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On p-adic comparison theorems for rigid analytic varieties, I

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Dedicated to Christopher Deninger on the occasion of his 60th birthday

Abstract. We compute, in a stable range, the arithmetic p-adic étale cohomology of smooth rigid analytic and dagger varieties (without any assumption on the existence of a nice integral model) in terms of differential forms using syntomic methods. The main technical input is a construction of a Hyodo–Kato cohomology and a Hyodo–Kato isomorphism with de Rham cohomology.

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1. INTRODUCTION

Let p be a prime. Let $\mathscr{O}_K$ be a complete discrete valuation ring of mixed characteristic $(0, p)$ with perfect residue field k and fraction field K. Let F be the fraction field of the ring of Witt vectors $\mathscr{O}_F = W(k)$ of k. Let $\overline{K}$ be

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an algebraic closure of K and let $C = \widehat{\overline{K}}$ be its p-adic completion; let $\mathscr{G}_K = \mathrm{Gal}(\overline{K}/K)$. Let F^{nr} be the maximal unramified extension of F in $\overline{K}$.

In a joint work with Gabriel Dospinescu [8, 9], we have computed the p-adic (pro-)étale cohomology of certain p-adic symmetric spaces. A key ingredient of these computations was a one-way (de Rham to étale) comparison theorem for rigid analytic varieties over K with a semi-stable formal model over $\mathscr{O}_K$ that allowed us to pass from (pro-)étale cohomology to syntomic cohomology and then to a filtered Frobenius eigenspace associated to de Rham cohomology.

The main goal of this paper is to define all the cohomologies that will be necessary for extending such comparison quasi-isomorphisms to all smooth rigid analytic varieties over K or C (without any assumption on the existence of a nice integral model). We will focus on the arithmetic case and leave the geometric case for the sequel of this paper [12].

1.1. **Main results.**

We are mainly interested in partially proper rigid analytic varieties. Since these varieties have a canonical overconvergent (or dagger) structure, we are led to study dagger varieties.[1] This is advantageous: for example, a dagger affinoid has de Rham cohomology that is a finite rank vector space with its natural Hausdorff topology while the de Rham cohomology of rigid analytic affinoids is, in general, infinite-dimensional and not Hausdorff.

Our first main result is the following theorem.

Theorem 1.1. *To any smooth dagger variety X over $L = K, C$, there are naturally associated:*[2]

(1) *A pro-étale cohomology $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r))$, $r \in \mathbf{Z}$. If X is partially proper, this agrees with the pro-étale cohomology of X considered as a rigid analytic variety.*

(2) *For $L = C$, a $\overline{K}$-valued rigid cohomology $\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X)$ and a natural strict quasi-isomorphism*[3]

$$\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X) \,\widehat{\otimes}^R_{\overline{K}}\, C \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

This defines a natural $\overline{K}$-structure on the de Rham cohomology.[4]

(3) *A Hyodo–Kato cohomology $\mathrm{R}\Gamma_{\mathrm{HK}}(X)$. This is a dg F-algebra if $L = K$, and a dg F^{nr}-algebra if $L = C$, equipped with a Frobenius φ and a monodromy operator N. For $L = C$, we have natural Hyodo–Kato strict*

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[1] Recall that a dagger variety is a rigid analytic variety equipped with an overconvergent structure sheaf. See [20] for the basic definitions and properties.

[2] All cohomology complexes live in the bounded below derived ∞-category of locally convex topological vector spaces over $\mathbf{Q}_p$. Quasi-isomorphisms in this category we call *strict quasi-isomorphisms.*

[3] See Proposition 5.19 for the definition of the tensor product.

[4] By the same procedure one can define a F^{nr}-valued rigid cohomology $\mathrm{R}\Gamma_{\mathrm{rig},F^{\mathrm{nr}}}(X)$ and a natural strict quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{rig},F^{\mathrm{nr}}}(X) \,\widehat{\otimes}^R_{F^{\mathrm{nr}}}\, C \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X)$.

quasi-isomorphisms:

$$\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(X)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X),$$

$$\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(X)\,\widehat{\otimes}^{R}_{F^{\mathrm{nr}}}\,C \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

(4) *For $L = K$, a syntomic cohomology* $\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))$*,* $r\in\mathbf{N}$*, that fits into a distinguished triangle*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) \longrightarrow [\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r} \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r, \tag{1.2}$$

and a natural period morphism

$$\alpha_r\colon \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))$$

that is a strict quasi-isomorphism after truncation $\tau_{\leq r}$*.*

We also prove an analogous theorem for smooth rigid analytic varieties.

Our second main result is the following corollary of Theorem 1.1.

Theorem 1.3. *Let X be a smooth dagger variety over K and let $r\geq 0$.*

(1) *For $1\leq i\leq r-1$, the boundary map induced by the distinguished triangle* (1.2)

$$\partial_r\colon \widetilde{H}^{i-1}_{\mathrm{dR}}(X) \to \widetilde{H}^{i}_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))$$

is an isomorphism. In particular, the cohomology $\widetilde{H}^{i}_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r))$ *is classical and it has a natural K-structure.*

(2) *We have long exact sequences*

$$0\to \widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r) \xrightarrow{\partial_r} \widetilde{H}^{r}_{\mathrm{pro\acute{e}t}}(X,\mathbf{Q}_p(r)) \to \widetilde{H}^{r}([\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r}) \xrightarrow{\iota_{\mathrm{HK}}} \widetilde{H}^{r}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r),$$

$$0\to \widetilde{H}^{r-1}_{\mathrm{HK}}(X)^{\varphi=p^{r-1}} \to \widetilde{H}^{r}([\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r}) \to \widetilde{H}^{r}_{\mathrm{HK}}(X)^{N=0,\varphi=p^r}\to 0.$$

Moreover, the cohomology $\widetilde{H}^{i}_{\mathrm{HK}}(X)$ *is classical.*

Here $\widetilde{H}$ refers to cohomology taken in the derived category of locally convex topological vector spaces over $\mathbf{Q}_p$ and "classical" means that the cohomology $\widetilde{H}$ is isomorphic to the algebraic cohomology H equipped with its natural quotient topology (very often this is equivalent to the natural topology on H being separated). If X is proper, we have the isomorphisms

$$H^{r-1}_{\mathrm{dR}}(X) \xrightarrow{\sim} \widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r),\quad H^{r}_{\mathrm{dR}}(X)/\Omega^r(X) \xrightarrow{\sim} \widetilde{H}^{r}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r).$$

If X is Stein, we get the isomorphisms

$$\widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)\simeq \Omega^{r-1}(X)/\operatorname{im} d_{r-1},\quad \widetilde{H}^{i}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)\simeq 0,\quad i\geq r.$$

Hence the cohomology $\widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)$ is classical.

We prove an analogous result in the case of smooth rigid analytic varieties over K and this generalizes the computations [10, Cor. 3.16] done for smooth affinoids with semi-stable reduction.

Remark 1.4. For a smooth proper scheme X over K, the analog of the map $\partial_r\colon \widetilde{H}^{i-1}_{\mathrm{dR}}(X)\to \widetilde{H}^i_{\text{proét}}(X,\mathbf{Q}_p(r))$ is a geometric incarnation of the Bloch–Kato exponential. See [31, Remark 2.14], [13, Prop. 3.8], [32, Th. 3.1] for a detailed discussion.

1.2. **Proof of Theorem 1.1.** We will now sketch how Theorem 1.1 is proved. The pro-étale cohomology in (1) is defined in the most naive way: if X is a smooth dagger affinoid with a presentation $\{X_h\}_{h\in\mathbf{N}}$ by a pro-affinoid rigid analytic variety,[5] we set $\mathrm{R}\Gamma_{\text{proét}}(X,\mathbf{Q}_p(r)) := \operatorname{hocolim}_h \mathrm{R}\Gamma_{\text{proét}}(X_h,\mathbf{Q}_p(r))$; then we globalize. From this description it is clear that we have a natural map $\mathrm{R}\Gamma_{\text{proét}}(X,\mathbf{Q}_p(r))\to \mathrm{R}\Gamma_{\text{proét}}(\widehat{X},\mathbf{Q}_p(r))$, where $\widehat{X}$ is the completion of X (a rigid analytic variety).

For the rest of Theorem 1.1, first we show that, using the rigid analytic étale local alterations of Hartl and Temkin [22, 39], the étale topology on X_L has a base consisting of semi-stable weak formal schemes (always assumed to be of finite type) over finite extensions of $\mathscr{O}_K$. This allows us to define sheaves by specifying them on such integral models and then sheafifying for the η-étale topology.[6] For example, for (2), we define $\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X) := \mathrm{R}\Gamma_{\text{ét}}(X,\mathscr{A}_{\mathrm{rig},\overline{K}})$, for a sheaf $\mathscr{A}_{\mathrm{rig},\overline{K}}$ induced from a presheaf assigning to a semi-stable model $\mathscr{Y}$ over $\mathscr{O}_C$ coming by base change from a semi-stable model $\mathscr{Y}_{\mathscr{O}_E}$ over $\mathscr{O}_E$, $[E:K]<\infty$, the complex[7] $\operatorname{hocolim}\mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{Y}_{\mathscr{O}_E,0})$, $\mathscr{Y}_{\mathscr{O}_E,0}$ is the special fiber of $\mathscr{Y}_{\mathscr{O}_E}$, where the homotopy colimit is taken over such models $\mathscr{Y}_{\mathscr{O}_E}$. In an analogous way, we define, for (3), the Hyodo–Kato cohomology using the overconvergent Hyodo–Kato cohomology of Grosse–Klönne that, for a semi-stable model $\mathscr{Y}$ over $\mathscr{O}_K$, is defined as $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{Y}_0) := \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{Y}_0/\mathscr{O}_F^0)$; the Hyodo–Kato quasi-isomorphism is induced from the one defined by Grosse–Klönne

$$\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{Y}_0/\mathscr{O}_F^0)\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{Y}_0/\mathscr{O}_F^{\times}).$$

Here $\mathscr{O}_K^{\times}$, $\mathscr{O}_K^0$ denote the (weak formal) scheme associated to $\mathscr{O}_K$ with the canonical and the induced by $\mathbf{N}\to\mathscr{O}_K$, $1\mapsto 0$, log-structure, respectively.

We define the syntomic cohomology in (4) in two different, but (non obviously) equivalent, ways. One definition is just as a homotopy fiber that yields the distinguished triangle (1.2). The other, for dagger affinoids with a presentation $\{X_h\}_{h\in\mathbf{N}}$, sets $\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) := \operatorname{hocolim}_h \mathrm{R}\Gamma_{\mathrm{syn}}(X_h,\mathbf{Q}_p(r))$. Here the syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,\mathbf{Q}_p(r))$ of a rigid analytic variety X_h is defined by η-étale descent, using the fact that semi-stable formal models form a base for the étale topology of X, from the crystalline syntomic cohomology of Fontaine–Messing. Recall that the latter is defined as the homotopy fiber $\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},\mathbf{Q}_p(r) := [F^r\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})\xrightarrow{\varphi-p^r}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})]$, where the crystalline cohomology is absolute (i.e., over $\mathbf{Z}_p$). The second definition works also for smooth dagger varieties over C.

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[5] See Section 3.2.1 for the definition of presentations.

[6] This construction mimics that of Beilinson in [2] done for algebraic varieties; here η-étale means topology induced from the étale topology of the generic fiber.

[7] We give here a rough definition; see Section 5.3 for a precise definition.

It is quite nontrivial to show that these two definitions agree. Along the way, we prove the main technical result of this paper:

Theorem 1.5. *Let $r \geq 0$. Let X be a smooth dagger variety over K. There is a natural morphism*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(\widehat{X}, \mathbf{Q}_p(r)).$$

It is a strict quasi-isomorphism if X is partially proper.

This theorem is proved by representing both sides of the morphism by means of the crystalline and the overconvergent Hyodo–Kato cohomology, respectively, then passing via Galois descent to X_C, and finally passing through the crystalline and overconvergent Hyodo–Kato quasi-isomorphisms (that need to be shown to be compatible) to the de Rham cohomology, where the result is known.

To define the period map in (4), for $L = K, C$, we first define it for rigid analytic varieties by the η-étale descent of the Fontaine–Messing period map $\alpha_r\colon \mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\text{ét}}(\mathscr{X}_L, \mathbf{Q}_p(r))$, for a semi-stable formal scheme $\mathscr{X}$ over $\mathscr{O}_L$. Then we use the second definition of syntomic cohomology and the period maps $\alpha_r\colon \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\text{ét}}(X_h, \mathbf{Q}_p(r))$ to get the period map α_r in Theorem 1.1. The fact that it is a strict quasi-isomorphism in a stable range follows from the computations of p-adic nearby cycles via syntomic complexes done in [40] in the geometric case and in [10] in the arithmetic case.

Remark 1.6. For an algebraic variety X over $L = K, C$, a well-behaved syntomic cohomology $\mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r))$, $r \geq 0$, was defined in [31]. A more conceptual definition was given in [13] but the approach in [31] is more concrete and this is the one we mimic in this paper. For $L = K$ and smooth X, there exists a natural map $\mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(X^{\mathrm{an}}, \mathbf{Q}_p(r))$, where X^{an} denotes the analytification of X. This should be a strict quasi-isomorphism if X is proper although we do not prove this in this paper.

Remark 1.7. Let $\mathscr{X}$ be a proper semi-stable scheme over $\mathscr{O}_K$ (we allow a horizontal divisor at infinity). Ertl and Yamada [15] have extended Grosse-Klönne's definition of the Hyodo–Kato morphism to this setting and defined the corresponding rigid syntomic cohomology by the defining property (1.2). See [43] for a more conceptual definition in the case when there is no horizontal divisor at infinity.

1.2.1. *Notation and conventions.* All formal schemes are p-adic. For a (weak formal or formal) scheme $\mathscr{X}$ over $\mathscr{O}_K$, we will denote by $\mathscr{X}_n$ its reduction modulo p^n, $n \geq 1$, and by $\mathscr{X}_0$ its special fiber.

We will denote by $\mathscr{O}_K$, $\mathscr{O}_K^{\times}$, and $\mathscr{O}_K^0$, depending on the context, the scheme $\mathrm{Spec}(\mathscr{O}_K)$ or the formal scheme $\mathrm{Spf}(\mathscr{O}_K)$ with the trivial, the canonical (i.e., associated to the closed point), and the induced by $\mathbf{N} \to \mathscr{O}_K, 1 \mapsto 0$, log-structure, respectively.

Definition 1.8. Let $N \in \mathbf{N}$. For a morphism $f\colon M \to M'$ of $\mathbf{Z}_p$-modules, we say that f is *p^N-injective* (resp. *p^N-surjective*) if its kernel (resp. its cokernel) is annihilated by p^N and we say that f is a p^N*-isomorphism* if it is p^N-injective and p^N-surjective. We define in the same way the notion of p^N*-distinguished triangle* or p^N*-acyclic complex* (a complex whose cohomology groups are annihilated by p^N) as well as the notion of p^N*-quasi-isomorphism* (map in the derived category that induces a p^N-isomorphism on cohomology).

Unless otherwise stated, we work in the derived (stable) ∞-category $\mathscr{D}(A)$ of left-bounded complexes of a quasi-abelian category A (the latter will be clear from the context). Many of our constructions will involve (pre)sheaves of objects from $\mathscr{D}(A)$. The reader may consult the notes of Illusie [25] and Zheng [44] for a brief introduction to how to work with such (pre)sheaves and [29, 30] for a thorough treatment.

We will use a shorthand for certain homotopy limits. Namely, if $f\colon C \to C'$ is a map in the derived ∞-category of a quasi-abelian category, we set

$$[\, C \xrightarrow{f} C' \,] := \operatorname{holim}(C \to C' \leftarrow 0).$$

And we set

$$\left[\begin{tikzcd} C_1 \arrow[r, "f"] \arrow[d] & C_2 \arrow[d] \\ C_3 \arrow[r, "g"] & C_4 \end{tikzcd}\right] := \big[[C_1 \xrightarrow{f} C_2] \to [C_3 \xrightarrow{g} C_4]\big],$$

for a commutative diagram (the one inside the large bracket) in the derived ∞-category of a quasi-abelian category.

2. An equivalence of topoi

Let X be a smooth rigid analytic variety over K, resp. C. In this section, we will show that the étale site of X has a base (in the sense of Verdier, see [41]) built from semi-stable formal schemes over finite extensions of $\mathscr{O}_K$, resp. over $\mathscr{O}_C$. We will show the same for smooth dagger spaces over K and C.

2.1. **A general criterium.**

In [1, 2.1] Beilinson generalized a well-known criterium of Verdier [41, 4.1] stating conditions under which one can change sites while preserving their topoi. While Verdier assumed the functor F below to be fully faithful, Beilinson allows it to be just faithful.

We will briefly summarize [1, 2.1]. Let $\mathscr{V}$ be an essentially small site and let $\mathrm{Sh}(\mathscr{V})$ be the corresponding topos. *A base for* $\mathscr{V}$ is a pair $(\mathscr{B}, F)$, where $\mathscr{B}$ is an essentially small category and $F\colon \mathscr{B} \to \mathscr{V}$ is a faithful functor, which satisfies the following property:

($\star$) For $V \in \mathscr{V}$ and a finite family of pairs $(B_\alpha, f_\alpha), B_\alpha \in \mathscr{B}, f_\alpha\colon V \to F(B_\alpha)$, there exists a set of objects $B'_\beta \in \mathscr{B}$ and a covering family $\{F(B'_\beta) \to V\}$ such that each composition $F(B'_\beta) \to V \to F(B_\alpha)$ lies in $\operatorname{Hom}(B'_\beta, B_\alpha) \subset \operatorname{Hom}(F(B'_\beta), F(B_\alpha))$.

Remark 2.1. (1) For the empty set of (B_α, f_α)'s the above means that every $V \in \mathscr{V}$ has a covering by objects $F(B)$, $B \in \mathscr{B}$. If F is fully faithful, then $(\star)$ is equivalent to this assertion.

(2) If $\mathscr{B}$ admits finite products and F commutes with finite products, then it suffices to check $(\star)$ for families (B_α, f_α) having ≤ 1 elements.

(3) In the general case, it suffices to check $(\star)$ for families (B_α, f_α) having ≤ 2 elements.

Let $(\mathscr{B}, F)$ be a base for $\mathscr{V}$. Define a covering sieve in $\mathscr{B}$ as a sieve whose F-image is a covering sieve in $\mathscr{V}$. The following proposition is proved by Beilinson [1, 2.1].

Proposition 2.2. (1) *Covering sieves in* $\mathscr{B}$ *form a Grothendieck topology on* $\mathscr{B}$.

(2) *The functor* $F\colon \mathscr{B} \to \mathscr{V}$ *is continuous.*

(3) F *induces an equivalence of topoi* $\mathrm{Sh}(\mathscr{B}) \xrightarrow{\sim} \mathrm{Sh}(\mathscr{V})$.

We call the above topology on $\mathscr{B}$ *the* F*-induced topology.*

Remark 2.3. (1) If F is fully faithful, the above proposition is [41, 4.1].

(2) Let $(F^s, F_s)\colon \mathrm{Sh}(\mathscr{B}) \leftrightarrows \mathrm{Sh}(\mathscr{V})$ be the usual adjoint functors. For a presheaf $\mathscr{F}$ on $\mathscr{V}$, we have $F_s(\mathscr{F}^a) = F_p(\mathscr{F})^a$, where F_p is the push-forward of presheaves and the subscript a means "associated sheaf".

(3) If $(\mathscr{B}, F)$ is a base for $\mathscr{V}$ and $(\mathscr{B}', F')$ is a base for the F-induced topology on $\mathscr{B}$, then $(\mathscr{B}', FF')$ is a base for $\mathscr{V}$.

2.2. **Categories of formal models.** We will show now that the étale site of smooth rigid analytic varieties over K, resp. over C, admits a base built from semi-stable formal schemes over finite extensions of $\mathscr{O}_K$, resp. over $\mathscr{O}_C$.

2.2.1. *Models.* Let $L = K, C$. A morphism of $\mathscr{O}_L$-schemes $f\colon Y \to X$ is called η-étale, an η-isomorphism, etc., if its generic fiber f_L is étale, an isomorphism, etc. An $\mathscr{O}_L$-scheme is *admissible* if it is flat and of finite type over $\mathscr{O}_L$. A formal $\mathscr{O}_L$-scheme $\mathscr{X}$ is *admissible* if it is flat and of finite type over $\mathrm{Spf}(\mathscr{O}_L)$. For an admissible formal $\mathscr{O}_L$-scheme $\mathscr{X}$, we denote by $\mathscr{X}_L$ (or $\mathscr{X}_\eta$) its rigid analytic generic fiber. We say that a morphism $\mathscr{Y} \to \mathscr{X}$ between admissible formal $\mathscr{O}_L$-schemes is *η-étale* if its generic fiber f_L (or f_η) is étale. Similarly, we define *η-smooth* morphisms.[8]

Let Sm_L be the category of smooth L-rigid varieties. We will consider categories $\mathscr{M}$ formed by semi-stable formal models of such varieties.

(a) *K-setting*: A *model over K* (a *K-model*) is an admissible formal $\mathscr{O}_K$-scheme $\mathscr{X}$. A formal scheme over $\mathscr{O}_K$ is called *semi-stable* if, locally for the Zariski topology, it admits an étale morphism to a formal scheme of the form

$$\mathrm{Spf}(\mathscr{O}_K\{X_1, \dots, X_l\}/(X_1 \cdots X_m - \varpi)), \quad 0 \leq m \leq l,$$

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[8] In a more traditional language we would call such morphisms "rig-étale", etc. However, since it is becoming standard to use η to denote the rigid generic fiber, we have elected to use η-étale in this paper.

for a uniformizer ϖ of $\mathscr{O}_K$ (we allow $m = 0$ just to get formal affine space – when the formal scheme is smooth). A K-model $\mathscr{X}$ is called *semi-stable* if it is semi-stable over $\mathscr{O}_E$ for a finite field extension E of K. In that case, assume that $\mathscr{X}_K$ is connected (which is equivalent to $\mathscr{X}$ being connected), and let $K_{\mathscr{X}}$ be the algebraic closure of K in $\Gamma(\mathscr{X}_K, \mathscr{O}_{\mathscr{X}_K})$ (note that $E \subset K_{\mathscr{X}}$). Then $\mathscr{O}_{K_{\mathscr{X}}}$ is the integral closure of $\mathscr{O}_K$ in $\Gamma(\mathscr{X}, \mathscr{O}_{\mathscr{X}})$ and $\mathscr{X}$ is semi-stable over $\mathscr{O}_{K_{\mathscr{X}}}$. We will say that $\mathscr{X}$ is *split* over $K_{\mathscr{X}}$.

Let $\mathscr{M}_K$ denote the category of K-models (morphisms are morphisms of formal schemes over $\mathscr{O}_K$) and let $\mathscr{M}_K^{\mathrm{ss}}$ be its full subcategory of semi-stable K-models.

(b) *C-setting*: A *model over C* (a *C-model*) is an admissible formal $\mathscr{O}_C$-scheme $\mathscr{X}$. It is called *semi-stable* if, locally for the Zariski topology, it admits an étale morphism to a formal scheme of the form

$$\mathrm{Spf}(\mathscr{O}_C\{X_1, \dots, X_l\}/(X_1 \cdots X_m - \varpi)), \quad 0 \le m \le l,$$

for $0 \neq \varpi \in \mathscr{O}_C$. It is called *basic semi-stable* if there exists a semi-stable model $\mathscr{X}'$ over $\mathscr{O}_E$, E a finite extension of K, and a C-point $\alpha\colon E \to C$ such that $\mathscr{X}$ is isomorphic to the base change $\mathscr{X}'_{\mathscr{O}_C}$. Let $\mathscr{M}_C$ denote the category of C-models and let $\mathscr{M}_C^{\mathrm{ss}}$, $\mathscr{M}_C^{\mathrm{ss},b}$ be its full subcategories of semi-stable and basic semi-stable C-models, respectively.

We note that if we equip the formal schemes in $\mathscr{M}_K^{\mathrm{ss}}$, $\mathscr{M}_C^{\mathrm{ss},b}$, and $\mathscr{M}_C^{\mathrm{ss}}$ with the log-structure associated to the special fiber over the ring over which they split, every map in these categories is a map of log-schemes. Warning: the maps in the category $\mathscr{M}_C^{\mathrm{ss},b}$ *do not* have to come from finite levels.

The K- and C-settings are connected by the base change functors

$$\begin{tikzcd} \mathscr{M}_C^{\mathrm{ss},b} \arrow[r] & \mathrm{Sm}_C \\ \mathscr{M}_K^{\mathrm{ss}} \arrow[r] \arrow[u] & \mathrm{Sm}_K, \arrow[u] \end{tikzcd} \tag{2.4}$$

where the right vertical arrow is the base change $(-)\widehat{\otimes}_K C$ and the left arrow assigns to a K-model $\mathscr{U}$ semi-stable over $\mathscr{O}_L$, L a finite extension of K, the disjoint union of semi-stable models $\mathscr{U} \mathbin{\widehat{\otimes}}_{\mathscr{O}_L,\alpha} \mathscr{O}_C$ over C-points $\alpha\colon L \to C$.

2.2.2. *Semistable reduction.* We say that an admissible formal $\mathscr{O}_L$-scheme $\mathscr{X}$ is *algebraizable* if it is isomorphic to the p-adic completion of an admissible $\mathscr{O}_L$-scheme X. The well-known algebraization theorem of Elkik [14] yields the following theorem.

Theorem 2.5 ([39, Th. 3.1.3]). *Any affine η-smooth admissible formal $\mathscr{O}_L$-scheme $\mathscr{X}$ is algebraizable. Moreover, we can find an affine η-smooth admissible $\mathscr{O}_L$-scheme X such that $\mathscr{X} \simeq \widehat{X}$.*

We quote two results of Temkin which generalize results of Hartl [22, Th. 1.4] (which works for complete discretely-valued fields) and Faltings [17, III.2] (see [39, Th. 2.5.2] for an algebraic analog and [4] for a refined algebraic analog).

Theorem 2.6 ([39, Th. 3.3.1])**.** *Let $\mathscr{X}$ be an η-smooth admissible formal scheme over $\mathscr{O}_L$. Then there exists a finite field extension E/L and an η-étale covering $\mathscr{X}' \to \mathscr{X} \otimes_{\mathscr{O}_L} \mathscr{O}_E$ such that $\mathscr{X}'$ is semi-stable over $\mathscr{O}_E$.*

Corollary 2.7 ([39, Cor. 3.3.2])**.** *Let X be a smooth qcqs rigid space over L. There exists a finite extension E/L and an étale covering $X' \to X \otimes_L E$ such that X' is affinoid and has a semi-stable affine formal model.*

Proof. Take an admissible formal model $\mathscr{X}$ of X (such a model exists by a theorem of Raynaud [5, Th. 4.1]). Take E/L and $\mathscr{X}' \to \mathscr{X} \otimes_{\mathscr{O}_L} \mathscr{O}_E$ as in Theorem 2.6. We can refine $\mathscr{X}'$ to make it affine. Then its generic fiber $\mathscr{X}'_E$ is affinoid and has $\mathscr{X}'$ for a semi-stable model. $\square$

2.2.3. *An equivalence of topoi.* Let $\mathscr{M}$ be any category from Section 2.2 and let F_η be the forgetful functor $\mathscr{X} \mapsto \mathscr{X}_\eta$. The main result of this section is the following.

Proposition 2.8. *If $\mathscr{M}$ is the category $\mathscr{M}_K$ or $\mathscr{M}_K^{\mathrm{ss}}$, then $(\mathscr{M}, F_\eta)$ is a base for $\mathrm{Sm}_{K,\text{ét}}$. If $\mathscr{M}$ is $\mathscr{M}_C$, $\mathscr{M}_C^{\mathrm{ss},b}$, or $\mathscr{M}_C^{\mathrm{ss}}$, then $(\mathscr{M}, F_\eta)$ is a base for $\mathrm{Sm}_{C,\text{ét}}$.*

Proof. Consider first the K-setting. We need to show that $\mathscr{M}_K$ satisfies condition $(\star)$ from Section 2.1. For that, assume that X is a rigid analytic variety over K and take a finite family[9] of K-models $\mathscr{U}_\alpha$ together with maps $f_\alpha \colon X \to \mathscr{U}_{\alpha,K}$. We need to find an étale covering $\pi \colon X' \to X$ and a K-model $\mathscr{X}'$ of X' such that every map $f_\alpha \pi$ extends to a map $\mathscr{X}' \to \mathscr{U}_\alpha$.

Replacing X by an affinoid admissible covering, we may assume that X is a disjoint union of affinoids. By a theorem of Raynaud [5, Th. 4.1], we can find a K-model of X. By [6, Lem. 5.6], this model can be modified by an admissible blow-up to a K-model $\mathscr{X}$ of X such that there exists a dotted arrow that makes the following diagram commute:

$$\begin{tikzcd}
\mathscr{X} \arrow[d, dotted] & X \arrow[l, hook', dashed] \arrow[d, "\prod_\alpha f_\alpha"] \\
\prod_\alpha \mathscr{U}_\alpha & \prod_\alpha \mathscr{U}_{\alpha,K}. \arrow[l, hook', dashed]
\end{tikzcd}$$

This is the model we wanted.

Now, to show that $(\mathscr{M}_K^{\mathrm{ss}}, F_\eta)$ is a base, it suffices, by Remark 2.3, to show that $(\mathscr{M}_K^{\mathrm{ss}}, \iota)$, for the natural functor $\iota \colon \mathscr{M}_K^{\mathrm{ss}} \hookrightarrow \mathscr{M}_K$, is a base of $\mathscr{M}_K$. Since ι is fully faithful, by Remark 2.1, it suffices to check that, for every K-model $\mathscr{U} \in \mathscr{M}_K$, there exists a map of K-models $\mathscr{U}' \to \mathscr{U}$ such that $\mathscr{U}'_K \to \mathscr{U}_K$ is étale and $\mathscr{U}'$ is semi-stable. But this follows from Theorem 2.6.

For the C-setting the argument is analogous in the case of $\mathscr{M}_C$ and $\mathscr{M}_C^{\mathrm{ss}}$. For $\mathscr{M}_C^{\mathrm{ss},b}$, since $\mathscr{M}_C^{\mathrm{ss},b} \hookrightarrow \mathscr{M}_C^{\mathrm{ss}}$ is fully faithful, by Remark 2.1, it suffices to check that, for every C-model $\mathscr{U} \in \mathscr{M}_C^{\mathrm{ss}}$, there exists a map of C-models $\mathscr{U}' \to \mathscr{U}$ such that $\mathscr{U}'_C \to \mathscr{U}_C$ is étale and $\mathscr{U}'$ is basic semi-stable. But this can be achieved by taking for $\mathscr{U}'$ a log-blow-up of $\mathscr{U}$ (see [35, Lem. 1.11]). $\square$

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[9] By Remark 2.1, we may assume that this family consists of one element.

We call the topology induced by F_η on the categories $\mathscr{M}$ the *η-étale* topology. The functors in (2.4) are continuous for the respective étale topologies. By Section 2.1 and Proposition 2.8, F_η identifies étale sheaves on Sm_K, resp. Sm_C, with η-étale sheaves on $\mathscr{M}_K$, $\mathscr{M}_K^{\mathrm{ss}}$, resp. $\mathscr{M}_C$, $\mathscr{M}_C^{\mathrm{ss},b}$, $\mathscr{M}_C^{\mathrm{ss}}$. We obtain the *étale localization* functors

$$\mathrm{Psh}(\mathscr{M}_K^?) \to \mathrm{Sh}(\mathrm{Sm}_{K,\text{ét}}), \quad \mathrm{Psh}(\mathscr{M}_C^?) \to \mathrm{Sh}(\mathrm{Sm}_{C,\text{ét}}),$$

which assign to any presheaf $\mathscr{F}$ on models the corresponding étale sheaf $\mathscr{F}^\sim$ viewed as an étale sheaf on varieties.

Remark 2.9. For any presheaf on $\mathscr{M}_K$ or $\mathscr{M}_C$, its η-étale sheafification is the same as the η-étale sheafification of its restriction to resp. $\mathscr{M}_K^{\mathrm{ss}}$ or $\mathscr{M}_C^{\mathrm{ss},b}$, $\mathscr{M}_C^{\mathrm{ss}}$.

Remark 2.10. In this paper we will use over and over again the following procedure to define an étale sheaf $\mathscr{F}$ on, say, Sm_K.

1. (*Local definition*). We define a functorial $\mathscr{F}(Y)$, $Y \in \mathscr{M}_K^{\mathrm{ss}}$.
2. (*Globalization*). We sheafify the so defined presheaf in η-étale topology. This yields an étale sheaf $\mathscr{F}$ on Sm_K (this notation is slightly abusive but hopefully will not cause problems in understanding).
3. (*Local-global compatibility*). We will often need to know that we have η-étale descent, i.e., that, for $Y \in \mathscr{M}_K^{\mathrm{ss}}$, the natural map $\mathscr{F}(Y) \to \mathrm{R}\Gamma_{\text{ét}}(Y_K, \mathscr{F})$ is a quasi-isomorphism.

2.3. Categories of weak formal models.

In this section, we will show that the étale site of smooth dagger varieties[10] over K, resp. over C, admits a base built from semi-stable weak formal schemes over finite extensions of $\mathscr{O}_K$, resp. over $\mathscr{O}_C$.

2.3.1. *Models.* Let $L = K, C$. A weak formal $\mathscr{O}_L$-scheme $\mathscr{X}$ is *admissible* if it is flat and of finite type over $\mathscr{O}_L$. For an admissible weak formal $\mathscr{O}_L$-scheme $\mathscr{X}$, we denote by $\mathscr{X}_L$ (or $\mathscr{X}_\eta$) its dagger generic fiber. We say that a morphism $f\colon \mathscr{Y} \to \mathscr{X}$ between admissible weak formal $\mathscr{O}_L$-schemes is *η-étale* if its generic fiber f_L (or f_η) is étale. Similarly, we define *η-smooth* morphisms.

Let $\mathrm{Sm}_L^\dagger$ be the category of smooth L-dagger varieties. We define the categories $\mathscr{M}_L^\dagger, \mathscr{M}_C^{\dagger,\mathrm{ss},b}$, and $\mathscr{M}_L^{\dagger,\mathrm{ss}}$, formed by weak formal models, basic semi-stable, and semi-stable weak formal models,[11] respectively, of such varieties in a similar way as in the rigid analytic case above. If we equip the weak formal schemes in $\mathscr{M}_L^{\dagger,\mathrm{ss}}$ with the log-structure associated to the special fiber over the ring over which they split, every map in these categories is a map of log-schemes. The functors $\mathscr{M}_L^{\dagger,\mathrm{ss}} \to \mathscr{M}_L^\dagger$, $\mathscr{M}_C^{\dagger,\mathrm{ss},b} \to \mathscr{M}_C^{\dagger,\mathrm{ss}}$ are fully faithful embeddings. The K- and C-settings are connected by the base change functors.

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[10] For basics on dagger (or overconvergent) varieties, we refer the reader to [20].

[11] Semistable weak formal schemes are defined by the same formulas as semi-stable formal schemes with the ring of convergent power series $\mathscr{O}_L\{X_1, \ldots, X_l\}$ replaced by the ring of overconvergent power series $\mathscr{O}_L[X_1, \ldots, X_l]^\dagger$.

2.3.2. *Semistable reduction.* We say that an admissible weak formal $\mathscr{O}_L$-scheme $\mathscr{X}$ is *algebraizable* if it is isomorphic to the weak completion of an admissible $\operatorname{Spec}(\mathscr{O}_L)$-scheme X. The algebraization theorem, Theorem 2.5, combined with the fact that, up to an isomorphism, there is a unique dagger structure on every rigid analytic affinoid [18, Cor. 7.5.10], yields the following corollary.

Corollary 2.11. *Any affine η-smooth admissible weak formal $\mathscr{O}_L$-scheme $\mathscr{X}$ is algebraizable. Moreover, we can find an affine η-smooth admissible $\mathscr{O}_L$-scheme X such that $\mathscr{X} \simeq X^\dagger$.*

This corollary allows us to prove the following.

Corollary 2.12.
1. *Let $\mathscr{X}$ be an η-smooth admissible weak formal scheme over $\mathscr{O}_L$. Then there exists a finite field extension E/L and an η-étale covering $\mathscr{X}' \to \mathscr{X} \otimes_{\mathscr{O}_L} \mathscr{O}_E$ such that $\mathscr{X}'$ is semi-stable over $\mathscr{O}_E$.*
2. *Let X a smooth qcqs dagger space over L. Then there exists a finite extension E/L and an étale covering $X' \to X \otimes_L E$ such that X' is a dagger affinoid and has a semi-stable affine weak formal model.*

Proof. For (1), having Corollary 2.11, Temkin's proof of Theorem 2.6 goes through. For (2), we modify the proof of Corollary 2.7 using the algebraization result from Theorem 2.5. □

2.3.3. *An equivalence of topoi.* Let $\mathscr{M}^\dagger$ be any category from Section 2.3.1 and let F_η be the forgetful functor $\mathscr{X} \mapsto \mathscr{X}_L$. The main result of this section is the following.

Proposition 2.13. *If $\mathscr{M}^\dagger$ is the category $\mathscr{M}_K^\dagger$ or $\mathscr{M}_K^{\dagger,\mathrm{ss}}$, then $(\mathscr{M}^\dagger, F)$ is a base for $\mathrm{Sm}^\dagger_{K,\text{ét}}$. If $\mathscr{M}^\dagger$ is $\mathscr{M}_C^\dagger$, $\mathscr{M}_C^{\dagger,\mathrm{ss},b}$, or $\mathscr{M}_C^{\dagger,\mathrm{ss}}$, then $(\mathscr{M}^\dagger, F_\eta)$ is a base for $\mathrm{Sm}^\dagger_{C,\text{ét}}$.*

Proof. Consider first the K-setting. Recall the following dagger version of Raynaud's theory of formal models of rigid analytic varieties:

Theorem 2.14 ([27])**.** *There is an equivalence of categories between*
1. *the category of quasi-paracompact admissible weak formal schemes over $\mathscr{O}_K$ localized by the class of weak formal blow-ups,*
2. *the category of quasi-separated quasi-paracompact K-dagger spaces.*

It is now easy to see that the proof of Proposition 2.8 goes through in our case with Raynaud's theory replaced by this dagger analog.

For the C-setting the argument is analogous to the one used in the proof of Proposition 2.8. □

We call the topology induced by F_η on the categories $\mathscr{M}^\dagger$ the *η-étale* topology. The base-change functors are continuous for the respective étale topologies. By Section 2.3 and Proposition 2.13, F_η identifies étale sheaves on $\mathrm{Sm}_K^\dagger$, resp. $\mathrm{Sm}^\dagger_C$, with η-étale sheaves on $\mathscr{M}^\dagger_K$, $\mathscr{M}^{\dagger,\mathrm{ss}}_K$, resp. $\mathscr{M}^\dagger_C$, $\mathscr{M}^{\dagger,\mathrm{ss},b}_C$, $\mathscr{M}^{\dagger,\mathrm{ss}}_C$. We obtain the *étale localization* functors

$$\mathrm{Psh}(\mathscr{M}^?_K) \to \mathrm{Sh}(\mathrm{Sm}^\dagger_{K,\text{ét}}), \quad \mathrm{Psh}(\mathscr{M}^?_C) \to \mathrm{Sh}(\mathrm{Sm}^\dagger_{C,\text{ét}}),$$

which assign to any presheaf $\mathscr{F}$ on weak formal models the corresponding étale sheaf $\mathscr{F}^\sim$ viewed as an étale sheaf on dagger varieties. Moreover, for any presheaf on $\mathscr{M}^\dagger_K$ or $\mathscr{M}^\dagger_C$, its η-étale sheafification is the same as the η-étale sheafification of its restriction to resp. $\mathscr{M}^{\dagger,\mathrm{ss}}_K$, $\mathscr{M}^{\dagger,\mathrm{ss},b}_C$, or $\mathscr{M}^{\dagger,\mathrm{ss}}_C$.

3. Pro-étale cohomology of dagger varieties

Let the base field L be K or C. Fix a pseudo-uniformizer $\varpi \in L$, i.e., an invertible, topologically nilpotent element. All the rigid analytic varieties considered are over L; we assume that they are separated and taut.[12]

The purpose of this section is to define the pro-étale cohomology of dagger varieties. We will do it in the most naive way: for a dagger affinoid, we will use its presentation of the dagger structure to define the pro-étale cohomology of the dagger affinoid as the homotopy colimit of pro-étale cohomologies of the (rigid) affinoids in the presentation; for a general dagger variety, we will globalize the construction for dagger affinoids via Čech coverings.

3.1. **Topology.**

Our cohomology groups will be equipped with a canonical topology. To talk about it in a systematic way, we will work rationally in the category of locally convex K-vector spaces and integrally in the category of pro-discrete $\mathscr{O}_K$-modules. We review here briefly the relevant basic definitions and facts. For details and further reading and references, the reader may consult [9, Section 2.1, 2.3].

3.1.1. *Derived category of locally convex K-vector spaces.* A topological K-vector space[13] is called *locally convex* (*convex* for short) if there exists a neighborhood basis of the origin consisting of $\mathscr{O}_K$-modules. We denote by C_K the category of convex K-vector spaces. It is a quasi-abelian category. Kernels, cokernels, images, and coimages are taken in the category of vector spaces and equipped with the induced topology. A morphism $f\colon E \to F$ is *strict* if and only if it is relatively open, i.e., for any neighborhood V of 0 in E, there is a neighborhood V' of 0 in F such that $f(V) \supset V' \cap f(E)$.

The category C_K has a natural exact category structure: the admissible monomorphisms are embeddings, the admissible epimorphisms are open surjections. A complex $E \in C(C_K)$ is called *strict* if its differentials are strict. There are truncation functors on $C(C_K)$

$$\begin{aligned} \tau_{\leq n}E &:= \cdots \to E^{n-2} \to E^{n-1} \to \ker(d_n) \to 0 \to \cdots, \\ \tau_{\geq n}E &:= \cdots \to 0 \cdots \to \mathrm{coim}(d_{n-1}) \to E^n \to E^{n+1} \to \cdots, \end{aligned}$$

---

[12] See [23, Def. 5.6.6] for the definition of “taut”.

[13] For us, a *K-topological vector space* is a K-vector space with a linear topology.

with cohomology objects

$$\widetilde{H}^n(E) := \tau_{\leq n}\tau_{\geq n}(E) = (\mathrm{coim}(d_{n-1}) \to \ker(d_n)).$$

We note that here $\mathrm{coim}(d_{n-1})$ and $\ker(d_n)$ are equipped naturally with the quotient and subspace topology, respectively. The cohomology $H^*(E)$ taken in the category of K-vector spaces we will call *algebraic* and, if necessary, we will always equip it with the sub-quotient topology.

We will denote the left-bounded derived ∞-category of C_K by $\mathscr{D}(C_K)$. A morphism of complexes that is a quasi-isomorphism in $\mathscr{D}(C_K)$, i.e., its cone is strictly exact, will be called a *strict quasi-isomorphism.* We will denote by $D(C_K)$ the homotopy category of $\mathscr{D}(C_K)$.

For $n \in \mathbf{Z}$, let $D_{\leq n}(C_K)$ (resp. $D_{\geq n}(C_K)$) denote the full subcategory of $D(C_K)$ of complexes that are strictly exact in degrees $k > n$ (resp. $k < n$). The above truncation functors extend to the truncation functors

$$\tau_{\leq n}\colon D(C_K) \to D_{\leq n}(C_K) \quad \text{and} \quad \tau_{\geq n}\colon D(C_K) \to D_{\geq n}(C_K).$$

The pair $(D_{\leq n}(C_K), D_{\geq n}(C_K))$ defines a t-structure on $D(C_K)$. The (left) heart $LH(C_K)$ is an abelian category: every object of $LH(C_K)$ is represented (up to equivalence) by a monomorphism $f\colon E \to F$, where F is in degree 0, i.e., it is isomorphic to a complex $0 \to E \xrightarrow{f} F \to 0$; *if f is strict*, this object is also represented by the cokernel of f (the whole point of this construction is to keep track of the two possibly different topologies on E: the given one and the one inherited by the inclusion into F).

We have an embedding $I\colon C_K \hookrightarrow LH(C_K)$, $E \mapsto (0 \to E)$, that induces an equivalence $D(C_K) \xrightarrow{\sim} D(LH(C_K))$ that is compatible with t-structures. These t-structures pull back to t-structures on the derived dg categories $\mathscr{D}(C_K)$, $\mathscr{D}(LH(C_K))$ and so does the above equivalence. There is a functor (the *classical part*) $C\colon LH(C_K) \to C_K$ that sends the monomorphism $f\colon E \to F$ to $\operatorname{coker} f$. We have $CI \simeq \mathrm{Id}_{C_K}$ and a natural epimorphism $e\colon \mathrm{Id}_{LH(C_K)} \to IC$.

We will denote by $\widetilde{H}^n\colon \mathscr{D}(C_K) \to \mathscr{D}(LH(C_K))$ the associated cohomological functors. Note that $C\widetilde{H}^n = H^n$ and we have a natural epimorphism $\widetilde{H}^n \to IH^n$. If, evaluated on E, this epimorphism is an isomorphism, we will say that the cohomology $\widetilde{H}^n(E)$ is *classical* (in most cases this is equivalent to $H^n(E)$ being separated).

3.1.2. *The category of pro-discrete $\mathscr{O}_K$-modules.* Objects in the category PD_K of pro-discrete $\mathscr{O}_K$-modules are topological $\mathscr{O}_K$-modules that are countable inverse limits, as topological $\mathscr{O}_K$-modules, of discrete $\mathscr{O}_K$-modules M^i, $i \in \mathbf{N}$. It is a quasi-abelian category. It has countable filtered projective limits. Countable products are exact functors.

Inside the category PD_K, we distinguish the category PC_K of pseudocompact $\mathscr{O}_K$-modules, i.e., pro-discrete modules $M \simeq \varprojlim_i M_i$ such that each M_i is of finite length (we note that if K is a finite extension of $\mathbf{Q}_p$, this is equivalent to M being profinite). It is an abelian category. It has countable exact products as well as exact countable filtered projective limits.

There is a functor from the category of pro-discrete $\mathscr{O}_K$-modules to convex K-vector spaces. Since $K \simeq \varinjlim(\mathscr{O}_K \xrightarrow{\varpi} \mathscr{O}_K \xrightarrow{\varpi} \mathscr{O}_K \xrightarrow{\varpi} \cdots)$, the algebraic tensor product $M \otimes_{\mathscr{O}_K} K$ is an inductive limit:

$$M \otimes_{\mathscr{O}_K} K \simeq \varinjlim(M \xrightarrow{\varpi} M \xrightarrow{\varpi} M \xrightarrow{\varpi} \cdots).$$

We equip it with the induced inductive limit topology. This defines a tensor product functor

$$(-) \otimes K \colon PD_K \to C_K, \quad M \mapsto M \otimes_{\mathscr{O}_K} K.$$

Since C_K admits filtered inductive limits, the functor $(-) \otimes K$ extends to a functor $(-) \otimes K\colon \operatorname{Ind}(PD_K) \to C_K$.

The functor $(-) \otimes K$ is right exact but not, in general, left exact.[14] For example, after tensoring with $\mathbf{Q}_p$, the short strict exact sequence

$$0 \to \prod_{i\geq 0} p^i\mathbf{Z}_p \xrightarrow{\text{can}} \prod_{i\geq 0} \mathbf{Z}_p \to \prod_{i\geq 0} \mathbf{Z}_p/p^i \to 0$$

is not costrict exact on the left (note that $(\prod_{i\geq 0} \mathbf{Z}_p/p^i) \otimes \mathbf{Q}_p$ is not Hausdorff). We will consider its (compatible) left derived functors

$$\begin{aligned}(-) \otimes^L K &\colon \mathscr{D}^-(PD_K) \to \operatorname{Pro}(\mathscr{D}^-(C_K)),\\ (-) \otimes^L K &\colon \mathscr{D}^-(\operatorname{Ind}(PD_K)) \to \operatorname{Pro}(\mathscr{D}^-(C_K)).\end{aligned}$$

The following fact will greatly simplify our computations.

Proposition 3.1 ([9, Prop. 2.6])**.** *If E is a complex of torsion-free and p-adically complete* (*i.e.*, $E \simeq \varprojlim_n E/p^n$) *modules from PD_K, then the natural map*

$$E \otimes^L K \to E \otimes K$$

is a strict quasi-isomorphism.

3.2. **Pro-étale cohomology of dagger varieties.** In this section we will define the pro-étale cohomology of dagger varieties and study its basic properties.

3.2.1. *Dagger varieties and pro-systems of rigid analytic varieties.* We will briefly review here the content of [42, Appendix]. Recall the following definition [42, Def. A.19]:

Definition 3.2. Let X be a rigid analytic affinoid. A *presentation of a dagger structure on* X is a pro-affinoid rigid variety $\{X_h\}$, $h \in \mathbf{N}$, where X and all X_h are rational subvarieties of X_1 such that $X \Subset X_{h+1} \Subset X_h$ and the pro-system is coinitial among rational subvarieties of X_1 containing X in their interiors.[15] A *morphism* of presentations between $\{X_h\}$ and $\{Y_k\}$ is a morphism of pro-objects, i.e., an element of $\varprojlim_k \varinjlim_h \operatorname{Hom}(X_h, Y_k)$.

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[14] We will call a functor F right exact if it transfers strict exact sequences $0 \to A \to B \to C \to 0$ to costrict exact sequences $F(A) \to F(B) \to F(C) \to 0$.

[15] Recall that, for an open immersion $X \subset Y$ of adic spaces over L, we write $X \Subset Y$ if the inclusion factors over the adic compactification of X over L (see [23, Th. 5.1.5]).

Example 3.3. Consider a rational inclusion $X = X_1(f_1/g, \dots, f_m/g) \Subset X_1$ of affinoid rigid varieties. We can write (see [42, Ex. A18])

$$\begin{aligned} \mathscr{O}(X_1) &= L\{\varpi\tau_1, \dots, \varpi\tau_n\}/I, \\ \mathscr{O}(X) &= L\{\tau_1, \dots, \tau_n, v_1, \dots, v_m\}/(I, (v_i g - f_i)_{1\leq i\leq m}). \end{aligned}$$

Let X_h be the rational subvariety of X_1 with

$$\mathscr{O}(X_h) = L\{\varpi^{1/h}\tau_1, \dots, \varpi^{1/h}\tau_n, \varpi^{1/h}v_1, \dots, \varpi^{1/h}v_m\}/(I, (v_i g - f_i)_{1\leq i\leq m}).$$

The pro-system $\{X_h\}$ of rational subvarieties of X_1 is a presentation of a dagger structure on X. We have

$$\varinjlim \mathscr{O}(X_h) \simeq L[\tau_1, \dots, \tau_n, v_1, \dots, v_m]^\dagger/(I, (v_i g - f_i)_{1\leq i\leq m}),$$

which is a dagger algebra.

The following proposition clarifies the relationship between presentations of dagger structures and dagger algebras.

Proposition 3.4 ([42, Prop. A.22])**.** *Let $\widehat{X} = \operatorname{Sp} \widehat{R}$ be a rigid affinoid and let $\{X_h\}$ be a presentation of a dagger structure on $\widehat{X}$. We have*

(1) *$R = \varinjlim \mathscr{O}(X_h)$ is a dagger algebra dense in $\widehat{R}$,*
(2) *the functor $\{X_h\} \mapsto \operatorname{Sp}^\dagger R$ induces an equivalence of categories between dagger affinoid varieties and their presentations.*

In fact, it is not hard to see that we have a functor pres: $X \mapsto \{X_h\}$ from dagger algebras to presentations of dagger structures (up to a unique isomorphism) that is the right inverse (on the nose) of the functor in the above proposition.

3.2.2. *Étale topology of dagger varieties.* For basic properties of dagger algebras and varieties and morphisms between them, see [20]. For basic properties of étale and smooth morphisms of dagger varieties, see [16]. We quote the following result.

Proposition 3.5 ([16, Th. 2.3])**.** *Let X be a dagger affinoid with completion $\widehat{X}$. We have a natural equivalence of étale topoi*

$$\operatorname{Sh}(\widehat{X}_{\text{ét}}) \xrightarrow{\sim} \operatorname{Sh}(X_{\text{ét}}).$$

One can promote the equivalence of categories between dagger spaces and their presentations in Proposition 3.4 to an equivalence of topoi.

Definition 3.6 ([42, Def. A.24])**.** (i) Let P be a property of morphisms of rigid analytic varieties. We say that a morphisms of pro-rigid varieties $\varphi\colon X \to Y$ has the property P if $X \simeq \{X_h\}, Y \simeq \{Y_k\}$ and $\varphi = \{\varphi_h\}$, with $\varphi_h\colon X_h \to Y_h$ having property P.

(ii) We say that a collection $\{\varphi_i\colon \{U_{ih}\} \to \{X_h\}\}_{i\in I}$ of morphisms of pro-rigid spaces is a *cover* if $X \Subset \bigcup_i \operatorname{im}(U_{ih})$ for all h.

In particular, one can define open immersions, smooth, and étale morphisms of presentations of dagger affinoids which agree with the corresponding notions for dagger affinoids. Since the morphisms $\widehat{X} \subset X_h$ are open immersions (hence étale), we deduce that if a morphism $X \to Y$ is an open immersion (resp. smooth, resp. étale), then so is the associated morphism $\widehat{X} \to \widehat{Y}$.

- From now on we will use the following convention: if X is a smooth dagger affinoid, the presentation $X \simeq \{X_h\}$ will be assumed to have all X_h smooth as well.

Corollary 3.7 ([42, Cor. A.28]). *Let X be a dagger affinoid with a presentation $\{X_h\}$. We have a natural equivalence of étale topoi*

$$\mathrm{Sh}(X_{\text{ét}}) \xrightarrow{\sim} \mathrm{Sh}(\{X_h\}_{\text{ét}}).$$

3.2.3. *Definition of pro-étale cohomology.* Let $r \in \mathbf{Z}$.

(i) *Local definition.* If $\{X_h\}$ is a pro-rigid analytic variety, we set

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\text{proét}}(\{X_h\}, \mathbf{Z}/p^n(r)) := \operatorname{hocolim}_h \mathrm{R}\Gamma_{\text{proét}}(X_h, \mathbf{Z}/p^n(r)) \\
\operatorname{hocolim}_h \mathrm{R}\Gamma_{\text{ét}}(X_h, \mathbf{Z}/p^n(r)) \arrow[u, "\wr"']
\end{tikzcd}$$

Let X be a dagger affinoid. We define its pro-étale cohomology as

$$\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Z}/p^n(r)) := \mathrm{R}\Gamma_{\text{proét}}(\mathrm{pres}(X), \mathbf{Z}/p^n(r)). \tag{3.8}$$

If the dagger affinoid X has a dagger presentation $\{X_h\}$, then

$$\mathrm{R}\Gamma_{\text{ét}}(X, \mathbf{Z}/p^n(r)) \xleftarrow{\sim} \operatorname{hocolim}_h \mathrm{R}\Gamma_{\text{ét}}(X_h, \mathbf{Z}/p^n(r))$$

and we have a natural quasi-isomorphism

$$\mathrm{R}\Gamma_{\text{ét}}(X, \mathbf{Z}/p^n(r)) \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Z}/p^n(r)). \tag{3.9}$$

We make similar definitions for $\mathbf{Z}_p$ and $\mathbf{Q}_p$ coefficients. We have the natural maps (note the direction of the second map)

$$\begin{aligned}
\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Z}_p(r)) &\to \mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)),\\
\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Z}_p(r)) &\to \mathrm{R}\Gamma_{\text{ét}}(X, \mathbf{Z}_p(r)).
\end{aligned}$$

The first map is a rational quasi-isomorphism. If the dagger affinoid X has dagger presentation $\{X_h\}$, then we define the second map in the following way:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Z}_p(r)) = \operatorname{hocolim}_h \mathrm{R}\Gamma_{\text{proét}}(X_h, \mathbf{Z}_p(r)) \\
\operatorname{hocolim}_h \mathrm{R}\Gamma_{\text{ét}}(X_h, \mathbf{Z}_p(r)) \arrow[u, "\wr"'] \arrow[d, equal] \\
\operatorname{hocolim}_h \operatorname{holim}_n \mathrm{R}\Gamma_{\text{ét}}(X_h, \mathbf{Z}/p^n(r)) \arrow[d] \\
\operatorname{holim}_n \operatorname{hocolim}_h \mathrm{R}\Gamma_{\text{ét}}(X_h, \mathbf{Z}/p^n(r)) \arrow[d, equal] \\
\operatorname{holim}_n \mathrm{R}\Gamma_{\text{ét}}(X, \mathbf{Z}/p^n(r)) & \mathrm{R}\Gamma_{\text{ét}}(X, \mathbf{Z}_p(r)). \arrow[l, "\sim"']
\end{tikzcd} \tag{3.10}$$

Here the second quasi-isomorphism holds because X_h is quasi-compact (cover X_h with a finite number of affinoids and use the quasi-isomorphism (3.9)).

(ii) *Topological issues.* We need to discuss topology. Let, for a moment, X be a rigid analytic variety over L. We equip the pro-étale and étale cohomologies $\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r))$ and $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, \mathbf{Q}_p(r))$ with a natural topology by proceeding as in [9, Section 3.3.2] by using as local data compatible $\mathbf{Z}/p^n$-free complexes.[16] If X is quasi-compact, we obtain in this way complexes of Banach spaces over $\mathbf{Q}_p$. In that case the natural continuous map $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r))$ is a strict quasi-isomorphism.

More precisely, we have

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) := \operatorname{hocolim} \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(U_\bullet, \mathbf{Q}_p(r)),$$

where the homotopy colimit is over étale quasi-compact hypercoverings[17] of X. Since all the complexes $\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(U_\bullet, \mathbf{Q}_p(r))$ are complexes of Fréchet spaces, all the arrows in the colimit are strict quasi-isomorphisms. Hence we can compute with any particular hypercovering.

Remark 3.11. We will often use the following simple observation. If X is a smooth rigid analytic variety, then we can find an increasing quasi-compact admissible covering $\{U_n\}_{n\in\mathbf{N}}$ of X such that U_i is contained in the relative interior of U_{i+1}. If X is moreover partially proper, we can assume that $U_i \Subset U_{i+1}$. We have analogous statements for dagger varieties.

It follows that, for a general smooth rigid analytic variety X, we have an increasing quasi-compact admissible covering $\{U_n\}_{n\in\mathbf{N}}$ of X, such that we have (in $\mathscr{D}(C_{\mathbf{Q}_p})$)

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \simeq \operatorname{holim}_n \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(U_n, \mathbf{Q}_p(r)).$$

Hence we have the short exact sequence

$$\begin{aligned} 0 \to H^1 \operatorname{holim}_n \widetilde{H}^{i-1}_{\mathrm{\acute{e}t}}(U_n, \mathbf{Q}_p(r)) \to \widetilde{H}^i \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \\ \to H^0 \operatorname{holim}_n \widetilde{H}^i_{\mathrm{\acute{e}t}}(U_n, \mathbf{Q}_p(r)) \to 0. \end{aligned}$$

If X is a dagger affinoid, its pro-étale cohomology acquires now natural topology by taking the homotopy colimit in (3.8) in $\mathscr{D}(C_{\mathbf{Q}_p})$.

(iii) *Globalization.* For a general smooth dagger variety X, we have the natural equivalence of analytic topoi

$$\mathrm{Sh}((\mathrm{SmAff}^\dagger_L/X_L)_{\mathrm{\acute{e}t}}) \xrightarrow{\sim} \mathrm{Sh}((\mathrm{Sm}^\dagger_L/X_L)_{\mathrm{\acute{e}t}}),$$

where $\mathrm{Sm}^\dagger_L/X_L$ is the category of smooth morphisms of dagger varieties to X_L and $\mathrm{SmAff}^\dagger_L/X_L$ is its full subcategory of affinoid objects. Using this equivalence, we define the sheaf $\mathscr{A}_{\mathrm{pro\acute{e}t}}(r)$, $r \in \mathbf{Z}$, on $X_{\mathrm{\acute{e}t}}$ as the sheaf associated

---

[16] Such complexes can be found, for example, by taking the system of étale hypercovers.

[17] Here and below, we use "colimit over hypercoverings" as a shorthand for "colimit over the filtered category of hypercoverings up to simplicial homotopy".

to the presheaf defined by $U \mapsto \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(U, \mathbf{Q}_p(r))$, $U \in \mathrm{SmAff}^{\dagger}_L$, $U \to X$ an étale map. We define the pro-étale cohomology of X as

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) := \mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(X, \mathscr{A}_{\mathrm{pro\acute{e}t}}(r)), \quad r \in \mathbf{Z}.$$

We equip it with topology by proceeding as in the case of pro-étale cohomology of rigid analytic varieties starting with the case of dagger affinoids that was described above.

(iv) *Local-global compatibility.* This definition is consistent with the previous definition:

Lemma 3.12. *Let X be a dagger affinoid with the presentation $\{X_h\}$. Then the natural map*

$$\mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(\{X_h\}, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(X, \mathscr{A}_{\mathrm{pro\acute{e}t}}(r)), \quad r \in \mathbf{Z},$$

is a strict quasi-isomorphism.

Proof. Set $\mathrm{R}\Gamma^{\sharp}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) := \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(\{X_h\}, \mathbf{Q}_p(r))$. It suffices to show that, for any étale affinoid hypercovering $U_\bullet$ of X, the natural map

$$\mathrm{R}\Gamma^{\sharp}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma^{\sharp}_{\mathrm{pro\acute{e}t}}(U_\bullet, \mathbf{Q}_p(r))$$

is a strict quasi-isomorphism (modulo taking a refinement of $U_\bullet$). For that, it suffices to show that, for any $k \in \mathbf{N}$, the map

$$\tau_{\leq k}\mathrm{R}\Gamma^{\sharp}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \to \tau_{\leq k}\mathrm{R}\Gamma^{\sharp}_{\mathrm{pro\acute{e}t}}(T, \mathbf{Q}_p(r)), \tag{3.13}$$

where $T = U_\bullet$, is a strict quasi-isomorphism. Since, for that, it is enough to work with the truncation $\tau_{\leq k+1}T$, we will assume that T is a finite hypercovering and has a finite number of affinoids in every degree.

Take the dagger presentation $X \simeq \{X_h\}, h \in \mathbf{N}$. We can represent T by a pro-system of hypercoverings $\{T_h \to V_h\}$, $V_h \subset X_h$, $h \in \mathbf{N}$, forming a dagger presentation of T degree-wise.[18] We note that then $V_{h+1} \Subset V_h$. From the universal property of $\{X_h\}$ and the quasi-compactness of V_h, we get that the two pro-rigid varieties $\{X_h\}$ and $\{V_h\}$ are equivalent. It follows that we have a natural strict quasi-isomorphism

$$\mathrm{hocolim}_h\, \mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(X_h, \mathbf{Q}_p(r)) \xrightarrow{\sim} \mathrm{hocolim}_h\, \mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(V_h, \mathbf{Q}_p(r)).$$

Hence the map (3.13) is represented by a composition

$$\begin{tikzcd}
\tau_{\leq k}\mathrm{R}\Gamma^{\sharp}_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) & \tau_{\leq k}(\mathrm{hocolim}_h\, \mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(V_h, \mathbf{Q}_p(r))) \arrow[l, "\sim"'] \arrow[d, "\wr"] \\
& \tau_{\leq k}(\mathrm{hocolim}_h\, \mathrm{R}\Gamma_{\acute{\mathrm{e}}\mathrm{t}}(T_h, \mathbf{Q}_p(r))) \simeq \tau_{\leq k}\mathrm{R}\Gamma^{\sharp}_{\mathrm{pro\acute{e}t}}(T, \mathbf{Q}_p(r)),
\end{tikzcd}$$

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[18] This uses the simple observation that if a collection of morphisms of pro-rigid spaces $\{\varphi_i \colon \{V_{ih}\} \to \{X_h\}\}_{i\in I}$ is an étale cover, then we can choose a subsequence $\{X_{k_h}\}$ of $\{X_h\}$ such that the pro-rigid spaces $\{V_{i,k_h} := V_{ih} \times_{X_h} X_{k_h}\}$ form an étale cover of $\{X_h\}$ and, moreover, all the maps $\{\varphi_i \colon \{V_{i,k_h}\} \to \{X_{k_h}\}\}_{i\in I}$ are étale covers (to see this use the "initial" part of the definition of presentations).

where the middle strict quasi-isomorphism follows from étale descent for rigid analytic varieties. This finishes our proof of the lemma. ☐

Remark 3.14. For a smooth dagger variety X, we can define similarly the integral pro-étale cohomology $\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Z}_p(r))$, $r \in \mathbf{Z}$. We have the natural maps

$$\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Z}_p(r)) \to \mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)),$$

$$\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Z}_p(r)) \to \mathrm{R}\Gamma_{\text{ét}}(X, \mathbf{Z}_p(r)) \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{ét}}(\widehat{X}, \mathbf{Z}_p(r)) \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{proét}}(\widehat{X}, \mathbf{Z}_p(r)).$$

For X quasi-compact, the first map becomes a strict quasi-isomorphism after tensoring with $\mathbf{Q}_p$; this is not the case for general X. The second map is a globalization of maps for dagger affinoids defined in (3.10).

3.2.4. *Comparison isomorphisms.* Let $L = K, C$. For $X \in \mathrm{Sm}_L^{\dagger}$, we have a natural map

$$\iota_{\text{proét}}\colon \mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\text{proét}}(\widehat{X}, \mathbf{Q}_p(r)). \tag{3.15}$$

It is obtained by globalization of such maps for dagger affinoids: if the dagger affinoid X has a dagger presentation $\{X_h\}$, then

$$\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)) = \mathrm{hocolim}_h\, \mathrm{R}\Gamma_{\text{proét}}(X_h, \mathbf{Q}_p(r))$$

and $\iota_{\text{proét}}$ is the canonical map

$$\mathrm{hocolim}_h\, \mathrm{R}\Gamma_{\text{proét}}(X_h, \mathbf{Q}_p(r)) \xrightarrow{\text{can}} \mathrm{R}\Gamma_{\text{proét}}(\widehat{X}, \mathbf{Q}_p(r)).$$

Proposition 3.16. *Let X be partially proper. Then the map* (3.15) *is a strict quasi-isomorphism.*

Proof. Since a partially proper smooth dagger variety is locally Stein, we can assume X to be Stein. Choose an admissible covering of X by an increasing sequence of dagger affinoids $\{U_n\}$, $n \in \mathbf{N}$, strictly contained in each other. Then the map $\iota_{\text{proét}}$ from (3.15) can be written as the composition

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)) \arrow[r, "\sim"] & \mathrm{holim}_n\, \mathrm{R}\Gamma_{\text{proét}}(U_n, \mathbf{Q}_p(r)) \arrow[d] & \\
& \mathrm{holim}_n\, \mathrm{R}\Gamma_{\text{proét}}(\widehat{U}_n, \mathbf{Q}_p(r)) & \mathrm{R}\Gamma_{\text{proét}}(\widehat{X}, \mathbf{Q}_p(r)) \arrow[l, "\sim"']
\end{tikzcd}$$

and we need to show that the middle map is a strict quasi-isomorphism. But, for every $n > 1$, the map $\widehat{U}_n \to \widehat{U}_{n-1}$ factorizes as $\widehat{U}_n \to \mathrm{pres}(U_n) \to \widehat{U}_{n-1}$, yielding the factorization

$$\mathrm{R}\Gamma_{\text{proét}}(\widehat{U}_{n-1}, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\text{proét}}(\mathrm{pres}(U_n), \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\text{proét}}(\widehat{U}_n, \mathbf{Q}_p(r)).$$

It follows that the prosystems

$$\{\mathrm{R}\Gamma_{\text{proét}}(U_n, \mathbf{Q}_p(r))\}, \quad \{\mathrm{R}\Gamma_{\text{proét}}(\mathrm{pres}(U_n), \mathbf{Q}_p(r))\}$$

are equivalent. Since $\mathrm{R}\Gamma_{\text{proét}}(U_n, \mathbf{Q}_p(r)) \xleftarrow{\sim} \mathrm{R}\Gamma_{\text{proét}}(\mathrm{pres}(U_n), \mathbf{Q}_p(r))$, we are done. ☐

4. Rigid analytic syntomic cohomology

In this section we define syntomic cohomology for smooth rigid analytic varieties over K or C by η-étale descent of the classical definition due to Fontaine–Messing. We show that the computations of syntomic cohomology from [10] done for rigid analytic varieties with semi-stable reduction generalize to all smooth rigid varieties. We also introduce the Hyodo–Kato cohomology for such varieties, prove that it satisfies Galois descent, and define the Hyodo–Kato morphism (that is a quasi-isomorphism over C). Finally, over K, we define the Bloch–Kato rigid analytic syntomic cohomology (built from Hyodo–Kato and de Rham cohomologies) and show that it is quasi-isomorphic to the rigid analytic syntomic cohomology.

4.1. **Definition of rigid analytic syntomic cohomology.** We define the syntomic cohomology of smooth rigid analytic varieties by étale descent of crystalline syntomic cohomology of semi-stable models.

Let $\mathscr{U}\in\mathscr{M}_K^{\rm ss}$. We consider it as a log-formal scheme with the log-structure associated to the special fiber. For $r\geq 0$, we have the mod p^n, completed, and rational absolute (i.e., over $\mathbf{Z}_p$) filtered crystalline cohomology

$$\mathrm{R}\Gamma_{\rm cr}(\mathscr{U}_n,\mathscr{J}^{[r]}),\quad \mathrm{R}\Gamma_{\rm cr}(\mathscr{U},\mathscr{J}^{[r]}):=\mathrm{holim}_n\,\mathrm{R}\Gamma_{\rm cr}(\mathscr{U}_n,\mathscr{J}^{[r]}),$$
$$\mathrm{R}\Gamma_{\rm cr}(\mathscr{U},\mathscr{J}^{[r]})_{\mathbf{Q}_p}:=\mathrm{R}\Gamma_{\rm cr}(\mathscr{U},\mathscr{J}^{[r]})\otimes^L_{\mathbf{Z}_p}\mathbf{Q}_p.$$

Here $\mathscr{J}^{[r]}$ denotes the r-th Hodge filtration sheaf. The corresponding η-étale sheafifications on $\mathscr{M}_K^{\rm ss}$ we will denote by $F^r\mathscr{A}_{{\rm cr},n}, F^r\mathscr{A}_{\rm cr}$, and $F^r\mathscr{A}_{{\rm cr},\mathbf{Q}_p}$. We make analogous definitions for crystalline cohomology of basic semi-stable models over $\mathscr{O}_C$ (see [2] for details).

For $r\geq 0$, define the mod p^n, completed, and rational crystalline syntomic cohomology

$$\begin{aligned}\mathrm{R}\Gamma_{\rm syn}(\mathscr{U},\mathbf{Z}/p^n(r))&:=\big[\mathrm{R}\Gamma_{\rm cr}(\mathscr{U}_n,\mathscr{J}^{[r]})\xrightarrow{p^r-\varphi}\mathrm{R}\Gamma_{\rm cr}(\mathscr{U}_n)\big]\\&\simeq\big[[\mathrm{R}\Gamma_{\rm cr}(\mathscr{U}_n)]^{\varphi=p^r}\xrightarrow{\rm can}\mathrm{R}\Gamma_{\rm cr}(\mathscr{U}_n)/\mathrm{R}\Gamma_{\rm cr}(\mathscr{U}_n,\mathscr{J}^{[r]})\big],\\ \mathrm{R}\Gamma_{\rm syn}(\mathscr{U},\mathbf{Z}_p(r))&:=\mathrm{holim}_n\,\mathrm{R}\Gamma_{\rm syn}(\mathscr{U},\mathbf{Z}/p^n(r)),\\ \mathrm{R}\Gamma_{\rm syn}(\mathscr{U},\mathbf{Z}_p(r))_{\mathbf{Q}_p}&:=\mathrm{R}\Gamma_{\rm syn}(\mathscr{U},\mathbf{Z}_p(r))\otimes^L_{\mathbf{Z}_p}\mathbf{Q}_p\\&\simeq\big[\mathrm{R}\Gamma_{\rm cr}(\mathscr{U},\mathscr{J}^{[r]})_{\mathbf{Q}_p}\xrightarrow{p^r-\varphi}\mathrm{R}\Gamma_{\rm cr}(\mathscr{U})_{\mathbf{Q}_p}\big].\end{aligned}$$

The corresponding η-étale sheafifications on $\mathscr{M}_K^{\rm ss}$ we will denote by $\mathscr{A}_{{\rm syn},n}(r)$, $\mathscr{A}_{\rm syn}(r)$, and $\mathscr{A}_{\rm syn}(r)_{\mathbf{Q}_p}$. We make analogous definitions for crystalline syntomic cohomology of basic semi-stable models over $\mathscr{O}_C$. We have the distinguished triangles

$$\mathscr{A}_{{\rm syn},n}(r)\to F^r\mathscr{A}_{{\rm cr},n}\xrightarrow{p^r-\varphi}\mathscr{A}_{{\rm cr},n},$$
$$\mathscr{A}_{{\rm syn},n}(r)\to\mathscr{A}_{{\rm cr},n}^{\varphi=p^r}\to\mathscr{A}_{{\rm cr},n}/F^r,$$

where we set $\mathscr{A}_{{\rm cr},n}^{\varphi=p^r}:=[\mathscr{A}_{{\rm cr},n}\xrightarrow{p^r-\varphi}\mathscr{A}_{{\rm cr},n}]$. Similarly for the completed and rational cohomology.

For $X \in \mathrm{Sm}_L$, $L = K, C$, we define two rational (rigid analytic) syntomic cohomologies:

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Z}_p(r))_{\mathbf{Q}_p} := \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{syn}}(r)) \otimes^L_{\mathbf{Z}_p} \mathbf{Q}_p,$$
$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) := \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{syn}}(r)_{\mathbf{Q}_p}).$$

From now on, to simplify the notation, we will write $(-)_{\mathbf{Q}_p}$ for $(-) \otimes^L_{\mathbf{Z}_p} \mathbf{Q}_p$; similarly for coefficients other than $\mathbf{Q}_p$. There is a canonical map

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Z}_p(r))_{\mathbf{Q}_p} \to \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)). \tag{4.1}$$

It follows immediately from the definitions that, for X quasi-compact, this is a quasi-isomorphism (but it is not so in general). By proceeding just as in [9, Section 3.3.1] (using crystalline embedding systems), we can equip both complexes in (4.1) with a natural topology for which they become complexes of Banach spaces over $\mathbf{Q}_p$ in the case X is quasi-compact[19] (and in that case the quasi-isomorphism (4.1) is strict). We do the same for the crystalline complexes involved in the definition of syntomic cohomology. We have distinguished triangles in $\mathscr{D}(C_{\mathbf{Q}_p})$:

$$\begin{aligned} \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Z}_p(r))_{\mathbf{Q}_p} &\to \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{cr}}^{\varphi = p^r})_{\mathbf{Q}_p} \to \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{cr}}/F^r)_{\mathbf{Q}_p}, \\ \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) &\to \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{cr},\mathbf{Q}_p}^{\varphi = p^r}) \to \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{cr},\mathbf{Q}_p}/F^r). \end{aligned} \tag{4.2}$$

We will show later (see Corollary 4.32) that if $X = \mathscr{X}_K$, for an admissible semi-stable formal scheme $\mathscr{X}$ over $\mathscr{O}_K$, then the canonical map

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r))$$

is a strict quasi-isomorphism.

4.1.1. *Rigid analytic de Rham cohomology.* Let $L = K, C$. Consider the presheaf $X \mapsto \mathrm{R}\Gamma_{\mathrm{dR}}(X)$ of filtered dg L-algebras on Sm_L. Let $\mathscr{A}_{\mathrm{dR}}$ be its étale sheafification on Sm_L. It is a sheaf of filtered L-algebras on $\mathrm{Sm}_{L,\text{ét}}$. For $X \in \mathrm{Sm}_L$, we have the natural filtered quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X) \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{dR}}).$$

We equip $\mathrm{R}\Gamma_{\mathrm{dR}}(X)$ with the topology induced by the canonical topology on affinoid algebras; we equip $\mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{dR}})$ with topology using étale descent as we did before. Then the above quasi-isomorphism is strict: sheaves of differential forms satisfy étale descent in the strict sense.

Let $X \in \mathrm{Sm}_L$. We will need to understand the cohomology groups in degrees $r-1$ and r of

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r \simeq \mathrm{R}\Gamma(X, \mathscr{O}_X \to \Omega^1_X \to \cdots \to \Omega^{r-1}_X).$$

To do that consider the distinguished triangle (in $\mathscr{D}(C_L)$)

$$0 \to \ker d_r[-r] \to \tau_{\leq r}\Omega^\bullet_X \to \Omega^{\leq r-1}_X \to 0, \tag{4.3}$$

---

[19] We note that $\mathscr{O}_K$ being syntomic over $\mathscr{O}_F$, all the integral complexes in sight are in fact p-torsion-free.

where $d_r\colon \Omega^r_X\to\Omega^{r+1}_X$ is the de Rham differential. It yields the long exact sequence

$$0\to \widetilde{H}^{r-1}_{\mathrm{dR}}(X)\to \widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)\to \widetilde{H}^{r}(X,\ker d_r[-r])\to \widetilde{H}^{r}_{\mathrm{dR}}(X).$$

Or, since $\widetilde{H}^r(X,\ker d_r[-r])=\Omega^r(X)^{d=0}$, the short exact sequence

$$0\to \widetilde{H}^{r-1}_{\mathrm{dR}}(X)\to \widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)\to \ker\pi\to 0,$$

where π is the natural map $\Omega^r(X)^{d=0}\to \widetilde{H}^r_{\mathrm{dR}}(X)$. We have a monomorphism $\operatorname{im} d_{r-1}(X)\hookrightarrow \ker\pi$.

The distinguished triangle (4.3) yields also the long exact sequence

$$0\to \operatorname{coker}\pi\to \widetilde{H}^{r}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)\to \widetilde{H}^{1}(X,\ker d_r)\to \widetilde{H}^{r+1}(X,\tau_{\leq r}\Omega^\bullet_X).$$

Remark 4.4. (a) If X is proper, all the Hodge and de Rham cohomology groups are classical (finite-dimensional vector spaces over K), the Hodge-de Rham spectral sequence degenerates at E_1 [37, Cor. 1.8], and we get the isomorphisms

$$H^{r-1}_{\mathrm{dR}}(X)\xrightarrow{\sim}\widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r),\quad H^{r}_{\mathrm{dR}}(X)/\Omega^r(X)\xrightarrow{\sim}\widetilde{H}^{r}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r).$$

(b) If X is Stein, we have $H^i(X,\Omega^j_X)=0$, $i\neq 0$, and all the de Rham cohomology groups are classical (Fréchet spaces). We have

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r\simeq(\mathscr{O}(X)\to\Omega(X)\to\cdots\to\Omega^{r-1}(X))$$

with strict differentials. Hence we get the isomorphisms

$$\widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)\simeq\Omega^{r-1}(X)/\operatorname{im} d_{r-1},\quad \widetilde{H}^{i}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)\simeq 0,\quad i\geq r.$$

Hence the cohomology $\widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)$ is classical.

Proposition 4.5. *Let $X\in\mathrm{Sm}_K$. Let $r\geq 0$. We have a canonical strict quasi-isomorphism*

$$\gamma_r\colon \mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r\xrightarrow{\sim}\mathrm{R}\Gamma_{\text{ét}}(X,\mathscr{A}_{\mathrm{cr},\mathbf{Q}_p}/F^r).$$

Proof. Let $\mathscr{X}$ be a quasi-compact semi-stable formal scheme over $\mathscr{O}_E$, with $[E:K]<\infty$. Recall that [31, Cor. 2.4] there exists a functorial and compatible with base-change quasi-isomorphism

$$\gamma_r\colon \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K)/F^r\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X},\mathscr{O}/\mathscr{J}^{[r]})_{\mathbf{Q}_p}.$$

This quasi-isomorphism is in fact strict: this is not completely evident because the integral version of the morphism is only a p^N-quasi-isomorphism for some constant N but can be seen by an argument identical to the one used at the end of the proof of [9, Prop. 6.1]. By η-étale descent, we get the strict quasi-isomorphism in the proposition. ☐

4.1.2. *Some computations.* Recall that, in a stable range and up to some universal constants, crystalline syntomic cohomology has a simple relation to de Rham cohomology. Let $\mathscr{X}$ be an affine semi-stable formal scheme over $\mathscr{O}_K$. Let $r \geq 0$. We note that $\tau_{\leq r-1}(\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})/F^r) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})/F^r$ and that the natural map $\tau_{\leq r+1}([\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})]^{\varphi=p^r}) \to [\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})]^{\varphi=p^r}$ is a p^{2r}-quasi-isomorphism (since $1-p^s\varphi$, $s \geq 1$, is invertible on differentials in degree $r+s$).

Proposition 4.6 ([10, Prop. 3.12]). (i) *The natural map*

$$\tau_{\leq r+1}\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},\mathbf{Z}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},\mathbf{Z}_p(r))$$

is a p^{2r}-quasi-isomorphism and

$$H^{r+1}\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},\mathbf{Z}_p(r)) \xrightarrow{\sim} H^{r+1}([\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})]^{\varphi=p^r}).$$

(ii) *The complex $\tau_{\leq r-1}([\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})]^{\varphi=p^r})$ is p^N-acyclic, for a constant $N = N(e,d,p,r)$, where $e=[K:F]$, $d = \dim \mathscr{X}/\mathscr{O}_K$. Hence the natural map $\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X})/F^r \to \tau_{\leq r-1}(\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},\mathbf{Z}_p(r))[1])$ is a p^N-quasi-isomorphism.*

(iii) *The above statements are valid also modulo p^n. Moreover, étale locally on $\mathscr{X}_n$, $H^{r+1}([\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_n)]^{\varphi=p^r})$ is p^N-trivial, for a constant $N=N(r)$.*

Let $X \in \mathrm{Sm}_K$, $r \geq 0$. The distinguished triangle (4.2) and Lemma 4.5 yield a natural map

$$\partial_r\colon (\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)[-1] \to \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)).$$

Corollary 4.7. (1) *For $i \leq r-1$, the map*

$$\partial_r\colon \widetilde{H}^{i-1}_{\mathrm{dR}}(X) \to \widetilde{H}^i_{\mathrm{syn}}(X,\mathbf{Q}_p(r))$$

is an isomorphism.

(2) *We have the exact sequence*

$$0 \to \widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r) \xrightarrow{\partial_r} \widetilde{H}^r_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) \to \widetilde{H}^r_{\text{ét}}(X,\mathscr{A}^{\varphi=p^r}_{\mathrm{cr},\mathbf{Q}_p}) \to \widetilde{H}^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r).$$

Proof. To prove the first claim, note that we have the long exact sequence

$$\widetilde{H}^{i-1}_{\text{ét}}(X,\mathscr{A}^{\varphi=p^r}_{\mathrm{cr},\mathbf{Q}_p}) \to \widetilde{H}^{i-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r) \to \widetilde{H}^i\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) \to \widetilde{H}^i_{\text{ét}}(X,\mathscr{A}^{\varphi=p^r}_{\mathrm{cr},\mathbf{Q}_p}).$$

If $i \leq r-1$, then $\widetilde{H}^{i-1}\mathrm{R}\Gamma_{\mathrm{dR}}(X) \xrightarrow{\sim} \widetilde{H}^{i-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)$ and (1) follows from Proposition 4.6 (which implies $\widetilde{H}^{i-1}_{\text{ét}}(X,\mathscr{A}^{\varphi=p^r}_{\mathrm{cr},\mathbf{Q}_p})=0$ and $\widetilde{H}^{i}_{\text{ét}}(X,\mathscr{A}^{\varphi=p^r}_{\mathrm{cr},\mathbf{Q}_p})=0$).

A similar argument shows that $\partial_r\colon \widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r) \to \widetilde{H}^r_{\mathrm{syn}}(X,\mathbf{Q}_p(r))$ is injective, which yields the second claim of the corollary. □

4.2. Arithmetic rigid analytic Hyodo–Kato cohomology.

We define here Hyodo–Kato cohomology of smooth rigid analytic varieties over K as well as a Hyodo–Kato morphism. We do it by η-étale descent of crystalline Hyodo–Kato cohomology and the Hyodo–Kato morphism for semi-stable models.

4.2.1. *Hyodo–Kato cohomology.* Let $\mathscr{A}_{\mathrm{HK}}$ be the η-étale sheafification of the presheaf $\mathscr{X} \mapsto \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) := \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^0_{F_L})_{\mathbf{Q}_p}$ on $\mathscr{M}^{ss}_K$. Here $\mathscr{X}$ is a semi-stable formal model over $\mathscr{O}_L$, $[L:K]<\infty$, $L=K_{\mathscr{X}}$, and F_L is the maximal absolutely unramified subfield of L. The sheaf $\mathscr{A}_{\mathrm{HK}}$ is a sheaf of dg F-algebras on $\mathrm{Sm}_{K,\text{ét}}$ equipped with a φ-action and a derivation N such that $N\varphi = p\varphi N$. For $X \in \mathrm{Sm}_K$, set $\mathrm{R}\Gamma_{\mathrm{HK}}(X) := \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{HK}})$. Equip it with a topology in the usual way, via η-étale descent, from the natural topology on $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0)$.

4.2.2. *Convergent cohomology.* Let $\mathscr{A}_{\mathrm{conv}}$ be the η-étale sheafification of the presheaf[20] $\mathscr{X} \mapsto \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_1/\mathscr{O}^\times_L)$, $L=K_{\mathscr{X}}$, on $\mathscr{M}^{\mathrm{ss}}_{K,\text{ét}}$. For $X \in \mathrm{Sm}_K$, we set $\mathrm{R}\Gamma_{\mathrm{conv}}(X) := \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{conv}})$. It is a dg K-algebra. We equip it with the topology induced by η-étale descent from the topology of the $\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_1/\mathscr{O}^\times_L)$'s. We have natural (strict) quasi-isomorphisms,

$$\mathscr{A}_{\mathrm{conv}} \simeq \mathscr{A}_{\mathrm{dR}}, \quad \mathrm{R}\Gamma_{\mathrm{conv}}(X) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X),$$

induced by the quasi-isomorphisms $\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_1/\mathscr{O}^\times_L) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)$ that hold because $\mathscr{X}$ is log-smooth over $\mathscr{O}^\times_L$.

4.2.3. *Hyodo–Kato morphism.* To define the Hyodo–Kato quasi-isomorphism, we will use the original Hyodo–Kato quasi-isomorphism defined for quasi-compact formal schemes in [24] (see also [34]). We will describe it now in some detail. Denote by r^+_F the algebra $\mathscr{O}_F[[T]]$ with the log-structure associated to T. Sending T to p induces a surjective morphism $r^+_F \to \mathscr{O}^\times_F$. We denote by r^{PD}_F the p-adic divided power envelope of r^+_F with respect to the kernel of this morphism. Frobenius is defined by $T \mapsto T^p$, monodromy is a $\mathscr{O}_F$-linear derivation given by $T \mapsto T$. We will skip the subscript F if there is no danger of confusion.

(i) *Local definition.* Assume that we have an admissible semi-stable formal scheme $\mathscr{X}$ over $\mathscr{O}_K$. We will work in the classical derived category. Recall that the Frobenius

$$r^{\mathrm{PD}}_{n,\varphi} \otimes^L_{r^{\mathrm{PD}}_n} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r^{\mathrm{PD}}_n) \to \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r^{\mathrm{PD}}_n),$$
$$\mathscr{O}_{F,n,\varphi} \otimes^L_{\mathscr{O}_{F,n}} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^0_{F,n}) \to \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^0_{F,n}),$$

has a p^N-inverse, for $N=N(d)$, $d=\dim \mathscr{X}_0$. This is proved in [24, 2.24]. Recall also that the projection $p_0\colon \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r^{\mathrm{PD}}_n) \to \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^0_{F,n})$, $T \mapsto 0$, has a functorial (for maps between formal schemes and a change of n) and Frobenius-equivariant p^{N_ι}-section, $N_\iota = N(d)$,

$$\iota_n\colon \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^0_{F,n}) \to \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r^{\mathrm{PD}}_n),$$

---

[20] Here $\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_1/\mathscr{O}^\times_L)$ (and later $\mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_1/\mathscr{O}^\times_L)$) are defined following the construction of Grosse–Klönne [21, 1.1–1.4] by taking rigid analytic tubes (resp. dagger tubes).

i.e., $p_0\iota_n = p^{N_\iota}$. This follows easily from the proof of [24, Prop. 4.13]; the key point being that the Frobenius on $\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^0_{F,n})$ is close to a quasi-isomorphism and the Frobenius on the PD-ideal of r^{PD} is close to zero. Moreover, the resulting map

$$\iota_n\colon \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^0_{F,n})\otimes^L_{\mathscr{O}_{F,n}} r^{\mathrm{PD}}_n \to \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r^{\mathrm{PD}}_n) \tag{4.8}$$

is a p^N-quasi-isomorphism, $N = N(d)$ (see [24, Lem. 5.2]), and so is the composite

$$p_p\iota_n\colon \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^0_{F,n}) \to \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^\times_{F,n}),$$

where the projection $p_p\colon \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r^{\mathrm{PD}}_n)\to\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^\times_{F,n})$ is defined by $T\mapsto p$. Taking holim_n of the last map, we obtain a map

$$p_p\iota\colon \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^0_F) \to \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^\times_F)$$

that is a p^N-quasi-isomorphism, $N = N(d)$.

We define the Hyodo–Kato map as the composition (the dotted arrow)

$$\begin{tikzcd}
\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^0_F)_F \arrow[r, "p^{-N_\iota}p_p\iota"] \arrow[d, dotted] & \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^\times_F)_F \arrow[r] & \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^\times_K)_K \\
\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_1/\mathscr{O}^\times_K) \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_0/\mathscr{O}^\times_K). \arrow[u, "\wr"']
\end{tikzcd} \tag{4.9}$$

The fourth map is actually a natural isomorphism by the invariance under infinitesimal thickenings of convergent cohomology [33, 0.6.1]. The induced map $\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^0_F)_F\otimes_F K\to\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K)$ is a strict quasi-isomorphism.

(ii) *Globalization.* Let now X be a smooth rigid analytic variety over K. Since the computation in [24, Prop. 4.13], leading to the existence of the section ι, can be done on the big topos as long as we can control the dimension of the schemes involved, the above Hyodo–Kato map can be lifted to a Hyodo–Kato map

$$\iota_{\mathrm{HK}}\colon \mathscr{A}_{\mathrm{HK}}\to\mathscr{A}_{\mathrm{dR}}$$

in the classical derived category of étale sheaves on X. It induces the Hyodo–Kato map

$$\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(X)\to\mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

Proposition 4.10 (Local-global compatibility)**.** *For a semi-stable formal scheme $\mathscr{X}$ over $\mathscr{O}_K$, the canonical map*

$$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0)\to\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_K) \tag{4.11}$$

is a strict quasi-isomorphism.

Proof. The proof of [31, Prop. 3.18] goes through practically verbatim. Key points: the de Rham analog of (4.11) holds plus we have Galois descent for both sides of (4.11) that allows us to deal with the field extensions appearing in the construction of local semi-stable models. ☐

Remark 4.12. The above definition of the Hyodo–Kato quasi-isomorphism was normalized (at p) so that it is functorial. A more customary definition depends on the uniformizer ϖ (one basically proceeds as above but using the PD-envelope of the map $\mathscr{O}_F\{T\} \to \mathscr{O}_K, T \mapsto \varpi$, instead of r_F^{PD}) and hence it is not functorial.

4.2.4. *Arithmetic r^{PD}-cohomology.* We define the r^{PD}-cohomology of smooth rigid analytic varieties over K by η-étale descent of the r^{PD}-cohomology of semi-stable models.

Let $\mathscr{A}_{\mathrm{PD}}$ be the η-étale sheafification of the presheaf $\mathscr{X} \mapsto \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r_L^{\mathrm{PD}})_{\mathbf{Q}_p}$ on $\mathscr{M}_K^{ss}$. Here $\mathscr{X}$ is an admissible semi-stable formal scheme over $\mathscr{O}_L$, $L = K_{\mathscr{X}}$. We wrote r_L^{PD} for the r^{PD}-ring corresponding to F_L. Let $\mathscr{R}^{\mathrm{PD}}$ be the η-étale sheafification of the presheaf $\mathscr{X} \mapsto r^{\mathrm{PD}}_{L,\mathbf{Q}_p}$ on $\mathscr{M}_K^{ss}$. The sheaf $\mathscr{A}_{\mathrm{PD}}$ is a sheaf of dg $\mathscr{R}^{\mathrm{PD}}_{\mathbf{Q}_p}$-algebras on $\mathrm{Sm}_{K,\mathrm{\acute{e}t}}$ equipped with a φ-action and a derivation N, compatible with the derivation on $\mathscr{R}^{\mathrm{PD}}$, such that $N\varphi = p\varphi N$. For $X \in \mathrm{Sm}_K$, set $\mathrm{R}\Gamma_{\mathrm{PD}}(X) := \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, \mathscr{A}_{\mathrm{PD}})$. Equip it with a topology in the usual way, via η-étale descent, from the natural topology on the $\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r_L^{\mathrm{PD}})_{\mathbf{Q}_p}$'s.

Proposition 4.13 (Local-global compatibility)**.** *For a semi-stable formal model $\mathscr{X}$ over $\mathscr{O}_K$, the canonical map*

$$\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r_K^{\mathrm{PD}})_{\mathbf{Q}_p} \to \mathrm{R}\Gamma_{\mathrm{PD}}(\mathscr{X}_K)$$

is a strict quasi-isomorphism.

Proof. It suffices to show that, for any η-étale hypercovering $\mathscr{U}_\bullet$ of $\mathscr{X}$ from $\mathscr{M}_K^{\mathrm{ss}}$ (we may assume that in every degree of the hypercovering we have a quasi-compact formal scheme), the natural map

$$\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r^{\mathrm{PD}})_{\mathbf{Q}_p} \to \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{U}_{\bullet,0}/r_{L_\bullet}^{\mathrm{PD}})_{\mathbf{Q}_p}$$

is a strict quasi-isomorphism (modulo taking a refinement of $\mathscr{U}_\bullet$). Recall that the p^N-quasi-isomorphism ι from (4.8) yields a strict quasi-isomorphism ($\widehat{\otimes}^R$ denotes the right derived functor of the tensor product)

$$s = p^{-N_\iota}\iota \colon \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) \,\widehat{\otimes}^R_F\, r^{\mathrm{PD}}_{K,\mathbf{Q}_p} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r_K^{\mathrm{PD}})_{\mathbf{Q}_p}. \tag{4.14}$$

Using it, we get the following commutative diagram:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r_K^{\mathrm{PD}})_{\mathbf{Q}_p} \arrow[r] & \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{U}_{\bullet,0}/r_{L_\bullet}^{\mathrm{PD}})_{\mathbf{Q}_p} \\
\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) \,\widehat{\otimes}^R_F\, r^{\mathrm{PD}}_{K,\mathbf{Q}_p} \arrow[r] \arrow[u, "s"', "\wr"] & \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{U}_{\bullet,0}) \,\widehat{\otimes}^R_{F_{L_\bullet}}\, r^{\mathrm{PD}}_{L_\bullet,\mathbf{Q}_p}. \arrow[u, "s_\bullet"', "\wr"]
\end{tikzcd}$$

Since $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{U}_{\bullet,0}) \,\widehat{\otimes}^R_{F_L}\, r^{\mathrm{PD}}_{L,\mathbf{Q}_p} \simeq \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{U}_{\bullet,0}) \,\widehat{\otimes}^R_F\, r^{\mathrm{PD}}_{K,\mathbf{Q}_p}$ and since, by Proposition 4.10, the natural map $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) \to \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{U}_{\bullet,0})$ is a strict quasi-isomorphism, so is the bottom map in the above diagram. It follows that the top map is also a strict-quasi-isomorphism, as wanted. ☐

4.3. Geometric rigid analytic Hyodo–Kato cohomology.

We will now define the Hyodo–Kato cohomology of smooth rigid analytic varieties over C. We will do it by η-étale descent of crystalline Hyodo–Kato cohomology of basic semi-stable models.

4.3.1. *Definition and basic properties.* Let $f\colon \mathscr{X} \to \operatorname{Spf}(\mathscr{O}_C)^\times$ be a semi-stable formal model. Suppose that f is the base change of a semi-stable formal model $f_L\colon \mathscr{X}_{\mathscr{O}_L} \to \operatorname{Spf}(\mathscr{O}_L)^\times$ by $\theta\colon \operatorname{Spf}(\mathscr{O}_C)^\times \to \operatorname{Spf}(\mathscr{O}_L)^\times$, for a finite extension L/K. That is, we have a map $\theta_L\colon \mathscr{X} \to \mathscr{X}_{\mathscr{O}_L}$ such that the square $(f, f_L, \theta, \theta_L)$ is Cartesian. In the algebraic setting (algebraic schemes and $\overline{K}$ in place of C) such data $(L, \mathscr{X}_{\mathscr{O}_L}, \theta_L)$ clearly form a filtered set. In our analytic case, this is also the case for the system

$$\Sigma = \big\{(L, \mathscr{X}_{\mathscr{O}_{L,1}}, \theta_L)\big\}$$

corresponding to the reduction modulo p of such data,[21] i.e., a system in which objects are reductions $(L, \mathscr{X}_{\mathscr{O}_{L,1}}, \theta_L)$ modulo p of the tuples $(L, \mathscr{X}_{\mathscr{O}_L}, \theta_L)$ as above but morphisms are morphisms between the reduced objects.

(i) *Hyodo–Kato cohomology.* For a morphism of tuples $(L', \mathscr{X}'_{\mathscr{O}_{L'},1}, \theta'_{L'}) \to (L, \mathscr{X}_{\mathscr{O}_L,1}, \theta_L)$ from Σ, we have a canonical base change identification compatible with φ-action (crystalline unramified base change)

$$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0}) \otimes_{F_L} F_{L'} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}'_{\mathscr{O}_{L'},0}).$$

We set[22]

$$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1) := \operatorname{hocolim}_\Sigma \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0}).$$

$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)$ is a dg F^{nr}-algebra[23] equipped with a φ-action and a derivation N such that $N\varphi = p\varphi N$. It is functorial with respect to $\mathscr{X}$: note that the restriction of a morphism $\mathscr{X} \to \mathscr{Y}$ to a morphism $\mathscr{X}_1 \to \mathscr{Y}_1$ is defined over a finite extension of K. Let $\mathscr{A}_{\mathrm{HK}}$ be the η-étale sheafification of the presheaf $\mathscr{X} \mapsto \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)$ on $\mathscr{M}_C^{\mathrm{ss},b}$. For $X \in \mathrm{Sm}_C$, we set $\mathrm{R}\Gamma_{\mathrm{HK}}(X) := \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{HK}})$. It is a dg F^{nr}-algebra equipped with a Frobenius, monodromy action, and a continuous action of $\mathscr{G}_K$ if X is defined over K (this action is smooth, i.e., the stabilizer of every element is an open subgroup of $\mathscr{G}_K$, if X is quasi-compact; in general, it is only "pro-smooth"). We equip it with the topology induced by η-étale descent from the topology of the $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0})$'s.

(ii) *Convergent cohomology.* Let $\mathscr{A}_{\mathrm{conv}}$ be the η-étale sheafification of the presheaf $\mathscr{X} \mapsto \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_1/\mathscr{O}_C^\times)$ on $\mathscr{M}_{C,\text{ét}}^{\mathrm{ss},b}$. For $X \in \mathrm{Sm}_C$, we set

$$\mathrm{R}\Gamma_{\mathrm{conv}}(X) := \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{conv}}).$$

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[21] This is because the schemes $\mathscr{X}_{\mathscr{O}_L,1}$ from above are algebraic.

[22] Everything here and below is done in the derived ∞-category $\mathscr{D}(C_{\mathbf{Q}_p})$.

[23] The field F^{nr} is equipped with the inductive limit topology. Later on we will use the same type of topology for $\overline{K}$.

It is a dg C-algebra equipped with a continuous action of $\mathscr{G}_K$. We equip it with the topology induced by η-étale descent from the topology of the $\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_1/\mathscr{O}_C^{\times})$'s. We have natural (strict) quasi-isomorphisms

$$\mathscr{A}_{\mathrm{conv}} \simeq \mathscr{A}_{\mathrm{dR}}, \quad \mathrm{R}\Gamma_{\mathrm{conv}}(X) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

Let $\mathscr{A}_{\mathrm{conv},\overline{K}}$ be the étale sheafification of the presheaf $\mathscr{X} \mapsto \mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{X}_1)$ on $\mathscr{M}^{\mathrm{ss},b}_{C,\text{ét}}$, where we set

$$\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{X}_1) := \mathrm{hocolim}_{\Sigma}\, \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_{\mathscr{O}_L,1}/\mathscr{O}_L^{\times})$$

in the notation from above. For $X \in \mathrm{Sm}_C$, we set

$$\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(X) := \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{conv},\overline{K}}).$$

It is a dg $\overline{K}$-algebra equipped with a continuous action of $\mathscr{G}_K$ if X is defined over K (this action is smooth if X is quasi-compact). We equip it with the topology induced by η-étale descent from the topology of the $\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_{\mathscr{O}_L,1}/\mathscr{O}_L^{\times})$'s. There are natural continuous morphisms

$$\mathscr{A}_{\mathrm{conv},\overline{K}} \to \mathscr{A}_{\mathrm{conv}}, \quad \mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(X) \to \mathrm{R}\Gamma_{\mathrm{conv}}(X).$$

Remark 4.15. Instead of $\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{X}_1)$ above, we could have used

$$\mathrm{R}\Gamma_{\mathrm{conv},F^{\mathrm{nr}}}(\mathscr{X}_1) := \mathrm{hocolim}_{\Sigma}\, \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_{\mathscr{O}_L,1}/\mathscr{O}_{F_L}^{\times}).$$

This would give a natural F^{nr}-structure on de Rham cohomology (see Proposition 4.22 below).

(iii) r^{PD}*-cohomology.* Let $\mathscr{A}_{\mathrm{PD}}$ be the η-étale sheafification of the presheaf $\mathscr{X} \mapsto \mathrm{R}\Gamma_{\mathrm{PD}}(\mathscr{X}_1)$ on $\mathscr{M}^{\mathrm{ss},b}_{C,\text{ét}}$, where we set

$$\mathrm{R}\Gamma_{\mathrm{PD}}(\mathscr{X}_1) := \mathrm{hocolim}_{\Sigma}\, \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_L,0}/r_L^{\mathrm{PD}})_{\mathbf{Q}_p}$$

in the notation from above. For $X \in \mathrm{Sm}_C$, we set $\mathrm{R}\Gamma_{\mathrm{PD}}(X) := \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{PD}})$. Set

$$r^{\mathrm{PD}}_{\overline{K}} := r^{\mathrm{PD}}_F \otimes_{\mathscr{O}_F} \mathscr{O}_{F^{\mathrm{nr}}} := \varinjlim_{L}(r^{\mathrm{PD}}_F \otimes_{\mathscr{O}_F} \mathscr{O}_{F_L}), \quad [L:K] < \infty.$$

$\mathrm{R}\Gamma_{\mathrm{PD}}(X)$ is a dg $r^{\mathrm{PD}}_{\overline{K},\mathbf{Q}_p}$-algebra equipped with a continuous action of $\mathscr{G}_K$ if X is defined over K (this action is smooth if X is quasi-compact). We equip it with the topology induced by η-étale descent from the topology of the $\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_L,0}/r_L^{\mathrm{PD}})_{\mathbf{Q}_p}$'s.

4.3.2. *Hyodo–Kato quasi-isomorphisms.* We keep the set-up from Section 4.3.1. The Hyodo–Kato morphisms from (4.9),

$$\begin{aligned} &\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0}) \to \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_{\mathscr{O}_L,1}/\mathscr{O}_L^{\times}),\\ &\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0}) \otimes_{F_L} L \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_{\mathscr{O}_L,1}/\mathscr{O}_L^{\times}),\end{aligned} \tag{4.16}$$

are compatible with morphisms in Σ and taking the homotopy colimit of the second one yields the first of the following two natural strict quasi-isomorphisms (called again the *Hyodo–Kato quasi-isomorphisms*):

$$\begin{aligned}(4.17)\qquad \iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)\otimes_{F^{\mathrm{nr}}}\overline{K} &\xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{X}_1)\\ &= \operatorname{hocolim}_{\Sigma}\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_{\mathscr{O}_L,1}/\mathscr{O}_L^{\times}),\\ \iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)\mathbin{\widehat{\otimes}}^R_{F^{\mathrm{nr}}} C &\xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C).\end{aligned}$$

By definition, $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)\otimes_{F^{\mathrm{nr}}}\overline{K} := \operatorname{hocolim}_L(\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)\otimes_{F^{\mathrm{nr}}} L)$, the homotopy colimit taken over fields L, $[L:F^{\mathrm{nr}}]<\infty$. We have $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)\otimes_{F^{\mathrm{nr}}}\overline{K}\simeq \operatorname{hocolim}_{\Sigma}(\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0})\otimes_{F_L} L)$. In the second Hyodo–Kato morphism in (4.17), by definition,[24]

$$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)\mathbin{\widehat{\otimes}}^R_{F^{\mathrm{nr}}} C := \operatorname{hocolim}_{\Sigma}(\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0})\mathbin{\widehat{\otimes}}^R_{F_L} C).$$

We note that all the maps in the homotopy colimits are strict quasi-isomorphisms. The Hyodo–Kato morphism itself is induced from the Hyodo–Kato strict quasi-isomorphism (4.16):

$$\operatorname{hocolim}_{\Sigma}(\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0})\mathbin{\widehat{\otimes}}^R_{F_L} C)\xrightarrow{\sim}\operatorname{hocolim}_{\Sigma}(\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_{\mathscr{O}_L,1}/\mathscr{O}_L^{\times})\mathbin{\widehat{\otimes}}^R_{L} C)$$

and the strict quasi-isomorphisms

$$\operatorname{hocolim}_{\Sigma}(\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_{\mathscr{O}_L,1}/\mathscr{O}_L^{\times})\mathbin{\widehat{\otimes}}^R_{L} C)\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_1/\mathscr{O}_C^{\times})\simeq\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C).$$

The first quasi-isomorphism is given by base change. We note here that, since $\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_{\mathscr{O}_L,1}/\mathscr{O}_L^{\times})$ is a complex of Banach spaces, the completed tensor product with C is exact.

Similarly, for $\mathscr{X}$ as at the beginning of Section 4.3.1, the strict quasi-isomorphism (4.14) yields a strict quasi-isomorphism

$$(4.18)\qquad s\colon \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)\mathbin{\widehat{\otimes}}^R_{F^{\mathrm{nr}}} r^{\mathrm{PD}}_{\overline{K},\mathbf{Q}_p}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{PD}}(\mathscr{X}_1),$$

where we set

$$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)\mathbin{\widehat{\otimes}}^R_{F^{\mathrm{nr}}} r^{\mathrm{PD}}_{\overline{K},\mathbf{Q}_p} := \operatorname{hocolim}_{\Sigma}(\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0})\mathbin{\widehat{\otimes}}^R_{F_L} r^{\mathrm{PD}}_{L,\mathbf{Q}_p}).$$

We also get ($T\mapsto 0$)

$$\mathrm{R}\Gamma_{\mathrm{PD}}(\mathscr{X}_1)\otimes_{r^{\mathrm{PD}}_{\overline{K},\mathbf{Q}_p}} F^{\mathrm{nr}}\simeq\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1),$$

where we set

$$\mathrm{R}\Gamma_{\mathrm{PD}}(\mathscr{X}_1)\otimes_{r^{\mathrm{PD}}_{\overline{K},\mathbf{Q}_p}} F^{\mathrm{nr}} := \operatorname{hocolim}_{\Sigma}(\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_{\mathscr{O}_L,0}/r^{\mathrm{PD}}_L)_{\mathbf{Q}_p}\mathbin{\widehat{\otimes}}^R_{r^{\mathrm{PD}}_{L,\mathbf{Q}_p}} F_L).$$

Varying $\mathscr{X}$ in the above constructions, we obtain the (Hyodo–Kato) maps

$$\iota_{\mathrm{HK}}\colon \mathscr{A}_{\mathrm{HK}}\to\mathscr{A}_{\mathrm{conv},\overline{K}},\quad \iota_{\mathrm{HK}}\colon \mathscr{A}_{\mathrm{HK}}\to\mathscr{A}_{\mathrm{dR}},\quad s\colon \mathscr{A}_{\mathrm{HK}}\to\mathscr{A}_{\mathrm{PD}}$$

---

[24] See [9, Section 2.1] for a quick review of basic facts concerning tensor products in the category $C_{\mathbf{Q}_p}$.

of sheaves on $\mathrm{Sm}_{C,\text{ét}}$. We claim that, for $X\in\mathrm{Sm}_C$, they induce the natural (Hyodo–Kato) strict quasi-isomorphisms

$$\begin{aligned}(4.19)\qquad &\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(X)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(X),\\ &\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(X)\,\widehat{\otimes}^R_{F^{\mathrm{nr}}}\,C\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{dR}}(X),\\ &\mathrm{R}\Gamma_{\mathrm{HK}}(X)\,\widehat{\otimes}^R_{F^{\mathrm{nr}}}\,r^{\mathrm{PD}}_{\overline{K},\mathbf{Q}_p}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{PD}}(X).\end{aligned}$$

Here we set[25]

$$\begin{aligned}(4.20)\qquad &\mathrm{R}\Gamma_{\mathrm{HK}}(X)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K}:=\mathrm{hocolim}((\mathrm{R}\Gamma_{\mathrm{HK}}\otimes_{F^{\mathrm{nr}}}\overline{K})(\mathscr{U}_{\bullet,1})),\\ &\mathrm{R}\Gamma_{\mathrm{HK}}(X)\,\widehat{\otimes}^R_{F^{\mathrm{nr}}}\,C:=\mathrm{hocolim}((\mathrm{R}\Gamma_{\mathrm{HK}}\,\widehat{\otimes}^R_{F^{\mathrm{nr}}}\,C)(\mathscr{U}_{\bullet,1})),\\ &\mathrm{R}\Gamma_{\mathrm{HK}}(X)\,\widehat{\otimes}^R_{F^{\mathrm{nr}}}\,r^{\mathrm{PD}}_{\overline{K},\mathbf{Q}_p}:=\mathrm{hocolim}((\mathrm{R}\Gamma_{\mathrm{HK}}\,\widehat{\otimes}^R_{F^{\mathrm{nr}}}\,r^{\mathrm{PD}}_{\overline{K},\mathbf{Q}_p})(\mathscr{U}_{\bullet,1})),\end{aligned}$$

where the homotopy colimit is taken over η-étale hypercoverings $\mathscr{U}_\bullet$ from $\mathscr{M}^{\mathrm{ss},b}_C$. We note that we have

$$\begin{aligned}(4.21)\qquad &\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(X)\simeq\mathrm{hocolim}\,\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{U}_{\bullet,1}),\\ &\mathrm{R}\Gamma_{\mathrm{PD}}(X)\simeq\mathrm{hocolim}\,\mathrm{R}\Gamma_{\mathrm{PD}}(\mathscr{U}_{\bullet,1}).\end{aligned}$$

Indeed, by Proposition 4.22 below (there is no circular reasoning here), we have

$$\begin{aligned}&\mathrm{hocolim}\,\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{U}_{\bullet,1})\xrightarrow{\sim}\mathrm{hocolim}\,\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{U}_{\bullet,C}),\\ &\mathrm{hocolim}\,\mathrm{R}\Gamma_{\mathrm{PD}}(\mathscr{U}_{\bullet,1})\xrightarrow{\sim}\mathrm{hocolim}\,\mathrm{R}\Gamma_{\mathrm{PD}}(\mathscr{U}_{\bullet,C}).\end{aligned}$$

Hence (4.21) follows from the fact that $\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(X)$ and $\mathrm{R}\Gamma_{\mathrm{PD}}(X)$ satisfy η-étale descent. Having (4.21), the first strict quasi-isomorphism in (4.19) follows from the first Hyodo–Kato strict quasi-isomorphism in (4.17). The second Hyodo–Kato strict quasi-isomorphism in (4.17) implies easily the second strict quasi-isomorphism we wanted. The third strict quasi-isomorphism follows from (4.18).

4.3.3. *Local-global compatibility and comparison results.* Having at our disposal the quasi-isomorphisms (4.19), we can prove the following comparison result (where the tensor products in (2) and (3) are defined as in (4.20)):

Proposition 4.22. (1) *Let* $\mathscr{X}\in\mathscr{M}^{\mathrm{ss},b}_C$. *The natural maps*

$$\begin{aligned}&\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)\to\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_C),\qquad \mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{X}_1)\to\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{X}_C),\\ &\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_1)\to\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_C),\qquad \mathrm{R}\Gamma_{\mathrm{PD}}(\mathscr{X}_1)\to\mathrm{R}\Gamma_{\mathrm{PD}}(\mathscr{X}_C)\end{aligned}$$

are strict quasi-isomorphisms.

(2) *For* $X\in\mathrm{Sm}_C$, *we have natural strict quasi-isomorphisms*

$$\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(X)\,\widehat{\otimes}^R_{\overline{K}}\,C\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{conv}}(X)\simeq\mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

---

[25] The notation is ad hoc and rather awful here but we hope that it is self-explanatory.

(3) *For $X \in \mathrm{Sm}_K$, we have a natural strict quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X) \mathbin{\widehat{\otimes}}_K \overline{K} \simeq \mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(X_C).$$

Proof. For the first claim, it suffices to show that, for any η-étale hypercovering $\mathscr{U}_\bullet$ of $\mathscr{X}$ from $\mathscr{M}_C^{\mathrm{ss},b}$, the natural maps

$$\mathrm{R}\Gamma_?(\mathscr{X}_1) \to \mathrm{R}\Gamma_?(\mathscr{U}_{\bullet,1}), \quad ? = \mathrm{HK}, \{\mathrm{conv}, \overline{K}\}, \mathrm{conv}, \mathrm{PD}, \tag{4.23}$$

are strict quasi-isomorphisms (modulo taking a refinement of $\mathscr{U}_\bullet$). We may assume that in every degree of the hypercovering we have a finite number of formal models. For the Hyodo–Kato case, it suffices to show the strict quasi-isomorphism after we tensor both sides with $\overline{K}$ over F^{nr}. But then we can use the Hyodo–Kato quasi-isomorphism (4.17) to reduce to the case of $\{\mathrm{conv}, \overline{K}\}$ in (4.23).

For that case, note that our map is strictly quasi-isomorphic to a map

$$\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L) \otimes_L \overline{K} \to (\mathrm{R}\Gamma_{\mathrm{dR}} \otimes_{L_\bullet} \overline{K})(\mathscr{U}_{\bullet,L_\bullet}).$$

The rather ugly notation for the hypercovering just underscores the fact that the field over which the particular formal schemes split varies. Passing to cohomology ($\widetilde{H}(-)$-cohomology) and then to a truncated hypercovering, we can assume that all the rigid spaces and maps involved are defined over a common field K', a finite extension of L. We get a strict quasi-isomorphism by étale descent for de Rham cohomology. The cases of PD- and conv-cohomology, can be reduced to that of Hyodo–Kato and de Rham cohomologies via the strict quasi-isomorphisms $\mathrm{R}\Gamma_{\mathrm{PD}}(X) \simeq \mathrm{R}\Gamma_{\mathrm{HK}}(X) \mathbin{\widehat{\otimes}}^R_{F^{\mathrm{nr}}} r^{\mathrm{PD}}_{\overline{K},\mathbf{Q}_p}$ and $\mathrm{R}\Gamma_{\mathrm{conv}}(X) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(X)$, respectively.

For the second claim of the proposition, it suffices to show that for an η-étale hypercovering $\mathscr{U}_\bullet$ of X from $\mathscr{M}_C^{\mathrm{ss},b}$, we have a strict quasi-isomorphism

$$(\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}} \mathbin{\widehat{\otimes}}^R_{\overline{K}} C)(\mathscr{U}_{\bullet,1}) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{U}_{\bullet,C}).$$

It suffices to argue degree-wise. Hence it to show that, for a semi-stable formal model $\mathscr{U}$ over $\mathscr{O}_E$, $[E:L] < \infty$, the first top horizontal arrow in the following diagram is a strict quasi-isomorphism:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{U}_{\mathscr{O}_C,1}) \mathbin{\widehat{\otimes}}^R_{\overline{K}} C \arrow[r] & \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{U}_{\mathscr{O}_C,1}) \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{U}_C) \\
\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{U}_{\mathscr{O}_E,1}) \mathbin{\widehat{\otimes}}^R_E C \arrow[u, "\wr"] \arrow[ur, "\sim"] \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{U}_{\mathrm{PD}}) \mathbin{\widehat{\otimes}}^R_E C. \arrow[ur, "\sim"] &
\end{tikzcd}$$

Since this diagram clearly commutes and the other arrows are strict quasi-isomorphisms, this is evident.

For the third claim of the proposition, it suffices to show that, for any η-étale hypercovering $\mathscr{U}_\bullet$ of X_C from $\mathscr{M}_C^{\mathrm{ss},b}$, the natural map

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X) \mathbin{\widehat{\otimes}}_K \overline{K} \to \mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\mathscr{U}_{\bullet,1}) \tag{4.24}$$

is a strict quasi-isomorphism (modulo taking a refinement of $\mathscr{U}_\bullet$). We can assume that $\mathscr{U}_\bullet$ has formal models in every degree. Then both sides of (4.24) can be computed by $(\mathrm{R}\Gamma_{\mathrm{dR}} \otimes_{L_\bullet} \overline{K})(\mathscr{U}_{\bullet,L_\bullet})$, proving what we wanted. $\square$

4.3.4. *Galois descent.* The following proposition shows that Hyodo–Kato cohomology satisfies Galois descent.

Proposition 4.25. *Let $X \in \mathrm{Sm}_K$. The natural projection $\varepsilon\colon X_{C,\text{ét}} \to X_{\text{ét}}$ defines pullback strict quasi-isomorphisms*

$$\begin{aligned}
&\varepsilon^*\colon \mathrm{R}\Gamma_{\mathrm{HK}}(X) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(X_C)^{\mathscr{G}_K},\\
&\varepsilon^*\colon \mathrm{R}\Gamma_{\mathrm{conv}}(X) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(X_C)^{\mathscr{G}_K},\\
&\varepsilon^*\colon \mathrm{R}\Gamma_{\mathrm{PD}}(X) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{PD}}(X_C)^{\mathscr{G}_K}.
\end{aligned}$$

Remark 4.26. We denoted by $\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)^{\mathscr{G}_K}$, etc., the complex obtained by taking the $\mathscr{G}_K$-fixed points of a representative of $\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)$. This definition makes sense, that is, two strictly quasi-isomorphic complexes representing $\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)$ give two strictly quasi-isomorphic complexes representing $\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)^{\mathscr{G}_K}$. Or, in other words , taking a cone of the given quasi-isomorphism, for a complex $T := T^0 \to T^1 \to T^2 \to \cdots$ such that each T^i is a direct sum of products of LB-spaces with a smooth action of $\mathscr{G}_K$, the complex $T^{\mathscr{G}_K}$ is strictly exact. Indeed, since the complex T is strictly exact, for all i, we have the strictly exact sequence

$$0 \to \ker d_i \to T^i \to \ker d_{i+1} \to 0, \tag{4.27}$$

and we need to show that the induced sequence

$$0 \to (\ker d_i)^{\mathscr{G}_K} \to (T^i)^{\mathscr{G}_K} \to (\ker d_{i+1})^{\mathscr{G}_K} \to 0 \tag{4.28}$$

is exact. We note that there exists a normalized trace function

$$\mathrm{tr}\colon T^i \to (T^i)^{\mathscr{G}_K}, \quad x \mapsto \varinjlim_{L\subset \overline{K}} \frac{1}{[L:K]} \sum_{\sigma\in \mathrm{Gal}(L/K)} \sigma(x).$$

This is well-defined because T^i is a finite direct sum of products of smooth $\mathscr{G}_K$-modules and on a smooth $\mathscr{G}_K$-module, the limit in the formula stabilizes. Let now $x \in (\ker d_{i+1})^{\mathscr{G}_K}$. Since the sequence (4.27) is exact, there exists $y \in T^i$ mapping to x. But then $\mathrm{tr}(y)$ maps to $\mathrm{tr}(x) = x$. Since $\mathrm{tr}(y) \in (T^i)^{\mathscr{G}_K}$, this means that the sequence (4.28) is exact, as wanted.

Proof of Proposition 4.25. By η-étale descent, we may assume that $X = \mathscr{X}_K$ for $\mathscr{X} \in \mathscr{M}_K^{\mathrm{ss}}$. Recall that the action of $\mathscr{G}_K$ on $\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)$, $\mathrm{R}\Gamma_{\mathrm{conv}}(X_C)$, and $\mathrm{R}\Gamma_{\mathrm{PD}}(X_C)$ is then smooth. We will prove only the first quasi-isomorphism – the proof of the others being analogous.

Passing to a finite extension of the splitting field L of $\mathscr{X}$, if necessary, we may assume that $\mathscr{X}$ is semi-stable over a finite Galois extension L of K. Consider the following commutative diagram (we added the base K and L in the definition of the arithmetic Hyodo–Kato cohomology to stress that we are working with the category $\mathscr{M}_K^{\mathrm{ss}}$ and $\mathscr{M}_L^{\mathrm{ss}}$, respectively):

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK}}(X/L) \arrow[r, "\varepsilon^*"] & \mathrm{R}\Gamma_{\mathrm{HK}}(X\otimes_L C)^{\mathscr{G}_L} \\
\mathrm{R}\Gamma_{\mathrm{HK}}(X/K) \arrow[r, "\varepsilon^*"] \arrow[u, "\wr"] & \mathrm{R}\Gamma_{\mathrm{HK}}(X\otimes_K C)^{\mathscr{G}_K}. \arrow[u]
\end{tikzcd}$$

By Proposition 4.10 and Proposition 4.22, the top horizontal map is quasi-isomorphic to the map

$$\varepsilon^*\colon \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0)\to(\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0)\otimes_{F_L}F^{\mathrm{nr}})^{\mathscr{G}_L},$$

which clearly is a quasi-isomorphism. Since $X\otimes_K C\simeq(X\otimes_L C)\times H$, for $H=\mathrm{Gal}(L/K)$, we have

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X\otimes_K C)\simeq\mathrm{R}\Gamma_{\mathrm{HK}}(X\otimes_L C)\times H.$$

Hence the right vertical map in the above diagram is a quasi-isomorphism as well. It follows that so is the bottom horizontal map, as wanted. ☐

4.4. **Passage to Bloch–Kato arithmetic rigid analytic syntomic cohomology.** Let $X\in\mathrm{Sm}_K$. Let $r\geq 0$. In this section, we define the Bloch–Kato rigid analytic syntomic cohomology:

$$\mathrm{R}\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(X,\mathbf{Q}_p(r)):=[[\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r}\xrightarrow{\iota'_{\mathrm{HK}}}\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r],$$

where the map ι'_{HK} is defined below, and we show that it is strictly quasi-isomorphic to the rigid analytic syntomic cohomology of X:

Proposition 4.29. *There is a natural strict quasi-isomorphism*

$$\iota_2\colon \mathrm{R}\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\simeq\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)).$$

Proof. (i) *Local definition.* Let $\mathscr{X}$ be an admissible semi-stable formal scheme over $\mathscr{O}_K$. We define a functorial strict quasi-isomorphism

$$\begin{aligned}\iota_2\colon \mathrm{R}\Gamma^{\mathrm{BK}}_{\mathrm{syn}}&(\mathscr{X},\mathbf{Q}_p(r))\\
&:=\big[[\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}_F^0)_F]^{N=0,\varphi=p^r}\xrightarrow{\iota'_{\mathrm{HK}}}\mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K)/F^r\big]\\
&\simeq\big[[\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_1/\mathscr{O}_F)_F]^{\varphi=p^r}\xrightarrow{\mathrm{can}}\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_1/\mathscr{O}_K^{\times})_K/F^r\big]\\
&\simeq\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X},\mathbf{Z}_p(r))_{\mathbf{Q}_p},\end{aligned}\tag{4.30}$$

by the following diagram:
(4.31)

$$\begin{tikzcd}
 & \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_1/\mathscr{O}_K^{\times})_K & \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \arrow[l, "\sim"'] \arrow[d, "\wr"'] \arrow[dd, bend left, "\wr"] \\
{[\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_1/\mathscr{O}_F)_F]^{\varphi=p^r}} \arrow[ur, "\mathrm{can}"] \arrow[d, "\wr"', "i^*"] & {[\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_1/\mathscr{O}_F)]^{\varphi=p^r}} \arrow[l, "\simeq"', "\varepsilon_1"] \arrow[r] \arrow[d, "\wr"', "i^*"] & \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_1/\mathscr{O}_K^{\times}) \arrow[ul, "\sim"] \arrow[d, "\wr"', "i^*"] \\
{[\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}_F)_F]^{\varphi=p^r}} \arrow[d, "\wr"] \arrow[dr] & {[\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_0/\mathscr{O}_F)]^{\varphi=p^r}} \arrow[l, "\simeq"', "\varepsilon_0"] \arrow[r] \arrow[dr] & \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_0/\mathscr{O}_K^{\times}) \\
{[\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r_F^{\mathrm{PD}})_{\mathbf{Q}_p}]^{N=0,\varphi=p^r}} \arrow[r, "p_p"] \arrow[drr, "p_0", "\sim"'] & \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}_F^{\times})_F & \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_0/\mathscr{O}_F^{\times}) \arrow[l, "\sim"'] \arrow[u] \\
 & & {[\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}_F^0)_F]^{N=0,\varphi=p^r}.} \arrow[uuuu, bend right=40, "\iota'_{\mathrm{HK}}"']
\end{tikzcd}$$

The vertical left bottom map is a quasi-isomorphism by [26, Lem. 4.2]. The map ι'_{HK} is defined by the zigzag in the diagram. The map p_0 is a quasi-isomorphism because Frobenius is highly nilpotent on T. The slanted map from the convergent to crystalline cohomology is a strict quasi-isomorphism because the log-scheme $\mathscr{X}_1$ is log-smooth over $\mathscr{O}_{K,1}^{\times}$. The two right maps i^* are strict quasi-isomorphisms (actually, natural isomorphisms) by the invariance of convergent cohomology under infinitesimal thickenings; the left map i^* is a quasi-isomorphism by a standard Frobenius argument (see the proof of [10, Lem. 5.9]). We claim that the maps $\varepsilon_1, \varepsilon_0$ are strict quasi-isomorphisms. Indeed, it suffices to check this for the second of the two maps and then it follows from the commutative diagram

$$\begin{tikzcd}
{[\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}_F)_F]^{\varphi=p^r}} \arrow[d, "\wr"] & {[\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_0/\mathscr{O}_F)]^{\varphi=p^r}} \arrow[l, "\varepsilon_0"'] \arrow[d, "\wr"] \\
{[\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r_F^{\mathrm{PD}})_{\mathbf{Q}_p}]^{N=0,\varphi=p^r}} \arrow[d, "p_0"', "\wr"] & {[\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_0/\hat{r}_F)]^{N=0,\varphi=p^r}} \arrow[l, "\varepsilon"'] \arrow[d, "p_0"', "\wr"] \\
{[\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}_F^0)_F]^{N=0,\varphi=p^r}} & {[\mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_0/\mathscr{O}_F^0)]^{N=0,\varphi=p^r},} \arrow[l, "\sim", "\varepsilon^0"']
\end{tikzcd}$$

since the map ε^0 is a strict quasi-isomorphism by the log-smoothness of the log-scheme $\mathscr{X}_0$ over k^0. Here $\hat{r}_F := \mathscr{O}_F\{T\}$ and the right vertical maps are strict quasi-isomorphisms by the same arguments as the left vertical maps.

(ii) *Globalization.* Let $\mathscr{A}^{\mathrm{BK}}_{\mathrm{syn}}$ be the η-étale sheafification of the presheaf $\mathscr{X} \to \mathrm{R}\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(\mathscr{X}, \mathbf{Q}_p(r))$ on $\mathscr{M}^{\mathrm{ss}}_{K,\mathrm{\acute{e}t}}$. We have

$$\begin{aligned}\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, \mathscr{A}^{\mathrm{BK}}_{\mathrm{syn}}) &\simeq [\mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, \mathscr{A}_{\mathrm{HK}})^{N=0,\varphi=p^r} \xrightarrow{\iota'_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, \mathscr{A}_{\mathrm{dR}})/F^r] \\ &\simeq [\mathrm{R}\Gamma_{\mathrm{HK}}(X)^{N=0,\varphi=p^r} \xrightarrow{\iota'_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r] \simeq \mathrm{R}\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(X, \mathbf{Q}_p(r)).\end{aligned}$$

Since $\mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) = \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, \mathscr{A}_{\mathrm{syn}})$, by η-étale descent, the strict quasi-isomorphisms ι_2 from (4.30) can be lifted to a strict quasi-isomorphism

$$\iota_2 \colon \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \simeq \mathrm{R}\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(X, \mathbf{Q}_p(r)),$$

as wanted. $\square$

Let us state the following corollary of the above computations.

Corollary 4.32 (Local-global compatibility). *Let $r \geq 0$. For a semi-stable formal scheme $\mathscr{X}$ over $\mathscr{O}_K$, the canonical map*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_K, \mathbf{Q}_p(r))$$

is a strict quasi-isomorphism.

Proof. By construction and Proposition 4.29, we have compatible strict quasi-isomorphisms

$$\begin{aligned}\iota_2 \colon \mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, \mathbf{Q}_p(r)) &\simeq \big[[\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0)]^{N=0,\varphi=p^r} \xrightarrow{\iota'_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K)/F^r\big], \\ \iota_2 \colon \mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_K, \mathbf{Q}_p(r)) &\simeq \big[[\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_K)]^{N=0,\varphi=p^r} \xrightarrow{\iota'_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K)/F^r\big].\end{aligned}$$

It suffice now to note that the natural map $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) \to \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_K)$ is a strict quasi-isomorphism by Proposition 4.10. $\square$

5. Overconvergent syntomic cohomology

In this section we define syntomic cohomology for smooth dagger varieties over K or C in two ways (yielding strictly quasi-isomorphic theories). Recall that in [9] syntomic cohomology of semi-stable weak formal schemes is defined as a homotopy fiber of a map from Frobenius eigenspaces of Hyodo–Kato cohomology to a filtered quotients of de Rham cohomology. By η-étale descent this yields the first definition of syntomic cohomology for smooth dagger varieties. For the second definition we take, for smooth dagger affinoids, the homotopy colimits of syntomic cohomologies of the rigid analytic affinoids forming a presentation of the dagger structure, and then we globalize. The second definition will allow us to define period maps to pro-étale cohomology.

To carry out the above, we introduce Hyodo–Kato cohomology for smooth dagger varieties, prove that it satisfies Galois descent, and define the Hyodo–Kato morphism (that is a strict quasi-isomorphism over C).

5.1. **Overconvergent de Rham cohomology.** Let $L = K, C$. Consider the presheaf $X \mapsto \mathrm{R}\Gamma_{\mathrm{dR}}(X)$ of filtered dg L-algebras on $\mathrm{Sm}^{\dagger}_{L}$. Let $\mathscr{A}_{\mathrm{dR}}$ be its étale sheafification. It is a sheaf of filtered L-algebras on $\mathrm{Sm}^{\dagger}_{L,\text{ét}}$. For $X \in \mathrm{Sm}^{\dagger}_{L}$, we have the filtered quasi-isomorphism $\mathrm{R}\Gamma_{\mathrm{dR}}(X) \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{dR}})$. We equip $\mathrm{R}\Gamma_{\mathrm{dR}}(X)$ with the topology induced by the canonical topology on dagger algebras; we equip $\mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{dR}})$ with topology using étale descent as we did before. Then the above quasi-isomorphism is strict: dagger differentials satisfy étale descent in the strict sense. The de Rham cohomology $H^i_{\mathrm{dR}}(X)$ is classical: it is a finite-dimensional K-vector space with its natural Hausdorff topology for X quasi-compact and a Fréchet space (a surjective limit of finite-dimensional K-vector spaces) for a general smooth X (use Remark 3.11). See the proof of Proposition 5.6 below for how this can be shown.

5.1.1. *The complex* $\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r$. Let $X \in \mathrm{Sm}^{\dagger}_{L}$. The cohomology groups of $\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r$ have the same description as their rigid analytic counterparts in Section 4.1.1. That is, the distinguished triangle (in $\mathscr{D}(C_L)$)

$$0 \to \ker d_r[-r] \to \tau_{\leq r}\Omega^{\bullet}_X \to \Omega^{\leq r-1}_X \to 0 \tag{5.1}$$

yields the strict short exact sequence

$$0 \to H^{r-1}_{\mathrm{dR}}(X) \to \widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r) \to \ker \pi \to 0,$$

where π is the natural map $\Omega^r(X)^{d=0} \to H^r_{\mathrm{dR}}(X)$. We have a strict monomorphism $\operatorname{im} d_{r-1}(X) \hookrightarrow \ker \pi$. We note that the cohomology $\widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)$ is classical (as an extension of classical objects).

The distinguished triangle (5.1) yields also the strict long exact sequence

$$0 \to \operatorname{coker} \pi \to \widetilde{H}^{r}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r) \to \widetilde{H}^{1}(X, \ker d_r) \to \widetilde{H}^{r+1}(X, \tau_{\leq r}\Omega^{\bullet}_X).$$

5.2. **Arithmetic overconvergent Hyodo–Kato cohomology.** We define the Hyodo–Kato cohomology of smooth dagger varieties over K by η-étale descent of overconvergent Hyodo–Kato cohomology of semi-stable models.

5.2.1. *Local definition.* Let X be a log-smooth scheme over k^0. The overconvergent Hyodo–Kato cohomology of X is defined (by Grosse–Klönne in [21]) as $\mathrm{R}\Gamma_{\mathrm{HK}}(X) := \mathrm{R}\Gamma_{\mathrm{HK}}(X/\mathscr{O}_F) := \mathrm{R}\Gamma_{\mathrm{rig}}(X/\mathscr{O}_F^0)$. It is a dg F-algebra, equipped with a φ-action and a monodromy operator N such that $N\varphi = p\varphi N$. We equip it with a topology as in [9, Section 3.1].

Let X be a semi-stable scheme over k^0. Recall that we have the Hyodo–Kato morphism

$$\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{rig}}(X/\mathscr{O}_F^0) \to \mathrm{R}\Gamma_{\mathrm{rig}}(X/\mathscr{O}_F^{\times}) \tag{5.2}$$

that is actually a strict quasi-isomorphism [9, Section 3.1.3]. We have chosen here the functorial version of this morphism as defined by Ertl=-Yamada [15, Prop. 2.5]: a combinatorial modification of the original morphism of Grosse–Klönne yields easy functoriality on most of the data; full functoriality is obtained by a coherent zigzag construction [15, Lem. 2.6].

Remark 5.3. For the convenience of the reader we will describe in more detail the constructions of Grosse–Klönne (see for details [9, Section 3.1.3]) and Ertl–Yamada. Let $\{X_i\}_{i\in I}$ be the irreducible components of X with the induced log-structure. Denote by $M_\bullet$ the nerve of the covering $\coprod_{i\in I} X_i \to X$. By [9, Lem. 3.8], the natural map

$$\mathrm{R}\Gamma_{\mathrm{rig}}(X/\mathscr{O}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(M_\bullet/\mathscr{O}), \quad \mathscr{O} = \mathscr{O}_F^0, \mathscr{O}_F^\times$$

is a strict quasi-isomorphism.

Let $\overline{X}$ be the log-scheme with boundary attached to X in [21]. It comes equipped with a natural map $M'_\bullet \hookrightarrow \overline{X}$, where $M'_\bullet$ is a slight combinatorial modification[26] of $M_\bullet$: there is a natural map $M_\bullet \to M'_\bullet$ that induces a strict quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{rig}}(M'_\bullet/\mathscr{O}) \to \mathrm{R}\Gamma_{\mathrm{rig}}(M_\bullet/\mathscr{O}).$$

We have the following commutative diagram, where $\mathscr{O}(0) = \mathscr{O}_F^0$, $\mathscr{O}(p) = \mathscr{O}_F^\times$, $a = 0$, p, and p_a is the map induced by $T \mapsto a$:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{rig}}(X/\mathscr{O}(a)) \arrow[r, "\sim"] \arrow[dr, "\sim"] & \mathrm{R}\Gamma_{\mathrm{rig}}(M'_\bullet/\mathscr{O}(a)) \arrow[d, "\wr"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}/r_F^\dagger) \arrow[l, "\sim"'] \arrow[ddd] \\
& \mathrm{R}\Gamma_{\mathrm{rig}}(M_\bullet/\mathscr{O}(a)) & \\
& \mathrm{R}\Gamma_{\mathrm{rig}}(M_\bullet/r_F^\dagger) \arrow[u, "p_a"] & \\
\mathrm{R}\Gamma_{\mathrm{rig}}(X/r_F^\dagger) \arrow[uuu, "p_a"] \arrow[ur, "\sim"] \arrow[rr, "\sim"] & & \mathrm{R}\Gamma_{\mathrm{rig}}(M'_\bullet/r_F^\dagger). \arrow[ul, "\sim"] \arrow[uuul, "p_a"']
\end{tikzcd}$$

We wrote here $r_F^\dagger := \mathscr{O}_F[T]^\dagger$ with the log-structure associated to T; Frobenius is defined by $T \mapsto T^p$, monodromy is the $\mathscr{O}_F$-linear derivation given by $T \mapsto T$. The Hyodo–Kato morphism (5.2) is now defined as the following composition:

$$\begin{tikzcd}
\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{rig}}(X/\mathscr{O}_F^0) \arrow[r, "\sim"] \arrow[dr, dotted] & \mathrm{R}\Gamma_{\mathrm{rig}}(M'_\bullet/\mathscr{O}_F^0) & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}/r_F^\dagger) \arrow[l, "\sim"'] \arrow[d, "\wr"] \\
& \mathrm{R}\Gamma_{\mathrm{rig}}(X/\mathscr{O}_F^\times) \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{rig}}(M'_\bullet/\mathscr{O}_F^\times).
\end{tikzcd}$$

For another semi-stable scheme Y over k^0 and a map of log-schemes $g\colon Y \to X$, Ertl–Yamada define a pullback morphism $g^*\colon \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}/r_F^\dagger) \to \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{Y}/r_F^\dagger)$ that makes ι_{HK} functorial [15, Lem. 2.6].

In what follows, to simplify the notation, we will write

$$\begin{aligned}
p_a &\colon \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}/r_F^\dagger) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(M'_\bullet/\mathscr{O}(a)) \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(X/\mathscr{O}(a)),\\
f_1 &\colon \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}/r_F^\dagger) \to \mathrm{R}\Gamma_{\mathrm{rig}}(M'_\bullet/r_F^\dagger) \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(X/r_F^\dagger).
\end{aligned}$$

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[26] We take the definition of Ertl–Yamada, which allows multiplicities in the index set, rather than the original definition of Grosse–Klönne, which does not allow them.

The above commutative diagram yields the functorial commutative diagram:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{rig}}(X/\mathscr{O}(a)) & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}/r_F^\dagger) \arrow[l, "\sim", "p_a"'] \arrow[dl, "f_1"] \\
\mathrm{R}\Gamma_{\mathrm{rig}}(X/r_F^\dagger). \arrow[u, "p_a"] &
\end{tikzcd}$$

If $\mathscr{X}$ is a semi-stable weak formal scheme over $\mathscr{O}_K$, we define the Hyodo–Kato map

$$\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) \to \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K)$$

as the following composition:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) \arrow[r, equal] & \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_0/\mathscr{O}_F^0) \arrow[r, "\iota_{\mathrm{HK}}"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_0/\mathscr{O}_F^\times) \arrow[d] & \\
 & & \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_0/\mathscr{O}_K^\times) \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{dR}}(X_K).
\end{tikzcd} \tag{5.4}$$

Note that this definition works also for base changes (with respect to $\mathscr{O}_K$) of semi-stable weak formal schemes over $\mathscr{O}_K$. Since the natural morphism $\mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_0/\mathscr{O}_F^\times)\otimes_F K \to \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_0/\mathscr{O}_K^\times)$ is a strict quasi-isomorphism, so is the induced morphism

$$\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0)\otimes_F K \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K).$$

5.2.2. *Globalization.* Let $\mathscr{A}_{\mathrm{HK}}$ be the η-étale sheafification of the presheaf $\mathscr{X} \mapsto \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0/\mathscr{O}_{F_L})$, $L = K_{\mathscr{X}}$, on $\mathscr{M}_K^{\dagger,ss}$; this is an étale sheaf of dg F-algebras on $\mathrm{Sm}_K^\dagger$ equipped with a φ-action and a derivation N such that $N\varphi = p\varphi N$. For $X \in \mathrm{Sm}_K^\dagger$, set $\mathrm{R}\Gamma_{\mathrm{HK}}(X) := \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{HK}})$. Equip it with a topology in the usual way, via η-étale descent, from the topology on the $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0/\mathscr{O}_{F_L})$'s.

Proposition 5.5 (Local-global compatibility)**.** *Let $\mathscr{X}$ be a semi-stable weak formal scheme over $\mathscr{O}_K$. Then the natural map*

$$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) \to \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_K)$$

is a strict quasi-isomorphism.

Proof. Same as the proof of Proposition 4.10. ☐

For $X \in \mathrm{Sm}_K^\dagger$, we define natural F-linear maps (*the overconvergent Hyodo–Kato morphisms*)

$$\iota_{\mathrm{HK}}\colon \mathscr{A}_{\mathrm{HK}} \to \mathscr{A}_{\mathrm{dR}}, \quad \iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(X) \to \mathrm{R}\Gamma_{\mathrm{dR}}(X),$$

by the η-étale sheafification of the Hyodo–Kato map

$$\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) \to \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K)$$

and its globalization, respectively.

5.2.3. *Topology.* We will now discuss topology in more detail.

Proposition 5.6. *Let X be a smooth dagger variety over K.*

1. *If X is quasi-compact, then $\widetilde{H}^*_{\mathrm{HK}}(X)$ is classical. It is a finite-dimensional F-vector space with its unique locally convex Hausdorff topology.*
2. *For a general X, the cohomology $\widetilde{H}^*_{\mathrm{HK}}(X)$ is classical. It is a Fréchet space, a limit of finite-dimensional F-vector spaces.*
3. *The endomorphism φ on $H^*_{\mathrm{HK}}(X)$ is a homeomorphism.*
4. *If k is finite and X is quasi-compact, then $H^*_{\mathrm{HK}}(X)$ is a mixed F-isocrystal, i.e., the eigenvalues*[27] *of φ are Weil numbers (if X is not quasi-compact, then $H^*_{\mathrm{HK}}(X)$ is a product of mixed F-isocrystals).*

Proof. In the case $X = \mathscr{X}_K$, for a semi-stable weak formal model $\mathscr{X}$ over $\mathscr{O}_K$, and for $\widetilde{H}^*_{\mathrm{HK}}(\mathscr{X}_0)$, this is [9, Prop. 3.2]. All algebraic statements concerning cohomology in the proposition follow from that by using η-étale descent and the quasi-isomorphism from Proposition 5.5.

We treat now the topological claims. For (1), we first use the η-étale descent and the fact that claim (1) holds in the case X has a semi-stable model over $\mathscr{O}_K$ to construct a filtration on the classical cohomology $H^i_{\mathrm{HK}}(X)$ with graded pieces finite rank vector spaces over F with their canonical Hausdorff topology. This implies that the natural topology on $H^i_{\mathrm{HK}}(X)$ is also Hausdorff. It remains to show that $\widetilde{H}^i_{\mathrm{HK}}(X)$ is classical. Take an η-étale hypercovering $\mathscr{U}_\bullet$ of X built from objects of $\mathscr{M}^{\dagger,\mathrm{ss}}_K$. Assume that in every degree we have a finite number of affine weak formal schemes (recall that X is quasi-compact). Then the complex $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{U}_{\bullet,0})$ is built from inductive limits of Banach spaces with injective and compact transition maps. Using the fact that these are strong duals of reflexive Fréchet spaces, we know that the kernels of the differentials and their coimages have the same property. In particular, they are LB-spaces. The cohomology $\widetilde{H}^i_{\mathrm{HK}}(X)$ is represented by the pair $\operatorname{coim} d_{i-1} \to \ker d_i$ and $H^i_{\mathrm{HK}}(X) = \ker d_i / \operatorname{im} d_{i-1}$ with the induced topology. Let W be a subspace of $\ker d_i$ that maps onto $H^i_{\mathrm{HK}}(X)$ and has the same rank as the latter. Then the map $\operatorname{coim} d_{i-1} \oplus W \to \ker d_i$ is a continuous map of LB-spaces that is an algebraic isomorphism hence, by the open mapping theorem, it is a topological isomorphism. Hence the map $\operatorname{coim} d_{i-1} \to \ker d_i$ is strict and the cohomology $\widetilde{H}^i_{\mathrm{HK}}(X)$ is classical.

A similar argument, using strong duals of reflexive Fréchet spaces, implies that a map between two Hyodo–Kato complexes associated to two (different) η-étale affine hypercoverings of X as above is a strict quasi-isomorphism. This implies that, for X quasi-compact, the cohomology of $\mathrm{R}\Gamma_{\mathrm{HK}}(X)$ is strictly quasi-isomorphic to the cohomology of $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{U}_{\bullet,0})$ for any η-étale affine hypercovering $\mathscr{U}_\bullet$ as above.

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[27] We define the eigenvalues of φ in $\mathbf{Q} \otimes F^*$ to be the s-th roots of the eigenvalues of φ^s, where s is any nonzero multiple of f for $|k| = p^f$. We note that this definition is stable under base change from F to F', $[F' : F] < \infty$.

To see that φ is a homeomorphism in (3), note that this is clear for quasi-compact X by the above remarks. For a general X, as in the case of pro-étale cohomology, cover it with an admissible increasing quasi-compact covering $\{U_n\}_{n\in\mathbf{N}}$. We obtain the exact sequence

$$0 \to H^1 \operatorname{holim}_n \widetilde{H}^{i-1}_{\mathrm{HK}}(U_n) \to \widetilde{H}^i_{\mathrm{HK}}(X) \to H^0 \operatorname{holim}_n \widetilde{H}^i_{\mathrm{HK}}(U_n) \to 0.$$

But, by (1), the cohomologies $\widetilde{H}^i_{\mathrm{HK}}(U_n)$ are classical and finite-dimensional over F. Hence, the cohomology $\widetilde{H}^i_{\mathrm{HK}}(X)$ is classical and we have

$$H^i_{\mathrm{HK}}(X) \xrightarrow{\sim} \varprojlim_n H^i_{\mathrm{HK}}(U_n).$$

Hence it is Fréchet, as wanted. We have proved (2), and (4) follows now trivially from (1). ☐

5.2.4. (φ, N)-*cohomology.* Let $X \in \mathrm{Sm}^\dagger_K$, $r \geq 0$. We will need to understand the cohomology of $[\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r}$. We have

$$[\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r} = \left[\begin{tikzcd} \mathrm{R}\Gamma_{\mathrm{HK}}(X) \arrow[r, "p^r-\varphi"] \arrow[d, "N"] & \mathrm{R}\Gamma_{\mathrm{HK}}(X) \arrow[d, "N"] \\ \mathrm{R}\Gamma_{\mathrm{HK}}(X) \arrow[r, "p^r-p\varphi"] & \mathrm{R}\Gamma_{\mathrm{HK}}(X) \end{tikzcd}\right].$$

This gives rise to a spectral sequence

$$E_2^{ij} = \widetilde{H}^i([H^j_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r}) \Rightarrow \widetilde{H}^{i+j}(\mathrm{R}\Gamma_{\mathrm{HK}}(X)^{N=0,\varphi=p^r}), \tag{5.7}$$

where $\widetilde{H}^*([H^j_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r})$ is the cohomology of the complex

$$\left[\begin{tikzcd} H^j_{\mathrm{HK}}(X) \arrow[r, "p^r-\varphi"] \arrow[d, "N"] & H^j_{\mathrm{HK}}(X) \arrow[d, "N"] \\ H^j_{\mathrm{HK}}(X) \arrow[r, "p^r-p\varphi"] & H^j_{\mathrm{HK}}(X) \end{tikzcd}\right].$$

That is, we can compute it by the sequence

$$H^j_{\mathrm{HK}}(X) \xrightarrow{(N,p^r-\varphi)} H^j_{\mathrm{HK}}(X) \oplus H^j_{\mathrm{HK}}(X) \xrightarrow{(p^r-p\varphi)-N} H^j_{\mathrm{HK}}(X).$$

The cohomology $\widetilde{H}^i([H^j_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r})$ is classical and a Fréchet space. This is because we can write naturally

$$H^i_{\mathrm{HK}}(X) \simeq \varprojlim_n H^i_{\mathrm{HK}}(U_n),$$

for an admissible increasing quasi-compact covering $\{U_n\}_{n\in\mathbf{N}}$ of X, and all the cohomologies $H^i_{\mathrm{HK}}(U_n)$ are finite-dimensional over F.

Hence, in the spectral sequence (5.7), the terms are classical and Fréchet spaces. Arguing by limits as above, we conclude that so is the abutment.

Remark 5.8. In the case when $H^j_{\mathrm{HK}}(X)$ is a finite (φ, N)-module (for example X quasi-compact), $H^*([H^j_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r}) \simeq \mathrm{Ext}^*_{\varphi,N}(F, H^j_{\mathrm{HK}}(X)\{r\})$, the Ext-groups in the category of finite (φ, N)-modules [2].

Proposition 5.9. *Let* $X \in \mathrm{Sm}_K^{\dagger}, r \geq 0$.

(1) *We have* $H^i([\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r}) = 0$ *for* $i \leq r-1$.

(2) *There is a strict short exact sequence*

$$0 \to H^{r-1}_{\mathrm{HK}}(X)^{\varphi=p^{r-1}} \to H^r([\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r}) \to H^r_{\mathrm{HK}}(X)^{N=0,\varphi=p^r} \to 0. \tag{5.10}$$

Proof. To see that, we note that the slopes of Frobenius on $H^i_{\mathrm{HK}}(X)$ are $\leq i$: it is enough to show this for X with a semi-stable reduction, where we can use the weight spectral sequence to reduce to showing that, for a smooth scheme Y over k, the slopes of Frobenius on the (classical) rigid cohomology $H^i_{\mathrm{rig}}(Y/F)$ are $\leq i$; but this is well known [7, Th. 3.1.2]. It follows that the morphism $\varphi - p^j$ is an isomorphism on $H^i_{\mathrm{HK}}(X)$ for $i < j$. Knowing that, we obtain both claims of the proposition from the spectral sequence (5.7). ☐

5.3. **Geometric overconvergent Hyodo–Kato cohomology.** We define the Hyodo–Kato cohomology of smooth dagger varieties over C by η-étale descent of overconvergent Hyodo–Kato cohomology of semi-stable models.

5.3.1. *Definition and basic properties.* Let $f\colon \mathscr{X} \to \mathrm{Spwf}(\mathscr{O}_C)^{\times}$ be a semi-stable weak formal model. Suppose that f is the base change of a semi-stable weak formal model $f_L\colon \mathscr{X}_{\mathscr{O}_L} \to \mathrm{Spwf}(\mathscr{O}_L)^{\times}$ over $\mathscr{O}_L$ by $\theta\colon \mathrm{Spwf}(\mathscr{O}_C)^{\times} \to \mathrm{Spwf}(\mathscr{O}_L)^{\times}$, for a finite extension L/K. That is, we have a map $\theta_L\colon \mathscr{X} \to \mathscr{X}_{\mathscr{O}_L}$ such that the square $(f, f_L, \theta, \theta_L)$ is Cartesian. Such data $\{(L, \mathscr{X}, \theta_L)\}$ reduced modulo p form a filtered set Σ (cf. Section 4.3.1).

(i) *Hyodo–Kato cohomology.* For a morphism of tuples $(L', \mathscr{X}'_{\mathscr{O}_{L'},1}, \theta'_{L'}) \to (L, \mathscr{X}_{\mathscr{O}_{L},1}, \theta_L)$ from Σ, we have a canonical base change identification compatible with φ-action (unramified base change)

$$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0}) \otimes_{F_L} F_{L'} \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}'_{\mathscr{O}_{L'},0}). \tag{5.11}$$

We set

$$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1) := \mathrm{hocolim}_{\Sigma}\, \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0}).$$

It is a dg F^{nr}-algebra[28] equipped with a φ-action and a derivation N such that $N\varphi = p\varphi N$. It is functorial with respect to $\mathscr{X}$: note that the restriction of a morphism $\mathscr{X} \to \mathscr{Y}$ to a morphism $\mathscr{X}_1 \to \mathscr{Y}_1$ is defined over a finite extension of K.

Let $\mathscr{A}_{\mathrm{HK}}$ be the η-étale sheafification of the presheaf $\mathscr{X} \mapsto \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)$ on $\mathscr{M}_C^{\dagger,\mathrm{ss},b}$. For $X \in \mathrm{Sm}_C^{\dagger}$, we set

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X) := \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{HK}}).$$

It is a dg F^{nr}-algebra equipped with a Frobenius, monodromy action, and a continuous action of $\mathscr{G}_K$ if X is defined over K (this action is smooth if X is

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[28] The field F^{nr} is equipped here with the inductive limit topology in C_F. In particular, a sequence $(x_n)_{n\in\mathbf{N}}$, of elements of F^{nr} converges if and only if there exists a finite extension L of F such that all $x_n \in L$ and the sequence $(x_n)_{n\in\mathbf{N}}$ converges inside L.

quasi-compact). We equip it with the topology induced, by η-étale descent, from the topology on the $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0})$'s.

Proposition 5.12. *Let X be a smooth dagger variety over C.*

1. *If X is quasi-compact, then $\widetilde{H}^*_{\mathrm{HK}}(X)$ is classical. It is a finite-dimensional F^{nr}-vector space with its natural topology.*
2. *The cohomology $\widetilde{H}^*_{\mathrm{HK}}(X)$ is classical. It is a limit (in C_F) of finite-dimensional F^{nr}-vector spaces.*
3. *The endomorphism φ on $H^*_{\mathrm{HK}}(X)$ is a homeomorphism.*
4. *If k is finite and X is quasi-compact, then $H^*_{\mathrm{HK}}(X)$ is a mixed F-isocrystal, i.e., the eigenvalues*[29] *of φ are Weil numbers (if X is not quasi-compact, then $H^*_{\mathrm{HK}}(X)$ is a product of mixed F-isocrystals).*

Proof. For claim (1), it suffices to show that, for every η-étale hypercovering $\mathscr{U}_\bullet$ of X from $\mathscr{M}_C^{\dagger,\mathrm{ss},b}$, the cohomology $\widetilde{H}^i_{\mathrm{HK}}(\mathscr{U}_{\bullet,C})$, $i \geq 0$, is classical and of finite rank over F^{nr}. Since we can assume that the weak formal schemes in every degree of the hypercovering are admissible, this follows immediately from Proposition 5.6 and the quasi-isomorphism (5.11).

Claim (2) follows easily from claim (1). Claim (3) and (4) follow by the same argument as claim (1). ☐

(i) *Rigid cohomology.* Let $\mathscr{A}_{\mathrm{rig}}$ be the η-étale sheafification of the presheaf $\mathscr{X} \mapsto \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_1/\mathscr{O}_C^\times)$ on $\mathscr{M}_C^{\dagger,\mathrm{ss},b}$. For $X \in \mathrm{Sm}_C^\dagger$, we set

$$\mathrm{R}\Gamma_{\mathrm{rig}}(X) := \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{rig}}).$$

It is a dg C-algebra equipped with a continuous action of $\mathscr{G}_K$ if X is defined over K. We equip it with the topology induced, by η-étale descent, from the topology on the $\mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_1/\mathscr{O}_C^\times)$'s. We have natural (strict) quasi-isomorphisms

$$\mathscr{A}_{\mathrm{rig}} \xrightarrow{\sim} \mathscr{A}_{\mathrm{dR}}, \quad \mathrm{R}\Gamma_{\mathrm{rig}}(X) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

Let $\mathscr{A}_{\mathrm{rig},\overline{K}}$ be the η-étale sheafification of the presheaf $\mathscr{X} \mapsto \mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(\mathscr{X}_1)$ on $\mathscr{M}_C^{\dagger,\mathrm{ss},b}$, where we set

$$\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(\mathscr{X}_1) := \mathrm{hocolim}_\Sigma \, \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_0/\mathscr{O}_L^\times).$$

For $X \in \mathrm{Sm}_C^\dagger$, we set $\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X) := \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}_{\mathrm{rig},\overline{K}})$. It is a dg $\overline{K}$-algebra equipped with a continuous action of $\mathscr{G}_K$ if X is defined over K (this action is smooth if X is quasi-compact). We equip it with the topology induced, by η-étale descent, from the topology on the $\mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_{\mathscr{O}_L,0})$'s. There are natural continuous morphisms

$$\mathscr{A}_{\mathrm{rig},\overline{K}} \to \mathscr{A}_{\mathrm{rig}}, \quad \mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X) \to \mathrm{R}\Gamma_{\mathrm{rig}}(X).$$

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[29] The cohomology $H^*_{\mathrm{HK}}(X)$ together with its Frobenius, a priori an F^{nr}-vector space of finite rank, is obtained by a base change from a finite rank F'-vector space V, where $[F' : F] < \infty$, equipped with a semilinear Frobenius, so we can use the definition of eigenvalues of Frobenius from the footnote to Proposition 5.6.

5.3.2. *Galois descent.* Again we have a Galois descent.

Proposition 5.13. *Let $X \in \mathrm{Sm}^\dagger_K$. The natural projection $\varepsilon\colon X_{C,\text{ét}} \to X_{\text{ét}}$ defines pullback quasi-isomorphisms*

$$\varepsilon^*\colon \mathrm{R}\Gamma_{\mathrm{HK}}(X) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(X_C)^{\mathscr{G}_K}, \quad \varepsilon^*\colon \mathrm{R}\Gamma_{\mathrm{dR}}(X) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X_C)^{\mathscr{G}_K}.$$

Proof. We can use the proof of Proposition 4.25 almost verbatim[30]. □

5.3.3. *Hyodo–Kato quasi-isomorphisms.*

(i) *Local definition.* Let $\mathscr{X} \to \mathrm{Spwf}(\mathscr{O}_C)^\times$ be as above. The Hyodo–Kato morphism from (5.4),

$$(5.14) \qquad \begin{aligned} \iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0}) &\to \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_{\mathscr{O}_L,0}/\mathscr{O}_L^\times),\\ \iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0}) \otimes_{F_L} L &\xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_{\mathscr{O}_L,0}/\mathscr{O}_L^\times),\end{aligned}$$

is compatible with the morphisms in Σ and taking its homotopy colimit yields the first of the following two natural strict quasi-isomorphisms (called again the *Hyodo–Kato quasi-isomorphisms*):

$$(5.15) \qquad \begin{tikzcd} \iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1) \otimes_{F^{\mathrm{nr}}} \overline{K} \arrow[r, dash, "\sim"] & \mathrm{hocolim}_\Sigma(\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0}) \otimes_{F_L} L) \arrow[d, "\wr"] \\ \mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(\mathscr{X}_1) : \arrow[r, equal] & \mathrm{hocolim}_\Sigma \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_{\mathscr{O}_L,0}/\mathscr{O}_L^\times), \end{tikzcd}$$

$$\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)\, \widehat{\otimes}^R_{F^{\mathrm{nr}}} C \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_1/\mathscr{O}_C^\times) \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C).$$

In the second Hyodo–Kato morphism, we set

$$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)\, \widehat{\otimes}^R_{F^{\mathrm{nr}}} C := \mathrm{hocolim}_\Sigma(\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0})\, \widehat{\otimes}^R_{F_L} C),$$

where all the maps in the homotopy limit are strict quasi-isomorphisms. This morphism is then defined as the composition

$$\begin{aligned} \mathrm{hocolim}_\Sigma(\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_{\mathscr{O}_L,0})\, \widehat{\otimes}^R_{F_L} C) &\xrightarrow{\iota_{\mathrm{HK}}} \mathrm{hocolim}_\Sigma(\mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_{\mathscr{O}_L,0}/\mathscr{O}_L^\times)\, \widehat{\otimes}^R_L C)\\ &\xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_1/\mathscr{O}_C^\times) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_C), \end{aligned}$$

where we have used the Hyodo–Kato quasi-isomorphism from (5.14), the second map is a strict quasi-isomorphism by base change. So the defined morphism is clearly a strict quasi-isomorphism.

(ii) *Globalization.* Varying $\mathscr{X}$ in the above constructions, we obtain the Hyodo–Kato maps

$$\iota_{\mathrm{HK}}\colon \mathscr{A}_{\mathrm{HK}} \to \mathscr{A}_{\mathrm{rig}}, \quad \iota_{\mathrm{HK}}\colon \mathscr{A}_{\mathrm{HK}} \to \mathscr{A}_{\mathrm{dR}}$$

of sheaves on $\mathrm{Sm}^\dagger_{C,\text{ét}}$. For $X \in \mathrm{Sm}^\dagger_C$, they induce the natural Hyodo–Kato strict quasi-isomorphisms

$$(5.16) \qquad \begin{aligned} \iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(X)\, \widehat{\otimes}_{F^{\mathrm{nr}}} \overline{K} &\xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X),\\ \iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma_{\mathrm{HK}}(X)\, \widehat{\otimes}^R_{F^{\mathrm{nr}}} C &\xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X).\end{aligned}$$

---

[30] Note that Remark 4.26 applies to this setting.

Here we set

$$\begin{aligned}(5.17)\qquad &\mathrm{R}\Gamma_{\mathrm{HK}}(X)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K} := \operatorname{hocolim}((\mathrm{R}\Gamma_{\mathrm{HK}}\otimes_{F^{\mathrm{nr}}}\overline{K})(\mathscr{U}_{\bullet,0})),\\ &\mathrm{R}\Gamma_{\mathrm{HK}}(X)\,\widehat{\otimes}^{R}_{F^{\mathrm{nr}}}\,C := \operatorname{hocolim}((\mathrm{R}\Gamma_{\mathrm{HK}}\,\widehat{\otimes}^{R}_{F^{\mathrm{nr}}}\,C)(\mathscr{U}_{\bullet,0})),\end{aligned}$$

where the homotopy colimit is taken over η-étale hypercoverings from $\mathscr{M}_C^{\dagger,\mathrm{ss},b}$. We note that

$$(5.18)\qquad \mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X) \simeq \operatorname{hocolim}\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(\mathscr{U}_{\bullet,1}).$$

This is because $\operatorname{hocolim}\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(\mathscr{U}_{\bullet,1}) \simeq \operatorname{hocolim}\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(\mathscr{U}_{\bullet,C})$, by Proposition 5.19 below (there is no circular reasoning here) and we have η-étale descent for $\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X)$. Having (5.18), the first strict quasi-isomorphism in (5.16) follows from the strict Hyodo–Kato quasi-isomorphism in (5.15). The latter also imply easily the second strict quasi-isomorphism we wanted.

(iii) *Local-global compatibility and comparison results.* The Hyodo–Kato quasi-isomorphisms allow us now to prove the following comparison result (where the tensor products in (2) and (3) are defined as in (5.17).

Proposition 5.19. (1) *Let $\mathscr{X}\in\mathscr{M}^{\dagger,\mathrm{ss},b}$. Then the natural maps*

$$\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_1)\to\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_C),\quad \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_1)\to\mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_C),$$
$$\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(\mathscr{X}_1)\to\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(\mathscr{X}_C)$$

are strict quasi-isomorphisms.

(2) *For $X\in\mathrm{Sm}_C^{\dagger}$, we have a natural strict quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X)\,\widehat{\otimes}^{R}_{\overline{K}}\,C \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{rig}}(X)\simeq\mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

(3) *For $X\in\mathrm{Sm}_K^{\dagger}$, we have a natural strict quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X)\,\widehat{\otimes}_K\,\overline{K}\simeq\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X_C).$$

Proof. The proof is almost verbatim the same as the proof of Proposition 4.22 (which contains analogous claims in the case of rigid analytic varieties), we just need to replace $\mathrm{R}\Gamma_{\mathrm{conv}}$ used there with $\mathrm{R}\Gamma_{\mathrm{rig}}$. □

Remark 5.20. Much of what we have described above in Section 5.3 goes through, with minimal changes, for $X\in\mathrm{Sm}_C$. Hence, working with formal schemes instead of weak formal schemes, we have the geometric Hyodo–Kato cohomology $\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)$. We wrote † to distinguished this cohomology from the geometric Hyodo–Kato cohomology $\mathrm{R}\Gamma_{\mathrm{HK}}(X)$ defined in Section 4.3. It is a dg F^{nr}-algebra equipped with a φ-action, derivation N such that $N\varphi=p\varphi N$, and a continuous action of $\mathscr{G}_K$ (which is smooth when X is quasi-compact). It has an arithmetic analogous that satisfies Galois descent of the type described in Proposition 5.13. We also have the Hyodo–Kato quasi-isomorphism

$$\iota_{\mathrm{HK}}\colon \mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K}\xrightarrow{\sim}\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X),$$

where the rigid cohomology is defined like its analog for dagger varieties.

If X is quasi-compact, the underlying isocrystal of $H^i\mathrm{R}\Gamma^{\dagger}_{\mathrm{HK}}(X)$ should be the one defined by Le Bras in [28].

5.4. **Arithmetic overconvergent syntomic cohomology.** We define now arithmetic overconvergent syntomic cohomology of smooth dagger varieties over K by η-étale descent of overconvergent syntomic cohomology of semi-stable weak formal models.

Let $\mathscr{X}$ be an admissible semi-stable weak formal scheme over $\mathscr{O}_L$, $[L:K]<\infty$. For $r \geq 0$, we define the overconvergent syntomic cohomology as

$$(5.21)\qquad \mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, \mathbf{Q}_p(r)) := \big[[\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0)]^{N=0,\varphi=p^r} \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_L)/F^r\big].$$

For a smooth dagger space X over K, we define the syntomic cohomology $\mathscr{A}_{\mathrm{syn}}(r)$ as the η-étale sheafification of the above complexes on $\mathscr{M}_K^{\mathrm{ss},\dagger}$; and we define the syntomic cohomology of X as

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) := \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, \mathscr{A}_{\mathrm{syn}}(r)).$$

We have the distinguished triangle

$$(5.22)\qquad \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \to [\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r} \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r.$$

Proposition 5.23 (Local-global compatibility)**.** *Let $r \geq 0$. Let $\mathscr{X}$ be a semi-stable weak formal scheme over $\mathscr{O}_K$. Then the natural map*

$$\mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}_K, \mathbf{Q}_p(r))$$

is a strict quasi-isomorphism.

Proof. Using the presentations of syntomic cohomology from (5.21) and (5.22) we reduce to proving that the natural map $\mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_0) \to \mathrm{R}\Gamma_{\mathrm{HK}}(\mathscr{X}_K)$ is a strict quasi-isomorphism. But this we know to be true by Proposition 5.5. ☐

5.4.1. *Examples.* We will discuss a couple of examples.

(i) *The closed ball.* Let $L = K, C$. Let $X_L := \mathbb{B}^d_L(\rho)$ be the overconvergent closed ball over L of dimension d and radius $\rho \in \sqrt{|L^\times|}$. Since $H^0_{\mathrm{dR}}(X_L) \simeq L$ and $H^i_{\mathrm{dR}}(X_L) = 0$, $i > 0$, and we have the Hyodo–Kato isomorphism $H^i_{\mathrm{HK}}(X_C) \otimes_{F^{\mathrm{nr}}} C \simeq H^i_{\mathrm{dR}}(X_C)$ and the Galois descent $H^i_{\mathrm{HK}}(X_K) \xrightarrow{\sim} H^i_{\mathrm{HK}}(X_C)^{\mathscr{G}_K}$, we get

$$H^i_{\mathrm{HK}}(\mathbb{B}^d_L(\rho)) \simeq \begin{cases} F_L & \text{if } i = 0,\\ 0 & \text{if } i \geq 1,\end{cases}$$

where $F_C = F^{\mathrm{nr}}$ and $F_K = F$.

From the exact sequence (5.10), we get

$$H^0([\mathrm{R}\Gamma_{\mathrm{HK}}(X_K)]^{N=0,\varphi=1}) \xrightarrow{\sim} H^0_{\mathrm{HK}}(X_K)^{N=0,\varphi=1},$$
$$H^0_{\mathrm{HK}}(X_K)^{\varphi=1} \xrightarrow{\sim} H^1([\mathrm{R}\Gamma_{\mathrm{HK}}(X_K)]^{N=0,\varphi=p}).$$

Hence, by the above,

$$H^i([\mathrm{R}\Gamma_{\mathrm{HK}}(\mathbb{B}^d_K(\rho))]^{N=0,\varphi=p^i}) \simeq \begin{cases} \mathbf{Q}_p & \text{if } i = 0,1,\\ 0 & \text{if } i \geq 2.\end{cases}$$

Let $r \geq 0$. By the triviality, in nonzero degrees, of the cohomology of coherent sheaves on $\mathbb{B}^d_K(\rho)$, we have

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)/F^r \simeq \mathscr{O}(X_K) \to \Omega(X_K) \to \cdots \to \Omega^{r-1}(X_K).$$

Hence $H^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)/F^r) = 0$, for $i \geq r$, and

$$H^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X_K)/F^r) \xleftarrow{\sim} \Omega^{r-1}(X_K)/\operatorname{im} d_{r-2} \simeq \Omega^r(X_K)^{d=0}.$$

From the definition of syntomic cohomology and the above computations, we get the long exact sequence

$$\begin{aligned} H^{r-1}([\mathrm{R}\Gamma_{\mathrm{HK}}(X_K)]^{N=0,\varphi=p^r}) &\to \Omega^{r-1}(X_K)/\operatorname{im} d_{r-2} \\ &\to H^r_{\mathrm{syn}}(X_K, \mathbf{Q}_p(r)) \to H^r([\mathrm{R}\Gamma_{\mathrm{HK}}(X_K)]^{N=0,\varphi=p^r}) \to 0 \end{aligned}$$

Hence

$$H^r_{\mathrm{syn}}(\mathbb{B}^d_K(\rho), \mathbf{Q}_p(r)) \simeq \begin{cases} \mathbf{Q}_p & \text{if } r = 0, \\ \Omega^{r-1}(\mathbb{B}^d_K(\rho))/\operatorname{im} d_{r-2} & \text{if } r \geq 2, \end{cases}$$

and, for $r = 1$, we get an extension

$$0 \to \mathscr{O}(\mathbb{B}^d_K(\rho)) \to H^1_{\mathrm{syn}}((\mathbb{B}^d_K(\rho), \mathbf{Q}_p(1)) \to \mathbf{Q}_p \to 0.$$

(ii) *The open ball.* Let $L = K, C$. Let $\mathbb{B}^{\mathrm{o},d}_L(\rho)$ be the overconvergent open ball over L of dimension d and radius ρ. Cover $\mathbb{B}^{\mathrm{o},d}_L(\rho)$ with an increasing union of overconvergent closed balls $\{U_n\}_{n\in\mathbf{N}}$. By the above example, we have $H^i_{\mathrm{HK}}(\mathbb{B}^{\mathrm{o},d}_L(\rho)) \simeq \varprojlim_n H^i_{\mathrm{HK}}(U_n)$. Hence

$$H^i_{\mathrm{HK}}(\mathbb{B}^{\mathrm{o},d}_L(\rho)) \simeq \begin{cases} F_L & \text{if } i = 0, \\ 0 & \text{if } i > 0. \end{cases}$$

The rest of the computations is exactly the same as for the closed ball in the first example (note that $\mathbb{B}^{\mathrm{o},d}_K(\rho)$ is Stein) yielding the same final formulas for $H^r_{\mathrm{syn}}((\mathbb{B}^{\mathrm{o},d}_K(\rho), \mathbf{Q}_p(r))$ (with $\mathbb{B}^{\mathrm{o},d}_K(\rho)$ in the place of $\mathbb{B}^d_K(\rho)$).

6. COMPARISON OF OVERCONVERGENT AND RIGID ANALYTIC ARITHMETIC SYNTOMIC COHOMOLOGY

We define a map from syntomic cohomology of a smooth dagger variety to syntomic cohomology of its completion. We show that it is a strict quasi-isomorphism when the variety is partially proper.

6.1. **Construction of the comparison morphism.** Let X be a smooth dagger space over K. We will now construct a functorial map

$$\iota\colon \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(\widehat{X}, \mathbf{Q}_p(r))$$

from the syntomic cohomology of X to the syntomic cohomology of its completion $\widehat{X}$. This will be done by first constructing a map ι_1 to the Bloch–Kato syntomic cohomology from Section 4.4:

$$\iota_1\colon \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(\widehat{X}, \mathbf{Q}_p(r)),$$

and then setting $\iota := \iota_2\iota_1$, for the map $\iota_2\colon \mathrm{R}\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(\widehat{X}, \mathbf{Q}_p(r)) \simeq \mathrm{R}\Gamma_{\mathrm{syn}}(\widehat{X}, \mathbf{Q}_p(r))$ that was defined in Proposition 4.29.

(i) *Local definition.* Let $\mathscr{X}$ be a semi-stable weak formal scheme of finite type over $\mathscr{O}_K$. First, we define a functorial morphism

$$\begin{aligned}(6.1)\qquad \iota_1\colon \mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{X}, \mathbf{Q}_p(r)) &= \big[[\mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_0/\mathscr{O}_F^0)]^{\varphi=p^r} \xrightarrow{\iota_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K)/F^r\big]\\ &\to \big[[\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}_F^0)_F]^{N=0,\varphi=p^r} \xrightarrow{\iota'_{\mathrm{HK}}} \mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{\mathscr{X}_K})/F^r\big].\end{aligned}$$

We use for that the following diagram (we note that all the terms in the first two columns carry a monodromy operator and that all the maps between these terms are compatible with the monodromy action)

(6.2)

$$\begin{tikzcd}
{[\mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_0/\mathscr{O}_F^0)]^{\varphi=p^r}} \arrow[dr, "\iota_{\mathrm{HK}}"] \arrow[rrr, bend left, "\iota_{\mathrm{HK}}"] \arrow[dddd, bend right=40] & & & \\
{[\mathrm{R}\Gamma_{\mathrm{rig}}(\overline{\mathscr{X}}_0/r_F^\dagger)]^{\varphi=p^r}} \arrow[u, "p_0", "\wr"'] \arrow[r, "p_p", "\sim"'] \arrow[d, "f_1"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_0/\mathscr{O}_F^\times) \arrow[r] \arrow[d] & \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_0/\mathscr{O}_K^\times) \arrow[d] & \mathrm{R}\Gamma_{\mathrm{dR}}(\mathscr{X}_K) \arrow[l, "\sim"] \arrow[d] \\
{[\mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_0/r_F^\dagger)]^{\varphi=p^r}} \arrow[ur, "p_p"] \arrow[d] \arrow[uu, bend left, "p_0"] & \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_0/\mathscr{O}_F^\times) \arrow[r] \arrow[d, "\wr"] & \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_0/\mathscr{O}_K^\times) & \mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{\mathscr{X}_K}) \arrow[l, "\sim"] \\
{[\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r_F^{\mathrm{PD}})_{\mathbf{Q}_p}]^{\varphi=p^r}} \arrow[r, "p_p"] \arrow[d, "p_0"', "\wr"] & \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}_F^\times)_F & & \\
{[\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}_F^0)_F]^{\varphi=p^r}.} \arrow[uurrr, bend right, "\iota'_{\mathrm{HK}}"'] & & &
\end{tikzcd}$$

The maps p_0, p_p are defined by sending T to $0, p$, respectively. The top triangle defines the overconvergent Hyodo–Kato morphism ι_{HK} as explained in Remark 5.3, where it is also shown that the maps p_0, p_p from $\overline{\mathscr{X}}_0$ commute with the ones from $\mathscr{X}_0$. The strict quasi-isomorphism between crystalline and convergent cohomology holds because $\mathscr{X}_0$ is log-smooth over k^0. The morphism between de Rham cohomologies is compatible with Hodge filtrations.

(ii) *Globalization.* We define the functorial map

$$\iota_1\colon \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma^{\mathrm{BK}}_{\mathrm{syn}}(\widehat{X}, \mathbf{Q}_p(r))$$

by lifting the map (6.1) via η-étale descent.

6.2. **A comparison result.**

We are now ready to prove our main comparison theorem:

Theorem 6.3. *Let X be a partially proper dagger space over K. The map*

$$\iota\colon \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(\widehat{X}, \mathbf{Q}_p(r))$$

is a strict quasi-isomorphism.

Proof. By the construction of the maps ι_1, ι_2, it suffices to show that the canonical maps

$$\mathrm{R}\Gamma_{\mathrm{dR}}(X) \to \mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X}), \quad [\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{\varphi=p^r} \to [\mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X})]^{\varphi=p^r}$$

are (filtered) strict quasi-isomorphisms. The first map is an isomorphism induced by the canonical identification of coherent cohomology of a partially proper dagger variety and its rigid analytic avatar [20, Th. 2.26]. For the second map, we will show that already the canonical map

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X) \to \mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}) \tag{6.4}$$

is a strict quasi-isomorphism. Our strategy is to pass to the geometric situation, where we can use the Hyodo–Kato isomorphisms to reduce to the de Rham cohomology. The main difficulty in this approach lies in showing the compatibility of the overconvergent and rigid analytic Hyodo–Kato isomorphisms.

(i) *Passage to de Rham cohomology.* We start with the passage to the geometric cohomologies. Since we have compatible strict quasi-isomorphisms (see Propositions 4.25 and 5.13)

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(X_C)^{\mathscr{G}_K}, \quad \mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}_C)^{\mathscr{G}_K},$$

to show that the map (6.4) is a strict quasi-isomorphism, it suffices to show that so is the canonical map

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \to \mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}_C). \tag{6.5}$$

Remark 6.6. Now, if we were to argue in analogy with the algebraic situation, we would use the following approach:

(1) We will prove the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \widehat{\otimes}^R_{F^{\mathrm{nr}}} C \arrow[r] \arrow[d, "\iota_{\mathrm{HK}}", "\wr"'] & \mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}_C) \widehat{\otimes}^R_{F^{\mathrm{nr}}} C \arrow[d, "\iota_{\mathrm{HK}}", "\wr"'] \\
\mathrm{R}\Gamma_{\mathrm{dR}}(X_C) \arrow[r, "\sim"] & \mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X}_C).
\end{tikzcd}$$

This is not an easy task, since the constructions of the rigid and the crystalline Hyodo–Kato maps are very different.

(2) The vertical arrows are the Hyodo–Kato quasi-isomorphisms (4.19) and (5.16), and the bottom arrow is a strict quasi-isomorphism because X_C is partially proper. Hence the top arrow is a strict quasi-isomorphism. The problem is that we do not know how to show that this implies the same for the map (6.5). So, below, we use instead the $\overline{K}$-Hyodo–Kato quasi-isomorphisms.

Consider the diagram

$$\begin{tikzcd}[column sep=large]
\mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \arrow[rr] \arrow[d, "\mathrm{can}"] & & \mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}_C) \arrow[d, "\mathrm{can}"] \\
\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K} \arrow[rr, "\beta"] \arrow[u, dotted, bend left, "\alpha"] \arrow[d, "\iota_{\mathrm{HK}}", "\wr"'] & & \mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}_C)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K} \arrow[u, dotted, bend left, "\hat{\alpha}"] \arrow[d, "\iota_{\mathrm{HK}}", "\wr"'] \\
\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X_C) \arrow[rr, "\beta'", "\sim"'] & & \mathrm{R}\Gamma_{\mathrm{conv},\overline{K}}(\widehat{X}_C) \\
\mathrm{R}\Gamma_{\mathrm{dR}}(X)\,\widehat{\otimes}_K\,\overline{K} \arrow[rr, "\sim"] \arrow[u, "\wr"] & & \mathrm{R}\Gamma_{\mathrm{dR}}(\widehat{X})\,\widehat{\otimes}_K\,\overline{K}. \arrow[u, "\wr"]
\end{tikzcd}\tag{6.7}$$

The maps $\alpha, \hat{\alpha}$ are the normalized trace maps, natural left inverses of the canonical vertical maps. The top squares (the dotted and the non-dotted one) commute. The bottom square clearly commutes. Its vertical maps are strict quasi-isomorrphisms by Proposition 4.22 and Proposition 5.19. The bottom map is a strict quasi-isomorphism because X is partially proper. It follows that the map β' is a strict quasi-isomorphism. We will show below that the middle square commutes on the level of ($\widetilde{H}$-)cohomology. This will imply that the map β is a cohomological isomorphism. This in turn will imply immediately that the map (6.5) is injective on cohomology level; we get its cohomological surjectivity by using the maps $\alpha, \hat{\alpha}$.

(ii) *Comparison of Hyodo–Kato quasi-isomorphisms.* Hence, it remains to show that the middle square in the above diagram commutes on cohomology level, or that the following diagram commutes:

$$\begin{tikzcd}
\widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K}) \arrow[r] \arrow[d, "\iota_{\mathrm{HK}}", "\wr"'] & \widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}_C)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K}) \arrow[d, "\iota_{\mathrm{HK}}", "\wr"'] \\
\widetilde{H}^i_{\mathrm{rig},\overline{K}}(X_C) \arrow[r, "\sim"] & \widetilde{H}^i_{\mathrm{conv},\overline{K}}(\widehat{X}_C).
\end{tikzcd}\tag{6.8}$$

We claim that we can assume that X is quasi-compact and argue just on the level of classical cohomology. Indeed, write X as an increasing union of quasi-compact open sets $\{U_n\}$, $n \geq 0$. Then we have

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K} \simeq \mathrm{holim}_n(\mathrm{R}\Gamma_{\mathrm{HK}}(U_{n,C}) \otimes_{F^{\mathrm{nr}}} \overline{K}).$$

This yields the exact sequence

$$\begin{aligned}
0 \to H^1\,\mathrm{holim}_n(\widetilde{H}^{i-1}_{\mathrm{HK}}(U_{n,C}) \otimes_{F^{\mathrm{nr}}} \overline{K}) &\to \widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K}) \\
&\to H^0\,\mathrm{holim}_n(\widetilde{H}^i_{\mathrm{HK}}(U_{n,C}) \otimes_{F^{\mathrm{nr}}} \overline{K}) \to 0.
\end{aligned}$$

By Proposition 5.12, the cohomology $\widetilde{H}^i_{\mathrm{HK}}(U_{n,C})$ is classical and finite rank over F^{nr}. This implies that the cohomology $\widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K})$ is classical as well and

$$H^i(\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K}) \xrightarrow{\sim} H^0\,\mathrm{holim}_n(H^i_{\mathrm{HK}}(U_{n,C}) \otimes_{F^{\mathrm{nr}}} \overline{K}).$$

Similarly, we can show that the cohomology $\widetilde{H}^i_{\mathrm{conv},\overline{K}}(\widehat{X}_C)$ is classical and we have

$$H^i_{\mathrm{conv},\overline{K}}(\widehat{X}_C) \xrightarrow{\sim} H^0 \operatorname{holim}_n H^i_{\mathrm{conv},\overline{K}}(\widehat{U}_{n,C}).$$

Indeed, arguing as above we get the exact sequence

$$\begin{aligned}(6.9)\qquad 0 \to H^1 \operatorname{holim}_n \widetilde{H}^{i-1}_{\mathrm{conv},\overline{K}}(\widehat{U}_{n,C}) \to \widetilde{H}^i_{\mathrm{conv},\overline{K}}(\widehat{X}_C)& \\ \to H^0 \operatorname{holim}_n \widetilde{H}^i_{\mathrm{conv},\overline{K}}(\widehat{U}_{n,C}) \to 0.&\end{aligned}$$

We note that the prosystems $\{\widetilde{H}^i_{\mathrm{conv},\overline{K}}(\widehat{U}_{n,C})\}_{n\in\mathbf{N}}$ and $\{\widetilde{H}^i_{\mathrm{rig},\overline{K}}(U_{n,C})\}_{n\in\mathbf{N}}$ are equivalent. This follows from the commutative diagram of prosystems

$$\begin{tikzcd}
\{\widetilde{H}^i_{\mathrm{conv},\overline{K}}(\widehat{U}_{n,C})\}_{n\in\mathbf{N}} \arrow[r, "\sim"] & \{\widetilde{H}^i_{\mathrm{conv},\overline{K}}(U^{\mathrm{o}}_{n,C})\}_{n\in\mathbf{N}} \\
\{\widetilde{H}^i_{\mathrm{rig},\overline{K}}(U_{n,C})\}_{n\in\mathbf{N}} \arrow[r, "\sim"] \arrow[u] & \{\widetilde{H}^i_{\mathrm{rig},\overline{K}}(U^{\mathrm{o},\dagger}_{n,C})\}_{n\in\mathbf{N}}. \arrow[u, "\wr"']
\end{tikzcd}$$

Here $U^{\mathrm{o},\dagger}$ denotes the rigid analytic space U^{o}, the interior of U, equipped with its canonical overconvergent structure. The horizontal equivalences are clear. The right vertical map is an isomorphism degree by degree because $U^{\mathrm{o},\dagger}$ is partially proper. This implies that the left vertical map is a an equivalence, as wanted.

Now, the cohomology $\widetilde{H}^i_{\mathrm{rig},\overline{K}}(U_{n,C})$ is classical and finite rank over $\overline{K}$ (it is strictly quasi-isomorphic to $H^i_{\mathrm{dR}}(U_n)\otimes_K \overline{K}$ by Proposition 5.19). Hence the term $H^1 \operatorname{holim}_n$ in the exact sequence (6.9) vanishes and we get our claim.

So, from now on, X is quasi-compact and we will show that the diagram (6.8) commutes on the level of classical cohomology. We have

$$\begin{aligned} H^i(\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K}) &\simeq H^i_{\mathrm{HK}}(X_C)\otimes_{F^{\mathrm{nr}}}\overline{K}\,, \\ H^i(\mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}_C)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,\overline{K}) &\simeq H^i_{\mathrm{HK}}(\widehat{X}_C)\otimes_{F^{\mathrm{nr}}}\overline{K}\,. \end{aligned}$$

Hence, we are reduced to showing that, for a quasi-compact $X\in \mathrm{Sm}_K$, the following diagram commutes:

(6.10)
$$\begin{tikzcd}
H^i_{\mathrm{HK}}(X_C) \arrow[r] \arrow[d, "\iota_{\mathrm{HK}}"] & H^i_{\mathrm{HK}}(\widehat{X}_C) \arrow[d, "\iota_{\mathrm{HK}}"] \\
H^i_{\mathrm{rig},\overline{K}}(X_C) \arrow[r] & H^i_{\mathrm{conv},\overline{K}}(\widehat{X}_C).
\end{tikzcd}$$

Assume first that X has an admissible semi-stable weak formal model $\mathscr{X}$ over $\mathscr{O}_L$, $[L:K]<\infty$, and consider the diagram

(6.11)
$$\begin{tikzcd}
 & \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_0/\mathscr{O}^0_{F_L}) \arrow[dr, "\iota_{\mathrm{HK}}"] \arrow[ddl, bend right] & \\
 & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{\mathscr{X}}_0/r_L^{\dagger}) \arrow[u, "p_0", "\wr"'] \arrow[r, "p_p"] \arrow[d, "f_1"] & \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_0/\mathscr{O}^{\times}_{F_L}) \arrow[d] \\
 & \mathrm{R}\Gamma_{\mathrm{rig}}(\mathscr{X}_0/r_L^{\dagger}) \arrow[ur, "p_p"] \arrow[u, bend left=60, "p_0"] \arrow[d] & \mathrm{R}\Gamma_{\mathrm{conv}}(\mathscr{X}_0/\mathscr{O}^{\times}_{F_L}) \arrow[d, "\wr"] \\
\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^0_{F_L})_{\mathbf{Q}_p} \arrow[r, bend left, "s"] \arrow[rr, bend right, "\iota_{\mathrm{HK}}"'] & \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r_L^{\mathrm{PD}})_{\mathbf{Q}_p} \arrow[l, "p_0"] \arrow[r, "p_p"] & \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^{\times}_{F_L})_{\mathbf{Q}_p}.
\end{tikzcd}$$

If we remove the section s (and hence also the bottom map ι_{HK}), the above diagram commutes. For a general quasi-compact and smooth X, take first a homotopy colimit of the above diagram (over L) and then glue by η-étale descent. We obtain the following diagram:

(6.12)
$$\begin{tikzcd}
 & \mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \arrow[dr, "\iota_{\mathrm{HK}}"] \arrow[ddl, bend right] & \\
 & \mathrm{R}\Gamma_{\mathrm{rig}}(\overline{X}_C/r^{\dagger}) \arrow[u, "p_0", "\wr"'] \arrow[r, "p_p", "\sim"'] \arrow[d, "f_1"] & \mathrm{R}\Gamma_{\mathrm{rig}}(X_C/\mathscr{O}^{\times}_{F}) \arrow[d] \\
 & \mathrm{R}\Gamma_{\mathrm{rig}}(X_C/r^{\dagger}) \arrow[ur, "p_p"] \arrow[u, bend left=60, "p_0"] \arrow[d] & \mathrm{R}\Gamma_{\mathrm{conv}}(\widehat{X}_C/\mathscr{O}^{\times}_{F}) \arrow[d, "\wr"] \\
\mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}_C) \arrow[r, bend left, "s"] \arrow[rr, bend right, "\iota_{\mathrm{HK}}"'] & \mathrm{R}\Gamma_{\mathrm{PD}}(\widehat{X}_C) \arrow[l, "p_0"] \arrow[r, "p_p"] & \mathrm{R}\Gamma_{\mathrm{cr}}(\widehat{X}_C/\mathscr{O}^{\times}_{F}).
\end{tikzcd}$$

The notation should be mostly self-explanatory: the cohomology complexes are defined by the homotopy colimit and the étale descent from the corresponding complexes in the diagram (6.11) following the procedure used in Section 5.3.1. The groups in the right column are F^{nr}-modules.

If we remove the section s, the above diagram commutes. To prove that the diagram (6.10) commutes, by the diagram (6.2), it suffices to show that so does, on the level of classical cohomology, the large round triangle,[31] in the diagram (6.12). For that, we note that we have the isomorphism

(6.13)
$$s\colon H^i_{\mathrm{HK}}(\widehat{X}_C)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\, r^{\mathrm{PD}}_{\overline{K},\mathbf{Q}_p} \xrightarrow{\sim} H^i_{\mathrm{PD}}(\widehat{X}_C).$$

---

[31] That is, the round triangle with vertices $\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)$, $\mathrm{R}\Gamma_{\mathrm{HK}}(\widehat{X}_C)$, and $\mathrm{R}\Gamma_{\mathrm{PD}}(\widehat{X}_C)$.

If $\widehat{X}$ has a quasi-compact semi-stable formal model $\mathscr{X}$ over $\mathscr{O}_L$, this arises from the p^N-quasi-isomorphism, $N=N(d)$, (see (4.8))

$$s\colon \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^0_{F_L})\,\widehat{\otimes}_{\mathscr{O}_{F_L}}\, r_L^{\mathrm{PD}}\to \mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/r_L^{\mathrm{PD}})$$

and the fact that $\mathrm{R}\Gamma_{\mathrm{cr}}(\mathscr{X}_0/\mathscr{O}^0_{F_L})\,\widehat{\otimes}_{\mathscr{O}_{F_L}}\, r_L^{\mathrm{PD}}$ is p-adically derived complete and $r_{L,n}^{\mathrm{PD}}$ is free over $\mathscr{O}_{F_L,n}$. For a general quasi-compact and smooth $\widehat{X}$ over K, the above argument goes through yielding the isomorphism (6.13), as wanted.

Now, to show that the round triangle in the diagram (6.12) commutes, consider the ideal

$$I_n:=\Big\{\sum_{i\geq p^n}\tfrac{a_i}{\lfloor i/e\rfloor !}T^i,\ \lim_{i\mapsto\infty}a_i=0\Big\}.$$

We have the exact sequence

$$0\to I_0\to r^{\mathrm{PD}}_{K,\mathbf{Q}_p}\to F^{\mathrm{nr}}\to 0.$$

The F^{nr}-linear and Frobenius equivariant section $s\colon H^i_{\mathrm{HK}}(\widehat{X}_C)\to H^i_{\mathrm{PD}}(\widehat{X}_C)$ of the projection p_0 satisfies

$$s(a)=\varphi^n\widetilde{\varphi^{-n}(a)}\mod H^i_{\mathrm{HK}}(\widehat{X}_C)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,I_n,\quad a\in H^i_{\mathrm{HK}}(\widehat{X}),n\geq 0,$$

where $\widetilde{b}$, for $b\in H^i_{\mathrm{HK}}(\widehat{X}_C)$, is a lifting of b via p_0. This is because, for any $a\in H^i_{\mathrm{HK}}(\widehat{X}_C)$, we have $s(a)=\varphi^n s(\varphi^{-n}(a))$ and $s(a)=\varphi^n\widetilde{\varphi^{-n}(a)}$ mod $H^i_{\mathrm{HK}}(\widehat{X}_C)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,I_0$. And we also have $\varphi^n(I_0)\subset I_n$.

Hence, to show that the round triangle in the diagram (6.12) commutes, it suffices to show that the intersection of the submodules $H^i_{\mathrm{HK}}(\widehat{X}_C)\,\widehat{\otimes}_{F^{\mathrm{nr}}}\,I_n$, $n\geq 0$, is trivial. But this is clear. ☐

6.3. **Overconvergent syntomic cohomology via presentations of dagger structures.** In this section we introduce a definition of overconvergent syntomic cohomology using presentations of dagger structures (see [42, Appendix], Section 3.2.1). We show that so defined syntomic cohomology, a priori different from the one defined in Section 5.4, is strictly quasi-isomorphic to it.

(i) *Local definition.* Let X be a dagger affinoid over $L=K,C$. Let $\mathrm{pres}(X)=\{X_h\}$. Define

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X,\mathbf{Q}_p(r)):=\mathrm{hocolim}_h\,\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,\mathbf{Q}_p(r)),\quad r\in\mathbf{N}.$$

Let $L=K$. We have a natural map

$$\iota^{\dagger}_{\mathrm{syn}}\colon \mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\to \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r))\tag{6.14}$$

defined as the composition

$$\begin{aligned}\mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X,\mathbf{Q}_p(r))&=\mathrm{hocolim}_h\,\mathrm{R}\Gamma_{\mathrm{syn}}(X_h,\mathbf{Q}_p(r))\\ &\xrightarrow{\sim}\mathrm{hocolim}_h\,\mathrm{R}\Gamma_{\mathrm{syn}}(X^0_h,\mathbf{Q}_p(r))\\ &\xleftarrow{\sim}\mathrm{hocolim}_h\,\mathrm{R}\Gamma_{\mathrm{syn}}(X^{0,\dagger}_h,\mathbf{Q}_p(r))\to \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)).\end{aligned}\tag{6.15}$$

The third quasi-isomorphism holds by Theorem 6.3 because X^0_h is partially proper.

(ii) *Globalization.* For a general smooth dagger variety X over L, using the natural equivalence of analytic topoi

$$\mathrm{Sh}(\mathrm{SmAff}^{\dagger}_{L,\mathrm{\acute{e}t}}) \xrightarrow{\sim} \mathrm{Sh}(\mathrm{Sm}^{\dagger}_{L,\mathrm{\acute{e}t}}),$$

we define the sheaf $\mathscr{A}^{\dagger}_{\mathrm{syn}}(r)$, $r \in \mathbf{N}$, on $X_{\mathrm{\acute{e}t}}$ as the sheaf associated to the presheaf defined by $U \mapsto \mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(U, \mathbf{Q}_p(r))$, $U \in \mathrm{SmAff}^{\dagger}_{L}$, $U \to X$ an étale map. We define[32]

$$\mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) := \mathrm{R}\Gamma_{\mathrm{\acute{e}t}}(X, \mathscr{A}^{\dagger}_{\mathrm{syn}}(r)), \quad r \in \mathbf{N}.$$

Globalizing the map $\iota^{\dagger}_{\mathrm{syn}}$ from (6.14), we obtain a natural map

$$\iota^{\dagger}_{\mathrm{syn}} \colon \mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)).$$

(iii) *A comparison quasi-isomorphism.*

Proposition 6.16. *The above map $\iota^{\dagger}_{\mathrm{syn}}$ is a strict quasi-isomorphism.*

Proof. By étale descent, we may assume that X is a smooth dagger affinoid. Looking at the composition (6.15) defining the map $\iota^{\dagger}_{\mathrm{syn}}$, we see that it suffices to show that the natural map

$$\mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{syn}}(X_h^{\mathrm{o},\dagger}, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r))$$

is a strict quasi-isomorphism. Or, from the definitions of both sides, that we have strict quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X) \xleftarrow{\sim} \mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{HK}}(X_h^{\mathrm{o},\dagger}), \quad \mathrm{R}\Gamma_{\mathrm{dR}}(X) \xleftarrow{\sim} \mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{dR}}(X_h^{\mathrm{o},\dagger}).$$

This is clear in the case of the second map, since this map factors as

$$\mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{dR}}(X_h^{\mathrm{o},\dagger}) \xrightarrow{\sim} \mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{dR}}(X_{h+1}) \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(X).$$

For the first map consider the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK}}(X) \arrow[d, "\wr"] & \mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{HK}}(X_h^{\mathrm{o},\dagger}) \arrow[l] \arrow[d, "\wr"] & \\
\mathrm{R}\Gamma_{\mathrm{HK}}(X_C)^{\mathscr{G}_K} & \mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{HK}}(X_{h,C}^{\mathrm{o},\dagger})^{\mathscr{G}_K} \arrow[l] \arrow[r, "\sim"] & (\mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{HK}}(X_{h,C}^{\mathrm{o},\dagger}))^{\mathscr{G}_K}.
\end{tikzcd}$$

Here the vertical maps are strict quasi-isomorphisms by Proposition 5.13. The horizontal map is a strict quasi-isomorphism because the prosystems $\{\mathrm{R}\Gamma_{\mathrm{HK}}(X_{h,C}^{\mathrm{o},\dagger})\}$ and $\{\mathrm{R}\Gamma_{\mathrm{HK}}(X_{h,C})\}$ are equivalent and the action of $\mathscr{G}_K$ on the terms of the last one is smooth. It suffices thus to show that the natural map

$$\mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \leftarrow \mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{HK}}(X_{h,C}^{\mathrm{o},\dagger})$$

---

[32] We will show below (see Remark 6.17) that this definition of $\mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X, \mathbf{Q}_p(r))$, for a smooth dagger affinoid X, gives an object naturally strictly quasi-isomorphic to the one defined above.

is a strict quasi-isomorphism. For that consider the following diagram:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \arrow[r] \arrow[d, shift left] & \mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{HK}}(X^{\mathrm{o},\dagger}_{h,C}) \arrow[d, shift left] \\
\mathrm{R}\Gamma_{\mathrm{HK}}(X_C) \widehat{\otimes}_{F^{\mathrm{nr}}} \overline{K} \arrow[u, bend left, "\alpha"] \arrow[d, "\iota_{\mathrm{HK}}"', "\wr"] & \mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{HK}}(X^{\mathrm{o},\dagger}_{h,C}) \widehat{\otimes}_{F^{\mathrm{nr}}} \overline{K} \arrow[l, "f_1"'] \arrow[u, bend left, "\mathrm{hocolim}_h \, \alpha_h"] \arrow[d, "\iota_{\mathrm{HK}}"', "\wr"] \\
\mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X_C) & \mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{rig},\overline{K}}(X^{\mathrm{o},\dagger}_{h,C}) \arrow[l, "f_2"'] \\
\mathrm{R}\Gamma_{\mathrm{dR}}(X) \widehat{\otimes}_K \overline{K} \arrow[u, "\beta"', "\wr"] & \mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{dR}}(X^{\mathrm{o},\dagger}_{h}) \widehat{\otimes}_K \overline{K} \arrow[l, "f_3"'] \arrow[u, "\mathrm{hocolim}_h \, \beta_h"', "\wr"] \\
(\mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{dR}}(X_h)) \otimes_K \overline{K} \arrow[u, "\wr"] \arrow[r, "\sim", "\gamma"'] & \mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{dR}}(X_h) \otimes_K \overline{K}. \arrow[u, "\wr"']
\end{tikzcd}$$

The maps α, α_h are left inverses of the canonical vertical maps (used already in the diagram (6.7)). The Hyodo–Kato morphisms are the ones from (5.16); they are strict quasi-isomorphisms. The maps β, β_h are those from Proposition 5.19; they are strict quasi-isomorphisms as well. The diagram clearly commutes. The strict quasi-isomorphism γ uses the fact that X_h is quasi-compact. It follows that the map f_3 is a quasi-isomorphism and then that so is the map f_1 and, finally, that so is the top horizontal map, as wanted. □

Remark 6.17. The above proof shows that, for a smooth dagger affinoid X over K with a dagger presentation $\{X_h\}$, the natural map

$$\mathrm{hocolim}_h \, \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\text{ét}}(X, \mathscr{A}^{\dagger}_{\mathrm{syn}}(r))$$

is a strict quasi-isomorphism. Hence the two definitions of $\mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X, \mathbf{Q}_p(r))$ that we gave above coincide.

7. ARITHMETIC p-ADIC PRO-ÉTALE COHOMOLOGY

We pass now to the computation of arithmetic p-adic pro-étale cohomology of smooth dagger and rigid analytic varieties.

7.1. **Syntomic period isomorphisms.** First, we will use the comparison theorem between syntomic complexes and p-adic nearby cycles from [10] to define period maps for smooth rigid analytic and dagger varieties.

Let $\mathscr{X}$ be a semi-stable formal model over $\mathscr{O}_K$. Recall that Fontaine–Messing [19] and Kato [26] have constructed period morphisms

$$\alpha^{\mathrm{FM}}_{r,n} \colon \mathscr{S}_n(r)_{\mathscr{X}} \to i^* Rj_* \mathbf{Z}/p^n(r)'_{\mathscr{X}_K}, \quad r \geq 0$$

($i \colon \mathscr{X}_0 \hookrightarrow \mathscr{X}, j \colon \mathscr{X}_K \hookrightarrow \mathscr{X}$), from syntomic cohomology to p-adic nearby cycles taken as complexes of sheaves on the étale site of $\mathscr{X}_0$. Here we set $\mathbf{Z}_p(r)' := \frac{1}{p^{a(r)}} \mathbf{Z}_p(r)$, for $r = (p-1)a(r) + b(r)$, $0 \leq b(r) \leq p-1$. The syntomic sheaf $\mathscr{S}_n(r)$ is associated to the presheaf $\mathscr{U} \mapsto \mathrm{R}\Gamma_{\mathrm{syn}}(\mathscr{U}, \mathbf{Z}/p^n(r))$, for formally étale $\mathscr{U} \to \mathscr{X}$.

Recall the following comparison result.

Theorem 7.1 ([10, Th. 1.1]). *For $0 \leq i \leq r$, consider the period map*

$$\alpha_{r,n}^{\mathrm{FM}} \colon \mathscr{H}^i(\mathscr{S}_n(r)_{\mathscr{X}}) \to i^* R^i j_* \mathbf{Z}/p^n(r)'_{\mathscr{X}_K}. \tag{7.2}$$

(i) *If K has enough roots of unity,*[33] *then the kernel and cokernel of this map are annihilated by p^{Nr+c_p} for a universal constant N (not depending on p, $\mathscr{X}$, K, n or r) and a constant c_p depending only on p (and d if $p=2$).*

(ii) *In general, the kernel and cokernel of this map are annihilated by p^N for an integer $N = N(e,p,r)$, which depends on e, r, but not on $\mathscr{X}$ or n.*

7.1.1. *Rigid analytic varieties.* The above comparison quasi-isomorphism globalizes easily to smooth rigid analytic varieties:

Corollary 7.3. *For $X \in \mathrm{Sm}_L$, $L = K, C$, the period maps*

$$\begin{aligned} \alpha_r \colon \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Z}_p(r))_{\mathbf{Q}_p} &\to \mathrm{R}\Gamma_{\text{ét}}(X, \mathbf{Q}_p(r)),\\ \alpha_r \colon \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) &\to \mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)) \end{aligned}$$

are strict quasi-isomorphisms after truncation $\tau_{\leq r}$.

Proof. Since both the domain and the target of the period maps satisfy η-étale descent, we may assume that X has a semi-stable model over $\mathscr{O}_K$. But in that case this follows from Theorem 7.1 as in analogous claims in the geometric setting in [9, Prop. 6.1, Cor. 3.46]. ☐

7.1.2. *Dagger varieties.* The comparison quasi-isomorphism (7.2) can also be extended to smooth dagger varieties. Let $X \in \mathrm{Sm}_K^{\dagger}$, $r \geq 0$. Define the period map

$$\alpha_r \colon \mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)) \tag{7.4}$$

as the composition

$$\mathrm{R}\Gamma_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \xleftarrow{\sim} \mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) \xrightarrow{\alpha_r^{\dagger}} \mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)),$$

where the first map is the map $\iota^{\dagger}_{\mathrm{syn}}$ from Proposition 6.16 and the second map is defined by globalizing the following map defined for X a dagger affinoid with presentation $\{X_h\}$:

$$\begin{tikzcd} \mathrm{R}\Gamma^{\dagger}_{\mathrm{syn}}(X, \mathbf{Q}_p(r)) = \operatorname{hocolim}_h \mathrm{R}\Gamma_{\mathrm{syn}}(X_h, \mathbf{Q}_p(r)) \arrow[d, "\alpha_r"] \\ \operatorname{hocolim}_h \mathrm{R}\Gamma_{\text{proét}}(X_h, \mathbf{Q}_p(r)) \simeq \mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(r)). \end{tikzcd}$$

Corollary 7.3 implies immediately the following result.

---

[33] See [10, Section 2.2.1] for what it means for a field to contain enough roots of unity. For any K, the field $K(\zeta_{p^n})$, for $n \geq c(K) + 3$, where $c(K)$ is the conductor of K, contains enough roots of unity.

Corollary 7.5. *For $X \in \mathrm{Sm}_K^\dagger$, the period map*

$$\alpha_r\colon \mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) \to \mathrm{R}\Gamma_{\text{proét}}(X,\mathbf{Q}_p(r))$$

is a strict quasi-isomorphism after truncation $\tau_{\leq r}$.

Remark 7.6. Let X be a smooth partially proper dagger variety over K. We claim that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{syn}}(X,\mathbf{Q}_p(r)) \arrow[r, "\alpha_r"] \arrow[d, "\iota"', "\wr"] & \mathrm{R}\Gamma_{\text{proét}}(X,\mathbf{Q}_p(r)) \arrow[d, "\iota_{\text{proét}}"', "\wr"] \\
\mathrm{R}\Gamma_{\mathrm{syn}}(\widehat{X},\mathbf{Q}_p(r)) \arrow[r, "\widehat{\alpha}_r"] & \mathrm{R}\Gamma_{\text{proét}}(\widehat{X},\mathbf{Q}_p(r)).
\end{tikzcd}$$

The map ι is the strict quasi-isomorphism from Theorem 6.3; the map $\iota_{\text{proét}}$ is the strict quasi-isomorphism from Proposition 3.16. The period maps $\widehat{\alpha}_r$, α_r are the ones defined above (we put hat above the rigid analytic period map to distinguish it from the dagger period map).

It suffices to show that this diagram naturally commutes étale locally. So we may assume that X is a smooth dagger affinoid. Then checking commutativity is straight-forward from the definitions (if tedious).

7.2. **Applications and Examples.**

We are now ready to list some applications of our computations and to discuss some examples of computations of p-adic pro-étale cohomology.

7.2.1. *Rigid analytic varieties.* We start with the rigid analytic case. Let $X \in \mathrm{Sm}_K$, $r \geq 0$. The distinguished triangle (4.2), Lemma 4.5, and the period map α_r above yield a natural map

$$\partial_r\colon (\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)[-1] \to \mathrm{R}\Gamma_{\text{proét}}(X,\mathbf{Q}_p(r)).$$

Theorem 7.7. *Let $X \in \mathrm{Sm}_K$, $r \geq 1$.*

(1) *For $1 \leq i \leq r-1$, the map*

$$\partial_r\colon \widetilde{H}^{i-1}_{\mathrm{dR}}(X) \to \widetilde{H}^i_{\text{proét}}(X,\mathbf{Q}_p(r))$$

is an isomorphism. In particular, the cohomology $\widetilde{H}^i_{\text{proét}}(X,\mathbf{Q}_p(r))$ is not, in general, classical.

(2) *We have the short exact sequence*

$$0 \to \widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r) \xrightarrow{\partial_r} \widetilde{H}^r_{\text{proét}}(X,\mathbf{Q}_p(r)) \\ \to \widetilde{H}^r([\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r}) \to \widetilde{H}^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r).$$

Proof. Corollary 7.3 allows us to pass (by the period map) to syntomic cohomology for which we have an analogous claim, with $\widetilde{H}^r_{\text{ét}}(X,\mathscr{A}^{\varphi=p^r}_{\mathrm{cr},\mathbf{Q}_p})$ in place of $\widetilde{H}^r([\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r})$, by Corollary 4.7. That the latter two are isomorphic follows from diagram (4.31). $\square$

7.2.2. *Dagger varieties.* Now we pass to the overconvergent case.

Let $X \in \mathrm{Sm}_K^\dagger$, $r \geq 0$. The distinguished triangle (5.22) and the period map α_r from (7.4) yield a natural map

$$\partial_r \colon (\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)[-1] \to \mathrm{R}\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)).$$

Theorem 7.8. *Let $X \in \mathrm{Sm}_K^\dagger$, $r \geq 1$.*

(1) *For $1 \leq i \leq r-1$, the map*

$$\partial_r \colon \widetilde{H}^{i-1}_{\mathrm{dR}}(X) \to \widetilde{H}^i_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r))$$

is an isomorphism. In particular, the cohomology $\widetilde{H}^i_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r))$ is classical.

(2) *We have the long exact sequence*

$$\begin{aligned} 0 \to \widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r) &\xrightarrow{\partial_r} \widetilde{H}^r_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \\ &\to \widetilde{H}^r([\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r}) \xrightarrow{\iota_{\mathrm{HK}}} \widetilde{H}^r(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r). \end{aligned}$$

Proof. For $i \leq r$, from the definition of syntomic cohomology and Corollary 7.5, we get the long exact sequence

$$\begin{aligned} \cdots \to \widetilde{H}^{i-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r) &\to \widetilde{H}^i_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \\ &\to \widetilde{H}^i([\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r}) \to \widetilde{H}^i(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r) \to \cdots. \end{aligned}$$

For the first claim of the theorem, it suffices to show that, for $i \leq r-1$, $\widetilde{H}^i([\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r}) = 0$ and $\widetilde{H}^{i-1}_{\mathrm{dR}}(X) \xrightarrow{\sim} \widetilde{H}^{i-1}(\mathrm{R}\Gamma_{\mathrm{dR}}(X)/F^r)$. The second isomorphism is clear and the first one follows from Proposition 5.9.

For the second claim of the theorem, we note that the injectivity on the left is implied by the fact that $\widetilde{H}^{r-1}([\mathrm{R}\Gamma_{\mathrm{HK}}(X)]^{N=0,\varphi=p^r}) = 0$ (see Proposition 5.9). The proof is complete. ☐

7.2.3. *Overconvergent balls.* Let X be the overconvergent open or closed ball over K of dimension $d \geq 0$ and radius $\rho \in \sqrt{|K^\times|}$. Using Corollary 7.5 and Example 5.4.1, we get

$$H^r_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(r)) \simeq \begin{cases} \mathbf{Q}_p & \text{if } r = 0, \\ \Omega^{r-1}(X)/\ker d_{r-1} \simeq \Omega^r(X)^{d=0} & \text{if } r \geq 2, \end{cases}$$

and, for $r = 1$, we get a strict exact sequence

$$0 \to \mathscr{O}(X) \to H^1_{\mathrm{pro\acute{e}t}}(X, \mathbf{Q}_p(1)) \to \mathbf{Q}_p \to 0.$$

For comparison, recall that, for the geometric pro-étale cohomology, we have a topological isomorphism [11]

$$\Omega^{r-1}(X_C)/\ker d_{r-1} \xrightarrow{\sim} H^r_{\mathrm{pro\acute{e}t}}(X_C, \mathbf{Q}_p(r)), \quad r \geq 1.$$

7.2.4. *Proper smooth rigid analytic varieties.* Let X be a proper smooth dagger variety over K (recall that every smooth proper rigid analytic variety over K has a canonical dagger structure). For $r \geq 1$, Theorem 7.8 and Section 4.1.1 imply that the cohomology $H^i_{\text{proét}}(X, \mathbf{Q}_p(r))$ is classical for $i \leq r$ and, since $H^i_{\text{proét}}(X, \mathbf{Q}_p(r)) \simeq H^i_{\text{ét}}(X, \mathbf{Q}_p(r))$ because X is proper, we have

$$H^{i-1}_{\text{dR}}(X) \simeq H^i_{\text{ét}}(X, \mathbf{Q}_p(r)), \quad 1 \leq i \leq r-1,$$

and we have a strict exact sequence

$$0 \to H^{r-1}_{\text{dR}}(X) \to H^r_{\text{ét}}(X, \mathbf{Q}_p(r)) \to E(r) \to 0,$$

where $E(r)$ is an extension

$$0 \to H^{r-1}_{\text{HK}}(X)^{\varphi = p^{r-1}} \to E(r) \to H^r_{\text{HK}}(X)^{N=0, \varphi = p^r} \cap \Omega^r(X) \to 0.$$

7.2.5. *The Drinfeld half-space.* Let $d \geq 1$ and let $\mathbb{H}^d_K$ be the Drinfeld half-space of dimension d, i.e.,

$$\mathbb{H}^d_K := \mathbb{P}^d_K \setminus \bigcup_{H \in \mathscr{H}} H,$$

where $\mathscr{H}$ denotes the set of K-rational hyperplanes. We set $G := \mathrm{GL}_{d+1}(K)$. For $1 \leq r \leq d$, denote by $\mathrm{Sp}_r(\mathbf{Q}_p)$ the generalized locally constant Steinberg $\mathbf{Q}_p$-representation of G equipped with a trivial action of $\mathscr{G}_K$ (for a definition see [9, Section 5.2.1]).

Corollary 7.9. (1) *For $0 \leq i \leq r$, the cohomology $\widetilde{H}^i_{\text{proét}}(\mathbb{H}^d_K, \mathbf{Q}_p(r))$ is classical.*

(2) *For $i \leq r-1$, there is a natural G-equivariant topological isomorphism*

$$H^i_{\text{proét}}(\mathbb{H}^d_K, \mathbf{Q}_p(r)) \simeq \mathrm{Sp}_{i-1}(K)^*.$$

(3) *We have a G-equivariant diagram of strict exact sequences*

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \mathrm{Sp}_{r-1}(\mathbf{Q}_p)^* \arrow[d] & \\
0 \arrow[r] & \Omega^{r-1}(\mathbb{H}^d_K)/\operatorname{im} d_{r-2} \arrow[r] & H^r_{\text{proét}}(\mathbb{H}^d_K, \mathbf{Q}_p(r)) \arrow[r] & \mathrm{E}(\mathbf{Q}_p) \arrow[r] \arrow[d] & 0. \\
 & & & \mathrm{Sp}_r(\mathbf{Q}_p)^* \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

Proof. Point (2) follows from Theorem 7.8 and the isomorphism $\widetilde{H}^i_{\text{dR}}(\mathbb{H}^d_K) \simeq \mathrm{Sp}_i(K)^*$ of Schneider–Stuhler [36].

For point (3), since $\mathbb{H}^d_K$ is Stein, by Section 4.1.1, we have

$$\widetilde{H}^{r-1}(\mathrm{R}\Gamma_{\text{dR}}(\mathbb{H}^d_K)/F^r) \simeq \Omega^{r-1}(\mathbb{H}^d_K)/\operatorname{im} d_{r-2}, \quad \widetilde{H}^r(\mathrm{R}\Gamma_{\text{dR}}(\mathbb{H}^d_K)/F^r) \simeq 0.$$

On the other hand, from (5.10), we get an exact sequence

$$\tag{7.10} \begin{aligned} 0 \to \widetilde{H}^{r-1}_{\mathrm{HK}}(\mathbb{H}^d_K)^{\varphi=p^{r-1}} &\to \widetilde{H}^r([\mathrm{R}\Gamma_{\mathrm{HK}}(\mathbb{H}^d_K)]^{N=0,\varphi=p^r}) \\ &\to \widetilde{H}^r_{\mathrm{HK}}(\mathbb{H}^d_K)^{N=0,\varphi=p^r} \to 0, \end{aligned}$$

where all the cohomologies are classical. But, by [9, Lem. 5.11], we have a G-equivariant isomorphism $\widetilde{H}^i_{\mathrm{HK}}(\mathbb{H}^d_K)^{\varphi=p^i} \simeq \mathrm{Sp}_i(\mathbf{Q}_p)^*$. Since the monodromy is trivial (see [9, Section 5.5]), (7.10) then yields an exact sequence

$$0 \to \mathrm{Sp}_{r-1}(\mathbf{Q}_p)^* \to \widetilde{H}^r([\mathrm{R}\Gamma_{\mathrm{HK}}(\mathbb{H}^d_K)]^{N=0,\varphi=p^r}) \to \mathrm{Sp}_r(\mathbf{Q}_p)^* \to 0.$$

Plugging the above computations into Theorem 7.8 and setting

$$E(\mathbf{Q}_p) := H^r([\mathrm{R}\Gamma_{\mathrm{HK}}(\mathbb{H}^d_K)]^{N=0,\varphi=p^r}),$$

we get point (2).

Point (1) follows now trivially from points (2) and (3). □

Remark 7.11. (1) We note that we have the strict exact sequence

$$0 \to H^{r-1}_{\mathrm{dR}}(\mathbb{H}^d_K) \to \Omega^{r-1}(\mathbb{H}^d_K)/\operatorname{im} d_{r-2} \xrightarrow{d_{r-1}} \Omega^r(\mathbb{H}^d_K)^{d=0} \to H^r_{\mathrm{dR}}(\mathbb{H}^d_K) \to 0$$

and that the two de Rham cohomology terms are topologically isomorphic to $\mathrm{Sp}_{r-1}(K)^*$ and $\mathrm{Sp}_r(K)^*$, respectively.

(2) It would be interesting to understand the computations in this example better. In particular, to describe the extensions of Steinberg representations that appear.

Remark 7.12. It is interesting to link the computation of the arithmetic cohomology $H^i_{\text{proét}}(\mathbb{H}^d_K, \mathbf{Q}_p(r))$ presented here to the computation of the geometric cohomology $H^i_{\text{proét}}(\mathbb{H}^d_C, \mathbf{Q}_p(r))$ done in [9, Th. 5.15]. The following argument would need to be made more precise but it shows that the two computations, the arithmetic and the geometric one, are compatible.

We have the Hochschild–Serre spectral sequence

$$\tag{7.13} H^n(\mathscr{G}_K, H^{i-n}_{\text{proét}}(\mathbb{H}^d_C, \mathbf{Q}_p(r))) \Longrightarrow H^i_{\text{proét}}(\mathbb{H}^d_K, \mathbf{Q}_p(r))$$

(Only $n = 0, 1, 2$ can possibly give a nonzero contribution.) Now, the exact sequence from [9, Th. 5.15] twisted by $(j-k)$, yields an exact sequence of $\mathscr{G}_K \times G$-modules

$$\begin{aligned} 0 \to C(j-k)\widehat{\otimes}_K(\Omega^{k-1}(\mathbb{H}^d_K)/\ker d_{k-1}) &\to H^k_{\text{proét}}(\mathbb{H}^d_C, \mathbf{Q}_p(j)) \\ &\to \mathrm{Sp}_k(\mathbf{Q}_p)^*(j-k) \to 0. \end{aligned}$$

Hence the computation of $H^n(\mathscr{G}_K, H^{i-n}_{\text{proét}}(\mathbb{H}_C, \mathbf{Q}_p(r)))$ will involve the groups $H^n(\mathscr{G}_K, \mathbf{Q}_p(r-i+n))$ and $H^n(\mathscr{G}_K, C(r-i+n))$.

Recall the following results of Tate and Bloch–Kato:

$$\tag{7.14}\begin{aligned}
&H^0(\mathscr{G}_K,\mathbf{Q}_p(j))\simeq\begin{cases}\mathbf{Q}_p & \text{if } j=0,\\ 0 & \text{if } j\geq 1,\end{cases}\\
&H^1(\mathscr{G}_K,\mathbf{Q}_p(j))\simeq\begin{cases}K\oplus\mathbf{Q}_p & \text{if } j=1,\\ K & \text{if } j\geq 2,\end{cases}\\
&H^2(\mathscr{G}_K,\mathbf{Q}_p(j))=0\quad\text{if } j\geq 2,\\
&H^0(\mathscr{G}_K,C(j))\simeq\begin{cases}K & \text{if } j=0,\\ 0 & \text{if } j\geq 1,\end{cases}\\
&H^1(\mathscr{G}_K,C(j))\simeq\begin{cases}K & \text{if } j=0,\\ 0 & \text{if } j\geq 1,\end{cases}\\
&H^2(\mathscr{G}_K,C(j))=0\quad\text{if } j\geq 0.
\end{aligned}$$

Using them, we see that the nonzero terms of the spectral sequence (7.13) contributing to $H^i_{\text{proét}}(\mathbb{H}^d_K,\mathbf{Q}_p(r))$, $i\leq r$, are the following: if $i=r$, we have an exact sequence

$$0\to\Omega^{i-1}(\mathbb{H}^d_K)/\ker d_{i-1}\to H^0(\mathscr{G}_K,H^i_{\text{proét}}(\mathbb{H}^d_C,\mathbf{Q}_p(r)))\to\mathrm{Sp}_i(\mathbf{Q}_p)^*\to 0,$$

and we have isomorphisms

$$\begin{aligned}
&H^1(\mathscr{G}_K,H^{i-1}_{\text{proét}}(\mathbb{H}_C,\mathbf{Q}_p(r)))\simeq(K\oplus\mathbf{Q}_p)\otimes_{\mathbf{Q}_p}\mathrm{Sp}_{i-1}(\mathbf{Q}_p)^* && \text{if } i=r,\\
&H^1(\mathscr{G}_K,H^{i-1}_{\text{proét}}(\mathbb{H}_C,\mathbf{Q}_p(r)))\simeq K\otimes_{\mathbf{Q}_p}\mathrm{Sp}_{i-1}(\mathbf{Q}_p)^*\simeq\mathrm{Sp}_{i-1}(K)^* && \text{if } i\leq r-1.
\end{aligned}$$

The above sequence is exact though (7.14) is not enough to ensure the surjectivity of the map $H^0(\mathscr{G}_K,H^i_{\text{proét}}(\mathbb{H}^d_C,\mathbf{Q}_p(r)))\to\mathrm{Sp}_i(\mathbf{Q}_p)^*$. It yields however the exact sequence

$$H^0(\mathscr{G}_K,H^i_{\text{proét}}(\mathbb{H}^d_C,\mathbf{Q}_p(r)))\to\mathrm{Sp}_i(\mathbf{Q}_p)^*\xrightarrow{\partial}\Omega^{i-1}(\mathbb{H}^d_K)/\ker d_{i-1}.$$

Now the boundary map ∂ is trivial by a representation theory argument: the map ∂ is continuous and G-equivariant, the G-smooth vectors are dense in $\mathrm{Sp}_i(\mathbf{Q}_p)^*$, but $\Omega^{i-1}(\mathbb{H}^d_K)/\ker d_{i-1}$ does not have any nonzero G-smooth elements, since it injects into $\Omega^i(\mathbb{H}^d_K)$.

Hence, for $0\leq i\leq r-1$, we get $H^i_{\text{proét}}(\mathbb{H}^d_K,\mathbf{Q}_p(r))\simeq\mathrm{Sp}_{i-1}(K)^*$ as in Corollary 7.9. For $i=r$, we get the diagram of exact sequences

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & & \Omega^{r-1}(\mathbb{H}^d_K)/\ker d_{r-1} \arrow[d] & \\
0 \arrow[r] & \begin{matrix}\mathrm{Sp}_{r-1}(K)^*\\ \oplus\\ \mathrm{Sp}_{r-1}(\mathbf{Q}_p)^*\end{matrix} \arrow[r] & H^r_{\text{proét}}(\mathbb{H}^d_K,\mathbf{Q}_p(r)) \arrow[r] & H^0(\mathscr{G}_K,H^r_{\text{proét}}(\mathbb{H}^d_C,\mathbf{Q}_p(r))) \arrow[r] \arrow[d] & 0. \\
 & & & \mathrm{Sp}_r(\mathbf{Q}_p)^* \arrow[d] & \\
 & & & 0 &
\end{tikzcd}$$

To compare this with Corollary 7.9, note that we have an exact sequence

$$0 \to H^{i-1}_{\mathrm{dR}}(\mathbb{H}^d_K) \to \Omega^{i-1}(\mathbb{H}^d_K)/\operatorname{im} d_{i-2} \to \Omega^{i-1}(\mathbb{H}^d_K)/\ker d_{i-1} \to 0$$

and the Schneider–Stuhler isomorphism

$$H^{i-1}_{\mathrm{dR}}(\mathbb{H}^d_K) \cong \mathrm{Sp}_{i-1}(K)^*.$$

Hence Corollary 7.9 and the above computation via Galois descent give us the same Jordan–Hölder components of $H^r_{\mathrm{pro\acute{e}t}}(\mathbb{H}^d_K, \mathbf{Q}_p(r))$ but they are put together in two different ways.

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