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On classification, UHF-stability, and tracial approximation of simple nuclear C^*-algebras

2013

Mathematik

On classification, UHF-stability, and tracial approximation of simple nuclear C^*-algebras

Inaugural-Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften im Fachbereich
Mathematik und Informatik
der Mathematisch-Naturwissenschaftlichen Fakultät
der Westfälischen Wilhelms-Universität Münster

vorgelegt von
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aus Toronto
- 2013 -

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Tag der mündlichen Prüfung:	29. Januar 2014
Tag der Promotion:	29. Januar 2014

Acknowledgments

I would like to express my gratitude to my supervisor Prof. Dr. Wilhelm Winter. Not only am I thankful for his help and support during my studies, I am also indebted to him for sparking my interest in the subject of C*-algebras and their classification.

Mein Dank gilt meinen Kollegen in Deutschland, insbesondere BJ, IK und SM. Sie haben meine Zeit in Münster sehr bereichert.

I would not have been able to complete this thesis without the encouragement of my family. My parents John and Trish and my sister Rebecca and my brother Ian have been unwavering in their support. I will always be deeply grateful.

Finally, my friends have been indispensable. I would like to give particular thanks to MvH and NBG who have been steadfast friends through many ups and downs, to NPB for always being willing to listen, and to AGS for being an endless source of creative inspiration and reminding me to "drink the second coffee, quit the self-analysis, walk around the Schloss and try to remember you generally enjoy what you're doing."

A (noncommutative) tale of two cities

After spending my entire life growing up in the city of Toronto, I finished high school, packed up my things and moved to Montréal. I loved the hurried bustle of a big city, but with all the usual boldness of a nineteen year old, I decided it was time to leave my hometown. As the next closest big city, Montréal was a logical destination. Not being terribly keen on my chosen path of engineering studies at McGill University, I spent as much time as I could exploring the streets, the parks, the metro stations. This is where my problems began. I became preoccupied with comparing each and every detail of this new city to Toronto. This neighbourhood is Montréal's Annex, I'd think. Rue Ste-Catherine is Yonge Street. Rue Saint Urbain, with its Orthodox Jewish community, is obviously Bathurst Street.

I moved out of the university residence on a whim (new experiences become addictive) and tried to describe my flat in Pointe-St-Charles: It's in a working class neighbourhood, I'd say. Montréal's Cabbagetown. No, no, not so far from the University; just across the Lachine Canal from the main city. It's like being on the other side of the Don Valley. (The canal is the Don River.)

I felt quite assured that these were the correct analogues. I continued to construct my map. Applied my morphism to Queen Street West (St. Laurent). The Islands (Île-St-Jacques). Eventually, it is no surprise, I ran into difficulty. The cities, of course, are not isomorphic. Montréal's Chinatown seemed so small, and was there only the one? And what in Toronto could possibly be compared to a Sunday afternoon in Mount Royal park, with its dancers and drum circles? My expectations of what makes a city a city were slowly corroded.

One winter evening, I set off exploring. In my haste to catch a train that had just arrived in the metro station, I ended up on the wrong metro line. It took me some time to realize my mistake, and I suddenly found myself all the way out at Olympic Stadium, nowhere near my intended destination. I exited the metro only to find out there were no trains heading back that evening. I wandered around outside the deserted stadium, trying my best to get my bearings, to find a street I knew or a bus stop from where I might catch a bus headed in the right direction. This would never happen in Toronto, I thought. This was certainly no SkyDome, not even Exhibition Stadium, or I'd have found my way home easily! As snow began to fall, I eventually found a taxi, and feeling lost and defeated, asked the driver to take me back to Pointe-St-Charles. When I finally reached my flat, I paid the driver and realized: I liked Montréal, but in my attempts to categorize the city, I was in fact searching for Toronto. I switched out of engineering, moved back to Toronto, and took up mathematics.

Classification is a natural methodology for understanding our surroundings. My attempt to classify cities might be seen as a failure. Certain aspects I expected to find in all cities broke down as soon as I left Toronto. Nevertheless, it did raise the questions: What makes a city a city? What makes it different from a town? How could I explain the difference between Montréal and Toronto?

In mathematics, classification is an indispensable tool for the proper understanding of mathematical objects, and correspondingly, it has played a central and recurring role in the subject. What properties can we expect to see in our mathematical objects? One may take certain properties for granted, only to find exotic and pathological examples where they do not occur. When do we have an isomorphism? Can we identify particular invariants that allow us to decide when we have an isomorphism without having to rely on a bare hands construction of such a map?

0. Introduction

In the theory of C*-algebras, it is the mandate of the Elliott classification programme to classify separable simple unital nuclear C*-algebras up to isomorphism by a computable set of invariants consisting of K-theory, the tracial state space, and the pairing between these objects.

The classification programmme in its current form was initiated by George Elliott after successful classification of approximately finite (AF) algebras by their scaled ordered K_0-group. He showed that for two AF algebras A and B, an order-preserving group isomorphism of $K_0(A) \to K_0(B)$ can be lifted to a *-isomorphism $A \to B$.

Further classification successes of the unital simple approximately interval (AI) and unital simple approximately circle (A$\mathbb{T}$) algebras required the addition of the K_1-group, the tracial state simplex, and the canonical pairing map of the tracial state space and the state space of the K_0-group. These successful classification results led Elliott to conjecture that all simple separable nuclear C*-algebras might be classified by these invariants.

Counterexamples to the original conjecture have since been constructed, resulting in both the development of new invariants and classification tools, as well as attempts to characterize those C*-algebras for which we can still expect classification via Elliott's original invariant. In particular, we now expect classification by Elliott invariants to hold when we restrict to those separable simple unital nuclear C*-algebras which are $\mathcal{Z}$-stable, that is, those which are isomorphic to themselves when tensored with the so-called Jiang–Su algebra $\mathcal{Z}$, constructed by X. Jiang and H. Su in [26].

A particularly elegant set of examples of C*-algebras are those which arise from mimimal dynamical systems. These examples have served as motivation for the work in this thesis. Let X be a compact metrizable space with a given homeomorphism $\alpha : X \to X$. Given a function $f \in \mathcal{C}(X)$, one can compose with the inverse of the homeomorphism, yielding another function $f \circ \alpha^{-1} \in \mathcal{C}(X)$. This induces an action of the integers on the C*-algebra $\mathcal{C}(X)$, and from this one can construct the crossed product C*-algebra $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$. In the case that α is a minimal homeomorphism, that is, X contains no proper α-invariant closed subsets, $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ is a simple separable unital nuclear C*-algebra.

There have been many spectacular results in the classification of C*-algebras of minimal dynamical systems. Notable examples include that of T. Giordano, I. Putnam and C. Skau in [22]. They show that one may define a K-group which distinguishes strong topological orbit equivalent systems for a minimal Cantor system (that is, a minimal dynamical system where X is the Cantor set). Moreover the K-group they define is order-isomorphic to the K-theory of the associated C*-algebra. Combining this with Elliott's classification for A$\mathbb{T}$ algebras, they show that two systems are strong orbit equivalent if and only if their associated C*-algebras are *-isomorphic.

The most wide-reaching classification result for the C*-algebras of minimal dynamical systems is given by A. Toms and W. Winter in [67, 66], where they prove that if X is an infinite compact metrizable space with finite covering dimension then $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ is $\mathcal{Z}$-stable whence it follows (by applying a special case of the main theorem of [60]) that, if projections separate traces in $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$, the C*-algebra is classifiable. In particular, this covers all C*-algebras associated to uniquely ergodic minimal dynamical systems with finite covering dimension.

Despite these successes, the classification problem for C*-algebras of minimal dynamical systems remains open. When X does not have finite covering dimension—for example, the minimal dynamical system (X, α) with $\dim(X) = \infty$ constructed by J. Giol and D. Kerr [21], which results in a non-$\mathcal{Z}$-stable C*-algebra—classification by Elliott invariants appears unlikely.

In [71], A. Windsor shows how to construct a minimal homeomorphism β of the n-sphere S^n, for $n \geq 3$ odd, which can have any finite number, countable number, or even a continuum of ergodic probability measures. These are considered by Connes in [11, Section 5] where he shows that the associated C*-algebras have no nontrivial projections but via the pairing of tracial states and β-invariant probability measures, a finite, countable or continuum number of extreme tracial states. In this case, projections cannot separate traces so these examples lie beyond the reach of current classification theorems.

Let $\mathcal{A}$ be a class of C*-algebras. If the class $\{A \otimes \mathcal{Z} \mid A \in \mathcal{A}\}$ can be classified by Elliott invariants, then we say that $\mathcal{A}$ is classified (by Elliott invariants) up to $\mathcal{Z}$-stability. It follows from above that in full generality one can likely only expect classification by Elliott invariants up to $\mathcal{Z}$-stability for the class of C*-algebras of minimal dynamical systems. An important task then is to find classification up to $\mathcal{Z}$-stability when projections do not necessarily separate tracial states so one does not have to make assumptions such as unique ergodicity. In many cases, the C*-algebras under consideration will turn out to be $\mathcal{Z}$-stable to begin with; in this case we have a complete classification result.

Since classification requires the lifting of maps between invariants to *-isomorphisms of C*-algebras, it is perhaps unsurprising that the situation becomes much more difficult when there are few projections, since the invariant—which includes the K_0-group as an essential ingredient—will contain less information. The main technique to get around such a problem in a C*-algebra A is to work with a related C*-algebra which has nicer structural properties while itself retaining enough information about the original C*-algebra A.

The approach in this thesis is to take a given class of C*-algebras and show that a related class of C*-algebras, given by tensoring with a UHF algebra, can be tracially approximated by interval algebras, that is, an algebras of the form $(\oplus_{k=1}^{K} \mathcal{C}([0,1]) \otimes M_{n_k}) \oplus (\oplus_{l=1}^{L} M_{n_l})$. The concept of tracial approximation was first suggested by H. Lin when he showed that simple separable unital nuclear tracially approximately finite algebras (TAF) which satisfy the universal coefficient theorem can be classified [33, 34]. Once this was known, one no longer had to use a bare hands approach to classification by showing a given C*-algebra has an inductive limit structure. Lin's definition for TAF gave a simplified set of requirements that one could check. Now, through classification, though this implies the existence of an inductive limit structure, the classifier need not find herself or himself caught in a tangle of connecting maps or faced with the task of organizing a cluttered pile of Hausdorff spaces and matrix algebras.

It was shown by W. Winter in [77] that classification results up to tensoring with UHF algebras, (that is, classification up to UHF-stability) could be used to deduce classification up to $\mathcal{Z}$-stability. Classification up to UHF-stability is often much easier to determine since UHF algebras supply many projections; it is then no surprise that K-theoretic data is more accessible to classification. Tensoring with a UHF algebra $\mathcal{U}$ effectively gives one more space in which to work: by tensoring with pairwise orthogonal projections, the original algebra A can be separated into arbitrarily many layers in the UHF algebra, making it easier to arrange $A \otimes \mathcal{U}$ in such away that one can verify it meets the requirements to be a tracially approximately interval algebra (TAI). This has proven quite successful. For example, the main result in [60], showed that C*-algebras associated to uniquely ergodic minimal dynamical systems of infinite compact metrizable spaces are classifiable up to $\mathcal{Z}$-stability (see also [67, 66]).

As an application to our classification up to $\mathcal{Z}$-stability of simple separable unital tracially approximately semihomogeneous C*-algebras with bounded dimension given in Chapter 3, if A and B are simple separable unital AH algebras with no dimension growth then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$. This result has already been established in [18], however it requires some highly technical results in the form of Gong's decomposition theorem [23]. This reduction theorem allows Elliott, Gong and Li to restrict their proof to the case where the dimension of the base spaces in the C*-algebras in the inductive limit have dimension no more than three. In contrast, Theorem 3.5.23, together with [36] entails the classification theorem directly, with no appeal to the decomposition theorem.

Appealing to known classification results, this also shows that a simple separable unital locally semihomogeneous C*-algebra with bounded dimension must in fact be an AH algebra. This is notable, as it is known to be false when A is not assumed to be simple; this was shown in [14]. Furthermore, we may also conclude that the class of simple separable unital AH algebras with slow dimension growth is closed under taking simple inductive limits.

It should be noted that H. Lin has announced a similar result in [37]. However, the proof given here is substantially different. In his paper, he classifies simple unital locally semihomogeneous C*-algebras with slow dimension growth and uses the invariant to deduce that such C*-algebras must be simple unital AH algebras with slow dimension growth. From this he can also conclude that such C*-algebras have tracial rank no more than one after tensoring with a UHF algebra. Here, on the other hand, Theorem 3.5.23 gives a direct proof that simple separable unital locally semihomogeneous C*-algebras of bounded dimension tensored with $\mathcal{Q}$ are TAI (which in turn implies they have tracial rank no more than one after tensoring with $\mathcal{Q}$ or any other UHF algebra of infinite type). Together with the classification for such "rationally TAI" algebras, the same result is established, though via a shorter route.

In Chapter 4 the main result gives classification for simple separable unital C*-algebras that can be locally approximated by a subclass of RSH algebras: Those with a decomposition into base spaces $X_0, X_1, \ldots, X_R$ that can be arranged so that a we can extend a projection from one space to the next in a suitable way. We also require the additional assumption that there are finitely many extreme tracial states all inducing the same state on the K_0-group. Using a similar technique to that given in Chapter 3, it is shown that such C*-algebras are TAI after tensoring with the universal UHF algebra $\mathcal{Q}$.

As an application, this gives classification of the examples of Elliott in [16] when there are finitely many extreme tracial states $\tau_0, \ldots, \tau_1$ all inducing the same state on the K_0-group. Note in particular that these classification results assume that projections do not separate tracial states. The application to Elliott's examples implies that the range of C*-algebras covered by the main classification theorem in Chapter 4 is in fact quite broad: it is shown in [16] that these constructions exhaust the Elliott invariant in the weakly unperforated case.

In the case of minimal dynamical systems we show that the main theorem of Chapter 4 gives another classification of some examples constructed by Lin and Matui in [38] of minimal homeomorphisms $\alpha : X \times \mathbb{T} \to X \times \mathbb{T}$ on the product of the Cantor set and the circle. In this case, the main theorem implies that a "large" subalgebra of $\mathcal{C}(X \times \mathbb{T}) \rtimes_\alpha \mathbb{Z}$, given by breaking the homeomorphism at a point, is rationally TAI. By using a generalization of [41, Lemma 4.2] given by [60, Theorem 4.6] (Lemma 4.3.26 below) this then implies $\mathcal{C}(X \times \mathbb{T}) \rtimes_\alpha \mathbb{Z}$ itself is rationally TAI.

Research acknowledgements

Chapters 3 and 4 are based on joint work with Wilhelm Winter. The work in Chapter 4 appears in the preprint [59], except for 4.3.23 to 4.3.28, which appears as joint work in [60]. Appendix A contains joint work appearing in [59], however as my contribution to this section was minimal, it has been included separately as an appendix.

Contents

1. Background

1.1. Notation

We briefly set notation that will be used throughout the thesis.

Let A be a C*-algebra.

(i) We denote by A_{sa} the subset of self-adjoint elements, $A_+ \subset A_{sa}$ the positive cone, A^1 the unit ball of A, and $\tilde{A}$ the (smallest) unitization of A. We also denote by A^1_{sa} and A^1_+ the subsets of A^1 consisting of self-adjoint elements, respectively positive elements.

(ii) $M_n(A)$ will denote $n \times n$ matrices over A and $M_\infty(A) = \overline{\cup_{n\in\mathbb{N}} M_n(A)}$.

1.1.1 To any partition of the interval $[0,1]$ we associated a partition of unity of sawtooth functions. For $K \in \mathbb{N} \setminus \{0\}$, if $\{0 = t_0 < t_{\frac{1}{K}} < \cdots < t_{\frac{K-1}{K}} < t_1 = 1\}$ is a given partition of $[0,1]$ into subintervals, let $\gamma_{\frac{k}{K}}, k = 0, \ldots, K \in \mathcal{C}([0,1])$ be the function defined as follows:

$$
\begin{aligned}
\gamma_0(t) &= \begin{cases} 1 & \text{if} \quad t = 0, \\ \text{linear} & \text{if} \quad 0 \leq t \leq t_{\frac{1}{K}}, \\ 0 & \text{if} \quad t \geq t_{\frac{1}{K}}; \end{cases} \\
\gamma_1(t) &= \begin{cases} 0 & \text{if} \quad t \leq t_{\frac{K-1}{K}}, \\ \text{linear} & \text{if} \quad t_{\frac{K-1}{K}} \leq t \leq 1, \\ 1 & \text{if} \quad t = 1; \end{cases} \\
\gamma_{\frac{k}{K}}(t) &= \begin{cases} 0 & \text{if} \quad t \leq t_{\frac{k-1}{K}} \text{ or } t \geq t_{\frac{k+1}{K}}, \\ 1 & \text{if} \quad t = t_{\frac{k}{K}}, \\ \text{linear} & \quad\text{elsewhere.} \end{cases}
\end{aligned}
$$

It will sometimes be the case that we require more than one partition at a time. To avoid overly cumbersome notation in such a circumstance, we will never reduce the fractions in the subscripts so that we can differentiate between a partiton with K subintervals and a partition with L subintervals. For example, in such a situation if we have $L = 4K$ then $\gamma_{\frac{k}{K}} \neq \gamma_{\frac{4k}{4K}}$.

1.1.2 Let $A = \varinjlim(\bigoplus_{n=1}^{N_i} A_{n,i}, \phi_i)$ be an inductive limit of C*-algebras $\bigoplus_{n=1}^{N_i} A_{n,i}$ with connecting maps $\phi_i : \bigoplus_{n=1}^{N_i} A_{n,i} \to \bigoplus_{n=1}^{N_{i+1}} A_{n,i+1}$. We denote the canonical map induced by the inductive limit by

$$\phi^{(i)} : \bigoplus_{n=1}^{N_i} A_{n,i} \to A.$$

1.1.3 Let $a \in A^1_+$ and let $0 < \epsilon < 1$. We denote by $(a-\epsilon)_+$ the functional calculus applied to a with respect to the function

$$(t-\epsilon)_+ = \begin{cases} 0 & \text{if} \quad 0 \leq t < \epsilon, \\ t-\epsilon & \text{if} \quad \epsilon \leq t \leq 1. \end{cases}$$

Notice that if a and b are orthogonal then $(a-\epsilon)_+ + (b-\epsilon)_+ = (a+b-\epsilon)_+$.

1.1.4 Let $\mathcal{Q}$ denote the universal UHF algebra, that is, the UHF algebra with $K_0(A) = \mathbb{Q}$. The unique tracial state on $\mathcal{Q}$ will be denoted by $\tau_{\mathcal{Q}}$. For any $m \in \mathbb{N}$ there are m pairwise orthogonal projections, each with normalized trace given by $1/m$. These will be used frequently and we will denote them by $q_{(0,m-1)}, q_{(1,m-1)}, \ldots, q_{(m-2,m-1)}, q_{(m-1,m-1)}$.

1.2. Classification of C*-algebras

1.2.1. The Elliott conjecture

Elliott's original conjecture says that separable simple nuclear C*-algebras can be classified, up to *-isomorphism, by their so-called Elliott invariants. Though this is now known not to hold in full generality, the question of when classification by Elliott invariants is possible remains open and the pursuit of further classification results continues to be an important and active area of research.

In this thesis, we are interested in the case of simple separable unital C*-algebras. In this case, we have the following definition.

1.2.5 Definition: Let A be a simple separable unital C*-algebra. The Elliott invariant of A is given by

$$\mathrm{Ell}(A) = (K_0(A), K_0(A)_+, [1_A], K_1(A), T(A), r_A),$$

where

- $(K_0(A), K_0(A)_+, [1_A])$ is the partially ordered K_0-group with positive cone $K_0(A)_+$ and order unit $[1_A]$,
- $K_1(A)$ is the K_1-group of A,
- $T(A)$ is the simplex of tracial states and
- $r_A : T(A) \to S(K_0(A))$ is the map given by $r_A(\tau)([p]-[q]) = \tau(p) - \tau(q)$.

1.2.6 For two C*-algebras A and B, an isomorphism $\Phi : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ consists of an order unit-preserving group homomorphism

$$\phi : (K_0(A), K_0(A)_+, [1_A]) \to (K_0(B), K_0(B)_+, [1_B]),$$

a group homomorphism

$$\psi : K_1(A) \to K_1(B),$$

and an affine homeomorphism

$$\gamma : T(B) \to T(A),$$

such that the following diagram commutes

$$\begin{tikzcd} T(B) \arrow[r, "\gamma"] \arrow[d, "r_B"] & T(A) \arrow[d, "r_A"'] \\ S(K_0(B)) \arrow[r, "\cdot\circ\phi"] & S(K_0(A)). \end{tikzcd}$$

If $\mathcal{A}$ is a class of C*-algebras, then we say that the C*-algebras in $\mathcal{A}$ are classified by Elliott invariants if, for any $A, B \in \mathcal{A}$ an isomorphism $\Phi : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ lifts to a *-isomorphism $\phi : A \to B$, and moreover that ϕ can be chosen to induce Φ at the level of the invariant.

1.2.2. The Jiang–Su algebra

Although classification by Elliott invariants continues to produce successful results, the construction of various counterexamples—for example [69, 54, 64, 21]—indicates the problem of classification is more curious than Elliott originally conjectured.

In 1999, X. Jiang and H. Su constructed a simple separable unital nuclear C*-algebra $\mathcal{Z}$ that is projectionless and infinite-dimensional. Despite the fact that $\mathcal{Z}$ is constructed as an inductive limit of easily described building blocks, the discovery of this seemingly innocuous C*-algebra has had many repercussions within the classification programme. The C*-algebra $\mathcal{Z}$ has unique trace and identical K-theory to the complex numbers. This alone suggests that the Elliott conjecture must be restricted to infinite-dimensional algebras, since here we now have an example of two C*-algebras, $\mathcal{Z}$ and $\mathbb{C}$, that are certainly not *-isomorphic, yet have identical Elliott invariants. Furthermore, one sees (via the Künneth Theorem for tensor products, for example) that the existence of this algebra, in conjunction with the Elliott conjecture, forces any pre-classifiable C*-algebra A to be $\mathcal{Z}$-absorbing, or $\mathcal{Z}$-stable, that is, $A \cong A \otimes \mathcal{Z}$.

The Jiang–Su algebra is itself $\mathcal{Z}$-absorbing; in fact one can say even more. Let $\phi, \psi : A \to B$ be unital *-homomorphisms between unital C*-algebras A and B. We say that ϕ and ψ are approximately unitarily equivalent if there exists a sequence of unitaries $(u_n)_{n\in\mathbb{N}} \subset B$ such that

$$\|\phi(a) - u_n\psi(a)u_n^*\| \to 0, \text{ as } n \to \infty \text{ for every } a \in A.$$

1.2.7 Definition: [65, Definition 1.3 (iv)] Let D be a separable unital C*-algebra. Then D is strongly self-absorbing if there exists a *-isomorphism $\phi : D \to D \otimes D$ such that ϕ is approximately unitarily equivalent to $\mathrm{id} \otimes 1_D$.

The fact that $\mathcal{Z}$ is strongly self-absorbing can be seen in the proof of [26, Theorem 8.7].

Jiang and Su show in their original paper that two classes of C*-algebras known to be classifiable by Elliott invariants are indeed $\mathcal{Z}$-stable: the unital simple AF algebras [26, Corollary 6.2] and the unital separable nuclear purely infinite C*-algebras (note that in this case the tracial state space is empty). Moreover, for all of the known counterexamples to classification by Elliott invariants, one can show the failure of $\mathcal{Z}$-stability.

Two paths of investigation have emerged. The first path leads us on the search for a new or extended invariant with can handle more examples than $\mathrm{Ell}(\cdot)$ yet which is also computable in a simpler way than directly verifying *-isomorphisms. Currently, the best candidate is the Cuntz semigroup.

If we venture down the second path, we take steps towards a restriction of the original conjecture to a subclass of simple separable nuclear C*-algebras, the obvious candidate being those that are $\mathcal{Z}$-stable. Since tensoring with $\mathcal{Z}$ results in a $\mathcal{Z}$-stable C*-algebra, one may now look for classification by Elliott invariants up to $\mathcal{Z}$-stability. Along this path we also seek out other regularity properties for those C*-algebras which we expect to be classifiable. Can one read from the invariant itself, for example, that A is $\mathcal{Z}$-stable? Are other structural properties indicative of when a class should be classifiable? So far, various ways of characterizing these C*-algebras have emerged.

1.2.3. Cuntz equivalence and the Cuntz semigroup

Though this thesis does not give classification by Cuntz semigroups, this object does appear frequently in the sequel and notions such as Cuntz comparison have become a standard tool used in the study of C*-algebras.

1.2.8 Definition: Let $a, b \in A_+$. The element a is Cuntz subequivalent to b, written $a \precsim b$, if there is a sequence $(z_n)_{n\in\mathbb{N}} \subset A$ such that $\|z_n^* b z_n - a\| \to 0$ as $n \to \infty$. If $a \precsim b$ and $b \precsim a$ then a and b are Cuntz equivalent, written $a \sim b$. The Cuntz semigroup of A is then given by

$$\mathrm{Cu}(A) = (A \otimes \mathcal{K})_+ / \sim .$$

When the C^*-algebra A has stable rank one, then we have the following useful proposition.

1.2.9 PROPOSITION: *[53, Proposition 2.4 (v)] Let A be a unital C^*-algebra with stable rank one. Then for any $a, b \in A_+$ the following are equivalent:*

(i) $a \precsim b$,

(ii) *for every $\epsilon > 0$ there is a unitary $u \in A$ such that $u^*(a-\epsilon)_+ u \in \mathrm{Her}(b)$.*

It was shown in [12] that $\mathrm{Cu}(A)$ belongs to the category **Cu**, and that this is the appropriate category (rather than simply the category of semigroups) in which Cuntz semigroups are well-behaved, for example with respect to inductive limits, or where one might expect morphisms to be liftable to *-homomorphisms under some additional assumptions. The map $A \mapsto \mathrm{Cu}(A)$ is a functor from the category of C*-algebras to the category **Cu** [12, Theorem 2].

1.2.10 Let A be a C*-algebra. Following the terminology in [50] we will say that **Cu** classifies homomorphisms from A if, for any unital C*-algebra B with stable rank one and any morphism

$$\alpha : \mathrm{Cu}(A) \to \mathrm{Cu}(B)$$

in **Cu** such that $\alpha([s_A]) \leq [s_B]$, with $s_A \in A_+$ and $s_B \in B_+$ strictly positive elements, there exists a *-homomorphism $\phi : A \to B$ such that $\mathrm{Cu}(\phi) = \alpha$ and that moreover, ϕ is unique up to approximate unitary equivalence.

1.2.4. Regularity properties

In order to accurately define a suitable class for which the Elliott conjecture will hold, one must consider structural regularity properties. We might think of the presence of these regularity properties as evidence that a C*-algebra is well-behaved with respect to classification. In particular their presence should exclude the pathological counterexamples that have already been produced.

There are three regularity properties which have emerged to play prominent roles: $\mathcal{Z}$-stability, strict comparison, and finite nuclear dimension. Indeed, it is conjectured that they should all be equivalent to one another as well as to the property of "being classifiable" by Elliott invariants.

1.2.11 DEFINITION: Let A be a simple separable unital exact C*-algebra. For any tracial state τ we may define a dimension function on the positive elements of $M_\infty(A)$ by

$$d_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}).$$

We say that A has strict comparison (of positive elements) if $d_\tau(a) < d_\tau(b)$ for all $\tau \in T(A)$ implies that $a \precsim b$.

1.2.12 Let A and B be C*-algebras. A completely positive (c.p.) map $\phi : A \to B$ is said to be order zero if it preserves orthogonality, that is, if $a, b \in A$ satisfy $ab = 0$ then $\phi(a)\phi(b) = 0$.

DEFINITION: [80, Definition 2.1] Let A be a C*-algebra. We say that A has nuclear dimension at most n, written $\dim_{\mathrm{nuc}}(A) \leq n$, if there exists a net $(F_\lambda, \psi_\lambda, \phi_\lambda)_{\lambda\in\Lambda}$ where F_λ are finite-dimensional C*-algebras, $\psi_\lambda : A \to F_\lambda$ and $\phi_\lambda : F_\lambda \to A$ are c.p. maps satisfying the following:

(i) ψ_λ is contractive,

(ii) for each $\lambda \in \Lambda$, F_λ decomposes into $n+1$ ideals $F_\lambda = F_\lambda^{(0)} \oplus \cdots \oplus F_\lambda^{(n)}$ such that $\phi_\lambda|_{F_\lambda^{(i)}}$ is c.p.c. order zero for $i \in \{0, \ldots, n\}$,

(iii) $\phi_\lambda \circ \psi_\lambda(a) \to a$ uniformly on finite subsets of A.

If no such n exists, than A is said to have infinite nuclear dimension, $\dim_{\mathrm{nuc}}(A) = \infty$.

The nuclear dimension should be thought of as a noncommutative analogue of topological covering dimension. Indeed, in the case of a commutative C*-algebra $\mathcal{C}_0(X)$ where X is a locally compact

Hausdorff space, the nuclear dimension is equal to the covering dimension of X [80, Proposition 2.4].

We will also frequently appeal to the related notion of decomposition rank.

1.2.13 DEFINITION: [29, Definition 3.1] Let A be a C*-algebra. We say that A has decomposition rank at most n, written $\mathrm{dr}(A) \leq n$, if there exists a net $(F_\lambda, \psi_\lambda, \phi_\lambda)_{\lambda\in\Lambda}$ as in Definition 1.2.12 which satisfies the additional requirement that for each $\lambda \in \Lambda$ the maps ϕ_λ are contractive.

Toms and Winter have made the following conjecture:

1.2.14 CONJECTURE: *Let A be a simple separable unital nuclear infinite-dimensional* C*-*algebra. Then the following are equivalent:*

(i) *A is $\mathcal{Z}$-stable.*

(ii) *A has strict comparison of positive elements.*

(iii) *A has finite nuclear dimension.*

Some implications are already known.

1.2.15 THEOREM: *Let A be a simple separable unital nuclear infinite-dimensional* C*-*algebra. Then the following hold:*

(i) *If A has finite nuclear dimension then A is $\mathcal{Z}$-stable* [78, Corollary 7.3].

(ii) *If A is $\mathcal{Z}$-stable then A has strict comparison* [55, Corollary 4.6].

(iii) *Suppose that the extreme boundary of $T(A)$ is compact and has finite covering dimension. Then if A has strict comparison, A is $\mathcal{Z}$-stable* [28, Corollary 7.9], [57, Corollary 1.2], [63, Corollary 4.7].

These regularity properties, as well as their known equivalences, are frequently employed in the sequel.

2. UHF stability and tracial approximation

2.1. Tensoring with a UHF algebra

Many examples of classification results require that a given class of C*-algebras contains "enough projections" in some sense. For example, one might ask for real rank zero, which implies an approximate unit of projections, or one might ask that there be enough projections to separate tracial states. That this is required is not surprising when one considers that classification relies on the Elliott invariant of a C*-algebra A containing enough information to recover A. Thus the more information in the K_0-group and the pairing map, the better.

Nevertheless, when restricting to the subclass of simple separable unital C*-algebras within the scope of the classification programme (that is, those which are $\mathcal{Z}$-stable), one finds examples without many projections (the Jiang–Su algebra itself is of course an example of projectionless C*-algebra. Note, however, that in this case, $\mathcal{Z}$ still has projections separating tracial states by virtue of its unique tracial state). Indeed, the range of the Elliott invariant shows that examples must exist. Explicit constructions can be given such as those in [16] or the C*-algebra crossed products coming from minimal dynamical systems of odd dimensional spheres constructed by Windsor in [71].

Let A be a unital C*-algebra. For any supernatural number $\mathfrak{p}$ of infinite type, one may consider the C*-algebra $A \otimes M_{\mathfrak{p}}$ where $M_{\mathfrak{p}}$ denotes the UHF algebra associated to $\mathfrak{p}$. When we do not need to keep track of $\mathfrak{p}$, a UHF algebra will be denoted $\mathcal{U}$ and it will be assumed, unless otherwise stated, that $\mathcal{U}$ is of infinite type. When $\mathfrak{p}$ consists of infinite powers of every prime, we denote the associated UHF algebra as $\mathcal{Q}$. This is the universal UHF algebra and satisfies $K_0(\mathcal{Q}) = \mathbb{Q}$.

Every UHF algebra of infinite type is strongly self-absorbing [65, Example 1.14(i)] so we refer to $A \otimes \mathcal{U}$ as a UHF-stable C*-algebra.

Tensoring A with a UHF algebra $\mathcal{U}$ introduces useful structural properties. At the same time, since the structure of $\mathcal{U}$ itself is not too complicated (it is, after all, not too far from being a matrix algebra), one should be able to keep track of the original C*-algebra A.

2.1.1 PROPOSITION: *Let A be a simple separable unital stably finite* C**-algebra and let $\mathcal{U}$ be a UHF algebra of infinite type. Then*

(i) *$A \otimes \mathcal{U}$ is simple.*

(ii) *$A \otimes \mathcal{U}$ has strict comparison.*

(iii) *$A \otimes \mathcal{U}$ is $\mathcal{Z}$-stable.*

(iv) *$A \otimes \mathcal{U}$ has stable rank one.*

(v) *$K_*(A \otimes \mathcal{U}) \cong (K_0(A) \otimes K_0(\mathcal{U})) \oplus (K_1(A) \otimes K_0(\mathcal{U}))$.*

(vi) $T(A \otimes \mathcal{U}) \cong T(A)$.

PROOF:

(i) For any $n \in \mathbb{N}$, $A \otimes M_n$ is simple and simplicity passes to inductive limits.

(ii) This is [53, Theorem 5.2].

(iii) Strongly self-absorbing C*-algebras are $\mathcal{Z}$-stable [76, Theorem 3.1].

(iv) This is [52, Corollary 6.6].

(v) Since $K_*(\mathcal{U})$ is torsion-free and $K_1(\mathcal{U}) = 0$, this follows from the Künneth Theorem for tensor products [2, Theorem 23.1.3].

(vi) Any UHF algebra has unique trace $\tau_{\mathcal{U}}$ and it is easy to check that $\tau \in T(A \otimes \mathcal{U})$ if and only if $\tau = \tau_A \otimes \tau_{\mathcal{U}}$ for some $\tau_A \in T(A)$.

∎

The UHF algebras of infinite type and the Jiang–Su algebra are among the (few) examples of unital strongly self-absorbing C*-algebras. In fact, their connection runs deeper still.

In [56], Winter and M. Rørdam showed that the Jiang–Su algebra can be written as a stationary inductive limit of $\mathcal{C}([0,1])$-algebras with UHF fibres. More specifically, if $\mathfrak{p}$ and $\mathfrak{q}$ are infinite supernatural numbers that are relatively prime then we define a generalized dimension drop algebra

$$\mathcal{Z}_{\mathfrak{p},\mathfrak{q}} = \{f \in \mathcal{C}([0,1], M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}) \mid f(0) \in M_{\mathfrak{p}} \otimes 1_{M_{\mathfrak{q}}}, f(1) \in 1_{M_{\mathfrak{p}}} \otimes M_{\mathfrak{q}}\}.$$

Then by [56, Theorem 3.4], if ϕ is a trace collapsing endomorphism on $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ we have

$$\mathcal{Z} = \varinjlim(\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}, \phi).$$

Winter's basic idea, originally appearing in [77], was to take isomorphisms (satisfying certain technical requirements) $A \otimes M_{\mathfrak{p}} \cong B \otimes M_{\mathfrak{p}}$ and $A \otimes M_{\mathfrak{q}} \cong B \otimes M_{\mathfrak{q}}$ and link them together in a suitable way along the interval $[0,1]$ to produce a particular $\mathcal{C}([0,1])$-isomorphism showing $A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \cong B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ which is in turn, via an intertwining argument, used to produce an isomorphism $A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$. Using this approach, one can take known classification results for UHF-stable C*-algebras to deduce classification results for $\mathcal{Z}$-stable algebras.

This idea was taken further by H. Lin in [36] and Lin and Z. Niu in [39, 40], where they gave an explicit way of determining when isomorphisms of A and B up to UHF-stability provide $\mathcal{Z}$-stable classification. This amounts to checking that $A \otimes \mathcal{Q}$ and $B \otimes \mathcal{Q}$ satisfy a tracial approximation property which will be discussed in detail in the next section.

2.2. Tracially approximately $\mathcal{S}$

The concept of tracial approximation was originally introduced by Lin in his paper on tracially approximately finite C*-algebras [33]. Up to that point, classification results tended to rely on determining an inductive limit structure of reasonable building blocks and using the continuity of the invariant and intertwining arguments to lift homomorphisms of invariants to *-homomorphisms of C*-algebras. Of course, not all C*-algebras appear as inductive limits of suitable building blocks and presenting a given C*-algebra in such a form, if it exists, can be technical and difficult. Rather than approximating via an inductive limit, Lin's approach was to considering approximating in a more measure-theoretical sense by using the tracial states in the C*-algebra.

Originally introduced were the tracially approximately finite (TAF) C*-algebras, which Lin showed could be classified in the separable simple unital nuclear case, providing the C*-algebras also satisfied the Universal Coefficient Theorem (UCT). Consequently, for a class $\mathcal{A}$ one no longer required a particular inductive limit for every $A \in \mathcal{A}$, but rather one need only check that each A satisfy the requirements to be TAF.

The notion of tracial approximation has since been generalized beyond the finite-dimensional case [35, 45, 19]. We record here the definition for a given class $\mathcal{S}$ of separable unital C*-algebras.

2.2.2 Definition:[cf. [31], [19]] Let $\mathcal{S}$ denote a class of separable unital C*-algebras. Let A be a simple unital C*-algebra. Then A is tracially approximately $\mathcal{S}$ (or TA$\mathcal{S}$) if the following holds. For every finite subset $\mathcal{F} \subset A$, every $\epsilon > 0$, and every nonzero positive element $c \in A$, there exists a projection $p \in A$ and a unital C*-subalgebra $B \subset pAp$ with $1_B = p$ and $B \in \mathcal{S}$ such that:

(i) $\|pa - ap\| < \epsilon$ for all $a \in \mathcal{F}$,

(ii) $\mathrm{dist}(pap, B) < \epsilon$ for all $a \in \mathcal{F}$,

(iii) $1_A - p$ is Murray–von Neumann equivalent to a projection in $\overline{cAc}$.

2.3. Tracial approximation of UHF-stable C^*-algebras

In this short section, we prove two useful lemmas that simplify the process of showing that a given UHF-stable C*-algebra is TA$\mathcal{S}$.

In this thesis we are mostly interested in tracial approximation of UHF-stable C^*-algebras by the class of interval algebras, I. An interval algebra is a C*-algebra A of the form

$$A = \bigoplus_{n=1}^{N} \mathcal{C}(X_n) \otimes M_{r_n}$$

for some $N \in \mathbb{N} \setminus \{0\}$, where $X_n = [0, 1]$ or X_n is a single point, and $r_n \in \mathbb{N} \setminus \{0\}$, $0 \leq n \leq N$.

Any C*-algebra in the class I can be written as a finitely presented universal C*-algebra (i.e. with finitely many generators and relations) and is semiprojective. In particular, any $A \in$ I has stable, hence weakly stable, relations [43]. Therefore we may make use of the following lemma, which says that to prove TAI after tensoring with $\mathcal{Q}$ it is enough to show that the approximating C*-algebras can always be chosen to have units that are bounded above zero in trace. The proof uses the same geometric series argument as the one given in [74, Lemma 3.2].

2.3.3 Lemma: *Let A be a separable simple unital stably finite exact* C*-*algebra and let $\mathcal{U}$ be a UHF algebra of infinite type. Suppose $\mathcal{S}$ is a class of* C*-*algebras that can be finitely presented with weakly stable relations (as universal* C*-*algebras), contains all finite-dimensional* C*-*algebras, and is closed under direct sums. Then $A \otimes \mathcal{U}$ is* TA$\mathcal{S}$ *if and only if there is an $n \in \mathbb{N}$ such that, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A \otimes \mathcal{U}$, there exist a projection $p \in A \otimes \mathcal{U}$ and a unital* C*-*subalgebra $B \subset p(A \otimes \mathcal{U})p$ with $1_B = p$ and $B \in \mathcal{S}$ such that:*

(i) $\|pa - ap\| < \epsilon$ *for all* $a \in \mathcal{F}$,

(ii) $\mathrm{dist}(pap, B) < \epsilon$ *for all* $a \in \mathcal{F}$,

(iii) $\tau(p) > 1/n$ *for all* $\tau \in T(A \otimes \mathcal{U})$.

Proof: If $A \otimes \mathcal{U}$ is TA$\mathcal{S}$ then (i) and (ii) are easily satisfied from the definition of TA$\mathcal{S}$. To show (iii) take a positive contraction $c \in A \otimes \mathcal{U}$ with $\tau(c) \leq 1/n$ for all $\tau \in T(A \otimes \mathcal{U})$ and then find p such that $1 - p$ is Murray–von Neumann equivalent to a projection in $\overline{cAc}$ to get $\tau(p) > 1/n$.

Now let a finite subset $\mathcal{F} \subset A \otimes \mathcal{U}$, $\epsilon > 0$ and a nonzero positive element $c \in A \otimes \mathcal{U}$ be given, and suppose that $A \otimes \mathcal{U}$ satisifies (i), (ii), (iii) with respect to some $n \in \mathbb{N}$. We show that $A \otimes \mathcal{U}$ is TA$\mathcal{S}$; the proof is almost identical to that of [74, Lemma 3.2]. First we note that $A \otimes \mathcal{U}$ has property (SP) (every nonzero hereditary C*-subalgebra has a nonzero projection) since $A \otimes \mathcal{U}$ has strict comparison and projections that are arbitrarily small in trace. Thus we find a projection $q \in \overline{c(A \otimes \mathcal{U})c}$, just as in [74, Lemma 3.2].

We inductively construct C*-algebras $B_i \subset A \otimes \mathcal{U}$ with each $B_i \in \mathcal{S}$.

As in [74, Lemma 3.2] the initial B_0 exists by assumption.

The construction of B_{i+1} from B_i is similar to the construction in Lemma 3.2 of [74]. We cannot apply Lemma 3.4 of [74] directly—even though $A \otimes \mathcal{U}$ is simple, unital and has the comparability property—since we do not want to make the assumption that $K_0(A\otimes\mathcal{U})_+$ has dense image in the positive affine functions on $T(A\otimes\mathcal{U})$. However, the result will still hold by choosing the projection e in [74, Lemma 3.4] to be of the form $(1_{A\otimes\mathcal{U}}-p)\otimes q$ (using the fact that $\mathcal{U}$ is strongly self-absorbing) for some projection $q\in\mathcal{U}$ satisfying

$$1/(t+1)<\tau_{\mathcal{U}}(q)<1/t$$

where $\tau_{\mathcal{U}}$ is the unique tracial state on $\mathcal{U}$. The projection e then satisfies the requirements of the projection in the proof, and the results of [74, Lemma 3.4] hold. Thus we get the finite-dimensional C*-algebras C_0, C_1 and D as in [74, Lemma 3.2].

Let $\mathcal{G}:=\{x_1,\dots,x_n,1_{B_i}\}\subset B_i$ where $x_1,\dots,x_n$ are generators for B_i. Let $\gamma>0$ be as in [74, Lemma 3.2]. Since B_i has weakly stable relations, there is a $\tilde{\vartheta}>0$ with the following property: If E is another C*-algebra, $p\in A$ a projection and $\phi:B_i\to E$ a *-homomorphism satisfying $\|p\phi(b)-\phi(b)p\|<\tilde{\vartheta}$ for all $b\in\mathcal{G}$, then there is a *-homomorphism $\tilde{\phi}:B_i\to pEp$ satisfying $\|\tilde{\phi}(b)-p\phi(b)p\|<\gamma$ for all $b\in\mathcal{G}$.

Now choose $0<\vartheta<\min\{\gamma,\tilde{\vartheta}\}$ such that the assertion of [74, Proposition 3.3] holds for the finite-dimensional algebra D. Set $\tilde{\mathcal{F}}:=\mathcal{F}\cup\mathcal{G}\cup\kappa(D)^1$ where $\kappa:D\to A\otimes\mathcal{U}$ is the *-homomorphism given by [74, Lemma 3.4] using $e=(1_{A\otimes\mathcal{U}}-p)\otimes q$ in place of the e in that proof.

By hypothesis there is a C*-algebra $F\subset A\otimes\mathcal{U}$, $F\in\mathcal{S}$ satisfying d), e), f) of [74, Lemma 3.4] with respect to $\tilde{\mathcal{F}}$ and ϑ. As in [74, Lemma 3.2], d) and the choice of ϑ provide the *-homomorphisms

$$\varrho:B_i\to(1_{A\otimes\mathcal{U}}-1_F)(A\otimes\mathcal{U})(1_{A\otimes\mathcal{U}}-1_F)$$

satisfying

$$\|\varrho(b)-(1_{A\otimes\mathcal{U}}-1_F)b(1_{A\otimes\mathcal{U}}-1_F)\|<\gamma \text{ for all } b\in\mathcal{G}$$

and

$$\bar{\kappa}:D\to(1_{A\otimes\mathcal{U}}-1_F)(A\otimes\mathcal{U})(1_{A\otimes\mathcal{U}}-1_F)$$

such that

$$\|\bar{\kappa}-(1_{A\otimes\mathcal{U}}-1_F)\kappa(d)(1_{A\otimes\mathcal{U}}-1_F)\|<\gamma\cdot\|d\| \text{ for all } 0\neq d\in D.$$

Set $B_{i+1}:=\varrho(B_i)\oplus F$. Then one easily checks that the same calculations given in [74, Proposition 3.3] can be used to complete the proof. ∎

The next lemma shows we need only consider finite subsets of $A\otimes\mathcal{Q}$ of a simplified form; essentially the only difficulty lies in approximating elements from A.

2.3.4 Lemma: *Let $\mathcal{S}$ denote a class of separable unital C*-algebras that is closed under tensoring with finite-dimensional C*-algebras. Let A be a separable unital C*-algebra with $T(A)\neq\emptyset$ and suppose there is $0<\eta\leq 1$ such that, for any $\epsilon>0$ and any finite subset $\mathcal{G}\subset A$, there are a projection $p\in A\otimes\mathcal{Q}$ and a unital C*-subalgebra $B\subset p(A\otimes\mathcal{Q})p$ with $1_B=p$ and $B\in\mathcal{S}$ such that*

(i) $\|p(a\otimes 1_{\mathcal{Q}})-(a\otimes 1_{\mathcal{Q}})p\|<\epsilon$ *for all* $a\in\mathcal{G}$,

(ii) $\mathrm{dist}(p(a\otimes 1_{\mathcal{Q}})p,B)<\epsilon$ *for all* $a\in\mathcal{G}$,

(iii) $\tau(p)\geq\eta$ *for all* $\tau\in T(A\otimes\mathcal{Q})$.

Then, for any $\epsilon>0$ and any finite subset $\mathcal{F}\subset A\otimes\mathcal{Q}$, there are a projection $q\in A\otimes\mathcal{Q}$ and a unital C-subalgebra $C\subset q(A\otimes\mathcal{Q})q$ with $1_C=q$ and $C\in\mathcal{S}$ such that*

(iv) $\|qa-aq\|<\epsilon$ *for all* $a\in\mathcal{F}$,

(v) $\mathrm{dist}(qaq,C)<\epsilon$ *for all* $a\in\mathcal{F}$,

(vi) $\tau(q)\geq\eta$ *for all* $\tau\in T(A\otimes\mathcal{Q})$.

Proof: The proof essentially appears in the proof of Lemma 4.3.26 ([60, Lemma 4.4]). Let $\epsilon>0$ and let $\mathcal{F}\subset A\otimes\mathcal{Q}$ be a finite subset. Using the identification

$$A\otimes\mathcal{Q}\cong A\otimes M_S\otimes\mathcal{Q}\cong A\otimes\mathcal{Q}\otimes M_S,$$

for $S \in \mathbb{N}$, we may assume that the finite set is of the form

$$(\{1_A\} \otimes \{1_{\mathcal{Q}}\} \otimes \mathcal{B}) \cup (\mathcal{G} \otimes \{1_{\mathcal{Q}}\} \otimes \{1_{M_S}\})$$

for some $S \in \mathbb{N}$ where $\mathcal{B}$ is a finite subset of M_S and $\mathcal{G}$ is a finite subset of A. We may further assume that $1_A \in \mathcal{G}$ and also that $1_{M_S} \in \mathcal{B}$. Then we have

$$\mathcal{F} = \mathcal{G} \otimes \{1_{\mathcal{Q}}\} \otimes \mathcal{B}.$$

By assumption, there exists a $B \in \mathcal{S}$ and a projection $p = 1_B$ satisfying properties (i) – (iii) for the finite set $\mathcal{G}$, with $\epsilon/\max\{1, \{\|b\| \mid b \in \mathcal{B}\}\}$ in place of ϵ.

Define $C = B \otimes M_S$ and $q := 1_C = p \otimes 1_{M_S} \in A \otimes \mathcal{Q} \otimes M_S$. The fact that q and C satisfy properties (iv) and (v) for $\tilde{\mathcal{F}}$ and ϵ is shown in the proof of Lemma 4.3.26 ([60, Lemma 4.4]).

To show (vi), simply observe that $\tau \in T(A \otimes \mathcal{Q} \otimes M_S)$ is of the form $\tau_1 \otimes \tau_2$ where $\tau_1 \in T(A \otimes \mathcal{Q})$ and $\tau_2 \in T(M_S)$. Then

$$\tau(q) = \tau(p \otimes 1_{M_S}) = \tau_1(p)\tau_2(1_{M_S}) = \tau(q) \geq \eta.$$

∎

3. Classification of tracially approximately semihomogeneous C^*-algebras

In this chapter we show that a simple separable unital tracially approximately semihomogeneous C^*-algebra of bounded dimension tensored with the universal UHF algebra $\mathcal{Q}$ is tracially approximately interval (TAI). Together with Huaxin Lin's classification theorem for such algebras, this gives a new proof for the classification by Elliott invariants of AH algebras with no dimension growth without appeal to Gong's reduction theorem [23]. Appealing to previous classification results, our result also shows that a simple separable unital locally semihomogeneous C^*-algebra of bounded dimension must be an AH algebra of slow dimension growth and that the class of AH algebras of slow dimension growth is closed under taking simple inductive limits. Such a local result is known to be false in the nonsimple case.

The chapter is organised as follows. In Section 3.1 we introduce the class of C*-algebras which will appear in the main theorem. Then, in Section 3.2 we give a few technical results that will be needed to cut up the base spaces of the approximating algebras in a tracially large way. In Section 3.3 we sketch a proof of the case where A is a locally approximately trivial homogeneous C*-algebra with two extreme tracial states. For the general case, we need to work harder to find an appropriate interval algebra to use in the TAI approximation; this is done in Section 3.4. The penultimate section, Section 3.5, contains the proof of the main theorem. In the final section, we give a number of applications to the classification project.

3.1. Approximation by semihomogeneous C^*-algebras

We first set the definitions for locally and tracially approximately semihomogeneous algebras and collect some observations about these C*-algebras.

3.1.1 Definition: We say that the C*-algebra A is locally semihomogeneous if, for every $\epsilon > 0$ and every finite subset $\mathcal{F} \subset A$, there is a separable semihomogeneous C*-subalgebra $B \subset A$ of the form

$$B = \bigoplus_{k=1}^{N} p_k(\mathcal{C}(X_k) \otimes M_{r_k})p_k,$$

where $r_k \in \mathbb{N}$, the X_k are compact metrizable spaces and the $p_k \in \mathcal{C}(X_k) \otimes M_{r_k}$ are projections satisfying

$$\mathrm{dist}(f, B) < \epsilon \text{ for all } f \in \mathcal{F}.$$

We will call A locally connected semihomogeneous if the approximating C*-subalgebra B can be chosen so that the X_k are connected.

If there exists an $L \in \mathbb{N}$ such that the approximating C*-subalgebra B can be chosen so that each of the base spaces $X_k, 1 \leq k \leq N$ has covering dimension less than or equal to L, then we say A is locally semihomogeneous of bounded dimension (at most L) and locally connected semihomogeneous of bounded dimension (at most L) if the X_k can be also chosen to be connected.

Note that sometimes the C*-algebras A of Definition 3.1.1 are called "locally homogeneous" [15, 14], however this term has often been used to denote the C*-algebras of the form of the approximating C*-subalgebras B (for example, in [3]) and we prefer to differentiate between the two. They have also been called "locally AH" [37] and "AH in the local sense" [5].

We now show that the modifier "connected" is in fact superfluous; we may always assume this to be true. This will make working with these algebras much less complicated.

3.1.2 Proposition: *A is a separable locally semihomogeneous* C*-*algebra of bounded dimension if and only if it is a separable locally connected semihomogeneous* C*-*algebra of bounded dimension.*

Proof: The "if" direction is clear. So assume that A is locally semihomogeneous of bounded dimension, but that the base spaces of the approximating C*-algebras are not necessarily chosen to be connected. Let $\mathcal{F} \subset A$ and $\epsilon > 0$ be given. Approximate $\mathcal{F}$ by a semihomogeneous algebra B within $\epsilon/2$. For every $a \in \mathcal{F}$, let $b_a \in B$ be some element such that $\|b_a - a\| < \epsilon/2$. Set $\mathcal{F}_b = \{b_a \mid a \in \mathcal{F}\}$. We have that $B = \bigoplus_{k=1}^{N} p_k(\mathcal{C}(X_k) \otimes M_{r_k})p_k$ with each X_k a compact metrizable space. It is a standard result that any compact metrizable space can be written as the inverse limit of connected finite-dimensional simplicial complexes with dimension equal to the original space. Thus each X_k can be written as the inverse limit of connected finite-dimensional simplicial complexes $(X_{k,n})_{n\in\mathbb{N}}$ with $\dim(X_{k,n}) = \dim(X_k)$ inducing an sequence of C*-algebras $(\mathcal{C}(X_{k,n}) \otimes M_{r_k})_{n\in\mathbb{N}}$ with (not necessarily injective) maps

$$\phi_{k,n} : \mathcal{C}(X_{k,n}) \otimes M_{r_k} \to \mathcal{C}(X_{k,n+1}) \otimes M_{r_k}$$

to give an inductive sequence with limit $\mathcal{C}(X_k) \otimes M_{r_k}$. For large enough $n \in \mathbb{N}$ we can find projections $p_{k,n} \in \mathcal{C}(X_{k,n}) \otimes M_{r_k}$ so that that in the limit we get $\lim_{n\to\infty} \phi^{(k,n)}(p_{k,n}) = p_k$ (here $\phi^{(k,n)} : \mathcal{C}(X_{k,n}) \otimes M_{r_k} \to \mathcal{C}(X_k) \otimes M_{r_k}$ are the canonical maps defined in 1.1.2), whence

$$B = \varinjlim\left(\bigoplus_{k=1}^{N} p_{k,n}(\mathcal{C}(X_{k,n}) \otimes M_{r_k})p_{k,n}, \oplus_{k=1}^{N} \phi_{k,n}|_{p_{k,n}(\mathcal{C}(X_{k,n})\otimes M_{r_k})p_{k,n}}\right)$$

with $X_{k,n}$ finite-dimensional simplicial complexes. Applying [17, Theorem 2.1], we see that B also has an inductive limit structure $B = \varinjlim \bigoplus_{k=1}^{M} q_{k,n}(\mathcal{C}(Y_{k,n}) \otimes M_{r_k})q_{k,n}$ for some $M \in \mathbb{N}$ with injective connecting homomorphisms, $Y_{k,n}$ connected and $\dim(Y_{k,n}) \leq \max\{\dim(X_{n,1}), \ldots, \dim(X_{k,n})\} < \infty$. Now just take n sufficiently large to approximate $\mathcal{F}_B$ by $\bigoplus_{k=1}^{M} q_{k,n}(\mathcal{C}(Y_{k,n}) \otimes M_{r_k})q_{k,n}$ within $\epsilon/2$ and the result follows. ∎

3.1.3 Let H be the class of semihomogeneous C*-algebras. If A is tracially approximately semihomogeneous (TAH) and there is an $L \in \mathbb{N}$ such that the approximating C*-subalgebras of the form $B = \bigoplus_{k=1}^{N} p_k(\mathcal{C}(X_k) \otimes M_{r_k})p_k$ from Definition 2.2.2 can be chosen so that the spaces X_k are of covering dimension less than or equal to L, then we will say that A is tracially approximately semihomogeneous of bounded dimension. Note that an AH algebra is locally semihomogeneous and that a locally semihomogeneous C*-algebra is tracially approximately semihomogeneous. It follows easily from Proposition 3.1.2 that we may assume that if A is TAH, the approximating C*-subalgebras can be chosen so that the X_k are connected.

3.2. Cutting up base spaces

A key part of the proof of the main theorem is chopping up the underlying spaces $X_1, \ldots, X_n$ of a suitably large semihomogeneous C*-subalgebra into open sets in a tracial way. We then move between these tracially large sets via a path of disjoint open sets, each chosen so that the elements of the finite subset $\mathcal{F}$, when restricted to one of the sets, can be approximated by functions taking constant matrix values. This will then be approximated by an interval in the sense that we will associate to each set a function in a partition of unity of $[0,1]$ and then twist the elements under the sets in a way that keeps the entire interval intact.

When we are dealing only with approximation by points instead of intervals, cutting out tracial pieces is straightforward.

3.2.4 LEMMA: *Let A be simple separable unital nuclear* C**-algebra with nonempty tracial state space. For any $\eta > 0$, if there are $N \in \mathbb{N}$ and nonzero pairwise orthogonal positive contractions $a_0, \ldots, a_N$ satisfying*

$$|\tau(a_n) - \tau'(a_n)| < \eta/(2(N+1))$$

for every $\tau, \tau' \in T(A \otimes \mathcal{Q})$, then there are partial isometries $v_n \in A \otimes \mathcal{Q}$ and a partial isometry $v = \sum_{n=0}^{N} v_n$ such that the $v_n^* v_n \in \mathrm{Her}(a_n)$, the $v_n v_n^*$ are pairwise orthogonal projections and

$$\tau(v^*v) \geq \tau(\textstyle\sum_{n=0}^{N} a_n) - \eta \text{ for every } \tau \in T(A \otimes \mathcal{Q}).$$

PROOF: For each $n = 0, \ldots, N$ we have $\tau(a_n) \in (0,1]$ for every $\tau \in T(A \otimes \mathcal{Q})$ so we can find a rational number $m_n \in (0,1]$ such that

$$m_n \in (\min_{\tau \in T(A\otimes\mathcal{Q})} \tau(a_n) - \eta/(2(N+1)), \min_{\tau \in T(A\otimes\mathcal{Q})} \tau(a_n)].$$

Let $p_n \in \mathcal{Q}$ be a projection such that $\tau_{\mathcal{Q}}(p_n) = m_n$. Since the a_n are pairwise orthogonal positive contractions, we must have $\tau(\sum_{n=0}^{N} a_n) \leq 1$ so that we can choose the m_n to satisfy $\sum_{n=0}^{N} m_n \leq 1$ and hence can arrange that the p_n are pairwise orthogonal.

By strict comparison and stable rank one (Proposition 2.1.1 (ii), (iv)) we find unitaries $u_0, \ldots, u_N \in A \otimes \mathcal{Q}$ such that $u_n^*(1_A \otimes p_n)u_n \in \mathrm{Her}(a_n)$. Put $v_n = (1_A \otimes p_n)u_n$ and $v = \sum_{n=0}^{N} v_n$. Then $v_n^* v_n \in \mathrm{Her}(a_n)$ and $v_n v_n^* = 1_A \otimes p_n$ are pairwise orthogonal. Note that if $n \neq m$ then

$$\|v_n^* v_m\|^4 = \|v_m^* v_n v_n^* v_m\|^2 = \|v_m^*(v_n v_n^*)(v_m v_m^*) v_n v_n^* v_m\| = 0$$

so $v^*v = \sum_{n=1}^{N} v_n^* v_n$ and

$$\begin{aligned} \tau(v^*v) &= \textstyle\sum_{n=0}^{N} \tau(v_n^* v_n) \\ &= \textstyle\sum_{n=0}^{N} \tau(v_n v_n^*) \\ &= \textstyle\sum_{n=0}^{N} m_n \\ &\geq \tau(\textstyle\sum_{n=0}^{N} a_n) - \eta, \end{aligned}$$

for every $\tau \in T(A \otimes Q)$. ▮

When we want to cut out intervals instead, we need to use positive elements instead of projections. These will be sawtooth functions in a partition of unity of $[0,1]$ as defined in 1.1.1. In this case, we have the added complication that functions supported on overlapping subintervals are not orthogonal.

3.2.5 LEMMA: *Let A be simple separable unital nuclear* C**-algebra with nonempty tracial state space. Given $0 < \eta < 1$, $K \in \mathbb{N}\backslash\{0\}$, nonzero pairwise orthogonal postive contractions $a_0, a_{\frac{1}{K}}, \ldots, a_{\frac{K-1}{K}}, a_1 \in A \otimes \mathcal{Q}$ and a *-homomorphism*

$$\psi : \mathcal{C}([0,1]) \to A \otimes \mathcal{Q}$$

such that

(i) *$\tau(\psi(\gamma_i)) > \tau(a_i) - \eta/2$ for every $\tau \in T(A \otimes \mathcal{Q})$, $i \in \{0,1\}$,*

(ii) *$0 < \tau(\psi(\gamma_{\frac{k}{K}})) < \tau(a_{\frac{k}{K}}) < \eta$ for every $\tau \in T(A \otimes \mathcal{Q})$, $k \in \{1, \ldots K-1\}$,*

the following holds:

For any $\delta > 0$ there are $s_{\frac{k}{K}} \in A \otimes \mathcal{Q}$ such that $s_{\frac{k}{K}} s_{\frac{k}{K}}^* = (\psi(\gamma_{\frac{k}{K}}) - \delta)_+$, $s_{\frac{k}{K}}^* s_{\frac{k}{K}} \in \mathrm{Her}(a_{\frac{k}{K}})$ and for $s = \sum_{k=0}^{K} s_{\frac{k}{K}}$ we have

(iii) $ss^* = \sum_{k=0}^{K} s_{\frac{k}{K}} s^*_{\frac{k}{K}}$,

(iv) $s^*s = \sum_{k=0}^{K} \sum_{\{k' \mid |k'-k| \leq 1\}} s^*_{\frac{k}{K}} s_{\frac{k'}{K}}$,

(v) $\tau(ss^*) > \tau(a_0 + a_1) - \eta - \delta$ *for every* $\tau \in T(A \otimes \mathcal{Q})$.

PROOF: Let $\delta > 0$. For each $k \in \{0, \dots, K\}$, since $A \otimes \mathcal{Q}$ has strict comparison and stable rank one, the tracial conditions imply there are unitaries $u_{\frac{k}{K}} \in A \otimes \mathcal{Q}$ such that $u^*_{\frac{k}{K}} \psi((\gamma_{\frac{k}{K}} - \delta)_+) u_{\frac{k}{K}} \in \mathrm{Her}(a_{\frac{k}{K}})$. Put

$$s_{\frac{k}{K}} = \psi((\gamma_{\frac{k}{K}} - \delta)_+^{1/2}) u_{\frac{k}{K}}.$$

Then $s_{\frac{k}{K}} s^*_{\frac{k}{K}} = \psi((\gamma_{\frac{k}{K}} - \delta)_+)$ and $s^*_{\frac{k}{K}} s_{\frac{k}{K}} \in \mathrm{Her}(a_{\frac{k}{K}})$.

Since the $a_{\frac{k}{K}}$ are pairwise orthogonal, we have that

$$\|s_{\frac{k}{K}} s^*_{\frac{k'}{K}}\|^4 = \|s_{\frac{k'}{K}} s^*_{\frac{k}{K}} s_{\frac{k}{K}} s^*_{\frac{k'}{K}}\|^2 = \|s_{\frac{k'}{K}} s^*_{\frac{k}{K}} s_{\frac{k}{K}} s^*_{\frac{k'}{K}} s_{\frac{k'}{K}} s^*_{\frac{k}{K}} s_{\frac{k}{K}} s^*_{\frac{k'}{K}}\| = 0, \tag{3.1}$$

whenever $k \neq k'$. It follows that

$$ss^* = \sum_{k=0}^{K} s_{\frac{k}{K}} s^*_{\frac{k}{K}}.$$

When $|k - k'| > 1$ the functions $\gamma_{\frac{k}{K}}$ and $\gamma_{\frac{k'}{K}}$ have disjoint support, whence $s^*_{\frac{k}{K}} s_{\frac{k}{K}} s^*_{\frac{k'}{K}} s_{\frac{k'}{K}} = 0$. A similar calculation to (3.1) then shows that $s^*_{\frac{k}{K}} s_{\frac{k'}{K}} = 0$ when $|k' - k| > 1$, and so

$$s^*s = \sum_{k=0}^{K} \sum_{\{k' \mid |k'-k| \leq 1\}} s^*_{\frac{k}{K}} s_{\frac{k'}{K}}.$$

Finally,

$$\begin{aligned} \tau(ss^*) &\geq \tau(\psi((\gamma_0 - \delta)_+) + \psi((\gamma_1 - \delta)_+)) \\ &\geq \tau(\psi(\gamma_0)) + \tau(\psi(\gamma_1)) - \delta \\ &> \tau(a_0 + a_1) - \eta - \delta. \end{aligned}$$

∎

In general, we will require that the tracial cutouts be attached to matrix algebras, since any finite set of functions in a semihomogeneous C*-algebra will take matrix values and these are what we must approximate in interval algebras. When we have tensored with the universal UHF algebra, finding the right size of matrix blocks is no difficulty. Choosing appropriate local matrix units in the semihomogeneous C*-algebra requires a bit more effort.

3.2.6 By local matrix units of a set $U \subset X$ we mean elements $e_{m,n} \in p(\mathcal{C}(X) \otimes M_r)p$ such that, for any $x \in U$ the maps

$$\mathrm{ev}_x : p(\mathcal{C}(X) \otimes M_r)p \to M_{\mathrm{rank}(p(x))} : f \mapsto f(x)$$

are surjective *-homomorphisms sending $e_{m,n}$ to the $(\mathrm{rank}(p(x)) \times \mathrm{rank}(p(x)))$ matrix with one in the $(m,n)^{\mathrm{th}}$-entry, zeros elsewhere. By Proposition 3.1.2 we can always assume our approximating semihomogeneous C*-algebras have connected base spaces. In this case, $\mathrm{rank}(p(x)) = R$ for some $R \in \mathbb{N} \setminus \{0\}$, for every $x \in X$.

Let $x \in X$, and for every $m, n \in \{1, \dots, R\}$ let $e^{(x)}_{m,n} \in p(C(X) \otimes M_n)p$ be the element sent to $e_{m,n} \in M_R$. Then by [20, Theorem 3.1] there is a neighbourhood U of x such that, for every $y \in U$ the map $e^{(x)}_{m,n}(y) \mapsto e_{m,n}$ is a *-isomorphism of M_R. Thus, restricted to U, the elements $e^{(x)}_{m,n}$ satisfy the relations of matrix units, that is, $e^{(x)}_{m,n} e^{(x)}_{m',n'}|_U = \delta_{n,m'} e^{(x)}_{m,n'}|_U$.

3.2.7 LEMMA: *Let A be simple separable unital nuclear C*-algebra with nonempty tracial state space. Suppose $p(\mathcal{C}(X) \otimes M_r)p \subset A$ where X is a compact connected metrizable space and $p \in \mathcal{C}(X) \otimes M_r$ is a projection with rank R. Let $K \in \mathbb{N} \setminus \{0\}$ and $a_0, a_{\frac{1}{K}}, \dots, a_{\frac{K-1}{K}}, a_1 \in p(\mathcal{C}(X) \otimes M_r)p \otimes \mathcal{Q}$ be pairwise orthogonal positive contractions supported on open sets $U_0, U_{\frac{1}{K}}, \dots, U_{\frac{K-1}{K}}, U_1$, respectively.*

Suppose further that that $e^{(\frac{k}{K})}_{m,n}$ *are local* $R \times R$ *matrix units defined over* $U_{\frac{k}{K}}$ *satisfying* $e^{(\frac{k}{K})}_{m,n}(x) = e^{(\frac{k+1}{K})}_{m,n}(x)$ *for every* $x \in U_{\frac{k}{K}} \cup U_{\frac{k+1}{K}}$.

Let $\psi : \mathcal{C}([0,1]) \to A \otimes \mathcal{Q}$ *be a* **-homomorphism satisfying the requirements of Lemma 3.2.5 and let* $K \in \mathbb{N} \setminus \{0\}$ *and* $s_{\frac{k}{K}} \in A \otimes \mathcal{Q}$ *be the elements provided by that lemma with respect to* $a_{\frac{k}{K}}, \psi$ *and any* $\delta > 0$. *Fix a* **-homomorphism* $\rho : M_R \hookrightarrow \mathcal{Q}$ *and define*

$$\phi : \mathcal{C}([0,1]) \otimes M_R \to A \otimes \mathcal{Q}$$

by $\phi(f \otimes a) = \psi(f) \otimes \rho(a)$. *Put*

$$s = \textstyle\sum_{m=1}^{R} \phi(1_{\mathcal{C}([0,1])} \otimes e_{m,1}) s_{\frac{k}{K}} e^{\frac{k}{K}}_{1,m},$$

where $R = \operatorname{rank}(p)$. *Then*

$$ss^* = \textstyle\sum_{k=0}^{K} \sum_{m=1}^{R} \phi(1_{\mathcal{C}([0,1])} \otimes e_{m,1}) s_{\frac{k}{K}} s^*_{\frac{k}{K}} \phi(1_{\mathcal{C}([0,1])} \otimes e_{1,m})$$

and

$$s^*s = \textstyle\sum_{k=0}^{K} \sum_{\{k', |k-k'| \leq 1\}} \sum_{m=1}^{R} e^{\frac{k}{K}}_{m,1} s^*_{\frac{k}{K}} s_{\frac{k'}{K}} e^{\frac{k'}{K}}_{1,m}.$$

Proof: Since the matrix units agree across adjacent $U_{\frac{k}{K}}$, $U_{\frac{k'}{K}}$ and this exactly where $s^*_{\frac{k'}{K}} s_{\frac{k}{K}} \neq 0$, it is easy to check that ss^* and s^*s can be computed by almost identical calculations to those in Lemma 3.2.5. ∎

We now turn to the task of fixing matrix units.

3.2.8 Lemma: *Let* X *be a compact connected metrizable space,* $r \in \mathbb{N} \setminus \{0\}$ *and* $p \in \mathcal{C}(X) \otimes M_r$ *a projection with* $\operatorname{rank}(p) = R$. *Given* $\eta > 0$, *a finite subset* $\mathcal{F} \subset p(\mathcal{C}(X) \otimes M_r)p$ *and* $x_0 \in X$ *there are open subsets* $U_0 \subset V_0 \subset X$ *with* $x_0 \in U_0$, $V_0 \setminus \overline{U_0} \neq \emptyset$, *and local* $R \times R$ *matrix units* $e^{(0)}_{m,n}$ *for* V_0 *induced by* ev_{x_0} *such that the following holds: If* $x_1 \in V_0 \setminus \overline{U_0}$ *there are open neighbourhoods* U_1 *and* V_1 *of* x_1 *with* $U_1 \subset V_1$ *such that*

(i) $U_1 \subset V_0$,

(ii) $U_0 \cap U_1 = \emptyset$,

(iii) $V_1 \setminus \overline{U_1} \neq \emptyset$,

(iv) ev_{x_1} *induces local* $R \times R$ *matrix units* $e^{(1)}_{m,n} \in p(\mathcal{C}(X) \otimes M_r)p$ *across* V_1,

(v) *for* $i = 0, 1$ *we have* $\|f(x)_{m,n} \cdot e^{(i)}_{m,n} - f(y)_{m,n} \cdot e^{(i)}_{m,n}\| < \eta$ *for every* $x, y \in V_i$ *and every* $f \in \mathcal{F}$.

Proof: The map $\mathrm{ev}_{x_0} : p(\mathcal{C}(X) \otimes M_r)p \to M_R$ is surjective. Let $e^{(0)}_{m,n} \in p(\mathcal{C}(X) \otimes M_r)p$ be functions that give a set of matrix units for M_R under this map. There is an open neighbourhood V_0 of x_0 such that $e^{(0)}_{m,n}(y)$ induces a set of matrix units for M_R for every $y \in V_0$. Shrinking V_0 if necessary, we may arrange that $\|f(x)_{m,n} \cdot e^{(0)}_{m,n} - f(y)_{m,n} \cdot e^{(0)}_{m,n}\| < \eta$ for every $x, y \in V_0$ and every $f \in \mathcal{F}$.

Let $U_0 \subset V_0$ be an open subset such that $V_0 \setminus \overline{U_0} \neq \emptyset$ and let $x_1 \in V_0 \setminus \overline{U_0}$. As above, there are an open neighbourhood V_1 of x_1 and functions $e^{(1)}_{m,n} \in p(\mathcal{C}(X) \otimes M_r)p$ such that $e^{(1)}_{m,n}(y)$ are matrix units for M_R for every $y \in V_1$, and shrinking V_1 if necessary we can arrange $\|f(x)_{m,n} \cdot e^{(1)}_{m,n} - f(y)_{m,n} \cdot e^{(1)}_{m,n}\| < \eta$ for every $x, y \in V_1$ for every $f \in \mathcal{F}$. Now it is easy to find an open set $U_1 \subset V_0$ containing x_1 that is disjoint from U_0. ∎

3.2.9 Lemma: *Let* X *be a compact connected metrizable space,* $r \in \mathbb{N} \setminus \{0\}$ *and* $p \in \mathcal{C}(X) \otimes M_r$ *a projection. Given* $\eta > 0$, $\mathcal{F} \subset p(\mathcal{C}(X) \otimes M_r)p$ *a finite subset and points* $x_0, x_1 \in X$, *let* $U_0 \subset V_0$, $U_1 \subset V_1$ *be open subsets (not necessarily disjoint) as given by Lemma 3.2.8 with respect to* $\eta > 0$, $\mathcal{F}$, x_0 *and* x_1. *Then there are* $K \in \mathbb{N} \setminus \{0\}$, $x_{\frac{k}{K}} \in X$, $k = 0, \ldots, K$, *open neighbourhoods* $U_{\frac{k}{K}}$ *of* $x_{\frac{k}{K}}$, *each equipped with local matrix units* $e^{(\frac{k}{K})}_{m,n}$, *satisfying*

(i) $e_{m,n}^{(\frac{k}{K})}(x) = e_{m,n}^{(\frac{k+1}{K})}(x)$ *for every* $x \in U_{\frac{k}{K}} \cup U_{\frac{k+1}{K}}$, $k \in \{0,\dots,K-1\}$

(ii) $U_{\frac{k}{K}} \cap U_{\frac{k'}{K}} = \emptyset$ *for* $k \neq k' \in \{0,\dots,K-1\}$

(iii) $\|f(x)_{m,n} \cdot e_{m,n}^{(\frac{k}{K})} - f(y)_{m,n} \cdot e_{m,n}^{(\frac{k+1}{K})}\| < \eta$ *for every* $x, y \in U_{\frac{k}{K}} \cup U_{\frac{k+1}{K}}$ *, every* $f \in \mathcal{F}$ *and every* $k \in \{0,\dots,K\}$.

Proof: Let $\operatorname{rank}(p) = R \leq r$. For every $x \in X$ we can find a small open neighbourhood of $V_x \subset X$ and $e_{m,n}^{(x)} \in \operatorname{ev}_x^{-1}(e_{m,n})$ such that $e_{m,n}^{(x)}(y) = e_{m,n}$ for every $y \in V_x$ are a set of matrix units for M_R. Shrinking the sets if necessary, we may assume that

$$\|f(y)_{m,n} \cdot e_{m,n}^{(y)} - f(z)_{m,n} \cdot e_{m,n}^{(z)}\| < \eta$$

for every $y, z \in V_x$ and every $f \in \mathcal{F}$. The set $\{V_x \mid x \in X\} \cup \{V_0, V_1\}$ is an open cover for X. Let $\mathcal{O}$ denote a finite subcover which contains V_0, V_1. We may assume that if $V, V' \in \mathcal{O}$ are two distinct sets that $V \setminus V' \neq \emptyset$, that is, no set of $\mathcal{O}$ is contained in another.

If $U_0 \subset V_1$ or $U_1 \subset V_0$, then by taking $e_{m,n}^{(x_1)}$ or $e_{m,n}^{(x_0)}$ as matrix units, we are done.

Otherwise, choose $\tilde{V}_1 \in \mathcal{O}$ such that $\tilde{V}_1 \cap V_0 \neq \emptyset$. Let $\tilde{U}_1 \in (V_0 \cap \tilde{V}_1) \setminus \overline{U_0 \cup U_1}$.

If $\tilde{V}_1 = V_1$ then set $K = 2$ and $U_{\frac{1}{K}} = \tilde{U}_1$. If $\tilde{V}_1 \neq V_1$, then there is $\tilde{V}_2 \in \mathcal{O} \setminus \{V_0, \tilde{V}_1\}$ such that $\tilde{V}_1 \cap \tilde{V}_2 \neq \emptyset$. Let $\tilde{U}_2 \subset (\tilde{V}_2 \cap \tilde{V}_1) \setminus \overline{U_0 \cup \tilde{U}_1 \cup U_1}$.

For $k \geq 1$, suppose we have sets $\tilde{U}_1, \dots, \tilde{U}_k$ and $\tilde{V}_1, \dots, \tilde{V}_k$ such that $\tilde{U}_i \subset \tilde{V}_{i-1} \cap \tilde{V}_i$ and $\tilde{U}_i \cap \tilde{U}_j = \emptyset$ when $j \neq i$. If U_1 is not contained in $\tilde{V}_k$ then there is $\tilde{V}_{k+1} \in \mathcal{O} \setminus \{V_0, \tilde{V}_1, \dots, \tilde{V}_k\}$ such that $\tilde{V}_k \cap \tilde{V}_{k+1} \neq \emptyset$. Let $\tilde{U}_{k+1} \subset (\tilde{V}_{k+1} \cap \tilde{V}_k) \setminus \overline{U_0 \cup \tilde{U}_1 \cup \cdots \cup \tilde{U}_k \cup U_1}$.

Since $\mathcal{O}$ is finite, eventually we find $K \in \mathbb{N} \setminus \{0\}$ such that $U_1 \subset \tilde{V}_K$.

Let $U_{\frac{k}{K}} := \tilde{U}_k$.

Each $U_{\frac{k}{K}}$ has a set of matrix units coming from distinguished sets of matrix units given on V_{k-1} and V_k, which were set above. Pick $x_k \in U_{\frac{k}{K}}$ and for $j \in \{k-1, k\}$, let

$$\tilde{\sigma}_{k,j} : p(\mathcal{C}(X) \otimes M_r)p \to M_r$$

be the map which sends the matrix units $e_{m,n}^{(j)}$ on V_j given by evaluation at x_k to the respective element $e_{m,n} \in M_R$.

Now define

$$\sigma_0 = \tilde{\sigma}_{0,0},$$

and recursively for $1 \leq k \leq K$

$$\sigma_{\frac{k}{K}} = (\sigma_{k-1} \circ \tilde{\sigma}_{k-1,k-1}^{-1}) \circ \tilde{\sigma}_{k,k-1}.$$

Notice that the composition of morphisms in the brackets is just an isomorphism of M_R. Thus $\sigma_{\frac{k}{K}}$ is a *-homomorphism from $p(\mathcal{C}(X) \otimes M_r)p \to M_R$. We also observe that for each $0 \leq k < K$ there is an isomorphism of $\omega \in M_r$ such that $\sigma_{\frac{k}{K}} = \omega \circ \tilde{\sigma}_{k,k}$ and $\sigma_{\frac{k+1}{K}} = \omega \circ \tilde{\sigma}_{k+1,k}$ (take $\omega = \sigma_{\frac{k}{K}} \circ (\tilde{\sigma}_{k,k})^{-1}$), and since $\tilde{\sigma}_{k,k+1}$ and $\tilde{\sigma}_{k+1,k+1}$ define the same matrix units on V_{k+1}, we get that adjacent sets $U_{\frac{k}{K}}, U_{\frac{k+1}{K}}$ are given the same matrix units, thus (i) is satisfied. We have (ii) by construction and (iii) follows from (i) together with the fact that $U_{\frac{k}{K}} \cup U_{\frac{k+1}{K}} \subset V_{k+1}$, which was chosen to be small with respect to $\mathcal{F}$ and its matrix units. ∎

We also have the following straightforward generalization.

3.2.10 Lemma: *Let X be a compact connected metrizable space with covering dimension at most $L < \infty$, let $r \in \mathbb{N}$ and let $p \in \mathcal{C}(X) \otimes M_r$ be a projection. Given $\eta > 0$, $\mathcal{F} \subset p(\mathcal{C}(X) \otimes M_r)p$ a finite subset and points $x_0, \dots, x_N \in X$, let $U_0 \subset V_0$, …, $U_N \subset V_N$ be open subsets as given by Lemma*

3.2.8 with respect to $\eta > 0$, $\mathcal{F}$ and $x_0, \dots, x_N$, coloured with $L+1$ colours so that any two sets with the same colour are disjoint. Then there are $K_1 \in \mathbb{N}$ and $K = K_1 N$, $x_{\frac{k}{K}} \in X$, $k = 0, \dots, K$, open neighbourhoods $U_{\frac{k}{K}}$ of $x_{\frac{k}{K}}$, each equipped with local matrix units $e_{m,n}^{(\frac{k}{K})}$ satisfying

(i) *$U_{\frac{nK_1}{K}} = U_n$ and $V_{\frac{nK_1}{K}} = V_n$ for $n = 0, \dots, N$,*

(ii) *$e_{m,n}^{(\frac{k}{K})}(x) = e_{m,n}^{(\frac{k+1}{K})}(x)$ for all $x \in U_{\frac{k}{K}} \cup U_{\frac{k+1}{K}}$,*

(iii) *If $U_{\frac{nK_1}{K}}$ and $U_{\frac{n'K_1}{K}}$ have the same colour then $U_{\frac{k}{K}} \cap U_{\frac{k'}{K}} = \emptyset$ for every $k \neq k' \in \{nK_1, \dots, (n+1)K_1 - 1\} \cup \{n'K_1, \dots, (n'+1)K_1 - 1\}$,*

(iv) *$\|f(x)_{m,n} \cdot e_{m,n}^{(\frac{k}{K})} - f(y)_{m,n} \cdot e_{m,n}^{(\frac{k+1}{K})}\| < \eta$ for every $x, y \in U_{\frac{k}{K}} \cup U_{\frac{k+1}{K}}$, every $f \in \mathcal{F}$, every $k = 0, \dots, K$.*

3.3. Sketch of proof with two extreme tracial states

In the Section 3.5, we prove the main theorem. The proof is quite lengthy in the general case and the relatively straightforward idea of chopping up the underlying space in a tracial way and exploiting the many orthogonal layers in the UHF algebra $\mathcal{Q}$ may be lost in the details. Thus to illuminate the main techniques behind the proof of Theorem 3.5.22 (and Theorem 4.3.15), we first sketch a proof of the special case that A is a simple unital locally connected trivial homogeneous C*-algebra with bounded dimension, that is, the approximating C*-subalgebras can be chosen to be of the form $\mathcal{C}(X) \otimes M_r$ with X a compact connected metric space, $\dim(X) \leq L < \infty$, together with the additional assumption that $T(A)$ has two extreme points.

Since the purpose of this next theorem is to illuminate the main idea behind the techniques used and as such we include only a sketch of the proof and make no attempts to pin down the estimates involved.

3.3.11 Theorem: *Let A be a simple unital locally connected homogeneous* C*-*algebra with bounded dimension and suppose $T(A)$ has two extreme points. Then $A \otimes \mathcal{Q}$ is TAI.*

Proof: [Sketch.] Let $L \in \mathbb{N}$ be such that a homogeneous C*-algebras $\mathcal{C}(X) \otimes M_r$ with $\dim(X) \leq L$ can be used to approximate A.

By Lemmas 2.3.3 and 2.3.4, it is enough to show that, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there is a projection $p \in A \otimes \mathcal{Q}$ and a unital C*-subalgebra $C \subset p(A \otimes \mathcal{Q})p$ with $1_C = p$ and $C \in I$ such that

(i) $\|p(a \otimes 1_{\mathcal{Q}}) - (a \otimes 1_{\mathcal{Q}})p\| < \epsilon$ for all $a \in \mathcal{F}$,

(ii) $\mathrm{dist}(p(a \otimes 1_{\mathcal{Q}})p, C) < \epsilon$ for all $a \in \mathcal{F}$,

(iii) $\tau(p) > 1/2(L+1)$ for all $\tau \in T(A \otimes \mathcal{Q})$.

Since A is locally connected homogeneous with bounded dimension, we may further assume, by taking a sufficiently good approximation, that $\mathcal{F} \subset \mathcal{C}(X) \otimes M_r \subset A$ where X is a compact connected metric space with $\dim(X) \leq L$.

To begin we divide up the underlying set X with respect to the finite set $\mathcal{F}$ and the two extreme tracial states of $T(A \otimes \mathcal{Q})$, which we label τ_0 and τ_1.

For every $x \in X$, we apply Lemma 3.2.8 to find an open set $U_x \subset X$ containing x that is small enough so that any $f \in \mathcal{F}$ restricted to U_x looks like the constant $r \times r$ matrix $f(x)$. (Note in this case, since our approximating algebra is a trivial homogeneous C*-algebra, there is no difficulty setting the appropriate matrix units; in fact we get Lemma 3.2.8 (iv) for free.)

The sets U_x cover X. Since X is compact and has covering dimension at most L, we can find an open cover $\mathcal{O} = \mathcal{O}^{(0)} \sqcup \dots \sqcup \mathcal{O}^{(L)}$ of finitely many of the sets U_x, coloured so that if $U, U' \in \mathcal{O}^{(l)}$ then $U \cap U' = \emptyset$.

Now we use the UHF algebra to separate the sets according to their colour. There are pairwise orthogonal projections $c_0, \dots, c_L \in \mathcal{Q}$ satisfying $\tau_{\mathcal{Q}}(c_l) = 1/(L+1)$ for all $0 \le l \le L$. For $U \in \mathcal{O}$, let f_U denote the function in the partition of unity supported on U subordinate to the cover $\mathcal{O}$. If U has colour l then we write $n(U) = l$. Consider the elements $f_U \otimes 1_{M_r} \otimes c_{n(U)}$, $U \in \mathcal{O}$. Setting $\eta > 0$ to be sufficiently close to zero, exactly one of the following holds for each such element:

(iv) $\tau_0(f_U \otimes 1_{M_r} \otimes c_{n(U)}) - \tau_1(f_U \otimes 1_{M_r} \otimes c_{n(U)}) \ge \eta$,

(v) $\tau_1(f_U \otimes 1_{M_r} \otimes c_{n(U)}) - \tau_0(f_U \otimes 1_{M_r} \otimes c_{n(U)}) \ge \eta$,

(vi) $|\tau_1(f_U \otimes 1_{M_r} \otimes c_{n(U)}) - \tau_0(f_U \otimes 1_{M_r} \otimes c_{n(U)})| < \eta$.

Thus we can write $\mathcal{O} = \mathcal{O}_0 \sqcup \mathcal{O}_1 \sqcup \mathcal{O}_2$ where $U \in \mathcal{O}_0$ (respectively $\mathcal{O}_1, \mathcal{O}_2$) if it is satisfies (iv) (respectively (v), (vi)). Sets in $\mathcal{O}_2$ will be cut out with projections. The sets in $\mathcal{O}_0$ will be paired with sets in $\mathcal{O}_1$, and these will be matched to opposite ends of an interval.

For $U \in \mathcal{O}_i$, $i \in \{0, 1\}$, we want $\tau_i(f_U \otimes 1_{M_r} \otimes c_{n(U)})$ to be "large" (sufficiently far from zero) while on the opposite tracial state, we want $\tau_{(i+1)\mathrm{mod}2}(f_U \otimes 1_{M_r} \otimes c_{n(U)})$ to be very close to zero. To do this, we subtract a projection in $\mathcal{Q}$, which we will denote p_U, of tracial size approximately equal to, but slightly less than, $\tau_{(i+1)\mathrm{mod}2}(f_U \otimes 1_{M_r} \otimes c_{n(U)} \otimes q_{(0,1)})$, where $q_{(0,1)} \in \mathcal{Q}$ is a projection as defined in 1.1.4. That is, for a suitably chosen $\delta > 0$ we find $p_U \in \mathcal{Q}$ satisfying

$$0 < \tau_{(i+1)\mathrm{mod}2}(f_U \otimes 1_{M_r} \otimes c_{n(U)} \otimes q_{(0,1)} - 1_{\mathcal{C}(X)} \otimes 1_{M_r} \otimes p_U) < \delta.$$

To set up the intervals, we require that $\mathcal{O}_0$ and $\mathcal{O}_1$ are symmetric with respect to the two extreme tracial states τ_0 and τ_1. Subdivide each $U \in \mathcal{O}_0 \sqcup \mathcal{O}_1$ by finding $m_U + 1$ pairwise orthogonal projections

$$q_{U,0}, \dots, q_{U,m_U} \in \mathcal{Q},$$

so that

$$\tau_i((f_U \otimes 1_{M_r} \otimes c_{n(U)} \otimes q_{(0,1)} - 1_{\mathcal{C}(X)} \otimes 1_{M_r} \otimes p_U) \otimes q_{U,m})$$

is approximately the same size for every $m \in \{0, \dots, m_U\}$ and every $U \in \mathcal{O}_0 \sqcup \mathcal{O}_1$.

For $U \in \mathcal{O}_i$, $i \in \{0, 1\}$, consider

$$(f_U \otimes 1_{M_r} \otimes c_{n(U)} \otimes q_{(0,1)} - 1_{\mathcal{C}(X)} \otimes 1_{M_r} \otimes p_U) \otimes q_{U,m}$$

For further symmetry, without loss of generalization, we may assume that there are an equal number, say M+1, of such elements when $i = 0$ and $i = 1$, and we label them as $a_0^{(m)}$ and $a_1^{(m)}$ where $m \in \{0, \dots, M\}$. Let the set U supporting $a_i^{(m)}$ be labelled $U_i^{(m)}$. Note that the sets $U_i^{(m)}$ are not necessarily distinct, however any duplicates will occur in disjoint layers of the UHF algebra. These will be the endpoints of a "discrete" version of an interval.

To fill in the discrete intervals, for each m, we use Lemma 3.2.9 to find "stepping stone" sets between the end points (again in this case setting up the matrix units becomes trivial): For each m, we have $K \in \mathbb{N}$ (which we may assume is the same for each m) and disjoint sets $U_0^{(m)}, U_{\frac{1}{K}}^{(m)}, \dots, U_{\frac{K-1}{K}}^{(m)}, U_1^{(m)}$ such that $\mathcal{F}$ doesn't vary much across adjacent sets and such that $U_0^{(m)}, U_{\frac{1}{K}}^{(m)}, \dots, U_{\frac{K-1}{K}}^{(m)}$ are pairwise disjoint.

We will now find an interval that reflects this tracial set up, copy it $M + 1$ times in layers of the UHF algebra, and use strict comparison to move it under these discrete models, see Figure 3.1.

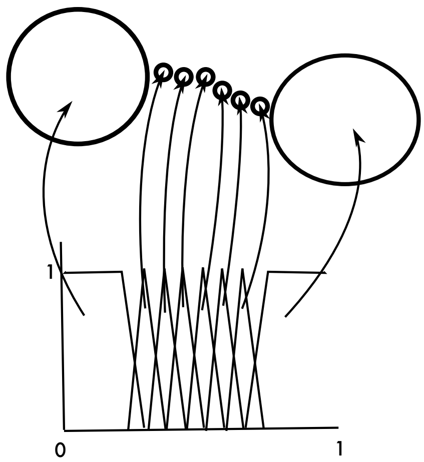
Let $\mathrm{Aff}_{\mathrm{b}}(T(A \otimes \mathcal{Q}))$ denote the set of $\mathbb{R}$-valued bounded affine functions on the tracial state space $T(A \otimes \mathcal{Q})$. Define a continuous function by

$$h(\tau_1 \otimes \tau_{\mathcal{Q}}) = 1,$$

and

$$0 < h(\tau_0 \otimes \tau_{\mathcal{Q}}) \approx 0.$$

Figure 3.1.: Embedding the interval.



Note that h is also strictly positive. Since A is simple and unital, by [7, Corollary 3.10], there is a positive contraction $b \in A_+$ satisfying

$$\tau(b) = h(\tau_i) \text{ for } i \in \{0,1\}.$$

We get a *-homomorphism from an interval algebra by taking the composition

$$\tilde{\phi} : \mathcal{C}([0,1]) \to \mathrm{C}^*(b,1) \hookrightarrow A \otimes \mathcal{Q},$$

Define, for $0 < \alpha < \beta < 1/2$, a function $g_{\alpha,\beta} \in \mathcal{C}([0,1])$ by

$$g_{\alpha,\beta}(t) = \begin{cases} 1, & 0 \le t \le \alpha, \\ \text{linear}, & \alpha \le t < \beta, \\ 0, & \beta \le t \le 1. \end{cases}$$

Put $\gamma_0 := g_{\alpha,2\alpha}$ and $\gamma_D := 1 - g_{1-2\alpha,1-\alpha}$, where α is suitably chosen so that $\tau_1(\tilde{\phi}(\gamma_0)) = \tau_2(\tilde{\phi}(\gamma_D)) \approx 1$. We then fill out to a partition of unity $\gamma_{\frac{1}{K}}, \ldots, \gamma_{\frac{k}{K-1}}$ (as given in 1.1.1). Define

$$a^{(m)}_{\frac{k}{K}} = f_{U^{(m)}_{\frac{k}{K}}} \otimes 1_{M_r} \otimes c_{n(U^{(m)}_0)} \otimes q_{(0,1)},$$

for $k \in \{1, \ldots, K-1\}$, where $q_{(0,1)}$ is a projection as in 1.1.4. Note that the the $a^{(m)}_{\frac{k}{K}}$, $k \in \{0, \ldots K\}$ are pairwise orthogonal. We also have, by having chosen a suitably small α, that the $\tilde{\phi}(\gamma_{\frac{k}{K}})$, $k \in \{1, \ldots, K-1\}$ are very small in trace, in particular we can arrange that they are smaller than the $a^{(m)}_{\frac{k}{K}}$ on every tracial state.

Let $d_0, \ldots, d_M \in \mathcal{Q}$ be pairwise orthogonal projections chosen so that

$$\tau_i(\tilde{\phi}(\gamma_i)) \otimes d_m \approx \tau_i(a^{(m)}_i),$$

for $i \in \{0,1\}$, and define

$$\phi^{(m)} : \mathcal{C}([0,1]) \to A \otimes \mathcal{Q}$$

by $\phi^{(m)}(f) = \tilde{\phi}(f) \otimes d_m$. Notice that the $\phi^{(m)}$ have orthogonal images.

For each $m \in \{0, \ldots, M\}$, apply Lemma 3.2.5 with respect to the pairwise orthogonal positive contractions $a^{(m)}_0, a^{(m)}_{\frac{1}{K}}, \ldots, a^{(m)}_{\frac{K-1}{K}}, a^{(m)}_1$, the *-homomorphisms $\phi^{(m)} : \mathcal{C}([0,1]) \to A \otimes \mathcal{Q}$, and a suitably small $\eta > 0$ to get $s^{(m)}_{\frac{k}{K}} \in A \otimes \mathcal{Q}$ with ${s^{(m)}_{\frac{k}{K}}}^* s^{(m)}_{\frac{k}{K}} \in \mathrm{Her}(a^{(m)}_{\frac{k}{K}})$ and $s^{(m)}_{\frac{k}{K}} {s^{(m)}_{\frac{k}{K}}}^* \approx \phi^{(m)}(\gamma_{\frac{k}{K}})$. Then applying Lemma 3.2.7 we get *-homomorphisms

$$\psi^{(m)} : \mathcal{C}([0,1]) \otimes M_r \to A \otimes \mathcal{Q}$$

and we can write

$$s_m = \sum_{n,k} \psi^{(m)}(1_{\mathcal{C}([0,1])} \otimes e_{n,1}) s^{(m)}_{\frac{k}{K}} (1_{\mathcal{C}(X)} \otimes e_{1,n}),$$

with $\sum_{m=0}^{M} s_m {s_m}^* \approx \oplus_{m=0}^{M} \psi^{(m)}(1_{\mathcal{C}([0,1])} \otimes 1_{M_r})$ and

$$\tau(\sum_{m=0}^{M} {s_m}^* s_m) \approx \tau(\sum_{U \in \mathcal{O}_0 \sqcup \mathcal{O}_1} f_U \otimes 1_{M_r} \otimes c_{n(U)} \otimes q_{(0,1)} - 1_{\mathcal{C}(X)} \otimes 1_{M_r} \otimes p_U),$$

for every $\tau \in T(A \otimes \mathcal{Q})$.

We are missing trace: that which comes from the sets $U \in \mathcal{O}_2$ and that which was lost when we subtracted the projections p_U, $U \in \mathcal{O}_0 \sqcup \mathcal{O}_1$.

Note that for $U \in \mathcal{O}_0 \sqcup \mathcal{O}_1$ by the choice of the projections p_U and strict comparison, we get that

$$\begin{aligned} 1_{\mathcal{C}(X)} \otimes 1_{M_r} \otimes p_U &\precsim f_U \otimes 1_{M_r} \otimes c_{n(U)} \otimes q_{(0,1)} \\ &\sim f_U \otimes 1_{M_r} \otimes \otimes c_{n(U)} \otimes q_{(1,1)} \end{aligned}$$

Thus we find $u_U \in A \otimes \mathcal{Q}$ with

$$u_U u_U^* = 1_{\mathcal{C}(X)} \otimes 1_{M_r} \otimes p_U$$

and

$$u_U^* u_U \in \mathrm{Her}(f_U \otimes 1_{M_r} \otimes c_{n(U)} \otimes q_{(1,1)}).$$

For the sets $U \in \mathcal{O}_2$ we use Lemma 3.2.4 to find partial isometries $u_U \in A \otimes \mathcal{Q}$ where $u_U u_U^* =: p_U$ are pairwise orthogonal projections and

$${u_U}^* u_U \in \mathrm{Her}(f_U \otimes 1_{M_r} \otimes c_{n(U)}),$$

and

$$\tau(\sum_{U \in \mathcal{O}_2} {u_U}^* u_U) \approx \tau(\sum_{U \in \mathcal{O}_2} f_U \otimes c_{n(U)}).$$

Since $\sum_{U \in \mathcal{O}} \tau(f_U \otimes 1_{M_R} \otimes c_{n(U)}) < 1$ we may choose the p_U, $U \in \mathcal{O}_2$ to be orthogonal to p_U, $U \in \mathcal{O}_1 \sqcup \mathcal{O}_2$.

Let

$$\tilde{s} = \sum_{m=0}^{M} s_m + \sum_{U \in \mathcal{O}} u_U.$$

Let F be the finite-dimensional C*-algebra generated by the projections p_U. One easily checks that

$$\left\| \tilde{s}\tilde{s}^* - (\sum_{m=0}^{M} \psi^{(m)}(1_{\mathcal{C}([0,1]) \otimes M_r})) + (1_{\mathcal{C}(X) \otimes M_R}) \otimes (\sum_{U \in \mathcal{O}} p_U) \right\|$$

can be made suitably small and hence be perturbed into an honest partial isometry, s.

We now are left to verify (i), (ii), (iii) with the projection $s^* s$ and the interval algebra given by $s^*(\oplus_{m=1}^{M} \psi^{(m)}(\mathcal{C}([0,1] \otimes M_r)) \oplus 1_{\mathcal{C}(X)} \otimes M_r \otimes F) s$, where F is the finite-dimensional C*-algebra generated by the p_U.

For (i) and (ii), we first approximate $f \in \mathcal{F}$ by taking $g \in \mathcal{C}(X) \otimes M_r \subset A \otimes \mathcal{Q}$ such that, for $U \in \mathcal{O}$ we have $g|_U = f(x_U)$ are constant $r \times r$ matrices given by evaluation at some $x_U \in U$, and writing

$$h = \sum_{m,k} \psi^{(m)}(\gamma_{\frac{k}{K}} \otimes g|_{U^{(m)}_{\frac{k}{K}}}) + \sum_{U \in \mathcal{O}} 1_{C(X)} \otimes g|_U \otimes p_U.$$

We leave the detailed calculations to the general case, which is similar.

For (iii), we use the fact that s^*s is close to $\tilde{s}^*\tilde{s}$ and provided all estimates have been carefully chosen,

$$\begin{aligned}
\tau(s^*s) \approx \tau(\tilde{s}^*\tilde{s}) &= \tau(\tilde{s}\tilde{s}^*) \\
&\approx \tau(\textstyle\sum_{m=0}^{M} a_0^m + a_1^m) + \tau(\textstyle\sum_{U \in \mathcal{O}_0 \sqcup \mathcal{O}_1} 1_{\mathcal{C}(X)\otimes M_r} \otimes p_U) \\
&\quad + \tau(\sum_{U \in \mathcal{O}_2} f_U \otimes 1_{M_r} \otimes c_{n(U)}) \\
&\geq \tau(\sum_{U \in \mathcal{O}} f_U \otimes 1_{M_r} \otimes c_{n(U)})/2 \\
&\approx 1/2(L+1) \\
&> 1/4(L+1)
\end{aligned}$$

for all $\tau \in T(A \otimes \mathcal{Q})$. ∎

3.4. Tracially large intervals from AI algebras

We now turn to proving the more general case: If A is a separable simple unital tracially approximately semihomogeneous C*-algebra of bounded dimension then A is TAI after tensoring with $\mathcal{Q}$. The main idea is the same, but there are significantly more technicalities involved. Unlike in the case of Theorem 3.3.11 we cannot produce an interval quite so easily when there are more than two extreme tracial states.

The idea behind the proof is to work within one of the approximating semihomogeneous C*-algebras which is chosen to be large enough to fill up most of A. We then want to look at the base spaces of this semihomogeneous C*-subalgebra and chop them up into sets of large trace which are narrow enough that the elements in the finite subset will not vary too much when restricted to any of these sets, as we did in Theorem 3.3.11. Once again, ideally, we want these sets to be disjoint. If we can find an interval with the right tracial distribution we will identify these sets with (tracially) corresponding disjoint bump functions on $[0, 1]$, as in the special case. Then we would be able to interpolate between the disjoint sets in our base spaces with narrow sets across which our given finite subset doesn't vary too much.

If we have exhausted enough of the trace in our interval $[0, 1]$ then we can fill out the bump functions to a partition of unity. If the trace has been correctly taken into account, we now just need to move these functions in the partition of unity under the sets in the base spaces using strict comparison. This moves the whole interval into the semihomogeneous algebra and now it is simply a matter of approximating our finite subset $\mathcal{F}$ by functions over the base space which are constant matrices across our sets, and then by these constant matrices tensored by our bump functions. (Getting the correct matrix size is not a problem because of $\mathcal{Q}$.)

Some force is needed to achieve the ideal situation set up above. Unlike in Theorem 3.3.11, since we do not want to make any restrictions on the tracial state space, we cannot isolate sets of large trace and easily find a single element to produce the appropriate interval reflecting the tracial set up. Once again, we take a cover of the base spaces and separate the sets in layers of the UHF algebra. This cuts the total trace by the dimension of the space. We find an interval which will give us bump functions of the appropriate trace by first embedding an AI algebra with the correct tracial state space, and then drop down to an interval algebra in its inductive limit. Though things become a bit more technical, this means we no longer have to add in the correcting projections p_U as were required in Theorem 3.3.11. This puts us in essentially the situation above and from here we are able to move the interval into $A \otimes \mathcal{Q}$ as we desire.

To prove that $A \otimes \mathcal{Q}$ is TAI we require a model interval algebra which will be used to tracially approximate $A \otimes \mathcal{Q}$. Certainly there are any number of intervals one can map into A (for example, by taking the C*-subalgebra generated by a positive contraction and the unit), however we require that the interval algebra be large with respect to all tracial states of A and lie in a particular position. The trick then is to find an approximating interval algebra which respects the traces of A. By the range results for AI algebras due to Thomsen [61, 62] and Villadsen in [68], any metrizable Choquet simplex can be realized as the tracial state space of an AI algebra. We will thus take our interval models to be AI algebras with the correct tracial state space and K_0-group. Using results about lifting maps from the Cuntz semigroup of such C*-algebras as in [9, 51, 10, 50], we are able to embed these AI algebras into $A \otimes \mathcal{Q}$ in a trace-preserving way and then use strict comparison to move them from a general to a particular position.

To begin the section, we show that a simple separable unital tracially approximately semihomogeneous C*-algbera has the correct invariant to be TAI after tensoring with $\mathcal{Q}$. This will also help us find the correct AI algebra to act as the interval model.

3.4.12 Proposition: *Let A be a simple separable unital tracially approximately semihomogeneous C*-algebra. Then $K_0(A)$ is rationally Riesz (cf. [40]).*

Proof: By [40, Proposition 5.7] it is enough to show that $K_0(A \otimes \mathcal{Q})$ has Riesz interpolation.

Let $x_1, x_2, y_1, y_2 \in K_0(A \otimes \mathcal{Q})$ with $x_i \leq y_j$ for $i, j \in \{1, 2\}$.

If $x_i = y_j$ for some $i, j \in \{1, 2\}$ then the result clearly holds. So assume that we have $x_i < y_j$, for $i, j \in \{1, 2\}$.

Without loss of generality, we may assume that $x_i, y_j \in K_0(A \otimes \mathcal{Q})_+$. Furthermore, semihomogeneous C*-algebras are stably finite, so A is also stably finite [19, Theorem 4.1]. It follows that $A \otimes \mathcal{Q}$ has stable rank one [52, Corollary 6.6], hence cancellation of projections. Thus it is enough to assume $x_i = [q_i]$ and $y_j = [r_j]$ for projections $q_i, r_j \in A \otimes \mathcal{Q}$.

Let $\delta < 6 \cdot \min\{\{|\tau(r_j - q_i)| \mid i, j \in \{1, 2\}, \tau \in T(A \otimes \mathcal{Q})\}, 1/8\}$.

Then for $\mathcal{F} = \{q_1, q_2, r_1, r_2\}$, $0 < \epsilon < \delta$ and $c \in (A \otimes \mathcal{Q})_+$ with $\tau(c) < \epsilon$ for all $\tau \in T(A \otimes \mathcal{Q})$, there is a unital semihomogeneous C*-subalgebra $B \subset 1_B(A \otimes \mathcal{Q})1_B$ of the form

$$B = \textstyle\bigoplus_{n=1}^{N} p_n(\mathcal{C}(X_n) \otimes M_{r_n})p_n$$

with

(i) $\|1_B a - a 1_B\| < \epsilon$ for all $a \in \mathcal{F}$,

(ii) $\mathrm{dist}(1_B a 1_B, B) < \epsilon$ for all $a \in \mathcal{F}$,

(iii) $1_A - 1_B$ is Murray–von Neumann equivalent to a projection in $\overline{c(A \otimes \mathcal{Q})c}$.

Note that (iii) implies that $\tau(1_A - 1_B) < \epsilon$.

Now, $\|(1_B a 1_B)^2 - 1_B a 1_B\| < \epsilon$ for all $a \in \mathcal{F}$ so there are projections $b_1, b_2, c_1, c_2 \in 1_B(A \otimes \mathcal{Q})1_B$ such that

$$\|b_i - 1_B q_i 1_B\|, \|c_j - 1_B r_j 1_B\| < 2\epsilon.$$

Furthermore, by (ii), there are $d_1, d_2, e_1, e_2 \in B$, which we may assume to be positive and self-adjoint, satisfying

$$\|d_i - b_i\|, \|e_j - c_j\| < 3\epsilon.$$

Thus we can find projections $s_1, s_2, t_1, t_2 \in B$ with

$$\|b_i - s_i\|, \|c_j - t_j\| < 6\epsilon,$$

hence

$$\|s_i - 1_B q_i 1_B\|, \|t_j - 1_B r_j 1_B\| < 8\epsilon.$$

By choice of ϵ, we have that s_i (respectively t_j) is Murray–von Neumann equivalent to $1_B q_i 1_B$ (respectively $1_B r_i 1_B$) [31, Lemma 2.5.4]. Hence

$$\tau(s_i) = \tau(1_B q_i 1_B) \text{ and } \tau(t_j) = \tau(1_B r_j 1_B)$$

for all $\tau \in T(A \otimes \mathcal{Q})$. By the choice of B we have

$$\begin{aligned}\|a - (1_B a 1_B + (1_A - 1_B)a(1_A - 1_B))\| &= \|a1_B - 1_B a 1_B + 1_B a - 1_B a 1_B\| \\ &< 2\epsilon\end{aligned}$$

for all $a \in \mathcal{F}$. It follows that

$$|\tau(a - (1_B a 1_B + (1_A - 1_B)a(1_A - 1_B)))| < 2\epsilon$$

for all $\tau \in T(A \otimes \mathcal{Q})$, all $a \in \mathcal{F}$. Since $\tau((1_A - 1_B)a(1_A - 1_B)) < \tau(c) < \epsilon$, we have

$$\tau(1_B a 1_B) \in [\tau(a) - 3\epsilon, \tau(a) + 3\epsilon]$$

for all $\tau \in T(A \otimes \mathcal{Q})$, all $a \in \mathcal{F}$. By choice of ϵ we get

$$\tau(q_i) \leq \tau(1_B q_i 1_B) + 3\epsilon < \tau(1_B r_j 1_B) - 3\epsilon \leq \tau(r_j)$$

for all $\tau \in T(A \otimes \mathcal{Q})$, $i, j \in \{1, 2\}$. Since $\tau(1_B q_i 1_B) = \tau(s_i)$ and $\tau(1_B r_j 1_B) = \tau(t_j)$, we also have

$$\tau(q_i) \leq \tau(s_i) + 3\epsilon < \tau(t_j) - 3\epsilon \leq \tau(r_j).$$

By Corollary 3.1.2 we may assume that the spaces X_n are connected, and so the maps given by $\tau|_{p_n(\mathcal{C}(X_n)\otimes M_n)p_n} \mapsto \tau(s_i|_{X_n})$ and $\tau|_{p_n(\mathcal{C}(X_n)\otimes M_n)p_n} \mapsto \tau(t_j|_{X_n})$ are constant. Since $K_0(\mathcal{Q}) = \mathbb{Q}$ we can find projections $a_n \in \mathcal{Q}$ satisfying

$$\tau(s_i|_{X_n}) + 3\epsilon/N < \tau(p_n \otimes a_n) < \tau(t_j|_{X_n}) - 3\epsilon/N$$

for all $\tau \in T(A \otimes \mathcal{Q})$, $n = 1, \ldots N$, $i, j \in \{1, 2\}$. Hence

$$\tau(q_i) \leq \tau(s_i) + 3\epsilon < \tau(\textstyle\sum_{n=1}^N p_n \otimes a_n) < \tau(t_j) - 3\epsilon \leq \tau(r_j)$$

for all $\tau \in T(A \otimes \mathcal{Q})$, $n = 1, \ldots N$, $i, j \in \{1, 2\}$. By strict comparison it follows that

$$x_i = [q_i] < [\textstyle\sum_{n=1}^N p_n \otimes a_n] < [r_j] = y_j$$

for $i, j \in \{1, 2\}$. ∎

3.4.13 Proposition: *Let A be a simple separable unital tracially approximately semihomogeneous* C*-*algebra. Then the pairing map*

$$r_A : T(A \otimes \mathcal{Q}) \to S(K_0(A \otimes \mathcal{Q}))$$

preserve extreme points.

Proof: By the results in [68], the pairing map of AH algebras is extreme-point preserving and [19, Theorem 4.19] shows that the property of having extreme-point preserving pairing maps passes from a class $\mathcal{S}$ of separable unital C*-algebras to the class of TA$\mathcal{S}$ C*-algebras. ∎

3.4.14 Corollary: *Let A be a simple separable unital tracially approximately semihomogeneous* C*-*algebra. There is a simple unital AI algebra B which is an inductive limit of direct sums of* C*-*algebras of the form $\mathcal{C}([0,1]) \otimes M_r$ with*

$$(K_0(B), T(B), r_B) \cong (K_0(A \otimes \mathcal{Q}), T(A \otimes \mathcal{Q}), r_{A \otimes \mathcal{Q}}).$$

Proof: By the Künneth Theorem for tensor products we have that

$$K_0(A \otimes \mathcal{Q}) \cong K_0(A) \otimes \mathbb{Q},$$

hence $K_0(A\otimes\mathcal{Q})$ is simple and torsion free. By the previous two propositions, we know that $K_0(A\otimes\mathcal{Q})$ is a Riesz group and also that the pairing map $r_{A\otimes\mathcal{Q}} : T(A\otimes\mathcal{Q}) \to S(K_0(A\otimes\mathcal{Q}))$ preserves extreme points. Thus it follows from the range results for AI algebras in [61, 62, 68] (see [68, Theorem 3.2] for an explicit statement of the range) that there exists a simple unital AI algebra B with the invariant

$$(K_0(B), T(B), r_B) \cong (K_0(A\otimes\mathcal{Q}), T(A\otimes\mathcal{Q}), r_{A\otimes\mathcal{Q}}).$$

∎

Below, we denote by $V(A)$ the semigroup of Murray–von Neumann equivalence classes of projections in A and by $\mathrm{LAff}(T(A))^{++}$ the semigroup of strictly positive (not necessarily bounded) lower semicontinuous affine functions on the tracial simplex $T(A)$.

3.4.15 Proposition: *Let A and B be separable simple unital nuclear* C*-*algebras such that $T(A)\neq\emptyset$, and B has stable rank one and is $\mathcal{Z}$-stable. Suppose that $(K_0(B), T(B), r_B) \cong (K_0(A\otimes\mathcal{Q}), T(A), r_{A\otimes\mathcal{Q}})$, that is, there is $\rho : K_0(B) \to K_0(A\otimes\mathcal{Q})$ is an order-unit preserving isomorphism, $\gamma : T(A\otimes\mathcal{Q}) \cong T(A) \to T(B)$ an affine homeomorphism, and that*

$$\begin{tikzcd} T(A\otimes\mathcal{Q}) \arrow[r, "\gamma"] \arrow[d, "r_{A\otimes\mathcal{Q}}"] & T(B) \arrow[d, "r_B"'] \\ S(K_0(A\otimes\mathcal{Q})) \arrow[r, "\cdot\circ\rho"] & S(K_0(B)) \end{tikzcd}$$

commutes. Suppose further that **Cu** *classifies homomorphisms from B (as in 1.2.10). Then there exists a *-homomorphism $\Psi : B \hookrightarrow A\otimes\mathcal{Q}$ and, in the category* **Cu**, *a morphism $\phi : \mathrm{Cu}(A\otimes\mathcal{Q}) \to \mathrm{Cu}(B)$ such that $(\gamma(\tau_A))(b) = (\tau_A\otimes\tau_{\mathcal{Q}})(\Psi(b))$ for all $b\in B$ and $\phi\circ\mathrm{Cu}(\Psi) = \mathrm{id}_{\mathrm{Cu}(B)}$.*

Proof: Since $T(A)\neq\emptyset$, A is stably finite thus $A\otimes\mathcal{Q}$ has stable rank one [52, Corollary 6.6] hence cancellation of projections [2, Proposition 6.5.1]. The C*-algebra B has stable rank one by assumption, hence cancellation of projections, and, also by assumption, is $\mathcal{Z}$-stable. Furthermore, $A\otimes\mathcal{Q}$ is $\mathcal{Z}$-stable (since $\mathcal{Q}$ is $\mathcal{Z}$-stable [26, Corollary 6.3]), thus by [8, Theorem 2.5] there are isomorphisms in **Cu** given by

$$\nu_A : \mathrm{Cu}(A\otimes\mathcal{Q}) \to V(A\otimes\mathcal{Q}) \sqcup \mathrm{LAff}(T(A\otimes\mathcal{Q}))^{++}$$

and similarly

$$\nu_B : \mathrm{Cu}(B) \to V(B) \sqcup \mathrm{LAff}(T(B))^{++},$$

where $\nu_A|_{V(A\otimes\mathcal{Q}\otimes\mathcal{K})} = \mathrm{id}_{V(A\otimes\mathcal{Q}\otimes\mathcal{K})}$, $\nu_B|_{V(B\otimes\mathcal{K})} = \mathrm{id}_{V(B\otimes\mathcal{K})}$ on $V(A\otimes\mathcal{Q}) = V(A\otimes\mathcal{Q}\otimes\mathcal{K})$ and $V(B) = V(B\otimes\mathcal{K})$, respectively. For an element $a\in(A\otimes\mathcal{Q}\otimes\mathcal{K})_+$, respectively $b\in(B\otimes\mathcal{K})_+$ not Cuntz equivalent to a projection the maps are given by $\nu_A([a])(\tau) = \sup_m \lim_{n\to\infty}\tau((p_m a p_m)^{1/n})$, (where $\tau\in T(A\otimes\mathcal{Q})$ is extended to $A\otimes\mathcal{Q}\otimes\mathcal{K}$), respectively $\nu_A([b])(\tau) = \sup_m \lim_{n\to\infty}\tau((q_m b q_m)^{1/n})$, ($\tau\in T(B)$ extended to $B\otimes\mathcal{K}$). Here $p_m = 1_{A\otimes\mathcal{Q}}\otimes e_m$ and $q_m = 1_B\otimes e_m$ where $(e_m)_{m\in\mathbb{N}}$ is any increasing sequence of projections in $\mathcal{K}$ having $\mathrm{rank}(e_m) = m$ [8].

Now let us define a map

$$\psi : V(B) \sqcup \mathrm{LAff}(T(B))^{++} \to V(A\otimes\mathcal{Q}) \sqcup \mathrm{LAff}(T(A\otimes\mathcal{Q}))^{++}$$

by

$$\psi([p]) = \rho|_{V(B)}([p])$$

for $p\in V(B)$ and

$$\psi([b])(\tau) = \sup_m \lim_{n\to\infty}(\gamma(\tau))((q_m b q_m)^{1/n})$$

for $b\in(B\otimes\mathcal{K})_+$ not equivalent to a projection and $\tau\in T(A\otimes\mathcal{Q})$.

We also have that

$$\begin{array}{ccc} T(B) & \xrightarrow{\gamma^{-1}} & T(A\otimes\mathcal{Q}) \\ \downarrow{\scriptstyle r_B} & & \downarrow{\scriptstyle r_{A\otimes\mathcal{Q}}} \\ S(K_0(B)) & \xrightarrow{\cdot\circ\rho^{-1}} & S(K_0(A\otimes\mathcal{Q})) \end{array}$$

commutes. Thus as above, we may define a map

$$\phi : V(A\otimes\mathcal{Q}) \sqcup \mathrm{LAff}(T(A\otimes\mathcal{Q}))^{++} \to V(B) \sqcup \mathrm{LAff}(T(B))^{++}$$

by

$$\phi([p]) = \rho^{-1}|_{V(A)}([p])$$

for $p \in V(A)$ and

$$\phi([a])(\tau) = \sup_m \lim_{n\to\infty} ((\gamma^{-1})(\tau))((p_m a p_m)^{1/n})$$

for $a \in (A\otimes\mathcal{Q}\otimes\mathcal{K})$ not equivalent to a projection and $\tau \in T(B)$.

It is straightforward to check that ψ and ϕ are morphisms in the category **Cu**, that is, semigroup maps preserving the zero element, the suprema of countable upward directed sets, and the relation $\ll$ (recall that $x \ll y$ if whenever (y_n) is an increasing sequence with $\sup_n y_n \geq y$ there is n such that $x \leq y_n$) [12, Section 2]; similar calculations can be found in [8].

We have that if $[p] \in V(B)$ then

$$\begin{aligned} \phi\circ\psi([p]) &= \rho^{-1}\circ\rho([p]) \\ &= [p]. \end{aligned}$$

Suppose that $[b] \in \mathrm{LAff}(T(B))^{++}$. Let $[a] \in \mathrm{LAff}(T(A\otimes\mathcal{Q}))^{++}$ be such that $\psi[b] = [a]$. Then for any $\tau_A \in T(A\otimes\mathcal{Q})$ we have

$$\sup_m \lim_{n\to\infty} \tau_A((p_m a p_m)^{1/n}) = [a](\tau_A) = \psi[b](\tau_A) = \sup_m \lim_{n\to\infty} \gamma(\tau_A)((q_m b q_m)^{1/n}).$$

Thus, for all $\tau \in T(B)$ we have

$$\begin{aligned} \phi\circ\psi[b](\tau_B) &= \phi[a](\tau_B) \\ &= \sup_m \lim_{n\to\infty} \gamma^{-1}(\tau_B)((p_m a p_m)^{1/n}) \\ &= \sup_m \lim_{n\to\infty} \gamma(\gamma^{-1}(\tau_B))((q_m b q_m)^{1/n}) \\ &= [b](\tau_b). \end{aligned}$$

Thus $\phi\circ\psi = \mathrm{id}_{\mathrm{Cu}(B)}$. Similarly one shows that $\psi\circ\phi = \mathrm{id}_{\mathrm{Cu}(A\otimes\mathcal{Q})}$.

Now **Cu** classifies homomorphisms from B and $A\otimes\mathcal{Q}$ has stable rank one [52, Corollary 6.6], so we can lift ψ to a *-homomorphism

$$\Psi : B \to A\otimes\mathcal{Q}$$

satisfying $\mathrm{Cu}(\Psi) = \psi$, whence $\phi\circ\mathrm{Cu}(\Psi) = \mathrm{id}_{\mathrm{Cu}(B)}$.

To check the condition on the traces, let $\tau \in T(A)$. Note that $\tau\circ\Psi \in T(B)$ since Ψ must be unital. Thus it is enough to show that $\tau\circ\Psi = \gamma(\tau)$. Since A and B are nuclear, all quasi-traces are traces [24], hence the lower semicontinuous dimension functions are in one-to-one correspondence with tracial states [4, Theorem II.2.2]. Therefore it is enough to check that $\lim_{n\to\infty} \tau\circ\Psi(b^{1/n}) = \lim_{n\to\infty} \gamma(\tau)(b^{1/n})$ for all $b \in B_+$.

Indeed,

$$\begin{aligned}\lim_{n\to\infty} \tau\circ\Psi(b^{1/n}) &= [\Psi(b)](\tau)\\ &= \psi([b])(\tau) = \sup_m \lim_{n\to\infty} \gamma(\tau)((q_m b q_m)^{1/n})\\ &= \lim_{n\to\infty}\gamma(\tau)(b^{1/n}).\end{aligned}$$

∎

3.4.16 LEMMA: *Let $a_0,\dots,a_N$ be pairwise orthogonal positive contractions in a simple separable unital tracially approximately semihomogeneous* C*-*algebra A. Then there are a simple unital AI algebra B, a *-homomorphism $\iota : B\hookrightarrow A\otimes\mathcal{Q}$ and pairwise orthogonal positive contractions $b_0,\dots,b_n$ satisfying*

$$\tau(\iota(b_n)) = \tau(a_n) \textit{ for every } \tau\in T(A\otimes\mathcal{Q}),\ n=0,\dots,N.$$

PROOF: Consider the C*-subalgebra of A generated by 1 and a_n, $0\le n\le N$. It is isomorphic to some quotient of a commutative C*-algebra with spectrum a tree. Let $A^{(0)}$ denote that commutative C*-algebra.

We have a map $\pi : A^{(0)}\to C^*(a_0,\dots,a_N,1)\hookrightarrow A\otimes\mathcal{Q}$ inducing a morphism in the category **Cu** at level of Cuntz semigroups

$$[\pi] : \mathrm{Cu}(A^{(0)})\to \mathrm{Cu}(A\otimes\mathcal{Q}).$$

By Corollary 3.4.14 there is a simple AI algebra B which is an inductive limit of direct sums of algebras of the form $\mathcal{C}([0,1])\otimes M_r$ with $(K_0(B)),T(B),r_B)\cong(K_0(A\otimes\mathcal{Q}),T(A\otimes\mathcal{Q}),r_{A\otimes\mathcal{Q}})$. Thus by Propostion 3.4.15 we get a map

$$\alpha : \mathrm{Cu}(A\otimes\mathcal{Q})\to\mathrm{Cu}(B)$$

such that the following diagram

$$\begin{tikzcd} \mathrm{Cu}(A^{(0)}) \arrow[r, "{[\pi]}"] \arrow[dr, "\alpha\circ[\pi]"'] & \mathrm{Cu}(A\otimes\mathcal{Q}) \arrow[d, "\alpha"] \\ & \mathrm{Cu}(B) \end{tikzcd}$$

commutes. Thus $\alpha\circ[\pi] : \mathrm{Cu}(A)\to\mathrm{Cu}(B)$ is a map sending $[\pi^{-1}(a_i)]$ to the image of $[a_i]$ under α.

Since $A^{(0)}$ is a quotient of an algebra of the form $\mathcal{C}(Y)$ for some tree Y, we have that **Cu** classifies homomorphisms from $A^{(0)}$ by the main results of [10]. Thus the **Cu** map $\alpha\circ[\pi]$ can be lifted to a *-homomorphism,

$$\Phi : A^{(0)}\to B.$$

Let $b_0,\dots,b_N\in B$ denote the images of the elements $\pi^{-1}(a_0),\ \dots,\ \pi^{-1}(a_N)$, respectively. Since $a_0,\dots,a_N$ are pairwise orthogonal, we may assume the same for $\pi^{-1}(a_0),\ \dots,\ \pi^{-1}(a_N)$ whence also $b_0,\dots,b_N$ are pairwise orthogonal. By Proposition 3.4.15 and the fact that **Cu** classifies homomorphisms from AI algebras [9], we also have an injective *-homomorphism lifting α^{-1}

$$\iota : B\hookrightarrow A\otimes\mathcal{Q}\,.$$

Let $\tau\in T(A\otimes\mathcal{Q})$. By the correspondence on traces given by Proposition 3.4.15 we have

$$\tau(\iota(b_n)) = \tau(a_n) \text{ for all } \tau\in T(A\otimes\mathcal{Q}).$$

∎

We now have three simple technical results which will allow us to perturb elements in an AI algebra in such a way that we don't lose too much trace. Given a finite set of positive contractions in an AI algebra B we first show that we can perturb them from the AI algebra to one of its approximating interval algebras, then that we may orthogonalize them as elements of $B \otimes \mathcal{Q}$. Finally, we separate them using the UHF algebra $\mathcal{Q}$ and a *-homomorphism from $\mathcal{C}([0,1])$ into $B \otimes \mathcal{Q}$ in such a way that we can squeeze further (tracially small) elements into the gaps with the result being a partition of unity whose image in $B \otimes \mathcal{Q}$ will be tracially large.

3.4.17 PROPOSITION: *Let B be a simple unital AI algebra with inductive limit structure $B = \varinjlim(B_i, \phi_i)$, with interval algebras B_i, and $\phi^{(i)} : B_i \to B$ be as defined in 1.1.2. Let $\eta > 0$ and let $\tilde{b}_0, \dots, \tilde{b}_N \in B$ be pairwise orthogonal positive contractions such that*

$$\tau(\textstyle\sum_{n=0}^N \tilde{b}_n) \geq \eta \text{ for all } \tau \in T(B).$$

Then for any $0 < \epsilon < \eta$ there are $i \in \mathbb{N}$ and pairwise orthogonal positive contractions $b_0, \dots, b_N \in B_i$ such that

$$\|\phi^{(i)}(b_n) - \tilde{b}_n\| < \epsilon$$

for every $n \in \{0, \dots, N\}$ and

$$\tau(\textstyle\sum_{n=0}^N b_n) \geq \eta - \epsilon \text{ for all } \tau \in T(B_i).$$

PROOF: Suppose not. Then there exists some $0 < \epsilon < \eta$ such that for every $i \in \mathbb{N}$ there is some tracial state $\tau_i \in T(B_i)$ with $\tau_i(\sum_{n=0}^N b_{i,n}) < \eta - \epsilon$ for any pairwise orthogonal positive contractions $b_{i,0}, \dots, b_{i,N} \in B_i$ with $\|\phi^{(i)}(b_{i,n}) - \tilde{b}_n\| < \epsilon$ for all $0 \leq n \leq N$. In particular, we may assume that $\|\phi^{(i)}(b_{i,n}) - \tilde{b}_n\| \to 0$ as $i \to \infty$.

Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. Consider the functional $\tilde{\tau} : \Pi_{i \in \mathbb{N}} B_i \to \mathbb{C}$ defined by $\tilde{\tau}((x_i)_{i \in \mathbb{N}}) = \lim_{i \to \omega} \tau_i(x_i)$. It is clearly continuous, linear, positive and tracial. Moreover, if $x = (x_i)_{i \in \mathbb{N}} \in \bigoplus_{i \in \mathbb{N}} B_i$ then since $\|x_i\| \to 0$, we have $\tilde{\tau}(x) = 0$. Since $B \subset \Pi B_i / \bigoplus_{i \in \mathbb{N}} B_i$ and $\tilde{\tau}(1_B) = \tilde{\tau}((1_{B_i})_{i \in \mathbb{N}}) = 1$, it follows that $\tilde{\tau}$ defines a tracial state on B.

Note that we may consider the sequence $(\tau_i(\sum_{n=0}^N b_{i,n}))_{i \in \mathbb{N}}$ as a continuous bounded function $f : \mathbb{N} \to [0,1]$ given by $f(i) = \tau_i(\sum_{n=0}^N b_{i,n})$. Hence it extends to a continuous function $f : \beta\mathbb{N} \to [0,1]$ where $f(\omega) = \lim_{i \to \omega} \tau_i(\sum_{n=0}^N b_n)$. Since $f(i) < \eta - \epsilon$ by continuity we must have $\tilde{\tau}(\sum_{n=0}^N b_n) = \lim_{i \to \omega} \tau_i(\sum_{n=0}^N b_{i,n}) = f(\omega) \leq \eta - \epsilon < \eta$, contradicting the assumption that $\tau(\sum_{n=0}^N b_n) \geq \eta$ for all $\tau \in T(B)$. ∎

3.4.18 LEMMA: *Let B be a simple separable unital AI algebra and let $b_0, \dots, b_N$ be pairwise orthogonal positive contractions in a unital subalgebra $B_i := \oplus_{m=1}^M \mathcal{C}([0,1]) \otimes M_{r_m} \subset B$ such that, for some $0 < \eta < 1$,*

$$\tau(\textstyle\sum_{n=0}^N b_n \otimes 1_{\mathcal{Q}}) \geq \eta \text{ for every } \tau \in T(B_i \otimes \mathcal{Q}).$$

Then, for every $0 < \epsilon < \eta/4$ there are $K_1, K_2 \in \mathbb{N}$, a partition of $\{0, \dots, K_1\} \times \{0, \dots, K_2\}$ into $N+1$ pieces $\mathcal{P}_0, \dots, \mathcal{P}_N$ and *-homomorphisms

$$\psi_{k_1} : \mathcal{C}([0,1]) \to B_i \otimes \mathcal{Q}, \quad k_1 \in \{0, \dots, K_1\}$$

with orthogonal images such that

(i) $\sum_{(k_1,k_2) \in \mathcal{P}_n} \psi_{k_1}(\gamma_{\frac{k_2}{K_2}}) \precsim_{B \otimes \mathcal{Q}} b_n \otimes 1_{\mathcal{Q}}$,

(ii) $\tau(\sum_{k_1=0}^{K_1} \psi_{k_1}(1_{\mathcal{C}([0,1])})) \geq \eta/4 - \epsilon$ *for every* $\tau \in T(B_i \otimes \mathcal{Q})$.

PROOF: Let $[0,1]_m$ denote the m^{th} copy of the interval in B_i. For every $m \in \{1, \dots, M\}$ and every $t \in [0,1]_m$, the map $\hat{t} : B_i \to \mathbb{C}$ given by $\hat{t}(f) = \tau_{M_{r_m}} f(t)$ is a tracial state on B_i. Let $\mathcal{F}_t = \{b \in \{b_0, \dots, b_N\} \mid \hat{t}(b) > 2\epsilon/(3N)\}$. At each $t \in [0,1]$ there is an interval $I_t \subset [0,1]$ such that

$\hat{s}(b) > 0$ for every $s \in I_t$ and every $b \in \mathcal{F}_t$. The subintervals I_t cover $[0,1]_m$ so we may find finitely many t such that I_t cover $[0,1]_m$. Adjusting the subintervals by making them smaller if necessary we find a partition $\{0 = t_{m,0} < t_{m,1} < \cdots < t_{m,K_m} = 1\}$ such that

$$\hat{s}(b) > \epsilon/(2N) \tag{3.2}$$

for every $s \in [t_{m,k-1}, t_{m,k+1}]$ and every $b \in \mathcal{F}_{t_{m,k}}$. Without loss of generality we may assume that $K_m = K_2$ and $t_{m,0} = t_0, t_{m,1} = t_1, \ldots, t_{m,K_M} = t_{K_2}$ for every $m \in \{1, \ldots, M\}$.

Set $t_{-1} = -\infty$ and $t_{K_2+1} = \infty$ and let $b_{n,m}$ denote the restriction of b_n to $[0,1]_m$.

Note that from (3.2) it follows that

$$\tau(\textstyle\sum_{m,k_2} \gamma_{\frac{k_2}{K_2}} \otimes \sum_{b_n \in \mathcal{F}_{t_{m,K_2}}} b_{n,m}(s)) \geq \tau(\sum_{m,k_2} \gamma_{\frac{k_2}{K_2}} \otimes \sum_{n=0}^{N} b_{n,m}(s)) - \epsilon/2 \tag{3.3}$$

for every $\tau \in T(B_i \otimes \mathcal{Q})$, every $s \in (t_{k_2-1}, t_{k_2+1}) \cap [0,1]_m$, $0 \leq k_2 \leq K_2$, $1 \leq m \leq M$.

Refining the partition further if necessary, we may assume that

$$\|\textstyle\sum_{m=1}^{M} \sum_{k_2=0}^{K_2} \gamma_{\frac{k_2}{K_2}} \otimes b_{n,m}(t_{k_2}) - b_n\| < \epsilon/(4N).$$

Then, by [27, Lemma 2.2],

$$\textstyle\sum_{m=1}^{M} \sum_{k_2=0}^{K_2} \gamma_{\frac{k_2}{K_2}} \otimes (b_{n,m}(t_{k_2}) - \epsilon/(2N))_+ = (\sum_{m=1}^{M} \sum_{k_2=0}^{K_2} \gamma_{\frac{k_2}{K_2}} \otimes b_{n,m}(t_{k_2}) - \epsilon/(2N))_+ \precsim b_n. \tag{3.4}$$

Now, using (3.2) and (3.3), we can find $D, D' \in \mathbb{N}$ satisfying the following:

$$1/(D+1) \leq \min_{\substack{m=1,\ldots,M \\ k_2=0,\ldots,K_2}} \{\hat{s}((b - \epsilon/(2N))_+) \mid b \in \mathcal{F}_{t_{k_2}}, s \in (t_{m,k_2-1}, t_{m,k_2+1}) \cap [0,1]\}, \tag{3.5}$$

$$\eta - 4\epsilon \leq D'/(D+1) \leq \eta - \epsilon/2. \tag{3.6}$$

and so that for each m, k_2 there is a partition of $\{0, \ldots, D'\}$ into $N+1$ pieces $\tilde{\mathcal{P}}_{0,m,k_2}, \ldots, \tilde{\mathcal{P}}_{N,m,k_2}$ such that

$$\textstyle\sum_{|\tilde{\mathcal{P}}_{n,m,k_2}|} 1/2((D+1)) < \tau_{M_{r_m}}((b_{m,n}(t_k) - \epsilon/(2N))_+). \tag{3.7}$$

whenever $b \in \mathcal{F}_{t_m,k_2}$.

Let $q_{(0,1)}, q_{(1,1)}, q_{(0,D)}, \ldots, q_{(D',D)}$ be projections as defined in 1.1.4 and let ι_m be the *-homomorphism given by

$$\mathcal{C}([0,1]) \otimes M_{r_m} \hookrightarrow B_i \otimes \mathcal{Q}\,.$$

Now by our choice of partition and (3.5), for every $0 \leq d \leq D'$ and every $b \in \mathcal{F}_{t_{k_2}}$,

$$\tau(\iota_m(\gamma_{\frac{k_2}{K_2}} \otimes 1_{M_{r_m}}) \otimes q_{(0,3)} \otimes q_{(d,D)}) \leq \tau(\iota_m(\gamma_{\frac{k_2}{K_2}} \otimes b_{n,m}(t_{k_2}) - \epsilon/(2N))_+) \otimes q_{(k_2 \mathrm{mod} 2,1)}), \tag{3.8}$$

for every $\tau \in T(B_i \otimes \mathcal{Q})$.

Thus by the choice of integers D and D' and strict comparison in $B \otimes \mathcal{Q}$ (any tracial state on $B \otimes \mathcal{Q}$ restricts to a tracial state on $B_i \otimes \mathcal{Q}$) we also have

$$\begin{aligned}
&\textstyle\sum_{m,k_2} \sum_{d \in \tilde{\mathcal{P}}_{n,m,k_2}} \iota_m(\gamma_{\frac{k_2}{K_2}} \otimes 1_{M_{r_m}}) \otimes q_{(0,3)} \otimes q_{(d,D)} \\
&\quad\precsim \quad \textstyle\sum_{m,k_2=0\mathrm{mod}2} \sum_{d \in \tilde{\mathcal{P}}_{n,m,k_2}} \iota_m(\gamma_{\frac{k_2}{K_2}} \otimes 1_{M_{r_m}}) \otimes q_{(0,3)} \otimes q_{(d,D)} \\
&\qquad\quad \oplus \textstyle\sum_{m,k_2=1\mathrm{mod}2} \sum_{d \in \tilde{\mathcal{P}}_{n,m,k_2}} \iota_m(\gamma_{\frac{k_2}{K_2}} \otimes 1_{M_{r_m}}) \otimes q_{(0,3)} \otimes q_{(d,D)} \\
&\quad\overset{(3.8),(3.7)}{\precsim} \quad \textstyle\sum_{m,k_2} \iota_m((\gamma_{\frac{k_2}{K_2}} \otimes (b_{n,m}(t_{k_2})) - \epsilon/(2N))_+) \otimes q_{(k_2 \mathrm{mod} 2,1)}),
\end{aligned} \tag{3.9}$$

since the summands are orthogonal. Define $K_1 := (D'+1)M$ *-homomorphisms

$$\psi_{d,m} : \mathcal{C}([0,1]) \to B_i \otimes \mathcal{Q}$$

by

$$\psi_{d,m}(f) = \iota_m(f \otimes 1_{M_{r_m}}) \otimes q_{(0,3)} \otimes q_{(d,D)},$$

$d \in \{0, \dots, D'\}, m \in \{1, \dots, M\}$. Then

$$\begin{aligned} &\textstyle\sum_{m,k_2} \sum_{d \in \tilde{\mathcal{P}}_{n,m,k_2}} \psi_{d,m}(\gamma_{\frac{k_2}{K_2}}) \\ \overset{(3.9)}{\precsim}\ & \textstyle\sum_{m,k_2} \iota_m(\gamma_{\frac{k_2}{K_2}} \otimes (b_{n,m}(t_{k_2}) - \epsilon/(2N))_+) \otimes q_{(k_2 \mathrm{mod} 2,1)} \\ \leq\ & \textstyle\sum_{m,k_2} \iota_m(\gamma_{\frac{k_2}{K_2}} \otimes (b_{n,m}(t_{k_2}) - \epsilon/(2N))_+) \otimes 1_{\mathcal{Q}}) \\ \overset{(3.4)}{\precsim}\ & b_n \otimes 1_{\mathcal{Q}} \end{aligned} \tag{3.10}$$

Reenumerate the maps $\psi_{d,m}$ by ψ_{k_1}, $k_1 \in \{0, \dots, K_1\}$, so that if $(m-1)(D'+1) \leq k_1 < m(D'+1)$ and $k_1 = d \bmod (D'+1)$ then $\psi_{k_1} = \psi_{d,m}$. We can then partition $\{0, \dots, K_1\} \times \{0, \dots, K_2\}$ into $\mathcal{P}_0, \dots, \mathcal{P}_N$ by setting $\mathcal{P}_n = \{(k_1, k_2) \mid (m-1)(D'+1) \leq k_1 < m(D'+1), k_1 = d \bmod (D'+1), m = 1, \dots, M; k_2 = 0, \dots, K_2; d \in \tilde{\mathcal{P}}_{n,m,k_2}\}$.

For every $n = 0, \dots, N$, by (3.10), we have

$$\textstyle\sum_{(k_1,k_2) \in \mathcal{P}_n} \psi_{k_1}(\gamma_{\frac{k_2}{K}}) \precsim b_n \otimes 1_{\mathcal{Q}},$$

showing (i).

For (ii) we have

$$\begin{aligned} \tau(\textstyle\sum_{k_1=0}^{K_1} \psi_{k_1}(1_{\mathcal{C}([0,1])})) &= \tau(\textstyle\sum_{k_1,k_2} \psi_{k_1}(\gamma_{\frac{k_2}{K_2}})) \\ &= \tau(\textstyle\sum_{d,m,k_2} \iota_m(\gamma_{\frac{k_2}{K_2}} \otimes 1_{M_{r_m}}) \otimes 1_{\mathcal{Q}} \otimes q_{(d,D)})/4 \\ &\overset{(3.6)}{>} \eta/4 - \epsilon, \end{aligned}$$

for every $\tau \in T(B_i \otimes \mathcal{Q})$. ∎

3.4.19 Lemma: *Let $A_0 = \oplus_{m=1}^M p_m(\mathcal{C}(X_m) \otimes M_{r_m})p_m$ be a semihomogeneous subalgebra of a simple separable unital tracially approximately semihomogeneous* C**-algebra A. Let $a_0, \dots, a_N \in A_0 \otimes \mathcal{Q}$ be pairwise orthogonal positive contractions. Suppose there is a $0 < \kappa < 1$ such that*

$$\tau(\textstyle\sum_{n=1}^N a_n) \geq \kappa \text{ for every } \tau \in T(A \otimes \mathcal{Q}).$$

Then, for every $\eta > 0$, there are $K_1, K_2 \in \mathbb{N}$, *-homomorphisms

$$\psi_{k_1} : \mathcal{C}([0,1]) \to A \otimes \mathcal{Q},$$

$0 \leq k_1 \leq K_1$ with orthogonal images and a partition of $\{0, \dots, K_1\} \times \{0, \dots, K_2\}$ into $N+1$ pieces $\mathcal{P}_0, \dots, \mathcal{P}_N$ and such that

(i) $\sum_{(k_1,k_2) \in \mathcal{P}_n} \psi_{k_1}(\gamma_{\frac{k_2}{K_2}}) \precsim a_n \otimes 1_{\mathcal{Q}}$,

(ii) $\tau(\sum_{k_1=0}^{K_1} \psi_{k_1}(1_{\mathcal{C}([0,1])})) \geq \kappa/4 - \eta$ *for every $\tau \in T(A \otimes \mathcal{Q})$.*

Proof: Using Lemma 3.4.16 applied to A we can find $b_0, \dots, b_N$ in a sub-AI-algebra with $\tau(b_n) = \tau(a_n)$ for every $n \in \{0, \dots, N\}$ and every $\tau \in T(A \otimes \mathcal{Q})$. Apply Proposition 3.4.17 to find $\tilde{b}_0, \dots, \tilde{b}_N \in B_i = \oplus_{m=1}^M \mathcal{C}([0,1]) \otimes M_{r_m}$, for some interval algebra B_i, satisfying

$$\|\phi^{(i)}(\tilde{b}_n) - b_n\| < \eta \tag{3.11}$$

and

$$\tau(\sum_{n=0}^{N} \tilde{b}_n) \geq \kappa - \eta \text{ for all } \tau \in T(B_i).$$

We may assume that $\phi^{(i)} : B_i \to B$ is unital. Note that by (3.11) we have

$$(\phi^{(i)}(\tilde{b}_n) - \eta)_+ \precsim b_n$$

by [27, Lemma 2.2]. Now we apply the previous lemma to the pairwise orthogonal elements $\phi^{(i)}((\tilde{b}_n - \eta)_+)$ and $\eta/2$ in place of η to find $K_1, K_2 \in \mathbb{N}$, a partition of $\{0, \dots, K_1\} \times \{0, \dots, K_2\}$ into $N+1$ pieces $\mathcal{P}_0, \dots, \mathcal{P}_N$ and *-homomorphisms

$$\tilde{\psi}_{k_1} : \mathcal{C}([0,1]) \to B_i \otimes \mathcal{Q}$$

with orthogonal images such that

(iii) $\sum_{(k_1,k_2)\in\mathcal{P}_n} \tilde{\psi}_{k_1}(\gamma_{\frac{k_2}{K}}) \precsim (\tilde{b}_n - \eta)_+ \otimes 1_{\mathcal{Q}}$,

(iv) $\tau(\sum_{k_1=0}^{K_1} \tilde{\psi}_{k_1}(1_{\mathcal{C}([0,1])})) \geq (\kappa - 2\eta)/4 - \eta/2 = \kappa/4 - \eta$ for every $\tau \in T(B_i \otimes \mathcal{Q})$.

Then taking $\psi_{k_1} = \phi^{(i)} \circ \tilde{\psi}_{k_1}$, the above shows (ii). From (iii) we see that

$$\sum_{(k_1,k_2)\in\mathcal{P}_n} \psi_{k_1}(\gamma_{\frac{k_2}{K}}) \precsim \phi^{(i)}((\tilde{b}_n - \eta)_+) \otimes 1_{\mathcal{Q}} \precsim b_n \otimes 1_{\mathcal{Q}} \sim a_n \otimes 1_{\mathcal{Q}},$$

where the last Cuntz equivalence holds since neither a_n nor b_n are Cuntz equivalent to a projection, so we can approximate $a_n \otimes 1_{\mathcal{Q}}$ (respectively $b_n \otimes 1_{\mathcal{Q}}$) by $(a_n \otimes 1_{\mathcal{Q}} - \epsilon_j)_+$ (respectively $(b_n \otimes 1_{\mathcal{Q}} - \epsilon_j)_+$) where $(\epsilon_j)_{j\in\mathbb{N}}$ is a sequence strictly decreasing to zero, and then use strict comparison. This shows (i). ∎

3.4.20 Lemma: *Let $N \in \mathbb{N}$ and $0 = t_0 < t_1 < \cdots < t_N = 1$ be a partition of $[0,1]$ into subintervals $[t_i, t_{i+1}]$ of the same length for all $0 \leq i \leq N-1$. Then for any $\epsilon > 0$ there is $\beta > 0$ and there are $\tilde{\gamma}_0, \dots, \tilde{\gamma}_N \in \mathcal{C}([0,1]) \otimes \mathcal{Q}$ satisfying*

(i) *$\tilde{\gamma}_n \perp \tilde{\gamma}_m$ for all $0 \leq m \neq n \leq N$,*

(ii) *For any subset $I \subset \{0 \dots, N\}$ we have $\beta \cdot \sum_{n\in I} \tilde{\gamma}_n \leq \sum_{n\in I} \gamma_{\frac{n}{N}} \otimes 1_{\mathcal{Q}}$ for all $0 \leq n \leq N$,*

(iii) *$\sum_{n=0}^{N} \gamma_{\frac{n}{N}} \otimes 1_{\mathcal{Q}}$ is a unit for $\tilde{\gamma}_0, \dots, \tilde{\gamma}_N$,*

(iv) *$\tau(\sum_{n=0}^{N} \tilde{\gamma}_n) \leq \tau(\sum_{n=0}^{N} \gamma_{\frac{n}{N}} \otimes 1_{\mathcal{Q}}) + \epsilon$ for all $\tau \in T(\mathcal{C}([0,1]) \otimes \mathcal{Q})$,*

Proof: Given $\epsilon > 0$, let $M \in \mathbb{N}$ be so large that $1/(M-1) < \epsilon$. For each i, $0 \leq i \leq N-1$, subdivide the intervals $[t_i, t_{i+1}]$ into subintervals of equal length by a partition $t_i = s_{i,0} < s_{i,1} < s_{i,2} < \cdots < s_{i,M} = t_{i+1}$.

Define, for $1 \leq i \leq N-1$, and $0 \leq j \leq M-2$, the following functions

$$g_{0,j}(t) = \begin{cases} 1 & \text{if } t \leq s_{0,M-2-j}, \\ \text{linear} & \text{if } s_{0,M-2-j} \leq t \leq s_{0,M-1-j}, \\ 0 & \text{if } t \geq s_{0,M-1-j}; \end{cases}$$

$$g_{N,j}(t) = \begin{cases} 0 & \text{if } t \leq s_{N,M-1-j}, \\ \text{linear} & \text{if } s_{N,M-1-j} \leq t \leq s_{N,M-j}, \\ 1 & \text{if } t \geq s_{N,M-j}; \end{cases}$$

$$g_{i,j}(t) = \begin{cases} 0 & \text{if } t \leq s_{i-1,M-1-j} \text{ or } t \geq s_{i,M-1-j}, \\ 1 & \text{if } s_{i-1,M-j} \leq t \leq s_{i,M-2-j}, \\ \text{linear} & \text{elsewhere}. \end{cases}$$

Then put

$$\tilde{\gamma}_i = \sum_{j=0}^{M-2} g_{i,j} \otimes q_{(j,M-2)}.$$

If $i \neq i'$ then

$$\begin{array}{rcl} \tilde{\gamma}_i\tilde{\gamma}_{i'} & = & (\sum_{j=0}^{M-2} g_{i,j} \otimes q_{(j,M-2)})(\sum_{j'=0}^{M-2} g_{i',j'} \otimes q_{(j',M-2)}) \\ & = & \sum_{j=0}^{M-2} g_{i,j}g_{i',j} \otimes q_{(j,M-2)} \\ & = & 0, \end{array}$$

showing (i).

Let $\beta = (t_1 - s_{0,M-2})/t_1$. We have that $\eta \cdot g_{i,j} \leq \gamma_{\frac{i}{N}}$ for all i, j and so, for any subset $I \subset \{0, \ldots, N\}$, we see

$$\beta \cdot \sum_{i \in I} \tilde{\gamma}_i = \sum_{i \in I} \sum_{j=0}^{M-2} \beta \cdot g_{i,j} \otimes q_{(j,M-2)} \leq \sum_{i \in I} \sum_{j=0}^{M-2} \gamma_{\frac{i}{N}} \otimes q_{(j,M-2)} = \sum_{i \in I} \gamma_{\frac{i}{N}} \otimes 1_{\mathcal{Q}},$$

showing (ii).

Clearly

$$\begin{array}{rcl} \tilde{\gamma}_{i'}(\sum_{i=0}^{N} \gamma_{\frac{i}{N}} \otimes 1_{\mathcal{Q}}) & = & (\sum_{j=0}^{M-2} g_{i,j} \otimes q_{(j,M-2)})(1_{\mathcal{C}([0,1])} \otimes 1_{M_r} \otimes 1_{\mathcal{Q}}) \\ & = & \tilde{\gamma}_{i'} \\ & = & (\sum_{i=0}^{N} \gamma_{\frac{i}{N}} \otimes 1_{\mathcal{Q}})\tilde{\gamma}_{i'}, \end{array}$$

so (iii) holds.

Finally, for (iv), note that

$$\sum_{i=0}^{N}\sum_{j=0}^{M-2}(1 - g_{i,j}) < 1_{\mathcal{C}([0,1])}.$$

Thus, for any $\tau \in T(\mathcal{C}([0,1]) \otimes \mathcal{Q})$ we have

$$\begin{array}{rcl} \tau(\sum_{i=0}^{N} \gamma_{\frac{i}{N}} \otimes 1_{\mathcal{Q}}) - \tau(\sum_{i=0}^{N} \tilde{\gamma}_i) & = & \tau(1_{\mathcal{C}([0,1])} \otimes 1_{\mathcal{Q}} - \sum_{i=0}^{N}\sum_{j=0}^{M-2} g_{i,j} \otimes q_{(j,M-2)}) \\ & = & \tau(\sum_{j=0}^{M-2} 1_{\mathcal{C}([0,1])} \otimes q_{(j,M-2)} - \sum_{i=0}^{N}\sum_{j=0}^{M-2} g_{i,j} \otimes q_{(j,M-2)}) \\ & = & \tau(\sum_{i=0}^{N}\sum_{j=0}^{M-2}(1 - g_{i,j}) \otimes q_{(j,M-2)}) \\ & \leq & (\sum_{j=0}^{M-2} \tau(\sum_{i=0}^{N}(1 - g_{i,j}) \otimes 1_{\mathcal{Q}})) \cdot 1/(M-1) \\ & < & 1/(M-1) \\ & < & \epsilon. \end{array}$$

∎

3.4.21 Lemma: *Let A be a unital separable* C*-*algebra with nonempty tracial state space, and let $0 = t_0 < \cdots < t_K = 1$ be a partition of $[0,1]$ for some $K \in \mathbb{N}$. For any $\delta > 0$, *-homomorphism $\phi : \mathcal{C}([0,1]) \to A$ and $\eta > 0$ such that $\eta < \tau(\phi(1_{\mathcal{C}([0,1])}))$ for every $\tau \in T(A \otimes \mathcal{Q})$, there is a *-homomorphism*

$$\psi : \mathcal{C}([0,1]) \to A \otimes \mathcal{Q}$$

satisfying

$$\tau(\psi(1_{\mathcal{C}([0,1])})) \geq \tau(\phi(1_{\mathcal{C}([0,1])})) - \eta \text{ for every } \tau \in T(A \otimes \mathcal{Q})$$

and a partition of unity $\gamma_{\frac{0}{4K}}, \gamma_{\frac{1}{4K}}, \ldots, \gamma_{\frac{4K-1}{4K}}, \gamma_{\frac{4K}{4K}}$ such that, for any subset $I \subset \{0, \ldots, K\}$, we have

$$\sum_{k \in I} \psi(\gamma_{\frac{4k}{4K}})) \precsim \sum_{k \in I} \phi(\gamma_{\frac{k}{K}}))$$

and

$$\tau(\psi(\gamma_{\frac{4k+j}{4K}})) < \delta \text{ for every } \tau \in T(A \otimes \mathcal{Q}),$$

for every $k \in \{0, \ldots, K\}$ and $j \in \{1, 2, 3\}$.

Proof: Apply Lemma 3.4.20 to the partition $0 = t_0 < \cdots < t_K = 1$ and $0 < \epsilon < \min\{\delta, \eta\}$ to get elements $\tilde{\gamma}_k \in \mathcal{C}([0,1]) \otimes \mathcal{Q}$, $k \in \{0 \ldots, K\}$. Let C be the C*-subalgebra generated by $(\phi \otimes \mathrm{id})(\tilde{\gamma}_k)$, $k \in \{0 \ldots, K\}$ and $(\phi \otimes \mathrm{id})(1_{\mathcal{C}([0,1])} \otimes 1_{\mathcal{Q}})$. Let $\gamma_{\frac{k}{2K}}$, $k \in \{0, \ldots, 2K\}$ denote the partition of unity with respect to $\{0 = t_0 < \cdots < t_{2K} = 1\}$, and define a map

$$\psi : \mathcal{C}([0,1]) \to C \hookrightarrow A \otimes \mathcal{Q}$$

by

$$\psi(\gamma_{\frac{2k}{2K}}) = (\phi \otimes \mathrm{id})(\tilde{\gamma}_k),$$

for $k \in \{0, \ldots, K\}$ and

$$\psi(\gamma_{\frac{2k+1}{2K}}) = (\phi \otimes \mathrm{id})((1_{\mathcal{C}([0,1])} - \textstyle\sum_{k=0}^{K} \tilde{\gamma}_k)|_{t_k, t_{k+1}}),$$

for $k \in \{0, \ldots, K-1\}$. Since the elements in a partition of unity together with the unit generate $\mathcal{C}([0,1])$ as a universal C*-algebra and the generators of C satisfy the same relations, it follows that ψ induces a well-defined *-homomorphism.

Let $\alpha < (1 - \epsilon/\delta)/2K$. Define the following functions $h_{0,\alpha}, \ldots, h_{2K,\alpha} \in \mathcal{C}([0,1])$ by

$$\begin{aligned}
h_{0,\alpha}(t) &= \begin{cases} 1 & \text{if} \quad t = 0, \\ \text{linear} & \text{if} \quad 0 \leq t \leq t_1 - \alpha, \\ 0 & \text{if} \quad t \geq t_1 - \alpha; \end{cases} \\
h_{2K,\alpha}(t) &= \begin{cases} 0 & \text{if} \quad t \leq t_{2K-1} + \alpha, \\ \text{linear} & \text{if} \quad t_{2K-1} + \alpha \leq t \leq 1, \\ 1 & \text{if} \quad t = 1; \end{cases} \\
h_{i,\alpha}(t) &= \begin{cases} 0 & \text{if} \quad t \leq t_{2i-1} + \alpha \text{ or } t \geq t_{2i+1} - \alpha, \\ 1 & \text{if} \quad t = t_{2i}, \\ \text{linear} & \qquad \text{elsewhere.} \end{cases}
\end{aligned}$$

Consider a new partition of $[0,1]$ given by

$$\begin{aligned}
&(0 = t_0 = s_0) < (s_1 = t_1 - \alpha) < (s_2 = t_1) < (s_3 = t_1 + \alpha) < (s_4 = t_2) < (s_5 = t_3 - \alpha) \\
&< \cdots < (s_{4K-1} = t_{2K-1} + \alpha) < (t_{2K} = s_{4K}),
\end{aligned}$$

and let $\gamma_{\frac{k}{4K}}$, $k \in \{0, \ldots, 4K\}$ denote the corresponding partition of unity. We have

$$\tau(\psi(1_{\mathcal{C}([0,1])})) \geq \tau(\phi(1_{\mathcal{C}([0,1])})) - \epsilon \geq \tau(\phi(1_{\mathcal{C}([0,1])})) - \eta$$

for every $\tau \in T(A \otimes \mathcal{Q})$ by the previous lemma. By (ii) of the previous lemma there is $\beta > 0$ such that for any subset $I \subset \{0, \ldots, K\}$ we have

$$\begin{aligned}
\textstyle\sum_{k \in I} \psi(\gamma_{\frac{4k}{4K}}) &< \textstyle\sum_{k \in I} \psi(\gamma_{\frac{2k}{2K}}) \\
&= \textstyle\sum_{k \in I} (\phi \otimes \mathrm{id})(\tilde{\gamma}_k) \\
&\leq \beta \cdot \textstyle\sum_{k \in I} (\phi \otimes \mathrm{id})(\gamma_{\frac{k}{K}} \otimes 1_{\mathcal{Q}}).
\end{aligned}$$

It follows that

$$\textstyle\sum_{k \in I} \psi(\gamma_{\frac{4k}{4K}}) \precsim \sum_{k \in I} \phi(\gamma_{\frac{k}{K}}).$$

We have that

$$(1 - \textstyle\sum_{i=0}^{2K} h_{i,\alpha}) \leq (1/(1 - 2\alpha K))(1 - \sum_{i=1}^{2K} \gamma_{\frac{21-1}{2K}}),$$

thus, for $k \in \{0, \dots, 4K\}$, $j \in \{1, 2, 3\}$, we have

$$\begin{aligned} \gamma_{\frac{4k+j}{4K}} &< (1_{\mathcal{C}([0,1])} - \textstyle\sum_{i=0}^{2K} \gamma_{\frac{4i}{4K}}) \\ &\leq (1/(1-2\alpha K))(1_{\mathcal{C}([0,1])} - \textstyle\sum_{i=1}^{2K} \gamma_{\frac{2i-1}{2K}}), \end{aligned}$$

and therefore

$$\begin{aligned} \tau(\psi(\gamma_{\frac{4k+j}{4K}})) &< \tau(\psi((1_{\mathcal{C}([0,1])} - \textstyle\sum_{i=0}^{2K} \gamma_{\frac{4i}{4K}})) \\ &\leq (1/(1-2\alpha K))\tau(1_{\mathcal{C}([0,1])} - \textstyle\sum_{i=1}^{2K} \gamma_{\frac{2i}{2K}}) \\ &< \epsilon/(1-2\alpha K) \\ &< \delta \end{aligned}$$

for every $\tau \in T(A \otimes \mathcal{Q})$. ∎

3.5. Approximation by TAH algebras with general tracial state spaces

We are now ready to prove the main theorem. Since we cannot isolate sets of large trace, we take a cover of the base spaces and separate the sets in layers of the UHF algebra. This cuts the total trace by the dimension of the space. Using the results of the previous section, we find an interval which will give us bump functions of the appropriate trace by first embedding an AI algebra with the correct tracial state space, and then drop down to an honest interval algebra in its inductive limit.

Once again, we interpolate between the sets in our set partition of the base space, and squeeze in corresponding tracially small functions into our partition of unity of $[0, 1]$. This then allows us to construct our partial isometry in a similar manner to the simplified case of Section 3.3.

3.5.22 THEOREM: *Let A be a separable simple unital tracially approximately semihomogeneous C^*-algebra of bounded dimension less than or equal to L. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A \otimes \mathcal{Q}$, there are a projection $p \in A \otimes \mathcal{Q}$, a unital C^*-algebra $C \subset p(A \otimes \mathcal{Q})p$ with $1_C = p$ and $C \in I$, satisfying the following properties:*

(i) $\|pa - ap\| < \epsilon$ *for every* $a \in \mathcal{F}$,

(ii) $\mathrm{dist}(pap, C) < \epsilon$ *for every* $a \in \mathcal{F}$,

(iii) $\tau(p) > 1/(8(L + 1))$ *for every* $\tau \in T(A \otimes \mathcal{Q})$.

PROOF: Let $\epsilon > 0$. By Lemma 2.3.4, it is enough to show (i)–(iii) for $\mathcal{F} = \mathcal{G} \otimes \{1_{\mathcal{Q}}\}$ where $\mathcal{G} \subset A$ is a finite subset.

Since A is tracially approximately semihomogeneous, we may assume that there is a semihomogeneous C^*-algebra A_0 such that for all $a \in \mathcal{G}$ we have $a = f_a + (1_A - 1_{A_0})a(1_A - 1_{A_0})$ for some $f_a \in A_0$ and that

$$\tau(1_A - 1_{A_0}) < 1/16 \text{ for every } \tau \in T(A \otimes \mathcal{Q}). \tag{3.12}$$

Write

$$A_0 = \oplus_{m=1}^{M} p_m(\mathcal{C}(X_m) \otimes M_{r_m})p_m$$

where the X_m are compact metrizable spaces with $\dim(X_m) \leq L$, $r_m \in \mathbb{N}$, and the $p_m \in \mathcal{C}(X) \otimes M_{r_m}$ are projections.

Let

$$\mathcal{G}_0 = \{f_a \mid a \in \mathcal{G}.\}$$

By Proposition 3.1.2, we may assume the X_m are connected, whence the p_m have constant rank. Let R_m denote the rank of $p_m(x)$ for any $x \in X$. Let $R = R_1 \cdot \dots \cdot R_m$ and $\hat{R}_m = R/R_m$. We have $M_{R_m} \otimes M_{\hat{R}_m} \cong M_R$, and for each m there is an isomorphism $M_{\hat{R}_m} \otimes \mathcal{Q} \cong \mathcal{Q}$. Thus we always have

$\mathcal{C}(Y) \otimes M_{R_m} \otimes \mathcal{Q} \cong \mathcal{C}(Y) \otimes M_R \otimes \mathcal{Q}$ for any compact Hausdorff space Y. In particular, we may assume that all the matrix sizes are the same size R and so for every $m \in \{1, \dots M\}$ and every $x \in X_m$ we have a surjective *-homomorphisms

$$\mathrm{ev}_x : A_0 \to M_R$$

given by evaluation at x.

Apply Lemma 3.2.8 with respect to $\epsilon/24$ and $\mathcal{G}_0$ to get open neighbourhoods $U_x \subset V_x$. Since X is compact, there are finitely many x such that the sets U_x cover X. We denote this finite cover by $\mathcal{O} = \{U_0, \dots, U_N\}$, along with the corresponding sets $V_0, \dots, V_N$ from Lemma 3.2.8. Moreover, since $\dim(X) \leq L$, we may assume that the cover $\mathcal{O}$ is $(L+1)$-colourable. Let $c_0, \dots, c_L \in \mathcal{Q}$ be pairwise orthogonal projections with $\tau_{\mathcal{Q}} = 1/(L+1)$. To each of the $L+1$ colours, we associate exactly one of the $c_0, \dots, c_L$. For $U \in \mathcal{O}$, we define

$$c(U) = c_l,$$

where c_l is the projection associated to the colour of U.

Let $\tilde{f}_n \in \mathcal{C}(X)$ denote the function in the partition of unity subordinate to the open cover $\mathcal{O}$, supported on U_n, $n \in \{0, \dots, N\}$ and let $f_n \in \mathcal{C}_0(U_n) \otimes M_R$ denote the matrix-valued bump function $f_n = \mathrm{diag}(\tilde{f}_n, \dots, \tilde{f}_n)$, where $\tilde{f}_n$ is copied R times down the diagonal. Then define

$$a_n := f_n \otimes c(U_n). \tag{3.13}$$

Then $\tau(\sum_{n=0}^{N} a_n) \geq 15/(16(L+1))$ for every $\tau \in T(A \otimes \mathcal{Q})$. Let

$$\eta \leq 7/(64(L+1)). \tag{3.14}$$

From Lemma 3.4.19 with $\kappa = 15/(16(L+1))$ and $\eta/2$ there are $K_1, K_0 \in \mathbb{N}$, a partition of $\{0, \dots, K_1\} \times \{0, \dots, K_0\}$ into $N+1$ pieces $\mathcal{P}_0, \dots, \mathcal{P}_N$ and *-homomorphisms

$$\tilde{\psi}_{k_1} : \mathcal{C}([0,1]) \to A \otimes \mathcal{Q}$$

$0 \leq k_1 \leq K_1$ with orthogonal images such that

(iii) $\sum_{k_1, k_0 \in \mathcal{P}_n} \tilde{\psi}_{k_1}(\gamma_{\frac{k_0}{K_0}}) \precsim a_n \otimes 1_{\mathcal{Q}}$,

(iv) $\tau(\sum_{k_1=0}^{K_1} \tilde{\psi}_{k_1}(1_{\mathcal{C}([0,1])})) \geq 15/(16(L+1)) - \eta/2$ for every $\tau \in T(A \otimes \mathcal{Q})$.

Let $k_1 \in \{0, \dots, K_1\}$. For every $k_0 \in \{0, \dots, K_0 - 1\}$, we have $(k_1, k_0) \in \mathcal{P}_n$ for some n.

If $(k_1, k_0) \in \mathcal{P}_n$ then denote $U_{k_1, \frac{k_0}{K_0}} := U_n$, and, correspondingly, $V_{k_1, \frac{k_0}{K_0}} := V_n$. For each k_1 apply Lemma 3.2.10 with $\epsilon/8$ in place of η starting from $U_{k_1,0} \subset V_{k_1,0}, U_{k_1,\frac{1}{K_0}} \subset V_{k_1,\frac{1}{K_0}}, \dots, U_{k_1,\frac{k_0}{K_0}} \subset V_{k_1,\frac{k_0}{K_0}}, \dots, U_{k_1,1} \subset V_{k_1,1}$. We find $K_2 = K_0 K \in \mathbb{N}$ and open sets $U_{k_1, \frac{k_2}{K_0 K}}$, $k_2 \in \{k_0 K + 1, \dots, (k_0+1)K - 1\}$ satisfying

(iv) $U_{k_1, \frac{k_0 K}{K_2}} = U_{k_1, \frac{k_0}{K_0}}$ for $k_2 \in \{0, \dots, K_0\}$,

(v) $e_{m,n}^{(k_1, \frac{k_2}{K_2})}(x) = e_{m,n}^{(k_1, \frac{k_2+1}{K_2})}(x)$ for all $x \in U_{k_1, \frac{k_2}{K_2}} \cup U_{k_1, \frac{k_2+1}{K_2}}$,

(vi) If the sets $U_{k_1, \frac{k_0 K}{K_2}}$ and $U_{k_1, \frac{k_0' K}{K_2}}$ have the same colour then we have $U_{k_1, \frac{k_2}{K_2}} \cap U_{k_1, \frac{k_2'}{K_2}} = \emptyset$ for every $k_2 \neq k_2' \in \{k_0 K, \dots, (k_0+1)K - 1\} \cup \{k_0' K, \dots, (k_0'+1)K - 1\}$,

(vii) $\| \sum_{m,n=1}^{r} f(x)_{m,n} e_{m,n}^{(k_1, \frac{k_2}{K_2})} - f(y)_{m,n} e_{m,n}^{(k_1, \frac{k_2+1}{K_2})} \| < \epsilon/16$ for every $x, y \in U_{k_1, \frac{k_2}{K_2}} \cup U_{k_1, \frac{k_2+1}{K_2}}$, every $f \in \mathcal{G}_0$, every $k_2 = 0, \dots, K_2$.

Note that we have assumed K to be the same in each of the $K_1 + 1$ applications of the lemma, which is possible by simply making the sets smaller when necessary. We also remark that the $U_{k_1, \frac{k_0 K}{K_2}}$ need not be distinct.

Let $f_{k_1,\frac{k_2}{K_2}}$ be the $(R\times R)$-matrix valued bump function for $U_{k_1,\frac{k_2}{K_2}}$, and define the elements by

$$a_{k_1,\frac{k_2}{K_2}} = f_{k_1,\frac{k_2}{K_2}} \otimes c(U_{k_1,\frac{k_2}{K_2}}).$$

(Note that $a_{k_1,\frac{k_0 K}{K_2}} = a_n$ if $(k_1,k_0)\in\mathcal{P}_n$.)

Then let

$$0<\delta<\min_{\substack{\tau\in T(A\otimes\mathcal{Q})\\ k_1,k_2}} \tau(a_{k_1,\frac{k_2}{K_2}}). \tag{3.15}$$

Note that we are able to take $\delta>0$ by simplicity of A.

Now apply Lemma 3.4.21 to each $\tilde{\psi}_{k_1}:\mathcal{C}([0,1])\to A\otimes\mathcal{Q}$, K_0, δ and $\eta/2$ to find *-homomorphisms

$$\hat{\psi}_{k_1}:\mathcal{C}([0,1])\to A\otimes\mathcal{Q}$$

satisfying

$$\tau(\hat{\psi}_{k_1}(1_{\mathcal{C}([0,1])}))\geq \tau(\tilde{\psi}_{k_1}(1_{\mathcal{C}([0,1])}))-\eta/2 \text{ for every } \tau\in T(A\otimes\mathcal{Q})$$

and a partition of unity $\gamma_{\frac{0}{4K_0}},\gamma_{\frac{1}{4K_0}},\dots,\gamma_{\frac{4K_0-1}{4K_0}},\gamma_{\frac{4K_0}{4K_0}}$ such that

$$\textstyle\sum_{\{k_0|(k_1,k_0)\in\mathcal{P}_n\}}\hat{\psi}_{k_1}(\gamma_{\frac{4k_0}{4K_0}})\precsim\sum_{\{k_0|(k_1,k_0)\in\mathcal{P}_n\}}\tilde{\psi}_{k_1}(\gamma_{\frac{k_0}{K_0}})$$

and

$$\tau(\hat{\psi}_{k_1}(\gamma_{\frac{4k_0+j}{4K_0}}))<\delta \text{ for every } \tau\in T(A\otimes\mathcal{Q}),$$

for every $k_0\in\{0,\dots,K_0\}$ and $j\in\{1,2,3\}$.

Putting this together with the above, we have

(viii) a partition of $\{0,\dots,K_1\}\times\{0,\dots,K_0\}$ into $N+1$ pieces $\mathcal{P}_0,\dots,\mathcal{P}_N$ such that

$$\textstyle\sum_{(k_1,k_0)\in\mathcal{P}_n}\hat{\psi}_{k_1}(\gamma_{\frac{4k_0}{4K_0}})\precsim a_n, \tag{3.16}$$

(ix) $\tau(\sum_{k_1,k_0}\hat{\psi}_{k_1}(\gamma_{\frac{4k_0}{4K_0}}))\geq 15/(64(L+1))-\eta$ for every $\tau\in T(A\otimes\mathcal{Q})$.

(x) For every $k_1\in\{0,\dots K_1-1\}, k_0\in\{0,\dots,K_0\}, j\in\{1,2,3\}$

$$\tau(\hat{\psi}_{k_1}(\gamma_{\frac{4k_0+j}{4K_0}}))<\delta \tag{3.17}$$

for every $\tau\in T(A\otimes\mathcal{Q})$.

There is a δ_0 such that, for every unital C*-algebra C, if $a\in C$ satisfies $\|aa^*-p\|<2\delta_0$ for some projection $p\in C$ then there is a partial isometry $v\in C$, $\|v-a\|<\epsilon/8$ and $vv^*=p$. By strict comparison and (3.16) for $\mathcal{P}_n$, we find unitaries $u_n\in A\otimes\mathcal{Q}$ such that

$$\begin{aligned} &u_n(\textstyle\sum_{(k_1,k_0)\in\mathcal{P}_n}\hat{\psi}_{k_1}(\gamma_{\frac{4k_0}{4K_0}})-\delta_0)_+{u_n}^*\\ &\quad= \quad u_n\textstyle\sum_{(k_1,k_0)\in\mathcal{P}_n}(\hat{\psi}_{k_1}(\gamma_{\frac{4k_0}{4K_0}})-\delta_0)_+u_n^*\in \mathrm{Her}(a_n),\end{aligned} \tag{3.18}$$

for $n\in\{0,\dots,N\}$

Now we further refine the partitions of $[0,1]$ by subdividing each subinterval $[4k_0+1,4(k_0+1)-1]$ into K subintervals. Then, for $(k_1,k_2)\in\{0,\dots,K_1\}\times\{k_2\in\{0,\dots,K_2\}\mid k_2\neq k_0'K\}$ applying (3.17) gives unitaries $u_{(k_1,k_2)}\in A\otimes\mathcal{Q}$ satisfying

$$u_{(k_1,k_2)}(\hat{\psi}_{k_1}(\gamma_{\frac{k_2}{K_2}})-\delta_0)_+{u_{(k_1,k_2)}}^*\in\mathrm{Her}(a_{k_1,\frac{k_2}{K_2}}). \tag{3.19}$$

Define

$$v_{(k_1,k_2)} := \begin{cases} u_n(\hat{\psi}_{k_1}(\gamma_{\frac{k_2}{K_0K}}) - \delta_0)_+^{1/2} & \text{if} \quad k_2 = k_0K, (k_1,k_2) \in \mathcal{P}_n \\ u_{(k_1,k_2)}(\hat{\psi}_{k_1}(\gamma_{\frac{k_2}{K_0K}}) - \delta_0)_+^{1/2} & \text{otherwise.} \end{cases}$$

Using an embedding $\rho : M_R \hookrightarrow \mathcal{Q}$, define

$$\phi_{k_1} : \mathcal{C}([0,1]) \otimes M_R \to A \otimes \mathcal{Q}$$

by

$$\phi_{k_1}(\gamma_{\frac{k_2}{K_2}} \otimes e_{m,n}) = \hat{\psi}_{k_1}(\gamma_{\frac{k_2}{K_2}}) \otimes \rho(e_{m,n}).$$

Put

$$v = \sum_{k_1,k_2} \sum_{m=1}^{R} \phi_{k_1}(1_{\mathcal{C}([0,1])} \otimes e_{m,1}) v_{(k_1,k_2)} e_{1,m}^{(k_1,\frac{k_2}{K_2})}.$$

It follows from Lemma 3.2.7 that

$$vv^* = \sum_{k_1,k_2} \sum_{m=1}^{R} \phi_{k_1}(1_{\mathcal{C}([0,1])} \otimes e_{m,1}) v_{(k_1,k_2)} v^*_{(k_1,k_2)} \phi_{k_1}(1_{\mathcal{C}([0,1])} \otimes e_{1,m}),$$

and that

$$v^*v = \sum_{k_1,k_2} \sum_{\{k_2' \mid |k_2'-k_2| \le 1\}} \sum_{m=1}^{R} e_{m,1}^{(k_1,\frac{k_2}{K_2})} v^*_{(k_1,k_2)} v_{(k_1,k_2')} e_{1,m}^{(k_1,\frac{k_2'}{K_2})}.$$

It is easy to check, by splitting the functions $\gamma_{\frac{k_2}{K_2}}$ into two sums of pairwise orthogonal functions, that,

$$\|vv^* - \textstyle\sum_{k_1=0}^{K_1} \phi_{k_1}(1_{\mathcal{C}([0,1])} \otimes 1_{M_R})\| < 2 \cdot \delta_0,$$

which, by choice of δ_0, gives an honest partial isometry $s \in A \otimes \mathcal{Q}$ satisfying

$$ss^* = \textstyle\sum_{k_1=0}^{K_1} \phi_{k_1}(1_{\mathcal{C}([0,1])} \otimes 1_{M_R})$$

and

$$\|s - v\| < \epsilon/8.$$

Let $p = s^*s$ and $C = s(\oplus_{k_1=0}^{K_1} \phi_{k_1}(\mathcal{C}([0,1]) \otimes M_R))s^*$. We will verify (i) – (iii).

Let $a \in \mathcal{F}$. Then $a = f_a + (1_A - 1_{A_0})a(1_A - 1_{A_0})$ for $f_a \in \mathcal{G}_0$. Since $v^*v \in (1_{A_0} \otimes 1_{\mathcal{Q}})(A \otimes \mathcal{Q})(1_{A_0} \otimes 1_{\mathcal{Q}})$, we have $va = vf_a$, so we need only consider what happen to the finite subset $\mathcal{G}_0$.

Let $a \in \mathcal{G}_0$. Note that

$$\begin{aligned} &\phi_{k_1}(1_{\mathcal{C}([0,1])} \otimes e_{m,1}) v_{(k_1,k_2)} e_{1,m}^{(k_1,\frac{k_2}{K_2})}(a \otimes 1_{\mathcal{Q}}) \\ &= \phi_{k_1}(1_{\mathcal{C}([0,1])} \otimes e_{m,1}) v_{(k_1,k_2)} e_{1,m}^{(k_1,\frac{k_2}{K_2})}(a|_{U_{k_1,\frac{k_2}{K_2}}} \otimes 1_{\mathcal{Q}}). \end{aligned}$$

Let $f \in A_0$ be a function satisfying

$$f|_{U_{k_1,\frac{k_2}{K_2}}}(y) = a(x_{k_1,\frac{k_2}{K_2}}) = (\lambda_{m,n}^{k_1,\frac{k_2}{K_2}})_{m,n} \in M_R$$

for every $y \in U_{k_1,\frac{k_2}{K_2}}$, for every k_1, k_2. From (vii) we can assume

$$\|f - a\| < \epsilon/8,$$

and

$$|\lambda_{m,n}^{k_1,\frac{k_2}{K_2}} - \lambda_{m,n}^{k_1,\frac{k_2+1}{K_2}}| < \epsilon/16$$

for every $1 \leq m, n \leq R$ and every k_1, k_2.

Then

$$
\begin{aligned}
& v^*vf \\
= \quad & \textstyle\sum_{k_1,k_2} \sum_{\{k_2' | |k_2'-k_2| \leq 1\}} \sum_{m=1}^{R} \lambda_{m,n}^{k_1,\frac{k_2'}{K_2}} \cdot e_{m,1}^{(k_1,\frac{k_2}{K_2 K_{l,4}})} v^*_{(k_1,k_2)} v_{(k_1,k_2')} e_{1,n}^{(k_1,\frac{k_2'}{K_2})},
\end{aligned} \tag{3.20}
$$

and

$$
\begin{aligned}
& fv^*v \\
= \quad & \textstyle\sum_{k_1,k_2} \sum_{\{k_2' | |k_2'-k_2| \leq 1\}} \sum_{m=1}^{R} \lambda_{m,n}^{k_1,\frac{k_2}{K_2}} \cdot e_{m,1}^{(k_1,\frac{k_2}{K_2})} v^*_{(k_1,k_2)} v_{(k_1,k_2')} e_{1,n}^{(k_1,\frac{k_2'}{K_2})},
\end{aligned}
$$

In the sum above, each value of k_1 corresponds to the maps ϕ_{k_1} and it follows from their construction and Lemma 3.4.19 that they have orthogonal images. It is easy to check (using the C*-equality, for example) that this implies

$$
e_{m,1}^{(k_1,\frac{k_2}{K_2})} v^*_{(k_1,k_2)} v_{(k_1,k_2')} e_{1,n}^{(k_1,\frac{k_2'}{K_2})} e_{m,1}^{(k_1',\frac{k_2''}{K_2})} v^*_{(k_1',k_2'')} v_{(k_1',k_2''')} e_{1,n}^{(k_1',\frac{k_2'''}{K_2})} = 0
$$

whenever $k_1 \neq k_1'$.

It follows that

$$
\begin{aligned}
& \|v^*vf - fv^*v\| \\
\leq \quad & \max_{k_1} \| \textstyle\sum_{k_2,\{k_2' | |k_2'-k_2| = 1\}} \sum_{m=1}^{R} (\lambda_{m,n}^{k_1,\frac{k_2'}{K_2}} - \lambda_{m,n}^{k_1,\frac{k_2}{K_2}}) \cdot e_{m,1}^{(k_1,\frac{k_2}{K_2})} v^*_{(k_1,k_2)} v_{(k_1,k_2')} e_{1,n}^{(l,k_1,\frac{k_2'}{K_2})} \| \\
\leq \quad & \max_{k_1} \| \textstyle\sum_{i=0}^{D_1} \sum_{m=1}^{R} (\lambda_{m,n}^{k_1,\frac{2i}{K_2}} - \lambda_{m,n}^{k_1,\frac{2i+1}{K_2}}) \cdot e_{m,1}^{(k_1,\frac{2i+1}{K_2})} v^*_{(k_1,2i+1)} v_{(k_1,2i)} e_{1,n}^{(l,k_1,\frac{2i}{K_2})} \\
& + (\lambda_{m,n}^{k_1,\frac{2i+1}{K_2}} - \lambda_{m,n}^{k_1,\frac{2i}{K_2}}) \cdot e_{m,1}^{(k_1,\frac{2i}{K_2})} v^*_{(k_1,2i)} v_{(k_1,2i+1)} e_{1,n}^{(l,k_1,\frac{2i+1}{K_2})} \| \\
& + \| \textstyle\sum_{i=0}^{D_2} \sum_{m=1}^{R} (\lambda_{m,n}^{k_1,\frac{2i+1}{K_2}} - \lambda_{m,n}^{k_1,\frac{2i+2}{K_2}}) \cdot e_{m,1}^{(k_1,\frac{2i+2}{K_2})} v^*_{(k_1,2i+2)} v_{(k_1,2i+1)} e_{1,n}^{(l,k_1,\frac{2i+1}{K_2})} \\
& + (\lambda_{m,n}^{k_1,\frac{2i+2}{K_2}} - \lambda_{m,n}^{k_1,\frac{2i+1}{K_2}}) \cdot e_{m,1}^{(k_1,\frac{k_2}{K_2})} v^*_{(k_1,2i+1)} v_{(k_1,2i+2)} e_{1,n}^{(l,k_1,\frac{2i+2}{K_2})} \|
\end{aligned}
$$

where

$$
D_1 = \begin{cases} \frac{K_2}{2} - 1, & \text{if } K_2 \text{ is even} \\ \frac{K_2-1}{2}, & \text{if } K_2 \text{ is odd,} \end{cases} \qquad D_2 = \begin{cases} \frac{K_2}{2} - 1, & \text{if } K_2 \text{ is even} \\ \frac{K_2-3}{2}, & \text{if } K_2 \text{ is odd.} \end{cases}
$$

Now one checks that if i and i' are both even or both odd, $i \neq i'$, then we have

$$
v^*_{(k_1,i)} x_0 v_{(k_1,i+1)} + v^*_{(k_1,i+1)} x_1 v_{(k_1,i)}
$$

orthogonal to

$$
v^*_{(k_1,i')} x_2 v_{(k_1,i'+1)} + v^*_{(k_1,i'+1)} x_3 v_{(k_1,i')}
$$

for any $x_0, x_1, x_2, x_2 \in A \otimes \mathcal{Q}$ since

$$
v_{(k_1,i+1)} v^*_{(k_1,i')} = v_{(k_1,i+1)} v^*_{(k_1,i'+1)} = v_{(k_1,i)} v^*_{(k_1,i')} = v_{(k_1,i)} v^*_{(k_1,i'+1)} = 0
$$

whenever $|i - i'| \geq 2$. Thus each summand in the norm estimates above are mutually orthogonal. Whence

$$
\begin{aligned}
\|v^*vf - fv^*v\| \quad & \leq \quad 4 \max_{k_1,k_2,m,n} |\lambda_{m,n}^{k_1,\frac{k_2}{K_2}} - \lambda_{m,n}^{k_1,\frac{k_2+1}{K_2}}| \\
& < \quad \epsilon/4.
\end{aligned}
$$

Now

$$\begin{aligned}\|s^*sa - as^*s\| &\leq \|s^*sf - fs^*s\| + 2\|a-f\|\\ &\leq \|v^*vf - fv^*v\| + 2\|s-v\| + 2\|a-f\|\\ &< \epsilon,\end{aligned}$$

showing (i).

Define $h \in \phi_{k_1}(\mathcal{C}([0,1]) \otimes M_R)$ by

$$h = \phi_{k_1}(\gamma_{k_2} \otimes (\lambda_{m,n}^{k_1,\frac{k_2}{K_2}})_{m,n}).$$

Now we have

$$\begin{aligned}v^*h &= (\textstyle\sum_{k_1,k_2,m} e_{m,1}^{(k_1,\frac{k_2}{K_2})} v^*_{(k_1,k_2)} \phi_{k_1}(1_{\mathcal{C}([0,1])} \otimes e_{1,m}))h\\ &= \textstyle\sum_{k_1,k_2,m} \sum_{\{k_2' \mid |k_2'-k_2|\leq 1\}} \lambda_{m,n}^{k_1,\frac{k_2'}{K_2}} \cdot e_{m,1}^{k_1,\frac{k_2}{K_2})} v^*_{(k_1,k_2)} \phi_{k_1}(\gamma_{k_2'} \otimes e_{1n}),\end{aligned}$$

and

$$\begin{aligned}fv^* &= \textstyle\sum_{k_1,k_2,m} \lambda_{m,n}^{(k_1,\frac{k_2}{K_2})} e_{m,1}^{(k_1,\frac{k_2}{K_2})} v^*_{(k_1,k_2)} \phi_{k_1}(1_{\mathcal{C}([0,1])} \otimes e_{1,m})\\ &= \textstyle\sum_{k_1,k_2,m,n} \sum_{\{k_2' \mid |k_2'-k_2|\leq 1\}} \lambda_{m,n}^{(k_1,\frac{k_2}{K_2})} \cdot e_{m,1}^{(k_1,\frac{k_2}{K_2})} v^*_{(k_1,k_2)} \phi_{k_1}(\gamma_{k_2}' \otimes e_{1,m})\end{aligned}$$

Thus

$$\begin{aligned}&\|fv^*v - v^*hv\|\\ &\leq \max_{k_1} \|\textstyle\sum_{k_2,m,n} \sum_{\{k_2' \mid |k_2'-k_2|\neq 1\}} (\lambda_{m,n}^{k_1,\frac{k_2'}{K_2}} - \lambda_{m,n}^{k_1,\frac{k_2}{K_2}}) \cdot e_{m,1}^{k_1,\frac{k_2}{K_2})} v^*_{(k_1,k_2)} \phi_{k_1}(\gamma_{k_2'} \otimes e_{1n}) v\|\\ &\leq \max_{k_1} (\|\textstyle\sum_{k_2 \text{ even},m,n} (\lambda_{m,n}^{k_1,\frac{k_2+1}{K_2}} - \lambda_{m,n}^{k_1,\frac{k_2}{K_2}}) \cdot e_{m,1}^{k_1,\frac{k_2}{K_2})} v^*_{(k_1,k_2)} \phi_{k_1}(\gamma_{k_2+1} \otimes e_{1n})\\ &\quad + (\lambda_{m,n}^{k_1,\frac{k_2}{K_2}} - \lambda_{m,n}^{k_1,\frac{k_2+1}{K_2}}) \cdot e_{m,1}^{k_1,\frac{k_2+1}{K_2})} v^*_{(k_1,k_2+1)} \phi_{k_1}(\gamma_{k_2} \otimes e_{1n})\|\\ &\quad + \|\textstyle\sum_{k_2 \text{ odd},m,n} (\lambda_{m,n}^{k_1,\frac{k_2+1}{K_2}} - \lambda_{m,n}^{k_1,\frac{k_2}{K_2}}) \cdot e_{m,1}^{k_1,\frac{k_2}{K_2})} v^*_{(k_1,k_2)} \phi_{k_1}(\gamma_{k_2+1} \otimes e_{1n})\\ &\quad + (\lambda_{m,n}^{k_1,\frac{k_2}{K_2}} - \lambda_{m,n}^{k_1,\frac{k_2+1}{K_2}}) \cdot e_{m,1}^{k_1,\frac{k_2+1}{K_2})} v^*_{(k_1,k_2+1)} \phi_{k_1}(\gamma_{k_2} \otimes e_{1n})\|),\end{aligned}$$

and one checks as above that the summands in both norm estimates are orthogonal, whence

$$\begin{aligned}\|fv^*v - v^*hv\| &\leq 4 \max_{k_1,k_2,m,n} |\lambda_{m,n}^{k_1,\frac{k_2}{K_2}} - \lambda_{m,n}^{k_1,\frac{k_2+1}{K_2}}|\\ &< \epsilon/4.\end{aligned}$$

Now for $a \in \mathcal{F}$ we have

$$\begin{aligned}\|s^*sas^*s - s^*hs\| &\leq \|fs^*s - s^*hs\| + \|a-f\|\\ &\leq \|fv^*v - v^*hv\| + 4\|s-v\| + \|a-f\|\\ &< \epsilon,\end{aligned}$$

showing (ii).

Finally, we show (iii).

We have

$$\begin{aligned}\tau(p) = \tau(s^*s) &= \tau(ss^*)\\ &= \tau(\textstyle\sum_{k_1=0}^{K_1} \phi_{k_1}(1_{\mathcal{C}([0,1])} \otimes 1_{M_R}))\\ &\geq 15/(64(L+1)) - \eta\\ &\stackrel{(3.14)}{\geq} 1/(8(L+1)).\end{aligned}$$

∎

Putting together the previous result with Lemma 2.3.3, we have proven the following theorem.

3.5.23 THEOREM: *Let A be a simple separable unital tracially approximately semihomogeneous* C*-*algebra of bounded dimension. Then $A \otimes \mathcal{Q}$ is TAI.*

3.5.24 COROLLARY: *Let A be a simple separable unital tracially approximately semihomogeneous* C*-*algebra of bounded dimension. Let $\mathfrak{p}$ be a supernatural number of infinite type and $M_{\mathfrak{p}}$ the associated UHF algebra. Then $A \otimes M_{\mathfrak{p}}$ is TAI.*

PROOF: This follows immediately from Theorem 3.5.23 and [40, Theorem 3.6] with [32, Theorem 7.1 (b)], which shows that a simple unital C*-algebra is TAI if and only if it has tracial rank less than or equal to one. However, [32, Theorem 7.1 (b)] uses Gong's decomposition theorem which we would prefer to avoid. Instead, we observe that tracial rank less than or equal to one can be replaced by TAI in the statements of Lemma 3.4 and Theorem 3.6 of [40] and that the proofs work in exactly the same way by simply replacing all C*-algebras of tracial rank less than or equal to one with C*-algebras that are TAI [35, Proposition 3.6]. ∎

3.6. Applications to classification

3.6.25 REMARK: If A is a simple separable unital AH algebra with finite decomposition rank, then Theorem 3.5.23 shows directly that $A \otimes \mathcal{Q}$ is TAI without requiring the full force of the classification programme.

3.6.26 COROLLARY: *Let A and B be simple separable unital tracially approximately semihomogeneous* C*-*algebras of bounded dimension satisfying the UCT. Then*

$$A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z}$$

if and only if

$$\mathrm{Ell}(A \otimes \mathcal{Z}) \cong \mathrm{Ell}(B \otimes \mathcal{Z}).$$

PROOF: Follows from Theorem 3.5.23 with [36, Corollary 11.9]. ∎

3.6.27 REMARK: Let A and B be simple separable unital AH algebras with no dimension growth. Then A and B have finite decomposition rank [29, Example 4.4] and hence are $\mathcal{Z}$-stable by [75, Theorem 5.1] and thus the previous corollary shows that $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$. This result was already shown in [18], however here we are able to get classification for such C*-algebras without appeal to the complicated reduction theorem of Gong [23].

3.6.28 PROPOSITION: *Let A be a separable simple unital approximately semihomogeneous* C*-*algebra. Then A satisifies the UCT.*

PROOF: For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$ we may approximate A by a semihomogeneous C*-algebra A. Clearly A satisfies the UCT. Therefore the result follows immediately by appealing to Theorem 1.1 of [13]. ∎

3.6.29 PROPOSITION: *Let A be a separable simple unital nonelementary approximately semihomogeneous* C*-*algebra with bounded dimension. Then A is $\mathcal{Z}$-stable.*

PROOF: By [75, Theorem 5.1] it is enough to show that A has finite decomposition rank. Since the approximating semihomogeneous C*-algebras can be chosen to have base spaces of covering dimension $\leq L$ for some $L \in \mathbb{N}$, we have that for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$ there is B with decomposition rank less than or equal to L [29, Remark 3.2 and Proposition 3.3]. and $\mathrm{dist}(\mathcal{F}, B) < \epsilon$. Thus the decomposition rank of A is also less than or equal to L; the proof of this is exactly the same as the proof of the case for the completely positive rank given in [72, Proposition 2.11]. ∎

3.6.30 COROLLARY: *Let A and B be separable simple unital approximately semihomogeneous* C*-*algebras of bounded dimension. Then*

$$A \cong B$$

if and only if

$$\mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

We require the next result to show that a separable simple unital approximately semihomogeneous C*-algebra of bounded dimension has the invariant of a separable simple AH algebra with slow dimension growth. The result is probably known, but we could not find a proof. Together with the classification result above, this allows us to show that simple AH algebras do in fact have a similar local property to that which is known in the AF case but is false in the nonsimple AH case; this is Corollary 3.6.32 below.

3.6.31 PROPOSITION: *Let A be a simple separable unital locally semihomogeneous* C*-*algebra with bounded dimension. Then if A is nonelementary, $K_0(A)/\mathrm{Tor}(K_0(A)) \neq \mathbb{Z}$.*

PROOF: We claim that since A is simple, for every $n \in \mathbb{N}$ there is an $\epsilon > 0$ and a finite subset $\mathcal{F} \subset A$ such that $A = \oplus_{k=1}^{N} p_k(\mathcal{C}(X_k) \otimes M_{r_k})p_k \subset A$ with $\mathrm{rank}(p_k) > n$, $1 \leq k \leq N$. Suppose not. Let $(a_i)_{i\in\mathbb{N}}$ be a dense subset in A and let $\mathcal{F}_i = \{a_0, \ldots, a_i\}$. We can choose $\epsilon_1 > \epsilon_2 > \cdots > 0$, $\epsilon_n \to 0$ as $n \to \infty$ such that $A_i = \oplus_{k=1}^{N} p_{i,k}(\mathcal{C}(X_{i,k}) \otimes M_{r_{i,k}})p_{i,k}$ has $\mathrm{rank}(p_{i,k}) \leq n$ for some $1 \leq k \leq N_i$. Passing to a subsequence if necessary, we may assume that $\mathrm{rank}(p_{i,k}) = m \leq n$ for all $i \in \mathbb{N}$.

For each $i \in \mathbb{N}$, let $x_i \in X_{i,k}$ (where k is chosen so that $\mathrm{rank}(p_{i,k}) = m$). Let

$$\mathrm{ev}_i : A_i \to M_m$$

be the representation given by evaluation at the point x_i. Let $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ be a free ultrafilter. Taking the limit along ω of $(\mathrm{ev}_i)_{i\in\mathbb{N}}$ gives a map

$$\pi : \Pi_{i\in\mathbb{N}} A_i / \oplus_{i\in\mathbb{N}} A_i \to M_m : (f_i)_{i\in\mathbb{N}} \mapsto \lim_{i\to\omega} \mathrm{ev}_i(f_i).$$

It is easy to check that since $\|f_i\| \to 0$ as $i \to \infty$ implies $\lim_{i\to\omega} \mathrm{ev}_i(f_i) = 0$, the map π is a well-defined *-homomorphism. Let $f \in A$. Write $f = \lim_i b_i$ where each $b_i \in \mathcal{F}_i$. Choose $f_i \in A_i$ such that $\|b_i - f_i\| < \epsilon_i$. Then since $\epsilon_i \to 0$ and $b_i \to f$ as $i \to \infty$, the map $f \mapsto (f_i)_{i\in\mathbb{N}}$ is a well-defined injective *-homomorphism from A into $\Pi_{i\in\mathbb{N}} A_i / \oplus_{i\in\mathbb{N}} A_i$. Thus we may restrict π to A. By simplicity of A, we see that $A \hookrightarrow M_m$. But A is nonelementary, contradiction. This proves the claim.

Next, we show that if $x \in K_0(A)/\mathrm{Tor}(K_0(A))$ generates the entire group then we may assume $x = [q]$ for some projection $q \in M_\infty(A)$. Since A satisfies the UCT by Proposition 3.6.28, it follows from [40, Proposition 10] that $(K_0(A), K_0(A)_+, [1_A])$ is weakly unperforated. Thus $K_0(A)/\mathrm{Tor}(K_0(A))$ is unperforated. Since x generates $K_0(A)/\mathrm{Tor}(K_0(A))$ and $[1_A] \in K_0(A)/\mathrm{Tor}(K_0(A))$ we have that $nx = [1_A] > 0$ for some $n \in \mathbb{Z}$. Clearly if $n = 0$ then $K_0(A)/\mathrm{Tor}(K_0(A)) \neq \mathbb{Z}$. Thus if $n > 0$ we have that $x > 0$, and otherwise, we may replace x with $-x$. Thus we may assume the generator $x > 0$, that is, $x = [q]$ for some projection $q \in M_\infty(A)$.

Finally, we will show that $x = [q]$ can be decomposed into a direct sum of smaller rank projections, showing that x cannot generate all of $K_0(A)/\mathrm{Tor}(K_0(A))$. To do this, let $(A_{\epsilon_i})_{i\in\mathbb{N}}$, $\epsilon_1 > \epsilon_2 > \cdots > 0$

with $\epsilon_n \to 0$ as $n \to 0$ be a sequence of semihomogeneous C*-algebras exhausting A with $\tau(1_A - 1_{A_{\epsilon_i}}) < \epsilon_i$ for all $i \in \mathbb{N}$, for all $\tau \in T(A)$. By taking ϵ_i to be sufficiently small, we can find projections $q_i \in A_{\epsilon_i}$ sufficiently close to p so that the they are all Murray–von Neumann equivalent (using, eg., [31, Lemma 2.5.1]).

Since $x = [q] = [q_i]$ generates $K_0(A)/\mathrm{Tor}(K_0(A))$, we must have that each p_i is supported over only one of the $X_{i,k}$ of the direct sum $\oplus_{k=1}^{N_i} p_{i,k}(\mathcal{C}(X_{i,k}) \otimes M_{r_{i,k}})p_{i,k}$. We may assume each $X_{i,k}$ to be a connected simplicial complex so that $\sigma_i \mapsto \sigma_i(q_i)$ is constant on $T(A_i)$. Thus for any $\sigma_i \in T(A_i)$ we get $\sigma_i(q_i) = \mathrm{rank}(q_i)/\mathrm{rank}(p_{i,k})$. In particular, for any $\tau \in T(A)$ we have that $1/(\tau(1_{A_{\epsilon_i}})) \cdot \tau$ is a tracial state on $T(A_i)$, so $\mathrm{rank}(q_i)/\mathrm{rank}(p_{i,k}) \cdot (1 - \epsilon_i) \leq \tau(q_i) \leq \mathrm{rank}(q_i)/\mathrm{rank}(p_{i,k}) \cdot (1 + \epsilon_i)$. By the above, $\mathrm{rank}(p_{i,k}) \to \infty$ so for sufficiently large i, we may represent x by q_i with $\mathrm{rank}(q_i) > (\dim(X_{i,k}) - 1)/2$. But then by [25, Proposition 9.1.1] $x = [q] = [p] + [r]$ for some projection $[p]$ of lesser rank. Thus x cannot generate $K_0(A)/\mathrm{Tor}(K_0(A))$ so $K_0(A)/\mathrm{Tor}(K_0(A))$ is non-cyclic. ∎

We remark that if A is a separable unital semihomogeneous C*-algebra that is not necessarily simple it need not be AH (even if it has bounded dimension) as is shown by the example in [14]. However, in the simple case, we see that this is true.

3.6.32 Corollary: *Let A be a simple separable unital locally semihomogeneous* C* *-algebra of bounded dimension. Then A is a simple unital AH algebra with slow dimension growth. Furthermore, the class of simple separable unital AH algebras with slow dimension growth is closed under simple unital inductive limits.*

Proof: By Proposition 3.6.31 with the range result for simple AH algebras in [70] we have that $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$ for some simple unital AH algebra B with slow dimension growth. Since [70, Theorem 2] in fact shows we can assume B is of bounded dimension less than or equal to three, we have $A \cong B$ by Corollary 4.3.20. For the second statement, let $A = \varinjlim A_n$ where A is unital and simple and each A_n is a simple separable unital AH algebra with slow dimension growth. By [78, Corollary 6.7] we may in fact assume that each A_n is $\mathcal{Z}$-stable and has no dimension growth. It thus follows from the classification above that we can assume each A_n has dimension bound of three. Then A must be a unital simple locally semihomogeneous with bounded dimension and hence a unital separable simple AH algebra with slow dimension growth. ∎

4. Classification of certain locally recursive subhomogeneous C^*-algebras

In an effort to generalize the results of the previous chapter, we develop techniques for certain C*-algebras which are locally subhomogeneous rather than locally semihomogeneous. The structure of a locally subhomogeneous C*-algebra comes with additional complications to those of the previous chapter. For example, the primitive ideal space of a subhomogeneous C*-algebra need not be Hausdorff. In this case, setting up the analogue of the "discrete" version of the interval as in the last section is significantly more complicated. For this reason, some extra restrictions are required. Nevertheless, we are able to arrive at a classification result for a class of C*-algebras where projections do not separate tracial states. In fact, we insist on the opposite: all tracial states must induce the same state on the K_0-group.

The chapter is organized as follows. In Section 4.1 we introduce notation for recursive subhomogeneous C*-algebras and definitions for $(\mathcal{F}, \eta)$-excisors and $(\mathcal{F}, \eta)$-bridges as well as statements of key results that will be required (the details for this section are given in Appendix A). Section 4.2 provides the technical results to find an interval that is large on all traces as well as a method for moving this interval from a general position and placing it underneath the discrete model given by the $(\mathcal{F}, \eta)$-path. The proof of the main result is then given in Section 4.3 where applications and an outlook are also discussed.

4.1. Recursive subhomogeneous C^*-algebras, $(\mathcal{F}, \eta)$-excisors, and $(\mathcal{F}, \eta)$-bridges

In [47], Phillips introduced the notion of recursive subhomogeneous algebras. These are subhomogeneous C*-algebras (that is, all irreducible representations are bounded in dimension) which arise as iterated pullbacks of homogeneous C*-algebras.

4.1.1 Definition: [47, Definition 1.1] A recursive subhomogeneous (RSH) algebra is a C*-algebra with the following recursive definition:

(i) Let X be a compact Hausdorff space and $n \in \mathbb{N}$. Then $\mathcal{C}(X, M_n)$ is a recursive subhomogeneous algebra.

(ii) Let A be a recursive subhomogeneous C*-algebra, X a compact Hausdorff space and $n \in \mathbb{N}$. Suppose $\Omega \subset X$ is a closed (possibly empty) subset, $\phi : A \to \mathcal{C}(\Omega, M_n)$ is a unital *-homomorphism, and let $\rho : \mathcal{C}(X, M_n) \to \mathcal{C}(\Omega, M_n)$ be the restriction map. Then the pullback

$$A \oplus_{\mathcal{C}(\Omega, M_n)} \mathcal{C}(X, M_n) = \{(a, f) \in A \oplus \mathcal{C}(X, M_n) \mid \phi(a) = \rho(f)\}$$

is a recursive subhomogeneous C*-algebra.

We will restrict to the case where the X in the above definition is metrizable so that the resulting C*-algebra is separable. Note also that Definition 4.1.1 implies that a recursive subhomogeneous C*-algebra is unital.

For a given C*-algebra, its recursive subhomogeneous decomposition (if it exists) is not unique; for us it will be important to keep track of the actual decompositions.

If B is a recursive subhomogeneous algebra, then we may write (cf. [47, Definition 1.5])

$$B = \Big(\dots\Big(\Big(C_0 \oplus_{C_1^{(0)}} C_1\Big) \oplus_{C_2^{(0)}} C_2\Big)\dots\Big) \oplus_{C_R^{(0)}} C_R,$$

where $C_l = \mathcal{C}(X_l)\otimes M_{n_l}$ for some compact metrizable X_l, some integer $n_l \geq 1$, and $C_l^{(0)} = \mathcal{C}(\Omega_l)\otimes M_{n_l}$ for a closed subset $\Omega_l \subset X_l$, $l \in \{0,\dots,R\}$.

For $0 \leq l \leq R$, define the l^{th}-stage of B to be the C*-algebra obtained by truncating the recursion after the l^{th} step,

$$B_l = \Big(\dots\Big(\Big(C_0 \oplus_{C_1^{(0)}} C_1\Big) \oplus_{C_2^{(0)}} C_2\Big)\dots\Big) \oplus_{C_l^{(0)}} C_l.$$

Note that $B_R = B$.

4.1.2 Definition: Let B be a unital recursive subhomogeneous C*-algebra. A recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$$

for B consists of compact Hausdorff spaces $\Omega_l \subset X_l$, $r_l \in \mathbb{N}$, unital C*-algebras B_l for $l \in \{1,\dots,R\}$, and of unital *-homomorphisms

$$\phi_l : B_l \to \mathcal{C}(\Omega_{l+1}) \otimes M_{r_{l+1}}$$

for $l \in \{1,\dots,R-1\}$, such that

$$\Omega_1 = \emptyset,\ B_1 = \mathcal{C}(X_1)\otimes M_{r_1},\ B_R = B,$$

and such that we have pullback diagrams

$$\begin{tikzcd} B_{l+1} \arrow[r, two heads] \arrow[d] & B_l \arrow[d, "\phi_l"] \\ \mathcal{C}(X_{l+1}) \otimes M_{r_{l+1}} \arrow[r, two heads] & \mathcal{C}(\Omega_{l+1}) \otimes M_{r_{l+1}}, \end{tikzcd} \tag{4.1}$$

where the lower horizontal map is restriction. We then have a canonical unital embedding

$$\iota_B : B \hookrightarrow \mathcal{C}(X_1)\otimes M_{r_1} \oplus \dots \oplus \mathcal{C}(X_R)\otimes M_{r_R},$$

canonical quotient maps

$$\psi_l : B \to B_l$$

and canonical embeddings

$$\iota_l : \mathcal{C}_0(X_l \setminus \Omega_l) \otimes M_{r_l} \hookrightarrow B_l$$

for $l \in \{1,\dots,R\}$.

We will usually assume that $X_{l+1} \setminus \Omega_{l+1} \neq \emptyset$, for otherwise the horizontal maps in (4.1) are just equalities.

4.1.3 Remark: In the situation above, if $x \in X_l \setminus \Omega_l$ for some $l \in \{1,\dots,R\}$, then the map $(\mathrm{ev}_x \otimes \mathrm{id}_{M_{r_l}}) \circ \iota_B : B \to M_{r_l}$ is surjective.

4.1.4 Definition: Let B be a unital recursive subhomogeneous C*-algebra with recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R.$$

We say that projections can be lifted along $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$, if for any $N \in \mathbb{N}$, any $l \in \{1, \ldots, R-1\}$ and any projection $p \in B_l \otimes M_N$ there is a projection $\bar{p} \in B_{l+1} \otimes M_N$ lifting p.

The following is shown in Appendix A (see Corollary A.1.0).

4.1.5 Proposition: *Let B be a unital recursive subhomogeneous* C**-algebra with decomposition*

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R.$$

Assume that $\dim X_l \leq 1$ *for* $l \geq 2$.

Then projections can be lifted along $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$.

Recall that a completely positive map has order zero when it preserves orthogonality, that is, a c.p. map $\phi : A \to B$ between the C*-algebras A and B such that, for any positive elements $a, b \in A$ with $ab = 0$ we have $\phi(a)\phi(b) = 0$ in B.

In the previous chapter, we constructed a discrete version of a tracially large interval by taking a number of point evaluations and then extending matrix units across small neighbourhoods of these points. Observe that this amounts to taking *-homomorphisms into finite-dimensional C*-algebras followed by order zero lifts which map back into the approximating semihomogeneous C*-algebra. Moreover, because we asked that the neighbourhoods be sufficiently small with respect to the finite subset, the image of the unit of the finite-dimensional C*-algebra multiplied with some $f \in \mathcal{F}$ was approximately equal to mapping f into the finite-dimensional algebra (by point evaluation) and then mapping back into the semihomogeneous C*-algebra via the extended matrix units.

Such a naïve approach is no longer successful in the case of approximately recursive subhomogeneous C*-algebras. The problem in this situation is that the RSH algebra can have nontrivial overlappings of base spaces which might not even be Hausdorff. If we tried to take a series of point evaluations and matrix unit extensions across one of these boundaries, we may run into trouble when we try to then approximate an element in $\mathcal{F}$.

To get around this problem, these ideas from the previous chapter are generalized to the concept of $(\mathcal{F},\eta)$-excisors and $(\mathcal{F},\eta)$-bridges. These were developed by Winter and will appear in our joint paper [59]. Further details are given in Appendix A.

4.1.6 Definition: Let B be a unital recursive subhomogeneous C*-algebra with recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R;$$

let $\mathcal{F} \subset B_+^1$ be a finite subset, where B_+^1 denotes the positive elements in the unit ball of B, and $\eta > 0$ be given.

An $(\mathcal{F},\eta)$-excisor (E, ρ, σ) for B consists of a finite-dimensional C*-algebra

$$E = \bigoplus_{l=1}^R E_l,$$

a unital *-homomorphism

$$\rho = \oplus_{l=1}^R \rho_l : B \to \bigoplus_{l=1}^R E_l = E$$

and an isometric c.p. order zero map

$$\sigma = \oplus_{l=1}^R \sigma_l : \bigoplus_{l=1}^R E_l = E \to B \otimes \mathcal{Q}$$

such that

$$\|\sigma(1_E)(b \otimes 1_{\mathcal{Q}}) - \sigma\rho(b)\| < \eta \text{ for } b \in \mathcal{F}.$$

We say (E, ρ, σ) is compatible with $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$, if each ρ_l factorizes through

$$\begin{tikzcd} B \arrow[r, "\rho_l"] \arrow[d, "\psi_l"'] & E_l \\ B_l \arrow[r, "\check{\psi}_l"] & \mathcal{C}(\check{X}_l) \otimes M_{r_l} \arrow[u, "\check{\rho}_l"'] \end{tikzcd}$$

for some compact $\check{X}_l \subset X_l \setminus \Omega_l$.

If (E, ρ, σ) is as above and

$$\kappa : E \to \mathcal{Q}$$

is a unital *-homomorphism, we say $(E, \rho, \sigma, \kappa)$ is a weighted $(\mathcal{F}, \eta)$-excisor compatible with the decomposition $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$.

4.1.7 Definition: Let B be a unital recursive subhomogeneous C*-algebra with recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R;$$

let $\mathcal{F} \subset B_+^1$ finite and $\eta > 0$ be given. Let $(E_i, \rho_i, \sigma_i, \kappa_i)$, $i \in \{0, 1\}$, be weighted $(\mathcal{F}, \eta)$-excisors (compatible with $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$).

An $(\mathcal{F}, \eta)$-bridge from $(E_0, \rho_0, \sigma_0, \kappa_0)$ to $(E_1, \rho_1, \sigma_1, \kappa_1)$ (compatible with $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$) consists of $K \in \mathbb{N}$ and weighted $(\mathcal{F}, \eta)$-excisors (each compatible with $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$)

$$(E_{\frac{j}{K}}, \rho_{\frac{j}{K}}, \sigma_{\frac{j}{K}}, \kappa_{\frac{j}{K}}),\ j \in \{1, \ldots, K-1\},$$

satisfying

$$\|\kappa_{\frac{j}{K}} \rho_{\frac{j}{K}}(b) - \kappa_{\frac{j+1}{K}} \rho_{\frac{j+1}{K}}(b)\| < \eta \text{ for } b \in \mathcal{F} \text{ and } j \in \{0, \ldots, K-1\}. \tag{4.2}$$

We write

$$(E_0, \rho_0, \sigma_0, \kappa_0) \sim_{(\mathcal{F},\eta)} (E_1, \rho_1, \sigma_1, \kappa_1)$$

if such an $(\mathcal{F}, \eta)$-bridge exists.

As in the semihomogeneous case where we took a point evaluation at a point x and extended matrix units across a small neighbourhood of x, we can define an $(\mathcal{F}, \eta)$-excisor at a given point x.

4.1.8 Notation: Let B be a unital recursive subhomogeneous C*-algebra with recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R.$$

If $l \in \{1, \ldots, R\}$ and $x \in X_l$, then

$$(\mathrm{ev}_x \otimes \mathrm{id}_{M_{r_l}}) \circ \iota_B : B \to M_{r_l}$$

factorizes through a sum of irreducible representations, say

$$B \xrightarrow{\rho_x} E_x \xrightarrow{\iota_{E_x}} M_{r_l}.$$

Upon fixing a unital embedding

$$M_{r_l} \to \mathcal{Q}$$

we obtain unital *-homomorphisms

$$B \xrightarrow{\rho_x} E_x \xrightarrow{\kappa_x} \mathcal{Q}$$

such that ρ_x is a sum of surjective irreducible representations and

$$\tau_{\mathcal{Q}} \kappa_x = \tau_{M_{r_l}} \iota_{E_x}.$$

4.1.9 Definition: Let B be a unital recursive subhomogeneous C*-algebra, and let $\mathcal{F} \subset B_+^1$ finite and $\eta > 0$ be given.

A recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$$

for B is $(\mathcal{F}, \eta)$-connected if the following holds:

If $l \in \{1, \ldots, R\}$ and $x, y \in X_l$, and if $(E_x, \rho_x, \sigma_x, \kappa_x)$ and $(E_y, \rho_y, \sigma_y, \kappa_y)$ are $(\mathcal{F}, \eta)$-excisors with $(E_x,\, \rho_x,\, \kappa_x)$ and $(E_y,\, \rho_y,\, \kappa_y)$ as in 4.1.8, then

$$(E_x, \rho_x, \sigma_x, \kappa_x) \sim_{(\mathcal{F},\eta)} (E_y, \rho_y, \sigma_y, \kappa_y).$$

4.1.10 LEMMA: *Let B be a separable unital recursive subhomogeneous C^*-algebra and let $\mathcal{F} \subset B^1_+$ finite and $\eta > 0$ be given.*

Suppose B has an $(\mathcal{F}, \eta)$-connected recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$$

along which projections can be lifted and such that $X_l \setminus \Omega_l \neq \emptyset$ for $l \geq 1$.

Let $\tau^{(0)}, \ldots, \tau^{(n-1)} \in T(B)$ be n faithful tracial states with

$$(\tau^{(0)})_* = \ldots = (\tau^{(n-1)})_*$$

(as states on the ordered $\mathrm{K}_0(B)$).

Then there are

$$0 = K_0 < K_1 < \ldots < K_{n-1} = K \in \mathbb{N}$$

and pairwise orthogonal weighted $(\mathcal{F}, \eta)$-excisors

$$(Q_{\frac{j}{K}}, \rho_{\frac{j}{K}}, \sigma_{\frac{j}{K}}, \kappa_{\frac{j}{K}}),\ j \in \{0, \ldots, K\},$$

implementing $(\mathcal{F}, \eta)$-bridges

$$\begin{aligned}(Q_{\frac{K_0}{K}}, \rho_{\frac{K_0}{K}}, \sigma_{\frac{K_0}{K}}, \kappa_{\frac{K_0}{K}}) \quad &\sim_{(\mathcal{F},\eta)} \cdots \sim_{(\mathcal{F},\eta)} \quad (Q_{\frac{K_m}{K}}, \rho_{\frac{K_m}{K}}, \sigma_{\frac{K_m}{K}}, \kappa_{\frac{K_m}{K}}) \\ &\sim_{(\mathcal{F},\eta)} \cdots \sim_{(\mathcal{F},\eta)} \quad (Q_{\frac{K_{n-1}}{K}}, \rho_{\frac{K_{n-1}}{K}}, \sigma_{\frac{K_{n-1}}{K}}, \kappa_{\frac{K_{n-1}}{K}}),\end{aligned}$$

and such that, for each projection $q \in Q_{\frac{K_m}{K}}$, $m \in \{0, \ldots, n-1\}$,

$$(\tau^{(m)} \otimes \tau_{\mathcal{Q}})\sigma_{\frac{K_m}{K}}(q) \geq \frac{1}{n+1} \cdot \tau_{\mathcal{Q}}\kappa_{\frac{K_m}{K}}(q). \tag{4.3}$$

4.2. Tracially large intervals

The technical foundation for the main result, Theorem 4.3.15, is laid out in Theorem 4.2.14. There we must find an interval in the C^*-algebra that is large on all traces and can be moved into position under the "discrete" version of the interval that will come from the $(\mathcal{F}, \eta)$-bridges of the previous section (see also Appendix A). The interval is twisted into place using a partial isometry obtained from strict comparison. To do this, we will tracially match the endpoints of the $(\mathcal{F}, \eta)$-bridges to functions in a partition of unity. This requires that the finitely many traces be separated along the interval. The next lemma shows that we can find an interval with each trace approximately concentrated at distinct points.

In Proposition 4.2.13, we find a positive element which acts as an "almost" partial isometry which takes order zero maps to order zero maps. In Theorem 4.2.14 such an element will be perturbed into an honest partial isometry (dependent on the finite subset $\mathcal{F} \subset A \otimes \mathcal{Q}$ and $\epsilon > 0$) and its support projection will be the unit for the approximating interval algebra. Proposition 4.2.13 below will furnish this unit with the appropriate properties so that (i) and (ii) of Definition 2.2.2 are satisfied.

The next two lemmas show how to find an interval that is large on all traces. In Lemma 4.2.11 we follow the techniques of A. Kishimoto in [30, Theorem 4.5], Matui and Y. Sato in [44, Lemma 3.2] and Toms, S. White and Winter in [63, Lemma 3.4] to move positive contractions of given tracial sizes that are approximately tracially orthogonal to positive contractions which are norm orthogonal and remain approximately the same tracial size as the original elements. In Lemma 4.2.12 we line up pairwise orthogonal elements, which, using Lemma 4.2.11, can be of a specified tracial size, in such a way as to generate an interval.

4.2.11 Lemma: *For every $\epsilon > 0$ and every $k \in \mathbb{N}$ there is $\delta > 0$ such that if A is a separable unital C*-algebra with $T(A) \neq \emptyset$ and $a_0, \ldots, a_k \in A$ are positive contractions satisfying*

$$\tau(a_i a_{i'}) < \delta \text{ for all } \tau \in T(A), i \neq i',$$

then there are pairwise orthogonal positive contractions $b_0, \ldots, b_k \in A$ satisfying

$$0 < \tau(a_i - b_i) < \epsilon \text{ for all } \tau \in T(A).$$

Proof: First of all, there is a $0 < \delta_0 < 1$ such that if A is a C*-algebra and $e_0, \ldots, e_k \in A_+$ are contractions satisfying $\|e_i e_{i'}\| < \delta_0$ when $i \neq i' \in \{0, \ldots, k\}$ then there are contractions $\tilde{e}_0, \ldots, \tilde{e}_k \in A_+$ such that $\|\tilde{e}_i - e_i\| < \epsilon/2$ and $\tilde{e}_i \tilde{e}_{i'} = 0$ for every $i \neq i' \in \{0, \ldots, k\}$ [31, Lemma 2.5.15].

Define a continuous function $f_{\delta_0} : (0, \infty] \to [0, 1]$ by

$$f_{\delta_0}(t) = \min(1, \frac{t}{\delta_0}).$$

Note that $(1 - f_{\delta_0}(t))t \leq \delta_0$ for every $t \geq 0$.

Let A be a separable unital C*-algebra with nonempty tracial state space $T(A)$. Choose

$$0 < \delta < \frac{\epsilon \cdot \delta_0}{2k}$$

and suppose that $a_0, \ldots, a_k \in A$ are positive contractions with $\tau(a_i a_{i'}) < \delta$ for every $\tau \in T(A)$ whenever $i \neq i' \in \{0, \ldots, k\}$.

For each $i \in \{0, \ldots, k\}$ define

$$g_i = a_i^{1/2} \left(\sum_{\substack{i' \in \{0, \ldots, k\} \\ i' \neq i}} a_{i'} \right) a_i^{1/2}. \tag{4.4}$$

Then

$$\tau(g_i) = \tau \left(\sum_{\substack{i' \in \{0, \ldots, k\} \\ i' \neq i}} a_i a_{i'} \right) < k\delta < \tfrac{\epsilon \cdot \delta_0}{2}$$

for every $\tau \in T(A)$.

For each $i \in \{0, \ldots, k\}$ define positive contractions

$$x_i = a_i^{1/2}(1 - f_{\delta_0}(g_i))a_i^{1/2}. \tag{4.5}$$

We have that $f_{\delta_0}(t) \leq \frac{t}{\delta_0}$ for every $t \in [0, k-1]$ from which it follows that

$$\begin{aligned} 0 \leq \tau(a_i - x_i) &= \tau(a_i^{1/2} f_{\delta_0}(g_i) a_i^{1/2}) \\ &\leq \tau(f_{\delta_0}(g_i)) \\ &\leq \frac{\tau(g_i)}{\delta_0} \\ &< \frac{\epsilon}{2}, \end{aligned}$$

for every $\tau \in T(A)$.

We compute

$$\begin{array}{rcl} \|x_{i'}x_i\|^2 & = & \|x_i x_{i'}^2 x_i\| \\ & \leq & \|x_i(\sum_{\substack{j\in\{0,\ldots,k\}\\ j\neq i}} x_j)x_i\| \\ & \overset{(4.5)}{=} & \|a_i^{1/2}(1-f_{\delta_0}(g_i))a_i^{1/2}(\sum_{\substack{j\in\{0,\ldots,k\}\\ j\neq i}} x_j)a_i^{1/2}(1-f_{\delta_0}(g_i))a_i^{1/2}\| \\ & \overset{(4.5)}{\leq} & \|a_i^{1/2}(1-f_{\delta_0}(g_i))a_i^{1/2}(\sum_{\substack{j\in\{0,\ldots,k\}\\ j\neq i}} a_j^{1/2}(1-f_{\delta_0}(g_j))a_j^{1/2})a_i^{1/2} \\ & & (1-f_{\delta_0}(g_i))a_i^{1/2}\| \\ & \leq & \|a_i^{1/2}(1-f_{\delta_0}(g_i))a_i^{1/2}(\sum_{\substack{j\in\{0,\ldots,k\}\\ j\neq i}} a_j)a_i^{1/2}(1-f_{\delta_0}(g_i))a_i^{1/2}\| \\ & \overset{(4.4)}{=} & \|a_i^{1/2}(1-f_{\delta_0}(g_i))g_i(1-f_{\delta_0}(g_i))a_i^{1/2}\| \\ & \leq & \|(1-f_{\delta_0}(g_i))g_i\| \\ & \leq & \delta_0. \end{array}$$

By the choice of δ_0 there are $b_0,\ldots,b_1 \in A$ pairwise orthogonal positive contractions with $\|b_i - x_i\| < \epsilon/2$. Thus

$$\begin{array}{rcl} \tau(a_i - b_i) & = & \tau(a_i - x_i) + \tau(x_i - b_i) \\ & < & \epsilon/2 + \|b_i - x_i\| \cdot \tau(1_A) \\ & < & \epsilon. \end{array}$$

∎

4.2.12 For $0 \leq \beta_1 < \beta_2 \leq 1$, define functions f_{β_1,β_2} and $g_{\beta_1,\beta_2} \in \mathcal{C}_0((0,1])$ by

$$f_{\beta_1,\beta_2}(t) = \begin{cases} 0, & 0 \leq t \leq \beta_1 \\ \text{linear}, & \beta_1 \leq t \leq \beta_2 \\ t, & \beta_2 \leq t \leq 1. \end{cases}$$

$$g_{\beta_1,\beta_2}(t) = \begin{cases} 0, & 0 \leq t \leq \beta_1 \\ \text{linear}, & \beta_1 \leq t \leq \beta_2 \\ 1, & \beta_2 \leq t \leq 1. \end{cases}$$

Note that if $\beta_1 < \beta_2 < \beta_3 \leq 1$ then

$$g_{\beta_1,\beta_2} f_{\beta_2,\beta_3} = f_{\beta_2,\beta_3} g_{\beta_1,\beta_2} = f_{\beta_2,\beta_3}. \tag{4.6}$$

LEMMA: *Let A be a separable simple unital nuclear* C^**-algebra with exactly $n > 0$ extreme tracial states $\tau_0,\ldots,\tau_{n-1} \in T(A)$. For $i \in \{0,\ldots,n-1\}$, define continuous functions on $[0,1]$ by*

$$\gamma_i(t) = \begin{cases} 0, & t \in [0,1] \cap ((-\infty, \frac{i-1}{n-1}] \cup [\frac{i+1}{n-1}, \infty)) \\ 1, & t = \frac{i}{n-1} \\ \textit{linear}, & \textit{elsewhere}. \end{cases}$$

Then for any $\delta > 0$ there is a *-homomorphism

$$\phi : \mathcal{C}([0,1]) \to A \otimes \mathcal{Q}$$

such that for $i \in \{0,\ldots,n-1\}$

$$\tau_i \otimes \tau_{\mathcal{Q}}(\phi(\gamma_i)) \geq 1 - \delta,$$

and

$$0 < \tau_j \otimes \tau_{\mathcal{Q}}(\phi(\gamma_i)) < \delta$$

when $j \neq i$.

Proof: Choose $0 < \beta < \min(\frac{1}{2}, \frac{\delta}{3})$ and from Lemma 4.2.11 obtain δ_0 for $n-1$ in place of k and $\epsilon < \frac{\delta}{6}$.

Let $\mathrm{Aff_b}(T(A \otimes \mathcal{Q}))$ denote the set of $\mathbb{R}$-valued bounded affine functions on the tracial state space $T(A \otimes \mathcal{Q})$. For each $i \in \{0, \ldots, n-1\}$ define continuous functions $\tilde{h}_i$ on the extreme boundary of $T(A \otimes \mathcal{Q})$ by

$$\tilde{h}_i(\tau_i \otimes \tau_{\mathcal{Q}}) = 1$$

and

$$0 < \tilde{h}_i(\tau_j) \leq \min(\tfrac{\delta}{6}, \delta_0) \text{ when } i \neq j.$$

Since the extreme boundary of $T(A \otimes \mathcal{Q})$ has only finitely many points and hence is compact, each $\tilde{h}_i$ extends to a continuous affine function $h_i \in \mathrm{Aff_b}(T(A \otimes \mathcal{Q}))$ satisfying $0 < h_i(\tau) \leq 1$ for all $\tau \in T(A \otimes \mathcal{Q})$ [1, Theorem II.3.12].

Note that the h_i are not only continuous but are also strictly positive. Since A is simple and unital, by [7, Corollary 3.10], there are positive contractions $a_i \in A_+$ satisfying

$$\tau(a_i) = h_i(\tau) \text{ for all } \tau \in T(A \otimes \mathcal{Q}).$$

This gives

$$\tau(a_i a_{i'}) < \delta_0 \text{ for all } \tau \in T(A \otimes \mathcal{Q}) \text{ and } i \neq i',$$

whence the previous lemma allows us to obtain pairwise orthogonal positive contractions $y_0, \ldots, y_{n-1} \in A \otimes \mathcal{Q}$ such that $\tau_i \otimes \tau_{\mathcal{Q}}(y_i) \geq 1 - \frac{\delta}{3}$ and $\tau_{i'} \otimes \tau_{\mathcal{Q}}(y_i) \leq \frac{\delta}{3}$ for $i \neq i' \in \{0, \ldots, n-1\}$.

Define the following positive elements:

$$\begin{aligned}
\tilde{b}_{n-1} &= y_{n-1} \\
b_{n-1} &= f_{\beta,2\beta}(\tilde{b}_{n-1}) \\
\tilde{b}_{n-2} &= g_{0,\beta}(\tilde{b}_{n-1}) + y_{n-2} \\
b_{n-2} &= f_{\beta,2\beta}(\tilde{b}_{n-2}) \\
&\vdots \\
\tilde{b}_1 &= g_{0,\beta}(\tilde{b}_2) + y_1 \\
b_1 &= f_{\beta,2\beta}(\tilde{b}_1) \\
b_0 &= 1.
\end{aligned}$$

Then we have

$$\|b_i\| \leq 1 \text{ and } b_i b_{i-1} = b_{i-1} b_i = b_i \text{ for every } i \in \{1, \ldots, n-1\}. \tag{$\mathcal{R}$}$$

Thus we obtain the map

$$\phi : \mathcal{C}([0,1]) \to \mathrm{C}^*(b_0, \ldots, b_{n-1})$$

satisfying

$$\phi(1_{\mathcal{C}([0,1])}) = b_0 \text{ and } \phi(g_{\frac{i-1}{n-1}, \frac{i}{n-1}}) = b_i \text{ for } i \in \{1, \ldots, n-1\},$$

since $\mathcal{C}([0,1])$ can be written as the universal C*-algebra generated by positive contractions satisfying the relations $(\mathcal{R})$.

For each $i = 1, \ldots, n-2$ we have that

$$\gamma_i = g_{\frac{i-1}{n-1}, \frac{i}{n-1}} - g_{\frac{i}{n-1}, \frac{i+1}{n-1}}$$

and also

$$\gamma_0 = 1 - g_{0,\frac{1}{n-1}}, \qquad \gamma_{n-1} = g_{\frac{n-2}{n-1},1}.$$

Thus $\phi(\gamma_i) = b_i - b_{i+1}$.

We note that

$$\begin{aligned}
\tau_i \otimes \tau_{\mathcal{Q}}(b_{i+1}) &\leq \tau_i \otimes \tau_{\mathcal{Q}}(\tilde{b}_{i+1}) \\
&= \tau_i \otimes \tau_{\mathcal{Q}}(g_{0,\beta}(\tilde{b}_{i+2})) + \tau_i \otimes \tau_{\mathcal{Q}}(y_{i+1}) \\
&\leq \tfrac{1}{\beta}\tau_i \otimes \tau_{\mathcal{Q}}(\tilde{b}_{i+2}) + \tau_i \otimes \tau_{\mathcal{Q}}(y_{i+1}) \\
&\vdots \\
&\leq \tfrac{1}{\beta^{n-i-1}} \cdot \tfrac{\delta}{3} + \tfrac{1}{\beta^{n-i-2}} \cdot \tfrac{\delta}{3} + \cdots + \tfrac{1}{\beta} \cdot \tfrac{\delta}{3} + \tfrac{\delta}{3} \\
&= \tfrac{\delta}{3} \cdot (1 - \tfrac{1}{\beta^{n-i}})/(1 - \tfrac{1}{\beta}) \\
&< \tfrac{\delta}{3}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\tau_i \otimes \tau_{\mathcal{Q}}(\phi(\gamma_i)) &= \tau_i \otimes \tau_{\mathcal{Q}}(b_i) - \tau_i \otimes \tau_{\mathcal{Q}}(b_{i+1}) \\
&\geq \tau_i \otimes \tau_{\mathcal{Q}}(y_i) - \beta - \tfrac{\delta}{3} \\
&\geq 1 - \delta.
\end{aligned}$$

Since $\sum_{j=0}^{n-1} \gamma_j = 1$, whenever $j \neq i$ we get

$$\gamma_j \leq 1 - \gamma_i,$$

whence

$$\tau_i \otimes \tau_{\mathcal{Q}}(\phi(\gamma_j)) \leq 1 - \tau_i \otimes \tau_{\mathcal{Q}}(\phi(\gamma_i)) \leq \delta.$$

$\blacksquare$

4.2.13 Recall that if F and A are separable C*-algebras with F unital, and $\sigma : F \to A$ is a c.p.c. order zero map, we may define a functional calculus for σ as follows. Let π_σ denote the supporting *-homomorphism of σ. Then for $f \in \mathcal{C}([0,1])$, we define $f(\sigma)(x) = f(\sigma(1_F))\pi_\sigma(x)$, and $f(\sigma)$ is a well-defined c.p. order zero map [79, Corollary 3.2], [75, 1.3].

Proposition: *Let A be a separable simple unital nuclear* C**-algebra with stable rank one and strict comparison. Let F be a finite-dimensional* C**-algebra. Let $0 < \alpha, \epsilon < 1$ and suppose*

$$\theta, \sigma : F \to A$$

are c.p. order zero maps satisfying

$$\tau(\sigma(p)) - d_\tau(\theta(p)) \geq \alpha$$

for every nonzero projection $p \in F$ and for every $\tau \in T(A)$. Then, for $0 < \beta_1 < \alpha/2$, there exists $s \in A$ satisfying, with $\beta_1 < \beta_2 < 1$, the following:

(i) $s^*s \in \mathrm{Her}(f_{\beta_1,\beta_2}(\sigma(1_F)))$,

(ii) $(\theta(x) - \epsilon)_+ ss^* = ss^*(\theta(x) - \epsilon)_+ = (\theta(x) - \epsilon)_+$ *for every $x \in F$,*

(iii) $s^*(\theta(x) - \epsilon)_+ s = g_{0,\beta_1}(x)(\sigma)s^*(\theta(1_F) - \epsilon)_+ s = s^*(\theta(1_F) - \epsilon)_+ s g_{0,\beta_1}(\sigma)(x)$ *for every $x \in F$.*

Proof: Let ϵ and α be given and let $\theta, \sigma : F \to A$ be c.p. order zero maps satisfying the statement of the proposition. Denote the supporting *-homomorphisms for the c.p. order zero maps θ and σ as π_θ and π_σ, respectively. By the functional calculus, $f_{\beta_1,\beta_2}(\sigma)$ is a well-defined order zero map for any choice of $0 < \beta_1 < \beta_2 < 1$.

We claim that $d_\tau(\theta(p)) < \tau(\sigma(f_{\beta_1,\beta_2})(p))$ for every projection $p \in F$ and every $\tau \in T(A)$.

Note that if σ is a *-homomorphism then $f_{\beta_1,\beta_2}(\sigma) = \sigma$ for any choice of $0 < \beta_1 < \beta_2 < 1$. In this case, the claim follows immediately.

Otherwise, we have

$$f_{\beta_1,\beta_2}(t) \geq t - \tfrac{\beta_1}{1-\beta_1} \cdot (1-t) \text{ for all } t \in [0,1],$$

thus

$$f_{\beta_1,\beta_2}(\sigma)(p) \geq \sigma(p) - \tfrac{\beta_1}{1-\beta_1} \cdot (1-\sigma(p)).$$

Note that if p is a nonzero projection then $\tau(\sigma(p)) \neq 0$ for any $\tau \in T(A \otimes \mathcal{Q})$ since A is simple. So $\tau(\sigma(p)) > 0$ for every $\tau \in T(A \otimes \mathcal{Q})$. Also, since $\alpha < 1$ we also have $\beta_1 < \frac{1}{2}$. Thus

$$\begin{array}{rcl} \tau(f_{\beta_1,\beta_2}(\sigma)(p)) & \geq & \tau(\sigma(p)) - \tfrac{\beta_1}{1-\beta_1} \cdot d_\tau(1-\sigma(p)) \\ & > & \tau(\sigma(p)) - 2\beta_1 \cdot d_\tau(1-\sigma(p)) \\ & > & \tau(\sigma(p)) - \alpha \\ & \geq & d_\tau(\theta(p)) \end{array} \tag{4.7}$$

for every $\tau \in T(A \otimes \mathcal{Q})$ and every nonzero projection $p \in F$, proving the claim.

Write

$$F = M_{r_1} \oplus \cdots \oplus M_{r_L},$$

and for $l = 1, \ldots, L$, let $e_{i,j}^{(l)}$ denote the partial isometry in F corresponding to the $(i,j)^{\text{th}}$ matrix unit in M_{r_l}. For $1 \leq l \leq L$, by (4.7) we have that

$$d_\tau(\theta(e_{1,1}^{(l)})) < \tau(f_{\beta_1,\beta_2}(\sigma)(e_{1,1}^{(l)})) \text{ for all } \tau \in T(A),$$

so by strict comparison it follows that $\theta(e_{1,1}^{(l)}) \precsim f_{\beta_1,\beta_2}(\sigma)(e_{1,1}^{(l)})$. By [53, Proposition 2.4], there are unitaries $u_l \in A$ such that

$$u_l(g_{\epsilon/2,\epsilon}(\theta)(e_{1,1}^{(l)}))u_l^* \in \mathrm{Her}(f_{\beta_1,\beta_2}(\sigma)(e_{1,1}^{(l)})). \tag{4.8}$$

Let

$$d_l = (g_{\epsilon/2,\epsilon}(\theta)(e_{1,1}^{(l)}))^{1/2}u_l^*. \tag{4.9}$$

Then d_l satisfies

$$d_l d_l^*(\theta(e_{1,1}^{(l)}) - \epsilon)_+ = g_{\epsilon/2,\epsilon}(\theta)(e_{1,1}^{(l)}))(\theta(e_{1,1}^{(l)}) - \epsilon)_+ = (\theta(e_{1,1}^{(l)}) - \epsilon)_+, \tag{4.10}$$

and similarly $(\theta(e_{1,1}^{(l)}) - \epsilon)_+ d_l d_l^* = (\theta(e_{1,1}^{(l)}) - \epsilon)_+$.

Furthermore, since $d_l^*(\theta(e_{1,1}^{(l)}) - \epsilon)_+ d_l \in \mathrm{Her}(f_{\beta_1,\beta_2}(\sigma)(e_{1,1}^{(l)}))$ by (4.8), we have

$$\begin{array}{rcl} g_{0,\beta_1}(\sigma)(e_{1,1}^{(l)})d_l^*(\theta(e_{1,1}^{(l)}) - \epsilon)_+ d_l & \overset{(4.6)}{=} & d_l^*(\theta(e_{1,1}^{(l)}) - \epsilon)_+ d_l \\ & \overset{(4.6)}{=} & d_l^*(\theta(e_{1,1}^{(l)}) - \epsilon)_+ d_l g_{0,\beta_1}(\sigma)(e_{1,1}^{(l)}). \end{array} \tag{4.11}$$

Set

$$s = \textstyle\sum_{l=1}^{L} \sum_{k=1}^{r_l} \pi_\theta(e_{k,1}^{(l)}) d_l \pi_\sigma(e_{1,k}^{(l)}). \tag{4.12}$$

Note that since $d_l^* d_l \in \mathrm{Her}(f_{\beta_1,\beta_2}(\sigma)(e_{1,1}^{(l)}))$ we have that

$$d_l \pi_\sigma(e_{1,k}^{(l)}) \overset{(4.6)}{=} d_l g_{0,\beta_1}(\sigma)(e_{1,1}^{(l)}) \pi_\sigma(e_{1,k}^{(l)}) = d_l g_{0,\beta_1}(\sigma)(e_{1,k}^{(l)}) \in A,$$

and similarly, since $d_l d_l^* \in \mathrm{Her}(g_{\epsilon/2,\epsilon}(\theta)(e_{1,1}^{(l)}))$ we have

$$\pi_\theta(e_{k,1}^{(l)})d_l \overset{(4.6)}{=} \pi_\theta(e_{k,1}^{(l)})g_{\epsilon/4,\epsilon/2}(\theta)(e_{1,1}^{(l)})d_l = g_{\epsilon/4,\epsilon/2}(\theta)(e_{k,1}^{(l)})d_l \in A$$

thus

$$\pi_\theta(e_{k,1}^{(l)})d_l\pi_\sigma(e_{1,k}^{(l)}) = g_{\epsilon/4,\epsilon/2}(\theta)(e_{k,1}^{(l)})d_l g_{0,\beta_1}(\sigma)(e_{1,k}^{(l)}) \in A,$$

and hence $s \in A$.

Since the hereditary C*-subalgebras $\mathrm{Her}(f_{\beta_1,\beta_2}(\sigma)(e_{1,1}^{(l)}))$ are pairwise orthogonal, we have that $d_l d_{l'}^* = 0$ when $l \neq l'$ and

$$ss^* = \textstyle\sum_{l=1}^{L}\sum_{k=1}^{r_l}\pi_\theta(e_{k,1}^{(l)})d_l d_l^*\pi_\sigma(e_{1,k}^{(l)}). \tag{4.13}$$

We have that $s^*s \in \mathrm{Her}(f_{\beta_1,\beta_2}(\sigma(\oplus_{l=1}^{L}e_{1,1}^{(l)})) \subset \mathrm{Her}(f_{\beta_1,\beta_2}(\sigma(1_F))$, showing (i).

For (ii), it is obviously enough to show that $(\theta(e_{i,j}^{(l)})-\epsilon)_+ss^* = (\theta(e_{i,j}^{(l)})-\epsilon)_+ = ss^*(\theta(e_{i,j}^{(l)})-\epsilon)_+$ for arbitrary i,j,l. Furthermore, since θ is order zero, it is clear that $(\theta(e_{i,j}^{(l)})-\epsilon)_+\pi_\theta(e_{k,1}^{(l')}) = 0$ when $l \neq l'$. Thus

$$\begin{aligned}
&(\theta(e_{i,j}^{(l)})-\epsilon)_+ss^* \\
&\overset{(4.13)}{=} (\theta(e_{i,j}^{(l)})-\epsilon)_+(\textstyle\sum_{k=1}^{r_l}\pi_\theta(e_{k,1}^{(l)})d_l d_l^*\pi_\theta(e_{1,k}^{(l)})) \\
&= (\theta(e_{i,j}^{(l)})-\epsilon)_+\pi_\theta(e_{j,1}^{(l)})d_l d_l^*\pi_\theta(e_{1,j}^{(l)}) \\
&= \pi_\theta(e_{i,1}^{(l)})(\theta(e_{1,1}^{(l)})-\epsilon)_+d_l d_l^*\pi_\theta(e_{1,j}^{(l)}) \\
&\overset{(4.10)}{=} \pi_\theta(e_{i,1}^{(l)})(\theta(e_{1,1}^{(l)})-\epsilon)_+\pi_\theta(e_{1,j}^{(l)}) \\
&= \pi_\theta(e_{i,1}^{(l)})(\theta(e_{1,j}^{(l)})-\epsilon)_+ \\
&= (\theta(e_{i,j}^{(l)})-\epsilon)_+.
\end{aligned}$$

The fact that $ss^*(\theta(e_{i,j}^{(l)})-\epsilon)_+ = (\theta(e_{i,j}^{(l)})-\epsilon)_+$ follows from a nearly identical calculation.

For (iii), again it suffices to show the case $x = e_{i,j}^{(l)}$.

$$\begin{aligned}
&s^*(\theta(e_{i,j}^{(l)})-\epsilon)_+s \\
&= (\pi_\sigma(e_{i,1}^{(l)})d_l^*\pi_\theta(e_{1,i}^{(l)}))(\theta(e_{i,j}^{(l)})-\epsilon)_+(\pi_\theta(e_{j,1}^{(l)})d_l\pi_\sigma(e_{1,j}^{(l)})) \\
&= \pi_\sigma(e_{i,1}^{(l)})d_l^*(\theta(e_{1,1}^{(l)})-\epsilon)_+d_l\pi_\sigma(e_{1,j}^{(l)}) \\
&\overset{(4.11)}{=} \pi_\sigma(e_{i,1}^{(l)})g_{0,\beta_1}(\sigma)(e_{1,1}^{(l)})d_l^*(\theta(e_{1,1}^{(l)})-\epsilon)_+d_l\pi_\sigma(e_{1,j}^{(l)}) \\
&= g_{0,\beta_1}(\sigma)(e_{i,j}^{(l)})\pi_\sigma(e_{j,1}^{(l)})d_l^*(\theta(e_{1,1}^{(l)})-\epsilon)_+d_l\pi_\sigma(e_{1,j}^{(l)}) \\
&= g_{0,\beta_1}(\sigma)(e_{i,j}^{(l)})\pi_\sigma(e_{j,1}^{(l)})d_l^*\pi_\theta(e_{1,j}^{(l)})(\theta(e_{j,j}^{(l)})-\epsilon)_+\pi_\theta(e_{j,1}^{(l)})d_l\pi_\sigma(e_{1,j}^{(l)}) \\
&= g_{0,\beta_1}(\sigma)(e_{i,j}^{(l)})s^*(\theta(1_F)-\epsilon)_+s.
\end{aligned}$$

Similarly, $s^*(\theta(e_{i,j}^{(l)})-\epsilon)_+s = s^*(\theta(1_F)-\epsilon)_+sg_{0,\beta_1}(\sigma)(e_{i,j}^{(l)})$. ∎

4.2.14 Theorem: *Let A be a separable simple unital locally recursive subhomogeneous C*-algebra. Suppose that an approximating recursive subhomogeneous algebra B can always be chosen to have an $(\mathcal{F},\eta)$-connected recursive subhomogeneous decomposition*

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^{R}$$

along which projections can be lifted and such that $X_l \setminus \Omega_l \neq \emptyset$ for $l \geq 1$. Suppose further that A has exactly n extreme tracial states $\tau_0,\ldots,\tau_{n-1} \in T(A)$ satisfying $(\tau_i)_* = (\tau_j)_*$ for every $i,j \in$

$\{0, \ldots, n-1\}$. *Then, for any finite subset* $\mathcal{F} \subset A^1_+$ *and any* $0 < \epsilon < 1$*, there are a partial isometry* $s \in A \otimes \mathcal{Q}$*, a finite-dimensional* C*-*subalgebra* $F \subset \mathcal{Q}$ *and a* **-homomorphism*

$$\Phi : \mathcal{C}([0,1]) \otimes F \to A \otimes \mathcal{Q}$$

such that

$$ss^* = \Phi(1_{\mathcal{C}([0,1])} \otimes 1_F)$$

and

(i) $\|s^*s(a \otimes 1_{\mathcal{Q}}) - (a \otimes 1_{\mathcal{Q}})s^*s\| < \epsilon$ *for all* $a \in \mathcal{F}$,

(ii) $\operatorname{dist}(s^*s(a \otimes 1_{\mathcal{Q}})s^*s, s^*\Phi(\mathcal{C}([0,1]) \otimes F)s) < \epsilon$ *for all* $a \in \mathcal{F}$,

(iii) $\tau \otimes \tau_{\mathcal{Q}}(s^*s) \geq \frac{1}{2(n+2)}$ *for all* $\tau \in T(A)$.

Proof: Let $\mathcal{F}$ and ϵ be given. Since A is locally recursive subhomogeneous, we may assume, by taking a sufficiently good approximation, that $\mathcal{F} \subset B$ for some recursive subhomogeneous C*-algebra B.

We may furthermore assume that $1_A \subset \mathcal{F}$ so that $\tau_i^{(B)} := \tau_i|_B \in T(B)$ are faithful and, as states on $K_0(B)$, satisfy $(\tau_i^{(B)})_* = (\tau_j^{(B)})_*$ for all $i, j \in \{0, \ldots, n-1\}$.

Let $\eta > 0$ and $\beta_1 < \frac{1}{8}$ be so small that

$$\eta < \frac{\beta_1}{6} \cdot \epsilon. \tag{4.14}$$

We may apply Lemma 4.1.10 with respect to η and $\mathcal{F}$ to get

$$0 = K_0 < K_1 < \ldots < K_{n-1} = K \in \mathbb{N}$$

and pairwise orthogonal weighted $(\mathcal{F}, \eta)$-excisors

$$(Q_{\frac{j}{K}}, \rho_{\frac{j}{K}}, \tilde{\sigma}_{\frac{j}{K}}, \kappa_{\frac{j}{K}}),\ j \in \{0, \ldots, K\},$$

implementing $(\mathcal{F}, \eta)$-bridges

$$\begin{aligned}(Q_{\frac{K_0}{K}}, \rho_{\frac{K_0}{K}}, \tilde{\sigma}_{\frac{K_0}{K}}, \kappa_{\frac{K_0}{K}}) \quad &\sim_{(\mathcal{F},\eta)} \cdots \sim_{(\mathcal{F},\eta)} \quad (Q_{\frac{K_m}{K}}, \rho_{\frac{K_m}{K}}, \tilde{\sigma}_{\frac{K_m}{K}}, \kappa_{\frac{K_m}{K}}) \\ &\sim_{(\mathcal{F},\eta)} \cdots \sim_{(\mathcal{F},\eta)} \quad (Q_{\frac{K_{n-1}}{K}}, \rho_{\frac{K_{n-1}}{K}}, \tilde{\sigma}_{\frac{K_{n-1}}{K}}, \kappa_{\frac{K_{n-1}}{K}}),\end{aligned}$$

and such that, for each projection $q \in Q_{\frac{K_i}{K}}$, $i \in \{0, \ldots, n-1\}$, from (72) we have

$$\tau_i \otimes \tau_{\mathcal{Q}}(\tilde{\sigma}_{\frac{K_i}{K}}(q)) \geq \frac{1}{n+1} \cdot \tau_{\mathcal{Q}} \kappa_{\frac{K_i}{K}}(q). \tag{4.15}$$

Let $0 < \alpha_1 < \alpha_2 < \frac{1}{2(n-1)}$ and choose

$$0 < \delta < \tfrac{2}{3} \tag{4.16}$$

to apply Lemma 4.2.12 with

$$0 < \delta_0 < \frac{(n-1)\delta\alpha_2}{2n} \tag{4.17}$$

to get a *-homomorphism

$$\phi : \mathcal{C}([0,1]) \to A \otimes \mathcal{Q}$$

satisfying

$$\tau_i \otimes \tau_{\mathcal{Q}}(\phi(\tilde{\gamma}_i)) \geq 1 - \delta_0,$$

and

$$0 < \tau_j \otimes \tau_{\mathcal{Q}}(\phi(\tilde{\gamma}_i)) < \delta_0$$

for $i \neq j \in \{0, \ldots, n-1\}$, where

$$\tilde{\gamma}_i(t) = \begin{cases} 0, & t \in [0,1] \cap ((-\infty, \frac{i-1}{n-1}] \cup [\frac{i+1}{n-1}, \infty)) \\ 1, & t = \frac{i}{n-1} \\ \text{linear}, & \text{elsewhere.} \end{cases}$$

For $i \in \{0, \ldots, n-1\}$, define $\hat{\gamma}_i \in \mathcal{C}([0,1])$ by

$$\hat{\gamma}_i = g_{\frac{i}{n-1}-\alpha_2, \frac{i}{n-1}-\alpha_1} - g_{\frac{i}{n-1}+\alpha_1, \frac{i}{n-1}+\alpha_2}$$

and for $i \in \{0, \ldots, n-2\}$ define $\gamma_{i,i+1} \in \mathcal{C}([0,1])$ by

$$\gamma_{i,i+1} = g_{\frac{i}{n-1}+\alpha_1, \frac{i}{n-1}+\alpha_2} - g_{\frac{i+1}{n-1}-\alpha_2, \frac{i+1}{n-1}-\alpha_1},$$

where we set $g_{-\alpha_2,-\alpha_1} = 1$ and $g_{1+\alpha_1,1+\alpha_2} = 0$. Note that

$$\hat{\gamma}_{n-1} + \textstyle\sum_{i=0}^{n-2} \hat{\gamma}_i + \gamma_{i,i+1} = 1_{\mathcal{C}([0,1])}$$

We will now estimate the traces of the $\phi(\hat{\gamma}_i)$, and $\phi(\gamma_{i,i+1})$. We have

$$0 \leq \hat{\gamma}_{i-1}(t), \gamma_{i-1,i}(t) \leq \frac{1}{(n-1)\alpha_2} \cdot \tilde{\gamma}_{i-1}(t)$$

for all $t \in [0,1]$, for all $i \in \{1, \ldots, n-1\}$, so

$$\begin{array}{rcl} \tau_i \otimes \tau_{\mathcal{Q}}(\phi(\gamma_{i-1,i})), \tau_i \otimes \tau_{\mathcal{Q}}(\phi(\hat{\gamma}_{i-1})) & \leq & \frac{1}{(n-1)\alpha_2} \cdot \tau_i \otimes \tau_{\mathcal{Q}}(\phi(\tilde{\gamma}_{i-1})) \\ & < & \frac{\delta_0}{(n-1)\alpha_2} \\ & \stackrel{(4.17)}{<} & \frac{\delta}{2n}. \end{array}$$

One similarly shows that

$$\tau_i \otimes \tau_{\mathcal{Q}}(\phi(\gamma_{i,i+1})), \tau_i \otimes \tau_{\mathcal{Q}}(\phi(\hat{\gamma}_{i+1})) \leq \tfrac{1}{(n-1)\alpha_2} \cdot \tau_i \otimes \tau_{\mathcal{Q}}(\phi(\tilde{\gamma}_{i+1})) < \tfrac{\delta}{2n}.$$

It follows that

$$\tau_i \otimes \tau_{\mathcal{Q}}(\phi(\hat{\gamma}_i)) = \tau_i \otimes \tau_{\mathcal{Q}}(1 - \textstyle\sum_{j=0}^{n-2} \phi(\gamma_{j,j+1}) - \sum_{j=0, j\neq i}^{n-1} \phi(\hat{\gamma}_j)) > 1 - \delta. \tag{4.18}$$

Let $t_0 < t_1 < \cdots < t_K$ be a partition of the interval $[0,1]$ satisfying

$$t_{K_{i-1}} = \tfrac{i-1}{n-1}, \quad t_{K_{i-1}+1} = \tfrac{i-1}{n-1} + \alpha_1, \quad t_{K_{i-1}+2} = \tfrac{i-1}{n-1} + \alpha_2$$

and

$$t_{K_i-2} = \tfrac{i}{n-1} - \alpha_2, \quad t_{K_i-1} = \tfrac{i}{n-1} - \alpha_1, \quad t_{K_i} = \tfrac{i}{n-1}$$

for $i \in \{1, \ldots, n-1\}$. When $j = K_i$ for some $i \in \{0, \ldots, n-1\}$, set

$$\gamma_{\frac{j}{K}} := \hat{\gamma}_j.$$

When $j \in \{0, \ldots, K\} \setminus \{K_0, \ldots, K_{n-1}\}$, define

$$\gamma_{\frac{j}{K}}(t) = \begin{cases} 0, & 0 \leq t \leq t_j \text{ and } t \geq t_{j+2} \\ 1, & t = t_{j+1} \\ \text{linear}, & t_j \leq t \leq t_{j+1} \text{ and } t_{j+1} \leq t \leq t_{j+2}, \end{cases}$$

so that the $\gamma_{\frac{j}{K}}$ are a partition of unity corresponding to $t_0 < t_1 < \cdots < t_K$.

Let $p \in \mathcal{Q}$ be a projection satisfying $\tau_{\mathcal{Q}}(p) = \frac{1}{n+2}$. Then by (4.15) and the choice of δ we have, for each $0 \leq j \leq K$, that

$$\tau \otimes \tau_{\mathcal{Q}}(\phi(\gamma_{\frac{j}{K}}) \otimes \kappa_{\frac{j}{K}}(q) \otimes p) < \tau \otimes \tau_{\mathcal{Q}}(\tilde{\sigma}_{\frac{j}{K}}(q))$$

for all $\tau \in T(A \otimes \mathcal{Q})$ and for all projections $q \in Q_{\frac{j}{K}}$.

Define c.p.c. order zero maps

$$\theta_{\frac{j}{K}} : Q_{\frac{j}{K}} \to A \otimes \mathcal{Q} \otimes \mathcal{Q} \otimes \mathcal{Q} \cong A \otimes \mathcal{Q}$$

by

$$\theta_{\frac{j}{K}}(a) = \phi(\gamma_{\frac{j}{K}}) \otimes \kappa_{\frac{j}{K}}(a) \otimes p. \tag{4.19}$$

Each finite-dimensional C*-algebra $Q_{\frac{j}{K}}$, $j \in \{0, \ldots, K\}$ can be written as a sum of $L^{(j)} \in \mathbb{N}$ matrix algebras, $Q_{\frac{j}{K}} = M_{r_1^{(j)}} \oplus \cdots \oplus M_{r_{L^{(j)}}^{(j)}}$ for some $r_1^{(j)}, \ldots, r_{L^{(j)}}^{(j)} \in \mathbb{N}$.

Note that for every $n \in \mathbb{N}$ and every $a \in (Q_{\frac{j}{K}})_+$ we have that

$$\tau \otimes \tau_{\mathcal{Q}}(\theta_{\frac{j}{K}}(a)^{1/n}) \leq \tau_{\mathcal{Q}} \otimes \tau_{\mathcal{Q}}((\kappa_{\frac{j}{K}}(a) \otimes p)^{1/n}) \text{ for all } \tau \in T(A).$$

Thus we see that for every projection $q \in Q_{\frac{j}{K}}$

$$\begin{aligned} d_\tau(\theta_{\frac{j}{K}}(q)) &< d_{\tau_{\mathcal{Q}} \otimes \tau_{\mathcal{Q}}}(\kappa_{\frac{j}{K}}(q) \otimes p) \\ &= \tau_{\mathcal{Q}} \otimes \tau_{\mathcal{Q}}(\kappa_{\frac{j}{K}}(q) \otimes p) \\ &= \tfrac{1}{n+2}\tau_{\mathcal{Q}}(\kappa_{\frac{j}{K}}(q)) \\ &< \tfrac{1}{n+1}\tau_{\mathcal{Q}}(\kappa_{\frac{j}{K}}(q)) \\ &\overset{(4.15)}{\leq} \tau(\tilde{\sigma}_{\frac{j}{K}}(q)) \end{aligned} \tag{4.20}$$

for all $\tau \in T(A \otimes \mathcal{Q})$.

Define order zero maps by

$$\sigma_{\frac{j}{K}} = g_{0,\beta_1}(\tilde{\sigma}_{\frac{j}{K}}).$$

Note that

$$g_{0,\beta_1}(t) \leq \tfrac{1}{\beta_1} t, \text{ for all } t \in [0,1], \tag{4.21}$$

thus

$$\begin{aligned} & \|\sigma_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}})(b \otimes 1_{\mathcal{Q}}) - \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(b))\| \\ = \; & \|g_{0,\beta_1}(\tilde{\sigma}_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}))\pi_{\tilde{\sigma}_{\frac{j}{k}}}(1_{Q_{\frac{j}{K}}})(b \otimes 1_{\mathcal{Q}}) - g_{0,\beta_1}(\tilde{\sigma}_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}))\pi_{\tilde{\sigma}_{\frac{j}{k}}}(\rho_{\frac{j}{K}}(b))\| \\ \overset{(4.21)}{\leq} \; & \tfrac{1}{\beta_1}\|\tilde{\sigma}_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}})(b \otimes 1_{\mathcal{Q}}) - \tilde{\sigma}_{\frac{j}{K}}(\rho_{\frac{j}{K}}(b))\| \\ \overset{(4.14)}{\leq} \; & \tfrac{\epsilon}{6}. \end{aligned} \tag{4.22}$$

Let $\eta_1 > 0$ be so small that if $a \in A$ and p is a projection such that $\|a^*a - p\| < \eta_1$ then there is $v \in A$ such that $v^*v = p$ and $\|v - a\| < \frac{\epsilon}{12}$.

Since (4.20) holds for $\theta_{\frac{j}{K}}$ and $\tilde{\sigma}_{\frac{j}{K}}$ for every $j \in \{0, \ldots, K\}$, we may apply Lemma 4.2.13 with

$$\eta_0 = \min\{\tfrac{\epsilon}{96}, \tfrac{1}{4}, \tfrac{\eta_1}{3}\} \tag{4.23}$$

in place of ϵ to each $j \in \{0, \ldots, K\}$ to get elements

$$s_{\frac{j}{K}} \in A \otimes \mathcal{Q}$$

satisfying

$$s^*_{\frac{j}{K}} s_{\frac{j}{K}} \in \mathrm{Her}(f_{\beta_1',\beta_2}(\tilde{\sigma}_{\frac{j}{K}})(1_{Q_{\frac{j}{K}}})) \tag{4.24}$$

with $0 < \beta_1' < \min\{(\tilde{\sigma}_{\frac{j}{K}}(q)) - d_\tau(\theta_{\frac{j}{K}}(q)), \beta_1\}$, and

$$\begin{aligned} & s_{\frac{j}{K}} s^*_{\frac{j}{K}} (\theta_{\frac{j}{K}}(a) - \eta_0)_+ \\ = \quad & (\theta_{\frac{j}{K}}(a) - \eta_0)_+ s_{\frac{j}{K}} s^*_{\frac{j}{K}} \\ = \quad & (\theta_{\frac{j}{K}}(a) - \eta_0)_+ \text{ for all } a \in Q_{\frac{j}{K}}, \end{aligned} \tag{4.25}$$

and

$$\begin{aligned} & \tilde{\sigma}_{\frac{j}{K}}(a) s^*_{\frac{j}{K}} (\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+ s_{\frac{j}{K}} \\ = \quad & s^*_{\frac{j}{K}} (\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+ s_{\frac{j}{K}} \tilde{\sigma}_{\frac{j}{K}}(a) \\ = \quad & s^*_{\frac{j}{K}} (\theta_{\frac{j}{K}}(a) - \eta_0)_+ s_{\frac{j}{K}} \text{ for all } a \in Q_{\frac{j}{K}}. \end{aligned} \tag{4.26}$$

If f is a continuous function then upon approximating by polynomials we have

$$s_{\frac{j}{K}} s^*_{\frac{j}{K}} f((\theta(1_{Q_{\frac{j}{K}}}) - \eta_0)_+) = f((\theta(1_{Q_{\frac{j}{K}}}) - \eta_0)_+) s_{\frac{j}{K}} s^*_{\frac{j}{K}} = f((\theta(1_{Q_{\frac{j}{K}}}) - \eta_0)_+). \tag{4.27}$$

Moreover

$$f(s^*_{\frac{j}{K}} \theta(1_{Q_{\frac{j}{K}}}) - \eta_0)_+ s_{\frac{j}{K}}) = s^*_{\frac{j}{K}} f((\theta(1_{Q_{\frac{j}{K}}}) - \eta_0)_+) s_{\frac{j}{K}},$$

hence

$$\sigma_{\frac{j}{K}}(a) s^*_{\frac{j}{K}} f((\theta(1_{Q_{\frac{j}{K}}}) - \eta_0)_+) s_{\frac{j}{K}} = s^*_{\frac{j}{K}} f((\theta(1_{Q_{\frac{j}{K}}}) - \eta_0)_+) s_{\frac{j}{K}} \sigma_{\frac{j}{K}}(a) \tag{4.28}$$

for all $a \in Q_{\frac{j}{K}}$. Now put

$$\tilde{s} = \textstyle\sum_{j=0}^K ((\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+)^{1/2} s_{\frac{j}{K}}. \tag{4.29}$$

Since $\mathcal{Q}$ is a UHF algebra, there is a finite-dimensional C*-algebra $F \subset \mathcal{Q}$ such that

$$\|1_F - 1_{\mathcal{Q}}\| < \eta_0 \tag{4.30}$$

and

$$\{\kappa_{\frac{j}{K}} \circ \rho_{\frac{j}{K}}(a) \mid a \in \mathcal{F}, 0 \le j \le K\} \subset_{\eta_0} F.$$

Let $\iota : F \hookrightarrow \mathcal{Q}$ be the inclusion map and set

$$\Phi := \phi(\cdot) \otimes \iota(\cdot) \otimes p : \mathcal{C}([0,1]) \otimes F \to A \otimes \mathcal{Q} \otimes \mathcal{Q} \otimes \mathcal{Q} \, (\cong A \otimes \mathcal{Q}). \tag{4.31}$$

Since the $\tilde{\sigma}_{\frac{j}{K}}$ have orthogonal images, it follows from (4.24) above that

$$s_{\frac{j}{K}} s^*_{\frac{j'}{K}} = 0, \tag{4.32}$$

and

$$s_{\frac{j}{K}} \sigma_{\frac{j'}{K}}(a) = 0 \text{ for all } a \in Q_{\frac{j'}{K}} \tag{4.33}$$

whenever $j \neq j'$.

It follows from (4.32) that

$$\begin{aligned} \tilde{s}\tilde{s}^* \quad & = \quad \textstyle\sum_{j=0}^K ((\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+)^{1/2} s_{\frac{j}{K}} s^*_{\frac{j}{K}} ((\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+)^{1/2} \\ & \overset{(4.25)}{=} \quad \textstyle\sum_{j=0}^K (\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+, \end{aligned}$$

and we estimate

$$\begin{aligned}
&\|\tilde{s}\tilde{s}^* - \Phi(1_{\mathcal{C}([0,1])} \otimes 1_F)\| \\
&\quad = \quad \|\textstyle\sum_{j=0}^K (\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+ - \Phi(1_{\mathcal{C}([0,1])} \otimes 1_F)\| \\
&\quad \overset{(4.19)}{\leq} \quad \|\textstyle\sum_{j=0}^K \phi(\gamma_{\frac{j}{K}}) \otimes \kappa_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) \otimes p - \Phi(1_{\mathcal{C}([0,1])} \otimes 1_F)\| + 2 \cdot \eta_0 \\
&\quad \overset{(4.31)}{=} \quad \|\phi(1_{\mathcal{C}([0,1]}) \otimes 1_{\mathcal{Q}} \otimes p - \phi(1_{\mathcal{C}([0,1])}) \otimes \iota(1_F) \otimes p\| + 2 \cdot \eta_0 \\
&\quad \overset{(4.30)}{=} \quad 3 \cdot \eta_0 \\
&\quad \overset{(4.23)}{<} \quad \eta_1.
\end{aligned}$$

By our choice of η_1 there is an honest partial isometry $s \in A \otimes \mathcal{Q}$ satisfying

$$ss^* = \Phi(1_{\mathcal{C}([0,1])} \otimes 1_F)$$

and

$$\|\tilde{s} - s\| < \tfrac{\epsilon}{12}. \tag{4.34}$$

Let $a \in \mathcal{F}$ and consider the element $\sum_{j=0}^K \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) \in A \otimes Q$. We will use this to estimate $\|a\tilde{s}^*\tilde{s} - \tilde{s}^*\tilde{s}a\|$. Note that since the functions $\gamma_{\frac{j}{K}}$ and $\gamma_{\frac{j'}{K}}$ are pairwise orthogonal whenever $|j-j'| \geq 2$ we have that $\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}})\theta_{\frac{j'}{K}}(1_{Q_{\frac{j'}{K}}}) = 0$ whenever $|j-j'| \geq 2$ whence

$$(\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2} s_{\frac{j}{K}})^*((\theta_{\frac{j'}{K}}(1_{Q_{\frac{j'}{K}}}) - \eta_0)_+^{1/2} s_{\frac{j'}{K}}) = 0 \tag{4.35}$$

whenever $|j-j'| \geq 2$. We calculate

$$\begin{aligned}
&(\textstyle\sum_{j=0}^K \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)))\tilde{s}^*\tilde{s} \\
&\overset{(4.29)}{=} \quad (\textstyle\sum_{j=0}^K \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)))(\sum_{j=0}^K s^*_{\frac{j}{K}}(\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2})(\sum_{j=0}^K (\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2} s_{\frac{j}{K}}) \\
&\overset{(4.33)}{=} \quad (\textstyle\sum_{j=0}^K \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) s^*_{\frac{j}{K}}(\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2}) \cdot (\sum_{j=0}^K (\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2} s_{\frac{j}{K}}) \\
&\overset{(4.35)}{=} \quad \textstyle\sum_{j=0}^K \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) s^*_{\frac{j}{K}}(\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2} \\
&\qquad \cdot (\textstyle\sum_{\{j' \mid |j-j'|<2\}} \theta_{\frac{j'}{K}}(1_{Q_{\frac{j'}{K}}}) - \eta_0)_+^{1/2} s_{\frac{j'}{K}}).
\end{aligned} \tag{4.36}$$

A similar calculation yields

$$\begin{aligned}
&\tilde{s}^*\tilde{s}(\textstyle\sum_{j=0}^K \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a))) \\
&\quad = \quad \textstyle\sum_{j=0}^K s^*_{\frac{j}{K}}(\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2} \cdot (\sum_{\{j' \mid |j-j'|<2\}} (\theta_{\frac{j'}{K}}(1_{Q_{\frac{j'}{K}}}) - \eta_0)_+^{1/2} s_{\frac{j'}{K}} \sigma_{\frac{j'}{K}}(\rho_{\frac{j'}{K}}(a))).
\end{aligned} \tag{4.37}$$

Thus

$$
\begin{aligned}
&\|\tilde{s}^*\tilde{s}(\textstyle\sum_{j=0}^{K} \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a))) - (\textstyle\sum_{j=0}^{K} \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)))\tilde{s}^*\tilde{s}\| \\
\overset{(4.36),(4.37)}{\leq}\quad & \|\textstyle\sum_{j=0}^{K} s^*_{\frac{j}{K}}(\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+ s_{\frac{j}{K}} \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) \\
& -\sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) s^*_{\frac{j}{K}}(\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+ s_{\frac{j}{K}}\| \\
& +\|\textstyle\sum_{j=0}^{K} \sum_{\{j' \mid |j-j'|=1\}} s^*_{\frac{j}{K}}(\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2} \\
& (\theta_{\frac{j'}{K}}(1_{Q_{\frac{j'}{K}}}) - \eta_0)_+^{1/2} s_{\frac{j'}{K}} \sigma_{\frac{j'}{K}}(\rho_{\frac{j'}{K}}(a)) - \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) s^*_{\frac{j}{K}}(\theta_{\frac{j}{K}}(1_{\mathcal{Q}_{\frac{j}{K}}}) - \eta_0)_+^{1/2} \\
& (\theta_{\frac{j'}{K}}(1_{\mathcal{Q}_{\frac{j'}{K}}}) - \eta_0)_+^{1/2} s_{\frac{j'}{K}}\| \\
\overset{(4.26)}{=}\quad & \|\textstyle\sum_{j=0}^{K} \sum_{\{j' \mid |j-j'|=1\}} s^*_{\frac{j}{K}}(\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2}(\theta_{\frac{j'}{K}}(\rho_{\frac{j'}{K}}(a)) - \eta_0)_+^{1/2} s_{\frac{j'}{K}} \\
& -s^*_{\frac{j}{K}}(\theta_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) - \eta_0)_+^{1/2}(\theta_{\frac{j'}{K}}(1_{Q_{\frac{j'}{K}}}) - \eta_0)_+^{1/2} s_{\frac{j'}{K}}\| \\
\leq\quad & \|\textstyle\sum_{i=0}^{D_1} s^*_{\frac{2i}{K}}(\theta_{\frac{2i}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2}(\theta_{\frac{2i+1}{K}}(\rho_{\frac{2i+1}{K}}(a)) - \eta_0)_+^{1/2} s_{\frac{2i+1}{K}} \\
& -s^*_{\frac{2i}{K}}(\theta_{\frac{2i}{K}}(\rho_{\frac{2i}{K}}(a)) - \eta_0)_+^{1/2}(\theta_{\frac{2i+1}{K}}(1_{Q_{\frac{2i+1}{K}}}) - \eta_0)_+^{1/2} s_{\frac{2i+1}{K}} \\
& +s^*_{\frac{2i+1}{K}}(\theta_{\frac{2i+1}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2}(\theta_{\frac{2i}{K}}(\rho_{\frac{2i}{K}}(a)) - \eta_0)_+^{1/2} s_{\frac{2i}{K}} \\
& -(\theta_{\frac{2i+1}{K}}(\rho_{\frac{2i+1}{K}}(a)) - \eta_0)_+^{1/2}(\theta_{\frac{2i}{K}}(1_{Q_{\frac{2i}{K}}}) - \eta_0)_+^{1/2} s_{\frac{2i}{K}}\| \\
& +\|\textstyle\sum_{i=0}^{D_2} s^*_{\frac{2i+1}{K}}(\theta_{\frac{2i+1}{K}}(1_{Q_{\frac{2i+1}{K}}}) - \eta_0)_+^{1/2}(\theta_{\frac{2i+2}{K}}(\rho_{\frac{2i+2}{K}}(a)) - \eta_0)_+^{1/2} s_{\frac{2i+2}{K}} \\
& -s^*_{\frac{2i+1}{K}}(\theta_{\frac{2i+1}{K}}(\rho_{\frac{2i}{K}}(a)) - \eta_0)_+^{1/2}(\theta_{\frac{2i+2}{K}}(1_{Q_{\frac{2i+2}{K}}}) - \eta_0)_+^{1/2} s_{\frac{2i+2}{K}} \\
& +s^*_{\frac{2i+2}{K}}(\theta_{\frac{2i+2}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2}(\theta_{\frac{2i+1}{K}}(\rho_{\frac{2i+1}{K}}(a)) - \eta_0)_+^{1/2} s_{\frac{2i+1}{K}} \\
& -s^*_{\frac{2i+2}{K}}(\theta_{\frac{2i+2}{K}}(\rho_{\frac{2i+2}{K}}(a)) - \eta_0)_+^{1/2}(\theta_{\frac{2i+1}{K}}(1_{Q_{\frac{2i}{K}}}) - \eta_0)_+^{1/2} s_{\frac{2i+1}{K}}\|,
\end{aligned}
$$

where

$$
D_1 = \begin{cases} \frac{D}{2} - 1, & \text{if } D \text{ is even} \\ \frac{D-1}{2}, & \text{if } D \text{ is odd,} \end{cases} \qquad D_2 = \begin{cases} \frac{D}{2} - 1, & \text{if } D \text{ is even} \\ \frac{D-3}{2}, & \text{if } D \text{ is odd.} \end{cases}
$$

Note that if i and i' are either both even or both odd, $i \neq i'$ we have

$$
\left(s^*_{\frac{i}{K}} x_0 s_{\frac{i+1}{K}} + s^*_{\frac{i+2}{K}} x_1 s_{\frac{i+1}{K}}\right) \cdot (s^*_{\frac{i'}{K}} x_2 s_{\frac{i'+1}{K}} + s^*_{\frac{i'+2}{K}} x_3 s_{\frac{i'+1}{K}}) = 0,
$$

for any $x_0, \ldots, x_3 \in A \otimes \mathcal{Q}$ since $|i + 1 - i'| > 2$ implies

$$
s_{\frac{i+1}{K}}(s_{\frac{i+1}{K}})^* s_{\frac{i'}{K}}(s_{\frac{i'}{K}})^* = 0.
$$

Thus each sum in the norm estimates above consists of mutually orthogonal summands, allowing us to estimate

$$\begin{aligned}
&\|\tilde{s}^*\tilde{s}(\textstyle\sum_{j=0}^K \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a))) - (\textstyle\sum_{j=0}^K \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)))\tilde{s}^*\tilde{s}\| \\
&\quad \leq \quad 2\cdot \max_{j=0\dots,K}(\|(\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2}(\theta_{\frac{j+1}{K}}((\rho_{\frac{j+1}{K}}(a)) - \eta_0)_+^{1/2} \\
&\qquad\qquad -(\theta_{\frac{j}{K}}((\rho_{\frac{j}{K}}(a)) - \eta_0)_+^{1/2})(\theta_{\frac{j+1}{K}}(1_{Q_{\frac{j+1}{K}}}) - \eta_0)_+^{1/2}\| \\
&\qquad\qquad +\|(\theta_{\frac{j+1}{K}}(1_{Q_{\frac{j+1}{K}}}) - \eta_0)_+^{1/2})(\theta_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) - \eta_0)_+^{1/2} \\
&\qquad\qquad -(\theta_{\frac{j+1}{K}}(\rho_{\frac{j+1}{K}}(a)) - \eta_0)_+^{1/2})(\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2}\|) \\
&\quad \leq \quad 4\cdot(4\eta_0^{1/2} + \max_{j=0,\dots,K}\|\theta_{\frac{j+1}{K}}(1_{Q_{\frac{j+1}{K}}})^{1/2}\theta_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a))^{1/2} - \theta_{\frac{j+1}{K}}(\rho_{\frac{j+1}{K}}(a))^{1/2}\theta_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}})^{1/2}\|) \\
&\quad \overset{(4.19)}{<} \quad 4\cdot(4\eta_0^{1/2} + \max_{j=0,\dots,K}\|\kappa_{\frac{j+1}{K}}(1_{Q_{\frac{j+1}{K}}})^{1/2}\kappa_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a))^{1/2} - \kappa_{\frac{j+1}{K}}(\rho_{\frac{j+1}{K}}(a))^{1/2}\kappa_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}})^{1/2}\|) \\
&\quad \overset{(4.2)}{<} \quad 16\eta_0^{1/2} + 4\eta^{1/2} \\
&\quad \overset{(4.23)}{<} \quad \epsilon/6 + \epsilon/12 \\
&\quad \overset{(4.14)}{<} \quad \epsilon/4.
\end{aligned} \tag{4.38}$$

It is straightforward to check that $s_{\frac{j}{K}} = s_{\frac{j}{K}}\sigma_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}})$. Then note that

$$\begin{aligned}
\|s_{\frac{j}{K}}\sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a))) - s_{\frac{j}{K}}(a\otimes 1_{\mathcal{Q}})\| \quad &\leq \quad \|\sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) - \sigma_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}})(a\otimes 1_{\mathcal{Q}})\| \\
&\overset{(4.22)}{<} \quad \epsilon/6.
\end{aligned} \tag{4.39}$$

Thus

$$\begin{aligned}
&\|s^*s(a\otimes 1_{\mathcal{Q}}) - (a\otimes 1_{\mathcal{Q}})s^*s\| \\
&\qquad \leq \qquad 4\cdot\|\tilde{s} - s\| + \|\tilde{s}^*\tilde{s}(a\otimes 1_{\mathcal{Q}}) - (a\otimes 1_{\mathcal{Q}})\tilde{s}^*\tilde{s}\| \\
&\qquad \overset{(4.34)}{\leq} \qquad \epsilon/3 + 2\cdot\max_j\|s_{\frac{j}{K}}\sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a))) - s_{\frac{j}{K}}(a\otimes 1_{\mathcal{Q}})\| \\
&\qquad\qquad\qquad +\|\tilde{s}^*\tilde{s}(\textstyle\sum_{j=0}^K \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a))) - (\textstyle\sum_{j=0}^K \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)))\tilde{s}^*\tilde{s}\| \\
&\qquad \overset{(4.39),(4.38)}{<} \qquad \epsilon/3 + \epsilon/3 + \epsilon/4 \\
&\qquad < \qquad \epsilon.
\end{aligned}$$

For (ii), we calculate, for $a \in \mathcal{F}$,

$$
\begin{aligned}
&\tilde{s}(\textstyle\sum_{j=0}^{K} \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)))\tilde{s}^* \\
&\overset{(4.33)}{=} \textstyle\sum_{j=0}^{K} (\theta(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2} s_{\frac{j}{K}} \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) s_{\frac{j}{K}}^* (\theta(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2} \\
&\overset{(4.27)}{=} \textstyle\sum_{j=0}^{K} s_{\frac{j}{K}} s_{\frac{j}{K}}^* (\theta(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2} s_{\frac{j}{K}} \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) s_{\frac{j}{K}}^* (\theta(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2} \\
&\overset{(4.28)}{=} \textstyle\sum_{j=0}^{K} s_{\frac{j}{K}} \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) s_{\frac{j}{K}}^* (\theta(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2} s_{\frac{j}{K}} s_{\frac{j}{K}}^* (\theta(1_{Q_{\frac{j}{K}}}) - \eta_0)_+^{1/2} \\
&\overset{(4.27)}{=} \textstyle\sum_{j=0}^{K} s_{\frac{j}{K}} \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) s_{\frac{j}{K}}^* (\theta(1_{Q_{\frac{j}{K}}}) - \eta_0)_+ s_{\frac{j}{K}} s_{\frac{j}{K}}^* \\
&\overset{(4.26)}{=} \textstyle\sum_{j=0}^{K} s_{\frac{j}{K}} s_{\frac{j}{K}}^* (\theta(\rho_{\frac{j}{K}}(a)) - \eta_0)_+ s_{\frac{j}{K}} s_{\frac{j}{K}}^* \\
&\overset{(4.25)}{=} \textstyle\sum_{j=0}^{K} (\theta(\rho_{\frac{j}{K}}(a)) - \eta_0)_+ \\
&\overset{(4.19)}{=} \textstyle\sum_{j=0}^{K} (\phi(\gamma_{\frac{j}{K}}) \otimes \kappa_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) \otimes p - \eta_0)_+.
\end{aligned}
$$

Define $h \in \mathcal{C}([0,1]) \otimes \mathcal{Q} \otimes \mathcal{Q}$ by

$$h := \textstyle\sum_{j=0}^{K} \phi(\gamma_{\frac{j}{K}}) \otimes \kappa_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) \otimes p.$$

Let $a_{\frac{j}{K}} \in F$ be elements satisfying

$$\|a_{\frac{j}{K}} - \kappa_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a))\| < \eta_0, \tag{4.40}$$

and put

$$h' := \textstyle\sum_{j=0}^{K} \Phi(\gamma_{\frac{j}{K}} \otimes a_{\frac{j}{K}}) \in \Phi(\mathcal{C}([0,1]) \otimes F).$$

Then

$$
\begin{aligned}
\|h - h'\| &\leq \|\textstyle\sum_{j \text{ even}} \phi(\gamma_{\frac{j}{K}}) \otimes (\kappa_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) - a_{\frac{j}{K}})\| \\
&\quad + \|\textstyle\sum_{j \text{ odd}} \phi(\gamma_{\frac{j}{K}}) \otimes (\kappa_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)) - a_{\frac{j}{K}})\| \\
&\overset{(4.40)}{<} 2 \cdot \eta_0 \\
&\overset{(4.23)}{<} \epsilon/48 \\
&< \epsilon/4.
\end{aligned}
$$

We calculate

$$
\begin{aligned}
&\|\tilde{s}^* h \tilde{s} - \tilde{s}^* \tilde{s}(\textstyle\sum_{j=0}^{K} \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)))\tilde{s}^* \tilde{s}\| \\
&\quad \leq \|h - \tilde{s}(\textstyle\sum_{j=0}^{K} \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)))\tilde{s}^*\| \\
&\quad \overset{(4.40)}{<} 2 \cdot \eta_0 \\
&\quad \overset{(4.23)}{<} \epsilon/4,
\end{aligned}
$$

so

$$
\begin{aligned}
\|s^* h' s - s^* s(a \otimes 1_{\mathcal{Q}}) s^* s\| &\leq 6 \cdot \|s - \tilde{s}\| + \|h - h'\| \\
&\quad + \|\tilde{s}^* h \tilde{s} - \tilde{s}^* \tilde{s}(\textstyle\sum_{j=0}^{K} \sigma_{\frac{j}{K}}(\rho_{\frac{j}{K}}(a)))\tilde{s}^* \tilde{s}\| \\
&< \epsilon/2 + \epsilon/4 + \epsilon/4 \\
&= \epsilon.
\end{aligned}
$$

This shows (ii).

Finally,

$$\begin{array}{rcl}\tau(s^*s) & = & \tau(ss^*) \\ & = & \tau(\phi(1_{\mathcal{C}([0,1])}) \otimes 1_F \otimes p) \\ & \stackrel{(4.30)}{>} & \frac{1-\eta_0}{n+2} \cdot \tau(\phi(1_{\mathcal{C}([0,1])})) \\ & \geq & \frac{1-\eta_0}{n+2} \cdot (\sum_{i=0}^{n-1} \tau(\phi(\gamma_i))) \\ & \stackrel{(4.18)}{\geq} & \frac{1-\eta_0}{n+2} \cdot (1-\delta) \\ & \stackrel{(4.16),(4.23)}{\geq} & \frac{1}{n+2} \cdot \frac{3}{4} \cdot \frac{2}{3} \\ & > & \frac{1}{2(n+2)},\end{array}$$

for all $\tau \in T(A \otimes \mathcal{Q})$, showing that (iii) holds. ▮

4.3. Main result, applications and outlook

4.3.15 Theorem: *Let A be a separable simple unital locally recursive subhomogeneous* C*-*algebra with exactly $n > 0$ extreme tracial states $\tau_0, \ldots, \tau_{n-1} \in T(A)$ satisfying $(\tau_i)_* = (\tau_j)_*$ for all $i, j \in \{0, \ldots, n-1\}$. Suppose that an approximating recursive subhomogeneous algebra B can always be chosen to have an $(\mathcal{F}, \eta)$-connected recursive subhomogeneous decomposition*

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$$

along which projections can be lifted and such that $X_l \setminus \Omega_l \neq \emptyset$ for $l \geq 1$. Then $A \otimes \mathcal{Q}$ is TAI.

Proof: The class I contains the finite-dimensional C*-algebras, is closed under direct sums and tensor products with finite-dimensional C*-algebras, and every C*-algebra in I can be written as a universal C*-algebra with weakly stable relations. Thus we may apply Lemma 2.3.3, and it is enough to show that there is an $m \in \mathbb{N}$ such that, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A \otimes \mathcal{Q}$, there exist a projection $p \in A \otimes \mathcal{Q}$ and a unital C*-subalgebra $B \subset p(A \otimes \mathcal{U})p$ and $B \in \mathcal{S}$ such that:

(i) $\|pb - bp\| < \epsilon$ for all $b \in \mathcal{F}$,

(ii) $\mathrm{dist}(pbp, B) < \epsilon$ for all $b \in \mathcal{F}$,

(iii) $\tau(p) > 1/m$ for all $\tau \in T(A \otimes \mathcal{Q})$.

By Lemma 2.3.4 we need only consider finite subsets of the form $\mathcal{F} = \mathcal{G} \otimes \{1_{\mathcal{Q}}\}$ for $\mathcal{G} \subset A$. Now the result follows from Theorem 4.2.14 with $m = 2(n+2)$. ▮

4.3.16 Corollary: *Let A be a separable simple unital locally recursive subhomogeneous* C*-*algebra with exactly $n > 0$ extreme tracial states $\tau_0, \ldots, \tau_{n-1} \in T(A)$ satisfying $(\tau_i)_* = (\tau_j)_*$ for all $i, j \in \{0, \ldots, n-1\}$. Suppose that an approximating recursive subhomogeneous algebra B can always be chosen to have a recursive subhomogeneous decomposition*

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$$

such that $\dim X_l \leq 1$ *for $l \geq 2$. Then $A \otimes \mathcal{Q}$ is TAI.*

Proof: Follows from Theorem 4.3.15 with Proposition 4.1.5. ▮

4.3.17 Notation: We let $\mathcal{A}$ denote the class of C*-algebras such that if $A \in \mathcal{A}$ then A is a unital separable simple locally recursive subhomogeneous C*-algebra such that the approximating recursive subhomogeneous algebra B can always be chosen to have an $(\mathcal{F}, \eta)$-connected recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$$

along which projections can be lifted and such that $X_l \setminus \Omega_l \neq \emptyset$ for $l \geq 1$.

4.3.18 Corollary: *Let $A \in \mathcal{A}$ with exactly $n > 0$ extreme tracial states $\tau_0, \ldots, \tau_{n-1} \in T(A)$ satisfying $(\tau_i)_* = (\tau_j)_*$ for all $i, j \in \{0, \ldots, n-1\}$. Let $\mathfrak{p}$ be a supernatural number and $M_{\mathfrak{p}}$ the associated UHF algebra. Then $A \otimes M_{\mathfrak{p}}$ is TAI.*

Proof: This proof is the same as that of Corollary 3.5.24. ∎

4.3.19 Proposition: *Let A be a separable simple unital locally recursive subhomogeneous* C*-*algebra. Then A satisfies the UCT.*

Proof: For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A$ we may approximate A by a subhomogeneous C*-algebra B. Since B is Type I, B satisfies the UCT. Therefore the result follows immediately by appealing to Theorem 1.1 of [13]. ∎

4.3.20 Corollary: *Let $A, B \in \mathcal{A}$ be* C*-*algebras, and let $n \in \mathbb{N} \setminus \{0\}$. Suppose there are exactly n extreme tracial states $\tau_0, \ldots, \tau_{n-1} \in T(A)$ satisfying $(\tau_i)_* = (\tau_j)_*$ for all $i, j \in \{0, \ldots, n-1\}$. Then*

$$A \otimes \mathcal{Z} \cong B \otimes \mathcal{Z} \text{ if and only if } \mathrm{Ell}(A \otimes \mathcal{Z}) \cong \mathrm{Ell}(B \otimes \mathcal{Z}).$$

If, in addition, A and B have finite decomposition rank, then

$$A \cong B \text{ if and only if } \mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

Proof: $A \otimes \mathcal{Q}$ and $B \otimes \mathcal{Q}$ are TAI by Theorem 4.3.15. Since A and B satisfy the UCT, the result follows by applying [36, Corollary 11.9]. Since A and B are separable, simple, nonelementary and unital, the second statement then follows from the fact that finite decomposition rank implies $\mathcal{Z}$-stability [75, Theorem 5.1]. ∎

In [16, Section 5], Elliott constructs examples of approximately subhomogeneous C*-algebras by attaching one-dimensional spaces to $\mathbb{T}$. These examples exhaust the Elliott invariant in the weakly unperforated case. In that paper, the Elliott invariant of these algebras is computed but classification results are not given. In the case of finitely many traces inducing the same state on K_0, we are able to obtain classification by the results above. In particular this shows that Elliott's examples, assuming the restriction to finitely many tracial states inducing the same state on K_0, agree with the examples of [40]; this was previously unknown.

4.3.21 Corollary: *Let A and B be inductive limits of building block algebras defined in [16, Section 5.1.2] and let $n \in \mathbb{N} \setminus \{0\}$. Suppose there are exactly n extreme tracial states $\tau_0, \ldots, \tau_{n-1} \in T(A)$ satisfying $(\tau_i)_* = (\tau_j)_*$ for all $i, j \in \{0, \ldots, n-1\}$ and exactly n extreme tracial states $\tau'_0, \ldots, \tau'_{n-1} \in T(B)$ satisfying $(\tau'_i)_* = (\tau'_j)_*$ for all $i, j \in \{0, \ldots, m-1\}$. Then $A \otimes \mathcal{Q}$ and $B \otimes \mathcal{Q}$ are TAI and we have*

$$A \cong B \text{ if and only if } \mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

Proof: By definition, A and B can be written as inductive limits $A = \varinjlim A_n$ and $B = \varinjlim B_n$ where A_n and B_n are recursive subhomogeneous C*-algebras of topological dimension less than or equal to one. It follows from Corollary 4.1.5 that $A, B \in \mathcal{A}$ and thus by the assumptions on the tracial state spaces, we have $A \otimes \mathcal{Q}$ and $B \otimes \mathcal{Q}$ are TAI by Corollary 4.3.16. Since the approximating algebras A_n and B_n all have dimension less than or equal to one, both A and B have finite decomposition rank. Thus classification follows from Corollary 4.3.20. ∎

At least to some extent, we are also able to apply Theorem 4.3.15 in the context of C*-algebras of minimal dynamical systems. In [38], Lin and Matui study minimal dynamical systems on the product of the Cantor set and $\mathbb{T}$. Let X be the Cantor set and let $\xi : X \to \mathbb{T}$ be a continuous map. Then we can define $R_\xi : X \to \mathrm{Homeo}(\mathbb{T})$ by $R_\xi(x)(t) = t + \xi(x)$ for $x \in X$ and $t \in \mathbb{T}$. If $\alpha : X \to X$ is a homeomorphism of the Cantor set X, then

$$\alpha \times R_\xi : X \times \mathbb{T} \to X \times \mathbb{T} : (x, t) \mapsto (\alpha(x), R_\xi(x)(t))$$

is a homeomorphism of $X \times \mathbb{T}$.

In the case that the homeomorphisms $\alpha \times R_\xi$ are minimal, Lin and Matui show that the crossed products $\mathcal{C}(X \times \mathbb{T}) \rtimes_{\alpha \times R_\xi} \mathbb{Z}$ are tracially approximately finite or have tracial rank one and hence are classifiable as they satisfy the UCT [38, Theorem 4.3]. Under the additional assumption of finitely many extreme tracial states, all of which induce the same state at the level of K_0, Theorem 4.3.15 offers an alternative route to the same result. First, we need to establish that it is enough to show that a certain large subalgebra of the crossed product is TAI. This appears in [60] and is a generalization of [41] (also [58]).

4.3.22 For a compact metric space X and a minimal homeomorphism $\alpha : X \to X$, put $A = \mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$. Let $y \in X$. We denote

$$A_{\{y\}} = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{y\})).$$

$A_{\{y\}}$ is a unital C^*-subalgebra of A, a generalization of those introduced by Putnam in [48]. This algebra carries much of the information contained in A while at the same time is significantly more tractable. In particular its K_0-group is isomorphic to that of A [46, Theorem 4.1(3)], and it can be written as an inductive limit of subhomogeneous algebras in a straightforward manner [42, Section 3]. There are natural bijections between the set of α-invariant probability measures on X, the set of tracial states on A and the set of tracial states on $A_{\{y\}}$ [42, Theorem 1.2].

4.3.23 Lemma: *Let X be an infinite compact metric space and $\alpha : X \to X$ a minimal homeomorphism. Let $y \in X$, and set $A_{\{y\}} = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{y\}))$, where u is the unitary in $\mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ implementing α. Let $\mathfrak{q}$ be a supernatural number of infinite type.*

Then, for any $\eta > 0$ and any open set $V \subset X$ containing y, there exists an open set $W \subset V$ with $y \in W$, functions $g_0 \in \mathcal{C}_0(W), g_1 \in \mathcal{C}_0(V)$, $0 \leq g_0, g_1 \leq 1$ and a projection $q_0 \in \overline{\mathcal{C}_0(V)A_{\{y\}}\mathcal{C}_0(V)} \otimes M_\mathfrak{q}$ such that

$$g_0(y) = 1, \quad g_1|_W = 1, \quad \textit{and} \quad \|q_0(g_1 \otimes 1) - g_1 \otimes 1\| \leq \eta.$$

Proof: We claim that there is a nonzero projection in $\overline{\mathcal{C}_0(V)A_{\{y\}}\mathcal{C}_0(V)} \otimes M_\mathfrak{q}$.

The set V is nonempty since $y \in V$. Thus $\mathcal{C}_0(V)$ is nonzero and hence we can find a nonzero positive contraction in $(\mathcal{C}_0(V)A_{\{y\}}\mathcal{C}_0(V)) \otimes M_\mathfrak{q}$, call it e. Since $A_{\{y\}}$ is simple by [41, 2.5], so is $A_{\{y\}} \otimes M_\mathfrak{q}$. Thus every tracial state $\tau \in T(A_{\{y\}} \otimes M_\mathfrak{q})$ is faithful, and in particular we have $\tau(e) > 0$ for every tracial state τ. Since $A_{\{y\}} \otimes M_\mathfrak{q}$ is unital, $T(A_{\{y\}} \otimes M_\mathfrak{q})$ is compact. Thus $\min_{\tau \in T(A_{\{y\}} \otimes M_\mathfrak{q})} \tau(e) > 0$. Furthemore, $d_\tau(e) > \tau(e)$ so the previous observations imply that $\min_{\tau \in T(A_{\{y\}} \otimes M_\mathfrak{q})} d_\tau(e) > 0$.

Since $A_{\{y\}} \otimes M_\mathfrak{q}$ has projections that are arbitrarily small in trace, there is a projection $p \in A_{\{y\}} \otimes M_\mathfrak{q}$ satisfying

$$\max_{\tau \in T(A_{\{y\}} \otimes M_\mathfrak{q})} \tau(p) < \min_{\tau \in T(A_{\{y\}} \otimes M_\mathfrak{q})} d_\tau(e).$$

By the above, for the projection p and any $\tau \in T(A_{\{y\}} \otimes M_\mathfrak{q})$, we have $d_\tau(p) = \tau(p) < d_\tau(e)$, so by strict comparison there are $x_n \in A_{\{y\}} \otimes M_\mathfrak{q}$ with $x_n e x_n^* \to p$. Let

$$a_n = e^{1/2} x_n^* x_n e^{1/2} \in (\mathcal{C}_0(V)A_{\{y\}}\mathcal{C}_0(V)) \otimes M_\mathfrak{q}.$$

Then a_n is self-adjoint and

$$\|a_n - a_n^2\| \to 0.$$

Disregarding any a_n such that $\|a_n - a_n^2\| \geq 1/4$, we obtain a sequence of projections b_n satisfying $\|b_n - a_n\| \leq 2\|a_n - a_n^2\| \to 0$ [31, Lemma 2.5.5]. Thus we obtain, for large enough n, a projection $b = b_n$ contained in $\overline{\mathcal{C}_0(V)A_{\{y\}}\mathcal{C}_0(V)} \otimes M_\mathfrak{q}$, proving the claim. Moreover, b is Murray–von Neumann equivalent to p, so $\min_\tau \tau(b) = \min_\tau \tau(p)$.

Let W be an open set contained in V such that $y \in W$ and small enough so that for every function $f \in \mathcal{C}_0(W)$ with $0 \leq f \leq 1$ we have $d_\tau(f \otimes 1_{M_\mathfrak{q}}) \leq \frac{1}{2} \min_\tau \tau(b)$ for every $\tau \in T(A_{\{y\}} \otimes M_\mathfrak{q})$. Choose $g_0, g_1 \in \mathcal{C}_0(W)$ such that $0 \leq g_0, g_1 \leq 1$, $g_0(y) = 1$ and $g_1 g_0 = g_0$. Then $d_\tau(g_1 \otimes 1) < d_\tau(b)$ for every

$\tau \in T(A_{\{y\}} \otimes M_{\mathfrak{q}})$, and so by the comparison of positive elements we have $(g_1 \otimes 1) \precsim b$ in $A_{\{y\}} \otimes M_{\mathfrak{q}}$ and hence also in $\overline{\mathcal{C}_0(V)A_{\{y\}}\mathcal{C}_0(V)} \otimes M_{\mathfrak{q}}$. Since $A_{\{y\}} \otimes M_{\mathfrak{q}}$ has stable rank one and $\overline{\mathcal{C}_0(V)A_{\{y\}}\mathcal{C}_0(V)} \otimes M_{\mathfrak{q}}$ is a full hereditary C*-subalgebra of $A_{\{y\}} \otimes M_{\mathfrak{q}}$, it also has stable rank one by [49, Theorem 3.6] with [6, Theorem 2.8]. By Proposition 2.4 of [53], for $\eta/2 > 0$ there is a unitary v in $(\overline{\mathcal{C}_0(V)A_{\{y\}}\mathcal{C}_0(V)} \otimes M_{\mathfrak{q}})^{\sim}$ such that $(g_1 \otimes 1 - \eta/2)_+ \leq vbv^*$ in $(\overline{\mathcal{C}_0(V)A_{\{y\}}\mathcal{C}_0(V)} \otimes M_{\mathfrak{q}})^{\sim}$, and hence in $\overline{\mathcal{C}_0(V)A_{\{y\}}\mathcal{C}_0(V)} \otimes M_{\mathfrak{q}}$. Put $q_0 = vbv^*$. Then

$$\begin{aligned} \|q_0(g_1 \otimes 1) - (g_1 \otimes 1)\| &< \|q_0(g_1 \otimes 1 - \eta/2)_+ - (g_1 \otimes 1)\| + \eta/2 \\ &= \|(g_1 \otimes 1 - \eta/2)_+ - (g_1 \otimes 1)\| + \eta/2 \\ &< \eta. \end{aligned}$$

∎

We will use the previous lemma to choose an initial projection in $A_{\{y\}} \otimes M_{\mathfrak{q}}$. However, since this projection actually only approximates the properties we would like it to have, we require the following easy lemma that pushes orthogonal projections into a C*-subalgebra. The proof is straightforward and hence omitted.

4.3.24 Lemma: *Given $\epsilon > 0$ and a positive integer n, there is a $\delta > 0$ with the following property. Let A be a C*-algebra, B a C*-subalgebra of A. Suppose that $p_1, \ldots, p_n$ are mutually orthogonal projections in A, the first k, $0 \leq k \leq n$, of which are contained in B, and that $a_{k+1}, \ldots, a_n$ are self adjoint elements of B such that*

$$\|p_i - a_i\| < \min(1/2, \delta), \quad i = k+1, \ldots, n.$$

Then there are mutuallly orthogonal projections $q_1, \ldots, q_n$ in B, where $q_i = p_i$ for $1 \leq i \leq k$, and for $k+1 \leq i \leq n$ we have

$$\|q_i - p_i\| < \epsilon.$$

Moreover, if A is unital then there are unitaries $u_i \in A$ such that $q_i = u_i p_i u_i^*$.

The following lemma will be required. The proofs are given in [58] and [60].

4.3.25 Lemma: *Let X be an infinite compact metric space with a minimal homeomorphism $\alpha : X \to X$. Let $A = \mathrm{C}^*(X) \rtimes_\alpha \mathbb{Z}$ and $A_{\{y\}} = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{y\}))$. Then*

$$K_0(A_{\{y\}} \otimes M_{\mathfrak{q}}) \cong K_0(A \otimes M_{\mathfrak{q}})$$

as ordered groups, with the isomorphism induced by the inclusion $\iota : A_{\{y\}} \to A$.

The following lemma generalizes [41, Lemma 4.2], due to H. Lin and N. C. Phillips and [58].

4.3.26 Lemma: *Let X be an infinite compact metric space, $\alpha : X \to X$ a minimal homeomorphism, $y \in X$ and $\mathfrak{q}$ be a supernatural number of infinite type. Let $A = \mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ and $A_{\{y\}} = \mathrm{C}^*(\mathcal{C}(X), u\mathcal{C}_0(X \setminus \{y\}))$, where u is the unitary implementing α in A. Then, for any finite subset $\mathcal{F} \subset A \otimes M_{\mathfrak{q}}$ and every $\epsilon > 0$, there is a projection p in $A_{\{y\}} \otimes M_{\mathfrak{q}}$ such that*

(i) *$\|pa - ap\| < \epsilon$ for all $a \in \mathcal{F}$,*

(ii) *$\mathrm{dist}(pap, p(A_{\{y\}} \otimes M_{\mathfrak{q}})p) < \epsilon$ for all $a \in \mathcal{F}$,*

(iii) *$\tau(1_{A \otimes M_{\mathfrak{q}}} - p) < \epsilon$ for all $\tau \in T(A \otimes M_{\mathfrak{q}})$.*

Proof: Let $\epsilon > 0$. We first show that there exists a projection satisfying properties (i) – (iii) of the lemma when $\mathcal{F}$ is assumed to be of the form

$$\mathcal{F} = (\mathcal{G} \otimes \{1_{M_{\mathfrak{q}}}\}) \cup \{u \otimes 1_{M_{q^\infty}}\}$$

where $\mathcal{G}$ is a finite subset of $\mathcal{C}(X)$.

Let $N_0 \in \mathbb{N}$ such that $\pi/(2N_0) < \epsilon/4$.

Let $\delta_0 > 0$ with $\delta_0 < \epsilon/4$ and sufficiently small so that for all $g \in \mathcal{G}$ we have $\|g(x_1) - g(x_2)\| < \epsilon/8$ as long as $d(x_1, x_2) < 4\delta_0$.

Choose $\delta > 0$ with $\delta < \delta_0$ and such that $d(\alpha^{-n}(x_1), \alpha^{-n}(x_2)) < \delta_0$ whenever $d(x_1, x_2) < \delta$ and $0 \leq n \leq N_0$.

Since α is minimal, there is an $N > N_0 + 1$ such that

$$d(\alpha^N(y), y) < \delta.$$

Let $R \in \mathbb{N}$ be sufficiently large so that

$$R > (N + N_0 + 1)/\min(1, \epsilon).$$

Minimality of α also implies that there is an open neighbourhood U of y such that

$$\alpha^{-N_0}(U), \alpha^{-N_0+1}(U), \ldots, U, \alpha(U), \ldots, \alpha^R(U)$$

are all disjoint. Making U smaller if necessary, we may assume that each $\alpha^n(U)$, $-N_0 \leq n \leq R$ has diameter less than δ. To apply Berg's technique, we only need U_n for $-N_0 \leq n \leq N$, however we require R to be larger in order to satisfy property (iii) of the lemma.

Let $\lambda = \max\{\|g\| \mid g \in \mathcal{G}\}$, and choose

$$0 < \epsilon_0 < \min(1/2, \epsilon/(2(N + 3N_0 + 1)), \epsilon/(32N(\lambda + \epsilon/4)), \tag{4.41}$$

$$0 < \epsilon_1 < \min(\epsilon_0/8, \delta_{\epsilon_0,N}/16) \tag{4.42}$$

and

$$0 < \eta < \min(2\epsilon, \delta_{\epsilon_1,N_0+N+1})$$

where $\delta_{\epsilon_1,N_0+N+1}$ is given by Lemma 4.3.24 with respect to ϵ_0 and $N_0 + N + 1$ in place of ϵ and n, respectively; similarly for $\delta_{\epsilon_0,N}$.

Let $f_0 : X \to [0, 1]$ be continuous with $\mathrm{supp}(f_0) \subset U$, and $f_0|_V = 1$ for some open set $V \subset U$ containing y.

By Lemma 4.3.23, there is an open set $W \subset V$ containing y, functions $g_0 \in \mathcal{C}_0(W)$, $g_1 \in \mathcal{C}_0(V)$, $0 \leq g_0, g_1 \leq 1$ and a projection $q_0 \in \overline{\mathcal{C}_0(V)A_{\{y\}}\mathcal{C}_0(V)} \otimes M_{\mathfrak{q}}$ such that

$$g_0(y) = 1, g_1|_W = 1 \text{ and } \|q_0(g_1 \otimes 1) - g_1 \otimes 1\| < \eta/2.$$

Consequently, $(f_0 \otimes 1)q_0 = q_0 = q_0(f_0 \otimes 1)$ and $\|q_0(g_0 \otimes 1) - g_0 \otimes 1\| < \eta/2$.

For $-N_0 \leq n \leq N$, set

$$q_n = (u^n \otimes 1)q_0(u^{-n} \otimes 1), \qquad f_n = u^n f_0 u^{-n} = f_0 \circ \alpha^{-n} \quad \text{and} \quad U_n = \alpha^n(U).$$

Then $\mathrm{supp}(f_n) \subset U_n$ and

$$(f_n \otimes 1)q_n = ((u^n f_0 u^{-n}) \otimes 1)(u^n \otimes 1)q_0(u^{-n} \otimes 1) = (u^n \otimes 1)(f_0 \otimes 1)q_0(u^{-n} \otimes 1) = q_n.$$

Similarly, $q_n(f_n \otimes 1) = q_n$. Since the f_n have disjoint support, it follows that the projections

$$q_{-N_0}, \ldots, q_{-1}, q_0, q_1, \ldots, q_N$$

are mutually orthogonal.

We claim that $q_{-N_0}, \ldots, q_{-1}, q_0 \in A_{\{y\}} \otimes M_{\mathfrak{q}}$ and that there are self-adjoint $c_1, \ldots, c_N \in A_{\{y\}} \otimes M_{\mathfrak{q}}$ such that $\|q_n - c_n\| < \eta$ for $1 \leq n \leq N$.

Let $1 \leq n \leq N_0$ and consider q_{-n}. We have $(uf_{-n} \otimes 1) \in A_{\{y\}} \otimes M_{\mathfrak{q}}$ for $1 \leq n \leq N_0$ since $U_{-n} \cap U_0 = \emptyset$. Let $a_n = f_0^n u^n \otimes 1$. Then

$$\begin{aligned} a_n = f_0^n u^n \otimes 1 &= (uu^{-1} f_0 u^2 u^{-2} f_0 u^3 u^{-3} \cdots u^n u^{-n} f_0 u^n) \otimes 1 \\ &= (uf_{-1} \otimes 1)(uf_{-2} \otimes 1) \cdots (uf_{-n} \otimes 1) \in A_{\{y\}} \otimes M_{\mathfrak{q}}. \end{aligned}$$

From this it follows that

$$\begin{aligned} q_{-n} &= (u^{-n} \otimes 1) q_0 (u^n \otimes 1) \\ &= (u^{-n} \otimes 1)(f_0^n \otimes 1) q_0 (f_0^n \otimes 1)(u^n \otimes 1) \\ &= a_n^* q_0 a_n \in A_{\{y\}} \otimes M_{\mathfrak{q}}. \end{aligned}$$

Note that $q_0(g_0 \otimes 1) = q_0(g_1 g_0 \otimes 1)$, since $g_1|_W = 1$ and $g_0 \in \mathcal{C}_0(W)$. Thus

$$\begin{aligned} \|(g_1 g_0 \otimes 1) - q_0(g_0 \otimes 1)\| &= \|(g_1 \otimes 1 - q_0(g_1 \otimes 1))(g_0 \otimes 1)\| \\ &< \eta/2. \end{aligned}$$

Also, $g_1 f_0 = g_1$ since $f_0|_V = 1$ and $g_1 \in \mathcal{C}_0(V)$. Hence

$$\begin{aligned} &\|(q_0 - g_1 \otimes 1)(f_0 \otimes 1 - g_0 \otimes 1) - (q_0 - g_1 \otimes 1)\| \\ &\quad = \|q_0(f_0 \otimes 1) - q_0(g_0 \otimes 1) - (g_1 f_0) \otimes 1 + (g_1 g_0) \otimes 1 - q_0 + g_1 \otimes 1\| \\ &\quad = \|(g_1 g_0 \otimes 1) - q_0(g_0 \otimes 1)\| \\ &\quad < \eta/2. \end{aligned}$$

Since $f_0(y) = 1 = g_0(y)$, we have that $u(f_0 - g_0) \otimes 1 \in A_{\{y\}} \otimes M_{\mathfrak{q}}$. Set

$$c_1 = (u(f_0 - g_0) \otimes 1)(q_0 - g_1 \otimes 1)(u(f_0 - g_0) \otimes 1)^* + (ug_1 u^* \otimes 1).$$

Then c_1 is a self-adjoint element in $A_{\{y\}} \otimes M_{\mathfrak{q}}$ and

$$\begin{aligned} &\|q_1 - c_1\| \\ &\quad = \|(u \otimes 1) q_0 (u^* \otimes 1) - (u(f_0 - g_0) \otimes 1)(q_0 - g_1 \otimes 1)(u(f_0 - g_0) \otimes 1)^* \\ &\qquad -(u \otimes 1)(g_1 \otimes 1)(u^* \otimes 1)\| \\ &\quad = \|q_0 - ((f_0 - g_0) \otimes 1)(q_0 - g_1 \otimes 1)((f_0 - g_0) \otimes 1) - g_1 \otimes 1\| \\ &\quad \leq \|(q_0 - g_1 \otimes 1) - (q_0 - g_1 \otimes 1)((f_0 - g_0) \otimes 1)\| \\ &\qquad + \|(q_0 - g_1 \otimes 1)((f_0 - g_0) \otimes 1) - ((f_0 - g_0) \otimes 1)(q_0 - g_1 \otimes 1)((f_0 - g_0) \otimes 1)\| \\ &\quad \leq \|(q_0 - g_1 \otimes 1) - (q_0 - g_1 \otimes 1)((f_0 - g_0) \otimes 1)\| \\ &\qquad + \|(q_0 - g_1 \otimes 1) - ((f_0 - g_0) \otimes 1)(q_0 - g_1 \otimes 1)\| \, \|((f_0 - g_0) \otimes 1)\| \\ &\quad < \eta. \end{aligned}$$

For $2 \leq n \leq N$, define

$$c_n = ((uf_{n-1} \cdots uf_1) \otimes 1) c_1 ((uf_{n-1} \cdots uf_1) \otimes 1)^*.$$

The c_n are self-adjoint elements in $A_{\{y\}} \otimes M_{\mathfrak{q}}$ since $f_{n-1}, \ldots, f_1$ all vanish at y. Furthermore,

$$\begin{aligned} \|q_n - c_n\| &= \|(u^n \otimes 1) q_0 (u^{-n} \otimes 1) - c_n\| \\ &= \left\|((u^n f_0^{n-1}) \otimes 1) q_0 ((f_0^{n-1} u^{-n}) \otimes 1) - c_n\right\| \\ &= \left\|((u^n f_0^{n-1} u^{-1}) \otimes 1) q_1 ((u f_0^{n-1} u^{-n}) \otimes 1) - c_n\right\| \\ &= \|((uf_{n-1} \cdots uf_1) \otimes 1) q_1 ((uf_{n-1} \cdots uf_1) \otimes 1)^* - c_n\| \\ &\leq \|q_1 - c_1\| \\ &< \eta. \end{aligned}$$

This proves the claim.

We now apply Lemma 4.3.24 to obtain projections $p_1, \ldots, p_N$ in $A_{\{y\}} \otimes M_{\mathfrak{q}}$ such that

$$q_{-N_0}, \ldots, q_{-1}, q_0, p_1, \ldots, p_N$$

are mutually orthogonal, and, for $1 \leq n \leq N$, we have

$$\|p_n - q_n\| < \epsilon_1 \tag{4.43}$$

and unitaries y_n such that

$$p_n = y_n q_n y_n^*.$$

Since $p_n \sim q_n$ and $q_n \sim q_0$, we have $[p_N] = [q_0]$ in $K_0(A \otimes M_{\mathfrak{q}})$. Since $K_0(\iota \otimes \mathrm{id}_{M_{\mathfrak{q}}}) : K_0(A_{\{y\}} \otimes M_{\mathfrak{q}}) \to K_0(A \otimes M_{\mathfrak{q}})$ is an isomorphism by Lemma 4.3.25, we also have $[p_N] = [q_0]$ in $K_0(A_{\{y\}} \otimes M_{\mathfrak{q}})$. Moreover, simplicity of $A_{\{y\}}$ [41, Proposition 2.5] implies $A_{\{y\}} \otimes M_{\mathfrak{q}}$ has stable rank one by Corollary 6.6 of [52]. Thus projections in matrix algebras over $A_{\{y\}} \otimes M_{\mathfrak{q}}$ satisfy cancellation, and there is a partial isometry $w \in A_{\{y\}} \otimes M_{\mathfrak{q}}$ such that $w^* w = q_0$ and $w w^* = p_N$.

For $t \in \mathbb{R}$, set

$$v(t) = \cos(\pi t/2)(q_0 + p_N) + \sin(\pi t/2)(w - w^*).$$

Then $v(t)$ is a unitary in the corner $(q_0 + p_N)(A_{\{y\}} \otimes M_{\mathfrak{q}})(q_0 + p_N)$. The matrix of $v(t)$ with respect to the obvious decomposition is

$$\begin{pmatrix} \cos(\pi t/2) & -\sin(\pi t/2) \\ \sin(\pi t/2) & \cos(\pi t/2) \end{pmatrix}.$$

For $0 \leq k \leq N_0$, define

$$w_k = (u^{-k} \otimes 1) v(k/N_0) u^k.$$

Also, let

$$w_k' = (a_k + b_k)^* v(k/N_0)(a_k + b_k)$$

where

$$\begin{aligned} a_k &= (f_0^k u^k) \otimes 1 = (u f_{-1} \cdots u f_{-k}) \otimes 1 \text{ (as above)} \\ b_k &= (f_N^k u^k) \otimes 1 = (u f_{N-1} \ldots u f_{N-k}) \otimes 1. \end{aligned}$$

Both a_k and b_k are in $A_{\{y\}} \otimes M_{\mathfrak{q}}$, hence

$$w_k' \in A_{\{y\}} \otimes M_{\mathfrak{q}}.$$

We show what w_k is close to w_k'. Define

$$x_k = (u^{-k} \otimes 1)(q_0 + q_N) v(k/N_0)(q_0 + q_N) u^k.$$

We have that

$$\begin{aligned} \|w_k - x_k\| &= \|(u^{-k} \otimes 1)(q_0 + p_N) v(k/N_0)(q_0 + p_N)(u^k \otimes 1) \\ &\quad -(u^{-k} \otimes 1)(q_0 + q_N) v(k/N_0)(q_0 + q_N)(u^k \otimes 1)\| \\ &= \|q_0 v(k/N_0) p_N - q_0 v(k/N_0) q_N + p_N v(k/N_0) q_0 - q_N v(k/N_0) q_0 \\ &\quad + p_N v(k/N_0) p_N - q_N v(k/N_0) q_N\| \\ &\leq 4\|p_N - q_N\| \\ &< 4\epsilon_1. \end{aligned} \tag{4.44}$$

Also,

$$\begin{aligned} &\|w_k' - (a_k + b_k)^*(q_0 + q_N) v(k/N_0)(q_0 + q_N)(a_k + b_k)\| \\ &\quad \leq \|a_k + b_k\|^2 \|(q_0 + p_N) v(k/N_0)(q_0 + p_N) - (q_0 + q_N) v(k/N_0)(q_0 + q_N)\| \\ &\quad \leq 4\|p_N - q_N\| \\ &\quad < 4\epsilon_1. \end{aligned} \tag{4.45}$$

But

$$\begin{aligned}&(a_k+b_k)^*(q_0+q_N)v(k/N_0)(q_0+q_N)(a_k+b_k)\\ =\;& (f_0^k u^k + f_N^k u^k)\otimes 1)^*(q_0+q_N)v(k/N_0)(q_0+q_N)(f_0^k u^k + f_N^k u^k)\otimes 1)\\ =\;& (u^{-k}\otimes 1)(q_0+q_N)v(k/N_0)(q_0+q_N)u^k\\ =\;& x_k.\end{aligned}$$

Thus

$$\|w_k - w_k'\| < 8\epsilon_1. \tag{4.46}$$

Comparing w_k to w_{k+1} conjugated by $u\otimes 1$ we have

$$\|(u\otimes 1)w_{k+1}(u^{-1}\otimes 1) - w_k\| = \|v((k+1)/N_0) - v(k/N_0)\| \leq \pi/(2N_0) < \epsilon/4 \tag{4.47}$$

for $0\leq k < N_0$. Define projections

$$e_0 = q_0, e_n = p_n \text{ for } 1\leq n < N - N_0$$

and

$$e_n = w_{N-n}q_{n-N}w_{N-n}^* \text{ for } N-N_0\leq n\leq N.$$

Also define

$$d_n = q_n \text{ for } 0\leq n < N-N_0$$

and

$$d_n = x_{N-n}q_{n-N}x_{N-n}^* \text{ for } N-N_0\leq n\leq N.$$

Note that this gives $d_N = x_0q_0x_0^* = (q_0+q_N)v(0)q_0v(0)^*(q_0+q_N) = q_0 = d_0$ and and $e_N = v(0)q_0v(0)^* = q_0 = e_0$. We also have that

$$x_k^*x_l = x_{k+1}^*(u\otimes 1)^*x_l = x_k^*(u\otimes 1)x_{l+1} = 0$$

when $k\neq l$. This follows from the fact that, if $k\neq l$, then q_{-k}, q_{N-k}, q_{-l} and q_{N-l}, $0\leq k\neq l\leq N_0$, are mutually orthogonal and

$$\begin{aligned}&(q_0+q_N)u^k(u^{-l}\otimes 1)(q_0+q_N)\\ =\;&(u^k\otimes 1)(q_{-k}+q_{N-k})(q_{-l}+q_{N-l})(u^{-l}\otimes 1)\\ =\;&0.\end{aligned}$$

Also, if $0 < m < N-N_0$ and $N-N_0\leq n\leq N$ then

$$q_m(u^{-(N-n)}\otimes 1)(q_0+q_N) = q_m(q_{-(N-n)}+q_n)(u^{-(N-n)}\otimes 1) = 0,$$

and similarly

$$(q_0+q_N)(u^{N-n}\otimes 1)q_m = 0.$$

From this it follows that

$$d_md_n = d_{m+1}(u\otimes 1)^*d_n =_m (u\otimes 1)d_{n+1} = 0 \tag{4.48}$$

for $0\leq m\neq n\leq N$.

For $1\leq n\leq N-N_0-1$ we have

$$\|e_n - d_n\| = \|p_n - q_n\| < \epsilon_1$$

and

$$\|e_0 - d_0\| = 0.$$

If $N - N_0 \leq n \leq N$, then

$$\begin{aligned}
\|e_n - d_n\| &= \|w_{N-n}q_{-(N-n)}w^*_{N-n} - x_{N-n}q_{-(N-n)}x^*_{N-n}\| \\
&= \|w_{N-n}q_{-(N-n)}w^*_{N-n} - w_{N-n}q_{-(N-n)}x^*_{N-n} + w_{N-n}q_{-(N-n)}x^*_{N-n} \\
&\quad -x_{N-n}q_{-(N-n)}x^*_{N-n}\| \\
&\leq \|w^*_{N-n} - x^*_{N-n}\| + \|w_{N-n} - x_{N-n}\| \\
&\overset{(4.44)(4.45)}{<} 4\epsilon_1 + 4\epsilon_1 \\
&= 8\epsilon_1. \qquad (4.49)
\end{aligned}$$

We now show that conjugating the d_n by $u \otimes 1$ acts approximately as a cyclic shift. For $1 \leq n \leq N - N_0 - 1$ we have $(u \otimes 1)d_{n-1}(u \otimes 1)^* = d_n$ since $d_n = q_n$.

If $n = N - N_0$, then

$$\begin{aligned}
d_{N-N_0} &= x_{N_0}q_{-N_0}x^*_{N_0} \\
&= (u^{-N_0} \otimes 1)(q_0 + q_N)v(1)(q_0 + q_N)q_0(q_0 + q_N)v(-1)(q_0 + q_N)(u^{N_0} \otimes 1) \\
&= (u^{-N_0} \otimes 1)(q_0 + q_N)p_N(q_0 + q_N)(u^{N_0} \otimes 1) \\
&= (u^{-N_0} \otimes 1)q_N p_N q_N(u^{N_0} \otimes 1).
\end{aligned}$$

Thus

$$\begin{aligned}
&\|(u \otimes 1)d_{N-N_0-1}(u^* \otimes 1) - d_{N-N_0}\| \\
&\quad = \|q_{N-N_0} - (u^{-N_0} \otimes 1)q_N p_N q_N(u^{N_0} \otimes 1)\| \\
&\quad = \|(u^{-N_0} \otimes 1)q_N(u^{N_0} \otimes 1) - (u^{-N_0} \otimes 1)q_N p_N q_N(u^{N_0} \otimes 1)\| \\
&\quad \overset{(4.43)}{\leq} \|q_N - p_N\| \\
&\quad < \epsilon_1.
\end{aligned}$$

When $N - N_0 < n \leq N$, first consider what happens to the e_n using the estimation made on the w_k above. We have

$$\begin{aligned}
&\|(u \otimes 1)e_{n-1}(u^* \otimes 1) - e_n\| \\
&\quad = \|(u \otimes 1)w_{N-(n-1)}(u^* \otimes 1)q_{n-N}(u \otimes 1)w^*_{N-(n-1)}(u^* \otimes 1) \\
&\qquad -w_{N-n}q_{n-N}w^*_{N-n}\| \\
&\quad \leq \|(u \otimes 1)w_{N-(n-1)}(u^* \otimes 1)q_{n-N}(u \otimes 1)w^*_{N-(n-1)}(u^* \otimes 1) - \\
&\qquad (u \otimes 1)w_{N-(n-1)}(u^* \otimes 1)q_{n-N}w^*_{N-n}\| \\
&\qquad +\|(u \otimes 1)w_{N-(n-1)}(u^* \otimes 1)q_{n-N}w^*_{N-n} - w_{N-n}q_{n-N}w^*_{N-n}\| \\
&\quad \leq \|(u \otimes 1)w^*_{N-(n-1)}(u^* \otimes 1) - w^*_{N-n}\| \\
&\qquad +\|(u \otimes 1)w_{N-(n-1)}(u^* \otimes 1) - w_{N-n}\| \\
&\quad \overset{(4.43)}{<} \epsilon/2.
\end{aligned}$$

From this we have

$$\begin{aligned}
\|(u \otimes 1)d_{n-1}(u^* \otimes 1) - d_n\| &< \|e_{n-1} - d_{n-1}\| + \|e_n - d_n\| + \epsilon/2 \\
&\overset{(4.49)(4.49)}{<} 16\epsilon_1 + \epsilon/2.
\end{aligned}$$

Now we use the fact that the w_k are almost in $A_{\{y\}} \otimes M_{\mathfrak{q}}$ to find projections in $A_{\{y\}} \otimes M_{\mathfrak{q}}$ that lie close to the e_n and hence also close to the d_n. When $0 \leq n \leq N-N_0-1$, we have $e_n = p_n \in A_{\{y\}} \otimes M_{\mathfrak{q}}$.

Also, since $e_N = q_0$, we only need to find projections when $N - N_0 \leq n \leq N-1$. In this case, we have

$$\begin{aligned} &\|e_n - w'_{N-n} q_{-(N-n)} (w'_{N-n})^* \| \\ &\quad = \quad \|w_{N-n} q_{-(N-n)} w^*_{N-n} - w'_{N-n} q_{-(N-n)} (w'_{N-n})^* \| \\ &\quad \overset{(4.46)}{<} \quad 16\epsilon_1. \end{aligned}$$

Since $w'_{N-n} q_{-(N-n)} (w'_{N-n})^* \in A_{\{y\}} \otimes M_{\mathfrak{q}}$, by Lemma 4.3.24 we find pairwise orthogonal projections $r_n \in A_{\{y\}} \otimes M_{\mathfrak{q}}$ with

$$\|r_n - e_n\| < \epsilon_0 \text{ and } r_n = z_n e_n z_n^*$$

for unitaries $z_n \in A \otimes M_{\mathfrak{q}}$. This also implies that

$$\|r_n - d_n\| < \epsilon_0 + 8\epsilon_1 < 2\epsilon_0. \tag{4.50}$$

For $1 \leq n \leq N - N_0 - 1$ put $r_n = e_n = p_n$ and put $r_N = e_N = q_0$. Then set

$$r = \textstyle\sum_{n=1}^N r_n \qquad \text{and} \qquad p = 1 - r.$$

We verify that the projection $p \in A_{\{y\}} \otimes M_{\mathfrak{q}}$ satisfies properties (i) – (iii) of the lemma.

Let $d = \sum_{n=1}^N d_n$. Note that

$$d - (u \otimes 1) d (u \otimes 1)^* = \textstyle\sum_{n=N-N_0}^N ((u \otimes 1) d_{n-1} (u \otimes 1)^* - d_n).$$

For $N - N_0 \leq m \neq n \leq N$, we have

$$\begin{aligned} &((u \otimes 1) d_{n-1} (u \otimes 1)^* - d_n)((u \otimes 1) d_{m-1} (u \otimes 1)^* - d_m) \\ &\quad = \quad (u \otimes 1) d_{n-1} d_{m-1} (u \otimes 1)^* - (u \otimes 1) d_{n-1} (u \otimes 1)^* d_m \\ &\qquad\quad - d_n (u \otimes 1) d_{m-1} (u \otimes 1)^* + d_n d_m \\ &\quad \overset{(4.48)}{=} \quad 0. \end{aligned}$$

Thus the terms in the sum are mutually orthogonal with norm at most $16\epsilon_1 + \epsilon/2$, hence

$$\|d - (u \otimes 1) d (u \otimes 1)^* \| < 16\epsilon_1 + \epsilon/2. \tag{4.51}$$

Now

$$\begin{aligned} &\|p - (u \otimes 1) p (u \otimes 1)^* \| \\ &\quad = \quad \|((u \otimes 1) r (u^* \otimes 1) - r) - ((u \otimes 1) d (u^* \otimes 1) - d) + ((u \otimes 1) d (u^* \otimes 1) - d)\| \\ &\quad \leq \quad 2\|r - d\| + 16\epsilon_1 + \epsilon/2 \\ &\quad \overset{(4.51)}{<} \quad \textstyle\sum_{n=1}^{N-N_0-1} 2\|p_n - q_n\| + \sum_{m=N-N_0}^{N-1} 2\|r_m - d_m\| + 16\epsilon_1 + \epsilon/2 \\ &\quad \overset{(4.43)(4.50)}{<} \quad 2(N - N_0 - 1)\epsilon_1 + 4N_0\epsilon_0 + 16\epsilon_1 + \epsilon/2 \\ &\quad \overset{(4.42)}{<} \quad (N - N_0 - 1)\epsilon_0 + 4N_0\epsilon_0 + 2\epsilon_0 + \epsilon/2 \\ &\quad \overset{(4.41)}{<} \quad \epsilon. \end{aligned}$$

Since $g_1(y) = 1$ it follows that $u(1 - g_1) \otimes 1 \in A_{\{y\}} \otimes M_{\mathfrak{q}}$. Thus we also have that $p(u \otimes 1)((1 - g_1) \otimes 1)(1 - q_0)p \in A_{\{y\}} \otimes M_{\mathfrak{q}}$. Note that $p \leq 1 - q_0$. Using this and the fact that $\|g_1 \otimes 1 - (g_1 \otimes 1) q_0\| < \eta/2 < \epsilon$, it follows that

$$\begin{aligned} &\|p(u \otimes 1) p - p(u \otimes 1)((1 - g_1) \otimes 1)(1 - q_0) p\| \\ &\quad = \quad \|p(u \otimes 1) p - p(u \otimes 1) p + p(u \otimes 1)(g_1 \otimes 1)(1 - q_0) p\| \\ &\quad \leq \quad \|p(u \otimes 1)(g_1 \otimes 1) p - p(u \otimes 1)(g_1 \otimes 1) q_0 p\| \\ &\quad < \quad \epsilon. \end{aligned}$$

This proves (i) and (ii) for the element $u \otimes 1 \in \mathcal{F}$.

Now consider $g \otimes 1 \in \mathcal{F}$, where $g \in \mathcal{C}(X)$.

Since $d(\alpha^N(y), y) < \delta$, we have $d(\alpha^n(y), \alpha^{n-N}(y)) < \delta_0$ for $N - N_0 \leq n \leq N$. It follows that $U_{n-N} \cup U_n$ has diameter less than $2\delta + \delta_0 \leq 3\delta_0$. The function $g \in \mathcal{G}$ varies by at most $\epsilon/8$ on sets of diameter less than $4\delta_0$, and since the sets

$$U_1, U_2, \ldots, U_{N-N_0-1}, U_{N-N_0} \cup U_{-N_0}, U_{N-N_0+1} \cup U_{-N_0+1}, \ldots, U_N \cup U_0$$

are open and pairwise disjoint, there is $\tilde{g} \in \mathcal{C}(X)$ which is constant on each of these sets and satisfies $\|g - \tilde{g}\| < \epsilon/4$. Let the values of $\tilde{g}$ on these sets be λ_1 on U_1 through to λ_N on $U_N \cup U_0$.

For $0 \leq n \leq N - N_0 - 1$ we have

$$\begin{aligned} \|(f_n \otimes 1)r_n - r_n\| &= \|(f_n \otimes 1)r_n - (f_n \otimes 1)q_n + q_n - r_n\| \\ &\leq 2\|q_n - p_n\| \\ &\overset{(4.43)}{<} 2\epsilon_1. \end{aligned}$$

Thus

$$\begin{aligned} \|(\tilde{g} \otimes 1)r_n - \lambda_n \cdot r_n\| &\leq \|(\tilde{g} \otimes 1)r_n - (\tilde{g} \otimes 1)(f_n \otimes 1)r_n\| \\ &\quad + \|(\tilde{g} \otimes 1)(f_n \otimes 1)r_n - \lambda_n \cdot r_n\| \\ &< 4\|\tilde{g}\|\epsilon_1. \end{aligned}$$

For $N - N_0 \leq n \leq N$, we have that $(f_{n-N} + f_n)x_{N-n} = x_{N-n}$, since we may write $x_{N-n} = (q_{n-N} + q_n)(u^{n-N} \otimes 1)v((N-n)/N_0)(u^{N-n} \otimes 1)(q_{n-N} + q_n)$. Similarly, $x^*_{N-n}(f_{n-N} + f_n) = x^*_{N-n}$. Thus $(f_{n-N} + f_n)d_n = d_n = d_n(f_{n-N} + f_n)$. It follows that

$$\begin{aligned} \|(f_{n-N} + f_n)r_n - r_n\| &= \|(f_{n-N} + f_n)r_n - (f_{n-N} + f_n)d_n + d_n - r_n\| \\ &< 4\epsilon_0. \end{aligned}$$

Thus, similar to the above, $\|(\tilde{g} \otimes 1)r_n - \lambda_n \cdot r_n\| < 8\|\tilde{g}\|\epsilon_0$.

Hence

$$\begin{aligned} \|(g \otimes 1)p - p(g \otimes 1)\| &< \|(\tilde{g} \otimes 1)p - p(\tilde{g} \otimes 1)\| + \epsilon/2 \\ &\leq \textstyle\sum_{n=1}^N \|(\tilde{g} \otimes 1)r_n - \lambda_n \cdot r_n + \lambda_n \cdot r_n - r_n(\tilde{g} \otimes 1)\| + \epsilon/2 \\ &\leq 2N(8\|\tilde{g}\|\epsilon_0) + \epsilon/2 \\ &< \epsilon. \end{aligned}$$

This shows property (i) of the lemma for $g \otimes 1$, $g \in \mathcal{G}$. The second condition is immediate since $g \otimes 1$ is an element of $A_{\{y\}} \otimes M_{\mathfrak{q}}$.

It remains to verify the third condition.

Since the sets $\alpha^{-N_0}(U), \alpha^{-N_0+1}(U), \ldots, U, \alpha(U), \ldots, \alpha^R(U)$ are all disjoint and $R > (N + N_0 + 1)/\min(1, \epsilon)$, it follows that

$$\textstyle\sum_{n=-N_0}^R u^n f_0 u^{-n} = f_0^{-N_0} + \cdots + f_0 + \cdots + f_0^R \leq 1$$

and hence $\tau_1(f_0) \leq \tau_1(1)/(R+N_0+1) < \epsilon/(N+N_0+1)$ for every $\tau_1 \in T(A)$. Since any $\tau \in T(A \otimes M_{\mathfrak{q}})$ is of the form $\tau = \tau_1 \otimes \tau_2$ for $\tau_1 \in T(A)$ and τ_2 the unique tracial state on $M_{\mathfrak{q}}$, we have $\tau(q_0) \leq \tau(f_0 \otimes 1) = \tau_1(f_0) < \epsilon/(N+N_0+1)$. For $1 \leq n \leq N - N_0 - 1$ each r_n is just q_0 conjugated by a unitary so $\tau(r_n) = \tau(q_0)$. For $N - N_0 \leq n \leq N$, $r_n = z_n e_n z_n^* = z_n w_{N-n} q_{-(N-n)} w_{N-n}^* z_n^*$. Thus

$$\begin{aligned} \tau(r_n) &= \tau(z_n w_{N-n} q_{-(N-n)} w_{N-n}^* z_n^*) \\ &= \tau(w_{N-n} q_{-(N-n)} w_{N-n}^*) \\ &= \tau(v((N-n)/N_0) q_0 v((N-n)/N_0)^*) \\ &= \tau((q_0 + p_N) q_0) \\ &= \tau(q_0). \end{aligned}$$

Thus

$$\tau(1-p) = \textstyle\sum_{n=1}^{N} \tau(r_n) = \sum_{n=1}^{N} \tau(q_0) < N\epsilon/(N+N_0+1) < \epsilon.$$

This proves the case where $\mathcal{F}$ is of the form $(\mathcal{G} \otimes \{1_{M_{\mathfrak{q}}}\}) \cup \{u \otimes 1_{M_{\mathfrak{q}}}\}$.

For the general case, let $\tilde{\mathcal{F}} \subset A \otimes M_{\mathfrak{q}}$ be a finite subset. Using the identification

$$A \otimes M_{\mathfrak{q}} \cong A \otimes M_{q^r} \otimes M_{\mathfrak{q}} \cong A \otimes M_{\mathfrak{q}} \otimes M_{q^r},$$

for $r \in \mathbb{N}$, we may assume that the finite set is of the form

$$(\{1_A\} \otimes \{1_{M_{\mathfrak{q}}}\} \otimes \mathcal{B}) \cup (\tilde{\mathcal{G}} \otimes \{1_{M_{\mathfrak{q}}}\} \otimes \{1_{M_{q^r}}\}) \cup (\{u\} \otimes \{1_{M_{\mathfrak{q}}}\} \otimes \{1_{M_{q^r}}\})$$

where $r \in \mathbb{N}$, $\mathcal{B}$ is a finite subset of M_{q^r} and $\mathcal{G}$ is a finite subset of $C(X)$.

We may further assume that $1_X = 1_A \in \mathcal{G}$ and also that $1_{M_{q^r}} \in \mathcal{B}$. Then $\mathcal{F} = (\mathcal{G} \otimes \{1_{M_{\mathfrak{q}}}\}) \cup \{u \otimes 1_{M_{\mathfrak{q}}}\}$ and $\tilde{\mathcal{F}} = \mathcal{F} \otimes \mathcal{B}$.

Let $\epsilon > 0$. By the above, there exists a projection $p \in A_{\{y\}} \otimes M_{\mathfrak{q}}$ satisfying properties (i) – (iii) of the lemma for the finite set $\mathcal{F} = \mathcal{G} \otimes \{1_{M_{\mathfrak{q}}}\} \cup \{u \otimes 1_{M_{\mathfrak{q}}}\}$, with $\epsilon/\max(\{\|b\| \mid b \in \mathcal{B}\}, 1)$ in place of ϵ.

Define $\tilde{p} := p \otimes 1_{M_{q^r}} \in A_{\{y\}} \otimes M_{\mathfrak{q}} \otimes M_{q^r}$. We now show that $\tilde{p}$ satisfies (i) – (iii) of the lemma for $\tilde{\mathcal{F}}$ and ϵ.

Let $\tilde{a} \in \tilde{\mathcal{F}}$. Then $\tilde{a} = a \otimes b$ for some $a \in \mathcal{F}$ and some $b \in \mathcal{B}$. We have

$$\begin{aligned} \|\tilde{p}\tilde{a} - \tilde{a}\tilde{p}\| &= \|(p \otimes 1)(a \otimes b) - (a \otimes b)(p \otimes 1)\| \\ &= \|(pa) \otimes b - (ap) \otimes b\| \\ &= \|(pa - ap) \otimes b\| \\ &= \|pa - ap\|\|b\| \\ &< \epsilon. \end{aligned}$$

By the special case above, for every $a \in \mathcal{F}$, there is some $x \in p(A_{\{y\}} \otimes M_{\mathfrak{q}})p$ such that $\|pap - x\| < \epsilon/(\max_{b \in \mathcal{B}} \|b\|)$. Thus $x \otimes 1 \in \tilde{p}(A_{\{y\}} \otimes M_{\mathfrak{q}} \otimes M_{q^r})\tilde{p}$. It is clear that $\tilde{p}(1 \otimes b)\tilde{p} \in \tilde{p}(A_{\{y\}} \otimes M_{\mathfrak{q}} \otimes M_{q^r})\tilde{p}$ for any $b \in \mathcal{B}$, and so $x \otimes b \in \tilde{p}(A_{\{y\}} \otimes M_{\mathfrak{q}} \otimes M_{q^r})\tilde{p}$. It follows that

$$\begin{aligned} \|\tilde{p}(a \otimes b)\tilde{p} - \tilde{p}(x \otimes b)\tilde{p}\| &= \|\tilde{p}(a \otimes 1)(1 \otimes b)\tilde{p} - \tilde{p}(x \otimes 1)(1 \otimes b)\tilde{p}\| \\ &= \|\tilde{p}(a \otimes 1)\tilde{p}(1 \otimes b) - \tilde{p}(x \otimes 1)\tilde{p}(1 \otimes b)\| \\ &= \|(pap - pxp) \otimes 1\|\|b\| \\ &< \epsilon. \end{aligned}$$

This shows that (i) and (ii) hold.

To prove (iii), simply observe that $\tau \in T(A \otimes M_{\mathfrak{q}} \otimes M_{q^r})$ is of the form $\tau_1 \otimes \tau_2$ where $\tau_1 \in T(A \otimes M_{\mathfrak{q}})$ and $\tau_2 \in T(M_{q^r})$. Then

$$\tau(1 - \tilde{p}) = \tau(1 \otimes 1 - p \otimes 1) = \tau\left((1-p) \otimes 1\right) = \tau_1(1-p)\tau_2(1) < \epsilon.$$

∎

4.3.27 Lemma: *Let $\mathcal{S}$ be a class of separable unital* C*-*algebras. Let A be a simple unital* C*-*algebra and $\mathfrak{q}$ be a supernatural number. Suppose that for every finite subset $\mathcal{F} \subset A \otimes M_{\mathfrak{q}}$, every $\epsilon > 0$, and every nonzero positive $c \in A \otimes M_{\mathfrak{q}}$, there exists a projection $p \in A \otimes M_{\mathfrak{q}}$ and a simple unital* C*-*subalgebra $B \subset p(A \otimes M_{\mathfrak{q}})p$ which is TA$\mathcal{S}$, satisfies $1_B = p$ and*

(i) $\|pa - ap\| < \epsilon$ *for all* $a \in \mathcal{F}$,

(ii) $\mathrm{dist}(pap, B) < \epsilon$ *for all* $a \in \mathcal{F}$,

(iii) $1_A - p$ *is Murray–von Neumann equivalent to a projection in* $\overline{c(A \otimes M_{\mathfrak{q}})c}$.

Then $A \otimes M_{\mathfrak{q}}$ is TA$\mathcal{S}$.

Proof: Although a TA$\mathcal{S}$ C*-algebra may not have property (SP), the C*-algebra $A \otimes M_{\mathfrak{q}}$ always will, since $A \otimes M_{\mathfrak{q}}$ has strict comparison and contains nonzero projections which are arbitrarily small in trace. After noting this, the proof is essentially the same as that of Lemma 4.4 of [41], replacing the C*-subalgebra of tracial rank zero with the TA$\mathcal{S}$ C*-subalgebra B, and replacing the finite-dimensional C*-subalgebra with a C*-subalgebra from the class $\mathcal{S}$. ∎

4.3.28 Theorem: *Let $\mathcal{S}$ be a class of separable unital* C*-*algebras such that the property of being a member of $\mathcal{S}$ passes to unital hereditary* C*-*subalgebras. Let X be an infinite compact metric space, $\alpha : X \to X$ a minimal homeomorphism, let u be the unitary implementing α in $A := \mathcal{C}(X) \rtimes_\alpha \mathbb{Z}$ and $\mathfrak{q}$ be a supernatural number. Suppose there is a $y \in X$ such that $A_{\{y\}} \otimes M_{\mathfrak{q}}$ is TA$\mathcal{S}$. Then $A \otimes M_{\mathfrak{q}}$ is TA$\mathcal{S}$.*

Proof: We show that $A \otimes M_{\mathfrak{q}}$ satisfies the conditions of Lemma 4.3.27.

Let $\epsilon > 0$, $\mathcal{F}$ a finite subset of $A \otimes M_{\mathfrak{q}}$ and a positive nonzero element c in $A \otimes M_{\mathfrak{q}}$ be given. Use Lemma 4.3.26 to find a projection $p \in A_{\{y\}} \otimes M_{\mathfrak{q}}$ with respect to the finite subset $\mathcal{F}$, c, and $\epsilon_0 = \min\{\epsilon, \min_{\tau \in T(A \otimes M_{\mathfrak{q}})} \tau(c)\}$. Put $B = p(A_{\{y\}} \otimes M_{\mathfrak{q}})p$. It is a unital simple C*-subalgebra of $p(A \otimes M_{\mathfrak{q}})p$ and is TA$\mathcal{S}$ by the assumptions made on $\mathcal{S}$ and Lemma 2.3 of [19]. Conditions (i) and (ii) of Lemma 4.3.27 are satisfied by the choice of p. Since $\tau(1_A - p) < \min_{\sigma \in T(A \otimes M_{\mathfrak{q}})} \sigma(c) < \tau(c)$ for every tracial state $\tau \in T(A \otimes M_{\mathfrak{q}})$, it follows from Theorem 5.2(a) of [53] that $1_A - p$ is Murray–von Neumann equivalent to a projection in $\overline{c(A \otimes M_{\mathfrak{q}})c}$. Thus $A \otimes M_{\mathfrak{q}}$ is TA$\mathcal{S}$ by Lemma 4.3.27. ∎

4.3.29 Corollary: *Let (X, α) and (Y, β) be Cantor dynamical systems, $\xi : X \to \mathbb{T}$ and $\zeta : Y \to \mathbb{T}$ continuous maps and suppose that $\alpha \times R_\xi$ and $\beta \times R_\zeta$ are minimal. Put $A := \mathcal{C}(X \times \mathbb{T}) \rtimes_{\alpha \times R_\xi} \mathbb{Z}$ and $B := \mathcal{C}(Y \times \mathbb{T}) \rtimes_{\alpha \times R_\zeta} \mathbb{Z}$. Suppose $T(A)$ and $T(B)$ each have finitely many extreme points such that $[\tau_A]_* = [\tau'_A]_*$ in $K_0(A)$ for every extreme point τ_A, τ'_A and $[\tau_B]_* = [\tau'_B]_*$ in $K_0(B)$ for every extreme point τ_B, τ'_B. Then*

$$A \cong B \text{ if and only if } \mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

Proof: Let A and B be as above. Let u and v be the canonical unitaries inducing the actions of $\mathbb{Z}$ in A and B, respectively. For $x \in X \times \mathbb{T}$ and $y \in Y \times \mathbb{T}$, let $A_{\{x\}}$ and $B_{\{y\}}$ be as above. By [42, Section 3] $A_{\{x\}}$ and $B_{\{y\}}$ can be written as inductive limits $A_{\{x\}} = \varinjlim A^{(n)}_{\{x\}}$ and $B_{\{y\}} = \varinjlim B^{(n)}_{\{y\}}$ where $A^{(n)}_{\{x\}}$ and $B^{(n)}_{\{y\}}$ are recursive subhomogeneous C*-algebras of topological dimensions $\dim(X \times \mathbb{T})$ and $\dim(Y \times \mathbb{T})$, respectively. Hence by Proposition A.3.0 the recursive subhomogeneous algebras can be chosen to have $(\mathcal{F}, \eta)$-connected decompositions with base spaces of dimension less than or equal to one. It follows from Corollary 4.1.5 that projections can be lifted along the recursive subhomogeneous decompositions.

We have affine homeomorphisms $T(A_{\{x\}}) \cong T(A)$, $T(B_{\{y\}}) \cong T(B)$ and order isomorphisms $K_0(A_{\{x\}}) \cong K_0(A)$, $K_0(B_{\{y\}}) \cong K_0(B_{\{y\}})$ [42, Theorem 1.2 (2), (4)] so the requirements for Corollary 4.3.16 are satisfied, hence with Corollary 4.3.18 we see that $A_{\{x\}} \otimes M_{\mathfrak{p}}$ and $B_{\{y\}} \otimes M_{\mathfrak{p}}$ are TAI for any supernatural number $\mathfrak{p}$. From [60, Theorem 4.5] this implies that $A \otimes M_{\mathfrak{p}}$ and $B \otimes M_{\mathfrak{p}}$ are both TAI.

Since A and B satisfy the UCT and are $\mathcal{Z}$-stable [66, Theorem B] (also see [67, Theorem 0.2]), as in the proof of Corollary 4.3.21, the result now follows from [36, Corollary 11.9]. ∎

A. $(\mathcal{F}, \eta)$-bridges via linear algebra

1. Lifting projections

A.1.1 Definition: Let B be a unital recursive subhomogeneous C*-algebra with recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R.$$

We say that projections can be lifted along $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$, if for any $N \in \mathbb{N}$, any $l \in \{1, \ldots, R-1\}$ and any projection $p \in B_l \otimes M_N$ there is a projection $\bar{p} \in B_{l+1} \otimes M_N$ lifting p.

A.1.2 Proposition: *Let X be compact metrizable with* $\dim X \leq 1$. *Let $k, r \in \mathbb{N}$, $\Omega \subset X$ a closed subspace and $p \in \mathcal{C}(\Omega, M_r)$ a projection with constant rank k.*

Then there is a projection $\bar{p} \in \mathcal{C}(X, M_r)$ extending p.

Proof: It is straightforward to find a closed neighborhood W of Ω and a projection

$$\tilde{p} \in \mathcal{C}(W, M_r)$$

extending p. Let $U \subset X$ be an open subset such that

$$\Omega \subset U \subset W.$$

Let $(W_\lambda)_\Lambda$ be a finite collection of open subsets of X such that

(i) $W \subset \bigcup_\Lambda W_\lambda$

(ii) $\|\tilde{p}(x) - \tilde{p}(x')\| \leq \frac{1}{2}$ whenever $x, x' \in \overline{W_\lambda}$ for some $\lambda \in \Lambda$

(iii) $W_\lambda \subset U$ if $W_\lambda \cap \Omega \neq \emptyset$.

From (ii) it is not hard to see that for each λ, $\tilde{p}|_{\overline{W_\lambda}}$ is homotopic to a constant projection of rank k; this yields projections

$$p_\lambda \in \mathcal{C}(\overline{W_\lambda} \times [0,1], M_r) \tag{1}$$

such that

$$p_\lambda(x,t) = \begin{cases} \tilde{p}(x), & \text{for } t \in [\frac{2}{3}, 1] \\ 1_k, & \text{for } t \in [0, \frac{1}{3}] \end{cases}$$

(where we think of 1_k as sitting in the upper left corner of M_r).

Since $\dim X \leq 1$, there is a finite open cover $(V_\gamma)_\Gamma$ of X refining the open cover consisting of $W_\lambda, \lambda \in \Lambda$, and $X \setminus W$, and such that

$$V_{\gamma_0} \cap V_{\gamma_1} \cap V_{\gamma_2} = \emptyset \tag{2}$$

whenever $\gamma_0, \gamma_1, \gamma_2 \in \Gamma$ are pairwise distinct. Let

$$(h_\gamma)_\Gamma$$

be a partition of unity subordinate to $(V_\gamma)_\Gamma$. Set

$$\Gamma' := \{\gamma \in \Gamma \mid V_\gamma \cap \Omega \neq \emptyset\}$$

and

$$\Gamma'' := \{\gamma \in \Gamma \setminus \Gamma' \mid V_\gamma \cap V_{\gamma'} \neq \emptyset \text{ for some } \gamma' \in \Gamma'\}$$

Note that by (ii) above, for any $\gamma' \in \Gamma'$ there is $\lambda(\gamma') \in \Lambda$ such that

$$V_{\gamma'} \subset W_{\lambda(\gamma')} \subset U \subset W. \tag{3}$$

We now define $\bar{p}$, observing that for each $x \in X$, by (2) there are at most two indices $\gamma, \gamma' \in \Gamma$ such that $h_\gamma(x), h_{\gamma'}(x) \neq 0$.

Case 1: There is only one index $\gamma \in \Gamma$ such that $h_\gamma(x) \neq 0$; in this case, $h_\gamma(x) = 1$.

Case 1a: If $\gamma \in \Gamma'$, set

$$\bar{p}(x) := \tilde{p}(x);$$

this is well defined by (3).

Case 1b: If $\gamma \in \Gamma \setminus \Gamma'$, set

$$\bar{p}(x) := 1_k.$$

Case 2: There are two distinct indices $\gamma, \gamma' \in \Gamma$ such that $h_\gamma(x), h_{\gamma'}(x) \neq 0$.

Case 2a: If $\gamma, \gamma' \in \Gamma'$, set

$$\bar{p}(x) := \tilde{p}(x);$$

again, this is well defined by (3).

Case 2b: If $\gamma, \gamma' \in \Gamma \setminus \Gamma'$, set

$$\bar{p}(x) := 1_k.$$

Case 2c: If $\gamma \in \Gamma \setminus \Gamma'$, $\gamma' \in \Gamma'$, then $h_\gamma(x) + h_{\gamma'}(x) = 1$, so $h_{\gamma'}(x) \in [0, 1]$ and by (3) and (1) we may set

$$\bar{p}(x) := p_{\lambda(\gamma')}(x, h_{\gamma'}(x)).$$

We have now defined a projection valued map

$$\bar{p} : X \to M_r$$

which by construction clearly extends p (note that if $x \in \Omega$, then only Cases 1a and 2a occur). It remains to check that $\bar{p}$ is continuous.

So let $x \in X$. In Case 2, there are $\gamma \neq \gamma' \in \Gamma$ with $h_\gamma(x), h_{\gamma'}(x) \neq 0$. But then $h_\gamma(y), h_{\gamma'}(y) \neq 0$ for all y in some small neighborhood V_x of x. In Case 2a, note that the map $y \mapsto \tilde{p}(y)$ is continuous; in Case 2b, $\bar{p}(y) = \bar{p}(x)$ for $y \in V_x$; in Case 2c, the map

$$y \mapsto p_{\lambda(\gamma')}(y, h_{\gamma'}(y))$$

is continuous on V_x since $h_{\gamma'}$ and $p_{\lambda(\gamma')}$ are.

In Case 1, we have $h_\gamma(x) = 1$. But then there is some neighborhood V_x of x such that $h_\gamma(y) \geq \frac{2}{3}$ for all $y \in V_x$, and we obtain

$$\bar{p}(y) = \begin{cases} \tilde{p}(y), & \text{if } \gamma \in \Gamma' \text{ (in Case 1a, 2a or 2c for } y \text{ in place of } x) \\ 1_k, & \text{if } \gamma \in \Gamma \setminus \Gamma' \text{ (in Case 1b for } y \text{ in place of } x) \end{cases}$$

for $y \in V_x$, whence $\bar{p}$ is continuous at x. ▮

A.1.3 Corollary: *Let B be a unital recursive subhomogeneous* C**-algebra with decomposition*

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R.$$

Assume that $\dim X_l \leq 1$ *for* $l \geq 2$.

Then projections can be lifted along $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$.

Proof: Obvious from Proposition A.1.0 and Definition 4.1.2. ▮

2. Approximately excising approximate paths

Recall that a completely positive map has order zero when it preserves orthogonality, that is, a c.p. map $\phi : A \to B$ between the C*-algebras A and B such that, for any orthogonal positive elements $a, b \in A$ with $ab = 0$ we have $\phi(a)\phi(b) = 0$ in B.

A.2.4 Definition: Let B be a unital recursive subhomogeneous C*-algebra with recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R;$$

let $\mathcal{F} \subset B_+^1$ be finite a subset, where B_+^1 denotes the positive elements in the unit ball of B, and $\eta > 0$ be given.

An $(\mathcal{F}, \eta)$-excisor (E, ρ, σ) for B consists of a finite dimensional C*-algebra

$$E = \bigoplus_{l=1}^R E_l,$$

a unital *-homomorphism

$$\rho = \oplus_{l=1}^R \rho_l : B \to \bigoplus_{l=1}^R E_l = E$$

and an isometric c.p. order zero map

$$\sigma = \oplus_{l=1}^R \sigma_l : \bigoplus_{l=1}^R E_l = E \to B \otimes \mathcal{Q}$$

such that

$$\|\sigma(1_E)(b \otimes 1_{\mathcal{Q}}) - \sigma\rho(b)\| < \eta \text{ for } b \in \mathcal{F}.$$

We say (E, ρ, σ) is compatible with $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$, if each ρ_l factorizes through

$$\begin{tikzcd} B \arrow[r, "\rho_l"] \arrow[d, "\psi_l"'] & E_l \\ B_l \arrow[r, "\check{\psi}_l"] & \mathcal{C}(\check{X}_l) \otimes M_{r_l} \arrow[u, "\check{\rho}_l"'] \end{tikzcd}$$

for some compact $\check{X}_l \subset X_l \setminus \Omega_l$.

If (E, ρ, σ) is as above and

$$\kappa : E \to \mathcal{Q}$$

is a unital *-homomorphism, we say $(E, \rho, \sigma, \kappa)$ is a weighted $(\mathcal{F}, \eta)$-excisor compatible with $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$.

A.2.5 Definition: Let B be a unital recursive subhomogeneous C*-algebra with recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R;$$

let $\mathcal{F} \subset B_+^1$ finite and $\eta > 0$ be given. Let $(E_i, \rho_i, \sigma_i, \kappa_i)$, $i \in \{0, 1\}$, be weighted $(\mathcal{F}, \eta)$-excisors (compatible with $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$).

An $(\mathcal{F},\eta)$-bridge from $(E_0,\rho_0,\sigma_0,\kappa_0)$ to $(E_1,\rho_1,\sigma_1,\kappa_1)$ (compatible with the decomposition $[B_l,X_l,\Omega_l,r_l,\phi_l]_{l=1}^R$) consists of $K\in\mathbb{N}$ and weighted $(\mathcal{F},\eta)$-excisors (each compatible with $[B_l,X_l,\Omega_l,r_l,\phi_l]_{l=1}^R$)

$$(E_{\frac{j}{K}},\rho_{\frac{j}{K}},\sigma_{\frac{j}{K}},\kappa_{\frac{j}{K}}),\ j\in\{1,\ldots,K-1\},$$

satisfying

$$\|\kappa_{\frac{j}{K}}\rho_{\frac{j}{K}}(b)-\kappa_{\frac{j+1}{K}}\rho_{\frac{j+1}{K}}(b)\|<\eta \text{ for } b\in\mathcal{F} \text{ and } j\in\{0,\ldots,K-1\}. \tag{4}$$

We write

$$(E_0,\rho_0,\sigma_0,\kappa_0)\sim_{(\mathcal{F},\eta)}(E_1,\rho_1,\sigma_1,\kappa_1)$$

if such an $(\mathcal{F},\eta)$-bridge exists.

A.2.6 Remarks: (i) Clearly, the relation $\sim_{(\mathcal{F},\eta)}$ defines an equivalence relation on the set of compatible weighted $(\mathcal{F},\eta)$-excisors (with fixed $\mathcal{F}$, η and recursive subhomogeneous decomposition).

(ii) If (E,ρ,σ,κ_i), $i\in\{0,1\}$, are $(\mathcal{F},\eta)$-excisors with κ_0 and κ_1 unitarily equivalent, $\kappa_0\approx_{\mathrm{u}}\kappa_1$, then

$$(E,\rho,\sigma,\kappa_0)\sim_{(\mathcal{F},\eta)}(E,\rho,\sigma,\kappa_1)$$

since κ_0 and κ_1 are in fact homotopic.

(iii) Let (E,ρ,σ,κ) and $(E',\rho',\sigma',\kappa')$ be $(\mathcal{F},\eta)$-excisors with an embedding

$$\iota: E'\to E$$

and such that

$$\rho'=\rho\circ\iota,\ \sigma\circ\iota=\sigma',\ \kappa'=\kappa\circ\iota.$$

Then

$$(E,\rho,\sigma,\kappa)\sim_{(\mathcal{F},\eta)}(E',\rho',\sigma',\kappa'),$$

with an $(\mathcal{F},\eta)$-bridge of length $K=1$.

A.2.7 Definition: Let B be a unital recursive subhomogeneous C*-algebra with recursive subhomogeneous decomposition

$$[B_l,X_l,\Omega_l,r_l,\phi_l]_{l=1}^R;$$

let $\mathcal{F}\subset B_+^1$ finite and $\eta>0$ be given.

(i) Let $(E_j,\rho_j,\sigma_j,\kappa_j)$, $j\in\{1,\ldots,L\}$, be weighted $(\mathcal{F},\eta)$-excisors. We say they are pairwise orthogonal if there are pairwise orthogonal projections

$$q_j\in\mathcal{Q},\ j\in\{1,\ldots,L\},$$

such that

$$\sigma_j(E_j)\subset B\otimes q_j\mathcal{Q}q_j\subset_{\mathrm{her}} B\otimes\mathcal{Q},\ j\in\{1,\ldots,L\}.$$

(ii) Let $(E_j,\rho_j,\sigma_j,\kappa_j)$, $j\in\{1,\ldots,L\}$, be pairwise orthogonal weighted $(\mathcal{F},\eta)$-excisors, and let

$$\gamma:\bigoplus_{j=1}^L\mathcal{Q}\to\mathcal{Q}$$

be a unital *-homomorphism.

We define the γ-direct sum

$$\bigoplus_\gamma(E_j,\rho_j,\sigma_j,\kappa_j):=(\bigoplus_{j=1}^L E_j,\bigoplus_{j=1}^L\rho_j,\bigoplus_{j=1}^L\sigma_j,\gamma\circ(\bigoplus_{j=1}^L\kappa_j)),$$

which is easily seen to be a weighted $(\mathcal{F},\eta)$-excisor.

If the $(E_j,\rho_j,\sigma_j,\kappa_j)$ are compatible with $[B_l,X_l,\Omega_l,r_l,\phi_l]_{l=1}^R$, then so is the γ-direct sum.

Since, up to unitary equivalence in $\mathcal{Q}$, the maps $\gamma \circ (\bigoplus_{j=1}^{L} \kappa_j)$ only depend on the positive rational weights $\nu_j := \tau_{\mathcal{Q}}(\gamma(1_j))$, we will sometimes neglect to explicitly specify γ and write

$$\bigoplus_{j=1}^{R} \nu_j \cdot \kappa_j$$

instead of $\gamma \circ (\bigoplus_{j=1}^{L} \kappa_j)$ and, similarly,

$$\bigoplus_{j=1}^{R} \nu_j \cdot (E_j, \rho_j, \sigma_j, \kappa_j)$$

instead of $\bigoplus_{\gamma}(E_j, \rho_j, \sigma_j, \kappa_j)$.

A.2.8 Proposition: *Let B be a unital recursive subhomogeneous* C*-*algebra with recursive subhomogeneous decomposition*

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^{R};$$

let $\mathcal{F} \subset B_+^1$ finite and $\eta > 0$ be given. Let $(E_j, \rho_j, \sigma_j, \kappa_j)$, $j \in \{1, \dots, L\}$, be weighted $(\mathcal{F}, \eta)$-excisors.

Then there are pairwise orthogonal weighted $(\mathcal{F}, \eta)$-excisors $(E_j, \rho_j, \dot{\sigma}_j, \kappa_j)$, $j \in \{1, \dots, L\}$, such that, for each j,

$$(E_j, \rho_j, \sigma_j, \kappa_j) \sim_{(\mathcal{F},\eta)} (E_j, \rho_j, \dot{\sigma}_j, \kappa_j). \tag{5}$$

If the $(E_j, \rho_j, \sigma_j, \kappa_j)$ are compatible with $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^{R}$, we may choose the $(E_j, \rho_j, \dot{\sigma}_j, \kappa_j)$ and the $(\mathcal{F}, \eta)$-bridges to be compatible as well.

Proof: Choose pairwise orthogonal nonzero projections $q_j \in \mathcal{Q}$, $j \in \{1, \dots, L\}$, and isomorphisms

$$\theta_j : \mathcal{Q} \to q_j \mathcal{Q} q_j;$$

set

$$\dot{\sigma}_j := (\mathrm{id} \otimes \theta_j) \circ \sigma_j.$$

It is clear that the $(E_j, \rho_j, \dot{\sigma}_j, \kappa_j)$ are pairwise orthogonal weighted $(\mathcal{F}, \eta)$-excisors. Now (5) holds, in fact with an $(\mathcal{F}, \eta)$-bridge of length $K = 1$, since passing from σ_j to $\dot{\sigma}_j$ does not affect (4). Also, changing the σ_j does not affect compatibility with the recursive subhomogeneous decomposition. ∎

A.2.9 Proposition: *Let B be a unital recursive subhomogeneous* C*-*algebra with recursive subhomogeneous decomposition*

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^{R};$$

let $\mathcal{F} \subset B_+^1$ finite and $\eta > 0$ be given. Let $(E_j, \rho_j, \sigma_j, \kappa_j)$, $j \in \{1, \dots, L\}$, be pairwise orthogonal weighted $(\mathcal{F}, \eta)$-excisors. Let $(E'_j, \rho'_j, \sigma'_j, \kappa'_j)$, $j \in \{1, \dots, L\}$, be another set of pairwise orthogonal weighted $(\mathcal{F}, \eta)$-excisors, and let

$$\gamma : \bigoplus_{j=1}^{L} \mathcal{Q} \to \mathcal{Q}$$

be a unital *-homomorphism. Suppose that

$$(E_j, \rho_j, \sigma_j, \kappa_j) \sim_{(\mathcal{F},\eta)} (E'_j, \rho'_j, \sigma'_j, \kappa'_j)$$

for each $j \in \{1, \dots, L\}$.

Then

$$\bigoplus_{\gamma}(E_j, \rho_j, \sigma_j, \kappa_j) \sim_{(\mathcal{F},\eta)} \bigoplus_{\gamma}(E'_j, \rho'_j, \sigma'_j, \kappa'_j).$$

If the $(E_j, \rho_j, \sigma_j, \kappa_j)$ and the $(E'_j, \rho'_j, \sigma'_j, \kappa'_j)$ are compatible with the decomposition $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^{R}$, we may choose the $(\mathcal{F}, \eta)$-bridge between the γ-direct sums to be compatible as well.

Proof: For each $j \in \{1,\ldots,L\}$ choose an $(\mathcal{F},\eta)$-bridge between $(E_j,\rho_j,\sigma_j,\kappa_j)$ and $(E'_j,\rho'_j,\sigma'_j,\kappa'_j)$; by repeating some of the steps, if necessary, we may assume that all of these have the same length, say K, and are given by weighted $(\mathcal{F},\eta)$-excisors $(E_{j,\frac{i}{K}},\rho_{j,\frac{i}{K}},\sigma_{j,\frac{i}{K}},\kappa_{j,\frac{i}{K}})$ with

$$(E_{j,0},\rho_{j,0},\sigma_{j,0},\kappa_{j,0}) = (E_j,\rho_j,\sigma_j,\kappa_j)$$

and

$$(E_{j,1},\rho_{j,1},\sigma_{j,1},\kappa_{j,1}) = (E'_j,\rho'_j,\sigma'_j,\kappa'_j).$$

As in the proof of Proposition A.2.0, choose pairwise orthogonal nonzero projections $q_j \in \mathcal{Q}$, $j \in \{1,\ldots,L\}$, as well as isomorphisms

$$\theta_j : \mathcal{Q} \to q_j\mathcal{Q}q_j.$$

Set

$$\dot{\sigma}_{j,\frac{i}{K}} := (\mathrm{id}\otimes\theta_j)\circ\sigma_{j,\frac{i}{K}},$$

then the sums

$$\bigoplus\nolimits_\gamma (E_{j,\frac{i}{K}},\rho_{j,\frac{i}{K}},\dot{\sigma}_{j,\frac{i}{K}},\kappa_{j,\frac{i}{K}})$$

are $(\mathcal{F},\eta)$-excisors implementing an $(\mathcal{F},\eta)$-bridge

$$\bigoplus\nolimits_\gamma (E_{j,0},\rho_{j,0},\dot{\sigma}_{j,0},\kappa_{j,0}) \sim_{(\mathcal{F},\eta)} \bigoplus\nolimits_\gamma (E_{j,1},\rho_{j,1},\dot{\sigma}_{j,1},\kappa_{j,1}).$$

As in the proof of A.2.0, it remains to observe that

$$\bigoplus\nolimits_\gamma (E_j,\rho_j,\sigma_j,\kappa_j) \sim_{(\mathcal{F},\eta)} \bigoplus\nolimits_\gamma (E_{j,0},\rho_{j,0},\dot{\sigma}_{j,0},\kappa_{j,0})$$

and

$$\bigoplus\nolimits_\gamma (E'_j,\rho'_j,\sigma'_j,\kappa'_j) \sim_{(\mathcal{F},\eta)} \bigoplus\nolimits_\gamma (E_{j,1},\rho_{j,1},\dot{\sigma}_{j,1},\kappa_{j,1}).$$

∎

A.2.10 Definition: Let B be a unital recursive subhomogeneous C*-algebra with recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R;$$

let $\mathcal{F} \subset B^1_+$ finite and $\eta > 0$ be given.

If $(E_j,\rho_j,\sigma_j,\kappa_j)$ and $(E_j,\rho_j,\dot{\sigma}_j,\kappa_j)$, $j \in \{1,\ldots,L\}$, are as in Proposition A.2.0, and if

$$\gamma : \bigoplus\nolimits_{j=1}^L \mathcal{Q} \to \mathcal{Q}$$

is a unital *-homomorphism, we say

$$\bigoplus\nolimits_\gamma (E_j,\rho_j,\dot{\sigma}_j,\kappa_j) \tag{6}$$

is a compatible γ-direct sum of the $(E_j,\rho_j,\sigma_j,\kappa_j)$.

A.2.11 Remark: Of course, the γ-direct sum in (6) depends on the choice of the $\dot{\sigma}_j$ in Proposition A.2.0, but for a different choice, say $\ddot{\sigma}_j$, it follows from Proposition A.2.0 that

$$\bigoplus\nolimits_\gamma (E_j,\rho_j,\dot{\sigma}_j,\kappa_j) \sim_{(\mathcal{F},\eta)} \bigoplus\nolimits_\gamma (E_j,\rho_j,\ddot{\sigma}_j,\kappa_j).$$

A.2.12 Proposition: *Let B be a unital recursive subhomogeneous* C**-algebra with recursive subhomogeneous decomposition*

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R;$$

let $\mathcal{F} \subset B^1_+$ finite and $\eta > 0$ be given.

Let

$$\left(E = \bigoplus\nolimits_{j=1}^L E_j, \rho, \sigma, \kappa\right)$$

be an $(\mathcal{F},\eta)$*-excisor and let*

$$\gamma : \bigoplus_{j=1}^{L} \mathcal{Q} \to \mathcal{Q}$$

be a unital **-homomorphism such that*

$$\tau_{\mathcal{Q}}(\gamma_j(1_{\mathcal{Q}})) = \tau_{\mathcal{Q}}(\kappa(1_{E_j})) \text{ for } j \in \{1,\dots,L\}.$$

Then there are pairwise orthogonal $(\mathcal{F},\eta)$*-excisors*

$$(E_j, \rho_j = \rho|_{E_j}, \dot{\sigma}_j, \dot{\kappa}_j : E_j \to \kappa(1_{E_j})\mathcal{Q}\kappa(1_{E_j}) \cong \mathcal{Q})$$

such that

$$\kappa \approx_{\mathrm{u}} \gamma \circ (\bigoplus_{j=1}^{L} \dot{\kappa}_j) \tag{7}$$

and such that

$$(E,\rho,\sigma,\kappa) \sim_{(\mathcal{F},\eta)} \bigoplus_{\gamma}(E_j,\rho_j,\dot{\sigma}_j,\dot{\kappa}_j). \tag{8}$$

If (E,ρ,σ,κ) *is compatible with* $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^{R}$*, we may choose the* γ*-direct sum and the* $(\mathcal{F},\eta)$*-bridge to be compatible as well.*

Proof: Let

$$\zeta_j : \kappa(1_{E_j})\mathcal{Q}\kappa(1_{E_j}) \to \mathcal{Q}$$

be an isomorphism for each $j \in \{1,\dots,L\}$, then the maps

$$\dot{\kappa}_j := \zeta_j \circ \kappa|_{E_j}$$

clearly satisfy (7). Choose pairwise nonzero orthogonal projections $q_j \in \mathcal{Q}$, $j \in \{1,\dots,K\}$, as well as isomorphisms

$$\theta_j : \mathcal{Q} \to q_j\mathcal{Q}q_j;$$

define

$$\dot{\sigma}_j := (\mathrm{id}_B \otimes \theta_j) \circ \sigma|_{E_j}.$$

It is then clear that the $(E_j,\rho_j,\dot{\sigma}_j,\dot{\kappa}_j)$ are pairwise orthogonal $(\mathcal{F},\eta)$-excisors and that

$$\bigoplus_{\gamma}(E_j,\rho_j,\dot{\sigma}_j,\dot{\kappa}_j) \sim_{(\mathcal{F},\eta)} (E,\rho,\sigma,\gamma \circ (\bigoplus_{j=1}^{L} \dot{\kappa}_j)).$$

Finally, (8) follows from (7) and Remark A.2.0(ii). ∎

A.2.13 We note the following lifting result, which will imply the existence of sufficiently many $(\mathcal{F},\eta)$-excisors, cf. Remark A.2.0(ii) below.

Proposition: *Let* B*,* F *be* C^**-algebras,* F *finite dimensional, and* $\pi : B \to F$ *a surjective* **-homomorphism; let* $\mathcal{F} \subset B_+^1$ *finite and* $\eta > 0$ *be given.*

Then there is an isometric c.p. order zero map

$$\sigma : F \to B$$

such that

$$\|\sigma(1_F)b - \sigma\pi(b)\| < \eta \text{ for } b \in \mathcal{F}. \tag{9}$$

Proof: Since F and $\mathcal{F}$ are separable, we may clearly also assume B to be separable, hence σ-unital. Recall that c.p.c. order zero maps are projective, whence there is a c.p. isometric order zero lift

$$\dot{\sigma} : F \to B.$$

Choose an approximate unit $(h_n)_{n\in\mathbb{N}}$ for $\ker \pi$ which is quasicentral for B. Define c.p.c. maps

$$\ddot{\sigma}_n : F \to B$$

by

$$\ddot{\sigma}_n(\,.\,) := (1_{B^\sim} - h_n)^{\frac{1}{2}}\dot{\sigma}(\,.\,)(1_{B^\sim} - h_n)^{\frac{1}{2}}.$$

The $\ddot{\sigma}_n$ clearly induce a c.p. isometric order zero map

$$\ddot{\sigma} : F \to B_\infty$$

which in turn lifts to a c.p. isometric order zero map

$$\bar{\sigma} : F \to \prod_{\mathbb{N}} B$$

with components $\bar{\sigma}_n$. Upon dropping finitely many components and rescaling, if necessary, we may assume each $\bar{\sigma}_n$ to be isometric. It is now straightforward to check that, for large enough N, $\sigma := \bar{\sigma}_N$ will satisfy (9). ∎

A.2.14 Notation: Let B be a unital recursive subhomogeneous C*-algebra with recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R.$$

If $l \in \{1,\ldots,R\}$ and $x \in X_l$, then

$$(\mathrm{ev}_x \otimes \mathrm{id}_{M_{r_l}}) \circ \iota_B : B \to M_{r_l}$$

factorizes through a sum of irreducible representations, say

$$B \xrightarrow{\rho_x} E_x \xrightarrow{\iota_{E_x}} M_{r_l}.$$

Upon fixing a unital embedding

$$M_{r_l} \to \mathcal{Q}$$

we obtain unital *-homomorphisms

$$B \xrightarrow{\rho_x} E_x \xrightarrow{\kappa_x} \mathcal{Q}$$

such that ρ_x is a sum of surjective irreducible representations and

$$\tau_{\mathcal{Q}}\kappa_x = \tau_{M_{r_l}}\iota_{E_x}.$$

A.2.15 Remarks: (i) The maps ρ_x and κ_x are uniquely determined by x up to unitary equivalence.

(ii) By Proposition A.2.0, for any finite subset $\mathcal{F} \subset B^1_+$ and $\eta > 0$, and for any $x \in X_l$, there is an isometric c.p. order zero map

$$\sigma_x : E_x \to B$$

such that $(E_x, \rho_x, \sigma_x, \kappa_x)$ is a weighted $(\mathcal{F},\eta)$-excisor (which is compatible with the decomposition $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$, provided $x \in X_l \setminus \Omega_l$).

(iii) It is clear that, if $x, x' \in X_l$ are such that

$$\|(\mathrm{ev}_x \otimes \mathrm{id}_{M_{r_l}}) \circ \iota_B(b) - (\mathrm{ev}_{x'} \otimes \mathrm{id}_{M_{r_l}}) \circ \iota_B(b)\| \le \eta$$

for all $b \in \mathcal{F}$, then

$$(E_x, \rho_x, \sigma_x, \kappa_x) \sim_{(\mathcal{F},\eta)} (E_{x'}, \rho_{x'}, \sigma_{x'}, \kappa_{x'}),$$

in fact via an $(\mathcal{F},\eta)$-bridge of length $K = 1$.

A.2.16 Proposition: *Let B be a unital recursive subhomogeneous* C**-algebra with recursive subhomogeneous decomposition*

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R;$$

let $\mathcal{F} \subset B^1_+$ finite and $\eta > 0$ be given.

Fix $l \in \{1,\ldots,R\}$ and $x \in X_l \setminus \Omega_l$, and let $(E_x,\rho_x,\sigma_x,\kappa_x)$ be an $(\mathcal{F},\eta)$-excisor, with $(E_x,\ \rho_x,\ \kappa_x)$ as in A.2.0 (note that $E_x \cong M_{r_l}$ since $x \in X_l \setminus \Omega_l$). Let

$$\gamma : \bigoplus_{j=1}^{L} \mathcal{Q} \to \mathcal{Q}$$

be a unital embedding for some $L \in \mathbb{N}$.

Then there are pairwise orthogonal $(\mathcal{F},\eta)$-excisors

$$(E_x,\rho_x,\sigma_{x,j},\kappa_x),\ j \in \{1,\ldots,L\},$$

such that

$$(E_x,\rho_x,\sigma_x,\kappa_x) \sim_{(\mathcal{F},\eta)} \bigoplus\nolimits_{\gamma}(E_x,\rho_x,\sigma_{x,j},\kappa_x). \tag{10}$$

Proof: Choose q_j and θ_j as in the proof of Proposition A.2.0. We take

$$\sigma_{x,j} := (\mathrm{id}_B \otimes \theta_j) \circ \sigma_x$$

(these correspond to the maps $\dot{\sigma}_x$ from Proposition A.2.0); as in the proof of A.2.0 one checks that

$$\bigoplus\nolimits_{\gamma}(E_x,\rho_x,\sigma_{x,j},\kappa_x) \sim_{(\mathcal{F},\eta)} (E_x,\rho_x,\sigma_x,\gamma\circ(\kappa_x^{\oplus L})).$$

Now observe that $\kappa_x \approx_{\mathrm{u}} \gamma\circ(\kappa_x^{\oplus L})$, and apply Remark A.2.0(ii) to obtain (10). ∎

3. $(\mathcal{F},\eta)$-connected decompositions

A.3.17 Definition: Let B be a unital recursive subhomogeneous C*-algebra, and let $\mathcal{F} \subset B_+^1$ finite and $\eta > 0$ be given.

A recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^{R}$$

for B is $(\mathcal{F},\eta)$-connected if the following holds:

If $l \in \{1,\ldots,R\}$ and $x,y \in X_l$, and if $(E_x,\rho_x,\sigma_x,\kappa_x)$ and $(E_y,\rho_y,\sigma_y,\kappa_y)$ are $(\mathcal{F},\eta)$-excisors with $(E_x,\ \rho_x,\ \kappa_x)$ and $(E_y,\ \rho_y,\ \kappa_y)$ as in A.2.0, then

$$(E_x,\rho_x,\sigma_x,\kappa_x) \sim_{(\mathcal{F},\eta)} (E_y,\rho_y,\sigma_y,\kappa_y).$$

A.3.18 Proposition: *Let B be a unital recursive subhomogeneous* C*-*algebra and let $\mathcal{F} \subset B_+^1$ finite and $\eta > 0$ be given.*

Then B has an $(\mathcal{F},\eta)$-connected recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^{R}.$$

If $\mathrm{dr} B \le n$, then $X_1,\ldots,X_R$ may be chosen so that $\dim X_l \le n$ for $l \in \{1,\ldots,R\}$.

Proof: This follows immediately from Remark A.2.0(iii) after decomposing each X_l in to pairwise disjoint closed subsets

$$X_l = \coprod_{k=1}^{N_l} X_{l,k}$$

such that, for each $l \in \{1,\ldots,R\}$, $k \in \{1,\ldots,N_l\}$ and $x_0,x_1 \in X_{l,k}$, there are $K \in \mathbb{N}$ and $x_{\frac{j}{K}} \in X_{l,k}$ such that

$$\|(\mathrm{ev}_{x_{\frac{j}{K}}} \otimes \mathrm{id}_{M_{r_l}})\circ\iota_B(b) - (\mathrm{ev}_{x_{\frac{j+1}{K}}} \otimes \mathrm{id}_{M_{r_l}})\circ\iota_B(b)\| \le \eta$$

for all $j \in \{0,\ldots,K-1\}$ and $b \in \mathcal{F}$.

The last statement follows from [73], since in this case the X_l (and hence the $X_{l,k}$) may be chosen to have dimension at most n. ∎

A.3.19 PROPOSITION: *Let B be a unital recursive subhomogeneous C^*-algebra; let $\mathcal{F} \subset B_+^1$ finite and $\eta > 0$ be given and suppose*

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$$

is an $(\mathcal{F},\eta)$-connected recursive subhomogeneous decomposition for B.

Let

$$\left(E_i = \bigoplus_{l=1}^R E_{i,l},\ \rho_i = \bigoplus_{l=1}^R \rho_{i,l},\ \sigma_i = \bigoplus_{l=1}^R \sigma_{i,l},\ \kappa_i\right),\ i \in \{0,1\}$$

be weighted $(\mathcal{F},\eta)$-excisors (compatible with the decomposition) satisfying

$$y_l := \tau_{\mathcal{Q}}(\kappa_0(1_{E_{0,l}})) = \tau_{\mathcal{Q}}(\kappa_1(1_{E_{1,l}})),\ l \in \{1,\ldots,R\}$$

and such that each $\rho_{i,l}$ factorizes as

$$\rho_{i,l} : B \to \mathcal{C}(X_l) \otimes M_{r_l} \to E_{i,l}.$$

Then (via a compatible $(\mathcal{F},\eta)$-bridge),

$$(E_0,\rho_0,\sigma_0,\kappa_0) \sim_{(\mathcal{F},\eta)} (E_1,\rho_1,\sigma_1,\kappa_1).$$

PROOF: For each $l \in \{1,\ldots,R\}$, choose $x_l \in X_l \setminus \Omega_l$. Set

$$E := \bigoplus_{l=1}^R E_{x_l},\ \rho := \bigoplus_{l=1}^R \rho_{x_l},\ \kappa := \gamma \circ \left(\bigoplus_{l=1}^R \kappa_{x_l}\right),$$

where E_{x_l}, ρ_{x_l}, κ_{x_l} are as in A.2.0 and

$$\gamma : \bigoplus_{l=1}^R \mathcal{Q} \to \mathcal{Q}$$

is a unital embedding such that

$$\tau_{\mathcal{Q}}(\gamma_l(1_{\mathcal{Q}})) = y_l,\ l \in \{1,\ldots,R\}.$$

Note that $E_{x_l} \cong M_{r_l}$, since $x_l \in X_l \setminus \Omega_l$ by Remark 4.1.3.

By Proposition A.2.0, there is an isometric c.p. order zero map

$$\sigma : E \to B \otimes \mathcal{Q}$$

such that (E,ρ,σ,κ) is a weighted $(\mathcal{F},\eta)$-excisor which is compatible with the decomposition $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$. By Proposition A.2.0, there are (compatible) pairwise orthogonal $(\mathcal{F},\eta)$-excisors of the form

$$(E_{x_l}, \rho_{x_l}, \dot{\sigma}_{x_l}, \kappa_{x_l}),\ l \in \{1,\ldots,R\},$$

such that (in a compatible way)

$$(E,\rho,\sigma,\kappa) \sim_{(\mathcal{F},\eta)} \bigoplus\nolimits_{\gamma} (E_{x_l}, \rho_{x_l}, \dot{\sigma}_{x_l}, \kappa_{x_l}). \tag{11}$$

Similarly, for $i \in \{0,1\}$ and $l \in \{1,\ldots,R\}$ there are (compatible) pairwise orthogonal $(\mathcal{F},\eta)$-excisors

$$(E_{i,l}, \rho_{i,l}, \dot{\sigma}_{i,l}, \dot{\kappa}_{i,l})$$

such that

$$\kappa_i \approx_{\mathrm{u}} \gamma \circ \left(\bigoplus_{l=1}^R \dot{\kappa}_{i,l}\right)$$

and such that (in a compatible way)

$$(E_i,\rho_i,\sigma_i,\kappa_i) \sim_{(\mathcal{F},\eta)} \bigoplus\nolimits_{\gamma}(E_{i,l},\rho_{i,l},\dot{\sigma}_{i,l},\dot{\kappa}_{i,l}). \tag{12}$$

Since $(E_{i,l},\rho_{i,l},\dot{\sigma}_{i,l},\dot{\kappa}_{i,l})$ is compatible with $[B_l,X_l,\Omega_l,r_l,\phi_l]_{l=1}^R$, there are $N_{i,l}\in\mathbb{N}$ and $x_{i,l,m}\in X_l\setminus\Omega_l$ for $m\in\{1,\ldots,N_{i,l}\}$ such that

$$(E_{i,l},\rho_{i,l},\dot{\sigma}_{i,l},\dot{\kappa}_{i,l}) = \left(\bigoplus\nolimits_{m=1}^{N_{i,l}} E_{x_{i,l,m}},\ \bigoplus\nolimits_{m=1}^{N_{i,l}} \rho_{x_{i,l,m}},\ \bigoplus\nolimits_{m=1}^{N_{i,l}} \dot{\sigma}_{x_{i,l,m}},\ \bigoplus\nolimits_{m=1}^{N_{i,l}} \dot{\kappa}_{x_{i,l,m}}\right);$$

note that

$$E_{x_{i,l,m}} \cong M_{r_l}$$

for all i,l,m.

Let

$$\gamma_{i,l} : \bigoplus\nolimits_{m=1}^{N_{i,l}} \mathcal{Q} \to \mathcal{Q}$$

be a unital embedding such that

$$\tau_{\mathcal{Q}}(\gamma_{i,l,m}(1_{\mathcal{Q}})) = \tau_{\mathcal{Q}}(\dot{\kappa}_{i,l}(1_{E_{i,l,m}})),\ m\in\{1,\ldots,N_{i,l}\}.$$

By Proposition A.2.0, there are pairwise orthogonal $(\mathcal{F},\eta)$-excisors

$$(E_{x_{i,l,m}},\rho_{x_{i,l,m}},\ddot{\sigma}_{x_{i,l,m}},\ddot{\kappa}_{x_{i,l,m}}),\ m\in\{1,\ldots,N_{i,l}\},$$

such that

$$\dot{\kappa}_{i,l} \approx_{\mathrm{u}} \gamma_{i,l}\circ\left(\bigoplus\nolimits_{m=1}^{N_{i,l}} \kappa_{i,l,m}\right)$$

and such that

$$(E_{i,l},\rho_{i,l},\dot{\sigma}_{i,l},\dot{\kappa}_{i,l}) \sim_{(\mathcal{F},\eta)} \bigoplus\nolimits_{\gamma_{i,l}}(E_{x_{i,l,m}},\rho_{x_{i,l,m}},\ddot{\sigma}_{x_{i,l,m}},\ddot{\kappa}_{x_{i,l,m}}). \tag{13}$$

By Proposition A.2.0, there are pairwise orthogonal $(\mathcal{F},\eta)$-excisors

$$(E_{x_l},\rho_{x_l},\dot{\sigma}_{x_l,m},\kappa_{x_l}),\ m\in\{1,\ldots,N_{i,l}\},$$

such that

$$(E_{x_l},\rho_{x_l},\dot{\sigma}_{x_l},\kappa_{x_l}) \sim_{(\mathcal{F},\eta)} \bigoplus\nolimits_{\gamma_{i,l}}(E_{x_l},\rho_{x_l},\dot{\sigma}_{x_l,m},\kappa_{x_l}). \tag{14}$$

Since $[B_l,X_l,\Omega_l,r_l,\phi_l]_{l=1}^R$ is $(\mathcal{F},\eta)$-connected, for each i,l,m we have

$$(E_{x_l},\rho_{x_l},\dot{\sigma}_{x_l,m},\kappa_{x_l}) \sim_{(\mathcal{F},\eta)} (E_{x_{i,l,m}},\rho_{x_{i,l,m}},\ddot{\sigma}_{x_{i,l,m}},\ddot{\kappa}_{x_{i,l,m}}).$$

By Proposition A.2.0, we have

$$\bigoplus\nolimits_{\gamma_{i,l}}(E_{x_l},\rho_{x_l},\dot{\sigma}_{x_l,m},\kappa_{x_l}) \sim_{(\mathcal{F},\eta)} \bigoplus\nolimits_{\gamma_{i,l}}(E_{x_{i,l,m}},\rho_{x_{i,l,m}},\ddot{\sigma}_{x_{i,l,m}},\ddot{\kappa}_{x_{i,l,m}})$$

which in turn yields

$$(E_{x_l},\rho_{x_l},\dot{\sigma}_{x_l},\kappa_{x_l}) \sim_{(\mathcal{F},\eta)} (E_{i,l},\rho_{i,l},\dot{\sigma}_{i,l},\dot{\kappa}_{i,l})$$

by (13) and (14).

Again by Proposition A.2.0, together with (11) and (12) this gives

$$(E,\rho,\sigma,\kappa) \sim_{(\mathcal{F},\eta)} (E_i,\rho_i,\sigma_i,\kappa_i),\ i\in\{0,1\},$$

from which we obtain

$$(E_0,\rho_0,\sigma_0,\kappa_0) \sim_{(\mathcal{F},\eta)} (E_1,\rho_1,\sigma_1,\kappa_1),$$

as desired. Of course all the $(\mathcal{F},\eta)$-bridges above may be chosen to be compatible with the given recursive subhomogeneous decomposition. ∎

4. Excising traces

A.4.20 Notation: Let B be a separable unital recursive subhomogeneous C*-algebra with (separable) recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R;$$

let $\tau \in T(B)$ be a tracial state. We inductively define positive tracial functionals

$$\tau_l, \bar{\tau}_l : B_l \to \mathbb{C},\ l \in \{1, \ldots, R\}$$

as follows:

For each l, let $0 \le h_l \le 1$ be a strictly positive element of $\mathcal{C}_0(X_l \setminus \Omega_l)$. Set

$$\tau_R := \tau : B \cong B_R \to \mathbb{C}.$$

If $\tau_l : B_l \to \mathbb{C}$ has been constructed, set

$$\bar{\tau}_l(b) := \lim_{n\to\infty} \tau_l((h_l^{\frac{1}{n}} \otimes 1_{M_{r_l}})b),\ b \in B_l.$$

(On positive elements b, the limit is over a bounded increasing sequence, hence exists; but then the limit also exists for general b).

If $\tau_l, \bar{\tau}_l$ have been constructed, set

$$\tau_{l-1}(b) = \tau_l(\hat{b}) - \bar{\tau}_l(\hat{b}),\ b \in B_{l-1},$$

where $\hat{b} \in B_l$ is a lift for b. It is easy to see that $\tau_l, \bar{\tau}_l,\ l \in \{1, \ldots, R\}$, are well-defined positive functionals which do not depend on the choice of the h_l, that $\bar{\tau}_l \le \tau_l$, that

$$y_l^\tau := \tau_l(1_{B_l}) - \tau_{l-1}(1_{B_{l-1}}) = \|\bar{\tau}_l\| (\le 1) \tag{15}$$

and that

$$\textstyle\sum_{l=1}^R y_l^\tau = 1. \tag{16}$$

Call the y_l^τ the weights of τ with respect to the decomposition $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$.

Now suppose

$$W \subset X_l \setminus \Omega_l$$

is a subset closed in X_l. Let

$$0 \le g_l \le 1$$

be a strictly positive element for $\mathcal{C}_0(X_l \setminus W)$, with $g|_{\Omega_l} \equiv 1$. It is not hard to check that, for all $b \in B_l$,

$$\lim_{n\to\infty} \tau_l(((1 - g_l^{\frac{1}{n}}) \otimes 1_{M_{r_l}})b) = \lim_{n\to\infty} \bar{\tau}_l(((1 - g_l^{\frac{1}{n}}) \otimes 1_{M_{r_l}})b)$$

(and, in particular, that the limits exist). As above, one may define a positive tracial functional

$$\tilde{\tau}_W : \mathcal{C}(W) \otimes M_{r_l} \to \mathbb{C}$$

by

$$\tilde{\tau}_W(b) = \lim_{n\to\infty} \tau_l((1 - g_l^{\frac{1}{n}})\hat{b}),$$

where $\hat{b} \in B_l$ is a lift of $b \in \mathcal{C}(W) \otimes M_{r_l}$.

A.4.21 Proposition: *Let B be a separable unital recursive subhomogeneous* C*-*algebra with (separable) recursive subhomogeneous decomposition*

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R;$$

let $\tau \in T(B)$ be a tracial state and let $\mathcal{F} \subset B_+^1$ finite and $\eta > 0$ be given.

Then, for any $\gamma > 0$, there is an $(\mathcal{F}, \eta)$-excisor

$$\left(E = \bigoplus_{l=1}^R E_l, \rho = \bigoplus_{l=1}^R \rho_l, \sigma = \bigoplus_{l=1}^R \sigma_l\right)$$

which is compatible with $[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$ and such that

$$(\bar{\tau}_l \otimes \tau_{\mathcal{Q}}) \circ (\psi_l \otimes \mathrm{id}_{\mathcal{Q}}) \circ \sigma_l(1_{E_l}) \geq y_l^\tau - \gamma,\ l \in \{1, \ldots, R\}.$$

Proof: It is straightforward to find, for each $l \in \{1, \ldots, R\}$, an $N_l \in \mathbb{N}$ and subsets $W_l \subset X_l \setminus \Omega_l$ satisfying the following:

(i) Each W_l is a disjoint union $W_l = \coprod_{n=1}^{N_l} W_{l,n}$ of closed subsets $W_{l,n} \subset X_l$, each containing a point $w_{l,n} \in W_{l,n}$,

(ii) $X_l \setminus W_l$ is an open neighborhood of Ω_l,

(iii) $y_l^\tau = \|\bar{\tau}_l\| \geq \|\tilde{\tau}_{W_l}\| \geq \|\bar{\tau}_l\| - \gamma$ (see A.4.0 for notation),

(iv) for any $b \in \mathcal{F}$, $l \in \{1, \ldots, R\}$, $n \in \{1, \ldots, N_l\}$ and $w, w' \in W_{l,n}$,

$$\|\mathrm{ev}_w \pi_{W_l}(b) - \mathrm{ev}_{w'} \pi_{W_l}(b)\| < \eta/2,$$

where

$$\pi_{W_l} : B_l \to \mathcal{C}(W_l) \otimes M_{r_l}$$

denote the canonical surjections.

Define

$$E_l := \bigoplus_1^{N_l} M_{r_l}, \tag{17}$$

$$\rho_l := \bigoplus_{n=1}^{N_l} \mathrm{ev}_{w_{l,n}} : B \to E_l$$

and

$$\tilde{\sigma}_l := \bigoplus_{n=1}^{N_l} 1_{W_{l,n}} \otimes \mathrm{id}_{M_{r_l}} : E_l \to \bigoplus_{n=1}^{N_l} \mathcal{C}(W_{l,n}) \otimes M_{r_l} \cong \mathcal{C}(W_l) \otimes M_{r_l}.$$

Note that

$$\tilde{\sigma} := \bigoplus_{l=1}^R \tilde{\sigma}_l : \bigoplus_{l=1}^R E_l \to \bigoplus_{l=1}^R \mathcal{C}(W_l) \otimes M_{r_l}$$

is a *-homomorphism, hence in particular c.p. order zero.

Let

$$\pi : \bigoplus_{l=1}^R \pi_{W_l} \circ \psi_l : B \to \bigoplus_{l=1}^R \mathcal{C}(W_l) \otimes M_{r_l}$$

Using projectivity of c.p.c. order zero maps together with an approximate unit for $\ker \pi \lhd B$ which is quasicentral for B, it is not hard to find a c.p.c. order zero lift

$$\sigma = \bigoplus_{l=1}^R \sigma_l : \bigoplus_{l=1}^R E_l \to B$$

with the right properties; the argument is essentially the same as in the proof of Proposition A.2.0, so we omit the details. ∎

5. $(\mathcal{F},\eta)$-bridges via linear algebra

A.5.22 PROPOSITION: *Let B be a separable unital recursive subhomogeneous C^*-algebra and let $\mathcal{F} \subset B^1_+$ finite and $\eta > 0$ be given. Suppose B has an $(\mathcal{F},\eta)$-connected recursive subhomogeneous decomposition*

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$$

along which projections can be lifted and such that $X_l \setminus \Omega_l \neq \emptyset$ for $l \geq 1$.

Let $\tau_0, \tau_1 \in T(B)$ be tracial states with

$$(\tau_0)_* = (\tau_1)_*$$

(as states on the ordered $\mathrm{K}_0(B)$) and let $0 < \bar{\beta} \leq 1$ be given.

Then there are $x_l \in X_l \backslash \Omega_l$ for $l \in \{1,\ldots,R\}$ and pairwise orthogonal weighted $(\mathcal{F},\eta)$-excisors

$$(E_{x_l}, \pi_{x_l}, \sigma_{x_l}, \kappa_{x_l}),\ l \in \{1,\ldots,R\},$$

as well as unital embeddings

$$\gamma_0, \gamma_1, \tilde{\gamma} : \bigoplus_{l=1}^R \mathcal{Q} \to \mathcal{Q}$$

and

$$\bar{\gamma} : \mathcal{Q} \oplus \mathcal{Q} \to \mathcal{Q}$$

such that, for

$$E := \bigoplus_{l=1}^R E_{x_l},\ \pi := \bigoplus_{l=1}^R \pi_{x_l},\ \sigma := \bigoplus_{l=1}^R \sigma_{x_l},$$

$$\bar{\kappa}_i := \gamma_i \circ \left(\bigoplus_{l=1}^R \kappa_{x_l}\right),\ \bar{\kappa} := \tilde{\gamma} \circ \left(\bigoplus_{l=1}^R \kappa_{x_l}\right), \tag{18}$$

the weighted $(\mathcal{F},\eta)$-excisors $(E,\pi,\sigma,\bar{\kappa}_i)$, $i \in \{0,1\}$, and $(E,\pi,\sigma,\bar{\kappa})$ satisfy

$$(E,\pi,\sigma,\bar{\gamma}\circ(\bar{\kappa}_0 \oplus \bar{\kappa})) \sim_{(\mathcal{F},\eta)} (E,\pi,\sigma,\bar{\gamma}\circ(\bar{\kappa}_1 \oplus \bar{\kappa})) \tag{19}$$

and such that

$$\bar{y}_{i,l} := \tau_{\mathcal{Q}}(\gamma_i(1_l)) \tag{20}$$

satisfy

$$|\bar{y}_{i,l} - y_l^{\tau_i}| < \bar{\beta} \tag{21}$$

for $i \in \{0,1\}$, $l \in \{1,\ldots,R\}$, where $y_l^{\tau_i}$ is defined as in 15.

PROOF: Choose $x_l \in X_l \setminus \Omega_l$, $l \in \{1,\ldots,R\}$; by Remark A.2.0(ii) and Proposition A.2.0 there are pairwise orthogonal weighted $(\mathcal{F},\eta)$-excisors $(E_{x_l}, \pi_{x_l}, \sigma_{x_l}, \kappa_{x_l})$; note that $E_{x_l} \cong M_{r_l}$ for all l.

<u>Claim 1:</u> For $l \in \{2,\ldots,R\}$, there are $L^{(l)} \in \mathbb{N}$ and pairwise disjoint nonempty subsets

$$\Omega_{l,1},\ldots,\Omega_{l,L^{(l)}} \subset \Omega_l$$

and

$$\nu^{(l)}_{m,k} \in \mathbb{Q}_+,\ m \in \{1,\ldots,R\},\ k \in \{1,\ldots,L^{(l)}\},$$

such that the following hold:

a) $\sum_{m=1}^R \nu^{(l)}_{m,k} = 1$ for $k \in \{1,\ldots,L^{(l)}\}$ and $\nu^{(l)}_{m,k} = 0$ if $m \geq l$,

b) $\bigcup_{k=1}^{L^{(l)}} \Omega_{l,k} = \Omega_l$,

c) for each $x \in \Omega_{l,k}$, $k \in \{1,\ldots,L^{(l)}\}$, there are finite subsets

$$Y_{l,x,m} \subset X_m \setminus \Omega_m,\ m \in \{1,\ldots,l-1\},$$

and, for each $y \in Y_{l,x,m}$, there is a positive integer

$$\mu_{l,x,y} \in \mathbb{N}$$

such that

$$\pi_x \approx_{\mathrm{u}} \bigoplus_{m=1}^{l-1} \Big(\bigoplus_{y \in Y_{l,x,m}} \big(\bigoplus_1^{\mu_{l,x,y}} \pi_y \big) \Big) \tag{22}$$

and

$$\sum_{y \in Y_{l,x,m}} \mu_{l,x,y} \cdot r_m = \nu_{m,k}^{(l)} \cdot r_l, \; m \in \{1, \dots, l-1\}; \tag{23}$$

moreover, we have

$$(E_x, \pi_x, \sigma_x, \kappa_x) \sim_{(\mathcal{F},\eta)} \bigoplus_{\substack{m \in \{1,\dots,l-1\} \\ y \in Y_{l,x,m}}} \tfrac{r_m \mu_{l,x,y}}{r_l} \cdot (E_y, \pi_y, \sigma_y, \kappa_y). \tag{24}$$

Proof of Claim 1: Note that we do not rule out $\Omega_l = \emptyset$. In this case, we set $L^{(l)} = 0$ and there is nothing to show.

Now for each $l \in \{2, \dots, R\}$ and $x \in \Omega_l$, π_x is unitarily equivalent to a direct sum of irreducible representations of B_{l-1}. More precisely, there are finite subsets $Y_{l,x,m} \subset X_m \setminus \Omega_m$, $m \in \{1, \dots, l-1\}$, and for each $y \in Y_{l,x,m}$ there is $\mu_{l,x,y} \in \mathbb{N}$ such that

$$\pi_x \approx_{\mathrm{u}} \Big(\bigoplus_{m=1}^{l-1} \Big(\bigoplus_{y \in Y_{l,x,m}} \big(\bigoplus_1^{\mu_{l,x,y}} \pi_y \big) \Big) \Big).$$

The ranks of the representations of B_{l-1} (with multiplicities) add up to the rank of π_x, so that

$$\sum_{m=1}^{l-1} \Big(\sum_{y \in Y_{l,x,m}} \mu_{l,x,y} \cdot r_m \Big) = r_l. \tag{25}$$

From this it follows that there are only finitely many, say $L^{(l)}$, values for tuples of the form

$$(\mu_{l,x,y})_{m \in \{1,\dots,l-1\}, y \in Y_{l,x,m}}$$

where x ranges over Ω_l. Decompose Ω_l into $L^{(l)}$ pairwise disjoint nonempty subsets $\Omega_{l,k}$, $k \in \{1, \dots, L^{(l)}\}$, such that the maps

$$x \mapsto (\mu_{l,x,y})_{m \in \{1,\dots,l-1\}, y \in Y_{l,x,m}}$$

are constant on each $\Omega_{l,k}$. For $k \in \{1, \dots, L^{(l)}\}$ and $m \in \{1, \dots, l-1\}$ set

$$\nu_{m,k}^{(l)} := \sum_{y \in Y_{l,x,m}} \tfrac{r_m \mu_{l,x,y}}{r_l}; \tag{26}$$

set

$$\nu_{m,k}^{(l)} := 0 \text{ for } m \geq l.$$

Then property a) of Claim 1 holds by (25); b) and c) hold by construction.

Claim 2: For $l \in \{1, \dots, R\}$ and $k \in \{1, \dots, L^{(l)}\}$ let

$$\kappa_k^{(l)} : E \to \mathcal{Q}$$

be a unital *-homomorphism such that

$$\tau_{\mathcal{Q}} \circ \kappa_k^{(l)}(1_{E_{x_m}}) = \nu_{m,k}^{(l)}, \; m \in \{1, \dots, R\} \tag{27}$$

(such $\kappa_k^{(l)}$ exist by Claim 1a)).

Then

$$(E, \pi, \sigma, \kappa_{x_l}) \sim_{(\mathcal{F},\eta)} (E, \pi, \sigma, \kappa_k^{(l)}), \tag{28}$$

where we have slightly misused notation by writing κ_{x_l} for the (canonical) extension of $\kappa_{x_l} : E_{x_l} \to \mathcal{Q}$ to all of E. Moreover (cf. A.2.0 for notation),

$$(E,\pi,\sigma,\kappa_k^{(l)}) \sim_{(\mathcal{F},\eta)} \bigoplus_{m=1}^{R} \nu_{m,k}^{(l)} \cdot (E,\pi,\sigma,\kappa_{x_m}). \tag{29}$$

Proof of Claim 2: Take $\Omega_{l,k}$ and $\nu_{m,k}^{(l)}$, $m \in \{1,\dots,l-1\}$ as in Claim 1; fix $x \in \Omega_{l,k}$ and let $Y_{l,x,m}$, $\mu_{l,x,y}$ be as in Claim 1c).

Note that since our recursive subhomogeneous decomposition is $(\mathcal{F},\eta)$-connected, we have

$$(E_x,\pi_x,\sigma_x,\kappa_x) \sim_{(\mathcal{F},\eta)} (E_{x_l},\pi_{x_l},\sigma_{x_l},\kappa_{x_l}) \tag{30}$$

and, for each $m \in \{1,\dots,l-1\}$ and $y \in Y_{l,x,m}$,

$$(E_y,\pi_y,\sigma_y,\kappa_y) \sim_{(\mathcal{F},\eta)} (E_{x_m},\pi_{x_m},\sigma_{x_m},\kappa_{x_m}), \tag{31}$$

with notation as in A.2.0.

Moreover note that

$$(E_{x_l},\pi_{x_l},\sigma_{x_l},\kappa_{x_l}) \sim_{(\mathcal{F},\eta)} (E,\pi,\sigma,\kappa_{x_l}) \tag{32}$$

by Remark A.2.0(iii). It follows from (22) that

$$\kappa_x \circ \pi_x \approx_{\mathrm{u}} \bigoplus_{\substack{m\in\{1,\dots,l-1\}\\ y\in Y_{l,x,m}}} \left(\tfrac{r_m}{r_l}\mu_{l,x,y}\right) \cdot \kappa_y \circ \pi_y,$$

cf. A.2.0 for notation.

But then by Proposition A.2.0 we have

$$\begin{aligned}
&(E_x,\pi_x,\sigma_x,\kappa_x)\\
&\overset{(24)}{\sim}_{(\mathcal{F},\eta)} \quad \bigoplus_{\substack{m\in\{1,\dots,l-1\}\\ y\in Y_{l,x,m}}} \left(\tfrac{r_m}{r_l}\mu_{l,x,y}\right) \cdot (E_y,\pi_y,\sigma_y,\kappa_y)\\
&\overset{(31)}{\sim}_{(\mathcal{F},\eta)} \quad \bigoplus_{\substack{m\in\{1,\dots,l-1\}\\ y\in Y_{l,x,m}}} \left(\tfrac{r_m}{r_l}\mu_{l,x,y}\right) \cdot (E_{x_m},\pi_{x_m},\sigma_{x_m},\kappa_{x_m})\\
&\overset{(26)}{\sim}_{(\mathcal{F},\eta)} \quad \bigoplus_{m\in\{1,\dots,l-1\}} \nu_{m,k}^{(l)} \cdot (E_{x_m},\pi_{x_m},\sigma_{x_m},\kappa_{x_m})\\
&\overset{(32)}{\sim}_{(\mathcal{F},\eta)} \quad \bigoplus_{m\in\{1,\dots,l-1\}} \nu_{m,k}^{(l)} \cdot (E,\pi,\sigma,\kappa_{x_m})\\
&\overset{(27)}{\sim}_{(\mathcal{F},\eta)} \quad (E,\pi,\sigma,\kappa_k^{(l)}).
\end{aligned}$$

Combining this with (30) and (32) now yields (28); it also shows (29). We have now verified Claim 2.

Claim 3: Let $p \in B$ be a projection such that

$$\frac{1}{r_l} \cdot \mathrm{rank}(p|_{X_l}) \equiv \xi_l \in \mathbb{Q} \tag{33}$$

is constant for each $l \in \{1,\dots,R\}$.

Then, for each $l \in \{2,\dots,R\}$, the ξ_l satisfy the relations

$$\xi_l = \textstyle\sum_{m=1}^{R} \nu_{m,k}^{(l)} \cdot \xi_m,\ k \in \{1,\dots,L^{(l)}\}, \tag{34}$$

where the $\nu_{m,k}^{(l)}$ are as in Claim 1.

Proof of Claim 3: For $l \in \{2,\dots,R\}$ and $k \in \{1,\dots,L^{(l)}\}$ choose $x \in \Omega_{l,k} \subset X_l$ and let $Y_{l,x,m}$ and $\mu_{l,x,y}$ be as in Claim 1c).

We have

$$
\begin{aligned}
\xi_l &\overset{(33)}{=} \tfrac{1}{r_l}\cdot \operatorname{rank}(\pi_x(p))\\
&\overset{(22)}{=} \tfrac{1}{r_l}\cdot\left(\sum_{m=1}^{l-1}\left(\sum_{y\in Y_{l,x,m}}\left(\sum_1^{\mu_{l,x,y}}\operatorname{rank}(\pi_y(p))\right)\right)\right)\\
&\overset{(33)}{=} \tfrac{1}{r_l}\cdot\left(\sum_{m=1}^{l-1}\left(\sum_{y\in Y_{l,x,m}}\mu_{l,x,y}\cdot r_m\cdot \xi_m\right)\right)\\
&\overset{(23)}{=} \sum_{m=1}^{l-1}\nu^{(l)}_{m,k}\cdot\xi_m,
\end{aligned}
$$

so (34) holds and Claim 3 is proven.

Before moving on to Claim 4, let us set

$$\bar{L} := \sum_{l=2}^{R} L^{(l)}$$

and define $\bar{L}\times R$ matrices

$$
T_+ := \begin{pmatrix}
\nu^{(2)}_{1,1} & 0 & \dots & & & & \\
\vdots & \vdots & & & & & \\
\nu^{(2)}_{1,L^{(2)}} & 0 & \dots & & & & \\
\vdots & & & & & & \\
\nu^{(l)}_{1,1} & \dots & \nu^{(l)}_{l-1,1} & 0 & \dots & & \\
\vdots & & \vdots & \vdots & & & \\
\nu^{(l)}_{1,L^{(l)}} & \dots & \nu^{(l)}_{l-1,L^{(l)}} & 0 & \dots & & \\
\vdots & & & & & & \\
\nu^{(R)}_{1,1} & & \dots & & & \nu^{(R)}_{R-1,1} & 0\\
\vdots & & & & & \vdots & \vdots\\
\nu^{(R)}_{1,L^{(R)}} & & \dots & & & \nu^{(R)}_{R-1,L^{(R)}} & 0
\end{pmatrix}
$$

and

$$
T_- := \begin{pmatrix}
0 & 1 & \dots & & & & \\
\vdots & \vdots & & & & & \\
0 & 1 & \dots & & & & \\
\vdots & & & & & & \\
0 & & \dots & 0 & 1 & \dots & \\
\vdots & & & \vdots & \vdots & & \\
0 & & \dots & 0 & 1 & \dots & \\
\vdots & & & & & & \\
0 & & & \dots & & 0 & 1\\
\vdots & & & \vdots & \vdots & & \\
0 & & & \dots & & 0 & 1
\end{pmatrix}
$$

and note that, with these definitions, $\xi_1,\dots,\xi_R$ satisfy the equation (34) for $l\in\{2,\dots,R\}$, $k\in\{1,\dots,L^{(l)}\}$ if and only if

$$\begin{pmatrix}\xi_1\\ \vdots\\ \xi_R\end{pmatrix}\in\ker(T_+-T_-). \tag{35}$$

Claim 4: Suppose we have

$$\underline{\xi} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_R \end{pmatrix} \in \ker(T_+ - T_-) \cap \mathbb{N}^R.$$

Then there are $\bar{N} \in \mathbb{N}$ and a projection

$$p \in B \otimes M_{\bar{N}} \subset \bigoplus_{l=1}^R \mathcal{C}(X_l) \otimes M_{r_l} \otimes M_{\bar{N}}$$

such that

$$\frac{1}{r_l} \cdot \operatorname{rank}(p|_{X_l}) \equiv \xi_l \tag{36}$$

for $l \in \{1, \ldots, R\}$.

Proof of Claim 4: Take a trivial projection p_1 in $B_1 = \mathcal{C}(X_1) \otimes M_{r_1} \otimes M_{\bar{N}}$ (for $\bar{N}$ large enough) with rank $r_1\xi_1$.

Now suppose we have constructed projections $p_1, \ldots, p_l$ in $B_1, \ldots, B_l$, respectively, such that

$$\frac{1}{r_m} \cdot \operatorname{rank}(p_{l'}|_{X_m}) \equiv \xi_m \text{ for } 1 \le m \le l' \le l \tag{37}$$

and

$$\psi'_{l'}(p_{l'+1}) = p_{l'} \text{ for } l' \in \{1, \ldots, l-1\},$$

where

$$\psi'_{l'} : B_{l'+1} \twoheadrightarrow B_{l'}$$

denotes the canonical surjection, cf. (4.1).

If $\Omega_{l+1} = \emptyset$, then

$$B_{l+1} \cong \mathcal{C}(X_{l+1}) \otimes M_{r_{l+1}} \oplus B_l$$

and we may define

$$p_{l+1} := q_{l+1} \oplus p_l,$$

where

$$q_{l+1} \in \mathcal{C}(X_{l+1}) \otimes M_{r_{l+1}} \otimes M_{\bar{N}}$$

is a trivial projection with rank $\xi_{l+1} r_{l+1}$.

If $\Omega_{l+1} \neq \emptyset$, then $\phi_l \otimes \mathrm{id}_{M_{\bar{N}}}(p_l)$ is a projection in $\mathcal{C}(\Omega_{l+1}) \otimes M_{r_{l+1}} \otimes M_{\bar{N}}$ and, for $x \in \Omega_{l+1}$, we have

$$\begin{aligned} \operatorname{rank}((\phi_l \otimes \mathrm{id}_{M_{\bar{N}}})(p_l)(x)) &\overset{(22)}{=} \sum_{m=1}^l \sum_{y \in Y_{l+1,x,m}} \mu_{l+1,x,y} \cdot \operatorname{rank}(p_l|_{X_m}) \\ &\overset{(37)}{=} \sum_{m=1}^l \sum_{y \in Y_{l+1,x,m}} \mu_{l+1,x,y} \cdot \xi_m r_m \\ &\overset{(23)}{=} \sum_{m=1}^l \nu_{m,k}^{(l+1)} \cdot r_{l+1} \xi_m \\ &\overset{(34)}{=} \xi_{l+1} r_{l+1}. \end{aligned}$$

But then by hypothesis, $(\phi_l \otimes \mathrm{id}_{M_{\bar{N}}})(p_l)$ lifts to a projection p'_{l+1} in $\mathcal{C}(X_{l+1}) \otimes M_{r_{l+1}} \otimes M_{\bar{N}}$; by changing p'_{l+1} on those components of X_{l+1} which do not intersect Ω_{l+1}, if necessary, we may assume that p'_{l+1} has constant rank $\xi_{l+1} r_{l+1}$ on X_{l+1}. Now

$$p_{l+1} := p'_{l+1} \oplus p_l \in B_{l+1} \otimes M_{\bar{N}} \subset \mathcal{C}(X_{l+1}) \otimes M_{r_{l+1}} \otimes M_{\bar{N}}$$

satisfies

$$\frac{1}{r_m} \cdot \operatorname{rank}(p_{l+1}|_{X_m}) \equiv \xi_m$$

for $1 \leq m \leq l+1$. Proceed inductively to construct $p_1, p_2, \ldots, p_R$, then

$$p := p_R$$

will be as desired. This proves Claim 4.

Let $\underline{\xi}$, $\bar{N}$ and p be as in Claim 4. Since $(\tau_0)_* = (\tau_1)_*$, we have

$$(\tau_0 \otimes \mathrm{tr}_{M_{\bar{N}}})(p) = (\tau_1 \otimes \mathrm{tr}_{M_{\bar{N}}})(p),$$

whence

$$\textstyle\sum_{l=1}^{R} \xi_l \cdot y_l^{\tau_0} = \sum_{l=1}^{R} \xi_l \cdot y_l^{\tau_1},$$

cf. A.4.0. But this just means that

$$\langle \underline{\xi}, \underline{y}^{(0)} \rangle = \langle \underline{\xi}, \underline{y}^{(1)} \rangle$$

or, equivalently,

$$\underline{\xi} \perp (\underline{y}^{(0)} - \underline{y}^{(1)}) \text{ in } \mathbb{R}^R, \tag{38}$$

where

$$\underline{y}^{(i)} = \begin{pmatrix} y_1^{\tau_i} \\ \vdots \\ y_R^{\tau_i} \end{pmatrix}, \; i \in \{0, 1\}. \tag{39}$$

Since $1_B \in B$ is a projection with $\frac{1}{r_l} \cdot \mathrm{rank}(1_B|_{X_l}) = 1$ for all l, we see from Claim 3 and (35) that

$$\underline{r} := \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \in \ker(T_+ - T_-) \cap \mathbb{Z}^R. \tag{40}$$

But positive integer multiples of $\underline{r}$ are also in $\ker(T_+ - T_-) \cap \mathbb{N}^R$, from which follows that

$$\ker(T_+ - T_-) \cap \mathbb{Z}^R = \ker(T_+ - T_-) \cap \mathbb{N}^R - \ker(T_+ - T_-) \cap \mathbb{N}^R. \tag{41}$$

Moreover, Claim 4 and (41) imply

$$\ker(T_+ - T_-) \cap \mathbb{Z}^R \perp (\underline{y}^{(0)} - \underline{y}^{(1)}) \text{ in } \mathbb{R}^R,$$

whence

$$\ker(T_+ - T_-) \cap \mathbb{Q}^R \perp (\underline{y}^{(0)} - \underline{y}^{(1)}) \text{ in } \mathbb{R}^R;$$

since T_+ and T_- have only rational coefficients, it follows that $\ker(T_+ - T_-) \cap \mathbb{Q}^R$ is dense in $\ker(T_+ - T_-)$, whence

$$\ker(T_+ - T_-) \perp (\underline{y}^{(0)} - \underline{y}^{(1)}) \text{ in } \mathbb{R}^R.$$

By elementary linear algebra we have

$$(\ker(T_+ - T_-))^{\perp} = \mathrm{Im}(T_+ - T_-)^*,$$

so there is $\underline{\zeta} \in \mathbb{R}^{\bar{L}}$ such that

$$(T_+ - T_-)^* \underline{\zeta} = \underline{y}^{(0)} - \underline{y}^{(1)}.$$

We may then write

$$\underline{\zeta} = \underline{\zeta}_+ - \underline{\zeta}_- \text{ with } \underline{\zeta}_+, \underline{\zeta}_- \in \mathbb{R}_+^{\bar{L}}$$

to obtain the equation

$$\underline{y}^{(0)} + T_+^* \underline{\zeta}_- + T_-^* \underline{\zeta}_+ = \underline{y}^{(1)} + T_+^* \underline{\zeta}_+ + T_-^* \underline{\zeta}_-$$

in $\mathbb{R}^R$, in which all vectors and matrices have only positive entries.

We wish to interpret the entries of the $\underline{y}^{(i)}$, $\underline{\zeta}_+$ and $\underline{\zeta}_-$ as coefficients of sums of $(\mathcal{F},\eta)$-bridges. To this end, we have to approximate them by rationals. Let us first set

$$\alpha := \frac{\bar{\beta}}{8R}(\leq 1). \tag{42}$$

Claim 5: There are

$$g^{(i)} = (g_l^{(i)})_{l\in\{1,\dots,R\}} \in \mathbb{Q}^R,\ i\in\{0,1\},$$

$$z_+ = (z_{+,k}^{(l)})_{\substack{l\in\{2,\dots,R\}\\ k\in\{1,\dots,L^{(l)}\}}} \in \mathbb{Q}_+^{\bar{L}}$$

and

$$z_- = (z_{-,k}^{(l)})_{\substack{l\in\{2,\dots,R\}\\ k\in\{1,\dots,L^{(l)}\}}} \in \mathbb{Q}_+^{\bar{L}}$$

satisfying

$$\|g^{(i)} - \underline{y}^{(i)}\|_{\max} \leq \alpha,\ i\in\{0,1\}, \tag{43}$$

$$\|z_+ - \underline{\zeta}_+\|_{\max},\ \|z_- - \underline{\zeta}_-\|_{\max} \leq \alpha,$$

$$\|g^{(0)} + T_+^* z_- + T_-^* z_+ - (g^{(1)} + T_+^* z_+ + T_-^* z_-)\|_{\max} \leq \alpha, \tag{44}$$

$$\langle \underline{r}, g^{(0)}\rangle = \langle \underline{r}, g^{(1)}\rangle \tag{45}$$

(with $\underline{r}$ as in (40)), and

$$\langle \underline{r}, T_+^* z_- + T_-^* z_+\rangle = \langle \underline{r}, T_+^* z_+ + T_-^* z_-\rangle. \tag{46}$$

Proof of Claim 5: Easy.

We now set

$$\underline{v}^{(0)} := T_+^* z_- + T_-^* z_+,\ \underline{v}^{(1)} := T_+^* z_+ + T_-^* z_-, \tag{47}$$

$$G := \langle \underline{r}, g^{(0)}\rangle = \textstyle\sum_{l=1}^R g_l^{(0)} \overset{(45)}{=} \langle \underline{r}, g^{(1)}\rangle, \tag{48}$$

$$Z_+ := \langle \underline{r}, T_-^* z_+\rangle = \sum_{\substack{m\in\{2,\dots,R\}\\ k\in\{1,\dots,L^{(m)}\}}} z_{+,k}^{(m)} \tag{49}$$

and

$$Z_- := \langle \underline{r}, T_-^* z_-\rangle = \sum_{\substack{m\in\{2,\dots,R\}\\ k\in\{1,\dots,L^{(m)}\}}} z_{-,k}^{(m)}. \tag{50}$$

Note that

$$|G-1| \overset{(39),(16)}{=} |\langle \underline{r}, g^{(0)} - \underline{y}^{(0)}\rangle| \overset{(43)}{\leq} R\alpha. \tag{51}$$

For any

$$z = (z_k^{(m)})_{\substack{m\in\{2,\dots,R\},\\ k\in\{1,\dots,L^{(m)}\}}} \in \mathbb{R}^{\bar{L}}$$

we compute (observing that $\nu_{l,k}^{(m)} = 0$ if $m \leq l$)

$$\begin{aligned}
\langle \underline{r}, T_+^* z\rangle &= \textstyle\sum_{l=1}^R \sum_{m=2}^R \sum_{k=1}^{L^{(m)}} \nu_{l,k}^{(m)} \cdot z_k^{(m)} \\
&= \textstyle\sum_{m=2}^R \sum_{k=1}^{L^{(m)}} \left(\sum_{l=1}^R \nu_{l,k}^{(m)}\right) \cdot z_k^{(m)} \\
&\overset{\text{Claim 1a)}}{=} \textstyle\sum_{m=2}^R \sum_{k=1}^{L^{(m)}} z_k^{(m)} \\
&= \langle \underline{r}, T_-^* z\rangle,
\end{aligned} \tag{52}$$

so that in particular

$$\langle \underline{r}, T_+^* z_+\rangle = Z_+ = \langle \underline{r}, T_-^* z_+\rangle$$

and

$$\langle \underline{r}, T_+^* z_- \rangle = Z_- = \langle \underline{r}, T_-^* z_- \rangle,$$

see (49), (50).

We set

$$V := \langle \underline{r}, \underline{v}^{(0)} \rangle \overset{(47)}{=} \langle \underline{r}, \underline{v}^{(1)} \rangle \overset{(49),(50)}{=} Z_+ + Z_-. \tag{53}$$

By (44), we may choose $w_+, w_- \in \mathbb{Q}_+^R$ such that

$$-(g^{(0)} + v^{(0)}) + (g^{(1)} + v^{(1)}) = w_+ - w_- \tag{54}$$

and such that

$$\|w_+\|_{\max},\ \|w_-\|_{\max} \le \alpha. \tag{55}$$

Set

$$W := \langle \underline{r}, w_+ \rangle \overset{(45),(46)}{=} \langle \underline{r}, w_- \rangle. \tag{56}$$

Note also that

$$\begin{aligned} |G+W-1| &\le \frac{1}{2}|\langle \underline{r}, g^{(0)}\rangle + \langle \underline{r}, g^{(1)}\rangle \\ &\quad +\langle \underline{r}, w_+\rangle + \langle \underline{r}, w_-\rangle \\ &\quad -\langle \underline{r}, \underline{y}^{(0)}\rangle - \langle \underline{r}, \underline{y}^{(1)}\rangle| \\ &\le \frac{1}{2}(|\langle \underline{r}, g^{(0)} - \underline{y}^{(0)}\rangle| + |\langle \underline{r}, g^{(1)} - \underline{y}^{(1)}\rangle| \\ &\quad +|\langle \underline{r}, w_+\rangle| + |\langle \underline{r}, w_-\rangle| \\ &\le 2R\alpha, \end{aligned}$$

whence

$$\tfrac{1}{G+W} \le 1 + 4R\alpha. \tag{57}$$

Let e_l denote the unit of the l^{th} copy of $\mathcal{Q}$ in $\oplus_{l=1}^R \mathcal{Q}$. We now choose unital *-homomorphisms

$$\gamma_0,\ \gamma_1,\ \tilde{\gamma} : \bigoplus_{l=1}^R \mathcal{Q} \to \mathcal{Q}$$

and

$$\bar{\gamma} : \mathcal{Q} \oplus \mathcal{Q} \to \mathcal{Q}$$

such that

$$\tau_{\mathcal{Q}} \circ \gamma_0(e_l) = \tfrac{g_l^{(0)} + w_{+,l}}{G+W}, \tag{58}$$

$$\tau_{\mathcal{Q}} \circ \gamma_1(e_l) = \tfrac{g_l^{(1)} + w_{-,l}}{G+W},$$

$$\tau_{\mathcal{Q}} \circ \tilde{\gamma}(e_l) = \tfrac{g_l^{(0)} + v_l^{(0)}}{G+V},$$

for $l \in \{1, \ldots, R\}$ and

$$\tau_{\mathcal{Q}} \circ \bar{\gamma}((1,0)) = \tfrac{G+W}{2G+V+W}, \tag{59}$$

$$\tau_{\mathcal{Q}} \circ \bar{\gamma}((0,1)) = \tfrac{G+V}{2G+V+W};$$

these exist by (48), (56) and (53).

Next observe that

$$\bar{y}_{0,l} \overset{(20)}{:=} \tau_{\mathcal{Q}}(\gamma_0(e_l)) \overset{(58)}{=} \tfrac{g_l^{(0)} + w_{+,l}}{G+W},$$

so

$$\begin{array}{rcl} |\bar{y}_{0,l} - y_l^{\tau_0}| & = & \left|\frac{1}{G+W}(g_l^{(0)} + w_{+,l}) - y_l^{\tau_0}\right| \\ & = & \frac{1}{G+W}|g_l^{(0)} + w_{+,l} - (G+W)y_l^{\tau_0}| \\ & \leq & \frac{1}{G+W}(|g_l^{(0)} - y_l^{\tau_0}| + |G-1|y_l^{\tau_0} + w_{+,l} + Wy_l^{\tau_0}) \\ & \overset{(15)}{\leq} & \frac{1}{G+W}(|g_l^{(0)} - y_l^{\tau_0}| + |G-1| + 2W) \\ & \leq & (1+4R\alpha)(\alpha + 3R\alpha) \\ & \overset{(42)}{<} & 8R\alpha \\ & \overset{(42)}{=} & \bar{\beta} \end{array}$$

and (21) holds. Here, for the third inequality we have used (57), (43), (39), (51) and (55).

Set

$$z := (z_k^{(l)})_{\substack{l\in\{2,\dots,R\} \\ k\in\{1,\dots,L^{(l)}\}}} \in \mathbb{Q}_+^{\bar{L}}$$

and

$$Z := \langle \underline{r}, T_+^* z\rangle;$$

note that

$$Z = \langle \underline{r}, T_-^* z\rangle;$$

by (52). We then compute

$$\begin{array}{ll} \multicolumn{2}{l}{\bigoplus_{m=1}^R \frac{1}{Z}(T_+^* z)_m \cdot (E,\pi,\sigma,\kappa_{x_m})} \\ = & \bigoplus_{m=1}^R \left(\sum_{l=2}^R \sum_{k=1}^{L^{(l)}} \nu_{m,k}^{(l)} \frac{z_k^{(l)}}{Z}\right) \cdot (E,\pi,\sigma,\kappa_{x_m}) \\ \overset{A.2.0}{\sim}_{(\mathcal{F},\eta)} & \bigoplus_{l=2}^R \bigoplus_{k=1}^{L^{(l)}} \frac{z_k^{(l)}}{Z} \cdot \left(\bigoplus_{m=1}^R \nu_{m,k}^{(l)} \cdot (E,\pi,\sigma,\kappa_{x_m})\right) \\ \overset{(29)}{\sim}_{(\mathcal{F},\eta)} & \bigoplus_{l=2}^R \bigoplus_{k=1}^{L^{(l)}} \frac{z_k^{(l)}}{Z} \cdot (E,\pi,\sigma,\kappa_k^{(l)}) \\ \overset{(28)}{\sim}_{(\mathcal{F},\eta)} & \bigoplus_{l=2}^R \bigoplus_{k=1}^{L^{(l)}} \frac{z_k^{(l)}}{Z} \cdot (E,\pi,\sigma,\kappa_{x_l}) \\ = & \bigoplus_{m=1}^R \frac{1}{Z}(T_-^* z)_m \cdot (E,\pi,\sigma,\kappa_{x_m}). \end{array} \tag{60}$$

As a consequence, we obtain

$$\begin{array}{ll} \multicolumn{2}{l}{\bigoplus_{m=1}^R \frac{1}{V}\underline{v}_m^{(0)} \cdot (E,\pi,\sigma,\kappa_{x_m})} \\ = & \bigoplus_{m=1}^R \frac{1}{V}((T_+^* z_-)_m + (T_-^* z_+)_m) \cdot (E,\pi,\sigma,\kappa_{x_m}) \\ \sim_{(\mathcal{F},\eta)} & \frac{Z_-}{V} \cdot \left(\bigoplus_{m=1}^R \frac{1}{Z_-}(T_+^* z_-)_m \cdot (E,\pi,\sigma,\kappa_{x_m})\right) \\ & \oplus \frac{Z_+}{V} \cdot \left(\bigoplus_{m=1}^R \frac{1}{Z_+}(T_-^* z_+)_m \cdot (E,\pi,\sigma,\kappa_{x_m})\right) \\ \overset{(60)}{\sim}_{(\mathcal{F},\eta)} & \frac{Z_-}{V} \cdot \left(\bigoplus_{m=1}^R \frac{1}{Z_-}(T_-^* z_-)_m \cdot (E,\pi,\sigma,\kappa_{x_m})\right) \\ & \oplus \frac{Z_+}{V} \cdot \left(\bigoplus_{m=1}^R \frac{1}{Z_+}(T_+^* z_+)_m \cdot (E,\pi,\sigma,\kappa_{x_m})\right) \\ \sim_{(\mathcal{F},\eta)} & \bigoplus_{m=1}^R \frac{1}{V}((T_-^* z_-)_m + (T_+^* z_+)_m) \cdot (E,\pi,\sigma,\kappa_{x_m}) \\ = & \bigoplus_{m=1}^R \frac{1}{V}\underline{v}_m^{(1)} \cdot (E,\pi,\sigma,\kappa_{x_m}). \end{array} \tag{61}$$

We finally compute

$$
\begin{aligned}
&(E,\pi,\sigma,\bar\gamma\circ(\bar\kappa_0\oplus\bar\kappa)) \\
\overset{(59),(58),(18)}{\sim_{(\mathcal{F},\eta)}}\quad & \bigoplus_{m=1}^{R}\tfrac{1}{2G+V+W}(g_m^{(0)}+w_{+,m}+g_m^{(0)}+\underline{v}_m^{(0)})\cdot(E,\pi,\sigma,\kappa_{x_m}) \\
\overset{(54)}{=}\quad & \bigoplus_{m=1}^{R}\tfrac{1}{2G+V+W}(g_m^{(1)}+\underline{v}_m^{(1)}+w_{-,m}+g_m^{(0)})\cdot(E,\pi,\sigma,\kappa_{x_m}) \\
\overset{A.2.0}{\sim_{(\mathcal{F},\eta)}}\quad & \tfrac{2G+W}{2G+V+W}\cdot\Big(\bigoplus_{m=1}^{R}\tfrac{1}{2G+W}(g_m^{(1)}+w_{-,m}+g_m^{(0)})\cdot(E,\pi,\sigma,\kappa_{x_m})\Big) \\
& \oplus\tfrac{V}{2G+V+W}\cdot\Big(\bigoplus_{m=1}^{R}\tfrac{1}{V}\underline{v}_m^{(1)}\cdot(E,\pi,\sigma,\kappa_{x_m})\Big) \\
\overset{A.2.0,(61)}{\sim_{(\mathcal{F},\eta)}}\quad & \bigoplus_{m=1}^{R}\tfrac{1}{2G+V+W}(g_m^{(1)}+w_{-,m}+g_m^{(0)}+\underline{v}_m^{(0)})\cdot(E,\pi,\sigma,\kappa_{x_m}) \\
\overset{(59),(58),(18)}{\sim_{(\mathcal{F},\eta)}}\quad & (E,\pi,\sigma,\bar\gamma\circ(\bar\kappa_1\oplus\bar\kappa)),
\end{aligned}
$$

thus establishing (19). ∎

A.5.23 Proposition: *Let B be a separable unital recursive subhomogeneous C^*-algebra with recursive subhomogeneous decomposition $[B_l,X_l,\Omega_l,r_l,\phi_l]_{l=1}^R$ and let $\mathcal{F}\subset B_+^1$ finite and $\eta,\delta>0$ be given.*

Let $(E,\pi,\sigma,\bar\kappa_0)$, $(E,\pi,\sigma,\bar\kappa_1)$ and $(E,\pi,\sigma,\bar\kappa)$ be weighted $(\mathcal{F},\eta)$-excisors and let

$$\bar\gamma:\mathcal{Q}\oplus\mathcal{Q}\to\mathcal{Q}$$

be a unital embedding such that

$$(E,\pi,\sigma,\bar\gamma\circ(\bar\kappa_0\oplus\bar\kappa))\sim_{(\mathcal{F},\eta)}(E,\pi,\sigma,\bar\gamma\circ(\bar\kappa_1\oplus\bar\kappa)) \tag{62}$$

(compatible with the decomposition).

Then there is a unital embedding

$$\gamma:\mathcal{Q}\oplus\mathcal{Q}\to\mathcal{Q}$$

such that

$$(E,\pi,\sigma,\gamma\circ(\bar\kappa_0\oplus\bar\kappa))\sim_{(\mathcal{F},\eta)}(E,\pi,\sigma,\gamma\circ(\bar\kappa_1\oplus\bar\kappa))$$

(also compatible with the decomposition) and

$$|\tau_{\mathcal{Q}}(\gamma((p,0)))-\tau_{\mathcal{Q}}(p)|<\delta$$

for every projection $p\in\mathcal{Q}$, in particular

$$\tau_{\mathcal{Q}}(\gamma((1_{\mathcal{Q}},0)))>1-\delta.$$

Proof: Choose $N\in\mathbb{N}$ so large that

$$\frac{\tau_{\mathcal{Q}}(\bar\gamma((0,1)))}{N\cdot\tau_{\mathcal{Q}}(\bar\gamma((1,0)))+\tau_{\mathcal{Q}}(\bar\gamma((0,1)))}<\delta$$

and a unital embedding

$$\theta:\mathbb{C}^{N+1}\otimes\mathcal{Q}\to\mathcal{Q}$$

such that

$$\tau_{\mathcal{Q}}(\theta(e_i\otimes 1_{\mathcal{Q}}))=\frac{\tau_{\mathcal{Q}}(\bar\gamma((1,0)))}{N\cdot\tau_{\mathcal{Q}}(\bar\gamma((1,0)))+\tau_{\mathcal{Q}}(\bar\gamma((0,1)))}$$

for $i\in\{1,\dots,N\}$ and

$$\tau_{\mathcal{Q}}(\theta(e_{N+1}\otimes 1_{\mathcal{Q}}))=\frac{\tau_{\mathcal{Q}}(\bar\gamma((0,1)))}{N\cdot\tau_{\mathcal{Q}}(\bar\gamma((1,0)))+\tau_{\mathcal{Q}}(\bar\gamma((0,1)))}.$$

Define

$$\gamma : \mathcal{Q} \oplus \mathcal{Q} \to \mathcal{Q}$$

by

$$\gamma := \theta \circ \left(\left(\left(\sum_{i=1}^{N} e_i \right) \otimes \mathrm{id}_{\mathcal{Q}} \right) \oplus (e_{N+1} \otimes \mathrm{id}_{\mathcal{Q}}) \right)$$

and

$$\gamma_j : \mathcal{Q} \oplus \mathcal{Q} \oplus \mathcal{Q} \oplus \mathcal{Q} \to \mathcal{Q}$$

by

$$\begin{aligned} \gamma_j \quad := \quad & \theta \circ \left(\left(\left(\sum_{i=1}^{j-1} e_i \right) \otimes \mathrm{id}_{\mathcal{Q}} \right) \oplus (e_j \otimes \mathrm{id}_{\mathcal{Q}}) \right. \\ & \left. \oplus \left(\left(\sum_{i=j+1}^{N} e_i \right) \otimes \mathrm{id}_{\mathcal{Q}} \right) \oplus (e_{N+1} \otimes \mathrm{id}_{\mathcal{Q}}) \right) \end{aligned}$$

for $j \in \{1, \ldots, N\}$.

We clearly have

$$\tau_{\mathcal{Q}}(\gamma((1_{\mathcal{Q}}, 0))) > 1 - \delta.$$

Note also that

$$\gamma \circ (\bar{\kappa}_0 \oplus \bar{\kappa}) = \gamma_N \circ (\bar{\kappa}_0 \oplus \bar{\kappa}_0 \oplus \bar{\kappa}_1 \oplus \bar{\kappa}) \tag{63}$$

and

$$\gamma \circ (\bar{\kappa}_1 \oplus \bar{\kappa}) = \gamma_1 \circ (\bar{\kappa}_0 \oplus \bar{\kappa}_1 \oplus \bar{\kappa}_1 \oplus \bar{\kappa}), \tag{64}$$

since

$$\gamma_N((0, 0, x, 0)) = \gamma_1((x, 0, 0, 0)) = 0$$

for $x \in \mathcal{Q}$.

We furthermore have

$$\gamma_j \circ (\bar{\kappa}_0 \oplus \bar{\kappa}_0 \oplus \bar{\kappa}_1 \oplus \bar{\kappa}) = \gamma_{j+1} \circ (\bar{\kappa}_0 \oplus \bar{\kappa}_1 \oplus \bar{\kappa}_1 \oplus \bar{\kappa}) \tag{65}$$

for $j \in \{1, \ldots, N-1\}$. From (62) and A.2.0 we obtain

$$(E, \pi, \sigma, \gamma_j \circ (\bar{\kappa}_0 \oplus \bar{\kappa}_0 \oplus \bar{\kappa}_1 \oplus \bar{\kappa})) \sim_{(\mathcal{F},\eta)} (E, \pi, \sigma, \gamma_j \circ (\bar{\kappa}_0 \oplus \bar{\kappa}_1 \oplus \bar{\kappa}_1 \oplus \bar{\kappa})). \tag{66}$$

We now have

$$\begin{aligned} (E, \pi, \sigma, \gamma \circ (\bar{\kappa}_0 \oplus \bar{\kappa})) \quad & \overset{(63)}{=} \quad (E, \pi, \sigma, \gamma_N \circ (\bar{\kappa}_0 \oplus \bar{\kappa}_0 \oplus \bar{\kappa}_1 \oplus \bar{\kappa})) \\ & \overset{(66)}{\sim}_{(\mathcal{F},\eta)} \quad (E, \pi, \sigma, \gamma_N \circ (\bar{\kappa}_0 \oplus \bar{\kappa}_1 \oplus \bar{\kappa}_1 \oplus \bar{\kappa})) \\ & \quad \vdots \\ & \overset{(65)}{=} \quad (E, \pi, \sigma, \gamma_j \circ (\bar{\kappa}_0 \oplus \bar{\kappa}_0 \oplus \bar{\kappa}_1 \oplus \bar{\kappa})) \\ & \sim_{(\mathcal{F},\eta)} \quad (E, \pi, \sigma, \gamma_j \circ (\bar{\kappa}_0 \oplus \bar{\kappa}_1 \oplus \bar{\kappa}_1 \oplus \bar{\kappa})) \\ & \quad \vdots \\ & = \quad (E, \pi, \sigma, \gamma_1 \circ (\bar{\kappa}_0 \oplus \bar{\kappa}_0 \oplus \bar{\kappa}_1 \oplus \bar{\kappa})) \\ & \sim_{(\mathcal{F},\eta)} \quad (E, \pi, \sigma, \gamma_1 \circ (\bar{\kappa}_0 \oplus \bar{\kappa}_1 \oplus \bar{\kappa}_1 \oplus \bar{\kappa})) \\ & \overset{(64)}{=} \quad (E, \pi, \sigma, \gamma \circ (\bar{\kappa}_1 \oplus \bar{\kappa})). \end{aligned}$$

∎

A.5.24 Proposition: *Let B be a separable unital recursive subhomogeneous* C*-*algebra and let $\mathcal{F} \subset B_+^1$ finite and $0 < \eta, \beta \leq 1$ be given.*

Suppose B has an $(\mathcal{F}, \eta)$-connected recursive subhomogeneous decomposition

$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$$

along which projections can be lifted and such that $X_l \setminus \Omega_l \neq \emptyset$ for $l \geq 1$.

Let $\tau_0, \tau_1 \in T(B)$ be tracial states with

$$(\tau_0)_* = (\tau_1)_*$$

(as states on the ordered $\mathrm{K}_0(B)$).

Then there are $x_l \in X_l \setminus \Omega_l$ for $l \in \{1, \ldots, R\}$ and pairwise orthogonal $(\mathcal{F}, \eta)$-excisors

$$(E_{x_l}, \pi_{x_l}, \sigma_{x_l}),\ l \in \{1, \ldots, R\};$$

in this case,

$$E_{x_l} \cong M_{r_l},\ l \in \{1, \ldots, R\}$$

and

$$\left(E := \bigoplus_{l=1}^R E_{x_l},\ \pi := \bigoplus_{l=1}^R \pi_{x_l},\ \sigma := \bigoplus_{l=1}^R \sigma_{x_l}\right) \tag{67}$$

is an $(\mathcal{F}, \eta)$-excisor.

Furthermore, there are unital embeddings

$$\kappa_i : E \to \mathcal{Q},\ i \in \{0, 1\},$$

such that

$$(E, \pi, \sigma, \kappa_0) \sim_{(\mathcal{F},\eta)} (E, \pi, \sigma, \kappa_1)$$

and such that

$$y_{i,l} := \tau_{\mathcal{Q}}(\kappa_i(1_{E_{x_l}})) \tag{68}$$

satisfy

$$|y_{i,l} - y_l^{\tau_i}| < \beta \tag{69}$$

for $i \in \{0, 1\}$, $l \in \{1, \ldots, R\}$, where the $y_l^{\tau_i}$ are as in (15).

Proof: Apply Proposition A.5.0 with

$$\bar{\beta} := \frac{\beta}{3}$$

to obtain $x_l \in X_l \setminus \Omega_l$, pairwise orthogonal weighted $(\mathcal{F}, \eta)$-excisors

$$(E_{x_l}, \pi_{x_l}, \sigma_{x_l}, \kappa_{x_l}),\ l \in \{1, \ldots, R\},$$

and unital embeddings

$$\gamma_0, \gamma_1, \tilde{\gamma} : \bigoplus_{l=1}^R \mathcal{Q} \to \mathcal{Q}$$

and

$$\bar{\gamma} : \mathcal{Q} \oplus \mathcal{Q} \to \mathcal{Q}.$$

Apply Proposition A.5.0 with

$$\delta := \frac{\beta}{3}$$

and with

$$\bar{\kappa}_i := \gamma_i \circ \left(\bigoplus_{l=1}^R \kappa_{x_l}\right),\ i \in \{0, 1\},$$

and

$$\bar{\kappa} := \tilde{\gamma} \circ \left(\bigoplus_{l=1}^R \kappa_{x_l}\right)$$

to obtain a unital embedding
$$\gamma : \mathcal{Q} \oplus \mathcal{Q} \to \mathcal{Q}$$
such that
$$|\tau_{\mathcal{Q}}(\gamma((p,0))) - \tau_{\mathcal{Q}}(p)| < \frac{\beta}{3} \tag{70}$$
for every projection $p \in \mathcal{Q}$, whence in particular
$$\tau_{\mathcal{Q}}(\gamma((0,1_{\mathcal{Q}}))) < \frac{\beta}{3}, \tag{71}$$
and such that
$$\kappa_i := \gamma \circ (\bar{\kappa}_i \oplus \bar{\kappa}),\ i \in \{0,1\}$$
satisfy
$$(E,\pi,\sigma,\kappa_0) \sim_{(\mathcal{F},\eta)} (E,\pi,\sigma,\kappa_1).$$
With
$$y_{i,l} = \tau_{\mathcal{Q}}(\kappa_i(1_{E_{x_l}})) = \tau_{\mathcal{Q}}(\gamma(\gamma_i(\kappa_{x_l}(1_{E_{x_l}})) \oplus \tilde{\gamma}(\kappa_{x_l}(1_{E_{x_l}}))))$$
and
$$\bar{y}_{i,l} := \tau_{\mathcal{Q}}(\gamma_i(\kappa_{x_l}(1_{E_{x_l}})))$$
we have
$$\begin{aligned} |y_{i,l} - y_l^{\tau_i}| &\leq |y_{i,l} - \bar{y}_{i,l}| + |\bar{y}_{i,l} - y_l^{\tau_i}| \\ &\leq |\tau_{\mathcal{Q}}(\gamma(\gamma_i(\kappa_{x_l}(1_{E_{x_l}})) \oplus 0)) - \tau_{\mathcal{Q}}(\gamma_i(\kappa_{x_l}(1_{E_{x_l}})))| \\ &\quad + \tau_{\mathcal{Q}}(\gamma(0 \oplus \tilde{\gamma}(\kappa_{x_l}(1_{E_{x_l}})))) \\ &\quad + |\bar{y}_{i,l} - y_l^{\tau_i}| \\ &\overset{(70),(71),(21)}{\leq} \frac{\beta}{3} + \frac{\beta}{3} + \frac{\beta}{3}. \end{aligned}$$

∎

A.5.25 Lemma: *Let B be a separable unital recursive subhomogeneous* C*-*algebra and let $\mathcal{F} \subset B_+^1$ finite and $\eta > 0$ be given.*

Suppose B has an $(\mathcal{F},\eta)$-connected recursive subhomogeneous decomposition
$$[B_l, X_l, \Omega_l, r_l, \phi_l]_{l=1}^R$$
along which projections can be lifted and such that $X_l \setminus \Omega_l \neq \emptyset$ for $l \geq 1$.

Let $\tau^{(0)}, \ldots, \tau^{(n-1)} \in T(B)$ be n faithful tracial states with
$$(\tau^{(0)})_* = \ldots = (\tau^{(n-1)})_*$$
(as states on the ordered $\mathrm{K}_0(B)$*).*

Then there are
$$0 = K_0 < K_1 < \ldots < K_{n-1} = K \in \mathbb{N}$$
and pairwise orthogonal weighted $(\mathcal{F},\eta)$-excisors
$$(Q_{\frac{j}{K}}, \rho_{\frac{j}{K}}, \sigma_{\frac{j}{K}}, \kappa_{\frac{j}{K}}),\ j \in \{0,\ldots,K\},$$
implementing $(\mathcal{F},\eta)$-bridges
$$\begin{aligned} (Q_{\frac{K_0}{K}}, \rho_{\frac{K_0}{K}}, \sigma_{\frac{K_0}{K}}, \kappa_{\frac{K_0}{K}}) &\sim_{(\mathcal{F},\eta)} \ldots \sim_{(\mathcal{F},\eta)} (Q_{\frac{K_m}{K}}, \rho_{\frac{K_m}{K}}, \sigma_{\frac{K_m}{K}}, \kappa_{\frac{K_m}{K}}) \\ &\sim_{(\mathcal{F},\eta)} \ldots \sim_{(\mathcal{F},\eta)} (Q_{\frac{K_{n-1}}{K}}, \rho_{\frac{K_{n-1}}{K}}, \sigma_{\frac{K_{n-1}}{K}}, \kappa_{\frac{K_{n-1}}{K}}), \end{aligned}$$

and such that, for each projection $q \in Q_{\frac{K_m}{K}}$, $m \in \{0,\dots,n-1\}$,

$$(\tau^{(m)} \otimes \tau_{\mathcal{Q}})\sigma_{\frac{K_m}{K}}(q) \geq \frac{1}{n+1} \cdot \tau_{\mathcal{Q}}\kappa_{\frac{K_m}{K}}(q). \tag{72}$$

PROOF: Let us first prove the lemma for $n = 2$. Choose

$$0 < \bar{\alpha}, \beta, \delta < \frac{1}{n}$$

such that

$$\left(\frac{1}{n} - \delta\right) \cdot (1 - 2\bar{\alpha}) \geq \frac{1}{n+1} \tag{73}$$

and

$$\beta < \bar{\alpha} \cdot \frac{y_l^{\tau^{(i)}}}{4} \text{ for all } i \in \{0,1\},\ l \in \{1,\dots,R\} \tag{74}$$

(this is possible since $X_l \setminus \Omega_l \neq \emptyset$ and the traces are faithful, whence $y_l^{\tau^{(i)}} > 0$).

Let $(E, \pi, \sigma, \kappa_i)$ and $y_{i,l} = \tau_{\mathcal{Q}(\kappa_i(1_{E_{x_l}}))}$ for $i \in \{0,1\}$, $l \in \{1,\dots,R\}$, be as in Proposition A.5.0.

Choose

$$0 < \gamma < \beta,$$

then

$$\begin{aligned} y_{i,l} - \gamma - \beta \geq y_{i,l} - 2\beta \quad &\overset{(74)}{\geq} \quad y_{i,l} - \bar{\alpha} \cdot \frac{y_l^{\tau^{(i)}}}{2} \\ &\overset{(74)}{\geq} \quad y_{i,l} - \bar{\alpha} \cdot (y_l^{\tau^{(i)}} - \beta) \\ &\overset{(69)}{\geq} \quad (1 - \bar{\alpha}) \cdot y_{i,l}. \end{aligned} \tag{75}$$

By Proposition A.4.0, there are $(\mathcal{F},\eta)$-excisors

$$(\dot{E}_i, \dot{\rho}_i, \dot{\sigma}_i),\ i \in \{0,1\},$$

compatible with the recursive subhomogeneous decomposition, with

$$\dot{E}_i = \bigoplus\nolimits_{l=1}^{R} \dot{E}_{i,l}$$

and each $\dot{E}_{i,l}$ a direct sum of copies of M_{r_l}, cf. (17), and such that

$$\begin{aligned} &(\bar{\tau}_{i,l} \otimes \tau_{\mathcal{Q}}) \circ (\psi_l \otimes \mathrm{id}_{\mathcal{Q}}) \circ \dot{\sigma}_{i,l}(1_{\dot{E}_{i,l}}) \\ &\geq \quad y_l^{\tau^{(i)}} - \gamma \\ &\overset{(69)}{\geq} \quad y_{i,l} - \gamma - \beta \\ &\overset{(75)}{\geq} \quad (1 - \bar{\alpha}) \cdot y_{i,l}. \end{aligned} \tag{76}$$

Choose unital *-homomorphisms

$$\dot{\kappa}_i : \dot{E}_i \to \mathcal{Q},\ i \in \{0,1\},$$

such that

$$\tau_{\mathcal{Q}} \circ \dot{\kappa}_i(1_{\dot{E}_{i,l}}) = y_{i,l}$$

and

$$(\tau^{(i)}\otimes\tau_{\mathcal{Q}})\circ\dot{\sigma}_i(q)\geq(\bar{\tau}_{i,l}\otimes\tau_{\mathcal{Q}})\circ(\psi_l\otimes\mathrm{id}_{\mathcal{Q}})\circ\dot{\sigma}_{i,l}(q)\geq(1-\bar{\alpha})\cdot\tau_{\mathcal{Q}}\circ\dot{\kappa}_i(q)$$

for all projections $q\in\dot{E}_{i,l}$; it follows that

$$(\tau^{(i)}\otimes\tau_{\mathcal{Q}})\circ\dot{\sigma}_i(q)\geq(1-\bar{\alpha})\cdot\tau_{\mathcal{Q}}\circ\dot{\kappa}_i(q) \tag{77}$$

for all projections $q\in\dot{E}_i$.

Now by Proposition A.3.0, we have

$$(\dot{E}_i,\dot{\rho}_i,\dot{\sigma}_i,\dot{\kappa}_i)\sim_{(\mathcal{F},\eta)}(E,\pi,\sigma,\kappa_i) \tag{78}$$

for $i\in\{0,1\}$. By Proposition A.5.0,

$$(E,\pi,\sigma,\bar{\kappa}_0)\sim_{(\mathcal{F},\eta)}(E,\pi,\sigma,\kappa_1),$$

so by transitivity,

$$(\dot{E}_0,\dot{\rho}_0,\dot{\sigma}_0,\dot{\kappa}_0)\sim_{(\mathcal{F},\eta)}(\dot{E}_1,\dot{\rho}_1,\dot{\sigma}_1,\dot{\kappa}_1), \tag{79}$$

with a bridge consisting of $(\mathcal{F},\eta)$-excisors

$$(\dot{E}_{\frac{j}{K}},\dot{\rho}_{\frac{j}{K}},\dot{\sigma}_{\frac{j}{K}},\dot{\kappa}_{\frac{j}{K}}),\ j\in\{0,\ldots,K\},$$

for some $K\in\mathbb{N}$.

Choose pairwise orthogonal projections

$$q_0,q_{1/K},\ldots,q_1\in\mathcal{Q}$$

such that

$$\textstyle\sum_{j=0}^{K}q_{\frac{j}{K}}=1_{\mathcal{Q}}$$

and

$$\tau_{\mathcal{Q}}(q_0)=\tau_{\mathcal{Q}}(q_1)=\frac{1}{2}-\delta$$

and

$$\tau_{\mathcal{Q}}(q_{j/K})=\frac{2\delta}{K-1},\ j\in\{1,\ldots,K-1\}.$$

Choose a *-isomorphism

$$\theta:\mathcal{Q}\otimes\mathcal{Q}\to\mathcal{Q}$$

and define, for $j\in\{0,\ldots,K\}$,

$$\begin{array}{rcl} Q_{\frac{j}{K}} & := & \dot{E}_{\frac{j}{K}}, \\ \rho_{\frac{j}{K}}(\,.\,) & := & \dot{\rho}_{\frac{j}{K}}(\,.\,), \\ \sigma_{\frac{j}{K}}(\,.\,) & := & (\mathrm{id}_B\otimes\theta)\circ(\dot{\sigma}_{\frac{j}{K}}(\,.\,)\otimes q_{\frac{j}{K}}), \\ \kappa_{\frac{j}{K}} & := & \dot{\kappa}_{\frac{j}{K}}. \end{array}$$

We check that the $(Q_{\frac{j}{K}},\rho_{\frac{j}{K}},\sigma_{\frac{j}{K}},\kappa_{\frac{j}{K}})$, $j\in\{0,\ldots,K\}$, have the right properties:

Each $\sigma_{\frac{j}{K}}$ is an isometric c.p. order zero map since $\dot{\sigma}_{\frac{j}{K}}$ is and since $q_{\frac{j}{K}}$ is nonzero. The $\sigma_{\frac{j}{K}}$ have pairwise orthogonal images, since the $q_{\frac{j}{K}}$ are pairwise orthogonal.

For $i\in\{0,1\}$ and $q\in Q_i$ a projection, we have

$$\begin{array}{rcl} (\tau^{(i)}\otimes\tau_{\mathcal{Q}})(\sigma_i(q)) & = & (\tau^{(i)}\otimes\tau_{\mathcal{Q}})(\mathrm{id}_B\otimes\theta)(\dot{\sigma}_i(q)\otimes q_i) \\ & = & (\tau^{(i)}\otimes\tau_{\mathcal{Q}}\otimes\tau_{\mathcal{Q}})(\dot{\sigma}_i(q)\otimes q_i) \\ & = & \tau_{\mathcal{Q}}(q_i)\cdot(\tau^{(i)}\otimes\tau_{\mathcal{Q}})(\dot{\sigma}_i(q)) \\ & \geq & (1/2-\delta)\cdot(1-\bar{\alpha})\cdot\tau_{\mathcal{Q}}\circ\kappa_i(q) \\ & \geq & \dfrac{1}{2+1}\cdot\tau_{\mathcal{Q}}\circ\kappa_i(q). \end{array} \tag{80}$$

For $j \in \{0,\ldots,K\}$ and $b \in \mathcal{F}$,

$$\begin{aligned}
&\|\sigma_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}})(b\otimes 1_{\mathcal{Q}}) - \sigma_{\frac{j}{K}}\rho_{\frac{j}{K}}(b)\| \\
&= \|((\mathrm{id}_B\otimes\theta)(\dot\sigma_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}})\otimes q_{\frac{j}{K}}))((\mathrm{id}_B\otimes\theta)(b\otimes 1_{\mathcal{Q}}\otimes 1_{\mathcal{Q}})) \\
&\quad -(\mathrm{id}_B\otimes\theta)(\dot\sigma_{\frac{j}{K}}\dot\rho_{\frac{j}{K}}(b)\otimes q_{\frac{j}{K}})\| \\
&= \|(\dot\sigma_{\frac{j}{K}}(1_{Q_{\frac{j}{K}}})(b\otimes 1_{\mathcal{Q}}))\otimes q_{\frac{j}{K}} - \dot\sigma_{\frac{j}{K}}\dot\rho_{\frac{j}{K}}(b)\otimes q_{\frac{j}{K}}\| \\
&< \eta,
\end{aligned}$$

so each $(Q_{\frac{j}{K}},\rho_{\frac{j}{K}},\sigma_{\frac{j}{K}},\kappa_{\frac{j}{K}})$ is a weighted $(\mathcal{F},\eta)$-excisor (which is clearly compatible with the given recursive subhomogeneous decomposition).

We now turn to the case of arbitrary n. Fix (E,π,σ) as in the first part of the proof, cf. (67). Choose $\bar\alpha,\beta,\delta$ as above; we may in addition assume that

$$(\tau^{(m)}\otimes\tau_{\mathcal{Q}})\sigma(q) \ge \frac{n\beta}{\delta} \tag{81}$$

for all $m\in\{0,\ldots,n-1\}$ and all nonzero projections $q\in E$.

We now apply the first part of the proof to each pair $\tau^{(m)},\tau^{(m+1)}$, $m\in\{0,\ldots,n-2\}$. This yields for each $m\in\{0,\ldots,n-1\}$ and $i\in\{0,1\}$ $(\mathcal{F},\eta)$-excisors

$$(E,\pi,\sigma,\kappa_i^{(m)})$$

and

$$y_{i,l}^{(m)} \overset{(68)}{=} \tau_{\mathcal{Q}}(\kappa_i^{(m)}(1_{M_{r_l}})),\ l\in\{1,\ldots,R\},$$

such that

$$|y_{i,l}^{(m)} - y_l^{\tau^{(m+i)}}| \overset{(69)}{<} \beta$$

as well as $(\mathcal{F},\eta)$-excisors

$$(\dot E_i^{(m)},\dot\rho_i^{(m)},\dot\sigma_i^{(m)},\dot\kappa_i^{(m)})$$

with

$$\begin{aligned}
(E,\pi,\sigma,\kappa_0^{(m)}) &\sim^{(78)}_{(\mathcal{F},\eta)} (\dot E_0^{(m)},\dot\rho_0^{(m)},\dot\sigma_0^{(m)},\dot\kappa_0^{(m)}) \\
&\sim^{(79)}_{(\mathcal{F},\eta)} (\dot E_1^{(m)},\dot\rho_1^{(m)},\dot\sigma_1^{(m)},\dot\kappa_1^{(m)}) \\
&\sim^{(78)}_{(\mathcal{F},\eta)} (E,\pi,\sigma,\kappa_1^{(m)})
\end{aligned} \tag{82}$$

and with

$$(\tau^{(m+i)}\otimes\tau_{\mathcal{Q}})\circ\dot\sigma_i^{(m)}(q) \overset{(77)}{\ge} (1-\bar\alpha)\cdot\tau_{\mathcal{Q}}\circ\dot\kappa_i^{(m)}(q)$$

for all projections $q\in\dot E_i^{(m)}$, $m\in\{0,\ldots,n-2\}$, $i\in\{0,1\}$.

But then it is not hard to find unital *-homomorphisms

$$\kappa^{(m)},\hat\kappa_1^{(m)},\hat\kappa_0^{(m+1)} : E\to\mathcal{Q}$$

such that

$$\tfrac{1-n\beta/2}{1-(n-1)\beta/2}\cdot\kappa^{(m)} \oplus \tfrac{\beta/2}{1-(n-1)\beta/2}\cdot\hat\kappa_1^{(m)} \approx_{\mathrm{u}} \kappa_1^{(m)} \tag{83}$$

and

$$\tfrac{1-n\beta/2}{1-(n-1)\beta/2}\cdot\kappa^{(m)} \oplus \tfrac{\beta/2}{1-(n-1)\beta/2}\cdot\hat\kappa_0^{(m+1)} \approx_{\mathrm{u}} \kappa_0^{(m+1)} \tag{84}$$

(here, we use notation as in A.2.0(ii) to denote weighted sums of *-homomorphisms $E\to\mathcal{Q}$).

Combining (82), (83) and (84) with Remark A.2.0(ii), one checks that

$$\begin{aligned}
&(1-(n-1)\tfrac{\beta}{2}) \cdot (E, \pi, \sigma, \kappa_0^{(m)}) \\
&\oplus \Big(\bigoplus\nolimits_{m' \in \{0,\ldots,n-1\} \setminus \{m\}} \tfrac{\beta}{2} \cdot (E, \pi, \sigma, \hat{\kappa}_0^{(m')})\Big) \\
&\quad \sim_{(\mathcal{F},\eta)} \quad (1-(n-1)\tfrac{\beta}{2}) \cdot (E, \pi, \sigma, \kappa_0^{(m+1)}) \\
&\qquad\qquad \oplus \Big(\bigoplus\nolimits_{m' \in \{0,\ldots,n-1\} \setminus \{m+1\}} \tfrac{\beta}{2} \cdot (E, \pi, \sigma, \hat{\kappa}_0^{(m')})\Big). \qquad (85)
\end{aligned}$$

Combining (82) with (85) we see that, for all $m \in \{0, \ldots, n-2\}$,

$$\begin{aligned}
&(1-(n-1)\tfrac{\beta}{2}) \cdot (\dot{E}_0^{(m)}, \dot{\rho}_0^{(m)}, \dot{\sigma}_0^{(m)}, \dot{\kappa}_0^{(m)}) \\
&\oplus \Big(\bigoplus\nolimits_{m' \in \{0,\ldots,n-1\} \setminus \{m\}} \tfrac{\beta}{2} \cdot (E, \pi, \sigma, \hat{\kappa}_0^{(m')})\Big) \\
&\quad \sim_{(\mathcal{F},\eta)} \quad (1-(n-1)\tfrac{\beta}{2}) \cdot (\dot{E}_0^{(m+1)}, \dot{\rho}_0^{(m+1)}, \dot{\sigma}_0^{(m+1)}, \dot{\kappa}_0^{(m+1)}) \\
&\qquad\qquad \oplus \Big(\bigoplus\nolimits_{m' \in \{0,\ldots,n-1\} \setminus \{m+1\}} \tfrac{\beta}{2} \cdot (E, \pi, \sigma, \hat{\kappa}_0^{(m')})\Big). \qquad (86)
\end{aligned}$$

Note that, for any projection $q \in \dot{E}_i^{(m)}$,

$$\begin{aligned}
&(1-(n-1)\tfrac{\beta}{2}) \cdot (\tau^{(m+i)} \otimes \tau_{\mathcal{Q}}) \circ \dot{\sigma}_i^{(m)}(q) \\
&\quad \geq \quad (1-(n-1)\tfrac{\beta}{2})(1-\bar{\alpha}) \cdot \tau_{\mathcal{Q}} \circ \dot{\kappa}_i^{(m)}(q) \\
&\quad \geq \quad (1-2\bar{\alpha}) \cdot \tau_{\mathcal{Q}} \circ \dot{\kappa}_i^{(m)}(q).
\end{aligned}$$

We may therefore assume that there are

$$0 = K_0 < K_1 < \ldots < K_{n-1} = K \in \mathbb{N}$$

and an $(\mathcal{F}, \eta)$-bridge consisting of $(\mathcal{F}, \eta)$-excisors

$$(Q_{\frac{j}{K}}, \rho_{\frac{j}{K}}, \bar{\sigma}_{\frac{j}{K}}, \kappa_{\frac{j}{K}}),\ j \in \{0, \ldots, K\}$$

with

$$\begin{aligned}
&(Q_{\frac{K_m}{K}}, \rho_{\frac{K_m}{K}}, \bar{\sigma}_{\frac{K_m}{K}}, \kappa_{\frac{K_m}{K}}) \\
&\quad = \quad (1-(n-1)\tfrac{\beta}{2}) \cdot (\dot{E}_0^{(m)}, \dot{\rho}_0^{(m)}, \dot{\sigma}_0^{(m)}, \dot{\kappa}_0^{(m)}) \\
&\qquad\qquad \oplus \Big(\bigoplus\nolimits_{m' \in \{0,\ldots,n-1\} \setminus \{m\}} \tfrac{\beta}{2} \cdot (E, \pi, \sigma, \hat{\kappa}_0^{(m')})\Big)
\end{aligned}$$

for $m \in \{0, \ldots, n-1\}$.

Choose pairwise orthogonal projections

$$q_0, q_{\frac{1}{K}}, \ldots, q_1 \in \mathcal{Q}$$

such that

$$\textstyle\sum_j q_{\frac{j}{K}} = 1_{\mathcal{Q}}$$

and such that each $q_{\frac{K_m}{K}}$ can be written as a sum of two projections

$$q_{\frac{K_m}{K}} = q'_{\frac{K_m}{K}} + q''_{\frac{K_m}{K}}$$

with

$$\tau_{\mathcal{Q}}(q'_{\frac{K_m}{K}}) = 1/n - \delta,$$

$$\tau_{\mathcal{Q}}(q''_{\frac{Km}{K}}) = \delta/2$$

and such that all other projections have the same tracial value $n\delta/2K$.

As in the first part of the proof, choose an isomorphism

$$\theta : \mathcal{Q} \otimes \mathcal{Q} \to \mathcal{Q}$$

and define

$$\sigma_{\frac{j}{K}} := (\mathrm{id}_B \otimes \theta) \circ (\bar{\sigma}_{\frac{j}{K}}(\,.\,) \otimes q_{\frac{j}{K}}).$$

Then the

$$(Q_{\frac{j}{K}}, \rho_{\frac{j}{K}}, \sigma_{\frac{j}{K}}, \kappa_{\frac{j}{K}})$$

clearly are $(\mathcal{F},\eta)$-excisors implementing an $(\mathcal{F},\eta)$-bridge. (72) is now checked in a similar manner as (80), using (73) and (81). ∎

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