Westfälische
Wilhelms-Universität
Münster

Tamino Meyhöfer

Optimal trading in order based markets with semi-rational noise traders

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Decay and reinforcement of irrational behaviour

2014

Mathematik

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Inauguraldissertation zur Erlangung des akademischen Grades
eines Doktors der Naturwissenschaften durch den
Fachbereich Mathematik und Informatik der
Westfälischen Wilhelms-Universität Münster

vorgelegt von

Tamino Meyhöfer

geboren in

Haltern am See

- 2014 -

Dekan:	Prof. Dr. Martin Stein
Erstgutachter:	PD Dr. Volkert Paulsen
Zweitgutachter:	Prof. Dr. Jörn Sass
Tag der mündlichen Prüfung:	09.07.2014
Tag der Promotion:	09.07.2014

Summary

In his seminal paper, Kyle [46] analyses a financial market under asymmetric information. In the presence of noise traders, a monopolistic, risk neutral insider who receives private information trades strategically with competitive, risk neutral market makers who try to infer the insider's signal. Subsequent extensions of this model stick to the assumption that noise traders act completely irrational, i.e. they are insensitive with respect to both, the price of the considered risky asset and its fundamental value. Their order flow is typically modelled by a Brownian motion in the continuous time framework.

The model developed in the present thesis relaxes the assumptions on the order flow of the noise traders in two different aspects. Firstly, it allows sensitivity with respect to a possible mispricing of the risky asset, i.e. the difference between its fundamental value and the market price, via a compensating drift term. Secondly, besides the well-known, continuous, Gaussian noise, a further, discontinuous noise that is correlated to jumps in the fundamental value process is introduced. Together, this model allows to describe a market with decay and reinforcement of irrational behaviour where the latter is caused by the arrival of new, unexpected information.

For both extensions the implications on a market equilibrium are studied by identifying necessary and sufficient conditions for its existence. This involves the analysis of optimal insider trading, on the one hand, and rationality of the market makers' inference of the noisy signal, on the other hand. Finally, the explicit form of a market equilibrium is determined.

For a start, the first extension is analysed separately for both cases, a risk neutral and a risk averse insider. It is shown that the existence of an equilibrium depends on the structure of the drift term where the fundamental value and the market price have to enter in a special form, weighted only by a deterministic intensity. This intensity in turn determines the equilibrium price pressure, i.e. the rate according to which variations of the total order affect the market price. As a result this leads to path dependent pricing rules. The case when the drift intensity approaches infinity at the end of the trading period is of special interest since the market gets infinitely deep and, in contrast to other insider equilibrium models, the market is already efficient in the absence of an insider. In particular, this places additional demands on optimal insider strategies, which are analysed. Furthermore, it is proved that the presence of noise drift allows equilibria for a risk averse insider beyond the framework of a linear pricing rule, which is typically a necessary condition for the existence of an equilibrium in other models.

In a second part, shot noise is incorporated in the analysis. In order to solve the market makers' inference problem, non-standard techniques for stochastic filtering are developed. The resulting equilibrium pricing rule exhibits a far more complex structure compared to the standard case, including a non-deterministic price pressure.

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Introduction

In financial markets, new and unexpected information often leads to abrupt changes in asset prices. Uncertainty, irrational behaviour, or wrong evaluation of this information may cause inappropriate adjustments of prices. This arises over- or underpricing with respect to the fundamental value of the particular asset. After some time the effect may fade away while the market comes back to more rational levels. Over the long run, this lets prices converge to their fundamental level. A rationally acting market participant should be able to take profit out of the resulting situation.

If all agents had full information and acted rationally, the market price should match the fundamental value of the risky asset according to the *efficient market hypothesis*. If this is not the case, one is facing a situation of optimal trading under asymmetric information, which is often referred to as *insider trading*. Such models have been widely discussed in financial literature, see e.g. [55], [6], [40] and [13]. But all these models have in common that prices are fixed exogenously and are not affected by the insider or only affected in the framework of a special structure of the price dynamics (cf. [29], [45]). In the latter case, the insider is often identified as *large investor*, i.e. an agent whose influence on the market is big enough to manipulate the price process dynamics by his decisions.

As Danilova [28, p. 2] points out, the assumptions of these models have two shortcomings from the market microstructure point of view:

> i) imposing strong efficiency of the markets even without an insider providing, through her trading, information to the market – that is, assuming a priori that the price will converge to the fundamental value
> ii) the less informed agents are not fully rational, since they do not try to infer the insider's private signal from market data (since there is no feedback from insider trading to equilibrium price).

However, price formation under asymmetric information considering the informational content of stock prices, and strategic insider trading, in an equilibrium framework, is discussed by dynamic equilibrium models, which were introduced by the seminal work of Kyle [46] and elaborated by Back [7], see also [25], [28], [27], [19] and further references in Section 1.2. In these models, the market price is a functional of the total order flow. More detailed, the original setup of these models considers a market with two assets: one risky asset and one bank account with interest rate zero. During a trading period $[0, T]$ the price P of the asset does not necessarily match its fundamental value V. This changes in T by a public announcement of the fundamental value. Thus, in general we have $P_t \neq V$, for $t < T$ and $P_t = V$, for $t \geq T$.

There are three different types of market participants: an insider, market makers, and uninformed traders (noise traders) who trade randomly. It is assumed that the informational advantage of the insider consists in prior knowledge of the exogenous variable V that is published to the market at time T. The order flow of the other market participants is independent of this variable. The market makers set the price and clear the market according to a pricing rule

$$P_t = H(t, Y_{[0,t]})$$

that depends on the observed path of the total order flow

$$Y_t = \theta_t + X_t$$

where θ denotes the trading strategy of the insider and X the cumulated demand of the noise traders. Indeed, the noise traders provide camouflage enabling the insider to hide his informed trading and to earn profits. The insider now faces the problem to maximise the expected utility of her profit, while the market makers try to infer the insider's information from the total order flow. Due to the dependence of the order strategy and the market price, the insider is revealing her information by trading. The pivotal question is whether there exists an equilibrium, i.e. a pair (θ, H), where θ is an optimal strategy given H, and H a rational pricing rule given θ. It turns out that an equilibrium exists and that the fundamental value is fully revealed right before the end of the trading period.

Holden and Subrahmanyam [39], Foster and Viswanathan [36] and Back et al. [9] introduce multiple insiders into this kind of equilibrium model. In Holden and Subrahmanyam [39] all insiders receive the same private signal. This competitive situation leads to agressive insider trading. As a consequence, their information is revealed immediately if the interval between auctions approaches zero (continuous trading). On the other hand, a *waiting game effect* caused by imperfectly correlated signals (the insiders try to infer each other's signals), considered in Foster and Viswanathan [36] and Back et al. [9], slows down the revelation of information.

All equilibrium models introduced above assume that there exists a group of market participants, called *noise traders*, which act irrationally in the sense that they are completely insensitive with respect to the price and the uncertainty over the fundamental value. Typically their order flow X is modelled by a Brownian motion in the continuous time case. The existence of such irrationally behaving noise traders can be explained by concepts of behavioural finance (cf. Barberis and Thaler [11] for a review, see also Dow and Gorton [30] for further treatment of noise trading). Nevertheless, the assumption of complete insensitivity with respect to the market price of the asset and the uncertainty over its fundamental value seems arbitrarily, justifiable only by analytic tractability (cf. [50], p. 360). In [59] and [50] a model with strategic noise traders who receive random endowment shocks is analysed in a one and two trading periods framework.

However, in the Kyle-Back model the presence of irrational agents is crucial since in a market populated only with rational participants the *no trade theorem* (cf. Brunnermeier [17], Chapter 2) states that it is impossible to profit from superior information by trading. The attendance of irrational traders leads to a *semi-strong efficient market*, i.e. a market that correctly reflects public, but not necessarily private information and where the trading on the basis of private information makes the latter public (cf. Fama [34]).

A second important assumption regards the public announcement of the fundamental value at some future date. If this was not about to happen, the insider could not benefit from dissolving her clandestinely aggregated portfolio since selling/buying again would lower/rise the market price.

Coming back to our initial example, we have to acknowledge that the efficiency (the price was assumed to return to its fundamental level at some time T) is not compatible with the existence of noise, on the one hand, and the absence of some announcement of the fundamental value, on the other hand. If we discard the assumption of totally irrational traders in the sense of the above discussed equilibria models, and replace these by traders who *become rational and aware of the fundamental value* (in a collective sense) as time passes by, indeed, this should result in a strong form efficient market because according to Fama [34], p. 388, "sufficient numbers of investors" have access to the privileged information.

Becoming rational should by no means happen suddenly since this again could only be justified by some exogenous event. Therefore, the order flow of the noise traders already should depend on the fundamental value before time T. Partially informed noise traders who may be "more rational than in the standard model" were already considered by Aase et al. [2] and modelled by a demand that is correlated to the fundamental value. However, the model in [2] only considers a one period trading framework.

In this thesis, we want to analyse the above situation in a continuous time setting. The order flow of the noise trader is described by

$$\mathrm{d}X_t = \mu(t, V, P_t)\,\mathrm{d}t + \sigma\,\mathrm{d}B_t, \quad t < T,$$

where B is a Brownian motion and μ such that $\operatorname{sgn}(\mu(t, V, P_t)) = \operatorname{sgn}(V - P_t)$. We decompose the trading into a noise part (uninformed trading), represented by the $\mathrm{d}B_t$ term, and an informed trading part, represented by the drift term. Since the informed part responds to the mispricing in a positive way, but is not necessarily rational (in a utility maximising sense), we call this kind of behaviour *semi-rational*. Hence, this model describes a situation that is located between the classical Kyle-Back model (one insider and uninformed noise traders, cf. [46], [7]) and the model of Holden and Subrahmanyam [39] (multiple insiders with homogeneous information). While the first model provides an equilibrium, this fails to hold true in the latter case if trading happens continuously. The pivotal question is, whether or not the insider accepts the competing noise drift such that there exists an equilibrium.

Indeed, we will point out a situation where an equilibrium is possible. The noise drift then is represented by a fixed functional depending on the total order and the fundamental value, and it is scaled by an arbitrary deterministic coefficient function called *noise drift intensity*. Letting this function go to infinity while approaching the trading horizon results in a model where the market is efficient even in absence of an insider. This provides the intuition of a market getting more rational or, conversely, a market getting less irrational.

In our initial example, uncertainty or irrationality arises from the sudden arrival of unexpected information. In a framework as described above, this raises the question if further incoming information could stop the decay of irrational behaviour or even reinforce it. In the first instance, new and relevant information should have an impact on the fundamental value of the traded asset. Besides some influence on the informed trading part, a *reinforcement of irrational behaviour* could become noticeable by the following two effects: Firstly, the period of time that is needed by the market to cool down could be prolonged. Secondly, further over- or underreactions are brought to the market by a sudden adjustment of the noise traders' portfolios.

Mathematically more detailed, this situation can be described by a fundamental value that is driven by a Poisson process N representing the arrival times of new information, i.e.

$$V_t = \sum_{i=0}^{N_t} V^{(i)},$$

and a demand process of the noise traders with dynamics

$$\mathrm{d}X_t = \mu(t, V_t, P_t)\mathrm{d}t + \sigma \mathrm{d}B_t + \mathrm{d}X_t^d$$

where X^d is some jump process that is correlated to V. The trading horizon T which is associated to the time when the market has cooled down completely now is a stopping time that depends on N, too.

Besides the usual, continuous noise, represented by a Brownian motion, the resulting model incorporates a second noise component. Due to its instantaneous appearance it may be called *shot noise*. On the other hand, we have a *noise compensating* drift term which can be identified with the decay of irrational behaviour. In the world of exogenously fixed price dynamics such a situation could be described by so-called *shot noise models* as in [58] or [5]. However, this thesis studies the process of price formation and its underlying equilibrium in the above motivated models. In order to do so we will proceed as follows:

In the first chapter, a definition of order based insider trading models in continuous time is given and the central concepts and problems are explained. This is done in a general way so that the above described extensions are covered by this framework. For an accurate classification of the proposed setting, a short review of existing literature regarding this type of equilibrium model is given. Significant extensions with respect to the standard model are described and the most important results are summarised.

As described above, the market makers have to infer the insider's signal. Mathematically this constitutes a problem of stochastic filtering. In general, such problems cannot be solved explicitly since this leads to an infinite dimensional system of SDE according to the *Kushner-Stratonovich equation* (cf. [10] Theorem 3.30, [48] Theorem 8.1). In the case of conditionally Gaussian signals, a closed system can be determined by taking advantage of a special relationship between conditional moments of higher order (cf. [49]). Typically, equilibrium models rely on the feature of a closed-form solution of the involved filter problem. In Chapter 2, we therefore reproduce these crucial results. Since they only incorporate Gaussian noise, non-standard techniques have to be developed for the case of additional shot noise. Presuming conditionally Gaussian shots, a finite dimensional filter for this jump diffusion problem can be derived from the (continuous) diffusion case.

In Chapter 3, we start the actual analysis of the model. As a first step, we only consider the extension regarding the drift term, called *noise drift*, and study the model up to a fixed horizon T, with a constant fundamental value. Both a risk neutral and a risk averse insider are allowed. Initially, a rigorous definition of the model is given. This also includes the fixing of a class of admissible pricing rules. We restrict the analysis to the case where the price is determined by a function depending on time and the path of the total order flow which is weighted according to a deterministic function, called *price pressure*. In the second section, we point out necessary conditions for the existence of an equilibrium under the assumption of absolutely continuous insider trading. This is done with the help of a *Hamilton-Jacobi-Bellman equation* approach. The derived conditions are related to the form of the noise drift. It turns out that dependences on the fundamental value and the market price have to exhibit a special structure that is scaled by some arbitrary, positive, deterministic coefficient function, also called *drift intensity*. We show that the price pressure has to be inversely related to the integrated drift intensity and thus decreasing, in particular. Primarily, this yields path dependent pricing rules. Furthermore, this leads to the interpretation that informed trading of the noise traders is accepted by the insider because the market gets deeper and hence future trades more profitable. Moreover, we can identify the case of infinite drift intensity with an infinitely deep market. In this case, we prove that the market is efficient by itself, i.e. the market price converges to the fundamental value even without insider trading. The latter places additional demands on the optimality of an insider strategy, which is studied in the third section. In the case of a bounded drift intensity, it is proved that an insider strategy is optimal if it is absolutely continuous and drives the market price to the asset's fundamental value at the end of the trading period. This corresponds to the results of the standard model and other extensions. If the drift intensity is unbounded, the above criteria obviously are not sufficient since the market is efficient by itself. To solve the resulting optimisation problem, new techniques have to be developed. It turns out that the extended optimality criterion relates to the convergence speed of the market price measured by the vanishing price pressure. Rationality of the pricing rule is analysed in the following section. In Section 3.6, the results of the preceding sections are brought together to state sufficient conditions for an equilibrium. With the help of the elaborated criteria it is possible to specify the explicit form of an equilibrium and prove its existence. The analysis is done for both a risk neutral insider as well as a risk averse insider. We show that, in contrast to other insider equilibrium models, the presence of noise drift also permits equilibria for non-linear price functions in the risk averse case.

Additionally incorporating the second extension, i.e. the reinforcing part, is the main task of Chapter 4. For the analysis of this model, we basically use methods analogous to those used in Chapter 3. Due to the far more complex structure induced by the jump parts, we restrict the setup to a bounded drift intensity and a risk neutral insider. We start with an heuristic transfer of the results derived in Chapter 3. This naive approach justifies further assumptions made in the rigorous definition of the model given thereafter, as for example a more complex structure of the pricing function or the different weighting of continuous and discontinuous changes in the total order flow. In the following analysis, it turns out that, in general, the price pressure (of the continuous changes) has to allow stochastic dependence. Again, optimality and rationality are studied and finally sufficient conditions for an equilibrium and its form are characterised.

Acknowledgement

I would like to express my gratitude to my supervisor PD Dr. Volkert Paulsen for the trust he placed in me by giving me the opportunity of research on this topic, and for instructive and fruitful discussions throughout this project. Moreover, my sincere thanks go to Prof. Dr. Matthias Löwe, from whose funds my employment was payed. I am also much obliged to all members of the Institut für Mathematische Statistik at the WWU Münster for a good working atmosphere, and friends and family for providing the necessary release from daily routine. Special thanks go to Anna-Lena Römisch for her continual support, especially during the last few month.

Notation

Filtered probability space and stochastic processes

All random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ satisfies the *usual hypothesis* (cf. [42], p. 5), i.e. $\mathbb{F}$ is right continuous and $\mathcal{F}_0$ contains all $(\mathbb{P}, \mathcal{F})$-negligible sets $\mathfrak{N}$. For any adapted stochastic càdlàg processes X we denote by

$$X_{t-} = \lim_{s \nearrow t} X_s$$

the left limit of X in t. A jump at time t is written as

$$\Delta X_t = X_t - X_{t-}.$$

For a semimartingale X with $\sum_{s\leq t} |\Delta X_s| < \infty$, for all $t \geq 0$, the process

$$X_t^c = X_t - \sum_{s\leq t} \Delta X_s, \quad t \geq 0,$$

is a continuous semimartingale. The discontinuous part of X, $(\sum_{s\leq t} \Delta X_s = X_t - X_t^c,\, t \geq 0)$, is denoted by X^d. In particular, $X_t = X_t^c + X_t^d$.

If not stated differently, for a given stochastic process X we will denote by $\mathbb{F}^X := (\mathcal{F}_t^X)_{t\geq 0}$ the completed filtration generated by the process X, i.e. for all $t \geq 0$

$$\mathcal{F}_t^X = \sigma(\{X_s : 0 \leq s \leq t\}) \cup \mathfrak{N}.$$

General notations, functions and functional spaces

For any real numbers x and y: $x \wedge y = \min(x, y)$, $x \vee y = \max(x, y)$.
For $x \in \mathbb{R}$ the signum function is defined by

$$\operatorname{sgn}(x) = \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0 \end{cases}.$$

For an integrable function $f : D \to \mathbb{R}$, $D \subset \mathbb{R}$ and $a, b \in D$, $a < b$, we write

$$\int_b^a f(x)\,\mathrm{d}x = -\int_a^b f(x)\,\mathrm{d}x.$$

For $k \in \mathbb{N}$, $\mathcal{C}^k(\mathcal{D})$ is the space of all real-valued continuous functions f on $\mathcal{D} \subset \mathbb{R}^n$ with continuous derivatives up to order k. $\mathcal{C}^0(\mathcal{D})$ is the space of continuous functions f on $\mathcal{D}$. For a given $f \in \mathcal{C}^2(\mathcal{D})$ the partial derivatives $\frac{\partial f}{\partial x_i}$, $\frac{\partial^2 f}{\partial x_i \partial x_j}$, $1 \leq i, j \leq n$, are also denoted by $\partial_{x_i} f$, $\partial_{x_i x_j} f$. An l-dimensional function $f = (f^{(1)}, \ldots, f^{(l)}) \in \mathcal{C}_l^0([0, T])$ if $f^{(i)} \in \mathcal{C}^0([0, T])$, for all $1 \leq i \leq l$.

Chapter 1

Strategic order based insider trading models in continuous time

This chapter is devoted to a basic description of the framework of strategic insider trading models presented in the introduction. This includes the main definitions and central problems that are discussed in such models. The second section gives a short overview of the existing literature regarding this type of equilibrium models.

1.1 Strategic order based insider trading: an introduction

For a detailed description recall that all random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. We consider a market with two assets, one risky asset and one bank account with interest rate equal to zero. During a trading period $[0, T)$ the *market price* P of the risky asset does not necessarily match its *fundamental value* V. V itself is a stochastic process, i.e. non-constant in general. The difference between market price on the one hand, and fundamental value on the other is caused by an informational asymmetry that is cleared at a (random) time T. This might be due to an announcement or some other event, as motivated in the introduction, which ensures completely homogeneous expectation of all market participants with respect to the value of the asset thereafter. It follows that from time T on, the market price P matches the fundamental value V, i.e. $P_t = V_t$, for all $t \geq T$.

Market participants

There are three different types of market participants: market makers, an insider, and noise traders.

- *Market makers:* The market makers are not able to observe the value process V but the *total order process* Y of the risky asset. In particular, they cannot distinguish between informed and uninformed trades but are aware of the presence of informed trading (for a model that incorporates uncertainty of informed trading we refer to [47]). Since they are risk neutral and competitive, they set the price and clear the market according to a *pricing* *rule*

$$P_t = \mathbb{E}\left(V_T \middle| \mathcal{F}_t^{\mathcal{M}}\right)$$

based on their observation $\mathbb{F}^{\mathcal{M}} = (\mathcal{F}_t^{\mathcal{M}})_{t\geq 0}$ where $\mathcal{F}_t^{\mathcal{M}} \supset \mathcal{F}_t^Y$ (typically it is assumed that $\mathbb{F}^{\mathcal{M}} = \mathbb{F}^Y$).

- *Insider:* In contrast to the market makers, the insider possesses privileged information that is the knowledge of the fundamental value process V. Additionally, she observes the market price process P. Her information can therefore be identified with the filtration $\mathbb{F}^{\mathcal{I}} := (\mathcal{F}_t^{\mathcal{I}})_{t\geq 0}$ where $\mathcal{F}_t^I$ contains the completion of $\sigma(\{P_s, V_s, T \wedge s, 0 \leq s \leq t\})$. The insider's strategy is represented by an $\mathbb{F}^{\mathcal{I}}$-adapted process θ.
- *Noise traders:* The noise traders do not act strategically. Their cumulated order flow, denoted by X, contains a random part (also called noise). The total order process Y, which is observed by the market makers, can thus be represented by $Y = X + \theta$.

The insider's wealth

As pointed out by Back [7] for a model with constant trading horizon or by Corcuera et al. [27] for the general case of a stopping time T, the insider's terminal wealth W_T^θ, corresponding to a certain strategy θ, can be calculated as follows:

First consider a discrete model with trading times $0 \leq t_1 < t_2 < \cdots < t_N = T$, $N \in \mathbb{N}$, where N might be random. Purchasing $\theta_{t_i} - \theta_{t_{i-1}}$ units of the financial asset at time t_i costs

$$P_{t_i}(\theta_{t_i} - \theta_{t_{i-1}})$$

since the orders are executed at the new price. After all trading periods this yields the total cost

$$\sum_{i=1}^N P_{t_i}(\theta_{t_i} - \theta_{t_{i-1}}).$$

Now, if the market price converges to the fundamental value right after t_N, this leads to an extra income $\theta_{t_N} V_{t_N}$. It follows

$$\begin{aligned} W_{N+}^\theta &= -\sum_{i=1}^N P_{t_i}(\theta_{t_i} - \theta_{t_{i-1}}) + \theta_{t_N} V_{t_N} \\ &= -\sum_{i=1}^N P_{t_{i-1}}(\theta_{t_i} - \theta_{t_{i-1}}) - \sum_{i=1}^N (P_{t_i} - P_{t_{i-1}})(\theta_{t_i} - \theta_{t_{i-1}}) + \theta_{t_N} V_{t_N}. \end{aligned}$$

Analogously, in continuous time,

$$W_{T+}^\theta = \theta_T V_T - \int_0^T P_{t-}\, \mathrm{d}\theta_t - [P, \theta]_T\,. \tag{1.1}$$

If we want to understand the above integral as Itô integral, we have to assume that θ as well as P are $\mathbb{F}^{\mathcal{I}}$-semimartingales. Observe that integration by parts (cf. [56], p. 68) would lead to

$$W^\theta_{T+} = \theta_T(V_T - P_T) + \int_0^T \theta_{t-}\,\mathrm{d}P_t.$$

P is an $\mathbb{F}^{\mathcal{M}}$-adapted process, but θ is not. Hence, $\int_0^T \theta_{t-}\,\mathrm{d}P_t$ is not well-defined as an Itô integral, in general. As pointed out by Aase et al. [1], the integral w.r.t. P could be understood as *forward integral*. However, in this work, P always is an $\mathbb{F}^{\mathcal{I}}$-semimartingale.

In the above argumentation, we used the fact that the convergence of the market price to the fundamental value happens right after T but not in T, i.e. we have $P_T \neq V_T$ in general, but $P_t = V_t$ for all $t > T$. If we assume that already $P_T = V_T$ holds, we get by an analogous argumentation in the discrete case

$$\begin{aligned} W^\theta_N &= -\sum_{i=1}^{N} P_{t_i}(\theta_{t_i} - \theta_{t_{i-1}}) + \theta_{t_N} V_{t_N} \\ &= -\sum_{i=1}^{N-1} P_{t_i}(\theta_{t_i} - \theta_{t_{i-1}}) - P_{t_N}(\theta_{t_N} - \theta_{t_{N-1}}) + \theta_{t_N} V_{t_N} \\ &= -\sum_{i=1}^{N-1} P_{t_i}(\theta_{t_i} - \theta_{t_{i-1}}) - V_{t_N}(\theta_{t_N} - \theta_{t_{N-1}}) + \theta_{t_N} V_{t_N} \\ &= -\sum_{i=1}^{N-1} P_{t_i}(\theta_{t_i} - \theta_{t_{i-1}}) + V_{t_N}\theta_{t_{N-1}}. \end{aligned}$$

In the continuous case, this corresponds to

$$W^\theta_T = \theta_{T-} V_T - \int_0^{T-} P_{t-}\,\mathrm{d}\theta_t - [P,\theta]_{T-}\,. \tag{1.2}$$

If θ is continuous, there indeed is no difference between the representation of the final wealth in (1.1) and (1.2). For reasons that will be explained later on in Chapter 3, we will always consider the latter case, i.e. (1.2).

Optimal insider strategies

The insider is assumed to be rational and utility maximising. Let W^θ_T denote her final wealth as calculated in (1.2), $\mathcal{U}$ be a fixed *utility function*, i.e. a strictly increasing function, and $\mathcal{S}$ the set of all *admissible* trading strategies. Then utility maximising means that the insider tries to solve the following optimisation problem:

$$\sup_{\theta\in\mathcal{S}} \mathbb{E}\left(\mathcal{U}(W^\theta_T)\right). \tag{1.3}$$

As pointed out above, $\mathcal{S}$ has to be a subset of all $\mathbb{F}^{\mathcal{I}}$-semimartingales. Depending on the particular model, further technical assumption are made to ensure the tractability of the model.

Definition 1.1. An admissible trading strategy $\theta \in \mathcal{S}$ is called *optimal* (in $\mathcal{S}$) if it solves the optimisation problem stated in (1.3).

Furthermore, we note that an optimal strategy in $\mathcal{S}_1$ might not be optimal in $\mathcal{S}_2$ if $\mathcal{S}_1 \subset \mathcal{S}_2$. For example, if θ is restricted to be absolutely continuous, i.e.

$$\theta_t = \theta_0 + \int_0^t \alpha_s \,\mathrm{d}s, \quad t \geq 0,$$

for some suitable integrable process α, then θ might not be optimal in a set of admissible strategies that allows quadratic variation or discontinuity. However, sometimes even optimality in the larger class can be proved, e.g. [7], [28], [26].

Admissible and rational pricing rules

Analogous to the insider strategies, the set of possible pricing rules is restricted to a certain class $\mathcal{P}$, called *admissible pricing rules.* Again the definition of admissible pricing rules gathers technical conditions that ensure the tractability of a certain model. Typically it is assumed that prices follow

$$P_{t\wedge T} = H(t \wedge T, \widetilde{Y}_{t\wedge T})$$

where $\widetilde{Y}$ is the weighted total order process, i.e.

$$\widetilde{Y}_t = \int_0^t \lambda_s \,\mathrm{d}Y_s$$

for some positive and $\mathbb{F}^{\mathcal{M}}$-adapted process λ, called *price pressure*, and H, called *pricing function*, is a sufficiently smooth function such that $H(t,\cdot)$ is strictly increasing for all t. In this case, the pair (H, λ) is also called pricing rule. Furthermore, a *rational pricing rule* is defined as follows:

Definition 1.2. An admissible pricing rule $P \in \mathcal{P}$ is called rational if

$$P_t = \mathbb{E}\left(V_T \middle| \mathcal{F}_t^{\mathcal{M}}\right), \quad \text{for all } t \geq 0. \tag{1.4}$$

Equilibrium

After introducing optimal insider strategies and rational pricing rules, we can state the following: The market price depends on the total order via a given pricing rule. Since the total order contains the informed trading of the insider, too, the price is influenced by the insider strategy. Conversely, the terminal wealth generated by a certain insider strategy depends on the market price. Hence, optimality of an insider strategy depends on the particular form of the pricing rule, and rationality of a pricing rule depends on the insider strategy. The central question is, whether there exists a pair comprising a pricing rule and an insider strategy that are rational and optimal respectively.

Definition 1.3. A pair $(\theta, P) \in \mathcal{S} \times \mathcal{P}$ is called *equilibrium* (in $(\mathcal{S}, \mathcal{P})$) if the following holds:

- P given θ is rational,
- θ given P is optimal.

1.2 A short review

This section gives a short summery about existing literature and models that follow the general framework presented in the last section. We start with the so-called *Kyle-Back model*, which is based on the seminal paper of Kyle [46] and was elaborated by Back [7]. Thereafter, different extensions of this standard model are presented and classified. For a further survey on equilibrium models under asymmetric information, including the Kyle-Back model and its extensions, we refer to [33].

1.2.1 The Kyle-Back model

The model developed by Kyle [46] and elaborated by Back [7] (cf. also [25]) can be summarised as follows: The order flow of the noise traders follows a Brownian motion B with constant volatility σ, i.e.

$$\mathrm{d}X_t = \sigma \mathrm{d}B_t.$$

The (constant) fundamental value V of the risky asset is to be published to the market at time 1. Furthermore, it is assumed that V can be described by $V = h(Z)$ where $Z \sim \mathcal{N}(0, 1)$. The price P_t is based on the total order up to time t

$$P_t = H(t, Y_t)$$

where H is required to be sufficiently smooth and $H(t, \cdot)$ is strictly increasing for each $t \in [0, 1]$. The insider is risk neutral, i.e. $\mathcal{U}(W) = W$. It turns out that an equilibrium can be achieved if

- H satisfies the PDE
$$\partial_t H + \frac{\sigma^2}{2} \partial_{yy} H = 0, \tag{1.5}$$
- θ is absolutely continuous and Y is an $\mathbb{F}^{\mathcal{M}}$-Brownian motion such that $H(1, Y_1) = V$ a.s.

For $\sigma = 1$, this is the case if

$$H(t,y) = \mathbb{E}h(y + X_1 - X_t)$$

and

$$\theta_t = (1-t)\int_0^t \frac{h^{-1}(V) - X_s}{(1-s)^2}\,\mathrm{d}s$$

(cf. [7], Theorem 1). Cho [25] considers a larger class of possible pricing functions

$$H(t, \int_0^t \lambda(s)\,\mathrm{d}Y_s) \tag{1.6}$$

for a positive, smooth function λ, called price pressure. However, in the risk neutral case it turns out that λ has to be constant.

1.2.2 The risk averse case

Cho [25] and Baruch [12] studied the Kyle-Back model under the presence of a risk averse insider. More detailed, it is assumed that the utility function $\mathcal{U}$ has an exponential structure

$$\mathcal{U}(W) = \beta \exp(\beta W), \quad \beta < 0.$$

It turns out that an equilibrium only exists in a linear framework, i.e. if the price is given by

$$P_t = H(t, \int_0^t \lambda(s)\,\mathrm{d}Y_s) = p_0 + q\int_0^t \lambda(s)\,\mathrm{d}Y_s.$$

This in turn requires V being Gaussian (cf. [25], Proposition 3 and Lemma 8). The equilibrium price pressure λ follows the dynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda(t)^{-1}\right) = -\beta\sigma^2\partial_y H(t,y).$$

In particular, λ is decreasing (recall that $\partial_y H > 0$). Hence, intuitively speaking, the insider's risk that the noise trading might move the price towards the asset value is compensated by a decreasing price pressure which makes later trades more favorable.

1.2.3 Imperfect dynamic information

Danilova [28] (see also [60] or [8] for additional time varying noise volatility) relaxes the strong assumption regarding the insider's knowledge of the fundamental value at the beginning. In contrast to a constant signal the privileged information is given by the conditional expectation $f(t, Z_t)$ of $V = h(Z_1)$ that can be expressed in terms of a sufficient statistics Z_t of the insider's information at time t. Here Z evolves according to

$$\mathrm{d}Z_t = \sigma_z(t)\mathrm{d}B_t^z$$

where B^z is a Brownian motion independent of the one that drives the noise traders' demand. Under certain assumptions on the volatility σ_z an equilibrium can be calculated. These assumptions ensure that the insider's signal is always at least as precise as the one of the market makers and even more precise close to the terminal time (cf. [28], Assumptions 2.2, 3.1, 3.2). For an equilibrium the same characterisation as in the Kyle-Back model holds. The equilibrium insider strategy is given by (cf. [28], Theorem 3.1)

$$\theta_t = \int_0^t \frac{Z_s - Y_s}{\int_0^s \sigma_z(u)^2 \,\mathrm{d}u - s + \sigma^2}\,\mathrm{d}s$$

where σ is the volatility of noise trading.

Campi et al. [20] generalise the above setup to a non-Gaussian model where

$$\mathrm{d}Z_t = \sigma_z(t)a(\Sigma(t), Z_t)\mathrm{d}B_t^z, \quad \Sigma(t) = c + \int_0^t \sigma_z(s)\,\mathrm{d}s.$$

1.2.4 Alternative noise dynamics

While the extensions in 1.2.2 and 1.2.3 mainly concerned the insider's point of view, extensions regarding the noise traders' order flow also have been discussed.

Fractional Brownian motion

Biagini et al. [15] replace the Brownian motion B in the noise trader dynamics by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$, i.e. the increments are positively correlated (cf. [14] for details on fractional Brownian motions). Heuristically this means that the noise traders have some memory. For the class of insider trading strategies of the form

$$\mathrm{d}\theta_t = (V - P_t)\alpha_t \mathrm{d}t$$

the optimal trading intensity α_t can only be calculated implicitly, but again is proved to ensure $P_T = V$ (cf. [15], Theorem 2.4).

Lévy noise

For the case of a constant signal V as in the Kyle-Back model, Corcuera et al. [26] consider general noise dynamics of the form

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t + \mathrm{d}L_t$$

where μ and σ are deterministic functions and L is a pure jump Lévy process independent of V and B. Given that admissible pricing rules take the form as in (1.6), it turns out that in the presence of a risk neutral insider there exists an equilibrium if and only if $L \equiv 0$ (cf. [26], Theorem 12 and Proposition 13). In contrast to the standard model PDE (1.5), the equilibrium pricing rule H satisfies the PDE

$$\partial_t H + \lambda_t \mu_t \partial_y H + \frac{\sigma_t^2 \lambda_t^2}{2} \partial_{yy} H = 0. \tag{1.7}$$

The additional term $\lambda_t \mu_t \partial_y H$ is due to the deterministic drift that is contained in the order flow of the noise traders.

1.2.5 Random time horizon

Caldentey and Stacchetti [18] assume that the announcement of the fundamental value $V_t = \sigma_v B_t^v$ (B^v Brownian motion) happens randomly at some time T that is exponentially distributed with scale parameter μ. In this case, the insider's expected payoff from time s onward is given by

$$W_T^\theta = \mathbb{E}\left(\int_s^\infty e^{-\mu(t-s)} (V_t - P_t) \, \mathrm{d}\theta_t + \int_s^\infty e^{-\mu(t-s)} \, \mathrm{d}\, [\theta, V - P]_t \right).$$

Despite the fact that T is not predictable for the insider, an (linear) equilibrium can be calculated (cf. [18], Theorem 3). Heuristically, the insider's risk of not having used all available information up to time T is compensated by a decreasing price pressure.

1.2.6 Defaultable bonds

Campi and Çetin [19] apply the Kyle-Back model to the case of a defaultable bond (that pays 1 unit of a currency at time 1), whose default time is known to the insider at time 0 and given by

$$\tau := \inf\{t > 0 : B_t = -1\}$$

where B is a Brownian motion. Similar to the Kyle-Back model it turns out that an equilibrium (H, θ) can be characterised by H verifying PDE (1.5) and Y being a Brownian motion w.r.t. $\mathbb{F}^{\mathcal{M}}$ such that $\lim_{t \to 1} H(\tau \wedge t, Y_{\tau \wedge t}) = \mathbb{1}_{[\tau > 1]}$ (cf. [19], Lemma 3.4). By assumption, the default time is totally inaccessible to the market. However, in equilibrium the default time gets predictable by the presence of a risk neutral insider revealing her information in order to maximise her expected profit. In [22] and [21] the assumption regarding the insider's information is relaxed.

Chapter 2

Preliminary results on stochastic filtering

In this chapter, we present some preliminary results that are crucial for the analysis of insider trading models. The first section describes common facts of stochastic filtering theory for conditionally Gaussian diffusions. These are taken from Liptser and Shiryaev [49]. In the second section, the results will be extended to a special case of conditionally Gaussian jump processes.

As we have seen in the introduction to order based insider trading models (cf. Section 1.1), the market makers are facing the problem to infer $V_t = h(Z_t)$ from the observation of the total order process

$$Y_t = \theta_t + X_t, \quad t \geq 0,$$

i.e. to derive $\mathbb{E}(h(Z_t)|\mathcal{F}_t^{\mathcal{M}})$. Mathematically this leads to a so-called *stochastic filtering problem* where the unobservable process Z is referred to as *signal process* and Y as *observation process*. If Z and Y are diffusion type processes, the filter problem is solved by the so-called *Kushner-Stratonovich equation* (cf. [10], Theorem 3.30, [48], Theorem 8.1). In the special case when Z is constant and Y defined as solution of

$$\mathrm{d}Y_t = a(t, Z)\mathrm{d}t + \mathrm{d}W_t$$

where W is a Brownian motion independent of Z and a some well-behaved function, this equation reads as

$$\pi_t(h(Z)) = \pi_0(h(Z)) + \int_0^t \pi_s(h(Z)a(s, Z)) - \pi_s(h(Z))\pi_s(a(s, Z))\,(\mathrm{d}Y_s - \pi_s(a(s, Z))\mathrm{d}s) \quad (2.1)$$

where $\pi_t(h(Z)) := \mathbb{E}(h(Z)|\mathcal{F}_t^Y)$. Similar results can be calculated for the case when Y and Z are jump diffusions, as done in [24], see also [37], [23]. Roughly speaking, the incoming observation $\mathrm{d}Y_t$ at t can be split into a *predicted* part $\pi_t(a(t, Z))\mathrm{d}t$ and an additional part $\mathrm{d}Y_t - \pi_t(a(t, Z)\mathrm{d}t$ containing new information that is independent of the current knowledge. Mathematically more detailed, the so-called *information process*

$$Y_t - \int_0^t \pi_s(a(s, Z))\,\mathrm{d}s, \quad t \geq 0,$$

is a Brownian motion w.r.t. the observation filtration $\mathbb{F}^Y$ (cf. [10], Proposition 2.30). Equation (2.1) can then be understood as proceeding update of the estimator of $h(Z)$ according to the incoming new information.

However, even in the special case stated above, in order to determine $\pi_t(h(Z))$ one has to know $\pi_t(h(Z)a(t,Z))$. For $\pi_t(h(Z)a(t,Z))$ in turn one needs $\pi_t(h(Z)a(t,Z)^2)$ and so on. Hence, a general solution can only be obtained by an infinite dimensional system of stochastic differential equations. For a closed form solution one has to search for an additional relation of the involved conditional moments. In the case of Gaussian or conditionally Gaussian processes

$$\pi_t(Z^3) = 3\pi_t(Z)\pi_t(Z^2) - 2(\pi_t(Z))^3$$

holds true. This leads to a closed system for $\pi_t(Z)$ and $\gamma_t(Z) = \pi_t(Z^2) - (\pi_t(Z))^2$. For continuous processes this has been pointed out in [49], Chapter 12.

Since this result will turn out to be crucial for the solution of an equilibrium in an insider trading model, we will reproduce it in the following section. As an application we will furthermore state an extended result for the case of conditionally Gaussian jump diffusions, which will be used in Chapter 4.

2.1 General results on optimal filtering for conditionally Gaussian diffusions

Theorem 2.1 (cf. [49], Theorem 12.6, p. 31). *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with right continuous filtration $(\mathcal{F}_t)_{0\le t\le T}$ and $W^{(1)}$ and $W^{(2)}$ be two mutually independent k and l-dimensional Brownian motions. Furthermore, let $Z = (Z_t^{(1)}, \dots, Z_t^{(k)}), Y = (Y_t^{(1)}, \dots, Y_t^{(l)})$, $0 \le t \le T$, be solutions to the following stochastic differential equations*

$$\mathrm{d}Z_t = \ (a_0(t,Y) + a_1(t,Y)Z_t)\ \mathrm{d}t + \sum_{i=1}^{2} b_i(t,Y)\,\mathrm{d}W_t^{(i)}, \tag{2.2}$$

$$\mathrm{d}Y_t = \ (A_0(t,Y) + A_1(t,Y)Z_t)\ \mathrm{d}t + \sum_{i=1}^{2} B_i(t,Y)\,\mathrm{d}W_t^{(i)} \tag{2.3}$$

where the elements of the vector functions

$$a_0(t,x) = (a_{01}(t,x), \dots, a_{0k}(t,x)), \quad A_0(t,x) = (A_{01}(t,x), \dots, A_{0l}(t,x))$$

and matrices

$$\begin{aligned} a_1(t,x) &= ||a_{ij}^{(1)}(t,x)||_{(k\times k)}, & A_1(t,x) &= ||A_{ij}^{(1)}(t,x)||_{(l\times k)}, \\ b_1(t,x) &= ||b_{ij}^{(1)}(t,x)||_{(k\times k)}, & b_2(t,x) &= ||b_{ij}^{(2)}(t,x)||_{(k\times l)}, \\ B_1(t,x) &= ||B_{ij}^{(1)}(t,x)||_{(l\times k)}, & B_2(t,x) &= ||B_{ij}^{(2)}(t,x)||_{(l\times l)} \end{aligned}$$

are assumed to be measurable non-anticipative functionals on

$$\left\{[0,T] \times \mathcal{C}_l^0([0,T]), \mathcal{B}([0,T]) \times \mathcal{B}_l([0,T])\right\}, \quad x = (x^{(1)}, \dots, x^{(l)}) \in \mathcal{C}_l^0([0,T]).$$

Furthermore, assume that for all i, j *and all* $x \in \mathcal{C}_l^0([0,T])$ *the following conditions hold* $\mathbb{P}$*-a.s.*

(1) $\displaystyle\int_0^T |a_{0i}(t,x)| + |a_{ij}^{(1)}(t,x)| + \sum_{n=1}^{2} (b_{ij}^{(n)}(t,x))^2 + (B_{ij}^{(n)}(t,x))^2 \,\mathrm{d}t < \infty,$

(2) $\displaystyle\int_0^T \left((A_{0i}(t,x))^2 + (A_{ij}^{(1)}(t,x))^2\right) \mathrm{d}t < \infty,$

(3) the matrix $B \circ B(t,x) := B_1(t,x)B_1^\top(t,x) + B_2(t,x)B_2^\top(t,x)$ *is uniformly non-singular, i.e. the elements of the reciprocal matrix are uniformly bounded,*

(4) if $g(t,x)$ *denotes any element of the matrices* $B_1(t,x)$ *and* $B_2(t,x)$*, then, for* $x, y \in \mathcal{C}_l^0([0,T])$

$$|g(t,x) - g(t,y)|^2 \le L_1 \int_0^t |x(s) - y(s)|^2 \,\mathrm{d}K(s) + L_2|x(t) - y(t)|^2,$$
$$g^2(t,x) \le L_1 \int_0^t (1 + |x(s)|^2) \,\mathrm{d}K(s) + L_2(1 + |x(t)|^2)$$

where $|x(t)|^2 = (x^{(1)}(t))^2 + \dots + (x^{(l)}(t))^2$ *and* $K(t)$ *is a non-decreasing right continuous function,* $0 \le K(t) \le 1$*,*

(5) $\displaystyle\int_0^T \mathbb{E}|A_{ij}^{(1)}(t,Y)Z_t^{(j)}| \,\mathrm{d}t < \infty,$

(6) $\mathbb{E}|Z_t^{(j)}| < \infty, \quad 0 \le t \le T,$

(7) $\displaystyle\mathbb{P}\left(\int_0^T \left(A_{ij}^{(1)}(t,Y)\eta_t^{(j)}\right)^2 \mathrm{d}t < \infty\right) = 1,$ *where* $\eta_t^{(j)} = \mathbb{E}(Z_t^{(j)}|\mathcal{F}_t^Y),$

(8) Z_0 *given* Y_0 *is Gaussian,* $\mathcal{N}(\eta_0, \gamma_0)$*, with* $\operatorname{Sp}\gamma_0 < \infty$ $\mathbb{P}$*-a.s.*

Then the process (Z,Y) *is conditionally Gaussian, i.e. for any* $0 \le t_0 < t_1 < \dots < t_n \le t$*, the conditional distribution of* $Z_{t_0}, \dots, Z_{t_n}$ *given* $\mathcal{F}_t^Y$ *is Gaussian.*

Theorem 2.2 (cf. [49], Theorem 12.7, p. 33)**.** *Given the assumptions of Theorem 2.1 and additionally*

(9) $|a_{ij}^{(1)}(t,x)| \le L, \quad |A_{ij}^{(1)}(t,x)| \le L,$

(10) $\displaystyle\int_0^T \mathbb{E}\left(a_{0i}^4(t,Y) + (b_{ij}^{(1)}(t,Y))^4 + (b_{ij}^{(2)}(t,Y))^4\right) \mathrm{d}t < \infty,$

(11) $\displaystyle\mathbb{E}\sum_{i=1}^{k} (Z_0^{(i)})^4 < \infty.$

Then the vector $\eta_t = \mathbb{E}\left(Z_t \middle| \mathcal{F}_t^Y\right)$ *and the matrix* $\gamma_t = \mathbb{E}\left((Z_t - \eta_t)(Z_t - \eta_t)^\top \middle| \mathcal{F}_t^Y\right)$ *are unique continuous* $\mathcal{F}_t^Y$*-measurable for any* t *solutions of the system of equations*

$$\begin{aligned}\mathrm{d}\eta_t &= (a_0(t,Y) + a_1(t,Y)\eta_t)\,\mathrm{d}t + \Big((b \circ B)(t,Y) + \gamma_t A_1^\top(t,Y)\Big)(B \circ B)^{-1}(t,Y)\\ &\quad\times\ (\mathrm{d}Y_t - (A_0(t,Y) + A_1(t,Y)\eta_t)\mathrm{d}t)\,,\end{aligned} \tag{2.4}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t &= a_1(t,Y)\gamma_t + \gamma_t a_1^\top(t,Y) + (b \circ b)(t,Y) - \Big((b \circ B)(t,Y) + \gamma_t A_1^\top(t,Y)\Big)\\ &\quad\times\ (B \circ B)^{-1}(t,Y)\Big((b \circ B)(t,Y) + \gamma_t A_1^\top(t,Y)\Big)^\top\end{aligned} \tag{2.5}$$

with initial conditions $\eta_0 = \mathbb{E}(Z_0|Y_0)$*,* $\gamma_0 = \mathbb{E}((Z_0 - \eta_0)(Z_0 - \eta_0)^\top|Y_0)$*. If in this case the matrix* γ_0 *is positive definite, then the matrices* γ_t*,* $0 \le t \le T$*, will have the same property.*

The following two corollaries are applications of the above theorems. The special situation described therein is used in the next section to identify the filter problem of a certain conditionally Gaussian jump diffusion with that of a multi-dimensional conditionally Gaussian diffusion.

Corollary 2.3. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a complete probability space with right continuous filtration* $(\mathcal{F}_t)_{0 \le t \le T}$*,* $Z = (Z_1, \ldots, Z_n)$ *be an* $\mathcal{F}_0$*-measurable* n*-dimensional random vector and* Y *be a* $(2n-1)$*-dimensional, adapted stochastic process on* $[0,T]$ *defined by*

$$Y_t = (Y_t^{(1)}, \ldots, Y_t^{(2n-1)})$$

where

$$\begin{aligned}\mathrm{d}Y_t^{(1)} &= A_0(t,Y) + \bar{A}_1(t,Y)Z\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^{(1)},\\ \mathrm{d}Y_t^{(i)} &= \sigma \mathrm{d}W_t^{(i)}, \quad i \in \{2, \ldots, 2n-1\}\end{aligned}$$

and

$$Y_0^{(i)} = V_i,\ i \in \{1, \ldots, n\}\,, \quad Y_0^{(n+i-1)} = T_i,\ i \in \{2, \ldots, n\}\,,$$

such that the following conditions hold:

(i) $(W^{(1)}, \ldots, W^{(2n-1)})$ *is an* $(2n-1)$*-dimensional Brownian motion,*

(ii) $V = (V_1, \ldots, V_n)$ *and* $\bar{T} = (T_2, \ldots, T_n)$ *are* $\mathcal{F}_0$*-measurable random variables such that* $\bar{T} \in (0,T)^{n-1}$*,* $T_i < T_{i+1}$*, for* $i \in \{2, \ldots, n-1\}$*, independent of* Z *and* V*, and* Z *given* V *is Gaussian,* $\mathcal{N}(\kappa, \chi)$*, with*

$$\chi = \begin{pmatrix} \chi_1 & 0 & \cdots & 0 \\ 0 & \chi_2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \chi_n \end{pmatrix}$$

where $\prod_{i=1}^{n} \chi_i < \infty$ $\mathbb{P}$-*a.s.,*

(iii) A_0 *and* $\bar{A}_1$ *are measurable non-anticipative functionals,*

(iv) $\bar{A}_1(t, Y) = A_1(t, Y) \left(\mathbb{1}_{[T_1,\infty)}, \mathbb{1}_{[T_2,\infty)}, \dots, \mathbb{1}_{[T_n,\infty)}\right)(t)$, *for some one dimensional functional* A_1 *with* $|A_1(t, Y)| \le L$, $T_1 := 0$,

(v) $\int_0^T (A_0(t, x))^2 \,\mathrm{d}t < \infty$, *for all* $x \in \mathcal{C}^0_{2n-1}([0, T])$,

(vi) $\mathbb{E}(Z_i)^4 < \infty$, $i \in \{1, \dots, n\}$.

Then the conditions of Theorem 2.1 and 2.2 are satisfied and Z *given* $\mathcal{F}^Y_t$ *is distributed normally with mean* η_t *and variance* γ_t *such that* η *and* γ *are unique solutions to*

$$\mathrm{d}\eta^{(i)}_t = \frac{A_1(t, Y)}{\sigma^2} \sum_{j=1}^{n} \gamma^{(ij)}_t \mathbb{1}_{[T_j,\infty)}(t) \left(\mathrm{d}Y^{(1)}_t - (A_0(t, Y) + A_1(t, Y) \sum_{j=1}^{n} \eta^{(j)}_t \mathbb{1}_{[T_j,\infty)}(t))\mathrm{d}t \right), \quad (2.6)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma^{(ij)}_t = -\frac{A_1(t, Y)^2}{\sigma^2} c^{(ij)}_t, \quad (2.7)$$

$$c^{(ij)}_t = \sum_{l=1}^{n}\sum_{k=1}^{n} \gamma^{(ik)}_t \gamma^{(lj)}_t \mathbb{1}_{[T_k \vee T_l,\infty)}(t),$$

for $1 \le i, j \le n$

The proof of Corollary 2.3 is provided in Appendix A.1. It mainly consists of verifying the conditions of Theorems 2.1 and 2.2 and applying Equations (2.4) and (2.5) to the special situation described above. The following corollary considers the same situation as in Corollary 2.3. It solves a filter problem for a jump process with $n - 1$ $\mathcal{F}^Y_0$-measurable jump times and $\mathcal{F}^Y_0$-measurable conditional jump sizes. Its proof is also provided in the appendix.

Corollary 2.4. *Let the assumptions of Corollary 2.3 be satisfied. Furthermore, for* $t \in [0, T]$, *define*

$$\widetilde{\eta}_t := \mathbb{E}\left(\sum_{i=1}^{n} Z_i \mathbb{1}_{[T_i,\infty)}(t) \middle| \mathcal{F}^Y_t \right), \qquad \widetilde{\gamma}_t := \mathbb{E}\left(\left(\sum_{i=1}^{n} Z_i \mathbb{1}_{[T_i,\infty)}(t) - \widetilde{\eta}_t \right)^2 \middle| \mathcal{F}^Y_t \right).$$

Then $\sum_{i=1}^{n} Z_i \mathbb{1}_{[T_i,\infty)}(t)$ *given* $\mathcal{F}^Y_t$ *is distributed normally with mean* $\widetilde{\eta}_t$ *and variance* $\widetilde{\gamma}_t$ *such that* $\widetilde{\eta}$ *and* $\widetilde{\gamma}$ *are unique solutions to*

$$\mathrm{d}\widetilde{\eta}_t = \widetilde{\gamma}_t \frac{A_1(t, Y)}{\sigma^2} \left(\mathrm{d}Y^{(1)}_t - (A_0(t, Y) + A_1(t, Y)\widetilde{\eta}_t) \,\mathrm{d}t \right) + \mathrm{d}\sum_{i=2}^{n} \kappa_i \mathbb{1}_{[T_i,\infty)}(t), \qquad \widetilde{\eta}_0 = \kappa_1,$$

$$\mathrm{d}\widetilde{\gamma}_t = -\left(\frac{A_1(t, Y)}{\sigma} \widetilde{\gamma}_t \right)^2 \mathrm{d}t + \mathrm{d}\sum_{i=2}^{n} \chi_i \mathbb{1}_{[T_i,\infty)}(t), \qquad \widetilde{\gamma}_0 = \chi_1.$$

2.2 Optimal filtering for conditionally Gaussian jump diffusions

In the last section, we presented general results on stochastic filtering for conditionally Gaussian diffusions. From these we now deduce an analogous result for special jump processes.

Proposition 2.5. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with right continuous filtration $(\mathcal{F}_t)_{0\le t\le T}$ and Z and Y be adapted, real-valued stochastic processes on $[0,T]$ defined by*

$$
\begin{aligned}
Z_t &= V_0 + \sum_{i=1}^{N_t} V_i, \\
Y_t &= U_0 + \int_0^t (A_0(s,Y) + A_1(s,Y)Z_s)\,\mathrm{d}s + \sigma B_t + \sum_{i=1}^{N_t} U_i,
\end{aligned}
$$

where the following conditions are satisfied:

(i) B *is a Brownian motion*

(ii) N *is a Poisson process with jump times* $T_1, T_2, \ldots$

(iii) $(V_i)_{i\in\mathbb{N}_0}$ *and* $(U_i)_{i\in\mathbb{N}_0}$ *are mutually independent sequences of random variables and for all* $i \in \mathbb{N}_0$, V_i *given* U_i *is* $\mathcal{N}(\kappa_i, \chi_i)$ *distributed, with* $\prod_{i=1}^n \chi_i < \infty$ $\mathbb{P}$*-a.s., for all* $n \in \mathbb{N}$,

(iv) A_0 *and* A_1 *are measurable non-anticipative functionals*

(v) $\int_0^T (A_0(t,x))^2\,\mathrm{d}t < \infty$, *for all piecewise continuous, bounded functions* x *on* $[0,T]$, *and* $|A_1(t,x)| \le L$,

(vi) $\mathbb{E}(V_n)^4 < \infty$, *for all* $n \in \mathbb{N}_0$.

Then Z_t *given* $\mathcal{F}_t^Y$ *is distributed normally with mean* $\eta_t := \mathbb{E}\left(Z_t \middle| \mathcal{F}_t^Y\right)$ *and variance* $\gamma_t := \mathbb{E}\left((Z_t-\eta_t)^2 \middle| \mathcal{F}_t^Y\right)$ *such that* η *and* γ *are unique solutions to*

$$
\mathrm{d}\eta_t = \gamma_t \frac{A_1(t,Y)}{\sigma^2}\left(\mathrm{d}Y_t^c - (A_0(t,Y) + A_1(t,Y)\eta_t)\,\mathrm{d}t\right) + \mathrm{d}\sum_{i=1}^{N_t} \kappa_i, \qquad \eta_0 = \kappa_0, \tag{2.8}
$$

$$
\mathrm{d}\gamma_t = -\left(\frac{A_1(t,Y)}{\sigma}\gamma_t\right)^2 \mathrm{d}t + \mathrm{d}\sum_{i=1}^{N_t} \chi_i, \qquad \gamma_0 = \chi_0, \tag{2.9}
$$

on $\{N_T < \infty\}$.

Proof. For $n \in \mathbb{N}$, define on $[0,T]$

$$
Z_t^n := V_0 + \sum_{i=1}^{N_t \wedge n} V_i
$$

and

$$Y_t^n := U_0 + \int_0^t (A_0(s,Y) + A_1(s,Y)Z_s^n)\,\mathrm{d}s + \sigma B_t + \sum_{i=1}^{N_t \wedge n} U_i,$$

i.e. Z^n and Y^n are the signal and observation process with a stopped jump part after T_n. In particular, Y^n is $\mathbb{F}^Y$-adapted. Furthermore, let $(\widetilde{\Omega}^n, \widetilde{\mathcal{F}}^n, \widetilde{\mathbb{P}}^n)$ be an enlargement of our probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with

$$\widetilde{\Omega}^n = \Omega \times \Omega^n, \quad \widetilde{\mathcal{F}}^n = \mathcal{F} \otimes \mathcal{F}^n, \quad \widetilde{\mathbb{P}}^n = \mathbb{P} \otimes \mathbb{P}^n$$

such that there exists a $2n$-dimensional Brownian motion W^n independent of Y and Z (where now all variables are defined on the enlarged probability space). Now, let $(\mathcal{F}_t^n)_{t\geq 0}$ be the filtration generated by W^n, and

$$\widetilde{\mathcal{F}}_t^n := \sigma(\mathcal{F}_t^{Y^n}, \mathcal{F}_T^{Y^{n,d}}, \mathcal{F}_t^n), \quad t \in [0,T].$$

In particular, for each $t \in [0,T]$, $\widetilde{\mathcal{F}}_t^n$ contains all information about $Y^{n,d}$, i.e. the discontinuous part of Y^n, up to time T. For any piecewise continuous and bounded function x on $[0,T]$ with n points of discontinuity $t_1, \ldots, t_n \in [0,T]$ there exist $y \in \mathcal{C}^0([0,T])$ and $u_1, \ldots, u_n \in \mathbb{R}$ with $x(t) = y(t) + \sum_{i=1}^n u_i \mathbb{1}_{[t_i,T]}(t)$. Hence, for A_0 and A_1 we can find functionals $\bar{A}_j$, $j \in \{0,1\}$, on $[0,T] \times \mathcal{C}_{2n+1}([0,T])$ with

$$\bar{A}_j(t,(y,u_1,\ldots,u_n,t_1,\ldots,t_n)) = A_j(t, y + \sum_{i=1}^n u_i \mathbb{1}_{[t_i,T]}).$$

In particular, there exist functionals $\bar{A}_j$, $j \in \{0,1\}$, such that

$$\bar{A}_j(t,(Y^{n,c},U_1,\ldots,U_n,T_1,\ldots,T_n)) = A_j(t,Y^n)$$

where $Y^{n,c}$ denotes the continuous part of Y^n (see notations for more details).

The filter problem induced by Z^n and Y^n on $[0,T]$ can now be identified with the situation of Corollary 2.3 (with probability space $(\widetilde{\Omega}^n, \widetilde{\mathcal{F}}^n, \widetilde{\mathbb{P}}^n)$ and observation filtration $(\widetilde{\mathcal{F}}_t^n)_{0\leq t\leq T}$). Denote by $\widetilde{\mathbb{E}}^n$ the expectation w.r.t. $\widetilde{\mathbb{P}}^n$ and define

$$\widetilde{\eta}_t^n := \widetilde{\mathbb{E}}^n(Z_t^n | \widetilde{\mathcal{F}}_t^n), \quad \widetilde{\gamma}_t^n := \widetilde{\mathbb{E}}^n((Z_t^n - \widetilde{\eta}_t^n)^2 \,|\widetilde{\mathcal{F}}_t^n).$$

Then, according to Corollary 2.4, $\widetilde{\eta}^n$ and $\widetilde{\gamma}^n$ are unique solutions to

$$\mathrm{d}\widetilde{\eta}_t^n = \widetilde{\gamma}_t^n \frac{A_1(t,Y^n)}{\sigma^2} (\mathrm{d}Y_t^{n,c} - (A_0(t,Y^n) + A_1(t,Y^n)\widetilde{\eta}_t^n)\,\mathrm{d}t) + \mathrm{d}\sum_{i=1}^n \kappa_i \mathbb{1}_{[T_i,\infty)}(t),$$

$$\mathrm{d}\widetilde{\gamma}_t^n = -\left(\frac{A_1(t,Y^n)}{\sigma}\widetilde{\gamma}_t^n\right)^2 \mathrm{d}t + \mathrm{d}\sum_{i=1}^{n} \chi_i \mathbb{1}_{[T_i,\infty)}(t),$$

with initial conditions $\widetilde{\eta}_0^n = \kappa_0$ and $\widetilde{\gamma}_0^n = \chi_0$. Due to the independence of W^n and (Y^n, Z^n), coming back to our original probability space, we have that Z_t given $\sigma(\mathcal{F}_t^{Y^n}, \mathcal{F}_T^{Y^{n,d}})$ is $\mathcal{N}(\eta_t^n, \gamma_t^n)$ distributed where η_t^n and γ_t^n have the same dynamics as $\widetilde{\eta}_t^n$ and $\widetilde{\gamma}_t^n$. Moreover, since Z^n as well as $Y^{n,d}$ have independent increments, we get

$$\eta_t^n = \mathbb{E}\left(Z_t^n \middle| \sigma(\mathcal{F}_t^{Y^n}, \mathcal{F}_T^{Y^{n,d}})\right) = \mathbb{E}\left(Z_t^n \middle| \mathcal{F}_t^{Y^n}\right)$$

and

$$\gamma_t^n = \mathbb{E}\left((Z_t^n - \eta_t^n)^2 \middle| \sigma(\mathcal{F}_t^{Y^n}, \mathcal{F}_T^{Y^{n,d}})\right) = \mathbb{E}\left((Z_t^n - \eta_t^n)^2 \middle| \mathcal{F}_t^{Y^n}\right).$$

Finally, $Z\mathbb{1}_{[0,T_n]} = Z^n\mathbb{1}_{[0,T_n]}$ $\mathrm{d}t\otimes\mathrm{d}\mathbb{P}$-a.s. and T_n is adapted to $(\mathcal{F}_t^{Y^n})_{0\le t\le T}$ as well as $(\mathcal{F}_t^{Y})_{0\le t\le T}$. Hence,

$$\eta_t\mathbb{1}_{[0,T_n]} = \eta_t^n\mathbb{1}_{[0,T_n]}, \quad \gamma_t\mathbb{1}_{[0,T_n]} = \gamma_t^n\mathbb{1}_{[0,T_n]}.$$

Since $T_n \to +\infty$ $\mathbb{P}$-a.s., for $n \to \infty$, we get the assertion. □

Corollary 2.6. *Let the assumptions of Proposition 2.5 be satisfied. Then,*

$$\gamma_t = \left(\int_0^t A_1(s,Y)^2\sigma^{-2}\,\mathrm{d}s + \sum_{i=0}^{N_t}\widetilde{\chi}_i\right)^{-1}, \tag{2.10}$$

with

$$\widetilde{\chi}_0 = \chi_0^{-1}, \quad \widetilde{\chi}_i = \frac{-\chi_i}{(\gamma_{T_i-} + \chi_i)\gamma_{T_i-}}, \quad \textit{for } i \in \mathbb{N}. \tag{2.11}$$

Proof. As SDE (2.9) only consists of a jump and a drift part, we can derive a pathwise solution. For fixed $\omega \in \Omega$ consider now the function defined in (2.10). For $t \in (T_i(\omega), T_{i+1}(\omega))$ we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t = \frac{-A_1(t,Y)^2\sigma^{-2}}{\left(\int_0^t A_1(s,Y)^2\sigma^{-2}\,\mathrm{d}s + \sum_{j=0}^{i}\widetilde{\chi}_j\right)^2} = -A_1(t,Y)^2\gamma_t^2\sigma^{-2}.$$

This corresponds to the drift part of SDE (2.9). It remains to show that $\Delta\gamma_{T_i} = \gamma_{T_i} - \gamma_{T_i-} = \chi_i$, for all $i \in \mathbb{N}$. This follows from

$$\widetilde{\chi}_i = \Delta\gamma_{T_i}^{-1} = \gamma_{T_i}^{-1} - \gamma_{T_i-}^{-1} = \frac{1}{\gamma_{T_i}} - \frac{1}{\gamma_{T_i-}} = \frac{\gamma_{T_i-} - \gamma_{T_i}}{\gamma_{T_i}\gamma_{T_i-}} = -\frac{\Delta\gamma_{T_i}}{(\gamma_{T_i-} + \Delta\gamma_{T_i})\gamma_{T_i-}}.$$

Together with (2.11) this proves the assertion. □

Chapter 3

A market with semi-rational noise traders and fixed time horizon

3.1 Introduction and model setup

As pointed out in the introduction, we want to analyse an order based insider trading model as introduced in Section 1.1 where the demand process of the noise traders includes a part of informed trading. The fundamental value V is assumed to be constant over a trading period $[0, T]$, $T \in \mathbb{R}$. As motivated in the introduction, from time T on the market price coincides with the fundamental value, i.e. $P_T = V$. The order process of the noise traders is given by

$$\mathrm{d}X_t = \bar{\mu}(t, P_t, V)\,\mathrm{d}t + \sigma(t)\,\mathrm{d}B_t, \quad t \in [0, T],$$

where B is a Brownian motion independent of V, σ some deterministic, continuously differentiable, positive function and $\bar{\mu}$ such that $\operatorname{sgn}(\bar{\mu}(t, P_t, V)) = \operatorname{sgn}(V - P_t)$. Such an order process is called *semi-rational*, meaning that the noise traders react on a possible under- or overpricing, but are not (completely) rational, i.e. do not choose a strategy that optimises their terminal wealth. As a result the demand process of the noise traders can be split into a part of informed trading, $\int_0^t \mu(s, P_s, V)\,\mathrm{d}s$, and a part of uniformed trading or noise trading, $\int_0^t \sigma(s)\,\mathrm{d}B_s$. Hence, referring to X as the demand process of the *noise traders* is not completely accurate. Nevertheless, we stick to this terminology referring to noise as not entirely rational behaviour.

As in [7] or [25] we assume that the trading horizon $T \in \mathbb{R}_+$ is fixed and the fundamental value V is constant on $[0, T]$. The insider's filtration $\mathbb{F}^{\mathcal{I}}$ is the (completed) filtration generated by P and V, i.e. $\mathbb{F}^{\mathcal{I}} = \mathbb{F}^{P,V}$. The market makers' information is reflected by $\mathbb{F}^{\mathcal{M}} = \mathbb{F}^{Y}$. Regarding V we start with the following assumption:

Assumption 3.1. *$V = h(Z)$ where h is a continuously differentiable and strictly increasing function, $Z \sim \mathcal{N}(0, 1)$ and $\mathbb{E}h(Z)^2 < \infty$, $\mathbb{E}(\partial_y h(Z))^2 < \infty$.*

The Gaussian distribution of Z ensures the analytical tractability of the filter problem, which is induced by insider trading and the market makers' ambition to find a rational price by observing the total order.

In contrast to the market makers, the insider observes a certain realisation $v \in \mathcal{V} := h(\mathbb{R})$ of V at time 0. As stated in Section 1.1, the insider's objective is to maximise the expected utility of the final wealth generated by her trading strategy θ, i.e. (cf. (1.3))

$$\sup_{\theta} \mathbb{E}\,\mathcal{U}\left(\theta_{T-}V - \int_0^{T-} P_{t-}\,\mathrm{d}\theta_t - [P,\theta]_{T-}\right).$$

We analyse this model for the risk neutral case, i.e.

$$\mathcal{U}(W) = W, \tag{3.1}$$

as well as a risk averse case with exponential utility, i.e.

$$\mathcal{U}(W) = \beta \exp\left(\beta W\right), \quad \beta < 0, \tag{3.2}$$

where we refer to $-\beta$ as *degree of risk aversion*. In order to characterise all possible utility functions by β, we use $\beta = 0$ to identify the risk neutral case with utility function as in (3.1).

As in [25], we assume that the price P is set in the following way

$$P_t = H(t, \widetilde{Y}_t), \qquad \widetilde{Y}_t = \int_0^t \lambda(s)\,\mathrm{d}Y_s, \qquad t \in [0,T), \tag{3.3}$$

for suitable functions H and λ. If λ and σ are bounded on $[0,T]$ there exists a unique strong solution $\xi^{s,y}$ to the SDE

$$\mathrm{d}\xi_t = \lambda(t)\sigma(t)\mathrm{d}B_t, \quad \xi_s = y, \tag{3.4}$$

for all initial conditions $(s,y) \in [0,T] \times \mathbb{R}$ (cf. [54], Section 1.3). More details and technical conditions for (H,λ) are given in the following definition of admissible pricing rules.

Definition 3.2. The pair (H,λ) is called *admissible pricing rule* if

- $\lambda : [0,T] \to \mathbb{R} \in \mathcal{C}^1([0,T])$ and $\lambda(t) > 0$ for all $t \in [0,T)$,
- $H \in \mathcal{C}^{1,2}\left((0,T) \times \mathbb{R}\right) \cap \mathcal{C}^{0,1}\left([0,T] \times \mathbb{R}\right)$ and $H(t,\cdot)$ is strictly increasing for all $t \in [0,T]$,
- $\sup_{t\in[0,T]} \mathbb{E}(y^*(t,V))^2 < \infty$ where y^* is the implicit function defined by
$$H(t, y^*(t,v)) = v, \qquad t \in [0,T],\ v \in \mathcal{V}, \tag{3.5}$$
- $\mathbb{E} \int_0^T H(t,\xi_t)^2 + (\partial_y H(t,\xi_t))^2\,\mathrm{d}t < \infty$,
- $\mathbb{E}H(T,\xi_T)^2 < \infty$.

The set of all admissible pricing rules is denoted by $\mathcal{P}$. In the sequel, we call H *pricing function* and λ *price pressure*. According to Definition 1.2, (H,λ) is called *rational* if

$$H(t,\widetilde{Y}_t) = \mathbb{E}\left(h(Z)\middle|\mathcal{F}_t^{\mathcal{M}}\right), \quad \text{for all } t \in [0,T).$$

The implicit function $y^*(t,v)$, $(t,v) \in [0,T]\times\mathcal{V}$, is well-defined and unique since $H \in \mathcal{C}^{0,1}([0,T]\times\mathbb{R})$ is strictly increasing for all $t \in [0,T]$.

We now fix the set of admissible trading strategies.

Definition 3.3. Let $-\beta \geq 0$ denote the degree of insider risk aversion. An insider strategy θ is called (H,λ)-*admissible* if θ is an $\mathbb{F}^{\mathcal{I}}$-semimartingale and

- $\mathbb{E}\displaystyle\int_0^T \lambda(t)^2\sigma(t)^2\partial_y H(t,\widetilde{Y}_t)^2\,\mathrm{d}t < \infty$,
- $\mathbb{E}\displaystyle\int_0^T \sigma(t)^2 H(t,\widetilde{Y}_t)^2\,\mathrm{d}t < \infty$,
- $\mathbb{E}\exp\left(\dfrac{1}{2}\displaystyle\int_0^T \beta^2\sigma(t)^2(H(t,\widetilde{Y}_t)-V)^2\,\mathrm{d}t\right) < \infty$.

The set of all (H,λ)-admissible, or short admissible, strategies is denoted by $\mathcal{S}(H,\lambda)$.

In the introduction of this section, we already claimed that the drift or informed part of the noise traders' order process is a function of the fundamental value V, the price P and the time. Due to our restriction on pricing rules in (3.3), we are able to rewrite the noise drift as

$$\bar{\mu}(t,P_t,V) = \mu(t,\widetilde{Y}_t,V)$$

for a suitable function $\mu : [0,T]\times\mathbb{R}\times\mathcal{V} \mapsto \mathbb{R}$. Moreover, we make the following assumptions related to the order process of the noise traders:

Assumption 3.4. *The order process of the noise traders is given by*

$$\mathrm{d}X_t = \mu(t,\widetilde{Y}_t,V)\mathrm{d}t + \sigma(t)\mathrm{d}B_t, \quad X_0 = 0,$$

where

(1) $\sigma \in \mathcal{C}^1([0,T])$, $\sigma(t) > 0$ *for all* $t \in [0,T]$,

(2) $\mu(t,\cdot,v) \in \mathcal{C}^0(\mathbb{R})$ *for all* $(t,v) \in [0,T)\times\mathcal{V}$,

(3) $\mu(t,\cdot,v)(H(t,\cdot)-v) \in \mathcal{C}^1(\mathbb{R})$ *for all* $(t,v) \in [0,T)\times\mathcal{V}$,

(4) $\mu(t,y,v)\begin{cases} >0, & \text{if } y < y^*(t,v) \\ =0, & \text{if } y = y^*(t,v) \\ <0, & \text{if } y > y^*(t,v) \end{cases}$, *for all* $(t,v) \in [0,T)\times\mathcal{V}$,

with $y^*(t,v)$ *as defined in (3.5).*

In the sequel, we refer to μ *as* noise drift.

Remark 3.5. Due to the monotonicity of H, condition (4) is equivalent to

$$\mu(t,y,v)(H(t,y)-v)\begin{cases} = 0, & \text{if } y = y^*(t,v) \\ < 0, & \text{if } y \neq y^*(t,v) \end{cases}, \quad \text{for all } (t,v) \in [0,T) \times \mathcal{V}.$$

3.2 Absolutely continuous insider trading: HJB equation and necessary conditions for equilibrium

After having introduced the model in the last section, we can now start with its analysis. The first question to answer is whether there exist necessary conditions, especially for μ, to ensure the existence of an equilibrium. We first assume that the insider strategy is absolutely continuous, i.e.

$$\theta_t = \int_0^t \alpha_s \,\mathrm{d}s, \quad t \in [0,T], \tag{3.6}$$

for a suitable $\mathbb{F}^{\mathcal{I}}$-adapted process α such that there exists a strong solution to Y for any initial condition $(t,y) \in [0,T] \times \mathbb{R}$ (denoted by $Y^{t,y}$) and

$$\mathbb{E}^v\, \mathcal{U}\left(\int_t^T \left|\left(V - H(s,\widetilde{Y}_s^{t,y})\right)\alpha_s\right| \mathrm{d}s\right) < \infty, \quad \text{for all } (t,y) \in [0,T] \times \mathbb{R},$$

where $\mathbb{E}^v$ is the expectation w.r.t. $\mathbb{P}^v = \mathbb{P}(\cdot|V=v)$, i.e. the measure under which V starts in $v \in \mathcal{V}$. By $\widetilde{\mathcal{S}}(t,y)$ let us denote the set of all such α. For the value function J, defined by

$$J(t,y,v) := \sup_{\alpha \in \widetilde{\mathcal{S}}(t,y)} \mathbb{E}^v\, \mathcal{U}\left(\int_t^T (V - H(s,\widetilde{Y}_s^{t,y}))\alpha_s \,\mathrm{d}s\right), \tag{3.7}$$

we can now calculate the *Hamilton-Jacobi-Bellman equation* (HJB equation). For details on the classical PDE approach to dynamic programming, we refer to Pham [54], Chapter 3. This approach has also been used in other insider trading models, see e.g. [25] and [26], to solve the optimisation problem of the insider. However, this section can be better understood as motivation for a class of admissible noise drift since optimality in the whole class $\mathcal{S}(H,\lambda)$ is analysed in the following section. In both cases, risk neutral and risk averse (exponential utility), we deduce necessary conditions on μ and H for the existence of an equilibrium.

3.2.1 The risk neutral case

In the case of risk neutrality, the definition of J in (3.7) reads as

$$J^0(t,y,v) = \sup_{\alpha \in \widetilde{\mathcal{S}}(t,y)} \mathbb{E}^v \int_t^T (V - H(s,\widetilde{Y}_s^{t,y}))\alpha_s \,\mathrm{d}s.$$

To shorten the notation, we also write $J^0(t,y)$ instead of $J^0(t,y,v)$. For any $t \in [0,T)$ we can split the integral at $t+\epsilon$, $\epsilon \in (0,T-t)$,

$$\begin{aligned} J^0(t,y) &= \sup_{\alpha \in \widetilde{\mathcal{S}}(t,y)} \mathbb{E}^v \left(\int_t^{t+\epsilon} (V - H(s,\widetilde{Y}_s^{t,y}))\alpha_s \,\mathrm{d}s + \int_{t+\epsilon}^T (V - H(s,\widetilde{Y}_s^{t,y}))\alpha_s \,\mathrm{d}s \right) \\ &= \sup_{\alpha \in \widetilde{\mathcal{S}}(t,y)} \mathbb{E}^v \left(\int_t^{t+\epsilon} (V - H(s,\widetilde{Y}_s^{t,y}))\alpha_s \,\mathrm{d}s + J^0(t+\epsilon, \widetilde{Y}_{t+\epsilon}^{t,y}) \right). \end{aligned} \tag{3.8}$$

Taking into account the dynamics of $\widetilde{Y}$ and assuming that J^0 is smooth enough, i.e. $J^0 \in C^{1,2}([0,T]\times\mathbb{R})$, we can apply Itô's formula (cf. [56], Theorem 33, Chapter II)

$$\begin{aligned} J^0(t+\epsilon, \widetilde{Y}_{t+\epsilon}) = J^0(t,\widetilde{Y}_t) &+ \int_t^{t+\epsilon} \lambda(s)\sigma(s)\partial_y J^0(s,\widetilde{Y}_s)\mathrm{d}B_s + \int_t^{t+\epsilon} \partial_t J^0(s,\widetilde{Y}_s)\,\mathrm{d}s \\ &+ \int_t^{t+\epsilon} \lambda(s)(\mu(s,\widetilde{Y}_s,V) + \alpha_s)\partial_y J^0(s,\widetilde{Y}_s) + \frac{\lambda(s)^2\sigma(s)^2}{2}\partial_{yy} J^0(s,\widetilde{Y}_s)\mathrm{d}s. \end{aligned} \tag{3.9}$$

Now, subtracting $J^0(t,y)$ on both sides of Equation (3.8) and inserting (3.9) into (3.8) leads to

$$\begin{aligned} 0 = \sup_{\alpha \in \widetilde{\mathcal{S}}(t,y)} \mathbb{E}^v \Bigg(& \int_t^{t+\epsilon} \lambda(s)\sigma(s)\partial_y J^0(s,\widetilde{Y}_s^{t,y})\mathrm{d}B_s + \int_t^{t+\epsilon} (V - H(s,\widetilde{Y}_s^{t,y}))\alpha_s + \partial_t J^0(s,\widetilde{Y}_s^{t,y}) \\ & + \lambda(s)(\mu(s,\widetilde{Y}_s^{t,y},V) + \alpha_s)\partial_y J^0(s,\widetilde{Y}_s^{t,y}) + \frac{\lambda(s)^2\sigma(s)^2}{2}\partial_{yy} J^0(s,\widetilde{Y}_s^{t,y})\mathrm{d}s \Bigg). \end{aligned}$$

Under certain regularity conditions on J^0, $(\int_0^t \lambda(s)\sigma(s)\partial_y J^0(s,\widetilde{Y}_s^{t,y})\mathrm{d}B_s,\ t \geq 0)$, is a true martingale. Then

$$\begin{aligned} 0 = \sup_{\alpha \in \widetilde{\mathcal{S}}(t,y)} \mathbb{E}^v \Bigg(& \int_t^{t+\epsilon} (V - H(s,\widetilde{Y}_s^{t,y}))\alpha_s + \partial_t J^0(s,\widetilde{Y}_s^{t,y}) \\ & + \lambda(s)(\mu(s,\widetilde{Y}_s^{t,y},V) + \alpha_s)\partial_y J^0(s,\widetilde{Y}_s^{t,y}) + \frac{\lambda(s)^2\sigma(s)^2}{2}\partial_{yy} J^0(s,\widetilde{Y}_s^{t,y})\mathrm{d}s \Bigg). \end{aligned}$$

Dividing this equation by ϵ and sending ϵ to zero, we get by the mean value theorem

$$0 = \sup_\alpha \left((v - H(t,y))\alpha + \partial_t J^0(t,y) + \lambda(t)(\mu(t,y,v) + \alpha)\partial_y J^0(t,y) + \frac{\lambda(t)^2\sigma(t)^2}{2}\partial_{yy} J^0(t,y) \right).$$

We note that the above HJB equation is linear in α. Hence, for a finite solution we necessarily need, for all $(t,y,v) \in (0,T)\times\mathbb{R}\times\mathcal{V}$,

$$\partial_y J^0(t,y,v) = \frac{H(t,y) - v}{\lambda(t)}, \tag{3.10}$$

$$\partial_t J^0(t,y,v) = -\lambda(t)\mu(t,y,v)\partial_y J^0(t,y,v) - \frac{1}{2}\sigma(t)^2\lambda(t)^2\partial_{yy} J^0(t,y,v). \tag{3.11}$$

With the help of Equations (3.10) and (3.11), we now derive necessary conditions for the existence of an equilibrium.

Proposition 3.6. *Let $(H, \lambda) \in \mathcal{P}$ and μ be as in Assumption 3.4. If there exists a function $J^0 \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$ such that (H, λ, J^0) is a solution to the system of Equations (3.10) and (3.11), then necessarily the following holds:*

- *H satisfies the PDE*

$$\partial_t H(t,y) + \frac{1}{2}\sigma(t)^2 \lambda(t)^2 \partial_{yy} H(t,y) = 0, \quad \textit{for all } (t,y) \in (0,T) \times \mathbb{R} \tag{3.12}$$

- *μ has the form*

$$\mu(t,y,v) = \begin{cases} -\frac{\lambda'(t)}{\lambda(t)^2} \frac{\int_y^{y^*(t,v)} H(t,x) - v \, \mathrm{d}x}{H(t,y) - v} & \textit{, if } y \neq y^*(t,v) \\ 0 & \textit{, if } y = y^*(t,v) \end{cases} \tag{3.13}$$

for all $(t,v) \in (0,T) \times \mathcal{V}$, where $y^*(t,v)$ is defined as in (3.5).

In particular, μ (as defined in (3.13)) always verifies conditions (2) and (3) of Assumption 3.4, and condition (4) if and only if λ is strictly decreasing.

Proof. Differentiation of (3.10) w.r.t. y and t yields

$$\partial_{yy} J^0(t,y) = \frac{\partial_y H(t,y)}{\lambda(t)}, \tag{3.14}$$

respectively

$$\partial_{ty} J^0(t,y) = \frac{\partial_t H(t,y)}{\lambda(t)} - (H(t,y) - v)\frac{\lambda'(t)}{\lambda(t)^2}. \tag{3.15}$$

By inserting (3.14) and (3.10) into (3.11), we obtain

$$\partial_t J^0(t,y) = -\mu(t,y,v)(H(t,y) - v) - \frac{1}{2}\sigma(t)^2 \lambda(t) \partial_y H(t,y). \tag{3.16}$$

Now differentiating (3.16) w.r.t. y yields

$$\partial_{yt} J^0(t,y) = -\partial_y(\mu(t,y,v)(H(t,y) - v)) - \frac{1}{2}\sigma(t)^2 \lambda(t) \partial_{yy} H(t,y). \tag{3.17}$$

Putting (3.15) and (3.17) together, we get for all $(t,y,v) \in (0,T) \times \mathbb{R} \times \mathcal{V}$ (v was arbitrarily fixed)

$$\frac{\partial_t H(t,y)}{\lambda(t)} + \frac{\sigma(t)^2 \lambda(t)}{2} \partial_{yy} H(t,y) = (H(t,y) - v)\frac{\lambda'(t)}{\lambda(t)^2} - \partial_y(\mu(t,y,v)(H(t,y) - v)). \tag{3.18}$$

Observe that the left hand site of (3.18) does not depend on v, since the pricing rule (H, λ) has to be independent of v. Now let

$$f(t,y) := \frac{1}{\lambda(t)}\left(\partial_t H(t,y) + \frac{1}{2}\sigma(t)^2\lambda(t)^2\partial_{yy}H(t,y)\right), \quad (t,y) \in (0,T) \times \mathbb{R},$$

and

$$\hat{\mu}(t,y,v) := \mu(t,y,v)(H(t,y)-v).$$

Following Assumption 3.4, $\hat{\mu}(t,\cdot,v) \in \mathcal{C}^1(\mathbb{R})$ for all $(t,v) \in [0,T) \times \mathcal{V}$. The PDE

$$\partial_y \hat{\mu}(t,y,v) = (H(t,y)-v)\frac{\lambda'(t)}{\lambda(t)^2} - f(t,y), \quad (t,y,v) \in (0,T) \times \mathbb{R} \times \mathcal{V}, \tag{3.19}$$

has the general solution

$$\hat{\mu}(t,y,v) = \int_{C_1(t,v)}^{y} \lambda'(t)\lambda(t)^{-2}(H(t,x)-v) - f(t,x)\,\mathrm{d}x + C_2(t,v)$$

for suitable functions C_1 and C_2 (independent of y). Without loss of generality, we can set $C_1(t,v) = y^*(t,v)$. For $y \neq y^*(t,v)$ it follows

$$\mu(t,y,v) = \frac{\int_{y^*(t,v)}^{y} \lambda'(t)\lambda(t)^{-2}(H(t,x)-v) - f(t,x)\,\mathrm{d}x + C_2(t,v)}{H(t,y)-v}. \tag{3.20}$$

Since $y \mapsto \int_{y^*(t,v)}^{y} H(t,x) - v\,\mathrm{d}x$ as well as $y \mapsto H(t,y)-v$ are differentiable functions and H is a strictly increasing function in y, i.e. $\partial_y H(t,y) > 0$ for all y, we can apply l'Hospital's rule in the following way

$$\lim_{y \to y^*(t,v)} \frac{\int_{y^*(t,v)}^{y} \frac{\lambda'(t)}{\lambda(t)^2}(H(t,x)-v) - f(t,x)\,\mathrm{d}x}{H(t,y)-v} = \lim_{y \to y^*(t,v)} \frac{\frac{\lambda'(t)}{\lambda(t)^2}(H(t,y)-v) - f(t,y)}{\partial_y H(t,y)}. \tag{3.21}$$

As we assumed μ to be continuous in y and $\mu(t,y^*(t,v),v) = 0$, we can directly conclude that both, $C_2(t,v)$ and $f(t,y^*(t,v))$ have to equal zero for all $(t,v) \in [0,T) \times \mathcal{V}$. Since the pricing rule (H,λ) has to be independent of v, this already leads to $f \equiv 0$. This proves (3.12) and (3.13).

Now, differentiability as well as continuity of μ in $y \neq y^*(t,v)$ is obvious since H is continuously differentiable. For $y = y^*(t,v)$ we get the continuity by (3.21) with $f = 0$. Lastly, the strict monotonicity of H and the definition of $y^*(t,v)$ ensure that

$$\int_{y}^{y^*(t,v)} H(t,x) - v\,\mathrm{d}x < 0, \quad \text{for all } y \in \mathbb{R}\backslash\{y^*(t,v)\}.$$

By Remark 3.5 we get that $\operatorname{sgn}(\mu(t,y,v)) = \operatorname{sgn}(y^*(t,v)-y)$ if and only if $\lambda'(t)\lambda(t)^{-2} < 0$. $\square$

Before we give an interpretation of the preceding results, we analyse the risk averse case to obtain analogous results. Their interpretation is presented thereafter.

3.2.2 The risk averse case

To analyse exponential utility with parameter $\beta < 0$, we consider the value function (cf. (3.7))

$$J^\beta(t,y) := J^\beta(t,y,v) := \sup_{\alpha \in \widetilde{\mathcal{S}}(t,y)} \mathbb{E}^v \, \beta \exp\left(\int_t^T \beta(V - H(s, \widetilde{Y}_s^{t,y}))\alpha_s \, \mathrm{d}s \right).$$

As in the risk neutral case, we start with calculating the HJB equation. As a first step we again split the integral at $t + \epsilon \in (t,T)$

$$\begin{aligned} J^\beta(t,y) = \sup_{\alpha \in \widetilde{\mathcal{S}}(t,y)} \mathbb{E}^v \beta \Bigg(&\exp\left(\int_{t+\epsilon}^T \beta(V - H(s, \widetilde{Y}_s^{t,y}))\alpha_s \, \mathrm{d}s \right) + \exp\left(\int_t^T \beta(V - H(s, \widetilde{Y}_s^{t,y}))\alpha_s \, \mathrm{d}s \right) \\ &\times \left(1 - \exp\left(\int_t^{t+\epsilon} -\beta(V - H(s, \widetilde{Y}_s^{t,y}))\alpha_s \, \mathrm{d}s \right) \right) \Bigg) \\ = \sup_{\alpha \in \widetilde{\mathcal{S}}(t,y)} \mathbb{E}^v \Bigg(&J^\beta(t+\epsilon, \widetilde{Y}_{t+\epsilon}^{t,y}) + J^\beta(t,y) \left(1 - \exp\left(- \int_t^{t+\epsilon} \beta(V - H(s, \widetilde{Y}_s^{t,y}))\alpha_s \, \mathrm{d}s \right) \right) \Bigg). \end{aligned}$$

If J^β is smooth enough, Itô's formula provides a representation of J^β as in Equation (3.9) (in place of J^0). Furthermore, if we again assume that the local martingale term is indeed a true martingale, we obtain

$$\begin{aligned} 0 = \sup_{\alpha \in \widetilde{\mathcal{S}}(t,y)} \mathbb{E}^v \Bigg(&J^\beta(t,y) \left(1 - \exp\left(- \int_t^{t+\epsilon} \beta(V - H(s, \widetilde{Y}_s^{t,y}))\alpha_s \, \mathrm{d}s \right) \right) \\ &+ \int_t^{t+\epsilon} \lambda(s)(\mu_s + \alpha_s)\partial_y J^\beta(s, \widetilde{Y}_s^{t,y}) + \partial_t J^\beta(s, \widetilde{Y}_s^{t,y}) + \frac{\lambda(s)^2\sigma(s)^2}{2} \partial_{yy} J^\beta(s, \widetilde{Y}_s^{t,y}) \mathrm{d}s \Bigg). \end{aligned}$$

Since

$$\lim_{\epsilon \searrow 0} \frac{1 - \exp\left(- \int_t^{t+\epsilon} \beta(V - H(s, \widetilde{Y}_s^{t,y}))\alpha_s \, \mathrm{d}s \right)}{\epsilon} = \beta(V - H(t,y))\alpha_t,$$

dividing by ϵ and sending ϵ to zero yields

$$\begin{aligned} 0 = \sup_\alpha \Bigg(&\beta J^\beta(t,y)(v - H(t,y))\alpha + \partial_t J^\beta(t,y) \\ &+ \lambda(t)(\mu(t,y,v) + \alpha)\partial_y J^\beta(t,y) + \frac{\lambda(t)^2\sigma(t)^2}{2} \partial_{yy} J^\beta(t,y) \Bigg). \end{aligned}$$

By linearity in α we get the following system of PDEs, for all $(t,y,v) \in (0,T) \times \mathbb{R} \times \mathcal{V}$,

$$0 = \frac{\beta(H(t,y) - v)}{\lambda(t)} J^\beta(t,y,v) - \partial_y J^\beta(t,y,v), \tag{3.22}$$

$$0 = \partial_t J^\beta(t,y,v) + \frac{\sigma(t)^2\lambda(t)^2}{2} \partial_{yy} J^\beta(t,y,v) + \lambda(t)\mu(t,y,v)\partial_y J^\beta(t,y,v). \tag{3.23}$$

Similarly to Subsection 3.2.1, the HJB approach leads to a system of two PDEs for J^β. We now again try to use the special dependences of these equations to derive necessary conditions on the pricing rule. We start with differentiating (3.22) w.r.t. y and t, respectively,

$$\begin{aligned} \partial_{yy} J^\beta(t,y) &= \frac{\beta}{\lambda(t)}\partial_y H(t,y) J^\beta(t,y) + \frac{\beta}{\lambda(t)}(H(t,y)-v)\partial_y J^\beta(t,y) \\ &\overset{(3.22)}{=} \frac{\beta}{\lambda(t)} J^\beta(t,y)\left(\partial_y H(t,y) + \frac{\beta}{\lambda(t)}(H(t,y)-v)^2\right), \\ \partial_{yt} J^\beta(t,y) &= \left(\frac{\beta}{\lambda(t)}\partial_t J^\beta(t,y) - \frac{\lambda'(t)}{\lambda(t)^2}\beta J^\beta(t,y)\right)(H(t,y)-v) + \frac{\beta}{\lambda(t)} J^\beta(t,y)\partial_t H(t,y). \end{aligned} \tag{3.24}$$

By combination of the latter equation with (3.23) we obtain

$$\begin{aligned} \partial_{yt} J^\beta(t,y) = &-\frac{\lambda'(t)}{\lambda(t)^2}\beta(H(t,y)-v)J^\beta(t,y) + \frac{\beta}{\lambda(t)} J^\beta(t,y)\partial_t H(t,y) \\ &+ \frac{\beta}{\lambda(t)}(H(t,y)-v)\left(-\frac{\sigma(t)^2\lambda(t)^2}{2}\partial_{yy} J^\beta(t,y) - \lambda(t)\mu(t,y,v)\partial_y J^\beta(t,y)\right). \end{aligned} \tag{3.25}$$

Now, by inserting (3.22) and (3.24) into (3.25) we get

$$\begin{aligned} \partial_{yt} J^\beta(t,y) = &-\frac{\lambda'(t)}{\lambda(t)^2}\beta(H(t,y)-v)J^\beta(t,y) + \frac{\beta}{\lambda(t)} J^\beta(t,y)\partial_t H(t,y) \\ &- \frac{\beta}{\lambda(t)}(H(t,y)-v)\left(\frac{\sigma(t)^2\lambda(t)^2}{2}\frac{\beta}{\lambda(t)} J^\beta(t,y)\left(\partial_y H(t,y) + \frac{\beta}{\lambda(t)}(H(t,y)-v)^2\right)\right. \\ &\left. + \beta\mu(t,y,v)(H(t,y)-v)J^\beta(t,y)\right) \\ = J^\beta(t,y)&\left(-\frac{\lambda'(t)}{\lambda(t)^2}\beta(H(t,y)-v) + \frac{\beta\partial_t H(t,y)}{\lambda(t)} - \frac{\sigma(t)^2\beta^3}{2\lambda(t)}(H(t,y)-v)^3\right. \\ &\left. - \frac{\sigma(t)^2\beta^2}{2}(H(t,y)-v)\partial_y H(t,y) - \frac{\beta^2}{\lambda(t)}\mu(t,y,v)(H(t,y)-v)^2\right). \end{aligned} \tag{3.26}$$

On the other hand, inserting (3.24) into (3.23) yields

$$\begin{aligned} 0 = \partial_t J^\beta(t,y) &+ \frac{\sigma(t)^2\lambda(t)\beta}{2} J^\beta(t,y)\left(\partial_y H(t,y) + \frac{\beta}{\lambda(t)}(H(t,y)-v)^2\right) \\ &+ \mu(t,y,v)\beta(H(t,y)-v)J^\beta(t,y) \end{aligned}$$

and differentiation w.r.t. y

$$\begin{aligned} 0 = \partial_{yt} J^\beta(t,y) &+ \frac{\sigma(t)^2\beta\lambda(t)}{2}\partial_y J^\beta(t,y)\left(\partial_y H(t,y) + \frac{\beta}{\lambda(t)}(H(t,y)-v)^2\right) \\ &+ \frac{\sigma(t)^2\lambda(t)\beta}{2} J^\beta(t,y)\left(\partial_{yy} H(t,y) + \frac{2\beta}{\lambda(t)}\partial_y H(t,y)(H(t,y)-v)\right) \\ &+ \beta\partial_y\left(\mu(t,y,v)(H(t,y)-v)\right)J^\beta(t,y) + \beta\mu(t,y,v)(H(t,y)-v)\partial_y J^\beta(t,y). \end{aligned}$$

Together with (3.22) we obtain

$$
\begin{aligned}
0 &= \partial_{yt} J^\beta(t,y) + \frac{\sigma(t)^2\beta^2}{2}(H(t,y)-v)J^\beta(t,y)\left(\partial_y H(t,y) + \frac{\beta}{\lambda(t)}(H(t,y)-v)^2\right) \\
&\quad + \frac{\sigma(t)^2\lambda(t)\beta}{2} J^\beta(t,y)\left(\partial_{yy}H(t,y) + \frac{2\beta}{\lambda(t)}\partial_y H(t,y)(H(t,y)-v)\right) \\
&\quad + \beta\partial_y\left(\mu(t,y,v)(H(t,y)-v)\right)J^\beta(t,y) + \frac{\beta^2}{\lambda(t)}\mu(t,y,v)(H(t,y)-v)^2 J^\beta(t,y) \\
&= \partial_{yt} J^\beta(t,y) + J^\beta(t,y)\left(\frac{\sigma(t)^2\beta^2}{2}(H(t,y)-v)\partial_y H(t,y) + \frac{\sigma(t)^2\beta^3}{2\lambda(t)}(H(t,y)-v)^3\right. \\
&\quad + \frac{\sigma(t)^2\lambda(t)\beta}{2}\partial_{yy}H(t,y) + \sigma(t)^2\beta^2\partial_y H(t,y)(H(t,y)-v) \\
&\quad \left. +\beta\partial_y\left(\mu(t,y,v)(H(t,y)-v)\right) + \frac{\beta^2}{\lambda(t)}\mu(t,y,v)(H(t,y)-v)^2\right).
\end{aligned}
\tag{3.27}
$$

Adding (3.26) and (3.27) together and then dividing by β and $J^\beta(t,y)$ finally yields

$$
\begin{aligned}
0 &= \frac{1}{\lambda(t)}\partial_t H(t,y) + \frac{\sigma(t)^2\lambda(t)}{2}\partial_{yy}H(t,y) - \frac{\lambda'(t)}{\lambda(t)^2}(H(t,y)-v) \\
&\quad + \sigma(t)^2\beta\partial_y H(t,y)(H(t,y)-v) + \partial_y\left(\mu(t,y,v)(H(t,y)-v)\right),
\end{aligned}
\tag{3.28}
$$

for all $(t,y,v) \in (0,T)\times\mathbb{R}\times\mathcal{V}$. This leads to the following characterisation.

Proposition 3.7. *Let $(H,\lambda) \in \mathcal{P}$ and μ be as in Assumption 3.4. If there exists a function $J^\beta \in \mathcal{C}^{1,2}((0,T)\times\mathbb{R})$ such that (H,λ,J^β) is a solution to the system of Equations (3.22) and (3.23), then necessarily the following holds:*

- *H satisfies the PDE*

$$
\partial_t H(t,y) + \frac{1}{2}\sigma(t)^2\lambda(t)^2\partial_{yy}H(t,y) = 0, \quad \textit{for all } (t,y) \in (0,T)\times\mathbb{R}, \tag{3.29}
$$

- *μ has the form*

$$
\mu(t,y,v) = \begin{cases} -\frac{\lambda'(t)}{\lambda(t)^2}\frac{\int_y^{y^*(t,v)}(H(t,x)-v)\,\mathrm{d}x}{H(t,y)-v} - \frac{\sigma(t)^2\beta}{2}(H(t,y)-v) & , \textit{if } y \neq y^*(t,v) \\ 0 & , \textit{if } y = y^*(t,v) \end{cases}, \tag{3.30}
$$

  for all $(t,v) \in (0,T)\times\mathcal{V}$.

In particular μ, as defined in (3.30), always verifies conditions (2) and (3) of Assumption 3.4, and condition (4) if and only if

$$
0 < -\frac{\lambda'(t)}{\lambda(t)^2}\int_{y^*(t,v)}^{y}(H(t,x)-v)\,\mathrm{d}x + \frac{\sigma(t)^2\beta}{2}(H(t,y)-v)^2 \tag{3.31}
$$

holds for all $(t,y,v) \in [0,T)\times\mathbb{R}\times\mathcal{V}$.

Proof. The proof of this proposition is nearly the same as the proof of Proposition 3.6. We only have to replace (3.18) by (3.28) and solve

$$\partial_y(\mu(t,y,v)(H(t,y)-v)) = (H(t,y)-v)\left(\frac{\lambda'(t)}{\lambda(t)^2} - \sigma(t)^2\beta\partial_y H(t,y)\right) - f(t,y),$$

for $(t,y,v) \in (0,T)\times\mathbb{R}\times\mathcal{V}$, instead of (3.19), where again

$$f(t,y) = \frac{1}{\lambda(t)}\partial_t H(t,y) + \frac{\lambda(t)\sigma(t)^2}{2}\partial_{yy}H(t,y).$$

By an analogous argumentation as in the proof of Proposition 3.6, we obtain that μ has to take the general form

$$\mu(t,y,v) = \frac{\int_{y^*(t,v)}^{y}(H(t,x)-v)\left(\frac{\lambda'(t)}{\lambda(t)^2} - \sigma(t)^2\beta\partial_y H(t,x)\right) - f(t,x)\,\mathrm{d}x + C_2(t,v)}{H(t,y)-v} \tag{3.32}$$

and, furthermore, that C_2 and f have to equal 0. Therefore, we get (3.29) and (3.30) since

$$\begin{aligned}\mu(t,y,v) &= \frac{\int_{y^*(t,v)}^{y}(H(t,x)-v)\left(\frac{\lambda'(t)}{\lambda(t)^2} - \sigma(t)^2\beta\partial_y H(t,x)\right)\mathrm{d}x}{H(t,y)-v} \qquad (3.33)\\ &= \frac{\int_{y^*(t,v)}^{y}(H(t,x)-v)\frac{\lambda'(t)}{\lambda(t)^2}\,\mathrm{d}x}{(H(t,y)-v)} - \sigma(t)^2\beta\frac{\frac{1}{2}(H(t,y)-v)^2}{(H(t,y)-v)}\\ &= \frac{\lambda'(t)}{\lambda(t)^2}\frac{\int_{y^*(t,v)}^{y}(H(t,x)-v)\,\mathrm{d}x}{H(t,y)-v} - \frac{\sigma(t)^2\beta}{2}(H(t,y)-v). \qquad (3.34)\end{aligned}$$

For the last part of the assertion observe that, according to Remark 3.5, condition (4) of Assumption 3.4 is satisfied if and only if $\mu(t,y,v)(v-H(t,y)) > 0$ for all $y \neq v$ (again $\mu(t,y^*(t,v),v) = 0$ holds by construction). Multiplying both sides of (3.34) with $(v-H(t,y))$ yields

$$\mu(t,y,v)(v-H(t,y)) = -\frac{\lambda'(t)}{\lambda(t)^2}\int_{y^*(t,v)}^{y}(H(t,x)-v)\,\mathrm{d}x + \frac{\sigma(t)^2\beta}{2}(H(t,y)-v)^2.$$

□

3.2.3 Admissible noise drift

The preceding Propositions 3.6 and 3.7 state necessary conditions for the existence of an equilibrium on μ and H for the risk neutral and risk averse case. In both cases

$$\partial_t H(t,y) + \frac{1}{2}\sigma(t)^2\lambda(t)^2\partial_{yy}H(t,y) = 0$$

has to hold. This is consistent with other insider trading models, for example [7], [25], [28], [19]. Furthermore, the above results suggest a class of *admissible noise drift* that is characterised in this subsection.

Characterisation of μ and connection to λ

If we identify $\beta = 0$ as the parameter for the risk neutral case, we can summarise that, according to Propositions 3.6 and 3.7, μ has to take the special form

$$\mu(t,y,v) = \begin{cases} m(t)\frac{\int_y^{y^*(t,v)} H(t,x)-v\,\mathrm{d}x}{H(t,y)-v} - \frac{\sigma(t)^2\beta}{2}(H(t,y)-v) & ,\text{if } y \neq y^*(t,v) \\ 0 & ,\text{if } y = y^*(t) \end{cases}, \tag{3.35}$$

where the second term vanishes in the risk neutral case ($\beta = 0$). Otherwise, there exists no equilibrium. Hence, a noise drift that preserves the possibility for the existence of an equilibrium having a form as in (3.35) can be called *admissible*. The terms of such a noise drift where V and $\widetilde{Y}_t$ enter,

$$\frac{\int_y^{y^*(t,v)} H(t,x)-v\,\mathrm{d}x}{H(t,y)-v} \quad \text{and} \quad \frac{\sigma(t)^2\beta}{2}(H(t,y)-v),$$

are fixed. Hence, admissible noise drift can be characterised by a deterministic positive coefficient $m(t)$ that is connected with the price pressure via

$$\frac{\mathrm{d}}{\mathrm{d}t}\lambda(t) = -m(t)\lambda(t)^2, \quad t \in [0,T), \quad \lambda(0) = \lambda_0 > 0. \tag{3.36}$$

Note that the particular noise drift determines only the dynamics of the price pressure and not its initial condition. The above Bernoulli type ODE (3.36) is solved by

$$\lambda(t) = \frac{1}{\lambda_0 + \int_0^t m(s)\,\mathrm{d}s}, \tag{3.37}$$

given that we have

$$\int_0^t m(s)\,\mathrm{d}s < \infty, \quad \text{for all } t \in [0,T).$$

This motivates the following assumption.

Assumption 3.8. *The noise drift μ has the form (3.35) for a positive function m, called* (noise) drift intensity*, such that m is bounded on $[0,t]$, for all $t \in [0,T)$.*

Remarks on the price pressure

In the special case when $\beta = 0$ and $m = 0$, we are in the situation of the classical model of Back [7], see also [25]. According to these articles, λ is constant in equilibrium. This is consistent with the form of λ in (3.37).

Although we have neither shown the existence of a function J^0 (or J^β) that is a solution to the system of Equations (3.10) and (3.11) ((3.22) and (3.23) respectively) nor any other part of the equilibrium, yet, we can state the following: If the equilibrium exists and the noise drift intensity m is not equal to zero, we are in a model with non-constant price pressure or, more detailed, with a (strictly) decreasing one.

Typically a decreasing price pressure would lead an insider to withhold her information, in order to profit from the higher market depth at the end of the trading period, i.e. a higher trading volume is required to reveal the information. In a market with partially informed noise traders, on the one hand, waiting is penalised by revelation of information through the noise traders. On the other hand, early trading (early revelation of information) is less profitable because of the higher price pressure. Intuitively, these two effects should be neutralised by a suitable chosen price pressure such that an equilibrium is possible. That this is indeed the case is shown in the following sections.

Coming back to our initial considerations in the introduction of this thesis, a market becoming less irrational should be reflected by an increasing drift intensity m. Alternatively, a decreasing price pressure might also be an intuitive reference for the informational asymmetry since a higher uncertainty about the real value of an asset should cause a more volatile market price. This intuitive connection is provided by this model. Furthermore, a price pressure reaching the state zero, i.e. $\lim_{t\to T}\lambda(t) = 0$, would reflect complete consensus of the market regarding the price of the asset. As a consequence of the above results this is the case if and only if

$$\int_0^t m(s)\,\mathrm{d}s \nearrow \infty, \quad \text{for } t \to T,$$

i.e. if the (integrated) drift intensity increases to infinity. In the following paragraph, we will see that in this case the market is efficient even in absence of an insider, i.e. we have

$$\lim_{t\to T} H(t, \widetilde{X}_t) = V$$

where

$$\widetilde{X}_t := \int_0^t \lambda(s)\,\mathrm{d}X_s,\ t \in [0,T],$$

is the total weighted order in absence of the insider. This is different to all other Kyle-Back type insider trading equilibria where efficiency is only provided by the insider. Furthermore, this supports the assumption that from time T on the former noise traders act rational and the market is strong form efficient.

Before we prove this important result, let us first come back to the initial considerations on the calculation of the insider's wealth in Chapter 1. We pointed out that the terminal wealth depends on the assumption whether we have $P_t = V_t$ fixed for $t \geq T$ or $t > T$, i.e. whether trades in T are executed at the fundamental value or the market price that might not be equal to the fundamental value. If we assume the latter, in the case of a vanishing price pressure the insider might have the possibility to gain infinite wealth from every trading strategy which leads the market price to $\lim_{t\to T} P_t \neq V$. Since the price pressure in T is equal to zero, the trading volume in T would have no influence on the price. Hence, exploiting the difference of V and P_T, the insider could generate infinite wealth. However, we assume that $P_T = V$. As we mentioned above this is plausible especially in case of an infinite trading intensity due to the following statement on market efficiency.

Market efficiency in absence of the insider

Since we allow the noise drift intensity to be unbounded, one could ask whether the market is already efficient without the presence of an insider. If m is bounded, this, quite obviously, is not the case. But, if $\lim_{t\to T} \int_0^t m(s)\,\mathrm{d}s = \infty$, the market is efficient even without an insider.

Proposition 3.9. *Let Assumptions 3.1 and 3.8 be satisfied such that* $\lim_{t\to T} \int_0^t m(s)\,\mathrm{d}s = \infty$. *Furthermore, suppose that for* $(H,\lambda) \in \mathcal{P}$*, (3.12), (3.37) and*

$$\sup_{t\in[0,T]} \mathbb{E} H(t,\widetilde{X}_t)^2 < \infty$$

hold. Then $H(T,\widetilde{X}_T) = V$ $\mathbb{P}$*-a.s.*

Proof. In order to prove the assertion, we show for

$$M(t,y) := \int_{y^*(t,V)}^{y} H(t,x) - V\,\mathrm{d}x \tag{3.38}$$

that

$$M(t,\widetilde{X}_t) \xrightarrow{\mathbb{P}} 0, \quad t \to T.$$

This is indeed sufficient since the monotonicity of H and the definition of y^* ensure

$$M(t,y) = 0 \quad \Leftrightarrow \quad y = y^*(t,V).$$

We want to apply Itô's formula to $M(t,\widetilde{X}_t)$. Therefore, we start with the calculation of the partial derivatives of M. Obviously,

$$\partial_y M(t,y) = H(t,y) - V, \quad \partial_{yy} M(t,y) = \partial_y H(t,y).$$

To calculate the partial derivative w.r.t. t, let us first consider the function

$$\widetilde{M}(t,y,z) := \int_z^y H(t,x) - V\,\mathrm{d}x, \quad t \in (0,T), (y,z) \in \mathbb{R}^2.$$

Together with PDE (3.12) satisfied by H this yields

$$\begin{aligned}
\partial_z \widetilde{M}(t,y,z) &= V - H(t,z),\\
\partial_t \widetilde{M}(t,y,z) &= \int_z^y \partial_t H(t,x)\,\mathrm{d}x = \int_z^y -\frac{1}{2}\sigma(t)^2\lambda(t)^2\partial_{yy}H(t,x)\,\mathrm{d}x\\
&= -\frac{1}{2}\sigma(t)^2\lambda(t)^2\left(\partial_y H(t,y) - \partial_y H(t,z)\right).
\end{aligned}$$

By differentiability of the implicit function $y^*(\cdot,v)$ (for all $v \in \mathcal{V}$), we get

$$\begin{aligned}
\partial_t M(t,y) &= \partial_t \widetilde{M}(t,y,y^*(t,V)) + \partial_z \widetilde{M}(t,y,y^*(t,V))\partial_t y^*(t,V)\\
&= -\frac{\sigma(t)^2\lambda(t)^2}{2}(\partial_y H(t,y) - \partial_y H(t,y^*(t,V))) + (V - H(t,y^*(t,V)))\partial_t y^*(t,V)\\
&= -\frac{\sigma(t)^2\lambda(t)^2}{2}(\partial_y H(t,y) - \partial_y H(t,y^*(t,V))). \qquad (3.39)
\end{aligned}$$

With the above calculated partial derivatives we are now able to apply Itô's formula

$$\begin{aligned}
\mathrm{d}M(t,\widetilde{X}_t) &= \partial_y M(t,\widetilde{X}_t)\mathrm{d}\widetilde{X}_t + \frac{1}{2}\partial_{yy}M(t,\widetilde{X}_t)\mathrm{d}\langle\widetilde{X}\rangle_t + \partial_t M(t,\widetilde{X}_t)\mathrm{d}t\\
&= \partial_y M(t,\widetilde{X}_t)\lambda(t)\mu(t,\widetilde{X}_t,V) + \frac{\sigma(t)^2\lambda(t)^2}{2}\partial_y H(t,y^*(t,V))\mathrm{d}t + \partial_y M(t,\widetilde{X}_t)\lambda(t)\sigma(t)\mathrm{d}B_t.
\end{aligned}$$

Since

$$\begin{aligned}
\mu(t,\widetilde{X}_t,V)\partial_y M(t,\widetilde{X}_t) &= \left(-m(t)\frac{M(t,\widetilde{X}_t)}{H(t,\widetilde{X}_t)-V} - \frac{\sigma(t)^2\beta}{2}\Big(H(t,\widetilde{X}_t)-V\Big)\right)\Big(H(t,\widetilde{X}_t)-V\Big)\\
&= -m(t)M(t,\widetilde{X}_t) - \frac{\sigma(t)^2\beta}{2}\Big(H(t,\widetilde{X}_t)-V\Big)^2,
\end{aligned}$$

we get

$$\begin{aligned}
\mathrm{d}M(t,\widetilde{X}_t) &= -m(t)M(t,\widetilde{X}_t)\lambda(t) - \frac{\lambda(t)\sigma(t)^2\beta}{2}\Big(H(t,\widetilde{X}_t)-V\Big)^2\mathrm{d}t\\
&\quad + \Big(H(t,\widetilde{X}_t)-V\Big)\lambda(t)\sigma(t)\mathrm{d}B_t + \frac{\sigma(t)^2\lambda(t)^2}{2}\partial_y H(t,y^*(t,V))\mathrm{d}t.
\end{aligned}$$

For $t \in [0,T)$, let

$$\rho(t) := \exp\left(\int_0^t m(s)\lambda(s)\,\mathrm{d}s\right).$$

In particular,

$$\mathrm{d}\rho(t) = \lambda(t)m(t)\rho(t)\mathrm{d}t.$$

Moreover, $\lim_{t\to T}\rho(t)=\infty$ because

$$\begin{aligned}\lim_{t\to T}\int_0^t m(s)\lambda(s)\,\mathrm{d}s &= \lim_{t\to T}\int_0^t \frac{m(s)}{\int_0^s m(u)\,\mathrm{d}u+\lambda_0^{-1}}\,\mathrm{d}s\\ &= \lim_{t\to T}\log\left(\int_0^t m(s)\,\mathrm{d}s+\lambda_0^{-1}\right)-\log(\lambda_0^{-1})=\infty.\end{aligned}$$

Now, integration by parts yields

$$\begin{aligned}\mathrm{d}M(t,\widetilde{X}_t)\rho(t) &= \rho(t)\mathrm{d}M(t,\widetilde{X}_t)+M(t,\widetilde{X}_t)\mathrm{d}\rho(t)\\ &= -\rho(t)\left(m(t)M(t,\widetilde{X}_t)\lambda(t)+\frac{\lambda(t)\sigma(t)^2\beta}{2}\left(H(t,\widetilde{X}_t)-V\right)^2\right)\mathrm{d}t\\ &\quad+\rho(t)\left(H(t,\widetilde{X}_t)-V\right)\lambda(t)\sigma(t)\mathrm{d}B_t\\ &\quad+\rho(t)\frac{\sigma(t)^2\lambda(t)^2}{2}\partial_y H(t,y^*(t,V))\mathrm{d}t+M(t,\widetilde{X}_t)\lambda(t)m(t)\rho(t)\mathrm{d}t\\ &= \rho(t)\left(H(t,\widetilde{X}_t)-V\right)\lambda(t)\sigma(t)\mathrm{d}B_t+\rho(t)\frac{\sigma(t)^2\lambda(t)^2}{2}\partial_y H(t,y^*(t,V))\mathrm{d}t\\ &\quad-\rho(t)\frac{\lambda(t)\sigma(t)^2\beta}{2}\left(H(t,\widetilde{X}_t)-V\right)^2\mathrm{d}t.\end{aligned}$$

Using integral notation and multiplying $\rho(t)^{-1}$ on both sides, we get

$$\begin{aligned}M(t,\widetilde{X}_t) &= M(0,0)\rho(t)^{-1}+\rho(t)^{-1}\int_0^t\rho(s)\left(H(s,\widetilde{X}_s)-V\right)\lambda(s)\sigma(s)\,\mathrm{d}B_s\\ &\quad+\rho(t)^{-1}\int_0^t\rho(s)\frac{\sigma(s)^2\lambda(s)}{2}\left(\lambda(s)\partial_y H(s,y^*(s,V))-\beta\left(H(s,\widetilde{X}_s)-V\right)^2\right)\mathrm{d}s.\end{aligned}\tag{3.40}$$

Now, Itô isometry (cf. [52], Corollary 3.1.7) yields

$$\begin{aligned}&\mathbb{E}\left(\rho(t)^{-1}\int_0^t\rho(s)\left(H(s,\widetilde{X}_s)-V\right)\lambda(s)\sigma(s)\,\mathrm{d}B_s\right)^2\\ &\quad=\mathbb{E}\left(\rho(t)^{-2}\int_0^t\rho(s)^2\left(H(s,\widetilde{X}_s)-V\right)^2\lambda(s)^2\sigma(s)^2\,\mathrm{d}s\right)\\ &\quad\le 2(\sup_{t\in[0,T]}\mathbb{E}H(t,\widetilde{X}_t)^2+\mathbb{E}V^2)\left(\rho(t)^{-2}\int_0^t\rho(s)^2\lambda(s)^2\sigma(s)^2\,\mathrm{d}s\right).\end{aligned}$$

If the integral term in the last line is bounded, the whole term vanishes for $t\to T$ since $\lim_{t\to T}\rho^{-1}(t)=0$. Otherwise, we get with l'Hospital's rule

$$\lim_{t\to T}\rho(t)^{-2}\int_0^t\rho(s)^2\lambda(s)^2\sigma(s)^2\,\mathrm{d}s=\lim_{t\to T}\frac{\rho(t)^2\lambda(t)^2\sigma(t)^2}{2\rho(t)^2m(t)\lambda(t)}=0.$$

Applying the same arguments to

$$\rho(t)^{-1} \int_0^t \rho(s)\lambda(s)\sigma(s)^2 \, \mathrm{d}s$$

this proves the assertion since

$$\lambda(t)\partial_y H(t, y^*(t, V)) - \beta \left(H(t, \widetilde{X}_t) - V \right)^2$$

is pathwise bounded. □

From the representation of $M(t, \widetilde{X}_t)$ in (3.40) we can conclude that a vanishing noise, i.e. $\sigma(t) \searrow 0$, for $t \to T$, would not lead to market efficiency since the terms on the right hand side only vanish if ρ^{-1} does, too. The latter happens if and only if $\int_0^t m(s) \, \mathrm{d}s \nearrow \infty$, $t \to T$. Hence, efficiency does not follow from decreasing noise trading, but from increasing informed trading.

The risk premium

Let us now again come back to the special form of admissible noise drift. Due to the additional part $\frac{\sigma(t)^2 \beta}{2}(H(t, y) - V)$ in the risk averse case, the price pressure

$$\lambda(t) = \frac{1}{\lambda_0^{-1} + \int_0^t m(s) \, \mathrm{d}s}, \quad t \geq 0,$$

is related to a slightly different noise drift compared to the risk neutral case. Since $\beta < 0$, this additional term makes the drift, roughly speaking, *less rational.* $\frac{\sigma(t)^2 \beta}{2}(H(t, y) - V)$ might therefore be seen as a risk premium on informed trading for the insider. However, we cannot guarantee that condition (3.31) holds in general. This strongly depends on H. It might rather happen that there exists no semi-rational noise drift at all which has the form as in Assumption 3.8. But, in contrast to Cho [25], equilibria are still possible beyond a linear framework. We now give a sufficient condition for the existence of an admissible semi-rational noise drift.

Proposition 3.10. *Let $\beta < 0$ and H be an admissible pricing function such that $\partial_y H(t, y) < C$ is bounded. Then there exists a semi-rational noise drift as in Assumption 3.8.*

Proof. Let m be an integrable function that is bounded from above on $[0, t]$ for all $t \in [0, T)$ and bounded from below by $-\sigma(t)^2 \beta C + \epsilon$ for some $\epsilon > 0$. Then (cf. Equation (3.33))

$$\mu(t, y, v) = \frac{\int_{y^*(t,v)}^y (H(t, x) - v) \left(-m(t) - \sigma(t)^2 \beta \partial_y H(t, x) \right) \mathrm{d}x}{H(t, y) - v}.$$

Due to our assumptions

$$-m(t) - \sigma(t)^2 \beta H(t, y) < -\epsilon \quad \text{for all } (t, y) \in [0, T] \times \mathbb{R}.$$

It follows

$$\mu(t,y,v)(H(t,y)-v) \leq -\int_{y^*(t,v)}^{y} (H(t,x)-v)\epsilon \,\mathrm{d}x < 0 \quad \text{for all } y \neq y^*(t,v).$$

Hence, μ is semi-rational (cf. Remark 3.5). □

We close this subsection with an example for the above situation.

Example 3.11. Let $H(t,y) := p_0 + qy$, $q > 0$. In the standard insider model analysed in [25] this is a necessary condition for the existence of an equilibrium in the risk averse case. We have

$$y^*(t,v) = \frac{v-p_0}{q}.$$

Furthermore,

$$\int_{y^*(t,v)}^{y} H(t,x) - v \,\mathrm{d}x = \int_{(v-p_0)q^{-1}}^{y} p_0 + qx - v \,\mathrm{d}x = \frac{1}{2q}\left(qy + p_0 - v\right)^2.$$

Hence, according to Proposition 3.7, μ is semi-rational if $m(t)q^{-1} + \sigma(t)^2\beta > 0$. If we choose $m(t) = -\sigma(t)^2\beta q$, we have $\mu = 0$. This is the case considered in [25]. Indeed, according to (3.36), λ^{-1} then has the dynamics

$$\mathrm{d}(\lambda(t)^{-1}) = -\sigma(t)^2\beta q \mathrm{d}t.$$

This coincides with the equilibrium price pressure in [25] (cf. Subsection 1.2.2).

3.3 Optimality: risk neutral insider

In the last section we have derived necessary conditions for the existence of an equilibrium in the case of absolutely continuous insider trading. We now enlarge the class of possible insider strategies to $\mathcal{S}(H,\lambda)$. In particular, we allow discontinuous trading strategies and strategies with quadratic variation. However, it turns out that an optimal trading strategy has to be absolutely continuous under the conditions that have been pointed out to be necessary for an equilibrium in the case of absolutely continuous insider trading. One of these conditions was the special form of μ as in Assumption 3.8. As we have seen in Subsection 3.2.3, $\lambda(T) = 0$ if and only if $\int_0^t m(s)\mathrm{d}s \nearrow \infty$, for $t \to T$. Mathematically, this case involves certain technical difficulties since the definition of the value function contains the term $\lambda(T)^{-1}$. Heuristically, one might wonder whether the fact that the market is efficient by itself (see Subsection 3.2.3) has an impact on the optimality of an insider strategy. Therefore, we start our analysis of optimality with the bounded case. This ensures that λ is bounded away from zero on the whole interval $[0,T]$. The case where $\lambda(T) = 0$ is considered separately thereafter.

3.3.1 Bounded drift intensity

First of all, we determine the value function J^0 considered in Subsection 3.2.1.

Lemma 3.12. *Let Assumptions 3.1 and 3.8 be satisfied such that m is bounded from above on $[0,T]$. Furthermore, suppose that for $(H,\lambda) \in \mathcal{P}$, (3.12) and (3.37) hold and for $(t,y,v) \in [0,T] \times \mathbb{R} \times \mathcal{V}$ define*

$$J^0(t,y,v) := \mathbb{E}^v \left(\int_t^T \mu(s, \xi_s^{t,y}, V)(H(s,\xi_s^{t,y}) - V)\,\mathrm{d}s + \lambda(T)^{-1} \int_{y^*(T,V)}^{\xi_T^{t,y}} (H(T,y) - V)\,\mathrm{d}y \right) \tag{3.41}$$

where $\xi^{t,y}$ is defined as in (3.4). Then the triple (H,λ,J^0) is a solution to the system of Equations (3.10) and (3.11). In particular, $J^0(T,y,v) \geq 0$, for all $y \in \mathbb{R}$, with equality if and only if $y = y^*(T,v)$.

Proof. First of all, we have to show that J^0 is well-defined. To see this, note that for all $(t,y) \in [0,T] \times \mathbb{R}$ we have $\mu(t,y,V)(H(t,y) - V) \leq 0$ (cf. Remark 3.5). Hence, we get with Fubini's theorem

$$\begin{aligned}
\mathbb{E}\left(\left| \int_0^T \mu(s,\xi_s,V)(H(s,\xi_s) - V)\,\mathrm{d}s \right| \right) &= \mathbb{E}\left(- \int_0^T \mu(s,\xi_s,V)(H(s,\xi_s) - V)\,\mathrm{d}s \right) \\
&= - \int_0^T \mathbb{E}\left(\mu(s,\xi_s,V)(H(s,\xi_s) - V) \right) \mathrm{d}s.
\end{aligned}$$

Now by (3.35) and the monotonicity of H

$$\begin{aligned}
- \int_0^T \mathbb{E}\left(\mu(s,\xi_s,V)(H(s,\xi_s) - V) \right) \mathrm{d}s &= \int_0^T m(s)\mathbb{E}\left(\int_{y^*(s,V)}^{\xi_s} H(s,x) - V\,\mathrm{d}x \right) \mathrm{d}s \\
&\leq \int_0^T m(s)\mathbb{E}\left((\xi_s - y^*(s,V))(H(s,\xi_s) - V) \right) \mathrm{d}s.
\end{aligned}$$

Let m be bounded by some constant M. This yields

$$\begin{aligned}
&\int_0^T m(s)\mathbb{E}\left((\xi_s - y^*(s,V))(H(s,\xi_s) - V) \right) \mathrm{d}s \\
&\qquad \leq M \int_0^T \mathbb{E}\left((\xi_s)^2 + (y^*(s,V))^2 + (H(s,\xi_s))^2 + V^2 \right) \mathrm{d}s.
\end{aligned}$$

Since $\xi_t = \int_0^t \lambda(s)\sigma(s)\,\mathrm{d}B_s$, for $t \in [0,T]$, and

$$\mathbb{E}\left(\int_0^t \lambda(s)\sigma(s)\,\mathrm{d}B_s \right)^2 = \int_0^t \sigma(s)^2\lambda(s)^2\,\mathrm{d}s \leq \lambda(0)^2 \int_0^T \sigma(s)^2\,\mathrm{d}s < \infty,$$

we get together with $\sup_{t\in[0,T]} \mathbb{E}(y^*(t,V))^2 < \infty$ that $\mathbb{E}\left((\xi_t)^2 + (y^*(t,V))^2\right)$ is uniformly bounded on $[0,T]$. Finally, the properties of a rational pricing function H ensure

$$\mathbb{E}\left(\left|\int_0^T \mu(s,\xi_s,V)(H(s,\xi_s)-V)\,\mathrm{d}s\right|\right)<\infty. \tag{3.42}$$

Furthermore, for the $\mathcal{C}^2$ function

$$y\mapsto \lambda(T)^{-1}\int_{y^*(T,V)}^{y} H(T,y)-V\,\mathrm{d}y$$

an analogous argumentation leads to

$$\mathbb{E}\left|\lambda(T)^{-1}\int_{y^*(T,V)}^{\xi_T} H(T,y)-V\,\mathrm{d}y\right|\leq \lambda(T)^{-1}\mathbb{E}\left((\xi_T)^2+(y^*(T,V))^2+H(T,\xi_T)^2+V^2\right)<\infty.$$

Now, in order to prove that J^0 satisfies the PDEs (3.10) and (3.11), we consider the equivalent system

$$\partial_y J^0(t,y,v)=\frac{H(t,y)-v}{\lambda(t)}, \tag{3.43}$$

$$\partial_t J^0(t,y,v)=-\mu(t,y,v)(H(t,y)-v)-\frac{1}{2}\sigma(t)^2\lambda(t)^2\partial_{yy}J^0(t,y,v). \tag{3.44}$$

The smoothness of J^0 follows from definition (see also [43], Section 4.3). Due to Feynman-Kac's formula (or the calculations made for the HJB equation in Subsection 3.2.1) we know that $J^0(t,y,v)$ is a solution to Equation (3.44) for all $(t,y)\in(0,T)\times\mathbb{R}$ if

$$\mathbb{E}^v\int_0^T \lambda(s)^2\sigma(s)^2\partial_y J^0(s,\xi_s,V)^2\,\mathrm{d}s<\infty,$$

which ensures that $\int_0^\cdot \lambda(s)\sigma(s)\partial_y J^0(s,\xi_s,v)\,\mathrm{d}B_s$ is a martingale. If (3.43) holds,

$$\mathbb{E}^v\int_0^T \lambda(s)^2\sigma(s)^2\partial_y J^0(s,\xi_s,V)^2\,\mathrm{d}s=\mathbb{E}^v\int_0^T \sigma(s)^2\left(H(s,\xi_s)-V\right)^2\,\mathrm{d}s<\infty.$$

It remains to show that the partial derivative of J^0 w.r.t. y verifies (3.43). Using dominated convergence and Fubini's theorem we get

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\mathbb{E}^v\left(\int_t^T m(s)\int_{\xi_s^{t,y}+\epsilon}^{y^*(s,V)} H(s,x)-V\,\mathrm{d}x\,\mathrm{d}s-\int_t^T m(s)\int_{\xi_s^{t,y}}^{y^*(s,V)} H(s,x)-V\,\mathrm{d}x\,\mathrm{d}s\right)\\
&=\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\mathbb{E}^v\left(\int_t^T m(s)\int_{\xi_s^{t,y}+\epsilon}^{\xi_s^{t,y}} H(s,x)-V\,\mathrm{d}x\,\mathrm{d}s\right)\\
&=\int_t^T m(s)\mathbb{E}^v\left(\lim_{\epsilon\downarrow 0}\frac{1}{\epsilon}\int_{\xi_s^{t,y}+\epsilon}^{\xi_s^{t,y}} H(s,x)-V\,\mathrm{d}x\right)\,\mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&= \int_t^T -m(s)\mathbb{E}^v \left(H(s,\xi_s^{t,y}) - V\right) \mathrm{d}s \\
&= \int_t^T -m(s)(H(t,y) - v) \, \mathrm{d}s \\
&= -(H(t,y) - v) \int_t^T m(s) \, \mathrm{d}s
\end{aligned}
$$

where in the second to last line we have used the fact that (due to PDE (3.12) and admissibility of (H, λ)) $(H(t, \xi_t), t \in [0,T])$ is a martingale. An analogous calculation for $\epsilon \uparrow 0$ yields the same result. Thus,

$$
\partial_y \mathbb{E}^v \left(\int_t^T m(s) \int_{\xi_s^{t,y}}^{y^*(s,V)} H(s,x) - V \, \mathrm{d}x \, \mathrm{d}s \right) = -(H(t,y) - v) \int_t^T m(s) \, \mathrm{d}s.
$$

With the same arguments we can deduce

$$
\partial_y \mathbb{E}^v \left(\lambda(T)^{-1} \int_{y^*(T,V)}^{\xi_T^{t,y}} (H(T,x) - V) \, \mathrm{d}x \right) = (H(t,y) - v)\lambda(T)^{-1}.
$$

Together this yields

$$
\partial_y J^0(t,y,v) = (H(t,y) - v) \left(\lambda(T)^{-1} - \int_t^T m(s) \, \mathrm{d}s \right) \overset{(3.37)}{=} (H(t,y) - v)\lambda(t)^{-1}.
$$

Last but not least, strict monotonicity of H directly implies $J^0(T,y,v) \geq 0$ for all $(y,v) \in \mathbb{R} \times \mathcal{V}$ with equality if and only if $y = y^*(T,v)$. This proves the second part of the assertion. $\square$

With the help of the preceding lemma, we are now able to derive sufficient conditions for the optimality of insider strategies. For necessity we have to show that there indeed exists a strategy that satisfies the below given criteria. This is done in Section 3.6. Furthermore, we note that the given optimality criteria correspond to those of other insider trading models, e.g. [7] or [28]. Hence, given that μ satisfies the conditions of Assumption 3.8 with a bounded drift intensity and that the price pressure is chosen correspondently, i.e. according to (3.37), the presence of partially informed noise traders has no influence on the structure of optimal strategies, so far.

Proposition 3.13. *Let Assumptions 3.1 and 3.8 be satisfied such that m is bounded from above on $[0,T]$. Furthermore, suppose that for $(H,\lambda) \in \mathcal{P}$, (3.12) and (3.37) hold. Then $\theta \in \mathcal{S}(H,\lambda)$ is optimal if*

(1) $H(T, \widetilde{Y}_T) = V$ $\mathbb{P}$*-a.s.,*

(2) θ *is of finite variation and continuous.*

Proof. In the sequel, we make use of the notation J_s^0 instead of $J^0(s, \widetilde{Y}_s, V)$ and H_s instead of $H(s, \widetilde{Y}_s)$. As pointed out in (1.2), the insider's wealth at time T corresponding to a strategy θ

can be written as

$$W_T^\theta = \int_0^{T-} V - H_{s-}\,\mathrm{d}\theta_s + [\theta, V-H]_{T-} \tag{3.45}$$

where we used the fact that V_t is constant on $[0,T]$. Since

$$H_t = H_0 + \int_0^t \partial_y H_{s-}\lambda(s)\,\mathrm{d}Y_s^c + \int_0^t \partial_t H_s\,\mathrm{d}s + \frac{1}{2}\int_0^t \partial_{yy}H_s\lambda(s)^2\,\mathrm{d}\langle Y^c\rangle_s + \sum_{s\le t}\Delta H_s,$$

the quadratic variation term $[\theta, V-H]_{T-} = -[\theta, H]_{T-}$ can be written as

$$[\theta, V-H]_{T-} = -\int_0^T \lambda(s)\partial_y H_s\,\mathrm{d}\langle\theta^c\rangle_s - \int_0^T \lambda(s)\sigma(s)\partial_y H_s\,\mathrm{d}\langle\theta^c, B\rangle_s - \sum_{s<T}\Delta H_s\Delta\theta_s.$$

Together with (3.45) this leads to the following representation of the insider's terminal wealth

$$\begin{aligned} W_T^\theta = &\int_0^{T-} V - H_{s-}\,\mathrm{d}\theta_s - \int_0^T \lambda(s)\partial_y H_s\,\mathrm{d}\langle\theta^c\rangle_s \\ &- \int_0^T \lambda(s)\sigma(s)\partial_y H_s\,\mathrm{d}\langle\theta^c, B\rangle_s - \sum_{s<T}\Delta H_s\Delta\theta_s. \end{aligned} \tag{3.46}$$

Consider now $J^0(s,\widetilde{Y}_s)$ with J^0 as in (3.41). According to Lemma 3.12

$$\lambda(s)\partial_y J_s^0 = H_s - V.$$

This implies

$$\Delta H_s = \Delta(H_s - V) = \Delta(\lambda(s)\partial_y J_s^0).$$

Inserting this into (3.46) yields

$$\begin{aligned} W_T^\theta &= -\int_0^{T-} \lambda(s)\partial_y J_{s-}^0\,\mathrm{d}\theta_s - \int_0^T \lambda(s)^2\partial_{yy}J_s^0\,\mathrm{d}\langle\theta^c\rangle_s \\ &\quad - \int_0^T \lambda(s)^2\sigma(s)\partial_{yy}J_s^0\,\mathrm{d}\langle\theta^c, B\rangle_s - \sum_{s<T}\Delta(\lambda(s)\partial_y J_s^0)\Delta\theta_s \\ &= -\int_0^T \lambda(s)\partial_y J_{s-}^0\,\mathrm{d}\theta_s^c - \int_0^T \lambda(s)^2\partial_{yy}J_s^0\,\mathrm{d}\langle\theta^c\rangle_s \\ &\quad - \int_0^T \lambda(s)^2\sigma(s)\partial_{yy}J_s^0\,\mathrm{d}\langle\theta^c, B\rangle_s - \sum_{s<T}\lambda(s)\partial_y J_s^0\Delta\theta_s. \end{aligned} \tag{3.47}$$

On the other hand, we have by Itô's formula

$$
\begin{aligned}
J_T^0 &= J_0^0 + \int_0^T \partial_y J_{s-}^0 \,\mathrm{d}\widetilde{Y}_s^c + \int_0^T \partial_t J_s^0 \,\mathrm{d}s + \frac{1}{2}\int_0^T \partial_{yy} J_s^0 \,\mathrm{d}\langle \widetilde{Y}^c\rangle_s + \sum_{s\le T} \Delta J_s^0 \\
&= J_0^0 + \int_0^T \lambda(s)\partial_y J_{s-}^0 \,\mathrm{d}Y_s^c + \int_0^T \partial_t J_s^0 \,\mathrm{d}s + \frac{1}{2}\int_0^T \lambda(s)^2\partial_{yy} J_s^0 \,\mathrm{d}\langle Y^c\rangle_s + \sum_{s\le T} \Delta J_s^0 \\
&= J_0^0 + \int_0^T \lambda(s)\sigma(s)\partial_y J_{s-}^0 \,\mathrm{d}B_s + \int_0^T \lambda(s)\partial_y J_{s-}^0 \,\mathrm{d}\theta_s^c + \int_0^T \lambda(s)^2\sigma(s)\partial_{yy} J_s^0 \,\mathrm{d}\langle \theta^c, B\rangle_s \\
&\quad + \frac{1}{2}\int_0^T \lambda(s)^2\partial_{yy} J_s^0 \,\mathrm{d}\langle \theta^c\rangle_s + \int_0^T \partial_t J_s^0 + \frac{\lambda(s)^2\sigma(s)^2}{2}\partial_{yy} J_s^0 + \lambda(s)\mu_s\partial_y J_s^0 \,\mathrm{d}s + \sum_{s\le T} \Delta J_s^0 .
\end{aligned}
$$

Again together with Lemma 3.12 (PDE (3.11)) we obtain

$$
\begin{aligned}
J_T^0 =& J_0^0 + \int_0^T \lambda(s)\sigma(s)\partial_y J_{s-}^0 \,\mathrm{d}B_s + \int_0^T \lambda(s)\partial_y J_{s-}^0 \,\mathrm{d}\theta_s^c \\
&+ \frac{1}{2}\int_0^T \lambda(s)^2\partial_{yy} J_s^0 \,\mathrm{d}\langle \theta^c\rangle_s + \int_0^T \lambda(s)^2\sigma(s)\partial_{yy} J_s^0 \,\mathrm{d}\langle \theta^c, B\rangle_s + \sum_{s\le T} \Delta J_s^0 .
\end{aligned} \tag{3.48}
$$

Now putting (3.47) and (3.48) together, we get

$$
W_T^\theta = J_0^0 - J_{T-}^0 + \int_0^T \sigma(s)\lambda(s)\partial_y J_{s-}^0 \,\mathrm{d}B_s - \int_0^T \frac{\lambda(s)^2}{2}\partial_{yy} J_s^0 \,\mathrm{d}\langle \theta^c\rangle_s + \sum_{s<T} \Delta J_s^0 - \lambda(s)\partial_y J_s^0 \Delta\theta_s .
$$

Since

$$
\mathbb{E}\int_0^T \sigma(s)^2 (H_s - V)^2 \,\mathrm{d}s < \infty,
$$

we have that

$$
\int_0^t \sigma(s)\lambda(s)\partial_y J_{s-}^0 \,\mathrm{d}B_s, \quad t\in[0,T],
$$

is a (true) martingale, in particular

$$
\mathbb{E}\int_0^T \sigma(s)\lambda(s)\partial_y J_{s-}^0 \,\mathrm{d}B_s = 0.
$$

Hence,

$$
\mathbb{E}W_T^\theta = \mathbb{E}\left(J_0^0 - J_{T-}^0 - \int_0^T \frac{1}{2}\lambda(s)^2\partial_{yy} J_s^0 \,\mathrm{d}\langle \theta^c\rangle_s + \sum_{s<T} \Delta J_s^0 - \lambda(s)\partial_y J_s^0 \Delta\theta_s \right). \tag{3.49}
$$

With the help of Equation (3.49), we now show that $\mathbb{E}W_T^\theta \le \mathbb{E}J_0^0$, for all $\theta \in \mathcal{S}(H,\lambda)$. First observe that

$$-\int_0^T \frac{1}{2}\lambda(s)^2 \partial_{yy} J_s^0 \,\mathrm{d}\langle\theta^c\rangle_s \leq 0$$

since

$$\partial_{yy} J_s^0 = \lambda(s)^{-1}\partial_y(H_s - V) = \lambda(s)^{-1}\partial_y H_s > 0,$$

with equality if and only if $\langle\theta^c\rangle_\cdot = 0$. On the other hand, by convexity of J^0 ($\partial_{yy}J^0 > 0$) together with continuity of λ we get

$$\begin{aligned}
&\Delta J_s^0 - \lambda(s)\partial_y J_s^0 \Delta\theta_s \\
&\quad = J^0(s, \widetilde{Y}_{s-} + \lambda(s)\Delta\theta_s, V) - J^0(s, \widetilde{Y}_{s-}, V) - \partial_y J^0(s, \widetilde{Y}_{s-} + \lambda(s)\Delta\theta_s, V)\lambda(s)\Delta\theta_s \leq 0
\end{aligned}$$

with equality if and only if $\Delta\theta_s = 0$. Last but not least, we can deduce from Lemma 3.12 that

$$\mathbb{E}(J_0^0 - J_{T-}^0) \leq \mathbb{E}J_0^0 - \mathbb{E}J_{T-}^0 \leq \mathbb{E}J_0^0,$$

with equality if and only if $\widetilde{Y}_{T-} = y^*(T, V)$, $\mathbb{P}$-a.s. If θ is continuous, this is equivalent to $H(T, \widetilde{Y}_T) = V$, $\mathbb{P}$-a.s. For an admissible strategy θ^* that satisfies conditions (1) and (2) and any other admissible strategy θ it follows

$$\mathbb{E}W_T^\theta \leq \mathbb{E}J_0^0 = \mathbb{E}W_T^{\theta^*}.$$

This proves the sufficiency of the criteria. $\square$

Remark 3.14. In the case where the terminal wealth of the insider is given by

$$W_T^\theta = \int_0^T V - H_{s-}\,\mathrm{d}\theta_s + [\theta, V - H]_T,$$

i.e. in a model where $P_t = V$ only for $t \in (T, \infty)$ but not necessarily $t = T$ (cf. (1.1)), we get by an analogous argumentation as in the preceding proof that

$$\mathbb{E}W_T^\theta \leq \mathbb{E}(J_0^0 - J_T^0) \leq \mathbb{E}J_0^0$$

with equality if and only if the conditions of Proposition 3.13 are satisfied. We note that the exact distinction of the trading horizon is irrelevant in the case of bounded drift intensity. This changes in the case of unbounded drift intensity, which is considered next.

3.3.2 Unbounded drift intensity

In this subsection, we want to generalise the results of Proposition 3.13 to the case where the (integrated) noise drift intensity m is not necessarily bounded. According to Proposition 3.9

we already have $H(T,\widetilde{X}_T)=V$ if $\lim_{t\to T}\int_0^t m(s)\,\mathrm{d}s=\infty$. Hence, this situation should place additional demands on an optimal strategy. Otherwise, the strategy $\theta=0$ already would be optimal according to Proposition 3.13.

While assuming the boundedness of m on $[0,T]$ in the previous subsection, we ensured that the corresponding price pressure

$$\lambda(t)=\frac{1}{\int_0^t m(s)\,\mathrm{d}s+\lambda_0^{-1}}$$

is bounded away from zero on the whole interval $[0,T]$. Now, if we allow

$$\int_0^t m(s)\,\mathrm{d}s\nearrow\infty,\quad \text{for } t\to T,$$

the price pressure λ converges to zero. Mathematically, this poses the problem that we cannot define the value function of our optimisation problem as we did in Lemma 3.12 by

$$J^0(t,y,v)=\mathbb{E}^v\left(\int_t^T\mu(s,\xi_s^{t,y},V)(H(s,\xi_s^{t,y})-V)\,\mathrm{d}s+\lambda(T)^{-1}\int_{y^*(T,V)}^{\xi_T^{t,y}}(H(T,y)-V)\,\mathrm{d}y\right)$$

since the factor $\lambda(T)^{-1}$ appears in the second part of the right hand side. Our solution to this problem is to consider auxiliary optimisation problems on the interval $[0,t]$ for $t<T$. These can be solved with the help of Proposition 3.13. In a second step, we show that the terminal wealth of the general optimisation problem can be represented as limit of the terminal wealth of the auxiliary problems.

Proposition 3.15. *Let Assumptions 3.1 and 3.8 be satisfied and suppose that for $(H,\lambda)\in\mathcal{P}$, (3.12) and (3.37) hold. Then $\theta\in\mathcal{S}(H,\lambda)$ is optimal if*

(1) θ is continuous and of finite variation,

(2) $\lim_{t\to T}\mathbb{E}J_t^{0,t}=0$, where $J_t^{0,t}=\lambda(t)^{-1}\int_{y^*(t,V)}^{\widetilde{Y}_t}H(t,y)-V\,\mathrm{d}y$, $t\in[0,T)$.

In particular, if θ verifies condition (1) and (2), $H(T,\widetilde{Y}_T)=V$ holds $\mathbb{P}$-a.s.

As stated above, we introduce some auxiliary models to analyse our optimisation problem. This is done with the following lemma.

Lemma 3.16. *Let Assumptions 3.1 and 3.8 be satisfied and suppose that for $(H,\lambda)\in\mathcal{P}$, (3.12) and (3.37) hold. For $t'\in[0,T)$ and for all $(t,y,v)\in[0,t']\times\mathbb{R}\times\mathcal{V}$ define*

$$J^{0,t'}(t,y,v):=\mathbb{E}^v\left(\int_t^{t'}\mu(s,\xi_s^{t,y},V)(H(s,\xi_s^{t,y})-V)\,\mathrm{d}s+\lambda(t')^{-1}\int_{y^*(t',V)}^{\xi_{t'}^{t,y}}(H(t',x)-V)\,\mathrm{d}x\right).$$

Then $J^{0,t'}$ satisfies the following system of PDEs for all $(t,y,v) \in (0,t') \times \mathbb{R} \times \mathcal{V}$

$$\partial_y J^{0,t'}(t,y,v) = \frac{H(t,y)-v}{\lambda(t)},$$
$$\partial_t J^{0,t'}(t,y,v) = -\lambda(t)\mu(t,y,v)\partial_y J^{0,t'}(t,y,v) - \frac{1}{2}\sigma(t)^2\lambda(t)^2\partial_{yy}J^{0,t'}(t,y,v).$$

Furthermore, for all $v \in \mathcal{V}$, the map $t \mapsto J^{0,t}(0,0,v)$ is positive, increasing and bounded from above, in particular, $\lim_{t\to T} J^{0,t}(0,0,v)$ exists and is finite.

Proof. Since $(H(t,\xi_t),\ t \geq 0)$ is a martingale, we have for all $t' \in [0,T)$

$$\mathbb{E}H(t',\xi_{t'})^2 \leq \mathbb{E}H(T,\xi_T)^2 < \infty.$$

Hence, H is admissible on $[0,t']$. Furthermore, m is bounded on $[0,t']$ (Assumption 3.8). In particular, the assumptions of Lemma 3.12 are satisfied on the intervall $[0,t']$ and the first part of the assertion follows with the same arguments.

For the second part again consider M as defined in (3.38). Together with the partial derivatives of M calculated in the proof of Proposition 3.9 Itô's formula yields

$$\begin{aligned}
\mathrm{d}M(s,\xi_s) &= \partial_y M(s,\xi_s)\mathrm{d}\xi_s + \frac{1}{2}\partial_{yy}M(s,\xi_s)\sigma(s)^2\lambda(s)^2\mathrm{d}s + \partial_t M(s,\xi_s)\mathrm{d}s \\
&= (H(s,\xi_s)-V)\,\lambda(s)\sigma(s)\mathrm{d}B_s + \frac{1}{2}\sigma(s)^2\lambda(s)^2\partial_y H(s,y^*(s,V))\mathrm{d}s.
\end{aligned}$$

Together with integration by parts we get

$$\begin{aligned}
J^{0,t}(0,0,v) &= \mathbb{E}^v\left(-\int_0^t M(s,\xi_s)\,\mathrm{d}\lambda(s)^{-1} + \lambda(t)^{-1}M(t,\xi_t)\right) \\
&= \mathbb{E}^v\left(\lambda(0)^{-1}M(0,0) + \int_0^t \lambda(s)^{-1}\,\mathrm{d}M(s,\xi_s)\right) \\
&= \mathbb{E}^v\left(\lambda(0)^{-1}M(0,0) + \int_0^t (H(s,\xi_s)-V)\,\sigma(s)\,\mathrm{d}B_s \right.\\
&\qquad \left. + \int_0^t \frac{\lambda(s)\sigma(s)^2}{2}\partial_y H(s,y^*(s,V))\,\mathrm{d}s\right) \\
&= \lambda(0)^{-1}\mathbb{E}^v M(0,0) + \int_0^t \lambda(s)\sigma(s)^2\partial_y H(s,y^*(s,v))\,\mathrm{d}s \\
&= \lambda(0)^{-1}\int_{y^*(0,v)}^0 H(0,x)-V\,\mathrm{d}x + \int_0^t \lambda(s)\sigma(s)^2\partial_y H(s,y^*(s,v))\,\mathrm{d}s.
\end{aligned}$$

Since $\lambda(s)\sigma(s)^2\partial_y H(s,y^*(s,v)) \geq 0$ and

$$\int_0^t \lambda(s)\sigma(s)^2\partial_y H(s,y^*(s,v))\,\mathrm{d}s \leq \int_0^T \lambda(s)\sigma(s)^2\partial_y H(s,y^*(s,v))\,\mathrm{d}s < \infty,$$

(recall that $\partial_y H \in \mathcal{C}^0([0,T]\times\mathbb{R})$) this proves the assertion. $\square$

With the help of Lemma 3.16, we are now in the position to prove Proposition 3.15.

Proof of Proposition 3.15. For any $t \in [0,T)$ and any admissible insider strategy $\theta \in \mathcal{S}(H,\lambda)$ define

$$\widetilde{W}_t^\theta := \left(V - H(t,\widetilde{Y}_t)\right)\theta_t + \int_0^t \theta_{s-}\,\mathrm{d}H(s,\widetilde{Y}_s). \tag{3.50}$$

In a model with announcement of V in $t+$, i.e. $P_s = V$ for all $s > t$, $\widetilde{W}_t^\theta$ is the terminal wealth of the insider strategy $\theta_{|[0,t]}$ (observe that all admissible trading strategies on $[0,T]$ are admissible in the reduced model on $[0,t]$ and all admissible pricing rules on $[0,T]$ are admissible pricing rules on $[0,t]$). Furthermore, $J^{0,t}$, as defined in Lemma 3.16, is the value function of the corresponding optimisation problem, and by Remark 3.14 and an analogous argumentation as in Proposition 3.13, with $\widetilde{W}_t^\theta$ in place of W_T^θ and $J^{0,t}$ instead of J^0, we get

$$\begin{aligned}\widetilde{W}_t^\theta &= J^{0,t}(0,0,V) - J^{0,t}(t,\widetilde{Y}_t,V) - \int_0^t \frac{1}{2}\lambda(s)^2\partial_{yy}J^{0,t}(s,\widetilde{Y}_s,V)\,\mathrm{d}\langle\theta^c\rangle_s \\ &\quad + \sum_{s\le t}\Delta J^{0,t}(s,\widetilde{Y}_s,V) - \lambda(s)\partial_y J^{0,t}(s,\widetilde{Y}_s,V)\Delta\theta_s + \int_0^t \sigma(s)(H(s,\widetilde{Y}_{s-}) - V)\,\mathrm{d}B_s \\ &\le J^{0,t}(0,0,V) + \int_0^t \sigma(s)(H(s,\widetilde{Y}_{s-}) - V)\,\mathrm{d}B_s\end{aligned}$$

In particular, for $t \in [0,T)$

$$\begin{aligned}J^{0,t}(0,0,V) &= \widetilde{W}_t^\theta + J^{0,t}(t,\widetilde{Y}_t,V) - \sum_{s\le t}\Delta J^{0,t}(s,\widetilde{Y}_s,V) - \lambda(s)\partial_y J^{0,t}(s,\widetilde{Y}_s,V)\Delta\theta_s \\ &\quad + \int_0^t \frac{1}{2}\lambda(s)^2\partial_{yy}J^{0,t}(s,\widetilde{Y}_s,V)\,\mathrm{d}\langle\theta^c\rangle_s - \int_0^t \sigma(s)(H(s,\widetilde{Y}_{s-}) - V)\,\mathrm{d}B_s.\end{aligned}$$

is uniformly integrable with respect to $\mathbb{P}^v$, for all $v \in \mathcal{V}$ (cf. Lemma 3.16). Since

$$\lim_{t\uparrow T}\widetilde{W}_t^\theta = \left(V - H(T-,\widetilde{Y}_{T-})\right)\theta_{T-} + \int_0^{T-}\theta_{s-}\,\mathrm{d}H(s,\widetilde{Y}_s) = W_T^\theta,$$

we get

$$\begin{aligned}\lim_{t\to T} J^{0,t}(0,0,v) &= \lim_{t\to T}\mathbb{E}^v J^{0,t}(0,0,V) \\ &= \mathbb{E}^v \lim_{t\to T}\Bigg(\widetilde{W}_t^\theta + J^{0,t}(t,\widetilde{Y}_t,V) - \sum_{s\le t}\Delta J^{0,t}(s,\widetilde{Y}_s,V) - \lambda(s)\partial_y J^{0,t}(s,\widetilde{Y}_s,V)\Delta\theta_s \\ &\qquad + \int_0^t \frac{1}{2}\lambda(s)^2\partial_{yy}J^{0,t}(s,\widetilde{Y}_s,V)\,\mathrm{d}\langle\theta^c\rangle_s - \int_0^t \sigma(s)(H(s,\widetilde{Y}_{s-}) - V)\,\mathrm{d}B_s\Bigg) \\ &\ge \mathbb{E}^v\left(\lim_{t\to T}\widetilde{W}_t^\theta - \int_0^T \sigma(s)(H(s,\widetilde{Y}_{s-}) - V)\,\mathrm{d}B_s\right) \\ &= \mathbb{E}^v W_T^\theta.\end{aligned}$$

In particular, (where $J_0^{0,t} = J^{0,t}(0,0,V)$)

$$\mathbb{E}W_T^\theta = \mathbb{E}\mathbb{E}^v W_T^\theta \leq \mathbb{E} \lim_{t\to T} J_0^{0,t}. \tag{3.51}$$

Furthermore, $\widetilde{W}_t^\theta$ corresponds to the terminal wealth (in T) of the strategy θ stopped at time t, i.e.

$$\begin{aligned} W_T^{\theta^t} &= V\theta_{T-}^t - \int_0^{T-} H(s-,\widetilde{Y}_{s-})\,\mathrm{d}\theta_s^t - [H(\cdot,\widetilde{Y}_\cdot),\theta^t]_{T-} \\ &= V\theta_t - \int_0^t H(s-,\widetilde{Y}_{s-})\,\mathrm{d}\theta_s - [H(\cdot,\widetilde{Y}_\cdot),\theta]_t \\ &= \widetilde{W}_t^\theta \end{aligned}$$

for $\theta_s^t := \theta_{t\wedge s}$, $s \in [0,T]$. For any optimal strategy θ and $t \in [0,T)$ obviously $\mathbb{E}W_T^{\theta^t} \leq \mathbb{E}W_T^\theta$ holds. Hence, from (3.51) it follows, for all $t \in [0,T)$,

$$\mathbb{E}\widetilde{W}_t^\theta \leq \mathbb{E}W_T^\theta \leq \mathbb{E} \lim_{t\to T} J_0^{0,t}. \tag{3.52}$$

On the other hand, with the same arguments as in Proposition 3.13, we have

$$\mathbb{E}\widetilde{W}_t^\theta \leq \mathbb{E}\left(J_0^{0,t} - J_t^{0,t}\right),$$

with equality if and only if $\theta_{|[0,t]}$ is continuous and of finite variation. And since $J_t^{0,t} \geq 0$, for all $t \in [0,T)$, we get for all those θ that

$$\mathbb{E}\widetilde{W}_t^\theta = \mathbb{E}\left(J_0^{0,t} - J_t^{0,t}\right) \leq \mathbb{E}J_0^{0,t}.$$

In particular, monotone convergence ($J_0^{0,t}$ is positive and increasing, cf. Lemma 3.16) then yields

$$\lim_{t\to T} \mathbb{E}\widetilde{W}_t^\theta = \lim_{t\to T} \mathbb{E}\left(J_0^{0,t} - J_t^{0,t}\right) \leq \mathbb{E} \lim_{t\to T} J_0^{0,t}$$

and

$$\lim_{t\to T} \mathbb{E}\widetilde{W}_t^\theta = \mathbb{E} \lim_{t\to T} J_0^{0,t}$$

if $\lim_{t\to T} \mathbb{E}J_t^{0,t} = 0$. Together with (3.52) we get the optimality of a strategy that satisfies conditions (1) and (2).

We finally have to show that $\mathbb{E}J_t^{0,t} \to 0$ implies $H(T,\widetilde{Y}_T) = V$ $\mathbb{P}$-a.s. From the definition of y^*, the strict monotonicity of $H(t,\cdot)$ and positivity of λ we get for all $t \in [0,T]$ that $\mathbb{E}J_t^{0,t} \geq 0$, with equality if and only if $\widetilde{Y}_t = y^*(t,V)$ or equivalently $H(t,\widetilde{Y}_t) = V$. Finally, the L_1 convergence

$$\lim_{t\to T} \mathbb{E}|J_t^{0,t}| = \lim_{t\to T} \mathbb{E}J_t^{0,t} = 0$$

implies convergence in probability of $J_t^{0,t}$ to 0. Since $\widetilde{Y}$ is continuous, we get $H(T, \widetilde{Y}_T) = V$ $\mathbb{P}$-a.s. $\square$

In the first instance, a comparison of the optimality criteria of Propositions 3.13 and 3.15 shows that a vanishing price pressure does not affect the general structure of an optimal insider strategy. In both cases the insider has to ensure that the price converges to V, i.e. to provide market efficiency, by using an absolutely continuous trading strategy. If the price pressure vanishes, the convergence has, roughly speaking, only to be fast enough w.r.t. λ. This is quite intuitive since, in this case, the market is already efficient without an insider (cf. Proposition 3.9). However, efficiency is provided faster in the presence of an insider, trading according to Proposition 3.15. To realise this, consider

$$J^{0,t}(t, \widetilde{X}_t) = \frac{M(t, \widetilde{X}_t)}{\lambda(t)}$$

with M as defined in (3.38). Analogous to the proof of Proposition 3.9, we can calculate the dynamics of this process

$$\begin{aligned} \mathrm{d}(M(t, \widetilde{X}_t)\lambda(t)^{-1}) &= \lambda(t)^{-1}\mathrm{d}M(t, \widetilde{X}_t) + M(t, \widetilde{X}_t)\mathrm{d}\lambda(t)^{-1} \\ &= (H(t, \widetilde{X}_t) - V)\sigma(t)\mathrm{d}B_t + \frac{\sigma(t)^2\lambda(t)}{2}\partial_y H(t, y^*(t, V))\mathrm{d}t. \end{aligned}$$

Obviously, it does not converge to V.

3.4 Optimality: risk averse insider

In the previous section, we studied optimality for a risk neutral insider. This section is devoted to analogous results for a risk averse insider.

3.4.1 Bounded drift intensity

We again start with determining the value function.

Lemma 3.17. *Let Assumptions 3.1 and 3.8 be satisfied such that m is bounded from above on $[0, T]$. Furthermore, suppose that for $(H, \lambda) \in \mathcal{P}$, (3.12) and (3.37) hold and define*

$$\begin{aligned} K^\beta(t, y, v) :=& \mathbb{E}^v \Bigg(\int_t^T \mu(s, \xi_s^{t,y}, V)(H(s, \xi_s^{t,y}) - V) + \frac{\sigma(s)^2\beta}{2}(H(s, \xi_s^{t,y}) - V)^2 \,\mathrm{d}s \\ &+ \lambda(T)^{-1} \int_{y^*(T,V)}^{\xi_T^{t,y}} (H(T, x) - V)\,\mathrm{d}x \Bigg), \end{aligned} \tag{3.53}$$

for $(t, y, v) \in [0, T] \times \mathbb{R} \times \mathcal{V}$, with $\xi_s^{t,y}$ as in (3.4) and

$$J^\beta(t, y, v) := \beta \exp\Big(\beta K^\beta(t, y, v)\Big). \tag{3.54}$$

Then the triple (H, λ, K^β) is a solution to the following system of PDEs:

$$0 = \frac{H(t,y) - v}{\lambda(t)} - \partial_y K^\beta(t,y,v), \tag{3.55}$$

$$0 = \partial_t K^\beta(t,y,v) + \frac{\sigma(t)^2\lambda(t)^2}{2}\partial_{yy}K^\beta(t,y,v) + \frac{\sigma(t)^2\beta}{2}(H(t,y)-v)^2 + \mu(t,y,v)(H(t,y)-v), \tag{3.56}$$

for all $(t, y, v) \in (0, T) \times \mathbb{R} \times \mathcal{V}$. In particular, $K^\beta(T, y, v) \geq 0$, for all $y \in \mathbb{R}$, with equality if and only if $y = y^*(T, v)$. Furthermore, the triple (H, λ, J^β) is a solution to the system of Equations (3.22) and (3.23) and $J^\beta(T, y, v) \geq \beta$, for all $y \in \mathbb{R}$, with equality if and only if $y = y^*(T, v)$.

Proof. An analogous proof of Lemma 3.12 where we replace

$$\int_t^T \mu(s, \xi_s^{t,y}, V)(H(s, \xi_s^{t,y}) - V)\,\mathrm{d}s$$

by

$$\int_t^T \mu(s, \xi_s^{t,y}, V)(H(s, \xi_s^{t,y}) - V) + \frac{\sigma(s)^2\beta}{2}(H(s, \xi_s^{t,y}) - V)^2\,\mathrm{d}s$$

shows that K^β is a solution to (3.55) and (3.56). Differentiation of (3.55) w.r.t. y yields

$$\partial_{yy}K^\beta(t,y,v) = \frac{\partial_y H(t,y)}{\lambda(t)} \tag{3.57}$$

and inserting in (3.56)

$$\partial_t K^\beta(t,y,v) = -\frac{\sigma(t)^2\lambda(t)}{2}\partial_y H(t,y) - \frac{\sigma(t)^2\beta}{2}(H(t,y)-v)^2 - \mu(t,y,v)(H(t,y)-v). \tag{3.58}$$

Now consider J^β as defined in (3.54). Using the partial derivatives of K^β calculated in (3.55), (3.57), and (3.58), yields

$$\partial_y J^\beta(t,y,v) = \beta\partial_y K^\beta(t,y,v)J^\beta(t,y,v) = \beta\frac{H(t,y)-v}{\lambda(t)}J^\beta(t,y,v), \tag{3.59}$$

$$\begin{aligned}\partial_{yy}J^\beta(t,y,v) &= \beta\partial_{yy}K^\beta(t,y,v)J^\beta(t,y,v) + \beta^2(\partial_y K^\beta(t,y,v))^2 J^\beta(t,y,v)\\ &= \beta\frac{\partial_y H(t,y)}{\lambda(t)}J^\beta(t,y,v) + \beta^2\left(\frac{H(t,y)-v}{\lambda(t)}\right)^2 J^\beta(t,y,v)\end{aligned}$$

and

$$\begin{aligned}\partial_t J^\beta(t,y,v) &= \beta\partial_t K^\beta(t,y,v)J^\beta(t,y,v)\\ &= \beta J^\beta(t,y,v)\left(-\frac{\sigma(t)^2\lambda(t)}{2}\partial_y H(t,y) - \frac{\sigma(t)^2\beta}{2}(H(t,y)-v)^2 - \mu(t,y,v)(H(t,y)-v)\right).\end{aligned}$$

From this we deduce that

$$0 = \partial_t J^\beta(t,y,v) + \frac{1}{2}\sigma(t)^2\lambda(t)^2\partial_{yy}J^\beta(t,y,v) + \lambda(t)\mu(t,y,v)\partial_y J^\beta(t,y,v).$$

Together with (3.59) J^β is a solution to the system of Equations (3.22) and (3.23). As in Lemma 3.12 we get that $K^\beta(T,y,v) \geq 0$, for all $y \in \mathbb{R}$, with equality if and only if $y = y^*(T,v)$. Hence, $J^\beta(T,y,v) \geq \beta$, for all $y \in \mathbb{R}$, with equality if and only if $y = y^*(T,v)$. □

Lemma 3.17 shows that the optimisation problem in the case of absolutely continuous insider trading is related to that in the risk neutral case because the risk averse value function of the HJB equation approach in Subsection 3.2.2 is a transform of the corresponding risk neutral value function. It seems to be plausible that this analogy persists in the case of general insider strategies.

Proposition 3.18. *Let Assumptions 3.1 and 3.8 be satisfied such that m is bounded from above on $[0,T]$. Furthermore, suppose that for $(H,\lambda) \in \mathcal{P}$, (3.12) and (3.37) hold. Then $\theta \in \mathcal{S}(H,\lambda)$ is optimal if*

(1) $H(T,\widetilde{Y}_T) = V$ $\mathbb{P}$-a.s.,

(2) θ is of finite variation and continuous.

Proof. The proof makes use of the same techniques as the proof of Proposition 3.13. We only have to take into account that we now have to maximise the utility $\mathcal{U}(W_T^\theta)$ of the final wealth W_T^θ generated by the trading strategy θ as in (3.45), where $\mathcal{U}(x) = \beta\exp(\beta x)$. With the same arguments as in the proof of Proposition 3.13 and using Lemma 3.17 we get

$$\begin{aligned} W_T^\theta = &- \int_0^T \lambda(s)\partial_y K_{s-}^\beta \,\mathrm{d}\theta_s^c - \int_0^T \lambda(s)^2\partial_{yy}K_s^\beta \,\mathrm{d}\langle\theta^c\rangle_s \\ &- \int_0^T \lambda(s)^2\sigma(s)\partial_{yy}K_s^\beta \,\mathrm{d}\langle\theta^c, B\rangle_s - \sum_{s<T}\lambda(s)\partial_y K_s^\beta \Delta\theta_s \end{aligned} \tag{3.60}$$

with $K_s^\beta = K^\beta(s,\widetilde{Y}_s,V)$ according to (3.53). On the other hand, Itô's formula yields

$$\begin{aligned} K_T^\beta = K_0^\beta &+ \int_0^T \lambda(s)\sigma(s)\partial_y K_{s-}^\beta \,\mathrm{d}B_s + \int_0^T \lambda(s)\partial_y K_{s-}^\beta \,\mathrm{d}\theta_s^c + \int_0^T \lambda(s)^2\sigma(s)\partial_{yy}K_s^\beta \,\mathrm{d}\langle\theta^c, B\rangle_s \\ &+ \int_0^T \lambda(s)\mu_s\partial_y K_s^\beta + \partial_t K_s^\beta + \frac{\lambda(s)^2\sigma(s)^2}{2}\partial_{yy}K_s^\beta \,\mathrm{d}s + \int_0^T \frac{\lambda(s)^2}{2}\partial_{yy}K_s^\beta \,\mathrm{d}\langle\theta^c\rangle_s + \sum_{s\leq T}\Delta K_s^\beta. \end{aligned}$$

Using PDE (3.56) (Lemma 3.17) this can be simplified to

$$\begin{aligned} K_T^\beta = &K_0^\beta + \int_0^T \lambda(s)\sigma(s)\partial_y K_{s-}^\beta \,\mathrm{d}B_s + \int_0^T \lambda(s)\partial_y K_{s-}^\beta \,\mathrm{d}\theta_s^c - \int_0^T \frac{\sigma(s)^2\beta}{2}\left(H_s - V\right)^2 \,\mathrm{d}s \\ &+ \int_0^T \frac{\lambda(s)^2}{2}\partial_{yy}K_s^\beta \,\mathrm{d}\langle\theta^c\rangle_s + \int_0^T \lambda(s)^2\sigma(s)\partial_{yy}K_s^\beta \,\mathrm{d}\langle\theta^c, B\rangle_s + \sum_{s\leq T}\Delta K_s^\beta. \end{aligned} \tag{3.61}$$

Putting (3.60) and (3.61) together, we get

$$\begin{aligned} W_T^\theta =& K_0^\beta - K_{T-}^\beta + \int_0^T \sigma(s)\lambda(s)\partial_y K_{s-}^\beta \,\mathrm{d}B_s - \int_0^T \frac{\sigma(s)^2\lambda(s)^2\beta}{2}\left(\partial_y K_s^\beta\right)^2 \mathrm{d}s \\ & - \int_0^T \frac{\lambda(s)^2}{2}\partial_{yy}K_s^\beta \,\mathrm{d}\langle\theta^c\rangle_s + \sum_{s<T} \Delta K_s^\beta - \lambda(s)\partial_y K_s^\beta \Delta\theta_s. \end{aligned} \tag{3.62}$$

Now, define

$$\widetilde{K}_T^\beta := K_0^\beta - K_{T-}^\beta - \int_0^T \frac{\lambda(s)^2}{2}\partial_{yy}K_s^\beta \,\mathrm{d}\langle\theta^c\rangle_s + \sum_{s<T} \Delta K_s^\beta - \lambda(s)\partial_y K_s^\beta \Delta\theta_s.$$

With the same arguments as in Proposition 3.13 we get $\widetilde{K}_T^\beta \leq K_0^\beta$ $\mathbb{P}$-a.s. with equality if and only if θ is absolutely continuous and $H_T = V$ $\mathbb{P}$-a.s. We conclude that

$$\beta \exp\left(\beta \widetilde{K}_T^\beta\right) \leq \beta \exp\left(\beta K_0^\beta\right) \quad \mathbb{P}\text{-a.s.}$$

with equality if and only if θ is absolutely continuous and $H_T = V$ $\mathbb{P}$-a.s. Hence,

$$\begin{aligned} \mathbb{E}\mathcal{U}(W_T^\theta) &= \beta\mathbb{E}\left(\exp\left(\beta\widetilde{K}_T^\beta + \int_0^T \beta\sigma(s)\lambda(s)\partial_y K_{s-}^\beta \,\mathrm{d}B_s - \int_0^T \frac{\sigma(s)^2\lambda(s)^2\beta^2}{2}\left(\partial_y K_s^\beta\right)^2 \mathrm{d}s\right)\right) \\ &\leq \beta\mathbb{E}\exp\left(\beta K_0^\beta\right)\mathbb{E}^v \exp\left(\int_0^T \beta\sigma(s)\lambda(s)\partial_y K_{s-}^\beta \,\mathrm{d}B_s - \int_0^T \frac{\sigma(s)^2\lambda(s)^2\beta^2}{2}\left(\partial_y K_s^\beta\right)^2 \mathrm{d}s\right). \end{aligned}$$

Since (admissibility of θ)

$$\mathbb{E}\left(\exp\left(\int_0^T \frac{\beta^2\sigma(s)^2\lambda(s)^2}{2}(\partial_y K_s^\beta)^2 \,\mathrm{d}s\right)\right) = \mathbb{E}\left(\exp\left(\int_0^T \frac{\beta^2\sigma(s)^2}{2}(H_s - V)^2 \,\mathrm{d}s\right)\right) < \infty,$$

Novikov's condition (cf. [42], Prop. 1.7.6.1) is satisfied. Hence,

$$\exp\left(\int_0^t \beta\sigma(s)\lambda(s)\partial_y K_{s-}^\beta \,\mathrm{d}B_s - \int_0^t \frac{\sigma(s)^2\lambda(s)^2\beta^2}{2}\left(\partial_y K_s^\beta\right)^2 \mathrm{d}s\right), \quad t \in [0,T],$$

is a uniformly integrable martingale. In particular,

$$\mathbb{E}^v \exp\left(\int_0^T \beta\sigma(s)\lambda(s)\partial_y K_{s-}^\beta \,\mathrm{d}B_s - \int_0^T \frac{\sigma(s)^2\lambda(s)^2\beta^2}{2}\left(\partial_y K_s^\beta\right)^2 \mathrm{d}s\right) = 1.$$

Altogether, we have

$$\mathbb{E}\mathcal{U}(W_T^\theta) \leq \beta\mathbb{E}\exp\left(\beta K_0^\beta\right)$$

with equality if θ is absolutely continuous and such that $H_T = V$. □

Remark 3.19. In the preceding proof we identified

$$\exp\left(\int_0^t \beta\sigma(s)(H(s,\widetilde{Y}_{s-})-V)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t \beta^2\sigma(s)^2(H(s,\widetilde{Y}_s)-V)^2\,\mathrm{d}s\right), \quad t\in[0,T],$$

as strictly positive and uniformly integrable martingale. By

$$\begin{aligned}\frac{\mathrm{d}\mathbb{P}^\beta}{\mathrm{d}\mathbb{P}} &= \exp\left(\int_0^T \beta\sigma(s)(H(s,\widetilde{Y}_{s-})-V)\,\mathrm{d}B_s - \frac{1}{2}\int_0^T \beta^2\sigma(s)^2(H(s,\widetilde{Y}_s)-V)^2\,\mathrm{d}s\right) \\ &=: \mathcal{E}\left(\int_0^T \beta\sigma(s)(H(s,\widetilde{Y}_{s-})-V)\,\mathrm{d}B_s\right)\end{aligned} \tag{3.63}$$

we can therefore define a probability measure $\mathbb{P}^\beta$ on $\mathcal{F}_T$, equivalent to $\mathbb{P}$.

3.4.2 Unbounded drift intensity

We now consider the case of an unbounded drift intensity in a risk averse setting. The following proposition is the risk averse analogue to Proposition 3.15.

Proposition 3.20. *Let Assumptions 3.1 and 3.8 be satisfied and suppose that for $(H,\lambda)\in\mathcal{P}$, (3.12) and (3.37) hold. Then $\theta\in\mathcal{S}(H,\lambda)$ is optimal if*

(1) θ is continuous and of finite variation,

(2) $\lim_{t\to T}\mathbb{E}^\beta\ \exp\left(-\beta J_t^{0,t}\right) = 1$, where $\mathbb{E}^\beta$ denotes the expectation w.r.t. $\mathbb{P}^\beta$ as defined in (3.63) and $J_t^{0,t}$ is defined as in Proposition 3.15.

In particular, if θ verifies condition (1) and (2), $H(T,\widetilde{Y}_T)=V$ holds $\mathbb{P}$-a.s.

Proof. Similarly to Lemma 3.16, we can define for $t'\in[0,T)$ and, $(t,y,v)\in[0,t']\times\mathbb{R}\times\mathcal{V}$

$$\begin{aligned}K^{\beta,t'}(t,y,v) := \mathbb{E}^v \int_t^{t'} &\mu(s,\xi_s^{t,y},V)(H(s,\xi_s^{t,y})-V) + \frac{\sigma(s)^2\beta}{2}(H(s,\xi_s^{t,y})-V)^2\,\mathrm{d}s \\ &+ \mathbb{E}^v\lambda(t')^{-1}\int_{y^*(t',V)}^{\xi_{t'}^{x,t}}(H(t',y)-V)\,\mathrm{d}y.\end{aligned} \tag{3.64}$$

With an analogous proof we get for all $t'\in[0,T)$ that $K^{\beta,t'}$ solves (3.55) and (3.56) on $[0,t')\times\mathbb{R}$ and that $K^{\beta,t}(0,0,V)$ is positive, increasing (for $t\in[0,T)$) and bounded from above. In particular, $\lim_{t\to T}K^{\beta,t}(0,0,V)<\infty$ exists. We then proceed with the same arguments as in Proposition 3.15. For

$$\widetilde{W}_t^\theta = \left(V - H(t,\widetilde{Y}_t)\right)\theta_t + \int_0^t \theta_{s-}\,\mathrm{d}H(s,\widetilde{Y}_s)$$

we get a representation corresponding to (3.62). Furthermore, using the uniform integrability of

$$\beta\exp\left(\beta K_0^{\beta,t}\right)\mathcal{E}\left(\int_0^t \beta\sigma(s)(H(s,\widetilde{Y}_{s-})-V)\,\mathrm{d}B_s\right), \quad t\in[0,T),$$

with respect to $\mathbb{P}^v$ (for all $v \in \mathcal{V}$), it follows that

$$\begin{aligned}
\mathbb{E}^v \mathcal{U}(W_T^\theta) &= \mathbb{E}^v \lim_{t\to T} \mathcal{U}(\widetilde{W}_t^\theta) \\
&\le \mathbb{E}^v \lim_{t\to T} \beta \exp\left(\beta K_0^{\beta,t}\right) \mathcal{E}\left(\int_0^t \beta\sigma(s)(H(s,\widetilde{Y}_{s-}) - V)\,\mathrm{d}B_s\right) \\
&= \lim_{t\to T} \beta \exp\left(\beta K^{\beta,t}(0,0,v)\right).
\end{aligned}$$

Hence,

$$\mathbb{E}\mathcal{U}(W_T^\theta) \le \mathbb{E} \lim_{t\to T} \beta \exp\left(\beta K_0^{\beta,t}\right).$$

On the other hand,

$$\mathbb{E}\mathcal{U}(\widetilde{W}_t^\theta) \le \beta \mathbb{E} \exp\left(\beta K_0^{\beta,t}\right),$$

for all $t \in [0,T)$, and

$$\lim_{t\to T} \mathbb{E}\mathcal{U}(\widetilde{W}_t^\theta) \le \mathbb{E} \lim_{t\to T} \beta \exp\left(\beta K_0^{\beta,t}\right)$$

with equality if θ is absolutely continuous and

$$\lim_{t\to T} \mathbb{E}\left(\exp\left(-\beta K_t^{\beta,t}\right)\mathcal{E}\left(\int_0^t \beta\sigma(s)(H(s,\widetilde{Y}_s) - V)\,\mathrm{d}B_s\right)\right) = \lim_{t\to T} \mathbb{E}^\beta \exp\left(-\beta K_t^{\beta,t}\right) = 1.$$

Since $K_t^{\beta,t} = J_t^{0,t}$, we get the optimality of θ if condition (1) and (2) are satisfied. It remains to show that the optimality criteria imply $H(T,\widetilde{Y}_T) = V$. With the same arguments as in the proof of Proposition 3.15 we obtain

$$J_t^{0,t} \xrightarrow{\mathbb{P}^\beta} 0, \quad \text{for } t \to T,$$

and thus $H(T,\widetilde{Y}_T) = V$ $\mathbb{P}^\beta$-a.s. Since $\mathbb{P}^\beta$ and $\mathbb{P}$ are equivalent probability measures, we also get the $\mathbb{P}$-almost sure equality. □

3.5 Rationality

In the previous sections, we derived criteria for the optimality of an insider strategy for the risk neutral as well as the risk averse case. In both cases, we get as one criterion the continuity and finite variation of the insider strategy. This motivates the following notation

$$\theta_t = \int_0^t \alpha_s\,\mathrm{d}s, \qquad 0 \le t \le T, \tag{3.65}$$

for some suitable $\mathbb{F}^{\mathcal{I}}$-adapted process α. As a next step, we present a characterisation of rationality of the pricing function. Recall that in the risk neutral case as well as the risk averse case

$$\partial_t H(t,y) + \frac{1}{2}\sigma(t)^2\lambda(t)^2\partial_{yy}H(t,y) = 0, \quad \text{for all } (t,y) \in (0,T) \times \mathbb{R} \tag{3.66}$$

was identified as a necessary condition for an equilibrium to exist under the assumption of absolutely continuous insider trading (cf. Propositions 3.6 and 3.7). The next proposition states equivalent conditions for the case when H is rational and satisfies (3.66). We can conclude that these conditions are also necessary for equilibrium. Furthermore, it turns out that these conditions characterise the insider strategy. More precisely, the resulting equivalent condition is the so-called *inconspicuousness of insider trading* and means that the market makers' expectation regarding the informed trading is zero or equivalently that Y is an integrated Brownian motion w.r.t. $\mathbb{F}^{\mathcal{M}}$. In the standard model (where noise trading is given by a Brownian motion) this reads as $\mathbb{E}\left(\alpha_t \middle| \mathcal{F}_t^{\mathcal{M}}\right) = 0$ (cf. [25] or [28]). In our setup this condition changes to $\mathbb{E}\left(\alpha_t + \mu_t \middle| \mathcal{F}_t^{\mathcal{M}}\right) = 0$, i.e. all informed trading has to be considered. Thus, we also have a slight change in the interpretation of an insider strategy that behaves as above. While in the standard case the insider has to trade such that his own strategy is not detected, now the insider has to take care that the whole informed trading is hidden behind the noise. One could characterise this more precisely as *inconspicuousness of informed trading.* We furthermore stress that the following characterisation does not depend on a special form of μ as in Assumption 3.8.

Proposition 3.21. *Let $(H,\lambda) \in \mathcal{P}$ be a rational pricing rule and $\theta \in \mathcal{S}(H,\lambda)$ an admissible trading strategy that is continuous and of finite variation. Then the following conditions are equivalent:*

(1) H satisfies (3.66),

(2) $\mathbb{E}(\alpha_t + \mu_t | \mathcal{F}_t^{\mathcal{M}}) = 0$ $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-a.s.,

(3) $\bar{Y}$ is a Brownian motion w.r.t. $\mathbb{F}^{\mathcal{M}}$ on $[0,T]$ where

$$\bar{Y}_t = \int_0^t \sigma(s)^{-1}\,\mathrm{d}Y_s, \quad t \in [0,T]. \tag{3.67}$$

Proof. For $t \in [0,T]$ apply Itô's formula to H and $\widetilde{Y}$ where $\widetilde{Y}$ is now continuous (recall that we use short notation H_t in place of $H(t,\widetilde{Y}_t)$ and μ_t instead of $\mu(t,\widetilde{Y}_t,V)$)

$$\begin{aligned}
H_t &= H_0 + \int_0^t \partial_t H_s\,\mathrm{d}s + \int_0^t \partial_y H_s\,\mathrm{d}\widetilde{Y}_s + \frac{1}{2}\int_0^t \partial_{yy}H_s\,\mathrm{d}\langle\widetilde{Y}\rangle_s \\
&= H_0 + \int_0^t \partial_t H_s + \frac{\lambda(s)^2\sigma(s)^2}{2}\partial_{yy}H_s + \partial_y H_s\lambda(s)(\mu_s+\alpha_s)\,\mathrm{d}s + \int_0^t \partial_y H_s\lambda(s)\sigma(s)\,\mathrm{d}B_s \\
&= H_0 + \int_0^t \partial_t H_s + \frac{\lambda(s)^2\sigma(s)^2}{2}\partial_{yy}H_s + \partial_y H_s\lambda(s)(\hat{\mu}_s+\hat{\alpha}_s)\,\mathrm{d}s + \int_0^t \partial_y H_s\lambda(s)\sigma(s)\,\mathrm{d}I_s
\end{aligned}$$

where

$$I_t := \int_0^t (\mu_s + \alpha_s - \hat{\mu}_s - \hat{\alpha}_s)\sigma(s)^{-1}\,\mathrm{d}s + B_t, \tag{3.68}$$

and

$$\hat{\mu}_t = \mathbb{E}\left(\mu_t \middle| \mathcal{F}_t^{\mathcal{M}}\right), \quad \hat{\alpha}_t = \mathbb{E}\left(\alpha_t \middle| \mathcal{F}_t^{\mathcal{M}}\right).$$

Obviously, for all $t \in [0,T]$,

$$I_t = \int_0^t \sigma(s)^{-1}\,\mathrm{d}Y_s - \int_0^t \sigma(s)^{-1}\left(\hat{\mu}_s + \hat{\alpha}_s\right)\mathrm{d}s. \tag{3.69}$$

All processes on the right hand side of Equation (3.69) are $\mathbb{F}^{\mathcal{M}}$-adapted. Hence, I is $\mathbb{F}^{\mathcal{M}}$-adapted. By Lévy's characterisation theorem (cf. [56], Theorem 39, Chapter II) I is a Brownian motion w.r.t. $\mathbb{F}^{\mathcal{M}}$ since I is continuous, $\langle I\rangle_t = \langle B\rangle_t = t$ and I is a martingale. To realise the latter, observe that for all $0 \le s < t \le T$ we have

$$\int_s^t \mathbb{E}\left(\alpha_u + \mu_u \middle| \mathcal{F}_s^{\mathcal{M}}\right)\mathrm{d}u = \int_s^t \mathbb{E}\left(\hat{\alpha}_u + \hat{\mu}_u \middle| \mathcal{F}_s^{\mathcal{M}}\right)\mathrm{d}u,$$

and therefore

$$\begin{aligned}
\mathbb{E}\left(I_t \middle| \mathcal{F}_s^{\mathcal{M}}\right) &= \mathbb{E}\left(B_t - B_s + \int_s^t (\alpha_u + \mu_u)\sigma(u)^{-1}\,\mathrm{d}u - \int_s^t (\hat{\alpha}_u + \hat{\mu}_u)\sigma(u)^{-1}\,\mathrm{d}u \middle| \mathcal{F}_s^{\mathcal{M}}\right) + I_s \\
&= \mathbb{E}\left(B_t - B_s \middle| \mathcal{F}_s^{\mathcal{M}}\right) + \int_s^t \sigma(u)^{-1}\mathbb{E}\left(\alpha_u + \mu_u \middle| \mathcal{F}_s^{\mathcal{M}}\right)\mathrm{d}u \\
&\quad - \int_s^t \sigma(u)^{-1}\mathbb{E}\left(\hat{\alpha}_u + \hat{\mu}_u \middle| \mathcal{F}_s^{\mathcal{M}}\right)\mathrm{d}u + I_s = I_s.
\end{aligned}$$

Now, by (3.69) we get the equivalence of (2) and $\bar{Y} = I$. Hence, (2) $\Rightarrow$ (3). On the other hand, if (2) did not hold, $\bar{Y}$ would be no $\mathbb{F}^{\mathcal{M}}$-martingale and therefore no $\mathbb{F}^{\mathcal{M}}$-Brownian motion. This yields the equivalence of (2) and (3). For the equivalence of (1) and (2) note that along with I, H is a martingale, too (rationality). Furthermore, by admissibility of θ we have that

$$\mathbb{E}\int_0^T (\partial_y H_s \lambda(s)\sigma(s))^2\,\mathrm{d}s < \infty.$$

Hence,

$$\int_0^t \partial_y H_s \lambda(s)\sigma(s)\,\mathrm{d}I_s, \quad t \in [0,T],$$

also is a martingale. It follows that necessarily

$$\int_0^t \partial_t H_s + \frac{\lambda(s)^2\sigma(s)^2}{2}\partial_{yy}H_s \,\mathrm{d}s + \int_0^t \partial_y H_s \lambda(s)(\hat{\mu}_s + \hat{\alpha}_s)\,\mathrm{d}s = 0, \qquad \mathrm{d}t\otimes\mathrm{d}\mathbb{P}\text{-a.s.}$$

Since $\partial_y H_s\lambda(s) > 0$, we get $\mathrm{d}t\otimes\mathrm{d}\mathbb{P}$-a.s. the required equivalence of

$$\partial_t H_t + \frac{\lambda(t)^2\sigma(t)^2}{2}\partial_{yy}H_t = 0 \qquad \text{and} \qquad \hat{\mu}_t + \hat{\alpha}_t = 0.$$

□

3.6 Equilibrium

In the last three sections, we derived sufficient conditions for the optimality of an insider strategy and necessary conditions for an equilibrium. Both we did for the risk neutral as well as for the risk averse case. We are now in the position to state sufficient conditions for an equilibrium. Recall that we associate $\beta = 0$ to the case of risk neutrality.

Proposition 3.22. *Let Assumptions 3.1 and 3.8 be satisfied, $(H,\lambda)\in\mathcal{P}$ and $\theta\in\mathcal{S}(H,\lambda)$. Then (H,λ,θ) is an equilibrium if the following conditions are fulfilled:*

(i) H satisfies (3.12), and λ is chosen as in (3.37),

(ii) θ is continuous and of finite variation,

(iii) $\bar{Y}$, as defined in (3.67), is an $\mathbb{F}^{\mathcal{M}}$-Brownian motion on $[0,T]$,

(iv) $\lim_{t\to T}\mathbb{E}J_t^{0,t} = 0$, if $\beta = 0$, and $\lim_{t\to T}\mathbb{E}^{\beta}\exp\left(-\beta J_t^{0,t}\right) = 1$, if $\beta < 0$, with $J_t^{0,t}$ as in Proposition 3.15.

If m is bounded, condition (iv) can be replaced by

(v) $H(T,\widetilde{Y}_T) = V$ $\mathbb{P}$-a.s.

Proof. We directly get the optimality of θ from condition (i), (ii), and (iv) and Proposition 3.15 if $\beta = 0$ or Proposition 3.20 if $\beta < 0$. For $t \geq 0$ now apply Itô's formula to $H(t,\widetilde{Y}_t)$. With condition (ii) and $\mathrm{d}\widetilde{Y}_t = \lambda(t)\sigma(t)\mathrm{d}\bar{Y}_t$ this yields

$$\begin{aligned} H_t =& H_0 + \int_0^t \partial_y H_s\,\mathrm{d}\widetilde{Y}_s + \int_0^t \partial_t H_s\,\mathrm{d}s + \frac{1}{2}\int_0^t \partial_{yy}H_s\,\mathrm{d}\langle\widetilde{Y}\rangle_s \\ =& H_0 + \int_0^t \partial_y H_s\sigma(s)\lambda(s)\,\mathrm{d}\bar{Y}_s + \int_0^t \partial_t H_s + \frac{1}{2}\partial_{yy}H_s\sigma(s)^2\lambda(s)^2\,\mathrm{d}s. \end{aligned}$$

Since $\bar{Y}$ is an $\mathbb{F}^{\mathcal{M}}$-Brownian motion and H satisfies PDE (3.12), it follows that $(H(t,\widetilde{Y}_t))_{t\geq 0}$ is an $\mathbb{F}^{\mathcal{M}}$-martingale with $H(T,\widetilde{Y}_T) = V$ (Proposition 3.15 or Proposition 3.20). Hence,

$$H(t,\widetilde{Y}_t) = \mathbb{E}\left(H(T,\widetilde{Y}_T)\middle|\mathcal{F}_t^{\mathcal{M}}\right) = \mathbb{E}\left(V\middle|\mathcal{F}_t^{\mathcal{M}}\right), \qquad \text{for all } t\in[0,T],$$

i.e. (H, λ) is a rational pricing rule. Together the optimal trading strategy θ and the rational pricing rule (H, λ) form an equilibrium. If m is bounded, the optimality already follows from (v) instead of (iv) with Proposition 3.13 or Proposition 3.18. $\square$

Remark 3.23. Since the assumptions of Proposition 3.22 match those of Proposition 3.21, condition (iii) could be replaced by $\mathbb{E}\left(\hat{\alpha}_t + \hat{\mu}_t \middle| \mathcal{F}_t^{\mathcal{M}}\right) = 0$.

With the help of Proposition 3.22, we are able to state the main result of this model – the existence and form of an equilibrium. The main task is to show that the controlled order process is an $\mathbb{F}^{\mathcal{M}}$-Brownian bridge with suitable terminal value. We start with the risk neutral case.

Theorem 3.24. *Let $\beta = 0$ and Assumptions 3.1 and 3.8 be satisfied. Define*

$$H(t, y) := \mathbb{E}\left(h(y + \xi_T - \xi_t)\right), \tag{3.70}$$

with ξ as in (3.4) and

$$\lambda(t)^{-1} := \int_0^t m(s)\,\mathrm{d}s + \lambda_0^{-1},$$

with $\lambda_0 > 0$ such that

$$\int_0^T \lambda(s)^2 \sigma(s)^2\,\mathrm{d}s = 1.$$

Furthermore, let

$$\theta_t := \int_0^t -\mu_s + a(s)(Z - \widetilde{Y}_s)\,\mathrm{d}s, \quad \textit{with} \quad a(t) := \frac{\lambda(t)\sigma(t)^2}{1 - \int_0^t \lambda(s)^2\sigma(s)^2\,\mathrm{d}s}. \tag{3.71}$$

If $\sup_{t\in[0,T]} \mathbb{E}(y^*(t, V))^2 < \infty$ and m is bounded, then the triple (H, λ, θ) defines an equilibrium. If additionally $\{J_t^{0,t},\ t \in [0, T)\}$ is uniformly integrable, $\lim_{t\to T} \lambda(t)a(t) = \infty$ and if there exist $T' < T$ and $\epsilon > 0$ such that $m(t)/a(t) \le 1 - \epsilon$ for any $t \in (T', T)$, then (H, λ, θ) defines an equilibrium even if m is unbounded. In particular, $\{J_t^{0,t},\ t \in [0, T)\}$ is uniformly integrable if

$$\sup_{t\in[0,T)} \mathbb{E}\left(\frac{y^*(T, V) - y^*(t, V)}{\lambda(t)}\right)^2 < \infty.$$

Proof. First of all, we show that there indeed always exists a $\lambda_0 > 0$ such that

$$\int_0^T \left(\int_0^t m(s)\,\mathrm{d}s + \lambda_0^{-1}\right)^{-2} \sigma(t)^2\,\mathrm{d}t = 1.$$

Obviously, the function

$$f : \mathbb{R}_+ \to \mathbb{R}_+, \quad x \mapsto \int_0^T \left(\int_0^t m(s)\,\mathrm{d}s + x\right)^{-2} \sigma(t)^2\,\mathrm{d}t$$

is continuous. On the one hand, we have

$$f(x) \leq \int_0^T x^{-2}\sigma(t)^2\,\mathrm{d}t.$$

Since σ is bounded, there exists an $x > 0$ such that $f(x) \leq 1$. On the other hand, for some $t \in (0, T)$ there exist constants $\epsilon, K \in \mathbb{R}_+$ with $\sigma(s) \geq \epsilon$ and $m(s) \leq K$ for all $s \in [0, t]$. Furthermore,

$$f(x) \geq \epsilon^2 \int_0^t (sK + x)^{-2}\,\mathrm{d}s = \epsilon^2 \frac{t}{x(Kt + x)}.$$

This shows the existence of $x > 0$ such that $f(x) \geq 1$. Altogether, we can find an $x > 0$ with $f(x) = 1$.

We conclude that λ is strictly positive on $[0, T)$. Furthermore, H satisfies the conditions of Definition 3.2. To see this, observe that $(H(t, \xi_t))_{t\in[0,T]}$ is a martingale. Hence, $(H(t, \xi_t)^2)_{t\in[0,T]}$ is a submartingale. Moreover, $\xi_T \overset{d}{=} Z$ since λ is chosen such that $\int_0^T \lambda(s)^2\sigma(s)^2\,\mathrm{d}s = 1$. In particular,

$$\mathbb{E}H(t, \xi_t)^2 \leq \mathbb{E}H(T, \xi_T)^2 = \mathbb{E}h(Z)^2 < \infty. \tag{3.72}$$

Smoothness and monotonicity follow directly from the definition (for smoothness see also [43], Section 4.3). Together, (H, λ) is an admissible pricing rule.

As a next step, we show that $\theta \in \mathcal{S}(H, \lambda)$. Due to Lemma 3.26 we know that for all $t \in [0, T]$

$$\widetilde{Y}_t \overset{d}{=} \xi_t.$$

Hence, as in (3.72), we get

$$\mathbb{E}H(t, \widetilde{Y}_t)^2 = \mathbb{E}H(t, \xi_t)^2 < \mathbb{E}h(Z)^2 < \infty. \tag{3.73}$$

Furthermore, due to the monotonicity and differentiability of h, we get by dominated convergence that

$$\partial_y H(t, y) = \partial_y \mathbb{E}h(y + \xi_T - \xi_t) = \mathbb{E}\partial_y h(y + \xi_T - \xi_t).$$

Thus, together with Jensen's inequality

$$\mathbb{E}(\partial_y H(t, \widetilde{Y}_t)^2) = \mathbb{E}(\partial_y H(t, \xi_t)^2) \leq \mathbb{E}\partial_y h(\xi_T)^2 = \mathbb{E}\partial_y h(Z)^2 < \infty.$$

In particular, θ is admissible.

Now, to prove our result, it suffices to check the corresponding conditions of Proposition 3.22. Obviously, λ verifies (3.37). (3.12) for H follows again from Feynman-Kac's formula. Moreover,

θ is continuous and of finite variation. In addition, $\bar{Y}$ is an $\mathbb{F}^{\mathcal{M}}$-Brownian motion according to Lemma 3.26. If m is bounded, it is enough to show that $H(T, \widetilde{Y}_T) = V$. Due to the special structure of H, this is the case if and only if $\widetilde{Y}_T = Z$. This is shown in Lemma 3.27. If m is unbounded, we may check condition (iv) of Proposition 3.22 instead of condition (v), i.e.

$$\lim_{t \to T} \mathbb{E} J_t^{0,t} = 0.$$

This follows from Lemma 3.28. □

Let us now consider the risk averse case.

Theorem 3.25. *Let $\beta < 0$ and Assumption 3.8 be satisfied such that m is bounded on $[0, T]$. Then, under the assumptions of Theorem 3.24, the triple (H, λ, θ) defines an equilibrium if*

$$\mathbb{E} \exp \left(\frac{1}{2} \int_0^T \beta^2 \sigma(t)^2 (H(t, \widetilde{Y}_t) - V)^2 \, \mathrm{d}t \right) < \infty.$$

If m is unbounded and the conditions of Theorem 3.24 are satisfied, (H, λ, θ) is an equilibrium if additionally $\{\exp(-\beta J_t^{0,t}),\ t \in [0, T)\}$ is uniformly integrable w.r.t. $\mathbb{E}^{\beta}$.

The proof is provided at the end of the next section.

3.7 Auxiliaries for the proofs of Theorem 3.24 and Theorem 3.25

Lemma 3.26. *In the situation of Theorem 3.24, $(\bar{Y}_t)_{0 \le t \le T}$, as defined in (3.67), is an $\mathbb{F}^{\mathcal{M}}$-Brownian motion.*

Proof. Due to the special form of θ in (3.71), on $[0, T]$ we get the following dynamics for the total order process Y

$$\mathrm{d}Y_t = a(t) \left(Z - \widetilde{Y}_t \right) \mathrm{d}t + \sigma(t) \, \mathrm{d}B_t, \quad Y_0 = 0,$$

the weighted order process $\widetilde{Y}$

$$\mathrm{d}\widetilde{Y}_t = \lambda(t) a(t) \left(Z - \widetilde{Y}_t \right) \mathrm{d}t + \sigma(t) \lambda(t) \, \mathrm{d}B_t, \quad \widetilde{Y}_0 = 0, \tag{3.74}$$

and the information process, defined in (3.68),

$$\mathrm{d}I_t = a(t) \sigma(t)^{-1} \left(Z - \mathbb{E}(Z | \mathcal{F}_t^{\mathcal{M}}) \right) \mathrm{d}t + \mathrm{d}B_t, \quad I_0 = 0. \tag{3.75}$$

Now, for $T' \in [0, T)$ the process $(Z, \widetilde{Y}_t)_{t \in [0, T']}$ defines a Gaussian filter problem w.r.t. the filtration $(\mathcal{F}_t^{\mathcal{M}})_{t \in [0, T']}$ (remember $Z = h^{-1}(V), Z \sim \mathcal{N}(0, 1)$). According to Theorem 2.2, we have for $t \in [0, T']$ (observe that a is bounded on $[0, T']$)

$$\mathbb{E}(Z | \mathcal{F}_t^{\mathcal{M}}) \sim \mathcal{N}(\eta_t, \gamma_t),$$

where

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t = -\left(\frac{\gamma_t \lambda(t) a(t)}{\lambda(t)\sigma(t)}\right)^2 = -\frac{\gamma_t^2 a(t)^2}{\sigma(t)^2}, \quad \gamma_0 = 1, \tag{3.76}$$

$$\mathrm{d}\eta_t = \frac{\gamma_t a(t)\lambda(t)}{\lambda(t)^2\sigma(t)^2}\left(\mathrm{d}\widetilde{Y}_t - \lambda(t)a(t)\left(\eta_t - \widetilde{Y}_t\right)\mathrm{d}t\right), \quad \eta_0 = 0. \tag{3.77}$$

Furthermore, ODE (3.76) is solved by

$$\gamma_t = \left(\int_0^t a(s)^2\sigma(s)^{-2}\,\mathrm{d}s + 1\right)^{-1}, \quad t \in [0,T].$$

Hence, (3.77) is equivalent to

$$\begin{aligned}
\mathrm{d}\eta_t &= \frac{\gamma_t a(t)}{\sigma(t)^2}\lambda(t)^{-1}\left(\lambda(t)\mathrm{d}Y_t - \lambda(t)a(t)\left(\eta_t - \widetilde{Y}_t\right)\mathrm{d}t\right) \\
&= \frac{\gamma_t a(t)}{\sigma(t)^2}\sigma(t)\,\mathrm{d}I_t \\
&= \frac{a(t)\sigma(t)^{-1}}{\int_0^t a(s)^2\sigma(s)^{-2}\,\mathrm{d}s + 1}\mathrm{d}I_t, \quad \eta_0 = 0.
\end{aligned}$$

Following Lemma A.1, a, as defined in (3.71), satisfies the integral equation

$$\frac{a(t)\sigma(t)^{-1}}{\int_0^t a(s)^2\sigma(s)^{-2}\,\mathrm{d}s + C} = \lambda(t)\sigma(t), \quad \text{for all } t \in [0,T).$$

Therefore, we can write η as

$$\eta_t = \int_0^t \lambda(s)\sigma(s)\,\mathrm{d}I_s. \tag{3.78}$$

(3.74), (3.75) and (3.78) together yield

$$\begin{aligned}
\mathrm{d}\widetilde{Y}_t &= \lambda(t)a(t)\left(Z - \widetilde{Y}_t\right)\mathrm{d}t + \lambda(t)\sigma(t)\,\mathrm{d}B_t \\
&= \lambda(t)a(t)\left(\eta_t - \widetilde{Y}_t\right)\mathrm{d}t + \lambda(t)\sigma(t)\,\mathrm{d}I_t \\
&= \lambda(t)a(t)\left(\int_0^t \lambda(s)\sigma(s)\,\mathrm{d}I_s - \widetilde{Y}_t\right)\mathrm{d}t + \lambda(t)\sigma(t)\,\mathrm{d}I_t, \quad \widetilde{Y}_0 = 0.
\end{aligned}$$

On the one hand, this SDE has a unique strong solution on $[0,T']$ given by

$$\widetilde{Y}_t = \int_0^t \lambda(s)\sigma(s)\,\mathrm{d}I_s.$$

On the other hand, we have for $t \in [0,T)$

$$\widetilde{Y}_t = \int_0^t \lambda(s)\,\mathrm{d}Y_s = \int_0^t \lambda(s)\sigma(s)\,\mathrm{d}\bar{Y}_s.$$

Hence, $I_t = \bar{Y}_t$ on $[0,T']$ and I is an $\mathbb{F}^{\mathcal{M}}$-Brownian motion. Since this holds true for all $T' \in [0,T)$ and $\bar{Y}$ is continuous, this proves the assertion. $\square$

Lemma 3.27. *For $t \in [0,T)$ let*

$$\rho(t) := \exp\left(-\int_0^t k(s)\,\mathrm{d}s\right), \qquad k(t) := \lambda(t)(a(t) - m(t)).$$

Then

$$\frac{Z - \widetilde{Y}_t}{\lambda(t)} \quad \sim \quad \mathcal{N}\left(0, \rho(t)^2\left(\lambda(0)^{-2} + \int_0^t \rho(s)^{-2}\sigma(s)^2\,\mathrm{d}s\right)\right),$$

for all $t \in [0,T)$, where

$$\lim_{t\to T} \rho(t)^2\left(\lambda(0)^{-2} + \int_0^t \rho(s)^{-2}\sigma(s)^2\,\mathrm{d}s\right) = 0.$$

In particular, $\widetilde{Y}_T = Z$ $\mathbb{P}$-a.s.

Proof. Since

$$\mathrm{d}\lambda(t)^{-1} = m(t)\mathrm{d}t,$$

we obtain with integration by parts

$$\begin{aligned}
\mathrm{d}\frac{Z - \widetilde{Y}_t}{\lambda(t)} &= -\lambda(t)^{-1}\mathrm{d}\widetilde{Y}_t + (Z - \widetilde{Y}_t)\mathrm{d}\lambda(t)^{-1} \\
&= (m(t) - a(t))(Z - \widetilde{Y}_t)\mathrm{d}t - \sigma(t)\mathrm{d}B_t.
\end{aligned}$$

Furthermore, obviously

$$\mathrm{d}\rho(t)^{-1} = \lambda(t)(a(t) - m(t))\rho(t)^{-1}\mathrm{d}t.$$

Again integration by parts yields for $t \in [0,T)$

$$\begin{aligned}
\rho(t)^{-1}\frac{Z - \widetilde{Y}_t}{\lambda(t)} = \frac{Z}{\lambda(0)} - \int_0^t \rho(s)^{-1}\sigma(s)\,\mathrm{d}B_s - \int_0^t \rho(s)^{-1}(\lambda(s)(a(s) - m(s))\frac{(Z - \widetilde{Y}_s)}{\lambda(s)}\,\mathrm{d}s \\
+ \int_0^t \rho(s)^{-1}\lambda(s)(a(s) - m(s))\frac{Z - \widetilde{Y}_s}{\lambda(s)}\,\mathrm{d}s.
\end{aligned}$$

Multiplying $\rho(t)$ on both sides results in

$$\frac{Z-\widetilde{Y}_t}{\lambda(t)} = \rho(t)\frac{Z}{\lambda(0)} - \rho(t)\int_0^t \rho(s)^{-1}\sigma(s)\,\mathrm{d}B_s.$$

For every $t \in [0,T)$, the deterministic function $\rho(\cdot)^{-1}$ is bounded on $[0,t]$. In particular, $\int_0^t \rho(s)^{-2}\sigma(s)^2\,\mathrm{d}s < \infty$ and hence

$$\rho(t)\int_0^t \rho(s)^{-1}\sigma(s)\,\mathrm{d}B_s \quad\sim\quad \mathcal{N}\left(0, \rho(t)^2\int_0^t \rho(s)^{-2}\sigma(s)^2\,\mathrm{d}s\right).$$

Since B is independent of Z,

$$\frac{Z-\widetilde{Y}_t}{\lambda(t)} \quad\sim\quad \mathcal{N}\left(0, \rho(t)^2\left(\lambda(0)^{-2} + \int_0^t \rho(s)^{-2}\sigma(s)^2\,\mathrm{d}s\right)\right).$$

With l'Hospital's rule we get

$$\lim_{t\to T}\rho(t)^2\int_0^t \rho(s)^{-2}\sigma(s)^2\,\mathrm{d}s = \lim_{t\to T}\frac{\sigma(t)^2\rho(t)^{-2}}{2k(t)\rho(t)^{-2}} = \lim_{t\to T}\frac{\sigma(t)^2}{2k(t)} = 0$$

since $\lim_{t\to T}k(t) = \infty$. If m is bounded and, in particular, $\lambda(T) > 0$, this is obvious. If m is unbounded, $\lim_{t\to T}k(t) = \lim_{t\to T}\lambda(t)a(t)(1-m(t)/a(t)) = \infty$ holds by assumption.

This shows the convergence in distribution of $\frac{Z-\widetilde{Y}_t}{\lambda(t)}$ to the constant 0 and therefore the convergence in probability. In particular $\widetilde{Y}_t$ converges to Z in probability. Due to the almost sure uniqueness of the limit we get the last part of our statement. □

In the case when $\lambda(T) > 0$, Lemma 3.27 already proves the optimality of our insider strategy. Nevertheless, for the general case, $\lambda(T) \geq 0$, more work has to be done.

Lemma 3.28. *Given the situation of Theorem 3.24 such that $\{J_t^{0,t},\ t\in[0,T)\}$ is uniformly integrable. Then $\mathbb{E}J_t^{0,t} \to 0$, $t\to T$. In particular, $\{J_t^{0,t},\ t\in[0,T)\}$ is uniformly integrable if*

$$\sup_{t\in(0,T)} \mathbb{E}\left(\frac{y^*(T,V)-y^*(t,V)}{\lambda(t)}\right)^2 < \infty. \tag{3.79}$$

Proof. Since, by assumption, $J_t^{0,t} > 0$ is uniformly integrable, the L_1 convergence follows from convergence in probability. For the latter first observe that

$$\int_{y^*(t,V)}^{\widetilde{Y}_t} H(t,x) - V\,\mathrm{d}x \leq \left|\widetilde{Y}_t - Z\right|\left|H(t,\widetilde{Y}_t) - V\right| + \int_{y^*(t,V)}^{y^*(T,V)} H(t,x) - V\,\mathrm{d}x \quad \mathbb{P}\text{-a.s.} \tag{3.80}$$

To realise this, consider the following cases:

(1) $Z \in [y^*(t,V), \widetilde{Y}_t]$ or $Z \in [\widetilde{Y}_t, y^*(t,V)]$

(2) $Z < \min(y^*(t,V), \widetilde{Y}_t)$

(3) $Z > \max(y^*(t,V), \widetilde{Y}_t)$

Due to the special form of H, we have $Z = y^*(T,V)$.

1. In the first case,

$$\left|H(t,\widetilde{Y}_t) - V\right| \geq |H(t,Z) - V|$$

due to the monotonicity of H. Hence,

$$\left|\int_Z^{\widetilde{Y}_t} H(t,x) - V \,\mathrm{d}x\right| \leq \left|\widetilde{Y}_t - Z\right| \left|H(t,\widetilde{Y}_t) - V\right|.$$

2. If $Z < \min(y^*(t,V), \widetilde{Y}_t)$, we get

$$\int_{y^*(t,V)}^{\widetilde{Y}_t} H(t,x) - V \,\mathrm{d}x \leq \left|\widetilde{Y}_t - y^*(t,V)\right| \left|H(t,\widetilde{Y}_t) - V\right| \leq \left|\widetilde{Y}_t - Z\right| \left|H(t,\widetilde{Y}_t) - V\right|.$$

3. If $Z > \max(y^*(t,V), \widetilde{Y}_t)$, we get

$$\int_{y^*(t,V)}^{\widetilde{Y}_t} H(t,x) - V \,\mathrm{d}x \leq \int_{y^*(t,V)}^{Z} H(t,x) - V \,\mathrm{d}x.$$

This finally proves (3.80).

We can now demonstrate the convergence in probability in two steps. Firstly, due to Lemma 3.27, $(\widetilde{Y}_t - Z)\lambda(t)^{-1}$ converges to 0. Secondly, for any $v \in \mathcal{V}$,

$$\lambda(t)^{-1} \int_{y^*(t,v)}^{y^*(T,v)} H(t,x) - v \,\mathrm{d}x \xrightarrow{t\to T} 0.$$

If $\lambda(T) > 0$, this follows from continuity of $y^*(\cdot, v)$. If $\lambda(T) = 0$, we get with l'Hospital's rule and (3.39)

$$\begin{aligned}
\lim_{t\to T} \lambda(t)^{-1} \int_{y^*(t,v)}^{y^*(T,v)} H(t,x) - v \,\mathrm{d}x &= \lim_{t\to T} \frac{\sigma(t)^2\lambda(t)^2 \left(\partial_y H(t, y^*(T,v)) - \partial_y H(t, y^*(t,v))\right)}{2m(t)\lambda(t)^2} \\
&= \lim_{t\to T} \frac{\sigma(t)^2}{2m(t)} \left(\partial_y H(t, y^*(T,v)) - \partial_y H(t, y^*(t,v))\right) \\
&= 0
\end{aligned}$$

since $\partial_y H \in \mathcal{C}^0([0,T])$. Altogether, this proves $J_t^{0,t} \to 0$ in probability.

It remains to show that $J_t^{0,t}$ is uniformly integrable if (3.79) holds. Since

$$\mathbb{E}\left(\lambda(t)^{-1} \int_{y^*(t,V)}^{\widetilde{Y}_t} H(t,x) - V \,\mathrm{d}x\right) \leq \mathbb{E}\frac{\widetilde{Y}_t - y^*(t,V)}{\lambda(t)} \left(H(t,\widetilde{Y}_t) - V\right),$$

Cauchy-Schwarz inequality yields

$$\mathbb{E}\left(\lambda(t)^{-1}\int_{y^*(t,V)}^{\widetilde{Y}_t} H(t,x)-V\,\mathrm{d}x\right)\le\left(\mathbb{E}\left(\frac{\widetilde{Y}_t-y^*(t,V)}{\lambda(t)}\right)^2\right)^{1/2}\left(\mathbb{E}\left(H(t,\widetilde{Y}_t)-V\right)^2\right)^{1/2}.$$

Due to (3.73), the second factor of the right hand side is uniformly bounded. It is therefore enough to show that

$$\sup_{t\in(0,T)}\mathbb{E}\left(\frac{\widetilde{Y}_t-y^*(t,V)}{\lambda(t)}\right)^2$$

is bounded, too. Since $Z=y^*(T,V)$, we obviously have

$$\begin{aligned}\mathbb{E}\left(\frac{\widetilde{Y}_t-y^*(t,V)}{\lambda(t)}\right)^2&\le\mathbb{E}\left(\frac{\widetilde{Y}_t-Z+y^*(T,V)-y^*(t,V)}{\lambda(t)}\right)^2\\&\le 2\mathbb{E}\left(\frac{\widetilde{Y}_t-Z}{\lambda(t)}\right)^2+2\mathbb{E}\left(\frac{y^*(T,V)-y^*(t,V)}{\lambda(t)}\right)^2.\end{aligned}$$

The first term in the last line converges to 0 due to Lemma 3.27. Now uniform integrability follows from (3.79). □

Proof of Theorem 3.25. Admissibility of (H,λ) and θ follow analogously to the proof of Theorem 3.24 under the additional condition that

$$\mathbb{E}\exp\left(\frac{1}{2}\int_0^T\beta^2\sigma(t)^2(H(t,\widetilde{Y}_t)-V)^2\,\mathrm{d}t\right)<\infty.$$

If m is bounded, the conditions of Proposition 3.22 can easily be verified (Lemmas 3.26 and 3.27). If m is unbounded, the convergence of $\mathbb{E}^\beta\exp\left(J_t^{0,t}\right)$ to 1 follows from uniform integrability and convergence in probability. Following Lemma 3.27,

$$\frac{Z-\widetilde{Y}_t}{\lambda(t)}=-\rho(t)\int_0^t\rho(s)\sigma(s)\,\mathrm{d}B_s+\frac{Z\rho(t)}{\lambda(0)}.$$

Due to Girsanov's theorem (cf. [56], Theorem 39, Chapter III)

$$B_t^\beta:=B_t-\int_0^t\beta\sigma(s)(H(s,\widetilde{Y}_s)-V)\,\mathrm{d}s,\quad t\in[0,T],$$

defines a Brownian motion w.r.t. $\mathbb{P}^\beta$. It follows that

$$\frac{Z-\widetilde{Y}_t}{\lambda(t)}+\rho(t)\int_0^t\rho(s)^{-1}\beta\sigma(s)^2(H(s,\widetilde{Y}_s)-V)\,\mathrm{d}s-\frac{Z}{\lambda(0)}\rho(t)$$

is distributed normally w.r.t. $\mathbb{P}^\beta$ with vanishing variance for $t\to T$. As in Lemma 3.27 we again get with l'Hospital's rule that the second part converges to 0 in probability ($\mathbb{P}^\beta$). From this it follows that $\frac{Z-\widetilde{Y}_t}{\lambda(t)}$ converges to 0 in probability, too. With an analogous argumentation as in Lemma 3.28 we conclude that $J_t^{0,t}$ converges to 0 in probability (w.r.t. $\mathbb{P}^\beta$). □

3.8 Remarks and example

3.8.1 Additional deterministic noise drift

One can show analogous results for the case when the noise drift contains a deterministic component, i.e.

$$\mu(t,y,V) = \mu_1(t,y,V) + \mu_2(t) \tag{3.81}$$

where μ_2 is some deterministic càdlàg function. For the existence of an equilibrium (cf. Propositions 3.6 and 3.7) we would get the necessary condition

$$\partial_t H(t,y) + \lambda(t)\mu_2(t)\partial_y H(t,y) + \frac{1}{2}\sigma(t)^2\lambda(t)^2\partial_{yy}H(t,y) = 0,$$

instead of (3.12) and μ_1 being as in Assumption 3.8. This PDE for H turns out to be the PDE of an equilibrium pricing rule in Corcuera et al. [26] (without jumps), see also (1.7), where an insider model with deterministic noise drift (not depending on V, i.e. $\mu_1 = 0$) is analysed. This particular model would be generalised by the consideration of an additional deterministic drift term in our model. However, we would no longer have semi-rationality of μ, i.e. $\operatorname{sgn}(\bar{\mu}(t,P_t,V)) = \operatorname{sgn}(V - P_t)$.

3.8.2 A larger class of admissible noise drift

It seems natural to ask whether there exists a possibility to extend our model to a larger class of noise drift. As plausible candidate consider

$$\mu(t,y,v) = \frac{\int_y^{y^*(t,v)} m(t,x)(H(t,x)-v)\,\mathrm{d}x}{H(t,y)-v}, \tag{3.82}$$

i.e. we preserve the general structure calculated in Section 3.2, but allow the drift intensity m to depend on the weighted total order. This could be covered by a price pressure λ that depends on the total order, too. To see this, we proceed analogously to the HJB equation approach made in Subsection 3.2.1 (in particular we consider the risk neutral case). Assuming $\sigma \equiv 1$, this means that we start with a system of equations corresponding to (3.10) and (3.11),

$$\partial_y J(t,y) = \frac{H(t,y)-v}{\lambda(t,y)}, \tag{3.83}$$

$$\partial_t J(t,y) = -\lambda(t,y)\mu(t,y,v)\partial_y J(t,y) - \frac{1}{2}\lambda(t,y)^2\partial_{yy}J(t,y). \tag{3.84}$$

The same arguments as in the proof of Proposition 3.6, i.e. differentiating (3.83) w.r.t. y and t,

$$\begin{aligned}\partial_{yy}J(t,y) &= \frac{\partial_y H(t,y)}{\lambda(t,y)} - (H(t,y)-v)\frac{\partial_y\lambda(t,y)}{\lambda(t,y)^2},\\ \partial_{yt}J(t,y) &= \frac{\partial_t H(t,y)}{\lambda(t,y)} - (H(t,y)-v)\frac{\partial_t\lambda(t,y)}{\lambda(t,y)^2},\end{aligned}$$

and combining this with the partial derivative of (3.84) w.r.t. y,

$$\partial_{ty}J(t,y) = -\frac{1}{2}\lambda(t,y)\partial_{yy}H(t,y) - \partial_y(\mu(t,y,v)(H(t,y)-v)) + \frac{1}{2}\partial_{yy}\lambda(t,y)(H(t,y)-v),$$

leads to

$$\begin{aligned}&\frac{\partial_t H(t,y)}{\lambda(t,y)} + \frac{\lambda(t,y)}{2}\partial_{yy}H(t,y)\\ &\qquad = (H(t,y)-v)\left(\frac{\partial_t\lambda(t,y)}{\lambda(t,y)^2} + \frac{\partial_{yy}\lambda(t,y)}{2} - \frac{\partial_y(\mu(t,y,v)(H(t,y)-v))}{(H(t,y)-v)}\right).\end{aligned}$$

Incorporating the semi-rationality of μ, this yields similarly to the proof of Proposition 3.6

$$m(t,y) = -\frac{\partial_t\lambda(t,y)}{\lambda(t,y)^2} - \frac{1}{2}\partial_{yy}\lambda(t,y),$$

for m as in (3.82).

We note that if we allowed dependence of λ w.r.t. $\widetilde{Y}_t$, we could even cover the case of more complex noise drift in the sense that these noise drift terms do not violate the initial necessary condition for the existence of an equilibrium. However, this complicates the optimisation problem a lot since the insider could take effect on the price pressure by her trading strategy. An approach as in Theorem 3.24 would require λ to ensure

$$\int_0^T \lambda(s,\widetilde{Y}_s)^2\,\mathrm{d}s = 1 \quad \mathbb{P}\text{-a.s.}$$

To make this accordable with the path dependence of λ, we have to permit further dependences, e.g. in $\Lambda_t := \int_0^t \lambda(s,\widetilde{Y}_s)^2\,\mathrm{d}s$. This in turn would again complicate the optimisation problem.

3.8.3 Example

We close this chapter with an example.

Example 3.29. Assume that $\beta = 0$ and $h(y) = \exp(y)$. Then

$$H(t,y) = \mathbb{E}\exp\left(y + \xi_T - \xi_t\right),$$

where

$$\xi_T - \xi_t \sim \mathcal{N}\left(0, \int_t^T \sigma(s)^2 \lambda(s)^2 \,\mathrm{d}s\right).$$

For notational convenience define

$$L(t) := \frac{1}{2}\int_t^T \sigma(s)^2\lambda(s)^2 \,\mathrm{d}s, \quad t \in [0,T].$$

Together with the moment generating function of the normal distribution it follows that

$$H(t,y) = \exp\left(y + L(t)\right).$$

We then get

$$y^*(t,v) = H(t,\cdot)^{-1}(v) \quad \Leftrightarrow \quad \exp\left(y^*(t,v) + L(t)\right) = v \quad \Leftrightarrow \quad y^*(t,v) = \log(v) - L(t).$$

Furthermore, we have

$$\begin{aligned}
\mu(t,y,v) &= m(t)\frac{\int_y^{y^*(t,v)} H(t,x) - v \,\mathrm{d}x}{H(t,y) - v} \\
&= m(t)\frac{\int_y^{\log(v)-L(t)} \exp\left(x + L(t)\right) - v \,\mathrm{d}x}{\exp\left(y + L(t)\right) - v} \\
&= m(t)\frac{v - \exp\left(y + L(t)\right) - v\left(\log(v) - L(t) - y\right)}{\exp\left(y + L(t)\right) - v} \\
&= m(t)\left(-1 - \frac{v\left(\log(v) - L(t) - y\right)}{\exp\left(y + L(t)\right) - v}\right)
\end{aligned}$$

as equilibrium noise drift. Incorporating the resulting total order flow, the market price is given by

$$P_t = \exp\left(L(0) + \int_0^t \lambda(s)a(s)\left(Z - \widetilde{Y}_s\right)\,\mathrm{d}s + \int_0^t \lambda(s)\sigma(s)\,\mathrm{d}B_s - \frac{1}{2}\int_0^t \sigma(s)^2\lambda(s)^2\,\mathrm{d}s\right),$$

and hence has the following dynamics

$$\mathrm{d}P_t = P_t\left(\lambda(t)a(t)\left(Z - \widetilde{Y}_t\right)\mathrm{d}t + \lambda(t)\sigma(t)\mathrm{d}B_t\right).$$

Since

$$\lambda(t)a(t)\left(Z - \widetilde{Y}_t\right) = \frac{\lambda(t)^2\sigma(t)^2}{1 - \int_0^t \lambda(s)^2\sigma(s)^2\,\mathrm{d}s}\left(Z - \log\left(P_t\right) + L(t)\right),$$

it holds that

$$\mathrm{d}P_t = P_t\left(\frac{\lambda(t)^2\sigma(t)^2}{2L(t)}\left(Z - \log\left(P_t\right) + L(t)\right)\mathrm{d}t + \lambda(t)\sigma(t)\mathrm{d}B_t\right). \tag{3.85}$$

For $\sigma = 1$, $T > 0$ and $n \in \mathbb{N}$ let

$$C = \sqrt{\frac{2n+1}{T^{2n+1}}}$$

and

$$m(t) = \frac{n}{C}(T-t)^{-(n+1)}.$$

Then

$$\int_0^t m(s)\,\mathrm{d}s = \frac{1}{C}(T-t)^{-n} - \frac{1}{C}T^{-n} \xrightarrow{t\to T} \infty.$$

Now for $\lambda_0^{-1} = \frac{1}{C}T^{-n}$, define λ by

$$\lambda(t) = \frac{1}{\lambda_0^{-1} + \int_0^t m(s)\,\mathrm{d}s} = \frac{1}{\frac{1}{C}(T-t)^{-n}} = C(T-t)^n.$$

In particular, $\lambda(T) = 0$ and

$$1 - \int_0^t \lambda(s)^2\,\mathrm{d}s = 1 - \frac{C^2}{2n+1}\left(T^{2n+1} - (T-t)^{2n+1}\right) = \frac{C^2}{2n+1}(T-t)^{2n+1}.$$

Hence,

$$1 - \int_0^T \lambda(s)^2\,\mathrm{d}s = 0.$$

We can now verify the conditions of Theorem 3.24. We have

$$a(t)\lambda(t) = \frac{\lambda(t)^2}{1 - \int_0^t \lambda(s)^2\,\mathrm{d}s} = \frac{C^2(T-t)^{2n}}{\frac{C^2}{2n+1}(T-t)^{2n+1}} = \frac{2n+1}{(T-t)} \xrightarrow{t\to T} \infty.$$

Furthermore

$$\frac{m(t)}{a(t)} = \frac{n}{C}(T-t)^{-(n+1)}\frac{1 - \int_0^t \lambda(s)^2\,\mathrm{d}s}{\lambda(t)} = \frac{n}{C}(T-t)^{-(n+1)}\frac{C}{2n+1}(T-t)^{n+1} = \frac{n}{2n+1} < 1$$

and again with l'Hospital's rule

$$\mathbb{E}\left(\frac{y^*(T,V) - y^*(t,V)}{\lambda(t)}\right)^2 = \left(\frac{L(t)}{\lambda(t)}\right)^2 \xrightarrow{t\to T} 0,$$

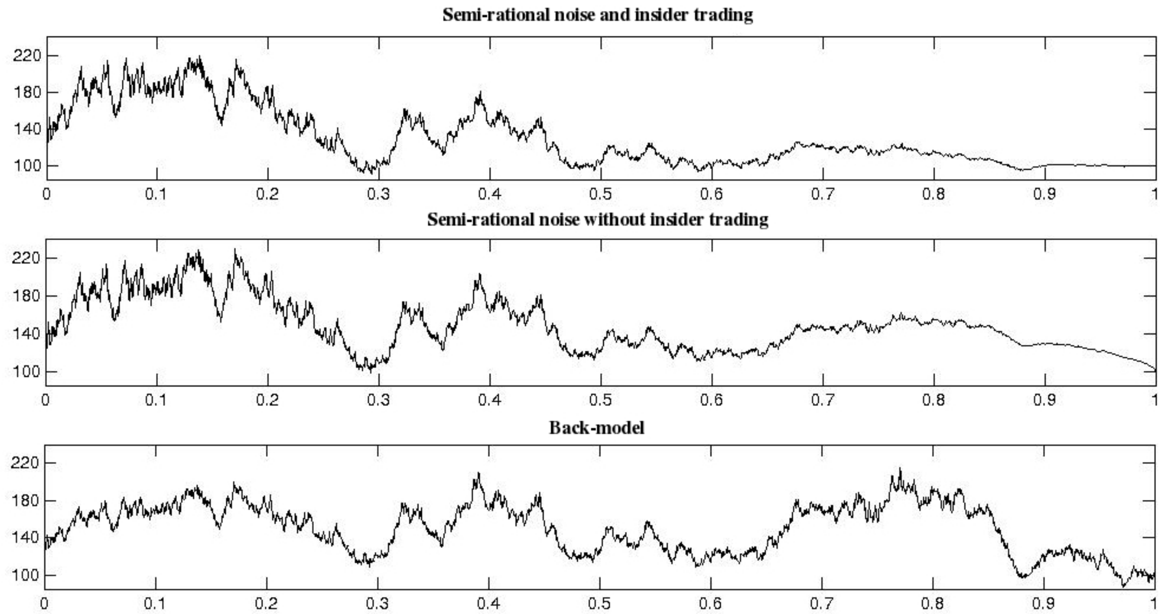

Figure 3.1. *Sample paths for different models with identical noise part and different informed trading and weighting.* $P(0) = 130$, $V = 100$, $h = \exp$, $T = 1$, $\sigma = 1$, $m(t) = C(T-t)^{-2}$ *(plot 1 and 2). The price in the first two plots gets less volatile due to the decreasing price pressure. The convergence of the price in a model with insider (plot 1) is faster than in a model without insider trading (plot 2).*

since

$$-\frac{\lambda(t)^2}{\lambda'(t)} = \frac{C^2(T-t)^{2n}}{nC(T-t)^{n-1}} = \frac{C}{n}(T-t)^{n+1} \xrightarrow{t\to T} 0.$$

Altogether, the conditions of Theorem 3.24 are satisfied.

Finally, inserting the derived coefficient functions into Equation (3.85)

$$\mathrm{d}P_t = P_t\left(\frac{2n+1}{T-t}\left(Z - \log(P_t) + \frac{C^2}{4n+2}(T-t)^{2n+1}\right)\mathrm{d}t + C(T-t)^n\mathrm{d}B_t\right).$$

If m were equal to 0 and $T = 1$, we had (compare Cho [25], Example 1 and 2)

$$\mathrm{d}P_t = P_t\left(\frac{1}{1-t}\left(Z - \log(P_t) + \frac{1-t}{2}\right)\mathrm{d}t + \mathrm{d}B_t\right).$$

Furthermore, it follows that

$$\begin{aligned}\alpha_t &= a(t)\left(Z - \widetilde{Y}_t\right) - \mu(t, \widetilde{Y}_t, V)\\ &= m(t)\left(\frac{2n+1}{n}(\log(V) - \widetilde{Y}_t) + 1 + \frac{V\left(\log(V) - L(t) - \widetilde{Y}_t\right)}{\exp\left(\widetilde{Y}_t + L(t)\right) - V}\right).\end{aligned}$$

Chapter 4

A market with reinforcing irrational behaviour

4.1 Introduction

We want to extend the model of Chapter 3 to a setting where the arrival of new information reinforces the irrational behaviour of the market and leads to further over- or underreaction. As we pointed out in the introduction, this reinforcement is expressed by an increasing time that is needed for the market to cool down from irrationality. If no further information arrives and the market cools down completely, the market price is assumed to coincide with the fundamental value according to the efficient market hypothesis. Hence, the trading horizon T is a stopping time, in general, and we have $P_{t \vee T} = V_{t \vee T}$, for all $t \geq 0$. Over- or underreactions are brought to the market via some shot noise that is correlated to the dynamics of the fundamental value. Information arrival is modelled with the help of a Poisson process N.

Primarily, new information should have an impact on the fundamental value. Hence, we consider value processes V that have the form

$$V_t = h\left(Z_t\right), \quad t \geq 0, \tag{4.1}$$

where h again is a strictly increasing function and $(Z_t)_{t \geq 0}$ is some pure jump process with mutually independent distributed increments and driven by a Poisson process N with jump times $0 < T_1 < T_2 < \dots$. Between the times of information arrival the noise traders act like in Chapter 3, i.e. orders are made according to a semi-rational drift and a noise part, given by a Brownian motion. If new information arrives while the market is still cooling down, this leads to an *order shock*. This shock is represented by a jump in the demand process that might again contain an over- or underreaction. Hence, the cumulated order flow of the noise traders is assumed to have the following dynamics

$$\mathrm{d}X_t = \mu_t \mathbb{1}_{[0,T]}(t)\mathrm{d}t + \sigma \mathrm{d}B_t + \mathrm{d}X_t^d, \quad t \geq 0, \tag{4.2}$$

where B again is a Brownian motion and μ a semi-rational noise drift. The additional part X^d is a jump process driven by N (i.e. with the same jump times as Z).

The market makers observe the jump times of N and the total order process

$$Y_t = \theta_t \mathbb{1}_{[0,T]}(t) + X_t, \quad t \geq 0,$$

but not the jumps of Z. In particular, if θ is continuous, the market makers observe the jumps of X. The conditional distribution of the jumps of Z with respect to the market makers' filtration at time T_i, i.e. ΔZ_{T_i} given $\Delta X^d_{T_i}$ is assumed to be Gaussian, $\mathcal{N}(\kappa_i, \chi_i)$.

As mentioned above, the trading horizon T is a stopping time. More detailed, we assume that

$$T = \inf\{t \geq 0 : S_t = 0\} \tag{4.3}$$

where

$$S_t = S_0 + \sum_{i=1}^{N_t} \Delta S_{T_i} - t, \quad t \geq 0, \quad S_0 > 0,$$

is the process associated to the market cooling, i.e. for $t \geq 0$, S_t denotes the time (from time t on) that is needed for the market to cool down completely if no further market shocks arrive till then. The increments ΔS_{T_i} are assumed to be $\mathcal{F}^{\mathcal{M}}_{T_i}$-measurable, non-negative random variables. As a consequence, S is an $\mathbb{F}^{\mathcal{M}}$-adapted process and thus T is an $\mathbb{F}^{\mathcal{M}}$-stopping time. In particular, the market makers' observation filtration can be described as $\mathbb{F}^{\mathcal{M}} = \mathbb{F}^{Y,S}$. The insider's filtration again contains additional information about the fundamental value process, i.e. $\mathbb{F}^{\mathcal{I}} = \mathbb{F}^{P,S,Z}$.

In contrast to the model analysed in Chapter 3, the present setting only allows constant volatility of the continuous noise part B. This is mainly for notational convenience. Additionally, with regard to the previous chapter, one could argue that the presence of a deterministic volatility does not effect the substancial structure of an equilibrium. Furthermore, we will restrict ourselves to the case of a risk neutral insider.

After this short introduction of the jump model's framework, we want to develop some intuition on the model's behaviour in equilibrium. This will provide a better understanding of the following approach regarding the dynamics and dependences of the pricing rule.

Let us therefore assume that we already derived sufficient conditions for an equilibrium and it turned out that, analogous to Chapter 3, we want to choose an insider strategy θ with

$$\theta_t = \int_0^{t \wedge T} -\mu_s + a_s \left(Z_s - \widetilde{Y}_s \right) \mathrm{d}s$$

where $\widetilde{Y}$ again denotes the weighted total order and a is an $\mathbb{F}^{\mathcal{M}}$-adapted process such that

- $\sigma^{-1} Y^c$ is an $\mathbb{F}^{\mathcal{M}}$-Brownian motion (where Y^c denotes the continuous part of Y) and
- $\widetilde{Y}_T = Z_T$ $\mathbb{P}$-a.s.

The resulting linear structure of the dynamics of Y^c in Z together with the conditional distribution of the increments then enables us to explicitly solve the corresponding filter problem w.r.t. $\mathbb{F}^{\mathcal{M}}$. On $[0,T]$, according to Proposition 2.5, Z_t given $\mathcal{F}_t^{\mathcal{M}}$ is distributed normally, $\mathcal{N}(\eta_t, \gamma_t)$, with

$$\mathrm{d}\eta_t = \frac{\gamma_t a_t}{\sigma^2}\Big(\mathrm{d}Y_t^c - (a_t(\eta_t - \widetilde{Y}_t))\mathrm{d}t\Big) + \mathrm{d}\sum_{i=1}^{N_t}\kappa_i, \quad \eta_0 = \kappa_0, \tag{4.4}$$

$$\mathrm{d}\gamma_t = -\Big(\frac{a_t\gamma_t}{\sigma}\Big)^2\mathrm{d}t + \mathrm{d}\sum_{i=1}^{N_t}\chi_i, \quad \gamma_0 = \chi_0. \tag{4.5}$$

According to Corollary 2.6 we have

$$\gamma_t = \left(\int_0^t a_s^2\sigma^{-2}\,\mathrm{d}s + \sum_{i=0}^{N_t}\widetilde{\chi}_i\right)^{-1}, \quad \text{with} \quad \widetilde{\chi}_i = -\frac{\chi_i}{(\gamma_{T_i-} + \chi_i)\gamma_{T_i-}}.$$

If we adopt the idea of Chapter 3, we choose a in a way that enables us to show, with the help of Equation (4.4), that $\sigma^{-1}Y^c$ is an $\mathbb{F}^{\mathcal{M}}$-Brownian motion. For this now assume that the continuous part of Y is again weighted according to a price pressure λ, i.e.

$$\widetilde{Y}_t^c = \int_0^t \lambda_s\,\mathrm{d}Y_s^c$$

and a is chosen such that

$$\frac{\gamma_t a_t}{\sigma^2} = \lambda_t. \tag{4.6}$$

Now, Equation (4.4) leads to the following representation of the continuous part η^c of η:

$$\eta_t^c = \int_0^t \lambda_s\sigma\,\mathrm{d}I_s$$

where

$$I_t = \int_0^t a_s\sigma^{-1}(Z_s - \eta_s)\,\mathrm{d}s + B_t$$

again is the so-called *information process* (w.r.t. the given filter problem) and an $\mathbb{F}^{\mathcal{M}}$-Brownian motion. We then obtain that

$$\begin{aligned}\mathrm{d}\widetilde{Y}_t^c &= \lambda_t a_t(Z_t - \widetilde{Y}_t)\mathrm{d}t + \lambda_t\sigma\mathrm{d}B_t\\ &= \lambda_t a_t(\eta_t - \widetilde{Y}_t)\mathrm{d}t + \lambda_t\sigma\mathrm{d}I_t.\end{aligned} \tag{4.7}$$

If the weighting of the jumps is chosen such that

$$\eta_t^d = \widetilde{Y}_t^d, \tag{4.8}$$

it follows

$$\eta_t - \widetilde{Y}_t = \eta_t^c - \widetilde{Y}_t^c = \int_0^t \lambda_s \sigma \,\mathrm{d}I_s - \widetilde{Y}_t^c$$

and hence, by (4.7),

$$\int_0^t \lambda_s \sigma \,\mathrm{d}I_s = \widetilde{Y}_t^c = \int_0^t \lambda_s \,\mathrm{d}Y_s^c.$$

This shows that $\sigma^{-1}Y^c$ indeed is an $\mathbb{F}^{\mathcal{M}}$-Brownian motion.

From the above considerations two consequences arise: First, the weighting of the jumps has to be chosen according to (4.8). In general, this requires a different weighting of continuous and discontinuous changes of the total order process. For the case of uniform weighting it has been shown by Corcuera et al. [26] that no equilibrium exists if there are jumps (uncorrelated to V) in the order flow of the noise traders. However, a different weighting in turn might offer additional arbitrage opportunities to an insider. To realise this, assume that the weighting of continuous changes were higher than the weighting of discontinuous ones. Then buying the asset according to a continuous strategy for an average price p_1 and selling the same amount discontinuously could result in a price $p_2 > p_1$. Therefore, we will only consider continuous insider strategies in this model. We also refer to Remark 4.11 for further analysis of discontinuous trading strategies. As second consequence, it turns out that a, as in (4.6), has the following form (cf. Lemma A.3)

$$a(t) = \frac{\lambda_t \sigma^2}{\sum_{i=0}^{N_t} \chi_i - \int_0^t \lambda_s^2 \sigma^2 \,\mathrm{d}s}.$$

Since $\widetilde{Y}_T$ has to equal Z_T, it seems plausible to require $a_t \to \infty$, for $t \to T$. Then λ has to be chosen such that

$$\sum_{i=0}^{N_T} \chi_i - \int_0^T \lambda_s^2 \sigma^2 \,\mathrm{d}s = 0, \quad \mathbb{P}\text{-a.s.} \tag{4.9}$$

If λ only depended on t, (4.9) would not hold in general. Therefore, it seems plausible that the dynamics of λ_t somehow depend on S_t and also

$$\Lambda_t := \sum_{i=0}^{N_t} \chi_i - \int_0^t \lambda_s^2 \sigma^2 \,\mathrm{d}s. \tag{4.10}$$

4.2 Model Setup

Let $N(\mathrm{d}t, \mathrm{d}\zeta)$, $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ be a Poisson random measure on $\mathbb{R}^3$ with finite Lévy measure

$$\nu(\mathrm{d}\zeta) := \nu_1(\mathrm{d}\zeta_1) \otimes \nu_2(\mathrm{d}\zeta_2) \otimes \nu_3(\mathrm{d}\zeta_3).$$

In the sequel, we will denote by

$$\bar{N}(\mathrm{d}t, \mathrm{d}\zeta) = N(\mathrm{d}t, \mathrm{d}\zeta) - \nu(\mathrm{d}\zeta)\mathrm{d}t$$

the *compensated Poisson random measure*, i.e. $\bar{N}(t, B)$ is a martingale for any Borel set $B \in \mathcal{B}^3$. Furthermore, $N_t := N(t, \mathbb{R}^3 \setminus \{0\})$ is the associated Poisson process with jump times $(T_i)_{i \in \mathbb{N}}$. For further details on Poisson random measures we refer to [53], Chapter 1.

The fundamental value process V, the trading horizon T and the cumulated order flow of the noise traders are defined as in (4.1), (4.3) and (4.2), respectively, such that Z, S and X are solutions to the following SDEs

$$\mathrm{d}Z_t = \int_{\mathbb{R}^3} \phi_t^Z(\zeta)\, N(\mathrm{d}t, \mathrm{d}\zeta), \quad t \geq 0, \tag{4.11}$$

$$\mathrm{d}S_t = -\mathrm{d}t + \int_{\mathbb{R}^3} \phi_t^S(\zeta)\, N(\mathrm{d}t, \mathrm{d}\zeta), \quad t \geq 0, \quad S_0 > 0, \tag{4.12}$$

$$\mathrm{d}X_t = \mu_t \mathbb{1}_{[0,T]}(t)\mathrm{d}t + \sigma \mathrm{d}B_t + \int_{\mathbb{R}^3} \phi_t^X(\zeta)\, N(\mathrm{d}t, \mathrm{d}\zeta), \quad t \geq 0, \quad X_0 = 0, \tag{4.13}$$

where

$$\phi_t^X(\zeta) = \phi^X(t, \zeta_2), \quad \phi_t^Z(\zeta) = \phi^Z(t, \zeta_1, \zeta_2), \quad \phi_t^S(\zeta) = \phi^S(t, S_{t-}, \zeta_2, \zeta_3),$$

such that $\phi^S \geq 0$ and, for given $(t, \zeta_2) \in \mathbb{R}_+ \times \mathbb{R}$,

$$\phi^Z(t, \zeta_1, \zeta_2) \frac{\nu_1(\mathrm{d}\zeta_1)}{\nu_1(\mathbb{R}^3)} \quad \sim \quad \mathcal{N}\left(\kappa(t, \phi^X(t, \zeta_2)), \chi(t, \phi^X(t, \zeta_2))\right)(\mathrm{d}\zeta_1) \tag{4.14}$$

for suitable functions

$$\kappa : [0, \infty) \times \mathbb{R} \mapsto \mathbb{R}, \quad \chi : [0, \infty) \times \mathbb{R} \mapsto \mathbb{R}_+,$$

i.e. the jumps of Z, ΔZ_{T_i}, $i \in \mathbb{N}$, are conditionally Gaussian given the jumps of X, ΔX_{T_i}, $i \in \mathbb{N}$, with mean $\kappa_i := \kappa(T_i, \Delta X_i)$ and variance $\chi_i := \chi(T_i, \Delta X_i)$.

Further technical conditions are summarised in the following assumption:

Assumption 4.1. *T, V, X are defined as above. Furthermore, the following conditions hold*

- $T < \infty$ $\mathbb{P}$*-a.s.,*
- Z_0 *given* $\mathcal{F}_0^{\mathcal{M}}$ *is* $\mathcal{N}(\kappa_0, \chi_0)$ *distributed for* $\kappa_0 \in \mathbb{R}$ *and* $\chi_0 \in \mathbb{R}_+$
- $\mathbb{E}(\Delta Z_{T_i})^4 < \infty$*, for all* $i \in \mathbb{N}_0$*,*
- $\mathbb{E}(h(Z_T)^2) < \infty$*, furthermore there exists a sufficiently smooth function* F *such that, for every* $t \geq 0$*,*

$$F(t \wedge T, Z_{t \wedge T}, S_{t \wedge T}) := \mathbb{E}\left(h(Z_T) \middle| \mathcal{F}_t^I\right). \tag{4.15}$$

Sufficiently smooth here means that F is continuously differentiable w.r.t. all variables.

- $\mathbb{E}\left(\int_0^T F(t, Z_t, S_t)^2 \,\mathrm{d}t\right) < \infty$,
- $\mathbb{E}\int_0^T \int_{\mathbb{R}^3} (F(t, Z_t + \phi_t^Z(\zeta), S_t + \phi_t^S(\zeta)) - F(t, Z_t, S_t))^2 \,\nu(\mathrm{d}\zeta)\mathrm{d}t < \infty$.

The assumption on the conditional distribution of the increments of the fundamental value process is the analogue to Assumption 3.1. It turns out to be crucial for the analytical tractability of the stochastic filter problem which is induced by insider trading and rationality of the market makers.

As in Chapter 3, we only allow pricing functions that have a special form. Despite the dependence on certain variables, as motivated in the introduction of this chapter, we again want the price at time t to depend on a functional $\widetilde{Y}_t$ of the total order process

$$Y_t = X_t + \theta_t \mathbb{1}_{[0,T]}(t) = Y_t^c + \int_0^t \int_{\mathbb{R}^3} \phi^X(s, \zeta_2)\, N(\mathrm{d}s, \mathrm{d}\zeta), \quad t \geq 0.$$

Note that in the above representation of the total order process we have used the continuity of θ. In particular, the jump part of the noise traders' order process, $\int_0^t \int_{\mathbb{R}^3} \phi^X(s, \zeta_2)\, N(\mathrm{d}s, \mathrm{d}\zeta)$, $t \geq 0$, is adapted w.r.t. $\mathbb{F}^{\mathcal{M}}$.

As we argued above, we should allow a different weighting of continuous and discontinuous changes of the total order process. We therefore consider weighted total order processes $\widetilde{Y}$ that have the following general form

$$\widetilde{Y}_t = \int_0^t \lambda_{s-}\, \mathrm{d}Y_s^c + \int_0^t \int_{\mathbb{R}^3} \varphi_s(\zeta)\, N(\mathrm{d}s, \mathrm{d}\zeta), \quad t \geq 0, \tag{4.16}$$

for suitable $\mathbb{F}^{\mathcal{M}}$-adapted processes φ and λ where

$$\mathrm{d}\lambda_t = l_t\, \mathrm{d}t + \int_{\mathbb{R}^3} \phi_t^\lambda(\zeta)\, N(\mathrm{d}t, \mathrm{d}\zeta),$$

such that a.s.

$$\int_0^T \lambda_s^2\, \mathrm{d}s < \infty.$$

Recall that we pointed out that Λ, as defined in (4.10), also plays an important role in the dynamics of the equilibrium price pressure. Adopting the notation of this section for Λ, we write

$$\mathrm{d}\Lambda_t = -\lambda_t^2 \sigma^2 \mathrm{d}t + \int_{\mathbb{R}^3} \phi_t^\Lambda(\zeta)\, N(\mathrm{d}t, \mathrm{d}\zeta), \quad t \geq 0, \quad \Lambda_0 = \chi_0, \tag{4.17}$$

where $\phi_t^\Lambda(\zeta) = \chi(t, \phi^X(t, \zeta_2))$ with χ as in (4.14).

Before we address the precise form of a pricing rule, let us introduce the following notations:

$$\bar{U}_t := (\Lambda_t, \lambda_t, S_t), \quad \phi_t^{\bar{U}}(\zeta) = \phi^{\bar{U}}(t, \bar{U}_{t-}, \zeta) := (\phi_t^{\Lambda}(\zeta), \phi_t^{\lambda}(\zeta), \phi_t^{S}(\zeta)) \tag{4.18}$$

and

$$U_t := (\bar{U}_t, Z_t, \widetilde{Y}_t), \quad \phi_t^{U}(\zeta) = \phi^{U}(t, U_{t-}, \zeta) = (\phi_t^{\bar{U}}(\zeta), \phi_t^{Z}(\zeta), \varphi_t(\zeta)). \tag{4.19}$$

Furthermore, let $\bar{\mathcal{D}}$ denote the state space of the process $(t \wedge T, \bar{U}_{t\wedge T})_{t\geq 0}$. In particular, we have for all $\bar{u} \in \bar{\mathcal{D}} \cap (\mathbb{R}^3 \times (0,\infty))$ that $t < T$ given $\bar{U}_t = \bar{u}$.

In Chapter 3, a pricing rule was characterised by a pair (H, λ) of a pricing function H and a price pressure λ. Since now Y^c and Y^d are weighted differently according to λ and φ, respectively, we need a triple (H, λ, φ) to completely characterise a pricing rule

$$P_t = H(t \wedge T, \widetilde{Y}_{t\wedge T}, \bar{U}_{t\wedge T}), \quad t \geq 0.$$

Here H is assumed to be a continuously differentiable function w.r.t. all variables and twice continuously differentiable with respect to the second variable. Furthermore, H again is supposed to be strictly increasing in the second variable. For λ and φ, as defined above, we assume that

$$\varphi_t(\zeta) = \varphi(t, \bar{U}_{t-}, \phi^X(t, \zeta_2))$$

and

$$\phi_t^{\lambda}(\zeta) = \phi^{\lambda}(t, \bar{U}_{t-}, \phi^X(t, \zeta_2), \phi^S(t, S_{t-}, \zeta_2, \zeta_3)), \quad l_t = l(t, \bar{U}_{t-}),$$

for suitable functions ϕ^{λ} and l such that $\lambda > 0$ $\mathbb{P}$-a.s., and that there exists a unique strong solution to the SDE

$$\mathrm{d}\xi_t = \lambda_{t-}\sigma \mathrm{d}B_t + \int_{\mathbb{R}^3} \varphi(t, \bar{U}_{t-}, \phi^X(t, \zeta_2))\, N(\mathrm{d}t, \mathrm{d}\zeta), \quad t \geq s, \quad \xi_s = x, \bar{U}_s = \bar{u}, \tag{4.20}$$

for all $(x, \bar{u}) \in \mathbb{R} \times \bar{\mathcal{D}}$. In particular, we can define $U_t^{\xi} := (\bar{U}_t, Z_t, \xi_t)$ and $\mathcal{D}^*$ as the state space of $(t \wedge T, U^{\xi}_{t\wedge T})_{t\geq 0}$. Observe that, analogous to Chapter 3, ξ denotes the *weighted total uninformed order*, i.e. the total order without the semi-rational part of X and without insider trading.

Further details and technical conditions regarding the admissibility of pricing rules are given in the following definition.

Definition 4.2. We call (H, λ, φ) *admissible pricing rule* ($(H, \lambda, \varphi) \in \mathcal{P}$) if H, λ and φ are defined as above such that for all $(t, u) = (t, \bar{u}, z, y) \in \mathcal{D}^*$ the following conditions hold:

- $|y^*(t, u)| < \infty$ where $y^*(t, u)$ is the implicit function defined by

$$H(t, y^*(t, u), \bar{u}) = F(t, z, S), \tag{4.21}$$

- $\mathbb{E}\left(\int_0^T \xi_t^2 + (y^*(t,U_t^\xi))^2 + (H(t,\xi_t,\bar{U}_t))^2\,\mathrm{d}t\right) < \infty$,
- $\mathbb{E}\left((\xi_T^2 + (y^*(T,U_T^\xi))^2 + (H(T,\xi_T,\bar{U}_T))^2 + F_T^2)\lambda_T^{-2}\right) < \infty$,
- $\mathbb{E}\left(\int_0^T (\partial_y H(t,\xi_t,\bar{U}_t))^2\,\mathrm{d}t\right) < \infty$,
- $\mathbb{E}\left(\int_0^T \int_{\mathbb{R}^3} \left(H(t,\xi_{t-}+\varphi_t(\zeta),\bar{U}_{t-}+\phi_t^{\bar{U}}(\zeta)) - H(t,\xi_{t-},\bar{U}_{t-})\right)^2 \nu(\mathrm{d}\zeta)\mathrm{d}t\right) < \infty$,
- $\mathbb{E}\left(\int_0^T \int_{\mathbb{R}^3} \left(\frac{H(t,\xi_{t-}+\varphi_t(\zeta),\bar{U}_{t-}+\phi_t^{\bar{U}}(\zeta))}{\lambda_{t-}+\phi_t^\lambda(\zeta)} - \frac{H(t,\xi_{t-},\bar{U}_{t-})}{\lambda_{t-}}\right)^2 \nu(\mathrm{d}\zeta)\mathrm{d}t\right) < \infty$,
- $\mathbb{E}\left(\int_0^T \int_{\mathbb{R}^3} \left(\frac{F(t,Z_{t-}+\phi_t^Z(\zeta),S_{t-}+\phi_t^S(\zeta))}{\lambda_{t-}+\phi_t^\lambda(\zeta)} - \frac{F(t,Z_{t-},S_{t-})}{\lambda_{t-}}\right)^2 \nu(\mathrm{d}\zeta)\mathrm{d}t\right) < \infty$.

According to Definition 1.2, a pricing rule (H,λ,φ) is said to be *rational* if

$$H(t\wedge T,\widetilde{Y}_{t\wedge T},\bar{U}_{t\wedge T}) = \mathbb{E}\left(h(Z_T)\middle|\mathcal{F}^{\mathcal{M}}_{t\wedge T}\right), \quad t\geq 0.$$

As in the previous chapter, we want to consider a semi-rational noise drift μ. In particular, μ depends on the price and the estimated fundamental value, i.e. $F(t,Z_t,S_t)$. Due to the dependences of H, we consider

$$\mu_t = \mu(t,U_t), \quad t\geq 0.$$

Semi-rationality now reads as

$$\operatorname{sgn}(\mu(t,U_t))\,\mathbb{1}_{[0,T]}(t) = \operatorname{sgn}\left(F(t,Z_t,S_t) - H(t,\widetilde{Y}_t,\bar{U}_t)\right)\mathbb{1}_{[0,T]}(t), \quad t\geq 0.$$

By analogy to Chapter 3, we develop a class of *admissible noise drift* in the following section via an HJB equation approach.

4.3 Absolutely continuous insider trading: HJB equation and necessary conditions for equilibrium

In this section, we again identify necessary conditions for equilibrium as we did in Section 3.2. These conditions will help us to derive optimality criteria for general insider trading. We again start with the assumption of absolutely continuous insider trading, i.e.

$$\theta_t = \int_0^{t\wedge T} \alpha_s\,\mathrm{d}s, \quad t\geq 0, \tag{4.22}$$

for a suitable $\mathbb{F}^{\mathcal{I}}$-adapted process $\alpha \in \widetilde{\mathcal{S}}(t,u)$ that ensures the existence of a unique strong solution to U for all initial conditions $(t,u) \in \mathcal{D}^*$ and

$$\mathbb{E}^{t,u} \int_t^T \left| \Big(F(s, Z_s, S_s) - H(s, \widetilde{Y}_s, \bar{U}_s) \Big) \alpha_s \right| \mathrm{d}s < \infty$$

where $\mathbb{E}^{t,u}$ denotes the conditional expectation given $U_t = u$. For the value function, defined by

$$J(t,u) := \sup_{\alpha \in \widetilde{\mathcal{S}}(t,u)} \mathbb{E}^{t,u} \left(\int_t^T (F(s, Z_s, S_s) - H(s, \widetilde{Y}_s, \bar{U}_s)) \alpha_s \, \mathrm{d}s \right), \quad (t,u) \in \mathcal{D}^*, \tag{4.23}$$

we can now derive the corresponding HJB equation, assuming J is smooth enough. For further details and a more general treatment of HJB equations for optimal control of jump diffusions we refer to Øksendal and Sulem [53], Chapter 3.

Due to the special structure of the stopping time T, for any $(t,u) \in \mathcal{D} := \mathcal{D}^* \cap (\mathbb{R}^3 \times (0,\infty) \times \mathbb{R}^2)$ there exists an $\epsilon^u > 0$ such that $T(\omega) > t + \epsilon^u$, for all $\omega \in \{U_t = u\}$. Hence,

$$\mathbb{E}^{t,u} \int_t^{(t+\epsilon)\wedge T} (F_s - H_s) \alpha_s \, \mathrm{d}s = \mathbb{E}^{t,u} \int_t^{t+\epsilon} (F_s - H_s) \alpha_s \, \mathrm{d}s, \quad \text{for all } \epsilon \in [0, \epsilon^u],$$

and

$$\mathbb{E}^{t,u} \int_{(t+\epsilon)\wedge T}^{T} (F_s - H_s) \alpha_s \, \mathrm{d}s = \mathbb{E}^{t,u} \int_{t+\epsilon}^{T} (F_s - H_s) \alpha_s \, \mathrm{d}s, \quad \text{for all } \epsilon \in [0, \epsilon^u],$$

where we use H_t and F_t as a short notation instead of $H(t, \widetilde{Y}_t, \bar{U}_t)$ and $F(t, Z_t, S_t)$, respectively. It follows

$$J(t,u) = \sup_{\alpha \in \widetilde{\mathcal{S}}(t,u)} \mathbb{E}^{t,u} \left(\int_t^{t+\epsilon} (F_s - H_s) \alpha_s \, \mathrm{d}s + J(t+\epsilon, U_{t+\epsilon}) \right), \quad \text{for all } \epsilon \in [0, \epsilon^u]. \tag{4.24}$$

Now, by Itô's formula (cf. [56], Theorem 33, Chapter II) we get, given $U_t = u$, $u \in \mathcal{D}$,

$$\begin{aligned}
&J(t+\epsilon, U_{t+\epsilon}) \\
&= J(t,u) + \int_t^{t+\epsilon} \lambda_{s-} \sigma \partial_y J(s, U_{s-}) \, \mathrm{d}B_s + \int_t^{t+\epsilon} \int_{\mathbb{R}^3} J(s, U_{s-} + \phi_s^U(\zeta)) - J(s, U_{s-}) \, \bar{N}(\mathrm{d}s, \mathrm{d}\zeta) \\
&\quad + \int_t^{t+\epsilon} \partial_t J(s, U_s) + l(s, \bar{U}_s) \partial_\lambda J(s, U_s) - \lambda_s^2 \sigma^2 \partial_\Lambda J(s, U_s) - \partial_S J(s, U_s) + \frac{\sigma^2 \lambda_s^2}{2} \partial_{yy} J(s, U_s) \, \mathrm{d}s \\
&\quad + \int_t^{t+\epsilon} \int_{\mathbb{R}^3} J(s, U_s + \phi_s^U(\zeta)) - J(s, U_s) \, \nu(\mathrm{d}\zeta) \, \mathrm{d}s + \int_t^{t+\epsilon} \lambda_s (\alpha_s + \mu(s, U_s)) \partial_y J(s, U_s) \, \mathrm{d}s
\end{aligned}$$

where we took into account that J was assumed to be smooth enough and that

$$\mathrm{d}\widetilde{Y}_t = \lambda_t (\alpha_t + \mu_t) \mathrm{d}t + \sigma \lambda_t \mathrm{d}B_t + \int_{\mathbb{R}^3} \varphi_t(\zeta) \, N(\mathrm{d}t, \mathrm{d}\zeta).$$

In combination with (4.24) this yields

$$\begin{aligned}0 = \sup_{\alpha \in \widetilde{\mathcal{S}}(t,u)} \mathbb{E}\Bigg(&\int_t^{t+\epsilon} \lambda_{s-}\sigma\partial_y J_{s-}\,\mathrm{d}B_s + \int_t^{t+\epsilon}\int_{\mathbb{R}^3} J(s,U_{s-}+\phi_s^U(\zeta)) - J(s,U_{s-})\,\bar{N}(\mathrm{d}s,\mathrm{d}\zeta)\\
&+ \int_t^{t+\epsilon} \alpha_s\left(\lambda_s\partial_y J_s + (F_s - H_s)\right)\mathrm{d}s + \int_t^{t+\epsilon}\int_{\mathbb{R}^3} J(s,U_s+\phi_s^U(\zeta)) - J(s,U_s)\,\nu(\mathrm{d}\zeta)\,\mathrm{d}s\\
&+ \int_t^{t+\epsilon} \partial_t J_s + \lambda_s\mu_s\partial_y J_s + \frac{\lambda_s^2\sigma^2}{2}\partial_{yy}J_s - \lambda_s^2\sigma^2\partial_\Lambda J_s - \partial_S J_s + l_s\partial_\lambda J_s\,\mathrm{d}s\Bigg).\end{aligned}$$

If the first two terms on the right hand side are true martingales, they cancel out in expectation. Dividing by ϵ and sending ϵ to zero leads to

$$\begin{aligned}\sup_\alpha\Bigg(&\partial_t J(t,u) + \lambda\mu(t,u)\partial_y J(t,u) + \lambda^2\sigma^2\Big(\frac{\partial_{yy}J(t,u)}{2} - \partial_\Lambda J(t,u)\Big) + l(t,\bar{u})\partial_\lambda J(t,u) - \partial_S J(t,u)\\
&+ \int_{\mathbb{R}^3} J(t,u+\phi^U(t,u,\zeta)) - J(t,u)\,\nu(\mathrm{d}\zeta) + \alpha\lambda\partial_y J(t,u) + (F(t,z,S) - H(t,y,\bar{u}))\alpha\Bigg) = 0\end{aligned}$$

for all $(t,u) = (t,\bar{u},z,y) = (t,\Lambda,\lambda,S,z,y) \in \mathcal{D}$. Due to the linearity of this equation, a finite solution can only be found if the following system of PIDEs is satisfied

$$0 = \frac{H(t,y,\bar{u}) - F(t,z,S)}{\lambda} - \partial_y J(t,u), \tag{4.25}$$

$$\begin{aligned}0 = {}&\partial_t J(t,u) + \mu(t,u)(H(t,y,\bar{u}) - F(t,z,s)) + \frac{\lambda^2\sigma^2}{2}\partial_{yy}J(t,u) - \lambda^2\sigma^2\partial_\Lambda J(t,u)\\
&+ l(t,\bar{u})\partial_\lambda J(t,u) - \partial_S J(t,u) + \int_{\mathbb{R}^3} J(t,u+\phi^U(t,u,\zeta)) - J(t,u)\,\nu(\mathrm{d}\zeta),\end{aligned} \tag{4.26}$$

for all $(t,u) = (t,\bar{u},z,y) = (t,\Lambda,\lambda,S,z,y,) \in \mathcal{D}$.

Like in Chapter 3, the HJB equation approach leads to a system of two equations that have to be satisfied by J for the existence of a finite equilibrium. The special dependences of these equations can be used to derive necessary conditions for the pricing rule.

We start with differentiating (4.25) w.r.t. y, Λ, λ, S and t, respectively:

$$\begin{aligned}&\partial_{yy}J(t,u) = \partial_y H(t,y,\bar{u})\lambda^{-1}, \quad \partial_{yyy}J(t,u) = \partial_{yy}H(t,y,\bar{u})\lambda^{-1},\\
&\partial_{y\Lambda}J(t,u) = \partial_\Lambda H(t,y,\bar{u})\lambda^{-1}, \quad \partial_{yt}J(t,u) = \left(\partial_t H(t,y,\bar{u}) - \partial_t F(t,z,S)\right)\lambda^{-1},\\
&\partial_{y\lambda}J(t,u) = \left(F(t,z,S) - H(t,y,\bar{u})\right)\lambda^{-2} + \partial_\lambda H(t,y,\bar{u})\lambda^{-1},\\
&\partial_{yS}J(t,u) = \left(\partial_S H(t,y,\bar{u}) - \partial_S F(t,z,S)\right)\lambda^{-1}.\end{aligned} \tag{4.27}$$

Differentiating (4.26) w.r.t. y yields

$$\begin{aligned}0 = {}&\partial_{ty}J(t,u) + \partial_y(\mu(t,u)(H(t,y,\bar{u}) - F(t,z,s))) + \frac{\lambda^2\sigma^2}{2}\partial_{yyy}J(t,u) - \lambda^2\sigma^2\partial_{\Lambda y}J(t,u)\\
&+ l(t,\bar{u})\partial_{\lambda y}J(t,u) - \partial_{Sy}J(t,u) + \int_{\mathbb{R}^3} \partial_y J(t,u+\phi^U(t,u,\zeta)) - \partial_y J(t,u)\,\nu(\mathrm{d}\zeta).\end{aligned}$$

Now, inserting the partial derivatives calculated in (4.27) leads to

$$\begin{aligned}0 &= \left(\partial_t H(t,y,\bar u) - \partial_t F(t,z,S)\right)\lambda^{-1} + \partial_y(\mu(t,u)(H(t,y,\bar u) - F(t,z,S))) + \frac{\lambda\sigma^2}{2}\partial_{yy}H(t,y,\bar u)\\ &\quad - \lambda\sigma^2\partial_\Lambda H(t,y,\bar u) + l(t,\bar u)\lambda^{-1}\partial_\lambda H(t,y,\bar u) - l(t,\bar u)\lambda^{-2}(H(t,y,\bar u) - F(t,z,S))\\ &\quad - \left(\partial_S H(t,y,\bar u) - \partial_S F(t,z,S)\right)\lambda^{-1} + \int_{\mathbb{R}^3} \frac{H(t,y+\varphi_t(\zeta),\bar u + \phi_t^{\bar U}(\zeta))}{\lambda + \phi_t^\lambda(\zeta)} - \frac{H(t,y,\bar u)}{\lambda}\,\nu(\mathrm{d}\zeta)\\ &\quad - \int_{\mathbb{R}^3} \frac{F(t,z+\phi_t^Z(\zeta),S+\phi_t^S(\zeta))}{\lambda+\phi_t^\lambda(\zeta)} - \frac{F(t,z,S)}{\lambda}\,\nu(\mathrm{d}\zeta). \end{aligned} \tag{4.28}$$

Despite the integral terms that occur due to the jumps of the involved variables and those which reflect the non-constant behaviour of F, i.e. $\partial_t F$ and $\partial_S F$, we have a similar structure as in Chapter 3 (cf. (3.18) and (3.28)). If we assume that for $(t,u) = (t,\bar u,z,y) = (t,\Lambda,\lambda,S,z,y) \in \mathcal{D}$

$$\partial_y\left(\mu(t,u)(H(t,y,\bar u) - F(t,z,S))\right) = \frac{l^{(1)}(t,\bar u)}{\lambda^2}\left(H(t,y,\bar u) - F(t,z,S)\right), \tag{4.29}$$

for a suitable function $l^{(1)}$, we get by an analogous argumentation as in the proof of Proposition 3.6 that a semi-rational μ (with analogous properties as in Assumption 3.4) has the following representation

$$\mu(t,u) = m(t,\bar u)\frac{\int_y^{y^*(t,u)} H(t,x,\bar u) - F(t,z,S)\,\mathrm{d}x}{H(t,y,\bar u) - F(t,z,S)} \tag{4.30}$$

where

$$m(t,\bar u) = -\frac{l^{(1)}(t,\bar u)}{\lambda^2} > 0$$

and $y^*(t,u)$ is as defined in (4.21). The above considerations motivate the following assumption:

Assumption 4.3. *μ takes the special form as in (4.30) where $m(t,\bar u)$ is some positive function bounded by $M \in (0,\infty)$.*

If this assumption is satisfied, Equation (4.28) allows the following conclusion:

Proposition 4.4. *Let Assumption 4.3 be satisfied and (H,λ,φ) be an admissible pricing rule. For $(t,\bar u) \in \bar{\mathcal{D}}$ define*

$$l^{(2)}(t,\bar u) := l(t,\bar u) + m(t,\bar u)\lambda^2.$$

If there exists a sufficiently smooth function J such that (H,λ,φ,J) is a solution to the system of Equations (4.25) and (4.26), then necessarily the following two PIDEs hold for all $(t,u) =$

$(t, \bar{u}, z, y) = (t, \Lambda, \lambda, S, z, y) \in \mathcal{D}$

$$\begin{aligned} 0 = \frac{\partial_t H(t,y,\bar{u})}{\lambda} + \frac{\lambda\sigma^2}{2}\partial_{yy}H(t,y,\bar{u}) - \frac{l^{(2)}(t,\bar{u})}{\lambda^2}H(t,y,\bar{u}) - \lambda\sigma^2\partial_\Lambda H(t,y,\bar{u}) - \frac{\partial_S H(t,y,\bar{u})}{\lambda} \\ + \frac{l(t,\bar{u})}{\lambda}\partial_\lambda H(t,y,\bar{u}) + \int_{\mathbb{R}^3} \frac{H(t, y+\varphi(t,\bar{u},\phi^X(t,\zeta_2)), \bar{u}+\phi^{\bar{U}}(t,\bar{u},\zeta))}{\lambda + \phi^\lambda(t,\bar{u},\phi^X(t,\zeta_2),\phi^S(t,S,\zeta_2,\zeta_3))} - \frac{H(t,y,\bar{u})}{\lambda}\,\nu(\mathrm{d}\zeta) \end{aligned} \tag{4.31}$$

and

$$\begin{aligned} 0 = -\frac{l^2(t,\bar{u})}{\lambda^2}F(t,z,S) + \frac{\partial_t F(t,z,S) - \partial_S F(t,z,S)}{\lambda} \\ + \int_{\mathbb{R}^3} \frac{F(t, z+\phi^Z(t,\zeta_1,\zeta_2), S+\phi^S(t,S,\zeta_2,\zeta_3))}{\lambda + \phi^\lambda(t,\bar{u},\phi^X(t,\zeta_2),\phi^S(t,S,\zeta_2,\zeta_3))} - \frac{F(t,z,S)}{\lambda}\,\nu(\mathrm{d}\zeta). \end{aligned} \tag{4.32}$$

Proof. Recall that H has to be independent on z. Inserting (4.29) in Equation (4.28) and separating the terms that depend on z and those which do not yields (4.31) and (4.32). ☐

4.4 Price pressure

If Assumption 4.3 is satisfied and $(H, \lambda, \varphi) \in \mathcal{P}$, Proposition 4.4 states that λ has to satisfy (4.32) and

$$\mathrm{d}\lambda_t = -m(t,\bar{U}_t)\lambda_t^2 + l^{(2)}(t,\bar{U}_t)\mathrm{d}t + \int_{\mathbb{R}^3} \phi_t^\lambda(\zeta)\, N(\mathrm{d}t, \mathrm{d}\zeta). \tag{4.33}$$

Since λ is assumed to be strictly positive, we can alternatively consider λ^{-1}. Due to Itô's formula λ^{-1} has the following dynamics:

$$\mathrm{d}\lambda_t^{-1} = m(t,\bar{U}_t) - l^{(2)}(t,\bar{U}_t)\lambda_t^{-2}\mathrm{d}t + \int_{\mathbb{R}^3} \phi^{\lambda^{-1}}(t,\bar{U}_{t-},\zeta_2,\zeta_3)\, N(\mathrm{d}t,\mathrm{d}\zeta) \tag{4.34}$$

where for $\bar{u} = (\Lambda, \lambda, S) \in \bar{\mathcal{D}}$

$$\phi^{\lambda^{-1}}(t,\bar{u},\zeta_2,\zeta_3) = -\frac{\phi^\lambda(t,\bar{u},\phi^X(t,\zeta_2),\phi^S(t,S,\zeta_2,\zeta_3))}{(\lambda + \phi^\lambda(t,\bar{u},\phi^X(t,\zeta_2),\phi^S(t,S,\zeta_2,\zeta_3)))\lambda}.$$

Using the martingale property of $(F_{t\wedge T})_{t\geq 0}$, PIDE (4.32) can be simplified as shown subsequently.

Proposition 4.5. *Let Assumption 4.1 be satisfied and λ^{-1} be as in (4.34). Then (4.32) is equivalent to*

$$\begin{aligned} 0 = -\frac{l^{(2)}(t,\bar{u})}{\lambda^2}F(t,z,S) \\ + \int_{\mathbb{R}^3} F(t, z+\phi^Z(t,\zeta_1,\zeta_2), S_t+\phi^S(t,S,\zeta_2,\zeta_3))\phi^{\lambda^{-1}}(t,\bar{u},\zeta_2,\zeta_3)\,\nu(\mathrm{d}\zeta), \quad (t,u) \in \mathcal{D}. \end{aligned} \tag{4.35}$$

Proof. Since $F(t \wedge T, Z_{t\wedge T}, S_{t\wedge T}), t \geq 0$, and

$$\int_0^{t\wedge T} \int_{\mathbb{R}^3} F(s, Z_{s-} + \phi_s^Z(\zeta), S_{s-} + \phi_s^S(\zeta)) - F(s, Z_{s-}, S_{s-})\, \bar{N}(\mathrm{d}s, \mathrm{d}\zeta), \quad t \geq 0,$$

are martingales (Assumption 4.1), an argumentation as in the previous section shows that

$$\begin{aligned} 0 &= \partial_t F(t,z,S) - \partial_S F(t,z,S) \\ &\quad + \int_{\mathbb{R}^3} F(t, z + \phi^Z(t,\zeta_1,\zeta_2), S + \phi^S(t,S,\zeta_2,\zeta_3)) - F(t,z,S)\, \nu(\mathrm{d}\zeta), \end{aligned} \tag{4.36}$$

for all $(t,u) = (t,\Lambda,\lambda,S,z,y) \in \mathcal{D}$. Taking into account that

$$\begin{aligned} &\int_{\mathbb{R}^3} \frac{F(t, z + \phi^Z(t,\zeta_1,\zeta_2), S + \phi^S(t,S,\zeta_2,\zeta_3))}{\lambda + \phi^\lambda(t,\bar{u},\phi^X(t,\zeta_2),\phi^S(t,S,\zeta_2,\zeta_3))} - \frac{F(t,z,S)}{\lambda}\, \nu(\mathrm{d}\zeta) \\ &= \lambda^{-1} \int_{\mathbb{R}^3} F(t, z + \phi^Z(t,\zeta_1,\zeta_2), S + \phi^S(t,S,\zeta_2,\zeta_3)) - F(t,z,S)\, \nu(\mathrm{d}\zeta) \\ &\quad + \int_{\mathbb{R}^3} F(t, z + \phi^Z(t,\zeta_1,\zeta_2), S + \phi^S(t,S,\zeta_2,\zeta_3)) \phi^{\lambda^{-1}}(t,\bar{u},\zeta_2,\zeta_3)\, \nu(\mathrm{d}\zeta), \end{aligned} \tag{4.37}$$

the equivalence of (4.32) and (4.35) follows from (4.36). □

Remark 4.6. Obviously, (4.35) is always satisfied if we choose $l^{(2)} = \phi^{\lambda^{-1}} = 0$. Together with the other drift part, $-m(t,\bar{U}_t)\lambda_t^{-2}$, λ would behave as in Chapter 3. Until now it is not clear why we should choose λ in a different way. But as we argued in the introduction of this chapter, the case where $\Lambda_T = 0$ will play an important role later on. However, it is not obvious whether non-trivial $l^{(2)}$ and $\phi^{\lambda^{-1}}$ exist such that

$$1 = \int_{\mathbb{R}^3} \frac{F(t, z + \phi^Z(t,\zeta_1,\zeta_2), S + \phi^S(t,S,\zeta_2,\zeta_3))}{F(t,z,S)} \frac{\phi^{\lambda^{-1}}(t,\bar{u},\zeta_2,\zeta_3)\lambda^2}{l^{(2)}(t,\bar{u})}\, \nu(\mathrm{d}\zeta)$$

since $l^{(2)}$ and $\phi^{\lambda^{-1}}$ are not allowed to depend on z or ζ_1. If for some $c \in \mathbb{R}$

$$c = \int_{\mathbb{R}^3} \frac{F(t, z + \phi^Z(t,\zeta_1,\zeta_2), S + \phi^S(t,S,\zeta_2,\zeta_3))}{F(t,z,S)}\, \nu(\mathrm{d}\zeta),$$

one might choose

$$\phi^{\lambda^{-1}}(t,\bar{u},\zeta_2,\zeta_3)\lambda^2 \nu(\mathbb{R}^3) = \frac{l^{(2)}(t,\bar{u})}{c}.$$

4.5 Optimality

For the HJB equation approach in Section 4.3 we assumed that an admissible θ is absolutely continuous. It turned out that the corresponding value function J has to satisfy Equations (4.25) and (4.26). Such a function will be specified in the following lemma.

Lemma 4.7. *Let Assumptions 4.1 and 4.3 be satisfied and $(H, \lambda, \varphi) \in \mathcal{P}$ such that (4.31), (4.33) and (4.35) hold. For $(t, u) \in \mathcal{D}^*$, the function*

$$
\begin{aligned}
J(t,u) := \mathbb{E}\Bigg(\int_t^T & \mu(s, U_s^\xi)(H(s, \xi_s, \bar{U}_s) - F(s, Z_s, S_s))\,\mathrm{d}s \\
& + \lambda_T^{-1} \int_{y^*(T, U_T^\xi)}^{\xi_T} (H(T, y, \bar{U}_T) - F(T, Z_T, S_T))\,\mathrm{d}y \Bigg| U_t^\xi = u \Bigg)
\end{aligned}
\tag{4.38}
$$

is well-defined. In particular, $J(T, U_T) \geq 0$, with equality if and only if $\widetilde{Y}_T = y^*(T, U_T)$. If furthermore J is smooth enough and

$$
\mathbb{E} \int_0^T \int_{\mathbb{R}^3} (J(t, U_{t-}^\xi + \phi^U(t, U_{t-}^\xi, \zeta)) - J(t, U_{t-}^\xi))^2\, \nu(\mathrm{d}\zeta)\mathrm{d}t < \infty, \tag{4.39}
$$

then (H, λ, φ, J) is a solution to the system of Equations (4.25) and (4.26).

Proof. In this proof, denote $H(t, \xi_t, \bar{U}_t)$ by H_t, $F(t, Z_t, S_t)$ by F_t, $\mu(t, U_t^\xi)$ by μ_t, etc. As a first step, we show that J is well-defined. Again we have that $\mu_t(H_t - F_t) < 0$, $0 \leq m_t \leq M$ and the monotonicity of H. An argumentation similar to that in Lemma 3.12 yields

$$
\begin{aligned}
\mathbb{E}\left(\left| \int_0^T \mu_s(H_s - F_s)\,\mathrm{d}s \right| \right) &\leq \mathbb{E}\left(M \int_0^T (\xi_s - y_s^*)(H_s - F_s)\,\mathrm{d}s \right) \\
&\leq \left(M\mathbb{E} \int_0^T (\xi_s)^2 + (y_s^*)^2 + H_s^2 + F_s^2\,\mathrm{d}s \right).
\end{aligned}
$$

Furthermore, we get with the same arguments

$$
\mathbb{E}\left(\int_{y_T^*}^{\xi_T} (H(T, x, \bar{U}_T) - F_T)\lambda_T^{-1}\,\mathrm{d}x \right) \leq \mathbb{E}\left(\frac{\xi_T^2 + (y_T^*)^2 + H_T^2 + F_T^2}{\lambda_T^2} \right).
$$

According to Assumption 4.1 and Definition 4.2 all terms in the above inequalities are bounded. Altogether J is well-defined.

As a second step, we have to verify (4.25) and (4.26). We start with (4.26). First observe that

$$
J(t \wedge T, U_{t\wedge T}^\xi) + \int_0^{t \wedge T} \mu(s, U_s^\xi)(H(s, \xi_s, \bar{U}_s) - F(s, Z_s, S_s))\,\mathrm{d}s, \quad t \geq 0,
$$

is a martingale by definition. Analogous to Section 4.3, we get for any $(t, u) \in \mathcal{D}$

$$
\begin{aligned}
J_{t+\epsilon} - J_t = & \int_t^{t+\epsilon} \partial_t J_s + l_s \partial_\lambda J_s - \lambda_s^2 \sigma^2 \partial_\Lambda J_s - \partial_S J_s + \frac{\lambda_s^2 \sigma^2}{2} \partial_{yy} J_s\,\mathrm{d}s \\
& + \int_t^{t+\epsilon} \int_{\mathbb{R}^3} J(s, U_{s-}^\xi + \phi^U(s, U_{s-}^\xi, \zeta)) - J(s, U_{s-})\, \nu(\mathrm{d}\zeta)\mathrm{d}s \\
& + \int_t^{t+\epsilon} \lambda_{s-} \sigma \partial_y J_{s-}\,\mathrm{d}B_s + \int_t^{t+\epsilon} \int_{\mathbb{R}^3} J(s, U_{s-}^\xi + \phi^U(s, U_{s-}^\xi, \zeta)) - J(s, U_{s-})\, \bar{N}(\mathrm{d}s, \mathrm{d}\zeta)
\end{aligned}
$$

for all $\epsilon \in (0, S)$. Due to condition (4.39) the integral w.r.t. $\bar{N}(\mathrm{d}t, \mathrm{d}\zeta)$ is a martingale (cf. [16], Corollary 4, Chapter VIII). Furthermore, if (4.25) holds, the $\mathrm{d}B$ term is a martingale, too. Then, with the same arguments used in Section 4.3 for the calculation of the HJB equation, we get that (4.26) holds. It remains to prove that J satisfies (4.25).

An argumentation similar to the one used in Lemma 3.12 yields

$$\partial_y J(t,u) = \mathbb{E}\left(\int_t^T -m_s(H_s - F_s)\,\mathrm{d}s + (H_T - F_T)\lambda_T \middle| U_t^\xi = u \right). \tag{4.40}$$

Furthermore, we get by Itô's formula in combination with PIDEs (4.31) and (4.35) and Proposition 4.5

$$\begin{aligned} \frac{H_T - F_T}{\lambda_T} &= \frac{H_{t\wedge T} - F_{t\wedge T}}{\lambda_{t\wedge T}} + \int_{t\wedge T}^T m_s(H_s - F_s)\,\mathrm{d}s \\ &\quad + \int_{t\wedge T}^T \sigma \partial_y H_{s-}\,\mathrm{d}B_s + \int_{t\wedge T}^T \int_{\mathbb{R}^3} \frac{H_s - F_s}{\lambda_s} - \frac{H_{s-} - F_{s-}}{\lambda_{s-}}\,\bar{N}(\mathrm{d}s, \mathrm{d}\zeta). \end{aligned} \tag{4.41}$$

Admissibility of (H, λ, φ) ensures that the last two integrals are uniformly integrable martingales (Itô-Lévy-Isometry, cf. [53], Theorem 1.17). In particular,

$$\mathbb{E}\left(\int_t^T \sigma \partial_y H_{s-}\,\mathrm{d}B_s + \int_t^T \int_{\mathbb{R}^3} \frac{H_s - F_s}{\lambda_s} - \frac{H_{s-} - F_{s-}}{\lambda_{s-}}\,\bar{N}(\mathrm{d}s, \mathrm{d}\zeta) \middle| U_t^\xi = u \right) = 0.$$

Finally, (4.41) in combination with (4.40) yields

$$\partial_y J(t,u) = \frac{H(t, y, \bar{u}) - F(t, z, S)}{\lambda}.$$

□

In contrast to the diffusion case of Chapter 3 we cannot conclude smoothness of J from its definition in (4.38). In the sequel, we therefore assume that this condition is satisfied.

Assumption 4.8. *There exists a sufficiently smooth function J, such that (4.38) and (4.39) hold. Sufficiently smooth here means that J is continuously differentiable w.r.t. all variables and twice continuously differentiable w.r.t. to the last variable.*

According to the HJB equation approach we would now be able to deduce optimality criteria from Lemma 4.7 for admissible insider strategies that are continuous and of finite variation. However, we want to characterise optimality in a larger class which is not restricted to finite variation strategies. As pointed out in the introduction of this chapter, discontinuous strategies might allow infinite wealth. Remark 4.11 illuminates this subsequent to the next proposition. We therefore exclude discontinuous strategies. Further details and technical conditions for admissibility of trading strategies are presented in the following definition.

Definition 4.9. A continuous $\mathbb{F}^{\mathcal{I}}$-semimartingale θ is called *admissible* trading strategy (w.r.t. (H, λ, φ)) if the following conditions hold:

- $\mathbb{E}\left(\int_0^T (\lambda_{t-}\sigma\partial_y J(t,U_{t-}))^2\,\mathrm{d}t\right) < \infty$,
- $\mathbb{E}\left(\int_0^T \int_{\mathbb{R}^3} (J(t,U_{t-}+\phi_t^U(\zeta)) - J(t,U_{t-}))^2\,\nu(\mathrm{d}\zeta)\mathrm{d}t\right) < \infty$,
- $\mathbb{E}\left(\int_0^T \int_{\mathbb{R}} (\theta_t(F(t,Z_{t-}+\phi_t^Z(\zeta),S_{t-}+\phi_t^S(\zeta)) - F(t,Z_{t-},S_{t-})))^2\,\nu(\mathrm{d}\zeta)\mathrm{d}t\right) < \infty$,
- $\mathbb{E}\left(\int_0^T \lambda_t^2\partial_y H(t,\widetilde{Y}_t,\bar{U}_t)^2\,\mathrm{d}t\right) < \infty$,
- $\mathbb{E}\left(\int_0^T \int_{\mathbb{R}^3} (H(t,\widetilde{Y}_{t-}+\varphi_t(\zeta),\bar{U}_{t-}+\phi_t^{\bar{U}}) - H(t,\widetilde{Y}_{t-},\bar{U}_{t-}))^2\,\nu(\mathrm{d}\zeta)\mathrm{d}t\right) < \infty$.

The set of all admissible strategies is denoted by $\mathcal{S}(H,\lambda,\varphi)$.

We are now in the position to characterise optimality.

Proposition 4.10. *Let Assumptions 4.1, 4.3, and 4.8 be satisfied and $(H,\lambda,\varphi)\in\mathcal{P}$ such that (4.31), (4.33) and (4.35) hold. Then, $\theta\in\mathcal{S}(H,\lambda,\varphi)$ is optimal if*

- *θ is of finite variation*
- *$H(T,\widetilde{Y}_T,\bar{U}_T) = h(Z_T)$ $\mathbb{P}$-a.s.*

Proof. To shorten the notation, we write H_t instead of $H(t,\widetilde{Y}_t,\bar{U}_t)$, F_t instead of $F(t,Z_t,S_t)$, etc., as long as this does not cause confusion.

As stated in (1.2), the final wealth corresponding to a certain strategy θ is given by

$$W_T^\theta = \theta_{T-}V_T - \int_0^{T-} P_{t-}\,\mathrm{d}\theta_t - [P,\theta]_{T-}\,.$$

Continuity of θ leads to

$$W_T^\theta = \theta_T V_T - \int_0^T P_{t-}\,\mathrm{d}\theta_t - \langle P,\theta\rangle_T.$$

Since $F(T,Z_T,S_T) = V_T$ by definition, integration by parts yields the following representation of the final wealth:

$$W_T^\theta = \int_0^T F_{s-} - H_{s-}\,\mathrm{d}\theta_s + \int_0^T \theta_{s-}\,\mathrm{d}F_s + \langle\theta, F-H\rangle_T. \tag{4.42}$$

Now, due to Itô's formula

$$H_{t\wedge T} = H_0 + \int_0^{t\wedge T} \lambda_{s-}\sigma\partial_y H_{s-}\,\mathrm{d}B_s + \int_0^{t\wedge T}\lambda_s\partial_y H_{s-}\,\mathrm{d}\theta_s + \sum_{s\le t\wedge T}\Delta H_s + \text{drift term},$$

where the so-called *drift term* is continuous and of finite variation. Hence,

$$\langle\theta,H\rangle_T = \int_0^T \lambda_s\partial_y H_s\,\mathrm{d}\langle\theta\rangle_s + \int_0^T \lambda_s\sigma\partial_y H_s\,\mathrm{d}\langle\theta,B\rangle_s,\quad \langle\theta,F\rangle_T = 0.$$

Putting this into (4.42) leads to

$$W_T^\theta = \int_0^T F_{s-} - H_{s-} \,\mathrm{d}\theta_s + \int_0^T \theta_{s-} \,\mathrm{d}F_s - \int_0^T \lambda_s \partial_y H_s \,\mathrm{d}\langle\theta\rangle_s - \int_0^T \lambda_s \sigma \partial_y H_s \,\mathrm{d}\langle\theta, B\rangle_s.$$

Consider now $J_s = J(s, U_s)$ with J as defined in (4.38). Due to Lemma 4.7 we have

$$\partial_y J_s = \frac{H_s - F_s}{\lambda_s}.$$

Inserting this in the above representation of the final wealth yields

$$W_T^\theta = -\int_0^T \lambda_{s-} \partial_y J_{s-} \,\mathrm{d}\theta_s + \int_0^T \theta_{s-} \,\mathrm{d}F_s - \int_0^T \lambda_s^2 \partial_{yy} J_s \,\mathrm{d}\langle\theta\rangle_s - \int_0^T \lambda_s^2 \sigma \partial_{yy} J_s \,\mathrm{d}\langle\theta, B\rangle_s. \quad (4.43)$$

On the other hand, we have by Itô's formula and $\langle Y^c\rangle_t = \sigma^2 t + 2\sigma\langle B, \theta\rangle_t + \langle\theta\rangle_t$ that

$$\begin{aligned}
J_T = J_0 &+ \int_0^T \partial_t J_s + l_s \partial_\lambda J_s - \lambda_s^2 \sigma^2 \partial_\Lambda J_s - \partial_S J_s \,\mathrm{d}s \\
&+ \int_0^T \lambda_{s-} \partial_y J_{s-} \,\mathrm{d}Y_s^c + \frac{1}{2} \int_0^T \lambda_s^2 \partial_{yy} J_s \,\mathrm{d}\langle Y^c\rangle_s + \sum_{s \le T} \Delta J_s \\
= J_0 &+ \int_0^T \partial_t J_s + l_s \partial_\lambda J_s - \lambda_s^2 \sigma^2 \partial_\Lambda J_s - \partial_S J_s + \frac{\sigma^2 \lambda_s^2}{2} \partial_{yy} J_s + \lambda_s \mu_s \partial_y J_s \,\mathrm{d}s \\
&+ \int_0^T \lambda_{s-} \sigma \partial_y J_{s-} \,\mathrm{d}B_s + \int_0^T \lambda_{s-} \partial_y J_{s-} \,\mathrm{d}\theta_s + \int_0^T \lambda_s^2 \sigma \partial_{yy} J_s \,\mathrm{d}\langle B, \theta\rangle_s \\
&+ \frac{1}{2} \int_0^T \lambda_s^2 \partial_{yy} J_s \,\mathrm{d}\langle\theta\rangle_s + \sum_{s \le T} \Delta J_s.
\end{aligned}$$

Using PIDE (4.26) (cf. Lemma 4.7) yields

$$\begin{aligned}
J_T = &J_0 + \int_0^T \lambda_{s-} \sigma \partial_y J_{s-} \,\mathrm{d}B_s + \int_0^T \lambda_{s-} \partial_y J_{s-} \,\mathrm{d}\theta_s + \int_0^T \lambda_s^2 \sigma \partial_{yy} J_s \,\mathrm{d}\langle B, \theta\rangle_s \\
&+ \frac{1}{2} \int_0^T \lambda_s^2 \partial_{yy} J_s \,\mathrm{d}\langle\theta\rangle_s + \sum_{s \le T} \Delta J_s \\
&- \int_0^T \int_{\mathbb{R}^3} J(s, U_{s-} + \phi^U(s, U_{s-}, \zeta)) - J(s, U_{s-}) \,\nu(\mathrm{d}\zeta)\mathrm{d}s.
\end{aligned} \quad (4.44)$$

Merging (4.43) and (4.44) leads to the following representation of the final wealth:

$$\begin{aligned}
W_T^\theta = J_0 - J_T &+ \int_0^T \lambda_{s-} \sigma \partial_y J_{s-} \,\mathrm{d}B_s + \int_0^T \theta_{s-} \,\mathrm{d}F_s - \int_0^T \frac{1}{2} \lambda_s^2 \partial_{yy} J_s \,\mathrm{d}\langle\theta\rangle_s \\
&+ \int_0^T \int_{\mathbb{R}^3} J(s, U_{s-} + \phi_s^U(\zeta)) - J(s, U_{s-}) \,\bar{N}(\mathrm{d}s, \mathrm{d}\zeta).
\end{aligned}$$

Since θ is admissible we have

$$\mathbb{E}\left(\int_0^T \int_{\mathbb{R}^3} J(s, U_{s-} + \phi_s^U(\zeta)) - J(s, U_{s-})\, \bar{N}(\mathrm{d}s, \mathrm{d}\zeta)\right) = 0$$

as well as

$$\mathbb{E}\left(\int_0^T \lambda_{s-}\sigma\partial_y J_{s-}\,\mathrm{d}B_s\right) = 0, \quad \text{and} \quad \mathbb{E}\left(\int_0^T \theta_{s-}\,\mathrm{d}F_s\right) = 0.$$

Hence,

$$\mathbb{E}(W_T^\theta) = \mathbb{E}\left(J_0 - J_T - \int_0^T \frac{1}{2}\lambda_s^2\partial_{yy}J_s\,\mathrm{d}\langle\theta\rangle_s\right). \tag{4.45}$$

With the same arguments as in Proposition 3.13 we obtain that

$$-\int_0^T \frac{1}{2}\lambda_s^2\partial_{yy}J_s\,\mathrm{d}\langle\theta\rangle_s \le 0,$$

with equality if and only if $\langle\theta\rangle_\cdot = 0$ and that

$$\mathbb{E}J_0 - J_T \le \mathbb{E}J_0 - \mathbb{E}J_T \le \mathbb{E}J_0 < \infty,$$

with equality if and only if $\widetilde{Y}_T = y^*(T, U_T)$ $\mathbb{P}$-a.s. Thus, the conditions of this proposition indeed are sufficient for optimality in $\mathcal{S}(H, \lambda, \varphi)$. □

Remark 4.11. As in Proposition 3.13, we could have considered more general, i.e. discontinuous trading strategies. Denote $\varphi(s, \bar{U}_{s-}, \Delta Y_s)$ by $\varphi(\Delta Y_s)$ and $\varphi(s, \bar{U}_{s-}, \Delta X_s)$ by $\varphi(\Delta X_s)$. A similar argumentation as in the proof of Proposition 3.13 would have led to

$$\begin{aligned}\mathbb{E}W_T^\theta = \mathbb{E}\Bigg(& J_0 - J_{T-} - \int_0^T \frac{1}{2}\lambda_s^2\partial_{yy}J_s\,\mathrm{d}\langle\theta^c\rangle_s \\ &+ \sum_{s<T} J(s, \bar{U}_s, Z_s, \widetilde{Y}_{s-} + \varphi(\Delta Y_s)) - J(s, \bar{U}_s, Z_s, \widetilde{Y}_{s-} + \varphi(\Delta X_s)) - \lambda_s\partial_y J_s \Delta\theta_s\Bigg)\end{aligned}$$

in place of Equation (4.45) (cf. Equation (3.49)).

For optimality in the larger class of trading strategies we then would have to show that additionally

$$\sum_{s<T} J(s, \bar{U}_s, Z_s, \widetilde{Y}_{s-} + \varphi(\Delta Y_s)) - J(s, \bar{U}_s, Z_s, \widetilde{Y}_{s-} + \varphi(\Delta X_s)) - \lambda_s\partial_y J_s \Delta\theta_s \le 0$$

with equality if and only if $\Delta\theta = 0$. Obviously, this does not hold true in general, but if

$\varphi(y) = \lambda_t y$, we get by the convexity of J in y that

$$\begin{aligned}
&J(s, \bar{U}_s, Z_s, \widetilde{Y}_{s-} + \varphi(\Delta Y_s)) - J(s, \bar{U}_s, Z_s, \widetilde{Y}_{s-} + \varphi(\Delta X_s)) - \lambda_s \partial_y J(s, \bar{U}_s, Z_s, \widetilde{Y}_{s-} + \varphi(\Delta Y_s))\Delta\theta_s \\
&\quad = J(s, \bar{U}_s, Z_s, \widetilde{Y}_{s-} + \lambda_s \Delta Y_s) - J(s, \bar{U}_s, Z_s, \widetilde{Y}_{s-} + \lambda_s \Delta(Y_s - \theta_s)) \\
&\qquad - \partial_y J(s, \bar{U}_s, Z_s, \widetilde{Y}_{s-} + \lambda_s \Delta Y_s))\lambda_s \Delta\theta_s \\
&\quad \le 0.
\end{aligned}$$

4.6 Rationality

In Proposition 4.4, we found out that PIDE (4.31) is a necessary condition for the existence of an equilibrium in the case of absolutely continuous insider trading. Furthermore, Proposition 4.10 characterises optimality of an insider strategy for the case when (4.31) is satisfied. We now again turn our focus to rational pricing functions and state equivalent conditions to this PIDE. The following proposition is the jump analogue to Proposition 3.21 and can be proved in a similar way. The differences in the particular PIDE that characterises H compared to that in Chapter 3 (cf. Equation (3.12)) are due to the more complex dependences in the jump case. As a remarkable difference to Chapter 3 and to other insider equilibrium models, for example, [25], [28], [26] or [19] (cf. Section 1.2), we do no longer necessarily have the inconspicuousness of informed trading, i.e. Y^c is not necessarily an $\mathbb{F}^{\mathcal{M}}$-martingale.

Proposition 4.12. *Let $(H, \lambda, \varphi) \in \mathcal{P}$, $\theta \in \mathcal{S}(H, \lambda, \varphi)$ and f some sufficiently smooth function on $[0, \infty) \times \mathbb{R}^4$. Then the following conditions are equivalent:*

(1) $\sigma^{-1}\left(Y^c - \int_0^{\cdot\wedge T} (\lambda_s \partial_y H(s, \widetilde{Y}_s, \bar{U}_s))^{-1} f(s, \widetilde{Y}_s, \bar{U}_s)\, \mathrm{d}s\right)$ *is an $\mathbb{F}^{\mathcal{M}}$-Brownian motion on $[0, T]$,*

(2) $\mathbb{E}((\alpha_t + \mu_t)\mathbb{1}_{[0,T]}(t) | \mathcal{F}_t^{\mathcal{M}}) = (\lambda_t \partial_y H(t, \widetilde{Y}_t, \bar{U}_t))^{-1} f(t, \widetilde{Y}_t, \bar{U}_t)\mathbb{1}_{[0,T]}(t)$*, for all $t \ge 0$,*

(3) for all $(t, u) = (t, \bar{u}, z, y) \in \mathcal{D}$, H satisfies

$$\begin{aligned}
0 = \partial_t H(t, y, \bar{u}) &+ \lambda^2 \sigma^2 \left(\frac{\partial_{yy} H(t, y, \bar{u})}{2} - \partial_\Lambda H(t, y, \bar{u}) \right) + l(t, \bar{u}) \partial_\lambda H(t, y, \bar{u}) - \partial_S H(t, y, \bar{u}) \\
&+ \int_{\mathbb{R}^3} H(t, y + \varphi(t, \bar{u}, \phi^X(t, \zeta_2)), \bar{u} + \phi^{\bar{U}}(t, \bar{u}, \zeta)) - H(t, y, \bar{u})\, \nu(\mathrm{d}\zeta) + f(t, y, \bar{u}).
\end{aligned} \tag{4.46}$$

If we furthermore assume that (H, λ, φ) satisfies PIDE (4.31) then $(1) - (3)$ *are equivalent to*

(4)
$$\begin{aligned}
\lambda^{-1} f(t, y, \bar{u}) = &-l^{(2)}(t, \bar{u}) \lambda^{-2} H(t, y, \bar{u}) \\
&+ \int_{\mathbb{R}^3} H(t, y + \varphi(t, \bar{u}, \phi^X(t, \zeta_2)), \bar{u} + \phi^{\bar{U}}(t, \bar{u}, \zeta)) \phi^{\lambda^{-1}}(t, \bar{u}, \zeta_2, \zeta_3) \nu(\mathrm{d}\zeta),
\end{aligned} \tag{4.47}$$

for all $(t, u) = (t, \bar{u}, z, y) \in \mathcal{D}$.

Proof. Again, we use the short notation H_t instead of $H(t, \widetilde{Y}_t, \bar{U}_t)$ and μ_t instead of $\mu(t, U_t)$, etc., as long as this does not lead to confusion. Taking in mind the dynamics of the involved proces, application of Itô's formula to H yields

$$\begin{aligned}
H_{t\wedge T} = H_0 &+ \int_0^{t\wedge T} \partial_t H_s + \frac{\sigma^2 \lambda_s^2}{2} \partial_{yy} H_s + l_s \partial_\lambda H_s - \lambda_s^2 \sigma^2 \partial_\Lambda H_s - \partial_S H_s \,\mathrm{d}s \\
&+ \int_0^{t\wedge T} \int_{\mathbb{R}^3} H(s, \widetilde{Y}_{s-} + \varphi_s(\zeta), \bar{U}_{s-} + \phi_s^{\bar{U}}(\zeta)) - H(s, \widetilde{Y}_{s-}, \bar{U}_{s-}) \,\nu(\mathrm{d}\zeta) \,\mathrm{d}s \\
&+ \int_0^{t\wedge T} \lambda_s (\hat{\mu}_s + \hat{\alpha}_s) \partial_y H_s \,\mathrm{d}s + \int_0^{t\wedge T} \lambda_{s-} \sigma \partial_y H_{s-} \,\mathrm{d}I_s \\
&+ \int_0^{t\wedge T} \int_{\mathbb{R}^3} H(s, \widetilde{Y}_{s-} + \varphi_s(\zeta), \bar{U}_{s-} + \phi_s^{\bar{U}}(\zeta)) - H(s, \widetilde{Y}_{s-}, \bar{U}_{s-}) \,\bar{N}(\mathrm{d}s, \mathrm{d}\zeta)
\end{aligned} \tag{4.48}$$

where

$$I_t := \int_0^{t\wedge T} (\mu_s + \alpha_s - \hat{\mu}_s - \hat{\alpha}_s) \sigma^{-1} \,\mathrm{d}s + B_t, \qquad t \geq 0, \tag{4.49}$$

and

$$\hat{\mu}_t := \mathbb{E}\left(\mu_t \middle| \mathcal{F}_t^{\mathcal{M}}\right), \quad \hat{\alpha}_t := \mathbb{E}\left(\alpha_t \middle| \mathcal{F}_t^{\mathcal{M}}\right).$$

Since T is an $\mathbb{F}^{\mathcal{M}}$-stopping time, we have

$$\begin{aligned}
\mathbb{E}\left(\int_s^{t\wedge T} \alpha_u + \mu_u \,\mathrm{d}u \middle| \mathcal{F}_s^{\mathcal{M}} \right) &= \int_s^t \mathbb{E}\left(\mathbb{E}\left((\alpha_u + \mu_u) \mathbb{1}_{[0,T]}(u) \middle| \mathcal{F}_u^{\mathcal{M}} \right) \middle| \mathcal{F}_s^{\mathcal{M}} \right) \mathrm{d}u \\
&= \int_s^t \mathbb{E}\left(\mathbb{E}\left((\alpha_u + \mu_u) \middle| \mathcal{F}_u^{\mathcal{M}} \right) \mathbb{1}_{[0,T]}(u) \middle| \mathcal{F}_s^{\mathcal{M}} \right) \mathrm{d}u \\
&= \mathbb{E}\left(\int_s^{t\wedge T} \hat{\alpha}_u + \hat{\mu}_u \,\mathrm{d}u \middle| \mathcal{F}_s^{\mathcal{M}} \right).
\end{aligned}$$

With the same arguments as in Proposition 3.21 where we replace

$$\mu_t + \alpha_t \quad \text{by} \quad (\mu_t + \alpha_t) \mathbb{1}_{[0,T]}(t), \qquad \text{and} \qquad \hat{\mu}_t + \hat{\alpha}_t \quad \text{by} \quad (\hat{\mu}_t + \hat{\alpha}_t) \mathbb{1}_{[0,T]}(t),$$

we can show that I is a Brownian motion w.r.t. $\mathbb{F}^{\mathcal{M}}$. Since

$$Y_t^c = \sigma I_t + \int_0^{t\wedge T} \hat{\mu}_s + \hat{\alpha}_s \,\mathrm{d}s, \quad t \geq 0,$$

we get the equivalence of (1) and (2) with a similar argumentation as in Proposition 3.21. For the equivalence of (2) and (3) observe that admissibility of θ ensures that the last two integrals in (4.48) are martingales. On the other hand, $(H_{t\wedge T})_{t\geq 0}$ itself is a martingale (recall that H is a rational pricing function). Again, with an analogous argumentation as in Proposition 3.21 we get the equivalence of (2) and (3).

To realise the equivalence of (3) and (4), observe that (similar to (4.37))

$$
\begin{aligned}
&\int_{\mathbb{R}^3} \frac{H(t, y + \varphi(t, \bar{u}, \phi^X(t, \zeta_2)), \bar{u} + \phi^{\bar{U}}(t, \bar{u}, \zeta))}{\lambda + \phi^\lambda(t, \bar{u}, \phi^X(t, \zeta_2), \phi^S(t, S, \zeta_2, \zeta_3))} - \frac{H(t, y, \bar{u})}{\lambda} \nu(\mathrm{d}\zeta) \\
&= \lambda^{-1} \int_{\mathbb{R}^3} H(t, y + \varphi(t, \bar{u}, \phi^X(t, \zeta_2)), \bar{u} + \phi^{\bar{U}}(t, \bar{u}, \zeta)) - H(t, y, \bar{u}) \, \nu(\mathrm{d}\zeta) \\
&\quad + \int_{\mathbb{R}^3} H(t, y + \varphi(t, \bar{u}, \phi^X(t, \zeta_2)), \bar{u} + \phi^{\bar{U}}(t, \bar{u}, \zeta)) \phi^{\lambda^{-1}}(t, \bar{u}, \zeta_2, \zeta_3) \, \nu(\mathrm{d}\zeta).
\end{aligned}
$$

Then the equivalence easily follows by multiplication of PIDE (4.46) with λ^{-1} and insertion of (4.47). □

4.7 Equilibrium

The following proposition gives sufficient conditions for a pair of an admissible pricing rule (H, λ, φ) and an admissible trading strategy θ to form an equilibrium. This is the jump part analogue to Proposition 3.22.

Proposition 4.13. *Let Assumptions 4.1, 4.3, and 4.8 be satisfied, $(H, \lambda, \varphi) \in \mathcal{P}$ and $\theta \in \mathcal{S}(H, \lambda, \varphi)$. Then $(H, \lambda, \varphi, \theta)$ is an equilibrium (in $\mathcal{P} \times \mathcal{S}(H, \lambda, \varphi)$) if the following conditions are fulfilled:*

(i) H, λ and F satisfy (4.31), (4.33), and (4.35),

(ii) θ is of finite variation,

(iii) $\sigma^{-1}\left(Y^c - \int_0^{\cdot\wedge T} (\lambda_s \partial_y H(s, \widetilde{Y}_s, \bar{U}_s))^{-1} f(s, \widetilde{Y}_s, \bar{U}_s) \, \mathrm{d}s\right)$ is an $\mathbb{F}^{\mathcal{M}}$-Brownian motion on $[0, T]$ where f is defined as in (4.47),

(iv) $H(T, \widetilde{Y}_T, \bar{U}_T) = h(Z_T)$ $\mathbb{P}$-a.s.

Proof. We conclude the optimality of θ from conditions (i), (ii), and (iv) with the help of Proposition 4.10.

For rationality of (H, λ, φ) first observe that $(H_{t\wedge T})_{t\geq 0}$ is an $\mathbb{F}^{\mathcal{M}}$-martingale. To see this, again use Itô's formula.

$$
\begin{aligned}
H_{t\wedge T} = H_0 &+ \int_0^{t\wedge T} \partial_t H_s + \frac{\sigma^2 \lambda_s^2}{2} \partial_{yy} H_s + l_s \partial_\lambda H_s - \lambda_s^2 \sigma^2 \partial_\Lambda H_s - \partial_S H_s \, \mathrm{d}s \\
&+ \int_0^{t\wedge T} \int_{\mathbb{R}^3} H(s, \widetilde{Y}_{s-} + \varphi_s(\zeta), \bar{U}_{s-} + \phi_s^{\bar{U}}(\zeta)) - H(s, \widetilde{Y}_{s-}, \bar{U}_{s-}) \, \nu(\mathrm{d}\zeta) \, \mathrm{d}s \\
&+ \int_0^{t\wedge T} \int_{\mathbb{R}^3} H(s, \widetilde{Y}_{s-} + \varphi_s(\zeta), \bar{U}_{s-} + \phi_s^{\bar{U}}(\zeta)) - H(s, \widetilde{Y}_{s-}, \bar{U}_{s-}) \, \bar{N}(\mathrm{d}s, \mathrm{d}\zeta) \\
&+ \int_0^{t\wedge T} \lambda_{s-} \partial_y H_{s-} \, \mathrm{d}Y_s^c.
\end{aligned}
\tag{4.50}
$$

Using PIDE (4.46), which follows from Proposition 4.12 and condition (iii), this simplifies to

$$\begin{aligned} H_{t\wedge T} &= H_0 + \int_0^{t\wedge T} \lambda_{s-}\partial_y H_{s-}\,\mathrm{d}Y_s^c - \int_0^{t\wedge T} f_s\,\mathrm{d}s \\ &\quad + \int_0^{t\wedge T}\int_{\mathbb{R}^3} H(s,\widetilde{Y}_{s-}+\varphi_s(\zeta),\bar{U}_{s-}+\phi_s^{\bar{U}}(\zeta)) - H(s,\widetilde{Y}_{s-},\bar{U}_{s-})\,\bar{N}(\mathrm{d}s,\mathrm{d}\zeta). \end{aligned}$$

Due to condition (iii) and the admissibility of (H,λ,φ) and θ, we conclude that $(H_{t\wedge T})_{t\geq 0}$ is an $\mathbb{F}^{\mathcal{M}}$-martingale. Finally, condition (iv) yields

$$H(t\wedge T,\widetilde{Y}_{t\wedge T},\bar{U}_{t\wedge T}) = \mathbb{E}\left(H(T,\widetilde{Y}_T,\bar{U}_T)\Big|\mathcal{F}_t^{\mathcal{M}}\right) = \mathbb{E}\left(h(Z_T)\big|\mathcal{F}_t^{\mathcal{M}}\right).$$

□

In the case when $f\equiv 0$ we are now able to state the form of an equilibrium in the following theorem. For a discussion of the non-inconspicuous case we refer to Section 4.9.

Up to now, we did not characterise the price pressure for the jumps, i.e. φ. For reasons that are due to the filter problem of the market makers, φ has to be chosen such that a jump of $\widetilde{Y}$ equals the estimated (w.r.t. the market makers information) corresponding jump of Z.

Theorem 4.14. *Suppose Assumptions 4.1, 4.3 and 4.8 are satisfied. Let*

$$\varphi(t,\bar{u},\phi^X(t,\zeta_2)) := \kappa(t,\phi^X(t,\zeta_2))$$

and λ be as in (4.33) such that

- *λ is pathwise bounded away from 0 and bounded from above on $[0,T]$*
- *(4.35) holds*
- *$\Lambda_t > 0$, for all $t\in[0,T)$, and $\Lambda_T = 0$ $\mathbb{P}$-a.s.*

Furthermore, let

$$H(t,y,\bar{u}) := \mathbb{E}\left(h(\xi_T)\big|\xi_t = y,\bar{U}_t = u\right), \tag{4.51}$$

and

$$\theta_t := \int_0^{t\wedge T} -\mu_s + a_s(Z_s-\widetilde{Y}_s)\,\mathrm{d}s, \quad \text{with} \quad a_t := \lambda_t\sigma^2\Lambda_t^{-1}. \tag{4.52}$$

If $(H,\lambda,\varphi)\in\mathcal{P}$, $\theta\in\mathcal{S}(H,\lambda,\varphi)$ and $f\equiv 0$ where f is defined as in (4.47), then $(H,\lambda,\varphi,\theta)$ defines an equilibrium.

Proof. The proof makes use of Proposition 4.13. Hence, we have to ensure that all conditions are satisfied. This is obviously the case for condition (ii). Condition (iii) is proved by Lemma 4.16. Due to the special structure of H, for condition (iv) it is sufficient to show the $\mathbb{P}$-a.s.

equality of $\widetilde{Y}_T$ and Z_T. This is done by Lemma 4.15. Last but not least, for condition (i), PIDEs (4.33) and (4.35) hold by assumption. Furthermore, since $f = 0$, it remains to show, according to Proposition 4.12, that H satisfies

$$0 = \partial_t H(t,y,\bar{u}) + \frac{\lambda^2\sigma^2}{2}\partial_{yy}H(t,y,\bar{u}) - \lambda^2\sigma^2\partial_\Lambda H(t,y,\bar{u}) + l(t,\bar{u})\partial_\lambda H(t,y,\bar{u}) - \partial_S H(t,y,\bar{u})$$
$$+ \int_{\mathbb{R}^3} H(t, y+\varphi(t,\bar{u},\phi^X(t,\zeta_2)), \bar{u} + \phi^{\bar{U}}(t,\bar{u},\zeta)) - H(t,y,\bar{u})\,\nu(\mathrm{d}\zeta)$$

for all $(t,u) \in \mathcal{D}$. This again follows from Feynman-Kac's formula as in Section 4.3 where we use the admissibility of (H,λ,φ). $\square$

4.8 Auxiliaries for the proof of Theorem 4.14

Lemma 4.15. *Let the assumptions of Theorem 4.14 be satisfied. Then $\widetilde{Y}_T = Z_T$ holds $\mathbb{P}$-a.s.*

Proof. For $\epsilon > 0$ define

$$\tau^\epsilon := \inf\{t \geq 0 : \; S_t \leq \epsilon\}.$$

Observe that for any $t \geq 0$ we have $T - (t \wedge \tau^\epsilon) \geq \epsilon$. Since λ is (pathwise) bounded away from 0 and bounded from above on $[0,T]$, there exist random variables $0 < \bar{\lambda}_1 \leq \bar{\lambda}_2 < \infty$ such that $\bar{\lambda}_1 \leq \lambda_{t\wedge T} \leq \bar{\lambda}_2$ $\mathbb{P}$-a.s. Furthermore, we have

$$\Lambda_{t\wedge\tau^\epsilon} \geq \int_{t\wedge\tau^\epsilon}^{\tau^\epsilon+\epsilon} \sigma^2\lambda_s^2\,\mathrm{d}s \geq \epsilon\sigma^2\bar{\lambda}_1 > 0, \quad \text{for all } t \geq 0.$$

It follows that $\lambda^2_{t\wedge\tau^\epsilon}\sigma^2\Lambda^{-1}_{t\wedge\tau^\epsilon}$ is pathwise uniformly (for $t \in [0,\infty)$) bounded away from zero and bounded from above.

Now, define

$$\rho_t^\epsilon := \exp\left(-\int_0^{t\wedge\tau^\epsilon} a_s\lambda_s\,\mathrm{d}s\right) = \exp\left(-\int_0^{t\wedge\tau^\epsilon} \frac{\lambda_s^2\sigma^2}{\Lambda_s}\,\mathrm{d}s\right).$$

For any $\epsilon > 0$, ρ^ϵ as well as $(\rho^\epsilon)^{-1}$ are well-defined, locally bounded, predictable processes of finite variation and

$$\mathrm{d}\frac{1}{\rho_t^\epsilon} = \frac{a_t\lambda_t}{\rho_t^\epsilon}\mathbb{1}_{[0,\tau^\epsilon]}(t)\mathrm{d}t, \quad t \geq 0, \quad \rho_0^\epsilon = 1.$$

By assumption we have

$$\mathrm{d}(Z_t - \widetilde{Y}_t) = -\lambda_t a_t\left(Z_t - \widetilde{Y}_t\right)\mathrm{d}t - \lambda_t\sigma\mathrm{d}B_t + \mathrm{d}\sum_{s\leq t}\Delta(Z_s - \widetilde{Y}_s).$$

Integration by parts yields (remember that $\widetilde{Y}_0 = 0$ and $\rho_0^\epsilon = 1$)

$$\begin{aligned}\frac{Z_{\tau^\epsilon} - \widetilde{Y}_{\tau^\epsilon}}{\rho^\epsilon_{\tau^\epsilon}} &= Z_0 + \int_0^{\tau^\epsilon} (Z_t - \widetilde{Y}_t)\,\mathrm{d}\frac{1}{\rho_t^\epsilon} + \int_0^{\tau^\epsilon} \frac{1}{\rho_t^\epsilon}\,\mathrm{d}(Z_t - \widetilde{Y}_t) \\ &= Z_0 + \int_0^{\tau^\epsilon} \frac{\lambda_t a_t}{\rho_t^\epsilon}(Z_t - \widetilde{Y}_t)\,\mathrm{d}t - \int_0^{\tau^\epsilon} \frac{\lambda_t a_t}{\rho_t^\epsilon}(Z_t - \widetilde{Y}_t)\,\mathrm{d}t - \int_0^{\tau^\epsilon} \frac{\lambda_t \sigma}{\rho_t^\epsilon}\,\mathrm{d}B_t \\ &\quad + \sum_{t\le\tau^\epsilon} \frac{\Delta(Z_t - \widetilde{Y}_t)}{\rho_t^\epsilon} \\ &= Z_0 - \int_0^{\tau^\epsilon} \frac{\lambda_t \sigma}{\rho_t^\epsilon}\,\mathrm{d}B_t + \sum_{t\le\tau^\epsilon} \frac{\Delta(Z_t - \widetilde{Y}_t)}{\rho_t^\epsilon}.\end{aligned}$$

Multiplying $\rho^\epsilon_{\tau^\epsilon}$ on both sides,

$$Z_{\tau^\epsilon} - \widetilde{Y}_{\tau^\epsilon} = \rho^\epsilon_{\tau^\epsilon}\left(Z_0 - \int_0^{\tau^\epsilon} \frac{\lambda_t \sigma}{\rho_t^\epsilon}\,\mathrm{d}B_t + \sum_{t\le\tau^\epsilon} \frac{\Delta(Z_t - \widetilde{Y}_t)}{\rho_t^\epsilon}\right). \tag{4.53}$$

By assumption we have that $T < \infty$ $\mathbb{P}$-a.s., and that N is a Poisson process. Let Ω_0 be the set of all $\omega \in \Omega$ with $N_T < \infty$. Then $\mathbb{P}(\Omega_0) = 1$. For $\omega \in \Omega_0$ let $n_\omega := N_T(\omega)$. It follows that

$$\delta_\omega := \inf_{0\le t\le T_{n_\omega}} \{S_t(\omega)\} = \min_{i=1,\dots,n_\omega} \{S_{T_i}(\omega)\} > 0,$$

where $T_i, i = 1, 2, \dots$ are the jump times of the related Poisson process. For all $\omega \in \Omega_0$ we get

$$T(\omega) - \tau^\epsilon(\omega) = \epsilon \quad \text{for all } \epsilon \in (0, \delta_\omega).$$

Hence, $\tau^\epsilon \nearrow T$ $\mathbb{P}$-a.s. for $\epsilon \searrow 0$. As Y and Z are càdlàg processes, we have $\mathbb{P}$-a.s. convergence of

$$\widetilde{Y}_{\tau^\epsilon} \xrightarrow{\epsilon\to 0} \widetilde{Y}_{T-}, \qquad Z_{\tau^\epsilon} \xrightarrow{\epsilon\to 0} Z_{T-}.$$

Moreover, $\mathbb{P}(\Delta Y_T = 0, \Delta Z_T = 0) = 1$. Altogether, it results

$$\widetilde{Y}_T = Z_T \quad \mathbb{P}\text{-a.s.}$$

if we show that the term on the right hand side of (4.53) vanishes while ϵ approaches zero. We prove this in three steps:

(1) $\rho^\epsilon_{\tau^\epsilon} Z_0 \to 0$

(2) $\rho^\epsilon_{\tau^\epsilon} \sum_{t\le\tau^\epsilon} (\rho_t^\epsilon)^{-1} \Delta(Z_t - \widetilde{Y}_t) \to 0$

(3) $\rho^\epsilon_{\tau^\epsilon} \int_0^{\tau^\epsilon} \lambda_t \sigma (\rho_t^\epsilon)^{-1}\,\mathrm{d}B_t \to 0$

in probability, for $\epsilon \searrow 0$.

(1) Let Ω_0 and δ_ω be as before, then for all $\omega \in \Omega_0$ we get for all $\epsilon \in (0, \delta_\omega)$ that $T_{n_\omega}(\omega) \leq \tau^\epsilon(\omega)$. The integral in the definition of ρ is defined pathwise. Using the special form of a, we get on the interval $[T_{n_\omega}(\omega), \tau^\epsilon(\omega)]$ that

$$\lambda_t(\omega) a_t(\omega) = \frac{\lambda_t^2(\omega)\sigma^2}{\Lambda_t(\omega)} = \frac{\lambda_t^2(\omega)\sigma^2}{\sum_{i=1}^{n_\omega} \chi_i(\omega) - \int_0^t \lambda_s^2(\omega)\sigma^2 \,\mathrm{d}s}.$$

Hence,

$$\begin{aligned}\left(\int_0^{\tau^\epsilon} \lambda_t a_t \,\mathrm{d}t\right)(\omega) &\geq \int_{T_{n_\omega}(\omega)}^{\tau^\epsilon(\omega)} \frac{\lambda_t^2(\omega)\sigma^2}{\sum_{i=1}^{n_\omega} \chi_i(\omega) - \int_0^t \lambda_s^2(\omega)\sigma^2 \,\mathrm{d}s} \,\mathrm{d}t \\ &= \log\left(\sum_{i=1}^{n_\omega} \chi_i(\omega) - \int_0^{T_{n_\omega}(\omega)} \lambda_t^2(\omega)\sigma^2 \,\mathrm{d}t\right) - \log\left(\sum_{i=1}^{n_\omega} \chi_i(\omega) - \int_0^{\tau^\epsilon} \lambda_t^2(\omega)\sigma^2 \,\mathrm{d}t\right) \\ &= \log\left(\Lambda_{T_{n_\omega}}\right)(\omega) - \log\left(\Lambda_{\tau^\epsilon}\right)(\omega).\end{aligned} \tag{4.54}$$

The first term in the last line of (4.54) is constant w.r.t. ϵ. By assumption we have that $\Lambda_t \to 0$ for $t \to T$. Hence, $\log(\Lambda_{\tau^\epsilon}) \searrow -\infty$, for $\epsilon \to 0$. Since $Z_0 < \infty$ a.s., we obtain altogether $Z_0 \rho_{\tau^\epsilon} \searrow 0$.

(2) For $0 \leq s \leq t$ define

$$\rho_{s,t}^\epsilon := \rho_t^\epsilon (\rho_s^\epsilon)^{-1} = \exp\left(-\int_{s\wedge\tau^\epsilon}^{t\wedge\tau^\epsilon} \lambda_u a_u \,\mathrm{d}u\right).$$

Since $\lambda_t a_t > 0$, it follows $\rho_{s_1,t}^\epsilon \leq \rho_{s_2,t}^\epsilon$ for $s_1 \leq s_2$. A calculation as in (4.54) yields

$$\rho_{T_{n_\omega},\tau^\epsilon}^\epsilon(\omega) = \exp\left(-\int_{T_{n_\omega}}^{\tau^\epsilon} \frac{\lambda_t^2\sigma^2}{\Lambda_t} \,\mathrm{d}t\right)(\omega) = \exp\left(\log(\Lambda_{\tau^\epsilon}) - \log(\Lambda_{T_{n_\omega}})\right)(\omega) = \frac{\Lambda_{\tau^\epsilon}}{\Lambda_{T_{n_\omega}}}(\omega). \tag{4.55}$$

In particular,

$$\rho_{T_{n_\omega},\tau^\epsilon}^\epsilon(\omega) \xrightarrow{\epsilon\to 0} 0. \tag{4.56}$$

With (4.56) and

$$\left|\sum_{i=1}^{n_\omega} \Delta(Z_{T_i} - \widetilde{Y}_{T_i})(\omega)\right| < \infty \quad \text{for almost all } \omega \in \Omega$$

we finally get

$$\begin{aligned}\left|\rho_{\tau^\epsilon} \sum_{t\leq\tau^\epsilon} (\rho_t^\epsilon)^{-1} \Delta(Z_t - \widetilde{Y}_t)\right|(\omega) &= \left|\sum_{t\leq\tau^\epsilon(\omega)} \rho_{t,\tau^\epsilon}(\omega) \Delta(Z_t - \widetilde{Y}_t)(\omega)\right| \\ &\leq \rho_{T_{n_\omega},\tau^\epsilon}(\omega) \left|\sum_{i=1}^{n_\omega} \Delta(Z_{T_i} - \widetilde{Y}_{T_i})(\omega)\right| \xrightarrow{\epsilon\searrow 0} 0.\end{aligned}$$

(3) Due to the independence of B and (λ, Λ, S), $(\rho^\epsilon)^{-1}$ given $(\lambda, \Lambda, S)_{t\in[0,\tau^\epsilon]}$ is a bounded, deterministic process and

$$\int_0^{\tau^\epsilon} (\rho_t^\epsilon)^{-1} \, \mathrm{d}B_t \sim \mathcal{N}\left(0, \int_0^{\tau^\epsilon} (\rho_t^\epsilon)^{-2} \, \mathrm{d}t\right).$$

It follows that

$$\rho_{\tau^\epsilon}^\epsilon \int_0^{\tau^\epsilon} (\rho_t^\epsilon)^{-1} \, \mathrm{d}B_t \sim \mathcal{N}\left(0, (\rho_{\tau^\epsilon}^\epsilon)^2 \int_0^{\tau^\epsilon} (\rho_t^\epsilon)^{-2} \, \mathrm{d}t\right).$$

Therefore, in order to prove (3), it suffices to show that $\mathbb{P}$-a.s.

$$(\rho_{\tau^\epsilon}^\epsilon)^2 \int_0^{\tau^\epsilon} (\rho_t^\epsilon)^{-2} \, \mathrm{d}t \xrightarrow{\epsilon \searrow 0} 0.$$

For all $\omega \in \Omega_0$ and $\epsilon \in (0, \delta_\omega)$

$$(\rho_{\tau^\epsilon}^\epsilon)^2(\omega) \int_0^{\tau^\epsilon(\omega)} (\rho_t^\epsilon)^{-2}(\omega) \, \mathrm{d}t = (\rho_{\tau^\epsilon}^\epsilon)^2(\omega) \left(\int_0^{T_{n_\omega}(\omega)} (\rho_t^\epsilon)^{-2}(\omega) \, \mathrm{d}t + \int_{T_{n_\omega}(\omega)}^{\tau^\epsilon(\omega)} (\rho_t^\epsilon)^{-2}(\omega) \, \mathrm{d}t \right).$$

Since $(\rho_t^\epsilon)^{-1}$ is increasing, we get

$$(\rho_{\tau^\epsilon}^\epsilon)^2(\omega) \int_0^{\tau^\epsilon(\omega)} (\rho_t^\epsilon)^{-2}(\omega) \, \mathrm{d}t \leq (\rho_{T_{n_\omega}, \tau^\epsilon}^\epsilon)^2(\omega) T_{n_\omega}(\omega) + (\rho_{\tau^\epsilon}^\epsilon)^2(\omega) \int_{T_{n_\omega}(\omega)}^{\tau^\epsilon(\omega)} (\rho_t^\epsilon)^{-2}(\omega) \, \mathrm{d}t. \tag{4.57}$$

Due to (4.56), the first term on the right hand side of (4.57) vanishes for $\epsilon \to 0$. For all $\epsilon \in (0, \delta_\omega)$ a calculation similar to that in (4.54) shows

$$(\rho_{\tau^\epsilon}^\epsilon)^2(\omega) \int_{T_{n_\omega}(\omega)}^{\tau^\epsilon(\omega)} (\rho_t^\epsilon)^{-2}(\omega) \, \mathrm{d}t = \Lambda_{\tau^\epsilon}^2(\omega) \int_{T_{n_\omega}(\omega)}^{\tau^\epsilon(\omega)} \Lambda_t^{-2}(\omega) \, \mathrm{d}t.$$

For $t \in (T_{n_\omega}(\omega), T(\omega))$ now consider

$$\Lambda_t^2(\omega) \int_{T_{n_\omega}(\omega)}^t \Lambda_s^{-2}(\omega) \, \mathrm{d}s.$$

Both factors are differentiable functions in t on $(T_{n_\omega}(\omega), T(\omega))$. By l'Hospital's rule we get

$$\lim_{t \to T(\omega)} \Lambda_t^2(\omega) \int_{T_{n_\omega}(\omega)}^t \Lambda_s^{-2}(\omega) \, \mathrm{d}s = \lim_{t \to T(\omega)} \frac{\Lambda_t^{-2}(\omega)}{2\sigma^2 \lambda_t^2(\omega) \Lambda_t^{-3}(\omega)} = \lim_{t \to T(\omega)} \frac{\Lambda_t(\omega)}{2\sigma^2 \lambda_t^2(\omega)} = 0.$$

This finally completes the proof of this lemma. □

Lemma 4.16. *Let the assumptions of Theorem 4.14 be satisfied. Then $\sigma^{-1} Y^c$ is a Brownian motion w.r.t. $\mathbb{F}^{\mathcal{M}}$ on the stochastic interval $[0, T]$.*

Proof. Analogous to Chapter 3, we want to apply Proposition 2.5 to obtain a certain representation of the dynamics of $\sigma^{-1}Y^c$. This representation allows us to conclude that $\sigma^{-1}Y^c$ is a Brownian motion w.r.t. the market makers filtration.

As seen before, the drift part of Y, i.e. $a_t(Z_t - \widetilde{Y}_t)$, does not match the assumptions of Proposition 2.5 since $a_t \xrightarrow{t\to T} \infty$. For this reason, we prove the assertion on a stochastic interval $[0,\tau^n]$, for $n \in \mathbb{N}$, where $(\tau^n)_{n\in\mathbb{N}}$ is a localising sequence such that $\lambda_t a_t \mathbb{1}_{[0,\tau^n]}(t)$ is bounded for any $n \in \mathbb{N}$ and $\tau^n \xrightarrow{n\to\infty} T$ $\mathbb{P}$-a.s.

Let us now define this sequence by

$$\tau^n := \inf\left\{t \geq 0 : \ \Lambda_t \leq n^{-1}, \ \lambda_t \geq n\right\} \wedge n, \quad n \in \mathbb{N}. \tag{4.58}$$

Obviously,

$$\left|a_t \mathbb{1}_{[0,\tau^n]}(t)\right| = \left|\frac{\lambda_t \sigma^2}{\Lambda_t} \mathbb{1}_{[0,\tau^n]}(t)\right| \leq \sigma^2 n^2$$

and $\tau^n \to T$ $\mathbb{P}$-a.s., for $n \to \infty$, since λ is bounded pathwise and $\Lambda_t > 0$ for all $t \in [0,T)$ and left-continuous in T.

Then consider the slightly modified filter problem on the interval $[0,n]$ induced by Z as signal process and (Y^n, S) as observation process where

$$Y_t^n := \theta_t^n + X_t, \qquad \theta_t^n := \int_0^t -\mu_s \mathbb{1}_{[0,T]}(s) + a_s(Z_s - \widetilde{Y}_s)\mathbb{1}_{[0,\tau^n]}(s)\,\mathrm{d}s.$$

Furthermore, define

$$\widetilde{Y}_t^n := \int_0^t \lambda_{s-}\,\mathrm{d}(Y_s^n)^c + \int_0^t \int_{\mathbb{R}^3} \kappa(s, \phi^X(t,\zeta_2)\,N(\mathrm{d}s,\mathrm{d}\zeta), \quad t \geq 0. \tag{4.59}$$

In particular, Y^n and $\widetilde{Y}^n$ coincide with Y and $\widetilde{Y}$, respectively, on $[0,\tau^n]$. We are now able to apply the results of Section 2.2. Due to Corollary 2.6,

$$\gamma_t = \left(\int_0^{t\wedge\tau^n} a_s^2\sigma^{-2}\,\mathrm{d}s + \sum_{i=0}^{N_t} \widetilde{\chi}_i\right)^{-1}, \tag{4.60}$$

with

$$\widetilde{\chi}_0 = \chi_0^{-1}, \quad \widetilde{\chi}_i = \frac{-\chi_i}{(\gamma_{T_i-} + \chi_i)\gamma_{T_i-}}, \quad \text{for } i \in \mathbb{N}. \tag{4.61}$$

Since (4.60) is defined pathwise, we can apply Lemma A.3 (pathwise) on

$$\frac{\gamma_t a_t}{\sigma^2} = \frac{a_t\sigma^{-2}}{\int_0^t a_s^2\sigma^{-2}\,\mathrm{d}s + \sum_{i=0}^{N_t}\widetilde{\chi}_i}$$

and obtain

$$\frac{\gamma_t a_t}{\sigma^2} = \lambda_t \tag{4.62}$$

if we show that

$$\widetilde{\chi}_i = -\frac{\chi_i}{\left(\frac{\lambda_{T_i-}\sigma^2}{a_{T_i-}} + \chi_i\right)\frac{\lambda_{T_i-}\sigma^2}{a_{T_i-}}} \quad \text{for all } i \in \mathbb{N}. \tag{4.63}$$

Obviously, if $\gamma_{T_i-}a_{T_i-} = \lambda_{T_i-}\sigma^2$ holds, we get (4.63) by (4.61). Hence, (4.62) follows by induction since $\gamma_0 a_0 = \lambda_0\sigma^2$ can easily be verified by the definition of γ and a.

Moreover, according to Proposition 2.5 we get for $\eta_t = \mathbb{E}(Z_t|\mathcal{F}_t^{Y^n,S})$ that

$$\begin{aligned}
\mathrm{d}\eta_t =& \frac{\gamma_{t-}a_{t-}}{\sigma^2}\mathbb{1}_{[0,\tau^n]}(t)\left(\mathrm{d}Y_t^{n,c} - a_t\mathbb{1}_{[0,\tau^n]}(t)\left(\eta_t - \widetilde{Y}_t^n\right)\mathrm{d}t\right) + \int_{\mathbb{R}^3}\kappa(t,\phi^X(t,\zeta_2)\,N(\mathrm{d}t,\mathrm{d}\zeta)\\
=& \frac{\gamma_{t-}a_{t-}}{\sigma^2}\mathbb{1}_{[0,\tau^n]}(t)\sigma\,\mathrm{d}I_t^n + \int_{\mathbb{R}^3}\kappa(t,\phi^X(t,\zeta_2))\,N(\mathrm{d}t,\mathrm{d}\zeta)
\end{aligned}$$

where

$$\mathrm{d}I_t^n = \mathrm{d}B_t + a_t\sigma^{-1}\mathbb{1}_{[0,\tau^n]}(t)(Z_t - \eta_t)\,\mathrm{d}t, \quad I_0^n = 0,$$

is the information process of the associated filter problem and a Brownian motion w.r.t. the observation filtration. Together with (4.62) this yields

$$\mathrm{d}\eta_t = \sigma\lambda_{t-}\mathbb{1}_{[0,\tau^n]}(t)\,\mathrm{d}I_t^n + \int_{\mathbb{R}^3}\kappa(t,\phi^X(t,\zeta_2))\,N(\mathrm{d}t,\mathrm{d}\zeta). \tag{4.64}$$

Now, on the one hand, for $Y^{n,c}$ (continuous part of Y^n) we get

$$\mathrm{d}Y_t^{n,c} = a_t\left(Z_t - \widetilde{Y}_t^n\right)\mathbb{1}_{[0,\tau^n]}(t)\,\mathrm{d}t + \sigma\mathrm{d}B_t = a_t\left(\eta_t - \widetilde{Y}_t^n\right)\mathbb{1}_{[0,\tau^n]}(t)\,\mathrm{d}t + \sigma\mathrm{d}I_t^n. \tag{4.65}$$

On the other hand, due to (4.59) and (4.64), we have

$$\eta_t - \widetilde{Y}_t^n = \int_0^t \sigma\lambda_{s-}\mathbb{1}_{[0,\tau^n]}(s)\,\mathrm{d}I_s^n - \int_0^t \lambda_{s-}\,\mathrm{d}Y_s^{n,c}.$$

Together this yields

$$\mathrm{d}Y_t^{n,c} = a_t\mathbb{1}_{[0,\tau^n]}(t)\left(\int_0^t \sigma\lambda_{s-}\mathbb{1}_{[0,\tau^n]}(s)\,\mathrm{d}I_s^n - \int_0^t \lambda_{s-}\,\mathrm{d}Y_s^{n,c}\right)\mathrm{d}t + \sigma\mathrm{d}I_t^n.$$

The unique strong solution to the above SDE with initial condition $Y_0^{n,c} = 0$ is given by

$$Y_t^{n,c} = \sigma I_t^n, \quad t \in [0,n].$$

We conclude that $\sigma^{-1}Y^{n,c}$ is a Brownian motion w.r.t. the observation filtration $\mathbb{F}^{Y^n,S}$. Furthermore, for any $n \in \mathbb{N}$ we have almost sure that

$$Y^{n,c}_{\cdot}\mathbb{1}_{[0,\tau^n]}(\cdot) = \sigma I^n_{\cdot}\mathbb{1}_{[0,\tau^n]}(\cdot) = \sigma I.\mathbb{1}_{[0,\tau^n]}(\cdot)$$

where I is a Brownian motion w.r.t. $\mathbb{F}^{\mathcal{M}}$. Since $\tau^n \to T$ $\mathbb{P}$-a.s., this proves the assertion. $\square$

4.9 Remarks and examples

Non-inconspicuous equilibria

In the situation of Theorem 4.14, assume that $f \neq 0$. Defining

$$H(t, y, \bar{u}) := \lambda\mathbb{E}\left(e^{\int_t^T -m(s,\bar{U}_s)\lambda_s\,\mathrm{d}s}h(\xi_T)\lambda_T^{-1}\middle|\xi_t = y, \bar{U}_t = u\right) \tag{4.66}$$

ensures that (4.31) is still satisfied even if $f \neq 0$. To realise this, first apply integration by parts to λ_t^{-1} and $H(t, \xi_t, \bar{U}_t)$. This yields (where we now write H_t for $H(t, \xi_t, \bar{U}_t)$)

$$\mathrm{d}H_t\lambda_t^{-1} = H_{t-}\mathrm{d}\lambda_t^{-1} + \lambda_{t-}^{-1}\mathrm{d}H_t + \mathrm{d}\left[H, \lambda^{-1}\right]_t. \tag{4.67}$$

Using the particular dynamics of H and λ^{-1}, calculated by Itô's formula, yields

$$\begin{aligned}&\lambda_t^{-1}\left(\partial_t H_t + \frac{\sigma^2\lambda_t^2}{2}\partial_{yy}H_t + l_t\partial_\lambda H_t - \sigma^2\lambda_t^2\partial_\Lambda H_t - \partial_S H_t\right) + H_t m_t - H_t l_t^{(2)}\lambda_t^{-2}\\&+ \int_{\mathbb{R}^3} H(t, \xi_t + \varphi_t(\zeta), \bar{U}_t + \phi_t^{\bar{U}_t}(\zeta))(\lambda_t^{-1} + \phi_t^{\lambda^{-1}}(\zeta)) - H(t, \xi_t, \bar{U}_t)\lambda_t^{-1}\,\nu(\mathrm{d}\zeta)\end{aligned}$$

as $\mathrm{d}t$ term in the representation of $\mathrm{d}H_t\lambda_t^{-1}$ in (4.67). Admissibility of H ensures that all other terms are martingale terms. Again applying integration by parts for $\lambda_t^{-1}H_t$ and $k_t := \exp\left(\int_0^t -m_s\lambda_s\,\mathrm{d}s\right)$ yields

$$\begin{aligned}&k_t\left(\lambda_t^{-1}\left(\partial_t H_t + \frac{\sigma^2\lambda_t^2}{2}\partial_{yy}H_t + l_t\partial_\lambda H_t - \sigma^2\lambda_t^2\partial_\Lambda H_t - \partial_S H_t\right) + H_t m_t - H_t l_t^{(2)}\lambda_t^{-2}\right.\\&\left.+ \int_{\mathbb{R}^3} H(t, \xi_t + \varphi_t(\zeta), \bar{U}_t + \phi_t^{\bar{U}_t}(\zeta))(\lambda_t^{-1} + \phi_t^{\lambda^{-1}}(\zeta)) - H(t, \xi_t, \bar{U}_t)\lambda_t^{-1}\,\nu(\mathrm{d}\zeta)\right) - H_t\lambda_t^{-1}m_t\lambda_t k_t\end{aligned} \tag{4.68}$$

$\mathrm{d}t$ term. The rest of the terms are again martingale terms since $-m_t\lambda_t < 0$, i.e. $k_t \leq 1$. Furthermore,

$$e^{\int_0^{t\wedge T} -m(s,\bar{U}_s)\lambda_s\,\mathrm{d}s}H_{t\wedge T}\lambda_{t\wedge T}^{-1}, \quad t \geq 0,$$

itself is a martingale by definition. We conclude that (4.68) has to equal 0 $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-a.s. on $[0,T]$. Dividing by k proves PIDE (4.31).

Nevertheless, an approach as used for Theorem 4.14 does not lead to an equilibrium. According to Proposition 4.13 we would have to ensure that

$$\sigma^{-1}\left(Y^c - \int_0^{\cdot} (\lambda_s \partial_y H(s, \widetilde{Y}_s, \bar{U}_s))^{-1} f(s, \widetilde{Y}_s, \bar{U}_s)\,\mathrm{d}s\right) =: \sigma^{-1} Y^{f,c}$$

is a Brownian motion. Defining θ as

$$\theta_t := \int_0^{t\wedge T} -\mu_s + \beta_s + a_s(Z_s - \widetilde{Y}_s)\,\mathrm{d}s, \quad \text{with} \quad a_t := \lambda_t \sigma^2 \Lambda_t^{-1}, \quad t \geq 0,$$

for a suitable $\mathbb{F}^{\mathcal{M}}$-adapted process β and proceeding as in Lemma 4.16 leads to (cf. (4.65))

$$\mathrm{d}Y_t^{f,c} = (a_t(\eta_t - \widetilde{Y}_t) + \beta_t)\mathrm{d}t + \sigma \mathrm{d}I_t - f_t(\lambda_t \partial_y H_t)^{-1}\mathrm{d}t$$

and

$$\eta_t - \widetilde{Y}_t = \int_0^t \sigma \lambda_{s-}\,\mathrm{d}I_s - \int_0^t \lambda_{s-}\,\mathrm{d}Y_s^{f,c} - \int_0^t f_s(\partial_y H_s)^{-1}\,\mathrm{d}s.$$

Together we have

$$\mathrm{d}Y_t^{f,c} = \left(a_t\left(\int_0^t \sigma\lambda_{s-}\,\mathrm{d}I_s - \int_0^t \lambda_{s-}\,\mathrm{d}Y_s^{f,c} - \int_0^t f_s(\partial_y H_s)^{-1}\,\mathrm{d}s\right) + \beta_t - \frac{f_t}{\lambda_t \partial_y H_t}\right)\mathrm{d}t + \sigma \mathrm{d}I_t.$$

Since $\sigma^{-1}Y^{f,c}$ is required to be a Brownian motion and, in particular, a martingale (i.e. the $\mathrm{d}t$ term has to vanish), we necessarily have to chose

$$\beta_t = a_t \int_0^t f_s(\partial_y H_s)^{-1}\,\mathrm{d}s + f_t(\lambda_t \partial_y H_t)^{-1}.$$

Now, a similar calculation as in the proof of Lemma 4.15 shows that

$$Z_{\tau^\epsilon} - \widetilde{Y}_{\tau^\epsilon} + \int_0^{\tau^\epsilon} \frac{f_t}{\partial_y H_t}\,\mathrm{d}t = \rho_{\tau^\epsilon}^\epsilon \left(Z_0 - \int_0^{\tau^\epsilon} \frac{\lambda_t \sigma}{\rho_t^\epsilon}\,\mathrm{d}B_t + \sum_{s \leq \tau^\epsilon} \frac{\Delta(Z_s - \widetilde{Y}_s)}{\rho_s^\epsilon}\right).$$

This leads to

$$Z_T = \widetilde{Y}_T - \int_0^T \frac{f_t}{\partial_y H_t}\,\mathrm{d}t.$$

However, according to Proposition 4.10, θ is not optimal.

Example: constant price pressure

Suppose that the conditional variance of the jumps of Z as well as the jumps of S are constant, i.e. $\chi_i = \chi \in \mathbb{R}_+$, for all $i \in \mathbb{N}_0$ and $\Delta S_{T_i} = S_0 = s \in \mathbb{R}_+$. We get

$$\Lambda_t = (N_t + 1)\chi - \int_0^t \lambda_u^2 \sigma^2 \,\mathrm{d}u.$$

In particular, λ has to be chosen such that for all $n \in \mathbb{N}$

$$n\chi - \sigma^2 \int_0^{sn} \lambda_u^2 \,\mathrm{d}u = 0.$$

This holds true for $\lambda = \sqrt{\chi(\sigma^2 s)^{-1}}$. According to Remark 4.6, (4.35) holds. Furthermore, if $\mu = 0$, λ satisfies the conditions of Theorem 4.14. Since $\lambda_t, \Lambda_t, S_t$ only depend on N_t, H admits the representation

$$H(t, \widetilde{Y}_t, \bar{U}_t) = \bar{H}(t, \widetilde{Y}_t, N_t)$$

for a suitable function $\bar{H}$. If furthermore $\kappa_i = \sqrt{\chi(\sigma^2 s)^{-1}}\Delta X_{T_i}$, for all $i \in \mathbb{N}$, we have a uniform price pressure for continuous and discontinuous changes of the total order process. According to Remark 4.11, an optimal strategy in $\theta \in \mathcal{S}(H, \lambda, \varphi)$ would also be optimal in the larger class where discontinuous strategies are allowed.

Appendix A

Auxiliary results

A.1 Proofs of Corollary 2.3 and Corollary 2.4

Proof of Corollary 2.3. We start with verifying conditions (1) - (11) of Theorems 2.1 and 2.2: (1) is obviously verified, since Z is constant and the volatility of $Y^{(i)}$ is constant, for $i = 1, \dots, 2n-1$. (2) follows directly from (iv) and (v). (3): obviously, the $(2n-1) \times (2n-1)$ diagonal matrix

$$\begin{pmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \sigma^2 \end{pmatrix}$$

is uniformly non-singular. (4) holds due to the constant volatility of $Y^{(i)}$. (5), (6) and (7) follow from (vi) and (iv) because

$$\mathbb{E}\left(\int_0^T (A_1(t,Y)\eta_t^{(j)})^2 \,\mathrm{d}t\right) \le L^2 \int_0^T \mathbb{E}(\eta_t^{(j)})^2 \,\mathrm{d}t \le L^2 \int_0^T \mathbb{E}Z_j^2 \,\mathrm{d}t < \infty.$$

(8) follows from (ii). (9), (10), and (11) are obviously satisfied, too. According to Theorem 2.1, Z given $\mathcal{F}_t^Y$ is $\mathcal{N}(\eta_t, \gamma_t)$-distributed. Furthermore, by inserting in (2.5) (Theorem 2.2) we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\gamma_t = &- A_1(t,Y)^2 \begin{pmatrix} \gamma_t^{(11)} & \cdots & \gamma_t^{(1n)} \\ \vdots & \ddots & \vdots \\ \gamma_t^{(n1)} & \cdots & \gamma_t^{(nn)} \end{pmatrix} \begin{pmatrix} \mathbb{1}_{[T_1,\infty)}(t) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{1}_{[T_n,\infty)}(t) & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sigma^2} & 0 & \cdots & 0 \\ 0 & \frac{1}{\sigma^2} & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & \frac{1}{\sigma^2} \end{pmatrix} \\ &\times \begin{pmatrix} \mathbb{1}_{[T_1,\infty)}(t) & \cdots & \mathbb{1}_{[T_n,\infty)}(t) \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} \gamma_t^{(11)} & \cdots & \gamma_t^{(1n)} \\ \vdots & \ddots & \vdots \\ \gamma_t^{(n1)} & \cdots & \gamma_t^{(nn)} \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&= -\frac{A_1(t,Y)^2}{\sigma^2}\begin{pmatrix} \gamma_t^{(11)} & \cdots & \gamma_t^{(1n)} \\ \vdots & \ddots & \vdots \\ \gamma_t^{(n1)} & \cdots & \gamma_t^{(nn)} \end{pmatrix}\begin{pmatrix} \mathbb{1}_{[T_1\vee T_1,\infty)}(t) & \cdots & \mathbb{1}_{[T_1\vee T_n,\infty)}(t) \\ \vdots & \ddots & \vdots \\ \mathbb{1}_{[T_n\vee T_1,\infty)}(t) & \cdots & \mathbb{1}_{[T_n\vee T_n,\infty)}(t) \end{pmatrix} \\
&\quad \times \begin{pmatrix} \gamma_t^{(11)} & \cdots & \gamma_t^{(1n)} \\ \vdots & \ddots & \vdots \\ \gamma_t^{(n1)} & \cdots & \gamma_t^{(nn)} \end{pmatrix}.
\end{aligned}
$$

Since

$$
\left(\gamma_t^{(i1)},\ldots,\gamma_t^{(in)}\right)\left(\mathbb{1}_{[T_j\vee T_1,\infty)}(t),\ldots,\mathbb{1}_{[T_j\vee T_n,\infty)}(t)\right)^\top = \sum_{k=1}^n \gamma^{(ik)}\mathbb{1}_{[T_j\vee T_k,\infty)}(t)
$$

and

$$
\begin{aligned}
&\left(\sum_{k=1}^n \gamma^{(ik)}\mathbb{1}_{[T_1\vee T_k,\infty)}(t),\ldots,\sum_{k=1}^n \gamma^{(ik)}\mathbb{1}_{[T_n\vee T_k,\infty)}(t)\right)\left(\gamma_t^{(1j)},\ldots,\gamma_t^{(nj)}\right)^\top \\
&\qquad = \sum_{l=1}^n\sum_{k=1}^n \gamma_t^{(ik)}\mathbb{1}_{[T_l\vee T_k,\infty)}(t)\gamma_t^{(lj)} = c_t^{(ij)},
\end{aligned}
$$

we get

$$
\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t = -\frac{A_1(t,Y)^2}{\sigma^2}\begin{pmatrix} c_t^{(11)} & \cdots & c_t^{(1n)} \\ \vdots & \ddots & \vdots \\ c_t^{(n1)} & \cdots & c_t^{(nn)} \end{pmatrix}.
$$

Furthermore, for $t\in[0,T]$, define

$$
\begin{aligned}
\widetilde{Y}_t^{(1)} &:= Y_t^{(1)} - \int_0^t A_0(s,Y) + \bar{A}_1(s,Y)\eta_s\,\mathrm{d}s \\
&= Y_t^{(1)} - \int_0^t A_0(s,Y) + A_1(s,Y)\sum_{i=1}^n \eta_s^{(i)}\mathbb{1}_{[T_i,\infty)}(s)\,\mathrm{d}s.
\end{aligned}
$$

Then we get by inserting in (2.4)

$$
\begin{aligned}
\mathrm{d}\begin{pmatrix} \eta_t^{(1)} \\ \vdots \\ \eta_t^{(n)} \end{pmatrix} &= \frac{A_1(t,Y)}{\sigma^2}\begin{pmatrix} \gamma_t^{(11)} & \cdots & \gamma_t^{(1n)} \\ \vdots & \ddots & \vdots \\ \gamma_t^{(n1)} & \cdots & \gamma_t^{(nn)} \end{pmatrix}\begin{pmatrix} \mathbb{1}_{[T_1,\infty)}(t) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathbb{1}_{[T_n,\infty)}(t) & 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} \mathrm{d}\widetilde{Y}_t^{(1)} \\ \mathrm{d}Y_t^{(2)} \\ \vdots \end{pmatrix} \\
&= \frac{A_1(t,Y)}{\sigma^2}\begin{pmatrix} \sum_{j=1}^n \gamma_t^{(1j)}\mathbb{1}_{[T_j,\infty)}(t) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sum_{j=1}^n \gamma_t^{(nj)}\mathbb{1}_{[T_j,\infty)}(t) & 0 & \cdots & 0 \end{pmatrix}\begin{pmatrix} \mathrm{d}\widetilde{Y}_t^{(1)} \\ \mathrm{d}Y_t^{(2)} \\ \vdots \end{pmatrix}
\end{aligned}
$$

$$= \frac{A_1(t,Y)}{\sigma^2} \begin{pmatrix} \sum_{j=1}^n \gamma_t^{(1j)} \mathbb{1}_{[T_j,\infty)}(t)\, \mathrm{d}\widetilde{Y}_t^{(1)} \\ \vdots \\ \sum_{j=1}^n \gamma_t^{(nj)} \mathbb{1}_{[T_j,\infty)}(t)\, \mathrm{d}\widetilde{Y}_t^{(1)} \end{pmatrix}.$$

□

Proof of Corollary 2.4. First observe that $\widetilde{\eta}$ and $\widetilde{\gamma}$ can be expressed in terms of η and γ of Corollary 2.3. For $\widetilde{\eta}$ we obviously have

$$\widetilde{\eta}_t = \sum_{i=1}^n \eta_t^{(i)} \mathbb{1}_{[T_i,\infty)}(t).$$

Using the linearity of the conditional mean and the fact that all T_i, $i \in \{1,\dots,n\}$, are $\mathcal{F}_0^Y$-measurable, we get

$$\begin{aligned} \widetilde{\gamma}_t &= \mathbb{E}\left(\left(\sum_{i=1}^n Z_i \mathbb{1}_{[T_i,\infty)}(t) - \mathbb{E}\left(\sum_{i=1}^n Z_i \mathbb{1}_{[T_i,\infty)}(t) \middle| \mathcal{F}_t^Y \right) \right)^2 \middle| \mathcal{F}_t^Y \right) \\ &= \mathbb{E}\left(\left(\sum_{i=1}^n \left(Z_i - \mathbb{E}(Z_i|\mathcal{F}_t^Y) \right) \mathbb{1}_{[T_i,\infty)}(t) \right)^2 \middle| \mathcal{F}_t^Y \right) \\ &= \sum_{i=1}^n \sum_{j=1}^n \gamma_t^{(ij)} \mathbb{1}_{[T_i \vee T_j,\infty)}(t). \end{aligned}$$

Now, the initial conditions are easily verified since $\widetilde{\eta}_t = \eta_t^{(1)}$ and $\widetilde{\gamma}_t = \gamma_t^{(11)}$ for $t \in [0,T_2)$ (remember that $T_1 = 0$). For $p \in \{1,\dots,n-1\}$ and $t \in (T_p, T_{p+1})$ we have

$$\sum_{j=1}^n \gamma_t^{(ij)} \mathbb{1}_{[T_j,\infty)} = \sum_{j=1}^p \gamma_t^{(ij)} \mathbb{1}_{[T_j,\infty)}.$$

Due to the dynamics of $\eta_t^{(i)}$, $i \in \{1,...,p\}$, it follows that on (T_p, T_{p+1})

$$\begin{aligned} \mathrm{d}\widetilde{\eta}_t =& \mathrm{d}\sum_{i=1}^p \eta_t^{(i)} = \left(\sum_{i=1}^p \frac{A_1(t,Y)}{\sigma^2} \sum_{j=1}^n \gamma_t^{(ij)} \mathbb{1}_{[T_j,\infty)} \right) \left(\mathrm{d}Y_t^{(1)} - (A_0(t,Y) + A_1(t,Y)\widetilde{\eta}_t)\,\mathrm{d}t \right) \\ =& \frac{A_1(t,Y)}{\sigma^2} \sum_{i=1}^p \sum_{j=1}^p \gamma_t^{(ij)} \left(\mathrm{d}Y_t^{(1)} - (A_0(t,Y) + A_1(t,Y)\widetilde{\eta}_t)\,\mathrm{d}t \right) \\ =& \frac{A_1(t,Y)}{\sigma^2} \widetilde{\gamma}_t \left(\mathrm{d}Y_t^{(1)} - (A_0(t,Y) + A_1(t,Y)\widetilde{\eta}_t)\,\mathrm{d}t \right). \end{aligned}$$

Furthermore, for $t \in (T_p, T_{p+1})$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\widetilde{\gamma}_t = \sum_{i=1}^p \sum_{j=1}^p \frac{\mathrm{d}}{\mathrm{d}t}\gamma_t^{(ij)} = -\frac{A_1(t,Y)^2}{\sigma^2} \sum_{i=1}^p \sum_{j=1}^p \sum_{l=1}^n \sum_{k=1}^n \gamma_t^{(ik)} \mathbb{1}_{[T_k,\infty)}(t) \gamma_t^{(lj)} \mathbb{1}_{[T_l,\infty)}(t)$$

$$
\begin{aligned}
&= -\frac{A_1(t,Y)^2}{\sigma^2}\sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p}\sum_{k=1}^{p}\gamma_t^{(ik)}\gamma_t^{(lj)} = -\frac{A_1(t,Y)^2}{\sigma^2}\sum_{j=1}^{p}\sum_{l=1}^{p}\gamma_t^{(lj)}\sum_{i=1}^{p}\sum_{k=1}^{p}\gamma_t^{(ik)} \\
&= -\frac{A_1(t,Y)^2}{\sigma^2}\left(\sum_{j=1}^{p}\sum_{l=1}^{p}\gamma_t^{(lj)}\right)^2 \\
&= -\frac{A_1(t,Y)^2}{\sigma^2}\left(\widetilde{\gamma}_t\right)^2.
\end{aligned}
$$

Since p was arbitrarily chosen, this proves the dynamics of the continuous part of $\widetilde{\eta}$ and $\widetilde{\gamma}$. It remains to show that $\Delta\widetilde{\eta}_{T_i} = \kappa_i$ and $\Delta\widetilde{\gamma}_{T_i} = \chi_i$.

As a first step, we show that for all $1 \le i, j \le n$, for all $t \in [0, T_i \vee T_j]$

$$
\gamma_t^{(ij)} = \begin{cases} 0, & \text{if } i \neq j \\ \chi_i, & \text{if } i = j \end{cases}.
$$

To see this, consider $(\gamma_t^{(1m)}, \dots, \gamma_t^{(m-1\,m)})$ for $1 \le m \le n$, $t \in [0, T_m)$. According to Equation (2.7) this vector is a solution to the following $(m-1)$-dimensional system of ODEs

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t^{(1m)} &= \sum_{l=1}^{m-1}\gamma_t^{(lm)}\mathbb{1}_{[T_l,\infty)}(t)\sum_{l=1}^{m-1}\gamma_t^{(1k)}\mathbb{1}_{[T_k,\infty)}(t), && \gamma_0^{(1m)} = 0, \\
&\vdots && \vdots \\
\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t^{(m-1\,m)} &= \sum_{l=1}^{m-1}\gamma_t^{(lm)}\mathbb{1}_{[T_l,\infty)}(t)\sum_{l=1}^{m-1}\gamma_t^{(m-1\,k)}\mathbb{1}_{[T_k,\infty)}(t), && \gamma_0^{(m-1\,m)} = 0.
\end{aligned}
$$

Hence,

$$
\gamma_t^{(im)} = 0, \quad \forall\, i \in \{1, \dots, m-1\}, \quad \forall\, t \in [0, T_m).
$$

Continuity of $\gamma^{(im)}$, $i \in \{1, \dots, m\}$, (cf. Theorem 2.2) yields the assertion on the whole interval $[0, T_m]$. Finally,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t^{(mm)} = \sum_{l=1}^{m-1}\gamma_t^{(lm)}\mathbb{1}_{[T_l,\infty)}(t)\sum_{k=1}^{m-1}\gamma_t^{(mk)}\mathbb{1}_{[T_k,\infty)}(t) = 0.
$$

Due to the initial condition $\gamma_0^{(mm)} = \chi_m$, we have $\gamma_t^{(mm)} = \chi_m$ for all $t \in [0, T_m]$.

Now, for $t \in [0, T_i)$, it follows

$$
\mathrm{d}\eta_t^{(i)} = \sum_{j=1}^{n}\gamma_t^{(ij)}\mathbb{1}_{[T_j,\infty)}(t)\frac{A_1(t,Y)}{\sigma^2}\,\mathrm{d}\widetilde{Y}_t^{(1)} = \sum_{j=1}^{i-1}\gamma_t^{(ij)}\mathbb{1}_{[T_j,\infty)}(t)\frac{A_1(t,Y)}{\sigma^2}\,\mathrm{d}\widetilde{Y}_t^{(1)} = 0
$$

where $\mathrm{d}\widetilde{Y}_t^{(1)} = \mathrm{d}Y_t^{(1)} - (A_0(t,Y) + A_1(t,Y)\widetilde{\eta}_t)\,\mathrm{d}t$. Hence, for all $i \in \{1,\dots,n\}$, $\eta_{T_i}^{(i)} = \kappa_i$ and

$$\Delta\widetilde{\eta}_{T_i} = \Delta\left(\sum_{j=1}^{n} \eta_{T_i}^{(j)} \mathbb{1}_{[T_j,\infty)}(T_i)\right) = \eta_{T_i}^{(i)} = \kappa_i.$$

This completes the proof of the dynamics of $\widetilde{\eta}$. For $\widetilde{\gamma}$ consider

$$\begin{aligned}\sum_{k=1}^{i+1}\sum_{l=1}^{i+1} \gamma_t^{(kl)} &= \sum_{k=1}^{i}\left(\sum_{l=1}^{i} \gamma_t^{(kl)} + \gamma_t^{(k\,i+1)}\right) + \sum_{l=1}^{i+1} \gamma^{(i+1\,l)} \\ &= \sum_{k=1}^{i}\sum_{l=1}^{i} \gamma_t^{(kl)} + 2\sum_{k=1}^{i} \gamma_t^{(k\,i+1)} + \gamma^{(i+1\,i+1)}.\end{aligned}$$

Hence,

$$\Delta\widetilde{\gamma}_{T_i} = 2\sum_{k=1}^{i-1} \gamma_{T_i}^{(ki)} + \gamma_{T_i}^{(ii)} = \chi_i.$$

□

A.2 Auxiliaries for Chapter 3

Lemma A.1. *Let λ and σ be positive differentiable functions on $[0,T)$ and C such that*

$$C^{-1} \geq \int_0^T \lambda(s)^2\sigma(s)^2\,\mathrm{d}s.$$

Define the function $a : [0,T) \mapsto \mathbb{R}$ by

$$a(t) := \frac{\lambda(t)\sigma(t)^2}{C^{-1} - \int_0^t \lambda(s)^2\sigma(s)^2\,\mathrm{d}s}. \tag{A.1}$$

Then a verifies the following integral equation for all $t \in [0,T)$

$$\frac{a(t)\sigma(t)^{-1}}{\int_0^t a(s)^2\sigma(s)^{-2}\,\mathrm{d}s + C} = \lambda(t)\sigma(t). \tag{A.2}$$

Proof. Due to the choice of C, a is well-defined with $a(t) \in (0,\infty)$, for all $t \in [0,T)$. Together with $\lambda(t) > 0$, $\sigma(t) > 0$, for all $t \in [0,T)$, we get

$$(A.2) \quad \Leftrightarrow \quad \int_0^t a(s)^2\sigma(s)^{-2}\,\mathrm{d}s + C = a(t)\sigma(t)^{-2}\lambda(t)^{-1}.$$

Now, let

$$b(t) := a(t)\sigma(t)^{-2}\lambda(t)^{-1}, \quad t \in [0, T).$$

Since a and λ are differentiable functions this also holds true for b. Hence,

$$\begin{aligned} (A.2) \quad &\Leftrightarrow \quad b(t) = \int_0^t \lambda(s)^2\sigma(s)^2 b(s)^2 \,\mathrm{d}s + C \\ &\Leftrightarrow \quad b'(t) = \lambda(t)^2\sigma(t)^2 b(t)^2, \quad b(0) = C. \end{aligned}$$

This special Bernoulli type ODE has the solution

$$b(t) = \left(C^{-1} - \int_0^t \lambda(s)^2\sigma(s)^2 \,\mathrm{d}s \right)^{-1}$$

which is equivalent to

$$a(t) = \frac{\lambda(t)\sigma(t)^2}{C^{-1} - \int_0^t \lambda(s)^2\sigma(s)^2 \,\mathrm{d}s}.$$

□

Lemma A.2. *Given the assumptions of Theorem 3.24. For $t \in [0, T)$ let*

$$k(t) := \lambda(t)(a(t) - m(t)). \tag{A.3}$$

Then $\lim_{t \to T} \int_0^t k(s) \,\mathrm{d}s = \infty$.

Proof. Since λ and a are continuous and positive and m is bounded on $[0, T']$, for all $T' \in [0, T)$,

$$k(t) = \lambda(t)a(t) \left(1 - \frac{m(t)}{a(t)} \right), \quad t \in [0, T),$$

is bounded on $[0, T']$, for all $T' \in [0, T)$. Hence, it suffices to show that there exists a $T' \in (0, T)$ such that

(1) $\displaystyle\lim_{t \to T} \int_{T'}^t \lambda(s)a(s) \,\mathrm{d}s = \infty$,

(2) $\displaystyle\frac{m(t)}{a(t)} < 1 - \epsilon$, $\epsilon > 0$, for all $t \in (T', T)$.

1. By definition of a,

$$\lambda(t)a(t) = \frac{\lambda(t)^2\sigma(t)^2}{1 - \int_0^t \lambda(s)^2\sigma(s)^2 \,\mathrm{d}s}$$

with λ such that for all $t \in [0,T)$

$$1-\int_0^t \lambda(s)^2\sigma(s)^2\,\mathrm{d}s > 0 \quad \text{and} \quad 1-\int_0^T \lambda(s)^2\sigma(s)^2\,\mathrm{d}s = 0.$$

Now, for an arbitrary but fixed $T' \in (0,T)$ we have for all $t \in (T',T)$

$$\int_{T'}^t \frac{\lambda(s)^2\sigma(s)^2}{1-\int_0^s \lambda(u)^2\sigma(u)^2\,\mathrm{d}u}\,\mathrm{d}s = -\log\left(1-\int_0^t \lambda(s)^2\sigma(s)^2\,\mathrm{d}s\right)+\log\left(1-\int_0^{T'} \lambda(s)^2\sigma(s)^2\,\mathrm{d}s\right).$$

In particular,

$$\int_{T'}^t \lambda(s)a(s)\,\mathrm{d}s \xrightarrow{t\to T} \infty.$$

2. If m is bounded, it follows that $\lambda(T) > 0$. In particular, $\lim_{t\to T} a(t) = \infty$. Hence, $\lim_{t\to T} m(t)/a(t) = 0$. If m is unbounded, (2) follows from the assumptions of Theorem 3.24.

□

A.3 Auxiliaries for Chapter 4

Lemma A.3. *Let $\sigma \in \mathbb{R}$, λ be a positive piecewise differentiable càdlàg function on $[0,T)$, with jumps in $t_i, i \in \{1,2,\dots,n\}, n \in \mathbb{N}$, with $0 = t_0 < t_1 < t_2 < ... < t_n < T < \infty$. Furthermore, let $\chi_i > 0$, $i \in \{0,1,\dots,n\}$, such that for all $t \in [0,T)$*

$$\sum_{i=0}^n \chi_i \mathbb{1}_{[t_i,T)}(t) - \int_0^t \lambda(s)^2\sigma^2\,\mathrm{d}s > 0.$$

Then the function $a : [0,T) \mapsto \mathbb{R}$, defined by

$$a(t) := \frac{\lambda(t)\sigma^2}{\sum_{i=0}^n \chi_i \mathbb{1}_{[t_i,T)}(t) - \int_0^t \lambda(s)^2\sigma^2\,\mathrm{d}s}, \tag{A.4}$$

verifies the following integral equation

$$\frac{a(t)\sigma^{-2}}{\int_0^t a(s)^2\sigma^{-2}\,\mathrm{d}s + \sum_{i=0}^n \widetilde{\chi}_i \mathbb{1}_{[t_i,T)}(t)} = \lambda(t), \quad t \in [0,T), \tag{A.5}$$

where

$$\widetilde{\chi}_0 = \chi_0^{-1} \quad \text{and} \quad \widetilde{\chi}_i = -\frac{\chi_i}{\left(\frac{\lambda(t_i-)\sigma^2}{a(t_i-)}+\chi_i\right)\frac{\lambda(t_i-)\sigma^2}{a(t_i-)}}. \tag{A.6}$$

Proof. The assertion follows from induction and Lemma A.1. More detailed, for $t = 0$ we have by definition, i.e. Equation (A.4),

$$a(0) = \frac{\lambda(0)\sigma^2}{\chi_0}$$

and by Equation (A.5)

$$\frac{a(0)\sigma^{-2}}{\widetilde{\chi}_0} = \lambda(0).$$

The above two equations are equivalent if $\widetilde{\chi}_0 = \chi_0^{-1}$. Now, for $t \in (0, t_1)$ we get (A.5) by Lemma A.1. Assume now that (A.5) holds for all $t \in [0, t_i)$, $i \geq 1$. In particular, we have

$$\frac{a(t_i-)\sigma^{-2}}{\int_0^{t_i} a(s)^2\sigma^{-2}\,\mathrm{d}s + \sum_{j=0}^{i-1}\widetilde{\chi}_j} = \lambda(t_i-) \quad \Leftrightarrow \quad \frac{a(t_i-)}{\lambda(t_i-)\sigma^2} = \int_0^{t_i} a(s)^2\sigma^{-2}\,\mathrm{d}s + \sum_{j=0}^{i-1}\widetilde{\chi}_j.$$

In order to satisfy (A.5) for t_i, $\widetilde{\chi}_i$ has to be chosen such that

$$\widetilde{\chi}_i = \frac{a(t_i)}{\lambda(t_i)\sigma^2} - \frac{a(t_i-)}{\lambda(t_i-)\sigma^2}.$$

Observe that due to (A.4)

$$\chi_i = \frac{\lambda(t_i)\sigma^2}{a(t_i)} - \frac{\lambda(t_i-)\sigma^2}{a(t_i-)}.$$

Then simple calculations and (A.6) show

$$\frac{a(t_i)}{\lambda(t_i)\sigma^2} - \frac{a(t_i-)}{\lambda(t_i-)\sigma^2} = \frac{\frac{\lambda(t_i-)\sigma^2}{a(t_i-)} - \frac{\lambda(t_i)\sigma^2}{a(t_i)}}{\frac{\lambda(t_i)\sigma^2}{a(t_i)}\frac{\lambda(t_i-)\sigma^2}{a(t_i-)}} = -\frac{\chi_i}{\left(\frac{\lambda(t_i-)\sigma^2}{a(t_i-)} + \chi_i\right)\frac{\lambda(t_i-)\sigma^2}{a(t_i-)}} = \widetilde{\chi}_i.$$

Hence, (A.5) holds for all $t \in [0, t_i]$. Again with the help of Lemma A.1, (A.5) holds for all $t \in [0, t_{i+1})$. This proves the assertion by induction. □

List of abbreviations

a.s.	almost surely
càdlàg	right continuous with left limits (continu à droite, limites à gauche)
w.r.t.	with respect to
HJB	Hamilton-Jacobi-Bellman
ODE	ordinary differential equation
PDE	partial differential equation
PIDE	partial integro-differential equation
SDE	stochastic differential equation

List of symbols

List of symbols: general model

α	instantaneous insider trading rate (absolutely continuous θ), 12
θ	insider strategy, 10
P	market price process of the risky asset, 9
T	trading horizon, 9
V	fundamental value process of the risky asset, 9
W_T^θ	terminal wealth of the insider, according to the strategy θ, 10
X	demand process of the noise traders, 10
Y	total order process, 9
$\mathcal{P}$	admissible pricing rules, 12
$\mathcal{S}$	admissible insider strategies, 11
$\mathcal{U}$	utility function (of the insider), 11
$\mathcal{F}_t^{\mathcal{I}}$	σ-algebra, insider's information at time t, 10
$\mathcal{F}_t^{\mathcal{M}}$	σ-algebra, market makers' information at time t, 10
$\mathbb{F}^{\mathcal{I}}$	filtration of the insider, $:= (\mathcal{F}_t^{\mathcal{I}})_{t\geq 0}$, 10
$\mathbb{F}^{\mathcal{M}}$	filtration of the market makers, $:= (\mathcal{F}_t^{\mathcal{M}})_{t\geq 0}$, 10

List of symbols: additional symbols in Chapter 3

$-\beta$	degree of risk aversion, 26
η_t	conditional mean of Z with respect to $\mathcal{F}_t^{\mathcal{M}}$, 64
γ_t	conditional variance of Z with respect to $\mathcal{F}_t^{\mathcal{M}}$, 64
$\lambda(t)$	price pressure, 26
$\mu(t, \widetilde{Y}_t, V)$	noise drift, informed part of X, 27
$\sigma(t)$	noise volatility, 25
ξ	weighted (according to λ) total uninformed order process, 26
B	Brownian motion, noise part of X, 25
$H(t, \widetilde{Y}_t)$	market price according to the pricing function H depending on the weighted total order, 26
h	value function of the risky asset, 25
I	information process (stochastic filtering), $\mathbb{F}^{\mathcal{M}}$-Brownian motion, 60

J	value function of the insider's optimisation problem for a general utility function, 28
J^0	value function of the insider's optimisation problem, risk neutral case, 28
J^β	value function of the insider's optimisation problem, exponential utility with parameter β, 32
$J^{0,t}$	risk neutral value function of an auxiliary optimisation problem with trading horizon t, 49
$m(t)$	noise drift intensity, 36
$y^*(t,V)$	implicit order volume of the fundamental value, i.e. $H(t,y^*(t,V)) := V$, 26
$\bar{Y}$	volatility adjusted total order process, 59
$\widetilde{Y}$	weighted total order process, according to λ, 26
Z	transformed fundamental value, $:= h^{-1}(V)$, 25
$\mathcal{S}(H,\lambda)$	set of (H,λ) admissible insider strategies, 27
$\mathcal{V}$	image of the value function h, $:= h(\mathbb{R})$, 26
$\mathbb{E}^v$	expectation with respect to $\mathbb{P}^v$, 28
$\mathbb{P}^v$	probability measure under which $V = v$, 28
$\mathbb{P}^\beta$	$\mathbb{P}$ equivalent probability measure induced by risk aversion (with degree $-\beta$), 57

List of symbols: additional symbols in Chapter 4

χ_i	conditional variance of ΔZ_{T_i} given ΔX_{T_i}, 79
η_t	conditional mean of Z_t with respect to $\mathcal{F}_t^{\mathcal{M}}$, 102
γ_t	conditional variance of Z_t with respect to $\mathcal{F}_t^{\mathcal{M}}$, 101
κ_i	conditional mean of ΔZ_{T_i} given ΔX_{T_i}, 79
Λ	important process used for determination of equilibrium, 80
λ	price pressure process for continuous changes in the order process, 80
$\mu(t,U_t)$	noise drift, informed part of X, 82
$\nu(\mathrm{d}\zeta)$	Lévy measure of $N(\mathrm{d}t,\mathrm{d}\zeta)$, 78
φ	price pressure process for jumps in the order process, 80
σ	volatility of continuous noise, 75
ξ	weighted (according to λ and φ) total uninformed order, 81
B	Brownian motion, continuous noise part of X, 75
$H(t,\widetilde{Y}_t,\bar{U}_t)$	market price according to the pricing function H, 81
I	continuous information process (stochastic filtering), $\mathbb{F}^{\mathcal{M}}$-Brownian motion, 94
J	value function of the insider's optimisation problem, 83
$m(t,\bar{U}_t)$	noise drift intensity, 85
$N(\mathrm{d}t,\mathrm{d}\zeta)$	poisson random measure on $\mathbb{R}^3$, drives the underlying jump process, 78
$\bar{N}(\mathrm{d}t,\mathrm{d}\zeta)$	compensated poisson random measure, $:=N(\mathrm{d}t,\mathrm{d}\zeta) - \nu(\mathrm{d}\zeta)\mathrm{d}t$, 79

N_t	$:= N(t, \mathbb{R}^3 \setminus \{0\})$, poisson process, 79
S	model horizon determining process (market cooling), 76
$T_i, i \in \mathbb{N}_0$	jump times of N, 79
U_t	$:= (\Lambda_t, \lambda_t, S_t, Z_t, \widetilde{Y}_t)$, 81
$\bar{U}_t$	$:= (\Lambda_t, \lambda_t, S_t)$, 81
U_t^ξ	$:= (\Lambda_t, \lambda_t, S_t, Z_t, \xi_t)$, 81
X^d	discontinuous part of X, shot noise, 75
$y^*(t, U_t)$	implicit order volume of the fundamental value, i.e. $H(t, y^*(t, U_t), \bar{U}_t) := V_t$, 81
$\widetilde{Y}$	weighted total order process, according to λ and φ, 80
Z	transformed fundamental value process of the risky asset, 75
$\bar{\mathcal{D}}$	state space of $(t \wedge T, \bar{U}_{t\wedge T})_{t\geq 0}$, 81
$\mathcal{D}^*$	state space of $(t \wedge T, U^\xi_{t\wedge T})_{t\geq 0}$, 81
$\mathcal{D}$	$:= \mathcal{D}^* \cap (\mathbb{R}^3 \times (0, \infty) \times \mathbb{R}^2)$, 83
$\mathcal{S}(H, \lambda, \varphi)$	set of (H, λ, φ) admissible insider strategies, 90

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Lebenslauf