Westfälische
Wilhelms-Universität
Münster

Jonas Bräuer

Entropies of algebraic $\mathbb{Z}^d$-actions and K-theory

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Entropies of algebraic $\mathbb{Z}^d$-actions and K-theory

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Jonas Bräuer
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Dekan:	Prof. Dr. Christopher Deninger
Erster Gutachter:	Prof. Dr. Christopher Deninger
Zweiter Gutachter:	Prof. Dr. Siegfried Echterhoff
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Chapter 1

Introduction

Let Γ be a discrete countable group and let X be a compact abelian group. An algebraic Γ-action on X is a homomorphism

$$\alpha : \Gamma \to \mathrm{Aut}(X)$$

of Γ into the group Aut(X) of continuous automorphisms of X.

In this thesis, two important notions in the theory of algebraic Γ-action play a central role:

The notion of expansiveness is a dynamical property of the action α. The action α is called expansive if there exists an open neighbourhood $\mathcal{U}$ of the identity such that

$$\bigcap_{\gamma \in \Gamma} \alpha^{\gamma}(\mathcal{U}) = 0.$$

The second important notion is the notion of entropy which should be thought of a measure of the chaos of the action α. The topological entropy $h(\alpha) \in [0, \infty]$ of the action α on X can be defined under the assumption that the group Γ be finitely generated, discrete and amenable.

In [Den06],[DS07], the entropy of expansive actions has been studied for the following algebraic Γ-actions:

Let $f \in M_r(\mathbb{Z}\Gamma)$ be an $r \times r$-matrix with entries in the group ring $\mathbb{Z}\Gamma$. The quotient $(\mathbb{Z}\Gamma)^r/(\mathbb{Z}\Gamma)^r f$ is a discrete abelian group with left Γ-action by multiplication. The Pontrjagin dual

$$X_f := \widehat{(\mathbb{Z}\Gamma)^r/(\mathbb{Z}\Gamma)^r f} := \mathrm{Hom}_{cont}((\mathbb{Z}\Gamma)^r/(\mathbb{Z}\Gamma)^r f, \mathbb{T})$$

is a compact abelian group with a left Γ-action by continuous group automorphisms. Here, $\mathbb{T}$ denotes the 1-dimensional torus $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$.

For example, if $r = 1$ and $f = \sum a_\gamma \gamma \in \mathbb{Z}\Gamma$, then X_f can be identified with the set of all sequences $(x_{\gamma'}) \in (\mathbb{R}/\mathbb{Z})^\Gamma$ which satisfy the equation

$$\sum_{\gamma' \in \Gamma} x_{\gamma'} a_{\gamma^{-1}\gamma'} = 0 \text{ in } \mathbb{R}/\mathbb{Z} \text{ for all } \gamma \in \Gamma \, .$$

The left Γ-action on X_f is given by $\gamma(x_{\gamma'}) = (x_{\gamma^{-1}\gamma'})$.

Recall that a countable group Γ is residually finite if there exists a sequence Γ_n of normal subgroups with finite index whose intersection is trivial. We will write $\Gamma_n \to e$ for such a sequence.

The main result in [DS07] is:

Theorem 1.1 ([DS07], Theorem 1.1)**.** *Let Γ be a countable discrete amenable and residually finite group and f an element of $\mathbb{Z}\Gamma$. Then the action of Γ on X_f is expansive if and only if f is a unit in $L^1(\Gamma, \mathbb{R})$. In this case the entropy $h(X_f)$ of X_f is given by*

$$h(X_f) = \log \det_{\mathcal{N}\Gamma} f.$$

Let us explain Theorem 1.1. Firstly, the dynamical property of the usual Γ-action on X_f to be expansive is translated into the algebraic property of the element f to be invertible in the convolution algebra $L^1(\Gamma, \mathbb{R})$ of infinite formal sums $\sum_{\gamma \in \Gamma} x_\gamma \gamma$ with real numbers x_γ such that $\sum_{\gamma \in \Gamma} |x_\gamma| < \infty$. Secondly, it expresses the entropy $h(X_f)$ as the logarithm of the Fuglede-Kadison determinant $\det_{\mathcal{N}\Gamma} f$ of f. The Fuglede-Kadison determinant is a homomorphism

$$\det_{\mathcal{N}\Gamma} : (\mathcal{N}\Gamma)^* \to \mathbb{R}_{>0}$$

from the units of the von Neumann algebra $\mathcal{N}\Gamma \supset L^1(\Gamma, \mathbb{R}) \supset \mathbb{Z}\Gamma$ into the positive real numbers.

The proof of this theorem involves on the one hand a description of the entropy of X_f as a renormalized logarithmic growth rate of the number of Γ_n-fixed points, i.e. one has

$$h(X_f) = h_{per}(X_f) := \lim_{n \to \infty} \frac{1}{(\Gamma : \Gamma_n)} \log |\mathrm{Fix}_{\Gamma_n}(X_f)| \tag{1.1}$$

independently of the choice of a sequence $\Gamma_n \to e$. On the other hand, one has to show that $\det_{\mathcal{N}\Gamma} f$ is the limit of certain finite dimensional determinants and that the values of these finite dimensional determinants are given by $|\mathrm{Fix}_{\Gamma_n}(X_f)|^{(\Gamma:\Gamma_n)}$.

Formula (1.1) motivates the following definition of what we call periodic p-adic entropy:

Let Γ be a countable discrete residually finite group acting on a set X. Let $\log_p : \mathbb{Q}_p^* \to \mathbb{Z}_p$ be the branch of the p-adic logarithm normalized by $\log_p(p) = 0$. Then by definition we say that the p-adic entropy of the Γ-action on X with respect to the sequence $\Gamma_n \to e$ exists if the limit

$$h_{p,\Gamma_n} := \lim_{n\to\infty} \frac{1}{(\Gamma : \Gamma_n)} \log_p |\mathrm{Fix}_{\Gamma_n}(X)| \tag{1.2}$$

exists, where $\mathrm{Fix}_{\Gamma_n}(X)$ denotes the set of points in X which are fixed by Γ_n. Changing slightly the terminology used in [Den09], we say that the periodic p-adic entropy $h_{p,per}(X)$ of the Γ-action exists, if the limit in (1.2) exists independently of the sequence $\Gamma_n \to e$ and always has the same value.

The main result in [Den09] is the following

Theorem 1.2 ([Den09], Theorem 2)**.** *Assume that the residually finite group Γ is elementary amenable and torsion-free. Let f be an element of $\mathbb{Z}\Gamma$ which is a unit in $c_0(\Gamma)$. Then the periodic p-adic entropy $h_{p,per}(X_f)$ of the Γ-action on X_f exists and we have*

$$h_{p,per}(X_f) = \log_p \det{}_\Gamma f.$$

Let us point out the analogies to Theorem 1.1. In the p-adic case, the convolution algebra $L^1(\Gamma, \mathbb{R})$ is replaced by the p-adic Banach algebra $c_0(\Gamma) := \{x = \sum_\gamma x_\gamma \gamma : x_\gamma \in \mathbb{Q}_p, |x_\gamma|_p \to 0 \text{ as } \gamma \to \infty \text{ in } \Gamma\}$, i.e. $c_0(\Gamma)$ consists of all formal series over Γ whose coefficients in $\mathbb{Q}_p$ converge to 0. The algebraic property $f \in c_0(\Gamma)^*$ guarantees that the periodic p-adic entropy $h_{p,per}(X_f)$ exists. Its value is given by the value of f under the so-called p-adic Fuglede-Kadison determinant

$$\log_p \det{}_\Gamma : c_0(\Gamma)^* \to \mathbb{Q}_p$$

which serves as a p-adic replacement of the homomorphism

$$\log \det{}_{\mathcal{N}\Gamma} : L^1(\Gamma, \mathbb{R})^* \subset (\mathcal{N}\Gamma)^* \to \mathbb{R}.$$

Theorem 1.2 provides an answer to a question which is motivated from the theory of expansive $\mathbb{Z}^d$-actions:

Let $f \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_d^{\pm 1}] =: R_d$. If we look at the topological entropy $h(X_f)$ of the Γ-action on X_f then it is known that $h(X_f)$ is given by the logarithmic Mahler measure $m(f)$ of f

$$h(X_f) = m(f) := \int_{\mathbb{T}^d} \log |f(z)| d\mu(z).$$

Here μ is the normalised Haar measure on the d-torus $\mathbb{T}^d$. The $\mathbb{Z}^d$-action on X_f is expansive if and only if f does not vanish in any point of $\mathbb{T}^d$ which is exactly the case if f is a unit in $L^1(\mathbb{Z}^d, \mathbb{R})$.

In analogy to this situation one has the notion of the p-adic Mahler measure $m_p(f)$. The p-adic Mahler measure of the Laurent polynomial f is only defined if f does not vanish in any point of the p-adic d-torus $T_p^d = \{z \in \mathbb{C}_p^d : |z_i|_p = 1\}$. Then $m_p(f) \in \mathbb{Q}_p$ is defined by the convergent Shnirelman integral

$$m_p(f) = \int_{T_p^d} \log f(z) \frac{dz}{z} := \lim_{\substack{N\to\infty,\\ (N,p)=1}} \frac{1}{N^d} \sum_{\zeta \in \mu_N^d} \log f(\zeta).$$

It was asked in [BD99] if $m_p(f)$ has an interpretation as a p-adically valued entropy. The answer is the following variant of Theorem 1.2 for the case $\Gamma = \mathbb{Z}^d$:

Theorem 1.3 ([Den09], Theorem 1)**.** *Assume that $f \in R_d$ does not vanish in any point of the p-adic d-torus. Then the periodic p-adic entropy $h_{p,per}(X_f)$ of the $\mathbb{Z}^d$-action on X_f exists and we have*

$$h_{p,per}(X_f) = m_p(f).$$

Three main open problems concerning dynamical systems and the notion of periodic p-adic entropy were formulated in [Den09]:

(1) Is there a notion of p-adic expansiveness for Γ-actions on compact spaces X which for dynamical systems X_f with $f \in M_r(\mathbb{Z}\Gamma)$ is equivalent to the condition $f \in \mathrm{GL}_r(c_0(\Gamma))$?

(2) Is it then possible to define a notion of p-adic entropy for all p-adically expansive dynamical systems which coincides with the periodic p-adic entropy of dynamical systems X_f, $f \in M_r(\mathbb{Z}\Gamma) \cap \mathrm{GL}_r(c_0(\Gamma))$?

(3) Is there a dynamical criterion for the existence of the limit defining periodic p-adic entropy?

In this thesis, we give an answer to questions (1) and (2) for algebraic $\mathbb{Z}^d$-actions. We also point out some problems that occur when one tries to solve problem (3).

We choose an algebraic approach to problems (1) and (2). Via Pontrjagin duality, algebraic $\mathbb{Z}^d$-actions correspond to modules over the ring R_d and dynamical properties of the $\mathbb{Z}^d$-action are reflected in algebraic properties of the dual module.

There are several reasons that suggest this approach. As stated in Theorem 1.2, given $f \in R_d$, we already have an algebraic criterion for the existence of the periodic p-adic entropy of X_f. As important, in order to define the p-adic Fuglede-Kadison determinant

$$\log_p \det_\Gamma : c_0(\Gamma)^* \to \mathbb{Q}_p,$$

Deninger constructs a homomorphism

$$\log_p \det_\Gamma : K_1(c_0(\Gamma, \mathbb{Z}_p)) \to \mathbb{Q}_p \tag{1.3}$$

defined for a certain class of groups Γ including the groups $\mathbb{Z}^d, d \geq 1$. Here, $c_0(\Gamma, \mathbb{Z}_p) = \{x \in c_0(\Gamma) : \max_{\gamma \in \Gamma} |x_\gamma|_p \leq 1\}$ and $K_1(c_0(\Gamma, \mathbb{Z}_p))$ is the first algebraic K-group of $c_0(\Gamma, \mathbb{Z}_p)$. We will use the homomorphism (1.3) to define a notion of p-adic entropy.

Let us give an overview of the main results. Let α be an algebraic $\mathbb{Z}^d$-action on the compact abelian group X. We denote by M^X the corresponding Pontrjagin dual R_d-module. Let S_p be the multiplicative set $S_p = R_d \cap c_0(\mathbb{Z}^d)^*$. We define the algebraic $\mathbb{Z}^d$-action α to be p-adically expansive if its dual module M^X belongs to the category $\mathcal{M}_{S_p}(R_d)$ of finitely generated R_d-modules which are S_p-torsion. Using the localisation sequence of algebraic K-theory

$$K_1(R_d) \to K_1(R_d[S_p^{-1}]) \to K_0(\mathcal{M}_{S_p}(R_d)) \to K_0(R_d) \to K_0(R_d[S_p^{-1}]) \to 0,$$

we attach to every p-adically expansive $\mathbb{Z}^d$-action on X an element

$$cl_p(X) \in K_1(R_d[S_p^{-1}])/R_d^*.$$

We prove:

Theorem 1.4. *There is a homomorphism*

$$\log_p \det_{\mathbb{Z}^d} : K_1(R_d[S_p^{-1}])/R_d^* \to \mathbb{Q}_p$$

which is given by the bottom row of the following commutative diagram:

$$\begin{array}{ccccccc}
 & & K_1(c_0(\mathbb{Z}^d, \mathbb{Z}_p))/R_d^* & \longrightarrow & & \longrightarrow & \mathbb{Q}_p \\
 & & \downarrow & & & & \| \\
K_1(R_d[S_p^{-1}])/R_d^* & \longrightarrow & K_1(c_0(\mathbb{Z}^d))/R_d^* & \xrightarrow{\det} & c_0(\mathbb{Z}^d)^*/R_d^* & \xrightarrow{\log_p \det_{\mathbb{Z}^d}} & \mathbb{Q}_p.
\end{array}$$

This enables us to define the p-adic entropy $h_p(X)$ of a p-adically expansive $\mathbb{Z}^d$-action on X as $\log_p \det_{\mathbb{Z}^d}(cl_p(X))$. We then show:

Theorem 1.5. *Let $f \in M_r(R_d) \cap GL_r(c_0(\mathbb{Z}^d))$. Then the usual $\mathbb{Z}^d$-action on X_f is p-adically expansive and we have*

$$h_p(X_f) = \log_p \det_{\mathbb{Z}^d}(f).$$

In particular, the periodic p-adic entropy of X_f coincides with the p-adic entropy of X_f:

$$h_p(X_f) = h_{p,per}(X_f).$$

In Section 5 we apply this K-theoretic approach to the theory of expansive algebraic $\mathbb{Z}^d$-action.

Let S_∞ be the multiplicative set $S_\infty = R_d \cap L^1(\mathbb{Z}^d, \mathbb{R})^*$ and let $M_{S_\infty}(R_d)$ be the category of finitely generated R_d-modules which are S_∞-torsion. We show the following characterization of expansiveness:

Theorem 1.6 (Algebraic criterion of expansiveness)**.** *Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. Then α is expansive if and only if $M^X \in \mathcal{M}_{S_\infty}(R_d)$.*

For an expansive $\mathbb{Z}^d$-action on X we then define an element

$$cl_\infty(X) \in K_1(R_d[S_\infty^{-1}])/R_d^* = \mathrm{SK}_1(R_d[S_\infty^{-1}]) \oplus (R_d[S_\infty^{-1}])/R_d^*.$$

Using the Fuglede-Kadison determinant we define a homomorphism

$$\log \det_{\mathcal{N}\mathbb{Z}^d} : K_1(R_d[S_\infty^{-1}])/R_d^* \to \mathbb{R}.$$

Then we show:

Theorem 1.7. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on a compact abelian group X. Then the topological entropy of the action α on X is given by*

$$h(X) = \log \det_{\mathcal{N}\mathbb{Z}^d}(cl_\infty(X)).$$

The K-theoretic approach to expansive $\mathbb{Z}^d$-actions leads naturally to the study of the group $\mathrm{SK}_1(R_d[S_\infty^{-1}])$. We show that the Fuglede-Kadison determinant vanishes on $\mathrm{SK}_1(R_d[S_\infty^{-1}])$. But in Section 5.2, we show the following result using topological K-theory:

Theorem 1.8. *Let $d \geq 5$. Then $SK_1(R_d[S_\infty^{-1}]) \neq 0$.*

Using this result one can show that there exist expansive $\mathbb{Z}^d$-actions on X such that the SK_1-component of $cl_\infty(X)$ is non-trivial.

Let us give an overview of the individual chapters of the thesis.

Chapter 2 consists of a short review of the papers [DS07] and [Den09]. Here, we introduce the algebraic dynamical systems of type X_f and give a short treatment of entropy. We define the group von-Neumann algebra $\mathcal{N}\Gamma$ and introduce the Fuglede-Kadison determinant. We state the main results expressing the entropy $h(X_f)$ of X_f in the expansive case as the value $\log \det_{\mathcal{N}\Gamma} f$. The last part of Chapter 2 is concerned with the periodic p-adic entropy and the p-adic Fuglede-Kadison determinant.

In Chapter 3 we discuss algebraic $\mathbb{Z}^d$-actions. In this case, there is a great interplay between dynamics and commutative algebra which gives us a deeper understanding of these actions. Via Pontrjagin duality, algebraic $\mathbb{Z}^d$-actions and modules over the ring R_d correspond to each other and dynamical properties of a dynamical system X can translated into algebraic properties of its dual module M^X. This provides a number of examples of algebraic $\mathbb{Z}^d$-action with specified properties. We discuss the structure of expansive $\mathbb{Z}$-actions on compact connected abelian groups and also algebraic $\mathbb{Z}^d$-actions which come from rings R_S of S-integers of algebraic number fields. The last part of Chapter 3 is concerned with the entropy of algebraic $\mathbb{Z}^d$-actions and its connection with the Mahler measure.

Chapter 4 and Chapter 5 contain the main results of this thesis. First, we provide some background material on algebraic K-theory. We introduce the notion p-adic expansiveness (Section 4.2) and define p-adic entropy for p-adically expansive $\mathbb{Z}^d$-actions (Section 4.3). In Section 4.4 we apply the theory developed in 4.2 and 4.3 to p-adically expansive $\mathbb{Z}$-actions on compact connected abelian groups and to the $\mathbb{Z}^d$-actions coming from rings R_S of S-integers of algebraic number fields as introduced in Section 3.3.

Section 5.1 contains the proofs of Theorem 1.6 and Theorem 1.7. In Section 5.2, we prove Theorem 1.8 using topological K-theory.

In Chapter 6 we determine the periodic p-adic entropy of the action of the discrete Heisenberg group Γ on X_f for a certain class of elements $f \in \mathbb{Z}\Gamma \cap c_0(\Gamma)^*$.

In Chapter 7 we discuss some open questions and problems. In particular, we provide a short discussion of our solution of Questions (1) and (2) as well as some comments on Question (3). We give an example to illustrate that there are $\mathbb{Z}^d$-actions where the periodic p-adic entropy does not exist but which can be treated with our method. Moreover, we give an example of an algebraic $\mathbb{Z}$-action where the periodic p-adic entropy exists for trivial reasons but which is not p-adically expansive in our sense.

Finally, we discuss some properties of the p-adic Banach algebra $c_0(\Gamma)$.

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Chapter 2

Algebraic Γ-actions, entropy and periodic p-adic entropy

In Section 2.1 we introduce the notions of algebraic Γ-actions, expansiveness and entropy. To elements $f \in M_r(\mathbb{Z}\Gamma)$ in the ring of $r \times r$-matrices with coefficients in the integral group ring $\mathbb{Z}\Gamma$ we attach a compact abelian group denoted by X_f which carries a natural algebraic Γ-action. The algebraic Γ-actions of type X_f and their dynamical properties are the central subject of Chapter 2.

In Section 2.2 we introduce the Fuglede-Kadison determinant and explain its connection to the topological entropy of expansive Γ-actions on X_f.

In Section 2.3 we define periodic p-adic entropy and review the construction of the p-adic Fuglede-Kadison determinant. It is shown in [Den09] that for a certain class of algebraic Γ-actions of type X_f the periodic p-adic entropy exists. We state the main results how periodic p-adic entropy is related to the p-adic Fuglede-Kadison determinant.

2.1 Algebraic Γ-actions and expansiveness

Definition 2.1. *Let Γ be a countable discrete group and let X be a compact abelian group. An algebraic Γ-action on X is a homomorphism $\alpha : \gamma \mapsto \alpha^\gamma$ from Γ into the group $\mathrm{Aut}(X)$ of continuous automorphisms of X.*

Definition 2.2. *An algebraic Γ-action on a compact abelian group X is expansive if there an expansive neighborhood $\mathcal{U}$ of the identity in X, i.e. if there exists an open neighborhood $\mathcal{U}$ of the identity with*

$$\bigcap_{\gamma\in\Gamma} \alpha^\gamma(\mathcal{U}) = 0.$$

Remark 2.3. In general, a Γ-action by homeomorphisms on a compact metrizable space X is called expansive if there exists a metric d defining the topology and an $\varepsilon > 0$ such that for all $x \neq y \in X$ we have $d(\gamma x, \gamma y) \geq \varepsilon$ for some $\gamma \in \Gamma$. For algebraic Γ-actions, this is equivalent to the existence of an expansive neighborhood $\mathcal{U}$ of the identity.

Let Γ be a countable discrete group and let $\mathbb{Z}\Gamma$ be the integral group ring of Γ consisting of all finite formal sums $f = \sum_{\gamma\in\Gamma} a_\gamma \gamma$ with coefficients in $\mathbb{Z}$. There are the following operations on $\mathbb{Z}\Gamma$:

For an element $f \in \mathbb{Z}\Gamma$ we denote by L_f (resp. R_f) the left (resp. right) multiplication with f. Furthermore, the ring $\mathbb{Z}\Gamma$ is equipped with an anti-involution

$$* : \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma, \quad \sum_{\gamma\in\Gamma} a_\gamma \gamma \mapsto \sum_{\gamma\in\Gamma} a_{\gamma^{-1}} \gamma. \tag{2.1}$$

Anti-involution means that we have $(f^*)^* = f$ and $(fg)^* = g^* f^*$.

If we replace more generally the ring $\mathbb{Z}\Gamma$ by the ring $M_r(\mathbb{Z}\Gamma)$, $r \geq 1$, of $r \times r$-matrices with entries in $\mathbb{Z}\Gamma$, matrix multiplication from the right with $f \in M_r(\mathbb{Z}\Gamma)$ defines on operation, also denoted by R_f,

$$R_f : (\mathbb{Z}\Gamma)^r \to (\mathbb{Z}\Gamma)^r, \; g \mapsto gf. \tag{2.2}$$

Using the anti-involution defined on $\mathbb{Z}\Gamma$ we define an anti-involution $*$ on $M_r(\mathbb{Z}\Gamma)$ by

$$* : M_r(\mathbb{Z}\Gamma) \to M_r(\mathbb{Z}\Gamma), \; f = (f_{ij})_{1\leq i,j\leq r} \mapsto f^* = (f^*_{ji})_{1\leq i,j\leq r}. \tag{2.3}$$

Identifying $M_r(\mathbb{Z}\Gamma)$ with the ring $M_r(\mathbb{Z})[\Gamma]$ of finite formal sums over Γ with coefficients in $M_r(\mathbb{Z})$ and $(\mathbb{Z}\Gamma)^r$ with $\mathbb{Z}^r[\Gamma]$, the operations R_f and $*$ take the following form: For $f = \sum_{\gamma\in\Gamma} a_\gamma \gamma \in M_r(\mathbb{Z})[\Gamma]$ it is

$$R_f : \mathbb{Z}^r[\Gamma] \to \mathbb{Z}^r[\Gamma], \; \sum_{\gamma\in\Gamma} b_\gamma \gamma \mapsto \sum_{\gamma\in\Gamma} \Big(\sum_{\gamma'\in\Gamma} b_{\gamma'} a_{\gamma'^{-1}\gamma} \Big) \gamma, \tag{2.4}$$

and

$$* : M_r(\mathbb{Z})[\Gamma] \to M_r(\mathbb{Z})[\Gamma], \; f \mapsto f^* = \sum_{\gamma\in\Gamma} a^*_{\gamma^{-1}} \gamma, \tag{2.5}$$

where $a^*_\gamma = a^t_\gamma$ is the transpose of the matrix $a_\gamma \in M_r(\mathbb{Z})$.

Let us now introduce an important class of algebraic Γ-actions. Let Γ be a countable discrete group and consider the compact abelian group

$\mathrm{Map}(\Gamma, (\mathbb{R}/\mathbb{Z})^r)$ consisting of all maps from Γ to $(\mathbb{R}/\mathbb{Z})^r$ with point-wise addition. We write elements $x \in \mathrm{Map}(\Gamma, (\mathbb{R}/\mathbb{Z})^r)$ as $x = (x_\gamma)$, where $x_\gamma \in (\mathbb{R}/\mathbb{Z})^r$ denotes the value of x at $\gamma \in \Gamma$. There are natural left and right Γ-shift actions λ and ρ on $\mathrm{Map}(\Gamma, (\mathbb{R}/\mathbb{Z})^r)$ given by

$$(\lambda^\gamma x)_{\gamma'} = x_{\gamma^{-1}\gamma'} \quad \text{and} \quad (\rho^\gamma x)_{\gamma'} = x_{\gamma'\gamma}, \quad \gamma \in \Gamma, \tag{2.6}$$

respectively. If we identify $\mathrm{Map}(\Gamma, (\mathbb{R}/\mathbb{Z})^r)$ with the group $(\mathbb{R}/\mathbb{Z})^r[[\Gamma]]$ of infinite formal sums with coefficients in $(\mathbb{R}/\mathbb{Z})^r$ the actions λ and ρ correspond to left multiplication with $\gamma \in \Gamma$ respectively right multplication with γ^{-1}:

$$\lambda^\gamma\Big(\sum_{\gamma'\in\Gamma} x_{\gamma'}\gamma'\Big) = \sum_{\gamma'\in\Gamma} x_{\gamma'}\gamma\gamma' \ , \quad \rho^\gamma\Big(\sum_{\gamma'\in\Gamma} x_{\gamma'}\gamma'\Big) = \sum_{\gamma'\in\Gamma} x_{\gamma'}\gamma'\gamma^{-1}. \tag{2.7}$$

The compact abelian group $\mathrm{Map}(\Gamma, (\mathbb{R}/\mathbb{Z})^r)$ with the left action λ is our first basic example of an algebraic Γ-action.

In order to describe more general algebraic Γ-actions, we make use of duality theory of locally compact abelian groups: We view $(\mathbb{Z}\Gamma)^r$ as discrete abelian group. Then the Pontrjagin dual group

$$\widehat{(\mathbb{Z}\Gamma)^r} := \mathrm{Hom}_{cont}((\mathbb{Z}\Gamma)^r, \mathbb{R}/\mathbb{Z})$$

is a compact abelian group. Thus, evaluation gives a natural pairing, the so-called Pontrjagin pairing,

$$\langle\ ,\ \rangle : (\mathbb{Z}\Gamma)^r \times \widehat{(\mathbb{Z}\Gamma)^r} \to \mathbb{R}/\mathbb{Z},\ (a, \chi) \mapsto \langle a, \chi\rangle := \chi(a).$$

The general Pontrjagin Duality Theorem says that for an abelian locally compact group G the homomorphism

$$G \to \widehat{\widehat{G}},\ a \mapsto \langle a, \cdot\rangle,$$

is an isomorphism of topological groups. In particular, we deduce that for an exact sequence

$$0 \to G' \xrightarrow{\sigma} G \xrightarrow{\tau} G'' \to 0$$

of locally compact abelian groups the sequence

$$0 \to \widehat{G''} \xrightarrow{\hat\tau} \widehat{G} \xrightarrow{\hat\sigma} \widehat{G'} \to 0$$

is exact, i.e.

$$\widehat{\ker(G \to G'')} \simeq \mathrm{coker}\,(\widehat{G''} \to \widehat{G}) \text{ and } \widehat{\mathrm{coker}\,(G' \to G)} \simeq \ker(\widehat{G} \to \widehat{G'}). \tag{2.8}$$

See [RV99], Chapter 3, for more details.

Under the identifications $(\mathbb{Z}\Gamma)^r = \mathbb{Z}^r[\Gamma]$ and $\widehat{(\mathbb{Z}\Gamma)^r} = (\mathbb{R}/\mathbb{Z})^r[[\Gamma]]$, it is for $\sum_{\gamma\in\Gamma} a_\gamma\gamma \in \mathbb{Z}^r[\Gamma]$ and $\sum_{\gamma\in\Gamma} x_\gamma\gamma \in (\mathbb{R}/\mathbb{Z})^r[[\Gamma]]$

$$\left\langle \sum_{\gamma\in\Gamma} a_\gamma\gamma, \sum_{\gamma\in\Gamma} x_\gamma\gamma \right\rangle = \sum_{\gamma,1\leq i\leq r} a_{\gamma,i}x_{\gamma,i} \in \mathbb{R}/\mathbb{Z}. \tag{2.9}$$

We also have a right multiplication with elements $f = \sum_{\gamma\in\Gamma} a_\gamma\gamma \in M_r(\mathbb{Z})[\Gamma]$ on $(\mathbb{R}/\mathbb{Z})^r[[\Gamma]]$:

$$R_f : (\mathbb{R}/\mathbb{Z})^r[[\Gamma]] \to (\mathbb{R}/\mathbb{Z})^r[[\Gamma]],\ \sum_{\gamma\in\Gamma} x_\gamma\gamma \mapsto \sum_{\gamma\in\Gamma}\left(\sum_{\gamma'\in\Gamma} x_{\gamma'}a_{\gamma'^{-1}\gamma}\right)\gamma. \tag{2.10}$$

We define

$$\rho_f = R_{f^*} : (\mathbb{R}/\mathbb{Z})^r[[\Gamma]] \to (\mathbb{R}/\mathbb{Z})^r[[\Gamma]]. \tag{2.11}$$

Then ρ_f is just the linear extension of the Γ-action ρ on $(\mathbb{R}/\mathbb{Z})^r[[\Gamma]]$ to elements $f \in M_r(\mathbb{Z})[\Gamma]$. The following formula holds for all $a \in \mathbb{Z}^r[\Gamma], f \in M_r(\mathbb{Z})[\Gamma]$ and $x \in (\mathbb{R}/\mathbb{Z})^r[[\Gamma]]$.

$$\langle af, x\rangle = \langle a, xf^*\rangle. \tag{2.12}$$

To prove equation (2.12), it suffices to check it on elements of the form $a = e_i\gamma, f = e_{jk}\gamma'$ and $x = \gamma''$, where $\gamma, \gamma', \gamma'' \in \Gamma$, $e_i \in \mathbb{Z}^r$ the i-th canonical basisvector and $e_{jk} \in M_r(\mathbb{Z})$ the matrix with zero entries everywhere except an 1 in the jk-th entry.

According to equation (2.12) the Pontrjagin dual of right multiplication with f on $\mathbb{Z}^r[\Gamma]$ is right multiplication with f^* on $(\mathbb{R}/\mathbb{Z})^r[[\Gamma]]$. Hence by equation (2.8) we have

$$\widehat{\mathbb{Z}^r[\Gamma]/(\mathbb{Z}^r[\Gamma])f} = \ker\big(\rho_f : (\mathbb{R}/\mathbb{Z})^r[[\Gamma]] \to (\mathbb{R}/\mathbb{Z})^r[[\Gamma]]\big).$$

Furthermore, since left multiplication and right multiplication with elements of Γ on $(\mathbb{R}/\mathbb{Z})^r[[\Gamma]]$ commute, the natural left Γ-action λ passes to X_f.

Definition 2.4. *Let Γ be a countable discrete group and let f be an element in $M_r(\mathbb{Z}\Gamma)$. We define the dynamical system X_f to be the compact abelian group*

$$X_f := \widehat{\mathbb{Z}^r[\Gamma]/(\mathbb{Z}^r[\Gamma])f} = \ker\big(\rho_f : (\mathbb{R}/\mathbb{Z})^r[[\Gamma]] \to (\mathbb{R}/\mathbb{Z})^r[[\Gamma]]\big)$$

with the Γ-action $\alpha_f := \lambda_{|X_f}$.

Example 2.5. According to equation (2.10) $X_f \subset \mathrm{Map}(\Gamma, (\mathbb{R}/\mathbb{Z})^r)$ consists of all sequences $(x_\gamma)_{\gamma\in\Gamma}$ with

$$\sum_{\gamma'\in\Gamma} x_{\gamma'} a^*_{\gamma^{-1}\gamma'} = 0 \quad \text{for all } \gamma \in \Gamma.$$

For example, if $\Gamma = \mathbb{Z}$ and $f = 2t^2 - t + 2 \in \mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$, then it is

$$X_f = \left\{x = (x_n) \in (\mathbb{R}/\mathbb{Z})^{\mathbb{Z}} : 2x_n - x_{n+1} + 2x_{n+2} = 0 \text{ in } \mathbb{R}/\mathbb{Z} \text{ for all } n \in \mathbb{Z}\right\}.$$

In the following, we give a description of the group of fixed points of X_f, $f \in M_r(\mathbb{Z}\Gamma)$, under a normal subgroup N of Γ.

Definition 2.6. *Let Γ be a countable group, X a compact group and let α be a Γ-action by automorphisms of X. For a subgroup $\Gamma' \subset \Gamma$ the subgroup of Γ'-invariant points in X is defined by*

$$\mathrm{Fix}_{\Gamma'}(X) := \{x \in X : \alpha^\gamma x = x \textit{ for all } \gamma \in \Gamma'\}.$$

Note that $\mathrm{Fix}_{\Gamma'}(X)$ is Γ-invariant if Γ' is a normal subgroup of Γ.

Let X_f be the dynamical system attached to some $f \in M_r(\mathbb{Z}\Gamma)$ as defined above. In this case, the group of fixed points under a normal subgroup N of Γ has the following description:

Let $^-\colon \Gamma \to \bar{\Gamma} := \Gamma/N$ be the quotient map. We have the induced quotient map $M_r(\mathbb{Z})[\Gamma] \to M_r(\mathbb{Z})[\bar{\Gamma}]$ and we denote by $\bar{f}$ the image of f under this map. Consider the natural isomorphism

$$(\mathbb{R}/\mathbb{Z})^r[[\bar{\Gamma}]] \xrightarrow{\sim} \mathrm{Fix}_N((\mathbb{R}/\mathbb{Z})^r[[\Gamma]]),\ \sum_{\delta\in\bar{\Gamma}} x_\delta \delta \mapsto \sum_{\gamma\in\Gamma} x_{\bar{\gamma}} \gamma.$$

Under this isomorphism, the action $\rho_{\bar{f}}$ on $(\mathbb{R}/\mathbb{Z})^r[[\bar{\Gamma}]]$ corresponds to the restriction of ρ_f to $\mathrm{Fix}_N((\mathbb{R}/\mathbb{Z})^r[[\Gamma]])$. Hence we have

$$\mathrm{Fix}_N(X_f) = \ker(\rho_{\bar{f}} : (\mathbb{R}/\mathbb{Z})^r[[\bar{\Gamma}]] \to (\mathbb{R}/\mathbb{Z})^r[[\bar{\Gamma}]]) = X_{\bar{f}}.$$

If we assume $\bar{\Gamma}$ to be finite, it follows that

$$\mathrm{Fix}_N(X_f) = \rho_{\bar{f}}^{-1}(\mathbb{Z}\bar{\Gamma})^r/(\mathbb{Z}\bar{\Gamma})^r,$$

where $\rho_{\bar{f}}$ on the right-hand side of the equation denotes the endomorphism of right multiplication by $\bar{f}^*$ on $(\mathbb{R}\bar{\Gamma})^r$. If we assume furthermore that $\rho_{\bar{f}}$ is an isomorphism of $(\mathbb{Q}\bar{\Gamma})^r$, then the order of $\mathrm{Fix}_N(X_f)$ is given by the index of the sublattice $\rho_{\bar{f}}((\mathbb{Z}\bar{\Gamma})^r)$ of $(\mathbb{Z}\bar{\Gamma})^r$. By the elementary divisors theorem, this index is $\pm \det \rho_{\bar{f}}$. We conclude:

Proposition 2.7. *Let N be a cofinite normal subgroup of Γ. Put $\bar{\Gamma} := \Gamma/N$. For $f \in M_r(\mathbb{Z})[\Gamma]$ we denote by $\bar{f}$ the image of f in $M_r(\mathbb{Z})[\bar{\Gamma}]$ under the natural quotient map. Assume that the endomorphism $\rho_{\bar{f}}$ of right multiplication with $\bar{f}^*$ on $(\mathbb{Q}\bar{\Gamma})^r$ is an isomorphism of $(\mathbb{Q}\bar{\Gamma})^r$. Then $\mathrm{Fix}_N(X_f)$ is finite and its order is given by*

$$|\mathrm{Fix}_N(X_f)| = \pm \det \rho_{\bar{f}}.$$

We want to finish Section 2.1 with a brief introduction of the notion of topological entropy. For more information, see for example the short survey on entropy in [Den06].

Definition 2.8. *A finitely generated discrete group Γ is called amenable, if it has a right Følner sequence $(F_n)_{n\in\mathbb{N}}$, i.e. Γ has a sequence $F_1, F_2, \ldots$ of finite subsets of Γ such that for every finite subset K of Γ, it is*

$$\lim_{n\to\infty} \frac{|F_n K \Delta F_n|}{|F_n|} = 0,$$

where $F_n K \Delta F_n := (F_n K \cup F_n) \setminus (F_n K \cap F_n)$ denotes the symmetric difference of $F_n K$ and F_n.

Example 2.9. The groups $\mathbb{Z}^d, d \geq 1$, are amenable. For integers $b_i \in \mathbb{Z}$ and $r_i > 0, 1 \leq i \leq d$, define the rectangle

$$Q((b_i, r_i)_{1\leq i\leq d}) := \prod_{i=1}^{d} [b_i, b_i + r_i - 1]_{\mathbb{Z}} \subset \mathbb{Z}^d,$$

where $[b_i, b_i + r_i - 1]_{\mathbb{Z}} := [b_i, b_i + r_i - 1] \cap \mathbb{Z}$ is the interval $[b_i, b_i + r_i - 1]$ intersected with $\mathbb{Z}$. Then any sequence $Q((b_i^{(n)}, r_i^{(n)})_{1\leq i\leq d})_{n\in\mathbb{N}}$ of rectangles with

$$\lim_{n\to\infty} \min_{1\leq i\leq d} \{r_i^{(n)}\} \to \infty$$

is a right Følner sequence. Since the idea of the proof is the same for every $d \geq 1$, we only prove the case $d = 1$ in order to keep the notation simple.

Let K be a finite subset of $\mathbb{Z}$. Let d_1 be the smallest integer and d_2 the largest integer such that $K \subset [d_1, d_2]_{\mathbb{Z}}$. Let $Q = Q((b, r)) = [b, b + r - 1]_{\mathbb{Z}}$ any rectangle in $\mathbb{Z}$ such that

(i) $r - 1 \geq \max\{|d_1|, |d_2|\}$ and

(ii) $r - 1 \geq d_2 - d_1$.

There are the cases (a) $d_1 \geq 0$, (b) $d_1 < 0$ and $d_2 > 0$, (c) $d_2 \leq 0$. We treat the case $d_1 \geq 0$, the other cases go similarly. Condition (ii) implies that $QK = [b + d_1, b + d_2 + r - 1]_{\mathbb{Z}}$ for any non-empty subset K of $[d_1, d_2]_{\mathbb{Z}}$. Then using (i) we get

$$QK \cup Q = [b, \ldots, b + d_1, \ldots, b + r - 1, \ldots, b + d_2 + r - 1]_{\mathbb{Z}} \text{ and}$$
$$QK \cap Q = [b + d_1, \ldots, b + r - 1]_{\mathbb{Z}}.$$

Thus,

$$|QK\Delta K| = |QK \cup K| - |QK \cap K| = d_2 + r - (r - d_1) = d_1 + d_2$$

independently of Q. Now, for any sequence $(Q_n = Q((b_n, r_n)))_{n \in \mathbb{N}}$ with $r_n \to \infty$ the conditions (i) and (ii) will be satisfied for any finite $K \subset \mathbb{Z}$ if n is large enough. Hence

$$\lim_{n \to \infty} \frac{|Q_n K \Delta Q_n|}{|Q_n|} = 0,$$

i.e.$(Q_n)_{n \in \mathbb{N}}$ is a right Følner sequence.

Assume that the finitely generated discrete amenable group Γ operates from the left by homeomorphisms on a compact metric space (X, d). Let $(F_n)_{n \in \mathbb{N}}$ be a right Følner sequence. The topological entropy of the Γ-action on X is defined as follows:

For an open cover $\mathcal{U}$ of X let $N(\mathcal{U})$ be the cardinality of a minimal subcover of $\mathcal{U}$. For a finite subset F of Γ let

$$\mathcal{U}^F = \bigvee_{\gamma \in F} \gamma \mathcal{U}$$

be the common refinement of the finitely many covers $\gamma\mathcal{U}$. Using [LW00], Theorem 6.1, one sees that the limit

$$h(\mathcal{U}) = \lim_{n \to \infty} \frac{1}{|F_n|} \log N(\mathcal{U}^{F_n})$$

exists and is independent of the Følner sequence $(F_n)_{n \in \mathbb{N}}$.

Definition 2.10. *Let Γ be a finitely generated discrete amenable group which operates from the left by homeomorphisms on a compact metric space (X, d). The topological entropy of the Γ-action on X is defined to be the quantity*

$$h_{cover} := \sup_{\mathcal{U}} h(\mathcal{U}).$$

Before we state the next result, let us recall the definition of a residually finite group.

Definition 2.11. *The group* Γ *is called residually finite, if there exists a sequence* $(\Gamma_n)_{n\in\mathbb{N}}$ *of normal subgroups of* Γ *of finite index whose intersection contains only the neutral element* e*. In this case, we write* $\Gamma_n \to e$ *for such a sequence.*

The following theorem is a central result. It tells us that for a countable residually finite amenable group Γ, the entropy of the algebraic Γ-action on X_f, $f \in M_r(\mathbb{Z}\Gamma)$, can be expressed as a certain logarithmic growth rate of the number of fixed points under the assumption that the action is expansive.

Theorem 2.12. *Let* Γ *be a countable residually finite amenable group and let* $\Gamma_n \to e$ *be a sequence of cofinite normal subgroups of* Γ *converging to* e*. Let* $f \in M_r(\mathbb{Z}\Gamma)$ *and assume that the algebraic* Γ*-action* α_f *is expansive. Then*

$$h(\alpha_f) = \lim_{n\to\infty} \frac{1}{|\Gamma/\Gamma_n|} \log |\mathrm{Fix}_{\Gamma_n}(X_f)|.$$

Proof. [DS07], Theorem 5.7 and [Mül08], Theorem 3.4.7. □

2.2 Entropy and the Fuglede-Kadison determinant

Let Γ be a discrete group and let $L^2(\Gamma,\mathbb{C})$ be the Hilbert space of square summable complex valued functions $x : \Gamma \to \mathbb{C}$. The group Γ acts isometrically from the left by the operation

$$\Gamma \times L^2(\Gamma,\mathbb{C}) \to L^2(\Gamma,\mathbb{C}) \ , \quad (\gamma, x) \mapsto \gamma x,$$

where the value of γx at $\gamma' \in \Gamma$ is given by $(\gamma x)'_\gamma := x_{\gamma^{-1}\gamma'}$.

Elements in $L^2(\Gamma,\mathbb{C})$ can be represented as formal sums $\sum_{\gamma\in\Gamma} x_\gamma \gamma$ with complex numbers x_γ such that $\sum_{\gamma\in\Gamma} |x_\gamma|^2 < \infty$. If we write elements of $L^2(\Gamma,\mathbb{C})$ as formal sums $\sum_{\gamma\in\Gamma} x_\gamma\gamma$, the left Γ-action is given by left multiplication by γ.

For a Banach space H let $\mathcal{B}(H)$ be the algebra of bounded linear operators of H into itself.

Definition 2.13. *The group von Neumann algebra* $\mathcal{N}\Gamma$ *of* Γ *is the algebra of* Γ*-equivariant bounded linear operators of* $L^2(\Gamma,\mathbb{C})$ *into itself,*

$$\mathcal{N}\Gamma := \mathcal{B}(L^2(\Gamma,\mathbb{C}))^\Gamma.$$

Definition 2.14. *The von Neumann trace on* $\mathcal{N}\Gamma$ *is the linear form*

$$tr_{\mathcal{N}\Gamma} : \mathcal{N}\Gamma \to \mathbb{C}, \quad tr_{\mathcal{N}\Gamma}(g) = (g(e), e),$$

where $e \in \Gamma \subset L^2(\Gamma, \mathbb{C})$ *is the unit in* Γ. *Here,* $(g(e), e)$ *denotes the inner product of the elements* $g(e)$ *and* e.

For a matrix $A = (a_{ij})_{1\le i,j\le r} \in M_r(\mathcal{N}\Gamma)$ *the von Neumann trace is defined as*

$$tr_{\mathcal{N}\Gamma}(A) := \sum_{i=1}^{r} tr_{\mathcal{N}\Gamma}(a_{i,i}).$$

The group Γ acts isometrically from the right on $L^2(\Gamma, \mathbb{C})$ by $(x\gamma)_{\gamma'} = x_{\gamma'\gamma^{-1}}$. This corresponds to right multiplication with γ if we view elements of $L^2(\Gamma, \mathbb{C})$ as formal sums. For $\gamma \in \Gamma$ define the operator $R_\gamma \in \mathcal{B}(L^2(\Gamma, \mathbb{C}))$ by $R_\gamma(x) := x\gamma$. This operator is Γ-equivariant and so defines an element in $\mathcal{N}\Gamma$. Then $\mathbb{C}\Gamma$ is embedded in $\mathcal{N}\Gamma$ by the injective $\mathbb{C}$-algebra homomorphism

$$r : \mathbb{C}\Gamma \to \mathcal{N}\Gamma, \ \sum_{\gamma\in\Gamma} a_\gamma \gamma \mapsto \sum_{\gamma\in\Gamma} a_\gamma R_{\gamma^{-1}}. \tag{2.13}$$

The adjoint f^* of $f = \sum_\gamma a_\gamma \gamma \in \mathbb{C}\Gamma \subset \mathcal{N}\Gamma$ is given by $f^* = \sum_\gamma \bar{a}_\gamma \gamma^{-1}$. More generally, we define

$$r_{n,n} : M_n(\mathbb{C}\Gamma) \to M_n(\mathcal{N}\Gamma), \ (f_{ij})_{1\le i,j\le n} \mapsto (r(f_{ij}))_{1\le i,j\le n}, \tag{2.14}$$

and

$$* : M_n(\mathbb{C}\Gamma) \to M_n(\mathbb{C}\Gamma), \ (f_{ij})_{1\le i,j\le n} \mapsto ((f_{ji})^*)_{1\le i,j\le n}. \tag{2.15}$$

Then because of $R_\gamma^* = R_{\gamma^{-1}}$ it is

$$r_{n,n}(f^*) = r_{n,n}(f)^*.$$

The L^1-convolution algebra $L^1(\Gamma, \mathbb{C})$ of Γ is the completion of $\mathbb{C}\Gamma$ in the $|| \, ||_1$-norm. We write elements in $L^1(\Gamma, \mathbb{C})$ as infinite formal sums $\sum_{\gamma\in\Gamma} x_\gamma \gamma$ with complex numbers x_γ which satisfy $\sum_{\gamma\in\Gamma} |x_\gamma| < \infty$. Right multiplication with elements in $L^1(\Gamma, \mathbb{C})$ on $L^2(\Gamma, \mathbb{C})$ is continuous because of the estimate $||\varphi \cdot f||_2 \le ||f||_1 ||\varphi||_2$ for all $\varphi \in L^2(\Gamma, \mathbb{C})$ and $f \in L^1(\Gamma, \mathbb{C})$. Thus, we obtain a natural injection

$$r : L^1(\Gamma, \mathbb{C}) \to \mathcal{N}\Gamma \text{ with } ||r(f)|| \le ||f||_1 \tag{2.16}$$

which extends the map (2.13) above. Similarly, we get an injection

$$r_{n,n} : M_n(L^1(\Gamma, \mathbb{C})) \to M_n(\mathcal{N}\Gamma) \tag{2.17}$$

which extends the homomorphism (2.14). In particular, units in $M_n(L^1(\Gamma, \mathbb{C}))$ give units in $M_n(\mathcal{N}\Gamma)$.

Definition 2.15. *The Fuglede-Kadison determinant of an element $u \in GL_r(\mathcal{N}\Gamma)$ is defined to be the real number*

$$\det_{\mathcal{N}\Gamma}(u) := \exp\Big(\frac{1}{2} tr_{\mathcal{N}\Gamma}(\log uu^*)\Big).$$

Here, the operator uu^* is positive and $\log uu^*$ is defined via functional calculus in $\mathcal{B}(L^2(\Gamma, \mathbb{C})^r)$

An important fact about the Fuglede-Kadison determinant is the following proposition which is proven in [Lüc02], Theorem 3.14.

Theorem 2.16. *The Fuglede-Kadison determinant is a homomorphism*

$$\det_{\mathcal{N}\Gamma} : GL_r(\mathcal{N}\Gamma) \to \mathbb{R}_{>0}.$$

Example 2.17. Let $\Gamma = \mathbb{Z}^d$. There is the following model for the von Neumann algebra $\mathcal{N}(\mathbb{Z}^d)$. Let $L^2(\mathbb{T}^d, \mathbb{C})$ be the Hilbert space of equivalence classes of L^2-integrable complex-valued functions on the d-torus $\mathbb{T}^d$, where two such functions are called equivalent if they differ on a subset of measure zero. Let $L^\infty(\mathbb{T}^d, \mathbb{C})$ be the ring of equivalence classes of essentially bounded measurable functions $f : \mathbb{T}^d \to \mathbb{C}$, where essentially bounded means that there exists a constant $C > 0$ such that the set $\{z \in \mathbb{T}^d : |f(z)| \geq C\}$ has measure zero. The group $\mathbb{Z}^d$ acts isometrically on $L^2(\mathbb{T}^d, \mathbb{C})$ by

$$\mathbb{Z}^d \times L^2(\mathbb{T}^d, \mathbb{C}), ((k_1, \ldots, k_d), f) \mapsto z_1^{k_1} \cdot \ldots \cdot z_d^{k_d} f.$$

Fourier transform yields an isometric $\mathbb{Z}^d$-equivariant isomorphism

$$L^2(\mathbb{Z}^d, \mathbb{C}) \xrightarrow{\sim} L^2(\mathbb{T}^d, \mathbb{C}).$$

Hence $\mathcal{N}(\mathbb{Z}^d) = \mathcal{B}(L^2(\mathbb{T}^d, \mathbb{C}))^{\mathbb{Z}^d}$.

Now sending a function $f \in L^\infty(\mathbb{T}^d, \mathbb{C})$ to the $\mathbb{Z}^d$-equivariant operator

$$M_f : L^2(\mathbb{T}^d, \mathbb{C}) \to L^2(\mathbb{T}^d, \mathbb{C}),\ g \mapsto g \cdot f,$$

gives an isomorphism

$$L^\infty(\mathbb{T}^d, \mathbb{C}) \xrightarrow{\sim} \mathcal{N}(\mathbb{Z}^d).$$

The Fuglede-Kadison determinant of an invertible element $f \in L^\infty(\mathbb{T}^d, \mathbb{C})$ is given by

$$\det_{\mathcal{N}\mathbb{Z}^d}(f) = \int_{\mathbb{T}^d} \log |f| d\mu,$$

where $d\mu$ is the normalized Haar measure one $\mathbb{T}^d$.

Example 2.18. Assume Γ is finite. Then it is $\mathbb{C}\Gamma = L^2(\Gamma, \mathbb{C}) = \mathcal{N}\Gamma$. For $f \in \mathrm{GL}_r(\mathbb{C}\Gamma)$ it is

$$\det_{\mathcal{N}\Gamma}(f) = |\det_{\mathbb{C}} R_f|^{\frac{1}{|\Gamma|}}.$$

Here, $\det_{\mathbb{C}} R_f$ is the determinant of the $\mathbb{C}$-linear endomorphism of $(\mathbb{C}\Gamma)^r$ given by right multiplication with f. Since the absolute value of the determinant of right muliplication with f is equal to the absolute value of the determinant of the endomorphism ρ_f of right multiplication with f^*, we also have

$$\det_{\mathcal{N}\Gamma}(f) = |\det_{\mathbb{C}} \rho_f|^{\frac{1}{|\Gamma|}}.$$

In the remainder of Section 2.2 we want to explain the connection of the Fuglede-Kadison determinant to the entropy of the usual Γ-action on $X_f, f \in M_r(\mathbb{Z}\Gamma)$, in the expansive case.

There is the following criterion for expansiveness of the usual Γ-action on X_f:

Theorem 2.19. *Let Γ be a countable group, $f \in M_r(\mathbb{Z}\Gamma)$, and let α_f be the Γ-action on X_f as in Definition 2.4. The following conditions are equivalent.*

(1) The action α_f is expansive.

(2) $f \in GL_r(L^1(\Gamma, \mathbb{R}))$.

Proof. See [DS07], Theorem 3.2, for the case $r = 1$ and [Mül08], Theorem 3.2.1, for the general case. □

Remark 2.20. Let Γ be residually finite and let $\Gamma_n \to e$ be a sequence of cofinite normal subgroups of Γ. Assume that $f \in M_r(\mathbb{Z}\Gamma) \cap \mathrm{GL}_r(L^1(\Gamma, \mathbb{R}))$ so that the Γ-action α_f is expansive. Then by Theorem 2.12, it is

$$h(\alpha_f) = \lim_{n\to\infty} \frac{1}{|\Gamma/\Gamma_n|} \log |\mathrm{Fix}_{\Gamma_n}(X_f)|. \tag{2.18}$$

Furthermore, for every $n \in \mathbb{N}$, the image $f^{(n)}$ of f in $M_r(\mathbb{Z}\Gamma^{(n)})$ lies in $\mathrm{GL}_r(L^1(\Gamma^{(n)}, \mathbb{R})) = \mathrm{GL}_r(\mathbb{R}\Gamma^{(n)})$, where $\Gamma^{(n)} = \Gamma/\Gamma_n$. This implies that the endomorphism $\rho_{f^{(n)}}$ of right multiplication with $f^{(n)^*}$ on $(\mathbb{Q}\Gamma^{(n)})^r$ is an isomorphism as an injective endomorphism of a finite dimensional $\mathbb{Q}$-vector space. By Proposition 2.7, it is

$$|\mathrm{Fix}_{\Gamma_n}(X_f)| = \pm \det \rho_{f^{(n)}}.$$

The last step to the main result is the following important approximation result of the Fuglede-Kadison determinant. It will give the connection of the Fuglede-Kadison determinant and the limit (2.18).

Theorem 2.21. *Let Γ be a countable residually finite discrete group and $\Gamma_n \to e$ a sequence of cofinite normal subgroups of Γ converging to $e \in \Gamma$. For $f \in GL_r(L^1(\Gamma, \mathbb{C}))$ it is*

$$\det{}_{\mathcal{N}\Gamma}(f) = \lim_{n\to\infty} \det{}_{\mathcal{N}\Gamma^{(n)}} f^{(n)}.$$

Proof. [Mül08], Theorem 3.5.2. □

Now, we can prove the main result of Section 2.2:

Theorem 2.22. *Let Γ be a countable discrete amenable and residually finite group and let $f \in M_r(\mathbb{Z}\Gamma)$. Then the Γ-action on X_f is expansive if and only if $f \in GL_r(L^1(\Gamma, \mathbb{R}))$. In this case*

$$h(\alpha_f) = \log \det{}_{\mathcal{N}\Gamma}(f).$$

Proof. By Theorem 2.19, the Γ-action on X_f is expansive if and only if $f \in \mathrm{GL}_r(L^1(\Gamma, \mathbb{R}))$. Then combining Theorem 2.21, Theorem 2.12, Example 2.18 and Proposition 2.7, we get

$$\begin{aligned} h(\alpha_f) &= \lim_{n\to\infty} \frac{1}{|\Gamma/\Gamma_n|} \log |\mathrm{Fix}_{\Gamma_n}(X_f)| = \lim_{n\to\infty} \log \det{}_{\mathcal{N}\Gamma^{(n)}} f^{(n)} \\ &= \log \det{}_{\mathcal{N}\Gamma}(f). \end{aligned}$$

□

2.3 Periodic p-adic entropy and the p-adic Fuglede-Kadison determinant

Let $\log_p : \mathbb{Q}_p^* \to \mathbb{Z}_p$ be the branch of the p-adic logarithm normalized by $\log_p(p) = 0$.

The field $\mathbb{C}_p$ is defined as the completion of an algebraic closure of $\mathbb{Q}_p$.

Definition 2.23. *Let* Γ *be a countable discrete residually finite group acting on a set* X*. Let* $\Gamma_n \to e$ *be a sequence of cofinite normal subgroups converging to* e*. The* p*-adic entropy of the* Γ*-action on* X *with respect to the sequence* Γ_n *is defined to be*

$$h_{p,\Gamma_n} := \lim_{n\to\infty} \frac{1}{(\Gamma : \Gamma_n)} \log_p |\mathrm{Fix}_{\Gamma_n}(X)|$$

if the limit exists.

If the limit exists independently of the choice of the sequence $\Gamma_n \to e$ *and the value is always the same we say the periodic entropy* $h_{p,per}$ *of the* Γ*-action on* X *exists.*

Let us give a short survey on the article [Den09]. The main results say that for a certain class of residually finite groups Γ the periodic p-adic entropy of the natural Γ-action on X_f, $f \in M_r(\mathbb{Z}\Gamma)$, exists under some assumption on the element f. Furthermore, the periodic p-adic entropy of X_f is expressed as the value of f under the so-called p-adic Fuglede-Kadison determinant.

Definition 2.24. *Let* Γ *be a countable discrete group. The* $\mathbb{Q}_p$*-algebra* $c_0(\Gamma)$ *is defined as*

$$c_0(\Gamma) := \left\{ x = \sum_{\gamma\in\Gamma} x_\gamma \gamma \in \mathbb{Q}_p[[\Gamma]] : |x_\gamma| \to 0 \text{ as } \gamma \to \infty \text{ in } \Gamma \right\}.$$

Here, $|x_\gamma| \to 0$ *as* $\gamma \to \infty$ *in* Γ *means that for every* $\varepsilon > 0$ *all but a finite number of the* x_γ *have absolute value less than* ε*.*

The algebra $c_0(\Gamma)$ with the supremum norm

$$\Big|\Big| \sum_{\gamma\in\Gamma} x_\gamma \gamma \Big|\Big| = \sup_{\gamma\in\Gamma} |x_\gamma|_p = \max_{\gamma\in\Gamma} |x_\gamma|_p$$

is a $\mathbb{Q}_p$-Banach algebra in the following sense.

Definition 2.25. *A* p*-adic Banach algebra over* $\mathbb{Q}_p$ *is a unital* $\mathbb{Q}_p$*-algebra* B *which is complete with respect to a norm* $||\ || : B \to \mathbb{R}_{\geq 0}$ *satisfying the following conditions:*

(i) $||x|| = 0$ *if and only if* $x = 0$*,*

(ii) $||x + y|| \leq \max(||x||, ||y||)$*,*

(iii) $||\lambda x|| = |\lambda|_p ||x||$ *for all* $\lambda \in \mathbb{Q}_p$*,*

(iv) $||xy|| \leq ||x|| \, ||y||$ *and* $||1|| = 1$.

Example 2.26. The algebra $M_r(c_0(\Gamma))$ of $r \times r$-matrices with entries in $c_0(\Gamma)$ is a p-adic Banach-algebra with norm $||(a_{ij})|| = \max_{ij} ||a_{ij}||$.

Let B be a p-adic Banach algebra whose norm $||\ ||$ takes values in $p^{\mathbb{Z}} \cup \{0\}$. Let $A = B^0 := \{b \in B : ||b|| \leq 1\}$ and let $U^1 = 1 + pA$ be the subgroup of 1-units in A^*. U^1 is indeed a subgroup of A^* because for $1 + pa \in U^1$ the element $\sum_{\nu=0}^{\infty} (-pa)^{\nu}$ is the inverse of $1 + pa$ in U^1. For example, if $B = c_0(\Gamma)$ it is

$$c_0(\Gamma)^0 = c_0(\Gamma, \mathbb{Z}_p) := \left\{ x = \sum_{\gamma \in \Gamma} x_\gamma \gamma \in c_0(\Gamma) : x_\gamma \in \mathbb{Z}_p \text{ for all } \gamma \in \Gamma \right\}.$$

The logarithmic series converges on U^1 and defines a continuous map

$$\log : U^1 \to A, \quad \log u = - \sum_{\nu=1}^{\infty} \frac{(1-u)^{\nu}}{\nu}.$$

Let $\mathrm{tr}_B : B \to \mathbb{Q}_p$ be a trace functional, i.e. tr_B is a continuous linear map such that $\mathrm{tr}_B(ab - ba) = 0$ for all $a, b \in B$. Then for $b \in B$ and $c \in B^*$ it is $\mathrm{tr}_B(cbc^{-1}) = \mathrm{tr}_B(b)$ and by [Den09], Theorem 13, the composition $\mathrm{tr}_B \log : U^1 \to \mathbb{Z}_p$ is a homomorphism.

We apply this in the situation where $B = M_r(c_0(\Gamma))$, $A = M_r(c_0(\Gamma, \mathbb{Z}_p))$ and $U^1 = 1 + pM_r(c_0(\Gamma, \mathbb{Z}_p))$. The trace functional that we want to consider is the compositum of the usual trace

$$\mathrm{tr} : M_r(c_0(\Gamma)) \to c_0(\Gamma)$$

and the trace functional

$$\mathrm{tr}_\Gamma : c_0(\Gamma) \to \mathbb{Q}_p, \ \sum_{\gamma \in \Gamma} a_\gamma \gamma \mapsto a_e.$$

We denote the compositum $\mathrm{tr}_\Gamma \circ \mathrm{tr}$ also by tr_Γ.

Theorem 2.27. *The map*

$$\log_p \mathrm{det}_\Gamma := tr_\Gamma \log : 1 + pM_r(c_0(\Gamma, \mathbb{Z}_p)) \to \mathbb{Q}_p$$

is a homomorphism.

If we assume Γ to be residually finite then the next step is to relate $\log_p \mathrm{det}_\Gamma(f)$ for $f \in 1 + pM_r(c_0(\Gamma, \mathbb{Z}_p))$ with the periodic p-adic entropy of X_f.

Proposition 2.28. *Let Γ be a residually finite countable discrete group and let $\Gamma_n \to e$ be a sequence of cofinite normal subgroups of Γ converging to e. For $f \in 1 + pM_r(c_0(\Gamma, \mathbb{Z}_p))$ consider its image $f^{(n)}$ in $1 + pM_r(\mathbb{Z}_p\Gamma^{(n)})$ where $\Gamma^{(n)}$ is the finite group $\Gamma^{(n)} = \Gamma/\Gamma_n$. Then we have*

$$\log_p \mathrm{det}_\Gamma f = \lim_{n\to\infty} \log_p \mathrm{det}_{\Gamma^{(n)}} f^{(n)} \ \text{in } \mathbb{Z}_p.$$

Proof. See [Den09], Proposition 17. □

Proposition 2.29. *Let Γ be finite. Then we have*

$$\log_p \mathrm{det}_\Gamma f = \frac{1}{|\Gamma|} \log_p \det_{\mathbb{Q}_p}(\rho_f)$$

for $f \in 1 + pM_r(\mathbb{Z}_p\Gamma)$, where ρ_f denotes the $\mathbb{Q}_p$-endomorphism of right multiplication with f^* on $(\mathbb{Q}_p\Gamma)^r$.

Proof. See [Den09], Proposition 15. □

As a corollary to the previous propositions we get:

Corollary 2.30. *Let Γ be a residually finite countable discrete group and f an element of $M_r(\mathbb{Z}\Gamma)$ which is a 1-unit in $M_r(c_0(\Gamma))$. Then the periodic p-adic entropy $h_p(X_f)$ of the Γ-action on X_f exists and we have*

$$h_{p,per}(X_f) = \log_p \mathrm{det}_\Gamma f \ \text{in } \mathbb{Z}_p.$$

Proof. By Proposition 2.7 and Proposition 2.29, it is

$$\begin{aligned} \frac{1}{(\Gamma:\Gamma_n)} \log_p |\mathrm{Fix}_{\Gamma_n}(X_f)| &= \frac{1}{(\Gamma:\Gamma_n)} \log_p \det_{\mathbb{Q}_p}(\rho_{f^{(n)}}) \\ &= \log_p \mathrm{det}_{\Gamma^{(n)}} f^{(n)}. \end{aligned}$$

Then the claim follows from Proposition 2.28. □

Next one would like to extend the map $\log_p \mathrm{det}_\Gamma$ to $c_0(\Gamma, \mathbb{Z}_p)^*$, or more generally, to $\mathrm{GL}_r(c_0(\Gamma, \mathbb{Z}_p))$. The first attempt to do so by using the exact sequence

$$0 \to 1 + pc_0(\Gamma, \mathbb{Z}_p) \to c_0(\Gamma, \mathbb{Z}_p)^* \to \mathbb{F}_p[\Gamma]^* \to 0$$

seems not to work since one does not know if $(\mathbb{F}_p[\Gamma]^*/\langle\Gamma\rangle)^{ab}$ is torsion. But for some groups Γ, it is known that the Whitehead group $Wh^{\mathbb{F}_p}(\Gamma) := K_1(\mathbb{F}_p[\Gamma])/\langle\Gamma\rangle$ over $\mathbb{F}_p$ of Γ is torsion, where $\langle\Gamma\rangle$ is the image of Γ under the canonical map $\mathbb{F}_p[\Gamma]^* \to K_1(\mathbb{F}_p[\Gamma])$. Recall that for a unital, not necessarily commutative ring R, it is

$$K_1(R) := \varinjlim_r \mathrm{GL}_r(R) / \varinjlim_r E_r(R),$$

where $\mathrm{GL}_r(R)$ is embedded in $\mathrm{GL}_{r+1}(R)$ via the homomorphism mapping a to $\left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $E_r(R)$ is the subgroup of $\mathrm{GL}_r(R)$ generated by elementary matrices. We refer to Section 4.1 for a more detailed review of K-theory.

Theorem 2.31. *Let Γ be a countable discrete residually finite group such that $Wh^{\mathbb{F}_p}(\Gamma)$ is torsion. Then there is a unique homomorphism*

$$\log_p \det_\Gamma : K_1(c_0(\Gamma, \mathbb{Z}_p)) \to \mathbb{Q}_p$$

with the following properties:

(i) For every $r \geq 1$ the composition

$$1 + pM_r(c_0(\Gamma, \mathbb{Z}_p)) \hookrightarrow GL_r(c_0(\Gamma, \mathbb{Z}_p)) \to K_1(c_0(\Gamma, \mathbb{Z}_p)) \overset{\log_p \det_\Gamma}{\to} \mathbb{Q}_p$$

coincides with the map $\log_p \det_\Gamma$ defined before.

(ii) On the image of Γ in $K_1(c_0(\Gamma, \mathbb{Z}_p))$ the map $\log_p \det_\Gamma$ vanishes.

Proof. See [Den09], Theorem 19. □

Definition 2.32. *We call the homomorphism*

$$\log_p \det_\Gamma : K_1(c_0(\Gamma, \mathbb{Z}_p)) \to \mathbb{Q}_p \tag{2.19}$$

of Theorem 2.31 as well as the homomorphisms

$$\log_p \det_\Gamma : GL_r(c_0(\Gamma, \mathbb{Z}_p)) \to \mathbb{Q}_p, r \geq 1,$$

derived from (2.19) *by composing the map $GL_r(c_0(\Gamma, \mathbb{Z}_p)) \to K_1(c_0(\Gamma, \mathbb{Z}_p))$ with the homomorphism* (2.19) *the p-adic Fuglede-Kadison determinant.*

For $f \in \mathrm{GL}_r(c_0(\Gamma, \mathbb{Z}_p))$ the value $\log_p \det_\Gamma f$ is then given a follows: There are integers $N \geq 1, s \geq r$, such that $f^N = i(\gamma) \cdot \varepsilon \cdot g$ in $M_s(c_0(\Gamma, \mathbb{Z}_p))$, where $i(\gamma)$ is the $s \times s$-matrix $\left(\begin{smallmatrix} \gamma & 0 \\ 0 & 1_{s-1} \end{smallmatrix}\right)$, $\varepsilon \in E_s(c_0(\Gamma, \mathbb{Z}_p))$ is an elementary matrix and $g \in 1 + pM_s(c_0(\Gamma, \mathbb{Z}_p))$. Then we have

$$\log_p \det_\Gamma f = \frac{1}{N} \log_p \det_\Gamma g,$$

where the homomorphism $\log_p \det_\Gamma$ on the right-hand side is the one of Theorem 2.27.

For groups Γ as in Theorem 2.31 whose group ring $\mathbb{F}_p[\Gamma]$ has no zero divisors it is possible to extend the definition of $\log_p \det_\Gamma$ from $c_0(\Gamma, \mathbb{Z}_p)^*$ to $c_0(\Gamma)^*$. Namely, by [Den09], Proposition 4, we know that

$$c_0(\Gamma)^* = p^{\mathbb{Z}} c_0(\Gamma, \mathbb{Z}_p)^* \text{ and } p^{\mathbb{Z}} \cap c_0(\Gamma, \mathbb{Z}_p)^* = 1.$$

Hence, there is a unique homomorphism

$$\log_p \det_\Gamma : c_0(\Gamma)^* \to \mathbb{Q}_p$$

which agrees with $\log_p \det_\Gamma$ defined on $c_0(\Gamma, \mathbb{Z}_p)^* = \mathrm{GL}_1(c_0(\Gamma, \mathbb{Z}_p))$ in Definition 2.32 and satisfies $\log_p \det_\Gamma(p) = 0$.

We have the following approximation result for the p-adic Fuglede-Kadison determinant.

Proposition 2.33. *Let Γ be a residually finite countable discrete group and let $\Gamma_n \to e$ be a family of cofinite normal subgroups converging to e. For f in $M_r(c_0(\Gamma))$ let $f^{(n)}$ be its image in $M_r(\mathbb{Q}_p\Gamma^{(n)})$. Then the formula*

$$\log_p \det_\Gamma f = \lim_{n\to\infty} \frac{1}{(\Gamma : \Gamma_n)} \log_p \det_{\mathbb{Q}_p}(\rho_{f^{(n)}})$$

holds whenever $\log_p \det_\Gamma f$ is defined. These are the cases

(i) where f is in $1 + pM_r(c_0(\Gamma, \mathbb{Z}_p))$

(ii) where $Wh^{\mathbb{F}_p}(\Gamma)$ is torsion and f is in $GL_r(c_0(\Gamma, \mathbb{Z}_p))$

(iii) where $Wh^{\mathbb{F}_p}(\Gamma)$ is torsion, $\mathbb{F}_p\Gamma$ has no zero divisors and f is in $c_0(\Gamma)^*$.

Proof. [Den09], Proposition 23. □

As an application to dynamical systems and periodic p-adic entropy we get:

Theorem 2.34. *Let Γ be a residually finite countable discrete group such that $Wh^{\mathbb{F}_p}(\Gamma)$ is torsion. Let f be an element of $M_r(\mathbb{Z}\Gamma) \cap GL_r(c_0(\Gamma, \mathbb{Z}_p))$. Then the periodic p-adic entropy $h_{p,per}(X_f)$ of the usual action of Γ on X_f exists and we have*

$$h_{p,per}(X_f) = \log_p \det_\Gamma f \text{ in } \mathbb{Q}_p.$$

Proof. See [Den09], Theorem 22. □

Let us give a short discussion of the previous results in the special case $\Gamma = \mathbb{Z}^d$. We denote by $R_d = \mathbb{Z}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ the integral group ring of $\mathbb{Z}^d$. Recall that the p-adic Mahler measure $m_p(f)$ of a Laurent polynomial which does not vanish in any point of the p-adic d-torus $T_p^d := \{z \in \mathbb{C}_p^d : |z_i|_p = 1, 1 \leq i \leq d\}$ is defined by the Shnirelman integral

$$m_p(f) := \int_{T_p^d} \log_p f(z) \frac{dz}{z} := \lim_{\substack{N \to \infty, \\ (N,p)=1}} \frac{1}{N^d} \sum_{\zeta \in \mu_N^d} \log f(\zeta),$$

where μ_N denotes the set of N-th roots of unity in $\mathbb{C}_p$. See Section 6.1 for more facts on the Shnirelman integral and the p-adic Mahler measure.

Theorem 2.35. *Let $f \in R_d \cap c_0(\mathbb{Z}^d)^*$. Then the periodic p-adic entropy $h_{p,per}(\alpha_f)$ of the $\mathbb{Z}$-action α_f on X_f is given by*

$$h_{p,per}(\alpha_f) = m_p(f).$$

Proof. Using Theorem 2.33, (iii), we see that $h_{p,per}(\alpha_f)$ exists. Choosing the sequence $\Gamma_n = (n\mathbb{Z}) \to 0$ with n prime to p gives the result (see [Den09], Theorem 9). □

In the case of an element $f \in M_r(R_d) \cap \mathrm{GL}_r(c_0(\mathbb{Z}^d))$, $r > 1$, Deninger proves the following result without using the p-adic Fuglede-Kadison determinant, see [Den09], Theorem 9:

Theorem 2.36. *Let $f \in M_r(R_d) \cap GL_r(c_0(\mathbb{Z}^d))$. Then the p-adic entropy with respect to the sequence $\Gamma_n = (n\mathbb{Z})^d \to 0$ with n prime to p of the $\mathbb{Z}^d$-action on X_f exists, and we have*

$$h_{p,\Gamma_n} = m_p(\det f).$$

In Chapter 4, we will show that the periodic p-adic entropy of X_f exists under the assumptions of the previous theorem using the p-adic Fuglede-Kadison determinant.

We want to finish this section with an example taken from [Den09]. Therefore, we need the following two propositions.

Proposition 2.37. *Let $f(t) = a_n t^n + \ldots + a_0$ be a polynomial in $\mathbb{C}_p$ with $a_n \cdot a_0 \neq 0$ whose zeroes ζ satisfy $|\zeta|_p \neq 1$. Then*

$$m_p(f) = \log_p a_0 - \sum_{0<|\zeta|_p<1} \log_p \zeta = \log_p a_n + \sum_{|\zeta|_p>1} \log_p \zeta \ .$$

Proof. See [BD99], Proposition 1.5. □

Proposition 2.38. *For $f \in \mathbb{Q}_p[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ the following properties are equivalent:*

(i) We have $f(z) \neq 0$ for every z in the p-adic d-torus T_p^d.

(ii) f is a unit in $c_0(\mathbb{Z}^d)$.

Proof. See [Den09], Proposition 6. □

Example 2.39. Consider the polynomial $f = 2t^2 - t + 2$. The zeroes of f in $\mathbb{Q}_2$ are given by $\alpha_\pm = \frac{1}{4}(1 \pm \sqrt{-15})$ with $|\alpha_+|_2 = 2$ and $|\alpha_-|_2 = 1/2$. By Proposition 2.38 f is a unit in $c_0(\mathbb{Z})$ and by Theorem 2.35 and Proposition 2.37 the periodic 2-adic entropy of X_f is given by

$$h_{2,per}(X_f) = \log_2 \alpha_+ \ .$$

Chapter 3

Algebraic $\mathbb{Z}^d$-actions

In this section we review some results on algebraic $\mathbb{Z}^d$-actions, i.e. actions of $\mathbb{Z}^d$ by continuous automorphisms on compact abelian groups.

The key to study algebraic $\mathbb{Z}^d$-actions is the connection with commutative algebra. Namely, via Pontrjagin duality algebraic $\mathbb{Z}^d$-actions correspond to modules over the ring $R_d := \mathbb{Z}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$. Dynamical properties of algebraic $\mathbb{Z}^d$-actions can be translated into algebraic properties of the dual module.

In the first part we give some examples and results on how the dynamics of the $\mathbb{Z}^d$-action on X interplays with algebraic properties of the dual module M^X. In particular, the geometro-algebraic criterion for expansiveness in terms of the associated prime ideals of M^X is important.

In Section 3.2 we discuss the structure of expansive algebraic $\mathbb{Z}$-actions on compact connected abelian groups.

In Section 3.3 we describe $\mathbb{Z}^d$-actions which correspond via Pontrjagin duality to rings R_S of S-integers of algebraic number fields.

In the last part of this chapter we provide some results on the entropy of algebraic $\mathbb{Z}^d$-actions with the connection to the Mahler measure.

3.1 Algebraic $\mathbb{Z}^d$-actions and the dual module

Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. Pontrjagin duality gives a dual left action $\hat{\alpha} : \mathbb{Z}^d \to \mathrm{Aut}(\hat{X})$ on the discrete additive group $\widehat{X}$. This makes $\widehat{X}$ to a R_d-module, and conversely, every R_d-module M gives an algebraic $\mathbb{Z}^d$-action on the compact abelian group $\widehat{M}$. We will also use the term dynamical system X for a compact abelian group X with an algebraic $\mathbb{Z}^d$-action α. We write M^X for the R_d-module corresponding to a dynamical system X.

Examples 3.1. Let $d \geq 1$.

(1) The algebraic $\mathbb{Z}^d$-action corresponding to the R_d-module R_d is the compact abelian group $X_{R_d} = \widehat{R_d} = (\mathbb{R}/\mathbb{Z})^{\mathbb{Z}^d} = (\mathbb{R}/\mathbb{Z})[[\mathbb{Z}^d]]$ with $\mathbb{Z}^d$-action α on $(\mathbb{R}/\mathbb{Z})^{\mathbb{Z}^d}$ given by $n \cdot (x_m) = (x_{m-n})$.

(2) The α-invariant closed subgroups of X_{R_d} correspond to ideals in R_d: Given an ideal $I \subset R_d$ the dual of R_d/I is the closed α-invariant subgroup

$$X_{R_d/I} = \{x \in X_{R_d} : \langle x, f\rangle = 1 \text{ for every } f \in I\},$$

where $\langle \ , \ \rangle$ denotes the Pontrjagin pairing. Conversely, given a closed α-invariant subgroup $Y \subset X_{R_d}$, the annihilator $Y^\perp$ of Y in R_d,

$$Y^\perp = \{f \in R_d : \langle y, f\rangle = 1 \text{ for every } y \in Y\},$$

is an ideal in R_d.

Since an algebraic $\mathbb{Z}^d$-action (X, α) is completely determined by its dual module M^X, one can in principle express all dynamical properties of α by properties of M^X. For dynamical systems X corresponding to noetherian R_d-modules many of the dynamical properties of the action α on X have been translated into algebraic properties of the module M^X. Note that by (2) in the examples above, the R_d-module M^X is noetherian if and only if the action α on X satisfies the descending chain condition, i.e. every strictly decreasing sequence

$$X \supsetneq X_1 \supsetneq X_2 \ldots$$

of closed, α-invariant subgroups of X is finite.

One fact used to translate dynamical properties of (X, α) into algebraic properties of M^X is that a noetherian R_d-module M admits a prime filtration, i.e. a sequence $M = M_r \supset M_{r-1} \supset \ldots \supset M_0 = \{0\}$ such that for $i = 1, \ldots, r$, the quotient M_i/M_{i-1} is isomorphic to $R_d/\mathfrak{p}_i$ for some prime ideal $\mathfrak{p}_i$ in R_d. Even better, certain dynamical properties of (X, α) can be expressed only in terms of the associated primes of M^X.

Let us recall the definition of an associated prime ideal of a module M over a commutative ring R. A prime ideal $\mathfrak{p} \subset R$ is said to be associated with M if $\mathfrak{p}$ is the annihilator of some element $m \in M$. This amounts to saying that M contains a submodule isomorphic to $R/\mathfrak{p}$. The set of associated primes of M is usually denoted by $\mathrm{Ass}_R(M)$ or just $\mathrm{Ass}(M)$. For later reference, we state the following result.

Proposition 3.2. *Let M be a noetherian R_d-module. Then the following holds:*

(i) The set Ass(M) is finite and non-empty.

(ii) There exists a prime filtration $M = M_s \supset \ldots \supset M_0 = \{0\}$ of M such that for every $i = 1, \ldots, s$, $M_i/M_{i-1} \cong R_d/\mathfrak{q}_i$, for some prime ideal $\mathfrak{q}_i \subset R_d$, and $\mathfrak{q}_i \supset \mathfrak{p}$ for some $\mathfrak{p} \in$ Ass(M).

Proof. For (i) see [Bou98], Chapter IV, §1.1, Corollary 1 and §1.4, Theorem 2. For (ii) see [Sch95], Corollary 6.2. □

For an ideal $I \subset R_d$ and an algebraically closed field K with char $K = 0$ we denote by

$$V_K(I) = \left\{c = (c_1, \ldots, c_d) \in (K^*)^d : f(c) = 0 \text{ for every } f \in I\right\}$$

the set of zeroes of I over K. For us, the fields $K = \mathbb{C}$ and $K = \overline{\mathbb{Q}}_p$ will be important. We denote by $\mathbb{T}^d$ the real d-torus, i.e. the set

$$\mathbb{T}^d = \{z \in \mathbb{C}^d : |z_i| = 1, 1 \leq i \leq d\}.$$

Let us return to algebraic $\mathbb{Z}^d$-actions on a compact abelian group X. The following theorem is part of [Sch95], Theorem 6.5.

Theorem 3.3 (Geometric criterion for expansiveness)**.** *Let α be an algebraic $\mathbb{Z}^d$-action on X. Assume the corresponding R_d-module M^X is noetherian. Then the $\mathbb{Z}^d$-action α is expansive if and only if*

$$V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{T}^d = \emptyset \textit{ for every } \mathfrak{p} \in \textit{Ass}(M^X).$$

In the last theorem we had to assume that M^X is noetherian. If, on the other hand, we start with an expansive algebraic $\mathbb{Z}^d$-action, we have:

Proposition 3.4. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on X. Then the dual module M^X is a noetherian R_d-torsion module.*

Proof. See [Sch95], Corollary 6.13. □

Let α be an algebraic $\mathbb{Z}^d$-action on X. For a subgroup Λ of $\mathbb{Z}^d$ recall that

$$\mathrm{Fix}_\Lambda(\alpha) = \{x \in X : \alpha^\gamma x = x \text{ for all } \gamma \in \Lambda\}.$$

We will also use the notation $\mathrm{Fix}_\Lambda(X)$ if no confusion on the action α on X can occur.

Theorem 3.5. *Let α be an algebraic $\mathbb{Z}^d$-action on X. Assume the corresponding R_d-module M^X is noetherian. Let $\Lambda \subset \mathbb{Z}^d$ be a subgroup of finite index. Let $I(\Lambda)$ be the ideal in R_d generated by*

$$(t_1^{n_1} \dots t_d^{n_d} - 1, \ (n_1, \dots, n_d) \in \Lambda).$$

The following conditions are equivalent.

(i) The set $\mathrm{Fix}_\Lambda(\alpha)$ *is finite.*

(ii) For every $\mathfrak{p} \in Ass(M^X)$

$$V_{\mathbb{C}}(\mathfrak{p}) \cap V_{\mathbb{C}}(I(\Lambda)) = \emptyset.$$

Proof. See [Sch95], Theorem 6.5. □

Remark 3.6. Note that by Theorems 3.3-3.5, if α is an expansive $\mathbb{Z}^d$-action on X and Λ a subgroup of $\mathbb{Z}^d$ of finite index, then $\mathrm{Fix}_\Lambda(\alpha)$ is finite because $V_{\mathbb{C}}(I(\Lambda)) \subset \mathbb{T}^d$.

Proposition 3.7. *Let α be an algebraic $\mathbb{Z}^d$-action on X and let M^X be the corresponding R_d-module. Then X is connected if and only if for every $\mathfrak{p} \in Ass(M^X)$ we have $V_{\mathbb{C}}(\mathfrak{p}) = \emptyset$.*

Proof. See [Sch95], Proposition 6.9. □

We finish this section with a comparison of the criterion for expansiveness of the Γ-action α_f on X_f, $f \in \mathbb{Z}\Gamma$, as stated in Theorem 2.19 applied to the case $\Gamma = \mathbb{Z}^d$ with the criterion of Theorem 3.3. For this, we need Wiener's famous result:

Theorem 3.8. *Let f be a continuous nowhere vanishing function on $\mathbb{T}^d$ which has Fourier coefficients in $L^1(\mathbb{Z}^d, \mathbb{C})$, then $1/f$ has Fourier coefficients in $L^1(\mathbb{Z}^d, \mathbb{C})$ as well.*

Proof. See for example [Kat04], VIII, Theorem 2.9, for a proof in the case $d = 1$. The generalisation to the case $d > 1$ is straightforward. □

Remark 3.9. Let $f \in R_d$. Then by Theorem 2.19, applied to the case $\Gamma = \mathbb{Z}^d$, the usual $\mathbb{Z}^d$-action on $X_f = \widehat{R_d/(f)}$ is expansive if and only if $f \in L^1(\mathbb{Z}^d, \mathbb{R})^*$. By Theorem 3.8 the element $f \in L^1(\mathbb{Z}^d, \mathbb{R})^*$ if and only if f considered as a function $f : \mathbb{T}^d \to \mathbb{C}$ has no zeroes on $\mathbb{T}^d$. Here we use that if $f \in L^1(\mathbb{Z}^d, \mathbb{R})$ and $f \in L^1(\mathbb{Z}^d, \mathbb{C})^*$ then f is already a unit in $L^1(\mathbb{Z}^d, \mathbb{R})$.

Now, the associated primes of the dual module $R_d/(f)$ of X_f are the prime ideals generated by the irreducible factors of f and, of course, f has no zeroes on $\mathbb{T}^d$ if and only if none of its prime factors has a zero on $\mathbb{T}^d$. So for $\Gamma = \mathbb{Z}^d$ and dynamical systems X_f, $f \in R_d$, Theorem 2.19 and Theorem 3.3 are equivalent.

3.2 Expansive $\mathbb{Z}$-actions on compact connected abelian groups

Let us recall the situation of an expansive $\mathbb{Z}$-action on a compact, connected, abelian group X.

Theorem 3.10. *Let α be a $\mathbb{Z}$-action on a compact, connected, abelian group X. The following conditions are equivalent.*

(i) α is expansive.

(ii) There exist primitive polynomials $f_1, \ldots, f_r$ in R_1 such that f_j divides f_{j+1} for $j = 1, \ldots, r-1$ and f_r has no roots of modulus 1 and a surjective morphism ϕ of dynamical systems

$$\phi : Y := Y_{f_1} \times \ldots \times Y_{f_r} \to X$$

with finite kernel.

Proof. By Proposition 3.4, the module M^X is a noetherian $\mathbb{Z}[t, t^{-1}]$-torsion module. By assumption, X is connected and so using Proposition 3.7, we deduce that M^X is torsion-free as an abelian group. This implies that M^X injects into $M^X \otimes_{\mathbb{Z}} \mathbb{Q}$. By the general theory of finitely generated torsion modules over principal ideal domains, we have an isomorphism

$$M^X \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}[t, t^{-1}]/(f_1) \times \ldots \times \mathbb{Q}[t, t^{-1}]/(f_r)$$

with elements $f_j \in \mathbb{Q}[t, t^{-1}], 1 \leq j \leq r$, such that f_j divides f_{j+1} for $j = 1, \ldots, r-1$.

We may assume that the f_j are in $\mathbb{Z}[t, t^{-1}]$ and that they are primitive. Moreover, because M^X is a finitely generated $\mathbb{Z}[t, t^{-1}]$-module, we may assume that the image of M^X under the composition

$$M^X \hookrightarrow M^X \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}[t, t^{-1}]/(f_1) \times \ldots \times \mathbb{Q}[t, t^{-1}]/(f_r)$$

lies in $\mathbb{Z}[t, t^{-1}]/(f_1) \times \ldots \times \mathbb{Z}[t, t^{-1}]/(f_r)$. Then there is an exact sequence of $\mathbb{Z}[t, t^{-1}]$-torsion modules

$$0 \to M^X \xrightarrow{\hat{\phi}} \mathbb{Z}[t, t^{-1}]/(f_1) \times \ldots \times \mathbb{Z}[t, t^{-1}]/(f_r) \to \operatorname{coker} \hat{\phi} \to 0 ,$$

such that the second arrow in the diagram is an isomorphism after tensoring with $\mathbb{Q}$. This implies that $\operatorname{coker} \hat{\phi}$ is a torsion group. Since $\mathbb{Z}[t, t^{-1}]/(f_1) \times$

$\ldots \times \mathbb{Z}[t,t^{-1}]/(f_r)$ is a finitely generated $\mathbb{Z}[t,t^{-1}]$-module we find a natural number $n \in \mathbb{N}$ such that

$$n \cdot (\mathbb{Z}[t,t^{-1}]/(f_1) \times \ldots \times \mathbb{Z}[t,t^{-1}]/(f_r)) \subset \hat{\phi}(M^X)$$

which implies that $\operatorname{coker}\hat{\phi}$ is annihilated by the natural number n. Then $\operatorname{coker}\hat{\phi}$ is a finitely generated $\mathbb{Z}[t,t^{-1}]/(n, f_1 \cdot \ldots \cdot f_r)$-module and so is finite.

Dualizing the short exact sequence, we get the surjective morphism $\phi : Y \to X$ with finite kernel. Now, the action α on X is expansive if and only if the canonical $\mathbb{Z}$-action on Y is expansive, and this is exactly the case if none of the f_j has a root of modulus 1. □

Now we come to a second description of expansive automorphisms on connected compact abelian groups.

Definition 3.11. *Given a matrix $A \in GL_n(\mathbb{Q})$ we define a closed shift-invariant subgroup X of $(\mathbb{T}^n)^{\mathbb{Z}}$ by*

$$X = \left\{x = (x_k) \in (\mathbb{T}^n)^{\mathbb{Z}} : mx_{k+1} = Bx_k \text{ for all } k \in \mathbb{Z}\right\},$$

where m is the smallest positive integer such that the matrix $B := mA$ has entries in $\mathbb{Z}$. We define $X^A = X^0$ to be the connected component of the identity.

Let us determine the dual module of X^A. We denote by A^t the transpose of the matrix A. We define the R_1-module M^A by

$$M^A := \mathbb{Z}^n[A^t, (A^{-1})^t] := \text{subgroup of } \mathbb{Q}^n \text{ generated by } \bigcup_{k\in\mathbb{Z}} (A^k)^t \mathbb{Z}^n,$$

where the variable $t \in R_1$ acts by multiplication with A^t on M^A. Note that by Proposition 3.7 the dual $\widehat{M^A}$ is connected because M^A is torsion-free as an abelian group.

Let η be the R_1-module homomorphism

$$\eta : \bigoplus_{\mathbb{Z}} \mathbb{Z}^n \to M^A, w = (w_k)_{k\in\mathbb{Z}} \mapsto \sum_{k\in\mathbb{Z}} (A^t)^k w_k.$$

Let $W \subset \bigoplus_{\mathbb{Z}} \mathbb{Z}^n$ be the subgroup generated by all $w = (w_k) \in \bigoplus_{\mathbb{Z}} \mathbb{Z}^n$ such that there exists an integer $l \in \mathbb{Z}$ with $B^t w_{l+1} = -mw_l$ and $w_k = 0$ for $k \notin \{l, l+1\}$. Again, m denotes the smallest positive integer such that the matrix $B := mA$ has entries in $\mathbb{Z}$.

Then $W \subset \ker\eta$. Furthermore, the quotient $\ker\eta/W$ is a torsion group. A computation shows that $X^A \subset W^{\perp}$ where here we have to use that X^A is connected. Furthermore, it is $\widehat{M^A} \subset X$. Dualizing these inclusions, we get a sequence

$$\bigoplus_{\mathbb{Z}} \mathbb{Z}^n/W \to M^{X^0} \to M^A \simeq \bigoplus_{\mathbb{Z}} \mathbb{Z}^n/\ker\eta$$

of surjetive R_1-module homomorphisms. If the homomorphism $M^{X^0} \to M^A$ was not an isomorphism, we would deduce that M^{X^0} has torsion elements because $\ker\eta/W$ is a torsion group. This cannot be true because X^0 is connected. It follows

$$\widehat{M^A} \simeq X^A.$$

Next, we want to determine under what conditions the shift action on X^A is expansive. First notice that by definition, the R_1-module M^A is finitely generated. Hence, to check expansiveness we need to determine the associated primes of M^A. Let χ_A be the characteristic polynomial of A and let k the smallest integer such that $k\chi_A \in R_1$. It is clear that $\mathrm{Ass}(M^X) = \{(f_1), \ldots, (f_r)\}$, where $k\chi_A = f_1 \cdot \ldots \cdot f_r$ is a prime decomposition in R_1. It follows that the shift action on X^A is expansive if and only if the characteristic polynomial χ_A has no zeroes in $\mathbb{T}$.

The next theorem states that any expansive automorphism of a compact connected group is in fact always conjugate to the shift action on X^A for some matrix $A \in \mathrm{GL}_n(\mathbb{Q})$, $n \geq 1$, without eigenvalues in $\mathbb{T}$.

Theorem 3.12. *An automorphism α of a compact, connected group X is expansive if and only if it is algebraically conjugate to the shift action on X^A for some matrix $A \in GL_n(\mathbb{Q})$, $n \geq 1$, without eigenvalues in $\mathbb{T}$.*

Proof. See [Sch95], Theorem 9.7. □

3.3 S-integer dynamical systems

Let $c = (c_1, \ldots, c_d) \in (\overline{\mathbb{Q}}^*)^d$. We denote by $\mathfrak{m}_c$ the vanishing ideal $\mathfrak{m}_c = \{f \in R_d : f(c) = 0\}$ of c. We want to study the algebraic $\mathbb{Z}^d$-action which corresponds to the R_d-module $R_d/\mathfrak{m}_c$ via Pontrjagin duality. It turns out that the module R_d-module $R_d/\mathfrak{m}_c$ is closely related to a ring $R_{P(c)}$ of S-integers determined by the point c. Here, $P(c)$ is a certain subset of the set of finite places of the number field $\mathbb{Q}(c)$. The ring $R_{P(c)}$ carries a natural R_d-module structure. In order to describe the algebraic $\mathbb{Z}^d$-action corresponding to the R_d-module $R_{P(c)}$ we need to provide some background material on adele rings.

Let K be an algebraic number field, i.e. a finite extension of $\mathbb{Q}$. An absolute value on K is a real valued function $|\ |$ on K such that

(i) $|x| \geq 0$ and $|x| = 0$ if and only if $x = 0$,

(ii) $|xy| = |x||y|$ and

(iii) $|x + y| \leq |x| + |y|$ for all $x, y \in K$.

The absolute value $| \ |$ is called non-archimedean if $|x + y| \leq \max\{|x|, |y|\}$ for all $x, y \in K$ and archimedean otherwise. Two absolute values $| \ |, | \ |'$ are said to be equivalent if they define the same topology on K which is exactly the case if there exists a positive real number s such that $| \ |' = | \ |^s$, i.e. $|x|' = |x|^s \ \forall x \in K$.

A place v of K is an equivalence class of non-trivial absolute values. Given a place v we denote by $| \ |_v$ an absolute value in the equivalence class of v and we denote by K_v the completion of K with respect to v. For example, every absolute value of $\mathbb{Q}$ is either equivalent to the usual archimedean absolute value $| \ |_\infty$ or to an absolute value $| \ |_p$, where p is a prime number, defined by

$$|0|_p = 0 \text{ and } |x|_p = p^{s-r} \text{ if } x = \frac{p^r m}{p^s n}, m, n \in \mathbb{Z} \setminus \{0\},$$

where m and n are both not divisible by p. Thus, the places of $\mathbb{Q}$ are indexed by the set $\mathcal{P} \cup \{\infty\}$, where $\mathcal{P} \subset \mathbb{N}$ is the set of prime numbers. The completion $\mathbb{Q}_\infty$ is the field $\mathbb{R}$ and for a prime number p the completion of $\mathbb{Q}$ is the field $\mathbb{Q}_p$ of p-adic numbers.

For a place v of K the restriction of v to $\mathbb{Q}$ is either equivalent to $| \ |_\infty$ or to $| \ |_p$ for some prime p. We write $v|p, p \in \mathcal{P} \cup \{\infty\}$, if the restriction of v to $\mathbb{Q}$ is equivalent to $| \ |_p$. If $v|\infty$, v is said to be infinite and in this case K_v is either $\mathbb{R}$ or $\mathbb{C}$. If $v|p, p \in \mathcal{P}$, v is said to be finite and in this case K_v is a finite extension of $\mathbb{Q}_p$. We write $P^K, P^K_{\mathfrak{f}}, P^K_\infty$ for the sets of places, finite places, and infinite places of K, respectively. Note that for each place p on $\mathbb{Q}$ there are only finitely many places v on K which lie above p, i.e. whose restriction to $\mathbb{Q}$ is p. In particular, P^K_∞ is a finite set.

For every $v \in P^K$ the set $R_v := \{x \in K_v : |x|_v \leq 1\}$ is a compact subset of K_v. If $v \in P^K_{\mathfrak{f}}$ then R_v is a subring of K_v which is open. We consider the ring

$$\mathbb{A}_K := \left\{ x \in \prod_{v \in P^K} K_v | \ x_v \in R_v \text{ for almost all } v \in P^K \right\}$$

with pointwise addition and multiplication. For every finite set $S \subset P^K$ containing all infinite places the product topology makes

$$\mathbb{A}_{K,S} := \prod_{v \in S} K_v \times \prod_{v \in P^K \setminus S} R_v$$

into a locally compact topological group. There exists a unique structure on $\mathbb{A}_K$ as a topological group such that the groups $\mathbb{A}_{K,S}$ are open topological subgroups of $\mathbb{A}_K$. With this topology $\mathbb{A}_K$ is a locally compact topological ring.

Definition 3.13. *The adele ring of K is defined to be the ring $\mathbb{A}_K$ with the locally compact topology described above.*

More generally, we define:

Definition 3.14. *Let K be a number field and let S be an arbitrary subset of the set $P_{\mathfrak{f}}^K$ of finite places of K. We define the locally compact ring $\mathbb{A}_K(S)$ as*

$$\mathbb{A}_K(S) := \left\{ x \in \prod_{v \in S \cup P_\infty^K} K_v \Big| \; x_v \in R_v \text{ for almost all } v \in S \cup P_\infty^K \right\}$$

with pointwise addition and multiplication and with the topology defined as follows. Let S' be a finite subset of S. Let

$$\mathbb{A}_{K,S'}(S) := \prod_{v \in S \cup P_\infty^K} K_v \times \prod_{v \in S \setminus S'} R_v.$$

We define a topology on $\mathbb{A}_K(S)$ by taking as a fundamental system of neighborhoods of 0 in $\mathbb{A}_K(S)$ the set of neighborhoods of 0 in $\mathbb{A}_{K,S'}(S)$.

Remark 3.15. (i) The ring $K_{\mathbb{A}}(S)$ is just the restricted direct product of the locally compact groups $(K_v)_{v \in S \cup P_\infty^K}$ with respect to the compact open subgroups $(R_v)_{v \in S}$ in the sense of Tate [Tat67], Section 3.

(ii) Obviously, for $S = P_{\mathfrak{f}}^K$ we get the full adele ring of K, i.e.

$$\mathbb{A}_K = \mathbb{A}_K(P_{\mathfrak{f}}^K).$$

We want to study the character group $\widehat{\mathbb{A}_K(S)} = \mathrm{Hom}_{cont}(\mathbb{A}_K(S), \mathbb{T})$ of $\mathbb{A}_K(S)$.

Proposition 3.16. *Let K be a number field and let $S \subset P_{\mathfrak{f}}^K$. Let $\widehat{K_v} = Hom_{cont}(K_v, \mathbb{T})$ be the character group of K_v. The homomorphism*

$$\left\{ (\chi_v)_{v \in S \cup P_\infty^K} \in \prod_{v \in S \cup P_\infty^K} \widehat{K_v} \Big| \; \chi_{v|R_v} = 0 \text{ for almost all } v \in S \cup P_\infty^K \right\} \to \widehat{\mathbb{A}_K(S)}$$

is an isomorphism of abstract groups. Its inverse is given by mapping $\chi \in \widehat{\mathbb{A}_K(S)}$ to the family $(\chi_{|K_v})_{v \in S \cup P_\infty^K}$ where $\chi_{|K_v} \in \widehat{K_v}$ is the restriction of χ to K_v.

Proof. See [Tat67], Lemma 3.2.1 and Lemma 3.2.2. □

We define the so-called standard character $\psi = (\psi_p)_{p \in P^{\mathbb{Q}}}$ on $\mathbb{A}_{\mathbb{Q}}$ by

$$\psi_\infty(x) = e^{-2\pi i x} \text{ for the archimedean prime } p = \infty \text{ and}$$
$$\psi_p = [\mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}/\mathbb{Z} \stackrel{e^{2\pi i}}{\to} \mathbb{T}] \text{ for non-archimedean } p.$$

Then for all non-archimedean places p, the character ψ_p is trivial on $\mathbb{Z}_p$ and so by Proposition 3.16, the family $(\psi_p)_{p \in P^{\mathbb{Q}}}$ defines a character on $\mathbb{A}_{\mathbb{Q}}$. Let K be a number field. Note that we have the trace homomorphism

$$\mathrm{tr} : \mathbb{A}_K \to \mathbb{A}_{\mathbb{Q}}, (x_v)_{v \in P^K} \mapsto \left(\sum_{v|p} \mathrm{tr}_v(x_v) \right)_{p \in P^{\mathbb{Q}}},$$

where $\mathrm{tr}_v : K_v \to \mathbb{Q}_p$ is the usual trace from the finite extension K_v of $\mathbb{Q}_p$ to $\mathbb{Q}_p$. The standard character ψ_K on $\mathbb{A}_K$ is defined by $\psi_K(x) = \psi(\mathrm{tr}(x))$. Then it is

$$\psi_K = \prod_{v \in P^K} \psi_{K,v} \text{ with } \psi_{K,v} = \psi_p \circ \mathrm{tr}_v \in \mathrm{Hom}_{cont}(K_v, \mathbb{T}), v|p.$$

For $S \subset P_{\mathfrak{f}}^K$, we define $\psi_{K,S} \in \widehat{\mathbb{A}_K(S)}$ by

$$\psi_{K,S} = \prod_{v \in P_\infty^K \cup S} \psi_{K,v}.$$

Definition 3.17. *Let K be a number field and let S be a subset of the set $P_{\mathfrak{f}}^K$ of finite places of K. The ring R_S of S-integers is defined as*

$$R_S = \{x \in K : |x|_v \le 1 \textit{ for every } v \notin S \cup P_\infty^K\}.$$

For example, if $S = P_{\mathfrak{f}}^K$ then $\mathbb{A}_K(S) = \mathbb{A}_K$ and $R_S = K$. The ring R_S injects into $\mathbb{A}_K(S)$ via the diagonal embedding

$$\Delta : R_S \to \mathbb{A}_K(S), x \mapsto (x, x, x, \ldots).$$

Let $\omega_1, \ldots, \omega_n$ an integral basis of K over $\mathbb{Q}$. Because $K \otimes_{\mathbb{Q}} \mathbb{R}$ is isomorphic to $\prod_{v \in P_\infty^K} K_v$ every $x \in \prod_{v \in P_\infty^K} K_v$ can be uniquely written as a sum $x = \sum_{i=1}^n a_i \omega_i$ with real numbers a_i. Here, we write again ω_i for the image of ω_i in $\prod_{v \in P_\infty^K} K_v$.

Lemma 3.18. *Let*

$$\overset{\infty}{D} = \prod_{v \in P_\infty^K} \left\{ x = \sum_{i=1}^{n} a_i \omega_i : 0 \le a_i < 1 \textit{ for } 1 \le i \le n \right\}$$

and

$$D_S = \overset{\infty}{D} \times \prod_{v \in S} R_v.$$

Then D_S is a fundamental domain of $\mathbb{A}_K(S)/R_S$, i.e. every element in $\mathbb{A}_K(S)/R_S$ has exactly one representative in D_S. In particular,

$$\mathbb{A}_K(S) = R_S + D_S.$$

Proof. To prove uniqueness, assume it is $x = y + d = y' + d'$ with $x \in \mathbb{A}_K(S)$ and $y, y' \in R_S, d, d' \in D_S$. Then from the equation $y - y' = d' - d$ we see that the element $y - y' \in R_S$ is in fact integral. As the projection of $d' - d$ to $\prod_{v \in P_\infty^K} K_v$ lies in $\prod_{v \in P_\infty^K} \left\{ x = \sum_{i=1}^{n} a_i \omega_i : -1 < a_i < 1 \text{ for } 1 \le i \le n \right\}$ it follows $y - y' = 0$. Then also $d = d'$ which proves uniqueness.

To prove that any element $x \in \mathbb{A}_K(S)$ can be written as a sum $x = y + d$ with $y \in R_S$, $d \in D_S$, we use the Chinese remainder theorem to find an element $y \in R_S$ such that for all finite places $v \in S$ it is $x - y \in R_v$. Then subtracting $x - y$ by an integral element y' of K, which is by definition contained in R_S, we may achieve that the infinite components of $x - (y + y')$ lie in $\overset{\infty}{D}$ without changing the property that $x - (y + y') \in R_v$ for all $v \in S$. □

Theorem 3.19. *Let $S \subset P_\mathfrak{f}^K$ and let $\mathbb{A}_K(S)$ be the locally compact topological ring as defined in 3.14. Let $\psi_{K,S}$ be the standard character in $\mathbb{A}_K(S)$. Then:*

(i) R_S is a discrete, cocompact subgroup of $\mathbb{A}_K(S)$.

(ii) The map

$$\mathbb{A}_K(S) \to \widehat{\mathbb{A}_K(S)}, \ a \mapsto \psi_{K,S,a},$$

where $\psi_{K,S,a}$ is the character $x \mapsto \psi_{K,S}(ax)$, is an isomorphism of topological groups.

(iii) The composition $R_S \to \mathbb{A}_K(S) \to \widehat{\mathbb{A}_K(S)}$ identifies R_S with the group $R_S^\perp$ of characters in $\widehat{\mathbb{A}_K(S)}$ which vanish on R_S. Thus, it is

$$\widehat{R_S} \simeq \mathbb{A}_K(S)/R_S.$$

Proof. That R_S is discrete follows from Lemma 3.18 because D_S has an interior. $\mathbb{A}_K(S)/R_S$ is compact because D is relatively compact. (ii) follows from [Tat67], Theorem 2.2.1, Lemma 2.2.3 and Theorem 3.2.1.

For (iii), one shows as in [Tat67], Corollary 4.1.1, that R_S is contained in $R_S^{\perp}$. As the Pontrjagin dual of the compact group $\mathbb{A}_K(S)/R_S$ the group $R_S^{\perp}$ is discrete. As a discrete subgroup of the compact group $\widehat{\mathbb{A}_K(S)}/R_S$ the quotient $R_S^{\perp}/R_S$ is finite. $R_S^{\perp}/R_S$ carries a R_S-module structure. But $R_S^{\perp}/R_S$ is torsion-free as R_S-module. Thus, the index $[R_S^{\perp} : R_S]$ cannot be greater than 1 because this would contradict the fact that $R_S^{\perp}/R_S$ is finite because R_S is not finite. □

Let us return to the algebraic $\mathbb{Z}^d$-actions that we are interested in. Let $c = (c_1, \ldots, c_d) \in (\overline{\mathbb{Q}}^*)^d$ and let $\mathfrak{m}_c$ be the vanishing ideal $\mathfrak{m}_c = \{f \in R_d : f(c) = 0\} \subset R_d$. We denote by $X_{R_d/\mathfrak{m}_c}$ the dynamical system $X_{R_d/\mathfrak{m}_c} = \widehat{R_d/\mathfrak{m}_c}$.

Definition 3.20. *Given a point $c = (c_1, \ldots, c_d) \in (\overline{\mathbb{Q}}^*)^d$, we define an algebraic $\mathbb{Z}^d$-action (Y_c, α_c) as follows. Let $K = \mathbb{Q}(c)$ and put*

$$F(c) := \{v \in P_{\mathfrak{f}}^K : |c_i|_v \neq 1 \text{ for some } i \in \{1, \ldots, d\}\}$$

and

$$P(c) := P_\infty^K \cup F(c) \ .$$

The abelian group $R_{P(c)} = \{x \in K : |x|_v \leq 1 \text{ for every } v \notin P(c)\}$ is an R_d-module under the action

$$\hat{\alpha}_c : R_d \times R_{P(c)} \to R_{P(c)}, (f, a) \mapsto f(c)a.$$

Dualizing, we get a $\mathbb{Z}^d$-action on $Y_c := \widehat{R_{P(c)}}$ which we denote by α_c.

Theorem 3.21. *There exists a surjective homomorphism*

$$\phi : Y_c \to X_{R_d/\mathfrak{m}_c}$$

with finite kernel which is compatible with the $\mathbb{Z}^d$-actions on Y_c and on $X_{R_d/\mathfrak{m}_c}$.

Proof. The natural evaluation homomorphism $R_d \to R_{P(c)}, f \mapsto f(c)$, induces an injective homomorphism $\hat{\phi} : R_d/\mathfrak{m}_c \to R_{P(c)}$. Dualizing this homomorphism, we get a surjective homomorphism $\phi : Y_c \to X_{R_d/\mathfrak{m}_c}$. For the proof that ϕ has finite kernel, i.e. that the quotient $R_{P(c)}/\hat{\phi}(R_d/\mathfrak{m}_c)$ is finite, see [Sch95], Theorem 7.1. □

The algebraic criterion for expansiveness in Theorem 3.3 gives the following result for the algebraic $\mathbb{Z}^d$-action on $X_{R_d/\mathfrak{m}_c}$.

Proposition 3.22. *The action α on $X_{R_d/\mathfrak{m}_c}$ is expansive if and only if the action α_c on Y_c is expansive. This is exactly the case if the orbit of c under the diagonal action of the Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on $(\overline{\mathbb{Q}}^*)^d$ does not intersect $\mathbb{T}^d$.*

Proof. See [Sch95], Proposition 7.2. □

3.4 The entropy of algebraic $\mathbb{Z}^d$-actions and the Mahler measure

In this section we want to give the reader a short overview on entropy of algebraic $\mathbb{Z}^d$-actions. For the definition of entropy we refer to Section 2.2.

The topological entropy of an algebraic $\mathbb{Z}^d$-action α on a compact group X is denoted by $h(\alpha)$. If no confusion on the action α can occur, we will also use the notation $h(X)$ for the topological entropy.

Theorem 3.23 (Yuzvinskii's addition formula)**.** *Let α_X be a $\mathbb{Z}^d$-action by automorphisms on a compact group X and let $Y \subset X$ be a normal, α-invariant subgroup. Let α_Y the restriction of α_X to Y and let $\alpha_{X/Y}$ be the induced action on X/Y. Then*

$$h(\alpha_X) = h(\alpha_Y) + h(\alpha_{X/Y}).$$

Proof. See [Sch95], Theorem 14.1. □

Remark 3.24. We interpret this result in the following way. If we only consider $\mathbb{Z}^d$-actions on compact abelian groups X and if we go from dynamical systems to their dual R_d-module, then the result says that entropy, viewed as a numerical invariant of R_d-modules, is additive in short exact sequences.

Definition 3.25. *The logarithmic Mahler measure of an element $f \in R_d$ is defined as*

$$m(f) := \int_{\mathbb{T}^d} \log |f(z)| d\mu(z),$$

where μ is the normalized Haar measure on the d-torus $\mathbb{T}^d$.

Proposition 3.26. *For every non-zero $f \in R_d$ it is $0 \leq m(f) < \infty$.*

Proof. [Sch95], Corollary 16.6. □

Now, we return to the entropy of algebraic $\mathbb{Z}^d$-actions. For $f \in R_d$, let α_f the usual $\mathbb{Z}^d$-action on X_f. The following holds:

Theorem 3.27. *For every $f \in R_d$, the entropy of the action α_f on X_f is given by $h(\alpha_f) = m(f)$.*

Proof. [Sch95], Theorem 18.1. □

Theorem 3.28. *Let $d \geq 1$ and let $\mathfrak{p} \subset R_d$ be a prime ideal. Then*

$$h(X_{R_d/\mathfrak{p}}) = \begin{cases} m(f) & \text{if } \mathfrak{p} = (f) \text{ is principal} \\ 0 & \text{if } \mathfrak{p} \text{ is not principal.} \end{cases}$$

Proof. We show that for a non-principal prime ideal $\mathfrak{p} \in R_d$ we have $h(\alpha_{R_d/\mathfrak{p}}) = 0$. The rest follows from Theorem 3.27.

Choose a prime element $f \in \mathfrak{p}$ and an element $g \in \mathfrak{p}$ which is not contained in the principal ideal (f). Then the following sequence is exact.

$$0 \to R_d/f \xrightarrow{\cdot g} R_d/f \to R_d/(f,g) \to 0.$$

Dualizing we get an exact sequence

$$0 \to X_{R_d/(f,g)} \to X_f \to X_f \to 0.$$

By Yuzvinskii's addition formula it is $h(\alpha_f) = h(\alpha_f) + h(\alpha_{R_d/(f,g)})$. By Proposition 3.26 it is $h(\alpha_f) < \infty$ so we deduce $h(\alpha_{R_d/(f,g)}) = 0$. But $X_{R_d/\mathfrak{p}}$ is a closed $\alpha_{R_d/(f,g)}$-invariant subgroup of $X_{R_d/(f,g)}$ and so $h(\alpha_{R_d/\mathfrak{p}}) \leq h(\alpha_{R_d/(f,g)}) = 0$. □

Example 3.29. Let $d > 1$, $c \in (\overline{\mathbb{Q}}^*)^d$, and let $\mathfrak{m}_c$ be the vanishing ideal of c as in Section 3.2. The ideal $\mathfrak{m}_c$ is prime and non-principal. By the previous Theorem 3.28 it follows $h(\alpha_{R_d/\mathfrak{m}_c}) = 0$.

Let $A \in \mathrm{GL}_n(\mathbb{Q})$ and let X^A be the compact connected abelian group with the $\mathbb{Z}$-action as defined in Section 3.2. In the next theorem we determine the entropy of this $\mathbb{Z}$-action.

Theorem 3.30. *Let $A \in GL_n(\mathbb{Q})$ and let X^A be the compact connected abelian group with the shift action σ as defined in Section 3.2. Let $\chi_A \in \mathbb{Q}[t]$ be the characteristic polynomial of A and let $a \in \mathbb{N}$ be the lowest common multiple of the denominators of the coefficients of χ_A. Then*

$$h(\sigma) = m(a\chi_A).$$

Proof. We know that $M^A = \widehat{X^A}$ is a noetherian R_1-module. As in the proof of 3.10 there exist primitive polynomials $f_1, \ldots, f_r \in R_1$ with $f_i | f_{i+1}, i = 1, \ldots, r-1$, and a surjective morphism $X_{f_1} \times \ldots \times X_{f_r} \to X^A$ with finite kernel. Then

$$h(\sigma) = h(\alpha_{f_1} \times \ldots \times \alpha_{f_r}) = \sum_{i=1}^{r} m(f_i) = m\Big(\prod_{i=1}^{r} f_i\Big).$$

But up to a unit in R_1, $a\chi_A$ equals the product $\prod_{i=1}^r f_i$ and so $m(a\chi_A) = m\big(\prod_{i=1}^r f_i\big)$. □

Chapter 4

p-adically expansive algebraic $\mathbb{Z}^d$-actions

In this chapter we introduce the notion of p-adically expansive algebraic $\mathbb{Z}^d$-actions and define p-adic entropy for these actions.

With our definition, the usual $\mathbb{Z}^d$-action on the compact group $X_f, f \in M_n(R_d)$, is p-adically expansive if and only if $f \in \mathrm{GL}_n(c_0(\mathbb{Z}^d))$, where R_d denotes the ring $R_d = \mathbb{Z}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ and $c_0(\mathbb{Z}^d) = \mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$ where

$$\mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle := \left\{ \sum_{\nu \in \mathbb{Z}^d} x_\nu t_1^{\nu_1} \dots t_d^{\nu_d} : x_\nu \in \mathbb{Q}_p, |x_\nu|_p \to 0 \text{ for } \sum_{i=1}^{d} |\nu_i| \to \infty \right\}.$$

As far as p-adic entropy is concerned, we do not know how to generalize periodic p-adic entropy for a greater class of algebraic $\mathbb{Z}^d$-actions. Instead, we will use the p-adic Fuglede-Kadison determinant to define p-adic entropy for the class of p-adically expansive $\mathbb{Z}^d$-actions. It will turn out that the connection between p-adically expansive $\mathbb{Z}^d$-actions and p-adic entropy can best be described in the framework of the lower algebraic K-groups and the localisation sequence of K-theory which gives a connection of the K-groups.

In Section 4.1 we provide some material on algebraic K-theory which will be used in the following sections.

In Section 4.2 we define p-adic expansiveness. We prove a criterion for p-adic expansiveness which is a p-adic analogue of the criterion for expansiveness presented in Section 3.1.

In Section 4.3 we attach to a p-adically expansive $\mathbb{Z}^d$-action on a compact abelian group X an element

$$cl_p(X) \in K_1(R_d[S_p^{-1}])/R_d^*,$$

where $R_d[S_p^{-1}]$ is the localisation of the ring R_d with respect to the multiplicative system $S_p = R_d \cap c_0(\mathbb{Z}^d)^*$. We define a homomorphism

$$\log_p \det_{\mathbb{Z}^d} : K_1(R_d[S_p^{-1}])/R_d^* \to \mathbb{Q}_p.$$

We then define the p-adic entropy $h_p(X)$ of a p-adically expansive $\mathbb{Z}^d$-action on X by $\log_p \det_{\mathbb{Z}^d}(cl_p(X))$. We show that

$$h_p(X_f) = h_{p,per}(X_f)$$

for p-adically expansive $\mathbb{Z}^d$-actions of the form X_f, $f \in M_n(R_d)$.

In Section 4.4 we give some applications.

4.1 Some basics in algebraic K-theory

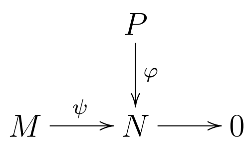
Definition 4.1. *An exact category is an additive category $\mathcal{C}$ embeddable as a full subcategory of an abelian category $\mathcal{A}$ such that $\mathcal{C}$ is equipped with a class $\mathcal{E}$ of short exact sequences $0 \to M' \to M \to M'' \to 0$ (I) satisfying*

(1) $\mathcal{E}$ is a class of sequences (I) in $\mathcal{C}$ that are exact in $\mathcal{A}$.

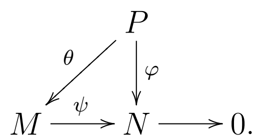
(2) $\mathcal{C}$ is closed under extensions in $\mathcal{A}$, i.e. if (I) is an exact sequence in $\mathcal{A}$ and $M', M'' \in \mathcal{C}$, then $M \in \mathcal{C}$.

Before we can introduce the exact categories that we will interested in we need the following definition.

Definition 4.2. *Let R be a not necessarily commutative unital ring. A projective left module over R is a R-module P with the property that whenever one has a diagram of R-modules with exact bottom row*



it can be completed to a commutative diagram



We will consider the following exact categories:

Examples 4.3. Let R be a not necessarily commutative unital ring.

(i) The category $\mathcal{P}(R)$ of finitely generated projective left R-modules is a full subcategory of the abelian category $\mathrm{Mod}(R)$ of left R-modules. Let $\mathcal{E}$ the class of all short sequences in $\mathcal{P}(R)$ which are exact in $\mathrm{Mod}(R)$. Then condition (1) of Definition 4.1 is satisfied. An exact sequence

$$0 \to P' \to M \to P \to 0$$

with P a projective module will split, i.e. $M \simeq P' \oplus P$. If we assume P' also to be projective, then M is projective too, so $\mathcal{P}(R)$ also satisfies condition (2) of 4.1. Thus, $\mathcal{P}(R)$ is an exact category.

(ii) Let S be a central multiplicative system in R, i.e. S is a subset of R which is closed under multiplication and for $s \in S$ it is $sr = rs$ for all $r \in R$. Then, the category $\mathcal{M}_S(R)$ of finitely generated S-torsion left R-modules with the class $\mathcal{E}$ of all short sequences in $\mathcal{M}_S(R)$ which are exact in $\mathrm{Mod}(R)$ is an exact category.

Definition 4.4. *For an exact category $\mathcal{C}$ such that isomorphism classes (C) of $\mathcal{C}$-objects form a set, define $K_0(\mathcal{C})$ to be the free abelian group on the isomorphism classes of $\mathcal{C}$-objects modulo the subgroup which is generated by all $(C) - (C') - (C'')$ for each short exact sequence $0 \to C' \to C \to C'' \to 0$ in $\mathcal{C}$.*

Definition 4.5. *Let R be a not necessarily commutative unital ring. Let $\mathcal{P}(R)$ be the exact category of finitely generated projective left modules. Define*

$$K_0(R) := K_0(\mathcal{P}(R)).$$

We think of $K_0(R)$ together with the assignment which sends a finitely generated projective R-module P to its class $[P]$ in $K_0(R)$ as the universal dimension for finitely generated projective R-modules. Namely, suppose we are given an abelian group A and an assignment d which associates to every finitely generated projective R-module an element $d(P) \in A$ and which is additive in short exact sequences, i.e. it is $d(P') + d(P'') = d(P)$ for any exact sequence $0 \to P' \to P \to P'' \to 0$ of finitely generated projective R-modules. Then there exists a unique homomorphism $\phi : K_0(R) \to A$ with $\phi([P]) = d(P)$.

If S is a central multiplicative in R, then we will also consider the K_0-group of the exact category $\mathcal{M}_S(R)$ of finitely generated S-torsion left R-modules. Then the analogous statement concerning the universal dimension for finitely generated S-torsion left R-modules holds for $K_0(\mathcal{M}_S(R))$.

Next, we want to introduce the abelian group $K_1(R)$ attached to a ring R. Let $\mathrm{GL}(R) = \cup_{n=1}^{\infty}\mathrm{GL}_n(R)$ be the infinite general linear group, where the inclusion $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R)$ is given by $a \mapsto \left(\begin{smallmatrix} a & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Let $E_n(R) \subset \mathrm{GL}_n(R)$ be the subgroup of elementary matrices, i.e. the group generated by matrices with 1's on the diagonal and at most one further non-zero entry. Let $E(R)$ be their union.

Proposition 4.6 (Whitehead Lemma)**.** *The subgroup $E(R) \subset GL(R)$ is precisely equal to the commutator subgroup of $GL(R)$.*

Proof. See [Mil71], Lemma 3.1. □

Definition 4.7. *For an unital ring R we define*

$$K_1(R) := GL(R)/E(R) = GL(R)^{ab}.$$

Let us assume that R is commutative. Then there are homomorphisms

$$\mathrm{rk} : K_0(R) \to H_0(R)$$

and

$$\det : K_1(R) \to R^*$$

which we will introduce now.

Let $\mathrm{Spec}(R)$ be as usual the prime spectrum of R with the Zariski topology. If P is a finitely generated projective R-module then for every prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ the localisation $P_\mathfrak{p}$ is a finitely generated free module over the local ring $R_\mathfrak{p}$ and thus has a well-defined rank $\mathrm{rk}_\mathfrak{p}(P_\mathfrak{p})$. If we endow $\mathbb{Z}$ with the discrete topology, then for every $P \in \mathcal{P}(R)$ the rank function

$$\mathrm{rk}(P) : \mathrm{Spec}(R) \to \mathbb{Z}, \quad \mathfrak{p} \mapsto \mathrm{rk}(P)(\mathfrak{p}) := \mathrm{rk}_\mathfrak{p}(P_\mathfrak{p})$$

is continuous, see [Bas68], Chapter III, Theorem 7.1. Let

$$H_0(R) := \big\{f : \mathrm{Spec}(R) \to \mathbb{Z} : f \text{ continuous}\big\}.$$

Then we have a natural homomorphism

$$\mathrm{rk} : K_0(R) \to H_0(R), \quad [P] \mapsto \mathrm{rk}(P).$$

Example 4.8. Let $R = \mathbb{Z}$. Because $\mathbb{Z}$ is an integral domain the topological space $\mathrm{Spec}(\mathbb{Z})$ connected. Thus, a continuous function $f : \mathrm{Spec}(\mathbb{Z}) \to \mathbb{Z}$ with target the discrete space $\mathbb{Z}$ will be constant. It follows $H_0(\mathbb{Z}) = \mathbb{Z}$. By the structure theorem of finitely generated modules over a principal ideal domain, two projective modules over $\mathbb{Z}$ are isomorphic if and only if they have the same rank. It follows that $\mathrm{rk} : K_0(\mathbb{Z}) \to \mathbb{Z}$ is an isomorphism.

To define the homomorphism

$$\det : K_1(R) \to R^*$$

just note that the usual determinant homomorphism $\mathrm{GL}_n(R) \to R^*$ is compatible with the inclusions $\mathrm{GL}_n(R) \hookrightarrow \mathrm{GL}_{n+1}(R)$. As R is commutative, the resulting homomorphism $\mathrm{GL}(R) \to R^*$ factors through $\mathrm{GL}(R)^{ab}$ which gives us the homomorphism $\det : K_1(R) \to R^*$.

Definition 4.9. *Let R be a commutative ring. The group $SK_1(R) \subset K_1(R)$ is defined as*

$$SK_1(R) := \ker(\det : K_1(R) \to R^*).$$

Note that $\det : K_1(R) \to R^*$ is a surjective homomorphism which is split by the inclusion $R^* = \mathrm{GL}_1(R) \to K_1(R)$. Thus,

$$K_1(R) = \mathrm{SK}_1(R) \oplus R^*.$$

Example 4.10. It can be shown that for an Euclidean ring R the group $\mathrm{SK}_1(R)$ vanishes, see, for example, [Ros94], Theorem 2.3.2. Hence, it is $K_1(\mathbb{Z}) = \mathbb{Z}^* = \{\pm 1\}$.

Next, we state the so-called Fundamental Theorem of algebraic K-theory and the Localisation sequence of K-theory. These deep results of algebraic K-theory will be fundamental in our approach to p-adic expansiveness and its connection to p-adic entropy. We will state the results only for the lower algebraic K-groups K_0 and K_1 in the special case where the ring R is a commutative regular ring because when we talk about algebraic $\mathbb{Z}^d$-actions this is the case which is interesting for us.

Recall that a commutative ring R is called regular if it is noetherian and if every finitely generated R-module M has a finite resolution with finitely generated projective R-modules, i.e. for every $M \in \mathcal{M}(R)$ there exists an exact sequence

$$0 \to P_n \to \ldots \to P_0 \to M \to 0$$

with $P_i \in \mathcal{P}(R), i = 0, \ldots, n$.

Theorem 4.11 (Fundamental Theorem of K_0 and K_1 of regular rings)**.** *Let R be a commutative regular ring. Then the following holds:*

(1) The inclusions $R \hookrightarrow R[t] \hookrightarrow R[t, t^{-1}]$ induce isomorphisms

$$K_0(R) \simeq K_0(R[t]) \simeq K_0(R[t, t^{-1}]).$$

(2) It is

$$K_1(R[t,t^{-1}]) \simeq K_1(R) \oplus K_0(R).$$

Proof. See [Bas68], Chapter XII, Theorem 3.1 and Theorem 7.4. □

An easy consequence of the Fundamental Theorem is the following proposition.

Proposition 4.12. *The homomorphism* det : $K_1(R_d) \to R_d^*$ *is an isomorphism.*

Proof. Using (1) and (2) of Theorem 4.11 iteratively, it follows

$$K_1(R_d) \simeq K_1(\mathbb{Z}) \oplus \underbrace{K_0(\mathbb{Z}) \oplus \ldots \oplus K_0(\mathbb{Z})}_{\text{d times}}.$$

By Example 4.8 and Example 4.10 it is

$$K_1(\mathbb{Z}) \oplus \underbrace{K_0(\mathbb{Z}) \oplus \ldots \oplus K_0(\mathbb{Z})}_{\text{d times}} \simeq \{\pm 1\} \oplus \mathbb{Z}^d,$$

which is isomorphic to R_d^*. Then the surjective homomorphism det : $K_1(R_d) \to R_d^*$ has to be an isomorphism. □

Theorem 4.13 (Localisation Sequence)**.** *Let R be a commutative regular ring and let S be a multiplicative system in R. Then there exist natural homomorphisms δ, ε such that the following sequence is exact:*

$$K_1(R) \to K_1(R_S) \xrightarrow{\delta} K_0(\mathcal{M}_S(R)) \xrightarrow{\varepsilon} K_0(R) \to K_0(R_S) \to 0,$$

where R_S is the localisation of R with respect to S.

Proof. For a proof see [Bas68], Chapter IX, Theorem 6.3 and Corollary 6.4.

For us it is important to know how the homomorphisms δ and ε are defined. Because R is regular, any $M \in \mathcal{M}_S(R)$ has a finite $\mathcal{P}(R)$-resolution, i.e. there exists an exact sequence $0 \to P_n \to \ldots \to P_0 \to M \to 0$ where $P_i \in \mathcal{P}(R)$. Define $\varepsilon([M]) = \sum(-1)^i[P_i] \in K_0(R)$. The map δ is defined as follows: if $\alpha \in \mathrm{GL}_n(R_S)$, let $s \in S$ be a common denominator for all entries of α such that $\beta = s\alpha$ has entries in R. Then $R^n/\beta R^n$ and R^n/sR^n have natural $\mathcal{P}(R)$-resolutions

$$0 \to R^n \xrightarrow{\beta} R^n \to R^n/\beta R^n \to 0 \quad \text{and}$$
$$0 \to R^n \xrightarrow{s} R^n \to R^n/sR^n \to 0.$$

Furthermore, R^n/sR^n and $R^n/\beta R^n$ are both S-torsion. For R^n/sR^n this is clear. To see that $R^n/\beta R^n$ is S-torsion, let $t \in S$ be such that $\alpha^{-1}t = \gamma$ has entries in R. Then $\gamma R^n \subset R^n$ implies that $tR^n \subset \alpha R^n$ and hence that $stR^n \subset s\alpha R^n = \beta R^n$. Then $st \in S$ annihilates $R^n/\beta R^n$. We now define $\delta([\alpha]) = [R^n/\beta R^n] - [R^n/sR^n]$. □

Let S be a multiplicative subset of the ring R_d. We end this section with two lemmata describing when a finitely generated R_d-module M is S-torsion.

Lemma 4.14. *Let I be an ideal in R_d and let S be a multiplicative subset of R_d. Then $R_d/I \in \mathcal{M}_S(R_d)$ if and only if $I \cap S \neq \emptyset$.*

Proof. The R_d-module R_d/I is S-torsion if and only if the unit $\bar{1} \in R_d/I$ is annihilated by some $s \in S$. This is exactly the case if $I \cap S \neq \emptyset$. □

Lemma 4.15. *Let M be a finitely generated R_d-module and let S be a multiplicative subset of R_d. Then $M \in \mathcal{M}_S(R_d)$ if and only if $S \cap \mathfrak{p} \neq \emptyset$ for every associated prime ideal $\mathfrak{p} \in Ass(M)$.*

Proof. If M is S-torsion then any submodule $M' \subset M$ is S-torsion. By definition, an associated prime ideal $\mathfrak{p}$ is the annihilator of some non-zero element $m \in M$, i.e. it is $\mathfrak{p} = \{f \in R_d; fm = 0 \text{ for some } m \in M \setminus \{0\}\}$. Then the submodule $\langle m \rangle \subset M$ generated by m is isomorphic to $R_d/\mathfrak{p}$. By the previous lemma this module is S-torsion if and only if $\mathfrak{p} \cap S \neq \emptyset$.

For the other implication assume $S \cap \mathfrak{p} \neq \emptyset$ for every $\mathfrak{p} \in \text{Ass}(M)$. There is a filtration $M = M_s \supset \ldots \supset M_0 = 0$ such that $M_r/M_{r-1} \simeq R_d/\mathfrak{q}_r$, $r = 1, \ldots, s$, where $\mathfrak{q}_r$ is a prime ideal lying above some associated prime ideal of M. By the previous lemma the quotients M_r/M_{r-1} are S-torsion. Whenever one has an exact sequence of R_d-modules

$$0 \to N' \to N \to N'' \to 0$$

such that N' and N'' are S-torsion then also N is S-torsion. So from the prime filtration of M it follows inductively that M is S-torsion. □

4.2 p-adically expansive $\mathbb{Z}^d$-actions

Let $R_d = \mathbb{Z}[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$ and let $S_p \subset R_d$ be the multiplicative system $S_p = R_d \cap c_0(\mathbb{Z}^d)^*$.

Definition 4.16. *An algebraic $\mathbb{Z}^d$-action on the compact abelian group X is called p-adically expansive if the R_d-module M^X is finitely generated and S_p-torsion, i.e. $M^X \in \mathcal{M}_{S_p}(R_d)$.*

Lemma 4.17. *Let $A \in M_n(R_d)$ and let $M = (R_d)^n/(A(R_d)^n)$. Then $M \in \mathcal{M}_{S_p}(R_d)$ if and only if $A \in GL_n(c_0(\mathbb{Z}^d))$.*

Proof. Assume $M \in \mathcal{M}_{S_p}(R_d)$. Then for any $x \in (R_d)^n$ there exists an element $s \in S$ such that sx lies in the image of A. It follows that the localised homomorphism $A_{S_p} : (R_d^n)_{S_p} \to (R_d^n)_{S_p}$ is surjective.

Consider the exact sequence

$$0 \to \ker A_{S_p} \to (R_d^n)_{S_p} \overset{A_{S_p}}{\to} (R_d^n)_{S_p} \to 0.$$

Because the quotient field $\mathrm{Frac}(R_d)$ of R_d is flat over $R_d[S_p^{-1}]$ this sequence stays exact after tensoring with $\otimes_{R_d[S_p^{-1}]}\mathrm{Frac}(R_d)$. It follows

$$\ker A_{S_p} \otimes_{R_d[S_p^{-1}]} \mathrm{Frac}(R_d) = 0.$$

But as $\ker A_{S_p}$ is a torsion-free $R_d[S_p^{-1}]$-module it follows $\ker A_{S_p} = 0$. So A_{S_p} is an isomorphism and thus $\det A \in S_p$ which shows $A \in \mathrm{GL}_n(c_0(\mathbb{Z}^d))$.

If we assume on the other hand that $M \in \mathrm{GL}_n(c_0(\mathbb{Z}^d))$ then $\det A \in S_p$. Let $\tilde{A} \in M_n(R_d)$ be the adjoint matrix of A. The matrix $\tilde{A}$ has entries $\tilde{a}_{ij} = (-1)^{i+j}\det(A_{ji})$ where A_{ji} is the $(n-1)\times(n-1)$-matrix obtained from A by deleting the j-th row and i-th column. It is a known fact from linear algebra that $A\tilde{A} = \tilde{A}A = \det A \cdot \mathrm{Id}$, where Id is the identity matrix. For any $m \in M$ it follows $\det A \cdot m = \tilde{A}(Am) = 0$, i.e. M is S_p-torsion. □

Recall that for an element $f \in M_n(R_d)$ the dynamical system X_f is the Pontrjagin dual of the module $(R_d)^n/(R_d)^n f$.

Corollary 4.18. *Let $f \in M_n(R_d)$. The $\mathbb{Z}^d$-action on X_f is p-adically expansive if and only if $f \in GL_n(c_0(\mathbb{Z}^d))$.*

Proof. To be precise, it is $M^{X_f} = (R_d)^n/(R_d)^n f = (R_d)^n/f^t(R_d)^n$, where f^t is the transpose of f. It is $f \in \mathrm{GL}_n(c_0(\mathbb{Z}^d))$ if and only if $f^t \in \mathrm{GL}_n(c_0(\mathbb{Z}^d))$. Now apply Lemma 4.17 to f^t. □

The next proposition gives a characterization of p-adically expansive $\mathbb{Z}^d$-actions which is analogous to the characterization of expansive $\mathbb{Z}^d$-actions. Recall that for an ideal $I \subset R_d$ the set of zeroes of I over $\overline{\mathbb{Q}}_p$ is defined as

$$V_{\overline{\mathbb{Q}}_p}(I) = \left\{ z \in (\overline{\mathbb{Q}}_p^*)^d : f(z) = 0 \text{ for all } f \in I \right\}.$$

The p-adic d-torus T_p^d is the set

$$T_p^d = \left\{ z \in \mathbb{C}_p^d : |z_i| = 1, 1 \le i \le d \right\}.$$

Proposition 4.19. *Let α be an algebraic $\mathbb{Z}^d$-action on X. Assume the corresponding R_d-module M^X is noetherian. Then the following properties are equivalent.*

(i) α is p-adically expansive.

(ii) For every prime ideal $\mathfrak{p} \in Ass(M^X)$ it is $V_{\overline{\mathbb{Q}}_p}(\mathfrak{p}) \cap T_p^d = \emptyset$.

Proof. First notice that for any ideal $I \subset R_d$, the maximal ideals of the algebra $\mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$ containing I correspond to the orbits of the $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-operation on $V_{\overline{\mathbb{Q}}_p}(I) \cap T_p^d$.

If α is p-adically expansive, i.e. $M^X \in \mathcal{M}_{S_p}(R_d)$, then by Lemma 4.15 every $\mathfrak{p} \in \mathrm{Ass}(M^X)$ contains an element which is a unit in $\mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$. This means that there is no maximal ideal in $\mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$ which contains $\mathfrak{p}$. It follows $V_{\overline{\mathbb{Q}}_p}(\mathfrak{p}) \cap T_p^d = \emptyset$.

Assume now that (ii) holds. This implies that for every $\mathfrak{p} \in \mathrm{Ass}(M^X)$ there is no maximal ideal in $\mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$ which contains $\mathfrak{p}$, i.e. every $\mathfrak{p} \in \mathrm{Ass}(M^X)$ generates the unit ideal in $\mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$. We show that this implies $\mathfrak{p} \cap S_p \neq \emptyset$. Then by Lemma 4.15 it follows that $M^X \in \mathcal{M}_{S_p}(R_d)$.

Let $f_1, \dots, f_r \in R_d$ be generators of the ideal $\mathfrak{p}$. Because $\mathfrak{p}$ generates the unit ideal in $\mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$ we find elements $g_i' \in \mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$ such that $\sum_{i=1}^r g_i' f_i = 1$. Then by multiplying with a suitable power of p, say p^n, we get an equality $\sum_{i=1}^r g_i f_i = p^n$ with $g_i \in \mathbb{Z}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$. Because R_d is dense in $\mathbb{Z}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$ we find $h_i \in R_d$ such that $h_i - g_i \in p^{n+1}\mathbb{Z}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$. Then the element

$$\sum_{i=1}^r h_i f_i \in p^n(1 + p\mathbb{Z}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle)$$

lies in $\mathfrak{p}$ and is a unit in $\mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$ because it is a product of the unit p^n with a 1-unit, i.e. an element in $1 + p\mathbb{Z}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$. □

Corollary 4.20. *A dynamical system of type $X_{R_d/I}$ with I generated by $f_1, \dots, f_r$ in R_d is p-adically expansive if and only if the $f_1, \dots, f_r$ generate the unit ideal in $c_0(\mathbb{Z}^d)$, i.e.*

$$V_{\overline{\mathbb{Q}}_p}(I) \cap T_p^d = \emptyset.$$

Proof. It is $V_{\overline{\mathbb{Q}}_p}(I) = \bigcup_{\mathfrak{p} \in \mathrm{Ass}(R_d/I)} V_{\overline{\mathbb{Q}}_p}(\mathfrak{p})$. It follows that $V_{\overline{\mathbb{Q}}_p}(I) \cap T_p^d = \emptyset$ if and only if $V_{\overline{\mathbb{Q}}_p}(\mathfrak{p}) \cap T_p^d = \emptyset$ for every $\mathfrak{p} \in \mathrm{Ass}(M^X)$. □

Proposition 4.21. *Let (X, α) be p-adically expansive. Then for every subgroup $\Lambda \subset \mathbb{Z}^d$ of finite index the set $Fix_\Lambda(\alpha)$ is finite.*

Proof. By Theorem 3.3, we have to show that for every $\mathfrak{p} \in \mathrm{Ass}(M^X)$ we have $V_{\mathbb{C}}(\mathfrak{p}) \cap V_{\mathbb{C}}(I(\Lambda)) = \emptyset$, where $I(\Lambda)$ is the ideal generated by all expressions $T_1^{n_1} \dots T_d^{n_d} - 1,\ (n_1, \dots, n_d) \in \Lambda$.

As the ideals $\mathfrak{p}$ and $I(\Lambda)$ are defined over the rationals it follows by the Nullstellensatz that this is exactly the case if $V_{\overline{\mathbb{Q}}}(\mathfrak{p}) \cap V_{\overline{\mathbb{Q}}}(I(\Lambda)) = \emptyset$ for every $\mathfrak{p} \in \mathrm{Ass}(M^X)$.

We fix an embedding $\overline{\mathbb{Q}} \subset \overline{\mathbb{Q}}_p$. Note that under any such embedding it is $V_{\overline{\mathbb{Q}}}(I(\Lambda)) \subset T_p^d$ because for any $z = (z_1, \dots, z_d) \in V_{\overline{\mathbb{Q}}}(I(\Lambda))$ the $z_i \in \overline{\mathbb{Q}}^*$ are of finite order.

So if we assume that there exists an ideal $\mathfrak{p} \in \mathrm{Ass}(M^X)$ such that $V_{\overline{\mathbb{Q}}}(\mathfrak{p}) \cap V_{\overline{\mathbb{Q}}}(I(\Lambda)) \neq \emptyset$ then $V_{\overline{\mathbb{Q}}_p}(\mathfrak{p}) \cap T_p^d \neq \emptyset$ which contradicts the assumption that (X, α) is p-adically expansive. □

Let us give a second characterization of p-adically expansive $\mathbb{Z}^d$-actions. We say an abelian group X has bounded p-torsion if there exists a natural number $n \in \mathbb{N}$ such that

$$X(p) := \bigcup_{i=1}^{\infty} \ker(p^i : X \to X) = \ker(p^n : X \to X).$$

Proposition 4.22. *Let α be an algebraic $\mathbb{Z}^d$-action on X. Then α is p-adically expansive if and only if M^X is noetherian and X has bounded p-torsion.*

Proof. First we prove that for any ideal $I \subset R_d$ the Pontrjagin dual $\widehat{R_d/I}$ has bounded p-torsion if and only if I generates the unit ideal in $\mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$.

Using this fact we then show that X has bounded p-torsion if and only if for every prime ideal $\mathfrak{p} \in \mathrm{Ass}(M^X)$ it is $V_{\overline{\mathbb{Q}}_p}(\mathfrak{p}) \cap T_p^d = \emptyset$ which gives the result by Proposition 4.19.

So let us assume that the ideal $I \subset R_d$ generates the unit ideal in $\mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$. Let $f_1, \dots, f_r \in R_d$ be generators of I and assume we have $1 = \sum_{i=1}^r f_r g_r'$ with elements $g_r' \in \mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$. Multiplying this relation by a suitable power of p, say p^n, we get $p^n = \sum_{i=1}^r f_i g_i$ with elements $g_i \in \mathbb{Z}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$, i.e.

$$p^n = \sum_{i=1}^r f_i g_i \in I \cdot \mathbb{Z}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle. \tag{4.1}$$

For any natural number $r \geq 1$, there is a natural isomorphism

$$\mathbb{Z}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]/(p^r) \cong \mathbb{Z}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle/(p^r).$$

It follows that

$$\mathbb{Z}[t_1^{\pm 1},\dots,t_d^{\pm 1}]/(p^r,I)\cong \mathbb{Z}_p\langle t_1^{\pm 1},\dots,t_d^{\pm 1}\rangle/(p^r,I).$$

Now for any $r\geq n$ consider the commutative diagram

$$\begin{array}{ccc} \mathbb{Z}[t_1^{\pm 1},\dots,t_d^{\pm 1}]/(p^r,I) & \xrightarrow{\sim} & \mathbb{Z}_p\langle t_1^{\pm 1},\dots,t_d^{\pm 1}\rangle/(p^r,I) \\ \downarrow & & \| \\ \mathbb{Z}[t_1^{\pm 1},\dots,t_d^{\pm 1}]/(p^n,I) & \xrightarrow{\sim} & \mathbb{Z}_p\langle t_1^{\pm 1},\dots,t_d^{\pm 1}\rangle/(p^n,I) \end{array} \quad ,$$

where the left vertical arrow is the canonical projection, the right vertical arrow is the identity by equation (4.1) and the horizontal arrows are isomorphism. We conclude that the projection

$$\mathbb{Z}[t_1^{\pm 1},\dots,t_d^{\pm 1}]/(p^r,I)\to \mathbb{Z}[t_1^{\pm 1},\dots,t_d^{\pm 1}]/(p^n,I)$$

is also an isomorphism for $r\geq n$, i.e. the ideals $(p^r,I)\subset R_d$ are equal for all $r\geq n$. Because

$$\ker(p^r:\widehat{R_d/I}\to\widehat{R_d/I})\cong \widehat{R_d/(p^r,I)}, \tag{4.2}$$

we see that $\widehat{R_d/I}$ has bounded p-torsion. In fact, we see that the bound on the p-torsion of $\widehat{R_d/I}$ is given by the smallest number $n\geq 0$ such that $p^n\in I\cdot\mathbb{Z}_p\langle t_1^{\pm 1},\dots,t_d^{\pm 1}\rangle$.

If on the other hand $\widehat{R_d/I}$ has bounded p-torsion, then by equation (4.2) there exist a natural number $n\in\mathbb{N}$ such that $(p^r,I)=(p^n,I)$ for all $r\geq n$. Then we can write

$$p^n=p^{n+1}g_0+\sum_{i=1}^{r}f_ig_i \text{ with elements } f_i\in I \text{ and } g_i\in R_d.$$

It follows that $p^n(1-pg_0)\in I$. But as a product of a unit in $\mathbb{Q}_p$ with the 1-unit $1-pg_0$ the element $p^n(1-pg_0)$ is a unit in $\mathbb{Q}_p\langle t_1^{\pm 1},\dots,t_d^{\pm 1}\rangle$. This proves that I generates the unit ideal in $\mathbb{Q}_p\langle t_1^{\pm 1},\dots,t_d^{\pm 1}\rangle$.

Now let us show that X has bounded p-torsion if $V_{\overline{\mathbb{Q}}_p}(\mathfrak{p})\cap T_p^d=\emptyset$ for all $\mathfrak{p}\in\mathrm{Ass}(M^X)$. By Proposition 3.2 we find a filtration $M=M_s\supset\ldots\supset M_0=\{0\}$ such that for every $i=1,\dots,s$, $M_i/M_{i-1}\cong R/\mathfrak{q}_i$ for some prime ideal $\mathfrak{q}_i\subset R$, and $\mathfrak{q}_i\supset\mathfrak{p}$ for some $\mathfrak{p}\in\mathrm{Ass}(M^X)$. As $\widehat{R/\mathfrak{q}_i}\hookrightarrow\widehat{R/\mathfrak{p}}$ and as $\widehat{R/\mathfrak{p}}$ has bounded p-torsion, the $\widehat{M_i/M_{i-1}}$ have bounded p-torsion. If we have an exact sequence

$$0\to N'\to N\to N''\to 0$$

of abelian groups such that N' and N'' have bounded p-torsion, so does N. That way we can deduce inductively that $X = \widehat{M}$ has bounded p-torsion.

For the converse implication let us assume that X has bounded p-torsion and that M^X is noetherian. As $\ker(p^n : X \to X) = \widehat{M^X/p^n M^X}$ saying that X has bounded p-torsion exactly means that the p-filtration

$$M^X \supset pM^X \supset p^2 M^X \supset \dots$$

is stable, i.e. $p^n M^X = p^{n+1} M^X$ for all n bigger than some fixed $n_0 \in \mathbb{N}$. We want to show that this implies that for every associated prime $\mathfrak{p} \in \mathrm{Ass}(M^X)$ we have $V_{\bar{\mathbb{Q}}_p}(\mathfrak{p}) \cap T_p^d = \emptyset$.

Let N be a submodule of M^X such that $N \simeq R_d/\mathfrak{p}$. If we can show that $p^n N = p^{n+1} N$ for all n large enough we are done: Because it is

$$p^n N/p^{n+1} N \simeq \frac{(p^n, \mathfrak{p})/\mathfrak{p}}{(p^{n+1}, \mathfrak{p})/\mathfrak{p}} \simeq (p^n, \mathfrak{p})/(p^{n+1}, \mathfrak{p}),$$

the equation $p^n N = p^{n+1} N$ would imply that the ideals $(p^n, \mathfrak{p})$ and $(p^{n+1}, \mathfrak{p})$ in R_d are equal. As before, this implies that $\mathfrak{p}$ contains an element which is a unit in $\mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$. Then $V_{\bar{\mathbb{Q}}_p}(\mathfrak{p}) \cap T_p^d = \emptyset$.

Thus, we have to show that the p-filtration on N is stable. By the Artin-Rees Lemma, see for example [AM69], Proposition 10.9, there exists a natural number $s \in \mathbb{N}$ such that for all $r \in \mathbb{N}$ we have $N \cap p^{r+s} M^X = p^r(N \cap p^s M^X)$. If we assume $n \geq \max\{s, n_0\}$ then

$$p^n N \supset p^n(N \cap p^n M^X) = N \cap p^{2n} M^X = N \cap p^n M^X \supset p^n N,$$

i.e. $p^n N = p^n(N \cap p^n M^X) = N \cap p^n M^X$. It follows

$$p^{n+1} N = N \cap p^{n+1} M^X = N \cap p^n M^X = p^n N.$$

□

We want to finish this section with a little observation made for dynamical systems X_f, where $f \in R_d$ is already a unit in $c_0(\mathbb{Z}^d, \mathbb{Z}_p)$. As the proof of Proposition 4.22 shows, the group X_f has no p-torsion in this case. Furthermore, we know that for every subgroup N of $\mathbb{Z}^d$ of finite index, $\mathrm{Fix}_N(X_f)$ is finite by Proposition 4.21. The next proposition tells us that the collection of $\mathrm{Fix}_N(X_f)$ for all cofinite N in $\mathbb{Z}^d$ already gives us some information on X_f concerning p-adic expansiveness. Before we can prove the proposition we need the following result.

Lemma 4.23. *Let N be a subgroup of finite index of $\mathbb{Z}^d$, i.e. it is $N = r_1\mathbb{Z} \times \ldots \times r_d\mathbb{Z}$ for some natural numbers $r_1, \ldots, r_d$. For any field K the kernel of the canonical surjective homomorphism*

$$\pi_N : K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}] \to K[\mathbb{Z}^d/N]$$

is the ideal $(t_1^{r_1} - 1, \ldots, t_d^{r_d} - 1)$. Thus, we have an isomorphism

$$K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]/(t_1^{r_1} - 1, \ldots, t_d^{r_d} - 1) \simeq K[\mathbb{Z}^d/N]$$

Proof. Obviously, it is $(t_1^{r_1} - 1, \ldots, t_d^{r_d} - 1) \subset \ker \pi_N$.

To prove $\ker \pi_N \subset (t_1^{r_1} - 1, \ldots, t_d^{r_d} - 1)$, first note that given integers $s_i, 1 \leq i \leq d$, the element

$$\prod_{i=1}^{d} t_i^{r_i s_i} - 1 = \sum_{i=1}^{d} \Big((t_i^{r_i s_i} - 1) \prod_{k>i} t_k^{r_k s_k} \Big) \in (t_1^{r_1} - 1, \ldots, t_d^{r_d} - 1) \tag{4.3}$$

is contained in the ideal $(t_1^{r_1} - 1, \ldots, t_d^{r_d} - 1)$ because the elements $t_i^{r_i s_i} - 1$ are contained in $(t_1^{r_1} - 1, \ldots, t_d^{r_d} - 1)$.

Let $\big\{[j_1, \ldots, j_d] : 0 \leq j_i < r_i,\ 1 \leq i \leq d\big\}$ be a full set of representatives of the elements in $\mathbb{Z}^d/N$. Given a multiindex $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{Z}^d$, we write it in the form $\nu = (j_1 + r_1 s_1, \ldots, j_d + r_d s_d)$. It is

$$a_\nu t_1^{\nu_1} \ldots t_d^{\nu_d} = \Big(\prod_{i=1}^{d} t_i^{r_i s_i} - 1 \Big) a_\nu t_1^{j_1} \ldots t_d^{j_d} + a_\nu t_1^{j_1} \ldots t_d^{j_d}. \tag{4.4}$$

By (4.3) and (4.4) it is for any $f = \sum_{\nu \in \mathbb{Z}^d} a_\nu t_1^{\nu_1} \ldots t_d^{\nu_d} \in K[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$

$$f = \sum_{\substack{0 \leq j_i < r_i, \\ 1 \leq i \leq d}} \Big(\sum_{\nu \in [j_1, \ldots, j_d]} a_\nu \Big) t_1^{j_1} \ldots t_d^{j_d} \mod (t_1^{r_1} - 1, \ldots, t_d^{r_d} - 1).$$

Now, it is $f \in \ker \pi_N$ if and only if $\sum_{\nu \in [j_1, \ldots, j_d]} a_\nu = 0$ for all $\nu \in \mathbb{Z}^d$. This implies $\ker \pi_N = (t_1^{r_1} - 1, \ldots, t_d^{r_d} - 1)$. □

Proposition 4.24. *Let $f \in R_d$, and let α_f be the usual $\mathbb{Z}^d$-action on X_f. The following conditions are equivalent.*

(i) f is invertible in $c_0(\mathbb{Z}^d, \mathbb{Z}_p)$.

(ii) The reduction $\bar{f}$ of f is invertible in $\mathbb{F}_p[t_1^{\pm 1}, \ldots, t_d^{\pm 1}]$.

(iii) The image $\bar{f}_N$ of f in $\mathbb{F}_p[\mathbb{Z}^d/N]$ is invertible for every subgroup N of finite index.

(iv) For every cofinite subgroup N of $\mathbb{Z}^d$ the group $\mathrm{Fix}_N(X_f)$ is finite and its order is not divisible by p.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is the special case $\Gamma = \mathbb{Z}^d$ of Lemma 7.17, where the analogue statement is proven for any residually finite group Γ.

The implication (ii) $\Rightarrow$ (iii) is clear. For the converse direction, we show that (iii) implies that $\bar{f}$ is not contained in any maximal ideal of $\overline{\mathbb{F}}_p[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$, where $\overline{\mathbb{F}}_p$ is an algebraic closure of $\mathbb{F}_p$.

Let us assume that $\bar{f}$ is contained in a maximal ideal of $\overline{\mathbb{F}}_p[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$, say $\bar{f} \in (t_1 - \alpha_1, \dots, t_d - \alpha_d)$ with $\alpha_1, \dots, \alpha_d \in (\overline{\mathbb{F}}_p)^*$. The α_i are of finite order in $\overline{\mathbb{F}}_p$, i.e. there are positive integers $r_1, \dots, r_d$ such that $\alpha_i^{r_i} = 1,\ 1 \leq i \leq d$. Then we consider the cofinite subgroup $N = r_1\mathbb{Z} \times \dots \times r_d\mathbb{Z}$ of $\mathbb{Z}^d$. By Lemma 4.23, it is

$$\mathbb{F}_p[t_1^{\pm 1}, \dots, t_d^{\pm 1}]/(t_1^{r_1} - 1, \dots, t_d^{r_d} - 1) \simeq \mathbb{F}_p[\mathbb{Z}^d/N].$$

The assumption (iii) implies that the ideal $(\bar{f}, t_1^{r_1} - 1, \dots, t_d^{r_d} - 1) \subset \overline{\mathbb{F}}_p[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ is the unit ideal. Furthermore, it is $(t_1^{r_1} - 1, \dots, t_d^{r_d} - 1) \subset (t_1 - \alpha_1, \dots, t_d - \alpha_d)$. But then

$$\begin{aligned} \overline{\mathbb{F}}_p[t_1^{\pm 1}, \dots, t_d^{\pm 1}] = (\bar{f}, t_1^{r_1} - 1, \dots, t_d^{r_d} - 1) \subset (\bar{f}, t_1 - \alpha_1, \dots, t_d - \alpha_d) \\ = (t_1 - \alpha_1, \dots, t_d - \alpha_d), \end{aligned}$$

which is a contradiction.

If (i) holds, we have proven that X_f has no p-torsion which is then of course also true for the subgroups $\mathrm{Fix}_N(X_f)$. By Proposition 4.21, $\mathrm{Fix}_N(X_f)$ is finite and thus (iv) follows. On the other hand, let $N \subset \mathbb{Z}^d$ be a subgroup of finite index. Then

$$\widehat{\ker(X_{f_N} \xrightarrow{\cdot p} X_{f_N})} \simeq \mathbb{Z}[\mathbb{Z}^d/N]/(f_N, p) \simeq \mathbb{F}_p[\mathbb{Z}^d/N]/(\bar{f}_N). \tag{4.5}$$

So $\mathrm{Fix}_N(X_f)$ has no p-torsion if and only if $\mathbb{F}_p[\mathbb{Z}^d/N]/(\bar{f}_N) = 0$, i.e. if $\bar{f}_N$ is a unit in $\mathbb{F}_p[\mathbb{Z}^d/N]$. Thus, (iv) implies (iii) and we are done. $\square$

Remark 4.25. Let $f_1, \dots, f_r \in \mathbb{Z}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]$ and let I be the ideal $I = (f_1, \dots, f_r)$. Then $X_{R_d/I}(p) = 0$ is equivalent to the geometric property that the fibre over p of $\operatorname{Spec} \mathbb{Z}[t_1^{\pm 1}, \dots, t_d^{\pm 1}]/(f_1, \dots, f_r)$ is empty.

4.3 The p-adic entropy of p-adically expansive algebraic $\mathbb{Z}^d$-actions

In this section we define a notion of p-adic entropy for all p-adically expansive $\mathbb{Z}^d$-actions.

To do so we use the localisation sequence of Theorem 4.13 to attach to every p-adically expansive $\mathbb{Z}^d$-action (X, α) an element

$$cl_p(X) \in K_1(R_d[S_p^{-1}])/R_d^*.$$

Then we use the homomorphism

$$\log_p \det_{\mathbb{Z}^d} : K_1(c_0(\mathbb{Z}^d, \mathbb{Z}_p)) \to \mathbb{Q}_p$$

discussed in Section 2.3 to define a homomorphism

$$K_1(R_d[S_p^{-1}])/R_d^* \to \mathbb{Q}_p$$

also denoted by $\log_p \det_{\mathbb{Z}^d}$. The p-adic entropy of a p-adically expansive $\mathbb{Z}^d$-action will then be defined as $\log_p \det_{\mathbb{Z}^d}(cl_p(X))$.

Theorem 4.26. *Consider the multiplicative system $S_p = R_d \cap c_0(\mathbb{Z}^d)^*$ in R_d. There is an isomorphism*

$$cl_p : K_0(\mathcal{M}_{S_p}(R_d)) \to K_1(R_d[S_p^{-1}])/R_d^*$$

such that

$$cl_p([(R_d)^n/f(R_d)^n]) = [f] \mod R_d^*$$

for all $f \in M_n(R_d) \cap GL_n(c_0(\mathbb{Z}^d))$.

Proof. The ring R_d is regular. Thus, by Theorem 4.13 there is an exact sequence

$$K_1(R_d) \to K_1(R_d[S_p^{-1}]) \xrightarrow{\delta} K_0(\mathcal{M}_{S_p}(R_d)) \xrightarrow{\varepsilon} K_0(R_d) \to K_0(R_d[S_p^{-1}]) \to 0.$$

By Proposition 4.12, it is $K_1(R_d) \simeq R_d^*$ and by Theorem 4.11, (1), we know that $K_0(R_d) \simeq K_0(\mathbb{Z}) \simeq \mathbb{Z}$.

Furthermore, $K_0(R_d[S_p^{-1}])$ contains a copy of $\mathbb{Z}$, because the homomorphism rk : $K_0(R_d[S_p^{-1}]) \to H_0(R_d[S_p^{-1}]) = \mathbb{Z}$ is split by the natural homomorphism

$$\mathbb{Z} \to K_0(R_d[S_p^{-1}]), n \mapsto [(R_d[S_p^{-1}])^{n+m}] - [(R_d[S_p^{-1}])^m],$$

where m is a positive integer such that $n + m > 0$.

It follows that the surjective homomorphism $K_0(R_d) \to K_0(R_d[S_p^{-1}])$ is also injective.

Then exactness of the sequence implies that the homomorphism δ is surjective and thus induces an isomorphism

$$\bar{\delta} : K_1(R_d[S_p^{-1}])/R_d^* \to K_0(\mathcal{M}_{S_p}(R_d)).$$

We define $cl_p := \bar{\delta}^{-1}$. Then it is clear that

$$cl_p([(R_d)^n/f(R_d)^n]) = [f] \mod R_d^*$$

for $f \in M_n(R_d) \cap \mathrm{GL}_n(c_0(\mathbb{Z}^d))$ because by definition of the homomorphism δ it is $\delta([f]) = [(R_d)^n/f(R_d)^n]$. □

The next step towards our definition of a notion of p-adic entropy is the construction of a homomorphism

$$K_1(R_d[S_p^{-1}])/R_d^* \to \mathbb{Q}_p$$

which is derived from the homomorphism

$$\log_p \det_{\mathbb{Z}^d} : K_1(c_0(\mathbb{Z}^d, \mathbb{Z}_p)) \to \mathbb{Q}_p$$

constructed in Section 2.3. To do so, we need the following result.

Lemma 4.27. *Let $[f] \in K_1(c_0(\mathbb{Z}^d, \mathbb{Z}_p))$ and let f be a representative of $[f]$ in some $GL_r(c_0(\mathbb{Z}^d, \mathbb{Z}_p))$. Let $\Gamma_n \to 0$ be a family of cofinite subgroups of $\mathbb{Z}^d$ converging to 0. Denote by $f^{(n)}$ the image of f in $M_r(\mathbb{Q}_p\Gamma^{(n)})$ and let $\rho_{f^{(n)}}$ the $\mathbb{Q}_p$-endomorphism of right multiplication with f^* on $(\mathbb{Q}_p\Gamma^{(n)})^r$. Assume that*

$$\det_{\mathbb{Q}_p}(\rho_{f^{(n)}}) = \pm 1 \text{ for all } n \in \mathbb{N}. \tag{4.6}$$

Then

$$\log_p \det_{\mathbb{Z}^d}[f] = 0.$$

In particular, the homomorphism

$$\log_p \det_{\mathbb{Z}^d} : K_1(c_0(\mathbb{Z}^d, \mathbb{Z}_p)) \to \mathbb{Q}_p$$

vanishes on $SK_1(c_0(\mathbb{Z}^d, \mathbb{Z}_p))$.

Proof. By Proposition 2.33, it is

$$\log_p \det_{\Gamma}[f] = \lim_{n\to\infty} \frac{1}{(\Gamma : \Gamma_n)} \log_p \det_{\mathbb{Q}_p}(\rho_{f^{(n)}}).$$

Thus, the assumptions made in the lemma imply that $\log_p \det_{\Gamma}[f] = 1$.

Assume now that $[f] \in \mathrm{SK}_1(c_0(\mathbb{Z}^d, \mathbb{Z}_p))$. For every $n \in \mathbb{N}$ we have a homomorphism

$$\mathrm{SK}_1(c_0(\mathbb{Z}^d, \mathbb{Z}_p)) \to \mathrm{SK}_1(c_0(\Gamma^{(n)}, \mathbb{Z}_p)) = \mathrm{SK}_1(\mathbb{Z}_p\Gamma^{(n)}).$$

The endomorphism $\rho_{f^{(n)}}$ on $(\mathbb{Q}_p\Gamma^{(n)})^r$ is $\mathbb{Q}_p\Gamma^{(n)}$-linear. By [Bou70], Chapitre 3, §9, Proposition 6, we have

$$\det_{\mathbb{Q}_p}(\rho_{f^{(n)}}) = N_{\mathbb{Q}_p\Gamma^{(n)}/\mathbb{Q}_p}(\det_{\mathbb{Q}_p\Gamma^{(n)}}\rho_{f^{(n)}}) \ , \tag{4.7}$$

where $N_{\mathbb{Q}_p\Gamma^{(n)}/\mathbb{Q}_p}$ denotes the norm from the finite dimensional $\mathbb{Q}_p$-algebra $\mathbb{Q}_p\Gamma^{(n)}$ to $\mathbb{Q}_p$.

To finish the proof, we show that $[f] \in \mathrm{SK}_1(c_0(\mathbb{Z}^d, \mathbb{Z}_p))$ implies that $\det_{\mathbb{Q}_p\Gamma^{(n)}}\rho_{f^{(n)}} = 1$ for all $n \in \mathbb{N}$. Then by equation (4.7), we have $\det_{\mathbb{Q}_p}(\rho_{f^{(n)}}) = 1$ for all $n \in \mathbb{N}$. Hence, by the first part of the lemma the homomorphism $\log_p\det_{\mathbb{Z}^d}$ vanishes on $\mathrm{SK}_1(c_0(\mathbb{Z}^d, \mathbb{Z}_p))$.

By definition, the endomorphism ρ_f is the right multiplication with f^* on $(c_0(\mathbb{Z}^d))^r$. If $f = (f_{i,j})_{1\le i,j\le r}$ then $f^* = (\mathrm{inv}(f_{j,i})_{1\le i,j\le r})$, where inv is the ring homomorphism

$$\mathrm{inv} : c_0(\mathbb{Z}^d) \to c_0(\mathbb{Z}^d), \sum_{\nu\in\mathbb{Z}^d} a_\nu z_1^{\nu_1}\dots z_d^{\nu_d} \mapsto \sum_{\nu\in\mathbb{Z}^d} a_\nu z_1^{-\nu_1}\dots z_d^{-\nu_d}.$$

Hence, it is

$$\det_{c_0(\mathbb{Z}^d)}\rho_f = \mathrm{inv}(\det_{c_0(\mathbb{Z}^d)}(f)) = 1,$$

and analogously $\det_{\mathbb{Q}_p\Gamma^{(n)}}\rho_{f^{(n)}} = 1$ for all $n \in \mathbb{N}$ which finishes the proof. □

Now, consider the homomorphism

$$\log_p\det_{\mathbb{Z}^d} : K_1(c_0(\mathbb{Z}^d, \mathbb{Z}_p)) \to \mathbb{Q}_p$$

defined in Section 2.3. By Lemma 4.27, the value $\log_p\det_{\mathbb{Z}^d}[f]$ does only depend on $\det[f] \in c_0(\mathbb{Z}^d, \mathbb{Z}_p)^*$. So to extend the homomorphism $\log_p\det_{\mathbb{Z}^d}$ to $K_1(c_0(\mathbb{Z}^d))$ we just apply the determinant to get an element in $c_0(\mathbb{Z}^d)^*$ and then use that there is a unique homomorphism

$$\log_p\det_{\mathbb{Z}^d} : c_0(\mathbb{Z}^d)^* \to \mathbb{Q}_p$$

which agrees with $\log_p\det_{\mathbb{Z}^d}$ previously defined on $c_0(\mathbb{Z}^d, \mathbb{Z}_p)^*$ and satisfies $\log_p\det_{\mathbb{Z}^d}(p) = 0$.

Using the homomorphism $K_1(R_d[S_p^{-1}]) \to K_1(c_0(\mathbb{Z}^d))$ induced by the canonical inclusion $R_d[S_p^{-1}] \hookrightarrow c_0(\mathbb{Z}^d)$ we have a well-defined homomorphism $K_1(R_d[S_p^{-1}]) \to \mathbb{Q}_p$. The latter homomorphism factorises through $K_1(R_d[S_p^{-1}])/R_d^*$ because elements in R_d^* satisfy the condition (4.6) of Lemma 4.27. We summarize:

Theorem 4.28. *There is a homomorphism*

$$\log_p \det_{\mathbb{Z}^d} : K_1(R_d[S_p^{-1}])/R_d^* \to \mathbb{Q}_p$$

which is given by the bottom row of the following commutative diagram:

$$\begin{array}{ccccccc} & & K_1(c_0(\mathbb{Z}^d, \mathbb{Z}_p))/R_d^* & \longrightarrow & & & \mathbb{Q}_p \\ & & \downarrow & & & & \| \\ K_1(R_d[S_p^{-1}])/R_d^* & \longrightarrow & K_1(c_0(\mathbb{Z}^d))/R_d^* & \xrightarrow{\det} & c_0(\mathbb{Z}^d)^*/R_d^* & \xrightarrow{\log_p \det_{\mathbb{Z}^d}} & \mathbb{Q}_p. \end{array}$$

Definition 4.29. *Let α be a p-adically expansive $\mathbb{Z}^d$-action on X. Then we define*

$$cl_p(X) := cl_p([M^X]) \in K_1(R_d[S_p^{-1}])/R_d^*.$$

Definition 4.30. *Let α be a p-adically expansive $\mathbb{Z}^d$-action on X. Then we define the p-adic entropy $h_p(X)$ of X by*

$$h_p(X) := \log_p \det_{\mathbb{Z}^d}(cl_p(X)) \in \mathbb{Q}_p.$$

Lemma 4.31. *Let $X_f = \widehat{(R_d)^n/(R_d)^n f}$ the $\mathbb{Z}^d$-action attached to some $f \in M_n(R_d) \cap GL_n(c_0(\mathbb{Z}^d))$. Then*

$$cl_p(X_f) = f^t.$$

Proof. This follows from

$$M^{X_f} = (R_d)^n/(R_d)^n f = (R_d)^n/f^t(R_d)^n$$

and Theorem 4.26. □

Theorem 4.32. *Let $f \in M_n(R_d) \cap GL_n(c_0(\mathbb{Z}^d))$. Then the usual $\mathbb{Z}^d$-action on X_f is p-adically expansive and we have*

$$h_p(X_f) := \log_p \det_{\mathbb{Z}^d}(f).$$

In particular, the periodic p-adic entropy of X_f coincides with the p-adic entropy of X_f as defined in 4.30:

$$h_p(X_f) = h_{p,per}(X_f).$$

Proof. If $f \in M_n(R_d) \cap \mathrm{GL}_n(c_0(\mathbb{Z}^d))$, then by Corollary 4.18 the $\mathbb{Z}^d$-action on X_f is p-adically expansive.

By Lemma 4.31, it is $h_p(X_f) = \log_p \det_{\mathbb{Z}^d}(cl_p(X_f)) = \log_p \det_{\mathbb{Z}^d}(f^t)$. But the value $\log_p \det_{\mathbb{Z}^d}(f^t)$ only depends on $\det(f^t) = \det(f)$. Thus, we have $h_p(X) = \log_p \det_{\mathbb{Z}^d}(f)$.

In order to show $h_p(X_f) = h_{p,per}(X_f)$, let $\Gamma_n \to 0$ be a sequence of cofinite subgroups of $\mathbb{Z}^d$ converging to 0. Using Theorem 2.33 we see that

$$\begin{aligned} h_{p,\Gamma_n}(X_f) &= \lim_{n\to\infty} \frac{1}{(\Gamma:\Gamma_n)} \log_p |\mathrm{Fix}_{\Gamma_n}(X_f)| = \lim_{n\to\infty} \frac{1}{(\Gamma:\Gamma_n)} \log_p \det_{\mathbb{Q}_p}(\rho_{f^{(n)}}) \\ &= \lim_{n\to\infty} \frac{1}{(\Gamma:\Gamma_n)} \log_p N_{\mathbb{Q}_p\Gamma^{(n)}/\mathbb{Q}_p}(\det_{\mathbb{Q}_p\Gamma^{(n)}} \rho_{f^{(n)}})) \\ &= \lim_{n\to\infty} \frac{1}{(\Gamma:\Gamma_n)} \log_p \det_{\mathbb{Q}_p}(\rho_{\det_{\mathbb{Q}_p\Gamma^{(n)}}(f^{(n)})}) \\ &= \lim_{n\to\infty} \log_p \det_{\Gamma^{(n)}}(\det_{\mathbb{Q}_p\Gamma^{(n)}} f^{(n)}) = \log_p \det_{\mathbb{Z}^d}(\det_{c_0(\mathbb{Z}^d)}(f)) \\ &= \log_p \det_{\mathbb{Z}^d}(f). \end{aligned}$$

Hence, the limit $\lim_{n\to\infty} \frac{1}{(\Gamma:\Gamma_n)} \log_p |\mathrm{Fix}_{\Gamma_n}(X_f)|$ exists for every $\Gamma_n \to 0$ and its value is given by $\log_p \det_{\mathbb{Z}^d}(f)$, i.e. $h_p(X_f) = h_{p,per}(X_f)$. □

Corollary 4.33. *Let $f \in M_r(R_d) \cap GL_r(c_0(\mathbb{Z}^d))$. Then the periodic p-adic entropy of the $\mathbb{Z}^d$-action on X_f exists and is given by*

$$h_{p,per}(X_f) = m_p(\det f) := \lim_{\substack{N\to\infty,\\ (N,p)=1}} \frac{1}{N^d} \sum_{\zeta \in \mu_N^d} \log f(\zeta).$$

Proof. By Theorem 4.32 we know that $h_{p,per}(X_f)$ exists. Choosing $\Gamma_n = (n\mathbb{Z})^d \to 0$ with n prime to p as in Theorem 2.36, we get $h_{p,per}(X_f) = m_p(\det f)$. □

4.4 Applications: p-adic expansiveness for automorphisms of compact connected abelian groups and dynamical systems defined by a point

This section contains a short discussion of p-adic expansiveness for $\mathbb{Z}$-actions on compact connected abelian groups and for $\mathbb{Z}^d$-actions attached to a point $c \in (\overline{\mathbb{Q}}^*)^d$, i.e. the $\mathbb{Z}^d$-action on the Pontrjagin dual of $R_d/\mathfrak{m}_c$ where $\mathfrak{m}_c$ is the vanishing ideal of the point c.

Recall that in Section 3.2 we gave a short account on expansive $\mathbb{Z}$-actions on compact connected abelian groups and in Section 3.3 we gave a criterion for expansiveness for $\mathbb{Z}^d$-actions attached to a point $c \in (\overline{\mathbb{Q}}^*)^d$. We want to point out that the results stated here are direct p-adic analogues of results stated in Section 3.2 and 3.3.

Proposition 4.34. *Let α be a p-adically expansive $\mathbb{Z}$-action on a compact connected abelian group X. Then there exist primitive polynomials $f_1, \ldots, f_r \in R_1$, such that $f_j | f_{j+1}$ for $j = 1, \ldots, r-1$ with $f_j \in c_0(\mathbb{Z}, \mathbb{Z}_p)^*, j = 1, \ldots, r$, and a surjective morphism η of dynamical systems*

$$\eta : Y := Y_{f_1} \times \ldots \times Y_{f_r} \to X$$

with finite kernel.

Proof. As in the proof of Theorem 3.10 we find primitive polynomials $f_1, \ldots, f_r$ with $f_j | f_{j+1}$ for $j = 1, \ldots, r-1$, such that we have an exact sequence of R_1-modules

$$0 \to M^X \to R_1/(f_1) \times \ldots \times R_1/(f_r) \to N \to 0,$$

where N is a finite.

The associated primes of $\prod_{j=1}^r R_1/(f_j)$ are the same as the associated primes of M^X and are generated by the prime factors of the f_j. Because α is p-adically expansive the associated primes of M^X do not vanish in any point of T_p. It follows $f_j \in c_0(\mathbb{Z}, \mathbb{Z}_p)^*$. □

Lemma 4.35. *Let M be a finite S_p-torsion R_1-module. Then*

$$[M] = 0 \in K_0(\mathcal{M}_{S_p}(R_1)).$$

Proof. Because M is a finite R_1-module, M has a composition series

$$0 = M_0 \subset \ldots \subset M_n = M$$

such that the quotients M_i/M_{i-1} are simple for every $1 \leq i \leq n$. Thus, we may assume that M is a simple module, i.e. $M \simeq \mathbb{Z}[t, t^{-1}]/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal. The ideal $\mathfrak{m}$ is generated by some prime number $l \in \mathbb{N}$ and an element $f \in \mathbb{Z}[t, t^{-1}]$ whose image $\bar{f} \in \mathbb{F}_l[t, t^{-1}]$ generates a maximal ideal. Then we have an exact sequence in $\mathcal{M}_{S_p}(R_1)$

$$0 \to \bar{f} \cdot \mathbb{F}_l[t, t^{-1}] \to \mathbb{F}_l[t, t^{-1}] \to M \to 0$$

where the $\mathbb{Z}[t, t^{-1}]$-modules $\bar{f} \cdot \mathbb{F}_l[t, t^{-1}]$ and $\mathbb{F}_l[t, t^{-1}]$ are isomorphic. It follows $[M] = 0 \in K_0(\mathcal{M}_{S_p}(R_1))$. □

Corollary 4.36. *Let α be a p-adically expansive automorphism of a finite abelian group X. Then $h_p(X) = 0$.*

Proof. M^X is a finite S_p-torsion R_1-module. Thus, it is $[M^X] = 0 \in K_0(\mathcal{M}_{S_p}(R_1))$ and $cl_p(X) = 1 \in K_1(R_1[S_p^{-1}])/(R_1)^*$. Then,

$$h_p(X) = \log_p \det_{\mathbb{Z}}(cl_p(X)) = 0.$$

□

Proposition 4.37. *Let α be a p-adically expansive $\mathbb{Z}$-action on a compact connected abelian group X and let*

$$\eta : Y := Y_{f_1} \times \ldots \times Y_{f_r} \to X$$

be as in Proposition 4.34. Then $cl_p(Y) = cl_p(X)$ and the p-adic entropy of X is given by

$$h_p(X) = h_p(Y) = \sum_{j=1}^{r} \log_p \det_{\mathbb{Z}}(f_j) = \log_p \det_{\mathbb{Z}}\Bigg(\prod_{j=1}^{r} f_j\Bigg).$$

Proof. By Proposition 4.34 there is an exact sequence

$$0 \to M^X \to M^Y \to N \to 0$$

where N is finite and M^X and M^Y are S_p-torsion, so N is also S_p-torsion. Then

$$[M^Y] = [M^X] + [N] = [M^X] \in K_0(\mathcal{M}_{S_p}(R_1))$$

by Lemma 4.35. In particular, we have $cl_p(X) = cl_p(Y) = [f_1 \cdot \ldots \cdot f_r]$ and the formula for the p-adic entropy follows from that. □

In Section 3.2 we gave a description of expansive $\mathbb{Z}$-action on compact connected abelian groups in terms of dynamical systems X^A associated to a matrix $A \in \mathrm{GL}_n(\mathbb{Q})$. Recall that the dual module of X^A is

$$M^A := \mathbb{Z}^n[A^t, (A^{-1})^t] := \text{subgroup of } \mathbb{Q}^n \text{ generated by } \bigcup_{k\in\mathbb{Z}} (A^k)^t \mathbb{Z}^n,$$

where the variable t acts by multiplication with the transpose A^t of A on M^A.

The next proposition is a p-adic analogue of Proposition 3.12.

Proposition 4.38. *An automorphism α of a compact connected abelian group X is p-adically expansive if and only if it is algebraically conjugate to the shift action σ on X^A for some matrix $A \in GL_n(\mathbb{Q})$, $n \geq 1$, without eigenvalues in T_p.*

Proof. The associated prime ideals of M^A are generated by the prime factors of $a\chi_A$, where χ_A is the characteristic polynomial of A and a is the least common multiple of the denominators of the coefficients of χ_A. By the discussion in Section 4.2, the shift action σ on X^A is p-adically expansive if and only if the matrix A has no eigenvalues in T_p.

If on the other hand the action α is p-adically expansive, then the module M^X is noetherian. Using the arguments of the proof of [Sch95], Theorem 9.7, X is conjugate to the shift action on X^A for some $A \in \mathrm{GL}_n(\mathbb{Q})$. As we assume the action to be p-adically expansive, the matrix A has no eigenvalues in T_p. □

Remark 4.39. This implies in particular that torus actions, i.e. actions of the form α^A with $A \in \mathrm{GL}_n(\mathbb{Z})$ cannot be p-adically expansive because the eigenvalues in $\overline{\mathbb{Q}}_p$ of a matrix $A \in \mathrm{GL}_n(\mathbb{Z})$ have absolute value 1.

Proposition 4.40. *Let $A \in GL_n(\mathbb{Q})$ and let X^A with the shift action σ as defined in 3.1. Let $\chi_A \in \mathbb{Q}[t]$ be the characteristic polynomial of A and let $a \in \mathbb{N}$ be the least common multiple of the denominators of the coefficients of χ_A. Assume that χ_A has no zeroes on T_p. Then*

$$h_p(X^A) = m_p(a\chi_A),$$

where $m_p(a\chi_A)$ is the p-adic Mahler measure of $a\chi_A$.

Proof. The proof of 3.30 shows that $cl_p(X^A) = a\chi_A \mod (R_1)^*$. By Theorem 4.32 and Theorem 2.35 it is $\log_p \det_{\mathbb{Z}}(a\chi_A) = m_p(a\chi_A)$. □

Next, we want to discuss p-adic expansiveness for $\mathbb{Z}^d$-actions attached to a point $c = (c_1, \ldots, c_d) \in (\overline{\mathbb{Q}}^*)^d$. Let $\mathfrak{m}_c$ be the vanishing ideal of c and let $(X = \widehat{R_d/\mathfrak{m}_c}, \alpha)$. Again, we denote by (Y_c, α_c) the dynamical system whose dual module is the ring of S-integers $R_{P(c)}$ where the set $P(c)$ is the union of the archimedean places in $K = \mathbb{Q}(c)$ and the set of finite places

$$F(c) := \left\{ v \in P_{\mathfrak{f}}^K : |c_i|_v \neq 1 \text{ for some } i \in \{1, \ldots, d\} \right\}$$

For more details see Section 3.3.

Proposition 4.41. *Let $d \geq 1$, $c = (c_1, \ldots, c_d) \in (\overline{\mathbb{Q}}^*)^d$, and let (X, α) and (Y_c, α_c) be as defined before. Then α is p-adically expansive if and only if α_c is p-adically expansive. This is the case if and only if the orbit of c under the diagonal action of the Galois group $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on $(\overline{\mathbb{Q}}^*)^d$ does not intersect T_p^d.*

Proof. The modules M^X and M^{Y_c} are both associated with the prime ideal $\mathfrak{m}_c$ defined by c. Thus, α is p-adically expansive if and only if α_c is p-adically expansive.

This is exactly the case if the ideal in $\mathbb{Q}_p\langle t_1^{\pm 1}, \dots, t_d^{\pm 1}\rangle$ generated by $\mathfrak{m}_c$ is the unit ideal which means that $\mathfrak{m}_c$ has no zero in T_p^d. But the zeroes of $\mathfrak{m}_c$ in T_p^d correspond to the orbit of the element c under the action of the group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ intersected with T_p^d. □

Example 4.42. Let us continue the discussion of Example 2.39. There we considered the 2-adically expansive dynamical system X_f attached to the polynomial $f = 2t^2 - t + 2$. The zeroes of f in $\mathbb{Q}_2$ are given by $\alpha_\pm = \frac{1}{4}(1 \pm \sqrt{-15})$ with $|\alpha_+|_2 = 2$ and $|\alpha_-|_2 = 1/2$. The periodic p-adic entropy of X_f is given by $h_{2,per}(X_f) = \log_2 \alpha_+ \in \mathbb{Z}_2$.

We want to understand this example from the adelic point of view. Let c be the point $c = \frac{1}{4}(1 + \sqrt{-15}) \in \overline{\mathbb{Q}}$. The corresponding algebraic number field is

$$K = \mathbb{Q}[(1/4)(1 + \sqrt{-15})] = \mathbb{Q}[\sqrt{-15}].$$

In order to determine Y_c or its dual module $R_{P(c)}$ we first determine the set $P(c)$.

Note that for any place $p \in P^{\mathbb{Q}}$, the inequivalent extensions of p to K correspond to the irreducible factors of $g = t^2 - \frac{1}{2}t + 1$ in $\mathbb{Q}_p$.

For $p = \infty$, the polynomial g is irreducible over $\mathbb{R}$ as $c \notin \mathbb{R}$. Thus, there is only one archimedean place on K extending $p = \infty$ which we also denote by ∞. It is $K_\infty = \mathbb{C}$.

For $p = 2$, there are two inequivalent extensions v_+ and v_- corresponding to $g = (t - \alpha_+)(t - \alpha_-) \in \mathbb{Q}_2[t]$. It is $K_{v_+} = \mathbb{Q}_2 = K_{v_-}$.

For $p \neq 2$, the polynomial g lies in $\mathbb{Z}_p[t]$. Even more, the coefficients of g lie in $\mathbb{Z}_p^* = \{z \in \mathbb{Z}_p : |z|_p = 1\}$. This implies that if v is an extension of $p \neq 2$ to K then $|c|_v = 1$. In particular, any $v \in P_{\mathfrak{f}}^K$ extending some $p \neq 2$ will not be contained in the set $F(c) = \{v \in P_{\mathfrak{f}}^K : |c|_v \neq 1\}$. So we find $F(c) = \{v_+, v_-\}$ and $P(c) = \{\infty, v_+, v_-\}$.

By Theorem 3.19, it is

$$\widehat{R_{P(c)}} = Y_c = \mathbb{C} \times \mathbb{Q}_2 \times \mathbb{Q}_2 / \Delta(R_{P(c)}).$$

We can lift the action α_c on Y_c to an action of the covering space $\mathbb{C} \times \mathbb{Q}_2 \times \mathbb{Q}_2$. The lifted action on the latter space is just given by multiplication with c on each component.

Chapter 5

Entropy of expansive $\mathbb{Z}^d$-actions and K-theory

In this chapter we want to apply the K-theoretical approach which we used to define p-adic expansiveness and p-adic entropy to the usual notions of expansiveness and entropy for $\mathbb{Z}^d$-actions.

We show that an algebraic $\mathbb{Z}^d$-action on the compact abelian group X is expansive if and only if M^X is a finitely generated S_∞-torsion R_d-module where $S_\infty = R_d \cap L^1(\mathbb{Z}^d, \mathbb{R})^*$. Then we attach to every expansive $\mathbb{Z}^d$-action on X an invariant

$$cl_\infty(X) \in K_1(R_d[S_\infty^{-1}])/R_d^* = \mathrm{SK}_1(R_d[S_\infty^{-1}]) \oplus R_d[S_\infty^{-1}]^*/R_d^*.$$

We show that the Fuglede-Kadison determinant defines a homomorphism

$$\log \det_{\mathcal{N}\mathbb{Z}^d} : K_1(R_d[S_\infty^{-1}])/R_d^* \to \mathbb{R}$$

such that the topological entropy $h(X)$ of an expansive algebraic $\mathbb{Z}^d$-action on X is given by $\log \det_{\mathcal{N}\mathbb{Z}^d}(cl_\infty(X))$.

In Section 5.2 we prove that for $d \geq 5$, the group $\mathrm{SK}_1(R_d[S_\infty^{-1}])$ is not trivial. This gives a new non-trivial additive invariant of expansive $\mathbb{Z}^d$-actions for $d \geq 5$.

5.1 A K-theoretic approach to entropy of expansive $\mathbb{Z}^d$-actions

Let $S_\infty \subset R_d$ be the multiplicative system $S_\infty = R_d \cap L^1(\mathbb{Z}^d, \mathbb{R})^*$. We denote by $\mathcal{M}_{S_\infty}(R_d)$ the category of finitely generated S_∞-torsion R_d-modules.

Lemma 5.1. *Let (X, α) be an algebraic $\mathbb{Z}^d$-action such that $M^X \in \mathcal{M}_{S_\infty}(R_d)$. Then α is expansive.*

Proof. First note that M^X is a noetherian module as it is finitely generated over the noetherian ring R_d. Then by Theorem 3.3 we have to show that $V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{T}^d = \emptyset$ for all $\mathfrak{p} \in \mathrm{Ass}(M^X)$.

$M^X \in \mathcal{M}_{S_\infty}(R_d)$ implies that every annihilator ideal of M^X contains an element f which is a unit in $L^1(\mathbb{Z}^d, \mathbb{R})$. In particular, this is true for all associated primes of M^X. By Theorem 3.8, f is a unit in $L^1(\mathbb{Z}^d, \mathbb{R})$ if and only if f does not vanish in any point of $\mathbb{T}^d$. It follows $V_{\mathbb{C}}(\mathfrak{p}) \cap \mathbb{T}^d = \emptyset$ for all $\mathfrak{p} \in \mathrm{Ass}(M^X)$. □

Theorem 5.2 (Algebraic criterion of expansiveness)**.** *Let α be an algebraic $\mathbb{Z}^d$-action on a compact abelian group X. Then α is expansive if and only if $M^X \in \mathcal{M}_{S_\infty}(R_d)$.*

Proof. If $M^X \in \mathcal{M}_{S_\infty}(R_d)$ then by Lemma 5.1 the action α is expansive.

For the reverse implication we show that for an expansive action α on X every $\mathfrak{p} \in \mathrm{Ass}(M^X)$ contains an element in S_∞. Then using Lemma 4.15 this implies $M^X \in \mathcal{M}_{S_\infty}(R_d)$.

Let $\mathfrak{p} \in \mathrm{Ass}(M^X)$ be generated by $f_1, \ldots, f_r \in R_d$. By Theorem 3.3, the f_i have no common zero on the d-torus $\mathbb{T}^d$. Define the element $g \in \mathfrak{p}$ by

$$g = \sum_{i=1}^{r} f_i f_i(t^{-1}) \quad \text{with } f_i(t^{-1}) = f_i(t_1^{-1}, \ldots, f_d^{-1}).$$

Because for an element $z \in \mathbb{T}$ the inverse z^{-1} is given by the complex conjugate $\bar{z}$, it is for $z = (z_1, \ldots, z_d) \in \mathbb{T}^d$

$$g(z) = \sum_{i=1}^{r} f_i(z) f_i(\bar{z}) = \sum_{i=1}^{r} f_i(z)\overline{f_i(z)} = \sum_{i=1}^{r} |f_i(z)|^2 \neq 0.$$

It follows that $g \in S_\infty$. □

Now that we have characterized expansive $\mathbb{Z}^d$-actions as those actions such that the dual module M^X is in $\mathcal{M}_{S_\infty}(R_d)$, we want to apply the K-theoretic formalism presented in Section 4.1 to expansive $\mathbb{Z}^d$-actions.

Theorem 5.3. *Consider the multiplicative system $S_\infty = R_d \cap L^1(\mathbb{Z}^d, \mathbb{R})^*$ in R_d. There is an isomorphism*

$$cl_\infty : K_0(\mathcal{M}_{S_\infty}(R_d)) \to K_1(R_d[S_\infty^{-1}])/R_d^*$$

such that

$$cl_\infty([(R_d)^n / f(R_d)^n]) = [f] \mod R_d^*$$

for all $f \in M_n(R_d) \cap GL_n(L^1(\mathbb{Z}^d, \mathbb{R}))$.

Proof. Using the localisation sequence of Theorem 4.13

$$K_1(R_d) \to K_1(R_d[S_\infty^{-1}]) \xrightarrow{\delta} K_0(\mathcal{M}_{S_\infty}(R_d)) \xrightarrow{\varepsilon} K_0(R_d) \to K_0(R_d[S_\infty^{-1}]) \to 0$$

we show with the same arguments as in the proof of Theorem 4.26 that δ induces an isomorphism

$$\bar{\delta} : K_1(R_d[S_\infty^{-1}])/R_d^* \to K_0(\mathcal{M}_{S_\infty}(R_d)).$$

Define cl_∞ as the inverse $\bar{\delta}^{-1}$ of $\bar{\delta}$. Then cl_∞ has the claimed property. $\square$

Definition 5.4. *Let (X, α) be an expansive algebraic $\mathbb{Z}^d$-action, i.e. $M^X \in \mathcal{M}_{S_\infty}(R_d)$. Then we define*

$$cl_\infty(X) := cl_\infty([M^X]) \in K_1(R_d[S_\infty^{-1}])/R_d^*.$$

Next, we want to show that the entropy $h(X)$ of an expansive $\mathbb{Z}^d$-action can be obtained by applying the Fuglede-Kadison determinant to $cl_\infty(X)$.

Lemma 5.5. *Let $f \in GL_r(L^1(\mathbb{Z}^d, \mathbb{R}))$. Assume that for all cofinite subgroups N of $\mathbb{Z}^d$ it is*

$$\det_{\mathbb{C}}(\rho_{\bar{f}}) = \pm 1,$$

where $\bar{f}$ is the image of f in $M_r(L^1(\mathbb{Z}^d/N, \mathbb{R}))$. Then

$$\log \det_{\mathcal{N}\mathbb{Z}^d} f = 0.$$

Proof. This follows from Theorem 2.21 and Example 2.18. $\square$

Corollary 5.6. *The Fuglede-Kadison determinant defines a homomorphism*

$$\log \det_{\mathcal{N}\mathbb{Z}^d} : K_1(R_d[S_\infty^{-1}])/R_d^* \to \mathbb{R}.$$

Proof. Using Lemma 5.5, we show that the Fuglede-Kadison determinant gives a well-defined homomorphism on $K_1(L^1(\mathbb{Z}^d, \mathbb{R}))/R_d^*$. The canonical homomorphism $K_1(R_d[S_\infty^{-1}])/R_d^* \to K_1(L^1(\mathbb{Z}^d, \mathbb{R}))/R_d^*$ will then give the stated map.

For every $r \geq 1$ and any cofinite subgroup N of $\mathbb{Z}^d$, the diagram

$$\begin{array}{ccc} \mathrm{GL}_r(L^1(\mathbb{Z}^d, \mathbb{R})) & \longrightarrow & \mathrm{GL}_r(L^1(\mathbb{Z}/N, \mathbb{R})) \\ \Big\downarrow{\scriptstyle \det} & & \Big\downarrow{\scriptstyle \det} \\ L^1(\mathbb{Z}^d, \mathbb{R})^* & \longrightarrow & L^1(\mathbb{Z}^d/N, \mathbb{R})^* \end{array}$$

commutes, where the horizontal arrows are the canonical reduction homomorphisms.

From Theorem 2.21, it follows that $\log\det_{\mathcal{N}\mathbb{Z}^d}$ is well-defined on the infinite general linear group $\mathrm{GL}(L^1(\mathbb{Z}^d,\mathbb{R}))$. The Fuglede-Kadison determinant passes to $K_1(L^1(\mathbb{Z}^d,\mathbb{R}))$ because elementary matrices have determinant 1 and the diagram above implies that for elementary matrices the condition of Lemma 5.5 is satisfied.

If $f \in R_d^*$, then for every cofinite subgroup N of $\mathbb{Z}^d$, the automorphism $\rho_{\bar{f}}$ is just a permutation of the canonical basis of $L^1(\mathbb{Z}^d/N,\mathbb{R})$ and so $\det\rho_{\bar{f}} = \pm 1$. Using again Lemma 5.5, the claim follows. □

Corollary 5.7. *The Fuglede-Kadison determinant vanishes on the subgroup* $SK_1(R_d[S_\infty^{-1}]) \subset K_1(R_d[S_\infty^{-1}])/R_d^*$. *Thus, the homomorphism*

$$\log\det_{\mathcal{N}\mathbb{Z}^d} : K_1(R_d[S_\infty^{-1}])/R_d^* \to \mathbb{R}$$

factorizes as

$$K_1(R_d[S_\infty^{-1}])/R_d^* \stackrel{\det}{\to} R_d[S_\infty^{-1}]^*/R_d^* \stackrel{\log\det_{\mathcal{N}\mathbb{Z}^d}}{\to} \mathbb{R}\ .$$

Proof. Let $f \in \mathrm{SL}_n(R_d[S_\infty^{-1}])$ be a representative of $[f] \in \mathrm{SK}_1(R_d[S_\infty^{-1}])$. Then as in the proof of Lemma 4.27 one shows that $\det_{\mathbb{C}}(\rho_{\bar{f}}) = \pm 1$ for every cofinite subgroup N of $\mathbb{Z}^d$. □

Theorem 5.8. *Let α be an expansive algebraic $\mathbb{Z}^d$-action on a compact abelian group X. Then the topological entropy of the action α on X is given by*

$$h(X) = \log\det_{\mathcal{N}\mathbb{Z}^d}(cl_\infty(X)).$$

Proof. Let $f' \in \mathrm{GL}_n(R_d[S_\infty^{-1}])$ be a representative of $cl_\infty(X)$ and let $s \in S_\infty$ such that the element $f = sf'$ is in $M_n(R_d)$. Then by the definition of the map δ in Theorem 4.13 it is

$$[M^X] = [(R_d)^n/f(R_d)^n] - [(R_d)^n/s(R_d)^n] \in K_0(\mathcal{M}_{S_\infty}(R_d))$$

and by Yuzvinskii's addition formula we know that $h(X)$ only depends on the class $[M^X] \in K_0(\mathcal{M}_{S_\infty}(R_d))$, i.e. it is $h(X) = h(X_{f^t}) - h(X_s)$, where X_s denotes the Pontrjagin dual of $(R_d)^n/s(R_d)^n$. By [Sch95], Chapter V, Example 18.7, (1), and by Example 2.17 we know that

$$h(X_f) = m(\det f) := \int_{\mathbb{T}^d} \log|\det f(z)| d\mu(z) = \log\det_{\mathcal{N}\mathbb{Z}^d}(\det(f)).$$

On the other hand, we know by Corollary 5.7 that for $f \in \mathrm{GL}_n(R_d[S_\infty^{-1}])$ it is $\log\det_{\mathcal{N}\mathbb{Z}^d}(f) = \log\det_{\mathcal{N}\mathbb{Z}^d}(\det f)$. Thus, writing Id_n for the identity matrix in $\mathrm{GL}_n(R_d[S_\infty^{-1}])$ we get

$$\begin{aligned}
\log\det_{\mathcal{N}\mathbb{Z}^d}(cl_\infty(X)) &= \log\det_{\mathcal{N}\mathbb{Z}^d}(\det(cl_\infty(X))) \\
&= \log\det_{\mathcal{N}\mathbb{Z}^d}(\det(f)) - \log\det_{\mathcal{N}\mathbb{Z}^d}(\det(s\cdot \mathrm{Id}_n)) \\
&= \log\det_{\mathcal{N}\mathbb{Z}^d}(\det(f^t)) - \log\det_{\mathcal{N}\mathbb{Z}^d}(\det(s\cdot \mathrm{Id}_n)) \\
&= h(X_{f^t}) - h(X_s) = h(X).
\end{aligned}$$

□

5.2 The group $\mathbf{SK}_1(R_d[S_\infty^{-1}])$

Given an expansive algebraic $\mathbb{Z}^d$-action (X,α) we defined an element

$$cl_\infty(X) \in K_1(R_d[S_\infty^{-1}])/R_d^* = \mathrm{SK}_1(R_d[S_\infty^{-1}]) \oplus (R_d[S_\infty^{-1}])^*/R_d^*.$$

So it is natural to study the group $\mathrm{SK}_1(R_d[S_\infty^{-1}])$ in order to understand expansive $\mathbb{Z}^d$-actions.

For so-called special normed commutative $\mathbb{R}$-algebras B, computations of $\mathrm{SK}_1(B)$ have been made in [Day76] using topological methods. Even though $R_d[S_\infty^{-1}]$ is not an $\mathbb{R}$-algebra, it lies densely in the commutative Banach algebra $C(\mathbb{T}^d,\iota)$ of continuous functions $f:\mathbb{T}^d\to\mathbb{C}$ which satisfy $\overline{f} = f\circ\iota$, where $\iota:\mathbb{T}^d\to\mathbb{T}^d$ is the involution given by complex conjugation and $\overline{f}$ is the composition of f with complex conjugation on $\mathbb{C}$.

We show that $\mathrm{SK}_1(R_d[S_\infty^{-1}])$ surjects onto $\mathrm{SK}_1(C(\mathbb{T}^d,\iota))$. Using topological K-theory we show that, for d large enough, $\mathrm{SK}_1(C(\mathbb{T}^d,\iota))$ is non-trivial which proves that $\mathrm{SK}_1(R_d[S_\infty^{-1}])\neq 0$.

We proceed as follows. First we shortly define topological K-theory of real Banach algebras and state some of the fundamental results of topological K-theory. For example, even though we are only interested in $\mathrm{SK}_1(C(\mathbb{T}^d,\iota))$ we need the Periodicity Theorem and the higher topological K-groups for our computations.

Next, we recall some results on SK_1 of a special normed commutative $\mathbb{R}$-algebra B. The main points here are that $\mathrm{SK}_1(B)$ equals the group of path components $\pi_0(\mathrm{SL}(B))$ of $\mathrm{SL}(B)$ and that $SK_1(B)$ is isomorphic to $\mathrm{SK}_1(B')$ if B and B' are special normed $\mathbb{R}$-algebras and B lies densely in B'.

Then we show that, for d large enough, $\mathrm{SK}_1(C(\mathbb{T}^d,\iota))\neq 0$.

Let us start with the definition of topological K-theory of real Banach algebras.

Definition 5.9. *Let A be a unital Banach algebra and let X be a compact Hausdorff space. An A-bundle over X is a locally trivial Banach space bundle whose fibers are finitely generated projective A-modules.*

Definition 5.10. *For a unital real Banach algebra A and a compact Hausdorff space X, let $\mathcal{P}(X;A)$ denote the category whose objects are A-bundles, and whose morphisms are A-linear bundle maps (between corresponding locally trivial Banach space bundles). The Grothendieck group of $\mathcal{P}(X;A)$ will be denoted by $K(X;A)$.*

Note that the additive structure of $K(X;A)$ is induced by taking the direct sum $E \oplus F$ of two A-bundles E, F over X.

In order to define K-groups for locally compact Hausdorff spaces we first introduce relative K-groups for compact pairs (X, Y), i.e. $Y \subset X$ and X, Y are compact Hausdorff spaces.

Definition 5.11. *Let (X, Y) be a pair of compact Hausdorff spaces, E_i, F_i A-bundles over X and $\alpha_i : E_i|_Y \to F_i|_Y, i = 1, 2$, A-bundle isomorphisms. The two triples (E_1, F_1, α_1) and (E_2, F_2, α_2) are isomorphic, provided that there are A-bundle isomorphisms $f : E_1 \to E_2$ and $g : F_1 \to F_2$ with*

$$\alpha_2 \circ f|_Y = g|_Y \circ \alpha_1.$$

Two triples are called stably isomorphic if they become isomorphic after adding elementary triples (a triple (E, F, α) is called elementary if $E = F$ and if α is homotopic to id_E in the set of A-bundle isomorphisms). The sum of two triples (E, F, α) and (E', F', α') is defined by $(E \oplus E', F \oplus F', \alpha \oplus \alpha')$. Equivalence classes of stably isomorphic triples form a group which will be denoted by $K(X, Y; A)$.

Definition 5.12. *If X is a locally compact Hausdorff space and $X^+ = X \cup \{\infty\}$ its one-point compactification, then we define*

$$K(X;A) = K(X^+, \{\infty\}; A).$$

For a closed subset $Y \subset X$ and $n \geq 0$, the higher (relative) K-groups are defined by

$$K^{-n}(X, Y; A) = K((X \setminus Y) \times \mathbb{R}^n; A).$$

We may now define the topological K-groups of a real Banach algebra A.

Definition 5.13. *For a unital real Banach algebra we define the n-th topological K-group of A for $n \geq 0$ by*

$$K_n^{top}(A) = K^{-n}(\{pt\}; A),$$

which gives
$$K_n^{top}(A) = K(\mathbb{R}^n; A) \simeq K(B^n, S^{n-1}; A),$$
where $B^n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$ *and* $S^{n-1} = \{x \in \mathbb{R}^n : ||x|| = 1\}$. *If* A *does not have a unit we define*
$$K_n^{top}(A) = \ker(K_n(\tilde{A}) \to K_n(\mathbb{R})),$$
where $\tilde{A} = A \times \mathbb{R}$ *with multiplication* $(a, x)(a', x') = (aa' + xa' + x'a, xx')$ *and the obvious addition is the unitization of* A.

Definition 5.14. *Let* A *be an unital commutative Banach algebra. Let* $\langle [A] \rangle$ *be the subgroup of* $K_0^{top}(A)$ *generated by the class* $[A]$ *of the trivial* A*-bundle. Note that* $\langle [A] \rangle$ *is isomorphic to* $\mathbb{Z}$. *The reduced* K_0^{top}*-group of* A *is defined as the quotient*
$$\tilde{K}_0^{top}(A) = K_0^{top}(A)/\langle [A] \rangle.$$

Theorem 5.15. *For any unital real Banach algebra* A *we have*
$$K_n^{top}(A) \simeq \pi_{n-1}(GL(A)),\ n > 0,$$
where $\pi_{n-1}(GL(A))$ *is the* $(n-1)$*-th homotopy group of* $GL(A)$.

Proof. See [Sch93], Theorem 1.4.6. □

Theorem 5.16 (Periodicity Theorem). *For a real Banach algebra* A *there are isomorphisms*
$$K_n^{top}(A) \simeq K_{n+8}^{top}(A),\ n \geq 0.$$

Proof. See [Kar78], III, 5.17. □

We say a sequence of Banach algebras and maps
$$0 \to A' \to A \to A'' \to 0$$
is exact if the underlying sequence of abelian groups is exact, i.e. A' is a two-sided ideal in A and A'' may be identified with A/A'.

Theorem 5.17. *Any short exact sequence of real Banach algebras*
$$0 \to A' \to A \to A'' \to 0$$
gives rise to a long exact sequence in K*-theory*
$$\ldots \to K_n^{top}(A') \to K_n^{top}(A) \to K_n^{top}(A'') \to K_{n-1}^{top}(A') \to \ldots\ .$$

Proof. See [Sch93], Theorem 1.4.14. □

In the following, it will turn out that topological K-theory provides a very useful tool to understand $\mathrm{SK}_1(R_d[S_\infty^{-1}])$. The reasons for this are basically that $R_d[S_\infty^{-1}]$ lies densely in the real Banach algebra $C(\mathbb{T}^d, \iota)$ and that for a Banach algebra A the group $\mathrm{SK}_1(A)$ has a topological description as the group of path components $\pi_0(\mathrm{SL}(A))$ of $\mathrm{SL}(A)$.

Definition 5.18. *Let A be a unitary commutative $\mathbb{R}$-algebra equipped with a norm $|| \ ||$. We say A is special if $||a|| < 1$ implies that $1 - a \in A^*$ for every $a \in A$. In this case, we will call A a special normed algebra for short.*

Example 5.19. Every commutative real Banach algebra A with unit is special. If A is a normed $\mathbb{R}$-algebra and $\hat{A}$ its completion, then the localisation of A with respect to all elements $a \in A$ which become invertible in $\hat{A}$ is special.

Lemma 5.20. *Let A be a special normed algebra. Then the group $E_n(A)$ generated by the elementary matrices is an open, path connected subgroup of the special linear group $SL_n(A)$.*

Proof. The proof is given in [Mil71], Lemma 7.1, for A a Banach algebra. The same proof works in the case of a special normed $\mathbb{R}$-algebra. □

Because $E_n(A)$ is path-connected, the group $E_n(A)$ is closed in $\mathrm{SL}_n(A)$. Hence, $E_n(A)$ is the component of the identity in $\mathrm{SL}_n(A)$, and the quotient $\mathrm{SL}_n(A)/E_n(A)$ can be identified with the group $\pi_0(\mathrm{SL}_n(A))$ of path components.

It is

$$\mathrm{SK}_1(A) = \mathrm{SL}(A)/E(A) = \varinjlim \mathrm{SL}_n(A)/E_n(A) = \varinjlim \pi_0 \mathrm{SL}_n(A).$$

Thus, if we give $\mathrm{SL}(A)$ the direct limit topology, then the group $\pi_0(\mathrm{SL}(A))$ of path components can be identified with $\varinjlim \pi_0 \mathrm{SL}_n(A)$. This proves the next result.

Corollary 5.21. *The group $SK_1(A)$ is isomorphic to the group $\pi_0(SL(A))$ of path components of $SL(A)$.*

Theorem 5.22. *Let B be a special normed algebra and let $A \subset B$ be a dense subring with the property $A \cap B^* = A^*$. Then*

(i) $SK_1(A) \to SK_1(B)$ is surjective.

If A is also a special normed $\mathbb{R}$-algebra, then the condition $A \cap B^* = A^*$ is automatically satisfied and *(ii) $SK_1(A) \to SK_1(B)$ is an isomorphism.*

Proof. Assume that A is special. We show that this implies $A \cap B^* = A^*$. If $a \in A$ is invertible in B we pick an element $c \in A$ which is close to the inverse of a. Then $||1 - ac|| < 1$ and because A is special it follows that ac and therefore a are invertible in A.

The proof of (ii) is given in [Day76], Theorem 2.7. To show surjectivity, the idea is the following: Let $b \in \mathrm{SL}_n(B)$. As B is special, $\mathrm{GL}_n(B)$ is open in $M_n(B)$. So we may pick an ε-ball $U_\varepsilon(b)$ around b which is contained in $\mathrm{GL}_n(B)$. Because A is dense in B, $U_\varepsilon(b)$ contains an element $a \in M_n(A)$. Let γ be the straight path in $U_\varepsilon(b) \subset \mathrm{GL}_n(B)$ connecting b and a, i.e. $\gamma(t) = ta+(1-t)b$, $t \in [0,1]$. Let $\delta(t)$ be the diagonal matrix with $\delta(t)_{11} = \det \gamma(t)^{-1}$ and $\delta(t)_{ii} = 1$ for $i \neq 1$. Then $\delta\gamma$ is a path in $\mathrm{SL}_n(B)$ connecting b and $\delta(1)a$. Now, as A is special, the element $\det a \in A \cap B^*$ is invertible in A. This implies that $\delta(1)a \in \mathrm{SL}_n(A)$. As $\mathrm{SK}_1(B) = \pi_0(\mathrm{SL}(B))$ it is $[b] = [\delta(1)a] \in \mathrm{SK}_1(B)$. This proves surjectivity of the homomorphism $\mathrm{SK}_1(A) \to \mathrm{SK}_1(B)$.

This proof uses only the fact that $A \cap B^* = A^*$ and the assumption that B is special for surjectivity. So the same proof shows that in (i) the map $\mathrm{SK}_1(A) \to \mathrm{SK}_1(B)$ is surjective. □

Corollary 5.23. *The inclusion $R_d[S_\infty^{-1}] \hookrightarrow C(\mathbb{T}^d, \iota)$ induces a surjective homomorphism*

$$SK_1(R_d[S_\infty^{-1}]) \to SK_1(C(\mathbb{T}^d, \iota)).$$

Proof. We want to apply Theorem 5.22, (i), to the case $A = R_d[S_\infty^{-1}]$ and $B = C(\mathbb{T}^d, \iota)$.

It is $R_d[S_\infty^{-1}] \cap C(\mathbb{T}^d, \iota)^* = R_d[S_\infty^{-1}]^*$. To show that $R_d[S_\infty^{-1}]$ lies densely in $C(\mathbb{T}^d, \iota)$, first note that by the Theorem of Stone-Weierstraß $\mathbb{C}[z_1^{\pm 1}, \dots, z_d^{\pm 1}]$ is dense in the algebra $C(\mathbb{T}^d, \mathbb{C})$ of continuous functions from $\mathbb{T}^d$ to $\mathbb{C}$. If $p \in \mathbb{C}[z_1^{\pm 1}, \dots, z_d^{\pm 1}]$ is an approximation of $f \in C(\mathbb{T}^d, \iota)$, then $\frac{1}{2}(p + \overline{p})$ is an approximation of f which lies in $\mathbb{R}[z_1^{\pm 1}, \dots, z_d^{\pm 1}]$. But any element in $\mathbb{R}[z_1^{\pm 1}, \dots, z_d^{\pm 1}]$ can be approximated by elements in $R_d[S_\infty^{-1}]$, so $R_d[S_\infty^{-1}]$ lies densely in $C(\mathbb{T}^d, \iota)$. □

We proved that there is a surjective homomorphism $\mathrm{SK}_1(R_d[S_\infty^{-1}]) \to \mathrm{SK}_1(C(\mathbb{T}^d, \iota))$. The next part will be concerned with the computation of the group $\mathrm{SK}_1(C(\mathbb{T}^d, \iota))$.

Lemma 5.24. *Let A be a commutative Banach algebra. The continuous map* $\det : GL(A) \to A^*$ *has a continuous section*

$$s : A^* \to GL(A),\ a \mapsto (a) \in GL_1(A) \subset GL(A).$$

The map $u : SL(A) \times A^* \to GL(A),\ (M,a) \mapsto Ms(a)$ is an isomorphism of topological spaces which induces an isomorphism of groups

$$\pi_0(GL(A)) = \pi_0(SL(A)) \oplus \pi_0(A^*).$$

Proof. The inverse of the continuous map u is given by

$$v : \mathrm{GL}(A) \to \mathrm{SL}(A) \times A^*,\ N \mapsto (N(\det N)^{-1}, \det N).$$

The continuous maps u and v induce maps

$$\pi_0(u) : \pi_0(\mathrm{SL}(A)) \times \pi_0(A^*) \to \pi_0(\mathrm{GL}(A))$$

and

$$\pi_0(v) : \pi_0(\mathrm{GL}(A)) \to \pi_0(\mathrm{SL}(A)) \times \pi_0(A^*)$$

which are inverse to each other.

Because the groups $\pi_0(\mathrm{GL}(A))$, $\pi_0(\mathrm{SL}(A))$ and $\pi_0(A^*)$ are abelian, it follows that $\pi_0(u)$ and $\pi_0(v)$ are group homomorphism. Thus, we get a direct sum decomposition of the group $\pi_0(\mathrm{GL}(A)) = \pi_0(\mathrm{SL}(A)) \oplus \pi_0(A^*)$. □

Recall that the algebraic K-group $K_1(R)$ for a commutative ring R splits as

$$0 \to \mathrm{SK}_1(R) \to K_1(R) \stackrel{\det}{\to} R^* \to 0.$$

As a corollary to Lemma 5.24 we get an analogous result for the group $K_1^{top}(A)$ for a commutative Banach algebra A:

Corollary 5.25. *Let A be a commutative Banach algebra. Define $SK_1^{top}(A) := \pi_0(SL(A)) = SK_1(A)$. Then we have a split exact sequence*

$$0 \to \pi_0(A^*) \to K_1^{top}(A) \to SK_1^{top}(A) \to 0.$$

Proof. By Theorem 5.15 the group $K_1^{top}(A)$ is isomorphic to $\pi_0(\mathrm{GL}(A))$. Then Lemma 5.24 gives the result. □

Lemma 5.26. *Let B be a special dense subalgebra of $C(\mathbb{T}^{n-1}, \iota)$. Then the natural map $B[z,z^{-1}]^* \to \pi_0(C(\mathbb{T}^n, \iota)^*)$ is surjective.*

Proof. It is proven in [Day76], Lemma 4.2, that if B' is a special dense subalgebra in the algebra $C(\mathbb{T}^{n-1}, \mathbb{C})$ of continuous complex-valued functions of the $n-1$-torus $\mathbb{T}^{n-1}$ then $B'[z,z^{-1}]^* \to \pi_0(C(\mathbb{T}^n)^*)$ is surjective. We show that the same proof works in the equivariant situation of our lemma.

Let $r : \mathbb{C}^* \to \mathbb{T}$ be the retraction $z \mapsto z/|z|$. Because the homotopy between the identity $\mathrm{Id} : \mathbb{C}^* \to \mathbb{C}^*$ and $\mathbb{C}^* \xrightarrow{r} \mathbb{T} \hookrightarrow \mathbb{C}^*$ given by

$$\mathbb{C}^* \times [0,1] \to \mathbb{C}^*, \ (z,t) \mapsto z \cdot \frac{1+t|z|}{|z|+t},$$

respects complex conjugation, we may identify $\pi_0(C(\mathbb{T}^n, \iota)^*)$ with the group $[\mathbb{T}^{n-1} \times \mathbb{T}, \mathbb{T}]^{equ}$ of equivariant homotopy classes of maps $\mathbb{T}^{n-1} \times \mathbb{T} \to \mathbb{T}$. By an equivariant homotopy H we mean a homotopy $H : \mathbb{T}^n \times [0,1] \to \mathbb{T}$ which satisfies $\overline{H(z,t)} = H(\iota(z), t)$ for all $t \in I = [0,1]$.

Thus, in order to prove the claim of the lemma it suffices to show that the map

$$B[z, z^{-1}]^* \to [\mathbb{T}^{n-1} \times \mathbb{T}, \mathbb{T}]^{equ}, \ b \mapsto r \circ b, \tag{5.1}$$

is surjective. First we note that if two equivariant maps $f, g : \mathbb{T}^n \to \mathbb{T}$ are close enough then there is an equivariant homotopy between them given by

$$\mathbb{T}^n \times I \to \mathbb{T}, \ (z,t) \mapsto \frac{(1-t)f(z) + tg(z)}{|(1-t)f(z) + tg(z)|}.$$

So let $f : \mathbb{T}^n \to \mathbb{T}$ with $f \circ \iota = \overline{f}$ be given. Then as proven in [Day76], there exists a homotopy between f and the function $g : \mathbb{T}^{n-1} \times \mathbb{T} \to \mathbb{T}, \ g(x,z) = f(x,1)z^n$ for some $n \in \mathbb{Z}$. Note that g satisfies $g \circ \iota = \overline{g}$. A homotopy between (fg^{-1}) and 1 is explicitly given by

$$F : \mathbb{T}^{n-1} \times \mathbb{T} \times I \to \mathbb{T}, \ (x, z, t) \mapsto e^{it\phi_x(z)},$$

where $\phi_x : \mathbb{T} \to \mathbb{R}$ is the unique continuous map such that $(fg^{-1})(x,z) = e^{i\phi_x(z)}$ for all $z \in \mathbb{T}$ and $\phi_x(1) = 0$. The uniqueness of ϕ_x implies

$$\phi_{\bar{x}}(\bar{z}) = -\phi_x(z) \text{ for all } x \in \mathbb{T}^{n-1} \text{ and for all } z \in \mathbb{T}. \tag{5.2}$$

Namely, if we define $\tilde{\phi}_x(z) = -\phi_{\bar{x}}(\bar{z})$, then $\tilde{\phi}_x(1) = 0$ and

$$e^{i\tilde{\phi}_x(z)} = \overline{(fg^{-1})(\bar{x}, \bar{z})} = (fg^{-1})(x,z).$$

Hence, by uniqueness, $\phi_{\bar{x}}(\bar{z}) = -\phi_x(z)$. Equation (5.2) implies that F is equivariant, so that there is an equivariant homotopy between f and g.

Now let $j : \mathbb{T}^{n-1} \to \mathbb{T}^{n-1} \times \mathbb{T}$ be defined by $j(x) = (x, 1)$. Because B is special dense in $C(\mathbb{T}^{n-1}, \iota)$ there is an element $h \in B^*$ such that $r \circ h$ is close to $f \circ j$, i.e. $[r \circ h] = [f \circ j]$ in $[\mathbb{T}^{n-1}, \mathbb{T}]^{equ}$. Thus,

$$[r \circ (hz^n)] = [(r \circ h)z^n] = [(f \circ j)z^n] = [g] = [f] \text{ in } [\mathbb{T}^{n-1} \times \mathbb{T}, \mathbb{T}]^{equ},$$

which proves that the element $hz^n \in B[z, z^{-1}]$ is mapped to $[f]$ under the map (5.1). $\square$

Theorem 5.27. *Let X be a compact Hausdorff space with an involution τ. We denote by $C(X,\tau)$ the Banach $\mathbb{R}$-algebra of continuous functions $f : X \to \mathbb{C}$ such that $f \circ \tau = \overline{f}$. Then*

$$K_n^{top}(C(X\times\mathbb{T},\tau\times\iota)) = K_n^{top}(C(X,\tau)) \oplus K_{n-1}^{top}(C(X,\tau)).$$

Proof. Let A' be the real Banach algebra

$$A' = \Big\{ f:\mathbb{R}\to C(X,\tau) : f \text{ cont.}, \lim_{|t|\to\infty} ||f(t)|| = 0, \overline{f_t(x)} = f_{-t}(\tau x) \Big\}.$$

Then the result comes from the exact K-theory sequence attached to the split exact sequence

$$0 \to A' \to C(X\times\mathbb{T},\tau\times\iota) \overset{f\mapsto f(x,1)}{\to} C(X,\tau) \to 0$$

and the fact that $K_n^{top}(A')$ is isomorphic to $K_{n-1}^{top}(C(X,\tau))$, see [Sch93], Theorem 1.5.4, for more details. □

We define $B_0 = \mathbb{R}$ and for $n \geq 1$ let $B_n := B_{n-1}[z,z^{-1}][S_{\infty,\mathbb{R}}^{-1}]$, where $S_{\infty,\mathbb{R}} = \mathbb{R}[z_1^{\pm 1},\ldots,z_n^{\pm 1}] \cap C(\mathbb{T}^n,\iota)^*$ (to be precise, $S_{\infty,\mathbb{R}}$ depends of course on n but for simplicity we omit the n in the notation). B_n is a special $\mathbb{R}$-algebra dense in $C(\mathbb{T}^n,\iota)$. There are natural homomorphisms

$$\begin{aligned}
&\sigma : K_1(B_{n-1}[z,z^{-1}]) \to K_1^{top}(C(\mathbb{T}^n,\iota)),\\
&\sigma'' : \mathrm{SK}_1(B_{n-1}[z,z^{-1}]) \to \mathrm{SK}_1^{top}(C(\mathbb{T}^n,\iota)),\\
&\sigma' : B_{n-1}[z,z^{-1}]^* \to \pi_0(C(\mathbb{T}^n,\iota)^*)
\end{aligned}$$

which all are induced by the inclusion $B_{n-1}[z,z^{-1}] \hookrightarrow C(\mathbb{T}^n,\iota)$.

In order to compute $\mathrm{SK}_1(C(\mathbb{T}^n,\iota))$ we compare the commutative diagrams
(5.3)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B_{n-1}[z,z^{-1}]^* & \longrightarrow & K_1(B_{n-1}[z,z^{-1}]) & \longrightarrow & \mathrm{SK}_1(B_{n-1}[z,z^{-1}]) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\sigma'} & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma''} & & \\
0 & \longrightarrow & \pi_0(C(\mathbb{T}^n,\iota)^*) & \longrightarrow & K_1^{top}(C(\mathbb{T}^n,\iota)) & \longrightarrow & \mathrm{SK}_1^{top}(C(\mathbb{T}^n,\iota)) & \longrightarrow & 0
\end{array}$$

and
(5.4)

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_1(B_{n-1}) & \longrightarrow & K_1(B_{n-1}[z,z^{-1}]) & \longrightarrow & K_0(B_{n-1}) & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\theta'} & & \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta''} & & \\
0 & \longrightarrow & K_1^{top}(C(\mathbb{T}^{n-1},\iota)) & \longrightarrow & K_1^{top}(C(\mathbb{T}^n,\iota)) & \longrightarrow & K_0^{top}(C(\mathbb{T}^{n-1},\iota)) & \longrightarrow & 0\ ,
\end{array}$$

where the homomorphisms θ, θ' and θ'' in the second diagram are again induced by the inclusions $B_{n-1}[z, z^{-1}] \subset C(\mathbb{T}^n, \iota)$ and $B_{n-1} \subset C(\mathbb{T}^{n-1}, \iota)$.

Only the commutativity of diagram (5.4) needs a justification. By Theorem 5.27 we know that $K_1^{top}(C(\mathbb{T}^n, \iota)) = K_1^{top}(C(\mathbb{T}^{n-1}, \iota)) \oplus K_0^{top}(C(\mathbb{T}^{n-1}, \iota))$, but to prove the commutativity of (5.4) we follow Swan's proof of the isomorphism

$$K_1^{top}(C(X \times \mathbb{T}, \mathbb{C})) = K_1^{top}(C(X, \mathbb{C})) \oplus K_0^{top}(C(X, \mathbb{C})),$$

where X is a compact Hausdorff space, in [Swa68].

Let F be a contravariant functor from topological spaces to groups. We say $a, b \in F(X)$ are homotopic if there exists some $g \in F(X \times I)$, $I = [0, 1]$, such that $F(i_0)(g) = a$ and $F(i_1)(g) = b$, where $i_0, i_1 : X \to X \times I$ are the inclusions $i_0(x) = (x, 0), i_1(x) = (x, 1)$. Then one can define a new functor be identifying homotopic elements in $F(X)$.

We want to show that the canonical homomorphism of algebraic K-groups

$$K_1(C(\mathbb{T}^{n-1}, \iota)[z, z^{-1}]) \to K_1(C(\mathbb{T}^n, \iota)),$$

which is induced by the inclusion $C(\mathbb{T}^{n-1}, \iota)[z, z^{-1}] \to C(\mathbb{T}^n, \iota)$, induces an isomorphism after identifying homotopic elements on both sides, i.e one has an isomorphism

(5.5)
$$K_1(C(\mathbb{T}^{n-1}, \iota)[z, z^{-1}])/(\text{hom.}) \to K_1(C(\mathbb{T}^n, \iota))/(\text{hom.}) = K_1^{top}(C(\mathbb{T}^n, \iota)).$$

Then the decomposition of $K_1(C(\mathbb{T}^{n-1}, \iota)[z, z^{-1}])$ into

$$K_1(C(\mathbb{T}^{n-1}, \iota)[z, z^{-1}]) = K_1(C(\mathbb{T}^{n-1}, \iota)) \oplus K_0(C(\mathbb{T}^{n-1}, \iota)) \oplus W,$$

where W is the subgroup of $K_1(C(\mathbb{T}^{n-1}, \iota)[z, z^{-1}])$ generated by elements $I + (z-1)N, I + (z^{-1} - 1)N$, I the identity matrix and N a nilpotent matrix with entries in $C(\mathbb{T}^{n-1}, \iota)$ yields the isomorphism

$$\begin{aligned} K_1(C(\mathbb{T}^{n-1}, \iota)[z, z^{-1}])/(\text{hom.}) &= K_1^{top}(C(\mathbb{T}^{n-1}, \iota)) \oplus K_0^{top}(C(\mathbb{T}^{n-1}, \iota)) \\ &= K_1^{top}(C(\mathbb{T}^n, \iota)) \end{aligned}$$

because the elements in W are homotopic to I and

$$K_0(C(\mathbb{T}^{n-1}, \iota))/(\text{hom.}) = K_0(C(\mathbb{T}^{n-1}, \iota)) = K_0^{top}(C(\mathbb{T}^{n-1}, \iota)).$$

(5.5) is proven in [Swa68], Lemmata 17.4-17.8, in the case $C(\mathbb{T}^n, \mathbb{C})$. But the same arguments work in the case $C(\mathbb{T}^n, \iota)$ which gives equation (5.5).

Because the summands $K_1(B_{n-1})$ and $K_1^{top}(C(\mathbb{T}^{n-1}, \iota))$ are embedded in $K_1(B_{n-1}[z, z^{-1}])$ and $K_1^{top}(C(\mathbb{T}^n, \iota))$ by the homomorphisms induced by the inclusions $B_{n-1} \subset B_{n-1}[z, z^{-1}]$ and $C(\mathbb{T}^{n-1}, \iota) \subset C(\mathbb{T}^n, \iota)$, respectively, and because we have shown that the splitting of $K_1^{top}(C(\mathbb{T}^n, \iota))$ comes from the splitting of the algebraic K-group $K_1(C(\mathbb{T}^{n-1}, \iota)[z, z^{-1}])$ we see that the arrows in diagram (5.4) commute.

We summarize what we know about the diagrams (5.3) and (5.4) in the following proposition.

Proposition 5.28. *The following holds:*

(i) The commutative diagrams (5.3) *and* (5.4) *have exact rows.*

(ii) θ' *and* σ' *are surjective.*

(iii) σ'' *is injective.*

Proof. (i) follows from the previous discussion.

By Lemma 5.26 σ' is surjective. It is $K_1(B_{n-1}) = B_{n-1}^* \oplus \mathrm{SK}_1(B_{n-1})$ and $K_1^{top}(C(\mathbb{T}^{n-1}, \iota)) = \pi_0(C(\mathbb{T}^{n-1}, \iota)^*) \oplus \mathrm{SK}_1^{top}(C(\mathbb{T}^{n-1}, \iota))$. Because B_{n-1} is special dense in $C(\mathbb{T}^{n-1}, \iota)$ the map $B_{n-1}^* \to \pi_0(C(\mathbb{T}^{n-1}, \iota)^*)$ induced by θ' is surjective. By Theorem 5.22, (ii), also the induced map $\mathrm{SK}_1(B_{n-1}) \to \mathrm{SK}_1^{top}(C(\mathbb{T}^{n-1}, \iota))$ is surjective. This shows that θ' is surjective.

For (iii) use [Day76], Proposition 4.1. □

Corollary 5.29. *There is an exact sequence*

$$0 \to SK_1(B_{n-1}[z, z^{-1}]) \to SK_1^{top}(C(\mathbb{T}^n, \iota)) \to \tilde{K}_0^{top}(C(\mathbb{T}^{n-1}, \iota)) \to 0.$$

Proof. We proved that $\sigma'' : \mathrm{SK}_1(B_{n-1}[z, z^{-1}]) \to \mathrm{SK}_1^{top}(C(\mathbb{T}^n, \iota))$ is injective. The exact sequences (5.3) and (5.4) induce exact sequences

$$\operatorname{coker} \sigma' \to \operatorname{coker} \sigma \to \operatorname{coker} \sigma'' \to 0$$

and

$$\operatorname{coker} \theta' \to \operatorname{coker} \theta \to \operatorname{coker} \theta'' \to 0.$$

Because θ' and σ' are surjective, we have $\operatorname{coker} \theta \simeq \operatorname{coker} \theta''$ and $\operatorname{coker} \sigma \simeq \operatorname{coker} \sigma''$. Using furthermore that $\theta = \sigma$, we may identify the cokernel of $\sigma'' : \mathrm{SK}_1(B_{n-1}[z, z^{-1}]) \to \mathrm{SK}_1^{top}(C(\mathbb{T}^n, \iota))$ with the cokernel of $\theta'' : K_0(B_{n-1}) \to K_0^{top}(C(\mathbb{T}^{n-1}, \iota))$.

The ring B_{n-1} is a localisation of $\mathbb{R}[z_1^{\pm 1}, \ldots, z_{n-1}^{\pm 1}]$. By Theorem 4.11 we know that $K_0(\mathbb{R}[z_1^{\pm 1}, \ldots, z_{n-1}^{\pm 1}]) = \mathbb{Z}$ and with the Localisation Sequence 4.13 we deduce that also $K_0(B_{n-1}) = \mathbb{Z}$. Thus, we may identify the cokernel of θ'' with the reduced K-group $\tilde{K}_0^{top}(C(\mathbb{T}^{n-1}, \iota))$. This gives the exact sequence of the lemma. □

Lemma 5.30. *It is* $SK_1(B_{n-1}) \simeq SK_1(B_{n-1}[z, z^{-1}])$.

Proof. By Theorem 4.11 it is

$$K_1(B_{n-1}[z, z^{-1}]) = K_0(B_{n-1}) \oplus K_1(B_{n-1}) = \mathbb{Z} \oplus K_1(B_{n-1}). \tag{5.6}$$

On the other hand, it is

$$K_1(B_{n-1}[z, z^{-1}]) = \mathrm{SK}_1(B_{n-1}[z, z^{-1}]) \oplus B_{n-1}^* \oplus \mathbb{Z}. \tag{5.7}$$

Comparing equations (5.6) and (5.7) we deduce that $\mathrm{SK}_1(B_{n-1})$ has to be isomorphic to $\mathrm{SK}_1(B_{n-1}[z, z^{-1}])$. □

Applying Theorem 5.27 to $C(\{pt\}, \mathrm{id}) = \mathbb{R}$ yields:

Proposition 5.31. *There is an isomorphism*

$$K_n^{top}(C(\mathbb{T}, \iota)) \simeq K_n^{top}(\mathbb{R}) \oplus K_{n-1}^{top}(\mathbb{R}).$$

Proposition 5.32. *The structure of* $K_n^{top}(\mathbb{R})$ *is given by*

$$K_n^{top}(\mathbb{R}) = \begin{cases} \mathbb{Z}, & n \equiv 0,4 \mod 8 \\ \mathbb{F}_2, & n \equiv 1,2 \mod 8 \\ 0, & n \equiv 3,5,6,7 \mod 8 \ . \end{cases}$$

Thus, by Proposition 5.31 we get

$$K_n^{top}(C(\mathbb{T}, \iota)) = \begin{cases} \mathbb{Z}, & n \equiv 0,4,5 \mod 8 \\ \mathbb{Z} \oplus \mathbb{F}_2, & n \equiv 1 \mod 8 \\ \mathbb{F}_2 \oplus \mathbb{F}_2, & n \equiv 2 \mod 8 \\ \mathbb{F}_2, & n \equiv 3 \mod \\ 0, & n \equiv 6,7 \mod 8 \ . \end{cases}$$

Proof. See [Kar78], III, Theorem 5.19. □

The next result follows from Theorem 5.27 by applying it to the Banach algebra $C(\mathbb{T}^{n-1}, \iota)$.

Proposition 5.33. *There is an isomorphism*

$$K_n^{top}(C(\mathbb{T}^n, \iota)) \simeq K_n^{top}(C(\mathbb{T}^{n-1}, \iota)) \oplus K_{n-1}^{top}(C(\mathbb{T}^{n-1}, \iota)).$$

Proposition 5.34. *Let* $n \geq 1$ *be a natural number. Then*

$$SK_1^{top}(C(\mathbb{T}^n, \iota)) = 0 \text{ for } n \leq 4.$$

For $n \geq 5$ *it is*

$$SK_1^{top}(C(\mathbb{T}^n, \iota)) \neq 0.$$

Proof. Because $\mathbb{R}[z, z^{-1}]$ is dense in $C(\mathbb{T}, \iota)$ we have

$$\mathrm{SK}_1(C(\mathbb{T}, \iota)) = \mathrm{SK}_1(\mathbb{R}[z, z^{-1}]) = 0.$$

For $n \geq 2$ we use the exact sequence of Corollary 5.29

$$0 \to \mathrm{SK}_1(B_{n-1}[z, z^{-1}]) \to \mathrm{SK}_1^{top}(C(\mathbb{T}^n, \iota)) \to \tilde{K}_0^{top}(C(\mathbb{T}^{n-1}, \iota)) \to 0,$$

the isomorphisms

$$\mathrm{SK}_1(B_{n-1}[z, z^{-1}]) \simeq \mathrm{SK}_1(B_{n-1}) \simeq \mathrm{SK}_1^{top}(C(\mathbb{T}^{n-1}, \iota)) \tag{5.8}$$

and Propositions 5.31, 5.32, 5.33 to compute $\mathrm{SK}_1^{top}(C(\mathbb{T}^n, \iota))$ inductively. We have

$$\begin{aligned}
\mathrm{SK}_1(C(\mathbb{T}^2, \iota)) &\simeq \tilde{K}_0^{top}(C(\mathbb{T}, \iota)) = 0 \\
\mathrm{SK}_1(C(\mathbb{T}^3, \iota)) &\simeq \tilde{K}_0^{top}(C(\mathbb{T}^2, \iota)) \simeq \tilde{K}_0^{top}(C(\mathbb{T}, \iota)) \oplus K_7^{top}(C(\mathbb{T}, \iota)) = 0 \\
\mathrm{SK}_1(C(\mathbb{T}^4, \iota)) &\simeq \tilde{K}_0^{top}(C(\mathbb{T}^2, \iota)) \oplus K_6^{top}(C(\mathbb{T}, \iota)) \oplus K_7^{top}(C(\mathbb{T}, \iota)) = 0 \\
\mathrm{SK}_1(C(\mathbb{T}^5, \iota)) &\simeq \tilde{K}_0^{top}(C(\mathbb{T}^4, \iota)) \simeq \tilde{K}_0^{top}(C(\mathbb{T}^3, \iota)) \oplus K_7^{top}(C(\mathbb{T}^3, \iota)) \\
&\simeq K_7^{top}(C(\mathbb{T}^2, \iota)) \oplus K_6^{top}(C(\mathbb{T}^2, \iota)) \simeq K_5^{top}(C(\mathbb{T}, \iota)) \simeq \mathbb{Z} .
\end{aligned}$$

With the exact sequence of Corollary 5.29 this implies $\mathrm{SK}_1^{top}(C(\mathbb{T}^n, \iota)) \neq 0$ for $n \geq 5$. □

These calculations now imply the main goal of the section. Namely, combining Corollary 5.23 and Proposition 5.34, we get the following result:

Theorem 5.35. *Let $d \geq 5$. Then $SK_1(R_d[S_\infty^{-1}]) \neq 0$.*

We finish this section with a short application of Theorem 5.35 to the theory of expansive $\mathbb{Z}^d$-actions.

Application 5.36. Using Theorem 5.35, we can show that there exist expansive $\mathbb{Z}^d$-actions on a compact abelian group X such that the SK_1- component of $cl_\infty(X) \in \mathrm{SK}_1(R_d[S_\infty^{-1}]) \oplus (R_d[S_\infty^{-1}])^*/R_d^*$ is non-trivial.

Namely, let $d \geq 5$ and let $f \in \mathrm{GL}_n(R_d[S_\infty^{-1}])$ be a representative of a non-zero element $[f] \in \mathrm{SK}_1(R_d[S_\infty^{-1}])$. Let $s \in S_\infty$ such that $sf \in M_n(R_d)$. Put $X_{sf} = \widehat{(R_d)^n/(R_d)^n sf}$. Then

$$cl_\infty(X_{sf}) = [f^t] \oplus [s^n] \in \mathrm{SK}_1(R_d[S_\infty^{-1}]) \oplus (R_d[S_\infty^{-1}])^*/R_d^*.$$

Open Problem 5.37. Let α be an expansive $\mathbb{Z}^d$-action on X. Let

$$cl_\infty(X) = [f_X] \oplus \det(cl_\infty(X)) \in \mathrm{SK}_1(R_d[S_\infty^{-1}]) \oplus (R_d[S_\infty^{-1}])^*/R_d^*.$$

The element $\det(cl_\infty(X))$ has a dynamical interpretation insofar that we know that the topological entropy of α is given by $h(\alpha) = \log \det_{\mathcal{N}\mathbb{Z}^d}(\det(cl_\infty(X)))$. It would be interesting to know if there is a dynamical interpretation of the element $[f_X] \in \mathrm{SK}_1(R_d[S_\infty^{-1}])$ or of some "mathematical object" which is derived from $[f_X]$.

Chapter 6

Periodic p-adic entropy in the case of the discrete Heisenberg group

Let $\Gamma \subset \mathrm{SL}(3, \mathbb{Z})$ be the discrete Heisenberg group, generated by the matrices

$$x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \ y = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ z = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with the commutation relations

$$xz = zx, \ yz = zy, \ y^l x^k = x^k y^l z^{kl}, \ k, l \in \mathbb{Z}.$$

In this chapter we want to compute the periodic p-adic entropy $h_{p,per}(X_f)$ of X_f for certain 1-units $f \in 1 + p\mathbb{Z}\Gamma$. By Corollary 2.30, we know that $h_{p,per}(X_f)$ exists in this case. The computation consists of two parts.

First, for a suitable sequence $\Gamma_n \to e$ of cofinite normal subgroups of Γ we need to compute the orders of the fixed point sets $\mathrm{Fix}_{\Gamma_n}(X_f)$. This has been done by K. Schmidt in order to compute the usual entropy of X_f in the expansive case. Then we have to determine the limit

$$h_{p,per}(X_f) = \lim_{n\to\infty} \frac{1}{(\Gamma : \Gamma_n)} \cdot \log_p |\mathrm{Fix}_{\Gamma_n}(X_f)|.$$

For the usual entropy of X_f Schmidt gets a formula involving the Mahler measure of some polynomials attached to f. In the case of the p-adic entropy we will get a formula involving the p-adic Mahler measure.

In Section 6.1 we introduce the Shnirelman integral and the p-adic Mahler measure attached to certain Laurent polynomials over $\mathbb{C}_p$. We prove a Fubini-like result for the Shnirelman integral (Lemma 6.4) and a result that states more or less that for a uniform covergent family of functions in $\mathbb{C}_p\langle z_1^{\pm 1}, \dots, z_n^{\pm 1}\rangle$ taking the limit function and Shnirelman integration commute (Lemma 6.3). We will use these results in the calculation of the periodic p-adic entropy in Section 6.2.

In Section 6.2 we first recall Schmidt's calculation of the number of fixed points in X_f under the action of certain subgroups Γ_n of Γ. Then we calculate $h_{p,per}(X_f)$.

6.1 The Shnirelman integral and the p-adic Mahler measure

Let $T_p^n = \{z \in \mathbb{C}_p^n : |z_i| = 1, 1 \leq i \leq n\}$ be the p-adic n-torus. The Shnirelman integral of a $\mathbb{C}_p$-valued function on T_p^n is defined by

$$\int_{T_p^n} f(z)\frac{dz}{z} := \lim_{\substack{N\to\infty,\\ (N,p)=1}} \frac{1}{N^n} \sum_{\zeta\in\mu_N^n} f(\zeta)$$

if the limit exists, where μ_N denotes the group of N-th roots of unity in T_p.

Notation: For a multiindex $\nu \in \mathbb{Z}^n$ we set $\min|\nu| = \min\{|\nu_1|, \dots, |\nu_n|\}$. For an element $f = \sum_{\nu\in\mathbb{Z}^n} a_\nu z_1^{\nu_1} \dots z_n^{\nu_n} \in \mathbb{C}_p\langle z_1^{\pm 1}, \dots, z_n^{\pm 1}\rangle$ we will just write $f = \sum_{\nu\in\mathbb{Z}^n} a_\nu z^\nu$ when there is no need to be more precise. The algebra $\mathbb{C}_p\langle z_1^{\pm 1}, \dots, z_n^{\pm 1}\rangle$ is defined as

$$\mathbb{C}_p\langle z_1^{\pm 1}, \dots, z_n^{\pm 1}\rangle := \left\{ \sum_{\nu\in\mathbb{Z}^n} x_\nu z_1^{\nu_1} \dots z_n^{\nu_n} : x_\nu \in \mathbb{C}_p, |x_\nu|_p \to 0 \text{ for } \sum_{i=1}^n |\nu_i| \to \infty \right\}.$$

Lemma 6.1. *Let $f = z_1^{\nu_1} \dots z_n^{\nu_n} \in \mathbb{C}_p[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ be a non-constant monomial. Assume $N > \min|\nu|$. Then*

$$\sum_{\zeta\in\mu_N^n} f(\zeta) = 0.$$

Proof. We may assume $|\nu_1| = \min|\nu|$. Since

$$\sum_{\zeta\in\mu_N^n} f(\zeta) = \sum_{\zeta_1\in\mu_N} \zeta_1^{\nu_1} \cdot \sum_{\zeta_2,\dots,\zeta_n\in\mu_N} \zeta_2^{\nu_2} \dots \zeta_n^{\nu_n}$$

the general case will follow from the case $n = 1$. Then we may assume that $f = z^\nu$ where ν is a positive integer because changing the sign of ν does not change the sum $\sum_{\zeta\in\mu_N} f(\zeta)$. Let $d = (\nu, N)$ be the greatest common divisor of ν and N and let $s = N/d$. From the exact sequence

$$0 \to \mu_d \to \mu_N \stackrel{\zeta\mapsto\zeta^\nu}{\to} \mu_s \to 0$$

we see that

$$\sum_{\zeta\in\mu_N} \zeta^\nu = d \cdot \sum_{\eta\in\mu_s} \eta = 0,$$

because the sum of all s-th roots of unity is zero which follows from comparing coefficients of $z^s - 1 = \prod_{\eta\in\mu_s}(z-\eta)$. □

Lemma 6.2. *Let $f(z) = \sum_{\nu\in\mathbb{Z}^n} a_\nu z^\nu \in \mathbb{C}_p\langle z_1^{\pm 1}, \dots, z_n^{\pm 1}\rangle$ be a convergent Laurent series on T_p^n. Then*

$$\int_{T_p^n} f(z)\frac{dz}{z} = a_0.$$

Proof. For any $N \in \mathbb{N}$ we may write f as a sum $f = f_{\min<N} + f_{\min\geq N}$ where

$$f_{\min<N} = \sum_{\substack{\nu\in\mathbb{Z}^n,\\ \min|\nu|<N}} a_\nu z_1^{\nu_1}\dots z_n^{\nu_n} \quad \text{and} \quad f_{\min\geq N} = \sum_{\substack{\nu\in\mathbb{Z}^n,\\ \min|\nu|\geq N}} a_\nu z_1^{\nu_1}\dots z_n^{\nu_n}.$$

Then by the previous lemma we have under the assumption $(N,p) = 1$

$$\left|\frac{1}{N^n}\sum_{\zeta\in\mu_N^n} f(\zeta) - a_0\right| = \left|\frac{1}{N^n}\sum_{\zeta\in\mu_N^n} f_{\min<N}(\zeta) - a_0 + \frac{1}{N^n}\sum_{\zeta\in\mu_N^n} f_{\min\geq N}(\zeta)\right| \leq$$

$$\left|\sum_{\zeta\in\mu_N^n}\sum_{\{\nu:\,\min|\nu|\geq N\}} a_\nu \zeta_1^{\nu_1}\dots\zeta_n^{\nu_n}\right| \leq \max_{\{\nu:\,\min|\nu|\geq N\}} |a_\nu|.$$

Since $\max_{\{\nu:\,\min|\nu|\geq N\}} |a_\nu| \to 0$ as $N \to \infty$ the assertion follows. □

Lemma 6.3. *Let $(f_i)_{i\in\mathbb{N}}, f_i = \sum_{\nu\in\mathbb{Z}^n} a_\nu^{(i)} z^\nu \in \mathbb{C}_p\langle z_1^{\pm 1}, \dots, z_n^{\pm 1}\rangle$, be a family of convergent Laurent series on T_p^n. Assume that for every $\varepsilon > 0$ there exists a natural number $r \in \mathbb{N}$ such that $|a_\nu^{(i)}| < \varepsilon$ for all ν with $\min|\nu| \geq r$ and for all $i \in \mathbb{N}$. Then*

$$\lim_{\substack{i,N\to\infty,\\ (N,p)=1}} \left(\frac{1}{N^n}\sum_{\zeta\in\mu_N^n} f_i(\zeta) - \int_{T_p^n} f_i(z)\frac{dz}{z}\right) = 0.$$

Proof. Under our assumptions the proof is the same as the proof of Lemma 6.2, i.e. we have for any $i \in \mathbb{N}$ and any $N \in \mathbb{N}$ with $(N,p)=1$

$$\left| \frac{1}{N^n} \sum_{\zeta \in \mu_N^n} f_i(\zeta) - \int_{T_p^n} f_i(z) \frac{dz}{z} \right| \leq \max_{\{\nu:\, \min|\nu| \geq N\}} |a_\nu^{(i)}|.$$

Since $\max_{\{\nu:\, \min|\nu|\geq N\}} |a_\nu^{(i)}| \to 0$ as $i, N \to \infty$ the assertion follows. $\square$

Lemma 6.4 (Fubini for the Shnirelman integral)**.** *Let f be a convergent Laurent series on T_p^n. Then*

$$\int_{T_p^n} f(z) \frac{dz}{z} = \int_{T_p} \cdots \left(\int_{T_p} f(z_1, \ldots, z_n) \frac{dz_1}{z_1} \right) \cdots \frac{dz_n}{z_n}.$$

Proof. For $z_2, \ldots, z_n \in T_p$ the function

$$f_{z_2,\ldots,z_n} : z_1 \mapsto f(z_1, z_2, \ldots, z_2) = \sum_{\nu_1 \in \mathbb{Z}} \Big(\sum_{\nu_2,\ldots,\nu_n \in \mathbb{Z}} a_{\nu_1,\nu_2,\ldots,\nu_n} z_2^{\nu_2} \ldots z_n^{\nu_n} \Big) z_1^{\nu_1}$$

is a convergent Laurent series on T_p in the variable z_1. By Lemma 6.2, we have

$$\int_{T_p} f_{z_2,\ldots,z_n}(z_1) \frac{dz_1}{z_1} = \sum_{\nu_2,\ldots,\nu_n \in \mathbb{Z}} a_{0,\nu_2,\ldots,\nu_n} z_2^{\nu_2} \ldots z_n^{\nu_n}.$$

Iteration gives the result. $\square$

Proposition 6.5. *Let*

$$f = a z_1^{\nu_1} \ldots z_n^{\nu_n} (1 + g(z)) \in \mathbb{C}_p[z_1^{\pm 1}, \ldots, z_n^{\pm 1}],$$

where $a \in \mathbb{C}_p^*$, $\nu \in \mathbb{Z}^n$ and $g \in \mathfrak{m}_p[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ be a Laurent polynomial which is a unit in $\mathbb{C}_p\langle z_1^{\pm 1}, \ldots, z_n^{\pm 1}\rangle$. Here, $\mathfrak{m}_p = \{x \in \mathbb{C}_p : |x|_p < 1\}$. Then the Shnirelman integral

$$m_p(f) := \int_{T_p^n} \log_p f \, \frac{dz}{z}$$

exists and is given by

$$m_p(f) = \log_p a + b_0$$

where b_0 is the 0-th coefficient in the Laurent expansion of $\log_p(1+g(z))$ on T_p^n.

Proof. The function $\log_p f : T_p^n \to \mathbb{C}_p$ is well-defined. Because it is $\log_p(xy) = \log_p(x) + \log_p(y)$ for all $x, y \in \mathbb{C}_p^*$, $\log_p(1) = 0$ and because $\mathbb{C}_p$ has no zero-divisors, we deduce that $\log_p$ vanishes on the set of all roots of unity. Using this and Lemma 6.2 we conclude

$$m_p(f) = m_p(a(1+g(z))) = \log_p a + \int_{T_p^n} \log_p(1+g(z))\frac{dz}{z} = \log_p a + b_0.$$

□

Definition 6.6. *Let f be as in Proposition 6.5. The value $m_p(f)$ is called the p-adic Mahler measure of the Laurent polynomial f.*

6.2 Calculation of the periodic p-adic entropy in some cases

Let us return to the discrete Heisenberg group Γ. We denote by $x, y, z \in \Gamma$ the generators of the discrete Heisenberg group as stated in the introduction of Chapter 6. First we note the following simple fact.

Lemma 6.7. *Every element γ in Γ has a unique expression of the form $\gamma = x^{m_1}y^{m_2}z^{m_3}$, $m_1, m_2, m_3 \in \mathbb{Z}$, i.e. there is a bijection of sets*

$$[\,] : \mathbb{Z}^3 \to \Gamma, \quad (m_1, m_2, m_3) \mapsto [m_1, m_2, m_3] := x^{m_1}y^{m_2}z^{m_3} \;.$$

Proof. Using the commutation relations we can write every $\gamma \in \Gamma$ in the form $\gamma = x^{m_1}y^{m_2}z^{m_3}$. For the uniqueness it is enough to note that $x^{m_1}y^{m_2}z^{m_3} = \mathrm{Id}$ if and only if $m_1 = m_2 = m_3 = 0$. This can be seen from the operation of $x^{m_1}y^{m_2}z^{m_3}$ on the standard basis e_1, e_2, e_3 of $\mathbb{Z}^3$. For example e_3 is mapped to $e_3 + m_1e_2 + m_3e_1$ under $x^{m_1}y^{m_2}z^{m_3}$. So $m_1 = m_3 = 0$ if $x^{m_1}y^{m_2}z^{m_3} = \mathrm{Id}$ and then $m_2 = 0$ follows immediately. □

Now, let $f = \sum_{\gamma\in\Gamma} a_\gamma\gamma \in \mathbb{Z}\Gamma \cap c_0(\Gamma)^*$. Recall that f^* is defined as $f^* = \sum_{\gamma\in\Gamma} a_{\gamma^{-1}}\gamma$. We may write f^* in the form

$$f^* = \sum_{m_1\in\mathbb{Z}} x^{m_1}\phi_{m_1}(y, z). \tag{6.1}$$

Here, the $\phi_{m_1} \in \mathbb{Z}[Y^{\pm1}, Z^{\pm1}]$, $m_1 \in \mathbb{Z}$, are integral Laurent polynomials in the variables Y, Z and $\phi_{m_1}(y, z)$ is the element in $\mathbb{Z}\Gamma$ obtained by substituting y and z for Y and Z, respectively. Note that by Lemma 6.7, the ϕ_{m_1} in equation (6.1) are uniquely determined.

We consider the dynamical system

$$X_f = \ker(\rho_f : (\mathbb{R}/\mathbb{Z})[[\Gamma]] \to (\mathbb{R}/\mathbb{Z})[[\Gamma]]),$$

where ρ_f is the right multiplication with f^* on the group $(\mathbb{R}/\mathbb{Z})[[\Gamma]]$ of infinite formal series with coefficients in $\mathbb{R}/\mathbb{Z}$. The algebraic Γ-action λ on X_f is given by left multiplication with elements $\gamma \in \Gamma$, see Section 2.1 for more details.

For every $q, r, s \geq 1$ we define a normal subgroup $\Gamma_{q,r,s} \subset \Gamma$ by

$$\Gamma_{q,r,s} = \Big\{ \left(\begin{smallmatrix} 1 & sqb & qc \\ 0 & 1 & rqa \\ 0 & 0 & 1 \end{smallmatrix} \right) : a, b, c \in \mathbb{Z} \Big\}. \tag{6.2}$$

Let us recall Schmidt's calculation of

$$\left|\mathrm{Fix}_{\Gamma_{q,r,s}}(X_f)\right| = \pm \det(\rho_f : \mathbb{C}_p[[\Gamma]]^{\Gamma_{q,r,s}} \to \mathbb{C}_p[[\Gamma]]^{\Gamma_{q,r,s}}),$$

where

$$\mathbb{C}_p[[\Gamma]]^{\Gamma_{q,r,s}} = \{ w \in \mathbb{C}_p[[\Gamma]] : \lambda^\gamma w = w \text{ for every } \gamma \in \Gamma_{q,r,s} \}.$$

This is done by decomposing $\mathcal{L} = \mathbb{C}_p[[\Gamma]]^{\Gamma_{q,r,s}}$ into irreducible subspaces of $\rho := \rho_f$ and calculating the determinant of ρ on each of these subspaces.

For every $\zeta, \eta, \theta \in T_p = \{c \in \mathbb{C}_p : \ |c| = 1\}$, we introduce the element $w^{(\zeta,\eta,\theta)} \in \mathrm{Map}(\Gamma, T_p)$ given by

$$w^{(\zeta,\eta,\theta)}_{[n_1,n_2,n_3]} = \zeta^{n_1}\eta^{n_2}\theta^{n_3}, \ (n_1, n_2, n_3) \in \mathbb{Z}^3.$$

The left and right shift actions λ and ρ of Γ act on $w^{(\zeta,\eta,\theta)}$ by

$$\lambda^{[m_1,m_2,m_3]} w^{(\zeta,\eta,\theta)} = \zeta^{-m_1}\eta^{-m_2}\theta^{m_1m_2-m_3} w^{(\zeta\theta^{-m_2},\eta,\theta)}, \tag{6.3}$$

$$\rho^{[m_1,m_2,m_3]} w^{(\zeta,\eta,\theta)} = \zeta^{m_1}\eta^{m_2}\theta^{m_3} w^{(\zeta,\eta\theta^{m_1},\theta)} \tag{6.4}$$

for every $(m_1, m_2, m_3) \in \mathbb{Z}^3$.

For every $q \geq 1$, every q-th root of unity θ and every $\zeta, \eta \in T_p$ we write

$$\mathcal{L}_{(\zeta,\eta,\theta)} = \langle \rho^\gamma w^{(\zeta,\eta,\theta)} : \gamma \in \Gamma \rangle = \langle w^{(\zeta,\eta\theta^k,\theta)} : k \in \mathbb{Z} \rangle$$

for the cyclic subspace of ρ generated by $w^{(\zeta,\eta,\theta)}$. We have $\dim_{\mathbb{C}_p}(\mathcal{L}_{(\zeta,\eta,\theta)}) = o(\theta)$, where $o(\theta)$ is the order of θ, and

$$\mathcal{L}_{(\zeta,\eta,\theta)} = \mathcal{L}_{(\zeta,\eta\theta^k,\theta)} \tag{6.5}$$

for every $k \in \mathbb{Z}$.

If $\theta = 1$, then

$$\det(\rho_f|_{\mathcal{L}_{(\zeta,\eta,1)}}) = f^*(\zeta, \eta, 1).$$

Again, the expression $f^*(\zeta, \eta, 1) \in \mathbb{C}_p$ means that in the equation (6.1), we substitute ζ, η and 1 for x, y and z, respectively.

If $q \geq 2$ and θ is a primitive q-th root of unity, then every $v \in \mathcal{L}_{(\zeta,\eta,\theta)}$ is of the form $v = \sum_{j=0}^{q-1} c_j v^{(j)}$ with $c_j \in \mathbb{C}_p$ and $v^{(j)} = w^{(\zeta,\eta\theta^j,\theta)}$ for $j = 0, \ldots, q-1$. Furthermore,

$$\rho_f v^{(i)} = \sum_{j=0}^{q-1} a_{i,j} v^{(j)}$$

with

$$a_{i,j} = \sum_{k,m_2,m_3 \in \mathbb{Z}} f^*_{(j+kq-i,m_2,m_3)} \zeta^{j+kq-i} (\eta\theta^i)^{m_2} \theta^{m_3} = \sum_{k \in \mathbb{Z}} \zeta^{j+kq-i} \phi_{j+kq-i}(\eta\theta^i, \theta)$$

for $i, j = 0, \ldots, q-1$. Hence

$$\det(\rho_f|_{\mathcal{L}_{(\zeta,\eta,\theta)}}) = \det(A^{(q)}_{(\zeta,\eta,\theta)}),$$

where

$$A^{(q)}_{(\zeta,\eta,\theta)} = \begin{pmatrix} a_{0,0} & \cdots & a_{0,q-1} \\ \vdots & & \vdots \\ a_{q-1,0} & \cdots & a_{q-1,q-1} \end{pmatrix}.$$

Note that

$$\det(A^{(q)}_{(\zeta,\eta,\theta)}) = \det(A^{(q)}_{(\zeta\theta^k,\eta,\theta)}) \tag{6.6}$$

$$\det(A^{(q)}_{(\zeta,\eta,\theta)}) = \det(A^{(q)}_{(\zeta,\eta\theta^{k'},\theta)}) \tag{6.7}$$

for every $k, k' \in \mathbb{Z}$ and every primitive q-th root of unity θ. Equation (6.6) can be deduced from the Leibniz expansion of the determinant, while equation (6.7) follows because the matrix $A^{(q)}_{(\zeta,\eta\theta^{k'},\theta)}$ describes the endomorphism ρ_f with respect to the basis $\tilde{v}^{(j)} = w^{(\zeta,\eta\theta^{k'+j},\theta)}, 0 \leq j \leq q-1$.

Lemma 6.8. *Let q, r, s be rational primes with $q \neq s$ and $q \neq r$. Consider the space $\mathcal{L} = \mathbb{C}_p[[\Gamma]]^{\Gamma_{q,r,s}}$ introduced earlier. Then $\mathcal{L}$ has the following decomposition into ρ-invariant subspaces:*

$$\mathcal{L} = \bigoplus_{\zeta \in \mu_{qr}, \eta \in \mu_{qs}} \mathcal{L}_{(\zeta,\eta,1)} \oplus \bigoplus_{\{\theta \neq 1 : \theta^q = 1\}} \bigoplus_{\zeta \in \mu_{qr}} \bigoplus_{\eta \in \mu_s} \mathcal{L}_{(\zeta,\eta,\theta)}. \tag{6.8}$$

It follows that

$$\mathrm{Fix}_{\Gamma_{q,r,s}}(X_f) = \pm \prod_{\zeta \in \mu_{qr}, \eta \in \mu_{qs}} f^*(\zeta, \eta, 1) \cdot \prod_{\{\theta \neq 1 : \theta^q = 1\}} \prod_{\zeta \in \mu_r} \prod_{\eta \in \mu_s} \det(A_{(\zeta,\eta,\theta)})^q.$$

Proof. By equation (6.3) the function $w^{(\zeta,\eta,\theta)}$ is an element of $\mathcal{L}$ for every ζ,η,θ with $\zeta^{qr}=\eta^{qs}=\theta^q=1$. Furthermore, it is $\dim(\mathcal{L})=|\Gamma^{q,r,s}|=q^3rs$. Thus, in order to show that the set

$$\{w^{(\zeta,\eta,\theta)}:\zeta,\eta,\theta\in\mathbb{C}_p,\zeta^{qr}=\eta^{qs}=\theta^q=1\}$$

spans $\mathcal{L}$, we only have to show that this set of functions is linearly independent.

This can be proven similarly to the way one proves linear independence of a set of distinct characters on a group: Assuming a non-trivial linear combination of 0 with a minimal number of functions $w^{(\zeta,\eta,\theta)}, \zeta\in\mu_{qr},\eta\in\mu_{qs},\theta\in\mu_q$, one uses the operators $\rho^{[0,1,0]}$ and $\rho^{[0,0,1]}$ to show that the functions involved do actually depend on the same parameters for η and θ; otherwise we would get a contradiction to minimality. But the linear independence of a family of distinct functions $w^{(\zeta,\eta,\theta)}$ with η,θ fixed and ζ running follows because these are characters on the subgroup of Γ generated by the element $x\in\Gamma$.

Using (6.5) and the fact that $\mu_q\times\mu_s\simeq\mu_{qs}$ we see that formula (6.8) gives the desired decomposition of $\mathcal{L}$ into ρ-invariant subspaces.

The formula on the fixed points follows by taking the determinant on each of the ρ-invariant subspaces. Here, we use that by formula (6.6) it is for η and $\theta\neq 1$ fixed

$$\prod_{\zeta\in\mu_{rq}}\det(A_{(\zeta,\eta,\theta)})=\prod_{\zeta\in\mu_r}\det(A_{(\zeta,\eta,\theta)})^q.$$

□

In the following we introduce some notation which will be useful when we consider the matrices $A^{(q)}_{(\zeta,\eta,\theta)}$ for varying choices of q or when we want to emphasize that for some of the parameters ζ,η,θ fixed we consider $A^{(q)}_{(\zeta,\eta,\theta)}$ or functions derived from $A^{(q)}_{(\zeta,\eta,\theta)}$ as a function of the remaining non-fixed parameters.

Definition 6.9. *Let $f^*=\sum_{m_1\in\mathbb{Z}}x^{m_1}\phi_{m_1}(y,z)\in\mathbb{Z}\Gamma$. For $q\in\mathbb{N}$ we define the matrix $A^{(q)}_{(X,Y,Z)}\in M_q(\mathbb{Z}[X^{\pm1},Y^{\pm1},Z^{\pm1}])$ by*

$$a^{(q)}_{i,j}=\sum_{k\in\mathbb{Z}}X^{j+kq-i}\phi_{j+kq-i}(YZ^i,Z)$$

for $i,j=0,\dots,q-1$. We define

$$g^{(q)}(X,Y,Z)=\det A^{(q)}_{(X,Y,Z)}\in\mathbb{Z}[X^{\pm1},Y^{\pm1},Z^{\pm1}].$$

For $\zeta, \eta, \theta \in T_p$ we define

$$\begin{aligned}
g_{\theta}^{(q)}(X,Y) &= \det A_{(X,Y,\theta)}^{(q)} \in \mathbb{Z}[\theta^{\pm 1}][X^{\pm 1}, Y^{\pm 1}], \\
g_{(\eta,\theta)}^{(q)}(X) &= \det A_{(X,\eta,\theta)}^{(q)} \in \mathbb{Z}[\theta^{\pm 1}, \eta^{\pm 1}][X^{\pm 1}], \\
g_{(\zeta,\eta,\theta)}^{(q)} &= \det A_{(\zeta,\eta,\theta)}^{(q)} \in \mathbb{Z}[\zeta^{\pm 1}, \theta^{\pm 1}, \eta^{\pm 1}].
\end{aligned}$$

In the following, we assume $f \in \mathbb{Z}\Gamma$ to be a 1-unit of the form

$$f = \sum_{i=0}^{k} x^i h_i(y,z).$$

The next step in the computation of the periodic p-adic entropy of X_f is the following result.

Lemma 6.10. *Let $f = \sum_{i=0}^{k} x^i h_i(y,z) \in 1 + p\mathbb{Z}\Gamma$ so that $f^* = \phi_0(y,z) + \ldots + x^{-k}\phi_{-k}(y,z)$. We denote by $f^*(X,Y,Z)$ the Laurent polynomial in three variables attached to f^*. Then the following holds:*

(1) It is $f^*(X,Y,1) \in 1 + p\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$. In particular, the p-adic Mahler measure $m_p(f^*(X,Y,1))$ exists and is given by

$$m_p(f^*(X,Y,1)) = m_p(\phi_0(Y,1)).$$

(2) We have

$$g_{(\zeta,\eta,\theta)}^{(q)} = \prod_{i=0}^{q-1} \phi_0(\eta\theta^i, \theta) + \sum_{l<0} \zeta^l R_l(\eta,\theta) \in 1 + p\mathbb{Z}[\zeta^{-1}, \eta^{\pm 1}, \theta^{\pm 1}],$$

where the R_l are Laurent polynomials. In particular, for $\eta, \theta \in T_p$ fixed, the p-adic Mahler measure of the function

$$g_{(\eta,\theta)}^{(q)}(X) : T_p \to \mathbb{C}_p, \quad \zeta \mapsto g_{(\eta,\theta)}^{(q)}(\zeta)$$

exists and is given by

$$m_p(g_{(\eta,\theta)}^{(q)}(X)) = \log_p\left(\prod_{i=0}^{q-1} \phi_0(\eta\theta^i, \theta)\right).$$

(3) Let $\theta \in \mu_q$ be fixed. Then the function

$$\eta \mapsto m_p(g^{(q)}_{(\eta,\theta)}(X))$$

is integrable over the p-adic torus T_p, and we have

$$\int_{T_p} m_p(g^{(q)}_{(\eta,\theta)}(X))\frac{d\eta}{\eta} = q \cdot m_p(\phi_0(Y,\theta)).$$

(4) The results (2) and (3) can be summarized in the way that for $\theta \in \mu_q$ fixed it is

$$m_p(g^{(q)}_\theta(X,Y)) = q \cdot m_p(\phi_0(Y,\theta)).$$

Proof. (1): It is $f^* = \phi_0(y,z)+\ldots+x^{-k}\phi_{-k}(y,z)$. As $\phi_0(y,z) \in 1+p\mathbb{Z}\Gamma$ and $\phi_i(y,z) \in p\mathbb{Z}\Gamma, i = -1,\ldots,-k$, we have $f^*(X,Y,1) \in 1 + p\mathbb{Z}[X^{\pm 1}, Y^{\pm 1}]$. By Proposition 6.5 we know that $m_p(f^*(X,Y,1))$ exists. Then by Lemma 6.4 we can calculate $m_p(f^*(X,Y,1))$ by first integrating $\log_p f^*(X,Y,1)$ with respect to the variable X and integrating with respect to Y afterwards. Again by Proposition 6.5 we know that the result of the first integration is the 0-th coefficient of the Laurent expansion of $\log(f^*(X,Y,1))$ with respect to the variable X. But as $f^*(X,Y,1)$ is a polynomial in X^{-1}, the 0-th coefficient of $\log_p f^*(X,Y,1)$ is just $\log_p(\phi_0(Y,1))$. Integrating this expression with respect to the variable Y, we get the result stated in (1).

(2): The entries in the matrix $A_{(\zeta,\eta,\theta)}$ are given by

$$a_{i,j} = \sum_{k\in\mathbb{Z}} \zeta^{j+kq-i}\phi_{j+kq-i}(\eta\theta^i,\theta)$$

for $i,j = 0,\ldots,q-1$. As $f^* = \phi_0(y,z)+\ldots+x^{-k}\phi_{-k}(y,z)$, the ϕ_{j+kq-i} in the definition of the $a_{i,j}$ are non-zero only for $j+kq-i \le 0$. Thus, we see that the $a_{i,j}$ are polynomials in ζ^{-1} with coefficients in $\mathbb{Z}[\eta^{\pm 1}, \theta^{\pm 1}]$. Furthermore, the $a_{i,j}$ have a constant ζ term if and only if $i = j$, which is then given by $\phi_0(\eta\theta^i,\theta)$. Then from the Leibniz expansion of $\det(A^{(q)}_{(\zeta,\eta,\theta)})$ we get the result about $\det(A^{(q)}_{(\zeta,\eta,\theta)})$ stated in (2). As the function $g^{(q)}_{(\eta,\theta)}(X)$ is a polynomial in X^{-1} with constant X-term equal to $\prod_{i=0}^{q-1}\phi_0(\eta\theta^i,\theta)$, it is

$$m_p(g^{(q)}_{(\eta,\theta)}(X)) = \log_p\left(\prod_{i=0}^{q-1}\phi_0(\eta\theta^i,\theta)\right).$$

(3): By (2) we have

$$\int_{T_p} m_p(g^{(q)}_{(\eta,\theta)}(X))\frac{d\eta}{\eta} = \sum_{i=0}^{q-1}\int_{T_p}\log\phi_0(\eta\theta^i,\theta)\frac{d\eta}{\eta}.$$

To calculate the right-hand side of the above equation, we have to expand the functions

$$\log_p \phi_0(\eta\theta^i, \theta) = \sum_{j\in\mathbb{Z}} \Phi_j(\theta^i, \theta)\eta^j$$

into Laurent series in the variable η and then take its 0-th coefficient. But it is

$$\Phi_0(\theta^i, \theta) = \Phi_0(1, \theta), \quad i = 1, \ldots, q-1,$$

because Φ_0 gives the constant part in the Laurent expansion of $\log_p \phi_0(\eta\theta^i, \theta)$ as a function of η, and θ^i and η are in the same argument of the function $\log \phi_0(\eta\theta^i, \theta)$. So we get

$$\sum_{i=0}^{q-1} \int_{T_p} \log_p \phi_0(\eta\theta^i, \theta)\frac{d\eta}{\eta} = q \int_{T_p} \log_p \phi_0(\eta, \theta)\frac{d\eta}{\eta} = q \cdot m_p(\phi(Y, \theta)).$$

(4) follows from (2) and (3) using Lemma 6.4.

□

Lemma 6.11. *Let* $f^* = \sum_{i=0}^{k} x^{-i}\phi_{-i}(y,z) \in 1 + p\mathbb{Z}\Gamma$. *For* $q \geq 1$, *let*

$$g^{(q)}(X,Y,Z) = \det A^{(q)}_{(X,Y,Z)} = \sum_{j=0}^{r} h_j^{(q)}(Y,Z)X^{-j} \in \mathbb{Z}[X^{-1}, Y^{\pm 1}, Z^{\pm 1}]\ .$$

and

$$f_q := \log g^{(q)} = \sum_{\nu\in\mathbb{Z}^3} a_\nu^{(q)} X^{\nu_1}Y^{\nu_2}Z^{\nu_3} \in \mathbb{Q}_p\langle X^{-1}, Y^{\pm 1}, Z^{\pm 1}\rangle\ .$$

Then the following holds:

(i) For all $q \geq 1$*:*

$$h_0^{(q)} - 1 \equiv 0 \mod p \text{ and}$$
$$h_j^{(q)} \equiv 0 \mod p^t \text{ for } (t-1)k + 1 \leq j \leq tk, t \geq 1.$$

(ii) The family $(f_q)_{q\in\mathbb{N}}$ *fulfills the condition of Lemma 6.3, i.e. for every* $\varepsilon > 0$ *there exists a natural number* $r \in \mathbb{N}$ *such that* $|a_\nu^{(q)}| < \varepsilon$ *for all* ν *with* $\min|\nu| \geq r$ *and for all* $q \in \mathbb{N}$.

Proof. That p divides $h_0^{(q)} - 1$ follows because $g^{(q)}$ is a 1-unit. For the second part of (i) note that the entries $a_{i,j}^{(q)}$ all have X-degree less or equal to $-k$. Furthermore, the summands in the Leibniz expansion of $g^{(q)}$ that contribute to the X-degree of $g^{(q)}$ are divisible by p. Part (i) follows from this.

For (ii) we just have to plug $g^{(q)}$ into the logarithmic series. Part (i) then implies that for $|\nu_1|$ large enough $|a_\nu^{(q)}|$ will be smaller than any $\varepsilon > 0$ independently of q. □

Theorem 6.12. *Let $f = h_0(y,z) + xh_1(y,z) + \ldots + x^k h_k(y,z) \in 1 + p\mathbb{Z}\Gamma$. Write $f^* = \phi_0(y,z) + \ldots + x^{-k}\phi_{-k}(y,z)$. Then the periodic p-adic entropy of X_f is given by*

$$h_p(X_f) = m_p(\phi_0).$$

Proof. We choose increasing sequences of prime numbers $(q_n), (r_n), (s_n)$, $n \in \mathbb{N}$, with $r_n = s_n \neq q_n$ for all $n \in \mathbb{N}$. We write Γ_n for the normal, cofinite subgroup Γ_{q_n,r_n,s_n} of Γ as defined in (6.2). Then $\Gamma_n \to e$ and according to the definition the periodic p-adic entropy of X_f is given by

$$h_{p,per}(X_f) = \lim_{n\to\infty} \frac{1}{(\Gamma : \Gamma_n)} \log_p |\mathrm{Fix}_{\Gamma_n}(X_f)|.$$

We will omit the index n in the following. By Lemma 6.8 it is

$$h_{p,per}(X_f) = \lim_{\substack{q,r\to\infty \\ q\neq r \text{ prime}}} \frac{1}{q^3r^2}\Bigg(\sum_{\zeta,\eta\in\mu_{qr}} \log_p f^*(\zeta,\eta,1) + q \sum_{\theta\in\mu_q\setminus\{1\}} \sum_{\zeta,\eta\in\mu_r} \log_p g^{(q)}_{(\zeta,\eta,\theta)} \Bigg).$$

In Lemma 6.10 we proved that $m_p(f^*(X,Y,1)) = m_p(\phi_0(Y,1))$. It follows

$$\text{(6.9)} \qquad \lim_{\substack{q,r\to\infty \\ q\neq r \text{ prime}}} \Bigg(\frac{1}{q^3r^2} \sum_{\zeta,\eta\in\mu_{qr}} \log_p f^*(\zeta,\eta,1) - \frac{1}{q} m_p(\phi_0(Y,1)) \Bigg) =$$

$$\lim_{\substack{q,r\to\infty \\ q\neq r \text{ prime}}} \frac{1}{q}\Bigg(\frac{1}{(qr)^2} \sum_{\zeta,\eta\in\mu_{qr}} \log_p f^*(\zeta,\eta,1) - m_p(\phi_0(Y,1)) \Bigg) = 0.$$

Now for $\theta \in \mu_q$ consider the family of functions $g^{(q)}_\theta(X,Y)$. We claim

$$\text{(6.10)} \qquad \lim_{\substack{q,r\to\infty \\ q\neq r \text{ prime}}} \Bigg(\frac{1}{r^2} \sum_{\zeta,\eta\in\mu_r} \log_p(g^{(q)}_\theta(\zeta,\eta)) - m_p(g^{(q)}_\theta(X,Y)) \Bigg) = 0.$$

Equation (6.10) follows using Lemma 6.11, (ii), and Lemma 6.3.

Now, equations (6.9) and (6.10) together with Lemma 6.10, (4), imply

$$\lim_{\substack{q,r\to\infty \\ q\neq r \text{ prime}}} \frac{1}{q^3r^2}\Bigg(\sum_{\zeta,\eta\in\mu_{qr}} \log_p f^*(\zeta,\eta,1) + q \sum_{\theta\in\mu_q\setminus\{1\}} \sum_{\zeta,\eta\in\mu_r} \log_p g^{(q)}_{(\zeta,\eta,\theta)} \Bigg) =$$

$$\lim_{\substack{q,r\to\infty \\ q\neq r \text{ prime}}} \frac{1}{q}\Bigg(m_p(\phi_0(Y,1)) + \sum_{\theta\in\mu_q\setminus\{1\}} m_p(\phi_0(Y,\theta)) \Bigg) =$$

$$\lim_{q\to\infty} \frac{1}{q} \sum_{\theta\in\mu_q} m_p(\phi_0(Y,\theta)) = m_p(\phi_0).$$

□

Chapter 7

Remarks on p-adic expansiveness, p-adic entropy and the p-adic Banach algebra $c_0(\Gamma)$

In this last chapter we address some open problems. Recall that one of the initial questions was, if there exists a dynamical criterion for the existence of the periodic p-adic entropy.

Section 7.1 deals with this question insofar that we give an example of an p-adically expansive algebraic $\mathbb{Z}$-action whose periodic p-adic entropy does not exist. On the other hand, we also provide an example of an algebraic $\mathbb{Z}$-action whose periodic p-adic entropy exists but which is not p-adically expansive.

We used the p-adic Fuglede-Kadison determinant to define a notion of p-adic entropy for p-adically expansive $\mathbb{Z}^d$-actions (see Chapter 4). Even though a general dynamical interpretation of the p-adic entropy remains open, our definitions were justified by the facts that for $f \in M_n(R_d) \cap \mathrm{GL}_n(c_0(\mathbb{Z}^d))$, the $\mathbb{Z}^d$-action on X_f is p-adically expansive and the periodic p-adic entropy $h_{p,per}(X_f)$ of X_f coincides with the p-adic entropy $h_p(X_f)$ of X_f.

Section 7.2 is concerned with the question whether there are several ways to define a notion of p-adic entropy which for systems $X_f, f \in M_n(R_d) \cap \mathrm{GL}_n(c_0(\mathbb{Z}^d))$, coincides with the periodic p-adic entropy of X_f. We observe that for expansive $\mathbb{Z}^d$-actions, the assignment which associates an expansive $\mathbb{Z}^d$-action its entropy is uniquely determined by additivity, monotonicity and the values of the entropies of the X_f's. In the p-adic case, the answer remains open but it leads to another interesting open problem.

In Section 7.3 we think of a possible generalisation of the notion of p-adic expansiveness for algebraic actions of countable abelian groups Γ. On the one hand, the case $\Gamma = \mathbb{Z}^d$ suggests to call an algebraic Γ-action on X p-adically expansive if and only if the dual module M^X is a finitely generated S_p-torsion module, where $S_p = \mathbb{Z}\Gamma \cap c_0(\Gamma)^*$. On the other hand, there is an algebraic criterion of expansiveness for algebraic actions of countable abelian groups Γ which has a direct translation into the p-adic setting. Section 7.3 contains a comparison of these two criteria.

In Section 7.4 we discuss some algebraic properties of the p-adic Banach algebra $c_0(\Gamma)$ for Γ a residually finite group.

7.1 Two examples concerning periodic p-adic entropy

Example 7.1. Let $p \neq 2$ be a prime number and consider the R_1-module

$$\mathbb{F}_{2^2} := \mathbb{F}_2[t, t^{-1}]/(t^2 + t + 1) = \mathbb{F}_2[t]/(t^2 + t + 1).$$

This is the finite field with 4 elements consisting of the elements $\overline{0}, \overline{1}, \overline{t}, \overline{t+1}$ which are mapped to $\overline{0}, \overline{t}, \overline{t+1}, \overline{1}$ under the action of $t \in R_1$, respectively. Because $\mathbb{F}_{2^2}$ is finite the Pontrjagin dual $\widehat{\mathbb{F}_{2^2}}$ of $\mathbb{F}_{2^2}$ is naturally isomorphic to $\mathbb{F}_{2^2}$.

Let S_p denote the multiplicative system $S_p = R_1 \cap c_0(\mathbb{Z})^*$. The $\mathbb{Z}$-action on $\widehat{\mathbb{F}_{2^2}}$ is p-adically expansive in the sense of Definition 4.16, i.e. $\mathbb{F}_{2^2}$ is an object of the category $\mathcal{M}_{S_p}(R_1)$ of finitely generated S_p-torsion R_1-modules. By Lemma 4.35, it is $[\mathbb{F}_{2^2}] = 0 \in K_0(\mathcal{M}_S(R_1))$ so that $h_p(\widehat{\mathbb{F}_{2^2}}) = 0$.

We show that the periodic p-adic entropy of the $\mathbb{Z}$-action on $\mathbb{F}_{2^2}$ does not exist.

It is $\mathrm{Fix}_{3\mathbb{Z}}(\widehat{\mathbb{F}_{2^2}}) = \widehat{\mathbb{F}_{2^2}}$. Let r_1 be a natural number not divisible by p and for $n \geq 1$ choose $r_{n+1} \in \mathbb{N}$ with $r_{n+1} > r_n$ so that the difference $r_{n+1} - r_n$ is not divisible by p. Then $(\frac{1}{3r_n} \log_p |\mathrm{Fix}_{3r_n\mathbb{Z}}(\widehat{\mathbb{F}_{2^2}})|)_{n\in\mathbb{N}}$ is not a Cauchy-sequence. Thus, for $\Gamma_n = (3r_n\mathbb{Z}) \to 0$ the limit

$$h_{p,\Gamma_n}(\widehat{\mathbb{F}_{2^2}}) = \lim_{n\to\infty} \frac{1}{3r_n} \log_p |\mathrm{Fix}_{3r_n\mathbb{Z}}(\widehat{\mathbb{F}_{2^2}})|$$

does not exist.

The next example illustrates that for some algebraic $\mathbb{Z}$-actions the periodic p-adic entropy exists for trivial reasons.

Example 7.2. Let α be the $\mathbb{Z}$-action on $X := \widehat{\mathbb{Q}}$ dual to multiplication by $3/2$ on $\mathbb{Q}$. For any natural number n it is

$$\widehat{\mathrm{Fix}_{n\mathbb{Z}}(X)} = \mathbb{Q}/((3/2)^n - 1)\mathbb{Q} = \{0\}.$$

So for any sequence of subgroups $\Gamma_n \to 0$ it is

$$h_{p,\Gamma_n}(\alpha) = 0.$$

Note that the topological entropy of α is $h(\alpha) = \log 3$, see [LW88].

Here, the R_d-module $\mathbb{Q}$ is not noetherian which implies that the action α on X cannot be p-adically expansive.

7.2 A comment on uniqueness of p-adic entropy

Let $\mathcal{M}_{S_\infty}(R_d)$ be the category of finitely generated S_∞-torsion R_d-modules, $S_\infty = R_d \cap L^1(\mathbb{Z}^d, \mathbb{R})^*$. By Theorem 5.2, an algebraic $\mathbb{Z}^d$-action α on the compact abelian group X is expansive if and only if $M^X \in \mathcal{M}_{S_\infty}(R_d)$.

We have the following uniqueness result concerning the entropy of expansive $\mathbb{Z}^d$-actions:

Proposition 7.3. *We identify the category of expansive $\mathbb{Z}^d$-actions with the category $\mathcal{M}_{S_\infty}(R_d)$ via Pontrjagin duality. Let v be an assignment which associates to every $M \in \mathcal{M}_{S_\infty}(R_d)$ a non-negative real number and satisfies the following conditions:*

(i) v is additive in short exact sequences.

(ii) v is monotone, i.e. if there exists a surjective homomorphism $M \to M' \to 0$ it is $v(M') \leq v(M)$.

(iii) It is $v(M) = \log_p \det_{\mathbb{Z}^d}(f)$ for $M = (R_d)^n/(f \cdot (R_d)^n)$, $f \in M_n(R_d) \cap GL_n(L^1(\mathbb{Z}^d, \mathbb{R}))$.

Then v equals the entropy h.

Proof. Let $M \in \mathcal{M}_{S_\infty}(R_d)$ and let $\{0\} = M_0 \subset \ldots \subset M_s = M$ be a prime filtration of M. It follows by additivity of v

$$v(M) = \sum_{i=1}^{s} v(R_d/\mathfrak{p}_i), \ \mathfrak{p}_i \in \mathrm{Spec}(R_d).$$

As in the proof of Theorem 3.28 one shows that for non-principal prime ideals $\mathfrak{p}$ it is $v(R_d/\mathfrak{p}) = 0 = h(R_d/\mathfrak{p})$. For a principal prime ideal $\mathfrak{p} = (f)$ we have by assumption $v(R_d/(f)) = h(R_d/(f))$. We conclude that $v(M) = h(M)$ for all $M \in \mathcal{M}_{S_\infty}(R_d)$. □

It is natural to ask if there is a similar result for p-adic entropy. We formulate the problem in an algebraic way:

Open Problem 7.4. Given a category $\mathcal{C}$ of $\mathbb{Z}^d$-modules which contains the class of all modules $(R_d)^n/f(R_d)^n, f \in M_n(R_d) \cap \mathrm{GL}_n(c_0(\mathbb{Z}^d))$. Assume $\mathcal{C}$ is equipped with an assignment v_p which associates to every $M \in \mathcal{C}$ a number $v_p(M) \in \mathbb{Q}_p$ and which satisfies the following properties:

(i) v_p is additive, i.e. for every short exact sequence

$$0 \to M' \to M \to M'' \to 0$$

of objects in $\mathcal{C}$, we have $v_p(M) = v_p(M') + v_p(M'')$.

(ii) $v_p((R_d)^n/f(R_d)^n) = \log_p \det_{\mathbb{Z}^d} f$ for $f \in M_n(R_d) \cap \mathrm{GL}_n(c_0(\mathbb{Z}^d))$.

Is the assignment v_p uniquely defined?

If we take $\mathcal{C} = \mathcal{M}_{S_p}(R_d)$ the category of finitely generated R_d-modules which are S_p-torsion, $S_p = R_d \cap c_0(\mathbb{Z}^d)^*$, this question is related to the following problem:

Open Problem 7.5. Let $d \geq 1$. Let $\mathfrak{p}$ be a non-principal prime ideal in R_d such that the algebraic $\mathbb{Z}^d$-action on $X = \widehat{R_d/\mathfrak{p}}$ is p-adically expansive. Is $h_p(X) = 0$?

7.3 Miles' criterion of expansive algebraic actions of countable abelian groups

In [Mil06] there is the following characterization of expansive algebraic actions of countable abelian groups:

Theorem 7.6. *Let α be an action of a countable abelian group Γ by automorphisms of a compact abelian group X. Then (X, α) is expansive if and only if M^X is a finitely generated $\mathbb{Z}\Gamma$-module and as $\mathfrak{a}$ runs through the annihilators of a set of generators for M^X, there is no ring homomorphism $\phi : \mathbb{Z}\Gamma/\mathfrak{a} \to \mathbb{C}$ for which the image of Γ in $\mathbb{C}$ is a subgroup of the unit circle.*

In this section Γ will always be a countable abelian group. Let S_p be the multiplicative system in $\mathbb{Z}\Gamma$ defined by $S_p = \mathbb{Z}\Gamma \cap c_0(\Gamma)^*$. Let $\mathcal{M}_{S_p}(\mathbb{Z}\Gamma)$ be the category of finitely generated $\mathbb{Z}\Gamma$-modules which are S_p-torsion.

Proposition 7.7. *Let (X, α) be an algebraic Γ-action such that the dual $\mathbb{Z}\Gamma$-module M^X is in $\mathcal{M}_{S_p}(\mathbb{Z}\Gamma)$. Then for every annihilator ideal $\mathfrak{a}$ there is no ring homomorphism $\phi : \mathbb{Z}\Gamma/\mathfrak{a} \to \mathbb{C}_p$ for which the image of Γ in $\mathbb{C}_p$ is a subgroup of T_p.*

Proof. The assumption that M^X is a finitely generated S_p-torsion module implies that for every annihilator ideal $\mathfrak{a}$ we have $\mathfrak{a} \cap S \neq \emptyset$.

Assume there exists a ring homomorphism $\phi : \mathbb{Z}\Gamma/\mathfrak{a} \to \mathbb{C}_p$ such that the image Γ is contained in T_p. Then we may define a ring homomorphism $\Phi : c_0(\Gamma)/\mathfrak{a}{\cdot}c_0(\Gamma) \to \mathbb{C}_p$ which extends ϕ as follows: Let ϕ' be the composition $\mathbb{Z}\Gamma \to \mathbb{Z}\Gamma/\mathfrak{a} \to \mathbb{C}_p$. Then we define

$$\Phi' : c_0(\Gamma) \to \mathbb{C}_p, \ \sum_{\gamma\in\Gamma} a_\gamma \gamma \mapsto \sum_{\gamma\in\Gamma} a_\gamma \phi'(\gamma).$$

This makes sense because $\mathbb{C}_p$ is complete and as $|a_\gamma| \to 0$ as $\gamma \to \infty$ in Γ and because $|\phi'(\gamma)| = 1$ for all $\gamma \in \Gamma$, the sum $\sum_{\gamma\in\Gamma} a_\gamma \phi'(\gamma)$ will converge in $\mathbb{C}_p$. Now because $\phi'(\mathfrak{a}) = 0$ the ring homomorphism Φ' factors through a ring homomorphism $\Phi : c_0(\Gamma)/\mathfrak{a} \cdot c_0(\Gamma) \to \mathbb{C}_p$. This is impossible because $\mathfrak{a}$ contains an element which is a unit in $c_0(\Gamma)$ so the quotient $c_0(\Gamma)/\mathfrak{a} \cdot c_0(\Gamma)$ is zero. □

Open Problem 7.8. When does the converse implication of Proposition 7.7 hold, i.e. if we assume $M^X \notin \mathcal{M}_{S_p}(\mathbb{Z}\Gamma)$, does there exist a ring homomorphism $\phi : \mathbb{Z}\Gamma/\mathfrak{a} \to \mathbb{C}_p$ such that the image of Γ lies in T_p?

Let us assume $M^X \notin \mathcal{M}_{S_p}(\mathbb{Z}\Gamma)$, i.e. there exists an annihilator ideal $\mathfrak{a}$ such that $\mathfrak{a} \cap S_p = \emptyset$. Then $\mathfrak{a}$ does not generate the unit ideal in $c_0(\Gamma)$ so that there exists a maximal ideal $\mathfrak{m} \in c_0(\Gamma)$ which contains $\mathfrak{a} \cdot c_0(\Gamma)$. Let us assume that $\mathfrak{m}$ has finite codimension. Note that $\mathfrak{m}$ is automatically closed in $c_0(\Gamma)$ because the group of units $c_0(\Gamma)^*$ is open in $c_0(\Gamma)$, see Lemma 7.11. Let Φ be the continuous homomorphism

$$\Phi : c_0(\Gamma) \to \mathbb{C}_p$$

given by composing the natural projection $c_0(\Gamma) \to c_0(\Gamma)/\mathfrak{m}$ with an $\mathbb{Q}_p$-linear embedding of $c_0(\Gamma)/\mathfrak{m}$ into $\mathbb{C}_p$. If we define

$$\phi : \mathbb{Z}\Gamma/\mathfrak{a} \to c_0(\Gamma)/\mathfrak{m} \to \mathbb{C}_p$$

as the composition of the natural homomorphism $\mathbb{Z}\Gamma/\mathfrak{a} \to c_0(\Gamma)/\mathfrak{m}$ with the embedding $c_0(\Gamma)/\mathfrak{m} \to \mathbb{C}_p$ the next lemma implies that $\phi(\Gamma) \subset T_p$.

Lemma 7.9. *Let $\Phi : c_0(\Gamma) \to \mathbb{C}_p$ be a continuous ring homomorphism which is $\mathbb{Q}_p$-linear. Then $\Phi(\Gamma) \subset T_p$.*

Proof. Continuity and $\mathbb{Q}_p$-linearity of Φ imply

$$\Phi\left(\sum_{\gamma\in\Gamma} x_\gamma \gamma\right) = \sum_{\gamma\in\Gamma} x_\gamma \Phi(\gamma) \in \mathbb{C}_p$$

for every $x = \sum_{\gamma\in\Gamma} x_\gamma\gamma \in c_0(\Gamma)$. In particular, the homomorphism Φ is determined by the values $(\Phi(\gamma))_{\gamma\in\Gamma}$. If γ is of finite order in Γ, the image $\Phi(\gamma)$ is a root of unity and so is in T_p.

Let now $\gamma \in \Gamma$ be of infinite order. If $\Phi(\gamma)$ was not contained in T_p, we may assume $|\Phi(\gamma)|_p > 1$. Let $(x_{\gamma^n})_{n\in\mathbb{N}}$ be a family of numbers in $\mathbb{Q}_p$ converging to zero such that $|x_{\gamma^n}\Phi(\gamma)^n|_p > 1$. Then the element $\sum_{n\in\mathbb{N}} x_{\gamma^n}\gamma^n$ is in $c_0(\Gamma)$ but $\sum_{n\in\mathbb{N}} x_{\gamma^n}\Phi(\gamma^n)$ does not exist which contradicts the assumption that Φ is continuous. We conclude that $\Phi(\Gamma) \subset T_p$. □

This short discussion leads to the problem for what groups Γ maximal ideals in $c_0(\Gamma)$ have finite codimension. For example, for $\Gamma = \mathbb{Z}^d$ all maximal ideals in $c_0(\mathbb{Z}^d)$ have finite codimension, see [BGR84], 6.1.2, Corollary 3.

7.4 Properties of the p-adic Banach algebra $c_0(\Gamma)$

Let Γ be a countable discrete residually finite group. In this section we discuss some algebraic properties of the $\mathbb{Q}_p$-Banach algebra $c_0(\Gamma)$.

Let B be a p-adic Banach algebra over $\mathbb{Q}_p$ as defined in Chapter 2, Definition 2.24. We assume that $|| \; ||$ takes values in $p^{\mathbb{Z}} \cup \{0\}$. We define

$$B^0 = \{x \in B : ||x|| \leq 1\} \text{ and } B^{\vee} = \{x \in B : ||x|| < 1\}.$$

B^0 is a subring of B which contains $B^{\vee}$ as an ideal. Furthermore, $B^{\vee}$ is open in B.

Example 7.10. Let $B = c_0(\Gamma)$. Then $A := B^0 = c_0(\Gamma, \mathbb{Z}_p)$ and $B^{\vee} = pA$. The quotient A/pA is isomorphic to $\mathbb{F}_p\Gamma$.

Lemma 7.11. *Let B be a p-adic Banach algebra. Then the group of units B^* is open in B.*

Proof. The set $1 + B^{\vee}$ an open neighborhood of $1 \in B^*$. Then for any unit $u \in B^*$, the set $u + uB^{\vee}$ is an open neighborhood of u. □

Proposition 7.12. *Let B be a p-adic Banach algebra over $\mathbb{Q}_p$ whose norm takes values in $p^{\mathbb{Z}} \cup \{0\}$ and set $A = B^0$. If the residue algebra has no zero divisors, then we have*

$$B^* = p^{\mathbb{Z}} A^* \quad \text{and} \quad p^{\mathbb{Z}} \cap A^* = 1.$$

Proof. [Den09], Proposition 4. □

For a discrete countable residually finite group Γ we denote by $COF(\Gamma)$ the set of cofinite normal subgroups of Γ.

Proposition 7.13. *Let Γ be a discrete countable residually finite group. Then the canonical homomorphism*

$$c_0(\Gamma) \to \prod_{N \in COF(\Gamma)} c_0(\Gamma/N)$$

is injective.

Proof. We show that for every $f \in c_0(\Gamma)$, there is a normal subgroup N of finite index such that $||f|| = ||f_N||$, where f_N is the image of f in $c_0(\Gamma/N)$.

Let $x_{\gamma_1}, \ldots, x_{\gamma_r}$ be the finitely many elements in Γ, such that $||f|| = |x_{\gamma_i}|_p,\ i = 1, \ldots, r$. As Γ is residually finite, we find a normal subgroup N of finite index such that

$$\{\gamma_i \gamma_j^{-1},\ i, j \in \{1, \ldots, r\}, j > i\} \cap N = \emptyset.$$

This means that for $i \neq j$, it is $\gamma_i \not\equiv \gamma_j \mod N$. It follows $||f|| = ||f_N||$, as the p-adic absolute value satisfies the strong triangle inequality. □

Corollary 7.14. *Let Γ be a discrete countable residually finite group.. For a normal subgroup N of Γ let $\pi_N : c_0(\Gamma) \to c_0(\Gamma/N)$ be the canonical reduction homomorphism. Then*

$$\bigcap_{N \in COF(\Gamma)} \ker(\pi_N) = 0.$$

Proof. It is

$$\bigcap_{N \in COF(\Gamma)} \ker(\pi_N) = \ker\left(c_0(\Gamma) \to \prod_{N \in COF(\Gamma)} c_0(\Gamma/N)\right) = 0.$$

□

Corollary 7.15. *For any $r \geq 1$, the canonical homomorphism*

$$M_r(c_0(\Gamma)) \to \prod_{N \in COF(\Gamma)} M_r(c_0(\Gamma/N))$$

is injective.

Proof. This follows from the case $r = 1$. □

Proposition 7.16. *Let Γ be a discrete countable residually finite group. Then the algebra $c_0(\Gamma)$ is von Neumann finite, i.e. if $g \cdot f = 1$ then $f \cdot g = 1$.*

Proof. Let us first assume that Γ is finite. Let $f, g \in c_0(\Gamma) = \mathbb{Q}_p\Gamma$. If $g \cdot f = 1$, then the endomorphisms ρ_f and ρ_g of finite dimensional $\mathbb{Q}_p$-vector spaces are invertible in $\mathrm{End}_{\mathbb{Q}_p}(\mathbb{Q}_p\Gamma)$. If $(\rho_f)^{-1}$ is the inverse endomorphism of ρ_f, then

$$\rho_g = \rho_g \circ \mathrm{Id} = \rho_g \circ (\rho_f \circ (\rho_f)^{-1}) = (\rho_f)^{-1}.$$

This implies $f \cdot g = 1$.

Let us assume now that Γ is residually finite. We consider the image of fg under the inclusion

$$c_0(\Gamma) \hookrightarrow \prod_{N \in COF(\Gamma)} c_0(\Gamma/N).$$

If we assume $g \cdot f = 1$, then by the first part of the proof fg is mapped to 1. By injectivity this implies $fg = 1$. □

Lemma 7.17. *For $f \in \mathbb{Z}\Gamma$ the following properties are equivalent:*

(i) f is a unit in $c_0(\Gamma, \mathbb{Z}_p)^*$.

(ii) The reduction $\bar{f}$ is invertible in $\mathbb{F}_p\Gamma$.

Proof. (i)⇒(ii) is clear.

For the converse implication consider the exact sequence

$$1 \longrightarrow 1 + pc_0(\Gamma, \mathbb{Z}_p) \longrightarrow c_0(\Gamma, \mathbb{Z}_p)^* \longrightarrow (\mathbb{F}_p\Gamma)^* \longrightarrow 1.$$

If $\bar{f}$ is a unit, there exists an element $g \in c_0(\Gamma, \mathbb{Z}_p)$ such that $fg \in 1 + pc_0(\Gamma, \mathbb{Z}_p) \subset c_0(\Gamma, \mathbb{Z}_p)^*$. Let $h \in c_0(\Gamma, \mathbb{Z}_p)$ be the inverse of fg, i.e. it is $fgh = 1$. By the previous lemma, it follows $ghf = 1$. □

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