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Abstract

We study the Rokhlin dimension of C*-dynamical systems that are induced by free actions of certain discrete groups on locally compact metric spaces. The concept of Rokhlin dimension was introduced by Hirshberg, Winter and Zacharias for actions of finite groups and the integers on unital C*-algebras. One of their key results is that actions with finite Rokhlin dimension posess a permanence property with respect to finite nuclear dimension, when passing to the crossed product C*-algebra.

We extend the notion of Rokhlin dimension to actions of residually finite groups on all C*-algebras, and show that an analogous permanence property with respect to nuclear dimension holds for a class of residually finite groups that contains all finitely generated and virtually nilpotent groups. For this, we use some ideas from coarse geometry, the most important ingredient being the so-called box spaces associated to residually finite groups.

As our main class of application, we consider free actions of $\mathbb{Z}^m$ on locally compact metric spaces with finite covering dimension. We show that these always yield C*-dynamical systems with finite Rokhlin dimension. In this setting, the aforementioned properties of Rokhlin dimension imply that the associated transformation group C*-algebra has finite nuclear dimension. Our approach employs a generalization of Gutman's marker property for aperiodic homeomorphisms, and a generalization of a technical result by Lindenstrauss that aperiodic homeomorphisms on finite-dimensional spaces satisfy a certain bounded version of the small boundary property. Finite Rokhlin dimension for free topological $\mathbb{Z}^m$-actions is then deduced from a controlled version of the marker property. We also consider more generally free actions of infinite, finitely generated and nilpotent groups on locally compact metric spaces with finite covering dimension, and show that the analogous result holds even in this setting. This employs the techniques from the case of $\mathbb{Z}^m$-actions and combines it with certain aspects of the geometric group theory of nilpotent groups.

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Contents

Introduction

Since its inception, the theory of operator algebras has been influenced in large part through ideas of a dynamical nature. The study of C*-dynamical systems, i.e. group actions of locally compact groups on C*-algebras, is interesting in many ways. The crossed product construction is of fundamental importance, which is a way to create a new C*-algebra that naturally incorporates the structure of a given C*-dynamical system. This construction has, by now, proved to be a virtually inexhaustible source of interesting examples of C*-algebras. In many cases, crossed products can turn out to be simple C*-algebras falling within the scope of the so-called Elliott classification program.

In 1976, Elliott established his seminal classification theorem of approximately finite-dimensional (AF) algebras [20] via K-theory, building on earlier work of Bratteli [9] on AF algebras and of Glimm [32] on UHF algebras. Elliott's classification theorem asserts that, up to isomorphism, AF algebras are classified by their scaled ordered K_0-groups. That is, two unital AF algebras A and B are isomorphic if and only if there exists an order-preserving isomorphism from $K_0(A)$ to $K_0(B)$ preserving also the K_0-classes of the respective units. Extending this result considerably, he later managed to prove that approximate circle (A$\mathbb{T}$) algebras of real rank zero are classified by their ordered K_0-groups and K_1-groups, see [21], and that simple, approximate interval (AI) algebras are classified by their K_0-groups, tracial state spaces and their pairings, see [19, 97]. In 1994, this led him to conjecture that a more general class of C*-algebras should be classifiable by K-theory and traces, see [22, 23]:

Conjecture (Elliott 1994)**.** Let A and B be two separable, unital, nuclear and simple C*-algebras. Then $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$, where Ell denotes the so-called Elliott-functor on unital C*-algebras given by

$$\mathrm{Ell}(A) = \big(K_0(A), K_0(A)^+, [\mathbf{1}_A], T(A), \rho_A, K_1(A)\big).$$

Here, $T(A)$ denotes the tracial state space of A and $\rho_A : K_0(A) \times T(A) \to \mathbb{R}$ is the natural pairing map given by $\rho_A([p], \tau) = (\mathrm{Tr}_n \otimes \tau)(p)$ for all projections $p \in M_n(A)$ and $\tau \in T(A)$, where Tr_n denotes the unique tracial state on M_n given by $\mathrm{Tr}_n\big((x_{i,j})_{1\leq i,j\leq n}\big)= \sum_{j=1}^{n} x_{j,j}$. The target category of this functor is the so-called Elliott category, with objects consisting of suitable tuples of (scalered, ordered) abelian groups, Choquet simplices and suitable pairing maps between these. The morphisms consist of tuples of (ordered, unit-preserving) group homomorphisms and affine maps compatible with the respective pairing maps. For a more thorough explanation, the reader is referred to the second chapter of Rørdam's book [81] on classification theory of C*-algebras.

The reason for the nuclearity assumption in Elliott's conjecture is not immediately obvious. However, the class of nuclear C*-algebras is very important and large enough to contain most of the C*-algebras that arise *naturally* in one way or another. Nuclear C*-algebras also have nice structural properties that are technically very useful in applications. A C*-algebra A is called nuclear, if for every C*-algebra B, there is a unique C*-cross norm on the algebraic tensor product of A and B. Surprisingly, the property of being nuclear has a variety of interesting equivalent characterizations. By combined work of Connes and Haagerup [13, 40], it is known that a C*-algebra is nuclear if and only if it is amenable as a Banach algebra. For this reason, nuclear C*-algebras are often called amenable. Combined work of Choi, Effros and Lance [11, 58] shows that a C*-algebra A is nuclear if and only if its enveloping von Neumann algebra A^{**} is injective. Perhaps the most important characterization for classification purposes, it is due to Choi, Effros and Kirchberg [12, 49] that a C*-algebra A is nuclear if and only if its identity map can be locally approximated by contractive completely positive maps factoring through finite-dimensional C*-algebras. In the separable case, Kirchberg has characterized nuclear C*-algebras A as those that are embeddable as C*-subalgebras into the Cuntz algebra of two generators $\mathcal{O}_2$ such that there exists a conditional expectation from $\mathcal{O}_2$ onto A, see [50]. In the case of group C*-algebras, nuclearity amounts to amenability of the group. That is, for any discrete group G, the universal C*-algebra $C^*(G)$ of unitary representations of G is nuclear if and only if G is amenable. For a proper treatement of nuclear C*-algebras in general and in particular its meaning in the setting of group C*-algebras, the reader is advised to consult Brown-Ozawa's book [10].

Since there exist various range results for the Elliott invariant, when restricted to special inductive limit C*-algebras (see for instance [23]), the Elliott conjecture would also entail that stably finite, classifiable C*-algebras should be expressible as inductive limits of particularly nice building blocks. These include AH algebras, i.e. inductive limits of finite direct sums of the form $p\mathcal{C}(X, M_n)p$ for a finite CW complex X and a projection $p \in \mathcal{C}(X, M_n)$. More generally, these also include ASH algebras, i.e. inductive limits of recursive subhomogeneous algebras, see [75]. In fact, the bulk of the earlier results within the Elliott program all presuppose a certain inductive limit decomposition into well-behaved building blocks for the C*-algebras under consideration.

For certain crossed product C*-algebras associated to natural classes of C*-dynamical systems, it is desirable to determine when they belong to a class of C*-algebras that is tractable enough to be covered by a classification theorem in the spirit of Elliott's conjecture. However, the task of showing that a crossed product of some C*-dynamical system admits such an inductive limit decomposition of this sort is a very challenging task.

A few years after Elliott proposed his conjecture, it was discovered that his conjecture in its original form was too much to ask for, see [101, 82, 98]. The reason behind it is, simply put, that Elliott's conjecture can fail for a pair of C*-algebras, if one of them is, in a sense, *regular*, but the other is not. This insight has led to several far-reaching new developments in C*-algebra theory, which, from today's point of view, yield more conceptual approaches to the Elliott program than the study of C*-algebras that are a priori given as inductive limits of some sort.

One important direction is the concept of tracial approximation. This notion has first been used very successfully by Lin for the study of tracially approximately finite-dimensional (TAF) algebras, see [59, 60]. The advantage of his classification theorem is that it relies on rather abstract properties of the involved C^*-algebras, compared to the assumption that it is given as an inductive limit of some sort. Moreover, the verification of these abstract properties of a C^*-algebra, in order to show classifiability, is expected to be much easier than directly obtaining an inductive limit decomposition. Driven most notably by Lin, Niu, Gong and Winter, there now exist very general classification theorems of this spirit, see [59, 60, 61, 108, 64, 34, 63, 35]. With these classification theorems at hand, one would often like to verify for a C^*-algebra that its stabilization by a certain UHF algebra is tracially approximated by either finite-dimensional C^*-algebras, interval algebras or more generally certain 1-NCCW-complexes. However, this easier task is in general still very difficult for C^*-algebras given as a crossed product of some C^*-dynamical system.

Another important conceptual approach to Elliott's program attacks the C^*-algebras of interest at a somewhat more basic level. At the heart of this recent progress is the discovery of various C^*-algebraic regularity properties, which have emerged as a necessary criterion for a C^*-algebra to be considered from the point of view of Elliott's conjecture in its original form. The study of these regularity properties has culminated into the so-called regularity conjecture of Toms and Winter, see [25, 107]:

Conjecture (Toms-Winter)**.** Let A be a separable, unital, simple and nuclear C^*-algebra that is not isomorphic to a finite matrix algebra. Then the following are equivalent:

(i) A has finite nuclear dimension.

(ii) A absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially.

(iii) A has strict comparison of positive elements.

As will be made more precise in the first chapter, finite nuclear dimension is a strengthening of nuclearity in a way that is motivated by the idea of covering dimension. In fact, it is a notion that generalizes covering dimension for locally compact metric spaces. Because of this, part (i) of the above conjecture is a rather topological statement, whereas parts (ii) and (iii) are much more algebraic in nature. The progress made on this conjecture during the last few years has been vast, see in particular [106, 107, 79, 68, 54, 99, 87, 69, 88].

If one looks back at the inital questions about the classifiability of crossed products, one can consider a natural problem. For now, let A be a C*-algebra, G a countable, discrete group and let $\alpha : G \to \mathrm{Aut}(A)$ be a group homomorphism. We call α a G-action on A and write $\alpha : G \curvearrowright A$.

Question 1. When is the crossed product $A \rtimes_\alpha G$ regular or classifiable? More modestly: under what conditions on α does regularity or classifiability pass from A to $A \rtimes_\alpha G$?

Of course, this question is interesting for all three versions of regularity, and for all kinds of nuclear C*-algebras A. Within this dissertation, however, we will restrict our focus to special cases of this question. The specification is two-fold; firstly, we will study this question for the regularity property of having finite nuclear dimension.

Question 2. Under what conditions on α does finite nuclear dimension pass from A to $A \rtimes_\alpha G$?

Secondly, the C*-dynamical systems under consideration are mainly examples coming from topological dynamics. So the question becomes:

Question 3. Let A be of the form $A = \mathcal{C}_0(X)$ for a finite-dimensional and locally compact metric space. When does the transformation group C*-algebra $\mathcal{C}_0(X) \rtimes_\alpha G$ have finite nuclear dimension?

Let us briefly review to what extent these questions have been answered for transformation group C*-algebras of the form $\mathcal{C}_0(X) \rtimes_\alpha G$ in the literature. For the most part, the current results concern the case that $G = \mathbb{Z}$ and X being compact. Early results of Putnam about characterizing crossed products of minimal homeomorphisms on the Cantor set as A$\mathbb{T}$ algebras [78] or of Elliott and Evans about irrational rotation algebras [24] have essentially set the stage for a deeper investigation of transformation group C*-algebras of this type. In a related context, Giordano, Putnam and Skau proved a celebrated result [31] asserting that two Cantor minimal systems yield isomorphic transformation group C*-algebras if and only if they are strongly orbit equivalent. In one of the more recent breakthroughs, Toms and Winter could show in [100] that a C*-algebra $\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$ associated to a minimal dynamical system $(X, \varphi, \mathbb{Z})$ has finite nuclear dimension, provided that X is compact and has finite covering dimension. Combining this fact with a result by Strung and Winter [91], they have shown that crossed products of uniquely ergodic minimal homeomorphisms on infinite compact metrizable spaces with finite covering dimension are classified by ordered K-theory. Before I began my doctoral studies, the combined work of Toms, Strung and Winter was the state-of-the-art result concerning the classifiability of transformation group C*-algebras of the form $\mathcal{C}(X) \rtimes \mathbb{Z}$.

For more general group actions, the situation becomes much more complicated. As it has been empirically observed in countless similar situations within mathematics, handling a problem concerning a fairly general class of groups can be much better accessible for the integer group $\mathbb{Z}$. Simply put, this stems from the fact that $\mathbb{Z}$ is the only singly-generated free group. As such, it is easier to prove something for $\mathbb{Z}$ than for other groups, because there are no group relations that one has to worry about. Moreover, the natural linear order on $\mathbb{Z}$ can be of technical use in many situations, whereas such a tool is mostly absent elsewhere. In the above context, a major problem is that already for $\mathbb{Z}^2$ it is not at all obvious how to define the large subalgebras as in [78] to *break orbits* within a crossed product of the form $\mathcal{C}(X) \rtimes \mathbb{Z}^2$. This is a considerable drawback even if one only wishes to follow Toms' and Winter's approach to show that certain C*-algebras of the form $\mathcal{C}(X) \rtimes \mathbb{Z}^2$ have finite nuclear dimension. As for the techniques of [91], it is at present still completely unclear how to proceed for non-integer group actions. However, the concept of *large subalgebras* pioneered by Phillips might shed some light on these questions in the future, see [76, 77].

Around the time that I began my doctoral studies, Hirshberg, Winter and Zacharias have introduced the concept of Rokhlin dimension in [42] as a tool for solving Question 2 in great generality. This notion has been introduced for finite group actions and integer actions on unital C*-algebras, but has the big advantage that it can be defined similarly for $\mathbb{Z}^m$-actions and even for other higher-rank groups. Integer actions with finite Rokhlin dimension have been shown to behave well with underlying C*-algebras of finite nuclear dimension. That is, the property of having finite nuclear dimension passes from the underlying C*-algebra to the crossed product. They have also provided a purely C*-algebraic proof that minimal $\mathbb{Z}$-actions on infinite and finite-dimensional compact metric spaces yield C*-dynamical systems with finite Rokhlin dimension. This, in turn, marked an alternative route towards showing that the transformation group C*-algebras $\mathcal{C}(X) \rtimes \mathbb{Z}$ of minimal actions have finite nuclear dimension, provided X is compact and finite-dimensional. With this approach, however, came the reasonable hope that these techniques could possibly carry over to actions of $\mathbb{Z}^m$ or even more general groups.

At this point, I would like to summarize my contributions to the second and third question from above, and simultaniously sketch how this dissertation is organized.

Towards making progress on Question 2, I have investigated to what extent the methods from the work of Hirshberg, Winter and Zacharias carry over to actions of higher-rank groups, in order to obtain a more general notion of Rokhlin dimension. I could indeed obtain a rather straightforward generalization to $\mathbb{Z}^m$-actions in [94], which posed only minor additional combinatorial difficulties. In later joint work with Wu and Zacharias [96], we could overcome the technical difficulties for non-abelian groups and were able to generalize Rokhlin dimension to cover actions of a subclass of residually finite groups that contains all finitely generated and virtually nilpotent groups. We will give a detailed treatment of all this in the first chapter.

Towards making progress on Question 3, I have examined under what conditions a topological action $\alpha : G \curvearrowright X$ on a finite-dimensional, locally compact metric space yields a transformation group C*-algebra with finite nuclear dimension. This will be the topic of the second and third chapter.

At the beginning of the second chapter, we revisit an approach of Toms and Winter showing that the C*-algebras $\mathcal{C}(X)\rtimes\mathbb{Z}$ have finite nuclear dimension, but under the assumption that the action is merely aperiodic instead of minimal. This approach employs the orbit-breaking subalgebras of [78] instead of arguments related to Rokhlin dimension. The so-called marker property, which is a dynamical property recently introduced by Gutman in [39] for aperiodic $\mathbb{Z}$-actions, turns out to be the right tool to carry out the original proof of Toms and Winter for aperiodic homeomorphisms. Following this, we use Gutman's marker property to obtain a purely dynamical generalization of the aforementioned result of Hirshberg, Winter and Zacharias that minimal $\mathbb{Z}$-actions on finite-dimensional, compact metric spaces have finite Rokhlin dimension. Although the marker property in itself is not a strong enough condition for this purpose, a formally stronger version of the marker property can be obtained directly from Gutman's proof. This stronger version will then enable us to give a comparably simple proof of finite Rokhlin dimension in the setting of aperiodic $\mathbb{Z}$-actions on finite-dimensional spaces.

In the third chapter, we investigate how one can extend Gutman's notion of the marker property in order to apply it to free actions of higher-rank groups, and in particular $\mathbb{Z}^m$. Although the suitable definition for $\mathbb{Z}^m$-actions is rather obvious, actually verifying this condition poses a big technical obstacle. It not only requires a suitable generalization of Gutman's proof itself, but of another technical result, which already served as a black box within Gutman's approach. Namely, a very strong version of the small boundary property is needed for the approach to work, which is something that Lindenstrauss has established in one of his earlier works [66] on $\mathbb{Z}$-actions on finite-dimensional spaces. Fortunately, the jump from $\mathbb{Z}$ to other groups is merely a question about more general combinatorial arguments, at least as far as Lindenstrauss' approach is concerned. In particular, we generalize this technical result even to free actions of countably infinite groups on finite-dimensional, locally compact metric spaces. As the main technical result of this dissertation, we then prove a general marker property lemma for such actions. In the particular case where the acting group is $\mathbb{Z}^m$, this general lemma implies an even stronger variant as a corollary, which is then sufficiently strong to prove finite Rokhlin dimension. As a culmination of this general analysis of free topological actions, we can answer Question 3 affirmatively in the case where $G = \mathbb{Z}^m$ and the action is free. This marks the main result of this dissertation. It should be pointed out that the results of the first half of the third chapter are essentially published in [94]; however, some of the partial results are carried out in greater generality here, for instance without the assumption that the underlying space is compact.

Following this, we explain how the aforementioned result for free $\mathbb{Z}^m$-actions can be extended even further to actions of not necessarily abelian groups. This is a result obtained within a collaboration with Wu and Zacharias [1], which is available as an arxiv preprint [96]. However, the approach presented in this dissertation will be somewhat different, more direct and also more general in that we treat the case of locally compact metric spaces.

In the first chapter, we have extended the notion of Rokhlin dimension to actions of a certain class of residually finite groups. This class of groups can be regarded as the groups having very low coarse-geometric complexity,

[1] Each author has contributed an approximately equal amount to this collaboration.

and includes all finitely generated and virtually nilpotent groups. So the natural candidate for a class of groups, for which Question 3 should have an affirmative answer, is given by finitely generated, nilpotent groups. Indeed, we will see that such groups acting freely on finite-dimensional, locally compact metric spaces automatically yield C*-dynamical systems with finite Rokhlin dimension. Compared to the case of $\mathbb{Z}^m$-actions, however, the proof is necessarily more technical and involves certain aspects of the geometric group theory of nilpotent groups.

Also worth mentioning is that shortly after I have obtained the main result of the third chapter, Wilhelm Winter has invented a new and conceptual way of verifying classifiability for certain C*-algebras, which is now known as *classification via embedding*, see [109]. The main motivation was to apply this to crossed product C*-algebras, and particularly to transformation group C*-algebras of the form $\mathcal{C}(X) \rtimes \mathbb{Z}^m$ associated to free and minimal actions on finite-dimensional, compact metric spaces. His main result asserts that classifiability is assured if the space of ergodic measures is compact and projections separate tracial states. Besides several new insights, his approach makes crucial use of two other results; one of them being Lin's deep theorem about AF-embeddability of transformation group C*-algebras from [62], and the other being the main result from the third chapter of this dissertation.

At least as far as uniquely ergodic, free and minimal $\mathbb{Z}^m$-actions are concerned, classifiability of their transformation group C*-algebras $\mathcal{C}(X) \rtimes \mathbb{Z}^m$ can also be obtained by combining the main result of the third chapter with the quasidiagonality result from [62] and the main result of [69].

Let us also mention that the C*-algebras associated to Penrose tilings or substitution tilings, as considered in [47], can often be expressed as transformation group C*-algebras of uniquely ergodic, free and minimal $\mathbb{Z}^m$-actions on the Cantor set. Since these objects appear naturally in the context of quasicrystals, the classifiability of these C*-algebras might even spark some applications in physics.

Apart from all of this, I have also obtained some other results during my doctoral studies, partly in collaboration with others, see [2, 95, 3]. While these results go in a similar direction as this dissertation, namely studying certain C*-dynamical systems and their crossed products with the help of Rokhlin type properties, they do not directly address the interplay between topological dynamics and classification theory of C*-algebras. In order to keep this dissertation self-contained, those results will therefore not be discussed at length here. However, we will conclude this dissertation in the fourth chapter by briefly reviewing the contents of [2, 95, 3] and by indicating how they relate to the initial outlined problems of this introduction.

The theory of Rokhlin dimension

The reader of this dissertation is supposed to be familiar with the general theory of C*-algebras, including the definition and basic properties of crossed product C*-algebras. We suggest [70, 103] as the standard references for the basic theory. Although we will recall the most importants bits in the first preliminary section, it may also be useful to be familiar with nuclearity and completely positive maps between C*-algebras, see [10]. It can also help to know some basic coarse geometry, see [71, 80] as standard references, in particular the definition and the most elementary properties of asymptotic dimension. Some needed definitions of this area will be treated in the second preliminary section. Before we begin, let us specify some notations:

Notation 1.0.1. Unless specified otherwise, we will stick to the following notations throughout this dissertation.

- A denotes a C*-algebra.
- For a C*-algebra A, we write A_+ for the set of positive elements in A.
- G denotes a group.
- α denotes a group action on a C*-algebra or a metric space. If $\alpha : G \curvearrowright X$ is a topological action on a locally compact space, then we denote by $\bar{\alpha} : G \curvearrowright \mathcal{C}_0(X)$ the naturally induced action on the continuous functions over X vanishing at infinity, given by $\bar{\alpha}_g(f) = f \circ \alpha_{g^{-1}}$ for all $g \in G$ and $f \in \mathcal{C}(X)$.
- If M is some set and $F \subset M$ is a finite subset, then we write $F \Subset M$.

- If $\varepsilon > 0$ is a positive number and a, b are elements in some normed space, then we write $a =_\varepsilon b$ as a shortcut for $\|a - b\| \leq \varepsilon$.

- Assume that "dim" is one of the notions of dimension that appear in this dissertation, and X a mathematical object on which this dimension can be evaluated. Then we sometimes use the convenient shortcut $\dim^{+1}(X) = 1 + \dim(X)$.

- Let (X, d) be a metric space. For $x \in X$ and $r > 0$, we denote the ball of radius r centered at x as

$$B^d_r(x) = \{y \in X \mid d(x, y) < r\}\,.$$

  In the case where the metric is known from context, we sometimes omit it from the notation for convenience, and just write $B_r(x)$.

- Let (X, d) be a metric space. If $Y \subset X$ is a subset, we denote its diameter as

$$\mathrm{diam}(Y) = \mathrm{diam}_d(Y) = \sup\{d(x, y) \mid x, y \in Y\}\,.$$

1.1 C*-algebraic preliminaries

In this section, we shall recollect some of the notions in the more current C*-algebra theory that are omnipresent in this dissertation.

Definition 1.1.1 (cf. [10, 1.5.1] and [110, 1.3])**.** Let A and B be C*-algebras. Let $\varphi : A \to B$ be a linear map.

- φ is called positive, if $\varphi(a) \geq 0$ for all $a \geq 0$ in A.

- φ is called completely positive, if for each $n \in \mathbb{N}$, the matrix amplification $\varphi^n = \mathrm{id}_{M_n} \otimes \varphi : M_n(A) \to M_n(B)$ of φ, given by $\varphi^n\big((a_{i,j})_{1\leq i,j\leq n}\big) = ((\varphi(a_{i,j}))_{1\leq i,j\leq n}$, is positive.

- If $\varphi : A \to B$ is completely positive, then φ is said to have order zero, if for all $a, b \in A_+$ the condition $ab = 0$ implies $\varphi(a)\varphi(b) = 0$. In other words, φ is called order zero if it preserves orthogonality of positive elements.

For brevity, we will call a completely positive map a c.p. map. A contractive and completely positive map between C*-algebras is abreviated as a c.p.c. map.

Order zero maps in particular appear in the definition of nuclear dimension (as introduced by Winter and Zacharias) and decomposition rank (as introduced by Kirchberg and Winter), which have become increasingly important regularity properties for the classification theory of nuclear C*-algebras.

Definition 1.1.2 (cf. [111, 2.1])**.** Let A be a C*-algebra. A is said to have nuclear dimension $r \in \mathbb{N}$, denoted by $\dim_{\mathrm{nuc}}(A) = r$, if r is the smallest natural number with the following property: For all $F \Subset A$ and $\varepsilon > 0$, there exists a finite-dimensional C*-algebra $\mathcal{F}$ and c.p. maps $\psi : A \to \mathcal{F}$ and $\varphi : \mathcal{F} \to A$ such that

- ψ is contractive.
- φ allows a decomposition $\varphi = \varphi^{(0)} + \cdots + \varphi^{(r)}$ into $r + 1$ c.p.c. order zero maps $\varphi^{(j)} : \mathcal{F} \to A$ for $j = 0, \ldots, r$.
- $\|\varphi \circ \psi(a) - a\| \leq \varepsilon$ for all $a \in F$.

The triple $(\mathcal{F}, \psi, \varphi)$ is then called an $(r+1)$-decomposable c.p. approximation of tolerance ε on F. If no such r exists, one writes $\dim_{\mathrm{nuc}}(A) = \infty$.

The notion of nuclear dimension is a noncommutative generalization of covering dimension invented by Winter and Zacharias. In fact, nuclear dimension recovers the theory of covering dimension for locally compact metric spaces:[1]

Proposition 1.1.3 (see [111, 2.4])**.** *For any locally compact metric space X, one has* $\dim_{\mathrm{nuc}}(\mathcal{C}_0(X)) = \dim(X)$.

Remark 1.1.4. The decomposition rank of a C*-algebra, as introduced by Kirchberg and Winter in [55], is defined in a similar manner as nuclear dimension. Namely, a C*-algebra A has decomposition rank at most $r \in \mathbb{N}$, if for every $F \Subset A$ and $\varepsilon > 0$, there exists an $(r + 1)$-decomposable c.p. approximation $(\mathcal{F}, \psi, \varphi)$ as in 1.1.2 of tolerance ε on F, with the additional property that φ is contractive.

[1] We will recall the notion of topological covering dimension in the next section.

However, we note that our focus in this dissertation lies on nuclear dimension rather than on decomposition rank.

Next, we shall recall the notion of sequence algebras:

Definition 1.1.5. Let A be a C*-algebra. We call

$$A_\infty = \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A) = \ell^\infty(\mathbb{N}, A)/\left\{(a_n)_n \in \ell^\infty(\mathbb{N}, A) \mid \lim_{n\to\infty} \|a_n\| = 0\right\}.$$

the sequence algebra of A. Then A embeds naturally into A_∞ as equivalence classes of constant sequences. We will frequently make use of this identification of A as a subalgebra of A_∞ without mention.

Remark. On A_∞, the norm is given by the limes superior on representatives. That is, if $x \in A_\infty$ is the image of a sequence $(x_n)_n \in \ell^\infty(\mathbb{N}, A)$, then $\|x\| = \limsup_{n\to\infty} \|x_n\|$.

The following observation has appeared frequently in Winter's work, and can be seen as a consequence of an important structure theorem for order zero maps established by Winter and Zacharias (see [110, 2.3 and 3.1]), paired with Loring's observation that cones over finite-dimensional C*-algebras are projective (see [67, 10.1.11 and 10.2.1]).

Theorem 1.1.6 (cf. [105, 1.2.4] and [55, 2.4])**.** *Let A be a* C**-algebra with an ideal $J \subset A$. Let $q : A \to A/J$ denote the quotient map. Let $\mathcal{F}$ be a finite-dimensional* C**-algebra. Then for any c.p.c. order zero map $\sigma : \mathcal{F} \to A/J$, there exists a c.p.c. order zero map $\bar{\sigma} : \mathcal{F} \to A$ with $q \circ \bar{\sigma} = \sigma$.*

In some technical steps later in this chapter, it will be useful to have a slightly more flexible characterization of nuclear dimension. In a very similar form, this has been used by Hirshberg, Winter and Zacharias in [42].

Proposition 1.1.7 (cf. [42, A.4])**.** *Let A be a* C**-algebra and $r \in \mathbb{N}$ a natural number. Then $\dim_{\mathrm{nuc}}(A) \leq r$, if and only if the following holds: For all $F \Subset A$ and $\delta > 0$, there exists a finite-dimensional* C**-algebra $\mathcal{F}$, a c.p.c. map $\psi : A \to \mathcal{F}$ and c.p.c. order zero maps $\varphi^{(0)}, \dots, \varphi^{(r)} : \mathcal{F} \to A_\infty$ such that*

$$a =_\delta \sum_{l=0}^{r} \varphi^{(l)} \circ \psi(a) \quad \text{for all } a \in F.$$

Proof. Since the 'only if' part is trivial, we show the 'if' part. Let $F \Subset A$ and $\delta > 0$ be arbitrary. Find $\mathcal{F}, \psi, \varphi^{(0)}, \ldots, \varphi^{(r)}$ as in the assertion.

By 1.1.6, we can find sequences of c.p.c. order zero maps $\varphi_n^{(l)} : \mathcal{F} \to A$ for $l = 0, \ldots, r$ with $\varphi^{(l)}(x) = [(\varphi_n^{(l)}(x))_n]$ for all $x \in \mathcal{F}$. In particular, it follows that $\limsup_{n\to\infty} \|a - \sum_{l=0}^{r} \varphi_n^{(l)} \circ \psi(a)\| \leq \delta$ for all $a \in F$. So there exists n with $a =_{2\delta} \sum_{l=0}^{r} \varphi_n^{(l)} \circ \psi(a)$ for all $a \in F$. Since F and δ were arbitrary, this shows that $\dim_{\mathrm{nuc}}(A) \leq r$. □

1.2 Geometric preliminaries

In this section, we recall some definitions and terminology from topology, coarse geometry of metric spaces and geometric group theory, which we need in this dissertation. First, let us fix some terminology concerning covers of spaces:

Definition 1.2.1. Let X be a set. A cover of X is a family $\mathcal{U}$ of subsets of X with $X = \bigcup \mathcal{U}$. We say that a cover $\mathcal{U}$ refines another cover $\mathcal{V}$, if for every $U \in \mathcal{U}$, there exists $V \in \mathcal{V}$ with $U \subset V$.

Moreover, we will use the following terminologies:

- We say that a cover $\mathcal{U}$ has finite multiplicity $m \in \mathbb{N}$, if the intersection of any $m+1$ distinct members of $\mathcal{U}$ is empty, and if m is the smallest number with this property.
- If X is a topological space, then an open cover is a cover consisting of open sets in X.
- If X is equipped with a metric d, we say that a cover $\mathcal{U}$ of X is uniformly bounded by $r > 0$, if $\mathrm{diam}(U) \leq r$ for all $U \in \mathcal{U}$.
- If X is equipped with a metric d, we say that a cover $\mathcal{U}$ of X has Lebesgue number $R > 0$, if every subset in X with diameter less than R is contained in a member of $\mathcal{U}$, and if R is the largest number with this property.

Now let us recall the notion of topological covering dimension:

Definition 1.2.2 (cf. [27, 1.6.7])**.** Let X be a normal topological space. The covering dimension (or Čech-Lebesgue dimension) of X, denoted $\dim(X)$, is the smallest natural number $d \in \mathbb{N}$ with the following property: For every finite open cover $\mathcal{U}$ of X, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that the multiplicity of $\mathcal{V}$ is at most $d+1$.

The following variation can be quite useful:

Proposition 1.2.3 (cf. [55, 1.6])**.** *Let X be a normal topological space and $d \in \mathbb{N}$ a natural number. Then $\dim(X) \leq d$ if and only if the following holds:*

For every finite open cover $\mathcal{U}$ of X, there exists an open refinement $\mathcal{V}$ of $\mathcal{U}$ such that one can write $\mathcal{V} = \mathcal{V}^{(0)} \cup \cdots \cup \mathcal{V}^{(d)}$ with the property

$$V_1 \cap V_2 = \emptyset \quad \textit{for all } V_1 \neq V_2 \textit{ in } \mathcal{V}^{(l)}$$

for all $l = 0, \ldots, d$.

We continue with some coarse geometry, and will follow Nowak and Yu's book [71] for the most part.

Definition 1.2.4 (cf. [71, 1.3.1 and 1.4.4])**.** Let (X, d_X) and (Y, d_Y) be two metric spaces. A map $f : X \to Y$ is called a coarse equivalence, if:

- There exists some $R > 0$ such that the image $f(X)$ interesects all balls of radius R in Y non-trivially.
- There exist functions $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ with $\lim_{t\to\infty} \rho_-(t) = \infty = \lim_{t\to\infty} \rho_+(t)$ such that we have

  $$\rho_-(d_X(x, y)) \leq d_Y(f(x), f(y)) \leq \rho_+(d_X(x, y))$$

  for all $x, y \in X$.

Two metric spaces (X, d_X) and (Y, d_Y) are called coarsely equivalent, if there exists a coarse equivalence from X to Y.

Definition 1.2.5 (cf. [71, 2.2.1])**.** Let (X, d) be a metric space. The asymptotic dimension of X is the smallest natural number $n \in \mathbb{N}$ such that the following holds: For any $R > 0$, there exists a uniformly bounded cover $\mathcal{U}$ of X, such that for any $x \in X$, the ball $B_R^d(x)$ intersects at most $n+1$ distinct members of $\mathcal{U}$. We write $\operatorname{asdim}(X) = n$ in this case. If no such number exists, we write $\operatorname{asdim}(X) = \infty$.

Recall that the asymptotic dimension of a metric space is an invariant of its coarse equivalence class:

Theorem 1.2.6 (see [71, 2.2.5])**.** *Two coarsely equivalent metric spaces have the same asymptotic dimension.*

For some purposes, it is useful to have the following equivalent characterization of asymptotic dimension, which can be obtained in a more or less straightforward manner by using the equivalent characterizations of asymptotic dimension exhibited in [4].

Proposition 1.2.7. *Let (X, d) be a metric space and let $n \in \mathbb{N}$ be a natural number. Then the following are equivalent:*

(1) $\operatorname{asdim}(X) \leq n$.

(2) *For every $R > 0$, there exist collections $\mathcal{U}^{(0)}, \ldots, \mathcal{U}^{(n)}$ of subsets of X, such that the following holds:*

- *for each $l = 0, \ldots, n$, any two distinct sets in $\mathcal{U}^{(l)}$ are disjoint;*
- *the family $\mathcal{U} = \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(n)}$ is a uniformly bounded cover of X with Lebesgue number at least R.*

(3) *For every $R > 0$, there exist collections $\mathcal{U}^{(0)}, \ldots, \mathcal{U}^{(n)}$ of subsets of X, such that the following holds:*

- *for each $l = 0, \ldots, n$, the distance between two distinct sets in $\mathcal{U}^{(l)}$ is at least R;*
- *the family $\mathcal{U} = \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(n)}$ is a uniformly bounded cover of X with Lebesgue number at least R.*

Proof. (1) $\implies$ (3) : Let $R > 0$ be a positive number and assume $\operatorname{asdim}(X) \leq n$. The equivalence "(2) $\Leftrightarrow$ (3)" within Theorem 19 from [4] shows that there exist collections $\mathcal{V}^{(0)}, \ldots, \mathcal{V}^{(n)}$ of subsets in X such that $\mathcal{V} = \mathcal{V}^{(0)} \cup \cdots \cup \mathcal{V}^{(n)}$ is a uniformly bounded cover and such that for each $l = 0, \ldots, n$, the distance between two distinct sets in $\mathcal{V}^{(l)}$ is greater than $3R$. Taking the memberwise R-neighbourhoods, we can consider the collections

$$\mathcal{U}^{(l)} = \left\{ B_R(V) \mid U \in \mathcal{V}^{(l)} \right\} \quad \text{with} \quad B_R(V) = \{x \in X \mid \operatorname{dist}(x, V) < R\}$$

for each $l = 0, \dots, n$. Then $\mathcal{U} = \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(n)}$ is clearly a uniformly bounded cover. By the triangle inequality, we have for each $l = 0, \dots, n$ that the distance between two distinct members in $\mathcal{U}^{(l)}$ is at least R. Now let $M \subset X$ be any subset with diameter less than R. Given any $x_0 \in M$, there is some $l \in \{0, \dots, n\}$ and $V \in \mathcal{V}^{(l)}$ with $x_0 \in V$. It follows that $M \subset B_R(x_0) \subset B_R(V) \in \mathcal{U}^{(l)}$. Thus, $\mathcal{U}$ has Lebesgue number at least R.

(3) $\Longrightarrow$ (1) : Let $R > 0$ be a positive number. Choose uniformly bounded collections $\mathcal{U}^{(0)}, \dots, \mathcal{U}^{(n)}$ of subsets of X with the required properties from (3) with respect to the number $2R+1$. Let $x \in X$. Then $B_R(x)$ has diameter less than $2R+1$ by the triangle inequality, and thus is contained in some $U \in \mathcal{U}^{(l)}$ for a number $l \in \{0, \dots, n\}$. But since for each $l = 0, \dots, n$, the collection $\mathcal{U}^{(l)}$ consists of sets with pairwise-distance at least $2R + 1$, it follows that $B_R(x)$ can intersect at most one set in $\mathcal{U}^{(l)}$. In particular, $B_R(X)$ can intersect at most $n + 1$ members of $\mathcal{U} = \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(n)}$. Since $R > 0$ was arbitrary, this shows that $\operatorname{asdim}(X) \leq n$.

(2) $\Longrightarrow$ (3): Let $R > 0$ be a positive number. Choose uniformly bounded collections $\mathcal{U}^{(0)}, \dots, \mathcal{U}^{(n)}$ of subsets of X such that $\mathcal{U} = \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(n)}$ is a cover with Lebesgue number at least $4R$, and such that for each $l = 0, \dots, n$, the collection $\mathcal{U}^{(l)}$ consists of pairwise disjoint sets. Given a subset $Z \subset X$, denote $B_{-R}(Z) = \{x \in X \mid B_R(x) \subset Z\}$. For each $l = 0, \dots, n$, set $\mathcal{V}^{(l)} = \{B_{-R}(U) \mid U \in \mathcal{U}^{(l)}\}$, and set $\mathcal{V} = \mathcal{V}^{(0)} \cup \cdots \cup \mathcal{V}^{(n)}$. Since $\mathcal{U}$ has Lebesgue number at least $4R$, it follows that given any $x \in X$, the set $B_{2R}(x)$ is contained in a member U of $\mathcal{U}$. In particular, $x \in B_{-R}(U)$, and thus $\mathcal{V}$ is a cover. Moreover, for x as before and each $y \in B_R(x)$, we have that $B_R(y) \subset B_{2R}(x)$ is also contained in the same set U. So by definition, it follows that $B_R(x) \subset B_{-R}(U) \in \mathcal{U}$. Since any set of diameter at most R is trivially contained in a ball of radius R, it follows that the Lebesgue number of $\mathcal{U}$ is at least R. Moreover, since any two distinct sets in $\mathcal{U}^{(l)}$ are pairwise disjoint, it follows from the construction that the distance between any two distinct sets in $\mathcal{V}^{(l)}$ is at least R.

(3) $\Longrightarrow$ (2): This is trivial. □

Definition 1.2.8 (cf. [71, 1.4.12]). Let (X_n, d_n) be a sequence of finite metric spaces. Consider the disjoint union $X = \bigsqcup_{n \in \mathbb{N}} X_n$ as a set, and let d_X be a metric on X satisfying the following two properties:

- for all $n \in \mathbb{N}$, the metric d_X restricted to X_n coincides with d_n;

- as $i \neq j$ and $i + j \to \infty$, we have $\mathrm{dist}_{d_X}(X_i, X_j) \to \infty$.

Then the pair (X, d_X) is called a coarse disjoint union of the sequence (X_n, d_n).

It seems to be a well-known fact in the field of coarse geometry that a metric as above always exists, and that the properties of such a metric, as required by 1.2.8, uniquely determine (X, d_X) up to coarse equivalence. However, I could not find any reference with an explicit proof of this fact. Hence, the detailed proofs are attached for the reader's convenience. But first, we establish a technical Lemma, which we will use several times throughout this chapter.

Lemma 1.2.9. *Let X be a set and $\mathcal{M}$ a family of metrics on X. Suppose that for any two metrics $d^{(1)}, d^{(2)} \in \mathcal{M}$, there exists a monotonously increasing function $\rho : [0, \infty) \to [0, \infty)$ with $\lim_{r\to\infty} \rho(r) = \infty$, and such that $d^{(2)}(x, y) \leq \rho(d^{(1)}(x, y))$ for all $x, y \in X$. Then for any two metrics $d^{(1)}, d^{(2)} \in \mathcal{M}$, the identity map on X is a coarse equivalence between $(X, d^{(1)})$ and $(X, d^{(2)})$.*

Proof. Since the identity map on X is bijective, the first condition of 1.2.4 is obviously satisfied. Given any two metrics $d^{(1)}, d^{(2)} \in \mathcal{M}$, let us construct the functions $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ as required by the second condition of 1.2.4.

First, find a monotonously increasing function $\rho_+ : [0, \infty) \to [0, \infty)$ with $\lim_{r\to\infty} \rho_+(r) = \infty$, and such that $d^{(2)}(x, y) \leq \rho_+(d^{(1)}(x, y))$ for all $x, y \in X$. Exchanging the roles of $d^{(1)}$ and $d^{(2)}$, also find a monotonously increasing function $\mu : [0, \infty) \to [0, \infty)$ with $\lim_{r\to\infty} \mu(r) = \infty$, and such that $d^{(1)}(x, y) \leq \mu(d^{(2)}(x, y))$ for all $x, y \in X$.

Define the map $\rho_- : [0, \infty) \to [0, \infty)$ via

$$\rho_-(r) = \inf \{s > 0 \mid r \leq \mu(s)\}.$$

Given any $R > 0$, the set $\mu([0, R])$ is bounded by some $r_0 > 0$, and thus we have $\rho_-(r) \geq R$ for all $r \geq r_0 + 1$. Thus we have $\lim_{r\to\infty} \rho_-(r) = \infty$. Lastly, by the choice of μ it follows that

$$\rho_-(d^{(1)}(x, y)) = \inf \left\{s > 0 \mid d^{(1)}(x, y) \leq \mu(s)\right\} \leq d^{(2)}(x, y)$$

for all $x, y \in X$. What this all amounts to is that we have found functions $\rho_-, \rho_+ : [0, \infty) \to [0, \infty)$ with $\lim_{r\to\infty} \rho_-(r) = \infty = \lim_{r\to\infty} \rho_+(r)$ satisfying the inequalities

$$\rho_-(d^{(1)}(x, y)) \leq d^{(2)}(x, y) \leq \rho_+(d^{(1)}(x, y))$$

for all $x, y \in X$. This shows that $\mathrm{id} : (X, d^{(1)}) \to (X, d^{(2)})$ is indeed a coarse equivalence. □

Proposition 1.2.10. *Let (X_n, d_n) be a sequence of finite metric spaces. Suppose that $d^{(1)}$ and $d^{(2)}$ are two metrics on the disjoint union $X = \bigsqcup_{n\in\mathbb{N}} X_n$ realizing both pairs $(X, d^{(1)})$ and $(X, d^{(2)})$ as coarse disjoint unions of the sequence (X_n, d_n). Then $(X, d^{(1)})$ and $(X, d^{(2)})$ are coarsely equivalent via the identity map from $(X, d^{(1)})$ to $(X, d^{(2)})$.*

Proof. Let $\mathcal{M}$ be the family of all metrics d on X realizing the pair (X, d) as a coarse disjoint union of the sequence (X_n, d_n). In order to show the statement, we appeal to 1.2.9.

So let $d^{(1)}, d^{(2)} \in \mathcal{M}$. Let $r > 0$ be any number. We claim that there are only finitely many pairs $(x, y) \in X \times X$ satisfying $d^{(1)}(x, y) < d^{(2)}(x, y)$ and $d^{(1)}(x, y) \leq r$. Indeed, by assumption, there exists $k_0 \in \mathbb{N}$ such that for all natural numbers $i \neq j$ and $i + j \geq k_0$, we have $\mathrm{dist}_{d^{(1)}}(X_i, X_j) > r$. Since both $d^{(1)}$ and $d^{(2)}$ restrict to the same metric on each set X_i, the condition $d^{(1)}(x, y) < d^{(2)}(x, y)$ implies that $x \in X_i$ and $y \in X_j$ for some $i \neq j$. But then we necessarily have $x, y \in \bigsqcup_{n\leq k_0} X_n$, which leaves only finitely many possibilites.

We can thus obtain a well-defined map $f : [0, \infty) \to \{-\infty\} \cup [0, \infty)$ given by

$$f(r) = \max \left\{ d^{(2)}(x, y) \;\middle|\; d^{(1)}(x, y) < d^{(2)}(x, y) \text{ and } d^{(1)}(x, y) \leq r \right\}.$$

Now set $\rho(r) = \max \{r, f(r)\}$ to obtain a monotonously increasing function $\rho : [0, \infty) \to [0, \infty)$ with $\lim_{r\to\infty} \rho(r) = \infty$. By construction of this map, we have the inequality $d^{(2)}(x, y) \leq \rho(d^{(1)}(x, y))$ for all $x, y \in X$. But his verifies the conditions in 1.2.9, and the claim follows. □

Proposition 1.2.11. *Let (X_n, d_n) be a sequence of finite metric spaces. For each $n \in \mathbb{N}$, let $x_n \in X_n$ be a point. Let $(a_n)_n$ be a sequence of positive* *numbers with* $\lim_{n\to\infty} a_n = \infty$*. Consider the disjoint union* $X = \bigsqcup_{n\in\mathbb{N}} X_n$*. Then the map* $d : X \times X \to [0, \infty)$ *given by*

$$d(y,z) = \begin{cases} d_n(y,z) & , \quad y,z \in X_n \\ d_k(y,x_k) + \sum_{j=k}^{n-1} a_j + d(x_n,z) & , \quad y \in X_k, z \in X_n \textit{ with } k < n \\ d_k(z,x_k) + \sum_{j=k}^{n-1} a_j + d(x_n,y) & , \quad z \in X_k, y \in X_n \textit{ with } k < n. \end{cases}$$

yields a metric on X*, which realizes the pair* (X, d) *as a coarse disjoint union of the sequence* (X_n, d_n)*.*

Proof. We first verify that d is a metric. By definition, it is symmetric, i.e. $d(y,z) = d(z,y)$ for all $y,z \in X$. Suppose that $d(y,z) = 0$ for some $y,z \in X$. Then we necessarily have $y,z \in X_n$ for some n, and $d_n(y,z) = 0$. Since each d_n is a metric, we obtain $y = z$. Now let $z_1, z_2, z_3 \in X$ and let us show the triangle inequality $d(z_1,z_3) \leq d(z_1,z_2) + d(z_2,z_3)$. Let $n_1, n_2, n_3 \in \mathbb{N}$ with $z_i \in X_{n_i}$ for $i = 1,2,3$. Let us fix the following notation: For any $k,l \in \mathbb{N}$, denote

$$b_{k,l} = \sum_{j=\min\{k,l\}}^{\max\{k,l\}-1} a_j.$$

Note that this defines a metric on $\mathbb{N}$, by declaring the distance between k and $k+1$ to be a_k for all $k \in \mathbb{N}$, and extending it with the linear order. Hence we have a triangle inequality, i.e. we have $b_{k,l} + b_{l,m} \leq b_{l,m}$ for all $k,l,m \in \mathbb{N}$.

Case 1: $n_1 = n_2 = n_3$**.** In this case, d restricts to the given metric on the finite subset X_{n_1}, and hence the triangle inequality is trivially satisfied.

Case 2: $n_1 \neq n_2 = n_3$**.** Then

$$\begin{aligned} d(z_1,z_2) + d(z_2,z_3) &= d_{n_1}(z_1,x_{n_1}) + b_{n_1,n_2} + d_{n_2}(x_{n_2},z_2) + d_{n_2}(z_2,z_3) \\ &\geq d_{n_1}(z_1,x_{n_1}) + b_{n_1,n_2} + d_{n_2}(x_{n_2},z_3) \\ &= d(z_1,z_3). \end{aligned}$$

Case 3: $n_1 \neq n_2 \neq n_3 = n_1$**.** Then

$$
\begin{aligned}
& d(z_1, z_2) + d(z_2, z_3) \\
& \quad = \quad d_{n_1}(z_1, x_{n_1}) + 2b_{n_1,n_2} + 2d_{n_2}(z_2, x_{n_2}) + d_{n_1}(z_3, x_{n_1}) \\
& \quad \geq \quad d_{n_1}(z_1, x_{n_1}) + d_{n_1}(z_3, x_{n_1}) \\
& \quad \geq \quad d_{n_1}(z_1, z_3) \ = \ d(z_1, z_3).
\end{aligned}
$$

Case 4: $n_1 \neq n_2 \neq n_3 \neq n_1$**.** Then

$$
\begin{aligned}
& d(z_1, z_2) + d(z_2, z_3) \\
& \quad = \quad d_{n_1}(z_1, x_{n_1}) + b_{n_1,n_2} + 2d_{n_2}(z_2, x_{n_2}) + b_{n_2,n_3} + d_{n_3}(z_3, x_{n_3}) \\
& \quad \geq \quad d_{n_1}(z_1, x_{n_1}) + b_{n_1,n_2} + b_{n_2,n_3} + d_{n_3}(z_3, x_{n_3}) \\
& \quad \geq \quad d_{n_1}(z_1, x_{n_1}) + b_{n_1,n_3} + d_{n_3}(z_3, x_{n_3}) \\
& \quad = \quad d(z_1, z_3).
\end{aligned}
$$

Thus we have verified that d defines a metric on X. It trivially satisfies the property that it recovers the metric d_n on each subset X_n. Moreover, note that for all $k < l$, we have

$$
\operatorname{dist}_d(X_k, X_l) \geq \sum_{j=k}^{l-1} a_j \geq a_{l-1} \geq a_{\lfloor \frac{l+k}{2} \rfloor}.
$$

As $k + l \to \infty$, this expression tends to infinity because of $\lim_{n\to\infty} a_n = \infty$. So we have shown that the pair (X, d) is indeed a coarse disjoint union of the sequence (X_n, d_n). This finishes the proof. □

Combining the previous two observations, we obtain:

Corollary 1.2.12. *Let (X_n, d_n) be a sequence of finite metric spaces. Then there exists a coarse disjoint union (X, d) for (X_n, d_n), and it is uniquely determined by the conditions given in 1.2.8 up to coarse equivalence.*

We now recall some basics of geometric group theory, in the sense of viewing groups as proper metric spaces. We will first stay in the more elementary setting of countable, discrete groups, mostly following [71, Chapter 1, Section 2].

Definition 1.2.13. Let (X, d) be a metric space. We say that d or (X, d) is proper, if for any $x \in X$ and $r > 0$, the closed r-ball $\{y \in X \mid d(x, y) \leq r\}$ is compact. This is equivalent to saying that the compact sets in X are given by the bounded and closed sets.

Definition 1.2.14 (cf. [71, 1.2.1])**.** Let G be a group. A length function on G is a function $\ell : G \to [0,\infty)$ such that for any $g, h \in G$ the following conditions are satisfied:

- $\ell(g) = 0$ if and only if g is the identity element;
- $\ell(g) = \ell(g^{-1})$;
- $\ell(gh) \leq \ell(g) + \ell(h)$.

Proposition 1.2.15 (cf. [71, 1.2.2])**.** *Every countable, discrete group admits a proper length function.*

For the following, see also the discussion after Proposition 1.2.2 in [71]:

Example 1.2.16. Let G be a finitely generated group, and let us fix a finite generating set $S \Subset G$ with $S = S^{-1}$. Then the map $\ell : G \to \mathbb{N}$ given by $\ell(g) = \min\{n \mid g = s_1 \cdots s_n \text{ with } s_i \in S\}$ defines a proper length function on G.

Proposition 1.2.17 (cf. [71, 1.2.5])**.** *Let G be a group. Given any length function ℓ, the map $d : G \times G \to [0,\infty)$ given by $d(x,y) = \ell(xy^{-1})$ defines a right-invariant metric. Conversely, given a right-invariant metric d on G, the map $\ell : G \to [0,\infty)$ given by $\ell(g) = d(g, 1_G)$ is a length function.*

This yields a one-to-one correspondence between the length functions on G and the right-invariant metrics on G, which restricts to a one-to-one correspondence between the proper length functions and the proper, right-invariant metrics on G.

In particular, it follows that any countable, discrete group G admits a proper, right-invariant metric.

We note that most authors in the standard literature are usually interested in left-invariant metrics on a group rather than right-invariant metrics. However, these are in a canonical one-to-one correspondence: Composing any left-invariant metric on a group with the group inversion yields a right-invariant metric and vice versa.

As it turns out, such a proper, right-invariant metric on a group is unique up to coarse equivalence:

Theorem 1.2.18 (cf. [71, 1.4.7])**.** *Let G be a countable, discrete group. Then for any two proper, right-invariant metrics $d^{(1)}$ and $d^{(2)}$ on G, there exists a monotonously increasing function $\rho : [0, \infty) \to [0, \infty)$ with $\lim\limits_{r\to\infty} \rho(r) = \infty$ and such that $d^{(2)}(g, h) \leq \rho(d^{(1)}(g, h))$ for all $g, h \in G$. Moreover, the metric spaces $(G, d^{(1)})$ and $(G, d^{(2)})$ are coarsely equivalent via the identity map.*

Proof. For $i = 1, 2$, the metric $d^{(i)}$ is given by a proper length function $\ell^{(i)}$ in the sense of 1.2.17. Thus, it suffices to show that there exists a monotonously increasing function $\rho : [0, \infty) \to [0, \infty)$ with $\lim_{r\to\infty} \rho(r) = \infty$, such that we have the inequality $\ell^{(2)}(g) \leq \rho(\ell^{(1)}(g))$ for all $g \in G$. Indeed, set

$$\rho(r) = \max\left\{\ell^{(2)}(g) \;\middle|\; g \in G \text{ with } \ell^{(1)}(g) \leq r\right\}.$$

Since the length functions $\ell^{(1)}$ and $\ell^{(2)}$ are proper, this yields a well-defined map. It is obviously monotonously increasing. Lastly, it follows directly from the definition that $\ell^{(2)}(g) \leq \rho(\ell^{(1)}(g))$ for every $g \in G$.

The second claim now follows by applying 1.2.9 to the family of all proper, right-invariant metrics on G. □

Let us also mention a more general existence result, due to Struble [90], for proper, left-invariant metrics in the case of locally compact groups:

Theorem 1.2.19 (cf. [90] and [15, 2.B.5])**.** *A locally compact group G is second-countable if and only if it is metrizable by a proper, left-invariant metric.*

Remark 1.2.20. By the canonical one-to-one correspondence between left-invariant metrics and right-invariant metrics via group inversion, it follows from 1.2.19 in particular that a locally compact, second-countable group G is metrizable by a proper, right-invariant metric.

To conclude this preliminary section, let us recall the notion of geometric group actions and associated orbit spaces, following the book of Drutu and Kapovich [18, Chapter 3]:

Definition 1.2.21 (cf. [18, 3.1.1 and 3.1.4])**.** Let (X, d) be a locally compact metric space and G a group. Let $\alpha : G \curvearrowright X$ be an action via homeomorphisms. Then α is called

- isometric, if α acts by isometries in the sense that $d(\alpha_g(x), \alpha_g(y)) = d(x, y)$ for all $g \in G$ and $x, y \in X$.

- properly discontinuous, if for every pair of compact subsets $K_1, K_2 \subset X$, there exist only finitely many $g \in G$ with $\alpha_g(K_1) \cap K_2 \neq \emptyset$.

- cobounded, if there exists a bounded set $B \subset X$ such that the collection $\{\alpha_g(B) \mid g \in G\}$ covers X.

- cocompact, if there exists a compact subset $K \subset X$ such that the collection $\{\alpha_g(K) \mid g \in G\}$ covers X.

- geometric, if it is isometric, properly discontinuous and cobounded.

Remark 1.2.22. Observe in the above definition that if the metric d on X is proper, then cobounded implies cocompact, since closures of bounded sets are compact.

Lemma 1.2.23 (cf. [18, 3.18])**.** *Let (X, d) be a locally compact, proper metric space and G a group. Let $\alpha : G \curvearrowright X$ be a faithful, isometric action. Then α is properly discontinuous if and only if G is discrete.*

Notation. Let $\alpha : G \curvearrowright X$ be an action of a group on some set. For each $x \in G$, denote the G-orbit of x by $\mathcal{O}_\alpha(x) = \{\alpha_g(x) \mid g \in G\}$.

Theorem 1.2.24 (cf. [18, 3.2, 3.3 and 3.20])**.** *Let (X, d) be a locally compact, proper metric space and G a countable, discrete group. Let $\alpha : G \curvearrowright X$ be a geometric action. Consider the orbit space $X/G = \{\mathcal{O}_\alpha(x) \mid x \in X\}$ with the natural surjection $\pi : X \to X/G, x \mapsto \mathcal{O}_\alpha(x)$. Then X/G, equipped with the quotient topology via π, is a compact Hausdorff space. Moreover, it is metrizable by measuring the distances of orbits, i.e. the map $\pi_*(d) : (X/G) \times (X/G) \to [0, \infty)$ given by*

$$\pi_*(d)(\mathcal{O}_\alpha(x), \mathcal{O}_\alpha(y)) = \inf \{d(\alpha_g(x), \alpha_h(y)) \mid g, h \in G\}$$

is a metric (called the push-forward metric of d) and induces the quotient topology on X/G.

Proposition 1.2.25. *Let X, G, α, d, π be as in 1.2.24. If α is free, then π is a local homeomorphism.*

Proof. Let $x \in X$. Choose a compact neighbourhood K of x. Since α is properly discontinuous, the set $F = \{g \in G \mid \alpha_g(K) \cap K \neq \emptyset\}$ is finite. Since α is free, we can find a compact neighbourhood K' of X such that $K' \cap \alpha_g(K') = \emptyset$ for all $g \in F \setminus \{1_G\}$. In particular, we can find some $r > 0$ such that

$$\overline{B}_r(x) \cap \alpha_g(\overline{B}_r(x)) = \emptyset \quad \text{for all } g \in G \setminus \{1_G\}\,.$$

By choosing r even smaller (for instance replacing it by $r/4$), we assume moreover that $d(y, \alpha_g(z)) \geq 3r$ for all $y, z \in \overline{B}_r(x)$ and all $g \neq 1_G$. But then it is obvious that π is injective on $\overline{B}_r(x)$. Since $\overline{B}_r(x)$ is compact, π restricts to a homeomorphism from $\overline{B}_r(x)$ onto its image. This shows that π is indeed a local homeomorphism. □

Of particular importance is the following special case:

Definition 1.2.26. Let G be a locally compact, second-countable group. A closed subgroup $H \subset G$ is called cocompact, if the action $\rho : H \curvearrowright G$ given by right-multiplication $\rho_h(g) = g \cdot h^{-1}$, is cocompact. That is, H is cocompact if and only if $G = \bigcup_{h \in H} K \cdot h$ for some compact subset $K \subset G$.

Remark 1.2.27. Observe that in the above situation, ρ-orbits are just left cosets with respect to H in the ordinary group-theoretic sense. That is, for each $g \in G$, we have $\mathcal{O}_\rho(g) = gH$. In particular, the orbit space of ρ agrees with the usual group theoretic quotient space G/H.

Corollary 1.2.28. *Let G be a locally compact, second-countable group, equipped with a proper, right-invariant metric d on G. Let $H \subset G$ be discrete, cocompact subgroup, and let $\pi : G \to G/H$ denote the quotient map. Then the quotient space G/H is a compact Hausdorff space, and is metrizable by the push-forward metric $\pi_*(d) : (G/H) \times (G/H) \to [0, \infty)$ given by*

$$\pi_*(d)(g_1H, g_2H) = \mathrm{dist}_d(g_1H, g_2H) = \inf\{d(g_1h_1, g_2h_2) \mid h_1, h_2 \in H\}$$

for all $g_1, g_2 \in G$. Moreover, π is a local homeomorphism.

Proof. Since d is chosen to be a right-invariant metric, the H-action on G by right-multiplication is isometric. It is trivially free. It is also properly discontinuous by 1.2.23 because H is discrete. Since H is cocompact by assumption, it follows that its induced action on G is cocompact. In particular, this action is geometric. The claim now follows directly from 1.2.27, 1.2.24 and 1.2.25. □

1.3 Box spaces and asymptotic dimension

In the next three sections, we will develop the theory of Rokhlin dimension for actions of residually finite groups on C*-algebras, extending such a theory for finite groups and the integers by Hirshberg, Winter and Zacharias from [42]. The content of these sections comes from a joint paper with Wu and Zacharias [2], which is available as an arxiv preprint in [96]. There is a substantial text overlap between the following sections of this chapter and [96], with large parts carried over verbatim with only minor changes. However, it should be pointed out that the second section of the current preprint version of [96], which corresponds to this section, contains several mistakes. For this reason, there are some adjustments in this section, some of which which we will indicate below after 1.3.1.

To avoid any confusion concerning the text overlap with [96], we will follow the convention to give the corresponding references to the preprint [96] for all partial results that were originally proved in [96], either verbatim or in a similar form. We note, however, that the Rokhlin dimension theory developed in [96] is more general than presented here because it treats also cocycle actions instead of only genuine group actions.

Before one delves into the details of Rokhlin dimension, it is important to understand a certain geometric invariant of discrete and residually finite groups, namely their box spaces. This notion is based on an idea of Roe from [80]. In this section, we will introduce box spaces of residually finite groups. A special focus will be on the asymptotic dimension of such spaces. The reason is (as we will see in the last section of this chapter) that the Rokhlin dimension theory for actions of a given group G is compatible with the notion of nuclear dimension, if G has a box space with finite asymptotic dimension.

Definition 1.3.1. Let G be a countable, discrete group. Let $G_n \trianglelefteq G$ be a decreasing sequence of normal subgroups with finite index, i.e.

[2] Each author has contributed an approximately equal amount to this collaboration.

$[G : G_n] < \infty$ and $G_{n+1} \subset G_n$ for all $n \in \mathbb{N}$. We say that the sequence $(G_n)_n$ is separating if $\bigcap_{n\in\mathbb{N}} G_n = \{1_G\}$. G is called residually finite, if it has such a separating sequence of normal subgroups with finite index. In what follows, such a decreasing and separating sequence is called a residually finite approximation.

Remark. At this point, the reader should be warned about a difference between this section and the analogous section in the current version of the preprint [96] on box spaces. Namely, residually finite approximations are not assumed in [96] to consist of subgroups that are normal. However, some assumptions on the subgroups in question are necessary. For instance, one can find counterexamples to the important technical Lemma 1.3.8 below within separating sequences of arbitrary subgroups with finite index. Since 1.3.8 is implicitely used in [96], the results presented in the second section of [96] are not correct in the presented generality.

Now, we will follow the ideas from [80, 11.24], [71, Chapter 4, Section 4] and the introduction of [48] to define box spaces of residually finite groups. However, unlike in these references, we do not require the groups to be finitely generated for the definition.

Definition 1.3.2. Let G be a countable, discrete, residually finite group and let $\sigma = (G_n)_n$ be a residually finite approximation of G. Let us equip G with a proper, right-invariant metric d. For each $n \in \mathbb{N}$, consider the quotient map $\pi_n : G \to G/G_n$. The box space $\Box_\sigma G$ associated to σ is defined as the coarse disjoint union of the sequence of finite metric spaces $(G/G_n, \pi_{n*}(d))$.

Since the metric d is chosen above to define the box space $\Box_\sigma G$, we have to justify why it does not appear in its notation. As it turns out, the coarse equivalence class of $\Box_\sigma G$ does not depend on the choice of d. In the finitely generated case, this is mentioned in the introduction of [48], but without an explicit proof.

Proposition 1.3.3. *Let G be a countable, discrete, residually finite group and let $\sigma = (G_n)_n$ be a residually finite approximation of G. For each $n \in \mathbb{N}$, consider the quotient map $\pi_n : G \to G/G_n$. Let $d^{(1)}$ and $d^{(2)}$ be two proper, right-invariant metrics on G. Then the coarse disjoint unions of the sequences $(G/G_n, \pi_{n*}(d^{(1)}))$ and $(G/G_n, \pi_{n*}(d^{(2)}))$ are coarsely equivalent via the identity map.*

Proof. Consider the family $\mathcal{M}$ of metrics d_B on the disjoint union $X = \bigsqcup_{n\in\mathbb{N}} G/G_n$ satisfying the requirements of 1.2.8 for the sequence of finite metric spaces $(G/G_n, \pi_{n*}(d))$, where d is an arbitrary proper, right-invariant metric on G. In order to show the claim, we appeal to 1.2.9.

So let $d^{(1)}$ and $d^{(2)}$ be two proper, right-invariant metrics on G, and $d_B^{(1)}, d_B^{(2)} \in \mathcal{M}$ two metrics on X such that for $i = 1, 2$ the metric $d_B^{(i)}$ satisfies the requirements of 1.2.8 for the sequence $(G/G_n, \pi_{n*}(d^{(i)}))$.

By 1.2.18, we can find a monotonously increasing function $\mu : [0,\infty) \to [0,\infty)$ with $\lim_{r\to\infty} \mu(r) = \infty$ and such that we have the inequality $d^{(2)}(g,h) \leq \mu(d^{(1)}(g,h))$ for all $g,h \in G$. We claim that for all $r \geq 0$, there are only finitely many pairs $x, y \in X$ satisfying both $\mu(d_B^{(1)}(x,y)) < d_B^{(2)}(x,y)$ and $\mu(d_B^{(1)}(x,y)) \leq r$.

For $n \in \mathbb{N}$, let $x, y \in G/G_n$ be two elements. Write $x = \pi_n(g_1)$ and $y = \pi_n(g_2)$ for elements $g_1, g_2 \in G$ such that $\pi_{n*}(d^{(1)})(x,y) = d^{(1)}(g_1,g_2)$. Then we have

$$\begin{aligned} \mu(d_B^{(1)}(x,y)) &= \mu\big(\pi_{n*}(d^{(1)})(x,y)\big) \\ &= \mu(d^{(1)}(g_1,g_2)) \\ &\geq d^{(2)}(g_1,g_2) \\ &\geq \pi_{n*}(d^{(2)})(x,y) \\ &= d_B^{(2)}(x,y). \end{aligned}$$

In particular, for any pair $x, y \in X$, the condition $\mu(d_B^{(1)}(x,y)) < d_B^{(2)}(x,y)$ implies that $x \in G/G_i$ and $y \in G/G_j$ for some $i \neq j$.

Choose $s_0 > 0$ such that $\mu(s) > r$ for all $s \geq s_0$. Then choose $k_0 \in \mathbb{N}$ such that for any $i \neq j$ and $i + j \geq k_0$, we have $\operatorname{dist}_{d_B^{(1)}}(G/G_i, G/G_j) \geq s_0$. Thus it follows that $\mu\big(\operatorname{dist}_{d_B^{(1)}}(G/G_i, G/G_j)\big) > r$ for all $i \neq j$ with $i + j \geq k_0$. In particular, for any two elements $x, y \in X$, the two conditions $\mu(d_B^{(1)}(x,y)) < d_B^{(2)}(x,y)$ and $\mu(d_B^{(1)}(x,y)) \leq r$ together imply that $x, y \in \bigsqcup_{i\leq k_0} G/G_i$, which leaves only finitely many possibilities.

We obtain a well-defined map $f : [0,\infty) \to \{-\infty\} \cup [0,\infty)$ given by

$$f(r) = \max\left\{ d_B^{(2)}(x,y) \;\middle|\; \mu(d_B^{(1)}(x,y)) < d_B^{(2)}(x,y) \text{ and } \mu(d_B^{(1)}(x,y)) \leq r \right\}.$$

Note that f is monotonously increasing. Now set $\rho(r) = \max\{\mu(r), f \circ \mu(r)\}$ to obtain a monotonously increasing function $\rho : [0,\infty) \to [0,\infty)$ with $\lim_{r\to\infty} \rho_+(r) = \infty$. Moreover, we have $d_B^{(2)}(x,y) \leq \rho(d_B^{(1)}(x,y))$ for all $x, y \in X$ by construction. This finishes the proof. □

Remark 1.3.4. It is well-known that the box spaces of G, as coarse metric spaces, encode important properties of G. For instance, assuming that G is finitely generated, the box space $\square_\sigma G$ has property A if and only if G is amenable, see [80, 11.39] and [71, 4.4.6]. On the other hand, property A is always implied by finite asymptotic dimension, see [71, 4.3.6]. When a box space has finite asymptotic dimension, one might therefore be tempted to think that this value encodes the geometric complexity of the group in some sense. This is demonstrated in the next Lemma:

Lemma 1.3.5 (cf. [96, 2.4]). *Let G be a residually finite group, and let $\sigma = (G_n)_n$ be a residually finite approximation of G. Then the following conditions are equivalent for all $s \in \mathbb{N}$:*

(1) *The box space $\square_\sigma G$ has asymptotic dimension at most s.*

(2) *For any $R > 0$, there exists $n \in \mathbb{N}$ and collections $\mathcal{U}^{(0)}, \dots, \mathcal{U}^{(s)}$ of subsets of G such that $\mathcal{U} = \mathcal{U}^{(0)} \cup \dots \cup \mathcal{U}^{(s)}$ is a uniformly bounded cover of G with Lebesgue number at least R, and such that for each $l \in \{0, \dots, s\}$, the collection $\mathcal{U}^{(l)}$ has mutually disjoint members and is G_n-invariant with respect to multiplication from the right.*

(3) *For every $\varepsilon > 0$ and $M \Subset G$, there exists $n \in \mathbb{N}$ and functions $\mu^{(l)} : G \to [0, 1]$ for $l = 0, \dots, s$ with the following properties:*

   (a) *For every $l = 0, \dots, s$, one has*

$$\operatorname{supp}(\mu^{(l)}) \cap \operatorname{supp}(\mu^{(l)})h = \emptyset \quad \textit{for all } h \in G_n \setminus \{1\}.$$

   (b) *For every $g \in G$, one has*

$$\sum_{l=0}^{s} \sum_{h \in G_n} \mu^{(l)}(gh) = 1.$$

   (c) *For every $l = 0, \dots, s$ and $g \in M$, one has*

$$\|\mu^{(l)} - \mu^{(l)}(g \cdot \_)\|_\infty \leq \varepsilon.$$

Remark. We note that condition (3)(a) above automatically forces the functions $\mu^{(l)}$ to be finitely supported, because each subgroup $G_n \subset G$ has, by definition, finite index.

Before we prove 1.3.5, we need a few observations:

Lemma 1.3.6. *Let G be a group and $H \subset G$ a subgroup with finite index. Let $\mathcal{V}$ be a collection of pairwise disjoint, finite subsets of G that is H-invariant with regard to multiplication from the right. Then there is a finite set $U \Subset G$ such that*

- *$U \cap Uh = \emptyset$ for all $h \in H \setminus \{1\}$.*
- *For every $V \in \mathcal{V}$, there exists some $h \in H$ with $V \subset Uh$.*

Proof. Let $V_1 \in \mathcal{V}$ be some set. If $\mathcal{V} = \{V_1 \cdot h \mid h \in H\}$, then we set $U = V_1$ and are done. So assume that this is not the case. As $\mathcal{V}$ is H-invariant and consists of mutually disjoint sets, there exists some $V_2 \in \mathcal{V}$ with $V_2 \cap V_1 h = \emptyset$ for all $h \in H$. But then, the set $U_2 = V_1 \cup V_2$ automatically satisfies $U_2 \cap U_2 h = \emptyset$ for all $h \in H \setminus \{1\}$. If

$$\mathcal{V} = \{V_i \cdot h \mid h \in H,\ i = 1, 2\},$$

then we can set $U = U_2$ and are done. So assume that this is not the case. Proceed inductively like this. Assume that for some $k \in \mathbb{N}$, we have found $V_1, \ldots, V_k \in \mathcal{V}$ such that $U_k = V_1 \cup \cdots \cup V_k$ satisfies $U_k \cap U_k h = \emptyset$ for all $h \in H \setminus \{1\}$, and such that

$$\{V_i \cdot h \mid h \in H,\ i = 1, \ldots, k\} \subsetneq \mathcal{V}.$$

Choosing some set $V_{k+1} \in \mathcal{V}$ in the complement, it again follows by H-invariance and pairwise disjointness of $\mathcal{V}$ that $U_{k+1} = U_k \cup V_{k+1}$ satisfies $U_{k+1} \cap U_{k+1} h = \emptyset$ for all $h \in H \setminus \{1\}$.

Notice that in every induction step, we are either done by setting $U = U_k$, or we can proceed with another step and strictly enlarge the set U_k to U_{k+1}. But at the same time, the condition $U_k \cap U_k h = \emptyset$ for all $h \in H \setminus \{1\}$ implies $|U_k| \leq [G : H] < \infty$. So the induction must stop after finitely many steps, and the claim follows. □

Proposition 1.3.7. *Let G be a countable group and $H \subset G$ a subgroup with finite index. Let $\pi : G \to G/H$ be the quotient map. Let d be a proper, right-invariant metric on G. Then for every $r > 0$ and $g \in G$, we have $\pi(B_r^d(g)) = B_r^{\pi_*(d)}(\pi(g))$.*

Proof. The "$\subseteq$" part is trivial because by the definition of the metric $\pi_*(d)$, the map π is contractive. If $x \in B_r^{\pi_{n*}(d)}(\pi_n(g))$, write $x = \pi_n(f)$ for some $f \in G$. By right-invariance of d, we have

$$\begin{aligned} r &> \pi_{n*}(d)(\pi_n(g), \pi_n(f)) \\ &= \inf \{d(gh_1, fh_2) \mid h_1, h_2 \in G_n\} \\ &= \inf \{d(g, fh) \mid h \in G_n\}. \end{aligned}$$

In particular, there exists $h \in G_n$ with $d(g, fh) < r$, which means $fh \in B_r^d(g)$, and moreover $\pi_n(fh) = \pi(f) = x$. This shows our claim. □

Lemma 1.3.8. *Let G be a residually finite group, and let $\sigma = (G_n)_n$ be a residually finite approximation of G. For each $n \in \mathbb{N}$, let $\pi_n : G \to G/G_n$ be the quotient map. Let d be a proper, right-invariant metric on G. For every $r > 0$, there exists $n_0 \in \mathbb{N}$, such that for each $n \geq n_0$ and $g \in G$, the restriction of π_n yields an isometric bijection between $B_r^d(g)$ and $B_r^{\pi_{n*}(d)}(\pi_n(g))$.*

Proof. Let $r > 0$. Since d is proper, $B_{4r}^d(1_G)$ is a finite set. So choose $n_0 \in \mathbb{N}$ so large that $B_{4r}^d(1_G) \cap G_{n_0} = \{1_G\}$. Then also $B_{4r}^d(1_G) \cap G_n = \{1_G\}$ for every $n \geq n_0$, because $(G_n)_n$ is decreasing. Given any $g_1, g_2 \in G$ with $d(g_1, g_2) \leq 2r$, this implies for all $h \in G_n$ that

$$d(g_1 h, g_2) = d(g_1 h g_1^{-1}, g_2 g_1^{-1}) \geq d(\underbrace{g_1 h g_1^{-1}}_{\in G_n}, 1_G) - 2r \geq 2r.$$

In particular,

$$d(g_1, g_2) \leq \min \{d(g_1 h, g_2) \mid h \in G_n\} = \pi_{n*}(d)(\pi_n(g_1), \pi_n(g_2)).$$

Since inequality in the other direction always holds, this shows that for every subset $X \subset G$ with $\operatorname{diam}_d(X) \leq 2r$, the restriction $\pi_n|_X$ is isometric. In particular, we have for every $g \in G$ and $n \geq n_0$ that $\pi_n|_{B_r^d(g)}$ is isometric. Applying 1.3.7, this finishes the proof. □

Proof of 1.3.5. For the entire proof, let us fix the following notation: For each $n \in \mathbb{N}$, denote by $\pi_n : G \to G/G_n$ the quotient map. Let d be a proper, right-invariant metric on G and let d_B be a metric on $\Box_\sigma G = \bigsqcup_{n \in \mathbb{N}} G/G_n$ satisfying the requirements of 1.2.8 with respect to the sequence of finite metric spaces $(G/G_n, \pi_{n*}(d))$.

(1) $\implies$ (2) : Given $R > 0$, use $\operatorname{asdim}(\Box_\sigma G) \leq s$ and 1.2.7(2) to find collections $\mathcal{V}^{(0)}, \ldots, \mathcal{V}^{(s)}$ of subsets of $\Box_\sigma G$ such that $\mathcal{V} = \mathcal{V}^{(0)} \cup \cdots \cup \mathcal{V}^{(s)}$ is a cover of $\Box_\sigma G$ uniformly bounded by some $r > 0$ and with Lebesgue number at least R. Let $\mathcal{V}_n = \mathcal{V}_n^{(0)} \cup \cdots \cup \mathcal{V}_n^{(s)}$ be the induced finite covers on the finite subspaces $G/G_n \subset \Box_\sigma G$.

Applying 1.3.8, choose $n_0 \in \mathbb{N}$ large enough such that for every $n \geq n_0$ and $g \in G$, the map π_n restricts to an isometric bijection between $B_r^d(g)$ and $B_r^{\pi_{n*}(d)}(\pi_n(g))$. By choosing n_0 even larger (if necessary), we may also assume that $B_{2r+1}(1_G) \cap G_n = \{1_G\}$ for all $n \geq n_0$, i.e. the distance between any two distinct elements in G_n is at least $2r + 1$. By the uniform boundedness, each $V \in \mathcal{V}_n$ is contained in $B_r^{\pi_{n*}(d)}(\pi_n(g))$ for some $g \in G$. Thus, we can find some $U_V \subset B_r^d(g)$ that is mapped isometrically onto V under π_n. Then, the preimage of V can be written as the disjoint union $\pi_n^{-1}(V) = \bigsqcup_{h \in G_n} U_V \cdot h$. Define

$$\mathcal{U}_n^{(l)} := \left\{ U_V \cdot h \mid V \in \mathcal{V}_n^{(l)},\ h \in G_n \right\}$$

for all $n \geq n_0$ and $l = 0, \ldots, s$. Then each $\mathcal{U}_n^{(l)}$ is G_n-invariant with regard to multiplication from the right and has mutually disjoint members by construction. Moreover, the collection $\mathcal{U}_n = \mathcal{U}_n^{(0)} \cup \cdots \cup \mathcal{U}_n^{(s)}$ covers G. We claim that the Lebesgue number of $\mathcal{U}$ is at least R. Let $X \subset G$ be a set with diameter at most R. Since the cover $\mathcal{V}_n$ of G/G_n has Lebesgue number at least R and π_n is contractive, it follows that $\pi_n(X) \subset V$ for some $V \in \mathcal{V}_n$. Hence

$$X \subset \pi_n^{-1}(\pi_n(X)) \subset \pi_n^{-1}(V) = \bigsqcup_{h \in G_n} U_V \cdot h.$$

By construction, the set U_V has diameter at most r, and the distance between two distinct elements in G_n is at least $2r + 1$. Thus, for $h_1 \neq h_2$ in G_n, we have for every $g_1, g_2 \in U_V$ that

$$d(g_1 h_1, g_2 h_2) = d(g_1 h_1 h_2^{-1} g_1^{-1}, g_2 g_1^{-1}) \geq d(\underbrace{g_1 h_1 h_2^{-1} g_1^{-1}}_{\in G_n}, 1_G) - r \geq r + 1.$$

This shows $\operatorname{dist}_d(U_V \cdot h_1, U_V \cdot h_2) \geq r + 1 > R$. But this implies that X must be entirely contained in $U_V \cdot h \in \mathcal{U}_n$ for some $h \in G_n$, which shows the claim.

(2) $\implies$ (3) : Let $\varepsilon > 0$ and $M \Subset G$ be given. Choose $R > \frac{2(2s+1)}{\varepsilon}$ large enough so that M is contained in $B^d_R(1_G)$. By assumption, there exists n and a uniformly bounded cover $\mathcal{U} = \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(s)}$ of G with Lebesgue number at least R, such that each $\mathcal{U}^{(l)}$ is G_n-invariant with regard to multiplication from the right and has mutually disjoint members. Upon applying 1.3.6 to $\mathcal{U}^{(l)}$ for each $l = 0, \ldots, s$, we may assume that $\mathcal{U}^{(l)}$ is of the form $\{U^{(l)} \cdot h \mid h \in G_n\}$ for some finite set $U^{(l)} \subset G$ with $U^{(l)} \cap (U^{(l)} \cdot h) = \emptyset$ for all $h \in G_n \setminus \{1_G\}$. Now define

$$\mu^{(l)} : G \to [0,1], \; g \mapsto \frac{\mathrm{dist}_d(g, G \setminus U^{(l)})}{\sum_{V \in \mathcal{U}} \mathrm{dist}_d(g, G \setminus V)}.$$

Then obviously $\mathrm{supp}(\mu^{(l)}) \subset U^{(l)}$ and $\{\mu^{(l)}(\_ \cdot h) \mid h \in G_n,\ l = 0, \ldots, s\}$ forms a partition of unity for G. This proves properties (a) and (b). By applying [71, 4.3.5], we see that our assumption on the Lebesgue number of $\mathcal{U}$ being at least R implies that each function $\mu^{(l)}$ is Lipschitz with regard to the constant $\frac{2(2s+1)}{R} \leq \varepsilon$, which shows that also condition (c) is met.

(3) $\implies$ (1) : Let $R > 0$ be given. Let $0 < \varepsilon < \frac{1}{(s+1)}$. Choose n and finitely supported functions $\mu^{(l)} : G \to [0,1]$ for $l = 0, \ldots, s$ satisfying the requirements of (3) with respect to the pair ε and $M = B^d_R(1_G)$. By choosing n large enough, we may also assume that the d_B-distance between any two sets of the form $\bigsqcup_{k=1}^{n-1} G/G_k$ and G/G_m, for $m \geq n$, is at least R. Define $U^{(l)} := \mathrm{supp}(\mu^{(l)})$. Then $U^{(l)} \cap (U^{(l)} \cdot h) = \emptyset$ for all $l = 0, \ldots, s$ and $h \in G_n \setminus \{1\}$ and $\{U^{(l)} \cdot h \mid h \in G_n,\ l = 0, \ldots, s\}$ covers G. For each $m \geq n$ and $l = 0, \ldots, s$, define the collection $\mathcal{V}^{(l)}_m = \{\pi_m(U^{(l)} \cdot h) \mid h \in G_n\}$, which consists of $[G_n : G_m]$ disjoint subsets of G/G_m. Define a cover $\mathcal{V} = \mathcal{V}^{(0)} \cup \cdots \cup \mathcal{V}^{(s)}$ of $\Box_\sigma G$ by

$$\mathcal{V}^{(0)} = \left\{ \bigcup_{k=1}^{n-1} G/G_k \right\} \cup \bigcup_{m=n}^{\infty} \mathcal{V}^{(0)}_m \text{ and } \mathcal{V}^{(l)} = \bigcup_{m=n}^{\infty} \mathcal{V}^{(l)}_m \quad \text{for } l = 1, \ldots, s.$$

The diameters of members of $\mathcal{V}$ are then bounded by

$$\max \left\{ \mathrm{diam}\Big(\bigcup_{m=1}^{n-1} G/G_m \Big), \mathrm{diam}(U^{(0)}), \ldots, \mathrm{diam}(U^{(s)}) \right\}.$$

Moreover, for every $l = 0, \ldots, s$, two distinct sets in $\mathcal{V}^{(l)}$ are disjoint. Let us now show that any ball of radius R is contained in some member of $\mathcal{V}$.

Given any point $x \in \Box_\sigma G$, our choice of n implies that $B_R^{d_B}(x)$ falls entirely in one of the subsets $\bigsqcup_{k=1}^{n-1} G/G_k$ or G/G_m for $m \geq n$. In the first case, $B_R^{d_B}(x) \subset \bigsqcup_{k=1}^{n-1} G/G_k \in \mathcal{V}^{(0)}$. In the case where $B_R^{d_B}(x) \subset G/G_m$ for $m \geq n$, choose some $g \in G$ with $\pi_m(g) = x$. By condition (b), g can be in the support of at most $s+1$ members of the partition of unity $\{\mu^{(l)}(\_ \cdot h) \mid h \in G_n,\ l = 0, \cdots, s\}$. Moreover, it follows that there exist $h \in G_n$ and $l \in \{0, \ldots, s\}$ such that $\mu^{(l)}(g \cdot h) \geq \frac{1}{s+1}$. For any $g' \in B_R^d(1_G)$, condition (3)(c) for $M = B_R^d(1_G)$ implies

$$\mu^{(l)}(g' \cdot g \cdot h) \geq \mu^{(l)}(g \cdot h) - \varepsilon > \frac{1}{s+1} - \frac{1}{s+1} = 0.$$

So $B_R^d(g)h = B_R^d(1_G)gh \subset \mathrm{supp}(\mu^{(l)}) = U^{(l)}$, which equivalently means $B_R^d(g) \subset U^{(l)}h^{-1}$. Since any set of diameter at most R is trivially contained in a ball of radius R, it follows that $\mathcal{U}$ has Lebesgue number at least R. This verifies $\mathrm{asdim}(\Box_\sigma G) \leq s$ by appealing to 1.2.7(2). □

Corollary 1.3.9 (cf. [96, 2.5])**.** *Let G be a countable, discrete, residually finite group and $\sigma = (G_n)_n$ a residually finite approximation. Let G be equipped with a proper, right-invariant metric. Then* $\mathrm{asdim}(G) \leq \mathrm{asdim}(\Box_\sigma G)$.

Proof. This follows immediately from 1.2.7(2) and 1.3.5(2). □

Corollary 1.3.10 (cf. [96, 2.6])**.** *Let G be a countable, discrete, residually finite group and $\sigma = (G_n)_n$ a residually finite approximation. Let $H \subset G$ be a subgroup. Then for the residually finite approximation $\kappa = (H \cap G_n)_n$ of H, we have* $\mathrm{asdim}(\Box_\kappa H) \leq \mathrm{asdim}(\Box_\sigma G)$.

Proof. This follow directly from 1.3.5(2) by restricting the covers given on G to H. □

Definition 1.3.11 (cf. [96, 2.7])**.** Let G be a countable, discrete and residually finite group. We denote by $\Lambda(G)$ the set of all residually finite approximations of G. The set $\Lambda(G)$ carries the following natural preorder: we write $(G_n)_n \precsim (H_n)_n$, if for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ with $H_m \subset G_n$. We call a residually finite approximation $(G_n)_n \in \Lambda(G)$ dominating, if it is dominating with respect to the above order, i.e. $(H_n)_n \precsim (G_n)_n$ for all $(H_n)_n \in \Lambda(G)$.

Proposition 1.3.12. *Let G be a countable, discrete and residually finite group. A residually finite approximation $(G_n)_n \in \Lambda(G)$ is dominating if and only if for every subgroup $H \subset G$ with finite index, there exists n such that $G_n \subset H$.*

Proof. It is obvious that a residually finite approximation $(G_n)_n$ is dominating if and only if for every normal subgroup $N \subset G$ with finite index, there exists n with $G_n \subset N$. The claim follows by recalling the well-known fact that any subgroup with finite index contains a normal subgroup with finite index: If H is a subgroup with finite index, let $F_H \Subset G$ be a finite set of representatives for G/H. This means that the map $F_H \times H \to G, (x,h) \mapsto x \cdot h$ is bijective. Then observe that the normal subgroup

$$\bigcap_{g \in G} gHg^{-1} = \bigcap_{x \in F_H, h \in H} (xh)H(xh)^{-1} = \bigcap_{x \in F_H} xHx^{-1}$$

is contained in H and has finite index in G as a finite intersection of subgroups with finite index. □

Example 1.3.13. Given $m \in \mathbb{N}$, the sequence $(n! \cdot \mathbb{Z}^m)_n$ is a dominating residually finite approximation of $\mathbb{Z}^m$.

Proof. Applying 1.3.12, it suffices to show that any finite index subgroup of $\mathbb{Z}^m$ contains a subgroup of the form $n\mathbb{Z}^m$ for some $n \in \mathbb{N}$. So let $H \subset \mathbb{Z}^m$ be a subgroup with finite index. Then H is free abelian with rank m. Let $v_1, \dots, v_m$ denote a set generators for H. If we view H and $\mathbb{Z}^m$ as subsets of $\mathbb{Q}^m$, then we claim that the vectors $v_1, \dots, v_m$ form a $\mathbb{Q}$-basis. Indeed, let $\lambda_1, \dots, \lambda_m \in \mathbb{Q}$ be given with $0 = \lambda_1 v_1 + \dots + \lambda_m v_m$. Then we can find a positive integer $k > 0$ such that $k\lambda_1, \dots, k\lambda_m \in \mathbb{Z}$, and we have $0 = (k\lambda_1)v_1 + \dots + (k\lambda_m)v_m$. Since H is free abelian of rank m, it follows that $k\lambda_i = 0$ for all i, which implies $\lambda_i = 0$ for all i. This shows that the vectors $v_1, \dots, v_m \in \mathbb{Q}^m$ are linearly independant in $\mathbb{Q}^m$, and thus form a basis.

Let $e_1, \dots, e_m$ denote the standard generators of $\mathbb{Z}^m$. We claim that for each $i = 1, \dots, m$, there is a positive integer k_i such that $k_i e_i \in H$. Indeed, we can find some $\lambda_1, \dots, \lambda_m \in \mathbb{Q}$ with $e_i = \lambda_1 v_1 + \dots + \lambda_m v_m$. If we choose k_i large enough such that $k_i \lambda_j \in \mathbb{Z}$ for all $j = 1, \dots, m$, then $k_i e_i = (k_i\lambda_1)v_1 + \dots + (k_i\lambda_m)v_m \in H$.

After having found these numbers, let n be the least common multiple of $k_1, \ldots, k_m$. Then for all $i = 1, \ldots, m$, it follows that $\frac{n}{k_i} \in \mathbb{Z}$ and thus $ne_i = \frac{n}{k_i} \cdot k_i \cdot e_i \in H$. In particular, we have $n\mathbb{Z}^m \subset H$. $\square$

Let us record the following variant of 1.3.5 in the special case of dominating residually finite approximations, which follows directly from 1.3.12:

Corollary 1.3.14. *Let G be a residually finite group, and let $\sigma = (G_n)_n$ be a dominating residually finite approximation of G. Then the following conditions are equivalent for all $s \in \mathbb{N}$:*

(1) *The box space $\square_\sigma G$ has asymptotic dimension at most s.*

(2) *For any $R > 0$, there exists some subgroup $H \subset G$ with finite index and collections $\mathcal{U}^{(0)}, \ldots, \mathcal{U}^{(s)}$ of subsets of G such that $\mathcal{U} = \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(s)}$ is a uniformly bounded cover of G with Lebesgue number at least R, and such that for each $l \in \{0, \ldots, s\}$, the collection $\mathcal{U}^{(l)}$ has mutually disjoint members and is H-invariant with respect to multiplication from the right.*

(3) *For every $\varepsilon > 0$ and $M \Subset G$, there exists a subgroup $H \subset G$ with finite index and functions $\mu^{(l)} : G \to [0,1]$ for $l = 0, \ldots, s$ with the following properties:*

   (a) *For every $l = 0, \ldots, s$, one has*

   $$\operatorname{supp}(\mu^{(l)}) \cap \operatorname{supp}(\mu^{(l)})h = \emptyset \quad \text{for all } h \in H \setminus \{1\}.$$

   (b) *For every $g \in G$, one has*

   $$\sum_{l=0}^{s} \sum_{h \in G_n} \mu^{(l)}(gh) = 1.$$

   (c) *For every $l = 0, \ldots, s$ and $g \in M$, one has*

   $$\|\mu^{(l)} - \mu^{(l)}(g \cdot \_)\|_\infty \leq \varepsilon.$$

Corollary 1.3.15 (cf. [96, 2.6])**.** *Let G be a residually finite group, and let $\sigma = (G_n)_n$ be a dominating residually finite approximation of G. Let $H \subset G$ be a subgroup of finite index, and let $\kappa = (H \cap G_n)_n$ be the induced residually finite approximation on H. Then* $\operatorname{asdim}(\square_\kappa H) = \operatorname{asdim}(\square_\sigma G)$*.*

Proof. The "$\leq$" part follows from 1.3.10, so let us show the inequality in the other direction. Denote $s = \operatorname{asdim}(\Box_\kappa H)$, and let d be a proper, right-invariant metric on G. Its restriction to H obviously yields a proper, right-invariant metric on H. Let $F_H \Subset G$ be a finite set of representatives for G/H, i.e. the map $F_H \times H \to G, (x, h) \mapsto x \cdot h$ is bijective. Let $R > 0$. Without loss of generality, assume that R is large enough such that $F_H \subset B^d_R(1_G)$.

Use 1.3.14(2) in order to find n and collections $\mathcal{W}^{(0)}, \ldots, \mathcal{W}^{(s)}$ of subsets of H such that $\mathcal{W} = \mathcal{W}^{(0)} \cup \cdots \cup \mathcal{W}^{(s)}$ is a uniformly bounded cover of H with Lebesgue number at least $8R$, and such that for each $l = 0, \ldots, s$, the collection $\mathcal{W}^{(l)}$ consists of pairwise disjoint sets and is $H \cap G_n$-invariant with respect to multiplication from the right. Since $(G_k)_k$ is dominating, we may choose n large enough according to 1.3.12 such that $G_n \subset H$, which implies $G_n \cap H = G_n$. For each $l = 0, \ldots, s$, define

$$\mathcal{V}^{(l)} = \left\{ B_{-4R}(W) \mid W \in \mathcal{W}^{(l)} \right\} \text{ with } B_{-4R}(W) = \left\{ h \in H \mid B^{d|_H}_{4R}(h) \subset W \right\}.$$

Then for each $l = 0, \ldots, s$, the collection $\mathcal{V}^{(l)}$ is again G_n-invariant with respect to multiplication from the right, and the distance between any two distinct sets is at least $4R$. Moreover, since $\mathcal{W}$ has Lebesgue number at least $8R$, any ball of radius $4R$ in H is contained in a member of $\mathcal{W}$. Therefore, it follows from the construction of $\mathcal{V}$ that it is a cover. Now for $l = 0, \ldots, s$, define

$$\mathcal{U}^{(l)} = \left\{ B^d_{2R}(1_G) \cdot V \mid V \in \mathcal{V}^{(l)} \right\}.$$

Since the distance between any two distinct sets in $\mathcal{V}^{(l)}$ is at least $4R$, it follows that any two distinct sets in $\mathcal{U}^{(l)}$ are disjoint. Moreover, it is obvious that $\mathcal{U}^{(l)}$ is G_n-invariant with respect to multiplication from the right. Moreover, we claim that $\mathcal{U} = \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(s)}$ is a cover with Lebesgue number at least R. Indeed, given $g \in G$, there are $x \in F_H$ and $h \in H$ with $g = xh$. Then $h \in V$ for some $V \in \mathcal{V}$. Since $x \in F_H \subset B^d_R(1_G)$, it follows that $g \in B^d_R(1_G) \cdot V$, and thus $B^d_R(g) = B^d_R(1_G)g \subset B^d_R(1_G) \cdot B^d_R(1_G) \cdot V \subset B^d_{2R}(1_G) \cdot V \in \mathcal{U}$. Since every set of diameter less than R is trivially contained in some ball of radius R, it follows that the Lebesgue number of $\mathcal{U}$ is indeed at least R. This finishes the proof. $\square$

In applications, the existence of a residually finite approximation can be quite useful. In the case of finitely generated groups, this turns out to be automatic:

Proposition 1.3.16. *Let G be a finitely generated and residually finite group. Then G has a dominating residually finite approximation.*

Proof. Let $S \Subset G$ be a finite generating set. Given some finite group E, there are at most as many group homomorphisms from G to E as there are maps from S to E, which there are $|E|^{|S|}$ of. Now up to isomorphism, there are only countably many finite groups. Since any normal subgroup of G with finite index arises as a kernel of a homomorphism into a finite group, this implies that G has at most countably many normal subgroups of finite index. Let $\{N_n\}_{n\in\mathbb{N}}$ be a respresentation of this set, and define $(G_n)_n \in \Lambda(G)$ recursively via $G_1 = N_1$ and $G_{n+1} = G_n \cap N_{n+1}$ for all $n \in \mathbb{N}$. By construction, any normal subgroup must eventually contain a member of this sequence as a subgroup, so this sequence is indeed dominating. $\square$

Remark 1.3.17. Let $\sigma, \kappa \in \Lambda(G)$ be two sequences with $\sigma \precsim \kappa$. Then it follows from 1.3.5(2) that $\operatorname{asdim}(\Box_\kappa G) \leq \operatorname{asdim}(\Box_\sigma G)$. In other words, the map

$$\Lambda(G) \longrightarrow \mathbb{N}, \quad \sigma \longmapsto \operatorname{asdim}(\Box_\sigma G)$$

is order-reversing. This implies that the values of asymptotic dimension for all box spaces associated to dominating sequences are the same, and they take the lowest value among all possible box spaces associated to residually finite approximations.

If $\sigma \in \Lambda(G)$ is a dominating residually finite approximation, we will call the associated box space $\Box_\sigma G$ a *standard* box space, and will sometimes denote it by $\Box_s G$ under the assumption that G is finitely generated. While there might a priori be some ambiguity, we will exclusively be interested in the asymptotic dimension of these spaces, so the above argument shows that there is no ambiguity concerning the asymptotic dimension of a standard box space.

1.4 Box spaces of nilpotent groups

In this section, we show that the standard box space of a finitely generated, virtually nilpotent group has finite asymptotic dimension. We first need to record a few technical observations:

Lemma 1.4.1. *Let G be a locally compact group with a proper, right-invariant metric d. Let $H \subset G$ be a discrete, cocompact subgroup. Suppose that G has finite covering dimension $m \in \mathbb{N}$. Then there exists a uniformly bounded, open cover $\mathcal{W}$ of G with positive Lebesgue number and a decomposition $\mathcal{W} = \mathcal{W}^{(0)} \cup \cdots \cup \mathcal{W}^{(m)}$, such that for each $l = 0, \ldots, s$, the collection $\mathcal{W}^{(l)}$ consists of mutually disjoint members and is H-invariant with respect to multiplication from the right.*

Proof. Consider the quotient map $\pi : G \to G/H$, which is a local homeomorphism by 1.2.28. Thus $m = \dim(G/H)$. Since $H \subset G$ is discrete, there exists $\eta > 0$ with $d(h, 1_G) \geq 3\eta$ for all $h \in H \setminus \{1_G\}$. By right-invariance of d, this implies $d(h_1, h_2) \geq 3\eta$ for all $h_1 \neq h_2$ in H.

For any $x \in G/H$, choose $g \in G$ with $x = \pi(g)$, and then choose an open neighbourhood U_x of g with diameter at most η. Now the collection $\{V_x\}_{x \in G/H}$ given by $V_x = \pi(U_x)$ is an open cover of G/H. By compactness, there exists some finite subcover. Using $m = \dim(G/H)$ and 1.2.3, we can choose an open cover $\mathcal{U}$ of G/H refining $\{V_x\}_{x \in G/H}$ with the following properties:

- There is a decomposition $\mathcal{U} = \mathcal{U}^{(0)} \cup \cdots \cup \mathcal{U}^{(m)}$ such that for each $l \in \{0, \ldots, m\}$, one has $U \cap V = \emptyset$ for all $U \neq V$ in $\mathcal{U}^{(l)}$.
- For all $V \in \mathcal{U}$, there exists an open set $U_V \subset G$ with $\pi(U_V) = V$ and $\operatorname{diam}(U_V) \leq \eta$.

Since G/H is compact, we can apply the Lebesgue number theorem (see [92, 7.2.12]) to deduce that this cover has some positive Lebesgue number $\mu > 0$ with respect to the push-forward metric $\pi_*(d)$. Set $r = \min(\mu, \eta)$.

Now consider the uniformly bounded, open cover $\mathcal{W}$ of G given by $\mathcal{W} = \mathcal{W}^{(0)} \cup \cdots \cup \mathcal{W}^{(m)}$, where

$$\mathcal{W}^{(l)} = \left\{ U_V \cdot h \mid V \in \mathcal{U}^{(l)}, h \in H \right\}.$$

Claim 1: For each $l = 0, \ldots, s$, the members in $\mathcal{W}^{(l)}$ are mutually disjoint.

So let $W_1 \neq W_2$ in $\mathcal{W}^{(l)}$. Let $V_1, V_2 \in \mathcal{U}^{(l)}$ and $h_1, h_2 \in H$ with $W_1 = U_{V_1} h_1$ and $W_2 = U_{V_2} h_2$.

Case 1: $W_1 \neq W_2$**.** In this case, we get $\pi(W_1) = V_1$ and $\pi(W_2) = V_2$, which are disjoint, so W_1 and W_2 have to be disjoint.

Case 2: $h_1 \neq h_2$**.** Then $d(h_1, h_2) \geq 3\eta$. Now W_1 is an open neighbourhood of h_1 with diameter at most η, and likewise W_2 is an open neighbourhood of h_2 with diameter at most η. It follows by triangle inequality that W_1 and W_2 have distance at least η, so they must be disjoint.

Claim 2: The Lebesgue number of $\mathcal{W}$ is at least r.

Let $X \subset G$ be some set of diameter less than r with respect to d. Then $\pi(X)$ has diameter less than r with respect to $\pi_*(d)$. Since $r \leq \mu$, there is some $V \in \mathcal{U}$ with $X \subset V$. But then we have

$$X \subset \pi^{-1}(X) \subset \pi^{-1}(V) = \bigcup_{h \in H} U_V \cdot h.$$

As we have observed earlier, for all $h_1 \neq h_2$, the distance between $U_V h_1$ and $U_V h_2$ is at least η with respect to d. Since $r \leq \eta$, it follows that X must be contained in exactly one open set of the form $U_V h$, which is a member of $\mathcal{W}$. □

The following will be our main method in showing that a given group admits a finite-dimensional box space:

Lemma 1.4.2. *Let G be a locally compact group G with a proper, right-invariant metric d. Let $H \subset G$ be a finitely generated, discrete and cocompact subgroup. Suppose that there exists a sequence of continuous automorphisms $\sigma_n \in \mathrm{Aut}(G)$ satisfying*

- *for all $n \in \mathbb{N}$, the map σ_n restricts to an endomorphism on H;*
- *for any compact set K and open neighbourhood U of 1_G, there exists $n \in \mathbb{N}$ with $K \subset \sigma_n(U)$.*

Then H is residually finite and $\mathrm{asdim}(\square_s H) \leq \dim(G)$.

Proof. Observe that each subgroup $H_n = \sigma_n(H) \subset H$ must have finite index in H. Indeed, since H_n is the image of H under an automorphism on G, the subgroup H_n is cocompact in G. Let $K \subset G$ be a compact subset with $G = \bigcup_{h \in H_n} K \cdot h$. Let $\pi_n : G \to G/H_n$ be the quotient map. Then the restriction of $\pi_n|_K$ is still surjective. Now we may identify H/H_n as a subset of G/H_n in the obvious way, and then the restriction of π_n yields a surjective map from $K \cap H$ onto H/H_n. But since $K \cap H$ is the intersection of a compact set and a discrete set, it is finite, and thus H/H_n is finite. Since H is discrete, there exists an open neighbourhood $U \subset G$ of the unit with $U \cap H = \{1_G\}$. Then given any predescribed finite set $F \Subset H$ containing the unit, there exists n with $F \subset \sigma_n(U)$, from which it follows that $F \cap H_n = \{1\}$. As F is arbitrary, it follows that $\bigcap_{n\in\mathbb{N}} H_n = \{1\}$, and in particular H is residually finite.

Without loss of generality, assume that G has finite covering dimension. Set $m = \dim(G)$. Apply 1.4.1 to choose a uniformly bounded, open cover $\mathcal{W}$ of G with positive Lebesgue number $\delta > 0$ and a decomposition $\mathcal{W} = \mathcal{W}^{(0)} \cup \cdots \cup \mathcal{W}^{(m)}$, such that for each $l = 0, \dots, s$, the collection $\mathcal{W}^{(l)}$ consists of mutually disjoint members and is H-invariant with respect to multiplication from the right.

For each n, consider the uniformly bounded cover $\mathcal{W}_n = \mathcal{W}_n^{(0)} \cup \cdots \cup \mathcal{W}_n^{(m)}$ of G given by $\mathcal{W}_n^{(l)} = \sigma_n(\mathcal{W}^{(l)}) = \left\{\sigma_n(U) \mid U \in \mathcal{W}^{(l)}\right\}$ for all $l = 0, \dots, m$. Then any two distinct members of $\mathcal{W}_n^{(l)}$ are disjoint. The first property of the σ_n ensures that for all n and $l = 0, \dots, m$, the collection $\mathcal{W}_n^{(l)}$ is right-invariant with respect to $H_n = \sigma_n(H) \subset H$. Now let $R > 0$. Since the metric d is proper, the second property of the sequence $(\sigma_n)_n$ ensures that we find some n such that $B_R(1_G) \subset \sigma_n(B_{\delta/3}(1_G))$. Since $\mathcal{W}$ has Lebesgue number δ, any ball of radius $\delta/3$ is contained in a member of $\mathcal{W}$. By our choice of n, any ball of radius R is therefore contained in a member of $\mathcal{W}_n$, which shows that the Lebesgue number of $\mathcal{W}_n$ is at least R.

If we now restrict the cover $\mathcal{W}_n = \mathcal{W}_n^{(0)} \cup \cdots \cup \mathcal{W}_n^{(m)}$ to a cover on H, we see that condition 1.3.14(2) is met for R. Since $R > 0$ was arbitrary, the claim follows. □

Corollary 1.4.3. *For all $m \in \mathbb{N}$, we have* $\operatorname{asdim}(\Box_s \mathbb{Z}^m) = m$.

Proof. Apply 1.4.2 for $G = \mathbb{R}^m, H = \mathbb{Z}^m$ and the sequence of automorphisms given by $\sigma_n(x) = n \cdot x$ to deduce that $\operatorname{asdim}(\Box_s \mathbb{Z}^m) \leq m$. On the other hand, we have $\operatorname{asdim}(\mathbb{Z}^m) = m$, so equality follows from 1.3.9. □

Let us now recall the definition of some important classes of groups:

Definition 1.4.4 (see the introduction of [102, Chapter 2])**.** Let G be a group. We call G polycyclic, if G admits a subnormal series of subgroups

$$\{1_G\} = G_0 \trianglelefteq G_1 \trianglelefteq \cdots \trianglelefteq G_r = G$$

of finite length $r \geq 1$, such that for each $k = 1, \ldots, r$, the subquotient G_k/G_{k-1} is cyclic. If we furthermore assume that these subquotients are infinite, we call G poly-$\mathbb{Z}$.

Definition 1.4.5 (cf. [102, p. 18])**.** Given a polycyclic group G, choose a subnormal series $\{G_k\}_{0\leq k\leq r}$ as above. Define the Hirsch length of G, written $\ell_{\mathrm{Hir}}(G)$, as the number of all k such that $G_k/G_{k-1} \cong \mathbb{Z}$. It is a well-known consequence of Schreier's refinement theorem (see [84, 5.11]) that this number is an invariant of the group G, and in particular does not depend on the choice of subnormal series.

Notation. Let G be a group. Given two subgroups $H_1, H_2 \subset G$, we denote the commutator subgroup by $[H_1, H_2] = \langle h_1 h_2 h_1^{-1} h_2^{-1} \mid h_1 \in H_1, h_2 \in H_2 \rangle$.

Definition 1.4.6 (cf. [84, p. 115])**.** Let G be a group. Define the collection $\{\gamma_n(G)\}_{n\in\mathbb{N}}$ of subgroups of G inductively by setting $\gamma_1(G) = [G, G]$ and $\gamma_{n+1}(G) = [\gamma_n(G), G]$. G is called nilpotent, if there exists $n \in \mathbb{N}$ with $\gamma_n(G) = \{1_G\}$.

Definition 1.4.7. Let G be a group. G is called virtually nilpotent, if it contains a nilpotent subgroup of finite index. Analogously, G is called virtually polycyclic, if it contains a polycyclic subgroup of finite index.

Remark 1.4.8. Let us now list a few well-known facts about polycyclic and nilpotent groups, which in particular make them interesting from the point of view of our previous section:

- Polycyclic groups are residually finite. (see [102, 2.10])
- A simple induction on the number $r \geq 1$ in 1.4.4 shows that polycyclic groups are finitely generated.
- A subgroup or quotient of a polycyclic group is polycyclic. A subgroup of a poly-$\mathbb{Z}$ group is poly-$\mathbb{Z}$. (see [72, 10.2.4])
- Let G be a polycyclic group. If $H \subset G$ is a subgroup, then $\ell_{\mathrm{Hir}}(H) \leq \ell_{\mathrm{Hir}}(G)$. If $N \trianglelefteq G$ is a normal subgroup, then $\ell_{\mathrm{Hir}}(G) = \ell_{\mathrm{Hir}}(N) + \ell_{\mathrm{Hir}}(G/N)$. (see [72, 10.2.10])
- A subgroup or quotient of a nilpotent group is nilpotent. (see [84, 5.35 and 5.36])

- A polycyclic group contains a poly-$\mathbb{Z}$ group of finite index. (see [72, 10.2.4] or [102, 2.6])
- Poly-$\mathbb{Z}$ groups are torsion-free. (see [102, p. 22])
- Finitely generated, nilpotent groups are polycyclic. (see [102, 2.13] or [72, 11.4.3(i)])
- The center of an infinite, finitely generated, nilpotent group is infinite. (see [72, 11.4.3(ii)])

In particular, finitely generated, nilpotent groups are residually finite. By 1.3.16, they admit a dominating residually finite approximation, and thus have a standard box space. For the rest of this section, we will show that such box spaces have finite asymptotic dimension.

Definition 1.4.9. Let R be a commutative, unital ring and $d \in \mathbb{N}$. The unitriangular matrix group of size d over R is defined by

$$U_d(R) = \{x = (x_{i,j})_{1\leq i<j\leq d} \mid x_{i,j} \in R\}.$$

The multiplication is given by

$$(x \cdot y)_{i,j} = \sum_{m=i}^{j} x_{i,m} y_{m,j} \quad \text{for all } 1 \leq i < j \leq d,$$

where the convention $x_{i,i} = 1$ is used for all $i = 1, \dots, d$.

Let us make a few observations:

Remark 1.4.10. The group $U_d(R)$ is generated by elements of the form $e^d_{i,j}(x) = (\delta_{k,i}\delta_{l,j}x)_{1\leq k<l\leq d}$ for $1 \leq i < j \leq d$ and $x \in R$. The assignment $e^d_{i,j}(x) \mapsto e^{d+1}_{i,j}$ then identifies $U_d(R)$ as a subgroup of $U_{d+1}(R)$. Moreover, these generators satisfy $e^d_{i,j}(x)^{-1} = e^d_{i,j}(-x)$, and the commuting relations

$$[e^d_{i,j}(x), e^d_{k,l}(y)] = \begin{cases} 1 & , \quad j \neq k \\ e_{i,l}(xy) & , \quad j = k \end{cases}$$

for all $1 \leq i < j,\ k < l \leq d$ and $x, y \in R$. From this, it can be seen that $[U_d(R), U_d(R)]$ is contained in the subgroup $U_{d-1}(R)$. Inductively, it hence follows that $\gamma_d(U_d(R)) = \{1\}$ (cf. 1.4.6), and thus $U_d(R)$ is nilpotent. In the case $R = \mathbb{Z}$, the group $U_d(\mathbb{Z})$ has a finite set of generators given by $\left\{e^d_{i,j}(1_\mathbb{Z}) \mid 1 \leq i < j \leq d\right\}$.

The finitely generated, nilpotent groups $U_d(\mathbb{Z})$ will play an important role for the remainder of this section. This is due to the following embedding theorem, which first appeared implicitely in Jenning's paper [45] on group rings of nilpotent groups, and was later explicitely proved by Swan in [93]:

Theorem 1.4.11 (see [45, 5.2] and [93])**.** *Let G be a finitely generated, torsion-free nilpotent group. Then G embeds as a subgroup into $U_d(\mathbb{Z})$ for some $d \geq 2$.*

Definition 1.4.12. Let $r > 0$ be a real number. Then the map $\alpha_r : U_d(\mathbb{R}) \to U_d(\mathbb{R})$ defined by

$$\alpha_r(x)_{i,j} = r^{j-i} x_{i,j} \quad \text{for all } 1 \leq i < j \leq d$$

yields a well-defined endomorphism. Moreover, it is immediate that $\alpha_{rs} = \alpha_r \circ \alpha_s$ for all $r, s > 0$ and $\alpha_1 = \mathrm{id}$, thus

$$\alpha : \mathbb{R}^{>0} \to \mathrm{Aut}(U_d(\mathbb{R})), \quad r \mapsto \alpha_r$$

yields a group action on $U_d(\mathbb{R})$.

Remark 1.4.13.

- Observe that if $r > 0$ is an integer, then α_r restricts to an endomorphism on $U_d(\mathbb{Z})$.
- It is a well-known fact that $U_d(\mathbb{Z})$ is a discrete and cocompact subgroup in $U_d(\mathbb{R})$. In fact, one has $U_d(\mathbb{R}) = K \cdot U_d(\mathbb{Z})$ for the compact set

$$K = \left\{ x = (x_{i,j})_{1 \leq i < j \leq d} \;\middle|\; |x_{i,j}| \leq \frac{1}{2} \right\}$$

in $U_d(\mathbb{R})$, see [1, 8.2.2].

Theorem 1.4.14 (cf. [96, 2.16])**.** *One has* $\mathrm{asdim}(\Box_s U_d(\mathbb{Z})) = d(d-1)/2$ *for all $d \geq 2$.*

Proof. We shall apply 1.4.2 for $G = U_d(\mathbb{R}), m = d(d-1)/2, H = U_d(\mathbb{Z})$ and $\sigma_n = \alpha_n$. Indeed, since $U_d(\mathbb{R})$ is homeomorphic to $\mathbb{R}^{d(d-1)/2}$ by evaluation of the matrix components, it has covering dimension $d(d-1)/2$. Moreover, $U_d(\mathbb{Z})$ is a discrete, cocompact subgroup such that each automorphism σ_n of $U_d(\mathbb{R})$ restricts to an endomorphism on $U_d(\mathbb{Z})$. Lastly, let $K \subset U_d(\mathbb{R})$ be any compact set and $U \subset U_d(\mathbb{R})$ any open neighbourhood of the unit. Then there are positive numbers $R, \delta > 0$ such that

$$K \subset \left\{ x = (x_{i,j})_{1 \leq i < j \leq d} \;\middle|\; |x_{i,j}| \leq R \right\}$$

and

$$\left\{ x = (x_{i,j})_{1 \leq i < j \leq d} \;\middle|\; |x_{i,j}| \leq \delta \right\} \subset U.$$

Then any natural number $n \geq \frac{R}{\delta}$ clearly satisfies $K \subset \sigma_n(U)$. Thus, we have verified the conditions in 1.4.2, and thus it follows that $\operatorname{asdim}(\Box_s U_d(\mathbb{Z})) \leq d(d-1)/2$. Equality follows from $\operatorname{asdim}(U_d(\mathbb{Z})) = d(d-1)/2$ and 1.3.9. $\Box$

Theorem 1.4.15 (cf. [96, 2.17])**.** *Let G be a finitely generated, virtually nilpotent group. Then* $\operatorname{asdim}(\Box_s G) < \infty$.

Proof. Let G' be a nilpotent subgroup of G with finite index. By 1.4.8, G' contains a torsion-free group H of finite index, which is then necessarily also finitely generated and nilpotent. By combining 1.4.11, 1.4.14 and 1.3.10, we obtain $\operatorname{asdim}(\Box_s H) < \infty$. Since H also has finite index as a subgroup in G, it follows from 1.3.15 that $\operatorname{asdim}(\Box_s G) < \infty$. $\Box$

1.5 Rokhlin dimension and permanence of finite nuclear dimension

We start this section by recalling the notion of relative central sequence algebras. We will adapt an idea of Kirchberg pioneered in [51], namely to divide out two-sided annihilators in non-unital relative commutants in order to obtain unital C*-algebras.

Definition 1.5.1 (cf. [51, 1.1])**.** Let A be a C*-algebra and $D \subset A_\infty$ a C*-subalgebra. Denote

$$\operatorname{Ann}(D, A_\infty) = \{ x \in A_\infty \mid \; xD = Dx = \{0\} \}.$$

Then $\mathrm{Ann}(D, A_\infty)$ is a closed, two-sided ideal inside the relative commutant

$$A_\infty \cap D' = \{x \in A_\infty \mid xd = dx \text{ for all } d \in D\}$$

and define

$$F(D, A_\infty) = (A_\infty \cap D')/\,\mathrm{Ann}(D, A_\infty)$$

the central sequence algebra of A relative to D. One writes $F_\infty(A) = F(A, A_\infty)$.

Here are some immediate observations, whose proof we will omit.

Remark 1.5.2. Let A be a C*-algebra and $D \subset A$ a separable C*-subalgebra.

- As D is separable, it admits a countable approximate unit e_n consisting of positve contractions. Then the positive contraction $e = [(e_n)_n] \in A_\infty$ satisfies $ed = d = de$ for all $d \in D$. In particular, the class $e + \mathrm{Ann}(D, A_\infty)$ of e defines a unit of $F(D, A_\infty)$, and hence this is a unital C*-algebra. (cf. [51, 1.9(3)])

- If $\varphi : A \to A$ is a $*$-endomorphism with $\varphi(D) \subset D$, then componentwise application of φ on representing sequences yields well-defined $*$-endomorphisms on both A_∞ and $F(D, A_\infty)$. Despite slight abuse of notation, both will be denoted by φ_∞.

- If $\alpha : G \curvearrowright A$ is an action and $D \subset A$ is α-invariant, then the family of automorphisms $\{\alpha_{g,\infty}\}_{g \in G}$ defines actions $\alpha_\infty : G \curvearrowright A_\infty$ and $\alpha_\infty : G \curvearrowright F(D, A_\infty)$.

Remark 1.5.3 (cf. [51, 1.1])**.** Let A be a C*-algebra and $D \subset A$ a separable C*-subalgebra. Then there is a canonical $*$-homomorphism

$$F(D, A_\infty) \otimes_{\max} D \to A_\infty \quad \text{via} \quad \big(x + \mathrm{Ann}(D, A_\infty)\big) \otimes a \mapsto x \cdot a.$$

The image of $\mathbf{1} \otimes a$ under this $*$-homomorphism is a for all $a \in D$.

Now if we assume additionally that $\alpha : G \curvearrowright A$ is an action of a discrete group such that D is α-invariant, then the above map becomes an equivariant $*$-homomorphism from $\big(F(D, A_\infty) \otimes_{\max} D, \alpha_\infty \otimes \alpha|_D\big)$ to $(A_\infty, \alpha_\infty)$.

In particular, it makes sense to multiply elements of $F(D, A_\infty)$ with elements of D to obtain an element in A_∞, and this multiplication is compatible with the naturally induced actions by α. We will implicitely make use of this observation throughout this section.

Notation 1.5.4. Let G be a discrete group and $H \subset G$ a subgroup. By the canonical G-shift action on $\mathcal{C}_0(G/H)$, we understand the action given by $g_1.f(g_2H) = f(g_1^{-1}g_2H)$ for all $g_1, g_2 \in G$ and $f \in \mathcal{C}_0(G/H)$. For brevity, we will often write $\bar{g} = gH \in G/H$ for $g \in G$.

We are now ready to start defining Rokhlin dimension. We first give the definitions together with some observations, and then discuss in what way this definition generalizes the original one from [42] by Hirshberg, Winter and Zacharias.

Definition 1.5.5 (cf. [96, 3.3])**.** Let A be a C*-algebra, G a countable group and $\alpha : G \curvearrowright A$ an action. Let $H \subset G$ be a subgroup with finite index. Let $d \in \mathbb{N}$ be a natural number. Then α is said to have Rokhlin dimension at most d with respect to H, written $\dim_{\mathrm{Rok}}(\alpha, H) \leq d$, if the following holds:

For all separable, α-invariant C*-subalgebras $D \subset A$, there exist equivariant c.p.c. order zero maps

$$\varphi_l : (\mathcal{C}(G/H), G\text{-shift}) \longrightarrow (F(D, A_\infty), \alpha_\infty) \quad (l = 0, \dots, d)$$

with $\varphi_0(\mathbf{1}) + \dots + \varphi_d(\mathbf{1}) = \mathbf{1}$.

Let us consider a few equivalent reformulations of the above definition.

Proposition 1.5.6. *Let A be a* C**-algebra, G a countable group and $\alpha : G \curvearrowright A$ an action. Let $H \subset G$ be a subgroup with finite index and $d \in \mathbb{N}$ a natural number. Then the following are equivalent:*

(1) $\dim_{\mathrm{Rok}}(\alpha, H) \leq d$.

(2) *For every separable, α-invariant* C**-subalgebra $D \subset A$, there exist positive contractions $(f_{\bar{g}}^{(l)})_{\bar{g} \in G/H}^{l=0,\dots,d}$ in $A_\infty \cap D'$ satisfying*

  (2a) $\left(\sum_{l=0}^{d} \sum_{\bar{g} \in G/H} f_{\bar{g}}^{(l)} \right) \cdot a = a$ *for all $a \in D$;*

  (2b) $f_{\bar{g}}^{(l)} f_{\bar{h}}^{(l)} \in \mathrm{Ann}(D, A_\infty)$ *for all $l = 0, \dots, d$ and $\bar{g} \neq \bar{h}$ in G/H;*

  (2c) $\alpha_{\infty,g}(f_{\bar{h}}^{(l)}) - f_{\overline{gh}}^{(l)} \in \mathrm{Ann}(D, A_\infty)$ *for all $l = 0, \dots, d$, $\bar{h} \in G/H$ and $g \in G$.*

(3) *For all $\varepsilon > 0$ and finite sets $M \subset\subset G$ and $F \subset\subset A$, there are positive contractions $(f_{\bar{g}}^{(l)})_{\bar{g}\in G/H}^{l=0,\dots,d}$ in A satisfying*

(3a) $\left(\sum_{l=0}^{d}\sum_{\bar{g}\in G/H} f_{\bar{g}}^{(l)}\right)\cdot a =_\varepsilon a$ *for all* $a \in F$*;*

(3b) $\|f_{\bar{g}}^{(l)} f_{\bar{h}}^{(l)} a\| \leq \varepsilon$ *for all* $a \in F,\ l = 0,\dots,d$ *and* $\bar{g} \neq \bar{h}$ *in* G/H*;*

(3c) $\alpha_g(f_{\bar{h}}^{(l)})\cdot a =_\varepsilon f_{\overline{gh}}^{(l)} \cdot a$ *for all* $a \in F,\ l = 0,\dots,d,\ \bar{h} \in G/H$ *and* $g \in M$*;*

(3d) $\|[f_{\bar{g}}^{(l)}, a]\| \leq \varepsilon$ *for all* $a \in F,\ l = 0,\dots,d$ *and* $\bar{g} \in G/H$*.*

Moreover, it suffices to check these conditions for M being contained in a given generating set of G.

Proof. (1) $\implies$ (2) : Assume $\dim_{\mathrm{Rok}}(\alpha, H) \leq d$. Choose equivariant c.p.c. order zero maps

$$\varphi_l : (\mathcal{C}(G/H), G\text{-shift}) \longrightarrow (F(D, A_\infty), \alpha_\infty) \quad (l = 0,\dots,d)$$

with $\varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1}) = \mathbf{1}$. Then, by definition, each positive contraction of the form $\varphi_l(\chi_{\{\bar{g}\}}) \in F(D, A_\infty)$ for $\bar{g} \in G/H$ has a representing element $f_{\bar{g}}^{(l)} \in A_\infty \cap D'$ with $f_{\bar{g}}^{(l)} + \mathrm{Ann}(D, A_\infty) = \varphi_l(\chi_{\{g\}})$. By functional calculus, we may assume that each $f_{\bar{g}}^{(l)}$ is a positive contraction. Then we have

$$\varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1}) = \sum_{l=0}^{d}\sum_{\bar{g}\in G/H} f_{\bar{g}}^{(l)} + \mathrm{Ann}(D, A_\infty),$$

and thus the property $\varphi_0(\mathbf{1})+\cdots+\varphi_d(\mathbf{1}) = \mathbf{1}$ translates to condition (2a) by 1.5.3. For each $l = 0,\dots,d$ and $\bar{g} \neq \bar{h}$ in G/H, the characteristic functions $\chi_{\bar{g}}, \chi_{\bar{h}} \in \mathcal{C}(G/H)$ are orthogonal. Since φ_l is an order zero map, we thus have $f_{\bar{g}}^{(l)} f_{\bar{h}}^{(l)} + \mathrm{Ann}(D, A_\infty) = \varphi_l(\chi_{\bar{g}})\varphi_l(\chi_{\bar{h}}) = 0 + \mathrm{Ann}(D, A_\infty)$. This implies condition (2b). Since φ_l is equivariant with respect to the G-shift on G/H, we get for all $g \in G,\ \bar{h} \in G/H$ that

$$\alpha_{\infty,g}(f_{\bar{h}}^{(l)}) + \mathrm{Ann}(D, A_\infty) = \alpha_{\infty,g}(\varphi_l(\chi_{\{\bar{h}\}})) = \varphi_l(\chi_{\{\overline{gh}\}}) = f_{\overline{gh}}^{(l)} + \mathrm{Ann}(D, A_\infty).$$

This implies condition (2c).

(2) $\implies$ (1) : Let $(f_{\bar{g}}^{(l)})_{\bar{g}\in G/H}^{l=0,\dots,d}$ be positive contractions in $A_\infty \cap D'$ satisfying the conditions (2a), (2b) and (2c). Then the same calculations as above, only read in the reverse order, show that conditions (2b) and

(2c) imply that for $l = 0, \ldots, d$, the linear map $\varphi_l : \mathcal{C}(G/H) \to F(D, A_\infty)$ given by $\varphi_l(\chi_{\{\bar{g}\}}) = f_{\bar{g}}^{(l)} + \mathrm{Ann}(D, A_\infty)$ is c.p.c. order zero and equivariant with respect to the G-shift and α_∞. Moreover, condition (2a) implies that $\mathbf{1} = \varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1})$ in $F(D, A_\infty)$.

(2) $\implies$ (3) : Let $\varepsilon > 0$, $M \Subset G$ and $F \Subset A$ be given. Let $D \subset A$ be a separable, α-invariant C*-subalgebra containing F. Choose positive contractions $(f_{\bar{g}}^{(l)})_{\bar{g} \in G/H}^{l=0,\ldots,d}$ in $A_\infty \cap D'$ satisfying the conditions (2a), (2b) and (2c). For each $l = 0, \ldots, d$ and $\bar{g} \in G/H$, the element $f_{\bar{g}}^{(l)}$ has a representing sequence $(f_{\bar{g},n}^{(l)}) \in \ell^\infty(\mathbb{N}, A)$, which we may assume by functional calculus to consist of positive contractions. Then conditions (2a), (2b) and (2c) translate to

- $\left(\sum_{l=0}^d \sum_{\bar{g} \in G/H} f_{\bar{g},n}^{(l)}\right) \cdot a \overset{n\to\infty}{\longrightarrow} a$ for all $a \in D$;
- $f_{\bar{g},n}^{(l)} f_{\bar{h},n}^{(l)} a \overset{n\to\infty}{\longrightarrow} 0$ for all $a \in D,\ l = 0, \ldots, d$ and $\bar{g} \neq \bar{h}$ in G/H;
- $(\alpha_g(f_{\bar{h},n}^{(l)}) - f_{\overline{gh},n}^{(l)}) \cdot a \overset{n\to\infty}{\longrightarrow} 0$ for all $a \in D,\ l = 0, \ldots, d,\ \bar{h} \in G/H$ and $g \in G$.

Moreover, the fact that $f_{\bar{g}}^{(l)} \in A_\infty \cap D'$ translates to

- $[f_{\bar{g},n}^{(l)}, a] \overset{n\to\infty}{\longrightarrow} 0$ for all $a \in D,\ l = 0, \ldots, d$ and $\bar{g} \in G/H$.

Since F is a subset of D, and both F and M are finite, we can choose some large number n such that the elements $(f_{\bar{g},n}^{(l)})_{\bar{g} \in G/H}^{l=0,\ldots,d}$ satisfy conditions (3a), (3b), (3c) and (3d).

(3) $\implies$ (2) : Let $S \subset G$ a be a generating set of G. Assume that (3) holds for all finite subsets M of S. Since S is countable, choose an increasing sequence of finite sets $M_n \Subset S$ with $S = \bigcup_{n \in \mathbb{N}} M_n$.

Let $D \subset A$ be a separable, α-invariant C*-subalgebra. Let $F_n \Subset D$ be an increasing sequence of finite subsets whose union $\bigcup_{k \in \mathbb{N}} F_n \subset D$ is dense. For every n, choose positive contractions $(f_{\bar{g},n}^{(l)})_{\bar{g} \in G/H}^{l=0,\ldots,d}$ in A satisfying conditions (3a), (3b), (3c) and (3d) for the triple $(\varepsilon, M, F) = (\frac{1}{n}, M_n, F_n)$. Set $f_{\bar{g}}^{(l)} = [(f_{\bar{g},n}^{(l)})_n]$ for each $l = 0, \ldots, d$ and $\bar{g} \in G/H$. Then by choice of the sequence $(f_{\bar{g},n}^{(l)})_n$, we have:

- $\left(\sum_{l=0}^d \sum_{\bar{g} \in G/H} f_{\bar{g},n}^{(l)}\right) \cdot a \overset{n\to\infty}{\longrightarrow} a$ for all $a \in \bigcup_{k \in \mathbb{N}} F_k$;
- $f_{\bar{g},n}^{(l)} f_{\bar{h},n}^{(l)} a \overset{n\to\infty}{\longrightarrow} 0$ for all $a \in \bigcup_{k \in \mathbb{N}} F_k,\ l = 0, \ldots, d$ and $\bar{g} \neq \bar{h}$ in G/H;

- $(\alpha_g(f^{(l)}_{\bar{h},n}) - f^{(l)}_{\overline{gh},n}) \cdot a \overset{n\to\infty}{\longrightarrow} 0$ for all $a \in \bigcup_{k\in\mathbb{N}} F_k,\ l = 0,\dots,d,\ \bar{h} \in G/H$ and $g \in \bigcup_{k\in\mathbb{N}} M_k = S$;
- $[f^{(l)}_{\bar{g},n}, a] \overset{n\to\infty}{\longrightarrow} 0$ for all $a \in \bigcup_{k\in\mathbb{N}} F_k,\ l = 0,\dots,d$ and $\bar{g} \in G/H$.

For the correspond elements in the sequence algebra, we thus have

(2a') $\left(\sum_{l=0}^{d} \sum_{\bar{g}\in G/H} f^{(l)}_{\bar{g}}\right) \cdot a = a$ for all $a \in \bigcup_{k\in\mathbb{N}} F_k$;

(2b') $f^{(l)}_{\bar{g}} f^{(l)}_{\bar{h}} a = 0$ for all $a \in \bigcup_{k\in\mathbb{N}} F_k,\ l = 0,\dots,d$ and $\bar{g} \neq \bar{h}$ in G/H;

(2c') $\alpha_g(f^{(l)}_{\bar{h}}) \cdot a = f^{(l)}_{\overline{gh}} \cdot a$ for all $a \in \bigcup_{k\in\mathbb{N}} F_k,\ l = 0,\dots,d,\ \bar{h} \in G/H$ and $g \in \bigcup_{k\in\mathbb{N}} M_k = S$;

(2d') $[f^{(l)}_{\bar{g}}, a] = 0$ for all $a \in \bigcup_{k\in\mathbb{N}} F_k,\ l = 0,\dots,d$ and $\bar{g} \in G/H$.

Since condition (2c') holds for all g in a generating set and this condition passes to products of elements, it follows that condition (2c') even holds for all $g \in G$. Since the union $\bigcup_{k\in\mathbb{N}} F_k \subset D$ is dense, it follows by continuity of multiplication that $f^{(l)}_{\bar{g}} \in A_\infty \cap D'$ for all $l = 0,\dots,d$ and $\bar{g} \in G/H$, and that the elements $(f^{(l)}_{\bar{g}})^{l=0,\dots,d}_{\bar{g}\in G/H}$ satisfy conditions (2a), (2b) and (2c). □

Remark 1.5.7. Let A be a C*-algebra, G a countable group and $\alpha : G \curvearrowright A$ an action. Let $H \subset G$ be a subgroup with finite index and $d \in \mathbb{N}$ a natural number. In the case that A is separable, observe that $\dim_{\mathrm{Rok}}(\alpha, H) \leq d$ if and only if there exist equivariant c.p.c. order zero maps

$$\varphi_l : (\mathcal{C}(G/H), G\text{-shift}) \longrightarrow (F_\infty(A), \alpha_\infty) \quad (l = 0,\dots,d)$$

with $\varphi_0(\mathbf{1}) + \cdots + \varphi_d(\mathbf{1}) = \mathbf{1}$.

Proof. In this case, A is trivially the maximal separable, α-invariant C*-subalgebra in itself. The "only if" part is obvious. For the "if" part, choose positive contractions $(f^{(l)}_{\bar{g}})^{l=0,\dots,d}_{\bar{g}\in G/H}$ in $A_\infty \cap A'$ satisfying the conditions (2a), (2b) and (2c) from 1.5.6 for A in place of D. Given any other separable, α-invariant C*-subalgebra D of A, we then have $f^{(l)}_{\bar{g}} \in A_\infty \cap D'$ and the conditions (2a), (2b) and (2c) hold for this subalgebra. Hence $\dim_{\mathrm{Rok}}(\alpha, H) \leq d$. □

Lemma 1.5.8. *Let A be a C*-algebra, G a countable group and $\alpha : G \curvearrowright A$ an action. If $H_2 \subset H_1 \subset G$ are two subgroups with finite index, then $\dim_{\mathrm{Rok}}(\alpha, H_1) \leq \dim_{\mathrm{Rok}}(\alpha, H_2)$.*

Proof. Since there exists an equivariant and unital $*$-homomorphism

$$(\mathcal{C}(G/H_1), G\text{-shift}) \hookrightarrow (\mathcal{C}(G/H_2), G\text{-shift}),$$

this follows directly from the definition 1.5.5. $\square$

Definition 1.5.9 (cf. [96, 3.7])**.** Let A be a C*-algebra, G a residually finite group and $\alpha : G \curvearrowright A$ an action. Let $\sigma = (G_n)_n \in \Lambda(G)$ be a residually finite approximation. We define

$$\dim_{\mathrm{Rok}}(\alpha, \sigma) = \sup_{n\in\mathbb{N}} \dim_{\mathrm{Rok}}(\alpha, G_n).$$

Moreover, we define

$$\dim_{\mathrm{Rok}}(\alpha) = \sup \{\dim_{\mathrm{Rok}}(\alpha, H) \mid H \subset G,\ [G : H] < \infty\}.$$

Remark 1.5.10. It follows from 1.3.12 and 1.5.8 that if σ is a dominating residually finite approximation, then $\dim_{\mathrm{Rok}}(\alpha, \sigma) = \dim_{\mathrm{Rok}}(\alpha)$.

Let us now discuss in what way this definition extends the ones due to Hirshberg, Winter and Zacharias in [42]:

Remark 1.5.11. Let G be a finite group, A a unital C*-algebra and $\alpha : G \curvearrowright A$ an action. Let $d \in \mathbb{N}$. Then $\dim_{\mathrm{Rok}}(\alpha) = \dim_{\mathrm{Rok}}(\alpha, \{1_G\})$ by 1.5.8. Apply 1.5.6(3) with $H = \{1_G\}$, $M = G$ and finite sets F that contain the unit $\mathbf{1}_A$. Then $\dim_{\mathrm{Rok}}(\alpha) \leq d$ if and only if the following holds:

For all $\varepsilon > 0$ and $1_A \in F \Subset A$, there are positive contractions $(f_g^{(l)})_{g\in G}^{l=0,\dots,d}$ in A satisfying

- $\sum_{l=0}^d \sum_{g\in G} f_g^{(l)} =_\varepsilon \mathbf{1}_A$ for all $a \in F$;
- $\|f_g^{(l)} f_h^{(l)}\| \leq \varepsilon$ for all $l = 0, \dots, d$ and $g \neq h$ in G;
- $\alpha_g(f_h^{(l)}) =_\varepsilon f_{gh}^{(l)}$ for all $l = 0, \dots, d$ and $g, h \in G$;
- $\|[f_g^{(l)}, a]\| \leq \varepsilon$ for all $a \in F,\ l = 0, \dots, d$ and $g \in G$.

In this way, we see that this notion of Rokhlin dimension for finite group actions recovers the definition [42, 1.1] of Hirshberg, Winter and Zacharias.

Remark 1.5.12. Let A be a unital C*-algebra and α an automorphism on A. We identify α with its induced $\mathbb{Z}$-action on A. Let $d \in \mathbb{N}$. Then by 1.3.13 and 1.5.10, we have

$$\dim_{\mathrm{Rok}}(\alpha) = \dim_{\mathrm{Rok}}(\alpha, \{n! \cdot \mathbb{Z}\}_{n\in\mathbb{N}}) = \sup\left\{\dim_{\mathrm{Rok}}(\alpha, n \cdot \mathbb{Z}) \mid n \in \mathbb{N}\right\}.$$

Apply 1.5.6(3) with $H = n \cdot \mathbb{Z}$, $M = \{1\} \subset \mathbb{Z}$ and finite sets F that contain the unit $\mathbf{1}_A$. For each $n \in \mathbb{N}$, we identify the elements in $\mathbb{Z}/n\mathbb{Z}$ with elements of the finite interval $\{0, \dots, n-1\}$ in the obvious way. Then $\dim_{\mathrm{Rok}}(\alpha) \leq d$ if and only if the following holds:

For all $n \in \mathbb{N}$, $\varepsilon > 0$ and $1_A \in F \Subset A$, there are positive contractions $(f_j^{(l)})_{j=0,\dots,n-1}^{l=0,\dots,d}$ in A satisfying

- $\sum_{l=0}^{d}\sum_{j=0}^{n-1} f_j^{(l)} =_\varepsilon \mathbf{1}_A$ for all $a \in F$;
- $\|f_j^{(l)} f_k^{(l)}\| \leq \varepsilon$ for all $l = 0, \dots, d$ and $1 \leq j \neq k \leq n-1$;
- $\alpha(f_j^{(l)}) =_\varepsilon f_{j+1}^{(l)}$ for all $l = 0, \dots, d$ and $1 \leq j < n-1$;
- $\alpha(f_{n-1}^{(l)}) =_\varepsilon f_0^{(l)}$ for all $l = 0, \dots, d$;
- $\|[f_j^{(l)}, a]\| \leq \varepsilon$ for all $a \in F,\ l = 0, \dots, d$ and $1 \leq j \leq n-1$.

Although this notion of Rokhlin dimension for integer actions does not coincide with the primary definition [42, 2.3a)] of Hirshberg, Winter and Zacharias, we see that it does recover a special case of what they have called Rokhlin dimension with single towers [42, 2.3c), 2.9]. The fact that finite Rokhlin dimension can also be characterized by the existence of certain c.p.c. order zero maps with approximately central images, was already observed by Hirshberg, Winter and Zacharias, see [42, 2.6]. If one compares two different versions of Rokhlin dimension for integer actions from [42], then it was shown in [42, 2.8, 2.9] that finiteness with respect to either version is equivalent.

Remark 1.5.13. In general, the value $\dim_{\mathrm{Rok}}(\alpha, \sigma)$ can indeed depend on the sequence σ. Hirshberg, Winter and Zacharias have first observed this phenomenon for integer actions, see [42, 2.4(vi)]. However, it appears to be unknown at present if there is any example of an action where the Rokhlin dimension associated to some residually finite approximation is finite, but the full Rokhlin dimension $\dim_{\mathrm{Rok}}$ is not.

Recall that one of the main results by Hirshberg, Winter and Zacharias in [42] is that actions with finite Rokhlin dimension enjoy a permanence property with respect to finite nuclear dimension:

Theorem (see [42, 1.3])**.** *Let G be a finite group, A a unital* C*-*algebra and $\alpha : G \curvearrowright A$ an action. Then*

$$\dim^{+1}_{\mathrm{Rok}}(A \rtimes_\alpha G) \leq \dim^{+1}_{\mathrm{Rok}}(\alpha) \cdot \dim^{+1}_{\mathrm{nuc}}(A)$$

and

$$\mathrm{dr}^{+1}(A \rtimes G) \leq \dim^{+1}_{\mathrm{Rok}}(\alpha) \cdot \mathrm{dr}^{+1}(A).$$

Theorem (see [42, 4.1] and its proof)**.** *Let A be a unital* C*-*algebra and α an automorphism on A. Then*

$$\dim^{+1}_{\mathrm{nuc}}(A \rtimes_\alpha \mathbb{Z}) \leq 2 \cdot \dim^{+1}_{\mathrm{Rok}}(\alpha) \cdot \dim^{+1}_{\mathrm{nuc}}(A).$$

It turns out that when one considers actions of a residually finite group with a finite-dimensional box space, then one has an analogous permanence property of actions with finite Rokhlin dimension with respect to C*-algebras of finite nuclear dimension. The following extends the above theorems and is the main result of this chapter:

Theorem 1.5.14. *Let A be a* C*-*algebra, G a countable, residually finite group and $\alpha : G \curvearrowright A$ an action. Let $\sigma = (G_n)_n$ be a residually finite approximation of G. Then*

$$\dim^{+1}_{\mathrm{nuc}}(A \rtimes_\alpha G) \leq \mathrm{asdim}^{+1}(\Box_\sigma G) \cdot \dim^{+1}_{\mathrm{Rok}}(\alpha, \sigma) \cdot \dim^{+1}_{\mathrm{nuc}}(A).$$

If G is finitely generated, we get the estimate

$$\dim^{+1}_{\mathrm{nuc}}(A \rtimes_\alpha G) \leq \mathrm{asdim}^{+1}(\Box_s G) \cdot \dim^{+1}_{\mathrm{Rok}}(\alpha) \cdot \dim^{+1}_{\mathrm{nuc}}(A)$$

as a special case.

Remark. This indeed extends the corresponding theorems [42, 1.3, 4.1] by Hirshberg, Winter and Zacharias. Note that for the case of finite groups G, every of its box spaces has asymptotic dimension zero by inserting the trivial cover $\{G\}$ of G in 1.3.5(2). So $\mathrm{asdim}^{+1}(\Box_s G) = 1$, and this factor does not appear. For integer actions, 1.4.3 shows that $\mathrm{asdim}(\Box_s \mathbb{Z}) = 1$, and hence the above formula recovers the factor $2 = \operatorname{asdim}^{+1}(\square_s\mathbb{Z})$ in the formula by Hirshberg, Winter and Zacharias.

Proof of 1.5.14. We may assume that all of the numbers $s = \operatorname{asdim}(\square_\sigma G)$, $r = \dim_{\mathrm{nuc}}(A)$ and $d = \dim_{\mathrm{Rok}}(\alpha, \sigma)$ are finite, or else the statement is trivial.

Let $F \Subset A \rtimes_\alpha G$ and $\delta > 0$ be given. In order to show the assertion, we show that there exists a finite-dimensional C*-algebra $\mathcal{F}$ with a c.p. approximation $(\mathcal{F}, \psi, \varphi)$ for F up to δ on $A \rtimes_\alpha G$ as required by 1.1.7.

Since the purely algebraic crossed product is dense in $A \rtimes_\alpha G$ and is linearly generated by terms of the form au_g for $a \in A$ and $g \in G$, we may assume without loss of generality that $F \subset \{au_g \mid a \in F', g \in M\}$ for two finite sets $F' \Subset A_1$ and $M \Subset G$.

The square root function $\sqrt{\cdot} : [0,1] \to [0,1]$ is uniformly continuous. So choose $\varepsilon > 0$ with $\varepsilon \leq \delta$ so small that $|\sqrt{s} - \sqrt{t}| \leq \delta$, whenever $|s - t| \leq \varepsilon$. Apply 1.3.5(3) to find $n \in \mathbb{N}$ and functions $\mu^{(j)} : G \to [0,1]$ for $j = 0, \dots, s$ such that

(d1) for every $j = 0, \dots, s$, and for all $h \in G_n \setminus \{1\}$ one has

$$\operatorname{supp}(\mu^{(j)}) \cap \operatorname{supp}(\mu^{(j)})h = \emptyset;$$

(d2) for every $g \in G$, we have

$$\sum_{j=0}^{s} \sum_{h \in G_n} \mu^{(j)}(gh) = \sum_{j=0}^{s} \sum_{h \in \bar{g}} \mu^{(j)}(h) = 1;$$

(d3) for every $j = 0, \dots, s$ and $g \in M$, one has

$$\|\mu^{(j)} - \mu^{(j)}(g^{-1} \cdot \_)\|_\infty \leq \varepsilon.$$

Consider the finite sets

$$B_n^{(j)} = \operatorname{supp}(\mu^{(j)}), \quad B_n = \bigcup_{j=0}^{s} B_n^{(j)} \Subset G \tag{e1}$$

and

$$\tilde{F} = \left\{\alpha_{h^{-1}}(a) \mid a \in F',\ h \in B_n\right\} \Subset A.$$

Let A act faithfully on a Hilbert space H and let $A \rtimes_\alpha G \cong A \rtimes_{r,\alpha} G$ be embedded into $\mathcal{B}(\ell^2(G) \otimes H)$ via the left-regular representation. Note that $\mathrm{asdim}(\Box_\sigma G) < \infty$ implies by 1.3.4 that G is amenable. Let $Q_j \in \mathcal{B}(\ell^2(G) \otimes H)$ be the projection onto the subspace $\ell^2(B_n^{(j)}) \otimes H$. Then the assignment $x \mapsto Q_j x Q_j$ defines a c.p.c. map $\Psi_j : A \rtimes_{r,\alpha} G \to M_{|B_n^{(j)}|}(A)$. We have for all $a \in A, g \in G$ and $j = 0, \dots, s$ that

$$\begin{aligned}
\Psi_j(au_g) &= Q_j \Big[\sum_{h \in G} e_{h,g^{-1}h} \otimes \alpha_{h^{-1}}(a) \Big] Q_j \\
&= \sum_{\substack{h \in B_n^{(j)}: \\ g^{-1}h \in B_n^{(j)}}} e_{h,g^{-1}h} \otimes \alpha_{h^{-1}}(a) \\
&= \sum_{h \in B_n^{(j)} \cap g B_n^{(j)}} e_{h,g^{-1}h} \otimes \alpha_{h^{-1}}(a). \qquad \text{(e2)}
\end{aligned}$$

For $j = 0, \dots, s$, define the diagonal matrices $E_j \in M_{|B_n^{(j)}|}(\mathbb{C})$ by $(E_j)_{h,h} = \mu^{(j)}(h)$. By our choice of ε and d3, we have

$$\Big\| \sqrt{\mu^{(j)}} - \sqrt{\mu^{(j)}(g^{-1} \cdot \_)} \Big\|_\infty \leq \delta \quad \text{for all } g \in M. \qquad \text{(e3)}$$

It follows for all $g \in M$ and $a \in A$ that

$$\begin{aligned}
&\|[\sqrt{E_j}, \Psi_j(au_g)]\| \\
&\overset{e2}{=} \Big\| \sum_{h \in B_n \cap g B_n} \Big(\sqrt{\mu^{(j)}(h)} - \sqrt{\mu^{(j)}(g^{-1}h)} \Big) e_{h,g^{-1}h} \otimes \alpha_{h^{-1}}(a) \Big\| \\
&\leq \max \Big\{ \Big| \sqrt{\mu^{(j)}(h)} - \sqrt{\mu^{(j)}(g^{-1}h)} \Big| \ \Big| \ h \in B_n \cap g B_n \Big\} \cdot \|a\| \\
&\overset{e3}{\leq} \delta \|a\|.
\end{aligned}$$

Define the c.p.c. map $\theta_j : A \rtimes_{r,\alpha} G \to M_{|B_n^{(j)}|}(A)$ by $\theta_j(x) = \sqrt{E_j}\Psi_j(x)\sqrt{E_j}$. By the previous calculation, we have

$$\|\theta_j(au_g) - E_j \Psi_j(au_g)\| \leq \delta \cdot \|a\| \quad \text{for all} \quad g \in M \text{ and } a \in A. \qquad \text{(e4)}$$

Observe that for $au_g \in F$, the matrix coefficients of $\theta_j(au_g) \in M_{|B_n^{(j)}|}(A)$ are all in $\tilde{F}$.

Choose an r-decomposable c.p. approximation $(\mathcal{F}, \psi, \varphi)$ for $\tilde{F}$ within A up to $\delta/[G : G_n]$, i.e. a finite-dimensional C*-algebra $\mathcal{F}$ and c.p. maps

$A \overset{\psi}{\to} \mathcal{F} \overset{\varphi^{(i)}}{\to} A$ for $i = 0, \dots, r$ such that ψ is c.p.c., the maps $\varphi^{(i)}$ are c.p.c. order zero and

$$\|x - (\varphi \circ \psi)(x)\| \leq \frac{\delta}{[G : G_n]} \quad \text{for all } x \in \tilde{F}, \tag{e5}$$

where φ denotes the sum $\varphi = \varphi^{(0)} + \cdots + \varphi^{(r)}$.

For all $j = 0, \dots, s$ and $i = 0, \dots, r$ let

$$\psi_j = \mathrm{id}_{|B_n^{(j)}|} \otimes \psi : M_{|B_n^{(j)}|} \otimes A \to M_{|B_n^{(j)}|} \otimes \mathcal{F} \tag{e6}$$

and

$$\varphi_{j,i} = \mathrm{id}_{|B_n^{(j)}|} \otimes \varphi^{(i)} : M_{|B_n^{(j)}|} \otimes \mathcal{F} \to M_{|B_n^{(j)}|} \otimes A \tag{e7}$$

denote the amplifications of ψ and $\varphi^{(i)}$.

Let $D \subset A$ be a separable, α-invariant C*-algebra containing $\tilde{F}$ and the image of $\varphi^{(i)}$ for each $i = 0, \dots, r$. Since $\dim_{\mathrm{Rok}}(\alpha, G_n) \leq d$, we can apply 1.5.6(2) and find positive contractions $(f_{\bar{h}}^{(l)})_{\bar{h} \in G/G_n}^{l=0,\dots,d}$ in $A_\infty \cap D'$ satisfying the relations

(R1) $\Big(\sum_{l=0}^{d} \sum_{\bar{h} \in G/G_n} f_{\bar{h}}^{(l)}\Big) a = a$ for all $a \in D$;

(R2) $f_{\bar{g}}^{(l)} f_{\bar{h}}^{(l)} a = 0$ for all $a \in D$, $\bar{g} \neq \bar{h}$ and all $l = 0, \dots, d$;

(R3) $\alpha_{\infty,g}(f_{\bar{h}}^{(l)}) a = f_{\overline{gh}}^{(l)} a$ for all $a \in D$, $l = 0, \dots, d$ and $g, h \in G$.

Now let $j \in \{0, \dots, s\}$ and $l \in \{0, \dots, d\}$. Define maps

$$\sigma_{j,l} : M_{|B_n^{(j)}|}(D) \to (A \rtimes_\alpha G)_\infty \quad \text{by} \quad \sigma_{j,l}(e_{g,h} \otimes a) = u_g f_{\bar{1}}^{(l)} a u_h^*. \tag{e8}$$

Note that these are indeed c.p. since $\sigma_{j,l}(x) = v_{j,l} x v_{j,l}^*$ for the $1 \times |B_n^{(j)}|$-matrix $v_{j,l} = (u_h f_{\bar{1}}^{(l)1/2})_{h \in B_n^{(j)}}$. We now show that these maps are order zero. For each $l = 0, \dots, d$, we denote $f^{(l)} = \sum_{\bar{g} \in G/G_n} f_{\bar{g}}^{(l)}$. Observe that by R3, this is an element fixed by α_∞ upon multiplying with an element in D. Let $h_1, h_2, h_3, h_4 \in B_n^{(j)}$ and $a, b \in D$. We have

$$\begin{aligned} \sigma_{j,l}(e_{h_1,h_2} \otimes a)\sigma_{j,l}(e_{h_3,h_4} \otimes b) &= u_{h_1} f_{\bar{1}}^{(l)} a u_{h_2}^* \cdot u_{h_3} f_{\bar{1}}^{(l)} b u_{h_4}^* \\ &= u_{h_1} a f_{\bar{1}}^{(l)} \cdot (\alpha_{h_3^{-1}h_2})_\infty (f_{\bar{1}}^{(l)}) u_{h_2^{-1}h_3} b u_{h_4}^* \end{aligned}$$

$$
\begin{aligned}
&\overset{\text{R3}}{=} \quad u_{h_1} a f^{(l)}_{\bar{1}} \cdot f^{(l)}_{\overline{h_3^{-1}h_2}} u_{h_2^{-1}h_3} b u^*_{h_4} \\
&\overset{\text{R2}}{=} \quad \delta_{\bar{h}_2,\bar{h}_3} \cdot u_{h_1} a (f^{(l)}_{\bar{1}})^2 b u^*_{h_4} \\
&\overset{\text{R2}}{=} \quad \delta_{\bar{h}_2,\bar{h}_3} \cdot u_{h_1} f^{(l)} f^{(l)}_{\bar{1}} a b u^*_{h_4} \\
&\overset{\text{R3,d1},e1}{=} \quad f^{(l)} \cdot \sigma_{j,l}\big((e_{h_1,h_2} \otimes a)(e_{h_3,h_4} \otimes b)\big).
\end{aligned}
$$

Note that for the last step of the above calculation, we have used d1 in that the canonical map $G \twoheadrightarrow G/G_n$ is injective on each set $B_n^{(j)}$. Since this calculation involves linear generators of the C*-algebra $M_{|B_n^{(j)}|}(D)$, we obtain the equation

$$\sigma_{j,l}(x)\sigma_{j,l}(y) = f^{(l)} \cdot \sigma_{j,l}(xy) \quad \text{for all } x, y \in M_{|B_n^{(j)}|}(D).$$

In particular, each map $\sigma_{j,l}$ is order zero and so is

$$\sigma_{j,l} \circ \varphi_{j,i} : M_{|B_n^{(j)}|} \otimes \mathcal{F} \to (A \rtimes_\alpha G)_\infty$$

for all $i = 0, \dots, r,\ j = 0, \dots, s$ and $l = 0, \dots, d$. Note that this composition makes sense because of our assumption that D contains the images of each $\varphi^{(i)}$, and so the image of each $\varphi_{j,i}$ is contained in the domain of $\sigma_{j,l}$.

As the next step, we would like to show that the maps

$$
\begin{array}{ccc}
A \rtimes_\alpha G & \longrightarrow & (A \rtimes_\alpha G)_\infty \\
{\scriptstyle \bigoplus_j \psi_j \circ \theta_j =: \Theta} \searrow & & \nearrow {\scriptstyle \sum_{j,l,i} \sigma_{j,l} \circ \varphi_{j,i} =: \Phi} \\
& \bigoplus_{j=0}^s M_{|G_n^{(j)}|}(\mathcal{F}) &
\end{array}
\qquad \text{(e9)}
$$

give rise to a good c.p. approximation of F. For this, we first calculate for every contraction $x \in D$ that

$$
\begin{aligned}
&\Big\| \sum_{j=0}^{s} \sum_{h \in B_n^{(j)} \setminus g B_n^{(j)}} \mu^{(j)}(h) f^{(l)}_{\bar{h}} x \Big\| \\
&\leq \quad (s+1) \cdot \max_{0 \leq j \leq s} \Big\| \sum_{h \in B_n^{(j)} \setminus g B_n^{(j)}} \mu^{(j)}(h) f^{(l)}_{\bar{h}} x \Big\| \\
&\overset{\text{R2}}{\leq} \quad (s+1) \cdot \max \Big\{ \mu^{(j)}(h) \mid h \in B_n^{(j)} \setminus g B_n^{(j)}, j = 0, \dots, s \Big\} \\
&\overset{e1}{\leq} \quad (s+1) \cdot \max \big\{ \|\mu^{(j)} - \mu^{(j)}(g^{-1} \cdot \_)\|_\infty \mid g \in M, j = 0, \dots, s \big\}
\end{aligned}
$$

$$\overset{\text{d3}}{\leq} \quad (s+1)\varepsilon \ \leq \ (s+1)\delta. \tag{e10}$$

Observe for all $l = 0, \dots, d$

$$\begin{aligned} f^{(l)} = \sum_{\bar{g} \in G/G_n} f_{\bar{g}}^{(l)} &\overset{\text{d2}}{=} \sum_{\bar{g} \in G/G_n} \sum_{j=0}^{s} \sum_{h \in \bar{g}} \mu^{(j)}(h) f_{\bar{g}}^{(l)} \\ &= \sum_{j=0}^{s} \sum_{h \in B_n^{(j)}} \mu^{(j)}(h) f_{\bar{h}}^{(l)}. \end{aligned} \tag{e11}$$

Note that the last equation follows from the fact that for each $j = 0, \dots, s$, the set $B_n^{(j)}$ is contained in a finite set of representatives of G/G_n by d1 and e1. It follows for all $g \in M$ and contractions $x \in D$ that

$$\begin{aligned} &\left(\sum_{l=0}^{d} \sum_{j=0}^{s} \sum_{h \in B_n^{(j)} \cap g B_n^{(j)}} \mu^{(j)}(h) f_{\bar{h}}^{(l)} \right) \cdot x \\ &\overset{e11,\text{R1}}{=} \qquad x - \left(\sum_{l=0}^{d} \sum_{j=0}^{s} \sum_{h \in B_n^{(j)} \setminus g B_n^{(j)}} \mu^{(j)}(h) f_{\bar{h}}^{(l)} \right) \cdot x \\ &\overset{e10}{=}_{(s+1)(d+1)\delta} \quad x. \end{aligned} \tag{e12}$$

Now let $au_g \in F$ and recall the definition of the maps Θ and Φ from the approximation diagram e9. We calculate that

$$\begin{aligned} &\Phi \circ \Theta(au_g) \\ &\overset{e9}{=} \quad \sum_{l=0}^{d} \sum_{j=0}^{s} \sum_{i=0}^{r} (\sigma_{j,l} \circ \varphi_{j,i} \circ \psi_j \circ \theta_j)(au_g) \\ &\overset{e4}{=}_{(s+1)(d+1)(r+1)\delta} \ \sum_{l=0}^{d} \sum_{j=0}^{s} \sum_{i=0}^{r} (\sigma_{j,l} \circ \varphi_{j,i} \circ \psi_j)(E_j \Psi_j(au_g)) \\ &\overset{e2}{=} \quad \sum_{l=0}^{d} \sum_{j=0}^{s} \sum_{i=0}^{r} (\sigma_{j,l} \circ \varphi_{j,i} \circ \psi_j) \left(\sum_{h \in B_n^{(j)} \cap g B_n^{(j)}} \mu^{(j)}(h) \cdot e_{h,g^{-1}h} \otimes \alpha_{h^{-1}}(a) \right) \end{aligned}$$

$$\overset{e6,e7}{=} \sum_{l=0}^{d}\sum_{j=0}^{s}\sum_{i=0}^{r}\sigma_{j,l}\left(\sum_{h\in B_n^{(j)}\cap gB_n^{(j)}}\mu^{(j)}(h)\cdot e_{h,g^{-1}h}\otimes(\varphi^{(i)}\circ\psi)\big(\alpha_{h^{-1}}(a)\big)\right)$$

$$\overset{e8}{=} \sum_{l=0}^{d}\sum_{j=0}^{s}\sum_{i=0}^{r}\sum_{h\in B_n^{(j)}\cap gB_n^{(j)}}\mu^{(j)}(h)\cdot u_h f_{\bar{1}}^{(l)}\Big[(\varphi^{(i)}\circ\psi)\big(\alpha_{h^{-1}}(a)\big)\Big]u_{h^{-1}g}$$

$$\overset{R3}{=} \sum_{l=0}^{d}\sum_{j=0}^{s}\sum_{h\in B_n^{(j)}\cap gB_n^{(j)}}\mu^{(j)}(h)f_{\bar{h}}^{(l)}\cdot u_h\Big[(\varphi\circ\psi)\big(\alpha_{h^{-1}}(a)\big)\Big]u_{h^{-1}g}$$

$$\overset{e5}{=}_{(s+1)(d+1)|B_n^{(j)}|\cdot\frac{\delta}{[G:G_n]}} \sum_{l=0}^{d}\sum_{j=0}^{s}\sum_{h\in B_n^{(j)}\cap gB_n^{(j)}}\mu^{(j)}(h)f_{\bar{h}}^{(l)}\cdot u_h\alpha_{h^{-1}}(a)u_{h^{-1}g}$$

$$= \left(\sum_{l=0}^{d}\sum_{j=0}^{s}\sum_{h\in B_n^{(j)}\cap gB_n^{(j)}}\mu^{(j)}(h)f_{\bar{h}}^{(l)}\right)\cdot au_g$$

$$\overset{e12}{=}_{(s+1)(d+1)\delta} au_g.$$

Summing up these approximation steps and using d1 in the form of the inequality $|B_n^{(j)}|\leq[G:G_n]$, it follows for all $au_g\in F$ that

$$au_g =_{3(s+1)(d+1)(r+1)\delta} \Phi\circ\Theta(au_g).$$

Now let us recall what we got. We have contructed a c.p. approximation

$$\left(\bigoplus_{j=0}^{s} M_{|B_n^{(j)}|}(\mathcal{F}),\ \Theta,\ \Phi\right)$$

of tolerance $3(s+1)(d+1)(r+1)\delta$ on F, where the map

$$\Phi=\sum_{i=0}^{r}\sum_{l=0}^{d}\sum_{j=0}^{s}\sigma_{j,l}\circ\varphi_{j,i}:\bigoplus_{j=0}^{s}M_{|B_n^{(j)}|}(\mathcal{F})\to(A\rtimes_\alpha G)_\infty$$

is a sum of $(s+1)(d+1)(r+1)$ c.p.c. order zero maps. Since d,s,r are constants and $F\Subset A\rtimes_\alpha G$ and $\delta>0$ were arbitrary, it follows from 1.1.7

that

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_\alpha G) \leq (s+1)(d+1)(r+1),$$

which is what we wanted to show. □

In the case of $\mathbb{Z}^m$-actions, the estimate of 1.5.14 simplifies nicely, since we know from 1.4.3 that the standard box space of $\mathbb{Z}^m$ has asymptotic dimension equal to m. We note that before the collaboration with Wu and Zacharias [96] took place, the following result was also proved in my own paper [94, 1.10] for unital C*-algebras, as a generalization of [42, 4.1], but with a worse estimate than given below.

Corollary 1.5.15. *Let A be a* C*-*algebra, $m \in \mathbb{N}$ and $\alpha : \mathbb{Z}^m \curvearrowright A$ an action. Then we have*

$$\dim_{\mathrm{nuc}}^{+1}(A \rtimes_\alpha \mathbb{Z}^m) \leq (m+1) \cdot \dim_{\mathrm{Rok}}^{+1}(\alpha) \cdot \dim_{\mathrm{nuc}}^{+1}(A).$$

Proof. This follows from 1.5.14 and 1.4.3. □

Crossed products by $\mathbb{Z}$ revisited

From this point on, we will mainly be interested in determining when transformation group C*-algebras by free actions of discrete groups have finite nuclear dimension. An important tool towards that will be the theory of Rokhlin dimension developed in the first chapter. However, we will first revisit the special case of $\mathbb{Z}$-actions in this chapter. The first reason is that this chapter is intended as a somewhat more pleasant starting point for most readers, where we do not begin with all the technicalities and complications that one has to get into in the third chapter. The second reason is that the case of $\mathbb{Z}$-actions, although easier to handle than higher-rank group actions, already showcases the need for the so-called marker property, which itself will play a major role in the third chapter. The two upcoming sections each handle a different approach towards showing that $\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$ has finite nuclear dimension, if X is a compact metric space with finite covering dimension and $\varphi : X \to X$ is aperiodic.

2.1 The Toms-Winter approach

In this section, we follow an approach by Toms and Winter from [100] to show that transformation group C*-algebras by certain $\mathbb{Z}$-actions have finite nuclear dimension. Their main result asserts that, if $\varphi : X \to X$ is a minimal homeomorphism on a compact metric space X with finite covering dimension, then $\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$ has finite nuclear dimension. We will extend this result to aperiodic homeomorphisms with the help of the marker property.

But first, we need to recall the notion of orbit-breaking algebras, as introduced originally for Cantor minimal systems by Putnam:

Definition 2.1.1 (cf. [78, Section 3])**.** Let X be a compact metric space and $\varphi : X \to X$ a homeomorphism. Consider the transformation group C*-algebra $A = \mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$ and denote by u the unitary implementing the action on $\mathcal{C}(X)$ that is induced by φ. If $Y \subset X$ is a closed subset, define its so-called orbit-breaking algebra via

$$A_Y = C^*\big(\mathcal{C}(X) \cup u \cdot \mathcal{C}_0(X \setminus Y)\big) \ \subset \ \mathcal{C}(X) \rtimes_\varphi \mathbb{Z}.$$

Next, we shall recall Phillips' notion of a recursive subhomogeneous algebra:

Definition 2.1.2 (cf. [75, 1.1])**.** A recursive subhomogeneous algebra is a C*-algebra given by the following recursive definition.

- If X is a compact Hausdorff space and $n \geq 1$, then $\mathcal{C}(X, M_n)$ is a recursive subhomogeneous algebra.
- If A is a recursive subhomogeneous algebra, X is a compact Hausdorff space, $X^{(0)} \subset X$ is closed, $\varphi : A \to \mathcal{C}(X^{(0)}, M_n)$ is any unital homomorphism, and $\rho : \mathcal{C}(X, M_n) \to \mathcal{C}(X^{(0)}, M_n)$ is the restriction homomorphism, then the pullback

  $$A \oplus_{\mathcal{C}(X^{(0)}, M_n)} \mathcal{C}(X, M_n) = \{(a, f) \in A \oplus \mathcal{C}(X, M_n) \mid \varphi(a) = \rho(f)\}$$

  is a recursive subhomogeneous algebra.

Remark 2.1.3 (cf. [75, 1.2])**.** Let R be a recursive subhomogeneous algebra. Then by definition, there exists some decomposition

$$R = \Big[\dots\Big[\Big[C_0 \oplus_{C_1^{(0)}} C_1\Big] \oplus_{C_2^{(0)}} C_2\Big] \dots\Big] \oplus_{C_l^{(0)}} C_l$$

for $C_k = \mathcal{C}(X_k, M_{n_k}), C_k^{(0)} = \mathcal{C}(X_k^{(0)}, M_{n_k})$ with natural numbers n_k and compact Hausdorff spaces X_k and closed subsets $X_k^{(0)} \subset X_k$. Given such a decomposition, we say that it has topological dimension at most $r \in \mathbb{N}$, if $\dim(X_k) \leq r$ for all r. By the main result of Winter's paper on the decomposition rank of subhomogeneous C*-algebras [104, 1.6], it follows that the decomposition rank (and so in particular the nuclear dimension) of R is bounded by the topological dimension of this decomposition.

Lemma 2.1.4. *Let X be a compact metric space and $\varphi : X \to X$ a homeomorphism. Let $Y \subset X$ be closed subset with $X = \bigcup_{j=0}^{N} \varphi^j(Y)$ for some $N \in \mathbb{N}$. Then its orbit-breaking algebra A_Y has a decomposition as a recursive subhomogeneous algebra with topological dimension at most* $\dim(X)$*. In particular, the decomposition rank of A_Y is at most the covering dimension of X.*

Proof. This fact was proved in [65, Section 3] in the case that φ is minimal and Y is closed with non-empty interior. However, that proof carries over verbatim with the given conditions. The key argument there only uses the fact that finitely many translates of Y cover all of X. □

Now we come to the definition of markers for a homeomorphism on a compact metric space. Conceptually, this notion was first defined for systems on zero dimensional spaces by Downarovicz in [17, Definition 2], being inspired by Krieger's marker Lemma [56, Lemma 2] in topological dynamics. In the presented generality, it was then defined by Gutman in [39].

Definition 2.1.5 (cf. [39, 5.1])**.** Let X be a compact metric space and $\varphi : X \to X$ a homeomorphism. Let $Z \subset X$ be a subset and $n \in \mathbb{N}$. We call Z an n-marker if $\overline{Z} \cap \varphi^j(\overline{Z}) = \emptyset$ for all $j = 1, \dots, n-1$ and $X = \bigcup_{j=0}^{N} \varphi^j(Z)$ for some N. The homeomorphism φ is said to have the marker property if there exist open n-markers for all n.

Gutman has asked the question whether every aperiodic homeomorphism on a compact metric space must necessarily have the marker property. Although this question is still unanswered in full generality, he proved the following partial result:

Theorem 2.1.6 (see [39, 6.1])**.** *Let X be a compact metric space with finite covering dimension and $\varphi : X \to X$ an aperiodic homeomorphism. Then φ has the marker property.*

Before we discuss how this can be applied, we need a technical Lemma that already appeared in an almost identical form in [100].

Lemma 2.1.7 (cf. [100, 1.1])**.** *Let $K \subset \mathbb{C}$ a compact subset and $\mathcal{G}$ finite set consisting of continuous functions $f : K \to \mathbb{C}$ with $f(0) = 0$, if $0 \in K$. For every $\varepsilon > 0$, there exists $\delta > 0$ with the following property: Whenever $x \in A$ is a normal element in a* C^**-algebra A with $\mathrm{Sp}(x) \subseteq K$, and $y \in A$ is a contraction with $\|[x, y]\| \leq \delta$ and $\|[x^*, y]\| \leq \delta$, then $\|[f(x), y]\| \leq \varepsilon$ for all $f \in \mathcal{G}$.*

Proof. Write $\mathcal{G} = \{f_1, \dots, f_k\}$. By the Weierstrass approximation theorem, we can find $N \in \mathbb{N}$ and polynomials $p_i(z) = \sum_{j=1}^N a_j^{(i)} z^j + c_j^{(i)} \overline{z}^j$ for $i = 1, \dots, k$ such that $\|f_i - p_i\|_{\infty,K} \leq \varepsilon/4$ for all $i = 1, \dots, k$. This implies $[f_i(x), y] =_{\varepsilon/2} [p_i(x), y]$ for all $i = 1, \dots, k$ and contractions $y \in A$. Using the basic properties of commutators, we calculate

$$\begin{aligned} \|[p_i(x), y]\| &= \Big\| \Big[\sum_{j=1}^N a_j^{(i)} x^j + c_j^{(i)} x^{*j}, y \Big] \Big\| \\ &\leq \sum_{j=1}^N j \cdot (|a_j^{(i)}| + |c_j^{(i)}|) \cdot \max \{ \|[x, y]\|, \|[x^*, y]\| \} . \end{aligned}$$

Now we observe that

$$\delta = \frac{\varepsilon}{2} \cdot \left(\sup_{i=1,\dots,k} \sum_{j=1}^N j \cdot (|a_j^{(i)}| + |c_j^{(i)}|) \right)^{-1}$$

yields the desired statement. □

As an application of 2.1.6, we can generalize one of the main results of [100], which was proved for minimal homeomorphisms. We note that the proof of the next result closely follows the original approach by Toms and Winter.

Theorem 2.1.8 (cf. [100, 3.3])**.** *Let X be a compact metric space and $\varphi : X \to X$ an aperiodic homeomorphism. Then the nuclear dimension of $\mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$ is at most $2\dim(X) + 1$.*

Proof. Set $d = \dim(X)$ and assume that it is finite, as there is otherwise nothing to show. Let $F \Subset A = \mathcal{C}(X) \rtimes_\varphi \mathbb{Z}$ be given. We have to show that there exist good $2(d+1)$-decomposable c.p. approximations of F through finite-dimensional C^*-algebras. As the purely algebraic crossed product

$\mathcal{C}(X) \rtimes_{alg} \mathbb{Z}$ is dense in A and is linearly generated by terms of the form fu^j for $f \in \mathcal{C}(X)$ and $j \in \mathbb{N}$, we may assume without loss of generality that

$$F = \left\{ fu^j \mid f \in F', -N \leq j \leq N \right\}$$

for some $N \in \mathbb{N}$ and $F' \Subset \mathcal{C}(X)_1$. So let $\varepsilon > 0$. Use 2.1.7 to choose $\delta > 0$ for this $\varepsilon > 0$ with respect to $K = [0, 1]$ and

$$\mathcal{G} = \left\{ x^{i/2j} \mid j = 1, \dots, N,\ i = 1, \dots, 2j - 1 \right\}.$$

Now choose a natural number $n \geq \frac{1}{\delta}$. Since φ has the marker property, we may choose an open $(2n+1)$-marker $Z \subset X$, i.e. we have $X = \bigcup_{j \in \mathbb{Z}} \varphi^j(Z)$ and $\varphi^j(\overline{Z}) \cap \varphi^l(\overline{Z}) = \emptyset$ for all $-n \leq j < l \leq n$. Since these pairwise disjoint closed sets have a minimal distance from each other, we can find an open neighbourhood Z' of $\overline{Z}$ that is still a $2n$-marker. Now apply the Urysohn-Tietze extension theorem to obtain a continuous function $g : X \to [0, 1]$ supported on Z' with $g|_Z = 1$. Since now the collection of positive contractions $\left\{ g \circ \varphi^j \mid j = -n, \dots, n \right\}$ is pairwise orthogonal, we can define a new positive contraction via

$$h = \sum_{j=-n}^{n} \frac{n - |j|}{n} \cdot g \circ \varphi^j \ \in\ \mathcal{C}(X).$$

For each $x = fu^j \in F$, we set $a_x = f \cdot (uh^{1/2|j|})^j$ and $b_x = f \cdot (u(\mathbf{1} - h)^{1/2|j|})^j$. We compute

$$\begin{aligned}
\|[u, h]\| &= \|h - uhu^*\| \\
&= \|h - h \circ \varphi^{-1}\| \\
&= \Big\| \sum_{j=-n}^{n} \frac{n - |j|}{n} \cdot g \circ \varphi^j - \sum_{j=-n}^{n} \frac{n - |j|}{n} \cdot g \circ \varphi^{j-1} \Big\| \\
&= \Big\| \sum_{j=-n}^{n} \frac{n - |j|}{n} \cdot g \circ \varphi^j - \sum_{j=1-n}^{n} \frac{n - |j+1|}{n} \cdot g \circ \varphi^j \Big\| \\
&\leq \frac{1}{n} \leq \delta.
\end{aligned}$$

By the choice of δ, it follows that $\|[u, h^{i/2j}]\| \leq \varepsilon$ and $\|[u, (\mathbf{1} - h)^{i/2j}]\| \leq \varepsilon$

for every $j = 1, \dots, N$ and $i = 1, \dots, 2j-1$. Observe that now

$$h^{1/2}xh^{1/2} =_{N\varepsilon} xh \quad \text{and} \quad (\mathbf{1}-h)^{1/2}x(\mathbf{1}-h)^{1/2} =_{N\varepsilon} x(\mathbf{1}-h) \tag{e1}$$

for all $x \in F$, by applying the inequality $\|[u, h^{1/2}]\| \leq \varepsilon$ at most N times. Secondly, observe that

$$h^{1/2}xh^{1/2} = h^{1/2}fu^jh^{1/2} =_{N\varepsilon} \begin{cases} h^{1/2}a_x & , \quad j \geq 0 \\ a_xh^{1/2} & , \quad j < 0 \end{cases} \tag{e2}$$

and analogously

$$(\mathbf{1}-h)^{1/2}x(\mathbf{1}-h)^{1/2} =_{N\varepsilon} \begin{cases} (\mathbf{1}-h)^{1/2}b_x & , \quad j \geq 0 \\ b_x(\mathbf{1}-h)^{1/2} & , \quad j < 0 \end{cases} \tag{e3}$$

for all $x \in F$. This involves passing a function of the form $h^{i/2j}$ from one side of a u or u^* to the other at most N times. By construction, h vanishes on $\varphi^n(\overline{Z})$ and $\mathbf{1}-h$ vanishes on $\overline{Z}$. In particular, $a_x \in A_{\varphi^n(\overline{Z})}$ and $b_x \in A_{\overline{Z}}$ and hence

$$h^{1/2}xh^{1/2} \in_{N\varepsilon} A_{\varphi^n(\overline{Z})} \quad \text{and} \quad (\mathbf{1}-h)^{1/2}x(\mathbf{1}-h)^{1/2} \in_{N\varepsilon} A_{\overline{Z}}$$

for all $x \in F$.

By 2.1.4, we know that both $A_{\overline{Z}}$ and $A_{\varphi^n(\overline{Z})}$ have decomposition rank at most d. Define

$$F_1 = \left\{ a_xh^{1/2} \mid x \in F \right\} \cup \left\{ h^{1/2}a_x \mid x \in F \right\} \Subset A_{\varphi^n(\overline{Z})}$$

and

$$F_2 = \left\{ b_x(\mathbf{1}-h)^{1/2} \mid x \in F \right\} \cup \left\{ (\mathbf{1}-h)^{1/2}b_x \mid x \in F \right\} \Subset A_{\overline{Z}}.$$

Choose two finite-dimensional C*-algebras $\mathcal{F}_1$ and $\mathcal{F}_2$, two c.p.c. maps $\theta_1 : A_{\varphi^n(\overline{Z})} \to \mathcal{F}_1$ and $\theta_2 : A_{\overline{Z}} \to \mathcal{F}_2$ and two sets of c.p.c. order zero maps $\psi_1^{(l)} : \mathcal{F}_1 \to A_{\varphi^n(\overline{Z})}$ and $\psi_2^{(l)} : \mathcal{F}_2 \to A_{\overline{Z}}$ for $l = 0, \dots, d$, such that

$$\sum_{l=0}^{d} \psi_i^{(l)} \circ \theta_i(y) =_{\varepsilon} y \tag{e4}$$

for all $x \in F_i$ and $i = 1, 2$. Moreover, we may assume that the sums $\sum_{l=0}^{d} \psi_i^{(l)}$ are contractive for $i = 1, 2$. By Arveson's theorem [10, 1.6.1], we can extend θ_1, θ_2 to two c.p.c. maps $\tilde{\theta}_i : A \to \mathcal{F}_i$ for $i = 1, 2$. Let $\iota_1 : A_{\varphi^n(\overline{Z})} \hookrightarrow A$ and $\iota_2 : A_{\overline{Z}} \hookrightarrow A$ be the two inclusion maps. We obtain for every $x \in F$ that

$$
\begin{aligned}
&\sum_{l=0}^{d} \iota_1 \circ \psi_1^{(l)} \circ \tilde{\theta}_1(h^{1/2} x h^{1/2}) + \iota_2 \circ \psi_2^{(l)} \circ \tilde{\theta}_2\big((\mathbf{1}-h)^{1/2} x (\mathbf{1}-h)^{1/2}\big) \\
\overset{e2,e3:\ \exists y_1 \in F_1, y_2 \in F_2}{=_{2N\varepsilon}} \quad & \sum_{l=0}^{d} \iota_1 \circ \psi_1^{(l)} \circ \theta_1(y_1) + \iota_2 \circ \psi_2^{(l)} \circ \theta_2(y_2) \\
\overset{e4}{=_{2\varepsilon}} \quad & \iota_1(y_1) + \iota_2(y_2) \\
=_{2N\varepsilon} \quad & h^{1/2} x h^{1/2} + (\mathbf{1}-h)^{1/2} x (\mathbf{1}-h)^{1/2} \\
\overset{e1}{=_{2N\varepsilon}} \quad & xh + x(\mathbf{1}-h) \;=\; x.
\end{aligned}
$$

In particular, this yields a $2(d+1)$-decomposable c.p. approximation

$$
\Big(\mathcal{F}_1 \oplus \mathcal{F}_2, \tilde{\theta}_1\big(h^{1/2} \cdot \_ \cdot h^{1/2}\big) \oplus \tilde{\theta}_2\big((\mathbf{1}-h)^{1/2} \cdot \_ \cdot (\mathbf{1}-h)^{1/2}\big), \sum_{i=1,2} \sum_{l=0}^{d} \iota_i \circ \psi_i^{(l)}\Big)
$$

of F up to $(6N+2)\varepsilon$. Since ε can be chosen arbitrarily small in comparison to N, this finishes the proof. □

Remark 2.1.9. By looking at the above proof, one might conjecture that the analogous statement of 2.1.8 is true for decomposition rank instead of nuclear dimension. However, this is not true. The reason is that the c.p. approximation constructed above does not even ensure that $\mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}$ is a finite C*-algebra, as the next example shows:

Example 2.1.10 (cf. [74, 8.8])**.** Let $X_1 = \mathbb{Z} \cup \{\pm\infty\}$ be the two-sided compactification of the integers. Consider the homeomorphism $\varphi_1 : X_1 \to X_1$ given by

$$
\varphi_1(x) = \begin{cases} x+1 & , \quad x \in \mathbb{Z} \\ x & , \quad x = \pm\infty. \end{cases}
$$

Let X_2 be another finite-dimensional compact metric space with some aperiodic homeomorphism $\varphi_2 : X_2 \to X_2$. (e.g. an Odometer action on the Cantor set.) Define $X = X_1 \times X_2$ and $\varphi = \varphi_1 \times \varphi_2 : X \to X$. Then φ is clearly aperiodic, so Theorem 2.1.8 applies and we have

$$\dim_{\mathrm{nuc}}(\mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}) \leq 1 + 2\dim(X) = 1 + 2\dim(X_2).$$

On the other hand, one has $\mathcal{C}(X_1) \rtimes_{\varphi_1} \mathbb{Z} \subset \mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}$ canonically. But if u is the canonical unitary implementing the action and $p \in \mathcal{C}(X_1)$ is the characteristic function associated to $\mathbb{N} \cup \{\infty\}$, then we have

$$\begin{aligned}(up + \mathbf{1} - p)^*(up + \mathbf{1} - p) &= p + (\mathbf{1} - p)up + \mathbf{1} - p \\ &= \mathbf{1} + (\mathbf{1} - p) \cdot p \circ \varphi^{-1} \cdot u \\ &= \mathbf{1}\end{aligned}$$

and

$$\begin{aligned}(up + \mathbf{1} - p)(up + \mathbf{1} - p)^* &= upu^* + \mathbf{1} - p \\ &= p \circ \varphi^{-1} + \mathbf{1} - p \\ &= \mathbf{1} - \chi_{\{0\}} \neq 0.\end{aligned}$$

Hence this transformation group C*-algebra contains a proper isometry and in particular, the decomposition rank of $\mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}$ is infinite.

2.2 The Rokhlin dimension approach

In the previous section, we saw how the marker property enables one to carry over a result for minimal homeomorphisms to aperiodic homeomorphisms, in that case the Toms-Winter approach for showing that a transformation group C*-algebra of the form $\mathcal{C}(X) \rtimes \mathbb{Z}$ has finite nuclear dimension. As it turns out, a stronger variant will allow us to do the same with the Rokhlin dimension approach, which was first carried out by Hirshberg, Winter and Zacharias in [42].

The following result arises upon analyzing Gutman's proof of 2.1.6 more carefully. Although we will not go over the proof at this point, we will prove a generalization of this result in the next chapter.

Theorem 2.2.1 (see [39, 6.1] with proof.)**.** *Let X be a compact metric space with finite covering dimension d and $\varphi : X \to X$ an aperiodic homeomorphism. For all $n \in \mathbb{N}$, we can find an open n-marker $Z \subset X$ with $X = \bigcup_{j=0}^{2(d+1)n-1} \varphi^j(Z)$.*

We will now demonstrate how this result can be used to obtain a comparably simple proof that these $\mathbb{Z}$-actions have finite Rokhlin dimension. The idea for the following proof is originally due to Wilhelm Winter:

Corollary 2.2.2. *Let X be a compact metric space and $\varphi : X \to X$ an aperiodic homeomorphism. Denote by $\bar{\varphi} : \mathcal{C}(X) \to \mathcal{C}(X)$ the induced automorphism given by $\bar{\varphi}(f) = f \circ \varphi^{-1}$. Then we have*

$$\dim_{\mathrm{Rok}}^{+1}(\bar{\varphi}) \leq 4 \dim^{+1}(X).$$

Proof. We may assume that X has finite covering dimension d, or else there is nothing to show. Let $k \in \mathbb{N}$ and $\varepsilon > 0$ be given. We will now construct at most $4(d+1)$ Rokhlin towers of length k as required by 1.5.12. To this end, choose a natural number $n \geq \frac{1}{\varepsilon}$ and use 2.2.1 to find an open set Z with $\overline{Z} \cap \varphi^j(\overline{Z}) = \emptyset$ for all $j = 1, \dots, 4nk-1$ and $X = \bigcup_{j=0}^{2(d+1)4nk-1} \varphi^j(Z)$. Since the pairwise disjoint sets $\left\{\varphi^j(\overline{Z}) \mid j = 0, \dots, 4nk-1\right\}$ have a minimal distance from each other, we can find an open neighbourhood $Z' \supset \overline{Z}$ that is still a $4nk$-marker. Use the Urysohn-Tietze extension theorem to find a continuous function $g : X \to [0,1]$ supported on Z' and with $g|_Z = 1$. Then the function

$$\begin{aligned} h &= \sum_{l=-n}^{-1} g \circ \varphi^{-lk} + \frac{n-|l|}{n} \cdot g \circ \varphi^{-(l-n)k} \\ &\quad + \sum_{l=0}^{n-1} g \circ \varphi^{-lk} + \frac{n-|l|}{n} \cdot g \circ \varphi^{-(l+n)k} \end{aligned}$$

yields a positive contraction in $\mathcal{C}(X)$. Observe that by the properties of the sets Z, Z' and the number n, this element satisfies the following properties:

- $h \cdot h \circ \varphi^{-j} = 0$ for all $j = 1, \dots, k-1$.
- $h \circ \varphi^{-k} =_\varepsilon h$.
- $h = 1$ on the set $\bigcup_{l=-n}^{n-1} \varphi^{lk}(Z)$.

For all $v = 0, \dots, k$, $i = 0, 1$ and $l = 0, \dots, 2d+1$, we define

$$h_v^{(i,l)} = \begin{cases} h \circ \varphi^{(-1-4l)nk-v} & , \quad i = 0 \\ h \circ \varphi^{(-3-4l)nk-v} & , \quad i = 1. \end{cases}$$

Then we can observe that

- $h_v^{(i,l)} \cdot h_w^{(i,l)} = 0$ for all $i = 0,1$, $l = 0,\dots,2d+1$ and $0 \le v < w \le k-1$.
- Up to ε, applying $\bar{\varphi}$ on $h_v^{(i,l)}$ results in a k-cyclic lower index shift.
- We have $\sum_{i=0,1}\sum_{l=0}^{2d+1}\sum_{v=0}^{k-1} h_v^{(i,l)} \ge 1$ on the set

$$\bigcup_{i=0,1}\varphi^{2nki}\left(\bigcup_{l=0}^{2d+1}\varphi^{4nkl}\left(\bigcup_{v=0}^{k-1}\varphi^{v}\left(\bigcup_{j=0}^{2n-1}\varphi^{jk}(Z)\right)\right)\right) = \bigcup_{j=0}^{2(d+1)4nk-1}\varphi^{j}(Z),$$

which is the whole space X by assumption.

In particular, the sum $S = \sum_{i=0,1}\sum_{l=0}^{2d+1}\sum_{v=0}^{k-1} h_v^{(i,l)}$ satisfies $\bar{\varphi}(S) =_{4(d+1)k\varepsilon} S$. Moreover, it is positive and its spectrum is in $[1, 4(d+1)]$. Now apply 2.1.7 with respect to $x = S$, $K = [1, 4(d+1)]$, $f(x) = x^{-1}$ and $y = u \in \mathcal{C}(X) \rtimes \mathbb{Z}$ the canonical unitary implementing $\bar{\varphi}$. Since we can choose ε arbitrarily small at the beginning of this construction, we may as well assume by 2.1.7 (by replacing ε with an even smaller number at the beginning) that we have $\bar{\varphi}(S^{-1}) =_{\varepsilon} S^{-1}$. Defining $f_v^{(i,l)} = S^{-1}h_v^{(i,l)}$ for all i, l, v, we observe that

- $f_v^{(i,l)} \cdot f_w^{(i,l)} = 0$ for all $i = 0,1$, $l = 0,\dots,2d+1$ and $0 \le v < w \le k-1$.
- Up to 2ε, applying $\bar{\varphi}$ on $f_v^{(i,l)}$ results in a k-cyclic lower index shift.
- $\sum_{i=0,1}\sum_{l=0}^{2d+1} f_v^{(i,l)} = 1$.

Since the collection $(f_v^{(i,l)})_{v=0,\dots,k-1}^{i=0,1,\ l=0,\dots,2d+1}$ of Rokhlin elements has $4(d+1)$ upper indices, and moreover k and ε were arbitrary, this verifies the conditions given in 1.5.12 and hence $\dim_{\mathrm{Rok}}^{+1}(\bar{\varphi}) \le 4(d+1)$. □

Corollary 2.2.3. *Let X be a compact metric space and $\varphi : X \to X$ an aperiodic homeomorphism. Then we have*

$$\dim_{\mathrm{nuc}}^{+1}(\mathcal{C}(X) \rtimes_{\varphi} \mathbb{Z}) \le 8\dim^{+1}(X)^2.$$

Proof. This follows directly by combining 2.2.2 and 1.5.15. □

Topological dynamics

From now on, we turn to the case of free actions of higher-rank groups on locally compact metric spaces. Although the main focus in terms of applications is, at least within this chapter, on actions of $\mathbb{Z}^m$ and nilpotent groups, the next two upcoming sections establish certain technical properties for actions of any countable group on finite-dimensional spaces. The main technical result will be that all free actions of countably infinite groups on finite-dimensional spaces satisfy the marker property. Moreover, the result will be of a form that allows us to show that, under suitable restrictions on the group, even a stronger variant of the marker property holds.

Note that this chapter has substantial text overlap with my paper [94], and large parts are carried over verbatim. However, the results in the following sections are carried out in greater generality than in [94], although it was already remarked in [94, 5.4] that one can remove the assumption that all spaces must be compact.

3.1 The topological small boundary property

In this section, we define a technical condition that we name the (bounded) topological small boundary property. Weaker versions of this were considered by Lindenstrauss in [66] and by Gutman in [39] and had connections to a dynamical system having mean dimension zero. It will turn out that we can in fact assume a stronger bounded variant, whenever we have a finite-dimensional underlying space. Unlike in [94], we will not restrict our focus only onto compact spaces, but rather work in the setting of locally compact metric spaces.

Definition 3.1.1 (cf. [66, 3.1])**.** Let X be a locally compact metric space, G a group and $\alpha : G \curvearrowright X$ an action. Let $M \subset G$ be a subset and $k \in \mathbb{N}$ be some natural number. We say that a set $E \subset X$ is (M,k)-disjoint, if for all distinct elements $\gamma(0), \dots, \gamma(k) \in M$ we have

$$\alpha_{\gamma(0)}(E) \cap \cdots \cap \alpha_{\gamma(k)}(E) = \emptyset.$$

We call E toplogically α-small if E is (G,k)-disjoint for some k.

Lemma 3.1.2. *Let X be a locally compact metric space with a group action $\alpha : G \curvearrowright X$. Let $F \Subset G$ be a finite subset and $n \in \mathbb{N}$ a natural number. If a compact subset $E \subset X$ is (F,n)-disjoint, then there exists an open, relatively compact neighbourhood V of E such that $\overline{V}$ is (F,n)-disjoint.*

Proof. Note that for all $S \subset F$ with $n = |S|$, we have

$$\emptyset = \bigcap_{\gamma \in S} \alpha_\gamma(E) = \bigcap_{\gamma \in S} \alpha_\gamma\Big(\bigcap_{\varepsilon > 0} \overline{B}_\varepsilon(E)\Big) = \bigcap_{\varepsilon > 0} \left(\bigcap_{\gamma \in S} \alpha_\gamma(\overline{B}_\varepsilon(E)) \right)$$

By compactness, there must exist some $\varepsilon(S) > 0$ such that $\overline{B}_{\varepsilon(S)}(E)$ is compact and

$$\bigcap_{\gamma \in S} \alpha_\gamma(\overline{B}_{\varepsilon(S)}(E)) = \emptyset.$$

If we set $\varepsilon = \min\{\varepsilon(S) \mid S \subset F, n = |S|\}$, then $V = B_\varepsilon(E)$ is a relatively compact, open neighbourhood of E whose closure is (F,n)-disjoint. ☐

Definition 3.1.3 (cf. [66, 3.2])**.** Let G be a group, $M \subset G$ a subset and $d \in \mathbb{N}$ a natural number. A topological dynamical system (X, α, G) has the (M,d)-small boundary property, if whenever $K \subset X$ is compact and $V \supset K$ is open, we can find a relatively compact, open set U with $K \subset U \subset V$ such that ∂U is (M,d)-disjoint.

We say moreover that (X, α, G) has the bounded topological small boundary property with respect to d, if it has the (G,d)-small boundary property.

The main goal of this section is to prove that free actions on finite-dimensional spaces have this property. The case $G = \mathbb{Z}$ and X compact has been done by Lindenstrauss in [66, 3.3]. It follows from his result that, if $\varphi : X \to X$ is an aperiodic homeomorphism, then (X, φ) has the bounded topological small boundary property with respect to $\dim(X)$. We will carry out his approach in a more general context, with a few modifications of rather combinatorial nature.

An important point is to observe that for a topological dynamical system, verifying the topological small boundary property with respect to d reduces to the (F, d)-small boundary property for arbitrarily large finite sets F inside the group G.

Lemma 3.1.4. *Let G be a countable group, X a locally compact metric space and $\alpha : G \curvearrowright X$ an action. Let $d \in \mathbb{N}$ be a natural number. Let*

$$F_1 \subset F_2 \subset F_3 \subset \dots \Subset G$$

be an increasing sequence of finite subsets such that $G = \bigcup_{k=1}^{\infty} F_k$. If (X, α, G) has the (F_k, d)-small boundary property for every $k \in \mathbb{N}$, then it has the topological small boundary property with respect to d.

Proof. Let $K \subset X$ be a compact subset and $V \subset X$ an open neighbourhood of K. We will show that there exists U as required by 3.1.3. For this, assume without loss of generality that V is relatively compact.

We will construct sequences of relatively compact, open sets $\{U_k\}_{k\in\mathbb{N}}$ and $\{V_k\}_{k\in\mathbb{N}}$ with the following properties for all k:

(1) $K \subset U_k \subset \overline{U}_k \subset U_{k+1} \subset V$.

(2) $\overline{V}_{k+1} \subset V_k$.

(3) $\overline{U}_{k+1} \subset U_k \cup V_k$.

(4) $\overline{V}_k$ is (F_k, d)-disjoint.

(5) $\partial U_k \subset V_k$.

First, apply the (F_1, d)-small boundary property to find U_1 such that $K \subset U_1 \subset \overline{U}_1 \subset V$ and ∂U_1 is (F_1, d)-disjoint. Apply 3.1.2 to find an open neighbourhood V_1 of ∂U_1 such that $\overline{V}_1 \subset V$ and $\overline{V}_1$ is (F_1, d)-disjoint. Clearly these sets satisfy (1)-(5) thus far. Suppose that the sets U_k, V_k have been defined for some k. Apply the (F_{k+1}, d)-small boundary property to find an open set U_{k+1} such that $\overline{U}_k \subset U_{k+1} \subset \overline{U}_{k+1} \subset U_k \cup V_k$ and ∂U_{k+1} is (F_{k+1}, d)-disjoint. Since V_k is an open neighbourhood of ∂U_{k+1}, we can find another open neighbourhood V_{k+1} of ∂U_{k+1} such that $\overline{V}_{k+1} \subset V_k$ is (F_{k+1}, d)-disjoint, by virtue of 3.1.2. Clearly these new sets satisfy properties (1)-(5) again.

Now set $U = \bigcup_{k=0}^{\infty} U_k$. It follows immediately from (1) that $K \subset U \subset V$. From condition (1), (2) and (3) it follows that $U_{k+r} \cup V_k = U_k \cup V_k$ for all k and $r > 0$, so in particular $U_k \subset U \subset U_k \cup V_k$ for all k. It follows that

$$\partial U \subset \overline{U_k \cup V_k} \setminus U_k \subset \overline{V}_k.$$

Since for each $k \in \mathbb{N}$, the set V_k is (F_k, d)-disjoint, we have that ∂U is (F_k, d)-disjoint for all k. In particular, it is (G, d)-disjoint. □

In this way, we have localized our problem by only having to consider how a group action behaves at given finite sets. In order to make general statements for actions on finite-dimensional spaces, we naturally need to apply dimension theory for topological spaces. More specifically, we shall now record some well-known facts about properties of covering dimension, which we will refer to throughout this section. These statements come up in [66, Section 3], but a detailed treatment can be found in [27], see in particular [27, 4.1.5, 4.1.7, 4.1.9, 4.1.14, 4.1.16]. All spaces in question are assumed to be separable metric spaces.

- D1 $A \subset B$ implies $\dim(A) \leq \dim(B)$.
- D2 If $\{B_i\}_{i\in\mathbb{N}}$ is a countable family of closed sets in A with $\dim(B_i) \leq k$, then $\dim(\bigcup B_i) \leq k$.
- D3 Let $E \subset A$ be a zero dimensional subset and $x \in U \subset A$ a point with an open neighbourhood. Then there exists some open set $U' \subset A$ with $x \in U' \subset U$ such that $\partial U' \cap E = \emptyset$.
- D4 If $A \neq \emptyset$, there exists a zero dimensional F_σ-set $E \subset A$ such that $\dim(A \setminus E) = \dim(A) - 1$.
- D5 Any countable union of k-dimensional F_σ-sets is a k-dimensional F_σ-set.

Lemma 3.1.5. *Let X be a locally compact metric space. Let $K \subset X$ be compact and $V \subset X$ an open neighbourhood of K. Let $E \subset X$ be a zero dimensional subset. Then there exists a relatively compact, open set U with $K \subset U \subset \overline{U} \subset V$ such that $\partial U \cap E = \emptyset$.*

Proof. Clearly ∂K is compact. For $x \in \partial K$, apply D3 and find relatively compact, open neighbourhoods $x \in B_x \subset \overline{B}_x \subset V$ such that $\partial B_x \cap E = \emptyset$. Choose a finite cover $\partial K \subset \bigcup_{i=1}^M B_i$ of such neighbourhoods and set $U = K \cup \bigcup_{i=1}^M B_i$. It is now immediate that U is relatively compact with $\overline{U} \subset V$ and that $\partial U \subset \bigcup_{i=1}^M \partial B_i$, so we have indeed $\partial U \cap E = \emptyset$. □

Definition 3.1.6. Let X be a locally compact metric space, G a group and $\alpha : G \curvearrowright X$ an action. Let $M \Subset G$ be a finite subset. We define

$$X(M) = \{x \in X \mid \text{the map } [M \ni g \mapsto \alpha_g(x)] \text{ is injective}\}.$$

By continuity, $X(M)$ is an open subset of X. The action α is then free if and only if one has $X(M) = X$ for every $M \Subset G$.

Definition 3.1.7 (following [57, Section 3] and [66, 3.4])**.** Let X be a metric space of finite covering dimension n. A family $\mathcal{B}$ of subsets in X is in general position, if for all finite subsets $S \Subset \mathcal{B}$ we have

$$\dim(\bigcap S) \leq \max(-1, n - |S|).$$

Lemma 3.1.8. *Let X be a locally compact metric space, G a group and $\alpha : G \curvearrowright X$ an action. Let $M \Subset G$ be finite subset. Let $K \subset X(M)$ be compact and let $V \subset X(M)$ be an open neighbourhood of K. Then there exists a relatively compact, open set U with $K \subset U \subset \overline{U} \subset V$ such that the family $\{\alpha_\gamma(\partial U)\}_{\gamma \in M}$ is in general position in X.*

Proof. Let n be the covering dimension of X. We prove this by induction in the variable $k = |M|$. The assertion trivially holds for $k = 1$. Now assume that the assertion holds for some natural number k. We show that it also holds for $k + 1$.

Let $M = \{\gamma(0), \dots, \gamma(k)\}$ be a set of cardinality $k + 1$ in G. Then obviously $X(M) \subset X(M')$ for every subset $M' \subset M$. Using the induction hypothesis, there exists a relatively compact, open set A_0 with $K \subset$

$A_0 \subset \overline{A}_0 \subset V$, such that the collection $\{\alpha_{\gamma(0)}(\partial A_0), \ldots, \alpha_{\gamma(k-1)}(\partial A_0)\}$ is in general position in X.

Since $\overline{A}_0 \subset V \subset X(M)$, we can find for every point $x \in \partial A_0$ a number $\eta(x) > 0$ such that $\overline{B}_{\eta(x)}(x) \subset V$ and such that the sets $\alpha_{\gamma(j)}(B_{\eta(x)}(x))$ are pairwise disjoint for $j = 0, \ldots, k$. Denote $\widehat{B}_x = B_{\eta(x)}(x)$ and $B_x = B_{\eta(x)/2}(x)$. Note that since A_0 was relatively compact, its boundary ∂A_0 is compact. So find some finite subcover $\partial A_0 \subset \bigcup_{i=1}^N B_i$. We will now construct relatively compact, open sets A_i for $i = 0, \ldots, N$ (A_0 is already defined) with the following properties:

(1) $\overline{A}_i \subset A_0 \cup \bigcup_{j=1}^N B_j$.

(2) $A_i \subset A_{i+1} \subset A_i \cup \widehat{B}_{i+1}$.

(3) The collection

$$\mathcal{A}_i = \{\alpha_{\gamma(j)}(\partial A_i)\}_{j<k} \cup \Big\{\alpha_{\gamma(k)}(\partial A_i \cap \bigcup_{j=1}^{i} B_j)\Big\}$$

is in general position.

Once we have done this construction, combining (1) with (3) implies that the set $U = A_N$ has the desired property. It remains to show how to construct the sets A_i.

So suppose that the set A_i has already been defined for $i < N$. According to D4, for all nonempty subsets $S \subset \mathcal{A}_i$, there exists a zero dimensional F_σ-set

$$E_S \subset \bigcap S \quad \text{with} \quad \dim(\bigcap S \setminus E_S) = \dim(\bigcap S) - 1.$$

Define

$$E := \bigcup_{\substack{\emptyset \neq S \subset \mathcal{A}_i \\ 0 \leq j \leq k}} \alpha_{\gamma(j)^{-1}}(E_S). \tag{e1}$$

By D5, E is a zero dimensional F_σ-set. Use 3.1.5 to find a relatively compact, open set W such that

$$\overline{A_i \cap B_{i+1}} \subset W \subset \overline{W} \subset \widehat{B}_{i+1} \cap (A_0 \cup \bigcup_{j=1}^{N} B_j) \tag{e2}$$

and

$$\partial W \cap E = \emptyset. \tag{e3}$$

Now set $A_{i+1} := A_i \cup W$. This clearly satisfies the properties (1) and (2). To show (3), let $\emptyset \neq S = \{S_1, \ldots, S_m\} \subset \mathcal{A}_{i+1}$ correspond to some subset $\{j_1, j_2, \ldots, j_m\} \subset \{0, \ldots, k\}$. Note that since $\partial A_{i+1} \subset \partial A_i \cup \partial W$, we have either

$$S_l = \alpha_{\gamma(j_l)}(\partial A_{i+1}) \subset \alpha_{\gamma(j_l)}(\partial A_i) \cup \alpha_{\gamma(j_l)}(\partial W) =: S_l^0 \cup S_l^1 \qquad (\text{if } j_l \neq k)$$

or

$$\begin{aligned} S_l &= \alpha_{\gamma(j_l)}(\partial A_{i+1} \cap \bigcup_{j=1}^{i+1} B_i) \\ &\subset \alpha_{\gamma(j_l)}((\partial A_i \setminus W) \cap \bigcup_{j=1}^{i+1} B_i) \cup \alpha_{\gamma(j_l)}(\partial W) \\ &\overset{e2}{\subset} \alpha_{\gamma(j_l)}(\partial A_i \cap \bigcup_{j=1}^{i} B_j) \cup \alpha_{\gamma(j_l)}(\partial W) \\ &=: S_l^0 \cup S_l^1 \qquad (\text{if } j_l = k). \end{aligned}$$

It follows that

$$\bigcap S \subset \bigcup_{a \in \{0,1\}^m} \left(\bigcap_{l=1}^{m} S_l^{a_l} \right).$$

Since $\overline{W} \subset \widehat{B}_{i+1}$, our choice of $\widehat{B}_{i+1}$ implies that the sets S_l^1 are pairwise disjoint. So it suffices to consider the case $a = (0, \ldots, 0)$ and, since we can change the order without loss of generality, the case $a = (1, 0, \ldots, 0)$. For $a = (0, \ldots, 0)$, note that $\{S_1^0, \ldots, S_m^0\}$ is a subset of $\mathcal{A}_i$, so we already have

$$\dim\Big(\bigcap_{l=1}^{m} S_l^0\Big) \quad \leq \quad \max(-1, n-m).$$

For $a = (1, 0, \ldots, 0)$, define $\hat{S} = \{S_2^0, \ldots, S_m^0\}$. This is a subset of $\mathcal{A}_i$, hence we know that it is in general position. Moreover, considering our choice of the set $E_{\hat{S}}$, recall that

$$\dim(\bigcap \hat{S} \setminus E_{\hat{S}}) \leq \dim(\bigcap \hat{S}) - 1 \leq \max(-1, n-(m-1)) - 1 \leq \max(-1, n-m).$$

By the choice of W we know that $\partial W \cap E = \emptyset$, see e3. Since $\alpha_{\gamma(j_1)^{-1}}(E_{\hat{S}}) \subset E$

(see e1), this implies $E_{\hat{S}} \cap \alpha_{\gamma(j_1)}(\partial W) = \emptyset$. In particular, it follows that

$$S_1^1 \cap \bigcap_{l=2}^{m} S_l^0 = \alpha_{\gamma(j_1)}(\partial W) \cap \bigcap \hat{S} \subset \bigcap \hat{S} \setminus E_{\hat{S}}.$$

Therefore we have established

$$\dim(S_1^1 \cap \bigcap_{l=2}^{m} S_l^0) \leq \max(-1, n-m).$$

If we combine these inequalities with D2, it follows that we have $\dim(\bigcap S) \leq \max(-1, n-m)$ as well. So $\mathcal{A}_{i+1}$ is in general position and we are done. □

Corollary 3.1.9. *Let X be a locally compact metric space with finite covering dimension $d \in \mathbb{N}$, G a group and $\alpha : G \curvearrowright X$ an action. Let $M \Subset G$ be finite subset. Let $K \subset X(M)$ be compact and let $V \subset X(M)$ be an open neighbourhood of K. Then there exists a relatively compact, open set U with $K \subset U \subset \overline{U} \subset V$ such that ∂U is (M, d)-disjoint.*

Proof. By 3.1.8, we can find a relatively compact, open set U with $K \subset U \subset \overline{U} \subset V$ such that the family $\{\alpha_g(\partial U)\}_{g \in M}$ is in general position. But this implies that the intersection of $d+1$ distinct members of this family has dimension at most -1, and hence is empty. So we see that ∂U is (M, d)-disjoint. □

Theorem 3.1.10. *Let X be a locally compact metric space with finite covering dimension d. Let G be a countable group, and $\alpha : G \curvearrowright X$ a free action. Then (X, α, G) has the bounded topological small boundary property with respect to d.*

Proof. Since α is free, we have $X(M) = X$ for every $M \Subset G$. So 3.1.9 implies that (X, α, G) has the (M, d)-small boundary property for every $M \Subset G$. The claim now follows from 3.1.4. □

3.2 A generalization of Gutman's marker property

The aim of this section is to use the technical results about dynamically small boundaries from the previous section to obtain a generalization of Gutman's marker property (see 2.2.1) for free countable group actions and free $\mathbb{Z}^m$-actions in particular. Very similarly to 2.2.2, it will follow that free $\mathbb{Z}^m$-actions have finite Rokhlin dimension. First we have to introduce the notion of markers and the marker property.

Definition 3.2.1. Let X be a locally compact metric space, G a group and $\alpha : G \curvearrowright X$ an action. Let $F \Subset G$ be a finite subset and $K \subset X$ a compact subset. We call a relatively compact set $Z \subset X$ an (F, K)-marker, if

- The family of sets $\left\{\alpha_g(\overline{Z}) \mid g \in F\right\}$ is pairwise disjoint.
- $K \subset \displaystyle\bigcup_{g \in M} \alpha_g(Z)$ for some $M \Subset G$.

We say that α has the marker property if there exist open (F, K)-markers for all $F \Subset G$ and compact $K \subset X$.

Remark 3.2.2. If X is compact, one can mostly neglect the parameter $K \subset X$ in the above definition. In that case, one can speak of F-markers, which are by definition (F, X)-markers. This terminology was used in my paper [94], where the focus was mainly on the case that X is assumed to be compact.

It is important to note that although the marker property is trivial if the action is assumed to be minimal, Gutman's result 2.2.1 gives a uniform bound (in relation to F) of how many copies one needs to cover the space with an F-marker, which is something new even in the minimal case. We would like to build on his ideas in the case $G = \mathbb{Z}$ to generalize his method of proof for the general case of countable group actions on locally compact metric spaces.

Lemma 3.2.3. *Let G be a group and $d \in \mathbb{N}$ a natural number. Let $F \Subset G$ be a finite subset and let $g_1, \dots, g_d \in G$ be group elements with the property that the sets*

$$F^{-1}F \ , \ g_1 F^{-1}F \ , \ \dots \ , \ g_d F^{-1}F$$

are pairwise disjoint. Using the notation $g_0 = 1_G$, set $M = \bigcup_{l=0}^{d} g_l F^{-1}F$.

Let X be a locally compact metric space and $\alpha : G \curvearrowright X$ be an action. Then the following holds:

Let $U, V \subset X$ be relatively compact, open sets such that

- *∂U is (M, d)-disjoint;*

- $\overline{U}$ *is* $(F,1)$*-disjoint;*
- $\overline{V}$ *is* $(M^{-1},1)$*-disjoint.*

Then there exists a relatively compact, open set $W \subset X$ *such that* $U \subset W$*,* $V \subset \bigcup_{g\in M} \alpha_g(W)$ *and* $\overline{W}$ *is* $(F,1)$*-disjoint.*

Proof. Set $R = \overline{V} \setminus \bigcup_{g\in M} \alpha_g(U)$. Observe that R is compact and $(M^{-1},1)$-disjoint, so apply 3.1.2 and choose $\rho > 0$ such that $\overline{B}_\rho(R)$ is compact and $(M^{-1},1)$-disjoint as well. We now claim that there exists a $\delta > 0$ such that

$$|\left\{g \in M |\ \alpha_g(\overline{U}) \cap \overline{B}_\delta(x) \neq \emptyset\right\}| \leq d \quad \text{for all } x \in R. \tag{e4}$$

Assume that this is not true. Let $x_n \in R$ be elements with $\delta_n > 0$ such that $\delta_n \to 0$ and

$$|\left\{g \in M |\ \alpha_g(\overline{U}) \cap \overline{B}_{\delta_n}(x_n) \neq \emptyset\right\}| \geq d+1 \quad \text{for all } n.$$

By compactness, we can assume that x_n converges to some $x \in R$ by passing to a subsequence. Moreover, since M has only finitely many subsets, we can also assume (again by passing to a subsequence if necessary) that there are distinct $\gamma(0), \dots, \gamma(d) \in M$ such that $\alpha_{\gamma(l)}(\overline{U}) \cap \overline{B}_{\delta_n}(x_n) \neq \emptyset$ for all n and all $l = 0, \dots, d$. But then $\delta_n \to 0$ implies

$$x \in R \cap \bigcap_{l=0}^{d} \alpha_{\gamma(l)}(\overline{U}) \subset \bigcap_{l=0}^{d} \alpha_{\gamma(l)}(\partial U) = \emptyset.$$

So this gives a contradiction to ∂U being (M,d)-disjoint. So we may choose a number $\delta \leq \rho$ satisfying e4. Moreover, choose some finite covering

$$R \subset \bigcup_{i=1}^{s} B_\delta(z_i) \quad \text{for some } z_1, \dots, z_s \in R.$$

Note that the right-hand side is relatively compact and $(M^{-1},1)$-disjoint by our choice of ρ. Since the sets $\left\{g_l F^{-1}F \mid l = 0, \dots, d\right\}$ are pairwise disjoint, observe that e4 enables us to define a map $c : \{1, \dots, s\} \to \{0, \dots, d\}$ such that

$$\alpha_g(\overline{U}) \cap \overline{B}_\delta(z_i) = \emptyset \quad \text{for all } g \in g_{c(i)}F^{-1}F. \tag{e5}$$

Finally, set

$$W = U \cup \bigcup_{i=1}^{s} \alpha_{g_{c(i)}^{-1}}(B_\delta(z_i)).$$

Obviously, W is a relatively compact, open set with $U \subset W$. Moreover, we have

$$\begin{aligned} V &\subset \bigcup_{g\in M} \alpha_g(U) \cup R \\ &\subset \bigcup_{g\in M} \alpha_g(U) \cup \bigcup_{i=1}^{s} \underbrace{B_\delta(z_i)}_{=\alpha_{g_{c(i)}}(\alpha_{g_{c(i)}^{-1}}(B_\delta(z_i)))} \\ &\subset \bigcup_{g\in M} \alpha_g(U) \cup \bigcup_{i=1}^{s} \alpha_{g_{c(i)}}(W) \quad \subset \bigcup_{g\in M} \alpha_g(W) \end{aligned}$$

At last we have to show that $\overline{W}$ is $(F,1)$-disjoint. Suppose that $\alpha_a(\overline{W}) \cap \alpha_b(\overline{W}) \neq \emptyset$ for some $a \neq b$ in F. That is, there exist $x, y \in \overline{W}$ such that $\alpha_a(x) = \alpha_b(y)$. Let us go through all the possible cases:

- $x, y \in \overline{U}$ is obviously impossible.
- $x \in \alpha_{g_{c(i_1)}^{-1}}(\overline{B}_\delta(z_{i_1}))$ and $y \in \alpha_{g_{c(i_2)}^{-1}}(\overline{B}_\delta(z_{i_2}))$ for some $1 \leq i_1, i_2 \leq s$. It follows that

  $$\alpha_a(x) = \alpha_b(y) \in \alpha_{ag_{c(i_1)}^{-1}}(\overline{B}_\delta(z_{i_1})) \cap \alpha_{bg_{c(i_2)}^{-1}}(\overline{B}_\delta(z_{i_2})),$$

  so

  $$\begin{aligned} \emptyset &\neq \alpha_{b^{-1}ag_{c(i_1)}^{-1}}(\overline{B}_\delta(z_{i_1})) \cap \alpha_{g_{c(2)}^{-1}}(\overline{B}_\delta(z_{i_2})) \\ &\subset \alpha_{b^{-1}ag_{c(i_1)}^{-1}}(\overline{B}_\rho(R))) \cap \alpha_{g_{c(i_2)}^{-1}}(\overline{B}_\rho(R)). \end{aligned}$$

  Observe that by $a \neq b$, we have $b^{-1}ag_{c(i_1)}^{-1} \neq g_{c(i_2)}^{-1}$ in M^{-1}. Since $\overline{B}_\rho(R)$ is $(M^{-1},1)$-disjoint, the right side of the above is empty. So this is impossible.
- $x \in \overline{U}$ and $y \in \alpha_{g_{c(i)}^{-1}}(\overline{B}_\delta(z_i))$ for some $1 \leq i \leq s$. Then it follows that

  $$\alpha_a(x) = \alpha_b(y) \in \alpha_a(\overline{U}) \cap \alpha_{bg_{c(i)}^{-1}}(\overline{B}_\delta(z_i)) \neq \emptyset.$$

  Or equivalently, $\alpha_{g_{c(i)}b^{-1}a}(\overline{U}) \cap \overline{B}_\delta(z_i) \neq \emptyset$, a contradiction to the definition of $c(i)$, see e5.

So we see that $\overline{W}$ is indeed $(F,1)$-disjoint. $\square$

The following Lemma constitues the main technical result of this chapter:

Lemma 3.2.4. *Let G be a group and $d \in \mathbb{N}$ a natural number. Let $F \Subset G$ be a finite subset and let $g_1, \dots, g_d \in G$ be group elements with the property that the sets*

$$F^{-1}F \ , \ g_1F^{-1}F \ , \ \dots \ , \ g_dF^{-1}F$$

are pairwise disjoint. Using the notation $g_0 = 1_G$, set $M = \bigcup_{l=0}^{d} g_lF^{-1}F$.

Let X be a locally compact metric space with an action $\alpha : G \curvearrowright X$ such that (X, α, G) has the (M, d)-small boundary property. Moreover, assume that $X(M^{-1}) = X$. Then given any compact subset $K \subset X$, there exists an open (F, K)-marker $Z \subset X$ with $K \subset \bigcup_{g \in M} \alpha_g(Z)$.

Proof. For all $x \in K$, use $X(M^{-1}) = X$ to choose a relatively compact, open neighbourhood U_x such that $\overline{U}_x$ is $(M^{-1}, 1)$-disjoint. By the (M, d)-small boundary property, we can also assume that ∂U_x is (M, d)-disjoint. Note that since $1_G \in F^{-1}F \subset M$, it follows that every $(M^{-1}, 1)$-disjoint set is also $(F, 1)$-disjoint.

Choose a finite subcovering $K \subset \bigcup_{i=0}^{s} U_i$. Apply 3.2.3 (with respect to $U = U_0, V = U_1$) to find a relatively compact, open set W_1 such that $U_0 \subset W_1, U_1 \subset \bigcup_{g \in M} \alpha_g(W)$ and such that $\overline{W}_1$ is $(F, 1)$-disjoint. Clearly we have $U_0 \cup U_1 \subset \bigcup_{g \in M} \alpha_g(W_1)$.

Now carry on inductively. If W_k is already defined, apply 3.2.3 (with respect to $U = W_k, V = U_{k+1}$) to find a relatively compact, open set W_{k+1} such that $W_k \subset W_{k+1}$ and $U_{k+1} \subset \bigcup_{g \in M} \alpha_g(W_{k+1})$ and such that $\overline{W}_{k+1}$ is $(F, 1)$-disjoint. Note also that if W_k had the property that

$$U_0 \cup \dots \cup U_k \subset \bigcup_{g \in M} \alpha_g(W_k),$$

then it follows that

$$\begin{aligned} U_0 \cup \dots \cup U_k \cup U_{k+1} \quad &\subset \quad \textstyle\bigcup_{g \in M} \alpha_g(W_k) \cup U_{k+1} \\ &\subset \quad \textstyle\bigcup_{g \in M} \alpha_g(W_k) \cup \bigcup_{g \in M} \alpha_g(W_{k+1}) \\ &= \quad \textstyle\bigcup_{g \in M} \alpha_g(W_{k+1}). \end{aligned}$$

So set $Z = W_s$. The set $\overline{Z}$ is compact and $(F, 1)$-disjoint by construction, and indeed an (F,K)-marker because

$$K \subset U_0 \cup \cdots \cup U_s \subset \bigcup_{g\in M} \alpha_g(Z).$$

□

Let us now list some obvious implications:

Corollary 3.2.5. *Let G be a group and $d \in \mathbb{N}$ a natural number. Let $F \Subset G$ be a finite subset and let $g_1, \dots, g_d \in G$ be group elements with the property that the sets*

$$F^{-1}F \ , \ g_1F^{-1}F \ , \ \dots \ , \ g_dF^{-1}F$$

are pairwise disjoint. Using the notation $g_0 = 1_G$, set $M = \bigcup_{l=0}^d g_lF^{-1}F$.

Let X be a locally compact metric space with covering dimension at most d. Let $\alpha : G \curvearrowright X$ be an action. Assume that $X(M) = X(M^{-1}) = X$. Then given any compact subset $K \subset X$, there exists an open (F,K)-marker $Z \subset X$ with $K \subset \bigcup_{g\in M} \alpha_g(Z)$.

Proof. Combine 3.1.9 and 3.2.4. □

Remark 3.2.6. It should be pointed out that ultimately, 3.2.5 is not really a statement about topological dynamical systems, but rather a statement about a property of *all* locally compact metric spaces with covering dimension at most some $d \in \mathbb{N}$. Since in the statement, G is allowed to be any group, we can just choose $G = \mathrm{Homeo}(X)$ and let it act on X in the canonical way. In this way, the statement boils down to a topological property of certain finite sets of homeomorphisms on X. The condition $X(M) = X(M^{-1}) = X$ in 3.2.5 translates to the fact that one requires the finite sets under consideration to move all points around in different ways. Conversely, if the statement of 3.2.5 holds for the canonical action of $G = \mathrm{Homeo}(X)$ on X, then it is not hard to see that 3.2.5 automatically holds for any group action on X.

Corollary 3.2.7. *Let X be a locally compact metric space of finite covering dimension d. Let G be a group and $\alpha : G \curvearrowright X$ a free action. Let $F \Subset G$ be a finite subset and let $g_1, \dots, g_d \in G$ be group elements with the property that the sets*

$$F^{-1}F \ , \ g_1F^{-1}F \ , \ \dots \ , \ g_dF^{-1}F$$

are pairwise disjoint. Using the notation $g_0 = 1_G$*, set* $M = \bigcup_{l=0}^{d} g_l F^{-1} F$*.*

Then given any compact subset $K \subset X$*, there exists an open* (F, K)*-marker* $Z \subset X$ *with* $K \subset \bigcup_{g \in M} \alpha_g(Z)$*.*

Corollary 3.2.8. *Let* X *be a locally compact metric space of finite covering dimension* d*. Let* G *be a countably infinite group. Then any free topological* G*-action on* X *has the marker property.*

Proof. If G is countably infinite, then for every finite subset $F \Subset G$, there exist $g_1, \dots, g_d$ as required by 3.2.7. This can be seen by fixing a proper length function on G (recall 1.2.15) and choosing the elements $g_1, \dots, g_d$ inductively with very large length compared to $F^{-1}F$. Hence 3.2.7 implies our claim. □

Lastly, let us prove that we have indeed extended Gutman's marker property result 2.2.1:

Corollary 3.2.9. *Let* X *be a compact metric space of finite covering dimension* $d \in \mathbb{N}$*, and let* $\alpha : X \to X$ *be an aperiodic homeomorphism. Then for all* $n \in \mathbb{N}$*, there exists an open set* $Z \subset X$ *such that*

$$\overline{Z} \cap \alpha^j(\overline{Z}) = \emptyset \quad \text{for all } j = 1, \dots, n-1$$

and

$$X = \bigcup_{j=0}^{2(d+1)n-1} \alpha^j(Z).$$

Proof. Set $F = \{0, \dots, n-1\}$ and $v_l = 2ln$ for $l = 0, \dots, d$. Then

$$F - F = \{1-n, \dots, 0, \dots, n-1\} \subset 1 - n + \{0, \dots, 2n-1\}.$$

It is clear that the sets $v_0(F - F), v_1(F - F), \dots, v_d(F - F)$ are pairwise disjoint. Their union is equal to

$$\begin{aligned} M &= \bigcup_{l=0}^{d} v_l + \{1-n, \dots, n-1\} \\ &\subset 1 - n + \bigcup_{l=0}^{d} 2ln + \{0, \dots, 2n-1\} \\ &= 1 - n + \{0, \dots, 2(d+1)ln - 1\}. \end{aligned}$$

Applying 3.2.7, we can find an open set $Z \subset X$ with

$$\alpha^j(\overline{Z}) \cap \alpha^k(\overline{Z}) = \emptyset \quad \text{for all } j \neq k \text{ in } F$$

and

$$X = \bigcup_{j \in M} \alpha^j(Z).$$

The first condition amounts to

$$\overline{Z} \cap \alpha^j(\overline{Z}) = \emptyset \quad \text{for all } j = 1, \ldots, n-1,$$

while the second condition implies

$$X \subset \bigcup_{j=0}^{2(d+1)n-1} \alpha^{1-n+j}(Z) = \alpha^{1-n}\left(\bigcup_{j=0}^{2(d+1)n-1} \alpha^j(Z) \right),$$

which in turn yields $X = \bigcup_{j=0}^{2(d+1)n-1} \alpha^j(Z)$. □

3.3 Rokhlin dimension of topological $\mathbb{Z}^m$-actions

In this section, we focus on free $\mathbb{Z}^m$-actions and show that on finite-dimensional spaces, their Rokhlin dimension is always finite. With the help of the topological results of the previous section, this will follow with a few technical, but straightforward calculations.

Notation 3.3.1. Let $m \in \mathbb{N}$. For the remainder of this chapter, let us denote $J_n = \{1-n, \ldots, n\}^m \subset \mathbb{Z}^m$ and $B_n = \{0, \ldots, n-1\}$. For all n and $a \in \{0,1\}^m$, defining

$$w_a = \left((-1)^{\delta_{1,a_j}} \cdot n \right)_{j=1,\ldots,m} \quad \in \quad \mathbb{Z}^m$$

yields 2^m distinct elements with the property that $J_{2n} = \bigcup_{a \in \{0,1\}^m} (w_a + J_n)$.

Proposition 3.3.2. *Let X be a locally compact metric space with finite covering dimension $d \in \mathbb{N}$ and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free action. Then given any compact set $K \subset X$ and $n \in \mathbb{N}$, we can find elements $v_l \in \mathbb{Z}^m$ for*

$1 \leq l \leq 2^m(d+1)$ *and an open* (J_n, K)*-marker* $Z \subset X$ *such that*

$$K \subset \bigcup_{l=1}^{2^m(d+1)} \bigcup_{v \in J_n} \alpha_{v_l+v}(Z).$$

Proof. For $n \in \mathbb{N}$, choose $x_1, \dots, x_d \in \mathbb{Z}^m$ such that (with $x_0 := 0$)

$$J_{2n} \ , \ x_1 + J_{2n} \ , \ \dots \ , \ x_d + J_{2n}$$

are pairwise disjoint. Define $M = \bigcup_{l=0}^{d}(x_l + J_{2n})$. Notice that J_{2n} contains the set $J_n - J_n$. Apply 3.2.7 to find a (J_n, K)-marker Z such that $K \subset \bigcup_{v \in M} \alpha_v(Z)$. Now use 3.3.1 to choose $w_1, \dots, w_{2^m} \in \mathbb{Z}^m$ so that $J_{2n} = \bigcup_j (w_j + J_n)$. It follows that

$$M = \bigcup_{l=0}^{d}(x_l + J_{2n}) = \bigcup_{l=0}^{d} \bigcup_{j=1}^{2^m}((x_l + w_j) + J_n),$$

so

$$K \subset \bigcup_{v \in M} \alpha_v(Z) = \bigcup_{l=0}^{d} \bigcup_{j=1}^{2^m} \bigcup_{v \in J_n} \alpha_{(x_l+w_j)+v}(Z).$$

Noticing that the family $\{x_l + w_j \mid l = 0, \dots, d, \ j = 1, \dots, 2^m\}$ has cardinality $2^m(d+1)$, this finishes the proof. □

The following two results constitute the main result of this chapter for $\mathbb{Z}^m$-actions:

Theorem 3.3.3. *Let* X *be a locally compact metric space and let* $\alpha : \mathbb{Z}^m \curvearrowright X$ *be a free action. Then*

$$\dim^{+1}_{\mathrm{Rok}}(\bar{\alpha}) \leq 4^m \dim^{+1}(X).$$

Proof. We may assume that the covering dimension d of X is finite, as there is otherwise nothing to show.

Let $L \in \mathbb{N}$, $\varepsilon > 0$ and $K \subset X$ compact. Choose n large enough such that $\frac{2}{n} \leq \varepsilon$. Apply 3.3.2 and find elements $v_l \in \mathbb{Z}^m$ for $1 \leq l \leq 2^m(d+1)$ and an open (J_{4Ln}, K)-marker $Z \subset X$ with $K \subset \bigcup_{l=1}^{2^m(d+1)} \bigcup_{v \in J_{4Ln}} \alpha_{v_l+v}(Z)$. By virtue of being a (J_{4Ln}, K)-marker, $\overline{Z}$ admits a relatively compact open neighbourhood Z_0 that is still a (J_{4Ln}, K)-marker. Choose some continuous function $g : X \to [0,1]$ with $g|_Z = 1$ and having support in Z_0.

For all $l = 1, \dots, 2^m(d+1)$, define functions $(f_v^{(l)})_{v \in B_L}$ via

$$f_v^{(l)}(x) = \begin{cases} g \circ \alpha_{-(v_l+w)}(x) & \text{, if } x \in \alpha_{v_l+w}(Z_0) \text{ for } \|w\|_\infty \leq 2Ln \\ & \quad \text{and } w = v \mod L\mathbb{Z}^m \\ \frac{3Ln - \|w\|_\infty}{Ln} \cdot g \circ \alpha_{-(v_l+w)}(x) & \text{, if } x \in \alpha_{v_l+w}(Z_0) \text{ for} \\ & \quad 2Ln < \|w\|_\infty \leq 3Ln \\ & \quad \text{and } w = v \mod L\mathbb{Z}^m \\ 0 & \text{, elsewhere.} \end{cases}$$

Now the properties of Z, Z_0 and g ensure that

- $f_v^{(l)} \cdot f_w^{(l)} = 0$ for $v \neq w$ in B_L.
- $\sum_{v \in B_L} f_v^{(l)}$ is constantly 1 on $\bigcup_{w \in J_{2Ln}} \alpha_{v_l+w}(Z)$.
- For $\|w\|_\infty = L$, we have $\|f_0^{(l)} - f_0^{(l)} \circ \alpha_{-w}\| \leq \frac{1}{n}$.
- For $v \in B_L$, we have $\|f_v^{(l)} - f_0^{(l)} \circ \alpha_{-v}\| \leq \frac{1}{n}$.
- Hence $\|f_w^{(l)} \circ \alpha_{-v} - f_{(v+w) \mod L\mathbb{Z}^m}^{(l)}\| \leq \frac{2}{n} \leq \varepsilon$ for all $v, w \in B_L$.

Now choose $\{a_j \mid j = 1, \dots, 2^m\} \subset \mathbb{Z}^m$ according to 3.3.1 such that

$$J_{4Ln} = \bigcup_{j=1}^{2^m} a_j + J_{2Ln}.$$

For $l = 1, \dots, 2^m(d+1)$, $j = 1, \dots, 2^m$ and $v \in B_L$, we define $f_v^{(l,j)} = f_v^{(l)} \circ \alpha_{-a_j}$.

Let us identify B_L with $\mathbb{Z}^m / L\mathbb{Z}^m$ in the obvious way. Then we have established that for all l and j, the functions $(f_v^{(l,j)})_{v \in B_L}$ satisfy the relations (3b), (3c) and (3d) of 1.5.6. Furthermore, $\sum_{v \in B_L} f_v^{(l,j)}$ is constantly 1 on $\bigcup_{w \in J_{2Ln}} \alpha_{v_l+a_j+w}(Z)$, so the choice of the a_j ensures that we have

$\sum_{j=1}^{2^m} \sum_{v \in B_L} f_v^{(l,j)} \geq 1$ on $\bigcup_{w \in J_{4Ln}} \alpha_{v_l+w}(Z)$, and hence

$$\sum_{l=1}^{2^m(d+1)} \sum_{j=1}^{2^m} \sum_{v \in B_L} f_v^{(l,j)} \geq 1 \quad \text{on} \quad \bigcup_{l=1}^{2^m(d+1)} \bigcup_{w \in J_{4Ln}} \alpha_{v_l+w}(Z) \supset K.$$

We see that the family $\left\{ f_v^{(l,j)} \mid l, j, v \right\}$ has $4^m \cdot (d+1)$ upper indices. Note that since the elements $\left\{ f_v^{(l,j)} \right\}_{v \in V}$ are pairwise orthogonal for fixed (l, j), the sum $\sum_{l=1}^{2^m(d+1)} \sum_{j=1}^{2^m} \sum_{v \in B_L} f_v^{(l,j)}$ is bounded by $4^m(d+1)$.

Now ε and $K \subset X$ were arbitrary and did not depend on L, so we can let ε go to zero and let the compact set K get larger and larger. This construction allows us to repeat the same argument as in the proof of "(3) $\Rightarrow$ (2) $\Rightarrow$ (1)" from 1.5.6 and find equivariant c.p.c. order zero maps

$$\varphi_l : \big(\mathcal{C}(\mathbb{Z}^m/L\mathbb{Z}^m), \mathbb{Z}^m\text{-shift}\big) \to \big(F_\infty(\mathcal{C}_0(X)), \bar{\alpha}_\infty\big) \quad (l = 0, \dots, 4^m(d+1)-1)$$

such that the element $S := \sum_{l=0}^{4^m(d+1)-1} \varphi_l(\mathbf{1})$ satisfies $\mathbf{1} \leq S \leq 4^m(d+1)\mathbf{1}$. Since S is fixed under $\bar{\alpha}_\infty$ and the target algebra of these maps is commutative, we may replace each φ_l by $S^{-1} \cdot \varphi_l$ and still have equivariant c.p.c. order zero maps. Hence we may assume without loss of generality that $\sum_{l=0}^{4^m(d+1)-1} \varphi_l(\mathbf{1}) = \mathbf{1}$. But this verifies $\dim_{\mathrm{Rok}}^{+1}(\bar{\alpha}, L\mathbb{Z}^m) \leq 4^m(d+1)$, for arbitrary $L \in \mathbb{N}$. Since $(L\mathbb{Z}^m)_{L \in \mathbb{N}}$ contains a dominating residually finite approximation of $\mathbb{Z}^m$ by 1.3.13, we obtain by 1.5.10 that

$$\dim_{\mathrm{Rok}}^{+1}(\bar{\alpha}) = \sup_{L \in \mathbb{N}} \dim_{\mathrm{Rok}}^{+1}(\bar{\alpha}, L\mathbb{Z}^m) \leq 4^m(d+1),$$

as desired. □

Combining everything, we obtain our main result:

Corollary 3.3.4. *Let X be a locally compact metric space and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free action. Then*

$$\dim_{\mathrm{nuc}}^{+1}(\mathcal{C}_0(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^m) \leq 4^m(m+1) \dim^{+1}(X)^2.$$

Proof. This follows directly from 1.1.3, 3.3.3 and 1.5.15. □

Corollary 3.3.5 (cf. [94, 5.3])**.** *Let X be a compact metric space and let $\alpha : \mathbb{Z}^m \curvearrowright X$ be a free action. Then*

$$\dim_{\mathrm{nuc}}^{+1}(\mathcal{C}(X) \rtimes_{\bar{\alpha}} \mathbb{Z}^m) \leq 4^m(m+1)\dim^{+1}(X)^2.$$

3.4 Finite Rokhlin dimension beyond $\mathbb{Z}^m$

As the title of this section suggests, we consider free actions of higher-rank, non-abelian groups, and study to what extent one can carry over a result like 3.3.4. We could already witness during the proof of 3.3.3 that even in the case of abelian groups, obtaining finite Rokhlin dimension from a marker-type property is a fairly technical endeavour. In the non-abelian case, this becomes more complicated. First of all, one still has to make restrictions on the group in question, because a result like 3.3.3 cannot be expected to hold for every residually finite group. In our case, we will restrict to finitely generated and nilpotent groups. Second, one has to turn these restrictions into a geometric statement that one can work with. In order to obtain finite Rokhlin dimension from these geometric and topological dynamical statements, it becomes useful to carry out certain calculations in a naturally induced product system.

Before we begin, let us mention that the results of this section essentially come from a collaboration with Wu and Zacharias [1], see [96]. However, the line of argument towards the main result is a bit different and more direct than in [96], because we forego the parallel study of amenability dimension in the sense of Guentner, Willett and Yu [37, 38]. However, the approach is still very much inspired by some techniques of [37, 38]. Moreover, the results presented here are more general than in [96] because we treat the case of actions on locally compact metric spaces, instead of only considering compact metric spaces.

Before we can study the actions themselves, we have to establish an important technical property of nilpotent groups, which basically amounts to their special geometry that comes from polynomial growth. Recall that by 1.4.8, every finitely generated and nilpotent group is poly-cyclic and thus has a Hirsch-length.

[1]Each author has contributed an approximately equal amount to this collaboration.

Lemma 3.4.1 (cf. [96, 5.1])**.** *Let G be a finitely generated, nilpotent group. Let ℓ be the Hirsch-length of G and set $m = 3^\ell$. Then for any finite subset $M \Subset G$, there is a finite subset $F \Subset G$ containing the identity and $g_1, \ldots, g_m \in G$, such that for any $g \in F^{-1}F$, we have $Mg \subset g_j F$ for some $j \in \{1, \ldots, m\}$.*

Proof. We apply induction by ℓ. If $\ell = 0$, then G is finite, and we may simply take $F = G$ and $m = 1$.

Now suppose G has Hirsch-length $\ell+1$ for some $\ell \geq 0$, and the statement has been proved for all nilpotent groups with Hirsch length at most ℓ. By 1.4.8, G has a central element $t \in G$ of infinite order. Set $H = G/\langle t \rangle$. This yields another finitely generated, nilpotent group of Hirsch-length ℓ. Denote by $\pi : G \to H$ the quotient map and set $m = 3^\ell$.

Now let $M \Subset G$ be a finite subset. We apply our induction hypothesis on H to get a finite set $F_0 \Subset H$ containing the identity and $h_1, \ldots, h_m \in H$ such that for each $h \in F_0^{-1}F_0$, we have $\pi(M)h \subset h_j F_0$ for some $j \in \{1, \ldots, m\}$.

Pick some cross section $\sigma : H \to G$ such that $\sigma(1_H) = 1_G$. Decompose M into

$$M = \bigsqcup_{k \in \pi(M)} M_k \cdot \sigma(k) \tag{e6}$$

for some finite subsets $M_k \Subset \langle t \rangle$. Define the finite sets

$$T = \bigcup_{k \in \pi(M)} \bigcup_{k_1, k_2 \in F_0} \sigma(k k_1^{-1} k_2)^{-1} \sigma(k) \sigma(k_1)^{-1} \sigma(k_2) M_k \tag{e7}$$

and

$$S = \bigcup_{j=1}^{m} \bigcup_{k \in \pi(M)} \bigcup_{k_1, k_2 \in F_0} \sigma(h_j^{-1} k k_1^{-1} k_2)^{-1} \sigma(h_j)^{-1} \sigma(k k_1^{-1} k_2). \tag{e8}$$

Since σ is a cross section, it follows that we have $S, T \subset \langle t \rangle$. Moreover, ST is finite, so there exists $n \in \mathbb{N}$ such that that

$$ST \subset \left\{ t^i \mid -n \leq i \leq n \right\}. \tag{e9}$$

We set

$$F_1 = \left\{ t^i \mid -3n \leq i \leq 3n \right\}, \quad F = \sigma(F_0) \cdot F_1 \tag{e10}$$

and

$$g_{i,j} = t^{4ni} \sigma(h_j) \tag{e11}$$

for $i = -1, 0, 1$ and $j = 1, \ldots, m$. Let us show that these satisfy the desired property of the assertion.

Let us choose an element $x \in F^{-1}F$. By e10, this element has the form

$$x = t^{l_1 - l_2}\sigma(k_1)^{-1}\sigma(k_2) \tag{e12}$$

for certain $k_1, k_2 \in F_0$ and $-3n \leq l_1, l_2 \leq 3n$. By assumption, we have

$$\pi(M)k_1^{-1}k_2 \subset h_{j_0}F_0 \tag{e13}$$

for some $j_0 \in \{1, \ldots, m\}$. For the number

$$i_0 = \begin{cases} -1 & , \quad \text{if} \quad -6n \leq l_2 - l_1 < -2n \\ 0 & , \quad \text{if} \quad -2n \leq l_2 - l_1 \leq 2n \\ 1 & , \quad \text{if} \quad 2n < l_2 - l_1 \leq 6n \end{cases}$$

we have

$$l_2 - l_1 + \{-n, \ldots, n\} \subset 4ni_0 + \{-3n, \ldots, 3n\}\,. \tag{e14}$$

Combining all this, observe that

$$\begin{aligned}
& Mx \\
\overset{e6,e12}{=} & \bigsqcup_{k\in\pi(M)} M_k \cdot \sigma(k)t^{l_1-l_2}\sigma(k_1)^{-1}\sigma(k_2) \\
= & \bigsqcup_{k\in\pi(M)} \sigma(k)\sigma(k_1)^{-1}\sigma(k_2)M_k t^{l_1-l_2} \\
= & \bigsqcup_{k\in\pi(M)} \sigma(kk_1^{-1}k_2)\sigma(kk_1^{-1}k_2)^{-1}\sigma(k)\sigma(k_1)^{-1}\sigma(k_2)M_k t^{l_1-l_2} \\
\overset{e7}{\subset} & \bigsqcup_{k\in\pi(M)} \sigma(kk_1^{-1}k_2)Tt^{l_1-l_2} \\
= & \bigsqcup_{k\in\pi(M)} \sigma(h_{j_0})\sigma(h_{j_0}^{-1}kk_1^{-1}k_2)\sigma(h_{j_0}^{-1}kk_1^{-1}k_2)^{-1}\sigma(h_{j_0})^{-1}\sigma(kk_1^{-1}k_2)Tt^{l_1-l_2} \\
\overset{e8}{\subset} & \bigsqcup_{k\in\pi(M)} \sigma(h_{j_0})\sigma(h_{j_0}^{-1}kk_1^{-1}k_2)STt^{l_1-l_2}
\end{aligned}$$

$$\overset{e13}{\subset} \quad \sigma(h_{j_0})\sigma(F_0)STt^{l_1-l_2}$$

$$\overset{e9,e14,e10}{\subset} \quad \sigma(h_{j_0})\sigma(F_0)F_1t^{4ni_0}$$

$$\overset{e10,e11}{=} \quad g_{i_0,j_0}\cdot F.$$

By the definition of the elements $g_{i,j}$ in e11, we see that there are $3m = 3^{\ell+1}$ lower indices. This finishes the induction step and the proof. □

Lemma 3.4.2. *Let X be a locally compact metric space with finite covering dimension d. Let G be an infinite, finitely generated, nilpotent group and $\alpha : G \curvearrowright X$ a free action.*

For any compact subset $K \subset X$, finite subset $M \Subset G$ and $\varepsilon > 0$, there exist finitely-supported maps $\mu^{(l,j)} : G \to \mathcal{C}_c(X)_{+,1}$ for $l = 0, \dots, d$ and $j = 1, \dots, m := 3^{\ell_{\mathrm{Hir}}(G)}$ satisfying:

(a) $\displaystyle\sum_{l=0}^{d}\sum_{j=1}^{m}\sum_{g\in G}\mu_g^{(l,j)} \leq \mathbf{1}$ *and* $\displaystyle\sum_{l=0}^{d}\sum_{j=1}^{m}\sum_{g\in G}\mu_g^{(l,j)}|_K = 1.$

(b) $\mu_g^{(l,j)}\mu_h^{(l,j)} = 0$ *for all $l = 0, \cdots, d$, $j = 1, \dots, m$ and $g \neq h$ in G.*

(c) $\mu_h^{(l,j)} \circ \alpha_{g^{-1}} =_\varepsilon \mu_{gh}^{(l,j)}$ *for all $l = 0, \cdots, d$, $j = 1, \dots, m$, $g \in M$ and $h \in G$.*

Proof. Let ℓ be the Hirsch-length of G and denote $m = 3^\ell$. Let $K \subset X$ be compact and $M \Subset G$ finite. Since G is amenable, we can choose a Følner set

$$J \Subset G \quad \text{with} \quad |J\Delta gJ| \leq \varepsilon|J| \quad \text{for all } g \in M. \tag{e15}$$

By 3.4.1, we can find a finite subset $F \Subset G$ containing the identity and $h_1, \dots, h_m \in G$ such that for every $x \in F^{-1}F$, we have $Jx \subset h_jF$ for some $j \in \{1, \dots, m\}$. Since G is infinite and nilpotent, so is its center by 1.4.8. Thus we can find $g_1, \dots, g_d$ in the center of G such that the sets

$$F^{-1}F \ ,\ g_1F^{-1}F \ ,\ \dots \ ,\ g_dF^{-1}F$$

are pairwise disjoint. Define the compact set

$$K_J = \bigcup_{g \in J} \alpha_{g^{-1}}(K) \subset X. \tag{e16}$$

By 3.2.7, there exists an (F, K_J)-marker $Z \subset X$ such that

$$K_J \subset \bigcup_{l=0}^{d} \bigcup_{g \in F^{-1}F} \alpha_{g_l g}(Z). \tag{e17}$$

For $l = 0, \ldots, d$ and $j = 1, \ldots, m$, define the finite sets

$$N^{(l,j)} = \{g \in G \mid Jg \subset g_l h_j F\} \subset\subset G.$$

For each $g \in N^{(l,j)}$, define the relatively compact, open set

$$Z^{(i,j,g)} = \alpha_g(Z) \subset X.$$

We now claim that

$$K_J \subset \bigcup_{i=0}^{d} \bigcup_{i=1}^{m} \bigcup_{g \in N^{(i,j)}} Z^{(i,j,g)}.$$

For any $x \in K_J$, we can apply e17 and find some $l \in \{0, \ldots, d\}$ and $g \in F^{-1}F$ with $x \in \alpha_{g_l g}(Z)$. By our choice of F and the elements $h_1, \ldots, h_m$, we can find $j \in \{1, \ldots, m\}$ with $Jg \subset h_j F$. But then $Jg_l g \subset g_l h_j F$, so we see that $g_l g \in N^{(l,j)}$ and $x \in Z^{(l,j,g_l g)}$.

Having this cover of K_J, we may find a partition of unity

$$\left\{\nu^{(l,j,g)} \mid g \in N^{(i,j)},\ l = 0, \ldots, d,\ j = 1, 2, \ldots, m\right\}$$

of K_J subordinate to the open cover

$$\left\{Z^{(l,j,g)} \cap K_J \mid g \in N^{(l,j)},\ l = 0, \ldots, d,\ j = 1, 2, \ldots, m\right\}.^2$$

By the Urysohn-Tietze extension theorem, we can extend each function $\nu^{(l,j,g)}$ to a continuous function on X with values in $[0,1]$ and with support inside $Z^{(l,j,g)}$, which we will (with slight abuse of notation) also denote by $\nu^{(l,j,g)}$. For the sake of convenience, let us set $\nu^{(l,j,g)} = 0$ for all $l = 0, \ldots, d$

[2] Note that we allow repetions with this notation.

and $j = 1, \dots, m$, whenever $g \notin N^{(l,j)}$.

For every $l = 0, \dots, d$ and $j = 1, \dots, m$, we now define the finitely supported maps $\mu^{(l,j)} : G \to \mathcal{C}_c(X)_{+,1}$ via

$$\mu_g^{(l,j)} = \frac{1}{|J|} \sum_{h \in J} \nu^{(l,j,h^{-1}g)} \circ \alpha_{h^{-1}}$$

We claim that these satisfy the desired properties. Firstly, the support of each map $\mu^{(l,j)}$ is contained in $J \cdot N^{(l,j)}$, and is hence finite. Secondly, since each $\nu^{(l,j,g)}$ has values in $[0,1]$, so does each $\mu_g^{(l,j)}$ by triangle inequality. We have

$$\begin{aligned} \sum_{i=0}^{d} \sum_{j=1}^{m} \sum_{g \in G} \mu_g^{(i,j)} &= \sum_{l=0}^{d} \sum_{j=1}^{m} \sum_{g \in G} \frac{1}{|J|} \sum_{h \in J} \nu^{(l,j,h^{-1}g)} \circ \alpha_{h^{-1}} \\ &= \frac{1}{|J|} \sum_{h \in J} \underbrace{\Big(\sum_{l=0}^{d} \sum_{j=1}^{m} \sum_{g \in G} \nu^{(l,j,h^{-1}g)} \Big)}_{=:S_h} \circ \alpha_{h^{-1}}. \end{aligned}$$

Now by construction of the maps $\nu^{(l,j,g)}$, we know that S_h is a compactly supported, continuous function on X with values in $[0,1]$, and which is constantly 1 on K_J. By the definition e16 of the set K_J, we see that $S_h \circ \alpha_{h^{-1}}$ is constantly 1 on K for each $h \in J$. So if we average these expressions via J, it follows that condition (a) holds.

Next we observe that for all $l = 0, \dots, d$, $j = 1, \dots, m$ and $g \in G$, the open support of $\mu_g^{(l,j)}$ satisfies

$$\begin{aligned} \mathrm{supp}(\mu_g^{(l,j)}) &\subset \bigcup_{h \in J} \alpha_h \big(\mathrm{supp}(\nu^{(i,j,h^{-1}g)}) \big) \\ &\subset \begin{cases} \bigcup_{h \in J} \alpha_h(\alpha_{h^{-1}g}(Z)) = \alpha_g(Z) & , \text{ if } g \in J \cdot N^{(l,j)} \\ \emptyset & , \text{ if } g \notin J \cdot N^{(l,j)}. \end{cases} \end{aligned}$$

By definition of the set $N^{(l,j)}$, we have $J \cdot N^{(l,j)} \subset g_l h_j F$. So let $g' \in G$ be an element different from g. If $g \notin J \cdot N^{(l,j)}$ or $g' \notin J \cdot N^{(l,j)}$, then we trivially have $\mu_g^{(i,j)} \mu_{g'}^{(i,j)} = 0$ because one of the functions is zero already. If $g, g' \in J \cdot N^{(l,j)} \subset g_l h_j F$, then we can write $g = g_l h_j f_1$ and $g' = g_l h_j f_2$ for some $f_1 \neq f_2$ in F. Since Z is an (F, K_J)-marker, it follows that $\alpha_{f_1}(Z) \cap \alpha_{f_2}(Z) = \emptyset$, and thus $\alpha_g(Z) \cap \alpha_{g'}(Z) = \emptyset$. In particular, the two functions

$\mu_g^{(l,j)}$ and $\mu_{g'}^{(l,j)}$ have disjoint open supports and are therefore orthogonal. This verifies condition (b).

Lastly, for all $l = 0, \dots, d,\ \ j = 1, 2, \dots, m,\ \ g \in M$ and $g' \in G$, we calculate

$$\begin{aligned}
&\|\mu_{g'}^{(l,j)} \circ \alpha_{g^{-1}} - \mu_{gg'}^{(l,j)}\| \\
=\ & \frac{1}{|J|}\Big\|\sum_{h\in J} \nu^{(l,j,h^{-1}g')} \circ \alpha_{h^{-1}} \circ \alpha_{g^{-1}} - \sum_{h\in J} \nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}}\Big\| \\
=\ & \frac{1}{|J|}\Big\|\sum_{h\in J} \nu^{(l,j,(gh)^{-1}gg')} \circ \alpha_{(gh)^{-1}} - \sum_{h\in J} \nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}}\Big\| \\
=\ & \frac{1}{|J|}\Big\|\sum_{h\in gJ} \nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}} - \sum_{h\in J} \nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}}\Big\| \\
=\ & \frac{1}{|J|}\Big\|\sum_{h\in gJ\setminus J} \nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}} - \sum_{h\in J\setminus gJ} \nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}}\Big\| \\
\leq\ & \frac{1}{|J|}\Big(\sum_{h\in gJ\setminus J} \|\nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}}\| + \sum_{h\in J\setminus gJ} \|\nu^{(l,j,h^{-1}gg')} \circ \alpha_{h^{-1}}\|\Big) \\
\leq\ & \frac{|J\Delta gJ|}{|J|} \overset{e15}{\leq} \varepsilon.
\end{aligned}$$

This verifies condition (c) and finishes the proof. □

This technical Lemma leads us to the main result of this section:

Theorem 3.4.3 (cf. [96, 5.4])**.** *Let G be an infinite, finitely generated, nilpotent group. Let X be a locally compact metric space and $\alpha : G \curvearrowright X$ a free action. Then for the induced* C**-algebraic action $\bar{\alpha} : G \curvearrowright \mathcal{C}_0(X)$, we have*

$$\dim_{\mathrm{Rok}}(\alpha) \leq 3^{\ell_{\mathrm{Hir}}(G)} \cdot \dim^{+1}(X).$$

Proof. Denote $m = 3^{\ell_{\mathrm{Hir}}(G)}$ and $d = \dim(X)$. Without loss of generality, we may assume that d is finite, as there is otherwise nothing to show.

Let $H \subset G$ be a subgroup of finite index. In order to show $\dim_{\mathrm{Rok}}(\bar{\alpha}, H) \leq m(d+1)$, we verify condition 1.5.6(3). Let $F \Subset \mathcal{C}_0(X)_1,\ \ M \Subset G$ and $\varepsilon > 0$ be given. Since F is finite and consists of continuous functions vanishing at infinity, we can choose a compact set $K \subset X$ such that

$$|a(x)| \leq \varepsilon \quad \text{for all } x \in X \setminus K \text{ and } a \in F. \tag{e18}$$

By 3.4.2, we can find finitely supported maps $\mu^{(l,j)} : G \to \mathcal{C}_c(X)_{+,1}$ for $l = 0, \dots, d$ and $j = 1, \dots, m$ with conditions (a), (b) and (c) from 3.4.2. For each $l = 0, \dots, d,\ j = 1, \dots, m$ and $\bar{g} \in G/H$, we define $f_{\bar{g}}^{(l,j)} = \sum_{h \in \bar{g}} \mu_h^{(l,j)}$. We claim that the collection $\left\{ f_{\bar{g}}^{(l,j)} \mid \bar{g} \in G/H,\ l = 0, \dots, d,\ j = 1, \dots, m \right\}$ satisfies conditions (3a), (3b), (3c) and (3d) from 1.5.6(3) for the triple $(F, M, 2\varepsilon)$.

Since $\mathcal{C}_0(X)$ is commutative, condition 1.5.6(3d) is redundant. For two elements $\bar{g} \neq \bar{h}$ in G/H, viewing them as subsets of G means $\bar{g} \cap \bar{h} = \emptyset$. Observe that thus condition (b) from 3.4.2 implies that for fixed l, j, the elements $\left\{ f_{\bar{g}}^{(l,j)} \mid \bar{g} \in G/H \right\}$ are pairwise orthogonal, so condition 1.5.6(3b) follows. Moreover, combining conditions (b) and (c) from 3.4.2 yields

$$\bar{\alpha}_g(f_{\bar{h}}^{(l,j)}) =_{2\varepsilon} f_{\overline{gh}}^{(l,j)} \quad \text{for all } \bar{h} \in G/H \text{ and } g \in M.$$

In particular, this verifies condition 1.5.6(3c).

Lastly, we have

$$\begin{aligned} \sum_{l=0}^{d} \sum_{j=1}^{m} \sum_{\bar{g} \in G/H} f_{\bar{g}}^{(l,j)} &= \sum_{l=0}^{d} \sum_{j=1}^{m} \sum_{\bar{g} \in G/H} \sum_{h \in \bar{g}} \mu_h^{(l,j)} \\ &= \sum_{l=0}^{d} \sum_{j=1}^{m} \sum_{g \in G} \mu_g^{(l,j)} \end{aligned}$$

Condition 3.4.2(a) says that this sum is a positive contraction and is constantly 1 on K. Thus, e18 implies that

$$\Big(\sum_{l=0}^{d} \sum_{j=1}^{m} \sum_{\bar{g} \in G/H} f_{\bar{g}}^{(l,j)} \Big) a =_{\varepsilon} a \quad \text{for all } a \in F.$$

This shows condition 1.5.6(3a) and finishes the proof. □

Combining this with the results of the first chapter, we obtain the most general result about transformation group C*-algebras within this dissertation:

Theorem 3.4.4 (cf. [96, 5.5, 5.6])**.** *Let G be an infinite, finitely generated, nilpotent group and X a locally compact metric space with finite covering dimension. Let $\alpha : G \curvearrowright X$ be a free action. Then the transformation group* C^**-algebra $\mathcal{C}_0(X) \rtimes_{\bar{\alpha}} G$ has finite nuclear dimension.*

Proof. This follows by combining 1.1.3, 3.4.3, 1.4.15 and 1.5.14. □

Actions on noncommutative C*-algebras

In this chapter, I will give a brief survey of the contents of the other papers [2, 3, 95] that I authored or co-authored over the course of my doctoral studies. These papers go in a somewhat similar direction as this dissertation, in that one studies certain C*-dynamical systems or their crossed products by means of considering certain Rokhlin-type properties. However, they do not address the interplay between topological dynamics and C*-algebra classification theory, which is the overall goal of this dissertation. That is why I do not discuss them at length. Each of the upcoming sections will describe one of these papers, outlining the underlying questions and problems and then presenting the main results.

Before we begin, let us mention what additional knowledge might be helpful for the reader in order to appreciate the results that are presented in this chapter. For a proper understanding of some of the results, it can be crucial to be somewhat familiar with Kasparov's KK-theory of C*-algebras, see the later chapters of [7] as a standard reference. To a more limited extent, it can be helpful to know basic E-theory and how it connects to KK-theory, see [14, 36]. Moreover, some results concern Rosenberg's and Schochet's universal coefficient theorem from [83], abbreviated as UCT, for C*-algebras.

4.1 The nuclear dimension of certain $\mathcal{O}_\infty$-absorbing C*-algebras

In [2], I have collaborated with Selçuk Barlak, Dominic Enders, Hiroki Matui and Wilhelm Winter. The paper is motivated by the regularity conjecture of Toms and Winter in the case of so-called Kirchberg algebras. A Kirchberg algebra is a separable, nuclear, simple and purely infinite C*-algebra. A simple C*-algebra is called purely infinite, if every non-trivial hereditary subalgebra contains an infinite projection, see [16]. For Kirchberg algebras, the Elliott conjecture has been confirmed with a very satisfying classification result due to Kirchberg and Phillips, see [50, 73].

Theorem (Kirchberg-Phillips)**.** *Let A and B be two unital Kirchberg algebras. Then A and B are isomorphic, if and only if there exists a KK-equivalence $\kappa \in KK(A,B)^{-1}$ preserving the K_0-class of the unit, i.e. for the map $\kappa_0 \in \mathrm{Hom}(K_0(A), K_0(B))$ induced by κ, one has $\kappa_0([\mathbf{1}_A]) = [\mathbf{1}_B]$.*

If both A and B satisfy the UCT, then A and B are isomorphic if and only if they have the same K-theoretic data, i.e.

$$\big(K_0(A), [\mathbf{1}_A], K_1(A)\big) \cong \big(K_0(B), [\mathbf{1}_B], K_1(B)\big).$$

Despite this classification result, it has remained unclear for a long time whether the Toms-Winter conjecture holds for all Kirchberg algebras. Since Kirchberg algebras are well-known to be both $\mathcal{Z}$-stable and having strict comparison of positive elements, the only open problem was if Kirchberg algebras have also finite nuclear dimension.

In the UCT case, there exist fairly tractable inductive limit model systems for Kirchberg algebras realizing all possible K-theoretic data. Winter and Zacharias have used these model systems to show in [111] that Kirchberg algebras satisfying the UCT do have finite nuclear dimension. In a groundbreaking paper of Matui and Sato [69], this question has been answered in the affirmative without the UCT assumption:

Theorem (Matui-Sato)**.** *The nuclear dimension of any Kirchberg algebra is at most three.*

In [2], two alternative lines of proof of this fact are presented, leading also to partial answers to the analogous problem in the non-simple case.

The first line of proof concerns the Rokhlin dimension of symmetries, i.e. $\mathbb{Z}_2$-actions, on Kirchberg algebras. The so-called Rokhlin property for finite group actions on unital C*-algebras, which coincides with Rokhlin dimension zero in our terminology from the first chapter, has been studied by Izumi in [43, 44]. From his work, it is well-known that the Rokhlin property is very restrictive, not least because there exist various K-theoretic obstructions to its existence. For instance, any two actions of a finite group with the Rokhlin property on $\mathcal{O}_2$ are conjugate, and in particular the crossed product of $\mathcal{O}_2$ by such an action is always isomorphic to $\mathcal{O}_2$. As it turns out, the notion of Rokhlin dimension at most one for symmetries is a much more flexible concept on Kirchberg algebras, so much so that it is automatic whenever one has a chance to expect it:

Theorem. *Any outer symmetry on a unital Kirchberg algebra has Rokhlin dimension at most one.*

On the other hand, Izumi has shown in [43] that many Kirchberg algebras can be realized as a crossed product of an outer symmetry on $\mathcal{O}_2$.

Theorem (Izumi)**.** *Let A be a Kirchberg algebra. Then there exists an outer symmetry α on $\mathcal{O}_2$ such that its crossed product $\mathcal{O}_2 \rtimes_\alpha \mathbb{Z}_2$ is stably isomorphic to $A \otimes M_{2^\infty}$.*

Combing all of this with a further technical dimension reduction step from [69], one obtains:

Theorem. *Let A be a Kirchberg algebra. Then* $\dim_{\mathrm{nuc}}(A \otimes M_{2^\infty}) \leq 3$ *and in fact* $\dim_{\mathrm{nuc}}(A) < \infty$*.*

The second line of proof towards finite nuclear dimension of Kirchberg algebras is a more direct one, not making use of any crossed product picture of the involved C*-algebras. Namely, for any C*-algebra A, it is shown in [2] that the nuclear dimensions of $A \otimes \mathcal{O}_\infty$ and $A \otimes \mathcal{O}_2$ are related by a formula. This marks the main result of [2]:

Theorem. *For any* C*-*algebra A, we have the inequality*

$$\dim_{\mathrm{nuc}}^{+1}(A \otimes \mathcal{O}_\infty) \leq 2 \dim_{\mathrm{nuc}}^{+1}(A \otimes \mathcal{O}_2).$$

Combining this general observation with Kirchberg's absorption theorems [52] and the fact from [111] that $\dim_{\mathrm{nuc}}(\mathcal{O}_2) = 1$, one recovers the

Matui-Sato result that a Kirchberg algebra has nuclear dimension at most three. However, the above inequality has the advantage that it holds in a very general context. Even in the non-simple case, it is known that tensoring a given C*-algebra with $\mathcal{O}_2$ has a great smoothening effect, leading to an immense loss of information of the original C*-algebra, see [53]. In particular, it is conceivable that any separable, nuclear and $\mathcal{O}_2$-stable C*-algebra might always have finite nuclear dimension. The above inequality then gives rise to the following question in the non-simple case:

Question. What is the nuclear dimension of a separable, nuclear and $\mathcal{O}_\infty$-absorbing C*-algebra?

In [2], a partial answer is given for C*-algebras that can be expressed as $\mathcal{C}(X)$-algebras with Kirchberg fibres. This follows from the above nuclear dimension formula and a dimension reduction argument from [53] for C*-algebras of the form $\mathcal{C}(X) \otimes \mathcal{O}_2$.

Theorem. *Let X be a compact metrizable space. Let A be an $\mathcal{O}_\infty$-absorbing, continuous $\mathcal{C}(X)$-algebra whose fibres are Kirchberg algebras. Then A has finite nuclear dimension.*

Concerning the nuclear dimension of Kirchberg algebras, we note that subsequent work of other authors yield an optimal nuclear dimension estimate. It has been shown in [85], building on a similar result in [26], that any Kirchberg algebra satisfying the UCT has nuclear dimension one. However, this approach used the entire power of Kirchberg-Phillips classification by using certain inductive limit model systems. Using sequence algebra techniques, a more direct proof was found in [8], not relying on the UCT assumption:

Theorem (Bosa et. al.)**.** *Any Kirchberg algebra has nuclear dimension one.*

4.2 The continuous Rokhlin property and the UCT

As mentioned in the previous section, Izumi has introduced the Rokhlin property of finite group actions in [43] within the general context of separable, unital C*-algebras. He has shown that such actions enjoy a number of structural well-behavedness properties. It was later also discovered by various authors that actions with the Rokhlin property preserve many C*-algebraic properties of interest, when passing to the crossed product. See [86] for a nice overview on these phenomena.

Hirshberg and Winter have then generalized the notion of the Rokhlin property to actions of compact groups on separable, unital C*-algebras, see [41]. Their main result asserts that the property of absorbing a given strongly self-absorbing C*-algebra passes from any unital C*-algebra to its crossed product by a Rokhlin action. In [28, 29, 30], Gardella has then undergone an investigation of compact group actions with the Rokhlin property and more specially circle actions with the Rokhlin property.

Despite the fact that many known results from the realm of finite group actions with the Rokhlin property carry over to the setting of compact groups, there remain some subtle difficulties concerning certain properties like the permanence of the UCT. It is known that in many cases, the UCT passes from a C*-algebra to its crossed product associated to a Rokhlin action of a finite group:

Theorem (cf. [86, Theorem 3])**.** *Let G be a finite group, A a separable* C*-*algebra and $\alpha : G \curvearrowright A$ an action with the Rokhlin property. Assume that A is nuclear and that every ideal of A satisfies the UCT. Then every ideal of $A \rtimes_\alpha G$ satisfies the UCT.*

At present, it is still unclear if such a permanence property holds for Rokhlin actions of compact groups as well. For example, when confronted with the problem of classifying Rokhlin actions of the circle on Kirchberg algebras by means of K-theory as in [28, 29], this proves to be a rather annoying obstacle. To overcome this, Gardella has introduced the continuous Rokhlin property for compact group actions on unital C*-algebras in [29]. It turns out that this stronger version is compatible with the UCT for circle actions on nuclear C*-algebras.

In the short note [95], the definition of the continuous Rokhlin property is extended to actions of metrizable compact groups on separable C*-algebras:

Definition. Let G be a compact group, A a separable C*-algebra and $\alpha : G \curvearrowright A$ a continuous action. Consider the path algebra of A defined by

$$A_{\mathfrak{c}} = \mathcal{C}_b\big([1,\infty), A\big)/\mathcal{C}_0\big([1,\infty), A\big).$$

Similarly as in 1.5.1, one defines the central path algebra of A as

$$F_{\mathfrak{c}}(A) = A_{\mathfrak{c}} \cap A' / \operatorname{Ann}(A, A_{\mathfrak{c}}).$$

In a completely analogous fashion to the case of sequence algebras treated in the first chapter, fibrewise application of the action α gives rise to a (not necessarily continuous) action $\alpha_{\mathfrak{c}}$ of G on $F_{\mathfrak{c}}(A)$. Then α is said to have the continuous Rokhlin property, if there exists a unital and equivariant $*$-homomorphism from $(\mathcal{C}(G), G\text{-shift})$ to $(F_{\mathfrak{c}}(A), \alpha_{\mathfrak{c}})$.

The main result of [95] is that an E-theoretic version of Gardella's UCT preservation theorem holds for actions of all metrizable compact groups on separable C*-algebras. This is done by using a somewhat more conceptual approach, enabling shorter and less technical proofs than in [29]:

Theorem. *Let G be a compact, metrizable group, A a separable* C**-algebra and $\alpha : G \curvearrowright A$ a continuous action with the continuous Rokhlin property. Assume that A satisfies the UCT in E-theory. Then both the fixed point algebra A^α and the crossed product $A \rtimes_\alpha G$ satisfy the UCT in E-theory.*

4.3 Rokhlin actions of finite groups on UHF-absorbing C*-algebras

In [3], I have collaborated with Selçuk Barlak. The paper serves as a source of examples of Rokhlin actions or locally representable actions of finite groups on C*-algebras satisfying a certain UHF-absorption condition. Over the last few years, Phillips has been steadily paving the way towards a classification theory of pointwise outer finite group actions on unital Kirchberg algebras, building on ideas of ordinary classification theory of Kirchberg algebras [73, 50] and making use of equivariant absorption theorems in the spirit of [52]. The key ideas of the latter have already been demonstrated in [33].

An important question related to such a classification theory is how large the range of the objects is that one wishes to classify. For example, if a certain class of C*-algebras is classified by K-theoretic data and one wishes to study finite group actions on such, this begs the question of whether every group action on the K-theory of a C*-algebra in this class can be realized by an honest group action on the C*-algebra. More generally, one can pose the following question:

Question. If A belongs to a class of C*-algebras that is classifiable by a functor Inv (in a suitable sense) and $\sigma : G \curvearrowright \mathrm{Inv}(A)$ is an action of a finite group on the invariant of A, does there exist an action $\alpha : G \curvearrowright A$ with $\mathrm{Inv}(\alpha) = \sigma$?

Compared to the recent progress in the Elliott classification program, satisfactory answers to this question are very scarce. Even within the setting of $G = \mathbb{Z}_n$ and A being an AF algebra, this question is still open, see [7, 10.11.3]. Only the case of actions on unital UCT Kirchberg algebras has been successfully studied in special cases of groups so far, in which the invariant boils down to K-theory, see [5, 89, 46].

In [3], another viewpoint of this problem is taken that is also suitable for actions on not necessarily classifiable C*-algebras. For this, let us fix some terminology. Write $\varphi \approx_{\mathrm{u}} \psi$, if two $*$-homomorphisms $\varphi, \psi : A \to B$ between separable C*-algebras are approximately unitarily equivalent by unitaries in the multiplier algebra $\mathcal{M}(B)$ of B.

Question. For a C*-algebra A and a finite group G, when can a homomorphism $G \to \mathrm{Aut}(A)/_{\approx_{\mathrm{u}}}$ lift to an honest action of G on A?

While we must certainly impose certain restrictions on A, it turns out that a sufficient criterion is common enough to produce a variety of interesting examples. Incidentally, the actions may all be chosen to have the Rokhlin property.

Theorem. *Let G be a finite group and A a separable* C**-algebra that absorbs the UHF algebra $M_{|G|^\infty}$ tensorially. Then any homomorphism $G \to \mathrm{Aut}(A)/_{\approx_{\mathrm{u}}}$ lifts to a Rokhlin action of G on A.*

Looking back to the above question about the range of the invariant of G-actions on classifiable C*-algebras, this more general viewpoint is weaker because the Elliott invariant alone is usually not strong enough to distinguish between approximate unitary equivalence classes of $*$-homomorphisms. However, all of the state-of-the-art classification results, when paired with the UCT, imply that for a classifiable C*-algebra A, the canonical map $\mathrm{Aut}(A)/_{\approx_{\mathrm{u}}} \to \mathrm{Aut}(\mathrm{Inv}(A))$ is not only surjective, but has a split. For example, this works for Kirchberg algebras or simple, nuclear TAF algebras, see [50, 73, 59, 60]. This enables one to reduce the above question about the range of the invariant of actions to the more elementary question of being able to lift homomorphisms $G \to \mathrm{Aut}(A)/_{\approx_{\mathrm{u}}}$ to honest group actions on A. In particular, the aforementioned theorem allows one to prove the following result:

Theorem. *Let G be a finite group and A a separable, unital, nuclear and simple* C^**-algebra with $A \cong M_{|G|^\infty} \otimes A$. Assume that A satisfies the UCT and is either purely infinite or TAF. Then Rokhlin actions of G on A exhaust all G-actions on the (ordered) K-theory of A.*

This result then becomes very useful for constructing interesting abelian group actions on classifiable C^*-algebra by considering the dual actions of Rokhlin actions. We now present the two most notable examples that are discussed in [3]:

Firstly, one can recover and extend Blackadar's famous construction [6] of certain symmetries on the CAR algebra having fixed point algebras with non-trivial K_1-groups, by combining the above existence result for Rokhlin actions with Lin's classification theory of TAF algebras [60]:

Theorem. *Let $p \geq 2$ be a natural number. Let (G_0, G_0^+, u) be a countable, uniquely p-divisible ordered abelian group with order unit, which is weakly unperforated and has the Riesz interpolation property. Let σ_0 be an ordered group automorphism of order p on G_0, such that $\ker(\mathrm{id} - \sigma_0)$ is isomorphic to $(\mathbb{Z}[\frac{1}{p}], \mathbb{Z}[\frac{1}{p}]^+, 1)$ as an ordered group with order unit. Let G_1 be a countable, uniquely p-divisible abelian group with an order p automorphism σ_1 such that $\ker(\mathrm{id} - \sigma_1) = 0$. Then there exists a $\mathbb{Z}_p$-action γ on M_{p^∞} such that $M_{p^\infty}^\gamma$ is a simple TAF* C^**-algebra satisfying the UCT and with $K_0(M_{p^\infty}^\gamma) \cong G_0$ as ordered groups and $K_1(M_{p^\infty}^\gamma) \cong G_1$.*

Secondly, one can reduce the UCT problem for separable, nuclear C^*-algebras to the question of whether one can leave the UCT class by passing to crossed product C^*-algebras of $\mathcal{O}_2$ by finite group actions. This is essentially done by realizing so many KK-theories of crossed products by finite group actions on $\mathcal{O}_2$ that it is sufficient to test the UCT question on these crossed products. Moreover, it is even sufficient to consider only locally representable actions of certain finite cyclic groups. We note that this relies crucially on Kirchberg's reduction of the UCT problem to Kirchberg algebras, see [51, 2.17].

Theorem. *Let $p, q \geq 2$ be two distinct prime numbers. The following are equivalent:*

(1) *Every separable, nuclear* C^**-algebra satisfies the UCT.*

(2) *If $\beta : \mathbb{Z}_p \curvearrowright \mathcal{O}_2$ and $\gamma : \mathbb{Z}_q \curvearrowright \mathcal{O}_2$ are pointwise outer, locally representable actions, then both $\mathcal{O}_2 \rtimes_\beta \mathbb{Z}_p$ and $\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_q$ satisfy the UCT.*

(3) *If $\gamma : \mathbb{Z}_{pq} \curvearrowright \mathcal{O}_2$ is a pointwise outer, locally representable action, then $\mathcal{O}_2 \rtimes_\gamma \mathbb{Z}_{pq}$ satisfies the UCT.*

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