

How to recognize a 4-ball when you see one

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Abstract. We apply the method of filling with holomorphic discs to a 4-dimensional symplectic cobordism with the standard contact 3-sphere as one convex boundary component. We establish the following dichotomy: either the cobordism is diffeomorphic to a ball, or there is a periodic Reeb orbit of quantifiably short period in the concave boundary of the cobordism. This allows us to give a unified treatment of various results concerning Reeb dynamics on contact 3-manifolds, symplectic fillability, the topology of symplectic cobordisms, symplectic nonsqueezing, and the nonexistence of exact Lagrangian surfaces in standard symplectic 4-space.

1. INTRODUCTION

Ever since the work of Hofer [24] on the Weinstein conjecture for overtwisted contact 3-manifolds, it has been a recurrent theme in symplectic and contact topology that the noncompactness of certain moduli spaces of holomorphic discs translates into the existence of periodic Reeb orbits. For recent work in this direction see for instance [2].

The inspiration for Hofer’s approach came from Eliashberg’s method of filling with holomorphic discs [11]. In [20] we gave a detailed discussion of that method in a moduli-theoretic framework. As had been observed by Eliashberg, a filling of the 4-ball D^4 by holomorphic discs adapted to a contactomorphism of the boundary 3-sphere S^3 yields a simple proof of Cerf’s theorem that every diffeomorphism of S^3 extends to a diffeomorphism of D^4 .

In the present paper we generalize the moduli-theoretic setup from [20] to a disc-filling of a symplectic cobordism that has the standard contact 3-sphere (S^3, ξ_{st}) as one convex boundary component. Our main result, which we shall refer to as the “ball theorem”, then says the following. Either the corresponding moduli space of holomorphic discs is compact, in which case the symplectic cobordism has to be the 4-ball, or there is noncompactness caused by bubbling-off of holomorphic discs or breaking, in which case there have to be periodic Reeb orbits in the concave boundary of the symplectic cobordism.

Energy estimates on the holomorphic discs give rise to estimates on the periods of these Reeb orbits.

This ball theorem may be regarded as a generalization of the following fundamental results in 4-dimensional symplectic resp. 3-dimensional contact topology:

- Existence of periodic Reeb orbits on star-shaped hypersurfaces in \mathbb{R}^4 , hypersurfaces of contact type, and overtwisted contact 3-manifolds (Rabinowitz, Viterbo, Hofer).
- Topology of symplectic fillings of (S^3, ξ_{st}) (Gromov, Eliashberg, McDuff).
- Tightness of weakly symplectically fillable contact structures (Gromov, Eliashberg).
- Nonexistence of exact Lagrangian surfaces in \mathbb{R}^4 (Gromov).

Indeed, all these results become straightforward consequences of the ball theorem.

Our methods also yield some new results on the existence of contractible periodic Reeb orbits. Moreover, our ball theorem allows us to define a symplectic capacity via the periods of Reeb orbits on contact type hypersurfaces. A simple computation of this capacity for the 4-ball and the cylinder over the 2-ball leads to a proof of

- Symplectic nonsqueezing (Gromov).

Conversely, this capacity can be used to give estimates on the shortest Reeb period. We recover some examples of Frauenfelder–Ginzburg–Schlenk and provide additional information about the periods of contractible orbits.

A precise description of the symplectic cobordisms we are considering is given in Section 2, which also contains the statement of the ball theorem, including a variant for symplectic cobordisms with an exact symplectic form. Various corollaries of the ball theorems, including the ones we just mentioned, will be proved in Section 3. The proof of the ball theorems is given in Section 4, subject to a compactness result for the relevant moduli space of holomorphic discs. This compactness result is proved in Section 6 after a brief discussion of the Hofer energy in Section 5. It is worth pointing out that the larger part of our compactness proof only involves classical bubbling-off analysis as in [24]; the new aspect here is that we have to deal with bubbling at the boundary. For the interior bubbling-off of spheres we rely on the more sophisticated compactness results from [5]. In a final section we give a brief sketch how the filling with holomorphic discs can be applied to weak symplectic fillings of $S^2 \times S^1$ with its standard contact structure; as in the case of S^3 this allows one to classify such fillings up to diffeomorphism.

The setup here is parallel to our previous paper [20]; it therefore seems opportune to list some minor corrections to that paper in an appendix to the present one. As in [20] we write $\mathbb{D} \subset \mathbb{C}$ and $\mathbb{H} \subset \mathbb{C}$ for the *closed* unit disc and upper half-plane, respectively. In [21] we extend the results of the present paper to higher dimensions.

2. THE BALL THEOREMS

We begin with a description of the specific symplectic cobordisms that form the setting of our main theorems; see Figure 1. For the basics of symplectic cobordisms cp. [19, Chap. 5].

Let $(M_{\pm}, \xi_{\pm} = \ker \alpha_{\pm})$ be two closed 3-dimensional contact manifolds, oriented by the volume forms $\alpha_{\pm} \wedge d\alpha_{\pm}$. The symplectic cobordisms (W, ω) we want to consider are compact, connected symplectic 4-manifolds, oriented by the volume form ω^2 , with the following properties:

- (C1) (W, ω) is minimal, i.e. does not contain symplectically embedded 2-spheres of self-intersection -1 (so-called exceptional spheres).
- (C2) The boundary of W equals

$$\partial W = \overline{M}_- \sqcup M_+ \sqcup S^3$$

as oriented manifolds, where \overline{M}_- denotes M_- with the reversed orientation. One or both of M_{\pm} may be empty, and they need not be connected.

- (C3) The restriction of ω to (the tangent bundle of) M_- equals $d\alpha_-$.
- (C4) The restriction of ω to the 2-plane field $\xi_+ = \ker \alpha_+$ on M_+ is positive.
- (C5) A neighborhood of $S^3 \subset \partial W$ in (W, ω) looks like a neighborhood of $S^3 = \partial D^4$ in D^4 with the standard symplectic form $\omega_{st} = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$.

Condition (C4), with the orientation condition (C2), says that (M_+, ξ_+) is a weakly convex boundary component of (W, ω) ; cp. [19, Chap. 5] for the various notions of convex resp. concave boundaries of symplectic manifolds. The choice of contact form α_+ defining the given ξ_+ is irrelevant for our purposes.

Let

$$\lambda_{st} := \frac{1}{2}(x_1 dy_1 - y_1 dx_1 + x_2 dy_2 - y_2 dx_2)$$

be the standard primitive of the symplectic form ω_{st} , and set $\alpha_{st} = \lambda_{st}|_{TS^3}$. Condition (C5) says that S^3 with its standard contact structure $\xi_{st} = \ker \alpha_{st}$ is a strongly convex boundary of (W, ω) , with a Liouville vector field Y for ω (i.e. $L_Y \omega = \omega$) defined near $S^3 \subset W$, pointing out of W , and such that $i_Y \omega$ restricts to the contact form α_{st} on S^3 . This condition on the induced contact form, together with condition (C3), serves to normalize the contact form α_- , which allows us to speak in quantitative terms about the Reeb dynamics of α_- .

Finally, condition (C3) can be read as saying that (M_-, ξ_-) is a strongly concave boundary of (W, ω) . This is well known and can best be seen with the help of relative de Rham cohomology. With U denoting a collar neighborhood of M_- in W , the relative de Rham cohomology group $H_{dR}^2(U, M_-)$ is trivial. From the definition of relative de Rham cohomology, cp. [4, p. 78], one finds a 1-form β on U which restricts to α_- on (the tangent bundle of) M_- and such that $\omega = d\beta$ on U . The vector field Y on U defined by $i_Y \omega = \beta$ is then a Liouville vector field for ω that induces α_- on M_- and points inward by the orientation condition (C2).

Recall that the Reeb vector field $R = R_{\alpha}$ of a contact form α is defined by the equations $i_R d\alpha = 0$ and $\alpha(R) = 1$.

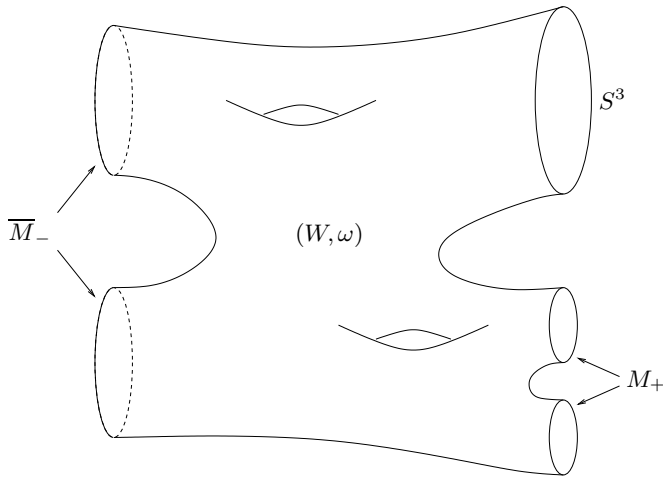


FIGURE 1. Might this be a 4-ball?

Notation. We write $\inf(\alpha)$ for the infimum of all positive periods of closed orbits of the Reeb vector field R_α . With $\inf_0(\alpha)$ we denote the infimum of all positive periods of *contractible* closed Reeb orbits.

Remark 2.1. An argument as in [26, p. 109] shows that both infima are minima, and in particular positive, unless the relevant set of Reeb orbits is empty and the infimum equal to ∞ .

We can now state our two main theorems, whose essence is the following: unless W is a 4-ball (and in particular M_\pm are empty), there must be a short Reeb orbit on (M_-, α_-) .

Theorem 2.2 (The ball theorem). *Let (W, ω) be a symplectic cobordism satisfying conditions (C1) to (C5). Then either $\inf(\alpha_-) \leq \pi$ or W is diffeomorphic to a 4-ball.*

Remark 2.3. In the case that W is a 4-ball, a theorem of Gromov [23, p. 311] implies that (W, ω) is actually *symplectomorphic* to (D^4, ω_{st}) .

The following version of the ball theorem sharpens the dichotomy under the additional requirement that the symplectic form have a suitable primitive.

Theorem 2.4 (The exact ball theorem). *Let $(W, \omega = d\lambda)$ be a symplectic cobordism satisfying conditions (C1) to (C5), and with $\lambda|_{TM_-} = \alpha_-$. Then either $\inf_0(\alpha_-) \leq \pi$ or W is diffeomorphic to a 4-ball.*

By the theorem of Stokes there are no symplectically embedded 2-spheres in an exact symplectic manifold, so condition (C1) is automatic in the exact case.

The dichotomy in the ball theorems can be exploited either way: from the nonexistence of short periodic Reeb orbits (e.g. when M_- is empty) one

can deduce topological information about symplectic cobordisms; conversely, topological information about a symplectic cobordism can be used to detect a short closed Reeb orbit. A number of such results will be derived in Section 3. In several of these corollaries the following concept and the construction that we shall describe presently play an important role.

Definition. Let (M_{\pm}, ξ_{\pm}) be closed 3-dimensional contact manifolds. A compact symplectic 4-manifold (W, ω) with oriented boundary $\partial W = \overline{M}_- \sqcup M_+$ is called a **Liouville cobordism** from M_- to M_+ if the symplectic form ω is exact and its primitive can be chosen as a contact form for ξ_{\pm} , i.e. $\omega = d\lambda$ with $\ker(\lambda|_{TM_{\pm}}) = \xi_{\pm}$.

The following example (also observed by Wendl [43]) shows that a symplectic cobordism with an exact symplectic form is not, in general, a Liouville cobordism.

Example. By the Weinstein tubular neighborhood theorem [41], the complement of a tubular neighborhood of a Lagrangian 2-torus in $(D^4, d\lambda_{\text{st}})$ is a strong symplectic cobordism from $(T^3, \ker(\cos \theta dx - \sin \theta dy))$ to (S^3, ξ_{st}) . By the exact ball theorem, however, there can be no Liouville cobordism, since the periodic Reeb orbits of the contact form $\cos \theta dx - \sin \theta dy$ are all noncontractible.

Remark 2.5. Given a symplectic cobordism (W, ω) with a weakly convex boundary component (M_+, ξ_+) and a symplectic form that is exact near M_+ , a construction of Eliashberg [13], cp. [18], allows one to modify the primitive λ of ω (defined in a neighborhood of M_+) in such a way that $\ker(\lambda|_{TM_+}) = \xi_+$. As the preceding example shows, such a modification is not possible, in general, at a concave end. Moreover, one can then arrange λ to equal a given contact form α_+ on M_+ up to a constant scale factor by first taking the symplectic completion of (W, ω) along M_+ , i.e. adding a cylindrical end of the form $([0, \infty) \times M_+, d(e^s \lambda|_{TM_+}))$, and then replacing $M_+ \equiv \{0\} \times M_+$ by a suitable graph in this cylindrical end.

Strong symplectic and Liouville cobordisms can be glued together (along a convex and a contactomorphic concave end) by using the Liouville vector field to define collar neighborhoods of the boundaries. This may require the rescaling of one of the symplectic forms by a constant and the insertion of a cylindrical tube, see [19, Prop. 5.2.5].

3. COROLLARIES OF THE BALL THEOREMS

3.1. Topology of symplectic cobordisms. Our first corollary was originally proved by McDuff [30, Thm. 1.2].

Corollary 3.2 (McDuff). *If (W, ω) is a compact symplectic 4-manifold with weakly convex boundary components only, and one of the boundary components is (S^3, ξ_{st}) , then the boundary is connected.*

Proof. After blowing down exceptional spheres in (W, ω) , cp. [31, Chap. 7], we may assume that (W, ω) is minimal. Since $H_{\text{dR}}^2(S^3) = 0$, the symplectic form ω is exact in a neighborhood of the boundary component S^3 . Hence, by Remark 2.5, ω can be modified in that neighborhood so that condition (C5) from Section 2 is satisfied (up to a constant scale factor). We are then in the situation of Theorem 2.2 with $M_- = \emptyset$. This implies $\inf(\alpha_-) = \infty$, and so the theorem tells us that $W \cong D^4$, which means $M_+ = \emptyset$. \square

This proof also yields the following variant of a result due to Gromov [23, p. 311], Eliashberg [11, Thm. 5.1] and McDuff [29, Thm. 1.7].

Corollary 3.3 (Gromov, Eliashberg, McDuff). *Any minimal weak symplectic filling of (S^3, ξ_{st}) is diffeomorphic to the 4-ball.* \square

Before we turn to the next corollary, we recall a statement about symplectic cobordisms that will be used in the proof of that and other corollaries. This statement is originally due to Etnyre–Honda [15]; here we give a proof based on the surgery presentation theorem [7] for contact 3-manifolds.

Theorem 3.4 (Etnyre–Honda). *Let $(M_{\text{ot}}, \xi_{\text{ot}})$ be an overtwisted contact 3-manifold, and (M, ξ) any contact 3-manifold, where both manifolds are assumed to be closed and connected. Then there is a Liouville cobordism from $(M_{\text{ot}}, \xi_{\text{ot}})$ to (M, ξ) .*

Proof. It suffices to show that (M, ξ) can be obtained from $(M_{\text{ot}}, \xi_{\text{ot}})$ by a sequence of Legendrian surgeries (or contact (-1) -surgeries in the sense of [7]), since such surgeries translate into a Liouville cobordism from the original to the surgered manifold.

By the classical surgery presentation theorem for 3-manifolds due to Lickorish and Wallace, there is a sequence of integer surgeries that gets us from M_{ot} to M . By a result of Eliashberg [12], cp. [19, Chap. 6.3], in an overtwisted contact manifold we can choose a Legendrian realization of the surgery link in such a way that the desired integer surgeries correspond to contact (-1) -surgeries along the components of the Legendrian link. This yields a Liouville cobordism from $(M_{\text{ot}}, \xi_{\text{ot}})$ to (M, ξ') , where ξ' is some contact structure on M . By adding homotopically trivial Lutz twists on $(M_{\text{ot}}, \xi_{\text{ot}})$ away from the surgery link (this does not change ξ_{ot} by Eliashberg’s classification [10] of overtwisted contact structures), we can ensure that ξ' is likewise overtwisted. This means that ξ' can be obtained from ξ by performing topologically trivial Lutz twists, which can be realized as contact $(+1)$ -surgeries [7]. Conversely, by the cancellation lemma from [7, Sec. 3], cp. [19, Prop. 6.4.5], (M, ξ) is obtained from (M, ξ') by contact (-1) -surgeries. \square

The part of the next corollary concerning overtwisted contact structures is contained in the work of Hofer [24].

Corollary 3.5 (Hofer). *If ξ_- is a contact structure on a closed 3-manifold M_- that can be defined by a contact form α_- without contractible periodic*

Reeb orbits, then there is no Liouville cobordism from (M_-, ξ_-) to a not necessarily connected contact 3-manifold with at least one overtwisted component. In particular, $\xi_- = \ker \alpha_-$ is tight, i.e. not overtwisted.

Proof. Suppose $(M_-, \xi_- = \ker \alpha_-)$ admits a Liouville cobordism to a contact manifold having an overtwisted component $(M_{\text{ot}}, \xi_{\text{ot}})$. According to Theorem 3.4, there is a Liouville cobordism from $(M_{\text{ot}}, \xi_{\text{ot}})$ to (S^3, ξ_{st}) . By gluing this Liouville cobordism to the given one (and modifying the boundaries inside the symplectic completion as in Remark 2.5), we obtain a cobordism as in the exact ball theorem, up to constant scale of the contact forms on the boundary. That theorem then guarantees the existence of a contractible periodic Reeb orbit for α_- . \square

Remark 3.6. Observe that the essence of Corollary 3.5 is that any contact form defining an overtwisted contact structure on a closed 3-manifold has a contractible periodic Reeb orbit.

3.7. Tightness and fillability. The following corollary belongs to Gromov [23, 2.4.D₂] and Eliashberg [9, Thm. 3.2.1].

Corollary 3.8 (Gromov, Eliashberg). *Let ξ be a contact structure on a closed 3-manifold M . If (M, ξ) is weakly symplectically fillable, then ξ is tight.*

Proof. We argue by contradiction. Assume that we have a weak symplectic filling (W_1, ω_1) of an overtwisted contact 3-manifold (M, ξ) . Let (W_2, ω_2) be a compact symplectic 4-manifold with disconnected boundary of contact type (i.e. a strong filling), as constructed in [30] or [17]. Take the boundary connected sum of (W_1, ω_1) with (W_2, ω_2) along (M, ξ) and one of the boundary components of (W_2, ω_2) . The result will be a weak symplectic filling of a disconnected contact manifold, one of whose components is an overtwisted contact manifold $(M_{\text{ot}}, \xi_{\text{ot}})$.

The Liouville cobordism from $(M_{\text{ot}}, \xi_{\text{ot}})$ to (S^3, ξ_{st}) from Theorem 3.4 is made up of symplectic 2-handles; by [19, Lemma 6.5.2] such a cobordism can also be attached to a weak filling. The resulting symplectic manifold contradicts Corollary 3.2. \square

3.9. Lagrangian surfaces in \mathbb{R}^4 . Let $i : \Sigma \hookrightarrow (W, \omega = d\lambda)$ be a Lagrangian embedding into an exact symplectic manifold, i.e. $i^*\omega = 0$, which means that $i^*\lambda$ is closed, and $\dim \Sigma = (\dim W)/2$. Such an embedding is called **exact** if $i^*\lambda$ is an exact 1-form.

Gromov [23, Cor. 2.3.B₂] has shown that there are no closed exact Lagrangian submanifolds in \mathbb{R}^{2n} with its standard symplectic structure. The exact ball theorem allows us to prove this result in $(\mathbb{R}^4, d\lambda_{\text{st}})$.

Corollary 3.10 (Gromov). *There are no closed exact Lagrangian surfaces in standard symplectic 4-space.*

Proof. A bundle-theoretic argument shows that a necessary condition for a closed surface Σ to admit a Lagrangian embedding in \mathbb{R}^4 is that Σ be a torus or

a nonorientable surface of Euler characteristic divisible by 4, cp. [3, Sec. 3.2]. Moreover, all these surfaces, except the Klein bottle, actually admit a Lagrangian embedding [22, 34, 38].

We prove the corollary by contradiction. Our argument applies to all surfaces of genus at least 1, which by the foregoing remark covers all potential cases. Thus, suppose that $i : \Sigma \hookrightarrow (\mathbb{R}^4, d\lambda_{\text{st}})$ is an exact Lagrangian embedding of such a surface Σ . Write $i^*\lambda_{\text{st}} = df$ for some smooth function f on Σ . By Weinstein's neighborhood theorem [41], a small tubular neighborhood U of $i(\Sigma)$ in $(\mathbb{R}^4, \omega_{\text{st}})$ may be identified symplectomorphically with a neighborhood of the zero section in the cotangent bundle $T^*\Sigma$ with its standard symplectic form $d\lambda_0$, where λ_0 is the Liouville 1-form $\mathbf{p} d\mathbf{q}$ (in local coordinates \mathbf{q} on Σ and their dual coordinates \mathbf{p}).

Under the identification provided by Weinstein's theorem, we regard λ_0 as a 1-form defined on $U \subset \mathbb{R}^4$. Equip Σ with the Riemannian metric of constant curvature $K \leq 0$ (here we use the genus restriction); this induces a bundle metric on $T^*\Sigma$. Choose a smaller tubular neighborhood U_0 that corresponds to a disc neighborhood $\{\|\mathbf{p}\| < \varepsilon\}$ in the Weinstein model and whose closure is contained in U . Then the boundary ∂U_0 is transverse to the radial Liouville vector field $\mathbf{p} \partial_{\mathbf{p}}$ for $d\lambda_0$ in the Weinstein model. In particular, the restriction of λ_0 to $T\partial U_0$ is a contact form.

We have $d(\lambda_{\text{st}} - \lambda_0) = 0$ on U . This implies that the 1-form $\lambda_{\text{st}} - \lambda_0$ represents a de Rham cohomology class $[\lambda_{\text{st}} - \lambda_0] \in H_{\text{dR}}^1(U) \cong H_{\text{dR}}^1(\Sigma)$. Moreover, we have $i^*\lambda_{\text{st}} = df$ and $i^*\lambda_0 = 0$, hence $i^*[\lambda_{\text{st}} - \lambda_0] = 0 \in H_{\text{dR}}^1(\Sigma)$. It follows that $[\lambda_{\text{st}} - \lambda_0] = 0 \in H_{\text{dR}}^1(U)$, so there is a smooth function g on U such that $\lambda_{\text{st}} - \lambda_0 = dg$. Let \tilde{g} be a smooth interpolation between 0 on U_0 and g near the boundary of U . Then $\lambda_0 + d\tilde{g}$ defines a primitive of ω_{st} that coincides with λ_0 on U_0 , and with λ_{st} near the boundary of U , and so extends to a global primitive λ of ω_{st} .

Now let $S_R^3 \subset \mathbb{R}^4$ be the sphere of radius R (centred at 0), where R is chosen so large that U is contained in the interior of S_R^3 . Then the complement of U_0 in the 4-ball D_R^4 of radius R with the symplectic form $d\lambda$ constitutes a Liouville cobordism between $(\partial U_0, \lambda_0|_{T\partial U_0})$ and $(S^3, R^2\alpha_{\text{st}})$. So the desired contradiction will follow from the exact ball theorem, provided we can show there are no contractible periodic Reeb orbits on ∂U_0 .

The Reeb flow on ∂U_0 corresponds to the geodesic flow on the unit tangent bundle of Σ , cp. [19, Thm. 1.5.2]. Hence, a contractible periodic Reeb orbit would correspond to a contractible closed geodesic on Σ , which in turn would lift to a closed geodesic on the universal cover of Σ . This is clearly impossible when that cover is the Euclidean or hyperbolic plane. \square

3.11. Reeb dynamics. In [37] Rabinowitz showed the existence of periodic solutions of the Hamiltonian equation on \mathbb{R}^{2n} on any star-shaped level surface of any given Hamiltonian function. This led Weinstein [42] to conjecture (in modern parlance) the existence of closed Reeb orbits on arbitrary contact type hypersurfaces in symplectic manifolds.

In dimension 3 this conjecture has been resolved positively by Taubes [39], using Seiberg–Witten–Floer theory. Our ball theorems allow us to retrace most of the earlier results in the history of the Weinstein conjecture, and they yield new existence statements about *contractible* periodic orbits.

Corollary 3.12 (Rabinowitz). *Let $S \subset \mathbb{R}^4$ be a smooth hypersurface bounding a domain star-shaped with respect to $0 \in \mathbb{R}^4$. Then the contact form $\lambda_{\text{st}}|_{TS}$ has a closed and obviously contractible Reeb orbit.*

Proof. Let $S_R^3 \subset \mathbb{R}^4$ be the sphere of radius R (centred at 0), where R is chosen so large that S is contained in the interior of S_R^3 . Then the region W between S and S_R^3 with the symplectic form $d\lambda_{\text{st}}$ constitutes a symplectic cobordism between $(S, \lambda_{\text{st}}|_{TS})$ and $(S^3, R^2\alpha_{\text{st}})$ as in the ball theorems. \square

The contact structure $\ker(\lambda_{\text{st}}|_{TS})$ in the theorem of Rabinowitz is always diffeomorphic to the standard tight contact structure ξ_{st} on S^3 . Hofer [24] was the first to prove the Weinstein conjecture for arbitrary contact forms on S^3 .

Corollary 3.13 (Hofer). *The Reeb vector field of any contact form on S^3 has a periodic Reeb orbit.*

Proof. For contact forms defining an overtwisted contact structure, this is contained in Corollary 3.5.

For tight contact structures one has the following argument from [24]. As shown by Eliashberg [14], there is a unique positive, (co-)oriented tight contact structure on S^3 up to isotopy. Thus, if α is a contact form defining any tight contact structure, there is a diffeomorphism φ of S^3 such that $\varphi^*\alpha = f\alpha_{\text{st}}$ for some smooth function $f : S^3 \rightarrow \mathbb{R}^+$.

Now consider the star-shaped hypersurface

$$S := \{\sqrt{f(p)}p \mid p \in S^3\}.$$

According to Corollary 3.12, the contact form $\lambda_{\text{st}}|_{TS}$ has a periodic Reeb orbit. Under the map $S^3 \rightarrow S$, $p \mapsto \sqrt{f(p)}p$, the 1-form $\lambda_{\text{st}}|_{TS}$ pulls back to $f\alpha_{\text{st}}$. So $f\alpha_{\text{st}}$ and hence α likewise have periodic Reeb orbits. \square

In a different direction, the result of Rabinowitz has been extended by Viterbo [40].

Definition. A hypersurface M in a symplectic manifold (W, ω) is said to be of **contact type** (or locally ω -convex) if there is a Liouville vector field for ω defined near and transverse to M . The hypersurface is said to be of **restricted contact type** (or globally ω -convex) if the Liouville vector field is defined on all of W .

Viterbo proved the Weinstein conjecture for compact contact type hypersurfaces in standard symplectic \mathbb{R}^{2n} . Our ball theorem covers this result in dimension 4.

Corollary 3.14 (Viterbo). *Let $M_- \subset (\mathbb{R}^4, \omega_{\text{st}})$ be a smooth compact hypersurface and Y a Liouville vector field for ω_{st} defined near and transverse to M_- . Then the contact form $\alpha_- := (i_Y \omega_{\text{st}})|_{TM_-}$ has a periodic Reeb orbit.*

Proof. Without loss of generality we take M_- to be connected. Then this hypersurface separates \mathbb{R}^4 into a bounded and an unbounded part. Choose a large sphere S_R^3 containing M_- in the interior, and write W for the part between M_- and S_R^3 .

The Liouville vector field Y near M_- points into W , otherwise Corollary 3.2 would be violated. So the ball theorem applies. \square

Remark 3.15. A neighborhood of $M_- \subset (\mathbb{R}^4, \omega_{\text{st}})$ looks like a neighborhood of $\{0\} \times M_-$ in the symplectization $(\mathbb{R} \times M_-, d(e^s \alpha_-))$. So we can form the symplectic manifold $(-\infty, 0] \times M_- \cup_{M_-} W$, with W as in the preceding proof and symplectic form $d(e^s \alpha_-)$ on $(-\infty, 0] \times M_-$. Any contact form defining the contact structure $\ker \alpha_-$ can be realized, up to a constant scale, on a graph in this half-symplectization. So the theorem holds for any such contact form.

For hypersurfaces of restricted contact type we get a stronger result.

Corollary 3.16. *Let $M_- \subset (\mathbb{R}^4, \omega_{\text{st}})$ be a smooth compact hypersurface and Y a Liouville vector field for ω_{st} defined on all of \mathbb{R}^4 and transverse to M_- . Then the contact form $\alpha_- := (i_Y \omega_{\text{st}})|_{TM_-}$ has a contractible periodic Reeb orbit.*

Proof. Choose S_R^3 as in the preceding proof. The symplectic form ω_{st} has the two global primitives $i_Y \omega_{\text{st}}$ and λ_{st} . Since $H_{\text{dR}}^1(S_R^3) = 0$, the difference $i_Y \omega_{\text{st}} - \lambda_{\text{st}}$ is exact in a neighborhood of S_R^3 . So we can easily construct a primitive of ω_{st} that coincides with $i_Y \omega_{\text{st}}$ near M_- and with λ_{st} near S_R^3 . Then the result follows from the exact ball theorem. \square

Implicit in that argument is the simple observation that a hypersurface of contact type with $H_{\text{dR}}^1 = 0$ is automatically of restricted contact type. But there are examples of hypersurfaces of restricted contact type with $H_{\text{dR}}^1 \neq 0$, for instance the connected sum of copies of $S^2 \times S^1$; this example can be constructed with the help of [28, Théorème 1]. So the existence of a *contractible* periodic orbit is not just a consequence of topology. On the other hand, the connected sum of copies of $S^2 \times S^1$ has nontrivial second homotopy group. So here the existence of a contractible periodic Reeb orbit also follows from Hofer's work [24, Thm. 9].

The next proposition gives a further surgical construction of contact manifolds having contractible periodic Reeb orbits.

Proposition 3.17. *Let (M, ξ) be a closed contact 3-manifold that is obtained from (S^3, ξ_{st}) by contact (+1)-surgery along a Legendrian link. Then every contact form defining ξ has a contractible periodic Reeb orbit.*

Proof. By the cancellation lemma [19, Prop. 6.4.5] the assumption of the proposition is equivalent to saying that (S^3, ξ_{st}) can be obtained from (M, ξ) by contact (-1)-surgeries. This means that there is a Liouville cobordism

from (M, ξ) to (S^3, ξ_{st}) . As described at the end of Section 2, this allows us to build a cobordism as in the exact ball theorem for any choice of contact form defining ξ . \square

Here are three examples to which this proposition applies.

Examples. (1) Contact (+1)-surgery on (S^3, ξ_{st}) along a standard Legendrian unknot yields $S^2 \times S^1$ with its standard contact structure (as described in Section 7 below), see [8, Lemma 4.3]. For this example the existence of a contractible periodic Reeb orbit also follows directly from [24], where Hofer proved the Weinstein conjecture for 3-manifolds with nontrivial second homotopy group.

(2) Contact (+1)-surgery on the Legendrian realization of the right-handed trefoil as in Figure 2 (showing the front projection of that Legendrian knot) produces a tight contact structure on the Brieskorn manifold $\Sigma(2, 3, 4)$ with the opposite of its natural orientation, see [36, p. 206]. The universal cover is S^3 , see [33], so $\Sigma(2, 3, 4)$ has trivial second homotopy group.

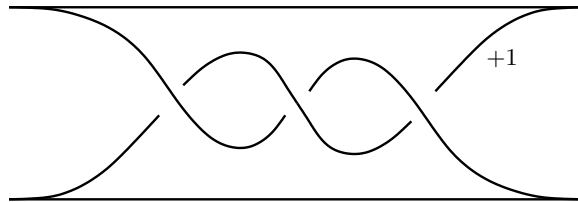


FIGURE 2. A tight contact structure on $\bar{\Sigma}(2, 3, 4)$.

(3) Figure 12.4 of [36] gives an example of a tight contact structure on the circle bundle of Euler number 2 over the torus, obtained by performing contact (+1)-surgeries on a Legendrian link in (S^3, ξ_{st}) . The second homotopy group of this manifold is trivial, since its universal cover is \mathbb{R}^3 .

Remark 3.18. Example (2) is finitely covered by the 3-sphere; example (3) is virtually overtwisted, i.e. finitely covered by an overtwisted contact manifold. On these covers, a contractible Reeb orbit is guaranteed by Corollaries 3.13 and 3.5, respectively. This implies the existence of a contractible periodic Reeb orbit downstairs. This contractible orbit may be a multiply covered one. In example (3) even the singly covered orbit will be contractible, since the fundamental group is torsion-free.

3.19. Capacities and nonsqueezing. Let (V, ω) be any 4-dimensional symplectic manifold. The manifold V may be noncompact and disconnected. For simplicity we assume that V does not have boundary; otherwise replace V by $\text{Int } V$ in the following definitions. We define the following symplectic invariant of (V, ω) :

$$c(V, \omega) := \sup_{(M, \alpha)} \{ \inf(\alpha) \mid \exists \text{ contact type embedding } (M, \alpha) \hookrightarrow (V, \omega) \}.$$

Here the supremum is taken over all closed, but not necessarily connected contact 3-manifolds (M, α) . By a contact type embedding $j : (M, \alpha) \hookrightarrow (V, \omega)$ we mean that there is a Liouville vector field Y for ω defined near $j(M)$ such that $j^*(i_Y \omega) = \alpha$.

When $\omega = d\lambda$ is exact, we can define the following invariant:

$$c_0(V, \lambda) := \sup_{(M, \alpha)} \{ \inf_0(\alpha) \mid \exists \text{ embedding } j : (M, \alpha) \hookrightarrow (V, d\lambda) \text{ with } j^* \lambda = \alpha \}.$$

In other words, here the supremum is taken over (M, α) admitting a restricted contact type embedding j into $(V, d\lambda)$, where the global primitive λ is fixed *a priori*.

In \mathbb{R}^4 with the standard symplectic form $\omega_{\text{st}} = d\lambda_{\text{st}}$ let B_r^4 be the open 4-ball of radius r and $Z_r = B_r^2 \times \mathbb{R}^2$ the cylinder over the open 2-ball of radius r . For $r = 1$ we simply write B^4 and Z , respectively.

Proposition 3.20. *The invariants $c(V, \omega)$ and $c_0(V, \lambda)$ are symplectic capacities, i.e. they satisfy the following axioms:*

Monotonicity: *If there exists a symplectic embedding $(V, \omega) \hookrightarrow (V', \omega')$, then $c(V, \omega) \leq c(V', \omega')$; similarly $c_0(V, \lambda) \leq c_0(V', \lambda')$ if there exists a symplectic embedding $(V, d\lambda) \hookrightarrow (V', d\lambda')$ pulling back λ' to λ .*

Conformality: *$c(V, a\omega) = a c(V, \omega)$ and $c_0(V, a\lambda) = a c_0(V, \lambda)$ for any $a \in \mathbb{R}^+$.*

Normalization: *$c(B^4) = c_0(B^4) = c(Z) = c_0(Z) = \pi$.*

Proof. Monotonicity and conformality are obvious from the definition. Write S_r^3 for the 3-sphere of radius r , and denote $\lambda_{\text{st}}|_{TS_r^3}$ by α_r . The Reeb vector field R_r of α_r is given by

$$R_r = \frac{2}{r^2} (x_1 \partial_{y_1} - y_1 \partial_{x_1} + x_2 \partial_{y_2} - y_2 \partial_{x_2});$$

this has length $2/r$. All the orbits of R_r are closed of length $2\pi r$, so the period is πr^2 . Since (S_r^3, α_r) for $r < 1$ has a (strict) contact type embedding into the four manifolds we are considering (with the symplectic form ω_{st} and the global primitive λ_{st}), all four capacities are bounded from below by π .

Suppose we have a (strict) contact type embedding $j : (M, \alpha) \hookrightarrow B^4$. Then we get a cobordism from $j(M)$ to S^3 as in the (exact) ball theorem, and these theorems tell us that $\inf(\alpha), \inf_0(\alpha) \leq \pi$, since the cobordism is not a ball. This concludes the proof of $c(B^4) = c_0(B^4) = \pi$.

If we have a (strict) contact type embedding $j : (M, \alpha) \hookrightarrow Z$, the image $j(M)$ is contained inside an ellipsoid

$$E(1, b) = \left\{ x_1^2 + y_1^2 + \frac{x_2^2 + y_2^2}{b^2} \leq 1 \right\}$$

for $b > 0$ sufficiently large. The boundary of this ellipsoid has a foliation by 2-dimensional ellipsoids $E^t := \partial E(1, b) \cap \{y_2 = t\}$, $t \in (-b, b)$, outside the two singular points $(0, 0, 0, \pm b)$, just as the foliation of S^3 by 2-spheres S^t that we are going to consider in the proof of the ball theorems in the next section.

Moreover, the relevant energy estimate in Proposition 5.1 below only depends on the fact that the projection of S^t to the x_1y_1 -plane is contained in the unit disc, which is also true for the projection of E^t . In other words, the ball theorems remain true with the convex boundary component S^3 replaced by $\partial E(1, b)$. Now, as before, this gives the upper bound π on the two capacities of Z and completes the proof of the proposition. \square

For a survey on other types of symplectic capacities see [6].

Remark 3.21. The same proof applies to show that $c_0(B, \lambda) = \pi$ for any primitive λ of ω_{st} with $\lambda = \lambda_{\text{st}}$ near ∂B^4 . For $\lambda = \lambda_{\text{st}}$ near ∂Z one can only deduce $c_0(Z, \lambda) \leq \pi$.

Gromov’s celebrated nonsqueezing theorem [23, p. 310] is now, in dimension 4, an immediate consequence of Proposition 3.20.

Corollary 3.22 (Gromov). *There is a symplectic embedding $B_r^4 \hookrightarrow Z_R$ if and only if $r \leq R$.*

Proof. The 4-ball B_r^4 with the symplectic form ω_{st} is symplectomorphic to the unit ball with the symplectic form $r^2\omega_{\text{st}}$. Hence $c(B_r^4) = \pi r^2$ by conformality. Similarly we have $c(Z_R) = \pi R^2$. Now the result follows from monotonicity. \square

3.23. Quantitative Reeb dynamics. With the capacities introduced in the preceding section we can derive some simple quantitative results on shortest Reeb orbits. Frauenfelder, Ginzburg and Schlenk [16, Rem. 1.13.3] show that an upper bound on the period of the shortest closed Reeb orbit on a compact hypersurface $M \subset (\mathbb{R}^{2n}, \omega_{\text{st}})$ of diameter $\text{diam}(M)$ is $\pi(\text{diam}(M))^2$. We recover their result in dimension 4, where we improve the constant by a reference to [27].

Corollary 3.24. *Let $(M, \alpha) \subset (\mathbb{R}^4, \omega_{\text{st}})$ be a compact hypersurface of contact type. Then $\inf(\alpha) \leq (2/5)\pi(\text{diam}(M))^2$.*

Proof. Since the symplectic form ω_{st} is translation-invariant, we have a contact type embedding of (M, α) into B_r^4 for any r greater than the circumradius of M , which by [27, p. 257] is (in dimension 4) at most equal to $\sqrt{2/5} \text{diam}(M)$. Hence

$$\inf(\alpha) \leq c(B_r^4) = \pi r^2 \text{ for any } r > \sqrt{2/5} \text{diam}(M). \quad \square$$

Remark 3.25. The upper bound $\sqrt{2/5} \text{diam}(M)$ for the circumradius (in dimension 4) is optimal; it is attained for the regular 4-simplex.

In view of Remark 3.21, the same argument with the capacity c replaced by c_0 gives the next corollary.

Corollary 3.26. *Let $(M, \alpha) \subset (\mathbb{R}^4, \omega_{\text{st}})$ be a compact hypersurface of restricted contact type. Then $\inf_0(\alpha) \leq (2/5)\pi(\text{diam}(M))^2$.* \square

For star-shaped hypersurfaces we have an alternative estimate.

Corollary 3.27. *On the star-shaped hypersurface*

$$S := \{\sqrt{f(p)}p \mid p \in S^3\} \subset \mathbb{R}^4,$$

with $f : S^3 \rightarrow \mathbb{R}^+$ a smooth function, we have $\inf_0(\lambda_{\text{st}}|_{TS}) \leq \pi \max f$. \square

Remark 3.28. As shown in [26, Sec. 3.5], on convex hypersurfaces in \mathbb{R}^{2n} the minimal period $\inf(\alpha)$ equals the Hofer–Zehnder capacity. In particular, this provides a lower bound on $\inf(\alpha)$ in terms of the inradius, i.e. the radius of the largest ball that can be embedded in the domain bounded by the hypersurface. The example of the “Bordeaux bottle” *loc.cit.* shows that for the class of star-shaped hypersurfaces there is no lower bound on $\inf(\alpha)$ in terms of the inradius.

4. PROOF OF THE BALL THEOREMS

Let (W, ω) be a symplectic cobordism as in one of the ball theorems. For the time being, only the conditions (C1) to (C5) on (W, ω) common to both theorems will be relevant. On some collar neighborhood $[0, \varepsilon) \times M_- \subset W$ of the strongly concave boundary M_- the symplectic form can be written as $\omega = d(e^s \alpha_-)$. We define a family of symplectic completions $(\widetilde{W}, \omega_\tau)$ of (W, ω) along M_- similar to [25, Sec. 2.2] as follows. Consider the family of functions

$$\mathcal{T} := \{\tau \in C^\infty((-\infty, \varepsilon), \mathbb{R}^+) \mid \tau' > 0, \tau(s) = e^s \text{ for } s \in [0, \varepsilon)\}.$$

Let \widetilde{W} be the manifold obtained from W by attaching infinite half-cylinders along the boundary M_- , i.e.

$$\widetilde{W} := (-\infty, 0] \times M_- \cup_{M_-} W,$$

where $M_- \subset \partial W$ is identified with $\{0\} \times M_-$ in the half-cylinder $(-\infty, 0] \times M_-$. Then define the symplectic form ω_τ on \widetilde{W} by

$$\omega_\tau := \begin{cases} \omega & \text{on } W, \\ d(\tau \alpha_-) & \text{on } (-\infty, 0] \times M_-. \end{cases}$$

Next we choose an almost complex structure J on \widetilde{W} compatible with each ω_τ and subject to the following conditions:

- (J1) Under the identification of a collar neighborhood of $S^3 \subset W$ in (W, ω) with a neighborhood of $S^3 = \partial D^4$ in $(D^4, \omega_{\text{st}})$, as stipulated by condition (C5), J looks like the standard complex structure on \mathbb{C}^2 .
- (J2) On the cylindrical end $(-\infty, \varepsilon) \times M_-$, the almost complex structure is cylindrical and symmetric in the sense of [5, p. 802, 807], i.e. it preserves ξ_- and satisfies $J\partial_s = R_{\alpha_-}$.
- (J3) Extend ξ_+ to a rank-2 distribution (still denoted ξ_+) in the tangent bundle TW over a neighborhood of M_+ in W such that $\omega|_{\xi_+} > 0$. Choose J on this neighborhood such that ξ_+ and its ω -orthogonal complement are J -invariant. In particular, the boundary M_+ is then J -convex.

(J4) Outside the regions described in (J1) to (J3), the almost complex structure is required to be chosen in such a way that J is regular for spheres (globally on \widetilde{W}) in the sense of [32, Def. 3.1.4], cp. the following remark (2).

Remarks 4.1. (1) By [20, Rem. 4.3], condition (J3) implies that M_+ can be written as the level set of a smooth function on W that is strictly plurisubharmonic in a neighborhood of M_+ . Then the maximum principle holds in that neighborhood.

(2) A choice of J as required by (J4) is possible by [32, Rem. 3.2.3]. By that remark, all that is required to achieve regularity for spheres is that no sphere lie entirely in the regions where J is prescribed by one of the conditions (J1) to (J3). Indeed, no such sphere can exist, since in all these regions the maximum principle applies. The proof of the relevant result [32, Thm. 3.1.5] only needs to be modified in one place in order to account for the noncompactness of \widetilde{W} : instead of requiring the condition $\|du\|_\infty \leq K$ (condition (3.2.3) in [32]) to hold globally, we only impose this condition on curves u with image in the subset $[-K, 0] \times M_- \cup_{M_-} W$. On each of these compact manifolds one has an open and dense set of regular almost complex structures, and one can then pass to the intersection of these sets over all $K > 0$ as in [32].

(3) According to [32, Thm. 3.1.5], the dimension of the moduli space of simple J -holomorphic spheres (quotiented by the 6-dimensional automorphism group of $S^2 = \mathbb{C}P^1$) in the homology class A is given by $2c_1(A) - 2$. Hence, if $A \in H_2(\widetilde{W}; \mathbb{Z})$ is represented by a nonconstant holomorphic sphere, then $c_1(A) \geq 1$.

We now want to introduce a moduli space of J -holomorphic discs in \widetilde{W} whose boundary is required to lie in $S^3 \subset \partial W$ (subject to a varying totally real boundary condition). For this we need to recall some notation from [20].

We begin with the unit sphere S^3 in \mathbb{C}^2 with complex Cartesian coordinates $(z_1 = x_1 + iy_1, z_2 = x_2 + iy_2)$. Let H be the height function on S^3 given by projection onto the y_2 -coordinate. For $t \in (-1, 1)$ the level sets $S^t := H^{-1}(t)$ define a smooth foliation of $S^3 \setminus \{(0, 0, 0, \pm 1)\}$ by 2-spheres. We regard the points

$$q_\pm^t := (0, 0, \pm\sqrt{1-t^2}, t)$$

as the poles of these 2-spheres.

This family of poles, together with the two poles $(0, 0, 0, \pm 1)$ of S^3 , forms an unknot

$$K := \{(0, 0, \pm\sqrt{1-t^2}, t) \mid t \in [-1, 1]\}$$

in S^3 . The complement $S^3 \setminus K$ is foliated by circles that bound holomorphic discs

$$D_s^t := D^4 \cap (\mathbb{C} \times \{x_2 = s, y_2 = t\}), \quad |t| < 1, \quad |s| < \sqrt{1-t^2}.$$

For each $t \in (-1, 1)$, the circles ∂D_s^t foliate the punctured 2-sphere $S^t \setminus \{q_\pm^t\}$.

For $|t| < 1$ and $|s| < \sqrt{1 - t^2}$ define a smooth real-valued function

$$\theta(s, t) := \frac{t}{2\sqrt{1 - t^2}} \cdot \ln \left(\frac{\sqrt{1 - t^2} + s}{\sqrt{1 - t^2} - s} \right).$$

For each t this defines a diffeomorphism from $(-\sqrt{1 - t^2}, \sqrt{1 - t^2})$ to \mathbb{R} . Now consider the parametrizations

$$u_s^t(z) := (\sqrt{1 - s^2 - t^2} \cdot e^{i\theta(s,t)} \cdot z, s, t), \quad z \in \mathbb{D},$$

of the holomorphic discs D_s^t . The rotation factor $e^{i\theta(s,t)}$ has been chosen in such a way that for each fixed $t \in (-1, 1)$ and $z \in \mathbb{D}$, the map $s \mapsto u_s^t(z)$, $|s| < \sqrt{1 - t^2}$, is a parametrization of a leaf of the characteristic foliation on $S^t \setminus \{q_{\pm}^t\}$ induced by the standard contact structure ξ_{st} on S^3 . The three leaves corresponding to $z = i^k$, $k = 0, 1, 2$, will be denoted by ℓ_k^t . These leaves will be used to put a restriction on three marked points of the holomorphic discs in our moduli space, which amounts to quotienting out the noncompact 3-dimensional automorphism group of \mathbb{D} .

For $|s|$ sufficiently close to $\sqrt{1 - t^2}$, the image of the holomorphic disc u_s^t will lie in the neighborhood of $S^3 \subset D^4$ that has been identified with a neighborhood of $S^3 \subset \widetilde{W}$. These discs define a relative homotopy class $A^t \in \pi_2(\widetilde{W}, S^t \setminus \{q_{\pm}^t\})$. We now always take the holomorphic identification between a neighborhood of $S^3 = \partial D^4$ in D^4 and $S^3 \subset \partial \widetilde{W}$ in \widetilde{W} for granted.

Definition. A t -level Bishop disc is a smooth (up to the boundary) J -holomorphic map

$$u^t : (\mathbb{D}, \partial\mathbb{D}) \longrightarrow (\widetilde{W}, S^t \setminus \{q_{\pm}^t\}),$$

i.e. a solution of the Cauchy–Riemann equation

$$\partial_x u + J(u)\partial_y u = 0,$$

satisfying the following conditions:

(D1) $[u^t] = A^t \in \pi_2(\widetilde{W}, S^t \setminus \{q_{\pm}^t\})$.

(D2) $u^t(i^k) \in \ell_k^t$, $k = 0, 1, 2$.

The collection

$$\mathcal{W} := \{u^t \mid t \in (-1, 1), u^t \text{ is a } t\text{-level Bishop disc}\}$$

of all such discs is the **moduli space of Bishop discs**.

For $\delta \in (0, 1)$ we define a neighborhood of the unknot $K \subset S^3$ by

$$\mathcal{U}^\delta := K \cup \{u_s^t(z) \mid z \in \partial\mathbb{D}, 1 - \delta < s^2 + t^2 < 1\} \subset S^3.$$

We choose δ so small that the holomorphic discs $D_s^t = u_s^t(\mathbb{D})$ with boundary in $\overline{\mathcal{U}}^\delta$ (i.e. with $1 - \delta \leq s^2 + t^2 \leq 1$) lie entirely in the neighborhood of S^3 in D^4 that has been identified holomorphically with a neighborhood of S^3 in \widetilde{W} . This allows us to regard those u_s^t as holomorphic discs in \widetilde{W} . Then, according to [20, Cor. 4.9], any t -level Bishop disc whose boundary meets the set $\overline{\mathcal{U}}^\delta$ is

one of the standard Bishop discs u_s^t . As in our previous paper, we can therefore introduce the following subset of the moduli space \mathcal{W} .

Definition. The **truncated moduli space** is

$$\mathcal{W}^\delta := \{u^t \mid t \in [-\sqrt{1-\delta}, \sqrt{1-\delta}],$$

$$u^t \text{ is a } t\text{-level Bishop disc such that } u^t(\partial\mathbb{D}) \subset S^3 \setminus \mathcal{U}^\delta\}.$$

The following statements from [20, Sec. 4], where such Bishop discs were studied for $W = D^4$, carry over to the present setting.

Proposition 4.2.

- (a) Every Bishop disc has Maslov index 2, i.e. $\mu(A^t) = 2$ for all $t \in (-1, 1)$.
- (b) All Bishop discs are embedded and mutually disjoint. □

The same arguments as in [20] apply to prove transversality, i.e. that the moduli space \mathcal{W} is a manifold; but see the appendix at the end of this paper.

In Section 6 we shall establish compactness for the truncated moduli space \mathcal{W}^δ under the assumption $\inf(\alpha_-) > \pi$ or, in the exact case, $\inf_0(\alpha_-) > \pi$. Thus, provided there are no short Reeb orbits for α_- , the truncated moduli space is a compact manifold with boundary. Then the proof of [20, Prop. 5.1] goes through unchanged; this result says the following.

Proposition 4.3. *Let (W, ω) be a symplectic cobordism as in the ball theorems. If $\inf(\alpha_-) > \pi$, or $\inf_0(\alpha_-) > \pi$ in the exact case, the evaluation map $\text{ev}_1 : u \mapsto u(1)$ defines a diffeomorphism*

$$\text{ev}_1 : \mathcal{W}^\delta \longrightarrow \{u_s^t(1) \mid s^2 + t^2 \leq 1 - \delta\} =: Q^\delta$$

between \mathcal{W}^δ and the closed 2-disc $Q^\delta \subset S^3$. □

It is now a simple matter to prove Theorems 2.2 and 2.4.

Proof of the ball theorems. Under the assumption $\inf(\alpha_-) > \pi$ or $\inf_0(\alpha_-) > \pi$, respectively, and hence with \mathcal{W}^δ being established as a closed disc by the preceding proposition, one can define an embedding

$$F : (\mathbb{D} \times \text{Int } \mathbb{D}, \partial\mathbb{D} \times \text{Int } \mathbb{D}) \longrightarrow (\widetilde{W} \setminus K, S^3 \setminus K)$$

as in [20, Sec. 5] by setting

$$F(z, s, t) = \begin{cases} \left(\text{ev}_1^{-1}(u_s^t(1))\right)(z) & \text{on } \mathbb{D} \times \{s^2 + t^2 \leq 1 - \delta\}, \\ u_s^t(z) & \text{on } \mathbb{D} \times \{1 - \delta \leq s^2 + t^2 < 1\}. \end{cases}$$

We also have the standard embedding

$$F_{\text{st}} : (\mathbb{D} \times \text{Int } \mathbb{D}, \partial\mathbb{D} \times \text{Int } \mathbb{D}) \longrightarrow (D^4 \setminus K, S^3 \setminus K)$$

given by $F_{\text{st}}(z, s, t) = u_s^t(z)$. Both F and F_{st} are holomorphic fillings of S^3 in the sense of [20, Def. 5.2], and they obviously coincide for $1 - \delta \leq s^2 + t^2 < 1$. It follows that the map $F \circ F_{\text{st}}^{-1}$, a priori defined on $D^4 \setminus K$, equals the identity

in a neighborhood of $S^3 \subset D^4$, and hence extends in the obvious way to an embedding

$$(D^4, S^3) \longrightarrow (\widetilde{W}, S^3),$$

which by the compactness of D^4 must be a diffeomorphism. \square

5. THE HOFER ENERGY

The compactness proof for the truncated moduli space \mathcal{W}^δ is based on energy estimates for holomorphic discs and spheres in the almost complex manifold (\widetilde{W}, J) . The following notion of energy is essentially the one introduced in [24, Sec. 3.2].

Definition. Let Σ be a Riemann surface (potentially noncompact or with boundary) and $u : \Sigma \rightarrow \widetilde{W}$ a J -holomorphic curve. The **Hofer energy** of u is

$$E(u) := \sup_{\tau \in \mathcal{T}} \int_{\Sigma} u^* \omega_{\tau}.$$

The Hofer energy of the holomorphic discs u_s^t in the standard holomorphic filling of $S^3 \subset \mathbb{C}^2$ is uniformly bounded, cp. [20, Sec. 2.4]. We now prove a sharp estimate for the energy of Bishop discs in \widetilde{W} .

Proposition 5.1. *The Hofer energy of the Bishop discs in \widetilde{W} is uniformly bounded by π , i.e. $E(u) \leq \pi$ for all $u \in \mathcal{W}$.*

Proof. Let $u = u^t$ be a t -level Bishop disc. Choose a function $\tau \in \mathcal{T}$. We want to estimate $\int_{\mathbb{D}} u^* \omega_{\tau}$.

By Proposition 4.2 the Bishop disc u is an embedding, hence

$$\int_{\mathbb{D}} u^* \omega_{\tau} = \int_{u(\mathbb{D})} \omega_{\tau}.$$

The boundary $u(\partial\mathbb{D})$ of the Bishop disc is contained in $S^t \setminus \{q_{\pm}^t\} \subset S^3$. The 2-sphere S^t is naturally oriented as the unit sphere in $x_1 y_1 x_2$ -space. Let D^t be the 2-disc in S^t (with the induced orientation) whose oriented boundary equals $u(\partial\mathbb{D})$; the disc D^t is characterized by the condition $q_{\pm}^t \in D^t$. The 2-discs $u(\mathbb{D})$ and D^t both represent the relative homotopy class $A^t \in \pi_2(\widetilde{W}, S^t \setminus \{q_{\pm}^t\})$, and they coincide along the boundary. Since ω_{τ} is exact near S^t (and closed on all of \widetilde{W}) it follows that

$$\int_{u(\mathbb{D})} \omega_{\tau} = \int_{D^t} \omega_{\tau} = \int_{D^t} \omega_{\text{st}}.$$

On TS^t we have $\omega_{\text{st}} = dx_1 \wedge dy_1$. So the integral of ω_{st} over a subset of S^t measures the area of the projection of that subset to the $x_1 y_1$ -plane, where the regions in the upper hemisphere $\{x_2 \geq 0\}$ are counted positively; those in the lower hemisphere, negatively. It follows that $\int_{D^t} \omega_{\text{st}}$, and hence $\int_{\mathbb{D}} u^* \omega_{\tau}$, is bounded above by $\pi(1 - t^2)$. \square

6. COMPACTNESS

We now want to show that the truncated moduli space \mathcal{W}^δ is compact. The basic setup is similar to [20, Sec. 6]. We equip \mathcal{W}^δ with the topology induced by the $W^{1,p}$ -norm, $p > 2$, on maps $\mathbb{D} \rightarrow \widetilde{W}$; in [20, Sec. 6] the range was $D^4 \subset \mathbb{C}^2$.

Proposition 6.1. *If $\inf(\alpha_-) > \pi$, or $\inf_0(\alpha_-) > \pi$ in the exact case, the truncated moduli space \mathcal{W}^δ is compact.*

Proof. Let (u_ν) be a sequence in \mathcal{W}^δ , where u_ν is of level t_ν . After passing to a subsequence we may assume that $t_\nu \rightarrow t_0 \in [-\sqrt{1-\delta}, \sqrt{1-\delta}]$. As in [20] we want to apply [32, Thm. B.4.2] in order to prove compactness, i.e. to find a converging subsequence of (u_ν) with respect to the $W^{1,p}$ -norm. This requires the following:

- (i) There is a uniform L^p -bound for the sequence $(|\nabla u_\nu|)$, where the norm is taken with respect to some *complete* metric on \widetilde{W} . We claim that the sequence $(|\nabla u_\nu|)$ is uniformly bounded even in the supremum norm on the closed disc \mathbb{D} . This part of the argument is to some extent parallel to [20]. In some places we need to invoke additional energy estimates to compensate for the lack of compactness of the range of our holomorphic discs. Extra care needs to be taken with potential bubbling at interior points, because we no longer have a global maximum principle on \widetilde{W} that would preclude spheres.
- (ii) The image $u_\nu(\mathbb{D})$ stays inside a fixed compact subset of \widetilde{W} for all ν .

First we notice that (ii) is a straightforward consequence of the bounds we establish in (i). Indeed, with the help of the mean value theorem the uniform C^0 -bound on the image $u_\nu(\mathbb{D})$ follows from the uniform bound on the supremum norm of $|\nabla u_\nu|$, together with the fact that $u_\nu(\partial\mathbb{D})$ stays in the compact subset $S^3 \subset \partial\widetilde{W}$ of \widetilde{W} (and the assumption that the metric be complete).

Arguing by contradiction, assume that there is no uniform bound (in $\nu \in \mathbb{N}$) on $\max_{z \in \mathbb{D}} |\nabla u_\nu(z)|$. We can then find a sequence of points $z_\nu \rightarrow z_0$ in \mathbb{D} such that $|\nabla u_\nu(z_\nu)| \rightarrow \infty$. We distinguish the cases $z_0 \in \partial\mathbb{D}$ and $z_0 \in \text{Int } \mathbb{D}$.

Case 1: $z_0 \in \partial\mathbb{D}$. Choose a conformal map from $\mathbb{H} \cup \{\infty\}$ to \mathbb{D} that sends 0 to z_0 and ∞ to $-z_0$. Subject to this conformal identification, we regard the u_ν as maps

$$u_\nu : (\mathbb{H}, \mathbb{R}) \longrightarrow (\widetilde{W}, S^{t_\nu} \setminus \{q_\pm^{t_\nu}\}),$$

and the sequence (z_ν) as a sequence in \mathbb{H} converging to 0, still satisfying

$$R_\nu := |\nabla u_\nu(z_\nu)| \rightarrow \infty.$$

As shown in the proof of [20, Prop. 6.1] (by an argument going back to Hofer), after passing to a subsequence of (u_ν) one can find a sequence $\varepsilon_\nu \searrow 0$ such that

- $\varepsilon_\nu R_\nu \rightarrow \infty$,

- $|\nabla u_\nu(z)| \leq 2R_\nu$ for all $z \in \mathbb{H}$ with $|z - z_\nu| \leq \varepsilon_\nu$,
- $R_\nu y_\nu \rightarrow r$ for some $r \in [0, \infty]$, where $z_\nu = x_\nu + iy_\nu$.

Case 1.1: $r < \infty$. Here the argument is largely analogous to that in [20]. One considers the rescaled sequence (w_ν) on \mathbb{H} , defined by

$$w_\nu(z) := u_\nu(x_\nu + z/R_\nu), \quad z \in \mathbb{H}.$$

The only issue to take care of is the noncompactness of the range \widetilde{W} of the w_ν . In fact, this does not cause any problems, since there is a uniform bound on the gradient of the w_ν , and the w_ν send $\mathbb{R} = \partial\mathbb{H}$ to the compact subset $S^3 \subset \partial\widetilde{W}$. So there is a C_{loc}^0 -bound on the w_ν , which allows us to apply [32, Thm. B.4.2] as in [20]. As there we then find a subsequence of (w_ν) (after a modification replacing the varying boundary condition by a varying almost complex structure) that converges in C_{loc}^∞ to a nonconstant J -holomorphic map

$$w : (\mathbb{H}, \mathbb{R}) \longrightarrow (\widetilde{W}, S^{t_0} \setminus \{q_\pm^{t_0}\}).$$

We now need to show that the singularity of w at ∞ can be removed, i.e. that w extends to an honest holomorphic disc

$$(\mathbb{D}, \partial\mathbb{D}) \longrightarrow (\widetilde{W}, S^{t_0} \setminus \{q_\pm^{t_0}\}).$$

Such a disc would have contradictory properties as in [20].

For this removal of singularities, it is again the noncompactness of \widetilde{W} that forces us to take extra care. Write $|\cdot|_\tau$ for the norm induced by the (incomplete) metric $g_\tau := \omega_\tau(\cdot, J\cdot)$. The **Dirichlet energy** of w is defined by

$$\frac{1}{2} \int_{\mathbb{H}} |\nabla w|_\tau^2 \, \text{dvol}_{\mathbb{H}} = \frac{1}{2} \int_{\mathbb{H}} (|\partial_s w|_\tau^2 + |\partial_t w|_\tau^2) \, ds \wedge dt,$$

where $z = s + it$. Since w is holomorphic for the ω_τ -compatible almost complex structure J , the Dirichlet energy of w equals its symplectic energy $\int_{\mathbb{H}} w^* \omega_\tau$, see [32, p. 21]. These energies are invariant under conformal reparametrizations, so Proposition 5.1 and the C_{loc}^∞ -convergence of the sequence (w_ν) yield the estimate

$$\frac{1}{2} \int_{\mathbb{H}} |\nabla w|_\tau^2 \, \text{dvol}_{\mathbb{H}} \leq \pi.$$

Finiteness of the Dirichlet energy is one of the conditions in the theorem on removal of singularities [32, Thm. 4.1.2]. In addition, that theorem requires the image of w to lie in a compact manifold and $w(\partial\mathbb{H})$ to lie in a Lagrangian submanifold with respect to a symplectic form taming J .

We first address the latter point. Choose a Riemannian metric g on \widetilde{W} such that J is orthogonal with respect to g , and such that J maps each tangent space of the totally real submanifold $L := S^{t_0} \setminus \overline{U}^{\delta/2}$ to its g -orthogonal complement; this is possible by a lemma of Frauenfelder, see [32, Lemma 4.3.3]. Define a nondegenerate 2-form σ on \widetilde{W} by $\sigma := g(J\cdot, \cdot)$. Then the pullback of σ to L vanishes identically, and J is σ -compatible.

The usual proof of Weinstein’s Lagrangian neighborhood theorem, see [31, Thm. 3.33], allows one to find a diffeomorphism ϕ from a neighborhood of L

in \widetilde{W} to a neighborhood of the zero section in the cotangent bundle T^*L such that the pullback $\omega_L := \phi^*(d\mathbf{p} \wedge d\mathbf{q})$ of the canonical symplectic form on T^*L coincides with σ on $T\widetilde{W}|_L$. In particular, J is tamed by ω_L on $T\widetilde{W}|_L$, and hence in a neighborhood of L . So ω_L is the desired symplectic form.

It remains to show that points in \mathbb{H} sufficiently close to ∞ are mapped by w into that neighborhood of L . By precomposing with the conformal equivalence

$$\mathbb{C} \supset \mathbb{R} \times [0, \pi] \longrightarrow \mathbb{H} \setminus \{0\}, \quad s + it \mapsto e^{s+it},$$

we may regard $w|_{\mathbb{H} \setminus \{0\}}$ as a J -holomorphic map defined on $\mathbb{R} \times [0, \pi] \subset \mathbb{C}$. In this parametrization, neighborhoods of the singular point are of the form $\{s > R\}$.

For $s \in \mathbb{R}$ set

$$\gamma_s(t) := w(s + it), \quad t \in [0, \pi].$$

The length $l(s)$ of γ_s with respect to the metric g_τ is

$$l(s) = \int_0^\pi |\dot{\gamma}_s|_\tau \, dt;$$

the energy $e(s)$ of γ_s we define as

$$e(s) = \int_0^\pi |\dot{\gamma}_s|_\tau^2 \, dt.$$

The Cauchy–Schwarz inequality gives

$$l(s)^2 \leq \pi e(s).$$

Since w maps $\mathbb{R} \times \{0\}$ and $\mathbb{R} \times \{\pi\}$ to the compact set $S^{t_0} \setminus \mathcal{U}^\delta \subset L$, the fact that w maps a neighborhood of the singularity to a neighborhood of L is a consequence of the following Courant–Lebesgue type lemma.

Lemma 6.2. $\lim_{s \rightarrow \infty} l(s) = 0$.

Proof. Since J is ω_τ -compatible and w is J -holomorphic, we have

$$|\dot{\gamma}_s|_\tau = |\partial_t w|_\tau = |\partial_s w|_\tau.$$

We now compute

$$\begin{aligned} \int_{-\infty}^R e(s) \, ds &= \int_{-\infty}^R \int_0^\pi |\dot{\gamma}_s|_\tau^2 \, dt \, ds \\ &= \frac{1}{2} \int_{-\infty}^R \int_0^\pi (|\partial_s w|_\tau^2 + |\partial_t w|_\tau^2) \, dt \, ds \\ &\leq \frac{1}{2} \int_{\mathbb{H}} |\nabla w|_\tau^2 \, d\text{vol}_{\mathbb{H}} \\ &= \int_{\mathbb{H}} w^* \omega_\tau \leq E(w) \leq \pi. \end{aligned}$$

It follows that $\lim_{s \rightarrow \infty} e(s) = 0$, and hence $\lim_{s \rightarrow \infty} l(s) = 0$. □

Case 1.2: $r = \infty$. In this case, again as in [20], we define the rescaled sequence (w_ν) by

$$w_\nu(z) := u_\nu(z_\nu + z/R_\nu) \text{ for } z = x + iy \text{ with } y \geq -y_\nu R_\nu.$$

Then $|\nabla w_\nu(0)| = 1$, the Dirichlet energy of the w_ν is bounded by π , and we have the uniform estimate

$$|\nabla w_\nu(z)| \leq 2 \text{ for all } z \in \mathbb{C} \text{ with } |z| \leq \varepsilon_\nu R_\nu \text{ and } y \geq -y_\nu R_\nu.$$

Case 1.2.a: The sequence $(w_\nu(0) = u_\nu(z_\nu))$ converges (after passing to a subsequence).

Because of the uniform bound $|\nabla w_\nu| \leq 2$ on the exhausting sequence

$$K_\nu := \{z \in \mathbb{C} \mid |z| \leq \varepsilon_\nu R_\nu, y \geq -y_\nu R_\nu\}$$

of compact subsets of \mathbb{C} , the convergence of $(w_\nu(0))$ implies that we have a C^0 -bound on w_ν on the compact set K_ν , which allows us again to apply [32, Thm. B.4.2]. This now gives us a subsequence of (w_ν) that converges in C_{loc}^∞ to a nonconstant holomorphic map $w : \mathbb{C} \rightarrow \widetilde{W}$ with $E(w) \leq \pi$. So w is a finite energy plane in the sense of [25].

By [25, Prop. 2.11] we now have the following alternative:

- (A1) A sphere bubbles off: the image of w is bounded, and w has a smooth extension over ∞ to a holomorphic sphere.
- (A2) A plane bubbles off: the image of w is unbounded, and there exists an $r_0 > 0$ such that

$$w(z) =: (a(z), f(z)) \in (-\infty, 0] \times M_- \text{ for } |z| \geq r_0.$$

Moreover, in the case of alternative (A2), there exists a sequence $r_\mu \rightarrow \infty$, $r_\mu \geq r_0$, and a negative number $T < 0$ such that

$$a(r_\mu e^{2\pi i t}) \rightarrow -\infty \text{ and } \gamma_\mu(t) := f(r_\mu e^{2\pi i t}) \rightarrow \gamma(Tt)$$

in $C^\infty(\mathbb{R}/\mathbb{Z}, (-\infty, 0])$ and $C^\infty(\mathbb{R}/\mathbb{Z}, M_-)$, respectively, for some $|T|$ -periodic Reeb orbit γ of α_- .

The next lemma is essentially contained in [24, Thm. 31], but we give a more direct proof in the present context.

Lemma 6.3. *The period $|T|$ in alternative (A2) satisfies $|T| \leq \pi$.*

Proof. For each $\mu \in \mathbb{N}$ we choose a compactly supported function

$$\tau_\mu \in C^\infty((-\infty, \varepsilon), \mathbb{R}_0^+)$$

with the following properties:

- (i) $\tau'_\mu \geq 0$,
- (ii) $\tau_\mu(s) = e^s$ for $s \in [0, \varepsilon)$,
- (iii) $\tau_\mu = 1 - 1/\mu$ on $a(\{|z| = r_\mu\})$.

Notice that τ_μ is an element of the C^∞ -closure of the set \mathcal{T} of functions used to define the Hofer energy.

We then compute

$$\begin{aligned} E(w) &\geq \int_{|z| \geq r_\mu} w^* \omega_{\tau_\mu} = \int_{|z| \geq r_\mu} w^*(d(\tau_\mu \alpha_-)) \\ &= -\left(1 - \frac{1}{\mu}\right) \int_{|z|=r_\mu} f^* \alpha_- \\ &= -\left(1 - \frac{1}{\mu}\right) \int_{\gamma_\mu} \alpha_- = -\left(1 - \frac{1}{\mu}\right) \int_0^1 \alpha_-(\dot{\gamma}_\mu(t)) dt \\ &\xrightarrow{\mu \rightarrow \infty} - \int_0^1 \alpha_-(T\dot{\gamma}(Tt)) dt \\ &= -T = |T|, \end{aligned}$$

where we have used the theorem of Stokes in the second line. Since $E(w) \leq \pi$, this proves the lemma. □

This lemma shows that our assumption $\inf(\alpha_-) > \pi$ in Proposition 6.1 precludes alternative (A2). In the exact case, where we only require $\inf_0(\alpha_-) > \pi$, we rule out (A2) as follows. Define the collar neighborhood $[0, \varepsilon) \times M_- \subset W$ of M_- by the flow of the Liouville vector field Y given by $i_Y d\lambda = \lambda$. Then λ on W and $\tau_\mu \alpha_-$ on $(-\infty, 0] \times M_-$ glue to a global primitive λ_μ of ω_{τ_μ} on \widetilde{W} . Under alternative (A2), the 2-form $w^* \omega_{\tau_\mu} = w^*(d\lambda_\mu)$ on \mathbb{C} would be compactly supported, hence

$$\int_{|z| \geq \tau_\mu} w^* \omega_{\tau_\mu} \leq \int_{\mathbb{C}} w^*(d\lambda_\mu) = 0$$

by the theorem of Stokes, which would imply $T = 0$ by the computation in the preceding lemma.

In the exact case, alternative (A1) is likewise impossible, since by Stokes there are no nonconstant holomorphic spheres in an exact symplectic manifold.

This concludes the discussion of Case 1.2.a, except for the potential bubbling of spheres in the nonexact case.

Case 1.2.b: The sequence $(w_\nu(0))$ is of the form $w_\nu(0) = (a_\nu(0), f_\nu(0)) \in (-\infty, 0] \times M_-$ (in the notation of alternative (A2)) with $a_\nu(0) \rightarrow -\infty$ (again possibly after passing to a subsequence).

In this case we use a trick from [25] to produce a finite energy plane in the symplectization $(\mathbb{R} \times M_-, d(e^s \alpha_-))$ of M_- . Let R'_ν be the maximal radius $\leq \varepsilon_\nu R_\nu$ such that with K_ν as in Case 1.2.a and

$$K'_\nu := K_\nu \cap \{|z| \leq R'_\nu\}$$

we have

$$w_\nu(K'_\nu) \subset (-\infty, 0] \times M_-.$$

Because of $\varepsilon_\nu R_\nu \rightarrow \infty$, the uniform estimate $|\nabla w_\nu| \leq 2$ on K_ν , and $a_\nu(0) \rightarrow -\infty$, the mean value theorem implies $R'_\nu \rightarrow \infty$.

Now consider the shifted sequence

$$(a_\nu - a_\nu(0), f_\nu) \in C^\infty(K'_\nu, \mathbb{R} \times M_-).$$

We continue to write (w_ν) for this sequence. By the compactness of M_- the sequence $(w_\nu(0))$ in $\{0\} \times M_-$ has a convergent subsequence, so just as in Case 1.2.a we may now apply [32, Thm. B.4.2] to obtain a finite energy plane $w : \mathbb{C} \rightarrow \mathbb{R} \times M_-$. Specifically, with the Hofer energy now defined as

$$E(w) = \sup_\tau \int w^* d(\tau \alpha_-),$$

where the supremum is taken over the set

$$\{\tau \in C^\infty(\mathbb{R}, [0, 1]) \mid \tau' \geq 0\},$$

we have $E(w) \leq \pi$.

This places us, once again, in the setting of [25, Prop. 2.11]. In the symplectization $(\mathbb{R} \times M_-, d(e^s \alpha_-))$ the maximum principle holds, so alternatives (A1) and (A2) are excluded; instead we must have the following, where $w = (a, f)$:

(A3) Breaking: there exists a sequence $r_\mu \rightarrow \infty$ and a positive number $T > 0$ such that

$$a(r_\mu e^{2\pi i t}) \rightarrow \infty \text{ and } \gamma_\mu(t) := f(r_\mu e^{2\pi i t}) \rightarrow \gamma(Tt)$$

in $C^\infty(\mathbb{R}/\mathbb{Z}, \mathbb{R})$ and $C^\infty(\mathbb{R}/\mathbb{Z}, M_-)$, respectively, for some T -periodic Reeb orbit γ of α_- .

Notice that this γ is now a *contractible* periodic orbit, since the whole energy plane can be projected into M_- . Next we estimate the symplectic energy by choosing τ to be identically equal to 1:

$$E(w) \geq \int_{|z| \leq r_\mu} w^*(d\alpha_-) = \int_{|z|=r_\mu} f^* \alpha_- \xrightarrow{\mu \rightarrow \infty} T.$$

This implies that the assumption $\inf_0(\alpha_-) > \pi$ suffices to rule out alternative (A3).

Case 2: $z_0 \in \text{Int } \mathbb{D}$. This case is completely analogous to Case 1.2.

Thus, at this point we have completed the proof of Proposition 6.1 in the exact case, and hence the proof of Theorem 2.4. In the nonexact case it still remains to show that no spheres can bubble off, i.e. that alternative (A1) in Case 1.2.a or Case 2 never happens.

Since the maximum principle applies near the convex boundary of \widetilde{W} , all potential bubbling spheres have to be disjoint from a neighborhood of that boundary. Moreover, our arguments have shown that no breaking or bubbling-off of planes can occur, in particular near the boundary. Therefore the compactness result [5, Thm. 10.2] for an almost complex manifold without boundary and with cylindrical ends applies to our situation (modulo a remark that we shall make presently). That compactness result says that the sequence (u_ν) has a subsequence convergent to a holomorphic building of height $k_-|1$ with $k_- \geq 0$. Any holomorphic building coming from a disc and having height $k_-|1$ with

$k_- > 0$ would contain at least one finite energy plane, whose existence we have excluded. So the limit is a holomorphic building of total height 1. In other words, the subsequence is Gromov-convergent to a stable J -holomorphic map $\{u^j\}_{j=0,\dots,n}$ in the sense of [32, Def. 5.1.1]. Here our labelling is chosen such that u^0 is a J -holomorphic disc, and u^1, \dots, u^n are J -holomorphic spheres. The disc and spheres form a bubble tree; in particular each sphere has at least one point of intersection with some other sphere or the disc.

Remark 6.4. The compactness theorem from [5] only applies in the case that α_- is nondegenerate (i.e. the linearized Poincaré return map along closed Reeb orbits of α_- , including multiples, does not have an eigenvalue 1).

If α_- is a degenerate contact form with $\inf(\alpha_-) > \pi$, we argue as in the final paragraph of [1]. Choose a sequence of smooth functions $f^{(\mu)} : M_- \rightarrow \mathbb{R}^+$ converging in C^∞ to the constant function 1 and with $f^{(\mu)}\alpha_-$ nondegenerate for all $\mu \in \mathbb{N}$. If $\inf(f^{(\mu)}\alpha_-) \leq \pi$ for all μ , then the argument in [1] would show that, likewise, $\inf(\alpha_-) \leq \pi$, contradicting our assumption. So we find a function f arbitrarily C^∞ -close to 1 with $f\alpha_-$ nondegenerate and $\inf(f\alpha_-) > \pi$. This contact form can be realized on the boundary M_- of the cobordism W by a small modification of W in a collar neighborhood of $M_- \subset W$ (after adding a small piece $(-\varepsilon, 0] \times M_-$ of the symplectization to W). Then the whole argument (including the part that follows below) can be applied to this modified W , showing W to be a 4-ball.

Returning to our purported bubble tree, we now compute with intersection numbers as in [44] in order to show that such a bubble tree (with $n \geq 1$) cannot exist. Arguing by contradiction, we assume $n \geq 1$, i.e. that at least one sphere bubbles off. Each disc u_ν represents a relative homotopy class $A^{t_\nu} \in \pi_2(\widetilde{W}, S^{t_\nu} \setminus \{q_\pm^{t_\nu}\})$ whose self-intersection number (as defined in [20, Sec. 8]) is $A^{t_\nu} \bullet A^{t_\nu} = 0$; see the proof of [20, Prop. 4.5]. Gromov convergence implies that, for ν large enough, the homotopy class of u_ν in $\pi_2(\widetilde{W}, S^3 \setminus K)$ equals that represented by the limiting bubble tree. For the purpose of computing intersection numbers we may assume that the bubble tree and the u_ν represent the same class $A \in \pi_2(\widetilde{W}, S^{t_0} \setminus \{q_\pm^{t_0}\})$ for some t_0 (since for discs in different levels we are back to classical intersection theory at interior points).

By positivity of intersections [20, Thm. 9.2] and $u_\nu \bullet u_\nu = 0$ we have

$$u^j \bullet u_\nu \geq 0 \text{ for } j = 0, \dots, n.$$

Since the intersection product is a homotopy invariant, we have

$$u^j \bullet A \geq 0 \text{ for } j = 0, \dots, n.$$

From

$$0 = A \bullet A = \sum_{j=0}^n u^j \bullet A \geq 0$$

we conclude $u^j \bullet A = 0, j = 0, \dots, n$.

Again by positivity of intersections we have $u^1 \bullet u^k \geq 0$ for $k = 0, 2, \dots, n$, and at least one of these intersection numbers (corresponding to a neighbor of u^1 in the bubble tree) is positive. Hence $u^1 \bullet u^1 < 0$. Beware that the intersection number in [20] is weighted differently from the standard intersection number of closed submanifolds, but the inequality $u^1 \bullet u^1 < 0$ remains true if \bullet is now interpreted in that standard way.

Without loss of generality we may assume that u^1 is simple, i.e. not multiply covered (otherwise apply the following argument to the corresponding simple sphere). Then the adjunction inequality [32, Thm. 2.6.4] says that

$$u^1 \bullet u^1 - c_1(u^1) + 2 \geq 0,$$

with equality if and only if u^1 is embedded. By Remark 4.1 (3) we have $c_1(u^1) \geq 1$, and hence $u^1 \bullet u^1 \geq -1$.

We conclude that $u^1 \bullet u^1 = -1$. Then further $c_1(u^1) = 1$, and equality holds in the adjunction formula. This means that u^1 is an exceptional sphere in $(\widetilde{W}, \omega_\tau)$ for any choice of $\tau \in \mathcal{T}$. If we take $\tau(s) = e^s$ on $(-\infty, \varepsilon)$, for instance, the vector field ∂_s is a Liouville vector field for ω_τ on $(-\infty, \varepsilon) \times M_-$, and its flow can be used to push u^1 into (W, ω) . Exceptional spheres in (W, ω) , however, are excluded by assumption (C1).

This finishes the proof of Proposition 6.1 and hence that of Theorem 2.2. \square

7. SYMPLECTIC FILLINGS OF $S^2 \times S^1$

An obvious strong symplectic filling of $S^2 \times S^1 \subset \mathbb{R}^3 \times S^1$ with its standard contact structure $\xi_{\text{st}} = \ker(\lambda_{\text{st}}|_{T(S^2 \times S^1)})$, where

$$\lambda_{\text{st}} := \frac{1}{2}(x dy - y dx) + z d\theta,$$

is given by $(D^3 \times S^1, dx \wedge dy + dz \wedge d\theta)$. The following result is implicit in [11]; for the uniqueness of the filling up to symplectic deformation equivalence see [35]. For more on the topology of symplectic and Stein fillings see [36, Chap. 12].

Theorem 7.1. *Any minimal weak symplectic filling of $(S^2 \times S^1, \xi_{\text{st}})$ is diffeomorphic to $D^3 \times S^1$.*

Proof. We only present the main outline of the argument; the details are then completely analogous to those used for the holomorphic filling of (\widetilde{W}, J) in Section 4.

The contact manifold $(S^2 \times S^1, \xi_{\text{st}})$ is foliated by the 2-spheres $S^\theta := S^2 \times \{\theta\}$, whose characteristic foliation looks like that of the level spheres S^t in Section 4. The singular points q_\pm^θ of these characteristic foliations form two circles $\{z = \pm 1\} \times S^1$.

Suppose (W, ω) is a weak symplectic filling of $(S^2 \times S^1, \xi_{\text{st}})$. Choose an almost complex structure J on W that satisfies conditions (J3) and (J4). In order to define θ -level Bishop discs, we formulate a condition analogous to (D2) by choosing three leaves of the characteristic foliation of the S^θ (in an S^1 -invariant family, say). As regards the homotopical condition (D1), *a priori*

we may have to consider two families of relative homotopy classes A_{\pm}^{θ} defined by standard Bishop discs in (W, J) near the singular points q_{\pm}^{θ} . Because of $A_{+}^{\theta} \bullet A_{-}^{\theta} = 0$ and positivity of intersections, however, it follows that any two θ -level Bishop discs in these two families are either disjoint or they coincide. This implies that it suffices to formulate (D1) only in terms of A_{+}^{θ} , say.

The corresponding truncated moduli space can then be shown to be diffeomorphic to $S^1 \times D^1$ via the evaluation map ev_1 just as in Proposition 4.3, and the proof then concludes like that of the ball theorems. The compactness argument remains unchanged; a suitable replacement for the bound π in the energy estimate in Proposition 5.1 is provided by $\max_{\theta} \int_{S^2 \times \{\theta\}} |\omega|$. \square

We close with a computation of the capacities c, c_0 in this context.

Proposition 7.2. *Let (V, ω) be a minimal strong symplectic filling of the manifold $(S^2 \times S^1, \alpha_{st})$, i.e. there is a Liouville vector field Y defined near and pointing outwards along $\partial V = S^2 \times S^1$ such that $(i_Y \omega)|_{T(S^2 \times S^1)} = \alpha_{st}$. Then $c(V, \omega) = \pi$. If, moreover, $\omega = d\lambda$ with $\lambda|_{T(S^2 \times S^1)} = \alpha_{st}$, then $c_0(V, \lambda) = \pi$.*

Proof. By assumption the symplectic form ω is the standard one near the boundary $S^2 \times S^1$, so in this case we actually have the bound π in the energy estimate in Proposition 5.1. The filling result above implies that any closed hypersurface M in $\text{Int } V$ is separating. The analogues of the ball theorems for the resulting cobordism from M (which has to be a concave end) to $S^2 \times S^1$ show that the capacities c, c_0 are at most equal to π . On the other hand, the Reeb vector field of α_{st} is given by

$$R_{\alpha_{st}} = \frac{2}{1+z^2}(x \partial_y - y \partial_x + z \partial_{\theta}),$$

whose minimal period is equal to π , corresponding to the contractible orbits

$$\gamma(t) = (\cos 2t, \sin 2t, 0, \theta_0) \in S^2 \times S^1 \subset \mathbb{R}^3 \times S^1, \quad t \in [0, \pi].$$

For $r < 1$ sufficiently close to 1 we have a (strict) contact type embedding of $S_r^2 \times S^1$ (with the contact form induced by λ_{st}) into (V, ω) . So the lower bound π on the capacities in question follows from an exhaustion argument as in the proof of Proposition 3.20. \square

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APPENDIX

Here are some corrections to our previous paper [20].

(1) In the first case, step (ii), of the proof of Proposition 6.1, u should be replaced by w in three instances.

(2) In the second case of the proof of Proposition 6.1, the uniform estimate $|\nabla w_{\nu}(z)| \leq 2$ holds on the set K_{ν} defined in Case 1.2.a of the present paper.

(3) Proposition 7.3 states that the space \mathcal{B} of level-preserving discs $(\mathbb{D}, \partial\mathbb{D}) \rightarrow (\mathbb{R}^4, S^t \setminus \{q_{\pm}^t\})$, $t \in (-1, 1)$, with the homotopical boundary condition (D1) from Section 4 above, is a Banach manifold. This statement is correct; the proof in [20] shows that it is a Banach manifold modelled on the Banach space of $W^{1,p}$ -sections η of $u^*(T\mathbb{R}^4, TS^3)$, where u is a $W^{1,p}$ -map $(\mathbb{D}, \partial\mathbb{D}) \rightarrow (\mathbb{R}^4, S^3 \setminus K)$, that satisfy the additional requirement

$$\langle \nabla H \circ u|_{\partial\mathbb{D}}, \eta|_{\partial\mathbb{D}} \rangle \equiv \text{const.},$$

with H the height function on S^3 as in Section 4.

In the proof of that Proposition 7.3 in [20] we tried to show more, namely, that \mathcal{B} is a Banach submanifold of the Banach manifold \mathcal{C} of all $W^{1,p}$ -maps $(\mathbb{D}, \partial\mathbb{D}) \rightarrow (\mathbb{R}^4, S^3 \setminus K)$ satisfying the corresponding homotopical boundary condition. This would require the subspace $T_u\mathcal{B} \subset T_u\mathcal{C}$ to split. Since solutions of the boundary value problem (P) in [20] need not be of class $W^{1,p}$, our argument does not prove the existence of a splitting. This stronger statement, however, is never used in [20] or the present paper.

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