Matthias Wulkau

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Matthias Wulkau
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Dekan :	Prof. Dr. Xiaoyi Jiang
Erster Gutachter :	Prof. Dr. Peter Schneider
Zweiter Gutachter :	Prof. Dr. Eugen Hellmann
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ABSTRACT

Based on work of Breuil and Schneider in [BrSch], we introduce crystalline representations of an extension of the absolute Galois group of a local field of mixed characteristic $(0, p)$ by a finite cyclic group of order $n \geq 2$ over a finite extension of $\mathbb{Q}_p$ on the one hand and weakly admissible $\frac{1}{n}\mathbb{Z}$-filtered isocrystals with coefficients in the same finite extension of $\mathbb{Q}_p$ on the other hand. Via analogues of the functors $\mathbb{D}_{\text{cris}}$ and $\mathbb{V}_{\text{cris}}$ we study the relationship between the categories consisting of these objects. In particular it will turn out that, in contrast to the main result of [CoFo], these analogues will in general not induce an equivalence between both categories.
Moreover, on the isocrystal side we transfer a result of Hellmann from [Hel] on the algebraic nature of geometric parameter spaces of weakly admissible $\mathbb{Z}$-filtered isocrystals with prescribed characteristic polynomial of a certain power of the Frobenius and prescribed filtration type to the context of $\frac{1}{n}\mathbb{Z}$-filtrations.

ZUSAMMENFASSUNG

Basierend auf Resultaten von Breuil und Schneider in der Arbeit [BrSch] führen wir kristalline Darstellungen einer Erweiterung der absoluten Galoisgruppe eines lokalen Körpers in gemischter Charakteristik $(0, p)$ um eine endliche zyklische Gruppe der Ordnung $n \geq 2$ über einer endlichen Erweiterung von $\mathbb{Q}_p$ einerseits sowie schwach zulässig $\frac{1}{n}\mathbb{Z}$-filtrierte Isokristalle mit Koeffizienten in der selben endlichen Erweiterung von $\mathbb{Q}_p$ andererseits ein. Via Analoga der Funktoren $\mathbb{D}_{\text{cris}}$ und $\mathbb{V}_{\text{cris}}$ studieren wir die Beziehung zwischen den Kategorien bestehend aus diesen Objekten. Es wird sich insbesondere herausstellen, dass, im Gegensatz zum Hauptresultat von [CoFo], diese Analoga im Allgemeinen keine Äquivalenz zwischen beiden Kategorien induzieren.
Auf Seite der Isokristalle übertragen wir darüberhinaus ein Resultat von Hellmann aus [Hel] über die algebraische Natur geometrischer Parameterräume für schwach zulässig $\mathbb{Z}$-filtrierte Isokristalle unter Vorgabe des charakteristischen Polynoms einer gewissen Potenz des Frobenius und des Filtrierungstyps in den Kontext von $\frac{1}{n}\mathbb{Z}$-Filtrierungen.

Contents

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1 Introduction

1.1 Conventions and notations

All rings are commutative with unit 1. Ring homomorphisms preserve units.
A choice of an algebraic closure of a field F is denoted $\overline{F}$.

The letter p stands for a rational prime. By L resp. K we denote two fixed finite extensions of $\mathbb{Q}_p$ (called base field resp. coefficient field), both contained in a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ and such that $[L : \mathbb{Q}_p]$ is equal to the number of $\mathbb{Q}_p$-algebra homomorphisms from L to K. By L_0 we denote the maximal unramified extension of $\mathbb{Q}_p$ in L. We define $f := [L_0 : \mathbb{Q}_p]$ and $e := [L : L_0]$.
The valuation on a finite extension of $\mathbb{Q}_p$ that sends p to 1 will be written v_p.
For any x in a finite extension of $\mathbb{Q}_p$ we set $|x|_p := p^{-v_p(x)} \in p^{\mathbb{Q}}$.
Whenever we index some expression by τ_0 resp. τ without further comment, the index set runs over all $\mathbb{Q}_p$-embeddings $\tau_0 : L_0 \hookrightarrow K$ resp. all $\mathbb{Q}_p$-embeddings $\tau : L \hookrightarrow K$.

For an object Z of a category and Z-objects $g : X \to Z$, $h : Y \to Z$ let $X \times_Z Y$ denote the fiber product with respect to g and h (if it exists). We usually omit the morphisms g and h in this notation.

Locally ringed spaces are usually denoted by their underlying topological space. The structure sheaf of a locally ringed space Y will be denoted by $\mathcal{O}_Y$.
If X and Y are Z-schemes and $Z = \mathrm{Spec}(C)$ is affine we also denote the fiber product of X and Y over Z by $X \times_C Y$. If additionally $Y = \mathrm{Spec}(B)$ is an affine scheme we also write $X \otimes_C B$. For general Z and $Y = \mathrm{Spec}(B)$ affine, the fiber product of X and Y over Z is also denoted by $X \otimes_Z B$.
For an arbitrary scheme Z and Z-schemes X and Y we write $\mathcal{E} \otimes_Z \mathcal{F}$ for the tensor product of the inverse images $p^*\mathcal{E} \otimes_{\mathcal{O}_W} q^*\mathcal{F}$ over $W := X \times_Z Y$ of an $\mathcal{O}_X$-module $\mathcal{E}$ and an $\mathcal{O}_Y$-module $\mathcal{F}$. In case $Z = \mathrm{Spec}(C)$ this tensor product will also be denoted by $\mathcal{E} \otimes_C \mathcal{F}$.
If Y is a locally ringed space, $y \in Y$ a point and $\mathcal{F}$ an $\mathcal{O}_Y$-module, we let $\mathcal{F}(y)$ denote the reduction of the stalk $\mathcal{F}_y$ of $\mathcal{F}$ at y by the maximal ideal of $\mathcal{O}_{Y,y}$.

If S is a subset of $\mathbb{R}$, $r \in \mathbb{R}$ and $* \in \{\leq, <, \geq, >\}$ then by $s \in S_{*r}$ we mean that simultaneously $s \in S$ and $s * r$ hold true.

Cross-references to specific results that occur within this text are given solely by their item number, e.g. “cf. 3.3.1”.

The following is a list of notations that are used throughout the text.

$\mathbf{Ad}(K)$	category of adic spaces over the adic space associated to $\mathrm{Spec}(K)$
$\mathbf{Alg}(R)$ (R a ring)	category of R-algebras
$\mathbf{Grp}$	category of groups
$\Gamma(U, \mathcal{F})$	sections of a (pre-)sheaf $\mathcal{F}$ over an open subset U of a topological space
$k(x)$	residue field of a point x of an adic space
$\kappa(x)$	residue field of a point x of a scheme
$\mathbf{lrs}$	category of locally ringed spaces
$\tilde{M}$	the quasi-coherent $\mathcal{O}_{\mathrm{Spec}(R)}$-module associated to a module M over a ring R
$\mathbf{Mod}(\mathcal{O}_Y)$	category of $\mathcal{O}_Y$-modules over a locally ringed space Y
$\mathbf{Mod}(R)$ (R a ring)	category of R-modules
$\mu_n(R)$ (R a ring, $n \in \mathbb{Z}_{>0}$)	the set $\{x \in R \mid x^n = 1\}$
O_F (F a valued field)	ring of integers of F with respect to the given valuation
$\mathbf{Sch}(R)$ (R a ring)	category of $\mathrm{Spec}(R)$-schemes
$\mathbf{Sch}(S)$ (S a scheme)	category of S-schemes
$\mathbf{Set}$	category of sets
$R^\times$ (R a ring)	invertible elements of the monoid $(R \setminus \{0\}, \cdot)$
$\mathbf{Vect}(F)$ (F a field)	category of F-vector spaces
$\#S$ (S a finite set)	cardinality of S

One aspect of p-adic Hodge theory is the description of p-adic Galois representations in terms of modules equipped with a semilinear operator ϕ and a $\mathbb{Z}$-filtration. Of particular interest is the class of weakly admissible modules on which the operator and the filtration satisfy a certain numerical relation. The representations corresponding to weakly admissible $\mathbb{Z}$-filtered ϕ-modules are called crystalline. The existence of this correspondence results from the theory developed in [Fo2] and in [CoFo]

A major part of this thesis is devoted to the study of relations between a category of certain representations of a group closely related to the absolute Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}_p|L)$ of L and a category of modules equipped with a semilinear operator and a filtration indexed by a certain subgroup of the rational numbers strictly containing $\mathbb{Z}$.

The motivation for our investigations originates from work of Breuil and Schneider in [BrSch]. In their paper the authors study the connection between de Rham and crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|L)$ over K and so-called locally-algebraic representations of a general linear group in the framework of the p-adic Langlands programme. In order to pass from the Galois side to the reductive group side, the aforementioned functorial correspondence between the category of crystalline $\mathrm{Gal}(\overline{\mathbb{Q}}_p|L)$-representations and the category of weakly admissible $\mathbb{Z}$-filtered ϕ-modules is used (cf. [BrSch, Corollary 3.3] where a sufficient criterion for the existence of a crystalline representation in terms of the existence of an invariant norm on a locally-algebraic representation is given). For technical reasons Breuil and Schneider have to assume that, on the reductive group side, half the sum of the positive roots of the group is an element of the integral weight lattice ([BrSch, §6]). In order to drop this assumption a corresponding construction on the Galois side is necessary. Hence, in [BrSch, §7], Breuil and Schneider introduce a character with values in $K^\times$ that is defined on a specific extension $G_{L,(2)}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|L)$ and whose square, under mild conditions on K, coincides with the restriction of the p-adic cyclotomic character to $\mathrm{Gal}(\overline{\mathbb{Q}}_p|L)$. Moreover, the authors show how the theory of Colmez and Fontaine ([CoFo]) is generalized to the setting of this bigger group. In particular, it is made precise what it means for a representation of this group to be crystalline.

While the major first part of this text, in which we construct a suitable generalization the correspondence between crystalline representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|L)$ and weakly admissible $\mathbb{Z}$-filtered ϕ-modules, is dominated by (semi-)linear algebraic methods, in the last section we focus on the concept of weak admissibility from an algebraic-geometric angle. Our motivation is given by the following discussion. The locus of points x in the L_0-scheme parametrizing flags of a given type μ (which for us is a finite increasing sequence of rational numbers encoding the jump indices with respective multiplicities of the filtrations to be considered) where a given ϕ-module over L_0 is a weakly admissible filtered ϕ-module over the residue field at x is not an open subscheme in general. The situation is improved when analytical methods are applied. Namely, in the context of Tate's rigid-analytic spaces (respectively in the context of Berkovich spaces), the "locus of weakly admissible points" is an admissible open (respectively analytic open) subset in the associated analytic flag scheme, cf. [RaZi, Proposition 1.36] (respectively [DOR, Proposition 8.2.1]). In Hellmann's approach of the structural investigation of the "weakly-admissible locus" for filtered ϕ-modules with coef-

ficients in residue fields of points of adic spaces, the prescribed data essentially consists of a tuple (c, μ) where c is a polynomial prescribing the characteristic polynomial of the f-th power of the Frobenius and μ is a filtration type whose members are integers (cf. [Hel, §5]).
A connection between this kind of data and weak admissibility was established by Breuil and Schneider in [BrSch, §3]. In [loc. cit., Proposition 3.2], where by assumption $c \in K[X]$ decomposes into linear factors and the filtration is given as a collection of ef full $\mathbb{Z}$-filtrations on the isotypical components of the underlying $L \otimes_{\mathbb{Q}_p} K$-module respectively, the authors prove that the existence of a weakly admissible $\mathbb{Z}$-filtered ϕ-module over L with coefficients in K such that the characteristic polynomial of the f-th power of the associated operator is equal to c and such that the associated filtration is of type μ is equivalent with the validity of certain numerical relations between the valuations of the zeros of c and the members of the filtration type.
In [Hel, Theorem 5.1, Proposition 5.2] the ground field L_0 remains fixed and the geometric objects that are of interest are defined over the coefficient field (which in Hellmann's case is $\mathbb{Q}_p$). If s denotes an integer ≥ 0, then these results show that the locus of points inside the adic fiber over a point of $(\mathbb{A}^s \times_{\mathbb{Q}_p} \mathbb{G}_m)^{\mathrm{ad}}$ where the universal $\mathbb{Z}$-filtered isocrystal is weakly admissible is (a base change of) the associated adic space of a quasi-projective $\mathbb{Q}_p$-scheme.

We now explain the contents of this thesis in more detail. Concerning notations, the first four sections are largely inspired by [BrSch, mainly §3 and §7].

In subsection 1.3 we give a condensed overview of the theory underlying [CoFo, Théorèm A], fixing ideas and notations in the process. Since this subsection contains no proofs, we refer to [Fo1],[Fo2] and [CoFo] for more detailed expositions.

In the first part of section 2, after fixing an integer $n \geq 2$, elementary results concerning $\frac{1}{n}\mathbb{Z}$-filtered ϕ-modules over L with coefficients in K are established. We introduce the category of $\frac{1}{n}\mathbb{Z}$-filtered isocrystals in this framework and define the concept of weak admissibility of (filtrations of) these objects, leading to the abelian tensor category $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$ (cf. 2.2.6, (2.)). From assumption 2.2.7 on we impose on the coefficient field K the condition that it contains an n-th root of every element in $\mathbb{Q}_p$. This is necessary for our construction of the group $G_{L,(n)}$ in the subsequent section. After introducing the full subcategory $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$ of $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$ and stating a result on the structure of its objects (cf. 2.2.14), section 2 ends with the discussion of an example due to Schneider (cf. 2.2.17). The filtered isocrystal constructed in this example is contained in $\mathbf{FIC}^{\mathrm{wa}}_{\mathbb{Q}_p,K,n}$ but not in $\mathbf{FIC}^{\mathrm{wa}}_{\mathbb{Q}_p,K,(n)}$. The existence of objects as in the example implies a fundamental difference between our and the classical theory which will become apparent in section 4.

In section 3 we introduce the group $G_{L,(n)}$ as a group-theoretic fiber product. Under the assumption made on K, this group comes equipped with a character with values in $K^\times$ whose n-th power coincides with the restriction of the p-adic cyclotomic character to $\mathrm{Gal}(\overline{\mathbb{Q}}_p|L)$. Based on [BrSch, Lemma 7.5], we prove an independent result which relates a splitting of the exact sequence induced by the construction of $G_{L,(n)}$ to the degree $[L : \mathbb{Q}_p]$ (cf. 3.1.7). The introduction of certain extensions $B_{\mathrm{cris},n}$ and $B_{\mathrm{dR},n}$ of Fontaine's period rings B_{cris} and B_{dR} together with an action of $G_{L,(n)}$ on the algebras $B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K$ and $B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K$ in subsection 3.2 then allows us to define the full abelian tensor subcategory consisting of crystalline representations among all finite-dimensional K-linear continuous $G_{L,(n)}$-representations V. This definition will be based on the relation between the K-dimension of V and the finite $L_0 \otimes_{\mathbb{Q}_p} K$-rank of the invariants

$$\mathbb{D}_{\mathrm{cris},n}(V) := ((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V)^{G_{L,(n)}}$$

with respect to the diagonal action (cf. subsection 3.3 and the discussion before 3.4.1).

After introducing the functor $\mathbb{V}_{\mathrm{cris},n}$ on the category of all filtered ϕ-modules over L with coefficients in K and studying its properties in subsection 4.1, in subsection 4.2 we formulate and prove one of our main results:

Theorem 1.2.1. *The functor $\mathbb{D}_{\mathrm{cris},n}$ induces an equivalence of tensor categories*

$$\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)}) \xrightarrow{\sim} \mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}.$$

A quasi-inverse is given by the restriction of $\mathbb{V}_{\mathrm{cris},n}$ to $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$.

The weakly admissible filtered isocrystal in Schneider's example from the end of section 2 raises the question of the behaviour of the inverse functor of this equivalence on the category $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$. Establishing results towards an answer to this question is what the third subsection of section 4 is about. In particular, we will see that $\mathbb{V}_{\mathrm{cris},n}$, restricted to $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$, takes values in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$ (cf. 4.3.5) and conclude that a weakly admissible object D does not lie in the essential image of $\mathbb{D}_{\mathrm{cris},n}$ if and only if the associated K-linear $G_{L,(n)}$-representation $\mathbb{V}_{\mathrm{cris},n}(D)$ has a K-dimension strictly smaller than the $L_0 \otimes_{\mathbb{Q}_p} K$-rank of D (cf. 4.3.6). In particular, the vector spaces underlying the representations associated to the isocrystals from 2.2.17 turn out to be $\{0\}$.

The final section 5, which is more or less independent from the results from the previous sections, is concerned with a description of the structure of geometric parameter spaces related to weakly admissible $\frac{1}{n}\mathbb{Z}$-filtered isocrystals with coefficients. We chose to keep the first part of this section expository in nature. It collects several well-known facts about representable functors on K-schemes arising in the context of weak admissibility on the one hand and about the functor "associated adic space" between K-schemes and adic spaces (in the sense of Huber) over $\mathrm{Spa}(K, O_K)$ on the other hand, thereby providing necessary notation and background for the second part of this section. In the latter, we present a modified definition of weak admissibility (cf. 5.2.4), in which the coefficient field is any valued field extension of K. In 5.2.9, we state and extensively prove an analogue of [Hel, Proposition 5.2] in the context of $\frac{1}{n}\mathbb{Z}$-filtrations.

Finally, in a short appendix we discuss the notion of Hodge-Tate weight in $\frac{1}{n}\mathbb{Z}$ for objects in $\mathbf{FIC}_{L,K,n}$ respectively in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$, assuming the notions and results up to (and including) subsection 4.2.

1.3 The classical case

One major result in p-adic Hodge theory is the proof of a conjecture by Fontaine which predicts an equivalence between a specific category of representations of the absolute Galois group of L and a specific category of semilinear algebra data over L_0 ([CoFo, Theorem A]). Since this result and the framework in which it was established are part of the motivation for this thesis, we give a short overview of the relevant aspects of the theory underlying the proof.

For any topological group G and any finite extension F of $\mathbb{Q}_p$ denote by $\mathbf{Rep}_F(G)$ the abelian category of continuous F-linear representations of G. Its objects are tuples (V, ρ), where V is a finite-dimensional F-vector space (with induced topology from F) and ρ is a homomorphism $G \to \mathrm{Aut}_F(V)$. Morphisms $f : (V, \rho) \to (V', \rho')$ are morphisms of F-vector spaces that satisfy $\rho'(g)(f(v)) = f(\rho(g)(v))$ for all $g \in G, v \in V$. Continuity means that

$$G \times V \to V,\ (g, v) \mapsto gv := \rho(g)(v)$$

is a continuous map. Together with

- $\otimes := \otimes_F$, with the diagonal action of G on the tensor product and
- the functor
$$\omega : \mathbf{Rep}_F(G) \to \mathbf{Vect}(F),\ (V, \rho) \mapsto V,$$

one knows that continuous representations make up a neutral Tannakian category over F (for a precise definition of this notion, cf. [DeMi, Definition 2.19]). A unit object is F with trivial G-action.
If (V, ρ) is an object of $\mathbf{Rep}_F(G)$, we often omit either V or ρ from the notation if it is clear which tuple is considered.

Let k_L denote the residue field of L and let σ_0 be the lift to L_0 of the field automorphism $k_L \to k_L, x \mapsto x^p$. For brevity, denote by $G_L := \mathrm{Gal}(\overline{\mathbb{Q}}_p|L)$ the absolute Galois group of L. Objects of the category $\mathbf{Rep}(G_L) := \mathbf{Rep}_{\mathbb{Q}_p}(G_L)$ are called p-adic representations of G_L or p-adic Galois representations in case the group is clear from the context. Within $\mathbf{Rep}(G_L)$, certain distinguished full subcategories are of particular interest. For example, the étale cohomology groups $H^{\cdot}_{\text{ét}}(X \otimes_L \overline{\mathbb{Q}}_p, \mathbb{Q}_p)$, where X is a proper smooth algebraic variety over L, naturally give rise to objects of these subcategories (see [Fo2, §6]). For their investigation and for comparison of various cohomology theories, Fontaine constructed in particular the topological $\mathbb{Q}_p$-algebras B_{cris} and B_{dR}. On both B_{cris} resp. B_{dR} the group G_L acts as ring automorphisms with invariant rings $B_{\text{cris}}^{G_L} = L_0$ resp. $B_{\text{dR}}^{G_L} = L$. With any p-adic representation V of G_L, one associates functorially objects of a linear-algebraic nature.

Definition 1.3.1. Let V be in $\mathbf{Rep}(G_L)$. Define

$$\mathbb{D}_{\text{cris}}(V) := (B_{\text{cris}} \otimes_{\mathbb{Q}_p} V)^{G_L}, \qquad \mathbb{D}_{\text{dR}}(V) := (B_{\text{dR}} \otimes_{\mathbb{Q}_p} V)^{G_L}$$

as the respective G_L-invariants with respect to the diagonal action on the tensor product.

Both are L_0- resp. L-vector spaces, whose dimensions are bounded above by $\dim_{\mathbb{Q}_p}(V)$. Of particular interest are those V for which the equality $\dim_{\mathbb{Q}_p}(V) = \dim_{L_0}(\mathbb{D}_{\mathrm{cris}}(V))$ holds true. They are called crystalline and they make up a full abelian subcategory of $\mathbf{Rep}(G_L)$, denoted $\mathbf{Rep}^{\mathrm{cris}}(G_L)$. The category $\mathbf{Rep}^{\mathrm{dR}}(G_L)$ of de Rham representations is defined analogously. Using the existence of an injective and G_L-equivariant morphism of rings $L \otimes_{L_0} B_{\mathrm{cris}} \to B_{\mathrm{dR}}$, via which B_{cris} is identified with a subring of the field B_{dR}, one can prove the following result.

Proposition 1.3.2. *Every crystalline representation of G_L is de Rham.*

The ring B_{cris} carries an injective ring endomorphism φ_0 induced by the Frobenius morphism on the ring of Witt vectors $W(\overline{k_L})$. The field B_{dR} is complete discretely valued with value group $\mathbb{Z}$. A uniformizer is given by any generator t of the rank one $\mathbb{Z}_p$-module $\mathbb{Z}_p(1) := \varprojlim_{n\in\mathbb{Z}_{>0}} \mu_{p^n}(\overline{\mathbb{Q}}_p) \subset B_{\mathrm{cris}}$. We note that φ_0 sends such an element t to pt. The filtration $(B^i_{\mathrm{dR}})_{i\in\mathbb{Z}}$ induced on B_{dR} by integral powers t^i of t is by C-vector spaces, where C is the completion of $\overline{\mathbb{Q}}_p$ with respect to the topology induced by the extension of v_p. Recall that the p-adic cyclotomic character $\chi_p : \mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p) \to \mathbb{Z}_p^\times$ is given by the $\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)$-action on $\mu_{p^n}(\overline{\mathbb{Q}}_p)$. The group G_L acts on $\mathbb{Z}_p(1)$ via the restriction of χ_p to G_L. One then has the so-called fundamental exact sequence

$$0 \to \mathbb{Q}_p \to B_{\mathrm{cris}}^{\varphi_0=1} \to B_{\mathrm{dR}}/B^0_{\mathrm{dR}} \to 0 \qquad (1)$$

of $\mathbb{Q}_p$-vector spaces with respect to the obvious maps. Here, the middle term is the subspace of B_{cris} consisting of those elements on which φ_0 operates trivially. Because of compatibility of the above algebraic structures with the G_L-action, they naturally carry over to $\mathbb{D}_{\mathrm{cris}}(V)$ and $\mathbb{D}_{\mathrm{dR}}(V)$ in the following sense: the former carries a σ_0-semilinear automorphism and the latter induces a decreasing, exhaustive and separated $\mathbb{Z}$-filtration by L-subspaces on $L \otimes_{L_0} \mathbb{D}_{\mathrm{cris}}(V)$.

Definition 1.3.3. Denote by $\mathbf{MF}_L$ the category whose objects are triples $\underline{D} := (D, \phi, F^\bullet D_L)$ consisting of a (not necessarily finite-dimensional) L_0-vector space D, a σ_0-semilinear endomorphism $\phi : D \to D$, and a decreasing, exhaustive and separated $\mathbb{Z}$-filtration $F^\bullet D_L$ consisting of L-vector spaces on $D_L := L \otimes_{L_0} D$. Morphisms are L_0-linear maps that are compatible with the semilinear endomorphisms and such that the induced morphism between L-vector spaces is a morphism of filtered vector spaces. Write $\mathbf{FIC}_L$ for the full subcategory consisting of those objects such that D is finite-dimensional over L_0 and ϕ is bijective. In this case, the operator ϕ is called Frobenius.

By setting $\underline{D}_1 \otimes \underline{D}_2 := (D_1 \otimes_{L_0} D_2, \phi_1 \otimes \phi_2, \text{tensor product filtration})$ the pair $(\mathbf{FIC}_L, \otimes)$ becomes a tensor category with unit object $(L_0, \sigma_0, F^1 = \{0\} \subset F^0 = L)$.
The Newton resp. Hodge numbers are numerical invariants assigned to (isomorphism classes of) objects of $\mathbf{FIC}_L$, depending on the Frobenius resp. the filtration. Both are additive on short exact sequences.
With respect to objects $\underline{D}$ of $\mathbf{FIC}_L$, two concepts of admissibility were introduced.

- On the one hand, weak admissibility expresses a numerical relationship between the Newton and the Hodge numbers attached to $\underline{D}$ and its Frobenius-invariant subspaces with induced filtration on the scalar extension to L. Weakly admissible objects form a full abelian subcategory $\mathbf{FIC}_L^{\mathrm{wa}}$ of $\mathbf{FIC}_L$.

- On the other hand, an object $\underline{D}$ is admissible if it lies in the essential image of the functor $\mathbb{D}_{\mathrm{cris}}$ restricted to $\mathbf{Rep}^{\mathrm{cris}}(G_L)$.

It was long known that both concepts are related by the fact that every admissible object is weakly admissible. The following result, established in [CoFo], proves a conjecture of Fontaine concerning the converse implication ([Fo1, §5.2.6]).

Theorem 1.3.4 (Colmez-Fontaine)**.** *Every object of* $\mathbf{FIC}_L^{wa}$ *is admissible. In other words, the functor* $\mathbb{D}_{\mathrm{cris}}$ *induces an equivalence of abelian categories*

$$\mathbf{Rep}^{cris}(G_L) \to \mathbf{FIC}_L^{wa}$$

respecting tensor products and unit objects. A quasi-inverse is

$$F^0(B_{\mathrm{cris}} \otimes_{L_0} D)^{\varphi_0 \otimes \phi = 1} \leftarrow\!\shortmid (D, \phi, F^\bullet D_L) : \mathbb{V}_{cris}$$

where on the left hand side those $\varphi_0 \otimes \phi$*-invariant elements are meant whose image lies in* $F^0(B_{\mathrm{dR}} \otimes_L (L \otimes_{L_0} D))$ *under the map induced by* $L \otimes_{L_0} B_{\mathrm{cris}} \hookrightarrow B_{\mathrm{dR}}$.

Hence, via the equivalence from the theorem, crystalline representations of G_L can be described by data given in terms of semilinear algebra, in particular by a specific relation between the Newton numbers and the Hodge numbers.

2 Weakly admissible filtered isocrystals with coefficients

Fix once and for all an integer $n \geq 2$. We set $R(L_0, K) := R := L_0 \otimes_{\mathbb{Q}_p} K$ and $R_L := L \otimes_{\mathbb{Q}_p} K = L \otimes_{L_0} R$. The ring R (resp. R_L) naturally is an algebra over L_0 (resp. over L) via $l \mapsto l \otimes 1$ (in both cases). Both rings also have a natural K-algebra structure via $k \mapsto 1 \otimes k$.

In [Fo1], Fontaine has developed a formalism of $\mathbb{Z}$-filtered ϕ-modules over a characteristic 0 field which is complete with respect to a discrete valuation and which has perfect residue field of positive characteristic. In this section we present basic constructions of a similar formalism within the class of $\frac{1}{n}\mathbb{Z}$-filtered ϕ-modules over the ring R.

2.1 Filtered isocrystals

Definition 2.1.1. Let A be a ring and M an A-module. Let Λ be a subgroup of the additive group of real numbers.

1. A decreasing Λ-filtration on M is a family $(F^\lambda M)_{\lambda\in\Lambda}$ of A-submodules of M, such that $F^\lambda M \supseteq F^\mu M$ whenever $\mu \geq \lambda$.
2. Let $(F^\lambda M)_{\lambda\in\Lambda}$ be a decreasing Λ-filtration on M. An element $\lambda \in \Lambda$ is called a filtration index. A filtration index $\lambda \in \Lambda$ is called jump if the quotient A-module
$$\mathrm{gr}^\lambda M := F^\lambda M / \bigcup_{\mu>\lambda} F^\mu M$$
is not reduced to zero.
3. A decreasing Λ-filtration on M is exhaustive resp. separated if
$$\bigcup_{\lambda\in\Lambda} F^\lambda M = M$$
resp. if
$$\bigcap_{\lambda\in\Lambda} F^\lambda M = \{0\}\,.$$

Unless otherwise mentioned we mean decreasing, exhaustive and separated Λ-filtration when we speak of a Λ-filtration. In case the group Λ is clear from the context, a specific filtration on an A-module M will also be abbreviated by $F^\bullet M$.
Let $F^\bullet M$ be a Λ-filtration on the A-module M and $M' \subseteq M$ be a submodule. Then $(F^\lambda M \cap M')_{\lambda\in\Lambda}$ is a Λ-filtration on M'. The filtration thusly obtained is called the induced filtration on M' by $F^\bullet M$ and also denoted by $F^\bullet M \cap M'$ if confusion is excluded.

Definition 2.1.2. A $\frac{1}{n}\mathbb{Z}$-filtered ϕ-module over R is a triple $\underline{D} := (D, \phi, F^\bullet D_L)$ consisting of an R-module D equipped with a $\sigma_0 \otimes K$-linear map $\phi : D \to D$ and a decreasing, exhaustive and separated $\frac{1}{n}\mathbb{Z}$-filtration $F^\bullet D_L$ by R_L-submodules on $D_L := L \otimes_{L_0} D \cong R_L \otimes_R D$.
A morphism $(D_1, \phi_1, F^\bullet D_{1,L}) \to (D_2, \phi_2, F^\bullet D_{2,L})$ between two $\frac{1}{n}\mathbb{Z}$-filtered ϕ-modules is an R-linear map $h : D_1 \to D_2$, such that $h(\phi_1(d)) = \phi_2(h(d))$ for all

$d \in D_1$ and such that the induced map

$$h_L := L \otimes h : D_{1,L} \to D_{2,L}$$

satisfies $h_L(F^i D_{1,L}) \subseteq F^i D_{2,L}$ for all $i \in \frac{1}{n}\mathbb{Z}$. With these notions $\frac{1}{n}\mathbb{Z}$-filtered ϕ-modules over R are a category which we denote by $\mathbf{MF}^{\phi}_{L,K,n}$.
An isomorphism in $\mathbf{MF}^{\phi}_{L,K,n}$ is a morphism which is an isomorphism on the underlying R-modules and which induces isomorphisms of R_L-modules $F^i D_{1,L} \cong F^i D_{2,L}$ for all $i \in \frac{1}{n}\mathbb{Z}$.
A sequence of morphisms

$$0 \to \underline{M}' \to \underline{M} \to \underline{M}'' \to 0$$

in $\mathbf{MF}^{\phi}_{L,K,n}$ is called exact if the sequence of morphisms between underlying R-modules is exact and if this sequence induces short exact sequences of R_L-modules between filtration steps with index i for all $i \in \frac{1}{n}\mathbb{Z}$.

Recall that semilinearity of ϕ with respect to $\sigma_0 \otimes K$ as in the definition means that ϕ is additive and that $\phi(rd) = (\sigma_0 \otimes K)(r)\phi(d)$ for all $r \in R$ and all $d \in D$. The latter property implies that $\phi^{[L_0:\mathbb{Q}_p]}$ is an R-linear map $D \to D$.
We remark that for any subgroup Λ of the additive group of the real numbers an analogue of the category $\mathbf{MF}^{\phi}_{L,K,n}$ with respect to Λ-filtrations can be made precise.

Remark 2.1.3.

1. For a ring A the following are equivalent:

   a) The ring A is semisimple.

   b) Every A-module is semisimple.

   c) Every A-module is projective.

   As a consequence, modules over a semisimple ring are flat. Note that the ring R is semisimple because

   $$\psi : R \to \bigoplus_{\tau_0} K_{\tau_0},\ x \otimes y \mapsto (\tau_0(x)y)_{\tau_0}$$

   is an isomorphism of rings and the direct sum decomposition is a decomposition into simple R-modules. Recall that the index set consists of all $\mathbb{Q}_p$-embeddings $L_0 \to K$ (equivalently, of the group $\mathrm{Gal}(L_0/\mathbb{Q}_p)$). The τ_0-isotypical component K_{τ_0} is the additive group K with L_0-vector space structure

   $$L_0 \times K \to K,\ (x,m) \mapsto \tau_0(x)m.$$

   We observe that the map induced by $x \otimes y \mapsto \sigma_0(x) \otimes y$ on R translates to the map $(m_{\tau_0})_{\tau_0} \mapsto (m_{\tau_0\sigma_0})_{\tau_0}$ on $\bigoplus_{\tau_0} K_{\tau_0}$ via ψ.
   By the same reasoning, the ring R_L is semisimple and the family $(K_\tau)_\tau$ forms a system of distinct representatives for isomorphism classes of simple R_L-modules.

2. Let $D_{\tau_0} := \{d \in D \mid (l \otimes 1)d = (1 \otimes \tau_0(l))d \text{ for all } l \in L_0\}$ be the τ_0-isotypical component of the underlying R-module of some $(D, \phi, F^\bullet D_L)$ in $\mathbf{MF}^{\phi}_{L,K,n}$

for some $\mathbb{Q}_p$-embedding $\tau_0 : L_0 \to K$. The K-vector space D_{τ_0} is equal to $\psi^{-1}(e_{\tau_0})D$ where $e_{\tau_0} \in \bigoplus_{\tau_0'} K_{\tau_0'}$ is the idempotent $(\delta_{\tau_0,\tau_0'})_{\tau_0'}$ in terms of the Kronecker symbol. Hence D is the direct sum of the D_{τ_0}.
Moreover, for every $l \in L_0, d \in D_{\tau_0}$ we have

$$\begin{aligned}(l \otimes 1)\phi(d) &= (\sigma_0\sigma_0^{-1}(l) \otimes 1)\phi(d)\\ &= \phi((\sigma_0^{-1}(l) \otimes 1)d)\\ &= \phi((1 \otimes \tau_0\sigma_0^{-1}(l))d) = (1 \otimes \tau_0\sigma_0^{-1}(l))\phi(d)\end{aligned}$$

where the second equality holds by semilinearity of ϕ, and the fourth is valid because ϕ is K-linear. It follows that $\phi(D_{\tau_0})$ is contained in $D_{\tau_0\sigma_0^{-1}}$ for all τ_0. Note that this generalizes the observation from the first part of the remark.

Lemma/Definition 2.1.4. *Let $\underline{D} := (D, \phi, F^\bullet D_L)$ and $\underline{D}' := (D', \phi', F^\bullet D'_L)$ be objects in $\mathbf{MF}^\phi_{L,K,n}$.*

1. *Together with*

$$\phi \oplus \phi' : D \oplus D' \to D \oplus D',\ (d, d') \mapsto (\phi(d), \phi'(d'))$$

and

$$F^i((D \oplus D')_L) := \eta(F^i D_L \oplus F^i D'_L),$$

where $\eta : D_L \oplus D'_L \tilde{\to} (D \oplus D')_L$ is the canonical isomorphism of R_L-modules, the direct sum $D \oplus D'$ is an object in $\mathbf{MF}^\phi_{L,K,n}$, denoted by $\underline{D} \oplus \underline{D}'$.

2. *Let D' be finitely generated over R. Together with the well-defined $\sigma_0 \otimes K$-linear map*

$$\phi \otimes \phi' : D \otimes_R D' \to D \otimes_R D',\ d \otimes d' \mapsto \phi(d) \otimes \phi'(d')$$

and the decreasing, exhaustive and separated $\frac{1}{n}\mathbb{Z}$-filtration defined by $F^i(D \otimes_R D')_L :=$

$$im\left(\bigoplus_{j \in \frac{1}{n}\mathbb{Z}} (F^j D_L \otimes_{R_L} F^{i-j} D'_L) \to D_L \otimes_{R_L} D'_L \tilde{\to} (D \otimes_R D')_L\right)$$

with respect to the composite of the canonical maps

$$(d_j)_j \mapsto \sum_j d_j \quad \textit{resp.} \quad (r \otimes d) \otimes (r' \otimes d') \mapsto (rr') \otimes (d \otimes d'),$$

the tensor product $D \otimes_R D'$ becomes an object of $\mathbf{MF}^\phi_{L,K,n}$, denoted by $\underline{D} \otimes \underline{D}'$. We call the filtration on this object the tensor product filtration.

3. *Together with $\sigma_0 \otimes K$ and*

$$F^\bullet R_L : F^0 R_L = R_L \supseteq F^{\frac{1}{n}} R_L = \{0\},$$

the R-module R is a left unit and a right unit with respect to $\otimes$, denoted by $\underline{R}$.
An object with only one filtration jump at 0 is said to have trivial filtration. Hence, the R_L-modules D_L together with the filtration associated with an object $\underline{D}$ of $\mathbf{MF}^{\phi}_{L,K,n}$ are modules over the trivially filtered ring R_L.

Proof. Ad 1.: It is clear that $\phi \oplus \phi'$ is $\sigma_0 \otimes K$-linear. Applying η to the identities

$$F^i D_L \oplus F^i D'_L \supseteq F^{i+\frac{1}{n}} D_L \oplus F^{i+\frac{1}{n}} D'_L \quad (i \in \frac{1}{n}\mathbb{Z}),$$

$$\bigcup_{i \in \frac{1}{n}\mathbb{Z}} (F^i D_L \oplus F^i D'_L) = D_L \oplus D'_L$$

and

$$\bigcap_{i \in \frac{1}{n}\mathbb{Z}} (F^i D_L \oplus F^i D'_L) = \{0\}\,,$$

one sees that $F^\bullet(D \oplus D')_L$ has the desired properties.
Ad 2.: The map

$$D \times D' \to D \otimes_R D', \ (d, d') \mapsto \phi(d) \otimes \phi'(d')$$

is R-balanced, so $\phi \otimes \phi'$ is a well-defined $\sigma_0 \otimes K$-linear endomorphism of $D \otimes_R D'$. Denote the canonical isomorphism $D_L \otimes_{R_L} D'_L \tilde{\to} (D \otimes_R D')_L$ by η'. Let $i \in \frac{1}{n}\mathbb{Z}$. Then for all $j \in \frac{1}{n}\mathbb{Z}$, we have inclusions of R_L-modules

$$F^j D_L \otimes_{R_L} F^{i-j} D'_L \subseteq F^{j-\frac{1}{n}} D_L \otimes_{R_L} F^{i-j} D'_L \subseteq \sum_{j' \in \frac{1}{n}\mathbb{Z}} F^{j'-\frac{1}{n}} D_L \otimes_{R_L} F^{i-j'} D'_L$$

where the first is valid by flatness of $F^{i-j} D'_L$ (see 2.1.3, 1.)). Now applying η' to

$$\sum_{j' \in \frac{1}{n}\mathbb{Z}} F^{j'} D_L \otimes_{R_L} F^{i-j'} D'_L \subseteq \sum_{j' \in \frac{1}{n}\mathbb{Z}} F^{j'-\frac{1}{n}} D_L \otimes_{R_L} F^{i-j'} D'_L$$

shows that $F^\bullet(D \otimes D')_L$ is decreasing.
Let $x := \sum_k \eta'(a_k \otimes b_k) \in (D \otimes_R D')_L$ with the $a_k \in D_L, b_k \in D'_L$ and where k runs through a finite index set. Since $F^\bullet D_L$ resp. $F^\bullet D'_L$ are exhaustive, all the a_k are contained in some $F^{j_1} D_L$ and all the b_k are contained in some $F^{j_2} D'_L$. It follows that $x \in F^{\min\{j_1, j_2\}}(D \otimes_R D')_L$. Thus $F^\bullet(D \otimes_R D')_L$ is exhaustive.
Let $\sum_k \eta'(a_k \otimes b_k) =: x \in \bigcap_{i \in \frac{1}{n}\mathbb{Z}} F^i(D \otimes_R D')_L$ where again $a_k \in D_L, b_k \in D'_L$ and where k runs through a minimal finite index set. Such minimal representations exist although they need not be unique. Since $F^\bullet D_L$ is separated, for every such representation there is a filtration index j maximal with the property that all the a_k are contained in $F^j D_L$. Since $F^\bullet D'_L$ is separated and D'_L is finitely generated, there is a sufficiently large index l, again dependent on the representation, such that $F^l D'_L = 0$. Suppose $x \neq 0$. By assumption, x is in particular contained in $F^m(D \otimes_R D')_L = \eta'(\sum_{r \in \frac{1}{n}\mathbb{Z}} F^r D_L \otimes_{R_L} F^{m-r} D'_L)$ for any $m > j + l$. If $r \leq j$ then $m - r \geq m - j > l$ and hence $F^{m-r} D'_L = 0$. If $r > j$ then not all a_k are contained in $F^r D_L$, hence x would have a representation of length smaller than the minimal one, which is a contradiction. Therefore, $F^\bullet(D \otimes_R D')_L$ is separated.

Ad 3.: Let $\underline{D} = (D, \phi, F^\bullet D_L)$ be in $\mathbf{MF}^{\phi}_{L,K,n}$. By 2.,

$$\sigma_0 \otimes \phi : R \otimes_R D \to R \otimes_R D, \; r \otimes d \mapsto \sigma_0(r) \otimes \phi(d)$$

is a well-defined $\sigma_0 \otimes K$-linear map. Then the canonical isomorphisms of R-modules $R \otimes_R D \cong D \cong D \otimes_R R$ induce natural isomorphisms $\underline{R} \otimes \underline{D} \cong \underline{D} \cong \underline{D} \otimes \underline{R}$ in $\mathbf{MF}^{\phi}_{L,K,n}$.

□

By abuse of notation we also write the tensor product filtration of objects in $\mathbf{MF}^{\phi}_{L,K,n}$ as

$$F^i(D \otimes_R D')_L = \sum_j F^j D_L \otimes_{R_L} F^{i-j} D'_L.$$

The content of the next proposition is to show how to consider $\mathbb{Z}$-filtered ϕ-modules as $\frac{1}{n}\mathbb{Z}$-filtered ϕ-modules via the ceiling function

$$\mathbb{R} \to \mathbb{Z}, \; x \mapsto \lceil x \rceil := \text{ the unique integer } s \text{ such that } x \leq s < x + 1.$$

.

Proposition 2.1.5. *Let $\mathbf{MF}^{\phi}_{L,K}$ denote the tensor category of $\mathbb{Z}$-filtered ϕ-modules. Then assigning to an object $\underline{D} = (D, \phi, F^\bullet D_L)$ in $\mathbf{MF}^{\phi}_{L,K}$ the object $I_n(\underline{D}) := (D, \phi, F^{\lceil \bullet \rceil} D_L)$ and to a morphism $f : \underline{D} \to \underline{E}$ in $\mathbf{MF}^{\phi}_{L,K}$ the morphism $I_n(f) : I_n(\underline{D}) \to I_n(\underline{E}), \; d \mapsto f(d)$ in $\mathbf{MF}^{\phi}_{L,K,n}$ defines a fully faithful functor $I_n : \mathbf{MF}^{\phi}_{L,K} \to \mathbf{MF}^{\phi}_{L,K,n}$. This functor is compatible with $\otimes$ and unit objects in the sense of 2.1.4.*

Proof. Let $(D, \phi, F^\bullet D_L)$ be in $\mathbf{MF}^{\phi}_{L,K}$. One easily sees that $F^{\lceil \bullet \rceil} D_L$ gives indeed a decreasing, exhaustive and separated $\frac{1}{n}\mathbb{Z}$-filtration on D_L. Let $i \in \mathbb{Z}$ be a jump of $F^\bullet D_L$. Then

$$F^{\lceil i \rceil} D_L = F^i D_L \supsetneq F^{i+1} D_L = F^{\lceil i + \frac{1}{n} \rceil} D_L,$$

so the jumps of $F^\bullet D_L$ are also jumps of $F^{\lceil \bullet \rceil} D_L$. For the reverse inclusion, let $x \in \frac{1}{n}\mathbb{Z}$ and $F^{\lceil x + \frac{1}{n} \rceil} D_L \subsetneq F^{\lceil x \rceil} D_L$. Then necessarily $\lceil x \rceil < \lceil x + \frac{1}{n} \rceil$ which is equivalent to x being in $\mathbb{Z}$. This implies that the above assignments indeed define a fully faithful functor $I_n : \mathbf{MF}^{\phi}_{L,K} \to \mathbf{MF}^{\phi}_{L,K,n}$ which preserves unit objects.

Let $\underline{D}, \underline{E}$ be in $\mathbf{MF}^{\phi}_{L,K}$ with E being finitely generated. On the one hand, the i-th filtration step of the filtration attached to $I_n(\underline{D} \otimes \underline{E})$ is

$$F^{\lceil i \rceil}(D \otimes_R E)_L = \sum_{j \in \mathbb{Z}} F^j D_L \otimes_{R_L} F^{\lceil i \rceil - j} E_L \tag{2}$$

for any $i \in \frac{1}{n}\mathbb{Z}$.

On the other hand, the i-th filtration step of the filtration attached to $I_n(\underline{D}) \otimes I_n(\underline{E})$ is

$$\sum_{k \in \frac{1}{n}\mathbb{Z}} F^{\lceil k \rceil} D_L \otimes_{R_L} F^{\lceil i - k \rceil} E_L \tag{3}$$

for any $i \in \frac{1}{n}\mathbb{Z}$. Let us justify that the right hand side of (2) and (3) agree for every $i \in \frac{1}{n}\mathbb{Z}$.
Note that $\lceil x+m \rceil = \lceil x \rceil + m$ for any $x \in \mathbb{R}, m \in \mathbb{Z}$. Using this property, it follows that the right hand side of (2) is contained in (3). To see the reverse inclusion, fix $j \in \mathbb{Z}$. For an integer s, the containment $s \in \{(j-1)n + r \mid 1 \leq r \leq n\}$ is equivalent to $\lceil \frac{s}{n} \rceil = j$. In this case the possible values of $\lceil i - \frac{s}{n} \rceil$ are contained in $\{-j + \lceil i \rceil, -j + 1 + \lceil i \rceil\}$ where both values are actually attained because $1 \leq r \leq n$. But then $\lceil \frac{s}{n} \rceil + \lceil i - \frac{s}{n} \rceil$ runs through $\{\lceil i \rceil, \lceil i \rceil + 1\}$. Hence the submodule of $D_L \otimes_{R_L} E_L$ generated by those $F^{\lceil \frac{s}{n} \rceil} D_L \otimes_{R_L} F^{\lceil i - \frac{s}{n} \rceil} E_L$ such that $\lceil \frac{s}{n} \rceil = j$ is contained in $F^j D_L \otimes_{R_L} F^{\lceil i \rceil - j} E_L$. Since this holds true for all $j \in \mathbb{Z}$, we conclude that the functor I_n is compatible with $\otimes$. □

The essential image of I_n consists of those objects $(D, \phi, F^\bullet D_L)$ for which the implication

$$F^x D_L / F^{x+\frac{1}{n}} D_L \neq 0 \Rightarrow x \in \mathbb{Z}$$

holds true whenever $x \in \frac{1}{n}\mathbb{Z}$. Via the functor I_n we consider $\mathbb{Z}$-filtered ϕ-modules as $\frac{1}{n}\mathbb{Z}$-filtered. If no confusion arises, we usually omit the explicit mentioning of I_n from the notation.
Using the same arguments as in the proof, the statement of the proposition is valid for ϕ-modules with Λ-filtration for any subgroup $\Lambda \subseteq \frac{1}{n}\mathbb{Z}$.

Proposition 2.1.6. *Let $\underline{D} := (D, \phi, F^\bullet D_L)$ be a non-zero object in $\mathbf{MF}^{\phi}_{L,K,n}$. Consider the statements:*

1. *The R-module D is free of finite rank.*
2. *The underlying L_0-vector space of D is finite-dimensional.*

Then statement 1. *implies statement* 2. *If, moreover, ϕ is bijective the converse also holds. In particular, under this latter condition, every ϕ-invariant R-submodule is automatically free.*

Proof. The implication "1. ⇒ 2." holds because of the identity $\dim_{L_0}(D) = \dim_{L_0}(R) \cdot \mathrm{rank}_R(D)$.
Let ϕ be bijective and assume statement 2. holds. With the notation of 2.1.3 we have $\phi(D_{\tau_0}) = D_{\tau_0 \sigma_0^{-1}}$ for all τ_0. By finite-dimensionality, there exists $s \in \mathbb{Z}_{\geq 1}$ such that the multiplicity of every isotypical component of D is s. This implies 1. □

Definition 2.1.7. Let $\underline{D} := (D, \phi, F^\bullet D_L)$ be in $\mathbf{MF}^{\phi}_{L,K,n}$, such that D is a free R-module of finite rank and such that ϕ is bijective. Then we call $\underline{D}$ a $\frac{1}{n}\mathbb{Z}$-filtered isocrystal over L with coefficients in K. The map ϕ is usually referred to as the Frobenius of $\underline{D}$. The full additive subcategory of $\mathbf{MF}^{\phi}_{L,K,n}$ of $\frac{1}{n}\mathbb{Z}$-filtered isocrystals over L with coefficients in K is denoted by $\mathbf{FIC}_{L,K,n}$.

Let $\underline{D} = (D, \phi, F^\bullet D_L)$ be in $\mathbf{FIC}_{L,K,n}$ and denote by $\mathrm{Hom}_R(D, R)$ (resp. $\mathrm{Hom}_{R_L}(D_L, R_L)$) the dual R-module of D (resp. the dual R_L-module of D_L). We have a canonical isomorphism of R_L-modules

$$\theta : R_L \otimes_R \mathrm{Hom}_R(D, R) \tilde{\to} \mathrm{Hom}_{R_L}(D_L, R_L),$$
$$a \otimes \psi \mapsto [a' \otimes d \mapsto \underbrace{(1 \otimes \psi(d))}_{\in R_L = L \otimes_{L_0} R} aa'],$$

where $a' \in R_L$ and $d \in D$. Then $\underline{D}$ gives rise to the object $\underline{D}^\vee := (D^\vee, \phi^\vee, F^\bullet(D^\vee)_L)$ whose underlying R-module is $D^\vee := \operatorname{Hom}_R(D, R)$. It is finitely generated free with rank equal to that of D. By definition,

$$\phi^\vee : D^\vee \to D^\vee,\ \xi \mapsto \left[d \mapsto (\sigma_0 \otimes K)(\xi(\phi^{-1}(d)))\right]$$

and

$$F^i(D^\vee)_L := \theta^{-1}(\left\{\xi \in \operatorname{Hom}_{R_L}(D_L, R_L) \,|\, \xi(F^{\frac{1}{n}-i}D_L) = 0\right\}),$$

where we view R_L as a trivially filtered R_L-module. Then $\phi^\vee(\xi)$ is clearly additive and it is R-linear because ϕ^{-1} is $\sigma_0^{-1} \otimes K$-linear. Hence $\phi^\vee$ is well-defined.
Let $r \in R$, $\xi_1, \xi_2 \in D^\vee$. Then we have for all $d \in D$

$$\begin{aligned}
\phi^\vee(r \cdot \xi_1)(d) &= ((\sigma_0 \otimes K) \circ (r \cdot \xi_1) \circ \phi^{-1})(d) \\
&= (\sigma_0 \otimes K)(r \underbrace{\xi_1(\phi^{-1}(d))}_{\in R}) \\
&= (\sigma_0 \otimes K)(r) \cdot \phi^\vee(\xi_1)(d) \\
&= \underbrace{((\sigma_0 \otimes K)(r) \cdot \phi^\vee(\xi_1))}_{\in D^\vee}(d),
\end{aligned}$$

and

$$\begin{aligned}
\phi^\vee(\xi_1 + \xi_2)(d) &= ((\sigma_0 \otimes K) \circ (\xi_1 + \xi_2) \circ \phi^{-1})(d) \\
&= ((\sigma_0 \otimes K) \circ \xi_1(\phi^{-1}(d)))(d) + ((\sigma_0 \otimes K)(\xi_2(\phi^{-1})))(d) \\
&= \phi^\vee(\xi_1)(d) + \phi^\vee(\xi_2)(d),
\end{aligned}$$

which shows that $\phi^\vee$ is $\sigma_0 \otimes K$-linear.
It is clear that $F^\bullet(D^\vee)_L$ is a family of submodules of $(D^\vee)_L$. To show that it is decreasing, let $\theta^{-1}(\xi) \in F^i(D^\vee)_L$. Then ξ vanishes on $F^{\frac{1}{n}-i}D_L$ hence on F^tD_L for all $t \in \frac{1}{n}\mathbb{Z}$ such that $t \geq \frac{1}{n} - i$. By definition this means that $\theta^{-1}(\xi) \in F^{\frac{1}{n}-t}(D^\vee)_L$. Since $\frac{1}{n} - t \leq i$, $F^\bullet(D^\vee)_L$ is decreasing. It is exhaustive because $F^\bullet D_L$ is separated and it is separated because $F^\bullet D_L$ is exhaustive. The object thus obtained is called the dual of $\underline{D}$. Hence we have proven part of the following result.

Proposition 2.1.8. *The category* $\mathbf{FIC}_{L,K,n}$ *is closed under the formation of tensor products and duals in the sense just defined.*

Proof. Concerning $\otimes$, note that the tensor product $D_1 \otimes_R D_2$ of two free R-modules of finite rank is again finite free over R. The tensor product filtration is decreasing, exhaustive and separated, as we have seen before. □

Let $\mathbf{FIC}_{L,K}$ denote the category of $\mathbb{Z}$-filtered isocrystals over L with coefficients in K. The following statement is proved in the same way as 2.1.5.

Proposition 2.1.9. *The functor* I_n *from proposition 2.1.5 restricts to a functor* $\mathbf{FIC}_{L,K} \to \mathbf{FIC}_{L,K,n}$ *with the same properties.*

In [Fo2] to (the isomorphism classes of) objects $\underline{D}$ of $\mathbf{FIC}_L$ (cf. section 1.3) is associated a pair of integers $(t_N(\underline{D}), t_H(\underline{D}))$ in order to define (weakly) admissible objects in $\mathbf{FIC}_L$. The notion of weak admissibility relates the Frobenius and the filtration of such a $\underline{D}$ and can numerically be expressed in terms of the functions t_N and t_H. In this subsection we introduce the corresponding invariants and study the concept of weak admissibility for objects in $\mathbf{FIC}_{L,K,n}$.
Let $\underline{D} = (D, \phi, F^\bullet D_L)$ be a non-zero object in $\mathbf{FIC}_{L,K,n}$ and set

$$h := \dim_{L_0}(D) = [K : \mathbb{Q}_p] \cdot \mathrm{rk}_R(D).$$

Then ϕ, as a σ_0-linear automorphism of D as an L_0-vector space via scalar restriction along the ring map $L_0 \to R$, $l \mapsto l \otimes 1$, induces on the top exterior power $\bigwedge^h D$ a bijective σ_0-linear map $\bigwedge^h \phi$. Since $\bigwedge^h D$ is one-dimensional, for every two elements $x, y \in \bigwedge^h D \setminus \{0\}$, there exist $a, b, c \in L_0^\times$ with $y = ax$ on the one hand and $(\bigwedge^h \phi)(x) = bx$ resp. $(\bigwedge^h \phi)(y) = cy$ on the other hand. Hence we obtain

$$cax = cy = (\bigwedge^h \phi)(y) = (\bigwedge^h \phi)(ax) = \sigma_0(a)bx$$

which implies $c = \sigma_0(a)ba^{-1}$. As both $v_p \circ \sigma_0$ and v_p extend the p-adic valuation on $\mathbb{Q}_p$ we have $v_p \circ \sigma_0 = v_p$ as functions from $L_0^\times$ to $\mathbb{Z}$. Therefore the following definition makes sense.

Definition 2.2.1. Let $\underline{D}$, x and b be as in the previous discussion. We set

$$t_N(\underline{D}) := v_p(b) \in \mathbb{Z}.$$

This is the Newton number of $\underline{D}$.

Remark 2.2.2.

1. The scalar b in the definition is equal to the determinant of the matrix representing ϕ after choosing an L_0-basis of D. The above considerations then imply that the definition of $t_N(\underline{D})$ is independent of this choice.

2. If there is no risk of confusion, we frequently denote the Newton number of $\underline{D}$ also as $t_N(D, \phi)$ or simply as $t_N(D)$.

3. Using the behavior of the determinant of an endomorphism under scalar restriction one checks that we have the equalities

$$t_N(\underline{D}) = \frac{1}{[L_0 : \mathbb{Q}_p]} v_p(\det_{L_0}(\phi^{[L_0:\mathbb{Q}_p]})) = [K : L_0]\, v_p(\det_K(\phi)) \qquad (4)$$

   where $\det_{L_0}$ resp. $\det_K$ means the determinant of an L_0-linear resp. of a K-linear map (cf. also [BouA$_1$, III, §9, no. 4, Proposition 6] and [SchT, discussion after Proposition 5.2]).

Keep the notations as before. Via scalar restriction, D_L is an h-dimensional $\frac{1}{n}\mathbb{Z}$-filtered L-vector space.

Definition 2.2.3. We set

$$t_H(\underline{D}) := \sum_{i \in \frac{1}{n}\mathbb{Z}} i \dim_L(\mathrm{gr}^i D_L) \in \frac{1}{n}\mathbb{Z}.$$

This is the Hodge number of $\underline{D}$.

Remark 2.2.4.

1. If there is no risk of confusion, we frequently denote the Hodge number of $\underline{D}$ also as $t_H(D_L, F^\bullet)$ or simply as $t_H(D)$.

2. For the Hodge number as defined in 2.2.3 we have

$$\begin{aligned} t_H(\underline{D}) &= [K : L] \sum_{i \in \frac{1}{n}\mathbb{Z}} i \dim_K(\mathrm{gr}^i D_L) \\ &= [K : L] \sum_{\tau} \sum_{i \in \frac{1}{n}\mathbb{Z}} i \dim_K \left[(F^i D_L \cap D_{L,\tau})/(F^{i+\frac{1}{n}} D_L \cap D_{L,\tau})\right] \end{aligned}$$

where the second equality uses semisimplicity of the ring $L \otimes_{\mathbb{Q}_p} K$ as explained in 2.1.3.

Definition 2.2.5. An object $\underline{D} = (D, \phi, F^\bullet D_L)$ in $\mathbf{FIC}_{L,K,n}$ is called weakly admissible if

$$t_H(\underline{D}) = t_N(\underline{D})$$

and if for every ϕ-invariant L_0-subspace $D' \subseteq D$ with induced filtration $F^\bullet D_L \cap D'_L$ we have

$$t_H((D'_L, F^\bullet D_L \cap D'_L)) \leq t_N((D', \phi|_{D'})).$$

Denote the full subcategory of weakly admissible objects in $\mathbf{FIC}_{L,K,n}$ by $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$. Also let $\mathbf{FIC}^{\mathrm{adm}}_{L,K}$ denote the category of weakly admissible $\mathbb{Z}$-filtered isocrystals over L with coefficients in K.

Remark 2.2.6.

1. We set $t_N(\{0\}) := 0$ and $t_H(\{0\}) := 0$. With this definition the zero object is weakly admissible.

2. In the category consisting of triples $(D, \phi, F^\bullet D_L)$ where D is a finite-dimensional L_0-vector space, ϕ is a σ_0-linear automorphism of D and $F^\bullet D_L$ is a decreasing, exhaustive and separated $\frac{1}{n}\mathbb{Z}$-filtration of D_L by L-subspaces, weakly admissible objects are defined as in 2.2.5. Then, by [Men, Theorem 2.2 and Satz 5.5], the category consisting of the weakly-admissible objects in this specific category is abelian and closed under the formation of tensor products and duals. From this one can conclude that the same is true for $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$. In particular the existence of kernels and cokernels of morphisms in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$ follows from Frobenius-invariance and the implication "2. $\Rightarrow$ 1." in 2.1.6.

3. Let $\underline{D}$ be in $\mathbf{FIC}_{L,K,n}$. Using the same arguments as in the proof of [BrMe, Prop. 3.1.1.5], one sees that it suffices to check the condition concerning weak admissibility on ϕ-invariant R-submodules of the underlying R-module of $\underline{D}$.

4. The functor I_n from proposition 2.1.9 restricts further to a functor $\mathbf{FIC}^{\mathrm{adm}}_{L,K} \to \mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$.

Assumption 2.2.7. *We will assume from now on that*

$$(K^\times)^n \text{ contains } \mathbb{Q}_p^\times \text{ and that } K \text{ contains all } n\text{-th roots of unity.}$$

That such a K of finite degree over $\mathbb{Q}_p$ with these properties exists follows essentially from the finiteness of the group index $[\mathbb{Q}_p^\times : (\mathbb{Q}_p^\times)^n]$. Assumption 2.2.7 implies that the polynomial $X^n - a \in K[X]$ splits completely for every $a \in \mathbb{Q}_p^\times$. Moreover, using this property,

we fix once and for all a root of $X^n - p \in K[X]$ *and denote it by* $p^{\frac{1}{n}}$.

The multiplicative inverse of $p^{\frac{1}{n}}$ is denoted by $p^{-\frac{1}{n}}$. Next we present two examples in which we apply the notions just introduced.

Example 2.2.8. The unit object $\underline{R}$ in $\mathbf{FIC}_{L,K,n}$ is weakly admissible since $\det_K(\sigma_0 \otimes K) \in \{-1, 1\}$.

Slightly less trivial is the following example which is inspired by the discussion in [BrSch, Section 7, after Lemma 7.3] for the case $n = 2$. It describes a "twist of the unit object from the previous example by $\frac{1}{n}$". Here and afterwards we sometimes denote the filtration associated with an object $\underline{D}$ of $\mathbf{FIC}_{L,K,n}$ by $F^\bullet \underline{D}$ if there is no risk of confusion.

Example 2.2.9. Define the filtered isocrystal $\underline{K}_n$ having as underlying R-module R, its Frobenius is $\sigma_0 \otimes p^{-\frac{1}{n}}$ and the filtration $F^\bullet \underline{K}_n$ is given as

$$F^{-\frac{1}{n}} \underline{K}_n := R_L \supsetneq F^0 \underline{K}_n := \{0\}\,.$$

Then $\underline{K}_n$ is weakly admissible in $\mathbf{FIC}_{L,K,n}$: via the isomorphism described in 2.1.3, we transport the action of the Frobenius on R to $\bigoplus_{\tau_0} K_{\tau_0}$ where it translates to a permutation of the components K_{τ_0} followed by a multiplication with $p^{-\frac{1}{n}} \in K$. Hence there are no proper Frobenius-invariant R-submodules. To conclude that $\underline{K}_n$ is weakly admissible, we therefore only need to compute $t_N(\underline{K}_n)$ and $t_H(\underline{K}_n)$. Setting $h := [K : \mathbb{Q}_p]$ we get, using (4),

$$\begin{aligned} t_N(\underline{K}_n) &= [K : L_0]\, v_p(\epsilon \cdot \det_K(R \to R, x \mapsto p^{-\frac{1}{n}} x)) \\ &= [K : L_0]\, v_p((p^{-\frac{1}{n}})^f) \\ &= -\frac{h}{n} \\ &= \sum_{j \in \frac{1}{n}\mathbb{Z}} j \, \dim_L(\mathrm{gr}^j R_L) = t_H(\underline{K}_n) \end{aligned}$$

where $\epsilon \in \{-1, 1\}$.
The object $\underline{K}_n^{\otimes n}$, the n-fold tensor product of $\underline{K}_n$ with itself, might not be well-defined a priori. But by associativity of the tensor product all the objects that arise are isomorphic via the canonical isomorphisms $R^{\otimes m} \xrightarrow{\sim} R$ (valid for any $m \geq 1$) and the fact that $F^\bullet \underline{K}_n$ has only one jump.

We *claim* that the canonical R-module isomorphism $R^{\otimes n} \tilde{\to} R$ is an isomorphism

$$\underline{K}_n^{\otimes n} \tilde{\to} \underline{K}$$

in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$, where $\underline{K}$ has underlying R-module R, its Frobenius is $\sigma_0 \otimes p^{-1}$ and the filtration $F^\bullet \underline{K}$ is given as

$$F^{-1}\underline{K} := R_L \supseteq F^{-\frac{n-1}{n}}\underline{K} := \{0\}\,.$$

Proof of claim. Note that $\underline{K}$ lies in the essential image of I_n: this can be seen by applying this functor to the object in $\mathbf{FIC}^{\mathrm{adm}}_{L,K}$ defined similarly as $\underline{K}$ with minus first filtration step being equal to R_L and with zeroth filtration step being the zero module. It follows that $\underline{K}$ lies in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$.
The underlying R-module of $\underline{K}_n^{\otimes n}$ is free of rank 1. The Frobenius of this object is

$$\otimes_{i=1}^n (l_i \otimes k_i) \mapsto \otimes_{i=1}^n (\sigma_0(l_i) \otimes p^{-\frac{1}{n}} k_i)$$

and it is compatible with $R^{\otimes n} \tilde{\to} R$. As for the filtration, let us first remark that for any $\underline{D}$ in $\mathbf{MF}^{\phi}_{L,K,n}$, the isomorphism $R^{\otimes n} \otimes_R D \tilde{\to} D$ induces an isomorphism of R_L-modules between $F^j(\underline{K}_n^{\otimes n-1} \otimes \underline{D})$ and $F^{j+\frac{n-1}{n}}\underline{D}$ for all $j \in \frac{1}{n}\mathbb{Z}$. Now the claim follows by setting $\underline{D} = \underline{K}_n$, by computing filtration steps for $j = -1$ resp. $j = -\frac{n-1}{n}$ and using the definition of the filtration associated with $\underline{K}$. □

Different choices of roots of $X^n - p$ yield the objects

$$\underline{K}_n^{\zeta} := (R, \sigma_0 \otimes \zeta p^{-\frac{1}{n}}, F^{-\frac{1}{n}} R_L := R_L \supseteq F^0 R_L := \{0\})$$

and isomorphisms

$$\underline{K}_n^{\zeta} \cong \underline{K}_n \otimes I_n((R, \sigma_0 \otimes \zeta, F^0 R_L := R_L \supseteq F^1 R_L := \{0\}))$$

in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$ where $\zeta \in \mu_n(K)$. The pairwise non-isomorphic objects $\underline{K}_n^{\zeta}$ are "n-th tensor roots" of $\underline{K}$ in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$, in the sense that $(\underline{K}_n^{\zeta})^{\otimes n}$ and $\underline{K}$ are isomorphic for all $\zeta \in \mu_n(K)$.

The previous example motivates the following definition.

Definition 2.2.10. Let $\underline{D}$ be in $\mathbf{MF}^{\phi}_{L,K,n}$.

1. Let $r \in \mathbb{Z}_{\geq 0}$. Define the twist of $\underline{D}$ by $\frac{r}{n}$ as $\underline{D}\left\langle \frac{r}{n} \right\rangle := \underline{K}_n^{\otimes r} \otimes \underline{D}$, where we set $\underline{M}^{\otimes 0} := \underline{R}$ for any $\underline{M}$ in $\mathbf{MF}^{\phi}_{L,K,n}$ (cf. 2.1.4).
2. Let $r \in \mathbb{Z}_{<0}$. Define the twist of $\underline{D}$ by $\frac{r}{n}$ as $\underline{D}\left\langle \frac{r}{n} \right\rangle := \left(\underline{K}_n^{\vee}\right)^{\otimes(-r)} \otimes \underline{D}$.

In particular, $\underline{R}\left\langle \frac{1}{n} \right\rangle \cong \underline{K}_n$ and, in view of 2.2.9, we find

$$\underline{R}\left\langle 1 \right\rangle = \underline{R}\left\langle \tfrac{n}{n} \right\rangle \overset{2.2.10}{=} \underline{K}_n^{\otimes n} \otimes \underline{R} \cong \underline{K}.$$

A simple computation using the properties of the objects $\underline{K}_n$ and $\underline{K}_n^{\vee}$ allows the following description of twists.

Lemma 2.2.11. *Let $\underline{D} = (D, \phi, F^\bullet D_L)$ be in $\mathbf{MF}^{\phi}_{L,K,n}$. Then we have for its twists an isomorphism*

$$\underline{D}\left\langle \tfrac{r}{n} \right\rangle \cong (D, (1 \otimes (p^{\frac{1}{n}})^{-r})\phi, (F')^\bullet D_L)$$

where $(F')^i D_L := F^{i+\frac{r}{n}} D_L$ for $r \in \mathbb{Z}$. Moreover, for all objects $\underline{D}, \underline{E}$ in $\mathbf{MF}^{\phi}_{L,K,n}$ and all $i, j \in \frac{1}{n}\mathbb{Z}$, there are canonical isomorphisms

$$\underline{D}\langle i \rangle \otimes \underline{E}\langle j \rangle \cong (\underline{D} \otimes \underline{E})\langle i + j \rangle,$$

in particular

$$(\underline{D}\langle i \rangle)\langle j \rangle \cong \underline{D}\langle i + j \rangle.$$

Remark 2.2.12.

1. In $\mathbf{MF}^{\phi}_{L,K,n}$, the objects $\underline{K}_n$ from 2.2.9 and $\underline{K}_n^{\vee} = \underline{R}\left\langle -\frac{1}{n} \right\rangle$ are mutually inverse with respect to the tensor product, by which we mean that there is an isomorphism $\underline{K}_n \otimes \underline{K}_n^{\vee} \cong \underline{R}$.

2. Let $\underline{D}$ be in $\mathbf{FIC}_{L,K,n}$ and let $\underline{D}'$ denote any twist of $\underline{D}$. On the one hand, the Frobenius maps of both objects differ by composition with a K-linear automorphism. On the other hand, the filtrations associated with both objects determine each other by a shift of filtration indices of the form $i \mapsto i + m$, $m \in \frac{1}{n}\mathbb{Z}$. It follows that Frobenius-invariant submodules of $\underline{D}$ with associated induced filtration are in bijective correspondence with Frobenius-invariant submodules of $\underline{D}'$ with associated induced filtration. The effect of twisting by some $j \in \frac{1}{n}\mathbb{Z}$ on the Newton and the Hodge numbers of an object $\underline{D} = (D, \phi, F^\bullet D_L)$ in $\mathbf{FIC}_{L,K,n}$ is

   $$t_N(\underline{D}\langle j \rangle) = t_N(\underline{D}) - \mathrm{rank}_R(D)[K : \mathbb{Q}_p]j$$

   and similarly

   $$t_H(\underline{D}\langle j \rangle) = t_H(\underline{D}) - \mathrm{rank}_R(D)[K : \mathbb{Q}_p]j.$$

   Hence we find that $\underline{D}$ is weakly admissible if and only if $\underline{D}\langle j \rangle$ is weakly admissible for any $j \in \frac{1}{n}\mathbb{Z}$ and that "twisting by $j \in \frac{1}{n}\mathbb{Z}$" induces a faithfully exact functor

   $$(-)\langle j \rangle : \mathbf{FIC}^{\mathrm{wa}}_{L,K,n} \to \mathbf{FIC}^{\mathrm{wa}}_{L,K,n},\ \underline{D} \mapsto \underline{D}\langle j \rangle$$

   in the following sense: let $\underline{\Delta}', \underline{\Delta}, \underline{\Delta}''$ be in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$. Then a sequence of morphisms in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$

   $$0 \to \underline{\Delta}' \stackrel{f}{\to} \underline{\Delta} \stackrel{g}{\to} \underline{\Delta}'' \to 0$$

   is exact if and only if the associated sequence

   $$0 \to \underline{\Delta}'\langle j \rangle \stackrel{f\langle j \rangle}{\to} \underline{\Delta}\langle j \rangle \stackrel{g\langle j \rangle}{\to} \underline{\Delta}''\langle j \rangle \to 0.$$

   is exact in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$ for all $j \in \frac{1}{n}\mathbb{Z}$.

For later purposes we introduce a specific subcategory of the abelian category

$\mathbf{FIC}^{\text{wa}}_{L,K,n}$.

Definition 2.2.13. Let $\underline{K}_n$ be as in 2.2.9. Define $\mathbf{FIC}^{\text{wa}}_{L,K,(n)}$:= the full subcategory of $\mathbf{FIC}^{\text{wa}}_{L,K,n}$ such that

- objects of $\mathbf{FIC}^{\text{adm}}_{L,K}$ are objects of $\mathbf{FIC}^{\text{wa}}_{L,K,(n)}$,
- all objects of $\mathbf{FIC}^{\text{wa}}_{L,K,n}$ isomorphic to $\underline{K}_n$ are objects of $\mathbf{FIC}^{\text{wa}}_{L,K,(n)}$,
- it is closed under (any finite combination of) the formation of $\oplus$, $\otimes$, duals and subquotients (:= quotient objects of subobjects) and
- whenever there is an isomophism $\underline{X} \to \underline{A}$ in $\mathbf{FIC}^{\text{wa}}_{L,K,n}$, $\underline{A}$ an object in $\mathbf{FIC}^{\text{wa}}_{L,K,(n)}$, then $\underline{X}$ is in $\mathbf{FIC}^{\text{wa}}_{L,K,(n)}$.

Moreover, we require $\mathbf{FIC}^{\text{wa}}_{L,K,(n)}$ to be minimal with these properties.

In the definition, by a subobject of $\underline{D}$ in $\mathbf{FIC}^{\text{wa}}_{L,K,n}$ we mean a Frobenius-invariant R-submodule of the underlying R-module D' of $\underline{D}$ together with its induced filtration such that the object induced by these data is again weakly admissible (i.e. $t_N(D') = t_H(D')$).

Proposition 2.2.14. *Let $\underline{D}$ be an object of $\mathbf{FIC}^{\text{wa}}_{L,K,(n)}$. Then there are objects $\underline{D}_0, \ldots, \underline{D}_{n-1}$ in $\mathbf{FIC}^{\text{adm}}_{L,K}$ and a decomposition, unique up to isomorphism,*

$$\underline{D} = \bigoplus_{i=0}^{n-1} \underline{D}_i \left\langle \tfrac{i}{n} \right\rangle .$$

Proof. Denote by $\mathfrak{T}_n$ the full subcategory of $\mathbf{FIC}^{\text{wa}}_{L,K,n}$ whose objects are those of $\mathbf{FIC}^{\text{wa}}_{L,K,n}$ which are isomorphic to one of the form

$$P(\underline{\Delta}_0, \ldots, \underline{\Delta}_{n-1}, \underline{K}_n)$$

Here P is an element of $\mathbb{Z}[X_1, \ldots, X_{n+1}]$ with coefficients ≥ 0 and the $\underline{\Delta}_i$ are objects in $\mathbf{FIC}^{\text{adm}}_{L,K}$ for $i = 0, \ldots, n-1$. Multiplication is to be interpreted as $\otimes$ and addition as $\oplus$. Then $\mathfrak{T}_n$ is a subcategory of $\mathbf{FIC}^{\text{wa}}_{L,K,(n)}$ since, by definition, the latter is closed under formation of objects of the kind just described. One checks that $\mathfrak{T}_n$ satisfies the first three properties of 2.2.13, so $\mathbf{FIC}^{\text{wa}}_{L,K,(n)}$ is a full subcategory of $\mathfrak{T}_n$, hence both categories coincide.
Note that $\underline{K} \cong \underline{K}_n^{\otimes n}$ is an object in $\mathbf{FIC}^{\text{adm}}_{L,K}$ (Example 2.2.9) and $\underline{K}_n^{\vee}$ is isomorphic with $\underline{K}_n^{\otimes n-1} \otimes \underline{K}^{\vee}$so after reordering according to tensor powers of $\underline{K}_n$, one sees that objects in $\mathbf{FIC}^{\text{wa}}_{L,K,(n)}$ are isomorphic to those from the statement. $\square$

Note that all objects $\underline{K}_n^{\zeta}$ from 2.2.9 lie in $\mathbf{FIC}^{\text{wa}}_{L,K,(n)}$ where $\zeta \in \mu_n(K)$.

Remark 2.2.15. Using the properties of twists with respect to $\otimes$ from 2.2.11, the tensor product of two objects $\underline{D} = \oplus_{i=0}^{n-1} \underline{D}_i \left\langle \frac{i}{n} \right\rangle$ and $\underline{E} = \oplus_{i=0}^{n-1} \underline{E}_i \left\langle \frac{i}{n} \right\rangle$ in

$\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$ may be computed as follows:

$$\begin{aligned}\underline{D}\otimes\underline{E} &\cong \bigoplus_{i,j}(\underline{D}_i\langle\tfrac{i}{n}\rangle\otimes\underline{E}_j\langle\tfrac{j}{n}\rangle)\\ &\cong \bigoplus_{i,j}(\underline{D}_i\otimes\underline{E}_j)\langle\tfrac{i+j}{n}\rangle\\ &\cong \bigoplus_{k=0}^{n-1}\bigoplus_{i+j\in k+n\mathbb{Z}}((\underline{D}_i\otimes\underline{E}_j)\langle\lfloor\tfrac{i+j}{n}\rfloor\rangle)\langle\tfrac{k}{n}\rangle.\end{aligned}$$

Here for any real number x, the expression $\lfloor x\rfloor$ denotes the largest integer y such that $x-1<y\le x$.
We will need this description of the tensor product in section 4.

Corollary 2.2.16. *Let $\underline{D}=\oplus_{r=0}^{n-1}\underline{D}_r\left\langle\frac{r}{n}\right\rangle$ be in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$. For $r=0,\ldots,n-1$, let $\{j_{r,1},\ldots,j_{r,k}\}$ denote the set of those jumps of $F^\bullet\underline{D}$ which are contained in $\frac{n-r}{n}+\mathbb{Z}$. Then this set coincides with the set of jumps of $F^\bullet\underline{D}_r\left\langle\frac{r}{n}\right\rangle$. With respect to associated graded pieces of the respective filtrations we have the equalities*

$$\mathrm{gr}^{j_{r,l}}\underline{D}=\mathrm{gr}^{j_{r,l}}\underline{D}_r\left\langle\tfrac{r}{n}\right\rangle$$

for all $l=1,\ldots,k$.

Proof. Both statements follow from the description of twists in 2.2.11 combined with the statement of 2.2.14. □

We conclude this section by discussing an example which shows that there exist objects in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$ that are not objects in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$ in general.

Example 2.2.17 (Schneider)**.** Let $L=\mathbb{Q}_p$ and consider the category $\mathbf{FIC}_{\mathbb{Q}_p,K,n}$. We first observe that an object $\underline{D}$ in $\mathbf{FIC}^{\mathrm{wa}}_{\mathbb{Q}_p,K,n}$ which has underlying onedimensional K-vector space must necessarily belong to $\mathbf{FIC}^{\mathrm{wa}}_{\mathbb{Q}_p,K,(n)}$: indeed, we have

$$t_H(\underline{D})=j[K:\mathbb{Q}_p]=t_N(\underline{D})\in\mathbb{Z}$$

where $j\in\frac{1}{n}\mathbb{Z}$ is the unique jump in the filtration associated with $\underline{D}$. By 2.2.11 we have $\underline{D}\cong(\underline{D}\langle-\lceil j\rceil+j\rangle)\langle\lceil j\rceil-j\rangle$. Note that $\underline{D}\langle-\lceil j\rceil+j\rangle$ is in $\mathbf{FIC}^{\mathrm{adm}}_{\mathbb{Q}_p,K}$ because the unique jump of the associated filtration of this object is $\lceil j\rceil$. Thus $\underline{D}$ lies in $\mathbf{FIC}^{\mathrm{wa}}_{\mathbb{Q}_p,K,(n)}$ by 2.2.14.
Additionally let K be such that it does not contain a square root of $p^{\frac{1}{n}}$. Then $K':=K[X]/(X^2-p^{\frac{1}{n}})K[X]$ is a field obtained by adjoining to K both square roots of $p^{\frac{1}{n}}$. For any onedimensional K-subspace $W\subset K'$ we define $\underline{K}'_W:=(K',x\mapsto\overline{X}x,F^\bullet K')$ in $\mathbf{FIC}_{\mathbb{Q}_p,K,n}$ where $\overline{X}:=$ residue class of X in K' and

$$F^jK':=\begin{cases}K' & j\le 0\\ W & j=\frac{1}{n}\\ 0 & j>\frac{1}{n}.\end{cases}$$

The matrix of the Frobenius with respect to the K-basis $\{1,\overline{X}\}$ of K' is

$$\begin{pmatrix}0 & p^{\frac{1}{n}}\\ 1 & 0\end{pmatrix},$$

having determinant $-p^{\frac{1}{n}}$ with respect to K. Its characteristic polynomial does not split into linear factors over K by assumption, whence there are no proper Frobenius-invariant K-subspaces of K'. We compute

$$t_N(\underline{K}'_W) = [K:\mathbb{Q}_p]\, v_p(-p^{\frac{1}{n}}) = \frac{[K:\mathbb{Q}_p]}{n}.$$

The only summand contributing to the Hodge number of $\underline{K}'_W$ is $\frac{1}{n}\dim_{\mathbb{Q}_p}(\mathrm{gr}^{\frac{1}{n}} K') = \frac{1}{n}\,[K:\mathbb{Q}_p]$, so

$$t_H(\underline{K}'_W) = \frac{[K:\mathbb{Q}_p]}{n}.$$

Hence $\underline{K}'_W$ is weakly admissible for any W. It cannot, however, be (isomorphic to) an object in $\mathbf{FIC}^{\mathrm{wa}}_{\mathbb{Q}_p,K,(n)}$. If it were, $\underline{K}'_W$ would necessarily decompose as in proposition 2.2.14 into two non-zero direct summands corresponding to the jumps $0 \in \mathbb{Z}$ and $\frac{1}{n} \in \frac{1}{n} + \mathbb{Z}$ of the filtration associated with it (cf. also 2.2.16). This would imply the existence of proper Frobenius-invariant K-subspaces of K'.

Similarly, the contradiction obtained by intersecting one of the direct summands of a finite direct sum $(\underline{K}'_W)^{\oplus m}$ ($m \in \mathbb{Z}_{\geq 1}$) with the Frobenius-invariant K-vector space underlying $\underline{D}_{n-1}\langle \frac{n-1}{n} \rangle$ shows that $(\underline{K}'_W)^{\oplus m}$ does not lie in $\mathbf{FIC}^{\mathrm{wa}}_{\mathbb{Q}_p,K,(n)}$.

3 Crystalline representations

We keep the notations and conventions of the previous section. In particular, we remain in the situation of 2.2.7.
As explained in the introduction, there is an equivalence between the abelian categories $\mathbf{Rep}^{\mathrm{cris}}(G_L)$ consisting of crystalline $\mathbb{Q}_p$-linear representations of $\mathrm{Gal}(\overline{\mathbb{Q}}_p|L)$ and $\mathbf{FIC}_L^{\mathrm{wa}}$ consisting of weakly admissible $\mathbb{Z}$-filtered isocrystals over L. In order to establish a result in this direction with respect to the categories introduced in the previous section, we now present corresponding necessary constructions on the representation-theoretic side.

3.1 The category $\mathbf{Rep}_K(G_{L,(n)})$

For the construction of the group $G_{L,(n)}$, we need a slight modification of the p-adic cyclotomic character χ_p.

Definition 3.1.1. Let $\mathbb{Q}_p \subseteq E$ be a finite field extension contained in $\overline{\mathbb{Q}}_p$. Denote by $\varepsilon_E : G_E := \mathrm{Gal}(\overline{\mathbb{Q}}_p|E) \to K^\times$ the continuous character obtained by composing the restriction of χ_p to G_E with the inclusions $\mathbb{Z}_p^\times \subseteq \mathbb{Q}_p^\times \subseteq K^\times$, where G_E has the profinite topology and $K^\times$ has the induced topology of the p-adic topology on K.

With respect to the following definition, we remark that in the category of topological groups all fiber products exist.

Definition 3.1.2. Let $\mathbb{Q}_p \subseteq E$ be a finite field extension contained in $\overline{\mathbb{Q}}_p$. Denote $(G_{E,(n)}, \varepsilon_{E,n} : G_{E,(n)} \to K^\times, \delta_{E,n} : G_{E,(n)} \to G_E)$ the fiber product of

$$\begin{array}{ccc} & & G_E \\ & & \downarrow{\scriptstyle \varepsilon_E} \\ K^\times & \xrightarrow[\;-^n\;]{} & K^\times \end{array}$$

with respect to ε_E and

$$-^n : K^\times \to K^\times,\ \lambda \mapsto \lambda^n.$$

In particular we obtain a continuous character $\varepsilon_{E,n}$.

Lemma 3.1.3. *Let $\mathbb{Q}_p \subseteq E$ be a finite field extension contained in $\overline{\mathbb{Q}}_p$ and let $G_{E,(n)}$ be the fiber product as in the previous definition. Then the morphism $\delta_{E,n}$ is surjective. Moreover, the subgroup $H_n := \ker(\delta_{E,n})$ is central in $G_{E,(n)}$ and $\varepsilon_{E,n}$ induces an isomorphism between H_n and $\mu_n(K)$.*

Proof. We identify the group $G_{E,(n)}$ with the subgroup of the direct product group $K^\times \times G_E$ consisting of the pairs (λ, σ) such that $\lambda^n = \varepsilon_E(\sigma)$. By 2.2.7 we have $\varepsilon_E(G_E) \subseteq (K^\times)^n$. Let $\sigma \in G_E$ and $\lambda \in K^\times$ such that $\lambda^n = \varepsilon_E(\sigma)$. Hence $(\lambda, \sigma) \in G_{E,(n)}$ and $\delta_{E,n}((\lambda,\sigma)) = \sigma$. This shows the first assertion.
The subgroup H_n is identified with $\{(\lambda, 1_{G_E}) \in K^\times \times G_E \mid \lambda^n = 1\}$. This simultaneously shows the second and third assertions. $\square$

Remark 3.1.4. Let $\mathbb{Q}_p \subseteq E$ be a finite field extension contained in $\overline{\mathbb{Q}}_p$. The lemma shows that we get a central extension

$$1 \to H_n \overset{\subseteq}{\to} G_{E,(n)} \overset{\delta_{E,n}}{\to} G_E \to 1 \qquad (\mathrm{Ext}_{E,n})$$

of G_E by a group which is isomorphic with $\mu_n(K)$.

1. In an appendix to this subsection we give a criterion concerning a splitting of $(\mathrm{Ext}_{E,n})$ in dependence of the degree of the field extension $E|\mathbb{Q}_p$.
2. Precomposition with $\delta_{E,n}$ induces an injective group homomorphism

$$\{\text{group homomorphisms } G_E \to K^\times\}$$
$$\to \{\text{group homomorphisms } G_{E,(n)} \to K^\times\}$$

under which ε_E is mapped to $\varepsilon_{E,n}^n : h \mapsto \varepsilon_{E,n}(h)^n$. Due to this fact, we may refer to $\varepsilon_{E,n}$ as an n-th root of ε_E.

From now on we focus on the case $E = L$ and in particular on the category of continuous finite-dimensional K-linear representations of $G_{L,(n)}$ which will be denoted by $\mathbf{Rep}_K(G_{L,(n)})$.
Let (V, ρ) be in $\mathbf{Rep}_K(G_{L,(n)})$. The elements of the cyclic group $\rho(H_n)$ are diagonalizable, as $X^n - 1$ factors completely over K with pairwise distinct roots. Denote by $X(H_n)$ the finite abelian group of characters $\chi : H_n \to K^\times$ and similarly for $X(\rho(H_n))$. Then precomposition with ρ induces an injective group homomorphism $X(\rho(H_n)) \to X(H_n)$ by which we identify $X(\rho(H_n))$ with a subgroup of $X(H_n)$. The eigenspace decomposition of V with respect to the elements of $X(H_n)$

$$V = \bigoplus_{\chi \in X(H_n)} V_\chi, \qquad (5)$$

where $V_\chi := \{v \in V \mid \rho(h)v = \chi(h)v \ \forall h \in H_n\}$ and $V_\chi := 0$ if $\chi \notin X(\rho(H_n))$ will be used throughout in the following. Due to centrality of H_n in $G_{L,(n)}$, the $V_\chi \subseteq V$ are K-subrepresentations of $G_{L,(n)}$. By the definition of morphisms in $\mathbf{Rep}_K(G_{L,(n)})$, the association $V \mapsto V_\chi$ is functorial in V for every $\chi \in X(H_n)$.
In the following, we write $\varepsilon_n := \varepsilon_{L,n}$ and

$$\chi_{n,j} := \text{ the restriction of } \varepsilon_n^{-j} \text{ to } H_n \quad (j = 0, \ldots, n-1).$$

With this notation we have

$$\chi_{n,j}^{-1} = \begin{cases} \chi_{n,0} & j = 0 \\ \chi_{n,j-1} & 1 \leq j \leq n-1. \end{cases}$$

Since ε_n is an isomorphism between H_n and $\mu_n(K)$, the restrictions of the n different elements ε_n^{-j} to H_n yield n different elements of the cyclic group $X(H_n)$ (a generator is $\chi_{n,1}$, for example). Hence equality (5) can be rewritten as

$$V = \bigoplus_{j=0}^{n-1} V_{\chi_{n,j}}.$$

If (K, χ) and (U, ρ) are in $\mathbf{Rep}_K(G_{L,(n)})$, we write $\chi \otimes U$ for the representation $(U, g \mapsto [u \mapsto \chi(g)(\rho(g)u)])$. We find that H_n operates trivially on $\varepsilon_n^j \otimes U_{\chi_{n,j}}$ for all $j = 0, \ldots, n-1$ and hence these are naturally objects of $\mathbf{Rep}_K(G_L)$.
If (V, ρ) is in $\mathbf{Rep}_K(G_{L,(n)})$, the tuple

$$(V^\vee, \rho^\vee) := (K\text{-vector space dual of } V,\ g \mapsto [\ell \mapsto [v \mapsto \ell(\rho(g^{-1})v)]])$$

is also in $\mathbf{Rep}_K(G_{L,(n)})$. Then one easily checks that the decomposition of (V, ρ) into H_n-eigenspaces is related to that of $(V^\vee, \rho^\vee)$ by $(V_{\chi_{n,j}})^\vee = (V^\vee)_{\chi_{n,j}^{-1}}$.

Remark 3.1.5. From now on, we ususally omit either the space or the group homomorphism in the datum of a representation of a group when there is no risk of confusion.

Appendix on the splitting of a sequence of the form ($\mathbf{Ext}_{E,n}$): The following discussion and the two subsequent results are inspired by [BrSch, Lemma 7.5]. Although interesting in their own right, these results will not be used in the sequel.

Let $\mathbb{Q}_p \subseteq E$ be a finite field extension contained in $\overline{\mathbb{Q}}_p$. We recall that isomorphism classes of central extensions with outer parts H_n and G_E as above are in canonical bijection with elements of the Galois cohomology group $H^2(G_E, \mu_n)$ [Mac, p. 112, discussion following Theorem 4.1] with law of composition written additively. In particular, such an extension splits if and only if it is represented by the trivial element in $H^2(G_E, \mu_n)$. By local class field theory, one has an isomorphism inv_E between the Galois cohomology group $H^2(G_E, \overline{\mathbb{Q}}_p^\times)$ and the absolute Brauer group of E which is isomorphic to $\mathbb{Q}/\mathbb{Z}$. Hence, the invariant map inv_E induces an isomorphism on n-torsion which implies $H^2(G_E, \mu_n) \cong \frac{1}{n}\mathbb{Z}/\mathbb{Z}$ (cf. [GiSz, Corollary 4.4.9]).
The bifunctor "second Galois cohomology" $H^2(-,-)$ is contravariant in the first argument, and the morphism induced by an inclusion of groups (but fixed Galois module) is the restriction. For the specific inclusion $G_E \hookrightarrow G_{\mathbb{Q}_p}$ and the induced restriction map, the class of $(\mathrm{Ext}_{\mathbb{Q}_p,n})$ corresponds to that of $(\mathrm{Ext}_{E,n})$ via this construction.
By one of the main theorems in local class field theory, the restriction induced by $G_E \hookrightarrow G_{\mathbb{Q}_p}$ is multiplication with $[E : \mathbb{Q}_p]$ on (torsion subgroups of) Brauer groups via the invariant map. Hence, we have

$$inv_E^{-1} \circ [E : \mathbb{Q}_p] \circ inv_{\mathbb{Q}_p}(\text{class of } (\mathrm{Ext}_{\mathbb{Q}_p,n}))$$

$$= \text{restriction}(\text{class of } (\mathrm{Ext}_{\mathbb{Q}_p,n})) = \text{class of } (\mathrm{Ext}_{E,n}).$$

Lemma 3.1.6. *With the notations from the preceding discussion, the sequence* $(\mathrm{Ext}_{\mathbb{Q}_p,n})$ *does not split.*

Proof. Recall that $G_{\mathbb{Q}_p,(n)}$ is by definition the fibre product of $K^\times$ and $G_{\mathbb{Q}_p}$ with respect to $-^n : K^\times \to K^\times$ and $\varepsilon_{\mathbb{Q}_p}$.
Suppose, that $(\mathrm{Ext}_{\mathbb{Q}_p,n})$ splits, with splitting $s : G_{\mathbb{Q}_p} \to G_{\mathbb{Q}_p,(n)}$. Let f be the composite of $\varepsilon_{\mathbb{Q}_p,n}$ with s. Then we have $f(h)^n = \varepsilon_{\mathbb{Q}_p}(h)$ for all $h \in G_{\mathbb{Q}_p}$ which for all $h \in \ker(\varepsilon_{\mathbb{Q}_p})$ implies $f(h) \in \mu_n(K)$. Since $\varepsilon_{\mathbb{Q}_p,n}$ induces an isomorphism of H_n with $\mu_n(K)$ we find that $s(h)$ is contained in H_n for all $h \in \ker(\varepsilon_{\mathbb{Q}_p})$.

Since s is an injective group homomorphism and $\ker(\varepsilon_{\mathbb{Q}_p})$ is not a finite group, we arrive at a contradiction. $\square$

Proposition 3.1.7. *Let $\mathbb{Q}_p \subseteq E$ be a finite field extension contained in $\overline{\mathbb{Q}}_p$. Define $c_{\mathbb{Q}_p} := inv_{\mathbb{Q}_p}$(class of $(\mathrm{Ext}_{\mathbb{Q}_p,n})$) in $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ and analogously for c_E. Then the following are equivalent:*

1. *The extension $(\mathrm{Ext}_{E,n})$ splits.*
2. *The order of $c_{\mathbb{Q}_p}$ divides $[E:\mathbb{Q}_p]$.*

Proof. The element $c_{\mathbb{Q}_p}$ generates a subgroup of $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ which does not consist of the unit element alone by 3.1.6. Then 1. holds if and only if the chain of equalities $0 = c_E = [E:\mathbb{Q}_p]c_{\mathbb{Q}_p}$ is valid in $\frac{1}{n}\mathbb{Z}/\mathbb{Z}$ (by the discussion before 3.1.6) if and only if 2. holds. $\square$

3.2 The rings $B_{\mathrm{dR},n}$ and $B_{\mathrm{cris},n}$

Via certain ring extensions of B_{cris} and B_{dR} we aim to single out a specific class of representations of $G_{L,(n)}$. In order to construct the ring extensions we fix a $\mathbb{Z}_p$-generator t of $\mathbb{Z}_p(1)$.
As one might expect, statements in this subsection and in the next section can often be proved by refering to the validity of a corresponding statement in the classical setting.

Definition 3.2.1. Define

$$B_{\mathrm{dR},n} := B_{\mathrm{dR}}[X]/(X^n - t)B_{\mathrm{dR}}[X]$$

as the quotient of the polynomial ring in one variable over B_{dR} modulo the ideal generated by the polynomial $X^n - t$.

Lemma 3.2.2. *The ring $B_{\mathrm{dR},n}$ is a field extension of B_{dR} which is Galois of degree n with Galois group isomorphic to $\mathbb{Z}/n\mathbb{Z}$. If α is a root of $X^n - t$ in an algebraic closure of B_{dR} then $B_{\mathrm{dR},n}$ and $B_{\mathrm{dR}}(\alpha)$ are isomorphic as fields via*

$$\overline{X} := \text{residue class of } X \mapsto \alpha$$

and α is a uniformizer of the totally ramified extension $B_{\mathrm{dR}}(\alpha)|B_{\mathrm{dR}}$.

Proof. By Eisenstein's criterion with t as prime element, the polynomial $X^n - t$ is irreducible in $B_{\mathrm{dR}}[X]$. The irreducibility holds if and only if the ideal generated by $X^n - t$ in $B_{\mathrm{dR}}[X]$ is maximal if and only if $B_{\mathrm{dR},n}$ is a field extension of B_{dR} of degree n. The zeros of $X^n - t$ are $(\zeta\overline{X})_{\zeta\in\mu_n(B_{\mathrm{dR}})}$ hence $B_{\mathrm{dR},n}|B_{\mathrm{dR}}$ is Galois. Every automorphism of $B_{\mathrm{dR},n}$ which leaves B_{dR} fixed is given by multiplication with a fixed $\zeta \in \mu_n(B_{\mathrm{dR}})$ on the set of zeros of $X^n - t$. Hence the Galois group of $B_{\mathrm{dR},n}|B_{\mathrm{dR}}$ is isomorphic to $\mathbb{Z}/n\mathbb{Z}$.
The remaining statements are proved in [FeVo, Proposition II.3.6]. $\square$

Remark 3.2.3. The definition of $B_{\mathrm{dR},n}$ is independent of the choice of t: indeed, let $t' \in \mathbb{Z}_p(1)$ be another $\mathbb{Z}_p$-generator. Then there exists $u \in \mathbb{Z}_p^\times$ such that $t' = ut$. Since $\overline{\mathbb{Q}}_p \subset B_{\mathrm{dR}}$, the field $B_{\mathrm{dR},n}$ contains all roots of the polynomial $X^n - u$. Let v be one such root. It follows that $(\zeta v\overline{X})^n = t'$ for any $\zeta \in \mu_n(B_{\mathrm{dR}})$. Thus, as ζ runs through $\mu_n(B_{\mathrm{dR}})$, the elements $\zeta v\overline{X}$ run through the different zeros of $X^n - t'$ which all lie in $B_{\mathrm{dR},n}$. Therefore we have $B_{\mathrm{dR},n} = B_{\mathrm{dR}}[X]/(X^n - t')B_{\mathrm{dR}}[X]$.

Definition 3.2.4. Define

$$B_{\mathrm{cris},n} := B_{\mathrm{cris}}[X]/(X^n - t)B_{\mathrm{cris}}[X]$$

as the quotient of the polynomial ring in one variable over B_{cris} modulo the ideal generated by the polynomial $X^n - t$.

The following result allows us to view $B_{\mathrm{cris},n}$ as a subring of $B_{\mathrm{dR},n}$.

Lemma 3.2.5. *Let A be a subring of a field E. Moreover, let the integer $d \geq 2$ and $a \in A^\times$ be such that the polynomial $h := X^n - a \in A[X]$ is irreducible in $E[X]$. Then $A[X]/hA[X]$ is identified with a subring of the field $E[X]/hE[X]$.*

Proof. We show that the kernel $hE[X] \cap A[X]$ of the composite

$$A[X] \hookrightarrow E[X] \twoheadrightarrow E[X]/hE[X].$$

of the canonical ring maps is equal to the principal ideal $hA[X]$. Then the statement follows from the homomorphism theorem for rings.
Let $f \in hE[X] \cap A[X]$. There exists $g := \sum_{j=0}^{m} b_j X^j \in E[X]$ with $b_m \neq 0$ such that $f = hg$. There are two possibilities. Assume first $m < n$. This means f can be written as

$$f = hg = -\sum_{j_1=0}^{m} ab_{j_1}X^{j_1} + \sum_{j_2=n}^{m+n} b_{n-j_2}X^{j_2}.$$

By looking at the right hand sum, it follows that g has coefficients in A whence $f \in hA[X]$.
Now let $m \geq n$. Then

$$f = hg = -\sum_{j_1=0}^{n-1} ab_{j_1}X^{j_1} + \sum_{j_2=n}^{m} (b_{j_2-n} - ab_{j_2})X^{j_2} + \sum_{j_3=m+1}^{m+n} b_{j_3-n}X^{j_3}.$$

Recall that $a \in A^\times$. Hence for the sum indexed over j_2 we get implications

$$(b_{j_2-n} - ab_{j_2}),\, b_{j_2-n} \in A \Rightarrow b_{j_2} \in A \quad (j_2 = n, \dots, m).$$

Again we find $f \in hA[X]$, and altogether $hE[X] \cap A[X] \subseteq hA[X]$. Since the reverse inclusion is trivial we get the statement of the lemma. □

Corollary 3.2.6. *The ring $B_{\mathrm{cris},n}$ is a subring of $B_{\mathrm{dR},n}$. In particular, $B_{\mathrm{cris},n}$ is an integral domain.*

Proof. The statements follow from applying 3.2.5 with $A = B_{\mathrm{cris}}$, $E = B_{\mathrm{dR}}$ and $a = t$. □

Definition 3.2.7. We denote by t_n the residue class of $X \in B_{\mathrm{dR}}[X]$ in $B_{\mathrm{dR},n}$.

The set $\{1, t_n, \ldots, t_n^{n-1}\}$ forms a basis of $B_{\mathrm{cris},n}$ as a module over B_{cris} and also a basis of $B_{\mathrm{dR},n}$ as a vector space over B_{dR}. Denote by v the normalized and discrete valuation on B_{dR} with respect to integer powers of $\mathfrak{m}_{B_{\mathrm{dR}}} = tB^0_{\mathrm{dR}} =: F^1B_{\mathrm{dR}}$ (cf. subsection 1.3). Then the unique extension of v to $B_{\mathrm{dR},n}$ is $w := \frac{1}{n} v \circ N_{B_{\mathrm{dR},n}|B_{\mathrm{dR}}}$, where $N_{B_{\mathrm{dR},n}|B_{\mathrm{dR}}}$ is the norm homomorphism. Since $nw(t_n) = v(t) = 1$, we have $w(t_n) = \frac{1}{n}$. The filtration on the field $B_{\mathrm{dR},n}$ induced by w is thus given via

$$F^r B_{\mathrm{dR},n} = \{y \in B_{\mathrm{dR},n} \mid w(y) \geq r\} \quad (r \in \tfrac{1}{n}\mathbb{Z}).$$

Let $r \in \frac{1}{n}\mathbb{Z}$ and $y = \sum_{i=0}^{n-1} b_i t_n^i \in F^r B_{\mathrm{dR},n} \setminus \{0\}$ with unique $b_i \in B_{\mathrm{dR}}$. For every index i such that $b_i \neq 0$, the number $w(b_i t_n^i)$ lies in $\frac{i}{n} + \mathbb{Z}$. Since $\frac{1}{n}\mathbb{Z}$ is disjointly covered by the sets $\frac{i}{n} + \mathbb{Z}$ $(i = 0, \ldots, n-1)$, there is a unique i' such that

$$w(b_{i'} t_n^{i'}) = \min_{i \in \{0, \ldots, n-1 | b_i \neq 0\}} w(b_i t_n^i).$$

Therefore we have

$$r \leq w(y) = w(b_{i'} t_n^{i'})$$

and this implies for all $i = 0, \ldots, n-1$ such that $b_i \neq 0$ the leftmost inequality in the string of equivalences

$$r \leq v(b_i) + \frac{i}{n} \Leftrightarrow r - \frac{i}{n} \leq v(b_i) \Leftrightarrow b_i \in F^{\lceil r - \frac{i}{n} \rceil} B_{\mathrm{dR}}$$

(for notation concerning the right hand term, cf. 2.1.5). Thus we have proven the following result.

Lemma 3.2.8. *Let $r \in \frac{1}{n}\mathbb{Z}$. Then $F^r B_{\mathrm{dR},n}$ decomposes as*

$$F^r B_{\mathrm{dR},n} = \bigoplus_{i=0}^{n-1} (F^{\lceil r - \frac{i}{n} \rceil} B_{\mathrm{dR}}) t_n^i.$$

Note that, in particular, t_n is contained in $F^{\frac{1}{n}} B_{\mathrm{dR},n}$ but is not contained in a filtration step with index strictly greater than $\frac{1}{n}$.

Remark 3.2.9. The filtration $F^\bullet B_{\mathrm{dR},n}$ described above restricts to the filtration of B_{dR} described in subsection 1.3. Whenever we speak of a filtration on $B_{\mathrm{dR},n}$ (resp. B_{dR}) without mentioning it explicitly, it is this filtration that we have in mind.

Let $* \in \{\mathrm{cris}, \mathrm{dR}\}$. Via the canonical ring homomorphism $B_* \otimes_{\mathbb{Q}_p} K \to B_{*,n} \otimes_{\mathbb{Q}_p} K$, the ring $B_{*,n} \otimes_{\mathbb{Q}_p} K$ becomes a free $B_* \otimes_{\mathbb{Q}_p} K$-module of rank n with basis $t_n^i \otimes 1$, $i = 0, \ldots, n-1$.
The formation of $B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K$ is independent of the choice of t. To see this, let $t' = ut$ where $u \in \mathbb{Z}_p^\times$. Due to assumption 2.2.7, K contains an element v with $v^n = u$. Then

$$B_{\mathrm{cris}}[X]/(X^n - t') \otimes_{\mathbb{Q}_p} K \to B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K,$$
$$(\sum_{i=0}^{n-1} b_i \overline{X}^i) \otimes z \mapsto \sum_{i=0}^{n-1} (b_i t_n^i \otimes v^i z)$$

is a ring isomorphism with inverse

$$\sum_{i=0}^{n-1}(c_i\overline{X}^i \otimes v^{-i}z') \leftarrow\!\shortmid (\sum_{i=0}^{n-1} c_i t_n^i) \otimes z'.$$

Recall that R (resp. R_L) stands for $L_0 \otimes_{\mathbb{Q}_p} K$ (resp. $L \otimes_{\mathbb{Q}_p} K$). Elements of the group $G_{L,(n)}$ act as R_L-algebra automorphisms on the left tensor factor of $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K$ via $\delta_n := \delta_{L,n} : G_{L,(n)} \to G_L$.

Proposition 3.2.10.

1. *The map*

$$\begin{aligned} G_{L,(n)} \times (B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K) &\to B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K, \\ (g, \sum_{i=0}^{n-1} b_i t_n^i \otimes 1) &\mapsto \sum_{i=0}^{n-1} [\delta_n(g)(b_i)] t_n^i \otimes \varepsilon_n^i(g) \end{aligned}$$

is well-defined and extends the action of G_L on $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K$ by R_L-algebra automorphisms to an action of $G_{L,(n)}$ on $B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K$ by R_L-algebra automorphisms.

2. *The map*

$$\begin{aligned} \tilde{\varphi}_0 : B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K &\to B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K \\ \sum_{i=0}^{n-1} b_i t_n^i \otimes 1 &\mapsto \sum_{i=0}^{n-1} \varphi_0(b_i) t_n^i \otimes (p^{\frac{1}{n}})^i \end{aligned}$$

is well-defined and extends the Frobenius $\varphi_0 : B_{\mathrm{cris}} \to B_{\mathrm{cris}}$ to an injective $\sigma_0 \otimes K$-linear ring endomorphism of $B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K$.

Proof. Ad 1.: Let $g \in G_{L,(n)}$ and consider the composite morphism of rings

$$\psi_g : B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K \to B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K \hookrightarrow (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K)[X]$$

where the first arrow is $b \otimes x \mapsto (\delta_n(g)(b)) \otimes x$. By the universal property of the polynomial ring in one variable, to the pair $(\psi_g, (1 \otimes \varepsilon_n(g))X)$ there corresponds a unique homomorphism of rings

$$\Phi_g : (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K)[X] \to (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K)[X]$$

with $\Phi_g(X) = (1 \otimes \varepsilon_n(g))X$ and such that

$$\psi_g = \Phi_g \circ (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K \hookrightarrow (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K)[X]).$$

Then the association $g \mapsto \Phi_g$ defines an action of the group $G_{L,(n)}$ on $(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K)[X]$ by R_L-algebra automorphisms that preserves the ideal generated by $X^n - (t \otimes 1)$. By the homomorphism theorem,

$$\underbrace{[(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K)[X] \twoheadrightarrow (B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K)[X]/(X^n - (t \otimes 1))]}_{:=\pi_g:=\text{canonical projection}} \circ \Phi_g$$

factorizes uniquely over π_g, via a ring automorphism $\tilde{\Phi}_g$ say. The assignment $g \mapsto \tilde{\Phi}_g$ is then the $G_{L,(n)}$-action which translates to the one on $B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K$ from the statement via the natural isomorphism of $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K$-algebras

$$(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K)[X]/(X^n - (t \otimes 1)) \cong B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K.$$

Ad 2.: This is shown similarly as the first part. By the universal property, the pair consisting of the ring homomorphism

$$\Phi := (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K \hookrightarrow (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K)\,[X]) \circ (\varphi_0 \otimes K)$$

and of the element $(1 \otimes p^{\frac{1}{n}})X \in (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K)\,[X]$ corresponds uniquely to a ring homomorphism $\Phi_0 : (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K)\,[X] \to (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K)\,[X]$ with $\Phi_0(X) = (1 \otimes p^{\frac{1}{n}})X$ and such that $\Phi_0 \circ (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K \hookrightarrow (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K)\,[X]) = \Phi$. The morphism Φ_0 is injective, $\sigma_0 \otimes K$-linear and preserves the ideal generated by $X^n - (t \otimes 1)$. Hence, as in the first part of the proof, we get a unique ring endomorphism $B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K \to B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K$ which is easily seen to be $\tilde{\varphi}_0$ from the statement. □

Remark 3.2.11. The $G_{L,(n)}$-action on $B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K$ restricts to an action by R-algebra automorphisms on the subring $B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K$.
Via the injective ring homomorphism

$$L \otimes_{L_0} B_{\mathrm{cris},n} \to B_{\mathrm{dR},n},\ l \otimes (\sum_i b_i t_n^i) \mapsto \sum_i (l b_i) t_n^i$$

we induce on $L \otimes_{L_0} B_{\mathrm{cris},n}$, and thus on $R_L \otimes_R (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K)$, a $\frac{1}{n}\mathbb{Z}$-filtration (cf. also point *2.* of the corollary below). Whenever we speak of a $\frac{1}{n}\mathbb{Z}$-filtration on $R_L \otimes_R (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K)$ without mentioning it explicitly, it is this filtration that we have in mind.

Corollary 3.2.12.

1. *As respective $G_{L,(n)}$-invariants for the action as in proposition 3.2.10 we get*

   $$(B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K)^{G_{L,(n)}} = L \otimes_{\mathbb{Q}_p} K \quad \text{and} \quad (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K)^{G_{L,(n)}} = L_0 \otimes_{\mathbb{Q}_p} K.$$

2. *The action of $G_{L,(n)}$ on $B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K$ as in proposition 3.2.10 is compatible with the decreasing, exhaustive and separated $\frac{1}{n}\mathbb{Z}$-filtration by $(F^0 B_{\mathrm{dR},n}) \otimes_{\mathbb{Q}_p} K$-submodules defined as*

   $$F^j(B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K) := (F^j B_{\mathrm{dR},n}) \otimes_{\mathbb{Q}_p} K \quad (j \in \tfrac{1}{n}\mathbb{Z}),$$

   i.e.

   $$g(F^j(B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K)) = F^j(B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K) \quad \text{for all } g \in G_{L,(n)},\ j \in \tfrac{1}{n}\mathbb{Z}.$$

3. *We have $g(\tilde{\varphi}_0(x)) = \tilde{\varphi}_0(g(x))$ for all $x \in B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K, g \in G_{L,(n)}$.*

Proof. Ad 1.: We only show the argument for $B_{\mathrm{dR},n}$, the one for $B_{\mathrm{cris},n}$ is literally the same, noting that $B_{\mathrm{cris}}^{G_L} = L_0$.
We use the identity

$$(B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K)^{G_{L,(n)}} = ((B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K)^{H_n})^{G_L}.$$

The action of H_n on B_{dR} and on K inside $B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K$ is trivial. On the $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K$-basis $(t_n^i \otimes 1)_{i=0,\dots,n-1}$, we have

$$h(t_n^i \otimes 1) = t_n^i \otimes \varepsilon_n(h)^i$$

for all $h \in H_n$ and $i = 0, \dots, n-1$. Since ε_n is not the trivial character on H_n, we deduce that the H_n-invariants are equal to the subring $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K$. It follows

$$(B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K)^{G_L} = (B_{\mathrm{dR}})^{G_L} \otimes_{\mathbb{Q}_p} K = L \otimes_{\mathbb{Q}_p} K.$$

Ad 2.: This is a consequence of the fact that the filtration of B_{dR} with respect to integer powers of its maximal ideal is preserved by the action of G_L, hence the action of $G_{L,(n)}$ on $t_n^i \otimes 1$ for $i = 0, \dots, n-1$ preserves the filtration steps of the filtration on $B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K$ as defined in the statement.
Ad 3.: This assertion follows by taking into account that the action of G_L on B_{cris} is commutes with φ_0. □

3.3 The functors $\mathbb{D}_{\mathrm{dR},n}$ and $\mathbb{D}_{\mathrm{cris},n}$

Let $f : V \to W$ be a morphism in $\mathbf{Rep}_K(G_{L,(n)})$. Denote by $(B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K) \otimes f$ the linear extension of the map

$$(B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V \to (B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K) \otimes_K W, \quad b \otimes v \mapsto b \otimes f(v)$$

to all of $(B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V$ and likewise for $(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes f$.

Definition 3.3.1. Let V be in $\mathbf{Rep}_K(G_{L,(n)})$.

1. Set

$$\mathbb{D}_{\mathrm{dR},n}(V) := \big((B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V\big)^{G_{L,(n)}}$$

and

$$\mathbb{D}_{\mathrm{cris},n}(V) := \big((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V\big)^{G_{L,(n)}}$$

as the invariants for the respective diagonal actions of $G_{L,(n)}$.

2. Let $f : V \to W$ be a morphism in $\mathbf{Rep}_K(G_{L,(n)})$. Set

$$\mathbb{D}_{\mathrm{dR},n}(f) := ((B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K) \otimes f)\big|_{\mathbb{D}_{\mathrm{dR},n}(V)}$$

and

$$\mathbb{D}_{\mathrm{cris},n}(f) := ((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes f)\big|_{\mathbb{D}_{\mathrm{cris},n}(V)}$$

as the restrictions of $(B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K) \otimes f$ and $(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes f$ to the respective $G_{L,(n)}$-invariants.

As in the classical case, both $\mathbb{D}_{\mathrm{dR},n}(V)$ and $\mathbb{D}_{\mathrm{cris},n}(V)$ inherit structural properties from $B_{\mathrm{dR},n}$ resp. $B_{\mathrm{cris},n}$.

Proposition 3.3.2. *With the above notations, the assignments*

$$V \mapsto \mathbb{D}_{\mathrm{dR},n}(V),\ [f : V \to W] \mapsto \mathbb{D}_{\mathrm{dR},n}(f)$$

define a covariant functor from $\mathbf{Rep}_K(G_{L,(n)})$ to the category of finitely generated R_L-modules which are equipped with a decreasing, exhaustive and separated $\frac{1}{n}\mathbb{Z}$-filtration. The assignments

$$V \mapsto \mathbb{D}_{\mathrm{cris},n}(V),\ [f : V \to W] \mapsto \mathbb{D}_{\mathrm{cris},n}(f)$$

define a covariant functor from $\mathbf{Rep}_K(G_{L,(n)})$ to $\mathbf{FIC}_{L,K,n}$.

Morphisms in the proclaimed target category of $\mathbb{D}_{\mathrm{dR},n}$ are R_L-module homomorphisms that respect the filtrations between source and target in the obvious sense.

Proof. Throughout this proof, $f : U \to V$ denotes a morphism in $\mathbf{Rep}_K(G_{L,(n)})$. Note that the $G_{L,(n)}$-action on $B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} V$ is R_L-linear hence $\mathbb{D}_{\mathrm{dR},n}(V)$ is an R_L-module. By a similar argument, $\mathbb{D}_{\mathrm{cris},n}(V)$ is an R-module.
Decomposing V into a direct sum of H_n-eigenspaces gives an R_L-module decomposition of $\mathbb{D}_{\mathrm{dR},n}(V)$ as

$$\begin{aligned}\mathbb{D}_{\mathrm{dR},n}(V) &= \big(((\oplus_{i=0}^{n-1} B_{\mathrm{dR}} t_n^i) \otimes_{\mathbb{Q}_p} K) \otimes_K (\oplus_{j=0}^{n-1} V_{\chi_{n,j}})\big)^{G_{L,(n)}} \\ &= \bigoplus_{i=0}^{n-1} ((B_{\mathrm{dR}} t_n^i \otimes_{\mathbb{Q}_p} K) \otimes_K V_{\chi_{n,i}})^{G_L}.\end{aligned}$$

The second equality holds because

$$[(B_{\mathrm{dR}} t_n^i \otimes_{\mathbb{Q}_p} K) \otimes_K V_{\chi_{n,j}}]^{H_n} = 0$$

in case $i \neq j$. When $i = j$ however, H_n acts trivially on the whole summand. The L-dimension of each such summand is bounded above by the $\mathbb{Q}_p$-dimension of $V_{\chi_{n,i}}$. It follows that $\mathbb{D}_{\mathrm{dR},n}(V)$ is a finitely generated R_L-module. The same argumentation for $B_{\mathrm{cris},n}$ shows that $\mathbb{D}_{\mathrm{cris},n}(V)$ is a finitely generated R-module. The R_L-linear map $\mathbb{D}_{\mathrm{dR},n}(f)$ maps to $\mathbb{D}_{\mathrm{dR},n}(V)$ by $G_{L,(n)}$-equivariance of f. Hence the covariance and the identity $\mathbb{D}_{\mathrm{dR},n}(g \circ f) = \mathbb{D}_{\mathrm{dR},n}(g) \circ \mathbb{D}_{\mathrm{dR},n}(f)$ for any morphism $g : V \to W$ in $\mathbf{Rep}_K(G_{L,(n)})$ follow. Trivially, we have $\mathbb{D}_{\mathrm{dR},n}(\mathrm{id}_V) = \mathrm{id}_{\mathbb{D}_{\mathrm{dR},n}(V)}$. The same properties with respect to morphisms are valid for $\mathbb{D}_{\mathrm{cris},n}$.
On the R_L-module $\mathbb{D}_{\mathrm{dR},n}(V)$ there is a natural decreasing, exhaustive and separated $\frac{1}{n}\mathbb{Z}$-filtration by virtue of

$$F^r \mathbb{D}_{\mathrm{dR},n}(V) := \big(F^r(B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V\big)^{G_{L,(n)}}$$

since the action of $G_{L,(n)}$ is compatible with the filtration on $B_{\mathrm{dR},n} \otimes_{\mathbb{Q}_p} K$ by 3.2.12. It follows immediately from the definitions that

$$\mathbb{D}_{\mathrm{dR},n}(f)(F^r \mathbb{D}_{\mathrm{dR},n}(U)) \subseteq F^r \mathbb{D}_{\mathrm{dR},n}(V).$$

On the R-module $(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V$, we have the injective $\sigma_0 \otimes K$-linear map $\tilde{\varphi}_0 \otimes V$ (cf. 3.2.10). It commutes with the diagonal action of $G_{L,(n)}$ because the G_L-action on B_{cris} commutes with φ_0. Thus $\tilde{\varphi}_0 \otimes V$ restricts to a $\sigma_0 \otimes K$-linear injective R-module endomorphism of $\mathbb{D}_{\mathrm{cris},n}(V)$. By finite-dimensionality over L_0, this endomorphism is actually bijective. Using 2.1.6 we see that $\mathbb{D}_{\mathrm{cris},n}(V)$ is a free R-module of finite rank.
The injective ring homomorphism

$$L \otimes_{L_0} B_{\mathrm{cris},n} \to B_{\mathrm{dR},n},\ l \otimes (\sum_i b_i t_n^i) \mapsto \sum (lb_i) t_n^i$$

induces an injection of R_L-modules

$$\begin{aligned} R_L \otimes_R \mathbb{D}_{\mathrm{cris},n}(V) &\cong L \otimes_{L_0} \mathbb{D}_{\mathrm{cris},n}(V) \\ &= (((L \otimes_{L_0} B_{\mathrm{cris},n}) \otimes_{\mathbb{Q}_p} K) \otimes_K V)^{G_{L,(n)}} \hookrightarrow \mathbb{D}_{\mathrm{dR},n}(V) \end{aligned}$$

where the third term denotes the $G_{L,(n)}$-invariants with respect to

$$l \otimes x \mapsto l \otimes g(x), \quad (l \in L, x \in (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V, g \in G_{L,(n)})$$

on elementary tensors. Now set for any $r \in \frac{1}{n}\mathbb{Z}$

$$F^r(L \otimes_{L_0} \mathbb{D}_{\mathrm{cris},n}(V)) := L \otimes_{L_0} \mathbb{D}_{\mathrm{cris},n}(V) \cap F^r \mathbb{D}_{\mathrm{dR},n}(V).$$

Hence $\mathbb{D}_{\mathrm{cris},n}(V)$ is an object of $\mathbf{FIC}_{L,K,n}$. Validity of the equality

$$\text{(restriction to } \mathbb{D}_{\mathrm{cris},n}(V) \text{ of } \tilde{\varphi}_0 \otimes V) \circ \mathbb{D}_{\mathrm{cris},n}(f)$$

$$= \mathbb{D}_{\mathrm{cris},n}(f) \circ \text{(restriction to } \mathbb{D}_{\mathrm{cris},n}(U) \text{ of } \tilde{\varphi}_0 \otimes U)$$

as well as compatibility of the induced R_L-module homomorphism with the filtration steps are immediate. The proof is finished. □

Note that, being composites of additive functors, both functors from the proposition are additive. The proof of the proposition shows in particular that, as R_L-modules, we have a decomposition

$$\mathbb{D}_{\mathrm{dR},n}(V) \cong \bigoplus_{i=0}^{n-1} \mathbb{D}_{\mathrm{dR}}(\varepsilon_n^i \otimes V_{\chi_{n,i}})$$

for any V in $\mathbf{Rep}_K(G_{L,(n)})$. In the following corollary, we summarize an analogous decomposition for $\mathbb{D}_{\mathrm{cris},n}(V)$ taking into account its structure as object of $\mathbf{FIC}_{L,K,n}$, using notation introduced in section 2.

Corollary 3.3.3. *Let V be in $\mathbf{Rep}_K(G_{L,(n)})$. Then $\mathbb{D}_{\mathrm{cris},n}(V)$ and*

$$\bigoplus_{i=0}^{n-1} \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V_{\chi_{n,i}}) \left\langle -\tfrac{i}{n} \right\rangle$$

are naturally isomorphic in $\mathbf{FIC}_{L,K,n}$.

Example 3.3.4. 1. We compute $\mathbb{D}_{\mathrm{cris},n}(K)$, where here K denotes the trivial $G_{L,(n)}$-representation. Obviously, the only H_n-eigenspace is $K_{\chi_{n,0}} =$

K. With 3.3.3 we see that

$$\mathbb{D}_{\mathrm{cris},n}(K) = \mathbb{D}_{\mathrm{cris}}(K) = (B_{\mathrm{cris}})^{G_L} \otimes_{\mathbb{Q}_p} K = R$$

as R-modules. Moreover, the Frobenius of $\mathbb{D}_{\mathrm{cris},n}(K)$ is $\sigma_0 \otimes K$ and the only jump in the filtration on $R_L \otimes_R R \cong R_L$ is at 0.
Hence, $\mathbb{D}_{\mathrm{cris},n}(K)$ is identified with the unit object $\underline{R}$ of $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$ (cf. 2.2.5 and the example following it).

2. We compute $\mathbb{D}_{\mathrm{cris},n}(\varepsilon_n^i)$ for any $1 \leq i \leq n-1$. With the notations from section 2 and 3.3.3 we get

$$\begin{aligned}\mathbb{D}_{\mathrm{cris},n}(\varepsilon_n^i) &\cong \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^{n-i} \otimes K_{\chi_{n,n-i}})\langle -\tfrac{n-i}{n}\rangle \\ &= \mathbb{D}_{\mathrm{cris}}(\varepsilon)\langle -\tfrac{n-i}{n}\rangle \\ &\cong \underline{K}\langle -\tfrac{n-i}{n}\rangle \qquad \text{(by [BrSch, Lemma 7.3])} \\ &\cong \underline{R}\langle 1\rangle\langle -\tfrac{n-i}{n}\rangle \cong \underline{K}_n^{\otimes i},\end{aligned}$$

where $\underline{K}$ is defined as in 2.2.9. Hence $\mathbb{D}_{\mathrm{cris},n}(\varepsilon_n^i)$ is naturally an object of $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$.

Before we define crystalline representations of $G_{L,(n)}$, we end this subsection with a general observation on an analogue of the crystalline comparison morphism. The properties of this morphism will be needed in the next section.

Lemma/Definition 3.3.5. *Let V be in $\mathbf{Rep}_K(G_{L,(n)})$. The family of maps*

$$(\alpha_{V,n} : (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R \mathbb{D}_{\mathrm{cris},n}(V) \to (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V,\ b \otimes d \mapsto bd)_V,$$

where V runs through $\mathbf{Rep}_K(G_{L,(n)})$, defines a natural transformation

$$\alpha_{\bullet,n} : (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R \mathbb{D}_{\mathrm{cris},n}(-) \to (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_K (-)$$

between functors from $\mathbf{Rep}_K(G_{L,(n)})$ to $\mathbf{Mod}(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K)$.

Proof. Let V be in $\mathbf{Rep}_K(G_{L,(n)})$. The fact that $\alpha_{V,n}$ is well-defined follows from the decompositions

$$(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V \cong \bigoplus_{i=0}^{n-1} ((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V_{\chi_{n,i}})$$

and

$$\mathbb{D}_{\mathrm{cris},n}(V) = \bigoplus_{i=0}^{n-1} ((B_{\mathrm{cris}} t_n^i \otimes_{\mathbb{Q}_p} K) \otimes_K V_{\chi_{n,i}})^{G_L} :$$

each summand $((B_{\mathrm{cris}} t_n^i \otimes_{\mathbb{Q}_p} K) \otimes_K V_{\chi_{n,i}})^{G_L}$ is a subset of $(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V_{\chi_{n,i}}$ hence $\mathbb{D}_{\mathrm{cris},n}(V)$ is a subset of $(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V_{\chi_{n,i}}$.
To see naturality of $\alpha_{\bullet,n}$, let $f : V \to W$ be a morphism in $\mathbf{Rep}_K(G_{L,(n)})$ and $b \otimes d \in (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R \mathbb{D}_{\mathrm{cris},n}(V)$. Then on elementary tensors we have

$$\begin{aligned}&(((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes f) \circ \alpha_{V,n})(b \otimes d) = ((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes f)(bd) \\ &= b((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes f)(d) = \alpha_{W,n}(b \otimes \mathbb{D}_{\mathrm{cris},n}(f)(d)) \\ &= (\alpha_{W,n} \circ ((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes \mathbb{D}_{\mathrm{cris},n}(f)))(b \otimes d)\end{aligned}$$

and the claim follows by extending linearly. □

On $(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R \mathbb{D}_{\mathrm{cris},n}(V)$ we naturally have additional structures: the group $G_{L,(n)}$ acts via $g(b \otimes d) = g(b) \otimes d$ and a $\sigma_0 \otimes K$-linear map is given by $b \otimes d \mapsto \tilde{\varphi}_0(b) \otimes (\tilde{\varphi}_0 \otimes V)(d)$. On the scalar extension over R to R_L we have a $\frac{1}{n}\mathbb{Z}$-filtration given by the tensor product filtration.
On $(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V$, we have the diagonal $G_{L,(n)}$-action. A $\sigma_0 \otimes K$-linear map and a $\frac{1}{n}\mathbb{Z}$-filtration are induced by the corresponding structures on the left tensor factor $B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K$.
The assertions in the following statement all refer to the structures just described.

Proposition 3.3.6. *Let V be in $\mathbf{Rep}_K(G_{L,(n)})$. Then $\alpha_{V,n}$ is an injective and $G_{L,(n)}$-equivariant morphism in $\mathbf{MF}^{\phi}_{L,K,n}$.*

Essentially all statements use validity of the corresponding property of the comparison morphism

$$B_{\mathrm{cris}} \otimes_{L_0} (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} W)^{G_L} \to B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} W$$

for p-adic Galois representations W.

Proof. We first introduce the auxiliary function

$$\begin{aligned} \{0, \dots, n-1\}^2 &\to \{0, \dots, n-1\}^2 \\ (i,j) &\mapsto ([i+j]_n, j), \end{aligned}$$

where $[i+j]_n$ denotes the residue of the division of $i+j$ by n. This function is bijective.
The domain of $\alpha_{V,n}$ decomposes in $\mathbf{MF}^{\phi}_{L,K,n}$ as

$$\bigoplus_{i,j=0}^{n-1} \underbrace{(B_{\mathrm{cris}} t_n^i \otimes_{\mathbb{Q}_p} K) \otimes_R \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^j \otimes V_{\chi_{n,j}})\langle -\tfrac{j}{n} \rangle}_{:=A_{ij}}$$

and the target of $\alpha_{V,n}$ decomposes in $\mathbf{MF}^{\phi}_{L,K,n}$ as

$$\bigoplus_{i,j=0}^{n-1} \underbrace{(B_{\mathrm{cris}} t_n^{[i+j]_n} \otimes_{\mathbb{Q}_p} K) \otimes_K V_{\chi_{n,j}}}_{B_{ij}}.$$

By the remark before the proof and the fact that $t_n \in B^{\times}_{\mathrm{cris},n}$, $\alpha_{V,n}$ restricts to an injective R-linear map $\alpha_{V,n} : A_{ij} \to B_{ij}$ for all $i, j = 0, \dots, n-1$.
The $G_{L,(n)}$-equivariance follows by a direct computation noting that $G_{L,(n)}$ acts on t via ε.
The operator $\tilde{\varphi}_0 \otimes (\tilde{\varphi}_0 \otimes V)$ restricts to a $\sigma_0 \otimes K$-linear endomorphism of A_{ij}. It is straightforward to check that $\alpha_{V,n}$ is compatible with $\tilde{\varphi}_0 \otimes (\tilde{\varphi}_0 \otimes V)$ and the restriction of $\tilde{\varphi}_0 \otimes V$ on B_{ij} for all $i, j = 0, \dots, n-1$ using $\tilde{\varphi}_0(t \otimes 1) = p(t \otimes 1)$.
Let $x \in \frac{1}{n}\mathbb{Z}$. Using similar arguments as in the proof of 2.1.5 and the identity

$$(1 \otimes t) F^s B_{\mathrm{cris},L} = F^{s+1} B_{\mathrm{cris},L} \quad (s \in \mathbb{Z})$$

from the classical setting, one sees that, denoting by $W_L := L \otimes_{L_0} W$ the extension of scalars from an L_0-vector space W to L, $\alpha_{V,n}$ induces an R_L-linear map from

$$\sum_{l \in \frac{1}{n}\mathbb{Z}} \bigoplus_{i=0}^{n-1} \bigoplus_{j=0}^{n-1} (F^{\lceil l - \frac{i}{n} \rceil} B_{\mathrm{cris},L}) t_n^i \otimes_L \left((F^{\lceil x-l-\frac{i}{n} \rceil} B_{\mathrm{cris},L}) t_n^j \otimes_{\mathbb{Q}_p} V_{\chi_{n,j}} \right)^{G_L}$$

to

$$\bigoplus_{i=0}^{n-1} \bigoplus_{j=0}^{n-1} \left((F^{\lceil x-\frac{i}{n} \rceil} B_{\mathrm{cris},L}) t_n^i \otimes_{\mathbb{Q}_p} V_{\chi_{n,j}} \right).$$

The upper R_L-module is identified with

$$F^x(R_L \otimes_R ((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R \mathbb{D}_{\mathrm{cris},n}(V)))$$

while the lower R_L-module is identified with

$$F^x(R_L \otimes_R ((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_K V)).$$

The proof is finished. □

3.4 The category $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$

Let $V = \bigoplus_{i=0}^{n-1} V_{\chi_{n,i}}$ be in $\mathbf{Rep}_K(G_{L,(n)})$. The theory of p-adic representations of G_L with coefficients in $\mathbb{Q}_p$ yields the estimate

$$\begin{aligned}[K:\mathbb{Q}_p] \cdot \mathrm{rank}_R(\mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V_{\chi_{n,i}}) \langle -\tfrac{i}{n} \rangle) &= \dim_{L_0}(\mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V_{\chi_{n,i}})) \\ &\leq \dim_{\mathbb{Q}_p}(V_{\chi_{n,i}}) \\ &= [K:\mathbb{Q}_p] \cdot \dim_K(V_{\chi_{n,i}})\end{aligned}$$

for all $i = 0, \ldots, n-1$ and thus

$$\begin{aligned}\mathrm{rank}_R(\mathbb{D}_{\mathrm{cris},n}(V)) &\overset{3.3.3}{=} \sum_{i=0}^{n-1} \mathrm{rank}_R(\mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V_{\chi_{n,i}})) \\ &\leq \sum_{i=0}^{n-1} \dim_K(V_{\chi_{n,j}}) = \dim_K(V).\end{aligned}$$

Those V whose associated filtered isocrystal $\mathbb{D}_{\mathrm{cris},n}(V)$ is of maximal R-rank make up the subcategory of $\mathbf{Rep}_K(G_{L,(n)})$ we are heading for.

Definition 3.4.1. We call V in $\mathbf{Rep}_K(G_{L,(n)})$ crystalline if

$$\mathrm{rank}_R(\mathbb{D}_{\mathrm{cris},n}(V)) = \dim_K(V).$$

The full subcategory of $\mathbf{Rep}_K(G_{L,(n)})$ consisting of crystalline representations is denoted by $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$.

Remark 3.4.2. The defining condition for being a crystalline representation of $G_{L,(n)}$ may equivalently be expressed as $\dim_{L_0}(\mathbb{D}_{\mathrm{cris},n}(V)) = \dim_{\mathbb{Q}_p}(V)$. By the above estimate, for V to lie in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$ it is necessary and sufficient that all the $\varepsilon_n^i \otimes V_{\chi_{n,i}}$ lie in $\mathbf{Rep}_K^{\mathrm{cris}}(G_L)$. Note that $\mathbf{Rep}_K^{\mathrm{cris}}(G_L)$ consists by definition of those W in $\mathbf{Rep}_K(G_L)$ such that the representation obtained by forgetting the K-vector space structure lies in $\mathbf{Rep}^{\mathrm{cris}}(G_L)$. Furthermore, by [CoFo, Theorem A] all the $\varepsilon_n^i \otimes V_{\chi_{n,i}}$ lie in $\mathbf{Rep}_K^{\mathrm{cris}}(G_L)$ if and only if all the $\mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V_{\chi_{n,i}})$ lie in $\mathbf{FIC}_{L,K}^{\mathrm{adm}}$. In this case, the direct summands in the decomposition of $\mathbb{D}_{\mathrm{cris},n}(V)$ from 3.3.3 are tensor products of objects in $\mathbf{FIC}_{L,K,(n)}^{\mathrm{wa}}$. The latter category is stable under tensor products and direct sums. It follows that $\mathbb{D}_{\mathrm{cris},n}$ restricts to a functor

$$\mathbb{D}_{\mathrm{cris},n} : \mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)}) \to \mathbf{FIC}_{L,K,(n)}^{\mathrm{wa}}.$$

Furthermore, again by [CoFo, Theorem A] and by 2.2.14, every object of $\mathbf{FIC}_{L,K,(n)}^{\mathrm{wa}}$ can be written uniquely (up to isomorphism) as a direct sum

$$\bigoplus_{i=0}^{n-1} \mathbb{D}_{\mathrm{cris}}(V_i)\langle \tfrac{i}{n} \rangle$$

for some V_i in $\mathbf{Rep}_K^{\mathrm{cris}}(G_L)$.

Example 3.4.3. According to 3.3.4, the representation ε_n^i lies in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$ for all $i = 0, \dots, n-1$.

Proposition 3.4.4.

1. *The natural transformation $\alpha_{\bullet,n}$ from 3.3.5 is an isomorphism between functors from $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$ to $\mathbf{MF}_{L,K,n}^{\phi}$.*

2. *Let V be in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$. Any subquotient of V (as an object of $\mathbf{Rep}_K(G_{L,(n)})$) is again an object of $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$. Therefore $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$ is an abelian category.*

3. *The restriction of $\mathbb{D}_{\mathrm{cris},n}$ to $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$ is exact and naturally compatible with the formation of direct sums, tensor products and duals (in $\mathbf{Rep}_K(G_{L,(n)})$ resp. in $\mathbf{FIC}_{L,K,n}$). Therefore $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$ is closed under these constructions and in particular a tensor category.*

Proof. Ad 1.: Let V be in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$. Recall that the $\varepsilon_n^i \otimes V_{\chi_{n,i}}$ are crystalline G_L-representations. The maps $\alpha_{V,n,i}$ from the proof of 3.3.6 are all bijective. The maps induced by scalar extension yield R_L-isomorphisms between the filtration steps of the corresponding filtrations.
Ad 2.: We use a similar argument as in the proof of [Fo2, Proposition 1.5.2]. Let $V' \subseteq V$ be a subrepresentation. Applying $\mathbb{D}_{\mathrm{cris},n}$ to the canonical short exact sequence induced by V', we get an exact sequence

$$0 \to \mathbb{D}_{\mathrm{cris},n}(V') \to \mathbb{D}_{\mathrm{cris},n}(V) \to \mathbb{D}_{\mathrm{cris},n}(V/V')$$

in $\mathbf{Mod}(R)$ whose morphisms are compatible with the respective induced semi-linear operators. Denote the image of the third arrow by D. By using that D is Frobenius-invariant and hence free of finite rank by 2.1.6, we get the string of (in-)equalities

$$\begin{aligned}\dim_K(V) &= \operatorname{rank}_R(\mathbb{D}_{\mathrm{cris},n}(V)) \\ &= \operatorname{rank}_R(\mathbb{D}_{\mathrm{cris},n}(V')) + \operatorname{rank}_R(D) \\ &\leq \dim_K(V') + \dim_K(V/V') = \dim_K(V)\end{aligned}$$

and one concludes that V' and V/V' are crystalline. Hence any subquotient of V is crystalline.
Ad 3.: We first show exactness of $\mathbb{D}_{\mathrm{cris},n}$. Let

$$0 \to V' \xrightarrow{f} V \xrightarrow{g} V'' \to 0$$

be a short exact sequence in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$. According to *2.*, what remains to be shown is that the sequence obtained by applying the functor $R_L \otimes_R \mathbb{D}_{\mathrm{cris},n}(-)$ to the above one is an exact sequence of R_L-modules and that the R_L-module homomorphisms $R_L \otimes \mathbb{D}_{\mathrm{cris},n}(f)$ and $R_L \otimes \mathbb{D}_{\mathrm{cris},n}(g)$ are compatible with the filtrations. This is equivalent to showing that for all $j \in \frac{1}{n}\mathbb{Z}$ the sequences of R_L-modules

$$0 \to F^j \mathbb{D}_{\mathrm{cris},n}(V')_L \to F^j \mathbb{D}_{\mathrm{cris},n}(V)_L \to F^j \mathbb{D}_{\mathrm{cris},n}(V'')_L \to 0$$

are exact.
First note that, for any $i \in \{0, \ldots, n-1\}$, we obtain an exact sequence of K-linear crystalline and hence de Rham G_L-representations

$$0 \to \varepsilon_n^i \otimes V'_{\chi_{n,i}} \to \varepsilon_n^i \otimes V_{\chi_{n,i}} \to \varepsilon_n^i \otimes V''_{\chi_{n,i}} \to 0$$

(cf. 1.3.2). Fix some $j \in \frac{1}{n}\mathbb{Z}$. Then there is $i \in \{0, \ldots, n-1\}$ such that $j \in \frac{i}{n} + \mathbb{Z}$. By [Fo2, Théorèm 3.8], the sequence of associated graded pieces

$$0 \to \mathrm{gr}^j \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V'_{\chi_{n,i}})_L \to \mathrm{gr}^j \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V_{\chi_{n,i}})_L \to \mathrm{gr}^j \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V''_{\chi_{n,i}})_L \to 0$$

is an exact sequence of R_L-modules. The graded pieces are formed with respect to the corresponding $\mathbb{Z}$-filtrations, twisted by $-\frac{i}{n}$. Thus we obtain for the dimensions of the underlying L-vector spaces

$$\begin{aligned}&\dim_L(F^j \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V_{\chi_{n,i}})_L) \\ = &\sum_{j' \in (\frac{i}{n}+\mathbb{Z})_{\geq j}} \dim_L(\mathrm{gr}^j \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V_{\chi_{n,i}})_L) \\ = &\sum_{j' \in (\frac{i}{n}+\mathbb{Z})_{\geq j}} \dim_L(\mathrm{gr}^j \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V'_{\chi_{n,i}})_L) \\ &\quad + \sum_{j' \in (\frac{i}{n}+\mathbb{Z})_{\geq j}} \dim_L(\mathrm{gr}^j \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V''_{\chi_{n,i}})_L) \\ = &\dim_L(F^j \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V'_{\chi_{n,i}})_L) + \dim_L(F^j \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V''_{\chi_{n,i}})_L).\end{aligned}$$

Now summation on both ends of this chain of equations over all $i = 0, \ldots, n-1$ and arguing similarly as in *2.* yields the desired exactness.
Compatibility with direct sums is clear by additivity of $\mathbb{D}_{\mathrm{cris},n}$.

Let V, W be in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$. The decomposition of $V \otimes W$ into H_n-eigenspaces is

$$V \otimes W = \bigoplus_{i=0}^{n-1} \underbrace{\bigoplus_{k=0}^{n-1} (V_{\chi_{n,k}} \otimes W_{\chi_{n,[i-k]}})}_{=(V\otimes W)_{\chi_{n,i}}}$$

where again $[i-k]_n$ denotes the unique representative in $\{0, \dots, n-1\}$ of the class $(i-k) + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$. Using 3.3.3 we obtain

$$\mathbb{D}_{\mathrm{cris},n}(V \otimes W) = \bigoplus_{i=0}^{n-1} [\bigoplus_{k=0}^{n-1} \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes (V_{\chi_{n,k}} \otimes W_{\chi_{n,[i-k]}}))]\langle -\tfrac{i}{n} \rangle.$$

We rewrite this double direct sum as

$$\bigoplus_{l=0}^{n-1} \bigoplus_{m=0}^{n-1} \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^{l+m} \otimes (V_{\chi_{n,l}} \otimes W_{\chi_{n,m}}))\langle -\tfrac{l+m}{n} \rangle,$$

where we make use of the isomorphism in $\mathbf{FIC}_{L,K}$

$$\begin{aligned} &\mathbb{D}_{\mathrm{cris}}(\varepsilon \otimes Y) \to \mathbb{D}_{\mathrm{cris}}(Y)\langle 1 \rangle, \\ &\sum_s b_s \otimes y_s \mapsto \sum_s b_s t \otimes y_s \quad (Y \text{ in } \mathbf{Rep}_K(G_L)) \end{aligned}$$

whenever $l + m \geq n$ for a pair of indices (l, m). By assumption on V and W, $\varepsilon_n^l \otimes V_{\chi_{n,l}}$ and $\varepsilon_n^m \otimes W_{\chi_{n,m}}$ lie in $\mathbf{Rep}_K^{\mathrm{cris}}(G_L)$ for every pair of indices (l, m). The functor $\mathbb{D}_{\mathrm{cris}}$ commutes with $\otimes$ on crystalline G_L-representations. Therefore the latter direct sum is isomorphic with

$$\bigoplus_{l=0}^{n-1} \bigoplus_{m=0}^{n-1} (\mathbb{D}_{\mathrm{cris}}(\varepsilon_n^l \otimes V_{\chi_{n,l}})\langle -\tfrac{l}{n} \rangle \otimes \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^m \otimes W_{\chi_{n,m}})\langle -\tfrac{m}{n} \rangle).$$

This is exactly the decomposition of $\mathbb{D}_{\mathrm{cris},n}(V) \otimes \mathbb{D}_{\mathrm{cris},n}(W)$ into H_n-eigenspaces. Comparing K-dimensions and R-ranks shows that $V \otimes W$ is crystalline.
As for duals, for V in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$ we compute on the one hand

$$\begin{aligned} \mathbb{D}_{\mathrm{cris},n}(V)^\vee &\cong \bigoplus_{i=0}^{n-1} \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V_{\chi_{n,i}})\langle -\tfrac{i}{n} \rangle^\vee && (6) \\ &\cong \bigoplus_{i=0}^{n-1} \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^{i-n} \otimes V_{\chi_{n,i}})\langle \tfrac{n-i}{n} \rangle^\vee && (7) \\ &\cong \bigoplus_{i=0}^{n-1} \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^{n-i} \otimes (V^\vee)_{\chi_{n,i}^{-1}})\langle -\tfrac{n-i}{n} \rangle. && (8) \end{aligned}$$

The isomorphism in (6) is induced by the compatibility of the functor $\mathrm{Hom}_R(-, R)$ with finite direct sums, in (7) the isomorphism in $\mathbf{FIC}_{L,K}$

$$\mathbb{D}_{\mathrm{cris}}(Y) \cong \mathbb{D}_{\mathrm{cris}}(\varepsilon^{-1} \otimes Y)\langle 1 \rangle \quad (Y \text{ in } \mathbf{Rep}_K(G_L))$$

was applied and in (8) we have used the compatibility of $\mathbb{D}_{\mathrm{cris}}$ with duals of crystalline representations of G_L.
On the other hand we naturally have

$$\mathbb{D}_{\mathrm{cris},n}(V^\vee) \cong \bigoplus_{i=0}^{n-1} \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes (V^\vee)_{\chi_{n,i}})\langle -\tfrac{i}{n}\rangle.$$

Up to order of the direct summands this is the decomposition of $\mathbb{D}_{\mathrm{cris},n}(V)^\vee$ from above. We conclude that $V^\vee$ is a crystalline representation of $G_{L,(n)}$. This finishes the proof. □

4 Relating $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$ and $\mathbf{Rep}^{\mathrm{cris}}_{K}(G_{L,(n)})$

Setup and the notations are the same as in the previous sections. We establish an equivalence between $\mathbf{Rep}^{\mathrm{cris}}_{K}(G_{L,(n)})$ and $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$ induced by $\mathbb{D}_{\mathrm{cris},n}$. Moreover, we discuss differences with respect to the classical equivalence between the categories of K-linear crystalline G_L-representations resp. weakly admissible $\mathbb{Z}$-filtered isocrystals over L with coefficients in K.

4.1 The functor $\mathbb{V}_{\mathrm{cris},n}$

In this subsection a quasi-inverse to the restriction of $\mathbb{D}_{\mathrm{cris},n}$ to $\mathbf{Rep}^{\mathrm{cris}}_{K}(G_{L,(n)})$ will be constructed.

Remark 4.1.1. Let $\underline{D} = (D, \phi, F^{\bullet}D_L)$ be in $\mathbf{MF}^{\phi}_{L,K,n}$. Then the object

$$((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R D, \tilde{\varphi}_0 \otimes \phi, \text{tensor product filtration})$$

lies in $\mathbf{MF}^{\phi}_{L,K,n}$, the filtration on $R_L \otimes_R (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K)$ being induced by the injective ring homomorphism

$$(B_{\mathrm{cris},n})_L \to B_{\mathrm{dR},n}, \quad l \otimes b \mapsto lb.$$

Setting $\underline{D} = \underline{R}$ and using 3.2.8, one checks that

$$(B_{\mathrm{cris}} t_n^i \otimes_{\mathbb{Q}_p} K, \text{restriction of } \tilde{\varphi}_0, \text{induced filtration})$$

is isomorphic to

$$(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K, \varphi_0 \otimes K, \text{filtration induced by } B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K)\langle -\tfrac{i}{n} \rangle$$

in $\mathbf{MF}^{\phi}_{L,K,n}$ for all $i \in \{0, \dots, n-1\}$.

Definition 4.1.2. Let $(D, \phi, F^{\bullet}D_L)$ be in $\mathbf{MF}^{\phi}_{L,K,n}$. We define

$\mathbb{V}_{\mathrm{cris},n}((D, \phi, F^{\bullet}D_L)) := \mathbb{V}_{\mathrm{cris},n}(D) :=$
the subset of $(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R D$ consisting of the elements x such that $(\tilde{\varphi}_0 \otimes \phi)(x) = x$ and such that $1 \otimes x$ is contained in $F^0\left((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R D\right)_L$.

Lemma 4.1.3. *The assignments $(D, \phi, F^{\bullet}D_L) \mapsto \mathbb{V}_{\mathrm{cris},n}(D)$ and*

$$[f : D \to D'] \mapsto \textit{restriction of } \left[(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes f\right] \textit{ to } \mathbb{V}_{\mathrm{cris},n}(D)$$

define a covariant functor from $\mathbf{MF}^{\phi}_{L,K,n}$ to the category of K-vector spaces with an action of $G_{L,(n)}$.

Proof. The K-vector space structure of $\mathbb{V}_{\mathrm{cris},n}(D)$ is obtained by restricting scalars along the natural morphism $K \to R$ (resp. $K \to R_L$) given as $x \mapsto 1 \otimes x$ in both cases. Then $\tilde{\varphi}_0 \otimes \phi$ is K-linear and the filtration steps of the tensor product filtration associated with $(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R D$ are K-vector spaces. The group $G_{L,(n)}$ acts K-linearly on $(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R D$ via its action on the left tensor factor. By compatibility of this action with $\tilde{\varphi}_0 \otimes \phi$ and the tensor

product filtration, it restricts to an action on $\mathbb{V}_{\mathrm{cris},n}(D)$.
Applying the definition of a morphism $f : D \to D'$ in $\mathbf{MF}^{\phi}_{L,K,n}$, one sees that $\mathbb{V}_{\mathrm{cris},n}(f)$ is a K-linear map $\mathbb{V}_{\mathrm{cris},n}(D) \to \mathbb{V}_{\mathrm{cris},n}(D')$. The identity $\mathbb{V}_{\mathrm{cris},n}(\mathrm{id}_D) = \mathrm{id}_{\mathbb{V}_{\mathrm{cris},n}(D)}$ and compatibility with composition of morphisms are immediate. □

Remark 4.1.4. Let $\underline{D}$ be an object of $\mathbf{FIC}^{\mathrm{adm}}_{L,K}$. By [CoFo, Théorèm A], [Fo2, Théorèm 5.3.5] and the same arguments from the above proof concerning the K-vector space structure of $\mathbb{V}_{\mathrm{cris},n}(D)$, there is an object V in $\mathbf{Rep}^{\mathrm{cris}}_K(G_L)$ together with isomorphisms

$$\mathbb{D}_{\mathrm{cris}}(V) \cong \mathbb{D}_{\mathrm{cris}}(\mathbb{V}_{\mathrm{cris}}(D)) \cong \underline{D} \qquad \text{in } \mathbf{Rep}^{\mathrm{cris}}_K(G_L)$$

and

$$V \cong \mathbb{V}_{\mathrm{cris}}(\mathbb{D}_{\mathrm{cris}}(V)) \cong \mathbb{V}_{\mathrm{cris}}(D) \qquad \text{in } \mathbf{FIC}^{\mathrm{adm}}_{L,K}.$$

It follows that $\mathbb{D}_{\mathrm{cris}}$ and $\mathbb{V}_{\mathrm{cris}}$ are mutually quasi-inverse functors between $\mathbf{Rep}^{\mathrm{cris}}_K(G_L)$ and $\mathbf{FIC}^{\mathrm{adm}}_{L,K}$.

Next we describe the restriction of the functor $\mathbb{V}_{\mathrm{cris},n}$ to the category $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$. For this we recall the definition of separability of an algebra over a field (cf. [BouA$_2$, V, §15, no. 2, Definition 1]) and formulate several auxiliary results.

Definition 4.1.5. Let A be a ring. It is called reduced if for an element $a \in A$ and an integer $m \geq 1$ the equality $a^m = 0$ always implies $a = 0$.
Let B be an algebra over a field E. Then B is called a separable E-algebra if the ring $B \otimes_E E'$ is reduced for every field extension E' of E.

Remark 4.1.6. If B is an algebraic field extension of E then B is separable as an E-algebra if and only if it is separable as an algebraic field extension of E.

Proposition 4.1.7. *Let B, B' be algebras over a field E. If B is a reduced ring and B' is separable then the tensor product $B \otimes_E B'$ is a reduced ring.*

Proof. This is [BouA$_2$, V, §15, no. 2, Proposition 5]. □

Example 4.1.8. Setting $E := \mathbb{Q}_p, B := B_{\mathrm{cris}}, B' := K$ in 4.1.7 we see that $B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K$ is reduced.

Lemma 4.1.9. *Let E be a field, B be an E-algebra and E' be an arbitrary field extension of E. Assume moreover that the underlying ring of B is a filtered ring with a decreasing $\mathbb{Z}$-filtration $F^\bullet B$ by E-vector spaces. Then the tensor product $B \otimes_E E'$ becomes a filtered ring with a decreasing $\mathbb{Z}$-filtration by E'-vector spaces via the definition*

$$F^m(B \otimes_E E') := (F^m B) \otimes_E E' \qquad (m \in \mathbb{Z}).$$

Proof. It is clear that the filtration steps of the tensor product are E'-vector spaces. Let k, l be integers. Choose E-bases $(b_{ik})_{i \in I_k}$ and $(b_{il})_{i \in I_l}$ of $F^k B$ and $F^l B$ respectively. Then for $x = \sum_{i \in I_k} b_{ik} \otimes \alpha_i \in F^k(B \otimes_E E')$ and $y \in \sum_{j \in I_l} b_{jl} \otimes \beta_j \in F^l(B \otimes_E E')$ we have

$$xy = \sum_{i \in I_k, j \in I_l} (b_{ik} b_{jl} \otimes \alpha_i \beta_j) \in F^{k+l}(B \otimes_E E')$$

since the containment is true for every single summand. □

We endow the ring $B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K$ with the $\mathbb{Z}$-filtration

$$F^\bullet(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K) := [\rho^{-1}(F^\bullet B_{\mathrm{dR}})] \otimes_{\mathbb{Q}_p} K.$$

Here ρ denotes the injective composite ring homomorphism

$$B_{\mathrm{cris}} \overset{\rho_1}{\to} L \otimes_{L_0} B_{\mathrm{cris}} \overset{\rho_2}{\to} B_{\mathrm{dR}}$$

with $\rho_1(b) = 1 \otimes b$ and $\rho_2(l \otimes b') = lb'$.
Moreover, for any $r \in \mathbb{Z}$ and $\lambda \in B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K$ we define the set

$$F^r(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K)^{\varphi_0 = \lambda} :=$$

$$\{x \in B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K \mid \varphi_0(x) = \lambda x, x \in F^r(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K)\}$$

Proposition 4.1.10. *The functor $\mathbb{V}_{\mathrm{cris},n}$ restricts to a functor $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)} \to \mathbf{Rep}^{\mathrm{cris}}_K(G_{L,(n)})$ compatible with $\otimes$ in both categories.*

Proof. Let $\underline{D} = (D, \phi, F^\bullet D_L) = \bigoplus_{j=0}^{n-1} \underbrace{(D_j, \phi_{D_j}, F^\bullet D_{j,L})}_{=:\underline{D}_j}\langle \frac{j}{n} \rangle$ in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$ where the $\underline{D}_j$ are in $\mathbf{FIC}^{\mathrm{adm}}_{L,K}$. Then $(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R D$ decomposes in $\mathbf{MF}^\phi_{L,K,n}$ as

$$\begin{aligned}(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R D &\cong \bigoplus_{i=0}^{n-1} \bigoplus_{j=0}^{n-1} ((B_{\mathrm{cris}} t_n^i \otimes_{\mathbb{Q}_p} K) \otimes_R D_j)\langle \tfrac{j}{n} \rangle \\ &\cong \bigoplus_{i=0}^{n-1} \bigoplus_{j=0}^{n-1} \underbrace{((B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K) \otimes_R D_j)\langle \tfrac{j-i}{n} \rangle}_{=:M_{ij}}.\end{aligned}$$

Hence, to any $v \in \mathbb{V}_{\mathrm{cris},n}(D)$ there corresponds a unique family (v_{ij}) of elements $v_{ij} \in M_{ij}$, each of which is invariant under the respective semilinear operator and at the same time is contained in the zeroth filtration step of the scalar extension to R_L. Let $i \neq j$. We *claim* that in this case the intersection of $\mathbb{V}_{\mathrm{cris},n}(D)$ with the filtered ϕ-submodule of $(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R D$ corresponding to M_{ij} under the above decomposition is $\{0\}$.
Proof of claim: Denote the corresponding ϕ-submodule also by M_{ij}. There exist $V_j \in \mathbf{Rep}^{\mathrm{cris}}_K(G_L)$ and isomorphisms between $\mathbb{D}_{\mathrm{cris}}(V_j)$ and $\underline{D}_j$ in $\mathbf{FIC}^{\mathrm{adm}}_{L,K}$ for all $j = 0, \ldots, n-1$. By [Fo2, Proposition 5.3.6] we may therefore identify M_{ij} and $((B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K) \otimes_K V_j)\langle \frac{j-i}{n} \rangle$ in $\mathbf{MF}^\phi_{L,K,n}$ and thus also the K-vector spaces $\mathbb{V}_{\mathrm{cris},n}(D) \cap M_{ij}$ and

$$F^{\lceil \frac{j-i}{n} \rceil}(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K)^{\varphi_0 = 1 \otimes p^{\frac{j-i}{n}}} \otimes_K V_j.$$

Note that the fractions $\frac{j-i}{n}$ attain all values in the set

$$[(\frac{1}{n}\mathbb{Z})_{\geq -\frac{n-1}{n}} \cap (\frac{1}{n}\mathbb{Z})_{\leq \frac{n-1}{n}}] \setminus \{0\}$$

as i and j from the assumption of the claim run through $\{0, \ldots, n-1\}$. The number $\lceil \frac{j-i}{n} \rceil$ is therefore either 0 or 1, depending on whether $j < i$ or $j > i$.

We show that $F^{\lceil \frac{j-i}{n} \rceil}(B_{\text{cris}} \otimes_{\mathbb{Q}_p} K)^{\varphi_0 = 1 \otimes p^{\frac{j-i}{n}}}$ is zero in both cases.
Let $j < i$ and

$$x \in F^{\lceil \frac{j-i}{n} \rceil}(B_{\text{cris}} \otimes_{\mathbb{Q}_p} K)^{\varphi_0 = 1 \otimes p^{\frac{j-i}{n}}} = F^0(B_{\text{cris}} \otimes_{\mathbb{Q}_p} K)^{\varphi_0 = 1 \otimes p^{\frac{j-i}{n}}}.$$

Since the filtration on $B_{\text{cris}} \otimes_{\mathbb{Q}_p} K$ defined above satisfies the assumptions of 4.1.9 with $E := \mathbb{Q}_p$, $B := B_{\text{cris}}$ and $E' := K$, we obtain $x^n \in F^0(B_{\text{cris}} \otimes_{\mathbb{Q}_p} K)^{\varphi_0 = p^{j-i}}$. Therefore we get

$$\begin{aligned}
(t^{i-j} \otimes 1)x^n &\in F^{i-j}(B_{\text{cris}} \otimes_{\mathbb{Q}_p} K)^{\varphi_0 = 1} \\
&\subseteq F^1(B_{\text{cris}} \otimes_{\mathbb{Q}_p} K)^{\varphi_0 = 1} \\
&= (F^1 B_{\text{cris}})^{\varphi_0 = 1} \otimes_{\mathbb{Q}_p} K = \{0\}.
\end{aligned}$$

The last equality holds by [CoFo, Proposition 1.3 i)]. Now the element $t^{i-j} \otimes 1$ is invertible in $B_{\text{cris}} \otimes_{\mathbb{Q}_p} K$ and therefore $x^n = 0$. According to 4.1.8 it follows that $x = 0$.
Let $i < j$ and

$$x \in F^{\lceil \frac{j-i}{n} \rceil}(B_{\text{cris}} \otimes_{\mathbb{Q}_p} K)^{\varphi_0 = 1 \otimes p^{\frac{j-i}{n}}} = F^1(B_{\text{cris}} \otimes_{\mathbb{Q}_p} K)^{\varphi_0 = 1 \otimes p^{\frac{j-i}{n}}}.$$

Therefore we obtain $x^n \in F^n(B_{\text{cris}} \otimes_{\mathbb{Q}_p} K)^{\varphi_0 = p^{j-i}}$. Recall that $n + i - j \geq 1$ by assumption whence

$$(t^{i-j} \otimes 1)x^n \in F^{n+i-j}(B_{\text{cris}} \otimes_{\mathbb{Q}_p} K)^{\varphi_0 = 1} \subseteq F^1(B_{\text{cris}} \otimes_{\mathbb{Q}_p} K)^{\varphi_0 = 1}.$$

As in the previous case we conclude that $x = 0$. The claim is proved.

We continue with the proof of 4.1.10. Taking into account the $G_{L,(n)}$- action on $B_{\text{cris}} t_n^i \otimes_{\mathbb{Q}_p} K$ for $i = 0, \ldots, n-1$ as well as the claim just proved we find the decomposition

$$\mathbb{V}_{\text{cris},n}(D) \cong \bigoplus_{i=0}^{n-1} \varepsilon_n^i \otimes \mathbb{V}_{\text{cris}}(D_i) \cong \bigoplus_{i=0}^{n-1} \varepsilon_n^i \otimes V_i,$$

whose direct summands are isomorphic with $\mathbb{V}_{\text{cris},n}(\underline{D}_i\langle \frac{i}{n} \rangle)$, respectively. Hence $\mathbb{V}_{\text{cris},n}$ restricted to $\mathbf{FIC}^{\text{wa}}_{L,K,(n)}$ naturally takes values in $\mathbf{Rep}_K(G_{L,(n)})$ (cf. also [BriCo, Proposition 9.3.6]). In order to see that $\mathbb{V}_{\text{cris},n}(D)$ lies in $\mathbf{Rep}^{\text{cris}}_K(G_{L,(n)})$ note that

$$\mathbb{V}_{\text{cris},n}(D)_{\chi_{n,i}} = \begin{cases} \mathbb{V}_{\text{cris}}(D_0) & i = 0 \\ \varepsilon_n^{n-i} \otimes \mathbb{V}_{\text{cris}}(D_{n-i}) & 1 \leq i \leq n-1. \end{cases}$$

Concerning R-ranks and K-dimensions we therefore obtain

$$
\begin{aligned}
&\operatorname{rank}_R(\mathbb{D}_{\mathrm{cris},n}(\mathbb{V}_{\mathrm{cris},n}(D)))\\
=&\sum_{i=0}^{n-1}\operatorname{rank}_R(\mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i\otimes\mathbb{V}_{\mathrm{cris},n}(D)_{\chi_{n,i}}))\\
=&\operatorname{rank}_R(\mathbb{D}_{\mathrm{cris}}(\mathbb{V}_{\mathrm{cris}}(D_0)))+\sum_{i=1}^{n-1}\operatorname{rank}_R(\mathbb{D}_{\mathrm{cris}}(\mathbb{V}_{\mathrm{cris}}(D_{n-i})))\\
=&\sum_{i=0}^{n-1}\dim_K(\mathbb{V}_{\mathrm{cris}}(D_i))\\
=&\dim_K(\mathbb{V}_{\mathrm{cris},n}(D)),
\end{aligned}
$$

so the restriction of $\mathbb{V}_{\mathrm{cris},n}$ to $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$ indeed takes values in $\mathbf{Rep}^{\mathrm{cris}}_K(G_{L,(n)})$. In the second equality, the identity $\operatorname{rank}_R(\mathbb{D}_{\mathrm{cris}}(V)) = \operatorname{rank}_R(\mathbb{D}_{\mathrm{cris}}(\varepsilon\otimes V))$ for V in $\mathbf{Rep}^{\mathrm{cris}}_K(G_L)$ is applied. In the third equality, the fact that all the $\mathbb{V}_{\mathrm{cris}}(D_i)$ $(i=0,\dots,n-1)$ are crystalline is used.
As for compatibility of $\mathbb{V}_{\mathrm{cris},n}$ with $\otimes$, let $\underline{D}=\bigoplus_{i=0}^{n-1}\underline{D}_i\langle\frac{i}{n}\rangle$ and $\underline{E}=\bigoplus_{i=0}^{n-1}\underline{E}_i\langle\frac{i}{n}\rangle$ be in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$. The functor $\mathbb{V}_{\mathrm{cris}}$ is additive and commutes with $\otimes$ on $\mathbf{FIC}^{\mathrm{adm}}_{L,K}$. We therefore get

$$
\begin{aligned}
\mathbb{V}_{\mathrm{cris},n}(\underline{D}\otimes\underline{E}) &\cong \mathbb{V}_{\mathrm{cris},n}\Big(\bigoplus_{k=0}^{n-1}\bigoplus_{i+j\in k+n\mathbb{Z}}((\underline{D}_i\otimes\underline{E}_j)\langle\lfloor\tfrac{i+j}{n}\rfloor\rangle)\langle\tfrac{k}{n}\rangle\Big)\\
&\cong \bigoplus_{k=0}^{n-1}\bigoplus_{i+j\in k+n\mathbb{Z}}[\varepsilon_n^k\otimes\varepsilon^{\lfloor\frac{i+j}{n}\rfloor}\otimes\mathbb{V}_{\mathrm{cris}}(D_i\otimes_R E_j)]\\
&\cong \bigoplus_{i,j=0}^{n-1}(\varepsilon_n^{i+j}\otimes\mathbb{V}_{\mathrm{cris}}(D_i\otimes_R E_j))\\
&\cong \Big[\bigoplus_{i=0}^{n-1}(\varepsilon_n^i\otimes\mathbb{V}_{\mathrm{cris}}(D_i))\Big]\otimes\Big[\bigoplus_{i=0}^{n-1}(\varepsilon_n^i\otimes\mathbb{V}_{\mathrm{cris}}(E_i))\Big]\\
&\cong \mathbb{V}_{\mathrm{cris},n}(\underline{D})\otimes\mathbb{V}_{\mathrm{cris},n}(\underline{E}).
\end{aligned}
$$

In the first isomorphism we use the computation from 2.2.15 and in the second line by a zeroth power of ε we mean the trivial G_L-representation with underlying space K.
It remains to be seen that $\mathbb{V}_{\mathrm{cris},n}$ respects unit objects. Evaluating $\mathbb{V}_{\mathrm{cris},n}$ on $\underline{R}$ (cf. 2.1.4) we find

$$
\begin{aligned}
\mathbb{V}_{\mathrm{cris},n}(\underline{R}) &= \left\{x\in B_{\mathrm{cris},n}\otimes_{\mathbb{Q}_p}K \mid x=\tilde{\varphi}_0(x), 1\otimes x\in F^0\left(B_{\mathrm{cris},n}\otimes_{\mathbb{Q}_p}K\right)_L\right\}\\
&= \left\{x\in B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}K \mid x=(\varphi_0\otimes 1)(x), 1\otimes x\in F^0\left(B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p}K\right)_L\right\}\\
&= \text{ the trivial } G_{L,(n)}\text{-representation } K.
\end{aligned}
$$

The second equality follows from the above claim. The third equality follows from the fundamental exact sequence of $\mathbb{Q}_p$-vector spaces

$$0 \to \mathbb{Q}_p \to B_{\mathrm{cris}}^{\varphi_0=1} \to B_{\mathrm{dR}}/F^0 B_{\mathrm{dR}} \to 0.$$

This finishes the proof. □

Example 4.1.11. The previous proposition and its proof yield

$$\mathbb{V}_{\mathrm{cris},n}(\underline{K}_n^{\otimes i}) \cong \varepsilon_n^i \otimes \mathbb{V}_{\mathrm{cris}}(\underline{R}) \cong \varepsilon_n^i$$

in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$ for all $i = 0, \ldots n-1$.

4.2 An equivalence between $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$ and $\mathbf{FIC}_{L,K,(n)}^{\mathrm{wa}}$

First we prove a result similar to 3.4.4 with respect to the functor $\mathbb{V}_{\mathrm{cris},n}$.

Proposition 4.2.1. *The family of maps*

$$(\beta_{D,n} : (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_K \mathbb{V}_{\mathrm{cris},n}(D) \to (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R D,\ b \otimes v \mapsto bv)_{\underline{D}},$$

where $\underline{D}$ runs through the category $\mathbf{FIC}_{L,K,(n)}^{\mathrm{wa}}$, defines a natural isomorphism

$$\beta_{\bullet,n} : (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_K \mathbb{V}_{\mathrm{cris},n}(-) \xrightarrow{\sim} (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes_R (-)$$

of functors from $\mathbf{FIC}_{L,K,(n)}^{\mathrm{wa}}$ to $\mathbf{MF}_{L,K,n}^{\phi}$. Every $\beta_{D,n}$ is $G_{L,(n)}$-equivariant.

Proof. This is accomplished in a similar manner as the one used in 3.4.4, so we only sketch the proof.
Let $(D', \phi, F^\bullet D'_L)$ be in $\mathbf{FIC}_{L,K}^{\mathrm{adm}}$. By [Fo2, Proposition 5.3.6] and [CoFo, Theorem A] the map

$$B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} \mathbb{V}_{\mathrm{cris}}(D') \to B_{\mathrm{cris}} \otimes_{L_0} D', \quad b \otimes d \mapsto bd$$

is an isomorphism in $\mathbf{MF}_{L,K}^{\phi}$ which is G_L-equivariant.
Let $\underline{D} = \bigoplus_{i=0}^{n-1} \underline{D}_i\langle \frac{i}{n} \rangle$ in $\mathbf{FIC}_{L,K,(n)}^{\mathrm{wa}}$. The domain of $\beta_{D,n}$ decomposes in $\mathbf{MF}_{L,K,n}^{\phi}$ as

$$\bigoplus_{i,j=0}^{n-1} \underbrace{(B_{\mathrm{cris}} t_n^i \otimes_{\mathbb{Q}_p} K) \otimes_K (\varepsilon_n^j \otimes \mathbb{V}_{\mathrm{cris}}(D_j))}_{:=A_{ij}}$$

while the target decomposes in $\mathbf{MF}_{L,K,n}^{\phi}$ as

$$\bigoplus_{i,j=0}^{n-1} \underbrace{(B_{\mathrm{cris}} t_n^{[i+j]_n} \otimes_{\mathbb{Q}_p} K) \otimes_R D_j\langle \tfrac{j}{n} \rangle}_{:=B_{ij}}.$$

Here $[i+j]_n$ denotes again the residue of $i+j$ after division by n. By the inclusion $\varepsilon_n^j \otimes \mathbb{V}_{\mathrm{cris}}(D_j) \subseteq (B_{\mathrm{cris}} t_n^j \otimes_K K) \otimes_R \underline{D}_j\langle \frac{j}{n} \rangle$ (which holds by construction) and by the classical situation described above, we find $\beta_{D,n}(A_{ij}) = B_{ij}$, compatible with the respective Frobenius and filtration structures on both sides for all $i, j = 0, \ldots, n-1$. It follows that $\beta_{D,n}$ is a $G_{L,(n)}$-equivariant isomorphism in $\mathbf{MF}_{L,K,n}^{\phi}$.

The $G_{L,(n)}$-equivariance of $\beta_{D,n}$ is straightforward to check on the respective restrictions of $\beta_{D,n}$ to A_{ij}. The definition of $\mathbb{V}_{\mathrm{cris},n}(h)$ for a morphism $h : \underline{D} \to \underline{D}'$ in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$ implies the identity

$$\beta_{D',n} \circ ((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes \mathbb{V}_{\mathrm{cris},n}(h)) = ((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes h) \circ \beta_{D,n}.$$

□

Now we prove one of our main results.

Theorem 4.2.2. *The functor*

$$\mathbb{D}_{\mathrm{cris},n} : \mathbf{Rep}^{\mathrm{cris}}_K(G_{L,(n)}) \to \mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$$

is an equivalence of tensor categories. A quasi-inverse is given by the restriction of $\mathbb{V}_{\mathrm{cris},n}$ *to* $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$.

Proof. For any category $\mathbf{C}$, denote by $\mathbb{1}_{\mathbf{C}} : \mathbf{C} \to \mathbf{C}$ the identity functor of $\mathbf{C}$. We construct natural isomorphisms

$$\tau_\bullet : \mathbb{V}_{\mathrm{cris},n} \circ \mathbb{D}_{\mathrm{cris},n} \tilde{\to} \mathbb{1}_{\mathbf{Rep}^{\mathrm{cris}}_K(G_{L,(n)})}, \quad \tau'_\bullet : \mathbb{D}_{\mathrm{cris},n} \circ \mathbb{V}_{\mathrm{cris},n} \tilde{\to} \mathbb{1}_{\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}}.$$

As for $\tau_\bullet$, let $f : V \to W$ be a morphism in $\mathbf{Rep}^{\mathrm{cris}}_K(G_{L,(n)})$. According to 3.4.4, $\alpha_{V,n}$ and $\alpha_{W,n}$ are isomorphisms in $\mathbf{MF}^{\phi}_{L,K,n}$ and the equality

$$\alpha_{W,n} \circ ((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes \mathbb{D}_{\mathrm{cris},n}(f)) = ((B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) \otimes f) \circ \alpha_{V,n}$$

holds true. Restricting both maps to $\mathbb{V}_{\mathrm{cris},n}(\mathbb{D}_{\mathrm{cris},n}(V))$ therefore yields a commutative square of morphisms between K-vector spaces

$$\begin{array}{ccc}
\mathbb{V}_{\mathrm{cris},n}(\mathbb{D}_{\mathrm{cris},n}(V)) & \xrightarrow{\mathbb{V}_{\mathrm{cris},n}(\mathbb{D}_{\mathrm{cris},n}(f))} & \mathbb{V}_{\mathrm{cris},n}(\mathbb{D}_{\mathrm{cris},n}(W)) \\
{\scriptstyle \alpha_{V,n}|_{\mathbb{V}_{\mathrm{cris},n}(\mathbb{D}_{\mathrm{cris},n}(V))}} \Big\downarrow \cong & & \cong \Big\downarrow {\scriptstyle \alpha_{W,n}|_{\mathbb{V}_{\mathrm{cris},n}(\mathbb{D}_{\mathrm{cris},n}(W))}} \\
F^0(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K)^{\tilde{\varphi}_0=1} \otimes_K V & \longrightarrow & F^0(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K)^{\tilde{\varphi}_0=1} \otimes_K W.
\end{array}$$

Here the filtration on $B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K$ is given via

$$F^\bullet(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K) := F^\bullet(B_{\mathrm{cris},n}) \otimes_{\mathbb{Q}_p} K$$

and is induced from the one on $B_{\mathrm{dR},n}$. From the proof of 4.1.10 we know that $F^0(B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K)^{\tilde{\varphi}_0=1}$ is canonically isomorphic to K, whence the vector spaces in the lower row of the diagram are canonically isomorphic to V resp. W, say via ξ_V resp. ξ_W. The latter isomorphisms yield the identity

$$\xi_W \circ \alpha_{W,n} \circ (\mathbb{V}_{\mathrm{cris},n}(\mathbb{D}_{\mathrm{cris},n}(f))) = f \circ \xi_V \circ \alpha_{V,n}|_{\mathbb{V}_{\mathrm{cris},n}(\mathbb{D}_{\mathrm{cris},n}(V))}$$

and the association

$$V \mapsto \tau_V := \xi_V \circ \alpha_{V,n}|_{\mathbb{V}_{\mathrm{cris},n}(\mathbb{D}_{\mathrm{cris},n}(V))}$$

is then a natural isomorphism $\tau_\bullet$ between the functors $\mathbb{V}_{\mathrm{cris},n} \circ \mathbb{D}_{\mathrm{cris},n}$ and $\mathbb{1}_{\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})}$.
The isomophism $\tau'_\bullet$ is constructed similarly. Let $g : \underline{D} \to \underline{E}$ be a morphism in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$. The map $\beta_{D,n}$ from 4.2.1, being $G_{L,(n)}$-equivariant, restricts to a bijection between R-modules

$$\beta_{D,n}|_{\mathbb{D}_{\mathrm{cris},n}(\mathbb{V}_{\mathrm{cris},n}(D))} : \mathbb{D}_{\mathrm{cris},n}(\mathbb{V}_{\mathrm{cris},n}(D)) \tilde{\to} (B_{\mathrm{cris},n} \otimes_{\mathbb{Q}_p} K)^{G_{L,(n)}} \otimes_R D.$$

The right-hand side is naturally isomorphic with D by 3.2.12, via η_D say and similarly for E and η_E. As in the first part of the proof this discussion yields the identity

$$\eta_E \circ \beta_{E,n} \circ \mathbb{D}_{\mathrm{cris},n}(\mathbb{V}_{\mathrm{cris},n}(g)) = g \circ \eta_D \circ \beta_{D,n}|_{\mathbb{D}_{\mathrm{cris},n}(\mathbb{V}_{\mathrm{cris},n}(D))}$$

The association

$$\underline{D} \mapsto \tau'_{\underline{D}} := \eta_D \circ \beta_D|_{\mathbb{D}_{\mathrm{cris},n}(\mathbb{V}_{\mathrm{cris},n}(D))}$$

is then a natural isomorphism $\tau'_\bullet$ between $\mathbb{D}_{\mathrm{cris},n} \circ \mathbb{V}_{\mathrm{cris},n}$ and $\mathbb{1}_{\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}}$. $\square$

Remark 4.2.3. The following direct calculations also make the equivalence from the statement explicit, using methods and techniques from previous (sub)sections. Let V be in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$. Then in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$ we have canonical isomorphisms

$$\begin{aligned}
\mathbb{V}_{\mathrm{cris},n}(\mathbb{D}_{\mathrm{cris},n}(V)) &\cong \mathbb{V}_{\mathrm{cris},n}\Big(\bigoplus_{i=0}^{n-1} \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V_{\chi_{n,i}})\langle -\tfrac{i}{n}\rangle\Big) \\
&\cong \bigoplus_{i=0}^{n-1} \mathbb{V}_{\mathrm{cris},n}(\mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V_{\chi_{n,i}})\langle -1\rangle\langle \tfrac{n-i}{n}\rangle) \\
&\cong \bigoplus_{i=0}^{n-1} (\varepsilon_n^{n-i} \otimes \mathbb{V}_{\mathrm{cris}}(\mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes V_{\chi_{n,i}})\langle -1\rangle)) \\
&\cong \bigoplus_{i=0}^{n-1} (\varepsilon_n^{n-i} \otimes \varepsilon^{-1} \otimes \varepsilon_n^i \otimes V_{\chi_{n,i}}) \cong V.
\end{aligned}$$

In the second line we have used that $\underline{M} \cong \underline{M}\langle -1\rangle\langle 1\rangle$ for an object $\underline{M}$ of $\mathbf{MF}^{\phi}_{L,K,n}$ while in the third line we have used that $\mathbb{V}_{\mathrm{cris},n}(\underline{M}\langle \frac{i}{n}\rangle) \cong \varepsilon_n^i \otimes \mathbb{V}_{\mathrm{cris}}(M)$ in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$ for any $\underline{M}$ in $\mathbf{FIC}^{\mathrm{adm}}_{L,K}$ and any $i \in \{0, \dots, n-1\}$.
Vice versa, let $\underline{D} = \oplus_{i=0}^{n-1} \underline{D}_i\langle \frac{i}{n}\rangle$ be in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$. Then in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$ we have canonical isomorphisms

$$\begin{aligned}
\mathbb{D}_{\mathrm{cris},n}(\mathbb{V}_{\mathrm{cris},n}(D)) &\cong \mathbb{D}_{\mathrm{cris},n}\Big(\bigoplus_{i=0}^{n-1} \mathbb{V}_{\mathrm{cris},n}(D)_{\chi_{n,i}}\Big) \\
&\cong \mathbb{D}_{\mathrm{cris}}(\mathbb{V}_{\mathrm{cris}}(D_0)) \oplus \bigoplus_{i=1}^{n-1} \mathbb{D}_{\mathrm{cris}}(\varepsilon_n^i \otimes (\varepsilon_n^{n-i} \otimes \mathbb{V}_{\mathrm{cris}}(D_{n-i})))\langle -\tfrac{i}{n}\rangle \\
&\cong \underline{D}_0 \oplus \bigoplus_{i=1}^{n-1} \mathbb{D}_{\mathrm{cris}}(\mathbb{V}_{\mathrm{cris}}(D_{n-i}))\langle \tfrac{n-i}{n}\rangle \cong \underline{D}.
\end{aligned}$$

Here in the second line the isomorphism $\varepsilon_n^{n-i} \otimes \mathbb{V}_{\mathrm{cris}}(D_{n-i}) \cong \mathbb{V}_{\mathrm{cris},n}(D)_{\chi_{n,i}}$, $i \in \{1, \dots, n-1\}$, was used.

□

To summarize, the essential image of the restriction of $\mathbb{D}_{\mathrm{cris},n}$ to the category of crystalline $G_{L,(n)}$-representations with coefficients in K is exactly $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$, which may be described as the Tannakian subcategory of $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$ generated by $\mathbf{FIC}^{\mathrm{adm}}_{L,K}$ and the object $\underline{K}_n$. However, as we have seen in 2.2.17, the "inclusion" of $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$ in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$ is strict in general.

4.3 The relation between $\mathbf{Rep}^{\mathrm{cris}}_K(G_{L,(n)})$ and $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$

We keep the notations from the previous subsections and also refer to the notations introduced in the appendix. As explained in 4.1.4, the functor $\mathbb{V}_{\mathrm{cris}}$ establishes an equivalence between the category of weakly admissible $\mathbb{Z}$-filtered isocrystals over L with coefficients in K and the category of crystalline K-linear G_L-representations. However, combination of 2.2.17 and 4.2.2 yields that "being weakly admissible" for an object of $\mathbf{FIC}_{L,K,n}$ is, in general, not equivalent to lying in the essential image of $\mathbb{D}_{\mathrm{cris},n}$. Up to now, we have almost exclusively considered the restriction of $\mathbb{V}_{\mathrm{cris},n}$ to this essential image. This subsection is therefore concerned with the study of the behaviour of $\mathbb{V}_{\mathrm{cris},n}$ on $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$.

Let $\underline{D} = (D, \phi, F^\bullet D_L)$ be in $\mathbf{FIC}_{L,K,n}$. On D_L we have the map

$$\begin{aligned} v := v_{F^\bullet D_L} : D_L &\to \tfrac{1}{n}\mathbb{Z} \cup \{\infty\} \\ x &\mapsto \begin{cases} \infty & x = 0 \\ \min\limits_{\tau : x_\tau \neq 0} \quad \sum_{j \in \frac{1}{n}\mathbb{Z}} j \, \dim_K(\mathrm{gr}^j K x_\tau) & x \neq 0, \end{cases} \end{aligned}$$

where the graded pieces are formed with respect to $(F^\bullet D_L)_\tau \cap K x_\tau$ for all τ. The map v has the following properties.

Lemma 4.3.1. *Let $\underline{D}$ and $v = v_{F^\bullet D_L}$ as before.*

1. *For $x \in D_L \setminus \{0\}$ we have*

$$v(x) = \min_{\tau : x_\tau \neq 0} - HT(K x_\tau) = \max\{j \in \frac{1}{n}\mathbb{Z} \mid x \in F^j D_L\},$$

where we refer to A.1 for the middle expression.

2. *The map v is a valuation on the abelian group $(D_L, +)$ in the sense that we have for all $x, y \in D_L$*
   - *$v(x) = \infty$ implies $x = 0$,*
   - *$v(-x) = v(x)$ and*
   - *$v(x + y) \geq \min\{v(x), v(y)\}$ with equality if $v(x) \neq v(y)$.*

3. *Let $x_1, \ldots, x_m$ be a finite family of elements in D_L such that there exists exactly one index $1 \leq k \leq m$ with $v(x_k) < v(x_l)$ for all $l = 1, \ldots, m$ and $l \neq k$. Then*
$$v(x_1 + \ldots + x_m) = v(x_k).$$

4. *For every $j \in \frac{1}{n}\mathbb{Z}$ we have $F^j D_L = \{x \in D_L \mid v(x) \geq j\}$.*

Proof. Ad 1.: This follows immediately from the definitions.
Ad 2.: The first two statements follow from the fact that $F^\bullet D_L$ is a separated filtration by R_L-submodules. As for the third statement, let $x, y \in D_L$ not both be equal to 0. Then they are contained in $F^{\min\{v(x),v(y)\}} D_L$. Again, since this is an R_L-submodule of D_L, $x+y$ is contained in $F^{\min\{v(x),v(y)\}} D_L$. If $v(x) < v(y)$ then $x + y$ cannot be contained in $F^{v(x)+\frac{1}{n}} D_L$ for otherwise x would also be contained in it.
Ad 3.: This is a formal consequence of 1. and follows by induction on m.
Ad 4.: The statement is obvious in case $F^j D_L = \{0\}$. So assume $0 \neq x \in F^j D_L$. Then, using the description of v in *1.*, we find that $x \in F^j D_L$ is equivalent with $v(x) \geq j$. □

We make use of the map v in the formulation of the following result, which is inspired by [Col, Lemma 10.9]. It shows that every object in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$ contains a Frobenius-invariant R-submodule such that the object obtained by restricting the Frobenius and by inducing the filtration on the scalar extension to R_L lies in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$ and which is moreover maximal with this property.

Proposition 4.3.2. *Let $\underline{D} = (D, \phi, F^\bullet D_L)$ be in $\mathbf{FIC}_{L,K,n}$ such that $t_H(D') \leq t_N(D')$ for every ϕ-invariant R-submodule $D' \subseteq D$ with induced filtration on D'_L. Then there exists a ϕ-invariant R-submodule $D^{(\mathrm{wa})}$ of D with the following properties:*

1. *The filtered isocrystal $\underline{D}^{(\mathrm{wa})} := (D^{(\mathrm{wa})}, \phi|_{D^{(\mathrm{wa})}}, F^\bullet D_L \cap D_L^{(\mathrm{wa})})$ lies in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$.*

2. *For every ϕ-invariant submodule $E \subseteq D$ such that $\underline{E} := (E, \phi|_E, F^\bullet D_L \cap E_L)$ lies in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$, we have $E \subseteq D^{(\mathrm{wa})}$.*

Remark 4.3.3. According to the assumption made on $\underline{D}$ in the lemma, an object $\underline{E}$ as in the statement is weakly admissible if and only if $t_H(E) = t_N(E)$.

Proof of 4.3.2. For every $i \in \{0, \ldots, n-1\}$ denote by S_i the set of all ϕ-invariant R-submodules $E \subseteq D$ which satisfy the following two conditions:

- The triple $\underline{E} = (E, \phi|_E, F^\bullet D_L \cap E_L)$ is weakly admissible.
- We have $v_{(F^\bullet D_L) \cap E_L}(E_L \setminus \{0\}) \subseteq \frac{i}{n} + \mathbb{Z}$.

Note that $\{0\} \in S_i$ and that S_i is partially ordered with respect to inclusion for all i.
For the moment, fix an arbitrary $i \in \{0, \ldots, n-1\}$. Let $E, E' \in S_i$. We *claim* that the short exact sequence

$$\begin{aligned} 0 \to E \cap E' \overset{\alpha}{\to} E \oplus E' &\overset{\beta}{\to} E + E' \to 0 \\ x \mapsto (x, x)& \\ (y, z) &\mapsto y - z \end{aligned}$$

in $\mathbf{Mod}(R)$ induces a short exact sequence in $\mathbf{FIC}_{L,K,n}$. Here the filtrations attached to $\underline{E}\cap\underline{E}'$ and $\underline{E}+\underline{E}'$ respectively are the filtrations induced by $F^\bullet D_L$. To *prove* the claim, let

$$\pi : E \oplus E' \to E \oplus E'/\alpha(E \cap E') =: Q$$

be the canonical projection and $\overline{\beta} : Q \tilde{\to} E + E'$ the canonical isomorphism of R-modules induced by β. Note that Q naturally inherits a Frobenius and that $\overline{\beta}$ is compatible with the Frobenius on source and target. We endow Q with the quotient filtration

$$F^\bullet Q_L := [F^\bullet(E \oplus E')_L + \alpha(E \cap E')]/\alpha(E \cap E').$$

With the structures just decribed, the sequence

$$0 \to \underline{E} \cap \underline{E}' \overset{\alpha}{\to} \underline{E} \oplus \underline{E}' \overset{\pi}{\to} \underline{Q} \to 0$$

is exact in $\mathbf{FIC}_{L,K,n}$. Since $\overline{\beta}$ is an isomorphism of R-modules, by [DOR, Lemma 1.1.12] we have $t_H(Q) \le t_H(E + E')$. This is an equality if and only if $\overline{\beta}$ induces isomorphisms of R_L-modules $F^j Q_L \tilde{\to} F^j(E + E')_L$ for all $j \in \frac{1}{n}\mathbb{Z}$. Now we compute

$$\begin{aligned} t_H(Q) &\le t_H(E + E') \\ &\le t_N(E + E') && \text{(assumption on } \underline{D}) \\ &= t_N(Q) && \text{(Frobenius-compatibility of } \overline{\beta}) \\ &= t_N(E \oplus E') - t_N(E \cap E') \\ & && \text{(additivity of } t_N \text{ on short exact sequences)} \\ &\le t_H(E \oplus E') - t_H(E \cap E') \\ & && \text{(weak admissibility of } E \oplus E' \text{ and assumption on } \underline{D}) \\ &= t_H(Q), && \text{(additivity of } t_H \text{ on short exact sequences)} \end{aligned}$$

hence equality holds throughout. In particular, the first inequality sign is an equality and this implies the statement of the claim.

By additivity of t_N and t_H on short exact sequences and by the assumptions made on D, E and E', we find that $\underline{E} + \underline{E}'$ is weakly admissible. By the claim just proved we have $v_{F^\bullet D_L}((E + E')_L \setminus \{0\}) \subseteq \frac{i}{n} + \mathbb{Z}$. This argumentation shows that the set S_i is closed under the formation of finite sums. By finite-dimensionality of D over L_0, every chain in S_i may be assumed to be finite and the union of all members of a chain is then an upper bound. By Zorn's lemma, S_i has a maximal element $D_i^{(\mathrm{wa})}$. It has the property that $M \subseteq D_i^{(\mathrm{wa})}$ whenever $M \in S_i$. Indeed, we have just seen that $M + D_i^{(\mathrm{wa})} \in S_i$. By maximality of $D_i^{(\mathrm{wa})}$, we have $M + D_i^{(\mathrm{wa})} = D_i^{(\mathrm{wa})}$. This implies $M \subseteq D_i^{(\mathrm{wa})}$.

After carrying out this argument for every $i = 0, \ldots, n-1$, we set

$$\underline{D}^{(\mathrm{wa})} := \sum_{i=0}^{n-1} \underline{D}_i^{(\mathrm{wa})}$$

where the $\underline{D}_i^{(\mathrm{wa})} := (D_i^{(\mathrm{wa})}, \phi|_{D_i^{(\mathrm{wa})}}, F^\bullet D_L \cap D_{i,L}^{(\mathrm{wa})})$. The sum is actually a direct

sum: indeed, let $d \in D_i^{(\mathrm{wa})} \cap \sum_{j\in\{1,\dots,n\}\setminus\{i\}} D_j^{(\mathrm{wa})}$ for some $i \in \{0,\dots,n-1\}$. Invoking the third part of 4.3.1 we see that $v_{F^\bullet D_L}(1\otimes d) \in (\frac{i}{n}+\mathbb{Z}\cup\{\infty\})\cap(\frac{i'}{n}+\mathbb{Z}\cup\{\infty\})$ for some $i' \in \{0,\dots,n-1\}\setminus\{i\}$. Using that $(\frac{i}{n}+\mathbb{Z})\cap(\frac{i'}{n}+\mathbb{Z}) = \emptyset$ for two different elements $i,i' \in \{0,\dots,n-1\}$, we find $1\otimes d = 0$ in $R_L \otimes_R D$, hence $d = 0$.
Property 1. of the assertion is clear if $D^{(\mathrm{wa})} = 0$. Assume $D^{(\mathrm{wa})} \neq 0$. Then, by construction of the $D_i^{(\mathrm{wa})}$, for any $i \in \{0,\dots,n-1\}$ there is $\underline{M}_i$ in $\mathbf{FIC}^{\mathrm{adm}}_{L,K}$ and an isomorphism in $\mathbf{FIC}_{L,K,n}$ between $\underline{M}_i\langle\frac{n-i}{n}\rangle$ and $\underline{D}_i^{(\mathrm{wa})}$. Hence $\underline{D}^{(\mathrm{wa})}$ lies in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$.
As for property 2., let $\{0\} \neq E \subseteq D$ be a ϕ-invariant R-submodule such that $\underline{E}$ is in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$, say $\underline{E} = \oplus_{i=0}^{n-1}\underline{E}_i\langle\frac{i}{n}\rangle$. Then for all $i \in \{0,\dots,n-1\}$ such that $\underline{E}_i\langle\frac{i}{n}\rangle \neq \{0\}$ we have

$$v_{F^\bullet D_L}(E_i\langle\tfrac{i}{n}\rangle_L \setminus \{0\}) \subseteq \tfrac{n-i}{n}+\mathbb{Z}.$$

Therefore the ϕ-invariant submodule of D corresponding to the summand $\underline{E}_i\langle\frac{i}{n}\rangle$ of $\underline{E}$ is contained in S_0 if $i = 0$ (resp. in S_{n-i} if $i > 0$). Hence said submodule is contained in $D_0^{(\mathrm{wa})}$ (resp. $D_{n-i}^{(\mathrm{wa})}$). This finishes the proof. □

Corollary 4.3.4. *Let $\underline{D}$ be in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$. Then the following statements are equivalent:*

1. *$\underline{D}$ lies in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$.*
2. *$\underline{D}^{(\mathrm{wa})} = \underline{D}$.*

Proof. Assume 1. We then have the decomposition $\underline{D} = \bigoplus_{i=0}^{n-1}\underline{D}_i\langle\frac{i}{n}\rangle$ where the $\underline{D}_i$ lie in $\mathbf{FIC}^{\mathrm{adm}}_{L,K}$. From the proof of 4.3.2 we see that $\underline{D}_0 = \underline{D}_0^{(\mathrm{wa})}$ and $\underline{D}_i\langle\frac{i}{n}\rangle = \underline{D}_{n-i}^{(\mathrm{wa})}$ if $i > 0$. This implies 2.
The reverse implication follows from the definition of $\underline{D}^{(\mathrm{wa})}$. □

With the subsequent result we can evaluate $\mathbb{V}_{\mathrm{cris},n}$ on all of $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$.

Proposition 4.3.5. *Let $(D,\phi_D,F^\bullet D_L)$ be in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$. Then $\mathbb{V}_{\mathrm{cris},n}(D) = \mathbb{V}_{\mathrm{cris},n}(D^{(\mathrm{wa})})$. In particular, $\mathbb{V}_{\mathrm{cris},n}(D)$ lies in $\mathbf{Rep}^{\mathrm{cris}}_K(G_{L,(n)})$.*

Proof. For $i = 0,\dots,n-1$ denote by $D\langle-\frac{i}{n}\rangle$ the underlying R-module of $(D,\phi_D,F^\bullet D_L)\langle-\frac{i}{n}\rangle$. We have a decomposition as $K[G_{L,(n)}]$-modules

$$\begin{aligned}
&\mathbb{V}_{\mathrm{cris},n}(D)\\
&=\bigoplus_{i=0}^{n-1}\{v \in ((t_n^i B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K)\otimes_R D)^{\tilde{\varphi}_0\otimes\phi=1} \mid 1\otimes v \in F^0((t_n^i B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p} K)\otimes_{R_L} D_L)\}\\
&=\bigoplus_{i=0}^{n-1} V_{n,i}(D),
\end{aligned}$$

where the underlying set of a summand $V_{n,i}(D)$ consists of those $\varphi_0 \otimes \phi_{D\langle-\frac{i}{n}\rangle}$-invariant elements $v \in (B_{\mathrm{cris}}\otimes_{\mathbb{Q}_p} K)\otimes_R D\langle-\frac{i}{n}\rangle$ such that $1\otimes v$ lies in the zeroth filtration step of the associated filtration. The $G_{L,(n)}$-action however is

$$g \times (b \otimes d) \mapsto \varepsilon_n^i(g) g(b) \otimes d$$

for $b \in B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K, d \in D\langle -\frac{i}{n}\rangle$. Then on $\varepsilon_n^{n-i} \otimes V_{n,i}(D)$ the group H_n acts trivially.
By proceeding exactly as in the proof of [CoFo, Proposition 4.5], for each $i = 0, \dots, n-1$ we find a $\phi_{D\langle -\frac{i}{n}\rangle}$-invariant R-submodule $E_i \subseteq D\langle -\frac{i}{n}\rangle$ such that

$$V_{n,i}(D) =$$

$$\varepsilon_n^i \otimes \{v \in ((B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K) \otimes_R E_i)^{\varphi_0 \otimes \phi_{E_i} = 1} \mid 1 \otimes v \in F^0((B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K) \otimes_{R_L} E_{i,L})\}$$

and such that the natural morphism in $\mathbf{MF}^{\phi}_{L,K,n}$

$$(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K) \otimes_K (\varepsilon_n^{n-i} \otimes V_{n,i}(D)) \to (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} (K \otimes \varepsilon)) \otimes_R D\langle -\tfrac{i}{n}\rangle, \quad a \otimes v \mapsto av,$$

which is compatible with the G_L-action on both sides, is injective and induces an isomorphism of $(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K)$-modules

$$(B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K) \otimes_K (\varepsilon_n^{n-i} \otimes V_{n,i}(D)) \tilde{\to} (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} (K \otimes \varepsilon)) \otimes_R E_i.$$

Passing to G_L-invariants, we obtain an isomorphism in $\mathbf{FIC}_{L,K,n}$

$$((B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K) \otimes_K (\varepsilon_n^{n-i} \otimes V_{n,i}(D)))^{G_L} \tilde{\to} E_i\langle 1\rangle,$$

and thus

$$\begin{aligned}
&\mathrm{rank}_R(((B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K) \otimes_K (\varepsilon_n^{n-i} \otimes V_{n,i}(D)))^{G_L}) \\
&= \mathrm{rank}_R(E_i) \\
&= \mathrm{rank}_{B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K}((B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K) \otimes_R E_i) \\
&= \mathrm{rank}_{B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K}((B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K) \otimes_K V_{n,i}(D)) \\
&= \dim_K(V_{n,i}(D)).
\end{aligned}$$

It follows that $\varepsilon_n^{n-i} \otimes V_{n,i}(D)$ lies in $\mathbf{Rep}_K^{\mathrm{cris}}(G_L)$ for $i = 0, \dots, n-1$, whence the triples $(E_i\langle \frac{i}{n}\rangle, \phi_D|_{E_i\langle \frac{i}{n}\rangle}, F^\bullet D_L \cap E_i\langle \frac{i}{n}\rangle_L)$ lie in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$. The Hodge-Tate weights of the latter (cf. A.1) are contained in $\frac{i}{n} + \mathbb{Z}$, respectively. By 4.3.2, we have $E_i\langle \frac{i}{n}\rangle \subseteq D_{n-i}^{(\mathrm{wa})}$ for $i = 1, \dots, n-1$ and $E_0 \subseteq D_0^{(\mathrm{wa})}$. Hence we obtain

$$\begin{aligned}
\mathbb{V}_{\mathrm{cris},n}(D) &= \bigoplus_{i=0}^{n-1} V_{n,i}(D) = \bigoplus_{i=0}^{n-1} \varepsilon_n^i \otimes \mathbb{V}_{\mathrm{cris}}(E_i) \\
&= \bigoplus_{i=0}^{n-1} W(E_i) \subseteq \mathbb{V}_{\mathrm{cris},n}(D^{(\mathrm{wa})}) \subseteq \mathbb{V}_{\mathrm{cris},n}(D),
\end{aligned}$$

where the K-vector spaces $W(E_i)$ in the second line denote

$$\{v \in ((t_n^i B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} K) \otimes_R E_i\langle \tfrac{i}{n}\rangle)^{\tilde{\varphi}_0 \otimes \phi_{E_i\langle \frac{i}{n}\rangle} = 1} \mid 1 \otimes v \in F^0((t_n^i B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} K) \otimes_{R_L} E_i\langle \tfrac{i}{n}\rangle_L)\},$$

respectively. This proves the first statement. The second statement of the proposition follows from the first together with the fact that $\mathbb{V}_{\mathrm{cris},n}$ restricted to $\mathbf{FIC}^{\mathrm{wa}}_{L,K,(n)}$ has values in $\mathbf{Rep}^{\mathrm{cris}}_K(G_{L,(n)})$. This finishes the proof. □

Corollary 4.3.6. *Let $\underline{D}$ be in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$. The following are equivalent:*

1. *$\underline{D}$ does not lie in the essential image of the restriction of $\mathbb{D}_{\mathrm{cris},n}$ to $\mathbf{Rep}^{\mathrm{cris}}_K(G_{L,(n)})$.*
2. *The inequality*

$$\dim_K(\mathbb{V}_{\mathrm{cris},n}(D)) < \mathrm{rank}_R(D)$$

   holds true.

Proof. Assume 1. According to the corollary to 4.3.2 we have a strict inclusion $D^{(\mathrm{wa})} \subsetneq D$. This implies by 4.3.5

$$\dim_K(\mathbb{V}_{\mathrm{cris},n}(D)) = \dim_K(\mathbb{V}_{\mathrm{cris},n}(D^{(\mathrm{wa})})) = \mathrm{rank}_R(D^{(\mathrm{wa})}) < \mathrm{rank}_R(D)$$

and therefore 2.
Assume 2. Suppose there exists V in $\mathbf{Rep}^{\mathrm{cris}}_K(G_{L,(n)})$ and an isomorphism between $\mathbb{D}_{\mathrm{cris},n}(V)$ and D in $\mathbf{FIC}^{\mathrm{wa}}_{L,K,n}$. Then, by 4.3.5 again, the contradiction

$$\begin{aligned}\mathrm{rank}_R(D) &= \mathrm{rank}_R(\mathbb{D}_{\mathrm{cris},n}(V)) \\ &= \mathrm{rank}_R(D^{(\mathrm{wa})}) \\ &= \dim_K(\mathbb{V}_{\mathrm{cris},n}(D)) < \mathrm{rank}_R(D)\end{aligned}$$

implies 1. □

Remark 4.3.7. Recall that the filtered isocrystal $\underline{K}'_W$ examined in 2.2.17 does not lie in the essential image of $\mathbb{D}_{\mathrm{cris},n}$ restricted to $\mathbf{Rep}^{\mathrm{cris}}_K(G_{\mathbb{Q}_p,(n)})$. Since the underlying K-vector space has no non-trivial Frobenius-invariant subspaces it follows that $(\underline{K}'_W)^{(\mathrm{wa})} = \{0\}$. By 4.3.5 we get $\mathbb{V}_{\mathrm{cris},n}(\underline{K}'_W) = \{0\}$.

5 Filtered isocrystals over schemes and adic spaces

We keep the setup and the notations from the previous sections. In particular our assumption on K (cf. 2.2.7) is still valid.
The present section focuses on weakly admissible $\frac{1}{n}\mathbb{Z}$-filtered isocrystals parametrized by points of schemes and adic spaces. Our study will mainly be based on the assumptions made in the statement of [Hel, Proposition 5.2] although the coefficients of the filtered isocrystals that we consider will be in residue fields of schemes and adic spaces defined over K. Throughout, we fix an integer $d \geq 0$.

5.1 Notations and preliminary considerations

In this auxiliary subsection we recollect necessary algebraic-geometric results and notation for later reference.

Some representable functors related with schemes If T is a scheme and $\mathcal{E}$ is an $\mathcal{O}_T$-module, let $\mathrm{Aut}_{\mathcal{O}_T}(\mathcal{E})$ denote the group of $\mathcal{O}_T$-module automorphisms of $\mathcal{E}$. Inverse resp. direct image functors associated with an arbitrary scheme morphism h will be denoted by h^* resp. h_*.

Proposition 5.1.1 (Automorphism group scheme). *Let T be a scheme and $\mathcal{E}$ be a locally free $\mathcal{O}_T$-module of finite rank. The contravariant functor*

$$\mathbf{Sch}(T) \to \mathbf{Grp}, \quad (h : S \to T) \mapsto \mathrm{Aut}_{\mathcal{O}_S}(h^*\mathcal{E})$$

is representable by an affine T-group scheme $GL(\mathcal{E})$.

Proof. This is [EGAInew, I, Proposition (9.6.4)]. □

Proposition 5.1.2 (Weil restriction). *Let $A \to A'$ be a morphism of rings via which A' becomes a finitely generated projective A-module. Let Z be a quasi-projective A'-scheme. The contravariant functor*

$$\mathbf{Sch}(A) \to \mathbf{Set}, \quad Y \mapsto \mathrm{Hom}_{\mathbf{Sch}(A')}(Y \otimes_A A', Z)$$

is representable by an A-scheme $\mathrm{Res}_{A'|A}(Z)$. If additionally A and A' are Noetherian rings, then $\mathrm{Res}_{A'|A}(Z)$ is a quasi-projective A-scheme.

Proof. As for representability of the functor in the statement, cf. [DeGa, I, §1, no. 6, paragraph 6.6] and [Oes, A.2.14]. Suppose A and A' are Noetherian. By assumption, A' is flat over A. Then [CGP, Proposition A.5.8] says that $\mathrm{Res}_{A'|A}(Z)$ is quasi-projective over A. □

In the following, let D denote a free R-module of rank $d+1$.

Remark 5.1.3. Let $T = \mathrm{Spec}(R)$ and $\mathcal{E} = \tilde{D}$ in 5.1.1 and let $A = K, A' = R$ and $Z = GL(\tilde{D})$ in 5.1.2. Then combining both results yields bijections

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{Sch}(K)}(Y, \mathrm{Res}_{R|K}(GL(\tilde{D}))) \\ \cong\ &\mathrm{Hom}_{\mathbf{Sch}(R)}(R \otimes_K Y, GL(\tilde{D})) \\ \cong\ &\mathrm{Aut}_{\mathcal{O}_{R\otimes_K Y}}(\tilde{D} \otimes_K \mathcal{O}_Y), \end{aligned}$$

natural in Y. Hence the affine K-scheme $\mathrm{Res}_{R|K}(GL(\tilde{D}))$ represents the contravariant functor

$$\mathbf{Sch}(K) \to \mathbf{Grp}, \quad Y \mapsto \mathrm{Aut}_{\mathcal{O}_{R\otimes_K Y}}(\tilde{D} \otimes_K \mathcal{O}_Y).$$

In particular, for every K-algebra A we have an isomorphism of groups

$$\mathrm{Res}_{R|K}(GL(\tilde{D}))(A) \cong \{R \otimes_K A\text{-linear automorphisms of } D \otimes_K A\},$$

natural in A. We identify $\mathrm{Spec}(R)$ with the disjoint union $\amalg_{\tau_0}\mathrm{Spec}(K)$. Therefore $GL(\tilde{D})$ is the disjoint union (indexed over the τ_0) of the fibers of the structural morphism $GL(\tilde{D}) \to \mathrm{Spec}(R)$, which are affine K-schemes. By comparing their functors of points, one finds that $\mathrm{Res}_{R|K}(GL(\tilde{D}))$ and the fiber product of affine K-schemes $\bigtimes_{\tau_0}(GL(\tilde{D}) \otimes_R K)$ are isomorphic. Thus the factor of the latter indexed by some τ_0 can be considered as representing the contravariant functor on K-schemes which sends a K-algebra A to the group of A-automorphisms of the free module of rank $d+1$ $D_{\tau_0} \otimes_K A$.

We present another instance of a Weil restriction $\mathrm{Res}_{R|K}$ that we are going to use. It relates to values of Grassmann functors with respect to the $\mathcal{O}_{\mathrm{Spec}(R)}$-module $\tilde{D}$ (cf. [EGAInew, I, Section 9.7]) on R-schemes of the form $R \otimes_K Y$, Y a K-scheme.

Proposition 5.1.4. *For each $i = 0, \ldots, d+1$, the association*

$$\begin{aligned}\mathcal{G}_i(\tilde{D}) : Y \mapsto \{&\mathcal{O}_{R\otimes_K Y}\text{-submodules } \mathcal{U} \subseteq \tilde{D} \otimes_K \mathcal{O}_Y \text{ such that the quotient} \\ &(\tilde{D} \otimes_K \mathcal{O}_Y)/\mathcal{U} \text{ is locally on } R \otimes_K Y \text{ free of rank } d+1-i\}\end{aligned}$$

defines a contravariant functor $\mathbf{Sch}(K) \to \mathbf{Set}$. *Each of these functors is representable by a projective K-scheme.*

Proof. Fix some $i \in \{0, \ldots, d+1\}$. For every morphism $h : Y' \to Y$ of K-schemes we define $\mathcal{G}_i(\tilde{D})(h) : \mathcal{G}_i(\tilde{D})(Y) \to \mathcal{G}_i(\tilde{D})(Y')$ as $\mathcal{V} \mapsto (R \otimes_K h)^*(\mathcal{V})$. Since $\mathcal{V}$ is of finite type, this map is indeed well-defined by [GoeWe, Proposition 8.10]. Let $Grass_{i,d+1}(\tilde{D})$ denote the projective R-scheme representing the contravariant functor

$$\begin{aligned}\mathbf{Sch}(R) &\to \mathbf{Set} \\ [h : X \to \mathrm{Spec}(R)] &\mapsto \{\mathcal{O}_X\text{-submodules } \mathcal{U} \subseteq h^*\tilde{D} \text{ such that the quotient} \\ &\qquad h^*\tilde{D}/\mathcal{U} \text{ is locally on } X \text{ free of rank } d+1-i\}.\end{aligned}$$

Then $\mathrm{Res}_{R|K}(Grass_{i,d+1}(\tilde{D}))$ exists, it is a quasi-projective K-scheme and it represents $\mathcal{G}_i(\tilde{D})$ by 5.1.2. The morphism of affine schemes corresponding to $K \to R = L_0 \otimes_{\mathbb{Q}_p} K$, $a \mapsto 1 \otimes a$ is étale whence $\mathrm{Res}_{R|K}(Grass_{i,d+1}(\tilde{D}))$ is proper over $\mathrm{Spec}(K)$ by [BLR, Proposition 7.6.5 f)]. Hence it is projective over $\mathrm{Spec}(K)$. □

Similary as for $GL(\tilde{D})$, each $\mathrm{Res}_{R|K}(Grass_{i,d+1}(\tilde{D}))$ is naturally isomorphic to a fiber product of projective K-schemes $\bigtimes_{\tau_0} Grass_{i,d+1}(\tilde{D}) \otimes_R K$ by [Oes, A.2.8, A.2.14]. Using the proposition, we conclude that

$\mathrm{Res}_{R|K}(Grass_{i,d+1}(\tilde{D}))(E)$ is canonically identified with the set of tuples of i-dimensional E-subspaces in the isotypical components of $D \otimes_K E$ for any field extension $K \subseteq E$. According to 2.1.6, a Frobenius-invariant R-submodule D' of an object $\underline{D}$ in $\mathbf{FIC}_{L,K,n}$ is automatically free. This freeness is equivalent to equidimensionality over K of the isotypical components of D'. Hence, if $\mathrm{rank}_R(D') = i$, such a submodule can naturally be considered as an element of $\mathrm{Res}_{R|K}(Grass_{i,d+1}(\tilde{D}))(K)$.

We recall representability of the functor of flags of a prescribed type on D_L.

Definition 5.1.5. Let $m \geq 1$ be an integer and let Λ be a subset of the real numbers endowed with its induced ordering. We denote by Λ_+^m the subset

$$\{\lambda = (\lambda_1, \ldots, \lambda_m) \mid \lambda_1 \leq \ldots \leq \lambda_m\}$$

of $\Lambda^m \subseteq \mathbb{R}^m$. Let $\lambda \in \Lambda_+^m$. The notation

$$\lambda = (\lambda(1)^{[m(\lambda(1))]}, \ldots, \lambda(s)^{[m(\lambda(s))]})$$

will mean that $\lambda(1) < \ldots < \lambda(s)$ are the different entries of λ with respective multiplicities $1 \leq m(\lambda(1)), \ldots, m(\lambda(s))$.

We will only be interested in the case $\Lambda = \frac{1}{n}\mathbb{Z}$. The description of the functor in the next proposition is based on the discussion in [DOR, beginning of II.1, p.31 f.].

Proposition 5.1.6 (Flag scheme)**.** *Let V be a finite-dimensional K-vector space which is not the zero vector space, let*

$$\lambda = (\lambda(1)^{[m(\lambda(1))]}, \ldots, \lambda(s)^{[m(\lambda(s))]}) \in \left(\frac{1}{n}\mathbb{Z}\right)_+^{dim_K(V)}$$

and set $\lambda(s+1) := \lambda(s) + \frac{1}{n}$. Then the functor $Fl_{\lambda,V}$ which assigns to a K-algebra A the set

{decreasing families $(F^{\lambda(j)}(V \otimes_K A))_{1 \leq j \leq s+1}$ of A-submodules in $V \otimes_K A$
such that $F^{\lambda(1)}(V \otimes_K A) = V \otimes_K A,\ F^{\lambda(s+1)}(V \otimes_K A) = 0$
and the quotient $F^{\lambda(r)}(V \otimes_K A)/F^{\lambda(r+1)}(V \otimes_K A)$ is locally on Spec(A) free of rank $m(\lambda(r))$ for all $r = 1, \ldots, s$}

is representable by a projective K-scheme called $\mathrm{Drap}(\lambda, V)$*.*

Proof. Let A be a K-algebra. The conditions concerning the ranks of the $F^{\lambda(r)}(V \otimes_K A)/F^{\lambda(r+1)}(V \otimes_K A)$ are equivalent to requiring that the quotient

$$V \otimes_K A/F^{\lambda(r)}(V \otimes_K A)$$

is locally on $\mathrm{Spec}(A)$ free of rank

$$\dim_K(V) - (m(\lambda(r)) + \ldots + m(\lambda(s)))$$

for all $r = 2, \ldots, s$. Noting that

$$\begin{aligned}\underline{m} :=(&\dim_K(V) - (m(\lambda(2)) + \ldots + m(\lambda(s))),\\ &\dim_K(V) - (m(\lambda(3)) + \ \ldots + m(\lambda(s))),\\ &\ldots,\\ &\dim_K(V) - m(\lambda(s)))\end{aligned}$$

is an increasing sequence of positive integers, we see that $Fl_{\lambda,V}$ is naturally isomorphic with the functor of flags of type $\underline{m}$ from [EGAInew, I, (9.9.2)]) on K-algebras. The latter functor is representable by a projective K-scheme by [EGAInew, I, Proposition 9.9.3]. □

Definition 5.1.7. Let A be a K-algebra, V a finite-dimensional K-vector space and $\lambda \in \left(\frac{1}{n}\mathbb{Z}\right)_+^{\dim_K(V)}$. Elements of $\mathrm{Drap}(\lambda, V)(A)$ are called filtrations of $V \otimes_K A$ of type λ.

As before, let D be a free R-module of rank $d+1$. We want to apply the previous considerations concerning filtrations to the R_L-module D_L and its isotypical components.
Let A be a K-algebra with structure morphism $f : K \to A$. Then f and the decomposition

$$D_L = \bigoplus_{\tau} D_{L,\tau}$$

of D_L into τ-isotypical components (cf. the remark at the end of the first part of 2.1.3) induce a decomposition

$$D_L \otimes_K A = \bigoplus_{\tau} (D_L \otimes_K A)_\tau.$$

Here $(D_L \otimes_K A)_\tau$ denotes the A-submodule consisting of those $c \in D_L \otimes_K A$ on which L acts via the $\mathbb{Q}_p$-algebra homomorphism $f \circ \tau$. Then $(D_L \otimes_K A)_\tau$ is free of rank $d+1$ for all τ since $(D_L \otimes_K A)_\tau$ is identified with $(D_{L,\tau}) \otimes_K A$. We get mutually inverse inclusion-preserving bijections

$$\begin{aligned} U &\longmapsto (U \cap (D_L \otimes_K A)_\tau)_\tau \\ L \otimes_{\mathbb{Q}_p} A \text{ -submodule generated by the } M_\tau &\longleftarrow (M_\tau)_\tau \end{aligned}$$

between the set of all $L \otimes_{\mathbb{Q}_p} A$-submodules of $D_L \otimes_K A$ and the set of tuples $(M_\tau)_\tau$ such that M_τ is an A-submodule of $(D_L \otimes_K A)_\tau$.

Definition 5.1.8. Let $\mu = (\mu_\tau)_\tau$ be a family of tuples where $\mu_\tau \in \left(\frac{1}{n}\mathbb{Z}\right)_+^{d+1}$ for all τ. Let A be a K-algebra. Suppose moreover that, for each τ, we have a filtration of type μ_τ on $(D_L \otimes_K A)_\tau$. This family of filtrations (or equivalently the corresponding filtration of $D_L \otimes_K A$ by $L \otimes_{\mathbb{Q}_p} A$-submodules) is called a filtration of type μ of $D_L \otimes_K A$. We let $\mathrm{Drap}(\mu, D_L)$ denote the projective K-scheme representing the contravariant functor

$$\mathbf{Sch}(K) \to \mathbf{Set},\ S \mapsto \{\text{Filtrations of type } \mu \text{ on } (D_L)^\sim \otimes_K \mathcal{O}_S\}.$$

Recall that $(D_L)^\sim$ denotes the free $\mathcal{O}_{\mathrm{Spec}(R)}$-module of rank $d+1$ associated to D_L. If no confusion about the R_L-module can arise, we also denote this scheme by $\mathrm{Drap}_L(\mu)$.

Via 5.1.6 and the discussion prior to 5.1.8 we see that the above functor sends an affine K-scheme $\mathrm{Spec}(A)$ to the set

$$\{(F_\tau^\bullet)_\tau \mid F_\tau^\bullet \in \mathrm{Drap}(\mu_\tau, D_{L,\tau})(A)\}$$

(in the notation of 5.1.7) and this functor is represented by the $[L : \mathbb{Q}_p]$-fold fiber product of projective K-schemes $\bigtimes_\tau \mathrm{Drap}(\mu_\tau, D_{L,\tau})$.

Adic spaces (in the sense of Huber) The main references for the material presented in this paragraph are [Hub2] and [Hub1].

By a totally ordered group Γ (with group law written multiplicatively) we mean a group whose underlying set is totally ordered with respect to a binary relation $\leq$ that satisfies $ac \leq bc$ and $ca \leq cb$ whenever $a \leq b$ for all $a, b, c \in \Gamma$. If Γ is a totally ordered abelian group, $\Gamma \cup \{0\}$ denotes Γ with an element 0 added and the order extended by $0a = 0 = a0$, $0 \leq a$ for all $a \in \Gamma \cup \{0\}$.

Remark 5.1.9. In the $\mathbb{Q}$-vector space $\Gamma \otimes_\mathbb{Z} \mathbb{Q}$ all elements can be written as elementary tensors. The additive law of composition is thus

$$(a \otimes \frac{1}{m}) + (b \otimes \frac{1}{n}) = a^n b^m \otimes \frac{1}{mn}$$

and scalar multiplication is

$$\frac{m}{n} \cdot (a \otimes \frac{1}{l}) = a^m \otimes \frac{1}{nl}.$$

Moreover, the total order of Γ is extended by setting

$$a \otimes \frac{1}{m} \leq b \otimes \frac{1}{n} :\Leftrightarrow a^n \leq b^m.$$

.

We recall the notion of valuation on a ring in the sense of [Hub2, (1.1.2)].

Definition 5.1.10. A valuation on a ring A with values in a totally ordered abelian group Γ is a multiplicative map $|\cdot|_A : A \to \Gamma \cup \{0\}$ satisfying

$$|0|_A = 0,$$
$$|x + y|_A \leq \max\{|x|_A, |y|_A\}.$$

The subgroup of Γ generated by $im(|\cdot|_A) \setminus \{0\}$ is denoted by $\Gamma_{|\cdot|_A}$ or by Γ_A. If A is a topological ring then a valuation $|\cdot|_A : A \to \Gamma \cup \{0\}$ is called continuous if $\{a \in A \mid |a|_A < \gamma\}$ is open for all $\gamma \in \Gamma$. Let $i : A \subseteq B$ be an injective homomorphism of rings and let $|\cdot|$ resp. $|\cdot|'$ be valuations on A resp. B. If there exists an injective homomorphism of totally ordered abelian groups $j : \Gamma_{|\cdot|} \cup \{0\} \to \Gamma_{|\cdot|'} \cup \{0\}$ such that $|i(a)|' = j(|a|)$ for all $a \in A$, the tuple $(i, j, |\cdot|, |\cdot|')$ is called an extension of rings with valuation. The other pieces of data being understood, we often simply refer to such an extension by i.

We will mostly consider the case where A and B are fields.

Example 5.1.11. The extension $\mathbb{Q}_p \subset K$ is an extension of fields with valuation $x \mapsto p^{-v_p(x)}$ since by assumption (cf. 2.2.7) we have

$$\Gamma_{\mathbb{Q}_p} = p^{\mathbb{Z}} \subset p^{\frac{1}{n}\mathbb{Z}} \subseteq \Gamma_K \subset \mathbb{R}_{>0}.$$

In case $A = B$ in the definition, having isomorphic totally ordered abelian groups $\Gamma_{|\cdot|} \cup \{0\}$ and $\Gamma_{|\cdot|'} \cup \{0\}$ defines an equivalence relation on the set of all valuations on A.

Definition 5.1.12. An adic space is a triple $(X, \mathcal{O}_X, (|\cdot|_x)_{x\in X})$ such that $(X, \mathcal{O}_X)$ is a locally ringed space, $|\cdot|_x$ is a representative of an equivalence class of valuations on the stalk $\mathcal{O}_{X,x}$ for all $x \in X$ and which is locally an affinoid adic space. The latter is a triple $(\mathrm{Spa}(A, A^+), \mathcal{O}_A, (|\cdot|_x)_{x\in \mathrm{Spa}(A,A^+)})$ where $\mathrm{Spa}(A, A^+)$ is a topological space having as underlying set

$$\{\text{Equivalence classes of continuous valuations } |\cdot|_A \text{ on } A$$
$$\text{such that } |a|_A \leq 1 \text{ whenever } a \in A^+\}.$$

Here A is a member of a certain class of topological rings and A^+ is an open integrally closed subring of A. Moreover $\mathcal{O}_A$ is a sheaf of complete topological rings on $\mathrm{Spa}(A, A^+)$ and $(|\cdot|_x)_{x\in\mathrm{Spa}(A,A^+)}$ is a family of representatives of equivalence classes of valuations on the stalks $\mathcal{O}_{A,x}$.
A morphism of adic spaces is a morphism of locally ringed spaces such that the induced morphism on stalks is compatible with the valuations.

If $\mathcal{O}_X$ is the structure sheaf of an adic space X and $x \in X$ is a point, the valuation $|\cdot|_x$ on the stalk at x maps those and only those elements of the maximal ideal of the local ring $\mathcal{O}_{X,x}$ to zero. Therefore the map

$$k(x) \to \Gamma_{\mathcal{O}_{X,x}} \cup \{0\}, \text{ residue class of } s \mapsto |s|_x$$

naturally endows the residue field of x with a valuation, which is also denoted $|\cdot|_x$.
Let Y be a K-scheme locally of finite type. The adic space

$$Y^{\mathrm{ad}} := Y \times_{\mathrm{Spec}(K)} \mathrm{Spa}(K, O_K)$$

is by definition the adic space associated to Y (for notation and explicit construction cf. [Hub1, Proposition 3.8, Remark 4.6 i)]). As the notation suggests the adic spaces $\mathrm{Spec}(K)^{\mathrm{ad}}$ and $\mathrm{Spa}(K, O_K)$ (whose underlying spaces consist of one point, respectively) can be identified. Denote by ℓ the forgetful functor which assigns to an adic space $(X, \mathcal{O}_X, (|\cdot|_x)_{x\in X})$ its underlying locally ringed space $(X, \mathcal{O}_X)$. To Y^{ad} naturally is attached a morphism of locally ringed spaces $ad_Y : \ell(Y^{\mathrm{ad}}) \to Y$ over $\mathrm{Spec}(K)$ such that for an adic space S over $\mathrm{Spa}(K, O_K)$ the map

$$\mathrm{Hom}_{\mathbf{Ad}(K)}(S, Y^{\mathrm{ad}}) \to \mathrm{Hom}_{\mathbf{lrs}(K)}(\ell S, Y), \; h \mapsto ad_Y \circ \ell(h)$$

is a bijection of sets, natural in S. Here $\mathbf{lrs}(K)$ denotes the category of locally ringed spaces whose structure sheaf is a sheaf of K-algebras. Hence Y^{ad}

represents the contravariant functor

$$\mathbf{Ad}(K) \to \mathbf{Set}, \ S \mapsto \mathrm{Hom}_{\mathbf{lrs}(K)}(\ell S, Y).$$

Let $f : Y \to Z$ be a morphism of K-schemes locally of finite type. Since $f \circ ad_Y = ad_Z \circ \ell(f^{\mathrm{ad}})$ for a unique morphism $f^{\mathrm{ad}} : Y^{\mathrm{ad}} \to Z^{\mathrm{ad}}$ of adic spaces over $\mathrm{Spa}(K, O_K)$, $(-)^{\mathrm{ad}} : Y \mapsto Y^{\mathrm{ad}}$ is a covariant functor from the category of K-schemes locally of finite type to the category of adic spaces over $\mathrm{Spa}(K, O_K)$. By [Hub1, Proposition 3.8], in $\mathbf{Ad}(K)$ there exists "the fibre product of Z^{ad} with Y over Z with respect to ad_Z and f", denoted by $Z^{\mathrm{ad}} \times_Z Y$. By construction it is an adic space over Z^{ad} and it is naturally isomorphic with Y^{ad}.
The category of K-schemes locally of finite type is closed under fiber products. Since this is also the fiber product in the category $\mathbf{lrs}(K)$, there is a natural isomorphism $(Y \times_X Z)^{\mathrm{ad}} \cong Y^{\mathrm{ad}} \times_{X^{\mathrm{ad}}} Z^{\mathrm{ad}}$ in $\mathbf{Ad}(K)$.

Example 5.1.13. Let Q denote the K-scheme

$$\mathbb{A}^d_K \times_K \mathbb{G}_{m,K} \cong \ \mathrm{Spec}(K[X_1, \ldots, X_{d+1}, X_{d+1}^{-1}]).$$

Then for any adic space S over K we have natural bijections

$$\mathrm{Hom}_{\mathbf{Ad}(K)}(S, Q^{\mathrm{ad}}) \cong \mathrm{Hom}_{\mathbf{lrs}(K)}(\ell S, Q) \cong \Gamma(S, \mathcal{O}_S)^d \times \Gamma(S, \mathcal{O}_S)^{\times}.$$

The second isomorphism is given via [EGAInew, I, Proposition 1.6.3] since Q is an affine scheme.

We collect two more necessary properties concerning fiber products of adic spaces and the functor $(-)^{\mathrm{ad}}$.

Lemma 5.1.14. *Let $f : Y \to Z$ be a morphism of K-schemes locally of finite type and let $z \in Z^{ad}$ a point of the underlying topological space of Z^{ad}.*

1. *Suppose f is an open immersion. Then the same is true for f^{ad}.*
2. *In $\mathbf{Ad}(K)$ there exists the fiber product of the adic spaces Y^{ad} and $Spa(k(z), k(z)^+)$ over Z^{ad} with respect to f^{ad} and the canonical morphism*

$$can_z : \ Spa(k(z), k(z)^+) \to Z^{ad}.$$

Let $(f^{ad})^{-1}(z)$ denote this fiber product. The projection to Y^{ad} induces a homoeomorphism from $(f^{ad})^{-1}(z)$ onto a topological subspace of Y^{ad}.

Proof. The first part is combination of the results in [BGR, §9.3], [Hub2, (1.1.11)] and [Hub1, Remark 4.6 (i)]).
The second part follows from [Hub2, 1.1.8, remark before 1.2.2, 1.2.4], using that morphisms locally of finite type are locally of weakly finite type. □

5.2 Weak admissibility revisited

Keeping the notations, we now connect the notions concerning schemes and adic spaces from the previous subsection with the concept of weak admissibility.

Let A be a K-algebra. We extend the ring automorphism

$$\sigma_0 \otimes A : L_0 \otimes_{\mathbb{Q}_p} A \to L_0 \otimes_{\mathbb{Q}_p} A, x \otimes y \mapsto \sigma_0(x) \otimes y$$

to a ring automorphism of the polynomial ring $(L_0 \otimes_{\mathbb{Q}_p} A)[X]$ resp. to a ring automorphism of the ring of $(d+1) \times (d+1)$-matrices with entries in $L_0 \otimes_{\mathbb{Q}_p} A$ by $X \mapsto X$ resp. by acting componentwise. After choosing a basis, a $(\sigma_0 \otimes A)$-linear bijective map from a free $L_0 \otimes A$-module of rank $d+1$ to itself corresponds to a unique element in $GL_{d+1}(L_0 \otimes_{\mathbb{Q}_p} A)$ and therefore to a unique $L_0 \otimes_{\mathbb{Q}_p} A$-linear automorphism g of this module (cf. 5.1.3). We frequently denote such a map by $g(\sigma_0 \otimes A)$ in order to emphasize $(\sigma_0 \otimes A)$-linearity.

Recall that $f = [L_0 : \mathbb{Q}_p]$ and that D denotes a free R-module of rank $d+1$.

Lemma 5.2.1. *Let A be a K-algebra. Let $\varphi : D \otimes_K A \to D \otimes_K A$ be a $(\sigma_0 \otimes A)$-linear and bijective map. Then the characteristic polynomial c_{φ^f} of the bijective $L_0 \otimes_{\mathbb{Q}_p} A$-linear map φ^f (f-fold composition) has coefficients in $A \subset (L_0 \otimes_{\mathbb{Q}_p} A)$, where we identify A with its image under the map $A \to L_0 \otimes_{\mathbb{Q}_p} A,\ a \mapsto 1 \otimes a$.*

Proof. By $(\sigma_0 \otimes A)$-conjugacy, the result of the subsequent argument does not depend on the choice of a specific $L_0 \otimes_{\mathbb{Q}_p} A$-basis of $D \otimes_K A$. Therefore let $e_1, \ldots, e_{d+1}$ be such a basis and denote by T the matrix of φ with respect to it. The product

$$C := T(\sigma_0 \otimes A)(T) \ldots (\sigma_0 \otimes A)^{f-1}(T)$$

is the matrix of φ^f. It satifies the identity

$$(\sigma_0 \otimes A)(C) = T^{-1}CT$$

and thus we get

$$\begin{aligned} c_{\varphi^f}(X) &= \det(XE_{d+1} - C) \\ &= \det(XE_{d+1} - T^{-1}CT) \\ &= \det(XE_{d+1} - (\sigma_0 \otimes A)(C)) = (\sigma_0 \otimes A)(c_{\varphi^f}(X)). \end{aligned}$$

Here E_{d+1} denotes the $(d+1) \times (d+1)$-unit matrix. Hence the coefficients of c_{φ^f} are $(\sigma_0 \otimes A)$-invariant. Now using that $(L_0 \otimes_{\mathbb{Q}_p} A)^{\sigma_0 \otimes A = 1} = A$ allows us to conclude. □

Corollary 5.2.2. *Let A be a K-algebra and let Q be the K-scheme from 5.1.13. The association*

$$\mathrm{char}_{f,A} : \mathrm{Res}_{R|K}(GL(\tilde{D}))(A) \to Q(A),\ g \mapsto (a_1, \ldots, a_{d+1}),$$

where

$$\begin{aligned} a_i := &\ \textit{coefficient of } X^{d+1-i} \textit{ of the characteristic polynomial of} \\ &\ \textit{the } f\textit{-fold composition } (g(\sigma_0 \otimes A))^f \\ &\ (i = 1, \ldots, d+1), \end{aligned}$$

is well-defined and natural in A.

Proof. Well-definedness follows from 5.2.1 and naturality is clear. □

Remark 5.2.3. In the situation of the corollary, the last entry a_{d+1} is equal to $(-1)^{d+1} det_{L_0 \otimes_{\mathbb{Q}_p} A}[(g(\sigma_0 \otimes A))^f] \in A^\times$.

We now generalize our previous notion of filtered isocrystal with coefficients following [Hel, §3]. Let $K \subseteq E$ be an extension of fields with valuation such that E becomes a topological field whose topology is defined by $|\cdot|_E$. Literally as in the case $E = K$ we define the category $\mathbf{FIC}_{L,E,n}$ but in order to define weakly-admissible objects for general E we reformulate the definition of t_N and t_H. For this, recall that $e = [L : L_0]$.

Definition 5.2.4. Let $K \subseteq E$ be an extension of fields with valuation and let $\underline{D} := (D, \phi, F^\bullet D_L)$ be in $\mathbf{FIC}_{L,E,n}$.

i) The Newton and Hodge numbers of $\underline{D}$ with values in Γ_E are defined as

$$t_N(D) := |\det_E(\phi)|_E^e \quad \text{and} \quad t_H(D) := |p|_E^{\sum_{j \in \frac{1}{n}\mathbb{Z}} j \dim_E(F^j D_L / F^{j+\frac{1}{n}} D_L)},$$

respectively.

ii) The object $\underline{D}$ is called weakly admissible if $t_N(D) = t_H(D)$ and if $t_N(D') \leq t_H(D')$ holds for every ϕ-invariant $L_0 \otimes_{\mathbb{Q}_p} E$-submodule D' with induced filtration on $D'_L = (L \otimes_{\mathbb{Q}_p} E) \otimes_{L_0 \otimes_{\mathbb{Q}_p} E} D'$.

As before let $\mathbf{FIC}^{\mathrm{wa}}_{L,E,n}$ denote the full subcategory of $\mathbf{FIC}_{L,E,n}$ consisting of weakly admissible objects.

Remark 5.2.5. Let $\underline{D} = (D, \phi, F^\bullet D_L)$ be in $\mathbf{FIC}_{L,E,n}$ and $\underline{D}'$ denote an object in $\mathbf{FIC}_{L,E,n}$ corresponding to a ϕ-invariant $L_0 \otimes_{\mathbb{Q}_p} E$-submodule $D' \subseteq D$ with induced filtration on D'_L.

i) In contrast to our former definition of weak admissibility, the inequality sign in the condition between t_N and t_H is reversed because both assignments take values in the multiplicative group Γ_E.

ii) If E is of finite degree over K, then $\underline{D}$ is weakly admissible in the sense of 5.2.4 if and only if it is weakly admissible in the sense of section 2.2 so the above generalizes the classical formula for weak admissibility. Indeed, let E be of finite degree over K and identify Γ_E with a subgroup of $p^{\mathbb{Q}}$. Then we obtain, using $e = [E : L]^{-1}[E : L_0]$ and equation (4),

$$\begin{aligned} t_N(D') &= p^{-[E:L]^{-1}[E:L_0] v_p(\det_E(\phi|_{D'}))} \\ &= p^{-([E:L]f)^{-1} v_p(\det_{L_0}(\phi^f|_{D'}))} \end{aligned}$$

for any $\underline{D}'$. The Hodge number is

$$\begin{aligned} t_H(D') &= p^{-\sum_{j \in \frac{1}{n}\mathbb{Z}} j \dim_E(F^j D'_L / F^{j+\frac{1}{n}} D'_L)} \\ &= p^{-[E:L]^{-1} \sum_{j \in \frac{1}{n}\mathbb{Z}} j \dim_L(F^j D'_L / F^{j+\frac{1}{n}} D'_L)}. \end{aligned}$$

Hence $t_N(D') \leq t_H(D')$ holds if and only if

$$\frac{1}{f} v_p(\det_{L_0}(\phi^f|_{D'})) \geq \sum_{j \in \frac{1}{n}\mathbb{Z}} j \dim_L(F^j D'_L / F^{j+\frac{1}{n}} D'_L)$$

holds with equality for $\underline{D} = \underline{D}'$.

iii) We have the equalities

$$\det_E(\phi|_{D'})^f = \mathrm{Norm}_{(L_0 \otimes_{\mathbb{Q}_p} E)|E}(\det_{L_0 \otimes_{\mathbb{Q}_p} E}(\phi^f|_{D'})) = \det_{L_0 \otimes_{\mathbb{Q}_p} E}(\phi^f|_{D'})^f,$$

the second because $\det_{L_0 \otimes_{\mathbb{Q}_p} E}(\phi^f|_{D'}) \in E^\times$, cf. 5.2.1.
Since $\det_{L_0 \otimes_{\mathbb{Q}_p} E}(\phi^f|_{D'}) \det_E(\phi|_{D'})^{-1} \in \mu_f(E)$ and $|\zeta|_E = 1_{\Gamma_E}$ for any root of unity ζ in E, we find

$$|\det_E(\phi|_{D'})|_E = |\det_{L_0 \otimes_{\mathbb{Q}_p} E}(\phi^f|_{D'})|_E.$$

Therefore, by definition, the object $\underline{D}$ is not weakly admissible if and only if

$$|\det_{L_0 \otimes_{\mathbb{Q}_p} E}(\phi^f)|_E^e < |p|_E^{\sum_{j \in \frac{1}{n}\mathbb{Z}} j \dim_E(F^j D_L / F^{j+\frac{1}{n}} D_L)}$$

or there exists a ϕ-invariant $L_0 \otimes_{\mathbb{Q}_p} E$-submodule $M \subseteq D$ such that

$$|\det_{L_0 \otimes_{\mathbb{Q}_p} E}(\phi^f|_M)|_E^e > |p|_E^{\sum_{j \in \frac{1}{n}\mathbb{Z}} j \dim_E(F^j M_L / F^{j+\frac{1}{n}} M_L)}.$$

The following result shows that weak admissibility is invariant under extension of scalars with respect to fields with valuation.

Proposition 5.2.6. *Let $K \subseteq E \subseteq E'$ be extensions of fields with valuation and let $\underline{D} := (D, \phi, F^\bullet D_L)$ denote an object of $\mathbf{FIC}_{L,E,n}$. Then $\underline{D}$ lies in $\mathbf{FIC}^{wa}_{L,E,n}$ if and only if the object $\underline{D} \otimes_E E' := (D \otimes_E E', \phi \otimes E', F^\bullet D_L \otimes_E E')$ obtained by extending scalars from E to E' lies in $\mathbf{FIC}^{wa}_{L,E',n}$.*

The statement of the propostion is [Hel, Corollary 3.22] for objects of slope one. There also a detailed proof is given.

Sketch of proof. Let $\underline{D} \otimes_E E'$ be weakly admissible. For all ϕ-invariant $L_0 \otimes_{\mathbb{Q}_p} E$-submodules $D' \subseteq D$ we have

$$\det_{E'}((\phi \otimes E')|_{D' \otimes_E E'}) = \det_E(\phi|_{D'}),$$

hence the values of t_N on D' and $D' \otimes_E E'$ coincide. By faithful flatness of vector spaces over fields the same is true for t_H. Therefore $\underline{D}$ is weakly admissible.
The reverse implication uses a slope theory as developed in [Hel, §3]: attached to a filtered isocrystal $\underline{\Delta}$ with coefficients in any extension of fields with valuation $K \subseteq E$ is a uniquely determined finite family of invariants $s_1 < \ldots < s_r \in \Gamma_E \otimes_{\mathbb{Z}} \mathbb{Q}$ (cf. 5.1.9) and the so-called HN-filtration of $\underline{\Delta}$ which is a decreasing family by r Frobenius-invariant $L_0 \otimes_{\mathbb{Q}_p} E$-submodules with induced filtration ([Hel, Proposition 3.19]) for an integer $r \geq 1$. This attached data has the property that validity of

$$r = 1 \quad \text{and} \quad s_1 = \text{ neutral element of } \Gamma_E \otimes_{\mathbb{Z}} \mathbb{Q}$$

is equivalent to weak admissibility of $\underline{\Delta}$, as follows from combination of [Hel, Lemma 3.13, Definition 3.15, Proposition 3.19].
Now in the case at hand one shows by a descent argument that the HN-filtration of $\underline{D} \otimes_E E'$ must be of the form

$$0 \subset \underline{D}_1 \otimes_E E' \subset \ldots \subset \underline{D}_r \otimes_E E' = \underline{D} \otimes_E E',$$

where the $\underline{D}_1, \ldots, \underline{D}_r = \underline{D}$ are the members of the HN-filtration of $\underline{D}$. By assumption, $r = 1$ and $s_1 =$ neutral element of $\Gamma_E \otimes_{\mathbb{Z}} \mathbb{Q}$. By invariance of s_1 under extension of scalars, $r = 1$ and $s_1 =$ neutral element of $\Gamma_{E'} \otimes_{\mathbb{Z}} \mathbb{Q}$ holds for $\underline{D} \otimes_E E'$. Hence $\underline{D} \otimes_E E'$ is weakly admissible. □

In what follows we write G_D for $\mathrm{Res}_{R|K}(GL(\tilde{D}))$ for any finitely generated free R-module D.

Definition 5.2.7. Let D be a finitely generated free R-module and let $\mu = (\mu_\tau)_\tau$ where $\mu_\tau \in (\frac{1}{n}\mathbb{Z})_+^{\mathrm{rank}_R(D)}$ for all τ. For any extension of fields with valuation $K \subseteq E$ we define $(G_D \times_K \mathrm{Drap}(\mu, D_L))(E)^{wa}$ as the subset of $G_D(E) \times \mathrm{Drap}(\mu, D_L)(E)$ consisting of those pairs $(g, F^\bullet)$ such that $(D \otimes_K E, g(\sigma_0 \otimes E), F^\bullet)$ lies in $\mathbf{FIC}^{\mathrm{wa}}_{L,E,n}$.

Remark 5.2.8. i) By 5.2.6 the definition of $(G_D \times_K \mathrm{Drap}(\mu, D_L))(E)^{wa}$ is functorial in E.

ii) In the situation of 5.2.7 let $d+1$ be the R-rank of D and let $E = K$. Assume moreover that for all τ we have $\mu_\tau \in \mathbb{Z}_+^{d+1}$ with

$$\mu_{\tau,1} < \ldots < \mu_{\tau,d+1}$$

(in particular, $\mathrm{Drap}(\mu, D_L)$ is the K-scheme whose K-valued points are in natural bijection with families $(F_\tau^\bullet)_\tau$ of full filtrations of $D_{L,\tau}$ of type μ_τ respectively). Let $c = X^{d+1} + c_1 X^d + \ldots + c_d X + c_{d+1} \in K[X]$ split completely into linear factors and moreover be such that $c_{d+1} \neq 0$. Then [BrSch, Proposition 3.2] characterizes non-emptyness of the intersection inside $(G_D \times_K \mathrm{Drap}(\mu, D_L))(K)$ between the set consisting of all K-valued points $(g, F^\bullet)$ of $G_D \times_K \mathrm{Drap}(\mu, D_L)$ such that $(g(\sigma_0 \otimes K))^f$ has characteristic polynomial c on the one hand and $(G_D \times_K \mathrm{Drap}(\mu, D_L))(K)^{wa}$ on the other hand in terms of the existence of certain numerical (in)equalities relating the valuations of the zeros of c and the entries of the μ_τ.

Let $K \subseteq E$ be an extension of fields with valuation. Then the Hodge number of any $\underline{D}$ in $\mathbf{FIC}_{L,E,n}$ whose associated filtration is of type $\mu = (\mu_\tau)_\tau$ is

$$t_H(D) = |p|_E^{\sum_\tau \sum_{i=1}^{d+1} \mu_{\tau,i}}.$$

Hence the possible values for Hodge numbers of R_L-submodules of D_L with induced filtration are finite in number. The following identity concerning the exponent on the right hand side will be useful in the proof of 5.2.9: let $a \in \frac{1}{n}\mathbb{Z}$

such that $a \leq \min_\tau\{\mu_{\tau,1}\}$. Then we have

$$\sum_\tau \sum_{i=1}^{d+1} \mu_{\tau,i} = a \dim_E(D_L) + \frac{1}{n} \sum_\tau \sum_{a<j} \dim_E(F^j D_{L,\tau})$$
$$= a \dim_E(D_L) + \frac{1}{n} \sum_{a<j} \dim_E(F^j D_L).$$

Here $F^j D_{L,\tau}$ denotes the image of $F^j D_L$ under the canonical projection $D_L \to D_{L,\tau}$. Validity of the second equation is clear. In order to see validity of the first one, fix any $\tau : L \hookrightarrow K$. By the choice of a we may write

$$\begin{aligned}\sum_{i=1}^{d+1} \mu_{\tau,i} &= \sum_{j \in \frac{1}{n}\mathbb{Z}} j \dim_E(F^j D_{L,\tau}/F^{j+\frac{1}{n}} D_{L,\tau}) \\ &= \sum_{a \leq j} j \dim_E(F^j D_{L,\tau}/F^{j+\frac{1}{n}} D_{L,\tau}) \\ &= \sum_{a \leq j} j \dim_E(F^j D_{L,\tau}) - \sum_{a \leq j} j \dim_E(F^{j+\frac{1}{n}} D_{L,\tau}) \\ &= \sum_{a \leq j} j \dim_E(F^j D_{L,\tau}) - \sum_{a \leq j} (j + \frac{1}{n}) \dim_E(F^{j+\frac{1}{n}} D_{L,\tau}) \\ &\quad + \frac{1}{n} \sum_{a \leq j} \dim_E(F^{j+\frac{1}{n}} D_{L,\tau}).\end{aligned}$$

In the last equality sign, we have used $j = (j + \frac{1}{n}) - \frac{1}{n}$. After reindexing, the second sum in the last expression is equal to

$$- \sum_{a+\frac{1}{n} \leq j} j \dim_E(F^j D_{L,\tau}).$$

We obtain

$$\sum_{i=1}^{d+1} \mu_{\tau,i} = a \dim_E(D_{L,\tau}) + \frac{1}{n} \sum_{a<j} \dim_E(F^j D_{L,\tau})$$

and hence, after summing over all τ, the stated equation (cf. also [DOR, Lemma 1.1.11], [CoFo, Formula (3.1)]).

Now let $m : S \to (G_D \times_K \mathrm{Drap}(\mu, D_L))^{\mathrm{ad}}$ be a morphism in $\mathbf{Ad}(K)$ where D is a free R-module of rank $d+1$ and $\mu = (\mu_\tau)_\tau$ with $\mu_\tau \in (\frac{1}{n}\mathbb{Z})_+^{d+1}$ for all τ. If $s \in S$ is a point with image $s' := (ad_{G_D \times_K \mathrm{Drap}(\mu, D_L)} \circ \ell m)(s) \in G_D \times_K \mathrm{Drap}(\mu, D_L)$, denote by $(g(s'), \mathcal{F}^\bullet(s'))$ the pullback of the universal element over $G_D \times_K \mathrm{Drap}(\mu, D_L)$ to $\mathrm{Spec}(\kappa(s'))$. Considering the fields K and $k(s)$ as an extension of fields with valuation, it is reasonable to ask whether

$$(g_s, \mathcal{F}_s^\bullet) := (g(s') \otimes k(s), \mathcal{F}^\bullet(s') \otimes_{\kappa(s')} k(s))$$

lies in $(G_D \times_K \mathrm{Drap}(\mu, D_L))(k(s))^{wa}$. The following result, whose statement and proof are based on [Hel, Proposition 5.2], provides structural information on points in fibers over rigid points of the morphism

$$\alpha := (char_f \circ \text{projection to } G_D)^{\text{ad}} : (G_D \times_K \text{Drap}(\mu, D_L))^{\text{ad}} \to Q^{\text{ad}}$$

regarding this question. Here we denote by $char_f : G_D \to Q$ the morphism of K-schemes corresponding to the natural transformation between the functors of points of G_D and Q described in 5.2.2.

Theorem 5.2.9. *Notations as in the preceeding discussion, let $x \in Q^{ad}$ be such that $k(x)$ is of finite degree over K. With m being the projection $\alpha^{-1}(x) \to (G_D \times_K \text{Drap}(\mu, D_L))^{ad}$ (cf. 5.1.14/2.), define the weakly admissible locus in $\alpha^{-1}(x)$ as*

$$\alpha^{-1}(x)^{wa} := \{z \in \alpha^{-1}(x) \mid (g_z, \mathcal{F}^\bullet_z) \in (G_D \times_K \text{Drap}(\mu, D_L))(k(z))^{wa}\}.$$

Then there exists a quasi-projective $k(x)$-scheme M such that

$$M^{ad} = \alpha^{-1}(x)^{wa}.$$

Proof. By the assumption on x, the field $k(x)$ is complete with respect to the unique extension of $|\cdot|_p$ and the equivalence class of this valuation is the unique point of the underlying space of $\text{Spa}(k(x), k(x)^+) = \text{Spa}(k(x), O_{k(x)})$.
We will prove the statement in several steps.
Step 1: The composite morphism of locally ringed spaces

$$ad_Q \circ \ell(can_x) : \ell(\text{Spa}(k(x), O_{k(x)})) \to Q$$

corresponds to a unique element $(c_1, \ldots, c_{d+1}) \in k(x)^d \times k(x)^\times$ by 5.1.13. Let c denote the polynomial $X^{d+1} + c_1X^d + \ldots + c_dX + c_{d+1} \in k(x)[X]$. In what follows we may and will assume that

$$|c_{d+1}|_p^e = p^{-\sum_\tau \sum_{i=1}^{d+1} \mu_{\tau,i}}$$

because in case the equality does not hold, $\alpha^{-1}(x)^{\text{wa}}$ is empty by definition of weak admissibility (cf. also 5.2.5 iii) and the discussion after 5.2.8) and by the equivalence

$$|c_{d+1}|_z^e \neq |p|_z^{\sum_\tau \sum_{i=1}^{d+1} \mu_{\tau,i}} \Leftrightarrow |c_{d+1}|_p^e \neq p^{-\sum_\tau \sum_{i=1}^{d+1} \mu_{\tau,i}}$$

for all $z \in \alpha^{-1}(x)$.

For the following auxiliary constructions, we fix an integer i with $0 \leq i \leq d+1$.

Step 2: Write $Gr_{R,i,D}$ for the projective K-scheme $\text{Res}_{R|K}(Grass_{i,d+1}(\tilde{D}))$ from 5.1.4. The family of maps

$$a_{i,A} : G_D(A) \times Gr_{R,i,D}(A) \to Gr_{R,i,D}(A), \quad (g, \mathcal{U}) \mapsto g(\sigma_0 \otimes A)\mathcal{U},$$

as A runs through all K-algebras, defines a morphism of K-schemes $a_i : G_D \times_K Gr_{R,i,D} \to Gr_{R,i,D}$. Note that the occurence of $\sigma_0 \otimes A$ in the definition of $a_{i,A}$

induces a permutation of the factors of $Gr_{R,i,D}$ (cf. the discussion subsequent to 5.1.4). Let $pr_{Gr_{R,i,D}} : G_D \times_K Gr_{R,i,D} \to Gr_{R,i,D}$ be the second projection. The universal property of the fiber product $Gr_{R,i,D} \times_K Gr_{R,i,D}$ yields a unique morphism of K-schemes $(a_i, pr_{Gr_{R,i,D}}) : G_D \times_K Gr_{R,i,D} \to Gr_{R,i,D} \times_K Gr_{R,i,D}$. Denote by $Z_{i,D}$ the fiber product of $G_D \times_K Gr_{R,i,D}$ and $Gr_{R,i,D}$ with respect to $(a_i, pr_{Gr_{R,i,D}})$ and the diagonal morphism $\Delta_{Gr_{R,i,D}}$. Note that $\Delta_{Gr_{R,i,D}}$ is projective since it is a closed immersion ([EGAII, 5.5.5 i)]). Hence, by stability under base change and composition ([EGAII, 5.5.5 ii), iii)]), the composition of projection morphisms $Z_{i,D} \to G_D \times_K Gr_{R,i,D} \to G_D$ is projective.
Step 3: Let S be a K-scheme. We write $\sigma_0 \otimes \mathcal{O}_S : \mathcal{O}_{R\otimes_K S} \to \mathcal{O}_{R\otimes_K S}$ for the inverse image of the morphism $\mathrm{Spec}(\sigma_0 \otimes K) : \mathcal{O}_{\mathrm{Spec}(R)} \to \mathcal{O}_{\mathrm{Spec}(R)}$ with respect to the projection to $\mathrm{Spec}(R)$.
The scheme $Z_{i,D}$ (cf. *Step* 2) represents the contravariant functor which sends a K-scheme S to the set of pairs $(g, \mathcal{U}) \in G_D(S) \times Gr_{R,i,D}(S)$ such that the $\sigma_0 \otimes \mathcal{O}_S$-linear endomorphism $g(\sigma_0 \otimes \mathcal{O}_S) : \tilde{D} \otimes_K \mathcal{O}_S \to \tilde{D} \otimes_K \mathcal{O}_S$ restricts to a semilinear endomorphism $g(\sigma_0 \otimes \mathcal{O}_S)|_{\mathcal{U}} : \mathcal{U} \to \mathcal{U}$. Pulling back to residue fields of points of $Y_{i,D} := Z_{i,D} \times_K \mathrm{Drap}(\mu, D_L) \subseteq G_D \times_K Gr_{R,i,D} \times_K \mathrm{Drap}(\mu, D_L)$ the universal elements on $Z_{i,D}$ respectively on $\mathrm{Drap}(\mu, D_L)$ along the canonical morphisms yields in particular a free $L_0 \otimes_{\mathbb{Q}_p} \kappa(y)$-submodule $\mathcal{V}_i(y) \subseteq D \otimes_K \kappa(y)$ of rank i and a filtration $\mathcal{F}^\bullet(y)$ of type μ of $D_L \otimes_K \kappa(y)$ where $y \in Y_{i,D}$. According to the semicontinuity principle (cf. [EGAIII$_2$, 7.6.9, (proof of) 7.7.5 (I)]), one finds that the function

$$
\begin{aligned}
&h_{ij} : Y_{i,D} \to \mathbb{Z},\\
&\qquad y \mapsto \dim_{\kappa(y)}(\mathcal{V}_i(y)_L \cap \mathcal{F}^j(y)) = \sum_\tau \dim_{\kappa(y)}(\mathcal{V}_i(y)_{L,\tau} \cap \mathcal{F}^j(y)_\tau)
\end{aligned}
$$

is upper semicontinuous for all $j \in \frac{1}{n}\mathbb{Z}$ [1]. It follows that the same is true for

$$
\begin{aligned}
&h_i : Y_{i,D} \to \frac{1}{n}\mathbb{Z},\\
&\qquad y \mapsto \sum_{j \in \frac{1}{n}\mathbb{Z}} j \dim_{\kappa(y)}[(\mathcal{V}_i(y)_L \cap \mathcal{F}^j(y))/(\mathcal{V}_i(y)_L \cap \mathcal{F}^{j+\frac{1}{n}}(y))]
\end{aligned}
$$

since we have

$$
h_i(y) = a[L : \mathbb{Q}_p]i + \frac{1}{n} \sum_{a<j} h_{ij}(y)
$$

by 5.2.8, ii), where $y \in Y_{i,D}$ and a is an arbitrary element in $\frac{1}{n}\mathbb{Z}$ with $a \leq \min_\tau\{\mu_{\tau,1}\}$.
It follows that the functions $h_{ij}^{\mathrm{ad}} := h_{ij} \circ ad_{Y_{i,D}}$ and $h_i^{\mathrm{ad}} := h_i \circ ad_{Y_{i,D}}$ are upper semicontinuous on $Y_{i,D}^{\mathrm{ad}}$. For $m, m' \in \frac{1}{n}\mathbb{Z}$ with $m' \geq m$ we therefore have inclusions of closed subsets

$$
\{y \mid h_i(y) \geq m'\} \subseteq \{y \mid h_i(y) \geq m\} \subseteq Y_{i,D}
$$

respectively

$$
\{z \mid h_i^{\mathrm{ad}}(z) \geq m'\} \subseteq \{z \mid h_i^{\mathrm{ad}}(z) \geq m\} \subseteq Y_{i,D}^{\mathrm{ad}}.
$$

[1] Here we call a function $f : S \to T$ from a topological space S to a totally ordered set T upper semicontinuous if the subsets $\{s \in S | f(s) \geq t\}$ are closed for all $t \in T$.

For each $m \in \frac{1}{n}\mathbb{Z}$, let $Y_{i,m,D}$ denote the reduced subscheme of $Y_{i,D}$ having as underlying topological space the set $\{y \mid h_i(y) \geq m\}$. As a closed immersion, the canonical inclusion $Y_{i,m,D} \to Y_{i,D}$ is proper. By stability under composition, the same is true for the composite

$$Y_{i,m,D} \to Y_{i,D} = Z_{i,D} \times_K \mathrm{Drap}(\mu, D_L) \to G_D \times_K \mathrm{Drap}(\mu, D_L).$$

Note that by [Köpf, 2.17] and [Hub2, 1.3.19] the corresponding morphism $Y_{i,m,D}^{\mathrm{ad}} \to (G_D \times_K \mathrm{Drap}(\mu, D_L))^{\mathrm{ad}}$ of adic spaces over $\mathrm{Spa}(K, O_K)$ is proper.
Step 4: We need to introduce new notations with respect to some of the previous constructions. If $(\eta_i, \mathcal{W}_i) \in G_D(Z_{i,D}) \times Gr_{R,i,D}(Z_{i,D})$ denotes the universal element of $Z_{i,D}$ and $p_{Z_{i,D}}$ is the projection $R \otimes_K Z_{i,D} \to Z_{i,D}$, let d_i denote the image of $p_{Z_{i,D},*}((\eta_i(\sigma_0 \otimes \mathcal{O}_{Z_{i,D}}))^f|_{\mathcal{W}_i})$ in $\Gamma(Z_{i,D}, \mathcal{O}_{Z_{i,D}})$ under the composition on global sections of the morphism of $\mathcal{O}_{Z_{i,D}}$-modules

$$\mathscr{H}om_{\mathcal{O}_{Z_{i,D}}}(p_{Z_{i,D},*}\mathcal{W}_i, p_{Z_{i,D},*}\mathcal{W}_i) \xrightarrow{\det} \mathscr{H}om_{\mathcal{O}_{Z_{i,D}}}(\bigwedge^{fi} p_{Z_{i,D},*}\mathcal{W}_i, \bigwedge^{fi} p_{Z_{i,D},*}\mathcal{W}_i) \xrightarrow{\sim} \mathcal{O}_{Z_{i,D}}.$$

The isomorphism on the right is given on open subsets of $Z_{i,D}$ by the map $[w \mapsto uw] \leftarrow\!\shortmid u$ (cf. [GoeWe, (7.20.7)]).
Let F_c be the splitting field of the polynomial $c \in k(x)[X]$ determined by $x \in Q^{\mathrm{ad}}$. If S_x denotes the fiber product of the morphisms $\mathrm{Spec}(k(x)) \to Q$ and

$$(char_f \circ \text{projection to } G_D) : G_D \times_K \mathrm{Drap}(\mu, D_L) \to Q,$$

let $T_{i,m,D}$ denote the F_c-scheme

$$Y_{i,m,D} \times_{G_D \times_K \mathrm{Drap}_L(\mu, D_L)} (S_x \otimes_{k(x)} F_c) \cong Y_{i,m,D} \times_Q F_c.$$

Step 5: With notations as before, we define for every $m \in \frac{1}{n}\mathbb{Z}$ a subset of $T_{i,m,D}$ by $S_{i,m,D} := \{y \in T_{i,m,D} \mid |d_i(y)|_p^e > p^{-m}\}$. Here we write $d_i(y)$ for the image of d_i in the residue field at y and we *claim* that $S_{i,m,D}$ is a union of connected components of $T_{i,m,D}$ for all $m \in \frac{1}{n}\mathbb{Z}$.
Proof of claim: By construction and the third part of 5.2.5, the element $d_i(y)$ is the product of an f-th root of unity and the determinant of an automorphism on a free $(L_0 \otimes_{\mathbb{Q}_p} \kappa(y))$-module of rank i such that the characteristic polynomial of this automorphism divides the polynomial c. Since $d_i(y)$ can thus be considered as an element of $F_c^\times$, the expression $|d_i(y)|_p^e$ is actually well-defined.
Associated to the image of d_i in $\Gamma(T_{i,m,D}, \mathcal{O}_{T_{i,m,D}})$ is a unique morphism of rings $\rho_{d_i} : F_c[X] \to \Gamma(T_{i,m,D}, \mathcal{O}_{T_{i,m,D}})$ which corresponds to a unique morphism of schemes

$$\theta_{d_i} : T_{i,m,D} \to \mathrm{Spec}(F_c[X]) \ , \ y \mapsto \ker(\rho_{d_i,y}).$$

Here $\rho_{d_i,y}$ denotes the composite of ρ_{d_i} with the canonical morphism from $\Gamma(T_{i,m,D}, \mathcal{O}_{T_{i,m,D}})$ to $\kappa(y)$. Hence $\rho_{d_i,y}$ sends a polynomial P to $P(d_i(y))$.
Let $\lambda_1, \ldots, \lambda_{d+1} \in F_c^\times$ denote the not necessarily distinct roots of c. Thus for every $y \in T_{i,m,D}$ there exists a subset J_y of $\{1, \ldots, d+1\}$ of cardinality i such that $d_i(y) \in (\prod_{j \in J_y} \lambda_j)\mu_f(F_c)$. It follows that the image of the morphism θ_{d_i}

is contained in the finite set

$$\{(X + \epsilon\zeta \prod_{j \in J} \lambda_j) F_c[X] \mid \epsilon \in \{\pm 1\}, J \subseteq \{1, \dots, d+1\}, \#J = i, \zeta \in \mu_f(F_c)\}$$

of closed points of $\mathbb{A}^1_{F_c}$. Hence θ_{d_i} is constant on connected components. Now we find

$$S_{i,m,D} = \bigcup_{\epsilon \in \{\pm 1\}} \bigcup_{\zeta \in \mu_f(F_c)} \bigcup_{J} \theta_{d_i}^{-1}(\{(X + \epsilon\zeta \prod_{j \in J} \lambda_j) F_c[X]\})$$

where the third index set runs over all subsets J of $\{1, \dots, d+1\}$ of cardinality i such that $|\prod_{j \in J} \lambda_j|_p^e > p^{-m}$. The claim is proved.
Note that $T_{i,m,D}$ is of finite type over F_c, therefore Noetherian and hence has only finitely many connected components. In particular, $S_{i,m,D}$ is a closed subset of $T_{i,m,D}$ for every $m \in \frac{1}{n}\mathbb{Z}$.
Step 6: There exists $m_0 \in \frac{1}{n}\mathbb{Z}$ such that

$$p^{-m_0} > \max\{|\prod_{j \in J} \lambda_j|_p^e \mid J \subseteq \{1, \dots, d+1\}, \#J = i\}.$$

Hence $S_{i,m,D}$ is empty for all $m \in \frac{1}{n}\mathbb{Z}$ such that $m \leq m_0$.
Let $\mathfrak{J}_i$ denote the set of all families $(J_\tau)_\tau$ of subsets of $\{1, \dots, d+1\}$ satisfying $\sum_\tau \#J_\tau = [L : \mathbb{Q}_p]i$. If $m_1 \in \frac{1}{n}\mathbb{Z}$ is such that

$$m_1 > \max\{\sum_\tau \sum_{j \in J_\tau} \mu_{\tau,j} \mid (J_\tau)_\tau \in \mathfrak{J}_i\},$$

then the set underlying $Y_{i,m,D}$ is empty for all $m \in \frac{1}{n}\mathbb{Z}$ such that $m \geq m_1$. It follows that $S_{i,m,D}$ is also empty in this case.
Step 7: Now applying these considerations to all $i \in \{0, \dots, d+1\}$ yields finiteness, hence closedness, of the union

$$\bigcup_{i=0}^{d+1} \bigcup_{m \in \frac{1}{n}\mathbb{Z}} \mathrm{pr}'_{i,m}(S_{i,m,D}) \subseteq S_x \otimes_{k(x)} F_c.$$

Here $\mathrm{pr}'_{i,m}$ denotes the base change to $T_{i,m,D}$ of the proper morphism $Y_{i,m,D} \to G_D \times_K \mathrm{Drap}(\mu, D_L)$ for all $i \in \{0, \dots, d+1\}$ and all $m \in \frac{1}{n}\mathbb{Z}$.
Let $s \in S_x \otimes_{k(x)} F_c$ be a point such that for $(D \otimes_K \kappa(s), g(s)(\sigma_0 \otimes \kappa(s)), \mathcal{F}^\bullet(s))$, where $(g(s), \mathcal{F}^\bullet(s))$ is the pullback to $\mathrm{Spec}(\kappa(s))$ of the universal element on $G_D \times_K \mathrm{Drap}(\mu, D_L)$, there exists a $g(s)(\sigma_0 \otimes \kappa(s))$-invariant $L_0 \otimes_{\mathbb{Q}_p} \kappa(s)$-submodule U of rank $r_U > 0$ with $|\det_{\kappa(s)}(g(s)(\sigma_0 \otimes \kappa(s))|_U^f)|_p^e > p^{-h_U}$. Here we have set

$$h_U := \sum_\tau \sum_{j \in \frac{1}{n}\mathbb{Z}} j \dim_{\kappa(s)}(\mathcal{F}^j(s)_\tau \cap U_{L,\tau} / \mathcal{F}^{j+1}(s)_\tau \cap U_{L,\tau}).$$

We remark that the same inequality is true with respect to any field extension E of $\kappa(s)$ and the corresponding submodule $U \otimes_{\kappa(s)} E$. The point s is contained in $\mathrm{pr}'_{r_U,h_U}(S_{r_U,h_U,D})$: indeed, a triple as above is a $\kappa(s)$-valued point of $Y_{r_U,D}$. Letting $y \in Y_{r_U,h_U,D}$ denote the image point, both s and y lie over the same point

in $G_D \times_K \mathrm{Drap}(\mu, D_L)$. Hence there exists $t \in T_{r_U,h_U,D}$ such that $\mathrm{pr}'_{r_U,h_U}(t) = s$ and therefore $s \in \mathrm{pr}'_{r_U,h_U}(S_{r_U,h_U,D})$ by the remark just made.
Step 8: Since the field extension $k(x) \subseteq F_c$ is finite, the projection morphism $q : S_x \otimes_{k(x)} F_c \to S_x$ is proper. Let M denote the open subscheme of S_x induced on the open subset

$$S_x \setminus \bigcup_{i=0}^{d+1} \bigcup_{m \in \frac{1}{n}\mathbb{Z}} q(\mathrm{pr}'_{i,m}(S_{i,m,D})).$$

Denote by $i_M : M \to S_x$ the canonical morphism of K-schemes.
The third point in the following claim uses notation from the discussion directly before the theorem.
Claim: The following statements are equivalent for a point $z \in \alpha^{-1}(x)$:
a) The point z is contained in M^{ad}, considered as a subset of $\alpha^{-1}(x)$ via i_M^{ad}.
b) The point $ad_{S_x}(z)$ is contained in M.
c) The pair $(g_z, \mathcal{F}_z^\bullet)$ is contained in $(G_D \times_K \mathrm{Drap}(\mu, D_L))(k(z))^{\mathrm{wa}}$.
d) The point z is contained in $\alpha^{-1}(x)^{\mathrm{wa}}$.
Proof of claim: For the equivalence of a) and b) we identify M^{ad} with $\alpha^{-1}(x) \times_{S_x} M$ in $\mathbf{Ad}(K)$. Then application of [Hub1, Proposition 3.9 ii)] to the equality $i_M \circ ad_M = ad_{S_x} \circ \ell((i_M)^{\mathrm{ad}})$ yields for every $s \in M$ and $y \in \alpha^{-1}(x)$ with $i_M(s) = ad_{S_x}(y)$ the existence of $t \in M^{\mathrm{ad}}$ such that $i_M^{\mathrm{ad}}(t) = y$ and $ad_M(t) = s$.
For the implication "b) $\Rightarrow$ c)" assume that $(g_z, \mathcal{F}_z^\bullet)$ is not contained in $(G_D \times_K \mathrm{Drap}(\mu, D_L))(k(z))^{\mathrm{wa}}$. The morphism q^{ad} is surjective. For $y \in (S_x \otimes_{k(x)} F_c)^{\mathrm{ad}}$ with $q^{\mathrm{ad}}(y) = z$, the pair $(g_y, \mathcal{F}_y^\bullet)$ is not contained in $(G_D \times_K \mathrm{Drap}(\mu, D_L))(k(y))^{\mathrm{wa}}$ by 5.2.6. Let $y' := ad_{S_x \otimes_{k(x)} F_c}(y)$. Restricting $|\cdot|_{k(y)}$ to $\kappa(y')$ and using the notation from the previous step, we see that $(D \otimes_K \kappa(y')), g(y')(\sigma_0 \otimes \kappa(y)), \mathcal{F}^\bullet(y'))$ is not weakly admissible. Hence, $y' = \mathrm{pr}'_{i,m}(t') \in \mathrm{pr}'_{i,m}(S_{i,m,D})$ for a suitable pair (i, m) and $t' \in T_{i,m,D}$ (cf. Step 7). But then $ad_{S_x}(z) = ad_{S_x}(q^{\mathrm{ad}}(y)) = q(y')$ is not contained in M.
For the implication "c) $\Rightarrow$ b)" assume that $ad_{S_x}(z) \notin M$. By surjectivity of q there exists $s \in \cup_{i=0}^{d+1} \cup_{m \in \frac{1}{n}\mathbb{Z}} \mathrm{pr}'_{i,m}(S_{i,m,D})$ such that $q(s) = ad_{S_x}(z)$. By [Hub1, Proposition 3.9 ii)] again there exists $t \in (S_x \otimes_{k(x)} F_c)^{\mathrm{ad}}$ with $q^{\mathrm{ad}}(t) = z$ and $ad_{S_x \otimes_{k(x)} F_c}(t) = s$. The latter equality and the same argument used to show equivalence of a) and b) imply that t is contained in the complement of the open subspace

$$((S_x \otimes_{k(x)} F_c) \setminus \bigcup_{i=0}^{d+1} \bigcup_{m \in \frac{1}{n}\mathbb{Z}} \mathrm{pr}'_{i,m}(S_{i,m,D}))^{\mathrm{ad}} \subseteq (S_x \otimes_{k(x)} F_c)^{\mathrm{ad}}.$$

This means there is a point t' in some $T_{i,m,D}^{\mathrm{ad}}$ with $(\mathrm{pr}'_{i,m})^{\mathrm{ad}}(t') = t$ and $|d_i(t')|_{k(t')}^e > |p|_{k(t')}^m$. The latter implies that $(g_{t'}, \mathcal{F}_{t'}^\bullet) \notin (G_D \times_K \mathrm{Drap}(\mu, D_L)(k(t'))^{\mathrm{wa}}$. The equality $q^{\mathrm{ad}}(t) = z$ together with 5.2.6 finally yield that $(g_z, \mathcal{F}_z^\bullet) \notin (G_D \times_K \mathrm{Drap}(\mu, D_L))(k(z))^{\mathrm{wa}}$ and hence b). The equivalence of c) and d) holds by definition. This proves the claim.

From the claim we finally conclude that

$$M^{\mathrm{ad}} = \alpha^{-1}(x)^{\mathrm{wa}}$$

holds, which finishes the proof. □

Remark 5.2.10. Taking into account the context of $\frac{1}{n}\mathbb{Z}$-filtrations, by an analogous reasoning as in the proof of [Hel, Theorem 5.1], the adic space $\alpha^{-1}(x)^{\mathrm{wa}}$ for general $x \in Q^{\mathrm{ad}}$ can be obtained as the fiber product of the adification of a quasi-projective scheme over a finite intermediate extension of K in $k(x)$ (as constructed in the above proof) with $\mathrm{Spa}(k(x), k(x)^+)$.

Example 5.2.11.

1. With notations from the proof, let $n = 2, d = 2, e = f = 1, \mu = (0, \frac{1}{2}, 1)$ and suppose $c = (X-1)(X-p^{\frac{1}{2}})(X-p) \in K[X]$. Then denoting by D a three-dimensional K-vector space we find

$$M = S_x \setminus (\mathrm{pr}'_{1,\frac{1}{2}}(S_{1,\frac{1}{2},D}) \cup \mathrm{pr}'_{1,1}(S_{1,1,D}) \cup \mathrm{pr}'_{2,1}(S_{2,1,D}) \cup \mathrm{pr}'_{2,\frac{3}{2}}(S_{2,\frac{3}{2},D})).$$

   In particular, a triple $(D = Ke_1 \oplus Ke_2 \oplus Ke_3, \phi, F^\bullet)$ with a K-linear map

$$\phi : e_1 \mapsto e_1, \quad e_2 \mapsto p^{\frac{1}{2}} e_2, \quad e_3 \mapsto pe_3$$

   and a filtration $F^\bullet$ of type μ on $Ke_1 \oplus Ke_2 \oplus Ke_3$ by K-vector spaces lies in the category $\mathbf{FIC}^{\mathrm{wa}}_{\mathbb{Q}_p,K,2}$ (and hence defines a K-valued point of M) if and only if $Ke_1 \cap F^{\frac{1}{2}} = 0$ and $(Ke_1 \oplus Ke_2) \cap F^1 = 0$. The set of filtrations satisfying these conditions is not empty.
   The object $Ke_1 \oplus Ke_2$ together with induced Frobenius and any induced weakly admissible filtration lies in $\mathbf{FIC}^{\mathrm{wa}}_{\mathbb{Q}_p,K,2}$. It lies in $\mathbf{FIC}^{\mathrm{wa}}_{\mathbb{Q}_p,K,(2)}$ if and only if the non-trivial filtration step is equal to the Frobenius-invariant subspace Ke_2. Regarding the results from the previous sections, in this case this object corresponds via $\mathbb{V}_{\mathrm{cris},2}$, up to isomorphism, to the direct sum of the trivial $G_{\mathbb{Q}_p,(2)}$-representation K with ε_2^{-1}. In all other cases, the functor $\mathbb{V}_{\mathrm{cris},2}$ sends this object to K.

2. Let $d = 1, f = 1, c = (X-1)(X-p^{\frac{1}{n}}) \in K[X]$ and $\mu = ((0, \frac{1}{n})_\tau)_\tau$. Thus let D be a two-dimensional K-vector space. We find that the underlying set of M in this case is identified with the subset of those points $y \in S_x$ such that for the corresponding pair $(g_y, \mathcal{F}^\bullet_y) \in (G_D \times_K \mathrm{Drap}(\mu, D_L))(\kappa(y))$ we have: the τ-isotypical component of the scalar extension to $L \otimes_{\mathbb{Q}_p} K$ of the g_y-eigenspace of 1 has trival intersection with $\mathcal{F}^{\frac{1}{n}}_{y,\tau}$ for every τ.

A Rational Hodge-Tate weights

In this short appendix we introduce the notion of Hodge-Tate weight in $\frac{1}{n}\mathbb{Z}$. We assume all notations and conventions up to section 4.2.

By the arguments given in 2.1.3, an R_L-module M is the same as a family $\bigoplus_\tau M_\tau$ of K-vector spaces made into a module over the ring $\bigoplus_\tau K_\tau$ by componentwise addition and componentwise scalar multiplication.
This holds in particular for the R_L-module D_L associated with an object $\underline{D}$ of $\mathbf{FIC}_{L,K,n}$. For any $x \in D_L$ and any $\mathbb{Q}_p$-algebra homomorphism $\tau' : L \to K$, let $x_{\tau'} := \pi_{\tau'}(x)$ where $\pi_{\tau'}$ is the projection $D_L \to D_L$ corresponding to the idempotent $(\delta_{\tau,\tau'})_\tau \in \bigoplus_\tau K_\tau$ (in terms of the Kronecker delta). Then

$$D_{L,\tau'} := \text{ image}(\pi_{\tau'}) = \{d \in D_L \mid (l \otimes 1)d = (1 \otimes \tau'(l))d \text{ for all } l \in L\}$$

and each $D_{L,\tau'}$ is naturally equipped with a decreasing, exhaustive and separated $\frac{1}{n}\mathbb{Z}$-filtration by K-vector spaces via $F^\bullet D_{L,\tau'} := (F^\bullet D_L)_{\tau'} := \pi_{\tau'}(F^\bullet D_L)$.

Definition A.1. Let $\underline{D}$ be in $\mathbf{FIC}_{L,K,n}$ and let $(F^\bullet D_{L,\tau})_\tau$ be the family of filtrations just described. A number $h \in \frac{1}{n}\mathbb{Z}$ is called a Hodge-Tate weight of $\underline{D}$ if there exists a $\mathbb{Q}_p$-algebra homomorphism $\tau : L \to K$ such that $\mathrm{gr}^{-h} D_{L,\tau} \neq 0$. The positive integer $\sum_\tau \dim_K(\mathrm{gr}^{-h} D_{L,\tau})$ is called the multiplicity of the Hodge-Tate weight h. Thus we obtain a multituple

$$HT(D_L) := HT(D) := (HT(D)_\tau)_\tau \in ((\frac{1}{n}\mathbb{Z})^{\mathrm{rank}_R(D)})_\tau$$

where each (unordered) $\mathrm{rank}_R(D)$-tuple $HT(D)_\tau$ contains those Hodge-Tate weights of $\underline{D}$ contributed by $D_{L,\tau}$ with respective multiplicity $\dim_K(\mathrm{gr}^{-h} D_{L,\tau})$.
Let V be in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$. We define the Hodge-Tate weights of V as being those of $\mathbb{D}_{\mathrm{cris},n}(V)$ and set $HT(V) := HT(\mathbb{D}_{\mathrm{cris},n}(V))$.

Note that the definition of Hodge-Tate weight depends only on the associated filtration and not on the Frobenius of $\underline{D}$. Hence non-isomorphic objects can induce the same Hodge-Tate tuple (cf. also the examples below).

Example A.2.

1. Recall the object $\underline{K}'_W$ from 2.2.17. Then $HT(\underline{K}'_W) = (0, -\frac{1}{n})$. The multiplicity of both Hodge-Tate weights is one.

2. Let $L = \mathbb{Q}_p$. The object $\underline{R} \oplus \underline{K}_n^\vee$ is an object of $\mathbf{FIC}^{\mathrm{wa}}_{\mathbb{Q}_p,K,(n)}$ and we find $HT(\underline{R} \oplus \underline{K}_n^\vee) = HT(K \oplus \varepsilon_n^{-1}) = (0, -\frac{1}{n})$. Again the multiplicity of both Hodge-Tate weights is one.

3. Concerning the characters ε_n^r $(r \in \mathbb{Z})$ we have $HT(\varepsilon_n^r) = ((\frac{r}{n}), \ldots, (\frac{r}{n}))$. Hence in any case there is only one Hodge-Tate weight with respective multiplicity $[L : \mathbb{Q}_p]$. In particular, $HT(\varepsilon) = ((1), \ldots, (1))$.

With the notions from the definition and keeping in mind the effect of twisting on Hodge-Tate weights, it is immediate that, if V is in $\mathbf{Rep}_K^{\mathrm{cris}}(G_{L,(n)})$, all Hodge-Tate weights of V are integers if and only if V is in $\mathbf{Rep}_K^{\mathrm{cris}}(G_L)$.

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