 1.1 and part (2) of [22, Thm. 5.1.12]. Start with a discrete left $\Omega(G)$-module M. There is a natural way to endow M with a decreasing filtration by letting $\mathrm{Fil}^j M = M[\mathrm{Fil}^{-j} \Omega(G)]$ for $j \leq 0$, cp (5.1.14) on [22, p. 399] which uses powers of the augmentation ideal J, however. Let $\mathrm{gr}\, M$ be the associated left module for $\mathrm{gr}\, \Omega(G) \xrightarrow{\sim} U(\mathfrak{g})$. In the latter isomorphism one has to replace $U(\mathfrak{g})$ by the p-restricted variant $U^{[p]}(\mathfrak{g})$ when $\Omega(G)$ is filtered by powers of J, see [12, App. A.2] and [17, p. 412]. We promote M to a bimodule by letting $\Omega(G)$ act on the right via the augmentation map $\epsilon \colon \Omega(G) \to k$. Using this as our input W in Theorem 1.1, the G-module W^{ad} is simply M viewed as a (left) G-module since $\epsilon(g) = 1$, so $g \bullet w = gw$. Similarly $\mathrm{gr}\, M$ becomes a bimodule for $U(\mathfrak{g})$ where the right-action is via $\epsilon \colon U(\mathfrak{g}) \to k$. Thus $(\mathrm{gr}\, W)^{\mathrm{ad}}$ is simply $\mathrm{gr}\, M$ viewed as a (left) $\mathfrak{g}$-module; indeed, $\epsilon(\xi) = 0$, so $\xi \bullet w = \xi w$ for $\xi \in \mathfrak{g}$. In this case, Theorem 1.1 gives a

spectral sequence

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, \operatorname{gr} M) \Longrightarrow H^{s+t}(G, M)$$

for any discrete left $\Omega(G)$-module M. This is exactly [22, Thm. 5.1.12 (2)] except they have to replace $H^{s,t}(\mathfrak{g}, \operatorname{gr} M)$ with the p-restricted Lie algebra cohomology $\operatorname{Ext}^{s,t}_{U^{[p]}(\mathfrak{g})}(k, \operatorname{gr} M)$ since they filter by J-powers. We have no nwanalog of part (1) of [22, Thm. 5.1.12], but we believe our method should work for Hochschild homology as well. We should also point out that the aforementioned application to the Lazard isomorphism (1) is already mentioned in [22, p. 401].

2. Basic notions in Lazard theory

Let (G, ω) be a p-valued group. Recall that a p-valuation on a group G is a function $\omega \colon G \backslash \{1\} \to (0, \infty)$ satisfying certain standard axioms which we will not repeat here. We refer the reader to [19, p. 169]. For any number $v > 0$, we introduce

$$G_v = \{g \in G \mid \omega(g) \geq v\}, \quad G_{v+} = \{g \in G \mid \omega(g) > v\}.$$

The aforementioned axioms guarantee both G_v and G_{v+} are normal subgroups of G. We always assume G is profinite in the topology defined by ω. Hence the subgroups G_v and G_{v+} are open balls for the ultrametric $d(g, g') = 2^{-\omega(g^{-1}g')}$ and $G = \varprojlim G/G_v$ as topological groups; in particular, G must be a pro-p-group since $\omega(g^p) = \omega(g) + 1$.

Definition 2.1. $\operatorname{gr} G = \bigoplus_{v>0} \operatorname{gr}_v G$, where $\operatorname{gr}_v G = G_v/G_{v+}$.

This is an $\mathbb{F}_p$-vector space and will be denoted additively. It has more structure. In fact, $\operatorname{gr} G$ is a module over the polynomial algebra $\mathbb{F}_p[\pi]$ by letting $\pi \colon \operatorname{gr}_v G \to \operatorname{gr}_{v+1} G$ be the map $gG_{v+} \mapsto g^p G_{(v+1)+}$. We always assume (G, ω) is of finite rank d say, which means $\operatorname{gr} G$ is finite free of rank d over $\mathbb{F}_p[\pi]$. Furthermore, $\operatorname{gr} G$ carries a Lie bracket induced by the group commutator which turns it into a graded Lie algebra over $\mathbb{F}_p[\pi]$. We refer to [19, Sections 23–25] for details.

By our finite rank assumption (G, ω) admits an ordered basis. That is a sequence $(g_1, \ldots, g_d)$ such that the map $(x_1, \ldots, x_d) \mapsto g_1^{x_1} \cdots g_d^{x_d}$ defines a bijection $\mathbb{Z}_p^d \xrightarrow{\sim} G$ and

$$\omega(g_1^{x_1} \cdots g_d^{x_d}) = \min\{v(x_i) + \omega(g_i) \mid i = 1, \ldots, d\}.$$

This corresponds to an ordered $\mathbb{F}_p[\pi]$-basis for $\operatorname{gr} G$ consisting of homogeneous elements $\sigma(g_i) = g_i G_{\omega(g_i)+}$.

Throughout the paper we fix a perfect field k of characteristic p and let $\mathcal{O} = W(k)$ be its ring of Witt vectors. Elements of the completed group algebra $\Lambda(G) = \mathcal{O}[\![G]\!]$ may be expressed as power series as follows. First introduce $\mathbf{b}_i = g_i - 1$ and $\mathbf{b}^\alpha = \mathbf{b}_1^{\alpha_1} \cdots \mathbf{b}_d^{\alpha_d}$ for a multi-index $\alpha \in \mathbb{N}^d$. Sending $X_i \mapsto \mathbf{b}_i$ determines an isomorphism of topological $\mathcal{O}$-modules

$$\mathcal{O}[\![X_1, \ldots, X_d]\!] \xrightarrow{\sim} \Lambda(G).$$

Thus an element $\lambda \in \Lambda(G)$ has a unique expansion $\lambda = \sum_{\alpha} c_{\alpha} \mathbf{b}^{\alpha}$ which we use to define the valuation $\tilde{\omega}$ on $\Lambda(G) \backslash \{0\}$ by the formula

$$\tilde{\omega}(\lambda) = \min_{\alpha} \left(v(c_{\alpha}) + \sum_{i=1}^{d} \alpha_i \omega(g_i) \right).$$

It is known that $\tilde{\omega}$ is multiplicative, independent of the choice of basis, and $\tilde{\omega}(g-1) = \omega(g)$, cp [19, Cor. 28.4–5]. By analogy to the G_v above, we introduce a filtration of $\Lambda(G)$ by two-sided ideals

$$\Lambda(G)_v = \{\lambda \in \Lambda(G) \mid \tilde{\omega}(\lambda) \geq v\}, \quad \Lambda(G)_{v+} = \{\lambda \in \Lambda(G) \mid \tilde{\omega}(\lambda) > v\}.$$

More explicitly $\Lambda(G)_v$ can be identified with the smallest closed $\mathcal{O}$-submodule of $\Lambda(G)$ containing all elements of the form $p^M(h_1 - 1) \cdots (h_N - 1)$ satisfying $M + \sum_{i=1}^{N} \omega(h_i) \geq v$, see [19, Thm. 28.3].

Definition 2.2. $\operatorname{gr} \Lambda(G) = \bigoplus_{v \geq 0} \operatorname{gr}_v \Lambda(G)$, where $\operatorname{gr}_v \Lambda(G) = \Lambda(G)_v / \Lambda(G)_{v+}$.

This defines a graded $\operatorname{gr} \mathcal{O}$-algebra, where $\operatorname{gr} \mathcal{O} = \bigoplus_{i=0}^{\infty} p^i \mathcal{O}/p^{i+1} \mathcal{O}$. Note that $\operatorname{gr} \mathcal{O} \xrightarrow{\sim} k[\pi]$. The following fundamental theorem relates $\operatorname{gr} \Lambda(G)$ to the universal enveloping algebra of $\operatorname{gr} G$.

Theorem 2.3 (Lazard). $\operatorname{gr} \mathcal{O} \otimes_{\mathbb{F}_p[\pi]} U(\operatorname{gr} G) \xrightarrow{\sim} \operatorname{gr} \Lambda(G)$.

We refer the interested reader to [19, Thm. 28.3] for its proof.

Our main interest in this paper is the mod p completed group algebra $\Omega(G) = k[\![G]\!]$ which we may identify with $\Lambda(G)/p\Lambda(G)$ and endow it with the quotient filtration $\Omega(G)_v$ (= the image of $\Lambda(G)_v$). This gives a graded k-algebra $\operatorname{gr} \Omega(G)$ as before. Letting $\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G = k \otimes_{\mathbb{F}_p} \operatorname{gr} G / \pi \operatorname{gr} G$, which is a graded Lie algebra over k, one easily deduces the mod p nwanalog of Theorem 2.3:

Corollary 2.4. $U(\mathfrak{g}) = k \otimes_{\mathbb{F}_p[\pi]} U(\operatorname{gr} G) \xrightarrow{\sim} \operatorname{gr} \Omega(G)$.

It is worthwhile to spell out the grading of $U(\mathfrak{g})$ in the previous result. The Lie algebra $\mathfrak{g}$ has a k-basis of vectors $\xi_i = 1 \otimes \sigma(g_i)$. By Poincaré–Birkhoff–Witt, the monomials $\xi^{\alpha} = \xi_1^{\alpha_1} \cdots \xi_d^{\alpha_d}$ form a k-basis for $U(\mathfrak{g})$. The isomorphism in Corollary 2.4 takes $\xi_i \mapsto \mathbf{b}_i + \Omega(G)_{\omega(g_i)+}$ and thus $\xi^{\alpha} \mapsto \mathbf{b}^{\alpha} + \Omega(G)_{(\sum_{i=1}^{d} \alpha_i \omega(g_i))+}$. Therefore in the grading of the universal enveloping algebra $\operatorname{gr}_v U(\mathfrak{g})$ is the k-span of all monomials ξ^{α} satisfying the equality $\sum_{i=1}^{d} \alpha_i \omega(g_i) = v$.

We say (G, ω) is equi-p-valued if it admits a basis $(g_1, \dots, g_d)$ for which $\omega(g_1) = \cdots = \omega(g_d) = \tau$. In this case all $\sigma(g_i) \in \operatorname{gr}_{\tau} G$ and, consequently, $\mathfrak{g} = \mathfrak{g}_{\tau}$ is concentrated in degree τ.

Any p-valuable group G admits a p-valuation ω with values in $\frac{1}{e}\mathbb{Z}$ for some $e \in \mathbb{N}$, see [19, Cor. 33.3]. Suppose ω has this property. Eventually we will work with spectral sequences so we need our filtrations to be indexed by integers. We reindex the filtration of $\Lambda(G)$ by letting

$$\operatorname{Fil}^i \Lambda(G) = \Lambda(G)_{\frac{i}{e}}, \quad i = 0, 1, 2, \dots.$$

The corresponding graded ring is $\operatorname{gr}\Lambda(G) = \bigoplus_{i=0}^{\infty} \operatorname{gr}^i \Lambda(G)$, where

$$\operatorname{gr}^i \Lambda(G) = \operatorname{gr}_{\frac{i}{e}} \Lambda(G) = \operatorname{Fil}^i \Lambda(G) / \operatorname{Fil}^{i+1} \Lambda(G).$$

We adopt similar notation for G and $\Omega(G)$ (for example, $G^i = G_{\frac{i}{e}}$ and analogously for $\operatorname{Fil}^i \Omega(G)$).

3. Hochschild cohomology versus group cohomology

We endow k with the discrete topology. Then by its definition $\Omega(G)$ is a pseudocompact k-algebra (meaning it is complete, Hausdorff, and has a neighborhood basis at 0 consisting of open two-sided ideals $I \subset \Omega(G)$, for which $\Omega(G)/I$ is finite-dimensional over k). We let $\operatorname{Pct}_{\Omega(G)}$ be the category of left pseudocompact $\Omega(G)$-modules V (= inverse limits of finite length quotients). This is an abelian category with enough projectives and exact inverse limits. In fact, Pontryagin duality gives an anti-equivalence between $\operatorname{Pct}_{\Omega(G)}$ and the category of smooth G-representations on k-vector spaces, see [10, Thm. 1.5].

In this section we will relate $\operatorname{Pct}_{\Omega(G)}$ to the category of pseudocompact $\Omega(G)$-bimodules W, which we may think of as $\operatorname{Pct}_{\Omega(G)^e}$, where

$$\Omega(G)^e = \Omega(G) \,\hat{\otimes}_k\, \Omega(G)^{\mathrm{op}}$$

is the completed enveloping algebra ($\hat{\otimes}_k$ denotes the completed tensor product). First, starting from an object W of $\operatorname{Pct}_{\Omega(G)^e}$, we define $R(W)$ to be the underlying vector space of W with G-action $g \bullet w = g w g^{-1}$ (the right-hand side uses the bimodule structure via $G \hookrightarrow \Omega(G)^{\times}$). This extends uniquely to a jointly continuous $\Omega(G)$-module structure which makes $R(W)$ an object of $\operatorname{Pct}_{\Omega(G)}$. Going the other way, starting with an object V of $\operatorname{Pct}_{\Omega(G)}$, we define $L(V)$ as the tensor product $V \,\hat{\otimes}_k\, \Omega(G)$. Its right $\Omega(G)$-module structure comes from the second factor, and it gets a left $\Omega(G)$-module structure by letting G act diagonally—on both V and $\Omega(G)$ from the left. This defines two exact functors:

$$L \colon \operatorname{Pct}_{\Omega(G)} \rightleftarrows \operatorname{Pct}_{\Omega(G)^e} : R.$$

This pair gives an adjunction between the two categories.

Lemma 3.1. *The functors (L, R) form a pair of adjoint functors, meaning there is a natural isomorphism of k-vector spaces—functorial in both V and W:*

$$\operatorname{Hom}_{\Omega(G)^e}(L(V), W) \xrightarrow{\sim} \operatorname{Hom}_{\Omega(G)}(V, R(W)).$$

(*Both* Hom*-spaces are taken in the respective categories of pseudocompact modules.*)

Proof. We give $\epsilon \colon V \to R(L(V))$ and $\eta \colon L(R(W)) \to W$, the two adjunction morphisms. Firstly, as a vector space $R(L(V))$ equals $V \,\hat{\otimes}_k\, \Omega(G)$ and ϵ is the map $\epsilon(v) = v \otimes 1$. Secondly, $L(R(W))$ is simply $W \,\hat{\otimes}_k\, \Omega(G)$ as a vector space, and η is given by $\eta(w \otimes \lambda) = w\lambda$. (One checks η is left G-linear.) It follows straight from their definition that ϵ and η define mutually inverse maps between the Hom-spaces in the usual way, cp [11, p. 29]; we leave the tedious calculations to the reader. $\square$

The goal of this section is the next result (known as Mac Lane isomorphism, cp [13, Prop. 7.4.2]). Here HH^n refers to *continuous* Hochschild cohomology as defined in [13, Section 1.5.12].

Proposition 3.2. $HH^n(\Omega(G), W) \xrightarrow{\sim} H^n(G, R(W))$.

Proof. A formal argument using the exactness of both L and R shows that the adjunction extends to Ext^n for all n. That is, there are functorial isomorphisms

$$\mathrm{Ext}^n_{\Omega(G)^e}(L(V), W) \xrightarrow{\sim} \mathrm{Ext}^n_{\Omega(G)}(V, R(W)).$$

(Indeed, start with a projective resolution $P^\bullet \to V$ in $\mathrm{Pct}_{\Omega(G)}$. Applying L gives a projective resolution $L(P^\bullet) \to L(V)$ in $\mathrm{Pct}_{\Omega(G)^e}$ by the exactness of L and R, and the adjointness which guarantees $L(P^\bullet)$ consists of projective objects. Now apply Lemma 3.1 and pass to cohomology.) For now we are only interested in the case where $V = k$ is the trivial G-representation, in which case $L(k) = \Omega(G)$. In this situation, $\mathrm{Ext}^n_{\Omega(G)}(k, R(W))$ is the same as the continuous cohomology $H^n(G, R(W))$ from [23, Section 2], for example. This fact is also noted in [12, Ch. V, Prop. 1.2.6]. Furthermore, the left-hand side $\mathrm{Ext}^n_{\Omega(G)^e}(\Omega(G), W)$ coincides with the (continuous) Hochschild cohomology $HH^n(\Omega(G), W)$ introduced in [13, Section 1.5.12] for instance. $\square$

We wish to give a completely explicit description of the isomorphism in Proposition 3.2. First of all, Hochschild cohomology is the cohomology of the Hochschild complex $C^\bullet(\Omega(G), W)$, which is

$$0 \longrightarrow W \xrightarrow{d_1} \mathrm{Hom}_k^{\mathrm{cts}}(\Omega(G), W) \xrightarrow{d_2} \mathrm{Hom}_k^{\mathrm{cts}}(\Omega(G)^{\hat{\otimes}2}, W) \xrightarrow{d_3} \cdots.$$

Here the coboundary map d_n is determined by the formula below, cp [13, p. 37]:

$$d_n(f)(\lambda_1 \otimes \cdots \otimes \lambda_n) = \lambda_1 f(\lambda_2 \otimes \cdots \otimes \lambda_n) + \sum_{i=1}^n (-1)^i f(\lambda_1 \otimes \cdots \otimes \lambda_i \lambda_{i+1} \otimes \cdots \otimes \lambda_n),$$

if we interpret the nth term as $(-1)^n f(\lambda_1 \otimes \cdots \otimes \lambda_{n-1})\lambda_n$. We have $\Omega(G^n) \simeq \Omega(G)^{\hat{\otimes}n}$, and thus we may identify the nth term of the Hochschild complex with all continuous maps $\mathcal{C}(G^n, W)$. In this realization, the coboundary map d_n becomes

$$d_n(f)(g_1, \ldots, g_n) = g_1 f(g_2, \ldots, g_n) + \sum_{i=1}^n (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_n),$$

with a similar interpretation of the nth term of the sum. On the other hand, continuous group cohomology $H^n(G, V)$ is the cohomology of the complex $C^\bullet(G, V)$ given by

$$0 \longrightarrow V \xrightarrow{\partial_1} \mathcal{C}(G, V) \xrightarrow{\partial_2} \mathcal{C}(G^2, V) \xrightarrow{\partial_3} \mathcal{C}(G^3, V) \longrightarrow \cdots,$$

where the coboundary map ∂_n is given by the standard formula:

$$\partial_n(f)(g_1, \ldots, g_n) = g_1 f(g_2, \ldots, g_n) + \sum_{i=1}^n (-1)^i f(g_1, \ldots, g_i g_{i+1}, \ldots, g_n),$$

where the nth term is now interpreted as $(-1)^n f(g_1, \ldots, g_{n-1})$. The two resulting complexes are canonically isomorphic when $V = R(W)$.

Proposition 3.3. $C^\bullet(\Omega(G), W) \xrightarrow{\sim} C^\bullet(G, R(W))$.

Proof. Let $V = R(W)$ for simplicity. To a continuous function $f \in \mathcal{C}(G^n, W)$ we associate the function $\tilde{f} \in \mathcal{C}(G^n, V)$ given by the formula

$$\tilde{f}(g_1, \ldots, g_n) = f(g_1, \ldots, g_n) \cdot (g_1 \cdots g_n)^{-1}.$$

(Here the right-hand side uses the bimodule structure of W.) A straightforward calculation, using the above formulas for the coboundary maps, shows that $\partial_{n+1}(\tilde{f}) = \widetilde{d_{n+1}(f)}$. We omit the details. $\square$

Lastly, we observe that in the presence of cup products we get an isomorphism of differential graded algebras in Proposition 3.3: Suppose W carries a continuous pairing $W \hat{\otimes}_k W \to W$ compatible with the $\Omega(G)$-bimodule structure, meaning the following identities hold for all $\lambda \in \Omega(G)$ and $w, w' \in W$:

$$\lambda(ww') = (\lambda w)w', \quad (w\lambda)w' = w(\lambda w'), \quad (ww')\lambda = w(w'\lambda).$$

As discussed in detail in [24, Rem. 1.3.4] for example, this gives a cup product $\smile$ on the Hochschild complex $C^\bullet(\Omega(G), W)$ turning it into a DGA (associative and unital when W is and $\lambda 1_W = 1_W \lambda$). We may view our pairing as a G-pairing $V \hat{\otimes}_k V \to V$ (as follows from the above identities) which gives rise to a cup product $\smile$ on $C^\bullet(G, V)$, as defined in [23, p. 259] for instance. Unwinding all definitions, a somewhat dreary but easy computation shows that the cup products correspond under the isomorphism in the proof of Proposition 3.3; $\tilde{f} \smile \tilde{f}' = \widetilde{f \smile f'}$ for all $f \in \mathcal{C}(G^n, W)$ and $f' \in \mathcal{C}(G^m, W)$. In other words, $f \mapsto \tilde{f}$ is an isomorphism of DGA's.

Of course, all the results of this section are true for any profinite group G. Unlike in Section 2, we do not need G to be p-valuable, or even pro-p.

4. Hochschild cohomology versus Lie algebra cohomology

Let $\mathrm{Mod}_{U(\mathfrak{g})}$ denote the category of left $\mathfrak{g}$-modules. We identify $\mathrm{Mod}_{U(\mathfrak{g})^e}$ with the category of $U(\mathfrak{g})$-bimodules. Note that there is a natural isomorphism $U(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g})^{\mathrm{op}}$ of k-algebras induced by the map $\mathfrak{g} \hookrightarrow U(\mathfrak{g})^{\mathrm{op}}$ sending $\xi \mapsto -\xi$. We will also need the composite k-algebra map

$$E \colon U(\mathfrak{g}) \xrightarrow{\Delta} U(\mathfrak{g}) \otimes_k U(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g}) \otimes_k U(\mathfrak{g})^{\mathrm{op}} = U(\mathfrak{g})^e,$$

where Δ is induced by the diagonal map $\mathfrak{g} \to \mathfrak{g} \oplus \mathfrak{g}$. Thus $E(\xi) = \xi \otimes 1 - 1 \otimes \xi$ for vectors $\xi \in \mathfrak{g}$. We always regard $U(\mathfrak{g})^e$ as a right $U(\mathfrak{g})$-module via the map E, which happens to be free as shown in the last paragraph of [2, p. 276]—verifying their condition (E.2).

By analogy with the previous section we now proceed to define functors (L, R). First, if W is a $U(\mathfrak{g})$-bimodule, we associate the $\mathfrak{g}$-module $R(W)$ whose underlying vector space is the same as that of W, but with $\mathfrak{g}$-action given by $\xi \bullet w = \xi w - w\xi$. Conversely, starting with a $\mathfrak{g}$-module V, we declare $L(V)$ to

be $U(\mathfrak{g})^e \otimes_{U(\mathfrak{g})} V$. By the aforementioned freeness result, L is exact (and R is trivially exact). These functors

$$L \colon \mathrm{Mod}_{U(\mathfrak{g})} \rightleftarrows \mathrm{Mod}_{U(\mathfrak{g})^e} : R$$

again give an adjunction between the two categories.

Lemma 4.1. (L, R) *form a pair of adjoint functors; there is an isomorphism functorial in V and W,*

$$\mathrm{Hom}_{U(\mathfrak{g})^e}(L(V), W) \xrightarrow{\sim} \mathrm{Hom}_{U(\mathfrak{g})}(V, R(W)).$$

Proof. The adjunction maps $\epsilon \colon V \to R(L(V))$ and $\eta \colon L(R(W)) \to W$ are given as follows. The first one is $\epsilon(v) = 1 \otimes v$, which is easily checked to be $\mathfrak{g}$-linear, noting that $\xi \bullet (\mu \otimes v) = E(\xi)\mu \otimes v$ for $\mu \in U(\mathfrak{g})^e$ and $v \in V$. The second is $\eta(\mu \otimes w) = \mu w$, which is trivially $U(\mathfrak{g})^e$-linear. Unwinding the definitions, it is easy to check that $R(\,\cdot\,) \circ \epsilon$ and $\eta \circ L(\,\cdot\,)$ give mutually inverse maps between the Hom-spaces. $\square$

This leads us to the key result of this section, cp [2, Ch. XIII, Thm. 5.1]. Here HH^n refers to *usual* Hochschild cohomology (as opposed to the continuous variant used earlier).

Proposition 4.2. $HH^n(U(\mathfrak{g}), W) \xrightarrow{\sim} H^n(\mathfrak{g}, R(W))$.

Proof. As in the proof of Proposition 3.2, the exactness of L and R shows that the above adjunction extends to Ext^n. That is, there are functorial isomorphisms

$$\mathrm{Ext}^n_{U(\mathfrak{g})^e}(L(V), W) \xrightarrow{\sim} \mathrm{Ext}^n_{U(\mathfrak{g})}(V, R(W)).$$

Taking $V = k$ to be the trivial $\mathfrak{g}$-module, with $U(\mathfrak{g})$ acting on k via augmentation, the right-hand side $\mathrm{Ext}^n_{U(\mathfrak{g})}(k, R(W))$ is exactly the Lie algebra cohomology $H^n(\mathfrak{g}, R(W))$. On the left-hand side,

$$L(k) = U(\mathfrak{g})^e \otimes_{U(\mathfrak{g})} k \simeq U(\mathfrak{g})^e / U(\mathfrak{g})^e E(I),$$

where $I = \ker(U(\mathfrak{g}) \to k)$. As shown on [2, p. 276]—verifying their condition (E.1)—the ideal $U(\mathfrak{g})^e E(I)$ coincides with the kernel of bimodule map $U(\mathfrak{g})^e \to U(\mathfrak{g})$ given by multiplication. We conclude that $L(k) \simeq U(\mathfrak{g})$, as bimodules, and hence $\mathrm{Ext}^n_{U(\mathfrak{g})^e}(L(k), W)$ coincides with $HH^n(U(\mathfrak{g}), W)$. $\square$

As in the previous section we would like a more precise and explicit result at the level of complexes. Here the Hochschild complex $C^\bullet(U(\mathfrak{g}), W)$ is

$$0 \longrightarrow W \xrightarrow{d_1} \mathrm{Hom}_k(U(\mathfrak{g}), W) \xrightarrow{d_2} \mathrm{Hom}_k(U(\mathfrak{g})^{\otimes 2}, W) \xrightarrow{d_3} \cdots,$$

with coboundary maps d_n as recalled earlier. On the other hand, Lie algebra cohomology can be computed from the Chevalley–Eilenberg complex $C^\bullet(\mathfrak{g}, V)$, which is

$$0 \longrightarrow V \xrightarrow{\partial_1} \mathrm{Hom}_k(\mathfrak{g}, V) \xrightarrow{\partial_2} \mathrm{Hom}_k\left(\bigwedge^2 \mathfrak{g}, V\right) \xrightarrow{\partial_3} \mathrm{Hom}_k\left(\bigwedge^3 \mathfrak{g}, V\right) \longrightarrow \cdots,$$

where the ∂_n are given by the usual formula, cp [2, p. 282].

Proposition 4.3. *There is a natural quasi-isomorphism $C^\bullet(U(\mathfrak{g}), W) \longrightarrow C^\bullet(\mathfrak{g}, R(W))$.*

Proof. Start with the standard resolution $P^\bullet \to k$ of the trivial $\mathfrak{g}$-module k by finite free $U(\mathfrak{g})$-modules:

$$\cdots \longrightarrow U(\mathfrak{g}) \otimes \bigwedge^3 \mathfrak{g} \longrightarrow U(\mathfrak{g}) \otimes \bigwedge^2 \mathfrak{g} \longrightarrow U(\mathfrak{g}) \otimes \mathfrak{g} \longrightarrow U(\mathfrak{g}) \longrightarrow k \longrightarrow 0.$$

See [2, Ch. XIII, Thm. 7.1] for the explicit differentials. Applying the (exact) functor L gives rise to a resolution of $L(k) \simeq U(\mathfrak{g})$ by finite free $U(\mathfrak{g})^e$-modules $L(P^\bullet)$. That is,

$$\cdots \longrightarrow U(\mathfrak{g})^e \otimes \bigwedge^3 \mathfrak{g} \longrightarrow U(\mathfrak{g})^e \otimes \bigwedge^2 \mathfrak{g} \longrightarrow U(\mathfrak{g})^e \otimes \mathfrak{g} \longrightarrow U(\mathfrak{g})^e \longrightarrow U(\mathfrak{g}) \longrightarrow 0.$$

Here the tensor products are still over k. On the other hand, $U(\mathfrak{g})$ has a standard resolution $B^\bullet$ by free $U(\mathfrak{g})^e$-modules, see [24, Section 1.1] for instance, where the differentials are made explicit,

$$\cdots \longrightarrow U(\mathfrak{g})^e \otimes U(\mathfrak{g})^{\otimes 2} \longrightarrow U(\mathfrak{g})^e \otimes U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})^e \longrightarrow U(\mathfrak{g}) \longrightarrow 0.$$

Furthermore, $\operatorname{Hom}_{U(\mathfrak{g})^e}(B^\bullet, W)$ is naturally isomorphic to the Hochschild complex $C^\bullet(U(\mathfrak{g}), W)$ as spelled out in [24, Section 1.1] for example. Also by the (L, R)-adjunction, we have isomorphisms of complexes

$$\operatorname{Hom}_{U(\mathfrak{g})^e}(L(P^\bullet), W) \simeq \operatorname{Hom}_{U(\mathfrak{g})}(P^\bullet, R(W)) \simeq C^\bullet(\mathfrak{g}, R(W)).$$

To finish the proof all we have to do is write down a morphism of resolutions $L(P^\bullet) \longrightarrow B^\bullet$ (which automatically has a homotopy inverse by basic homological algebra). Once we do that, applying the functor $\operatorname{Hom}_{U(\mathfrak{g})^e}(-, W)$ gives the desired quasi-isomorphism. To construct the morphism between resolutions take the n-fold tensor product of $\mathfrak{g} \hookrightarrow U(\mathfrak{g})$ and anti-symmetrize. This gives a k-linear map

$$A\colon \bigwedge^n \mathfrak{g} \longrightarrow U(\mathfrak{g})^{\otimes n}, \quad A(x_1 \wedge \cdots \wedge x_n) = \sum_{\gamma \in S_n} \operatorname{sign}(\gamma)(x_{\gamma(1)} \otimes \cdots \otimes x_{\gamma(n)}).$$

Tensoring with $U(\mathfrak{g})^e$ produces a map $L(P^\bullet) \to B^\bullet$. The fact that the anti-symmetrization maps are compatible with the differentials is essentially the content of [2, Ch. XIII, Thm. 7.1]. $\qquad\square$

Going through the steps of the previous proof shows that the quasi isomorphism of Proposition 4.3 is given by composition with the anti-symmetrization map,

$$\operatorname{Hom}_k(U(\mathfrak{g})^{\otimes n}, W) \longrightarrow \operatorname{Hom}_k\Big(\bigwedge^n \mathfrak{g}, R(W)\Big), \quad f \mapsto f \circ A.$$

Suppose W comes with a pairing $W \otimes_k W \to W$ which is compatible with the $U(\mathfrak{g})$-bimodule structure (in the sense discussed in Section 3). Then $C^\bullet(U(\mathfrak{g}), W)$ carries a cup product $\smile$ turning it into a DGA (associative and unital when W is). Let $V = R(W)$ to simplify notation and view the pairing

as a map $\beta\colon V\otimes_k V\to V$. The latter is a $\mathfrak{g}$-pairing in the sense that for all $x,y\in V$ and $\xi\in\mathfrak{g}$, we have the identity

$$\beta(\Delta(\xi)(x\otimes y))=\beta((\xi\bullet x)\otimes y+x\otimes(\xi\bullet y))=\xi\bullet\beta(x\otimes y).$$

This defines a cup product $\smile$ on $C^\bullet(\mathfrak{g},V)$ in a standard fashion. We refer the reader to [2, p. 284] for the explicit formula in terms of shuffles. A technical calculation, which we skip here, verifies that the morphism in Proposition 4.3 is a quasi-isomorphism of DGA's.

Finally, suppose $\mathfrak{g}=\mathfrak{g}^0\oplus\mathfrak{g}^1\oplus\cdots$ is a graded Lie algebra. Then $\bigwedge^n\mathfrak{g}$ is graded as well by letting

$$(2)\qquad\operatorname{gr}^j\Big(\bigwedge^n\mathfrak{g}\Big)=\bigoplus_{j_1+\cdots+j_n=j}\mathfrak{g}^{j_1}\wedge\cdots\wedge\mathfrak{g}^{j_n}.$$

Assume, moreover, that V is a $\mathbb{Z}$-graded $\mathfrak{g}$-module. Then it follows that the space $\operatorname{Hom}_k(\bigwedge^n\mathfrak{g},V)$ inherits the $\mathbb{Z}$-grading

$$\operatorname{Hom}_k\Big(\bigwedge^n\mathfrak{g},V\Big)=\bigoplus_{s\in\mathbb{Z}}\operatorname{Hom}_k^s\Big(\bigwedge^n\mathfrak{g},V\Big),$$

where Hom_k^s represents the homogeneous k-linear maps of degree s, cp [4, Lem. 4.2]. One checks easily that the coboundaries ∂_n preserve the grading. In other words, the Chevalley–Eilenberg complex breaks up as a direct sum of subcomplexes $\operatorname{gr}^s C^\bullet(\mathfrak{g},V)$. Passing to cohomology gives a bigrading of Lie algebra cohomology.

Definition 4.4. $H^{s,t}(\mathfrak{g},V)=H^{s+t}(\operatorname{gr}^s C^\bullet(\mathfrak{g},V))$.

Example 4.5. As a special case suppose $\mathfrak{g}=\mathfrak{g}^\ell$ is concentrated in a single degree $\ell>0$. Note that $\mathfrak{g}$ is then necessarily abelian $[\mathfrak{g},\mathfrak{g}]\subset\mathfrak{g}^{2\ell}=0$. Then $\bigwedge^n\mathfrak{g}$ is concentrated in degree ℓn. If we take $V=k$ to be the trivial $\mathfrak{g}$-module (concentrated in degree 0), then obviously $\operatorname{Hom}_k(\bigwedge^n\mathfrak{g},k)$ is concentrated in degree $-\ell n$. Consequently, $H^n(\mathfrak{g},k)=\bigwedge^n\mathfrak{g}^*$ is concentrated in bidegree $s=-\ell n$ and $t=(\ell+1)n$.

Similarly for Hochschild cohomology. First $T_n(\mathfrak{g})=\mathfrak{g}^{\otimes n}$ is graded analogously to (2). This gives a grading of $T(\mathfrak{g})$ and thus a grading of its quotient $U(\mathfrak{g})$ (by a homogeneous ideal). Let us make $\operatorname{gr}^j U(\mathfrak{g})$ more explicit. Choose an ordered k-basis $(\xi_1,\ldots,\xi_d)$ for $\mathfrak{g}$ consisting of homogeneous vectors $\xi_i\in\mathfrak{g}^{\ell_i}$. By PBW, the monomials $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_d^{\alpha_d}$ form a basis for $U(\mathfrak{g})$, and

$$\operatorname{gr}^j U(\mathfrak{g})=\operatorname{span}_k\Big\{\xi^\alpha\ \Big|\ \sum_{i=1}^d\alpha_i\ell_i=j\Big\}.$$

In turn this gives a grading of $U(\mathfrak{g})^{\otimes n}$ by a formula similar to (2). Let W be a $\mathbb{Z}$-graded $U(\mathfrak{g})$-bimodule, and consider the space of degree s maps $\operatorname{Hom}_k^s(U(\mathfrak{g})^{\otimes n},W)$. This defines a subcomplex of the Hochschild complex $C^\bullet(U(\mathfrak{g}),W)$, which we denote $\operatorname{gr}^s C^\bullet(U(\mathfrak{g}),W)$. Beware that the direct sum

$\bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_k^s(U(\mathfrak{g})^{\otimes n}, W)$ may not exhaust $\mathrm{Hom}_k(U(\mathfrak{g})^{\otimes n}, W)$, since $U(\mathfrak{g})$ is ∞-dimensional. We let

$$^*\mathrm{Hom}_k(U(\mathfrak{g})^{\otimes n}, W) = \bigoplus_{s \in \mathbb{Z}} \mathrm{Hom}_k^s(U(\mathfrak{g})^{\otimes n}, W)$$

and define the graded subcomplex $^*C^\bullet(U(\mathfrak{g}), W)$ accordingly. Its cohomology is bigraded.

Definition 4.6. $HH^{s,t}(U(\mathfrak{g}), W) = H^{s+t}(\mathrm{gr}^s C^\bullet(U(\mathfrak{g}), W))$.

We observe that the comparison isomorphism in Proposition 4.2 respects the bigradings.

Proposition 4.7. $HH^{s,t}(U(\mathfrak{g}), W) \xrightarrow{\sim} H^{s,t}(\mathfrak{g}, R(W))$.

Proof. Unwinding all the gradings, it is trivial to check that the quasi-isomorphism $f \mapsto f \circ A$ sends

$$\mathrm{gr}^s C^\bullet(U(\mathfrak{g}), W) \longrightarrow \mathrm{gr}^s C^\bullet(\mathfrak{g}, R(W)).$$

We are to show this morphism is still a quasi-isomorphism. Passing to H^n and summing over $s \in \mathbb{Z}$ shows it suffices to check the inclusion $^*C^\bullet(U(\mathfrak{g}), W) \to C^\bullet(U(\mathfrak{g}), W)$ is a quasi-isomorphism (using Proposition 4.2 of course). As done earlier, we may identify the Hochschild complex with $\mathrm{Hom}_{U(\mathfrak{g})^e}(B^\bullet, W)$, where $B^\bullet \to U(\mathfrak{g})$ is the standard resolution $(B^{-n} = U(\mathfrak{g})^e \otimes U(\mathfrak{g})^{\otimes n})$. Under this identification, $\mathrm{Hom}_k^s(U(\mathfrak{g})^{\otimes n}, W)$ corresponds to $\mathrm{Hom}_{U(\mathfrak{g})^e}^s(B^{-n}, W)$. On the other hand, $U(\mathfrak{g})^e \simeq U(\mathfrak{g} \oplus \mathfrak{g})$ is Noetherian, see [15, Cor. 1.7.4] for instance, so $U(\mathfrak{g})$ admits a resolution $R^\bullet \to U(\mathfrak{g})$ by finite free $U(\mathfrak{g})^e$-modules $R^{-n} = U(\mathfrak{g})^e \otimes X_n$, where X_n is a finite-dimensional graded k-vector space. Any choice of homotopy equivalence $R^\bullet \to B^\bullet$ preserving the gradings gives quasi-isomorphisms

- $\mathrm{Hom}_{U(\mathfrak{g})^e}(B^\bullet, W) \longrightarrow \mathrm{Hom}_{U(\mathfrak{g})^e}(R^\bullet, W)$,
- $^*\mathrm{Hom}_{U(\mathfrak{g})^e}(B^\bullet, W) \longrightarrow {}^*\mathrm{Hom}_{U(\mathfrak{g})^e}(R^\bullet, W)$.

However, since $\dim_k X_n < \infty$, we have the middle equality below for all n:

$$^*\mathrm{Hom}_{U(\mathfrak{g})^e}(R^{-n}, W) = {}^*\mathrm{Hom}_k(X_n, W)$$
$$= \mathrm{Hom}_k(X_n, W) = \mathrm{Hom}_{U(\mathfrak{g})^e}(R^{-n}, W).$$

Altogether this shows $^*C^\bullet(U(\mathfrak{g}), W) \longrightarrow C^\bullet(U(\mathfrak{g}), W)$ is a quasi-isomorphism, as desired. $\qquad\square$

As one last remark of this section we note that $\smile$ is compatible with the (s,t)-bigradings of Hochschild and Lie algebra cohomology when W comes with a pairing $W \otimes W \to W$ of degree zero.

5. A spectral sequence for Hochschild cohomology

We return to the setting of Section 2. Thus (G, ω) is a p-valued group where ω takes values in $\frac{1}{e}\mathbb{Z}$, and the completed group algebra $\Omega(G) = k[\![G]\!]$ carries the

filtration $\operatorname{Fil}^i \Omega(G) = \Omega(G)_{\frac{i}{e}}$. We have the graded Lie algebra $\mathfrak{g} = k \otimes_{\mathbb{F}_p[\pi]} \operatorname{gr} G$ with universal enveloping algebra $U(\mathfrak{g}) \xrightarrow{\sim} \operatorname{gr} \Omega(G)$ by Corollary 2.4.

We are concerned with $HH^*(\Omega(G), W)$ for an object W of $\operatorname{Pct}_{\Omega(G)^e}$, but now we are assuming that W is finite-dimensional and comes with a filtration by $\Omega(G)^e$-submodules $\operatorname{Fil}^i W$ indexed by $i \in \mathbb{Z}$ such that

$$\operatorname{Fil}^i \Omega(G) \times \operatorname{Fil}^j W \times \operatorname{Fil}^k \Omega(G) \longrightarrow \operatorname{Fil}^{i+j+k} W.$$

Moreover, we assume the filtration is exhaustive and separated, so $\operatorname{Fil}^i W = W$ for $i \ll 0$ and $\operatorname{Fil}^i W = 0$ for $i \gg 0$. We say W is finite filtered. We associate the $\operatorname{gr} \Omega(G)$-bimodule $\operatorname{gr} W$ and consider its Hochschild cohomology $HH^*(\operatorname{gr} \Omega(G), \operatorname{gr} W)$ with its bigrading discussed in the previous section. Our goal in this section is to find a spectral sequence with $E_1^{s,t} = HH^{s,t}(\operatorname{gr} \Omega(G), \operatorname{gr} W)$ converging to $HH^{s+t}(\Omega(G), W)$.

We identify the nth term of the Hochschild complex $C^\bullet(\Omega(G), W)$ with $\operatorname{Hom}_k^{\mathrm{cts}}(\Omega(G)^{\otimes n}, W)$, where we take the actual tensor product $\otimes$ over k, and not the completed one $\hat{\otimes}$. An element hereof is a k-linear function $f : \Omega(G)^{\otimes n} \to W$ factoring through $(\Omega(G)/I)^{\otimes n}$ for some open two-sided ideal $I \subset \Omega(G)$.

Our next goal is to turn $C^\bullet(\Omega(G), W)$ into a filtered complex. We first endow $\Omega(G)^{\otimes n}$ with the tensor product filtration

$$\operatorname{Fil}^i(\Omega(G)^{\otimes n}) = \sum_{i_1 + \cdots + i_n = i} \operatorname{Fil}^{i_1} \Omega(G) \otimes \cdots \otimes \operatorname{Fil}^{i_n} \Omega(G).$$

The initial step is to identify the graded pieces $\operatorname{gr}^i(\Omega(G)^{\otimes n})$ for this filtration.

Lemma 5.1. *Fix an $i \geq 0$. Then the natural map*

$$\bigoplus_{i_1 + \cdots + i_n = i} \operatorname{gr}^{i_1} \Omega(G) \otimes \cdots \otimes \operatorname{gr}^{i_n} \Omega(G) \longrightarrow \operatorname{gr}^i(\Omega(G)^{\otimes n})$$

is an isomorphism. In particular, $(\operatorname{gr} \Omega(G))^{\otimes n} \xrightarrow{\sim} \operatorname{gr}(\Omega(G)^{\otimes n})$ as graded vector spaces.

Proof. This is essentially the content of [20, Lem. A.1]. For convenience we will give a rough outline of the argument. Since k is a field, we may split the filtration. That is, write $\operatorname{Fil}^j \Omega(G) = \Delta^j \oplus \operatorname{Fil}^{j+1} \Omega(G)$ for all j for some vector space complement Δ^j. We keep our $i \geq 0$ fixed and decompose $\Omega(G)$ as a direct sum $\Delta^0 \oplus \cdots \oplus \Delta^i \oplus \operatorname{Fil}^{i+1} \Omega(G)$. Then $\Omega(G)^{\otimes n}$ decomposes as

$$\Omega(G)^{\otimes n} = \Big(\bigoplus_{i_1 + \cdots + i_n \leq i} \Delta^{i_1} \otimes \cdots \otimes \Delta^{i_n} \Big) \oplus \operatorname{Fil}^{i+1}(\Omega(G)^{\otimes n}),$$

as checked in detail in [20]. It follows that $\operatorname{gr}^i(\Omega(G)^{\otimes n}) = \bigoplus_{i_1 + \cdots + i_n = i} \Delta^{i_1} \otimes \cdots \otimes \Delta^{i_n}$, and we are done by observing that $\operatorname{gr}^j \Omega(G) \simeq \Delta^j$ for all j. $\square$

For the remainder of this section we will just write $C^\bullet$ for the Hochschild complex $C^\bullet(\Omega(G), W)$. It has a natural filtration defined below.

Definition 5.2. $\operatorname{Fil}^s C^n = \{f \mid f(\operatorname{Fil}^i \Omega(G)^{\otimes n}) \subset \operatorname{Fil}^{i+s} W \text{ for all } i \geq 0\}$.

Clearly, $\mathrm{Fil}^s C^n = 0$ for $s \gg 0$. Furthermore, by the continuity assumption on $f \in C^n$, the filtration is exhaustive ($C^n = \bigcup_{s\in\mathbb{Z}} \mathrm{Fil}^s C^n$). Also there is an inclusion $d_n(\mathrm{Fil}^s C^{n-1}) \subset \mathrm{Fil}^s C^n$, so we get a subcomplex $\mathrm{Fil}^s C^\bullet$. Thus $C^\bullet$ is a filtered complex. We first identify the associated graded complex $\mathrm{gr}\, C^\bullet = \bigoplus_{s\in\mathbb{Z}} \mathrm{gr}^s C^\bullet$.

Lemma 5.3. $\mathrm{gr}^s C^n \xrightarrow{\sim} \mathrm{Hom}_k^s(\mathrm{gr}(\Omega(G)^{\otimes n}), \mathrm{gr}\, W) \xrightarrow{\sim} \mathrm{gr}^s C^n(\mathrm{gr}\,\Omega(G), \mathrm{gr}\, W)$.

Proof. This is of the same flavor as [20, Lem. A.2]. Any $f \in \mathrm{Fil}^s C^n$ defines a k-linear degree s map $\mathrm{gr}(\Omega(G)^{\otimes n}) \to \mathrm{gr}\, W$ in the obvious fashion. This gives an injection
$$\mathrm{gr}^s C^n \hookrightarrow \mathrm{Hom}_k^s(\mathrm{gr}(\Omega(G)^{\otimes n}), \mathrm{gr}\, W).$$
To show surjectivity suppose we are given a collection of k-linear maps $\psi^i : \mathrm{gr}^i(\Omega(G)^{\otimes n}) \to \mathrm{gr}^{i+s} W$ for $i \geq 0$. As in the proof of Lemma 5.1, we pick linear complements $\mathrm{Fil}^i \Omega(G)^{\otimes n} = \nabla^i \oplus \mathrm{Fil}^{i+1} \Omega(G)^{\otimes n}$ for all $i \geq 0$, and choose lifts $\Psi^i \colon \nabla^i \to \mathrm{Fil}^{i+s} W$ of ψ^i. Pick a j large enough that $\psi^{j+1} = 0$. Decompose
$$\Omega(G)^{\otimes n} = \nabla^0 \oplus \cdots \oplus \nabla^j \oplus \mathrm{Fil}^{j+1} \Omega(G)^{\otimes n}.$$
Define $f \colon \Omega(G)^{\otimes n} \to W$ by declaring that $f = 0$ on $\mathrm{Fil}^{j+1} \Omega(G)^{\otimes n}$ and that $f = \Psi^i$ on ∇^i for $i \leq j$. By construction, $f \in \mathrm{Fil}^s C^n$ reduces to the collection $(\psi^i)_{i\geq 0}$, thereby showing bijectivity. The second isomorphism in the lemma follows from Lemma 5.1. $\square$

The spectral sequence of the filtered complex $C^\bullet$ takes the following form.

Proposition 5.4. *There is a convergent spectral sequence collapsing at a finite stage,*
$$E_1^{s,t} = HH^{s,t}(\mathrm{gr}\,\Omega(G), \mathrm{gr}\, W) \Longrightarrow HH^{s+t}(\Omega(G), W).$$

Proof. We find [21, 12.22] to be a very useful reference for the spectral sequence of a filtered complex, and we adopt its conventions and terminology. The standard construction gives a cohomological spectral sequence $(E_r, \partial_r)_{r\geq 1}$ starting from $E_1 = \oplus E_1^{s,t}$, where
$$E_1^{s,t} = H^{s+t}(\mathrm{gr}^s C^\bullet) = HH^{s,t}(\mathrm{gr}\,\Omega(G), \mathrm{gr}\, W),$$
by Lemma 5.3. The induced filtration on $H^n(C^\bullet) = HH^n(\Omega(G), W)$ is given by
$$\mathrm{Fil}^s H^n(C^\bullet) = \mathrm{im}\big(H^n(\mathrm{Fil}^s C^\bullet) \longrightarrow H^n(C^\bullet)\big).$$
We may identify $E_1^{s,t}$ with $H^{s,t}(\mathfrak{g}, R(\mathrm{gr}\, W))$ by Corollary 2.4 and Proposition 4.7. Since $H^*(\mathfrak{g}, R(\mathrm{gr}\, W))$ is finite-dimensional and concentrated in degrees $[0, d]$ the spectral sequence is bounded and collapses at a finite stage ($E_{r_0} = E_{r_0+1} = \cdots = E_\infty$). On the other hand, we may identify $H^n(C^\bullet)$ with $H^n(G, R(W))$, by Proposition 3.2. Since $\Omega(G)$ is Noetherian and of global dimension $\leq d$ (see [19, Thm. 33.4] for example), $H^*(G, R(W))$ is also finite-dimensional and concentrated in degrees $[0, d]$. It follows that $\mathrm{Fil}^s H^n(C^\bullet) = H^n(C^\bullet)$ for $s \ll 0$ and $\mathrm{Fil}^s H^n(C^\bullet) = 0$ for $s \gg 0$. In particular, the spectral sequence converges, cp [21, Lem. 12.22.13]; therefore $\mathrm{gr}^s H^n(C^\bullet) \simeq E_\infty^{s,n-s}$ for

all $s \in \mathbb{Z}$ and $n \geq 0$ ("weak convergence") and each $H^n(C^\bullet)$ is finite filtered. See [21, Def. 12.22.9]. □

As pointed out in the previous proof, the comparison results (Propositions 4.7 and 3.2) allow us to rephrase Proposition 5.4 as follows.

Theorem 5.5. *There is a convergent spectral sequence collapsing at a finite stage,*

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, R(\mathrm{gr}\, W)) \Longrightarrow H^{s+t}(G, R(W)).$$

When stated this way, there is no mention of Hochschild cohomology at all. The latter serves as a go-between relating Lie algebra cohomology to group cohomology. However, Theorem 5.5 applies only to finite-dimensional $\Omega(G)$-bimodules W; not arbitrary G-modules.

Finally, suppose W comes with a pairing $W \otimes_{\Omega(G)} W \to W$ mapping $\mathrm{Fil}^i W \times \mathrm{Fil}^j W \to \mathrm{Fil}^{i+j} W$. Passing to graded spaces gives a degree zero pairing $\mathrm{gr}\, W \otimes_{\mathrm{gr}\, \Omega(G)} \mathrm{gr}\, W \to \mathrm{gr}\, W$. In particular, $C^\bullet = C^\bullet(\Omega(G), W)$ becomes a filtered DGA. Indeed it is easily verified that $\mathrm{Fil}^s C^n \times \mathrm{Fil}^{s'} C^{n'} \overset{\smile}{\longrightarrow} \mathrm{Fil}^{s+s'} C^{n+n'}$. Moreover, in Lemma 5.3 we get an isomorphism of DGA's $\mathrm{gr}\, C^\bullet \overset{\sim}{\longrightarrow} C^\bullet(\mathrm{gr}\, \Omega(G), \mathrm{gr}\, W)$. In particular, the spectral sequences in Theorems 5.4 and 5.5 become multiplicative. This means each sheet E_r has a multiplication $E_r \otimes E_r \to E_r$ compatible with the (s,t)-bigrading and satisfying the Leibniz formula. Furthermore, $H^*(E_r) \simeq E_{r+1}$ as algebras. On E_∞ the multiplication is compatible with the cup product on $H^*(C^\bullet)$ in the sense that the diagram below commutes.

$$
\begin{array}{ccc}
E_\infty^{s,n-s} \otimes E_\infty^{s',n'-s'} & \longrightarrow & E_\infty^{s+s',n+n'-s-s'} \\
\simeq \big\downarrow & & \big\downarrow \simeq \\
\mathrm{gr}^s H^n(C^\bullet) \otimes \mathrm{gr}^{s'} H^{n'}(C^\bullet) & \longrightarrow & \mathrm{gr}^{s+s'} H^{n+n'}(C^\bullet).
\end{array}
$$

6. Examples and applications

6.1. **The trivial bimodule.** The simplest case is the trivial bimodule $W = k$, with $\Omega(G)$ acting from both sides via the augmentation map $\Omega(G) \to k$. We filter k such that $\mathrm{gr}\, k$ is concentrated in degree 0. In this case $R(W)$ is the trivial G-module k, and $R(\mathrm{gr}\, W)$ is the trivial $\mathfrak{g}$-module k. Thus in this case we have a convergent multiplicative spectral sequence

$$(3) \qquad\qquad E_1^{s,t} = H^{s,t}(\mathfrak{g}, k) \Longrightarrow H^{s+t}(G, k).$$

Often one assumes $\mathrm{gr}\, G$ is abelian. In fact, this can always be achieved by perturbing ω and replacing it by $\omega_C = \omega - C$ for small enough $C > 0$, cp [19, Lem. 26.13]. In this case $\mathfrak{g}$ is of course also abelian and the E_1-page has terms $H^{s,t}(\mathfrak{g}, k) = \mathrm{Hom}_k^s(\bigwedge^{s+t} \mathfrak{g}, k)$.

We wish to quantify at which stage (3) collapses.

Proposition 6.2. $E_r^{s,t} \simeq E_{r+1}^{s,t}$, *provided*

$$r \geq e(s+t)\big(\max_i \omega(g_i) - \min_i \omega(g_i)\big) = e(s+t)\mathrm{amp}(\omega).$$

Proof. Let $\mathfrak{g} = \mathfrak{g}_{\tau_1} \oplus \cdots \oplus \mathfrak{g}_{\tau_b}$, where $\tau_1 < \cdots < \tau_b$ denote the values $\omega(g_i)$ in increasing order, and without repetitions. Write $\tau_i = \frac{f_i}{e}$ for positive integers $f_1 < \cdots < f_b$. Recall our upper-numbering $\mathfrak{g}^{f_i} = \mathfrak{g}_{\tau_i}$ of the graded pieces of $\mathfrak{g}$.

Fix an $n \in [0,d]$. Note that $\mathrm{gr}^j(\bigwedge^n \mathfrak{g}) = 0$ unless j is an n-sum from $\{f_1, \ldots, f_b\}$. In particular, $\bigwedge^n \mathfrak{g}$ is supported in degrees $[nf_1, nf_b]$. It follows that $\mathrm{Hom}_k^s(\bigwedge^n \mathfrak{g}, k) = 0$ unless $-s$ is an n-sum from $\{f_1, \ldots, f_b\}$. We infer that $E_1^{s,n-s} = 0$ unless s belongs to the interval $[-nf_b, -nf_1]$. Thus the E_1-page is concentrated in a conic sector of the second quadrant. Drawing the picture shows that all outgoing and incoming differentials at $E_r^{s,n-s}$ vanish provided $r \geq n(f_b - f_1)$. This shows the result. $\qquad\square$

In particular, $E_r = E_\infty$ provided $r \geq ed \cdot \mathrm{amp}(\omega)$.

When (G,ω) is equi-p-valued, $\mathrm{amp}(\omega) = 0$ and the spectral sequence collapses already on the E_1-page. In this case we recover the following celebrated theorem of Lazard, cp [12, Ch. V, Prop. 2.5.7.1]:

Corollary 6.3 ("Lazard's isomorphism"). *Let (G,ω) be an equi-p-valued group. Then there is an isomorphism of algebras*

$$H^*(G,k) \simeq \bigwedge \mathfrak{g}^*.$$

Proof. As noted in Example 4.5 the Lie algebra $\mathfrak{g} = \mathfrak{g}^\ell$ is abelian and $H^n(\mathfrak{g},k) = \bigwedge^n \mathfrak{g}^*$ is concentrated in bidegree $s = -\ell n$ and $t = (\ell+1)n$. Consequently, the filtration of $H^n(G,k)$ only has one jump at $s = -\ell n$. Altogether we have the following string of isomorphisms

$$H^n(G,k) = \mathrm{gr}^{-\ell n} H^n(G,k) \simeq E_\infty^{-\ell n, (\ell+1)n} = E_1^{-\ell n, (\ell+1)n} = \bigwedge^n \mathfrak{g}^*.$$

As n varies this gives an isomorphism of algebras, since the spectral sequence (3) is multiplicative. $\qquad\square$

6.4. **Quotients of the Iwasawa algebra.** As another application we may take $W = \Omega(G)/I$ for an open two-sided ideal $I \subset \Omega(G)$. This becomes an $\Omega(G)$-bimodule via $\Omega(G) \to \Omega(G)/I$ and multiplication on the quotient. For simplicity we will take $I = \mathrm{Fil}^N \Omega(G)$ for an $N \geq 0$ fixed throughout this section. Thus W gets the quotient filtration $\mathrm{Fil}^i W = (\mathrm{Fil}^i \Omega(G) + I)/I$, which makes it a finite filtered $\Omega(G)$-bimodule. Indeed, $\mathrm{Fil}^0 W = W$ and $\mathrm{Fil}^N W = 0$. The corresponding $\mathrm{gr}\,\Omega(G)$-bimodule is the truncated algebra

$$\mathrm{gr}\,W = \mathrm{gr}^{<N} \Omega(G) \simeq \mathrm{gr}\,\Omega(G)/\mathrm{gr}^{\geq N} \Omega(G).$$

In this situation we prefer to keep $HH^*(\Omega(G), \Omega(G)/I)$ instead of identifying it with group cohomology; this makes the cup product more transparent. We next unwind the E_1-page of Theorem 5.4.

Proposition 6.5. *Let (G, ω) be equi-p-valued (with value $\tau = \frac{1}{e}$). Then for $n \in [0, d]$ and $s \in \mathbb{Z}$,*

$$E_1^{s,n-s} = \begin{cases} \bigwedge^n \mathfrak{g}^* \otimes S^{n+s}(\mathfrak{g}) & \text{if } 0 \le n + s < N, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We identify $\operatorname{gr} W$ with the $U(\mathfrak{g})$-bimodule $U(\mathfrak{g})^{<N}$ via the isomorphism $\operatorname{gr} \Omega(G) \xrightarrow{\sim} U(\mathfrak{g})$. Since $\mathfrak{g}$ is abelian, $U(\mathfrak{g})$ is the symmetric algebra $S(\mathfrak{g})$, and thus $\mathfrak{g}$ acts trivially on $R(\operatorname{gr} W) = S(\mathfrak{g})^{<N}$ via the $\bullet$-action (because $S(\mathfrak{g})$ is commutative). Consequently,

$$E_1^{s,t} = H^{s,t}(\mathfrak{g}, R(\operatorname{gr} W)) = \operatorname{Hom}_k^s \Big(\bigwedge^{s+t} \mathfrak{g}, S(\mathfrak{g})^{<N} \Big).$$

By assumption $\mathfrak{g} = \mathfrak{g}_{\frac{1}{e}} = \mathfrak{g}^1$ is concentrated in degree one, so $\bigwedge^{s+t} \mathfrak{g}$ is concentrated in degree $s + t$. Hence $E_1^{s,t}$ consists of all k-linear maps from $\bigwedge^{s+t} \mathfrak{g}$ to the degree $2s + t$ component of $S(\mathfrak{g})^{<N}$. $\qquad\square$

In summary, when (G, ω) is equi-p-valued, there is a multiplicative convergent spectral sequence

$$(4) \qquad E_1^{s,t} = \bigwedge^{s+t} \mathfrak{g}^* \otimes S^{2s+t}(\mathfrak{g}) \Longrightarrow HH^{s+t}\big(\Omega(G), \Omega(G)/\operatorname{Fil}^N \Omega(G)\big)$$

collapsing at a finite stage, with the understanding that $E_1^{s,t} = 0$ when $2s + t$ is negative or $\ge N$.

6.6. Cohomology of quaternion groups.

Let D be the central division algebra over $\mathbb{Q}_p$ of dimension n^2 and invariant $\frac{1}{n}$. The valuation v on $\mathbb{Q}_p$ extends uniquely to a valuation $\tilde{v} \colon D^\times \to \frac{1}{n}\mathbb{Z}$ by the formula $\tilde{v}(x) = \frac{1}{n} v(\operatorname{Nrd}_{D/\mathbb{Q}_p}(x))$, and the valuation ring $\mathcal{O}_D = \{x \mid \tilde{v}(x) \ge 0\}$ is the maximal compact subring of D. It is local with maximal ideal $\mathfrak{m}_D = \{x \mid \tilde{v}(x) > 0\}$ and residue field $\mathbb{F}_D \xrightarrow{\sim} \mathbb{F}_{p^n}$. When we write $\mathbb{F}_{p^n}$ below we really mean the residue field $\mathbb{F}_D$. We may pick an element ϖ_D satisfying $\tilde{v}(\varpi_D) = \frac{1}{n}$ ("uniformizing parameter"); then $\mathfrak{m}_D = \varpi_D \mathcal{O}_D = \mathcal{O}_D \varpi_D$. In fact, we may and will pick ϖ_D such that $p = \varpi_D^n$. Indeed, $\mathcal{O}_D$ has the following presentation, cp [6, 3.1.1]:

$$\mathcal{O}_D = W(\mathbb{F}_{p^n})[\varpi_D], \quad \varpi_D^n = p, \quad \operatorname{Frob}(x) = \varpi_D x \varpi_D^{-1} \quad \text{for all } x \in W(\mathbb{F}_{p^n}).$$

The commutation relation is essentially what it means that D has invariant $\frac{1}{n}$, cp [16, p. 29].

Our main interest in this section is the mod p cohomology ring of the unit group $\mathcal{O}_D^\times$. We consider its Sylow pro-p subgroup $\tilde{G} = 1 + \mathfrak{m}_D$. The Teichmüller lift $\mathbb{F}_D^\times \to W(\mathbb{F}_{p^n})^\times$ gives a (non-central) splitting, and we may thus factor $\mathcal{O}_D^\times$ as a semi-direct product $\tilde{G} \rtimes \mathbb{F}_D^\times$. By a classical result of Nakaoka, the Hochschild–Serre spectral sequence collapses on the first page and we get an

isomorphism

$$
\begin{aligned}
(5) \qquad H^*(\mathcal{O}_D^\times, \mathbb{F}_p) &\simeq H^*(\tilde{G} \rtimes \mathbb{F}_D^\times, \mathbb{F}_p) \\
&\simeq H^*(\mathbb{F}_D^\times, H^*(\tilde{G}, \mathbb{F}_p)) \\
&\simeq H^0(\mathbb{F}_D^\times, H^*(\tilde{G}, \mathbb{F}_p)) \\
&\simeq H^*(\tilde{G}, \mathbb{F}_p)^{\mathbb{F}_D^\times}.
\end{aligned}
$$

The second isomorphism above is Nakaoka's theorem, see [3, Thm. 5.3.1] and the discussion in [3, Section 7.1]. The third isomorphism follows from the observation that $|\mathbb{F}_D^\times|$ is prime-to-p and annihilates all higher cohomology, see [3, Ex. 2.1.1]. At least for $n = 2$ the $\mathbb{F}_D^\times$-action on $H^*(\tilde{G}, \mathbb{F}_p)$ is understood (and nontrivial)—see [7, Prop. 7, Part (b)] for a complete description.

Now on $\tilde{G}$ we define the function $\omega(g) = \tilde{v}(g - 1)$. This gives a p-valuation, provided $\frac{1}{n} > \frac{1}{p-1}$. In other words, when $p > n + 1$. Under this assumption our theory applies. First, the filtration of $\tilde{G}$ is given by the subgroups $\tilde{G}^i = \tilde{G}_{\frac{i}{n}} = 1 + \mathfrak{m}_D^i$. As for the graded pieces, [16, Prop. 1.8] tells us that the map $1 + x\varpi_D^i \mapsto \bar{x}$ (the reduction of x modulo $\mathfrak{m}_D$) gives an isomorphism of $\mathbb{F}_p$-vector spaces

$$
\operatorname{gr}^i \tilde{G} = \operatorname{gr}_{\frac{i}{n}} \tilde{G} = (1 + \mathfrak{m}_D^i)/(1 + \mathfrak{m}_D^{i+1}) \overset{\sim}{\longrightarrow} \mathbb{F}_D
$$

for $i > 0$. Thus $\operatorname{gr} \tilde{G} = \mathbb{F}_D \oplus \mathbb{F}_D \oplus \cdots$. Since we are assuming $p > n + 1$, the $\mathbb{F}_p[\pi]$-module structure is very simple. By [6, Lem. 3.1.4], the π-operator just shifts the components of $\operatorname{gr} \tilde{G}$ by degree 1. Also the Lie bracket can be made very explicit. By [6, Lem. 3.1.4] or [16, Lem. 1.8], we have the formula

$$
(6) \qquad [\bar{x}, \bar{y}] = \bar{x}\bar{y}^{p^i} - \bar{y}\bar{x}^{p^j},
$$

valid for $\bar{x} \in \operatorname{gr}^i \tilde{G} \overset{\sim}{\longrightarrow} \mathbb{F}_D$ and $\bar{y} \in \operatorname{gr}^j \tilde{G} \overset{\sim}{\longrightarrow} \mathbb{F}_D$. Altogether this gives a very simple model for the graded Lie algebra $\tilde{\mathfrak{g}}$ over $\mathbb{F}_p$. Namely, $\tilde{\mathfrak{g}} = \mathbb{F}_D \oplus \cdots \oplus \mathbb{F}_D$ is concentrated in degrees $1, 2, \ldots, n$, with Lie bracket given by (6) above; with the understanding that $[\bar{x}, \bar{y}] = 0$ when $i + j > n$.

There is another reduction one can make by passing to the subgroup $\operatorname{SL}_1(D) = \ker(\operatorname{Nrd}_{D/\mathbb{Q}_p})$. It is well known that $\operatorname{Nrd}_{D/\mathbb{Q}_p}$ maps $1 + \mathfrak{m}_D$ onto $1 + p\mathbb{Z}_p$, cp [16, Lem. 1.7] for example. We define the subgroup G by the short exact sequence:

$$
1 \longrightarrow G \longrightarrow 1 + \mathfrak{m}_D \overset{\operatorname{Nrd}}{\longrightarrow} 1 + p\mathbb{Z}_p \longrightarrow 1.
$$

The sequence splits. The inequality $p > n + 1$ implies that $\frac{1}{n} \in \mathbb{Z}_p$, so $(\cdot)^{\frac{1}{n}}$ makes sense on $1 + p\mathbb{Z}_p$ viewed as a central subgroup of $1 + \mathfrak{m}_D$. Thus we break up $\tilde{G}$ as a product $G \times (1 + p\mathbb{Z}_p)$, and by Künneth,

$$
H^*(\tilde{G}, \mathbb{F}_p) \simeq H^*(G, \mathbb{F}_p) \otimes H^*(1 + p\mathbb{Z}_p, \mathbb{F}_p) \simeq H^*(G, \mathbb{F}_p) \otimes \mathbb{F}_p[\varepsilon],
$$

with $\mathbb{F}_p[\varepsilon]$ denoting the dual numbers ($\varepsilon^2 = 0$). The identification $H^*(1 + p\mathbb{Z}_p, \mathbb{F}_p) \simeq \mathbb{F}_p[\varepsilon]$ can be viewed as a very special case of Corollary 6.3. The real problem is to understand $H^*(G, \mathbb{F}_p)$. We may of course restrict ω and get

a p-valuation on G. Then $G^i = G_{\frac{i}{n}} = (1 + \mathfrak{m}_D^i)^{\mathrm{Nrd}=1}$, and the graded pieces are again given by [16, Prop. 1.8]:

$$\mathrm{gr}^i G = \mathrm{gr}_{\frac{i}{n}} G \xrightarrow{\sim} \begin{cases} \mathbb{F}_D, & i \not\equiv 0 \bmod n, \\ \mathbb{F}_D^{\mathrm{Tr}=0}, & i \equiv 0 \bmod n. \end{cases}$$

By $\mathbb{F}_D^{\mathrm{Tr}=0}$ we mean the kernel of the trace $\mathrm{Tr}_{\mathbb{F}_D/\mathbb{F}_p}$. Thus $\mathfrak{g} \subset \tilde{\mathfrak{g}}$ is a codimension one Lie subalgebra, namely, $\mathfrak{g} = \mathbb{F}_D \oplus \cdots \oplus \mathbb{F}_D^{\mathrm{Tr}=0}$. Still concentrated in degrees $1, 2, \ldots, n$ and with Lie bracket given by (6). (One can easily check that indeed $[\bar{x}, \bar{y}]$ has trace zero when $i + j = n$.)

For the remainder of this section we assume $n = 2$. Thus D is the division quaternion algebra over $\mathbb{Q}_p$ for a prime $p > 3$. Our goal is to make the mod p cohomology algebra of $G = (1 + \mathfrak{m}_D)^{\mathrm{Nrd}=1}$ explicit. In the quaternion case, $\mathbb{F}_D \xrightarrow{\sim} \mathbb{F}_{p^2}$ and the Lie algebra $\mathfrak{g} = \mathbb{F}_D \oplus \mathbb{F}_D^{\mathrm{Tr}=0}$ sits in degrees $1, 2$. The Lie bracket is given by $[\bar{x}, \bar{y}] = \bar{x}\bar{y}^p - \bar{y}\bar{x}^p$ for any two $\bar{x}, \bar{y} \in \mathbb{F}_D$ of degree one. In particular, $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}_D^{\mathrm{Tr}=0}$, and therefore $H_1(\mathfrak{g}, \mathbb{F}_p) = \mathfrak{g}^{\mathrm{ab}} = \mathbb{F}_D$.

Remark 6.7. To add a few more details, the obvious inclusion $[\mathfrak{g}, \mathfrak{g}] \subseteq \mathbb{F}_D^{\mathrm{Tr}=0}$ is an equality for dimension reasons. (Both subspaces of $\mathbb{F}_D$ are one-dimensional over $\mathbb{F}_p$ since D is now a quaternion algebra.)

We first want to understand the bigrading of $H^1(\mathfrak{g}, \mathbb{F}_p) = \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_D, \mathbb{F}_p)$. Clearly we must have $s = -1$ or $s = -2$, since $\mathbb{F}_p$ sits in degree 0. However, $H^{-2,3}(\mathfrak{g}, \mathbb{F}_p) = 0$ as there is no nontrivial map $\mathfrak{g} \to \mathbb{F}_p$ of degree $s = -2$ vanishing on all commutators. Consequently,

$$H^1(\mathfrak{g}, \mathbb{F}_p) = H^{-1,2}(\mathfrak{g}, \mathbb{F}_p) = \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_D, \mathbb{F}_p).$$

Moreover, since $\bigwedge^3 \mathfrak{g} \simeq \mathbb{F}_p$ is concentrated in degree $1 + 1 + 2 = 4$, the top cohomology $H^3(\mathfrak{g}, \mathbb{F}_p) \simeq \mathbb{F}_p$ sits in bidegree $(-4, 7)$. To summarize, we have

$$H^3(\mathfrak{g}, \mathbb{F}_p) = H^{-4,7}(\mathfrak{g}, \mathbb{F}_p) = \mathbb{F}_p.$$

Obviously $H^0(\mathfrak{g}, \mathbb{F}_p) = H^{0,0}(\mathfrak{g}, \mathbb{F}_p) = \mathbb{F}_p$. It remains to understand $H^2(\mathfrak{g}, \mathbb{F}_p)$ and its bigrading. We note that the Poincaré duality holds for Lie algebra cohomology, see [5, p. 27]. Thus, as $\mathbb{F}_p$-vector spaces,

$$H^2(\mathfrak{g}, \mathbb{F}_p) = H^{-3,5}(\mathfrak{g}, \mathbb{F}_p) = H^{-1,2}(\mathfrak{g}, \mathbb{F}_p)^\vee = \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_D, \mathbb{F}_p)^\vee = \mathbb{F}_D.$$

In particular, (3) collapses on the E_1-page. Via the trace pairing $(\bar{x}, \bar{y}) \mapsto \mathrm{Tr}_{\mathbb{F}_D/\mathbb{F}_p}(\bar{x}\bar{y})$, we have an $\mathbb{F}_p$-linear isomorphism $\mathbb{F}_D \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}_p}(\mathbb{F}_D, \mathbb{F}_p) = H^1(\mathfrak{g}, \mathbb{F}_p)$.

We may summarize our findings as an isomorphism of graded $\mathbb{F}_p$-algebras (where we view $\mathbb{F}_D$ simply as an $\mathbb{F}_p$-vector space):

$$(7) \qquad\qquad H^*(G, \mathbb{F}_p) \simeq \mathbb{F}_p \oplus \mathbb{F}_D \oplus \mathbb{F}_D \oplus \mathbb{F}_p.$$

Here the cup product $H^1(G, \mathbb{F}_p) \times H^2(G, \mathbb{F}_p) \to H^3(G, \mathbb{F}_p)$ between the middle factors corresponds to the trace pairing $\mathbb{F}_D \times \mathbb{F}_D \to \mathbb{F}_p$, and the only other interesting cup product $H^1(G, \mathbb{F}_p) \times H^1(G, \mathbb{F}_p) \to H^2(G, \mathbb{F}_p)$ vanishes. (One way to see this is to choose a basis $\{e_1, e_2\}$ for $H^1(G, \mathbb{F}_p)$ and let $\{e_1^*, e_2^*\}$ be

the dual basis for $H^2(G, \mathbb{F}_p)$. Thus $e_i \smile e_j^* = e_j^* \smile e_i = \delta_{ij}\eta$ for a fixed choice of a nonzero $\eta \in H^3(G, \mathbb{F}_p)$. By graded-commutativity, it suffices to show $e_1 \smile e_2 = 0$. Expand $e_1 \smile e_2 = Ae_1^* + Be_2^*$. Cupping with e_1 and e_2 shows that $A = 0$ and $B = 0$, respectively, since $e_i \smile e_i = 0$.)

The result in (7) independently confirms [7, Prop. 7]. Henn's proof is different. Instead of passing to Lie algebra cohomology he works directly with $H^*(G_{\frac{i}{2}}, \mathbb{F}_p)$ and uses that the subgroup $G_{\frac{1}{2}}$ is equi-p-valued. See also Ravenel's calculation [18, Thm. 6.3.22].

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