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Proper Actions, Nonlinearity and Homotopy Theory

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Rue nia ni binni peru runiná ni naa.
Ndaani xquidxe xtubi ri'enecabe naa,
xiindi gune chupa dxiiña gapa guidxiiche;
ngaca anna de dxiqu' ñaca' naquiite,
yanna ti bacaanda' reedachite naa...

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Chapter 1

Introduction

Stable Homotopy Theory deals with stark properties of topological spaces which are preserved under a controlled ascent of dimensions. Far from its original goals and initiating questions (like Freudenthal's first remarks on stability), stable homotopy theory became a dominant discourse in some branches of pure mthematics today. Stable Homotopy Theory has been after decades a succesful deliverer of conceptual frameworks for crucial developings of, for instance, algebraic K-theory, giving at the same time powerful and explicit computation techniques like spectral sequences for the gathering of evidence.

Writing a thesis about homotopy theory from the viewpoint of a school which is more devoted to K-theory, as the Topology Group in Münster has to be oriented towards applications and open to other disciplines. This is a constant interest ofthe author of this work.

In many fields of modern research, the notion of symmetry plays a crucial role. Regularity, periodicity and recurrence are the intuitions which lie behind the modern notion of a group action. This works deals mainly with the interaction of symmetry and the methods of stable homotopy theory. It is motivated by the fact that the highly developed Equivariant Stable Homotopy Theory, as it is exposed for example in [May96], restricts itself to actions of finite or compact Lie groups. This problem lies in the extreme dependence of their methods and defnitions to the representation theory of the acting groups. The aim of this work is precisely to enlight the dialectic of this particularly strong interaction in finite group actions and independence in more general contexts, which is for the first time introduced in this thesis. By the end of 2009, relativelly parallel to the finishing time of this work, the paper [Kit09] adressed a definition of equivariant K-theory for proper actions of other class of noncompact groups, not considered here. We follow this developement very closely and see in these developements a confirmation of the need of a systematical developement of equivariant homotopy theory in the context of infinite groups.

This work should be understood as the first attempt to bring homotopy theory to the contexts where it is needed, namely, where non-compact groups act properly on possibly non-compact spaces. This is the main theoretical goal here. As the first test for the relevance and accuracy of the proposals for generalized definitions, a fundamental result in homotopy theory is brought into new realms. That is the 1984 proven Segal conjecture. The Segal conjecture is a statement which was made for the first time in 1970 in the International

Congress of Mathematicians and is considered to be the capital achievement of the stable homotopy theory of the first part of the eighth decade of last century. It involves a ring, the Burnside ring, and equivariant homotopy groups of a classyfying space. Both notions are extended to non compact groups in this thesis, and so are severals results in this directions in the developement of Chapter 4. Also, the frontiers of these approaches and results are recogninsed in several counterexamples in the body of the text. So, this work provides the homotopy theoretical community with extensions of their notions to the context of proper actions, as well as the extension of a highlighting theorem, which occupied by more than a decade many of the best topologists. This is an aknowledgement to a vaste field of research which the author of this thesis just begins to see and appreciate in all its complexity.

There is a third small spring flooding the lands of homotopy theory. One which is very often overseen by the leading researchers in this field. That is, applications and active contributions from other branches of mathematics. This plays also a role in the developement of our ideas. Notions coming from index theory, operator algebras, functional analysis and gauge topology are presented as an example of what stable homotopy theory can do out of their artificially delimited terrains. But they also contribute actively to its developement. The best example is the theory of nonlinear perturbations of fredholm operators. This is a topic motivated by the qualitative analysis of partial differential equations, and which did not gained alot of interest among leading homotopy theorists. A second aim of this work is to reverse this, providing a modest application of the methods of homotopy theory in these index theoretical contexts. The author of this work thanks the Gauge theoretical community in Germany, which was always *hilfsbereit*, specially Markus Szymik, Stefan Bauer and Raphael Zentner, who never hesitate to share ideas to achieve this program. Also, an active dialogue with the differential geometry and noncommutative geometry groups in Münster is behind these developements.

The author wants to express his deep gratitude to all institutions which contributed to the succes of this project in Münster. The DAAD-Conacyt scholarship was a crucial aspect for the developement of the projects here. The author expresses his concern about the future of this program in these hard days for his homeland, Mexico. He is also convinced that public investment in science and technology made this effort succesful, and this should be a priority for developing countries.

The *Westfälische Wilhelms-Universität Münster* has to be gratefully recognised for his hospitality and cooperation, reflected in several succesful programs on which the author of this work participated, in particular the people involved in the *Graduiertenkolleg Analytische Topologie und Metageometrie*, and the *Sonderforschungsbereich Geometrische Strukturen in der Mathematik*. In a more broad sense, the support of the university in Münster was seen in *die Brücke* and the work of autonomous organs inside the *Allgemeine Studierendenausschuss.* Other institutions which contributed to this work in some exent include the Max Planck Institute, the Oberwolfach Mathematical Research Institute, the Studientsttiftung des Deutschen Volkes, and the National University of Mexico, with particular engagement of Prof. Carlos Prieto.

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Chapter 2

Infinite discrete groups

In this work we shall extend clasical definitions and results of equivariant stable homotopy theory to the context of proper actions of discrete groups. The first approach we propose is given in terms of some infinite loop spaces. In order to construct them, we make use of Segal's infinite loop space machine, Γ-spaces. Although Γ -spaces are certainly limited, in that they define only connective spectra, this is no problem for us, since we are dealing with a flexible generalization of the sphere spectrum, the canonical example of a connective spectrum.

We recall briefly the definition of Γ-spaces, in the sense of Segal. For the expert, all Γ-spaces we deal with are called in the modern literature *special* Γ-spaces. We also use Segal's original notation since it is technically more convenient for our purposes.

A Γ-space can be understood from many points of view, the most useful for us being a technically easier to handle substitute for a connective spectrum. Or, more precisely, of its infinite loop space. We follow the classical definition, [Seg74].

Definition 1. Let Γ be the category whose objects are finite sets and where a morphism $F : S \to T$ is a function $S \to \mathcal{P}(T)$ such that $F(s) \cap F(s^{'}) = \phi$ for $s \neq s^{'}$. Denote by $\overline{n}$ the object of Γ which consists of $\{1, \ldots, n\}$. A Γ-space is a contravariant functor $\Gamma \to$ Spaces such that:

1. $A(0) = \{*\}$
2. For each $\overline{n}$, the map $A(\overline{n}) \to \Pi_{i=1}^{n} A_{\overline{1}}$ induced by the inclusions $\kappa_i : \overline{1} \longrightarrow \overline{n}$, $\overline{1} \xrightarrow{\kappa_i} \overline{i}$ is a homotopy equivalence.

For a finite set S denote by Set_S the category of *S-partitioned* finite sets. That is, its objects are finite subsets X of $\mathbb{N}^\infty = \coprod_{n \in \mathbb{N}} \mathbb{N}$ together with a decomposition of the form $X = \coprod\limits_{s \in S} X_s$, where $X_s \subset \mathbb{N}$ is a finite set. A morphism is an isomorphism of sets which preserves the decomposition. Let $\theta : S \to T$ be a morphism in Γ. It induces a functor $\mathrm{Set}_T \to \mathrm{Set}_S$ by sending $\coprod_{t \in T} X_t$ to $Y = \coprod_{s \in S} Y_s$, where $Y_s = \coprod_{t \in \theta(s)} X_s$, giving a functor $\Gamma^{op} \to \mathrm{Set}$.

Remark 1. Indeed, the previous functor is an instance of what Segal calls a Γ-category. A Γ-category is a functor defined in the cetegory Cat of small categories $F : \Gamma^{op} \to \mathrm{Cat}$ such that $F(0)$ is equivalent to the category of one object and one morphism, and $F(\overline{n})$ is equivalent to the category $\underbrace{\mathcal{C}(1) \times \ldots \times \mathcal{C}(1)}_{\text{n−times}}$, the equivalence being induced by the morphisms $i_k : \overline{1} \to \overline{n}$. Our main example is the following. Define for a symmetric monoidal category $\mathcal{C}, \coprod$ the category $\mathcal{C}(S)$ whose objects consist of sums indexed by the finite set S and a morphism is an isomorphism which preserves the indexing. Then $S \mapsto \mathcal{C}(S)$ is such a functor.

Our interest on Γ-spaces comes is focused around their role as infinite loop spaces, that is, representing objects for cohomology theories. The main result concerning this is the following proposition due to Graeme Segal, which is called the group completion theorem.

Proposition 1. Let A be a Γ-space such that $\pi_0(A(\overline{1}))$ contains a cofinal free abelian monoid. Denote by K_A the contravariant functor defined in the category of compact spaces by $X \mapsto [X, \Omega\mathrm{B}A]$. Then the transformation $[X, A] \to K_A$, induced as ajoint to the map $\Sigma A(1) \to \mathrm{B}A(1)$ is universal among transformations $\Theta : [\quad, A] \to F$, where F is a representable abelian-group valued functor on compact spaces and Θ is a transformation of monoid valued functors.

Proof. See [Seg74], proposition 4.1 □

In the following discussion G denotes a discrete group. Let $\mathcal{E}(G)$ be the transport category on the group G, that is, the category whose objects are the elements of the group G, with exactly a morphism between each two elements. Note that G acts on $\mathcal{E}(G)$. The action of a on the object x gives the object ax. It sends the morphism $l_{g,x} : g \to x$ to the morphism $l_{ag,ax}$.

Denote by $\mathcal{B}(G)$ the category consisting of one object and morphism set the elements of G. For a subgroup H of G, the *transport category* $\mathcal{E}(G)/H$ is the category whose objects are elements in G/H and where a morphism is a traslation $l_a : gH \to agH$. The reason of the notation is the following technical result

Lemma 1. Let H be a subgroup. Then, the following holds

1. $\mathcal{E}(G)/H$ is naturally equivalent to $\mathcal{B}(H)$.

2. $\mid \mathcal{E}G \mid \approx$ EG, where $\mid \quad \mid$ stays for the geometric realization and EG stays for the total space of a universal principal G-bundle.

3. For every small category $\mathcal{C}$, there is an action of G in $\mathrm{Fun}(\mathcal{E}(G), \mathcal{C})$. Moreover, for every subgroup H, the H-fixed points of the action form a subcategory which is equivalent to the category $\mathrm{Fun}(\mathcal{E}(G)/H, \mathcal{C})$.

Proof. 1. Define the functor $F : \mathcal{E}(G)/H \to \mathcal{B}(H)$ as follows. It is constant on objects with value $*$. To describe its behavior in morphisms, let $g_\alpha H$ be a partition of the set G/H. Choose set isomorphisms $\alpha : g_\alpha H \to eH$. Given a morphism $l_a : gH \to agH$, note that the morphism $eH \to eH$ given by $eH \underset{\alpha_g}{\to} gH \underset{l_a}{\to} agH \underset{\alpha_{ag}}{\to} eH$ is given by translation by some $h_{g,a} \in H$.

The inclusion functor $E : eH \to \mathcal{E}(G)/H$ satisfies that $E \circ F$ and $F \circ E$ are naturally equivalent to the identity functors.

2. G acts freely and simplicially over EG. On the other side, the map $G \to e$ and the inclusion $e \to G$ induce a pair of adjoint functors, proving that $\mid \mathcal{E}G \mid$ is contractible.

3. The first part is clear. Define the functor $F_e : (\mathcal{E}(G)/H, \mathcal{C}) \to \mathrm{Fun}(\mathcal{E}(G), \mathcal{C})^H$ to be the functor which assigns to a functor $f : \mathcal{E}G/H \to C$ the constant functor with value $f(eH)$ on objects,and where a morphism represented by the pair of elements gH, $g^{'}H$ is asigned to the map batween a pair of representatives $g, g^{'}$. This functor lies in the H-fixed point set. Let $G : \mathrm{fun}(\mathcal{E}G, \mathcal{C})^H \to \mathrm{fun}(\mathcal{E}G/H, \mathcal{C})$ be the functor which is determined on objects by the induced map of sets. There is a natural transformation between the compositions of F and G and the identities.

□

Definition 2. Let G be a discrete group. The Γ-space of G-sets is the space defined by the geometrical realization of the category of G-sets. In symbols,

$$\underline{G\text{−Set}}(S) = \mid \mathrm{Func}(\mathcal{E}(G), \mathrm{Set}_S) \mid$$

The following result justifies our notation for $\underline{G\text{−Set}}$

Proposition 2. For every subgroup H, there is a homotopy equivalence

$$\mid \underline{G\text{−Set}}\ (\overline{n}) \mid^H \simeq \underbrace{\mid \underline{G\text{−Set}}\ \mid^H \times \ldots \times \mid \underline{G\text{−Set}}\ \mid^H}_{\text{n−times}}$$

Proof. After point 3. of previous proposition, one interprets the left side as the nerve of the category which consists of functors from $\mathcal{E}(G)$ into n-tuple disjoint unions of finite sets in the category of functors $\mathcal{E}(G)/H \to$ Set. On the other hand, the right side can be handled as the nerve of the category consisting of functors from $\mathcal{E}(G)/H$ into n-tuple disjoint unions of finite sets in Set, which is contained in the previous one. Since every element on the first category can be decomposed as a sum of elements in the second one, there is a equivalence of categories, giving the claimed homotopy equivalence. □

Definition 3. A Γ-*space* with an action of the group G is a Γ-space A consisting of G-spaces such that the fixed point sets $A(\bar{n})^H$ are Γ-spaces for all n.

Recall that Γ spaces have a geometric realization as simplicial sets. Precisely, denote by BG−Set the Γ-space which is given by $S \mapsto \mid T \mapsto A(S \times T) \mid$, where the bars denote realization as simpicial set.

Definition 4. The *equivariant infinite loop space* is the group completion of G−Set. In symbols,

$$Q_G = \Omega \mathrm{B}G\text{−Set}$$

Our reason for the nomenclature is the fact that for every finite subgroup H, the H-fixed points of Q_G^H classify H-equivariant stable cohomotopy. The following discussion is our first try to make precise this claim.

By means of the category equivalence $\mathcal{E}(G)/H \leftrightarrow \mathcal{B}(H)$ one can handle a functor $\mathcal{E}(G)/H \longrightarrow$ Set as a *finite H-set.* For the image of the unique object in $\mathcal{B}H$ under the functor in the category of finite sets yields a finite set of points $S \subset \mathbb{N}^\infty$ provided with a group homomorphism $H \to \mathrm{Iso}_{\mathrm{Set}}(S)$. Denote by

Σ_S the group of automorphisms (= H-equivariant permutations) of such an object. Then the realization of the Γ-space $\underline{G\!-\!\mathrm{Set}}^H$ can be identified with the classifying space of the topological monoid consisting of the classifying spaces of the group of automorphisms of such objects. In symbols,

$$\coprod_{\text{H }-\text{Isomorphism classes of S}} \mathrm{B}\Sigma_S$$

Where the multiplication is defined by means of the partial assignments $\mathrm{B}\Sigma_S \times \mathrm{B}\Sigma_{S'} \longrightarrow \mathrm{B}\Sigma_{S\coprod S'}$ induced by the group homomorphisms $\Sigma_S \times \Sigma_{S'} \longrightarrow \Sigma_{S\coprod S'}$. Under this interpretation, the following result is immediate and amounts to an equivariant version of the Barrat-Priddy-Quillen-Segal theorem.

Proposition 3. Let H be a finite subgroup. There is a homotopy equivalence

$$Q_G^H \simeq \Omega\mathrm{B}(\coprod_{\text{H}-\text{Isomorphism classes of S}} \Sigma_S)$$

where on the right side B stands for the classifying space of the monoid.

We recall the following classical definitions. See [tD87], [Hau77] for more details. Let H be a finite group. A family of representations Ξ closed under sums is called *cofinal* if it contains a representative of each class of irreducible representations. Let S^V denote the one-point-compactification of the H-module V. Put $\Sigma^V X = X_* \wedge S^V$. If X is any space denote by $\Omega^V(X)$ the space of continuous, equivariant pointed maps $\mathrm{Map}_G(S^V, X_+)$. If X is a H-space, denote by $\Omega_H^V(X)$ the subspace of the H-equivariant maps. Let Ξ be a cofinal system consisting in the *set* of irreducible representations which have as underlying space $\mathbb{C}^n$ for some $n \geq 0$, ordered with direct sums. Define $\{S, S\} = \mathrm{colim}_\Xi \Omega^V S^V$.

Proposition 4. There is a H-equivariant homotopy equivalence $Q_G^H \simeq \{S, S\}_*^H$

Proof. Since the spaces in consideration have the H-equivariant homotopy type of an H-CW complex, one has to prove that the K-fixed points have the same weak homotopy groups for every subgroup $K \leq H$. Hence, we get a situation of the form

$$\pi_r \Omega\mathrm{B}(\coprod_S \mathrm{B}\Sigma_S) \longrightarrow \pi_r\{S, S\}^K \tag{2.1}$$

where S runs over the isomorphism classes of finite K-sets. Note that the right side admits a splitting [Hau77], [Seg71]. Precisely, denote by $W_{K',K} = N_{K',K}/K'$ the Weyl group of a subgroup $K' \leq K$. Let $ccs(G)$ be the set of conjugacy classes of subgroups of G. Then the right side of 2.1 is isomorphic to

$$\bigoplus_{K' \in \mathrm{ccs} K} \pi_r^{\mathrm{st}}(\mathrm{B}W_{K',K})$$

Our task now is to conveniently decompose the left side. Note for this that a finite K-set admits a decomposition into irreducible orbits $S = \coprod n_i K/K_i$, where n_i stands for the n_i-tuple disjoint union of the corresponding orbit. This allows a description of the automorphism group Σ_S as $\Pi_i W K_i^{n_i} \int \Sigma_{n_i}$. This group acts freely on the space $\coprod_{k,K} \mathrm{E}\Sigma_k \times \mathrm{E}W_{K'}^k$, the quotient being $\coprod_{k,K} \mathrm{E}\Sigma_\mathrm{n} \underset{\Sigma_\mathrm{n}}{\times} \mathrm{BW}_{\mathrm{K}'}^{\mathrm{n}}$. Hence, we can identify the disjoint union of classifying spaces with the product

$\underset{K'}{\Pi} \coprod_{n\in\mathbb{N}} \mathrm{E\Sigma_n} \underset{\Sigma_n}{\times} \mathrm{BW_{K'}}^{\mathrm{n}}$. Finally, there exists for any connected, well pointed space a homotopy equivalence [Seg73], [Sch07], theorem 12.

$$\mathrm{B}(\coprod_{n} \mathrm{E\Sigma_n} \underset{\Sigma_n}{\times} \mathrm{X^n}) \approx \Omega^{\infty-1}\Sigma^{\infty}\mathrm{X}$$

hence, we get a homotopy equivalence $\tau : \Omega\mathrm{B}(\coprod_{n\in\mathbb{N}} \mathrm{E\Sigma_n} \underset{\Sigma_n}{\times} \mathrm{BW_{K'}}^{\mathrm{n}}) \longrightarrow \mathrm{Q}(\mathrm{BW_{K'}})$. Putting all this together,

$$\pi_r\Omega\mathrm{B}(\coprod_{S\in\mathrm{K-sets}} \mathrm{B}\Sigma_S) \cong \prod_{K'\in\mathrm{ccs}K} \pi_r^{\mathrm{st}}(\mathrm{B}W_{K'}) \cong \pi_r\{S,S\}_K$$

□

2.1 Comparison with the approach of Lück

In [Lüc05a], a geometrical definition for equivariant stable cohomotopy is proposed in the context of proper actions of discrete groups on finite G-CW complexes. We shall extend this construction for all G-CW complexes by means of the infinite loop spaces constructed in the previous section. Let us recal briefly the construction of Lück.

An equivariant (real) vector bundle over a proper G-CW complex X is a (real) vector bundle $\xi : E \longrightarrow X$ such that the *traslations* $l_g : E \to E$ are fiberwise linear isomorphisms. A *map* of real vector bundles from $\xi_0 : E_0 \to X_0$ to $\xi_1 : E_1 \to X_1$ is a pair of G-equivariant maps $\bar{f} : E_0 \to E_1$, $f : X_0 \to X_1$ such that $\xi_1 \circ \bar{f} = f \circ \xi_0$ with the property that $\bar{f}$ is fiberwise a linear map. From a real vector bundle we form the *associated sphere bundle* by means of a one-point compactification process on every fiber. Precisely, we have

Definition 5. Let $\xi : E \to X$ be a real equivariant vector bundle. The associated *sphere bundle*, $S^{\xi} \to X$ is the locally trivial G-bundle whose fiber over $x \in X$ is S^{E_x}.

Given a pair of sphere bundles ξ, μ, we form the G-bundles over X whose fiber are given by the wedge, respectively the smash product of the fibers. We denote them by $S^{\xi} \vee_X S^{\mu}$, $S^{\xi} \wedge_X S^{\mu}$.

Fix an equivariant, proper G-CW complex. Form the category $\mathrm{SPHB}^G(X)$ having as objects the G-sphere bundles over X. A morphism from $\xi : E \to X$ to $\mu : F \to X$ is a bundle map $S^{\xi} \to S^{\mu}$ covering the identity in X, which preserves fiberwise the basic points. A homotopy between the morphisms u_0, u_1 is a G-bundle map $h : S^{\xi} \times [0,1] \to S^{\mu}$ from the bundle $S^{\xi} \times [0,1] \to [0,1] \times X$ to the bundle S^{μ} covering the projection $X \times [0,1] \longrightarrow X$ and preserving the base points on every fiber such that its restriction to $X \times \{i\}$ is u_i for $i = 0, 1$. Let $\underline{\mathbb{R}^n}$ be the trivial vector bundle over X, which is furnished with the trivial action of G. Two morphisms of the form

$$u_i : S^{\xi_i \oplus \underline{\mathbb{R}^{k_i}}} \to S^{\xi_i \oplus \underline{\mathbb{R}^{k_i+n}}}$$

are said to be equivalent if there are objects μ_i in $\mathrm{SPHB}^G(X)$ and an isomorphism of vector bundles $\nu : \mu_0 \oplus \xi_0 \cong \mu_1 \oplus \xi_1$ such that the following diagram of morphisms in $\mathrm{SPHB}^G(X)$ commutes up to homotopy

$$\begin{tikzcd}
S^{\mu_0\oplus\underline{\mathbb{R}^{k_1}}}\wedge_X S^{\xi_0\oplus\underline{\mathbb{R}^{k_0}}} \arrow[r, "\mathrm{id}\wedge_X \mathrm{u}_0"] \arrow[d, "\sigma_1"'] & S^{\mu_0\oplus\underline{\mathbb{R}^{k_1}}}\wedge_X S^{\xi_0\oplus\underline{\mathbb{R}^{k_0+n}}} \arrow[d, "\sigma_2"] \\
S^{\mu_0\oplus\xi_0\oplus\underline{\mathbb{R}^{k_0+k_1}}} \arrow[d, "S^{\nu\oplus\mathrm{id}}"'] & S^{\mu_0\oplus\xi_0\oplus\underline{\mathbb{R}^{k_0+k_1+n}}} \arrow[d, "S^{\nu\oplus\mathrm{id}}"] \\
S^{\mu_1\oplus\xi_1\oplus\underline{\mathbb{R}^{k_0+k_1}}} \arrow[d, "\sigma_3"'] & S^{\mu_1\oplus\xi_1\oplus\underline{\mathbb{R}^{k_0+k_1+n}}} \arrow[d, "\sigma_4"] \\
S^{\mu_1\oplus\underline{\mathbb{R}^{k_0}}}\wedge_X S^{\xi_1\oplus\underline{\mathbb{R}^{k_1}}} \arrow[r, "\mathrm{id}\wedge_X u_1"'] & S^{\mu_1\oplus\underline{\mathbb{R}^{k_0}}}\wedge_X S^{\xi_1\oplus\underline{\mathbb{R}^{k_1+n}}}
\end{tikzcd}$$

where the isomorphisms σ_i are determined by the fiberwise defined homeomorphism $S^{V\oplus W}\approx S^V\wedge S^W$ and the associativity of smash products, which holds for every pair of representations V,W.

Definition 6. Let X be a G-CW complex. We define its n-th G-equivariant stable cohomotopy group, $\pi^n_G(X)$ as the set of homotopy clases of equivalence classes of morphisms $u: S^{\xi\oplus\underline{\mathbb{R}^k}}\to S^{\xi\oplus\underline{\mathbb{R}^{k+n}}}$ under the above mentioned relation. For a G-CW pair, (X,A) we define $\pi^n_G(X,A)$ as the equivalence classes of morphisms which are trivial over A, i.e. those which are given by a representative $u: S^{\xi\oplus\underline{\mathbb{R}^k}}\to S^{\xi\oplus\underline{\mathbb{R}^{k+n}}}$ which satisfies that over every point $a\in A$, the map $u_a: S^{\xi_a\oplus\mathbb{R}^k}\to S^{\xi_a\oplus\underline{\mathbb{R}^{k+n}}}$ is constant with value the base point. For a pair of bundle morphisms $u: S^{\xi\oplus\underline{\mathbb{R}^k}}\to S^{\xi\oplus\underline{\mathbb{R}^{k+n}}}$, $v: S^{\xi'\oplus\underline{\mathbb{R}^k}}\to S^{\xi'\oplus\underline{\mathbb{R}^{k+n}}}$, the sum is defined as the homotopy class of the morphism

$$u: S^{\xi\oplus\xi'\oplus\underline{\mathbb{R}^k}}\wedge_X S^{\mathbb{R}} \overset{\mathrm{id}\wedge_X\nabla}{\to} S^{\xi\oplus\xi'\oplus\underline{\mathbb{R}^k}}\wedge_X (S^{\mathbb{R}}\vee_X S^{\mathbb{R}}) \overset{\sigma_3}{\to}$$
$$(S^{\xi\oplus\underline{\mathbb{R}^k}}\wedge_X S^{\mathbb{R}})\vee_X(S^{\xi'\oplus\underline{\mathbb{R}^k}}\wedge_X S^{\mathbb{R}}) \overset{(u\wedge_X id)\vee_X(v\wedge_X id)}{\to}$$
$$S^{\xi\oplus\xi'\oplus\underline{\mathbb{R}^{k+n}}}\wedge_X S^{\mathbb{R}}$$

where σ_3 is the canonical isomorphism given by the fiberwise distributivity and associativity isomorphisms and ∇ denotes the pinching map $S^{\mathbb{R}}\to S^{\mathbb{R}}\vee S^{\mathbb{R}}$. The relative version for elements lying in the group of a pair, $\pi^n_G(X,A)$ translates word by word when one sets all sphere bundles and morphisms to be trivial over A. The multiplicative structure is introduced fiberwise as well. Precisely, given $a\in\pi^n_G(X,A)$, $b\in\pi^m_G(X,B)$, choose representatives $u: S^{\xi\oplus\underline{\mathbb{R}^k}}\to S^{\xi\oplus\underline{\mathbb{R}^{k+n}}}$, $v: S^{\eta\oplus\underline{\mathbb{R}^l}}\to S^{\eta\oplus\underline{\mathbb{R}^{l+m}}}$ and define their product to be the element in $\pi^{n+m}_G(X,A\cup B)$ which is represented by the morphism

$$S^{\xi\oplus\eta\oplus\underline{\mathbb{R}^k}\oplus\underline{\mathbb{R}^l}} \overset{\sigma_4}{\to}$$
$$S^{\xi\oplus\underline{\mathbb{R}^k}}\wedge_X S^{\eta\oplus\underline{\mathbb{R}^l}} \overset{u\wedge_X v}{\to} S^{\xi\oplus\underline{\mathbb{R}^{k+n}}}\wedge_X S^{\eta\oplus\underline{\mathbb{R}^{l+m}}} \overset{\sigma_5}{\to}$$
$$S^{\xi\oplus\eta\oplus\underline{\mathbb{R}^{l+k+m+n}}}$$

One obtains the long exact sequence of a pair after defining a coboundary operator $\delta^n_G(X,A): \pi^n_G(A)\to\pi^{n+1}_G(X,A)$. This is done as follows. One proves the existence of a natural isomorphism $\sigma_G^n(X,A) : \pi_G^n(X,A) \to \pi_G^{n+1}(X \times I, A \times \{0,1\})$. The coboundary map is then defined as the composition

$$\begin{aligned}
\pi_G^n(A) \xrightarrow{\sigma_G^n(A)} {} & \pi_G^{n+1}(A \times I, A \times \{0,1\}) \\
& \xrightarrow{\pi_G^{n+1}(i_1)^{-1}} \pi_G^{n+1}(X \cup_{A\times\{0\}} A \times I, X \coprod A \times \{1\}) \\
& \quad \xrightarrow{\pi_G^{n+1}(i_2)} \pi_G^{n+1}(X \cup_{A\times\{0\}} A \times I, A \times \{1\}) \\
& \qquad\qquad \xrightarrow{\pi_G^{n+1}(\mathrm{pr}_1)^{-1}} \pi_G^{n+1}(X,A)
\end{aligned}$$

Where the map $\pi_G^{n+1}(i_1)$ is bijective by excision and $\pi_G^{n+1}(\mathrm{pr}_1)$ is bijective because of homotopy invariance.

The following result is proved in[Lüc05a]

Theorem 1. Equivariant stable cohomotopy defines an equivariant cohomology theory with multiplicative structure for finite proper equivariant CW-complexes. For every finite subgroup H of the group G the abelian groups $\pi_G^n(G/H)$ and π_H^n are isomorphic.

Now, we shall construct a natural transformation of monoid-valued functors between our invariants defined via Γ-spaces and the groups of Lück. We shall restrict it to a transformation of G-cohomology theories in the category of finite proper G-CW complexes. In order to present this construction, we give to our functor an interpretation as the monoid of isomorphism classes of equivariant covering maps. Precisely,

Definition 7. A covering map $p : \tilde{X} \to X$ is a *G-covering* if there are actions on $\tilde{X}$ and X such that p is G-equivariant and every $g \in G$ acts on $\tilde{X}$ and X by means of deck transformations. A *map* of G-equivariant coverings over X is a map of fibered bundles $(\bar{f}, f)$ consisting of G-equivariant maps. A G-covering map $\tilde{X} \to X$ is called trivial if it is G-isomorphic to a covering map of the type $X \times \bar{n}$, where $\bar{n}$ carries the trivial G-action.

We now recall the following technical result, which is a consequence of the slice theorem [Pal61].

Lemma 2. Let $p : \tilde{X} \to X$ be a G-covering map over a proper G-CW complex. Then the following fact holds:

1. For every point $x \in X$, there is a finite G_x-set S, a neighborhood U of x and a map $U \to G/G_x$ such that $p\mid_U$ is isomorphic to the pullback of the canonical bundle $G \underset{G_x}{\times} S \to G/G_x$.

In analogy with vector bundles, covering maps can be fibrewise "added" and "multiplied". Intuitively, this corresponds to a process of taking disjoint unions and cartesian products of equivariant sets over every point of X. Precisely,

Definition 8. Let $p_i : \tilde{X}_i \to X_{i=0,1}$ be G-covering maps. Their *sum* $p_0 \coprod_X p_1$is the locally trivial G-bundle whose fiber over x is $p_0^{-1}(x) \coprod p_1^{-1}(x)$. Their *product* $p_0 \times_X p_1$ is the locally trivial bundle whose fiber on x is $p_0^{-1}(x) \times p_1^{-1}(x)$.

Definition 9. Two G-covering maps p_0, p_1 are called *stable equivalent* if there is a trivial G-covering map q such that

$$p_0 \coprod_X q \cong p_1 \coprod_X q$$

yields.

Definition 10. The monoid of stable G-covering maps, $G-\mathrm{Cov}(X)/\sim$ consists of the set of stable isomorphism classes of G-coverings over X together with the sum as operation.

The functor $X \to G-\mathrm{Cov}(X)$ is a contravariant functor in the category of abelian monoids. Imprecisely, our construction for equivariant stable cohomotopy is in some sense the universal extension to the category of abelian groups. This admits a precise formulation following the lines of the group completion theorem, proposition 1. But in order to apply this, we need to exhibit a relationship between isomorphism classes of coverings, the construction of Lück, and our construction. The first step in this direction is the

Lemma 3. Let $p : \tilde{X} \to X$ be a finite G-covering map over the proper finite G-CW complex X. Then, there is a vector bundle $\xi : E \to X$ and a G-map $\tilde{X} \to E$ injective on fibers covering the identity on X.

Proof. Let $p : E \to X$ be a G-locally trivial fiber bundle with fiber F. By means of local trivializations, one can find a cover U_i consisting of G-invariant neighborhoods together with G-maps $U_i \xrightarrow{r_i} G/H_i$ such that $p\mid_{p^{-1}(U_i)}$ is isomorphic to the pullback of the bundle $G \underset{H_i}{\times} F \to G/H_i$ along r_i. Let $\Gamma \subset \mathrm{Homeo}(F), H \subset G$ be a pair of subgroups. Write $\mathrm{Rep}_\Gamma(H) = Hom(H, \Gamma)/\mathrm{Inn}(\Gamma)$. A bundle of the form $G \underset{H}{\times} F \to G/H$ determines in a canonical way an element in $\mathrm{Rep}_\Gamma(H)$. Since the existence of a G-map $G/H \to G/K$ yields a map between bundles over orbits (elements in $\mathrm{Rep}_\Gamma(H)$), a cocycle for the bundle E is given by functions $g_{U_i,U_j} : U_i \cap U_j \to \mathrm{Aut}(\varprojlim_{H\in\mathcal{F}} \mathrm{Rep}_\Gamma(H))$, where $\mathcal{F}$ represents the family of isotropy subgroups of points lying in X. Analogous to the non-equivariant situation, cocycles give the gluing instructions to get the total space from the trivializations. Precisely, denote by $E_{i,H}$ the total space of $r_i^*(G \underset{H}{\times} F)$, handled as a subspace of the product $U_i \times G \underset{H}{\times} F$. Then, the space

$$\coprod_{i\in I, H\in\mathcal{F}} E_{i,H} \diagup (x,s) \sim (x, g_{U_i,U_j}(s))_{x\in U_i\cap U_j, s\in G\underset{H}{\times}F}$$

is G-homeomorphic to the total space E. Now, let us consider the case of a G-covering. That is, $F = \bar{n}$ and $\Gamma = \Sigma_n$, the symmetric group on n letters. We are given a family of cocycles $g_{U_i,U_j} : U_i \cap U_j \to \mathrm{Aut} \varprojlim_{H\in\mathcal{F}} \mathrm{Rep}_{\Sigma_n}(H))$. The inclusion $\Sigma_n \to O_n$ induces a map $\mathrm{Aut} \varprojlim_{H\in\mathcal{F}} \mathrm{Rep}_{\Sigma_n}(H) \to \mathrm{Aut} \varprojlim_{H\in\mathcal{F}} \mathrm{Rep}_{O_n}(H)$, which can be interpreted as the map which assigns to a given finite H-set the permutation representation with basis on it. Consider over G/H_i the bundle $E_i := G \underset{H_i}{\times} \mathbb{R} < S_i > \to G/H_i$ and form the vector bundle E determined by the aboye described cocycles. Note that we have an inclusion $\tilde{X}_{p^{-1}(U_i)} \subset E_i$. This induces after identification a well defined map $\tilde{X} \to E$, which is fibrewise injective. $\square$

By means of the above described embedding we can handle the fiber of a G-covering map as an embedded subset into a representation of the stabilizer group. This is a point in a so-called configuration space.

Definition 11. Let H be a finite group, V be an H-representation. The equivariant configuration space of V, $C_H(V)$ is the space of finite H-sets embedded in V. This space is topologized as follows. Let V_k be the subspace of $\underbrace{V \times \ldots \times V}_{\text{k times}}$ consisting of sequences $(v_i)_i$ with $v_i \neq v_j$. We restrict ourselves to the subspace V_{kH} of such finite sets S which are invariant under the action of H. Let $C^k_H(V)$ denote the quotient of this space by the action of the symmetric group in k-letters. $C_H(V)$, the space of H-equivariant configurations, is topologized as the disjoint union of the ${C^k}_H(V)$'s. We identify an embedding of a finite set with its image as a configuration in $C(V)$.

Let $S \to V$ be such an embedding. Following Segal [Seg73], a map in a loop space is constructed. This is done as follows. Let d be an H-invariant metric on V, S be the image of such an embedding and $\epsilon > 0$. Define $C^\epsilon_H(V) = \{S \in C_H(V) \mid d(s_i, s_j) > 2\epsilon\, s_i, s_j \in S\}$ and $S_\epsilon = \{v \in V \mid d(v, S) > \epsilon\}$. Finally, let $\epsilon > 0$ be sufficiently small so that $V - S_\epsilon$ is an H-equivariant tubular neighborhood of S and $S \in C^\epsilon_H(V)$. Let $s_i \in S$ and $h_i : B_\epsilon(s_i) \to V$ the canonical diffeomorphism. Define the map $\Theta_S : V \cup \infty \to V \cup \infty$ by

$$v \mapsto \begin{cases} \infty & \text{if } v \in S_\epsilon \\ d(v, S)h_i(v) & \text{if } v \in B_\epsilon(s_i) \end{cases}$$

The Segal map $T_H : C_H(V) \to Map_H(S^V, S^V)$ is defined to be $S \mapsto \Theta_S$.

For more details on this see [Hau80] or [RS00], alternatively [CW85], where a related map is described in the context of compact lie groups.

Definition 12. Let $\tilde{X} \subset \xi : E \to X$ be a fiberwise embedding of the finite G-covering $\tilde{X}$ into a G-vector bundle ξ defined over the proper G-CW complex X. Let d be a G-invariant metric for ξ (cfr. [Pal61], theorem 4.3.1). The parametrized Segal map is the element of $\pi^0_G(X)$ which is given fibrewise as

$$S_{G_x}(v) = \begin{cases} \infty & \text{if } v \in S_{\epsilon_x} \\ d_x(v, S_x)h_{i_x}(v) & \text{if } v \in B_\epsilon(s_{i_x}) \end{cases}$$

where S_x is the image of the embedding of the fiber of $\tilde{X}$ in ξ, handled as a G_x-configuration and h_{i_x} is a G_x-equivariant local diffeomorphism into the representation ξ_x.

Lemma 4. The parametrized Segal map is well defined.

Proof. Let $r_n : (\tilde{X} \to E_n)_{n \in 0,1}$ be embeddings of the covering $p : \tilde{X} \to X$. We can assume $E_0 = EW_1$ Form the sum $E = E_0 \oplus E_1$ and the map $E \times I \to E$ which is given on every fiber as $(x_0, x_1, t) \mapsto (\cos(\frac{\pi t}{2})x_0, \sin(\frac{\pi t}{2})x_1)$. The fiberwise embedding $\tilde{X} \times I \to E \times I$ given fiberwise by $\cos(\frac{\pi t}{2})r_0 + \sin(\frac{\pi t}{2})r_1$ allows us to apply the parametrized segal map, obtaining a bundle map $S^E \times I \to S^E$ covering the projection $X \times I \to X$. This gives a homotopy between the spherical morphism $S^{E_0 \oplus E_1} \overset{S(p) \wedge_X \mathrm{id}}{\longrightarrow} S^{E_0 \oplus E_1}$ and $S^{E_0 \oplus E_1} \overset{\mathrm{id} \wedge_X S_p}{\longrightarrow} S^{E_0 \oplus E_1}$, which are equivalent to the spheric bundle morphisms obtained by the embeddings in E_0, respectively, E_1. Finally note that given stably equivalent covering maps p_0, p_1, one can find a G-vector bundle E with an embedding : $\tilde{X}_0 \coprod_X \mathbf{n} \cong \tilde{X}_1 \coprod_X \mathbf{n} \to E$. Finally note that the spheric bundle morphism $K_{p_i \coprod_X \mathbf{n}}$ is equivalent to K_{p_i}. □

Lemma 5. The parametrized Segal map is additive

Proof. Let $r_i : \tilde{X}_i \to E_i$ be a pair of embeddings. In view of the independence of particular embeddings, let us choose an embedding of $X_0 \coprod X_1$ into $E_0 \oplus E_1 \oplus \mathbb{R}$ as follows. Over every point $x \in X$, $p_0^{-1}(x) \subset E_0 \times 0 \times \mathbb{R}^+$ and $p_1^{-1}(x) \subset 0 \times E_1 \times \mathbb{R}^-$. After applying the parametrized Segal map to the corresponding configuration bundle, one gets a map $S_{p_0 \coprod p_1}$ which fits into the following commutative diagram

$$\begin{tikzcd}
S^{E_0 \oplus E_1} \wedge_X S^{\mathbb{R}} \arrow[rr, "S_{p_0 \coprod p_1}"] \arrow[d, "\mathrm{id} \wedge_X \nabla"'] & & S^{E_0 \oplus E_1} \wedge_X S^{\mathbb{R}} \arrow[dd] \\
S^{E_0 \oplus E_1} \wedge_X S^{\mathbb{R}} \vee_X S^{\mathbb{R}} \arrow[d] & & \\
S^{E_0 \oplus E_1} \wedge_X S^{\mathbb{R}} \vee S^{E_0 \oplus E_1} \wedge_X S^{\mathbb{R}} \arrow[dr, "(S_{p_0} \wedge_X \mathrm{id} \wedge_X \mathrm{id}) \vee_X (\mathrm{id} \wedge_X S_{p_1} \wedge_X \mathrm{id})"'] & & S^{E_0 \oplus E_1} \wedge_X S^{\mathbb{R}} \\
& S^{E_0 \oplus E_1} \wedge_X S^{\mathbb{R}} \vee_X S^{E_0 \oplus E_1} \wedge_X S^{\mathbb{R}} \arrow[ur, "\mathrm{id} \vee_X \mathrm{id}"'] &
\end{tikzcd}$$

where the unlabeled arrows are given by the canonical homeomorphisms. □

Now we shall describe the relationship with the construction proposed previously. Essentially, our argument consists in proving that the space $\underline{G\text{−Set}}$ classifies finite G-coverings. In order to state this precisely, we have to construct the universal coverings which are going to be pulled back, which is the purpouse of the following discussion.

Choose a functor X from the category of finite sets with isomorphisms into the category whose objects are the natural numbers, and morphism the identities $\bar{n} \to \bar{n}$. That is, a *labeling for finite sets.*

The category Set_n consists of the full subcategory of Set consisting of objects of cardinality n.

The category of n-frames of sets, Frame_n consists of pairs $(S, x(S))$ where $S \subset \mathrm{Set}_n$ and X is a labeling. Thus n-frames are pairs consisting of a finite set of cardinality n and a numeration of its elements.

We are mainly interested in the equivariant version of these categories. Put $G-\mathrm{Frame}_n$ for the category of functors from $\mathcal{E}(G)$ to Frame_n. Write $\underline{G-\mathrm{Frame}}_n$ for its classifying space. This space carries an action of $G \times \Sigma_n$ given by translation and permutation of the labeling.

Proposition 5. $\underline{G-\mathrm{Frame}}_n$ is a universal space for those $G \times \Sigma_n$-complexes where $\Sigma_n \cong 1 \times \Sigma_n$ acts freely

Proof. Since the space in question is a $G \times \Sigma_n$-CW-complex, it suffices to prove that the H-fixed points are contractible for every subgroup of the product. A subgroup H subconjugate to $1 \times \Sigma_n$ is essentially of two shapes. Either H satisfies $H \subset 1 \times \Sigma_n$, in which case, $\underline{G-\mathrm{Frame}}_n^H = \phi$, since Σ_n acts freely, or H is the graph of a group homomorphism $\varphi : H^{'} \to \Sigma_n$ for some subgroup $H^{'} \subset G$ hence,the category in question can be identified with the category of φ-equivariant functors from $\mathcal{E}(G)$ to Frame_n. Since there is a unique morphism between every pair of objects in $\underline{G-\mathrm{Frame}}_n$, every object is terminal. Hence, the classifying space is contractible. ☐

Since Σ_n acts freely on $\underline{G-\mathrm{Frame}}_n$, the forgetful functor $G-\mathrm{Frame}_n \to G-\mathrm{Set}_n$ induces a G-equivariant homeomorphism $\underline{G-\mathrm{Frame}}_n/\Sigma_n \approx \underline{G-\mathrm{Set}}_n$. It follows that the G-bundle $\underline{G-\mathrm{Frame}}_n \times_{\Sigma_n} \bar{n} \to \underline{G-\mathrm{Set}}_n$ is a universal n-sheeted covering. Hence, we have

Proposition 6. There is a isomorphism of sets $G-\mathrm{Cov}(X) \cong [X, \underline{G-\mathrm{Set}}]$.

Proof. ☐

We can arrange this in X natural transformations into a diagram of the shape

$$\begin{tikzcd}
G-\mathrm{Cov}(X)/\sim \arrow[r, "S"] & \pi_G^0(X) \\
{[X, \underline{G-\mathrm{Set}}]} \arrow[u] &
\end{tikzcd}$$

Where S is the morphism determined by the parametrized Segal map, the unlabeled arrow is the map determined by the pullback of the canonical covering, followed with the identification of stable equivalent coverings. Note in particular that this can be done for G-CW pairs (X, A) by taking covering maps which are trivial over A, inducing up to a homotopy a map in $[X/A, \underline{G-\mathrm{Set}}]$. We extend this to morphisms $[\Sigma^n X/A, \underline{G-\mathrm{Set}}] \to \pi_G^n(X, A)$ and by means of the map $\underline{G-\mathrm{Set}} \to Q_G$, to morphisms $S_n^{(X,A)} : [\Sigma^n X/A, Q_G] \to \pi_G^n(X, A)$. Let us restrict ourselves to the category of finite proper G-CW complexes. We shall show that the value of the functors agree. Since both functors are in this case equivariant cohomology theories, we can recover the data from Mayer-Vietoris sequences for squares of the form

$$\begin{tikzcd}
G/H \times S^{m-1} \arrow[r] \arrow[d, "\varphi"'] & G/H \times D^m \arrow[d] \\
X \arrow[r] & G/H \times D^m \cup_\varphi X
\end{tikzcd}$$

hence, it suffices to prove the isomorphism for complexes of the form $G/H \times Y$ where Y carries a trivial G-action. On the one hand we have

$$[G/H \times Y, Q_G]_G \cong [Y, Q_G^H]$$

whereas on the other hand

$$\pi^0_G(G/H \times Y) \cong \pi^0_H(Y)$$

because of the induction structure for the inclusion $H \to G$. Since $\pi_0(\underline{G\text{--}\mathrm{Set}}^H)$ is a monoid (the monoid of isomorphism classes of H-sets), proposition 1 allows us to conclude that $[X, Q_G^H]$ is universal among representable functors from compact spaces to abelian groups extending $[X, \underline{G\text{--}\mathrm{Set}}^H]$. Since $\pi^0_H(X)$ is representable, it is the universal functor. Hence

$$[G/H \times Y, Q_G]_G \cong [Y, Q_G^H] \cong \pi^0_H(Y) \cong \pi^0_G(G/H \times Y)$$

Now, we extend the definitions in degree $n \leq 0$ by

$$\Pi_G^{-n}(X, A) = [\Sigma^n X/A, Q_G]_G$$

This agrees with the definition of Lück because of the following argument. Choose a representative of a map $\Sigma^1 X/A \to \underline{G\text{--}\mathrm{Set}}$. By means of the canonical identification $X \times I \to \Sigma X/A$, we pull the canonical covering back to a covering on $X \times I$ which is trivial on $(X, A) \times (I, \{0, 1\})$. The procedure applied to this yields an element of $\pi^0_G((X, A) \times (I, \{0, 1\})) \underset{\sigma_1(X,A)}{\cong} \pi_G^{-1}(X)$, where $\sigma_1(X, A)$ stays for the suspension isomorphism of [Lüc05a]. The general case follows then by induction.

In order to get a definition in degrees $n \geq 0$, we recall that the geometric realizations $\mathrm{B}^n A$ of a Γ-space A behave like an Ω-spectrum away of the grade zero. Hence, it makes sense to define $Q_G^n := \Omega\mathrm{B}^{n+1}\underline{G\text{--}\mathrm{Set}}$ and consequently

$$\Pi^n_G(X/A) = [X/A, Q^n_G]$$

We now check that this definition agrees with that of Lück. For this, we need some notation. Put $\mathcal{L}(X/A) = Map_G((S^1, \{1\}), (X, A))$, and define the map $X/A \to \mathcal{L}(X/A)$ by assigning the constant map to a point in X. We define a map $[X/A, \mathrm{B}\underline{G\text{--}\mathrm{Set}}]_G \to \pi^1_G(X, A)$ which by the universal property of group completion extends to the required map $[X/A\Omega\mathrm{B}^2\underline{G\text{--}\mathrm{Set}}]_G \to \pi^1_G(X/A)$. To do this we consider the following sequence of maps

$$\begin{aligned}
&[X/A, \mathrm{B}\underline{G\text{--}\mathrm{Set}}]_G \to [\mathcal{L}(X/A), \Omega\mathrm{B}\underline{G\text{--}\mathrm{Set}}]_G \to \\
&\qquad \pi_e^G(S^1) \times [S^1 \times \mathcal{L}(X/A), \Omega\mathrm{B}\underline{G\text{--}\mathrm{Set}}]_G \to \\
&\qquad [S^1 \times \mathcal{L}(X/A), \Omega\mathrm{B}\underline{G\text{--}\mathrm{Set}}]_G \to_{S_{S^1 \times \mathcal{L}(X/A)}} \\
&\qquad\qquad \pi^1_G(S^1 \times \mathcal{L}(X/A)) \to \pi^1_G(X)
\end{aligned}$$

the first map is given by the inclusion of constant loops, the second one is given by the inclusion on the right coordinate, the third map is the map which assigns a map f an the generator $g \in \pi^0_e(S^1)$ the map $g \times f$, the labeled map is the parametrized segal map, and the last map is the map induced by pullback of the constant loop map.

The Barrat-Puppe sequences for the inclusion $i : A \to X$ give rise to exact sequences of the form

$$\begin{gathered}
\longrightarrow \dots [\Sigma^n A, Q_G]_G \overset{d_G^{n+1}(X,A):=p^*}{\longrightarrow} [\Sigma^{n-1} X \cup_A \mathrm{cone}(A), Q_G] \xrightarrow{j^*} \\
[\Sigma^{n-1} X, Q_G]_G \xrightarrow{i^*} [\Sigma^{n-1} A, Q_G] \longrightarrow \dots \\
\dots \longrightarrow [A, Q_G] \cong [\Sigma A, Q_G^1] \overset{d_G^1(X,A):=p^*}{\longrightarrow} [X \cup_A \mathrm{cone}(A), Q_G^1]_G \\
\xrightarrow{j^*} [X, Q_G^1]_G \xrightarrow{i^*} [A, Q_G^1]_G \longrightarrow \dots
\end{gathered}$$

Where $p : X \cup_A \mathrm{cone}A \to \Sigma A$ is the canonical projection and we identify the isomorphic sets of homotopy classes $[X/A, Z] \cong [X \cup_A \mathrm{cone}A, Z]$.

Lemma 6. There is a natural transformation of G-cohomology theories $\Pi_G^* \to \pi_G^*$ consisting of isomorphisms for every finite proper G-CW pair (X, A).

Proof. Note that the following diagram commutes for any n

$$\begin{tikzcd}
\cdots \arrow[r] & \Pi_G^n(A) \arrow[r, "d_G^{n+1}(X,A)"] \arrow[d, "S_G^n(A)"'] & \Pi_G^{n+1}(X,A) \arrow[r, "j^*"] \arrow[d, "S_G^{n+1}(X,A)"'] & \Pi_G^{n+1}(X) \arrow[r, "i*"] \arrow[d, "S_G^{n+1}(X)"'] & \Pi_G^{n+1}(A) \arrow[r] \arrow[d, "S_G^{n+1}(A)"'] & \cdots \\
\cdots \arrow[r] & \pi_G^n(A) \arrow[r, "\delta_G^{n+1}(X,A)"'] & \pi_G^{n+1}(X,A) \arrow[r, "j^*"'] & \pi_G^{n+1}(X) \arrow[r, "i*"'] & \pi_G^{n+1}(A) \arrow[r] & \cdots
\end{tikzcd}$$

□

Let us explain now how further structures in the construction of Lück carry out into our context. The first point to be considered shall be the multiplicative structure. We shall produce a pairing of the infinite loop spaces involved in our construction. We point out for the homotopy theoretic-oriented reader, that we are by no means trying to construct a strict monoidal structure in a category of equivariant spectra. Our pairings are only defined up to homotopy.

The idea behind this is that one can define another monoidal structure in the category of finite sets, which is given by the cartesian product. Let $\coprod_{s \in S} X_s \in \mathrm{Set}_S$, $\coprod_{t \in T} Y_t$ be partitioned sets. By means of an isomorphism $\alpha : \mathbb{N} \times \mathbb{N} \cong \mathbb{N}$, the set $X_s \times Y_t \subset \mathbb{N} \times \mathbb{N}$ can be handled as a finite set in $\mathbb{N}$. Hence, for every pair of finite sets S, T we get a functor $\alpha_* : \mathrm{Set}_S \times \mathrm{Set}_T \to \mathrm{Set}_{S \times T}$ given by $(\coprod_{s \in S} X_s, \coprod_{t \in T} Y_t) \mapsto \coprod_{(s,t) \in S \times T} \alpha_*(X_s \times Y_t)$. For any pair of groups G, H, this extends to functors $G{-}\mathrm{Set}_S \times H{-}\mathrm{Set}_T \to G \times H{-}\mathrm{Set}_{S \times T}$. By considering the geometrical realization, this gives a $G \times H$- equivariant map $\underline{G{-}\mathrm{Set}}_S \times \underline{H{-}\mathrm{Set}}_T \to \underline{G \times H{-}\mathrm{Set}}_{S \times T}$. Since the restriction of the map to $\underline{G{-}\mathrm{Set}}_S \vee \underline{H{-}\mathrm{Set}}_T$ is constant with value on the basis point (the product with the empty set is empty), there is a map $\underline{G{-}\mathrm{Set}}_S \wedge \underline{H{-}\mathrm{Set}}_T \to \underline{G \times H{-}\mathrm{Set}}_{S \times T}$. Note that both the target and the source of this map have two compatible structures of Γ-spaces, namely, those which are given by letting S, respectively T, run over the category of finite sets. After considering the separate realization of this Γ-spaces as simplicial sets we get a map $\mathrm{B}\underline{G{-}\mathrm{Set}} \wedge \mathrm{B}\underline{H{-}\mathrm{Set}} \to \mathrm{B}^2\underline{G \times H{-}\mathrm{Set}}$. And at the level of loop spaces, $\Omega^2(\mathrm{B}\underline{G{-}\mathrm{Set}} \wedge \mathrm{B}\underline{H{-}\mathrm{Set}}) \xrightarrow{\alpha_*} \Omega^2\mathrm{B}^2\underline{G \times H{-}\mathrm{Set}}$. Now define the pairing $Q_G \wedge Q_H \to Q_{G \times H}$ by

$$\begin{aligned}Q_G \wedge Q_H = \Omega \mathrm{B}\underline{G\mathrm{-Set}} \wedge \Omega\mathrm{B}\underline{H\mathrm{-Set}} \to \\ \Omega^2(\mathrm{B}\underline{G\mathrm{-Set}} \wedge \mathrm{B}\underline{H\mathrm{-Set}}) \overset{\Omega^2|\alpha_*|}{\to} \Omega^2\mathrm{B}^2\underline{G\times H\mathrm{-Set}} \simeq \Omega\mathrm{B}\underline{G\times H\mathrm{-Set}} \\ = Q_{G\times H}\end{aligned}$$

This map does not depend on the choice of the isomorphism α as a consequence of the following

Lemma 7. Let $\alpha : \mathbb{N}^\infty \to \mathbb{N}^\infty$ be an injective map. Then, the induced map $\alpha_* : G\mathrm{-Set} \to G\mathrm{-Set}$ defined by composition with $\mathrm{Set} \overset{\alpha}{\to} \mathrm{Set}$ is G-homotopic to the identity.

Proof. There is a natural transformation of the functor α_* to the identity. ☐

Note that there is more than one homotopy equivalence $\Omega^2\mathrm{B}^2\underline{G\mathrm{-Set}} \to \Omega\mathrm{B}\underline{G\mathrm{-Set}}$. The following technical result denies a role of this choice in our discussion.

Lemma 8. Let A be any equivariant Γ-space. Then, for any n and $0 \le k \le n$, the maps $i_n^k : \Omega^{n+1}\mathrm{B}^{n+1}A \to \Omega^n\mathrm{B}^nA$ induced by the inclusions $\{1\} \to \{1,\dots,n\}$ differ up to weak G-equivariant stable equivalence.

Proof. Note that all i_n^k differ by permutation of the coordinates of B^nA as a simplicial set and a switch of the looping. Write $\sigma_* : \Omega^n\mathrm{B}^nA \to \Omega^n\mathrm{B}^nA$ for the map induced by such a permutation $\sigma \in \Sigma_n \subset \Sigma_{n+1}$. It suffices to show that all σ_* are homotopic to the identity. Consider for this the following diagram

$$\begin{tikzcd}
\Omega\mathrm{B}A \arrow[r, "\varphi"] \arrow[d, "\mathrm{id}"'] & \Omega^{n+1}\mathrm{B}^{n+1}A \arrow[d, "1\times\sigma_*"'] & \Omega^n\mathrm{B}^nA \arrow[l, "i_n^n"'] \arrow[d, "\sigma_*"] \\
\Omega\mathrm{B}A \arrow[r, "\varphi"'] & \Omega^{n+1}\mathrm{B}^{n+1}A & \Omega^nB^nA \arrow[l, "i_n^n"]
\end{tikzcd}$$

where $\varphi = i_n^0 \circ \dots \circ i_n^n$ is induced by identifying $A(S)$ with $A(S,1,\dots,1)$. All maps are weak G-homotopy equivalences. Hence, $(1\times\sigma_*)$ and $(\sigma)_*$ are weakly G-homotopic to the identity. ☐

The ring structure on $\Pi^*(X)$ is given as follows. Let $\Delta^* : [X, Q_{G\times G}]_G \to [X, Q_G]_G$ be the map induced by the restriction to the diagonal subcategory $\mathcal{E}(G) \subset \mathcal{E}(G\times G)$. Now compose it with the pairing obtained before: $[X,Q_G]\times[X,Q_G] \to [X,Q_{G\times G}]_G \overset{\Delta^*}{\to} [X,Q_G]_G$.

Lemma 9. For any discrete group G and any G-space X, the following square commutes

$$\begin{tikzcd}
{[X,Q_G]\times[X,Q_G]} \arrow[r, "S_X\times S_X"] \arrow[d, "\alpha_*"'] & \pi_X^0(X)\times\pi_G^0(X) \arrow[d, "\times"] \\
{[X,Q_G]} \arrow[r, "S_X"'] & \pi_G^0(X)
\end{tikzcd}$$

where × stands for the above mentioned products.

Proof. Let $\mu : \mathrm{B}\underline{G{-}\mathrm{Set}} \wedge \mathrm{B}\underline{G{-}\mathrm{Set}} \to \mathrm{B}^2\underline{G{-}\mathrm{Set}}$ be the map determined by the pairing constructed above and f_1, f_2 be functions $X \to \underline{G{-}\mathrm{Set}}$. Denote by p_1, respectively p_2 the pullback of a finite covering $\tilde{X_G} \to \underline{G{-}\mathrm{Set}}$ along f_1, respectively f_2. Consider the map p defined as the following composition

$$X \overset{f_1 \wedge f_2}{\to} \mathrm{B}\underline{G{-}\mathrm{Set}} \wedge \mathrm{B}\underline{G{-}\mathrm{Set}} \overset{\mu_*}{\to} \mathrm{B}^2\underline{G{-}\mathrm{Set}} \to \underline{G{-}\mathrm{Set}}$$

where the last map is induced up to homotopy by iteration of the projection $\Sigma\underline{G{-}\mathrm{Set}} to \mathrm{B}\underline{G{-}\mathrm{Set}} \to \underline{G{-}\mathrm{Set}}$. For $i = 1, 2$, define the map q_i to be the universal map which fits into the following pullback diagram.

$$\begin{tikzcd}
p^*(\tilde{X_G}) \arrow[rd, dashed, "q_i"] \arrow[rrd, bend left, "\pi_{X_G}"] \arrow[rdd, bend right] & & \\
& p_1 \arrow[r] \arrow[d] & \tilde{X_G} \arrow[d] \\
& X \arrow[r] & \underline{G{-}\mathrm{Set}}
\end{tikzcd}$$

where the unlabeled arrows are projections, respectively canonical maps. Note that the canonical map

$$\begin{tikzcd}
p^*(\tilde{X_G}) \arrow[rd, dashed] \arrow[rrd, bend left, "q_1"] \arrow[rdd, bend right] & & \\
& p_1 \times_X p_2 \arrow[r] \arrow[d] & p_1 \arrow[d] \\
& p_2 \arrow[r] & X
\end{tikzcd}$$

is an isomorphism of finite G-coverings over X. Now a fiberwise inspection, which runs analogous to 5 gives the result. For the covering $p_1 \times_X p_2$ embeds naturally into $\xi_1 \oplus \xi_2$, where $p_i \subset \xi_i$. After taking the parametrized Segal map, one obtains an element representing the product in $\pi^0_G(X)$, from where the result follows.

□

Let us resume our results on multiplicative structure in the following

Theorem 2. For any discrete group G and any proper G-complex X the pairings α_X define a structure of graded ring on $\Pi^*_G(X)$

Proof. Let us verify the existence of a unit. Denote by $P^1 \in \underline{G{-}\mathrm{Set}}$ the vertex for the constant functor $\mathcal{E}(G) \to \mathrm{Set}_1$. Put $[P^1]_\Omega = i^0_0 \in \Omega\mathrm{B}\underline{G{-}\mathrm{Set}}$. Then the following diagram commutes

$$\begin{tikzcd}
\Omega\mathrm{B}\underline{G{-}\mathrm{Set}} \arrow[r, "{[P^1]\wedge-}"] \arrow[d, "\mathrm{id}"'] & \underline{G{-}\mathrm{Set}} \wedge \Omega\mathrm{B}\underline{G{-}\mathrm{Set}} \arrow[r, "\alpha_*"] \arrow[d, "\simeq"', "i^0_0"] & \Omega\mathrm{B}\underline{G{-}\mathrm{Set}} \arrow[d, "\simeq"', "i^0_1"] \\
\Omega\mathrm{B}\underline{G{-}\mathrm{Set}} \arrow[r, "{[P^1]_\Omega\wedge-}"'] & \Omega\mathrm{B}\underline{G{-}\mathrm{Set}} \wedge \Omega\mathrm{B}\underline{G{-}\mathrm{Set}} \arrow[r, "\alpha_*"'] & \Omega^2\mathrm{B}^2\underline{G{-}\mathrm{Set}}
\end{tikzcd}$$

the upper row being homotopic to the identity after lemma 9. The commutativity of the multiplication in $\Pi^0_G(X)$ is obtained as consequence of the fact that $\alpha_* : \mathrm{B}\underline{G{-}\mathrm{Set}} \wedge \mathrm{B}\underline{G{-}\mathrm{Set}} \to \mathrm{B}^2\underline{G{-}\mathrm{Set}}$ commutes up to homotopy because of lemma 9. For the map $\Omega\mathrm{B}\underline{G{-}\mathrm{Set}} \to \Omega^2\mathrm{B}^2\underline{G{-}\mathrm{Set}}$ is unique. Associativity follows from the fact that triple products are induced by maps

$$\begin{tikzcd}
\Omega\mathrm{B}\underline{G{-}\mathrm{Set}}^{\wedge 3} \arrow[r] & \Omega^3(\mathrm{B}\underline{G{-}\mathrm{Set})^3}) \arrow[r, shift left, "\Omega^3|\alpha_*(\alpha_*\wedge\mathrm{id})|"] \arrow[r, shift right, "\Omega^3|\alpha_*(\mathrm{id}\wedge\alpha_*)|"'] & \Omega^3\mathrm{B}^3\underline{G{-}\mathrm{Set}} & \Omega\mathrm{B}\underline{G{-}\mathrm{Set}} \arrow[l, "\simeq"']
\end{tikzcd}$$

where the middle maps are homotopic by lemma 9, and the last one is any of the three possible maps. □

Now let us look at the induction structure. Let $\alpha : H \to G$ be a group homomorphism and X an H-space. We check the induction structure in degree zero in the case of a G-CW complex X. This extends in the usual way to non-zero groups of pairs. Our results on this matter are a consequence of the following

Lemma 10. Let $\varphi : H \to G$ be a group homomorphism. The composition with the induced functor $\mathcal{E}(H) \to \mathcal{E}(G)$ induces an H-equivariant map $\underline{H{-}\mathrm{Set}} \to \underline{G{-}\mathrm{Set}}$ of Γ-spaces and hence a G-equivariant map $\varphi^* : Q_H \to Q_G$. For any subgroup L such that $L \cap \ker(\varphi) = 1$, φ^* restricts to a homotopy equivalence $Q^L_H \simeq Q^{\varphi(L)}_G$.

Proof. Note that for such subgroups, $L \cong \varphi(L)$ holds. Hence, we get an equivalence of categories $\mathcal{E}(H)/L \cong \mathcal{E}(G)/\varphi(L)$, for both are equivalent to $\mathcal{B}(L)$. Hence $G{-}\mathrm{Set}^{\varphi}(L)(S) =\mid \mathrm{Fun}(\mathcal{E}(H)/L, \mathrm{Set}(S)) \mid$ is homotopy equivalent to $H{-}\mathrm{Set}^L(S) =\mid \mathrm{Fun}(\mathcal{E}(G)/f(L), \mathrm{Set}(S)) \mid$ for each object S of Γ. □

Proposition 7. Let X be a space on which $\ker(\alpha)$ acts freely. Then, there is an induction isomorphism $\Pi^0_G(\mathrm{ind}_\alpha X) \cong \Pi^0_H(X)$.

Proof. There is an isomorphism $[\mathrm{ind}_\alpha X, Q_G]_G \cong [X, Q_G]_H$. And the induced map $\alpha^* : Q_G \to Q_H$ restricts after lemma 10 to a homotopy equivalence of fixed points $Q^L_G \to Q^L_H$ for finite subgroups L satisfying $L \cap \ker(\alpha) = e$. Since the isotropy groups of the complex X lie in this family, we conclude (Theorem 3.4 in [DL98]) that

$$[\mathrm{ind}_\alpha X, Q_G]_G \cong [X, Q_G]_H \xrightarrow{\alpha^*} [X, Q_H]_H$$

□

Proposition 8. The homomorphism $[\mathrm{ind}_\alpha X, Q_G]_G \cong [X, Q_G]_H \xrightarrow{\alpha^*} [X, Q_H]_H$ defines an induction structure.

Proof.

1. Bijectivity. Its clear in view of the previous proposition.
2. Compatibility with boundary. Follows from lemma 6.
3. Functoriality. Let $\beta : G \to K$ be a group homomorphism. Note that the following diagram commutes after lemma 10:

$$\begin{tikzcd}
{[\Sigma^n \mathrm{ind}_\alpha \mathrm{ind}_\beta (X/A), Q_H]_H} \arrow[r, "\mathrm{ind}_\beta"] & {[\Sigma^n \mathrm{ind}_\alpha X/A, Q_G]_G} \arrow[d, "\mathrm{ind}_\alpha"] \\
{[\Sigma^n \mathrm{ind}_{\beta\circ\alpha} (X/A), Q_H]_H} \arrow[u, "\pi^n_G(f_1)"'] \arrow[r, "\mathrm{ind}_{\beta\circ\alpha}"'] & {[\Sigma^n X/A, Q_K]_K}
\end{tikzcd}$$

where $f_1 : \mathrm{ind}_\beta \mathrm{ind}_\alpha (X/A) \to \mathrm{ind}_{\beta\circ\alpha} X/A$ is the canonical H- homeomorphism given by $(k, g, x) \mapsto k\beta(g), x)$.

4. Compatibility with conjugation. Follows from element chasing in the diagram

$$\begin{tikzcd}
{[\Sigma^n \mathrm{ind}_{c(g)} X/A, Q_G]_G} \arrow[r, "\mathrm{ind}_{c(g)}"] & {[\Sigma^n X/A, Q_G]_G} \arrow[dl, "\pi^n_G(f_2)"'] \\
{[\Sigma^n \mathrm{ind}_{c(g)} X/A, Q_G]_G} \arrow[u, "="] &
\end{tikzcd}$$

where $f_2 : X/A \to \mathrm{ind}_{c(g)} X/A$ is given by $x \mapsto (1, g^{-1}x)$ and $c(g)(g) = gg^{'}g^{-1}$.

□

Proposition 9. Let $H \leq G$ be a subgroup. Then there exists an in X natural homomorphism

$$\mathrm{res}^G_H : \Pi^n_G(X) \longrightarrow \Pi^n_H(X \mid_H)$$

where $X \mid_H$ carries the obvious action obtained by restriction.

Proof. Let $i : H \to G$ be the inclusion. It induces a functor $\mathcal{E}(H) \to \mathcal{E}(G)$, which gives an H- equivariant map $Q_G \to Q_H$. This gives a homomorphism

$$[X, Q_G]^*_G \overset{\mathrm{res}^G_H}{\longrightarrow} [X \mid_H, Q_H]^*_H$$

One extens to positive and negative degrees and for any G-CW pair. □

We resume the results of this section in the following

Theorem 3. Equivariant stable cohomotopy $\Pi^*_?$ as defined in the first section defines an equivariant cohomology theory with multiplicative structure in the category of proper G-CW-complexes. It restricts to the equivariant stable cohomotopy of Lück [Lüc05a] on the category of finite proper G-CW complexes.

In view of this result, we drop the notation $\Pi^*_?$.

2.2 A map to equivariant topological K-theory

In the paper [LO01a], a construction for equivariant topological K-theory is proposed in the context of proper actions of a discrete group on G-CW complexes. It is defined in an analogous way to the one followed here for stable cohomotopy. Namely, one considers an equivariant Γ-space $\underline{G-\mathrm{Vec}}$ obtained from the symmetric monoidal category of finite dimensional vector spaces in $\mathbb{F}^\infty$, where F stands for $\mathbb{R}$, respectively $\mathbb{C}$. Precisely, one defines the functor

$$G\text{−Vec} : S \longmapsto \mid \mathrm{Fun}(\mathcal{E}(G), \oplus_{s\in S} V_s) \mid$$

One gets after group completion a G-space $K_G = \Omega \mathrm{B}\underline{G\text{−Vec}}$. One defines for the pair (X, A) and $n \leq 0$

$$K_G^n(X, A) = [\Sigma^n X/A, K_G]$$

Under the assumption of finiteness of the G-CW complex X, this agrees with the Grothendieck group of isomorphism classes of equivariant vector bundles over X, $\mathbb{K}_G(X)$ as defined in [LO01b]. The following result was proved in [Lüc05a]

Theorem 4. There is a natural transformation of equivariant cohomology theories with multiplicative structure for pairs of equivariant proper finite CW complexes, given by maps

$$\psi_G^* : \pi_?^* \to K_?^*$$

If $H \subset G$ is a finite subgroup of the group G, then the map

$$\psi_G^n : \pi_G^n(G/H) \to K_G^0(G/H)$$

is trivial for $n \geq 1$ and agrees for $n = 0$ under the identifications $\pi_G^0(G/H) = \pi_H^0$ and $K_G^0(G/H) = K_G^0(\{*\}) = R_{\mathbb{C}}(H)$ with the ring homomorphismus

$$A(H) \to R_{\mathbb{C}}(H) \quad [S] \mapsto [\mathbb{C}[S]]$$

which assigns to a finite H-set the associated complex permutation representation.

Something about the construction of the transformation shall be said. It is given as the restriction $\psi_G^n = \phi_G^{n,0}(a, 1_X)$ of pairings of equivariant cohomology theories

$$\phi_G^{m,n}(X; A, B) : \pi_G^m(X, A) \times K_G^n(X, B) \to K_G^{m+n}(X, A \cup B)$$

In order to describe them, let $a \in \pi_G^n(X, A)$ be an element represented by an over A trivial morphism $u : S^{\xi\oplus\underline{\mathbb{R}^k}} \to S^{\xi\oplus\underline{\mathbb{R}^{k+n}}}$ such that $k \geq 0$, $m + k \geq 0$. Denote now by v the morphism

$$S^{\xi\oplus\xi\oplus\underline{\mathbb{R}^k}} \to S^\xi \wedge_X S^{\xi\oplus\underline{\mathbb{R}^k}} \overset{\mathrm{id}\wedge_X u}{\to} S^\xi \wedge_X S^{\xi\oplus\underline{\mathbb{R}^{k+n}}} \overset{\sigma^{-1}}{\to} S^{\xi\oplus\xi\oplus\underline{\mathbb{R}^{n+k}}}$$

and notice that v is another representative of u. The bundle $\xi \oplus \xi$ carries a canonical structure of a complex vector bundle. Denote this bundle by $\xi_{\mathbb{C}}$. Let $\sigma^k(X; A \cup B) : K_G^{m+n}(X, A \cup B) \overset{\cong}{\to} K_G^{m+n+k}((X, A \cup B) \times (D^k, S^k))$ be the suspension isomorphism. Let $pr_k : X \times D^k \to X$ be the projection and $pr_k^*(\xi_{\mathbb{C}})$ be the vector bundle obtained from it by the pullback construction. Associated to it there is a Thom isomorphism

$$\begin{aligned} T_{pr_k^*(\xi_{\mathbb{C}})}^{m+n+k} &: K_G^{n+m+k}((X, A \cup B) \times D^k) \\ &\overset{\cong}{\to} K_G^{m+n+k+2\dim(\xi)}(S^{pr_k^*\xi_{\mathbb{C}}}, S^{pr_k^*\xi_{\mathbb{C}}|_{X\times S^{k-1}\cup(A\cup B)\times D^k}} \cup (X \times D^k)_\infty) \end{aligned}$$

where $(X \times D^k)_\infty$ stands for the copy of $X \times D^k$ given by the various points an infinity. Let

$$\begin{aligned} p_k : (S^{pr_k^*\xi_{\mathbb{C}}}, S^{pr_k^*\xi_{\mathbb{C}}|_{X\times S^{k-1}\cup(A\cup B)\times D^k}} \cup (X \times D^k)_\infty)) \\ \longrightarrow (S^{\xi\oplus\xi\oplus\underline{\mathbb{R}^k}}, S^{\xi\oplus\xi\oplus\underline{\mathbb{R}^k}|_{A\cup B}} \cup X_\infty) \end{aligned}$$

be the obvious projection, which induces by excision an isomorphism on K_G^*. Define an isomorphism

$$\begin{aligned} \mu^{m+n.m+n+k+2\dim(\xi)} : K_G^{m+n}(X, A \cup B) \\ \longrightarrow K_G^{m+n+k+2\dim(\xi)}(S^{\xi\oplus\xi\oplus\underline{\mathbb{R}^k}}, S^{\xi\oplus\xi\oplus\underline{\mathbb{R}^k}|_{A\cup B}} \cup X_\infty) \end{aligned}$$

by the composite $K_G^{m+n+k+2\dim(\xi)}(p_k)^{-1} \circ T^{n+m+k}_{pr_k^*\xi_{\mathbb{C}}} \circ \sigma^k(X, A \cup B)$. Define

$$\begin{aligned} \mu^{n.m+n+k+2\dim(\xi)} : K_G^n(X; A, B) \\ \longrightarrow K_G^{m+n+k+2\dim(\xi)}(S^{\xi\oplus\xi\oplus\underline{\mathbb{R}^k}}, S^{\xi\oplus\xi\oplus\underline{\mathbb{R}^k}|_B} \cup X_\infty) \end{aligned}$$

analogously. Let the desired map $\phi_G^{m,n}(X, A)(a)$ be the composite

$$\begin{aligned} & K_G^n(B) \xrightarrow{\mu^{n.m+n+k+2\dim(\xi)}} \\ & \qquad K_G^{m+n+k+2\dim(\xi)}(S^{\xi\oplus\xi\oplus\underline{\mathbb{R}^k}}, S^{\xi\oplus\xi\oplus\underline{\mathbb{R}^k}} \,|_B \cup X_\infty) \\ & \xrightarrow{K_G^{m+n+k+2\dim(\xi)}(v)} K_G^{m+n+k+2\dim(\xi)}(S^{\xi\oplus\xi\oplus\underline{\mathbb{R}^k}}, S^{\xi\oplus\xi\oplus\underline{\mathbb{R}^k}|_{A\cup B}} \cup X_\infty) \\ & \qquad\qquad \xrightarrow{\mu^{n.m+n+k+2\dim(\xi)^{-1}}} K_G^{m+n}(X, A \cup B) \end{aligned}$$

The maps $\phi_G^{m,n}(X, A \cup B)(a)$ do not depend of the choices of k and u. They define for the various elements $a \in \pi_G^m(X, A)$ pairings

$$\phi_G^{m+n}(A, B) : \pi_G^m(X, A) \times K_G^n(X, B) \to K_G^{m+n}(X, A \cup B)$$

Now we shall transcribe this result into our context.

Theorem 5. There is a natural transformation of equivariant cohomology theories with multiplicative structure for pairs of equivariant proper CW complexes

$$\psi_?^* : \pi_G^* \to K_?^*$$

Proof. Let $< \;\; >$: Sets $\to \mathbb{C} -$ Mod the functor which associates to a finite set the finite dimensional complex vector space with basis on it. This gives for any finite set S and any discrete group a functor

$$\mathrm{Fun}(\mathcal{E}(G), \mathrm{Set}_S) \longrightarrow \mathrm{Fun}(\mathcal{E}(G), \mathrm{Vec}_S)$$

Where Vec_S denotes the category of S-partitioned finite dimensional complex vector spaces. This gives for any objects S, T a map

$$\underline{G\text{−}\mathrm{Set}}_S \wedge \underline{G\text{−}\mathrm{Vec}}_T \to \underline{G \times G\text{−}\mathrm{Vec}}_{S\times T}$$

which after iterated geometrical realization turns into

$$\mathrm{B}G{-}\mathrm{Set} \wedge \mathrm{B}G{-}\mathrm{Vec} \to \mathrm{B}^2G \times G{-}\mathrm{Vec}$$

the argument used to define the multiplicative structure gives rise to a pairing

$$Q_G \wedge K_G \to K_G$$

determining for any $n, m, k, l \in \mathbb{Z}$, any proper G-CW pairs (X, A), (X, B) an in $(X; A, B)$ natural ring homomorphism

$$\Phi_G^{l-m,k-n} : [\Sigma^n X/A, Q_G^k]_G \times [\Sigma^m X/B, K_G^l]_G \to [\Sigma^{n+m} X/A \cup B, K_G^{k+l}]_G$$

We notice that this coincides with the pairing constructed in [Lüc05a]. Let $\varphi : \pi_G^0(X) \to [X, Q_G]_G$, $\beta : \mathbb{K}_G(X) \to [X, K_G]_G$ be the in X natural isomorphisms described above. Denote by $\mu : [X, \underline{G{-}\mathrm{Set}}]_G \times [X, G{-}\mathrm{Vec}]_G \to [X, \mathrm{B}^2G{-}\mathrm{Vec}]_G$ the morphism determined by the above described pairing and the composition with the diagonal map $\Delta : \underline{G \times G{-}\mathrm{Vec}} \to \underline{G{-}\mathrm{Vec}}$ Notice that for any finite proper G-CW complex X the following diagram is commutative

$$\begin{tikzcd}
{[X, \underline{G{-}\mathrm{Set}}]_G \times [X, \underline{G{-}\mathrm{Vec}}]_G} \arrow[r, "\mu"] \arrow[d] & {[X, \mathrm{B}^2G{-}\mathrm{Vec}]_G} \arrow[d] \\
{[X, Q_G]_G \times [X, K_G]_G} \arrow[r, "\Phi_G^{0,0}"'] & {[X, \Omega\mathrm{B}G{-}\mathrm{Vec}]_G \cong [X, \Omega^2\mathrm{B}^2G{-}\mathrm{Vec}]_G}
\end{tikzcd}$$

where the unlabeled arrows are the obvious structure maps. The naturality of φ, β, implies that we can complete the previous diagram with a lower block of the form

$$\begin{tikzcd}
{[X, Q_G]_G \times [X, K_G]_G} \arrow[r, "\Phi_G^{0,0}"] \arrow[d, "\varphi\times\beta^{-1}"'] & {[X, \Omega\mathrm{B}G{-}\mathrm{Vec}]_G} \arrow[d, "\beta^{-1}"] \\
\pi_G^0(X) \times \mathbb{K}_G^G(X) \arrow[r, "\phi_G^{0,0}"'] & \mathbb{K}_G^0(X)
\end{tikzcd}$$

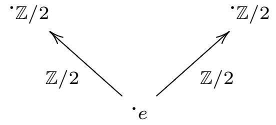
where ϕ stands for the pairing defined in [Lüc05a]. □

Recall that the classyfying space for proper actions is a proper G-CW complex characterized up to G-equivariant homotopy by the fact that the fixed point sets of finite subgroups are contractible and empty in other case. The classifying space for proper actions always exist, and there are many interesting examples (see for instance section 3 of the following chapter or [Lüc05c]).

Definition 13 (A Burnside Ring for arbitrary groups)**.** Let G be a discrete group and let $\underline{E}G$ be a model for the classifying space for proper actions for G, in the sense of [Lüc05c]. The homotopy theoretical Burside ring of G is defined to be the 0th-equivariant stable cohomotopy group of $\underline{E}G$. In symbols:

$$A^{ho}(G) = \pi_G^0(\underline{E}G)$$

Example 1. Let $D_\infty = \mathbb{Z}/2 * \mathbb{Z}/2$ be the infinite dihedral group. Recall that a model for $\underline{\mathrm{E}D_\infty}$ is one dimensional and schematically given by the following picture:



It follows that $A^{ho}(D_\infty) = \pi^0_{\mathbb{Z}/2}(\{*\}) \times \pi^0_{\mathbb{Z}/2}(\{*\})$.

We shall give more examples on the following chapter, where we deal with a generalization to the case of lie groups. In the third chapter of this work, we will study the Burnside ring from the point of view of completion theorems. We finally point out that we are even able to define a Burnside ring for arbitrary locally compact groups, as defined in chapter five.

Chapter 3

An Analytical Approach: lie groups

Stable homotopy theory and equivariant topology are nowadays a fundamental tool in algebraic topology. Until now, very few of the main methods of this discipline have been extended to the context of actions of non-compact groups. The aim of this work is to extend the definition of equivariant cohomotopy to the settig of proper actions of lie groups, as well as to present two applications of this.

In the first section of this work, we define equivariant cohomotopy in terms of certain nonlinear perturbations of fredholm morphisms of hilbert bundles over a certain proper G-CW complex. The reason for doing this is twofold: first, the seek for applications in analysis, and -perhaps more natural from the point of view of topology-, the fact that the technical difficulties involved in the proof of excision for equivariant cohomology theories, where the equivariance group is not discrete, cannot be solved with constructions using finite dimensional G-vector bundles. This is a phenomenon discovered in connection with the Baum-Connes Conjecture, see [Phi89], chapter 9 for a detailed description. We prove (Theorem 6 that our invariants generalize previous definitions, such as that of W. Lück [Lüc05a] in the context of proper actions of discrete groups on finite G-CW complexes. The proof has its roots in methods employed in the qualitative analysis of Partial Differential Equations.

After some computational remarks in section 3, we illustrate the applications of our methods in section 4. We generalize one Gauge theoretical invariant of 4-dimensional manifolds, due to Bauer and Furuta to allow proper actions of lie groups on four-manifolds and a refined invariant which also takes the group action in account. Finally, we introduce a Burnside ring in operator theoretical terms and as a test for the suitability of this definition. After some computational remarks we verify the extension of a weak form of the Segal Conjecture for a certain class of non-compact lie groups.

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3.1 Cocycles for Equivariant Cohomotopy

We begin by describing cocycles for equivariant cohomotopy. They are defined in terms of certain nonlinear operators on real G-Hilbert bundles, so we briefly recall first some well known facts in linear functional analysis parametrized over a space. A comprehensive treatment of them is given in the book [Phi89]. For matters of reality and functional analysis we remit to the text [Sch93], in particular to connections with Kasparov theory.

We point out that these constructions require proper actions of locally compact, second countable groups as an input. In the first part of this work we restrict our attention to proper actions of lie groups, though we also explain the possibility of modifying Phillips' slightly more general arguments to this setting.

We recall first some basic definitions and technical facts of equivariant topology.

Definition 14. Let G be a second countable, locally compact hausdorff group. Recall that a G-space is proper if the map

$$\underset{(g,x)\mapsto(x,gx)}{G\times X \underset{\theta_X}{\to} X\times X}$$

is proper.

Remark 2. In the case of lie groups, a proper action amounts to the fact that all isotropy subgroups are compact ant that a local triviality condition, coded in the Slice Theorem is satisfied [Pal61]. Specializing to discrete groups acting on well behaved spaces (see below), this conditions boils down to the fact that all stabilizers are finite.

The notion of a proper G-CW complex [Lüc89], p. 8 will be of relevance in this work. Recall that a G-CW complex structure on the pair (X, A) consists of a filtration of the G-space $X = \cup_{-1\leq n} X_n$ beginning mit A and for which every space is inductively obtained from the previous one by attaching cells in pushout diagrams of the type

$$\begin{tikzcd}
\coprod_i S^{n-1}\times G/H_i \arrow[r] \arrow[d] & X_{n-1} \arrow[d] \\
\coprod_i D^n \times G/H_i \arrow[r] & X_n
\end{tikzcd}$$

We say that a proper G-CW complex is finite if it consists of a finite number of cells $G/H \times D^n$.

The following result enumerates some facts which we will need in this work, which are proven in chapter one of [Lüc89]:

Proposition 10. Let (X, A) be a proper G-CW pair

1. The inclusion $A \to X$ is a closed cofibration.
2. A is a neighborhood G-deformation retract, in the sense that there exists a neighborhood $A \subset U$, which is a G-equivariant deformation retract. The neighborhood can be chosen to be closed or open.

Some equivariant Hilbert bundles are the main objects on which our constructions begin. Let us recall basic terminology of [Phi89], which works in a slightly more general context.

Definition 15. Let X be a locally compact, hausdorff proper G-space. A Banach bundle over X is a locally trivial fiber bundle E with fiber on a Banach space H, whose structure group is the set of all isometric linear bijections of H with the strong topology.

If H is a (real) Hilbert space, we will speak of a Hilbert bundle. A G-Hilbert bundle is a map $p : E \to X$ such that there is a continuous action of the locally compact group G in the total space, the map p -called the projection- is assumed to be G-equivariant and the action on the total space is given by linear isometries, in the sense that for any g, the traslation $E_x \to E_{gx}$ is a linear isometry for any x.

Let E and F be hilbert bundles over X. A linear morphism from E to F is an equivariant, continuous function $t : E \to F$ covering the identity on X, and for which the fiberwise adjoint t^* defined by $< t^*x, y >=< x, ty >$ is continuous. The support of a morphism t is the set $\{x \in X \mid tx \neq 0\}$

A linear morphism $t : E \to F$ is a compact morphism if it is fiberwise compact in the usual sense (that is, for every x, $t_x.E_x \to F_x$ maps bounded sets to relatively compact sets) and in adition, for every point $x \in X$, there exist local trivializations $a : E \mid_U \to U \times E_x$, $b : F \mid_U \to U \times F_x$ such that $bta^{-1} : U \times E_x \to U \times F_x$ is given by an expresion $(y, x) \mapsto (y, \psi(x))$ for a norm continuous map $\psi : U \to L(E_x, F_x)$ into the bounded linear maps between the fibers over x.

A linear morphism $t : E \to F$ is said to be fredholm if there exists a morphism $s : F \to E$ such that $st - 1$ and $ts - 1$ are compact morphisms with compact supports. A fredholm morphism is said to be essentially unitary if one can take $t = s^*$ in the definition. We recall the existence of G-invariant riemannian metrics on vector bundles over proper G-spaces, which is proved for instance in [Pal61] as consecuence of the slice theorem.

Remark 3.
1. Note that we choose the structure group of our bundles to be the isometric bijections with the strong operator topology. Other choices, like the weak topology would not give *enough vector bundles*, as pointed out by Phillips in [Phi89], chapter 9.
2. Note that we forced the existence of adjoints for our morphisms. This is technically convenient. We also assume that adjoint operators are always continuous in the operator norm, and the same for morphisms of Hilbert bundles.

We now ennumerate a collection of facts on linear morphisms and G-Hilbert Bundles which we will use later.

Proposition 11 (Proper stabilization theorem)**.** Let E be a G-Hilbert bundle over a proper G-space. Denote by $\mathcal{H}$ the numerable Hilbert space consisting of the numerable sum of the space of square integrable functions in G, in symbols $\mathcal{H} = \oplus_{n=1}^{\infty} L^2(G)$. Let $\mathcal{H} \times X$ be the associated trivial Hilbert G-bundle. There exists an equivariant linear isomorphism of vector bundles

$$E \oplus \mathcal{H} \times X \cong \mathcal{H} \times X$$

.

Proof. Theorem 2.9, p. 29 of [Phi89], Theorem 2.1.4 , p. 58 of [Sch93]. We point out that Phillips realizes this isomorphism to be an adjointable morphism between the G-$C_0(X)$- hilbert modules $\Gamma(E)\otimes_{C_0(X)}\mathcal{H}\times X$ and $\mathcal{H}\otimes C_0(X)$. The identification of such a morphism with an isomorphism of G-Hilbert bundles is consequence of lemma 1.9 in [Phi89]. □

Next, we modify Phillip's definition of complex equivariant K-theory for proper actions of locally compact groups, [Phi89] to allow real cocycles. The main reference for technical isssues concernig the passage to real K-theory is [Sch93].

Definition 16. The real equivariant K-theory of the proper and finite G-CW complex X $KO_0^G(X)$ is represented by cocycles (E, F, l), where E and F are real G-Hilbert bundles and $l : E \to F$ is a fiberwise *linear* real fredholm morphism. A cocycle is said to be *trivial* if l is fibrewise unitary. Two cocycles $(E_i, F_i, l_i)_{i=0,1}$ are equivalent if there exists a trivial cocycle τ such that $l_0 \oplus \tau = l_1 \oplus \tau$ is homotopic to a trivial morphism.

There exists at least one equivalent approach to the definition of equivariant K-theory in this context. We just recall one of them as it is needed as a technical modificationi in one of our arguments.

- $KO_0^G(X)$ can be realized as the set of essentially unitary fredholm linear morphisms. Two such cocycles are said to be equivalent if they become homotopic via an essentially unitary homotopy after adding a trivial cocycle (Theorem 4.7, page 56 in [Phi89]) for the complex case.

We now enumerate two consequences of the proper stabilization theorem, which are fundamental for veryfing excision in Phillips' construction of equivariant K-theory.

Proposition 12. Let $i : U \to X$ be the inclusion of a G-invariant, open subset of X. Let (E, F, t) be a linear cocycle over U such that t is fibrewise bounded and has a bounded fredholm inverse. Then, there exists a linear cocycle $(X \times \mathcal{H}, X \times \mathcal{H}, r)$ $i^*(r)$ and t agree after adding a unitary linear cocycle.

Proof. Proposition 5.9, p. 74 in [Phi89]. We recall that the constructed classes agree after application of the proper stabilization theorem, for the real modification see [Sch93], theorem 2.1.4 p.58 .

□

Proposition 13. Let X be a proper G-CW complex and $\varphi : E \to E$ be a fibrewise fredholm operator defined on the space $X\times I$. Denote by $\varphi_0 : E_0 \to E_0$ the restriction to $X \times \{0\}$. Then:

1. There exists a unitary cocycle ρ between (E, E, φ) and (E_0, E_0, φ_0). Moreover, the isomorphism can be taken to be unitary over a fixed, invariant subspace $A \subset X$.
2. Let $A \subset X$ be a G-subcomplex and $l : F \mid_A \to E \mid_A$ be a bounded morphism. Then, there exists a linear cocycle $(l^{'}, E, F)$, defined over X such that $i^*(l)$ and l are equivalent .

3. Let $A \subset X$ be a G-invariant closed subset and $U \subset X$ an open neighborhood of which A is a deformation retract. Suppose that $(U \times K, U \times K, l)$ is an essentially unitary cocycle over U, where K is a strong unitary G-representation in a hilbert space. Then, there exists an essentially unitary cocycle $(X \times H, X \times H, F)$ such that $i^*(F) = l$.

Proof. 1. As the involved bundles are locally trivial, the total space of E carries the weak topology with respect to the set $p^{-1}(X_i)$, where X_i is an element of the filtration in the basis G-CW complex structure cfr. lemma 1.26 in [Lüc89]. Hence, the statement reduces to the case where $(X, A) = (G/H \times D^n, G/H \times S^{n-1})$ and E has the form $G \underset{H}{\times} \mathcal{H}$, for a given strong, unitary and continuous H-representation $\mathcal{H}$ in a separable real hilbert space. Let $\mathcal{U}_c(\mathcal{H})$ be the subspace of the H-equivariant, unitary operators u in $\mathcal{H}$, for which the conjugation with an arbitrary element $h^{-1}uh$ is a continuous operator (recall that this is a contractible space after results of Segal, it is an ANR). Giving an isomorphism ρ as described above amounts to give a map $\rho \in \mathrm{Map}(D^n \times I, \mathcal{U}_c(\mathcal{H}))$ which is the identity on $G/H \times S^{n-1} \times \{0\}$. There is no obstruction for doing this because the inclusion $G/H \times S^{n-1} \to G/H \times D^n$ is a cofibration and $\mathcal{U}_c(\mathcal{H})$ is contractible.

□

Remark 4. Note the technical difficulties arising in the extension of linear morphisms over arbitrary spaces. As explained in point 3, we are only able to extend morphisms which are defined on globally trivial hilbert bundles using the proper stabilization theorem. The attempt to overcome this difficulty by extending a section of a bundle instead of a morphisms breaks down essentially because the morphisms as defined here are the sections of a bundle whose fiber is the space of operators $L(H_1, H_2)$ with the $*$-strong topology, which is not even a Banach bundle.

The second ingredient for our construction of cocycles for stable cohomotopy are some basic notions of nonlinear functional analysis. This is a complete subject itself, whose developements are certainly less known by orthodoxe homotopy theorists. More or less encyclopedical references to this topic are the books [Ber77] and [Dei85]. The reader interested in its relationship to Hopf bifurcation problems should read [Ize05].

Relying on our interests, we shall change between two equivalent approaches to this. One of them , the “no -zeros on the boundary- picture” will be useful in applications and has its origin in the theory of partial differential equations. For our purposes in algebraic topology, we shall give the following, technically more convenient

Definition 17. Supose that $k : H \to H$ is a (possibly nonlinear) compact map in a (real) Hilbert space, in the sense that k sends bounded sets into relatively compact sets. A map $f = t + k : H \to H$ is a compact perturbation of the real fredholm linear map f if the preimages of bounded sets under f are bounded. Let $t : E \to F$ be a fredholm morphism between the G-hilbert bundles over the proper G- space X. A compact perturbation of t is a continuous, equivariant map $f : E \to F$, which has fibrewise the form $t + k$ for a compact perturbation of f.

Definition 18. Let X be a locally compact, proper G-CW complex and let G be a locally compact group. Let l be a real linear cocycle representing a class in $KO^0_G(X)$ in the sense of Phillips.

A cocycle for the equivariant cohomotopy theory of X, $\Pi^l_G(X)$ is a four-tuple (E, F, l, c) where

- E is a real G- Hilbert bundle over X, with a linear, selfadjoint fredholm morphism $\tilde{l} : E \to F$, which is equivalent to l in the sense of Phillips.
- A compact perturbation of l, c, such that the map $l + c$ is proper and extends to a map between one-point compactification bundles.

Two cocycles (E, F, l, c) and $(E^{'}, F^{'}, l^{'}, c^{'})$ are equivalent if there is a linear, unitary cocycle $(H, H^{'}, k)$ such that $E \oplus H, F \oplus H, l \oplus k$ and $E^{'} \oplus H^{'}, F^{'} \oplus H^{'} l^{'} \oplus k$ are unitary equivalent as linear cocycles, by an isomorphism which preserves the compact perturbations.

Two cocycles E, F, l, c and $E, F, l, c^{'}$ are homotopic if there exists a homotopy $H : S^E \times I \to S^F$, pointed over every fiber and relative to l. such that $H\mid_0 = l + c$, $H\mid_1 = l + c^{'}$.

The set $\Pi^l_G(X)$ is called the G-equivariant cohomotopy of X in degree l.

Remark 5. Note that our groups are naturally graded by $KO^0_G(X)$. This generalizes the fact that for compact groups certain classical definitions of equivariant cohomology theories [May96] are assumed to be $RO(G)$-graded. This is included in our definition in the case of compact groups, where the representation V is associated to a Fibrewise equivariant fredholm morphism whit index the trivial bundle $X \times V$.

Remark 6 (The "no-zeros in the boundary" -picture)**.** Let X be a locally compact, Hausdorff proper G-space. Let E and F be real G- Hilbert bundles over X. Let $t : E \to F$ be a real fredholm morphism between them. A compact perturbation of t is a continuous equivariant map which is fibrewise of the form $t+k$, where k is a *non necessary linear*, compact fibrewise map (it maps fibrewise bounded sets to relatively compact subsets) defined on the unitary disk $D(E) = \{x \in E \mid |x| \leq 1\}$. We introduce the notation $C_t(D, F)$ for the set of compact perturbations of the fredholm operator t, with the topology which is given by the supremum norm, and obviate the corresponding fibrewise definition. Moreover, we shall restrict us to the subspace $C^{\neq 0}_t(D, F)$ consisting of *strongly non zero* compact perturbations, that is those c for which $cx = 0$ has no solutions on the boundary of the unit disk D. In the "no- zeros on the boundary" -picture, stability up to unitary linear morphisms carries over. The homotopies are not only assumed to be compact, but also strongly non- zero.

Let $f = t + k$ be a compact perturbation of a fredholm map, as given in the definition. Then f extends to a map of one point compactifications of the Hilbert space $f : S^H \to S^H$. Moreover, homotopy classes of pointed nonlinear maps are in correspondence with strongly nonzero compact homotopy classes of perturbations defined on the unitary disk. The same holds parametrized and equivariant over a proper G-space.

This is essentially due to the fact that the maps defined in the previuos construction amount to maps of pairs $(D(H), \partial D(H)) \to (H, H - \{0\})$, which are equivalent via excision and homotopy equivalence to (S^H, S^H), via the intermediary pairs $(S^H, H - \{0\})$ and $(S^H, S^H - \mathrm{Int}D(H))$. Note that the point at infinity in the pointed picture is identified with the point $(0,1)$ in the *no-zeros on the boundary* picture, if we identify the pair $D(H) \subset H \oplus \mathbb{R}$ in the usual way, and everything holds fibrewise and equivariant over a proper G-space.

We describe now an additive structure in equivariant cohomotopy theory.

Let $(E_0, F_0 l_0, c_0)$ and $(E_1, F_1 l_1, C_1)$ be cocycles in the equivariant cohomotopy theory of a given degree l.

Let us suppose without loss of generality that we have representatives of the form (E_0, F_0, l, c_0) and (E_0, F_0, l, c_1). Let $X \times \mathbb{R} \to X \times \mathbb{R}$ be the trivial bundle and . Denote by $S^{\mathbb{R}}$ the one point compactification bundle. Define the pinching map $S^{\mathbb{R}} \to S^{\mathbb{R}} \vee S^{\mathbb{R}}$ as the obvius extension of the map which sends the positive ray under the map $ln(x)$ and $-ln(-x)$ in the negative ray.

The sum of two cocycles is represented by the cocycle $(E_0 \oplus \mathbb{R}, F_0 \oplus \mathbb{R}, l \oplus \mathrm{id}, c)$, where $l \oplus \mathrm{id} + c : S^{E_0 \oplus \mathbb{R}} \to S^{F_0 \oplus \mathbb{R}}$ is given as the following composition

$$S^{E_0} \wedge_X S^{\mathbb{R}} \overset{\mathrm{id} \wedge_X \nabla}{\to} S^{E_0} \wedge_X S^{\mathbb{R}} \vee_X S^{\mathbb{R}} \overset{\approx}{\to}$$

$$S^{E_0 \oplus \mathbb{R}} \vee_X S^{E_0 \oplus \mathbb{R}} \overset{(l \oplus \mathrm{id} + c_0) \wedge_X (l \oplus \mathrm{id} + c_1)}{\to} S^{F_0 \oplus \mathbb{R}}$$

The zero element is represented by a cocycle $(E \oplus \mathbb{R}, F \oplus \mathbb{R}, l, c)$ such that c extends to a map sending fibrewise S^E to the point at infinity. .

The inverse of an element (E, F, l, c) is represented by the element $(E, F, l, -c)$. We have the following result:

Proposition 14. The operations described above turn the equivariant cohomotopy theory in degree l into an abelian group.

There is a relative version for pairs (X, A) of proper G-CW complexes. An element in Π_l is represented by a compact perturbation of a fibrewise perturbation of a fredholm operator $\varphi + c : E \to F$, which extends to the one-point compactification bundles beeing constant over the subspace A, with the value at infinity. Note that this is consistent with the usual identification of X with the pair (X, ϕ).

Alternatively, in the "no-zeros in the boundary"-picture, a cocycle for the equivariant Π-theory can be chosen to be a conpact perturbation of a fredholm map $\varphi + c : DE \to F$ whithout zeros on the boundary, which is constant over A with value $(0,1)$.

We now construct a multiplicative structure on the equivariant cohomotopy theory.

$$\cup : \Pi_G^{l_1}(X, A_1) \times \Pi_G^{1_2}(X, A_2) \to \Pi_G^{l_1 + l_2}(X, A_1 \cup A_2)$$

Consider for this representing elements $u_i = \varphi_i + c_i \in \Pi_G^{l_i}(X, A_i)$ for $i \in \{1, 2\}$, where c_i is a compact map accepting fibrewise an extension to the one-point compactification, constant over A_i with the value at infinity. $u_1 \cup u_2$ is the cocycle defined as $(E_1 \oplus E_2, \varphi_1 \oplus \varphi_2, C)$ where the map C is such that $C : (e_1, e_2) \mapsto (c_1(e_1), c_2(e_2))$. Note that this map allows an extension to the one-point compactification.

Proposition 15. Let (X, A) be a proper G-CW pair. There exists a natural sequence

$$\Pi_G^l(X, A) \overset{\rho^*}{\to} \Pi_G^l(X) \overset{i^*}{\to} \Pi_G^l(A)$$

which is exact in the middle, where ρ and i denote the inclusion of A into X and X into (X, A), respectively.

Proof. That $i^* \circ \rho^* = 0$ is clear form the definitions. Let now $\varphi + c : E \to F$ be a cocycle for which $i^*([\varphi + c])$ is compactly homotopic to the trivial morphism over A.

In view of proposition 11, we can choose a representative (which we denote by the same symbols) for which both E and F are the trivial G-Hilbert bundle $\mathcal{H} \times X \to X$ and c is constant over A with value ∞. Using proposition 12, we can assume up to equivalence that the fredholm linear operator extends to a onoe $\tilde{\varphi}$ in all of X.

Suppose that there is a homotopy $h_t : i^*S^E \times I \to i^*S^F$ defined over A which begins with $i^*[\varphi + c]$ and ends with a map $\varphi + c$ which sends the space E to the base point at infinity. As $X - A$ is build up of a finite number of equivariant cells, one can argue inductively to extend h to a map $H : S^E \to S^F$, defined on all X such that $h \mid_{A \times I} = h$. H determines a homotopy betweeen certain element $\rho^*(\tilde{\varphi} + \tilde{c})$ defined over X and $\varphi + c$.

□

Proposition 16. Let (X, B) be a proper, finite G-CW pair obtained as the pushout with respect to the cellular map $(f, F) : (X_0, A) \to (X, B)$ as in the following diagram:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d] & B \arrow[d] \\
X_0 \arrow[r, "F"'] & X
\end{tikzcd}$$

then the map $(f, F)^* : \Pi^l_G(X, B) \to \Pi^l_G(X_0, A)$ induces a natural isomorphism.

Proof. Let $(E, F\varphi + C) \in \Pi^l_G(X_0, A)$. Due to propositon 12, it is possible to assume that there exists a linear morphism $\tilde{\varphi} : E \oplus E^{'} \to F \oplus E^{'}$ defined oxer X such that $F^*(\tilde{\varphi})$ and φ differ by addition of an unitary cocycle. As $c \mid_A = \infty$, it is posible to extend c to a map $\tilde{c}$ defined on X for which $\tilde{c} \mid_B = \infty$.Then $(F^*, f^*) : \pi^*_G(X, B) \to \pi^*_G(X_0, A)$ sends $\tilde{\varphi} + \tilde{c}$ to $\varphi + c$. This proves surjectivity. To prove injectivity, recall that if $\varphi + h_t : E \times I \to E$ is a nullhomotopy starting with $(F^*, f^*)(\varphi + c)$, ending with the constant ∞. As before, as c is constant over B and $(F^*, f^*)(c)$ is trivial over A, it is possible to extend the map h_t to a homotopy $\tilde{h}_t$ defined over X which is trivial over B and which begins with $\varphi + c$, ends with a constant map. this shows that the map is injective. □

The induction structure is defined as follows. To illustrate the proof and avoid technicalities, we restrict ourselves to the case where G is a Lie group.

Proposition 17. Let $\alpha : H \to G$ be a lie group homomorphism. Then there exists a group homomorphisn

$$\Pi^*_G(\mathrm{ind}_\alpha(X, A)) \to \Pi^*_H(X, A)$$

satisfying

1. Bijectivity. If $\ker(\alpha)$ acts freely on (X, A), then the map is an isomorphism.

2. Compatibility with the boundary homomorphisms. $\delta^n_H \circ \mathrm{Ind}_\alpha = \mathrm{ind}_\alpha \circ \delta^n_G$.

3. Functoriality. If $\beta : G \to K$ is a group homomorphism, then the diagram commutes:

4. Compatibility with conjugation.For any $g \in G$, the homomorphism

$$\mathrm{ind}_{c(g):G\to G}\Pi^n_G(X, A) \to \Pi^n_G(\mathrm{ind})_{\mathrm{c(g):G\to G}}(\mathrm{X, A}))$$

   agrees with the map $\pi^n_G(f_2)$, where $f_2 : (X, A) \to \mathrm{ind}_{c(g):G\to G}$ sends x to $(1, g^{-1}x)$ and $c(g)$ is the conjugation isomorphism in G

Proof. 1. Let $\varphi + c : E \to E$ be a compact perturbation over the space $(\mathrm{ind}_\alpha X, A)$. The map $\mathrm{i} : X \to \mathrm{ind}_\alpha(X)$ $(x \mapsto (1_G, X)$ induces a group homomorphism $\mathrm{Fix}^*_G(\mathrm{ind}_\alpha X, A) \to \mathrm{Fix}^*_H(X, A)$. An inverse is given by the map which associates to a linear cocycle (E, F, φ) the cocycle $(E/H, F/H, \varphi/H)$. It is easy to show that this is the case for the perturbation and that this still satisfies the boundedness condition. We point out that this only makes sense for lie groups, and cannot be generalized to the more general context of locally compact groups, see remark below.

2. Follows from the naturality of the induced bundle construcctions.

3. Follows from the functoriality of the induced vector bundle construction.

4. Compatibility with conjugation. Follows from element chasing in the diagram

$$\begin{tikzcd}
\Pi^l_G(\mathrm{ind}_{c(g)}(X, A)) \arrow[r, "\mathrm{ind}_{c(g)}"] & \Pi^l_G(X, A) \\
\Pi^l_G(\mathrm{ind}_{c(g)}(X, A)) \arrow[u, "="] \arrow[ur, "\Pi^n_G(f_2)"'] &
\end{tikzcd}$$

   where $f_2 : (X, A) \to \mathrm{ind}_{c(g)}(X, A)$ is given by $x \mapsto (1, g^{-1}x)$ and $c(g)(g) = gg^{'}g^{-1}$.

□

Remark 7. N.C. Phillips proves in [Phi89], corollary8.5 , p. 131 a more general result for equivariant k-theory, allowing proper actions of locally compact groups as input, instead of only lie groups. It is certainly plausible to try to generalize them, though we prefer to keep the arguments in this note as easy as possible, with technicalities only in the background. The main problematic point in this is that we cannot guarantee the local triviality of bundles $E/H \to X/H$ unless H is lie, see [Pal61].

We shall prove following properties which turn the equivariant cohomotopy theory into a G-cohomology theory, the first of which is

Proposition 18 (Homotopy invariance)**.** Let $f_0, f_1 : (X, A) \to (Y, B)$ be two G-maps of pairs of proper G-CW complexes. If they are homotopic, then $\Pi^*_G(f_1) = \Pi^*_G(f_2)$.

Proof. in view if the naturality of the construction, this amounts to prove that $\Pi^*_G(h) = \mathrm{id}$ for the map $h : (X, A) \times I \to (X, A) \times I$ given by $(x, t) \mapsto (x, 0)$. Let (E, F, l, c) be a nonlinear cocycle representing an element in $\mathrm{Fix}^*_G(X, A \times I)$. In the notation of proposition 13, there exist unitary morphisms $u : E \to h^*(E)$, $v : F \to h^*(F)$ covering the identity $X \times I \to X \times I$ such that the restrictions to E_0, F_0 are the respective identities. Note that the composition $f = h^*(E) \overset{u^{-1}}{\to} E \overset{\varphi+c}{\to} F \overset{v^{-1}}{\to} h^*(F)$ is homotopic to $h^*(\varphi + c)$ relative to $h^*(\varphi)$. After checking out the conditions for the definition of $\mathrm{Fix}^*_{\mathrm{G}}$, one has that the equivalence classes $\Pi^*_G(h)(\varphi + c) = [f] = [\varphi + c]$ agree. □

We construct a suspension isomorphism

$$\sigma^{X,A}_n : \Pi^n_G(X, A) \to \Pi^{n+1}_G((X, A) \times (I, \{0, 1\}))$$

Given $\varphi + c : E \to F \in \pi^n_G((X, A)$, form the bundle $E^{'} = E \oplus \mathbb{R}$, denote by p the fibrewise projection on $E^{'} \to E$ and define the map $\sigma_n(\varphi + c) : DE^{'} \times I \to E^{'} \times I$ defined as $(e, t) \mapsto p \circ \varphi + (\log(t) - (\log(-t)))(c(p(v))$. By this means, we obtain a compact map which extends to the fibrewise one-point compactifications, beeing trivial on the required subspace. Given an element $\varphi + c \in \pi^{n+1}_G((X, A) \times I, \{0, 1\})$, consider a unitary, fibrewise linear cocycle $u : E \to E_0 \times I$ $v : F \to F_0$ covering the identity $X \times I \to X \times I$, which restricted over the subspace $A \times I \cup X \times \{0, 1\}$ is the identity map. The map constructed as $E_0 \times I \overset{u^{-1}}{\to} E \overset{\varphi+c}{\to} F \overset{v}{\to} F_0 \times I$ determines an inverse for $\sigma_n X, A)$.

A coboundary map is defined as the composition

$$\begin{aligned}\Pi^n_G(A) &\xrightarrow{\sigma^n_G(A)} \Pi^{n+1}_G(A \times I, A \times \{0, 1\}) \\ &\xrightarrow{\Pi^{n+1}_G(i_1)^{-1}} \Pi^{n+1}_G(X \cup_{A\times\{0\}} A \times I, X \coprod A \times \{1\}) \\ &\xrightarrow{\Pi^{n+1}_G(i_2)} \Pi^{n+1}_G(X \cup_{A\times\{0\}} A \times I, A \times \{1\}) \\ &\xrightarrow{\Pi^{n+1}_G(\mathrm{pr}_1)^{-1}} \Pi^{n+1}_G(X, A)\end{aligned}$$

Where the map $\Pi^{n+1}_G(i_1)$ is bijective by excision and $\Pi^{n+1}_G(\mathrm{pr}_1)$ is bijective because of homotopy invariance.

Several approaches have been proposed towards the definition of equivariant cohomotopy theory for proper actions. The work of Lück [Lüc05a] using finite dimensional bundles deals with the difficulties appearing in the case where a discrete group acts on a finite G-CW complex. We briefly recal this approach

Fix an equivariant, proper G-CW complex. Form the category

$\mathrm{SPHB}^G(X)$ having as objects the G-sphere bundles over X. A morphism from $\xi : E \to X$ to $\mu : F \to X$ is a bundle map $S^\xi \to S^\mu$ covering the identity in X, which preserves fiberwise the basic points. A homotopy between the morphisms u_0, u_1 is a G-bundle map $h : S^\xi \times [0, 1] \to S^\mu$ from the bundle $S^\xi \times [0, 1] \to [0, 1] \times X$ to the bundle S^μ covering the projection $X \times [0, 1] \longrightarrow X$ and preserving the base points on every fiber such that its restriction to $X \times \{i\}$ is u_i for $i = 0, 1$. Let $\underline{\mathbb{R}^n}$ be the trivial vector bundle over X, which is furnished with the trivial action of G. Two morphisms of the form

$$S^{\xi_i \oplus \underline{\mathbb{R}^{k_i}}} \to S^{\xi_i \oplus \underline{\mathbb{R}^{k_i+n}}}$$

are said to be equivalent if there are objects μ_i in $\mathrm{SPHB}^G(X)$ and an isomorphism of vector bundles $\nu : \mu_0 \oplus \xi_0 \cong \mu_1 \oplus \xi_1$ such that the following diagram of morphisms in $\mathrm{SPHB}^G(X)$ commutes up to homotopy

$$\begin{tikzcd}
S^{\mu_0\oplus\underline{\mathbb{R}^{k_1}}} \wedge_X S^{\xi_0\oplus\underline{\mathbb{R}^{k_0}}} \arrow[r, "\mathrm{id}\wedge_X u_0"] \arrow[d, "\sigma_1"'] & S^{\mu_0\oplus\underline{\mathbb{R}^{k_1}}} \wedge_X S^{\xi_0\oplus\underline{\mathbb{R}^{k_0+n}}} \arrow[d, "\sigma_2"] \\
S^{\mu_0\oplus\xi_0\oplus\underline{\mathbb{R}^{k_0+k_1}}} \arrow[d, "S^{\nu\oplus\mathrm{id}}"'] & S^{\mu_0\oplus\xi_0\oplus\underline{\mathbb{R}^{k_0+k_1+n}}} \arrow[d, "S^{\nu\oplus\mathrm{id}}"] \\
S^{\mu_1\oplus\xi_1\oplus\underline{\mathbb{R}^{k_0+k_1}}} \arrow[d, "\sigma_3"'] & S^{\mu_1\oplus\xi_1\oplus\underline{\mathbb{R}^{k_0+k_1+n}}} \arrow[d, "\sigma_4"] \\
S^{\mu_1\oplus\underline{\mathbb{R}^{k_0}}} \wedge_X S^{\xi_1\oplus\underline{\mathbb{R}^{k_1}}} \arrow[r, "\mathrm{id}\wedge_X u_1"'] & S^{\mu_1\oplus\underline{\mathbb{R}^{k_0}}} \wedge_X S^{\xi_1\oplus\underline{\mathbb{R}^{k_1+n}}}
\end{tikzcd}$$

where the isomorphisms σ_i are determined by the fiberwise defined homeomorphism $S^{V\oplus W} \approx S^V \wedge S^W$ and the associativity of smash products, which holds for every pair of representations V,W. We recall now W. Lück's definition of equivariant cohomotopy:

Definition 19. Let X be a G-CW complex, where G is a discrete group and X is finite. We define its n-th G-equivariant stable cohomotopy group, $\pi^n_G(X)$ as the set of homotopy clases of equivalence classes of morphisms $u : S^{\xi\oplus\underline{\mathbb{R}^k}} \to S^{\xi\oplus\underline{\mathbb{R}^{k+n}}}$ under the above mentioned relation. For a G-CW pair, (X, A) we define $\pi^n_G(X, A)$ as the equivalence classes of morphisms which are trivial over A, i.e. those which are given by a representative $u : S^{\xi\oplus\underline{\mathbb{R}^k}} \to S^{\xi\oplus\underline{\mathbb{R}^{k+n}}}$ which satisfies that over every point $a \in A$, the map $u_a : S^{\xi_a\oplus\mathbb{R}^k} \to S^{\xi_a\oplus\underline{\mathbb{R}^{k+n}}}$ is constant with value the base point. For a pair of bundle morphisms $u : S^{\xi\oplus\underline{\mathbb{R}^k}} \to S^{\xi\oplus\underline{\mathbb{R}^{k+n}}}$, $v : S^{\xi'\oplus\underline{\mathbb{R}^k}} \to S^{\xi'\oplus\underline{\mathbb{R}^{k+n}}}$, the sum is defined as the homotopy class of the morphism

$$u : S^{\xi\oplus\xi'\oplus\underline{\mathbb{R}^k}} \wedge_X S^{\mathbb{R}} \overset{\mathrm{id}\wedge_X\nabla}{\to} S^{\xi\oplus\xi'\oplus\underline{\mathbb{R}^k}} \wedge_X (S^{\mathbb{R}} \vee_X S^{\mathbb{R}}) \overset{\sigma_3}{\to}$$
$$(S^{\xi\oplus\underline{\mathbb{R}^k}} \wedge_X S^{\mathbb{R}}) \vee_X (S^{\xi'\oplus\underline{\mathbb{R}^k}} \wedge_X S^{\mathbb{R}}) \overset{(u\wedge_X id)\vee_X(v\wedge_X id)}{\to}$$
$$S^{\xi\oplus\xi'\oplus\underline{\mathbb{R}^{k+n}}} \wedge_X S^{\mathbb{R}}$$

where σ_3 is the canonical isomorphism given by the fiberwise distributivity and associativity isomorphisms and ∇ denotes the pinching map $S^{\mathbb{R}} \to S^{\mathbb{R}} \vee S^{\mathbb{R}}$. The relative version for elements lying in the group of a pair, $\pi^n_G(X, A)$ translates word by word when one sets all sphere bundles and morphisms to be trivial over A.

Our approach extends this notion and solves the principal problem of this construction, namely: the the lack of finite dimensional G-vector bundles to represent excisive G-cohomology theories. The crucial result in this is the construction of an index theory in the context of parametrized nonlinear analysis. The immediate goal is

Theorem 6. Let G be a discrete group acting on a G-CW pair (X, A). Denote by l a selfadjoint fredholm morphism whose fibrewise index is a trivial virtual vector bundle of dimension p. The *parametrized Schwartz index* defines an isomorphism

$$\Pi^l_G(X,A) \to \pi^p_G(X,A)$$

We give two proofs of theorem 6, relying in the two approaches to nonlinear analysis: the *no-zeros on the boundary* and the proper maps one. Because of its simplicity, we give now the proof relying on the first one. We point out that the techniques in the first case can only be used for fredholm operators of positive fredholm index, whereas the second one is valid more generally, at the price of technicalities. In both cases, the initial point is an approximation lemma for compact perturbation of fredholm maps, which takes the following form:

Proposition 19. Suppose that $U \subset E$ is a locally trivial bundle with fiber a bounded subset of the hilbert bundle E. Suppose that $c : U \to F$ is a bounded, fibrewise compact map with target on a G-Banach bundle. Given $\epsilon > 0$, there exist a continuous , bounded fibrewise mapping $t_\epsilon : U \to Z_\epsilon$ such that

- Z_ϵ is a subset of a finite dimensional subspace of F.
- $\mid t_\epsilon x - tx \mid \leq \epsilon$ on every fiber and the image of f_ϵ is contained in the convex hull of $f(U)$.

Proof. It is essentially proved in page 89 of [Ber77] in the non- equivariant and non parametrized case, so we just sketch the modification we need. First of all, we assume the existence of an invariant riemmanian metric on F, which can be done since the action on the basis space is proper and the group is locally compact.

Since the closure of the image of U under c is compact, the image over every point can be covered by a finite union of balls of radius ϵ and centers $y_1, \dots y_n$. Moreover, in presence of the invariant riemannian metric, this can be done fibrewise over X. Let Z_ϵ be the G-subbundle of Y which is fibrewise spanned by the y_i. We now construct an invariant partition of unity on U as follows. Over a point in X, define for $0 \geq i \leq k$ and $x \in U$ $\mu_i(x) = \max\{0, \epsilon - \mid cx - y_i \mid\}$. We note that this is a G-invariant function since the norm is it. Put $\lambda_i = \frac{\mu_i}{\Sigma \mu_i}$. and notice that $\Sigma \lambda_i = 1$. Define $c_\epsilon(x) = \Sigma_i \lambda_i y_i$. This defines an equivariant map because of the linear action of the group on every fiber. Now, notice that

$$\mid cx - c_\epsilon x \mid = \mid \Sigma_i \lambda_i(x)(cx - y_i) \mid \leq \Sigma_i \lambda_i x \mid cx - y_i \mid \leq \epsilon$$

□

We are now able to state our main result in this section, whose unparametrized and non-equivariant version can be found in [Šva64] or [Ber77], p. 257 for a modern reference in english.

Before we begin with the proof of theorem 6, we point out that the equivariant cohomotopy groups for discrete groups and finite G-CW complexes were defined in [Lüc05a] as certain equivalence classes of morphisms between one-point compactifications of finite dimensional vector bundles. By technical reasons which will become clear in the proof of the theorem, we find it easier to handle this groups as equivalence classes of spherical bundles inside finite dimensional vector bundles with a G-invariant, riemannian metric.

Proof.

1. Step 1. Let $f \in C_l^{\neq 0}(D, E)$. We can assume that $f = l + c$, where c has a finite dimensional range contained in a finite dimensional subbundle l_{n+1} of dimension $n+1$ in E, and l is fiberwise surjective ($n \geq p+1$).

2. Since l is fiberwise fredholm with index p, we may write $E = \mathrm{Ker} l \oplus X_1$, where the Kernel is a finite dimensional vector bundle ξ of dimension p and $l : X_1 \to E$ is a fibrewise linear bounded homeomorphism with inverse l^{-1}.

3. Restrict now f to $\partial D_{n+p} \cap \{\mathrm{Ker} l \oplus l^{-1}(l_{n+1})\}$. Note that since f is nonzero in the boundary and compact, there exists $\alpha > 0$ such that $\inf_{x \in \partial D} fx \geq \alpha$ on every fiber. This follows because a compact perturbation of a fredholm operator is locally proper, see [Sma65], theorem 1.6. So we can consider the spherical morphism $\tilde{f} : S^{\xi \oplus l_{n+1}} \to S^{l_{n+1}}$ which is fibrewise given as $\frac{f}{|f|}$. This does not depend on the choices of subspaces containing the image of ∂D, for if l_n and l_m are finite dimensional bundles containing $C(\partial D)$, and $\tilde{f}$ and $\tilde{g}$ are the spherical morphisms obtained by the previous procedure, they can be seen to be of the form $h \wedge_x \mathrm{id}_{\eta_n} : S^{l_n \cap l_m \oplus \eta_n} \to S^{l_n}$, respectively $h \wedge_x \mathrm{id}_{\eta_m} : S^{l_n \cap l_m \oplus \eta_m} \to S^{l_n}$, where the morphism $h : \partial D \cap \{\mathrm{Ker} l \cap L^{-1}(l_n \cap l_m)\} \to S^{l_n \cap l_m}$ is obtained by the normalization procedure. (That is, f and g are suspensions of the same mapping by certain vector bundles). This is well defined at the level of compact homotopy classes, for a compact homotopy $h = l + c$ joining f_0 and f_1 can be always be choosen to have a fixed finite dimensional range for all $t \in I$. The restriction to the corresponding finite dimensional bundle gives a fibrewise homotopy between the spherical morphisms associated to f_0 and f_1.

4. To finish the proof, we verify that two perturbations f_0, f_1 with the same associated spherical morphism are homotopic. We can assume, both the range of $f_0 - l$ and $f_1 - l$ are defined in the same finite dimensional bundle l. We use the homotopy h between the spherical morphisms to produce a fibrewise compact homotopy $H = L + h : S^{l \oplus \xi} \times I \to S^l$ between f_0 and f_1. We have to extend the compact homotopy on the finite dimensional sphere to a compact homotopy in ∂D. For this, we use a fibrewise and equivariant extension theorem. We begin with the equivariant (not fiberwise!) extension theorem of Antonyan [Ant85]. Let $x \in X$ and consider the map defined on the fibre of $h_x : S_x^{l \oplus \xi} \to E_x$. Note that $S_x^{l \oplus \xi} \times I$ is a closed invariant subset of the metrizable G-space ∂D, where of course, the metric is determined by an invariant scalar product on the hilbert space E_x. As E_x is a locally convex space, the main theorem of [Ant85] applies and gives an extension $H : \partial D \times I \to E_x$ such that $\sup_{\partial D_x \times I} \mid H \mid = \sup_{S_x^{\xi \oplus l}} \mid h \mid$. This gives a compact homotopy between f_{0_x} and g_{0_x}. We use now a G-invariant partition of unity over X to glue the homotopies together and we get the fibrewise compact homotopy.

□

This finishes the first proof of theorem 6. In the second case, the process of *normalizing* is replaced by the following retraction lemma:

Lemma 11. Let $\tau \subset E$ be a finite dimensional G- vector bundle over the proper G-space X. Let $l : E \to E$ be a linear fredholm morphism. Denote by

$S^{\tau^\perp}$ the fiber bundle which is determined by the unit sphere in the orthogonal complement of τ. Then, there exists a fibrewise retraction $\rho_\tau : l^{-1}(\tau) \to S^E - \tau^\perp$.

Proof. ☐

The approximation lemma is now as follows:

Proposition 20. Let $f = l + c : E \to E$ be a fibrewise compact perturbation of the linear fredholm morphism l defined in a G- hilbert bundle E over the proper G-space X. Then, there exist finite dimensional G-vector bundles $\xi \subset E$ such that

- $\xi \oplus l(E) \cong E$.
- Given a fibrewise inclusion as orthogonal summand in a finite dimensional vector bundle $\xi \to \tau$, $\xi \oplus \zeta = \tau$, the restricted map $f : S^{l^{-1}(\xi)} \to S^E$ sends the unit sphere $S^{\xi^\perp}$ to the orthogonal complement of ξ. The map $\rho_\xi(f)$ has image in a
- The maps $\rho_\tau f \mid_{l^{-1}(\tau)}$ and $id_\zeta \wedge_X \rho_\xi$ are G-homotopic over X.

Proof.
1. Denote by D the unit ball bundle in E. Due to the boundedness condition, the map $f^{-1}(D_x)$ is bounded over any point $x \in X$. Hence, the closure C_x of its image under the compact map c is fibrewise compact. Cover C_x with a finite number of balls of radius $\epsilon_x \leq \frac{1}{4}$ centered at points v_{i_x}. Construct the vector bundle η which is fibrewise spanned by these vectors. Form the finite dimensional G-vector bundle $\xi = \eta \oplus l(E)^\perp$. It is clear that $\xi \oplus l(E) \cong E$.

2. If $w_x \in S_x^{\tau^\perp}$ is in the image of $f \mid_{S^{l^{-1}(\tau)}}$, then $f^{-1}(w_x)) \cap l^{-1}(\tau)$ will be mapped under $f \mid_{l^{-1}}$ to a subspace of $\tau_x + C_x$. So, w_x will be contained in $S^{\tau^\perp} \cap \tau_x + C_x$, which is not possible, because the distance between these subspaces is greater than $1 - \epsilon_x > \frac{3}{4}$.

3. In view of the slice theorem and the local triviality of the G-hilbert bundles involved, we can cover the space X with invariant neighborhoods for which there is a map $U_x \to G/H$, and the bundle over U_x is the pullback of the bundle $G \underset{H}{\times} E_x \to G/H.$, where E_x is some strong, norm continuous representation of H in a hilbert space. Hence, we can restrict ourselves to bundles over an orbit. In the notation of part 2, there is a retraction $\rho_{\tau_x} : S^{\tau_x} \to S^{E_x} - S^{\tau_x^\perp}$, and consider the isomorphism $l^{-1}(\tau_x) \cong \zeta_x \oplus l^{-1}(\xi_x)$ given by $w_x^{'} \mapsto (l \circ (1 - \mathrm{pr}_{l^{-1}(\xi_x)} w_x^{'}, \mathrm{pr}_{l^{-1}(\xi_x)} w_x^{'})$. We claim that after this isomorphism, the maps $\mathrm{id}_{\zeta_x} \rho_{\xi_x}(f \mid_{l^{-1}(\xi_x)})$ and $f \mid_{S^{l^{-1}(\tau_x)}}$ are homotopic. Consider for this a ball $D \subset E$ which contains the inverse image $f^{-1}(D_1(0))$ of the unitary ball We define the homotopy $h : D \times I \to S^E - S^{\tau^\perp}$ as follows

$$h(w_x, t) = \begin{cases} l + [(1-3t)\mathrm{id}_{E_x} + (3t)\mathrm{pr}_{\xi_x}] \circ c & t \in [0, \frac{1}{3}] \\ l + \mathrm{pr}_{\xi_x} \circ c[(2-3t)\mathrm{id}_{l^{-1}(\xi_x)} + (3t-1)\mathrm{pr}_{l^{-1}(\xi_x)}] & t \in [\frac{1}{3}, \frac{2}{3}] \\ \mathrm{pr}_{\zeta_x} \circ l + [(3-3t)\mathrm{pr}_{\xi_x} + (3t-2)\rho_{\zeta_x} \circ (l+c) \circ \mathrm{pr}_{l^{-1}(\zeta_x)}] & t \in [\frac{2}{3}, 1] \end{cases}$$

Since $S^E_x - D \cap \tau_x^\perp$ is contractible, the homotopy above can be extended to a homotopy $S^{l^{-1}(\tau)} \times I \to S^E - S^{\tau^\perp} \simeq S^\tau$, as needed.

□

Definition 20. Let G be a discrete group acting on the proper G-CW complex X. Denote by (E, F, l, k) a non-linear cocycle for the equivariant cohomotopy -theory over the proper G-space X, where l is a linear morphism whose index bundle is trivial of dimension p. Let ξ be a finite dimensional G-vector bundle as constructed in proposition 20. The parametrized Schwartz index is the element.

$$[p_\xi(l+k) \mid_{l^{-1}(\xi)}] : S^{l^{-1}(\xi)} \to S^{l(E)^\perp} \in \pi_G^{\mathrm{ind}}(l)(X)$$

in the equivariant cohomotopy group as, introduced by Lück in [Lüc05a]. This construction is well defined as consequence of part 3 of proposition 20. and gives an isomorphism of both theories,

This finishes the second proof of theorem 6

In case of compact lie groups equivariant cohomotopy is a $RO(G)$- graded equivariant cohomology theory, in the sense of [May96]. The definition is as follows:

Definition 21. Let G be a compact lie group and X a G-CW complex. For any representation W, form the one-point compactification S^W and define the set of equivariant and pointed maps $\Omega^W S^W = \mathrm{Map}_G(S^W, S^W)$. The equivariant cohomotopy group in grade V, where V is a virtual representation, is defined to be the abelian group constructed out of the homotopy sets of maps

$$\pi_G^V(X) = \mathrm{colim}_W [S^V \wedge X_+, \Omega W S^W]$$

where the systems runs along a complete G-universe, that is a hilbert space containing as subspaces all irreducible representations, where the trivial representation appears infinitely often.

Remark 8. In case of compact groups, the stability condition allows to suppose that the index bundle ker −coker has the form of a trivial bundle $X \times V$. The equivariant Schwartz index identifies this with the usual definition for equivariant cohomotopy groups for finite G-complexes, in such a way that our proof for discrete groups is formally the same, with the classical definition instead of that of Lück.

3.2 Computational Remarks

In order to do computations, we introduce some basic tools from homotopy theory.

Remark 9 (The spectral sequence)**.** Let X be a proper G-CW complex. There is an equivariant Atiyah-Hirzebruch spectral sequence which converges to $\pi^n_G(X)$ and whose E^2-term is given in terms of Bredon cohomology

$$E_2^{p,q} = H^p_{\mathbb{Z}\mathrm{Or}G}(X, \pi^q_?)$$

Applied to the universal proper G-space $\underline{E}G$:

$$E_2^{p,q} = H^p_{\mathbb{Z}\mathcal{SUB}_{\mathcal{COM}(G)}}(\underline{\mathrm{E}}G, \pi^q_?)$$

where $\pi^0_?$ is the contravariant coefficient system $H \mapsto \pi^0_H$.
There is a canonical identification

$$H^0_{\mathbb{Z}\mathcal{SUB}_{\mathcal{COM}(G)}}(\underline{\mathrm{E}}G, \pi^0_?) \cong \varprojlim{}_{H \in \mathcal{COM}} \pi^0_H$$

The edge homomorphism of the spectral sequence defines a map $\mathrm{edge}^G : \pi^0_G(\underline{\mathrm{E}}G) \to A_{inv}(G)$. Several known results of the spectral sequence go trough. Among them, as in case of discrete groups, the edge homomorphism is a rational isomorphism.

We now define a Burnside ring in operator theoretical terms for non compact lie groups. We first recall the definition for compact lie groups, which was first indtroduced by Tom Dieck in [tD75].

Definition 22. Let G be a compact lie group. Consider the folowing equivalence relation on the collection of finite G-CW complexes. X ∼Y if and only if for all $H \subset G$, the spaces X^H and X^H have the same Euler characteristic. Let $A(G)$ be the set of equivalence classes. Disjoint union and cartesian product of complexes are compatible with this equivalence relation and induce composition laws on $A(G)$. It is easy to verify that $A(G)$ together with these composition laws is a commutative ring with identity. The zero element is represented by a complex X such that the Euler characteristic $\chi(X^H)$ is zero for each $H \subset G$. If K is a space with trivial G-action and $\chi(K) = -1$, then $X \times K$ represents the additive inverse of X in $A(G)$.

We collect some information about the algebraic structure of the burnside ring in the following the crucial point which is used to motivate our definition in the following results, which have been published by Tom dieck in [tD87], p 240 and 250, and 256 , respectively

Proposition 21.
1. As abelian group $A(G)$ is the free abelian group on G/H, where $H \in \Phi(G)$ and $\Phi(G)$ denotes the set of conjugacy classes of subgroups such that $N(H,G)/H$ is finite, where $N(H,G)$ denotes the normalizer of H in G.
2. There is a character map $\mathrm{char}_G : A(G) \to \mathrm{Map}(\Phi(G), \mathbb{Z})$, where $\Phi(G)$, the space of closed subgroups of G carries the Hausdorff metric (in particular it is a compact Hausdorff space). And $\mathrm{char}_G(X)$ is defined by $H \mapsto X^H$.

Proposition 22. By means of the character map, the elements of the burnside ring can be identified with sums

$$\sum_K n(H,K)\chi(X^K) \cong 0 \bmod \mid NH/H \mid \qquad (*)$$

where the sum is over conjugacy classes (K) such that H is normal in K, $K/H \subset N_{H,G}/H$ is cyclic, the integer numbers $n(H,K)$ are defined to be

$$n(h,K) = \mid \mathrm{Gen}(K/H) \mid \mid W_{H,G}/N_{W_{K/H},W_{H,G}} \mid$$

and $\mathrm{Gen}(Z)$ denotes the cardinality of the generators of the finite cyclic group Z. In particular, the rationalized Burnside ring $A(G) \otimes \mathbb{Q}$ can be identified with the ring of continuous rational functions defined in $\Phi(G)$

Theorem 7. Let G be a compact lie group. there is an isomorphism

$$\pi^0_G(\{*\}) \to A(G)$$

Definition 23 (An operator theoretical Burnside Ring)**.** Let G be a locally compact group. The operator theoretical burnside ring of G, $A^{\mathrm{op}}(G)$ is the 0-dimensional equivariant cohomotopy theory of the classifying space of proper actions $\underline{\mathrm{E}}G$. In symbols

$$A^{\mathrm{op}}(G) = \Pi^0_G(\underline{\mathrm{E}}G)$$

The augmentation ideal $\hat{\mathbf{I}}_G \subset \Pi^0_G(\underline{\mathrm{E}}G)$ is defined to be the kernel of the composition of the restriction to the oth- skeleton of the classifying space and the restriction to the trivial group

$$\Pi^0_G(\underline{\mathrm{E}}G) \to \Pi^0_G(\underline{\mathrm{E}}G_0) \to \Pi^0_{\{e\}}(\underline{\mathrm{E}}G_0)$$

Example 2 (The group $Sl_2(\mathbb{R})$)**.** Recall that the group $Sl_2(\mathbb{R})$ is defined to be the group of real 2×2-matrices with determinant 1. It is a Lie group of dimension 3 and has one connected component. The maximal compact subgroup is $S^1 = SO_2$.

As $Sl_2(\mathbb{R})$ is almost connected, a model for $\mathrm{E}_{\mathcal{COM}}\mathrm{Sl}_2$ is $Sl_2(\mathbb{R})/SO_2 \approx \mathbb{R}^2$, which can be handled as the upper-half plane model for the 2-dimensional hyperbolic space. Note that this is a zero-dimensional proper CW-complex. From the equivariant Atiyah- Hirzebruch Spectral Sequence follows that the edge homomorphism

$$\mathrm{edge}^{Sl_2(\mathbb{R})} : \pi^0_{Sl_2(\mathbb{R})}(\underline{\mathrm{E}}GSl_2(\mathbb{R})) \to \lim_{\mathrm{inv} H \in \mathcal{COM}(Sl_2(\mathbb{R}))} \pi_H(pt)$$

is an isomorphism. On the other hand, since S^1 is a final object in the category of compact subgroups of $Sl_2(\mathbb{R})$, we have

$$A^{\mathrm{op}}(Sl_2(\mathbb{R})) \cong A(S^1)$$

$$A(S^1) \cong \mathbb{Z}$$

is a well known fact.

3.3 An example in the topology of four manifolds

We illustrate now an example where our constructions in terms of perturbation of fredholm morphisms appear in a natural way. This topics motivated the interest of the author in nonlinear analiysis, and are concentraterd around some results in Gauge theory due to Markus Szymik, (not yet published) Stefan Bauer [BF04] and Mikio Furuta. The author is deeply grateful to the Gauge theoretical community in Germany, in particular Markus Szymik and Raphael Zentner, who were always avalaible for questions and shared their knowledge. An introduction to this topic is the work of Moore [Moo01], the book [Nic00] and the notes [Tel].

We recall briefly that in the context of smooth, riemannian oriented manifolds, the existence of a $\mathrm{Spin}^c(4)$-structure is always guaranteed. This amounts to a map from a Spin^c principal bundle Q together with a bundle map $Q \to P$ to the frame bundle of the tangential bundle. The identification of the group $\mathrm{Spin}^c(4)$ with the sugroup $\{u_+, u_- \mid u_-, u_+ \in \mathrm{U}(2), \det(u_+) = \det(u_-)\}$ allows to define positive, respectively, negative spinor bundles $S^+, S^- = Q \underset{\rho^{+,-}}{\times} \mathbb{C}^2$, where $\rho_{+,-} : Spin^c \to \mathrm{U}(2)$ are the respective projections. Using quaternionic multiplication, it is posible to furnish $S := S^+ \oplus S^-$ with the structure of a module over the Clifford algebra of the cotangential bundle $T^*(X) \times S^{+,-} \to S^{-,+}$. Clifford identities give a linear map $\rho : \Lambda^2 \to \mathrm{End}_C(S^+)$ whose kernel is the bundle of anti-selfdual 2-forms and whose image is the bundle of trace free skew hermintian endomorphisms. For any spin^c-connection A, define the associated Dirac operator D as the composition $\Gamma(S^+) \underset{\nabla_{A+a}}{\rightarrow} \Gamma(S^+) \otimes \Lambda^1(T^*M) \xrightarrow{\gamma} \Gamma(S^-)$, where γ denotes Clifford multiplication.

The *monopole map* $\tilde{\mu}$ is defined for four-tuples (A, ϕ, a, f) of a Spin^c connection A, a positive spinor ϕ, a 1-form a and a locally constant function f on M as

$$\mu : \mathrm{Conn} \times \Gamma(S^+) \oplus \Omega^1(M) \oplus H^0(M) \to \mathrm{Conn} \times \Gamma(S^-) \oplus \Omega^+(M) \oplus \Omega^0(M) \oplus H^1(M)$$

$$(A, \phi, a, f) \mapsto (A, D_{A+a}\phi, F^*_{A+a} - \sigma(\phi), d^*(a) + f, a_{\mathrm{harm}})$$

where σ is the tracefree endomorphism $(-i)(\phi \otimes \phi^*) - \frac{1}{2} \mid \phi \mid^2 \dot{\mathrm{id}}$. Given a point in M, the based gauge group $\mathcal{G}_x$ is the kernel of the evaluation map at x. $\mathrm{map}(X, \mathbb{S}^1) \to S^1$. The subspace $A + ker(d)$ is invariant under the free action of the based Gauge group. The quotient is isomorphic to the *Picard torus*, $\mathfrak{Pic}(X) = H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$. Let $\mathcal{A}$ and $\mathcal{B}$ be the quotients

$$a + \ker d \times \Gamma(S^+) \oplus \Omega^1(X) \oplus H^0(X, \mathbb{R})/G_x$$

respectively

$$a + \ker d \times \Gamma(S^-) \oplus \Omega^+(X) \oplus \Omega^0(X, \mathbb{R}) \oplus H^1(X, \mathbb{R})/G_x$$

the quotient map $\mu : \tilde{\mu}/G_x : \mathcal{A} \to \mathcal{B}$ has by definition a presentatioon as a fibrewise compact perturbation of a fredholm operator. It is proper after a result of Bauer and Furuta, [BF04], which essentially uses estimates determined by the Weitzenböck formula. This gives rise to a cocycle. $\mathcal{A}, \mathcal{B}, D_A + d + d^*, c = F^+_A + a \cdot \phi + \sigma(\phi)$, where σ is the selfdual tracefree endomorphism $\phi \mapsto (-i\phi \otimes \phi^* - \frac{1}{2} \mid \phi \mid)$.

Example 3 (A generalized Bauer-Furuta invariant for proper actions on 4-manifolds)**.** Suppose G is a (possibly noncompact) Lie group acting properly and cocompactly on the smooth Spin^c- manifold M. assume furthermore, that the group preserves the orientation and by means of isomorphisms of complex spin structures, and respecting the connection. As the action is proper, it can be assumed that G preserves the metric. Let $\mathbb{G}$ be the group of pairs (φ, u), where φ is a G-equivariant diffeomorphism which preserves both the metric and the orientation and $u : f^*(\sigma_M) \to \sigma_M$ is an isomorphism of the spin-c principal bundle. In particular, this gives a description of $\mathbb{G}$ in the middle of the following exact sequence

$$1 \to S^1 \to \mathbb{G} \to G \to 1$$

In this situation, the class of μ is denoted by $m_{\mathbb{G}}(X, \sigma_X) \in \Pi_G^{\mathrm{ind}(\lambda)}(\mathfrak{Pic}(X))$ and is called the parametrized Bauer-Furuta invariant. The restriction map

$$\mathrm{res}_{\mathbb{G}}^{S^1} \Pi_{\mathbb{G}}^{\mathrm{ind}(l)}(\mathfrak{Pic}(X)) \to \Pi_{S^1}^{\mathrm{ind}(l)}(\mathrm{Pic}(X)) \cong \pi_{S^1}^{\mathrm{ind}(l)}(\mathfrak{Pic}(X))$$

maps $m_{\mathbb{G}}$ to the S^1-equivariant cohomotopical Bauer-Furuta invariant defined in [BF04]. If, moreover, vector bundles do suffice to represent the equivariant KO-theory of the Picard torus, the element λ can be identified after choice of an invariant riemannian metric on the manifold with the difference of the complex virtual index bundle of the dirac operator and the trivial bundle of the space of selfdual harmonic two-forms.

Chapter 4

The Segal Conjecture

In this chapter we shall transcribe to the context of proper actions of lie groups one of the outstanding developements of equivariant topology in the last century, the proof of the Segal conjecture. Originally formulated in the context of finite groups, the conjecture is a statement relating the completion of the Burnside ring and the stable homotopy groups of the classifying space of the group. It was definitively proved by Gunnar Carlsson [Car84] in the form of the following

Theorem 8. [Segal conjecture for finite groups]

Let G be a finite group and X be any finite G-CW complex. There exists an isomorphism

$$\pi_G^n(X)_{\hat{\mathbf{I}}_G} \xrightarrow{\cong} \pi_e^n(\mathrm{EG} \underset{\mathrm{G}}{\times} \mathrm{X})$$

in particular we get in the case $X = \{*\}$ and $n = 0$ an isomorphism

$$A(G)_{\hat{\mathbf{I}}_G} \xrightarrow{\cong} \pi_e^0(\mathrm{B}G) \tag{4.1}$$

The analogous statement for compact Lie groups is known to be false (see example 4). A weaker form of the result assuming certain hypotheses on the action of the Weyl group on the maximal torus was proved by Mark Feshbach [Fes87].

This was simplified by Stefan Bauer [Bau89], who also provided a counterexample which shows the necessity of the hypothesis concerning the action of the Weyl group. The result is

Theorem 9. [Segal conjecture for compact lie groups]

Let G be a compact Lie group with maximal torus T of dimension n and Weyl group $W = N_{T,G}/T$. Let $\rho : W \to Gl_n(\mathbb{Z})$ be a representation which gives rise to the action of W on $T \approx \mathbb{R}^n/\mathbb{Z}^n$. Suppose that ρ does not originate at a generalized quaternion group of order 2^n. Then the map

$$A(G)_{\hat{\mathbf{I}}_G} \to \pi_e^0(\mathrm{B}G)$$

has dense image in the skeletal filtration.

As an application of the techniques developed in the previous chapters, we prove a slightly more general version of this result.

Theorem 10. [Segal Conjecture for almost connected Lie groups] The Segal conjecture is true for (non compact) Lie groups with finitely many components. That is, there is a map

$$A^{\mathrm{ho}}(G)_{\hat{\mathbf{I}}_G} \to \pi^0_e(\mathrm{B}G)$$

with dense image in the skeletal filtration whenever a maximal compact subgroup of G satisfies the hypotheses of theorem 9.

Our proof makes essential use of Feshbach and Bauer's results. We shall briefly recall their arguments on the way to the proof. Before we begin with the proof, we notice that in the case of discrete groups a sharper, much more beautiful result due to Wolfgang Lück, [Lüc08] is possible:

Theorem 11. Let G be a discrete group and X be any finite G-CW complex. Suppose that there is a finite model for $\underline{\mathrm{E}}G$. Then there is an isomorphism

$$\pi^n_G(X)_{\hat{\mathbf{I}}_G} \cong \pi^n_{\{e\}}(\mathrm{EG} \underset{\mathrm{G}}{\times} \mathrm{X})$$

We prove a generalization of this result to the context of infinite discrete groups and families of finite subgroups. This takes the form of the following

Theorem 12. Let G be a discrete group and $\mathcal{F}$ be a family of finite subgroups closed under conjugation and intersection. Suppose that X is a finite proper G-CW complex. Let $f : X \longrightarrow L$ be a G-map to a proper G-CW complex having an upper bound on the cardinality of isotropy subgroups denote by $\mathrm{E}_{\mathcal{F}}$ the classifying space for the family $\mathcal{F}$. Then there is an ideal $\hat{\mathbf{I}}_{\mathcal{F},f} \subset \pi^0_G(L)$ and an isomorphism

$$\pi^n_G(X)_{\hat{\mathbf{I}}_{\mathcal{F},f}} \cong \pi^n_G(\mathrm{E}_{\mathcal{F}} \times \mathrm{X})$$

4.1 Proof of the main theorem

The idea behind our arguments is to reduce the result to a compact subgroup. An essential role in this procedure is played by the following

Proposition 23. Let G be a Lie group with finitely many components. Then

1. There is up to conjugacy a unique maximal compact subgroup K of G. Any other compact subgroup is subconjugated to K.
2. There exist diffeomorphisms $G \approx G/K \times K$ and $G/K \approx \mathbb{R}^k$.

Proof. See [Hoc65], Theorem 3.1 p. 180. □

We mainly use the following consequence of this fact:

Corollary 1. Let G be an almost connected lie group (that is, G/G_0 is finite). Then there exists a maximal compact subgroup K, unique up to conjugacy such that

1. $\mathrm{B}G \simeq \mathrm{B}K$.
2. G/K is a model for $\mathrm{E}_{\mathcal{COMP}}G$. In particular, the induction isomorphism gives an isomorphism $A^{\mathrm{ho}}(G) \cong A(K)$, where $A(K)$ stands for the Burnside ring in the sense of Tom Dieck.

We now recall Feshbach's proof of the Segal conjecture and show how the previous facts fit into. The first step towards the result is given by transfer maps. In order to state precisely this, we need the following

Remark 10. Let G be a compact lie group. Denote by $\rho_{H,G}$ the inclusion of a closed subgroup H. If $\mathcal{H}$ is a representable cohomology theory and $g \in G$ is an arbitrary element, there exists a conjugation isomorphism $C_g : \mathcal{H}(\mathrm{B}H) \to \mathcal{H}(\mathrm{B}H^g)$. An element $x \in \mathcal{H}(B(H)$ is *stable* with respect to $\rho_{H,G}$ if

$$\rho^*_{H \cap H^g,H}(x) = \rho^*(H \cap H^g, H^g)C_g(x)$$

we shall denote the set of stable elements of $\mathcal{H}(\mathrm{B}H)$ by $\mathcal{H}(\mathrm{B}H)^s$. In the particular situation of the maximal torus of a compact Lie group there is an action of the Weyl group $W = N_{T,G}/T$ on $\mathcal{H}(\mathrm{B}N)$. We shall denote the invariant elements of this action by $\mathcal{H}^s(\mathrm{B}N)$. We point out that for normal groups, this is exactly invariance.

Proposition 24. Let N be the normalizer of a maximal torus T in K. Then there exist isomorphisms

1. $\pi^0(\mathrm{B}K) \cong \pi^0(\mathrm{B}N)^s$.
2. $A(K)_{\hat{\mathbf{I}}_K} \cong A(N)^s_{\hat{\mathbf{I}}_N}$, where the definition of an invariant element in the burnside ring is analogous to that of elements in $\mathcal{H}(\mathrm{B}K)$.

Proof. 1. Consider the fibration $K/N \to \mathrm{B}N \xrightarrow{i_*} \mathrm{B}K$. Because K/N is compact, one has a transfer map [Dwy96]. This induces an isomorphism $tr_p \circ p : \pi^0(\mathrm{B}K) \to \pi^0(\mathrm{B}N)^s \otimes (\chi(K/N))$. Since $\chi(K/N)$ is a unit, the result follows.

2. Consider the induction map $A(N) \to A(K)$. It is easy to see that the image of $\mathrm{ind}_{N,K} \circ \mathrm{res}_{K,N}$ consists of invariant elements.

[Fes79] □

The essential point in Feshbach's proof is the approximation of a compact lie group by a distinguished family of finite subgroups. Precisely,

Proposition 25. There exists a nested sequence F_i of finite subgroups of N with $F_i/F_i \cap T \cong W$ such that $\cup_i(F_i) \cap T$ consists of the subgroup $T_{\mathbb{Q}}$ generated by the torsion elements in T.

Proof. Theorem 1.1 in [Fes87]

□

The following result is crucial in Feshbach's argument and is proved in [Fes87] , p. 6

Proposition 26.

$$\pi^q(\mathrm{B}N) \cong \lim_i \pi^q(\mathrm{B}F_i)$$

sketch. Denote by $\pi^0_e(\tilde{\mathrm{B}}N) = \mathrm{invlim}_n \pi^0_e(\tilde{\mathrm{B}}N)/n\pi^0_e(\tilde{\mathrm{B}}N)$ the abelian completion. (This is given at the level of spectra by smashing with a shifted Moore spectrum $\Sigma^{-1}M(\mathbb{Q}/\mathbb{Z})$). Because of the finiteness of the stable homotopy groups of spheres, $\pi^0_e(\tilde{\mathrm{B}}N) = \pi^0_e(\mathrm{B}N)$. Put $N_{\mathbb{Q}} = \cup_i F_i$. It is proved that $\pi^0_e(\mathrm{B}N_{\mathbb{Q}}) = \pi^0_e(\tilde{\mathrm{B}}N)$, the main argument beeing that the inclusion induces an isomorphism of $\mathbb{Q}/\mathbb{Z}$-homology isomorphism at the level of classifying spaces, together eith the fact that $\pi^0_e(\mathrm{B}N_{\mathbb{Q}})$ coincides with its abelian completion. The standard lim^1-argument gives the identification $\pi^0_e(\mathrm{B}N_{\mathbb{Q}}) = \mathrm{invlim}_i \pi^0_e(\mathrm{B}F_i)$. □

As a consequence of the Segal conjecture for finite groups, one has $A(F_i)_{\hat{\mathbf{I}}} \cong \pi^0(\mathrm{B}F_i)$. Hence the Segal conjecture reduces to the understanding of the restriction map $A(G)\hat{\mathbf{I}}_G \to \lim_i A(F_i)_{\hat{\mathbf{I}}_{F_i}}$. As explained in example 4, this map is neither injective nor surjective in the simple case of $O(2)$. Under the hypotheses of theorem 9, it is posibly to prove that the image is dense in the skeletal filtration. Otherwise, example 5 contradices the Segal conjecture even in this weaker formulation.

4.2 Examples and counterexamples

Example 4. [Feshbach's counterexample to the original segal conjecture] Recall the description of the Burnside ring of the 2-dimensional orthogonal group. It is generated as abelian group by the homogeneous spaces $O(2)/O(2)$, $O(2)/SO(2)$ and $\{O(2)/D_n\}_{n\in\mathbb{N}}$, where D_n is the subgroup generated by the subgroup of $SO(2)$ which is isomorphic to $\mathbb{Z}/n$ and an element $\alpha \ni SO(2)$ (This is a generalized dihedral group isomorphic to a semidirect product of the form $\mathbb{Z}/k \rtimes_{\varphi} D^{'}_{2n}$, where $D^{'}$ represents the usual group of symmetries of a n-gon.

Denote by $\hat{\mathbf{I}}(G)$ the completed augmentation ideal. We claim that the restriction map $\hat{\mathbf{I}}(O(2)) \to \lim_n \hat{\mathbf{I}}(D_n)$ is neither surjective nor injective. This follows from two claims.

First, the I-adical completion coincides with the 2-adical completion. An argument to see this is the fact that $[O(2)/D_n] \times [O(2)/D_m]$ can be decomposed as the union of two orbits of type $D_{\mathrm{gcd(m,n)}}$ and more orbits of type $\mathbb{Z}/k$. Since $\mathbb{Z}/k$ has infinite index in its normalizer, they are zero in the Burnside Ring. An analogous argument shows that $[O(2)/SO(2)] \times [O(2)/D_k] = 0$, and since $SO(2)$ has index two on $O(2)$, we get $[O(2)/SO(2)]^2 = 2[O(2)/SO(2)]$.

On the other hand, the semidirect product structure of D_n implies that $\lim_n \hat{\mathbf{I}}(D_n) = \lim_n \hat{\mathbf{I}}(P_n)$, where $P_n = D(2^n)$.

This implies for instance that the image of $[O(2)/D_m]$ under the restriction map only depends on the two-order of m, therefore, the restriction map is not injective. In order to describe an element which is not hit by the restriction map let us remark that $A(P_n)$ is generated by the elements $\{P_n/T_k \mid 0 \le k \le n\} \cup \{P_n/P^{'}_j \mid 0 \le j \le n-1\}$, where $T_k = SO(2) \cap P_k =< t >$ and $P^{'}_j =< T_j, t >$. put $v_{j,k} = [P_k/P_j] + [P_j/P^{'}_j] - [P_j/T_k]$ for $j < k$ and $v_{k,k} = 2 - [P_k/T_k]$. Define

$$w_{j,k} = \begin{cases} v_{j,k} - v_{j-1,k} & \text{if } j \le k \\ 0 & \text{otherwise} \end{cases}$$

and put $w_j = \lim_k w_{j,k} \in \lim I(P_k)$ and $x = \lim v_{k,k}$. Then, one has

$$\lim_n \hat{\mathbf{I}}(P_n) = \prod (\hat{\mathbb{Z}}_2[w_j]/w_j^2 = 2w_j) \oplus \hat{\mathbb{Z}}_2[x]$$

as abelian groups. The element $y = \prod w_j$ does not lie on the image of the restriction map.

Example 5. [Bauer's counterexample to the density of restriction maps.] Let $\Pi = \langle a, b \mid a^{2n-1} = b^2, b^4 = 1, aba = b \rangle$ be a generalized quaternion group. Let $\mathbb{Z}\Pi \to \mathbb{Q}(\zeta_{2^n}) \oplus \mathbb{Q}(\zeta_{2^n})j = V$ be the morphism determined by $a \mapsto \zeta_{2^n}$, $b \mapsto j$, for ζ_{2^n} a primitive root of unity. Denote by Λ the image of $\mathbb{Z}\Pi$ inside V. Accord to a theorem of representation theory, there exists a Λ-lattice N inside V such that $N_{\hat{2}} = M_{\hat{2}} \oplus M_{\hat{2}}$ for some other lattice M. Let $T = N^*$ the pontrjagyin dual of the discrete group N. It is a 2-torus for which the image of the restricton map is not dense.

Remark 11. The seek for weaker statements generalizing the Segal conjecture to compact Lie groups is an active research field. An approach using trace methods is for classifying spaces of tori is proposed in the preprint [CDD].

4.3 Families of finite subgroups in discrete groups

We first describe the context where our completion theorem will take place. We briefly recall some standard notation for handling the algebraic part. Let R be an associative ring with unit. A promodule indexed by the integers is an inverse system of R-modules.

$$M_0 \overset{\alpha_1}{\leftarrow} M_0 \overset{\alpha_2}{\leftarrow} M_1 \overset{\alpha_3}{\leftarrow} M_2, \ldots$$

We write $\alpha_n^m = \alpha_{m+1} \circ \ldots \circ \alpha_n : M_n \to M_m$ for $n > m$ and put $\alpha_n^n = \mathrm{id}_{M_n}$.

A *strict* pro-homomorphism $\{M_n, \alpha_n\} \to \{N_n, \beta_n\}$ consists of a collection of homomorphisms $\{f_n : M_n \to N_n\}$ such that $\beta_n \circ f_n = f_{n-1} \circ \alpha_n$ holds for each $n \geq 2$. A pro R-module $\{M_n, \alpha_n\}$ is called pro-trivial if for each $m \geq 1$ there is some $n \geq m$ such that $\alpha_n^m = 0$. A strict homomorphism f as above is called a pro isomorphism if $\ker(f)$ and coker(f) are both pro-trivial. A sequence of strict homomorphisms

$$\{M_n, \alpha_n\} \overset{\{f_n\}}{\to} \{M_n^{'}, \alpha_n^{'}\} \overset{\{g_n\}}{\to} \{M_n^{''}, \alpha_n^{''}\}$$

is called pro-exact if $g_n \circ f_n = 0$ holds for $n \geq 1$ and the pro-R-module $\{\ker(g_n)/\mathrm{im}(f_n)\}$ is pro-trivial. The following lemmas are proved in [AM69], section 2:

Lemma 12. Let $0 \to \{M^{'}, \alpha_n^{'}\} \to \{M_n, \alpha_n\} \to \{M_n^{''}, \alpha_n^{''}\} \to 0$ be a pro-exact sequence of pro-R-modules. Then there is a natural exact sequence

$$0 \to \mathrm{invlim} M_n^{'} \overset{\mathrm{invlim} f_n}{\longrightarrow} \mathrm{invlim} M_n \overset{\mathrm{invlim} g_n}{\longrightarrow} \mathrm{invlim} M_n^{''} \overset{\delta}{\to}$$
$$\mathrm{invlim}^1 M_n^{'} \overset{\mathrm{invlim}^1 f_n}{\longrightarrow} \mathrm{invlim}^1 M_n \overset{\mathrm{invlim}^1 g_n}{\longrightarrow} \mathrm{invlim}^1 M_n^{''}$$

In particular, a pro-isomorphism $\{f_n\} : \{M_n, \alpha_n\} \to \{N_n, \beta_n\}$ induces isomorphisms

$$\mathrm{invlim}_{n \geq 1} f_n : \mathrm{invlim}_{n \geq 1} \overset{\cong}{\to} \mathrm{invlim}_{n \geq 1} N_n$$
$$\mathrm{invlim}^1_{n \geq 1} f_n : \mathrm{invlim}^1_{n \geq 1} \overset{\cong}{\to} \mathrm{invlim}^1_{n \geq 1} N_n$$

Lemma 13. Fix any commutative noetherian ring R and any ideal $I \subset R$. Then, for any exact sequence $M^{'} \to M \to M^{''}$ of finitely generated R-modules, the sequence
$$\{M^{'}/I^nM^{'}\} \to \{M/I^nM\} \to \{M^{''}/I^nM^{''}\}$$
of pro-R-modules is pro-exact.

We now set up the topological context we are going to study. We consider an action of a discrete group G on a finite proper G-CW complex X and a map $f : X \to Z$, where Z is a proper G-CW complex with an upper bound on the cardinality of the isotropy subgroups. We begin by defining the augmentation modules

Definition 24. Let Z be a proper G-CW complex with an upper bound on the cardinality of isotropy subgroups. Denote by Z^0 the equivariant 0-skeleton of Z. The augmentation module for the family $\mathcal{F}$, in symbols $I^n_{G,\mathcal{Z}}(Z)$ is defined to be the kernel of the map

$$\pi^n_G(Z) \xrightarrow{i^*} \pi^n_G(Z^0) \xrightarrow{\Pi \text{res}} \prod_{H\in\mathcal{F}} \pi_H(Z^0)$$

If $n = 0$, the augmentation module is an ideal in $\pi^0_G(Z)$.

We now recall the following easy but crucial fact

Lemma 14. Let X be an n- dimensional proper G-CW complex and $I = \ker(\pi^*_G(X) \overset{i^*}{\to} \pi^*_G(X^0))$. Then one has $I^{n+1} = 0$.

Proof. Fix elements $x \in I^n$, $y \in I$. We inductively assume that x vanishes in $\pi^*_G(X^{n-1})$. Hence, there exists a lift $x^{'} \in \pi^*_G(X, X^{n-1})$. We note that $\pi^*_G(X, X^{n-1})$ is a π^*_G-module. Now $I \cdot \pi^*_G(X, X^{n-1}) = 0$, since I vanishes in orbits. Hence $yx^{'} = 0$ and $yx = 0$ □

As a consequence of the previous lemma, for any $n \geq 0$ the composite

$$\mathrm{I}^n_{G,\mathcal{F}}(Z)\pi^*_G(X) \subset \pi^*_G(X) \xrightarrow{\mathrm{proj}^*} \pi^*_G(\mathrm{E}_\mathcal{F} \times \mathrm{X}) \xrightarrow{\mathrm{i}^*} \pi^*_\mathrm{G}((\mathrm{E}_\mathcal{F} \times \mathrm{X})^{\mathrm{n}-1})$$

is zero. Hence, we can define a homomorphism of pro-modules

$$\lambda^m_{X,\mathcal{F},f} : \left\{\pi^m_G(X)\big/\mathrm{I}^n_{G,\mathcal{F}} \cdot \pi^m_G(X)\right\}_n \to \left\{\pi^m_G(\mathrm{E}_\mathcal{F} \times \mathrm{X}^{\mathrm{n}-1})\right\}_\mathrm{n}$$

We formulate the following

Theorem 13 (Segal Conjecture for families of finite subgroups)**.** Let G be a discrete group and $\mathcal{F}$ be a family of finite subgroups of G closed under conjugation and under subgroups. Fix a finite proper G-CW complex X, a finite dimensional proper G-CW complex Z whose isotropy subgroups lie in $\mathcal{F}$ and have bounded order. Let $f : X \to Z$ be a G-map. Regard $\pi^0_G(X)$ as a module over $\pi^0_G(Z)$ and set

$$I = I_{\mathcal{F},Z} = \ker(\pi^0_G(Z) \xrightarrow{\text{res}^H_G \circ i^*} \prod_{H\in\mathcal{F}} \pi^0_H(Z^0))$$

then

$$\lambda^m_{X,\mathcal{F},f} : \left\{\pi^m_G(X)\big/I^n \cdot \pi^m_G(X)\right\} \to \left\{\pi^m_G(\mathrm{E}_{\mathcal{F}} \times \mathrm{X}^{\mathrm{n}-1})\right\}$$

is an isomorphism of pro-groups. Also, the inverse system

$$\left\{\pi^m_G((\mathrm{E}_{\mathcal{F}}(\mathrm{G}) \times \mathrm{X})^{\mathrm{n}})\right\}_{\mathrm{n}\geq 1}$$

satisfies the Mittag-leffler condition. In particular

$$\lim{}^1 \pi^m_G((\mathrm{E}_{\mathcal{F}} \times \mathrm{X})^{\mathrm{n}}) = 0$$

and $\lambda_{X,\mathcal{F},f}$ induces an isomorphism

$$\pi^m_G(X)_{\hat{\mathbf{I}}} \xrightarrow{\cong} \pi^m_G(\mathrm{E}_{\mathcal{F}} \times \mathrm{X}) \cong \lim_{\mathrm{n}} \pi^{\mathrm{m}}_{\mathrm{G}}((\mathrm{E}_{\mathcal{F}} \times \mathrm{X})^{\mathrm{n}})$$

Proof. **Step 1**. Assume first that $X = G/H$ for some finite subgroup $H \subset G$ and consider the following commutative diagram:

$$\begin{tikzcd}
\pi^0_G(Z) \arrow[r, "f^*"] & \pi^m_G(G/H) \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\mathrm{ind}^{\cong}_{H\to G}"'] & \pi^m_G(\mathrm{E}_{\mathcal{F}}(\mathrm{G}) \times \mathrm{G/H}) \arrow[d, "\mathrm{ind}^{\cong}_{H\to G}"] \\
A(H) \arrow[r, "\cong"'] & \pi^0_H(\{*\}) \arrow[r, "\mathrm{pr}_2"'] & \pi^m_H(\mathrm{E}_{\mathcal{F}|_{\mathrm{H}}} \times \mathrm{X})
\end{tikzcd}$$

Denote by $I_{\mathcal{F}}(H)$ the kernel of the map

$$A(H) \longrightarrow \prod_{L \in \mathcal{F}|_H} A(L)$$

and note that the ideal $I^{'}$, defined as $I^{'} := \mathrm{ind}_{H\to G}(I)$ is contained in $I_{\mathcal{F}}(H)$. Now, by the completion theorem of Adams, Jackowski, Haeberly and May [AHJM88], pr_2 induces an isomorphism of progroups

$$\left\{\pi^m_G(\{*\})/I_{\mathcal{F}}(H)^n\right\}_n \longrightarrow \left\{\pi^m(\mathrm{E}_{\mathcal{F}|_{\mathrm{H}}})^{\mathrm{n}-1})\right\}$$

We now claim that pr_1 induces an isomorphism of progroups

$$\left\{\pi^m_G(G/H)/I^n\pi^m_H(G/H)\right\}_n \longrightarrow \left\{\pi^m_G(\mathrm{E}_{\mathcal{F}}(\mathrm{G}) \times \mathrm{G/H}^{\mathrm{n}-1})\right\}$$

It suffices to show that for some k, $I_{\mathcal{F}}(H)^k \subset I^{'}$, equivalently, that the ideal $I_{\mathcal{F}}(H)/I^{'}$ is nilpotent. Since $A(H)$ is a noetherian ring [AM69], proposition 1.8, this is equivalent to the fact that it is contained in all prime ideals of $A(H)/I^{'}$. hence, we must show that every prime ideal $\mathcal{P}$ which contains $I^{'}$ also contains $I_{\mathcal{F}}(H)$. We recall for this that prime ideals on the Burnside ring of a finite group are detemined as the inverse image of prime ideals in $\mathbb{Z}$ along a character map $\varphi_K : A(H) \to \mathbb{Z}$ for a subgroup $K \subset H$ (compare [Dre69]), proposition 1. We therefore write $\mathcal{P}_{K,p}$ for such an ideal.

In view of proposition 27, which we found convenient to state separately, we need to show that $\mathcal{P}$ contains the image of the structure map for H of the inverse limit

$$\mathrm{invlim}_{L\in\mathcal{F}} I_L \to I_H$$

Let n_0 be a natural number divided by the cardinality of all subgroups in $\mathcal{F}$. Note that the H-set S formed by the disjoint union of $n_0/\mid H \mid$ copies of H is free. Choose an isomorphism $S \overset{u}{\cong} \{1, \dots, r\}$ and a group homomorphism $H \overset{\rho_u}{\to} S_r$ into the set of automorphisms (=permutations) of this set. This defines an action of H on S_r. Denote by $S_r[\rho_u]$ the H-set obtained by this procedure and notice that this does not depend on the isomorphism to $\{1, \dots, r\}$. Let $Syl_p(S_r)$ be the p-sylow subgroup of S_r. We point out that the action of H on S_r defines an action on the homogeneous space $S_r/Syl_p(S_r)$ by $h \cdot \bar{\sigma} = \overline{h \cdot \sigma}$. Denote by $S_r/Syl_p(S_r)[\rho_u]$ the H- set given by this construction. This does not depend on the choice of the particular isomorphism to $\{1, \dots, r\}$. Hence we denote $[S_r] := S_r[\rho_u]$ and $[S_r/Syl_p(S_r)] := S_r/Syl_p(S_r)[\rho_u]$.

The elements $\{[S_r]\}$ form a compatible system in $\lim_{L \in \mathcal{F}} A(L)$. For if a group homomorphism $H_0 \to H_1$ is injective, then the restriction homomorphism $A(H_1) \to A(H_0)$ sends $[S_r]$ to $[S_r]$. The same holds for $[S_r/Syl_p(S_r)]$ hence, we can choose elements

$$\big\{[S_r] - \mid S_r \mid\big\}$$

and

$$\big\{[S_r/Syl_p(S_r)] - \mid S_r/Syl_p(S_r) \mid\big\}$$

in $\lim_{L \in \mathcal{F}} I(L)$. The image under the structure map for the subgroup H is $[S_r] - \mid S_r \mid [H/H]$ and $[S_r/Syl_p(S_r)] - \mid S_r/Syl_p(S_r) \mid [H/H]$. Both elements are in the prime ideal $\mathcal{P}_{K,p}$. Since $\varphi_K : A(H) \to \mathbb{Z}$ sends both elements to $p\mathbb{Z}$, and $\varphi_K([S_r] - \mid S_r \mid) = - \mid S_r$ for $k \neq \{1\}$, we conclude that $K = \{1\}$ or $p \neq 0$. If $K = \{1\}$, then $I_H = \mathcal{P}_{\{1\},0} \subset \mathcal{P}_{\{1\},p}$. If $K \neq \{1\}$, then $p \neq 0$ is a prime and $\varphi_K([S_r/Syl_p(S_r)] - \mid S_r/Syl_p(S_r) \mid \in p\mathbb{Z}$. Since $\mid S_r/Syl_p(S_r) \mid$ is prime to p, and p divides the difference $\mid (S_r/Syl_p(S_r)^K \mid - \mid S_r/Syl_p(S_r) \mid$, we conclude that K is a p-group and $\mathcal{P}_{K,p} = \mathcal{P}_{\{1\},p}$. Hence $I_H = \mathcal{P}_{\{1\},0} \subset \mathcal{P}_{\{1\},p}$. This finishes the proof for the case $X = G/H$.

Step 2 Suppose now inductively that the completion theorem is proved on complexes of dimension at most $n-1$. Let X be an n-dimensional complex and write X as the pushout

$$\begin{tikzcd}
\coprod G/H \times S^{n-1} \arrow[r] \arrow[d] & \coprod G/H \times D^n \arrow[d] \\
X^{n-1} \arrow[r] & X
\end{tikzcd}$$

The Mayer-Vietoris sequence associated to this pushout gives rise to a situation of the form:

$$\begin{tikzcd}
\cdots \arrow[d] & \cdots \arrow[d] \\
\left\{\pi_G^{m-1}(G/H\times S^{n-1})/I_{\mathcal{F},f}(L)^k\right\}_k \arrow[r, "\lambda_{\mathcal{F},f}(G/H\times S^{n-1})"'] \arrow[d] & \left\{\pi_G^{m-1}((\mathrm{E}_{\mathcal{F}}\times \mathrm{G/H}\times \mathrm{S}^{\mathrm{n}-1})^{\mathrm{k}-1})\right\}_{\mathrm{k}} \arrow[d] \\
\left\{\pi_G^{m}(X)/I_{\mathcal{F},f}(L)^k\right\}_k \arrow[r, "\lambda_{\mathcal{F},f}(X)"'] \arrow[d] & \left\{\pi_G^{m}((\mathrm{E}_{\mathcal{F}}\times \mathrm{X})^{\mathrm{k}-1})\right\}_{\mathrm{k}} \arrow[d] \\
\begin{array}{c}\left\{\pi_G^{m}(G/H\times D^n)/I_{\mathcal{F},f}(L)^k\right\}_k\\ \bigoplus \\ \left\{\pi_G^{m}(X^{n-1})/I_{\mathcal{F},f}(L)^k\right\}_k\end{array} \arrow[r, "\begin{array}{c}\lambda_{\mathcal{F},f}(G/H\times D^n)\\ \oplus\\ \lambda_{\mathcal{F},f}(X^{n-1})\end{array}"'] \arrow[d] & \begin{array}{c}\left\{\pi_G^{m}((\mathrm{E}_{\mathcal{F}}\times \mathrm{G/H}\times \mathrm{D}^{\mathrm{n}})^{\mathrm{k}-1})\right\}_{\mathrm{k}}\\ \bigoplus \\ \left\{\pi_G^{m}((\mathrm{E}_{\mathcal{F}}\times \mathrm{X}^{\mathrm{n}-1})^{\mathrm{k}-1})\right\}_{\mathrm{k}}\end{array} \arrow[d] \\
\left\{\pi_G^{m}(G/H\times S^{n-1})/I_{\mathcal{F},f}(L)^k\right\}_k \arrow[r, "\lambda_{\mathcal{F},f}(G/H\times S^{n-1})"'] \arrow[d] & \left\{\pi_G^{m}((\mathrm{E}_{\mathcal{F}}\times \mathrm{G/H}\times \mathrm{S}^{\mathrm{n}-1})^{\mathrm{k}-1})\right\}_{\mathrm{k}} \arrow[d] \\
\cdots & \cdots
\end{tikzcd}$$

where we can assume inductively that the completion theorem holds for X^{n-1}, $G/H\times S^{n-1}$ and $G/H\times D^n \simeq G/H$. The result follows from a standard application of the 5-lemma for progroups. □

Proposition 27. Let L be a l-dimensional proper G-CW complex with isotropy in a family $\mathcal{F}$ of finite subgroups of G, let $\mathcal{H}$ be an equivariant cohomology theory. Suppose that for $r = 2, \dots, l-1$ the differential of the equivariant Atiyah-Hirzebruch spectral sequence for L and $\mathcal{H}^*_G$, $d_r^{0,0} : E_r^{0,0} \to E_r^{r,1-r}$ vanishes rationally. Let $H \in \mathcal{F}(L)$, $\mathcal{P} \subset I(H)$ be any prime ideal and $f : G/H \to L$ be any G-map. Then the image of the augmentation ideal under f is contained in $\mathcal{P}$ if $\mathcal{P}$ contains the image of the structure map for H

$$\phi_H : \mathrm{invlim}_{K\in \mathrm{Sub}_{G,\mathcal{F}}} I_K \to I_H$$

Proof. It is proved in[Lüc08] using the equivariant Atiyah-Hirzebruch spectral sequence. We recover here his main argument. The crucial point is to note that since the differentials $d_r : E_r^{0,0} \to E_r^{r,1-r}$ vanish rationally, for any $x \in H_{\mathbb{Z}\mathrm{Or}(G)}(L, \mathcal{H}^0_G(G/?))$, there exists a positive integer k such that x^k is contained in the image of the edge homomorphism $\mathcal{H}^0_G(L) \to H^0_{\mathbb{Z}\mathrm{Or}(G),\mathcal{F}}$. Consider $a \in \mathrm{invlim}_{K\in\mathcal{F}} I_K$ let x be its image under the following composition of maps

$$\begin{aligned}\mathrm{invlim}_{K\in\mathcal{F}} I_K \to \mathcal{H}^0_K(\{*\}) \xrightarrow{\cong} \mathcal{H}^0_{\mathbb{Z}\mathrm{Or}G}(\mathrm{E}\mathcal{F}, \mathcal{H}^0_G(G/?))&\\ \xrightarrow{u^*} H_{\mathbb{Z}\mathrm{Or}(G),\mathcal{F}}(L, H^0_G(G/?))&\end{aligned}$$

where the first map is unduced by the inclusions, the second one by the canonical isomorphism and the third one by the classifying map $u : L \to \mathrm{E}\mathcal{F}$. So, there is $y \in \mathcal{H}^0_G(L)$ with $\mathrm{edge}^{0,0} = x^k$. Now, since the composite

$$\mathcal{H}^0_G(L) \xrightarrow{f^*} \mathcal{H}^0_G(G/H) \overset{\mathrm{ind}_{H\to G}}{\longrightarrow} \mathcal{H}_H(\{*\})$$

maps y to $\mathcal{P}$, ϕ_h maps a^k to $\mathcal{P}$.

$\square$

Chapter 5

The Bivariant Theory

We now extend Lück's and the previously constructed proper equivariant cohomology theory to a bivariant theory. We first sketch the definitions for a discrete group acting on proper cocompact finite CW complexes, although the general arguments outlined in chapter 2 also can be considered here. We extend the bivariant theory for cocompact complexes and discrete groups to infinite complexes by means of straightforward generalizations of our techniques developed around the parametrized Segal map.

5.1 Homotopy over the universal proper space.

The first construction of the bivariant theory involves finite dimensional G-Vector bundles over the classifying space of proper actions $\underline{\mathrm{E}}G$. As it is well described in the literature ([LO01b]), constructions of vector bundles over this space can only be performed under the assumption of finiteness. We shall therefore suppose in this section that the group has a finite model for $\underline{\mathrm{E}}G$. Altough this hypothesis is satisfied in many interesting cases (Mapping class groups, Word Hyperbolic Groups, One Relator groups, Coxeter Groups, Cat(0)-groups, etc.), we shall drop it out in the following section.

Loosely speaking, the idea is to use vector bundles over $\underline{\mathrm{E}}G$ to stabilize maps between proper G-CW complexes.

Let X be a proper cocompact G-CW complex and $r_X : X \to \underline{\mathrm{E}}G$ a representative of the classifying map. Given a finite dimensional G-vector bundle $\xi : E \to \underline{\mathrm{E}}G$, We form the pointed suspension along ξ, $\Sigma^\xi X$ as the space

$$S^\xi \underset{\underline{\mathrm{E}}G}{\times} X \coprod \underline{\mathrm{E}}G$$

with the topology whose basis is determined by the open sets of X and sets of the shape $r_X^{-1}(W) - K \coprod W$, where W is an open set in $\underline{\mathrm{E}}G$ and $K \subset W$ is such that $r_X(K)$ is compact in $\underline{\mathrm{E}}G$. We note that this space has a G-map $p : \Sigma^\xi X \to \underline{\mathrm{E}}G$ called the projection, as well as $s : \underline{\mathrm{E}}G \to \Sigma^\xi X$, called the section such that $p \circ s = \mathrm{id}_{\underline{\mathrm{E}}G}$. Slightly more general, If X, Y, Z are spaces with a map to $\underline{\mathrm{E}}G$,(as it is the case of proper G-CW complexes and fibre bundles over them), we form their product over $\underline{\mathrm{E}}G$ as the pullback $X \underset{\underline{\mathrm{E}}G}{\times} Y$. The one point compactification over $\underline{\mathrm{E}}G$ of Z, Z_+ is defined to be the space $Z \coprod \underline{\mathrm{E}}G$ with the topology whose basis is determined by the open sets of Z and sets of the shape $r_Z^{-1}(W) - K\coprod W$, in an analogous notation to the pointed suspension above. The reduced product over $\underline{E}G$, $X \wedge_{\underline{E}G} Y$ is defined to be the one point compactification over $\underline{E}G$ of the product over $\underline{E}G$. On the other hand, given a space with a map to $\underline{E}G$, we can always add a disjoint copy of $\underline{E}G$ in order to have a section. If X,Y are spaces with a projection and a section, we form the space $X \vee_{\underline{E}G} Y$ by identifying on $X\coprod Y$ the corresponding sections.

Of course, if $f : X \to Z$ and $g : Y \to Z$ are section and projection preserving maps, they determine a map $f \wedge_{\underline{E}G} g : X \wedge_{\underline{E}G} Y \to Z($ if they agree on the section), as well as a map $f \vee_{\underline{E}G} g : X \vee_{\underline{E}G} Y \to Z$.

Let Y be a proper finite G-CW complex and ξ,μ finite dimensional G-vector bundles over $\underline{E}G$. A $\mu - \xi$- *stable map* over $\underline{E}G$ is a map $f : \Sigma^\xi X \to \Sigma^\mu Y$ such that $p_X = p_Y \circ f$, $s_Y = s_X \circ f$, where the subscripts are meant to distinguish the corresponding sections and projections.

Two maps $f_i : \Sigma^\xi X \to \Sigma^\mu Y$ are homotopic over $\underline{E}G$ if there is a map $H : \Sigma^\xi X \times I \to \Sigma^\mu Y$ which is compatible with the sections and projections and for which $H(\quad, i) = f_i$ holds.

Let $f_i : \Sigma^{\xi_i \oplus \underline{\mathbb{R}^{k_i}}} X \to \Sigma^{\xi_i \oplus \underline{\mathbb{R}^{k_i+n}}}$ be stable maps over $\underline{E}G$ for $i = 0, 1$. They are equivalent if there are vector bundles μ_0, μ_1 over $\underline{E}G$ and a vector bundle isomorphism $\nu : \xi_0 \oplus \mu_0 \cong \xi_1 \oplus \mu_1$ such that following diagram commutes up to homotopy over $\underline{E}G$:

$$\begin{tikzcd}
\Sigma^{\xi_0 \oplus \underline{\mathbb{R}^{k_0}} \oplus \mu_0} X \arrow[r, "f_0 \wedge_{\underline{E}G} \mathrm{id}"] \arrow[d, "\sigma_1"'] & \Sigma^{\xi_0 \oplus \underline{\mathbb{R}^{k_0+n}} \oplus \mu_0} Y \arrow[d, "\sigma_2"] \\
\Sigma^{\xi_0 \oplus \mu_0 \oplus \underline{\mathbb{R}^{k_0}}} X \arrow[d, "\sigma_3"'] & \Sigma^{\xi_0 \oplus \mu_0 \oplus \underline{\mathbb{R}^{k_0+n}}} Y \arrow[d, "\sigma_4"] \\
\Sigma^{\xi_1 \oplus \mu_1 \oplus \underline{\mathbb{R}^{k_1}}} X \arrow[d, "\sigma_5"'] & \Sigma^{\xi_1 \oplus \mu_1 \oplus \underline{\mathbb{R}^{k_1+n}}} Y \arrow[d, "\sigma_6"'] \\
\Sigma^{\xi_1 \oplus \underline{\mathbb{R}^{k_1}} \oplus \mu_1} X \arrow[r, "f_1 \wedge_{\underline{E}G} \mathrm{id}"] & \Sigma^{\xi_1 \oplus \underline{\mathbb{R}^{k_1+n}} \oplus \mu_1} X
\end{tikzcd}$$

Where σ_3 and σ_4 are determined by the vector bundle isomorphism ν and the other maps are determined by transposition and canonical isomorphisms.

For any $n \in \mathbb{Z}$, we define the n-dimensional bivariant homotopy groups of X and Y to be the set of equivalence classes of stable maps over $\underline{E}G$ of the shape $f : \Sigma^{\xi \oplus \mathbb{R}^k} X \to \Sigma^{\xi \oplus \mathbb{R}^{k+n}} Y$ after dividing out equivalence in the previous sense and homotopy over $\underline{E}G$. We denote them by $\omega^n_{\underline{E}G}(X, Y)$.

Proposition 28. For any finite proper G-CW complex X, there is an in X natural isomorphism of sets

$$\omega^*_{\underline{E}G}(X, \underline{E}G) \cong \pi^*_G(X)$$

Proof. We first illustrate this for 0-stable maps. Let $f : \Sigma^\xi X \to \Sigma^\xi \underline{E}G$ be a representative of an element in $\omega_{\underline{E}G}(X, \underline{E}G)$. Let $p : S^{r_X^* \xi} \to X$ be the projection and note that the universal property of the pullback gives rise to a map $u : S^{r_X^*} \to S^{r_X^*}$. It is straightforward to check that u is a bundle map covering the identity on X, hence a representative of an element in $\pi^0_G(X)$. In the other direction, let $u : S^\xi \to S^\xi$ be a representative of an element in $\pi^0_G(X)$. After lemma 3.7 of[LO01b], there exists a vector bundle μ over $\underline{\mathrm{E}}G$ such that ξ is a direct summand of $r^*_X(\mu)$. Hence, we can form a bundle map $S^{r^*_X(\mu)} \to S^\xi \overset{u \wedge id}{\to} S^\xi \to S^\mu$ which obviously covers r_X. This extends to the one-point compactification and the assignments are compatible with homotopy and stability conditions. ☐

We now summarize some properties of the homotopy sets of stable maps over $\underline{\mathrm{E}}G$.

Lemma 15. 1. There is an abelian group structure on $\omega^*_{\underline{\mathrm{E}}G}(X, Y)$.

2. There is a composition product $\omega^*_{\underline{\mathrm{E}}G}(X, Y) \times \omega^*_{\underline{\mathrm{E}}G}(Y, Z) \to \omega^*_{\underline{\mathrm{E}}G}(X, Z)$.

3. The assignment is $X \to \omega^*_{\underline{\mathrm{E}}G}(X, Y)$ is a contravariant homotopy functor of X.

Proof. 1. Let $u : S^\xi \to S^\xi$ be a map covering the identity over $\underline{\mathrm{E}}G$ which is fibrewise the collapse onto the basis point at infinity. Let $r_X : X \to \underline{\mathrm{E}}G$ the classifying map. The zero element is represented by a map $f = u \wedge_{\underline{\mathrm{E}}G} r_X : S^\xi \wedge_{\underline{\mathrm{E}}G} X \to S^\xi \wedge_{\underline{\mathrm{E}}G} Y$. Let f_i for $i = 0, 1$ be representatives of elements in $\omega^n_{\underline{\mathrm{E}}G}(X, Y)$. $f_i : \Sigma^{\xi_0 \oplus \underline{\mathbb{R}^{n_i}}} X \to \Sigma^{\xi_0 \oplus \underline{\mathbb{R}^{n_i+n}}} Y$. Extending the corresponding notion for cohomotopy, We define their sum to be the map over $\underline{\mathrm{E}}G$ which is represented by

$$\Sigma^{\xi_0 \oplus \xi_1 \oplus \underline{\mathbb{R}^k}} X \wedge_{\underline{\mathrm{E}}G} S^{\mathbb{R}} \overset{\mathrm{id} \wedge_{\underline{\mathrm{E}}G} \nabla}{\to} \Sigma^{\xi_0 \oplus \xi_1 \oplus \underline{\mathbb{R}^k}} X \wedge_{\underline{\mathrm{E}}G} (S^{\mathbb{R}} \vee_{\underline{\mathrm{E}}G} S^{\mathbb{R}}) \overset{\sigma_3}{\to}$$
$$\Sigma^{\xi_0 \oplus \underline{\mathbb{R}^k}} X \wedge_{\underline{\mathrm{E}}G} S^{\mathbb{R}} \vee_{\underline{\mathrm{E}}G} \Sigma^{\xi_1 \oplus \underline{\mathbb{R}^k}} X \wedge_{\underline{\mathrm{E}}G} S^{\mathbb{R}} \overset{(f_0 \wedge_{\underline{\mathrm{E}}G} id) \vee_{\underline{\mathrm{E}}G} (f_1 \wedge_{\underline{\mathrm{E}}G} id)}{\to}$$
$$\Sigma^{\xi_0 \oplus \xi_1 \oplus \underline{\mathbb{R}^{k+n}}} Y \wedge_{\underline{\mathrm{E}}G} S^{\mathbb{R}}$$

where $\nabla : S^{\mathbb{R}} \to \mathbb{R} \vee \mathbb{R}$ is the pinching map.

The additive inverse of an element represented by a map f is given by the class of the composition

$$\Sigma^{\xi \oplus \underline{\mathbb{R}^{k+1}}} X \overset{\sigma_1}{\to} \Sigma^{\xi \oplus \underline{\mathbb{R}^k}} X \wedge_{\underline{\mathrm{E}}G} \Sigma^{\mathbb{R}} \overset{f \wedge_{\underline{\mathrm{E}}G} - \mathrm{id}}{\to} \Sigma^{\xi \oplus \underline{\mathbb{R}^{k+n}}} Y \wedge_{\underline{\mathrm{E}}G} S^{\mathbb{R}} \overset{\sigma_2}{\to} \Sigma^{\xi \oplus \underline{\mathbb{R}^{k+n+1}}} Y$$

2. It is clear.

3. It is clear.

☐

5.2 A Parametrized Fixed Point Index

One of the most remarkable features of stable homotopy is the close interaction with fixed point theory. It is hard to trace the authors of the first steps in this direction, but in the modern formulation the theory is connected with the names of Solomon Lefschetz, Heinz Hopf and Albrecht Dold. This interaction has led to a fruitful exchange of techniques on both theories. Nonlinear analysis and Manifold Topology are among the branches of mathematics on which notions of a fixed point index and methods from stable homotopy theory accelerated interesting discoverings. We shall describe how this ideas get into our context, in the hope of being useful and interesting for mathematicians working in other fields.

To the knowledge of the author, the approach to parametrized fixed point theory by equivariant methods was first outlined in the context of compact Lie groups by Hanno Ulrich [Ulr88], [PU91]. A reference with some recent applications is [Cra07]. We shall here present a somewhat ad-hoc approach, the main immediate objective beeing the identification of the parametrized Segal map constructed in this work with a parametrized fixed point index. We assume in this section proper finiteness on the basis complexes.

Definition 25. Let X be a proper finite G-CW complex and G be a discrete (possibly infinite) group. A fiberwise G-euclidean Neighborhood over X is a locally trivial G-fiber bundle U over X together with a fibrewise open inclusion $U \subset \xi$ into a finite dimensional G-vector bundle over X.

A G-Euclidean Neighborhood retract over X, denoted $\mathrm{G}-\mathrm{ENR}_X$ is a locally trivial G-fiber bundle M over X together with fibrewise maps $i : M \to U$, $r : M \to U$ into a euclidean neighborhood over X such that $r \circ i = \mathrm{id}$.

We now describe the main object of study of parametrized fixed point theory

Definition 26. Let $U \subset M$ be an open, G-invariant subset of a fibrewise $\mathrm{G}-\mathrm{ENR}_X$. and suppose $f : U \to M$ is a fibrewise G-map over X which is *compactly fixed* in the sense that the fixed subset

$$\mathrm{Fix}(f)\{x \in U \mid f(x) = x\} \subset U$$

is compact and there is an open G-invariant neighborhood V of $\mathrm{fix}(f)$ such that $f(V)$ has compact closure in M.

Definition 27. The parametrized fixed point index of a compact fixed map is a stable map

$$L_X(f,U) \in \omega^0_{\underline{\mathrm{E}}G}(X, U_+)$$

defined explicitely as a map over $\underline{\mathrm{E}}G$

$$q : \Sigma^\xi X \to \Sigma^\xi U_+$$

constructed as follows

1. Step 1. By the compactness of $\bar{V} - V$, there is a real number ϵ such that $\|f(x) - x\| \geq \epsilon$ for all $x \in \bar{V} - V$. The collapse map $c : \xi \to S^\xi$

$$c(v) = \begin{cases} (\epsilon^2 - \|v\|^2)^{-\frac{1}{2}} v & v \in \bar{V} \\ \infty & \text{if } v \notin V \end{cases}$$

   maps the open ball of radius ϵ and centre 0 homeomorphically onto ξ and collapses the complement to the point at infinity.

2. Denote by U_+ the fibrewise one-point compactification of the G-bundle with fiber U the map $q : S^\xi \to U_+$ given fiberwise by

$$q(x) = \begin{cases} c(x - f(x)) & x \in \bar{V} \\ \infty & x \notin V \end{cases}$$

is proper over every point in X. It determines hence a map of one-point compactifications over $\underline{E}G$, $q : \Sigma^{\xi} X \to \Sigma^{\xi} U_+$.

We just state the following properties of the parametrized fixed point index:

Proposition 29. [PU91] Let $f.U \to M$ be a compactly fixed map over a proper G-CW complex X. The parametrized point index has the following properties:

1. (Naturality). Let $\alpha.X \to X^{'}$ be a G map to a finite G-CW complex. Then
$$L_{X'}(\alpha^* f, \alpha^*(U)) = L_X(f, U)$$
2. (Localization). Let $U^{'} \subset U$ be an open G-subspace of U containing the fixed point set of f. Then $L_X(f \mid_{U'}, U^{'})$ maps to $L_X(f, U)$ under the map
$$\omega_{\underline{E}G}(X, U^{'}_+) \to \omega_{\underline{E}G}(X, U_+)$$
3. (Additivity). Supose that U is the disjoint union of two open G-subspaces U_1 and U_2. Then
$$L_X(f, U) = (i_{1_*}) L_X(f \mid_{U_1}, U_1) + (i_{2_*}) L_X(f \mid_{U_2}, U_2)$$
4. (Homotopy invariance). Suppose that $f : I \times U \to I \times M$ is a fibrewise G-map over $I \times X$ with compact fixed-point set. Denote by $L_X(f_t, U)$ the restriction of f to the subspace $\{t\} \times U$. Then $L_X(f_t, U)$ is independent of t.
5. (Contraction). Let $p : N \to M$ be a fibrewise G-map of $\mathrm{G} - \mathrm{ENR}_X$. Let $h : U \to N$ be a fibrewise G-map defined on an open subspace U of M. Write $V = p^{-1}(U) \subset N$, $f = p \circ h : U \to M$, $g = h \circ p : V \to N$ as in the following diagram:
$$\begin{tikzcd}
p^{-1}(U) = V \arrow[r, "g"] \arrow[d, "p|"] & N \arrow[d, "p"] \\
U \arrow[ur, "h"] \arrow[r, "f"'] & M
\end{tikzcd}$$
Supppse that $\mathrm{Fix}(f)$ is compact and that h maps a compact neigborhood of $\mathrm{Fix}(f)$ into a compact subspace of N. Then f and g are compactly fixed and p maps $L_X(g, V)$ to $L_X(f, U)$.

Remark 12 (The parametrized Segal map as a fixed point index)**.** Indeed, the parametrized segal map as constructed in definition 12 is a fixed point index for the very special situation of a compact, discrete $\mathrm{G} - \mathrm{ENR}_X$ (=a finite G-covering) and a map going out of a parametrized tubular neighborhood of it into itself, giving rise to a fixed point situation which involves the retraction of the tubular neighborhood. The fixed point sets over X can be identified with the covering itself.

5.3 Dropping out the finiteness condition

As it has been previously done for equivariant cohomotopy, we shall extend the bivariant theory for arbitrary proper G-CW complexes. Our strategy for doing this is to construct an appropiate analogon of the infinite loop space QX associated to a space X. We generalize and conveniently modify ideas which go back to Graeme Segal [Seg74] in the nonequivariant setting and a number of mathematicians including Costenoble, Waner, and Hauschild among others in the equivariant setting. Let us briefly recall Segal's original idea in the following

Construction 1. Let Y be an unpointed space. The category of finite sets over Y, $\mathcal{C}_Y$ has the following description. Objects are pairs (S, φ), where S is a finite set and $\varphi : S \to Y$ is a bijective function. Morphisms from (S, φ) to (T, ψ) consist of an injective map $\theta : S \to T$ such that $\theta^*(\psi) = \varphi$. This is a topological category, where the space of morphisms carries the discrete topology, and the space of objects is topologized as the configuration space $C_*(X)$, see [Seg73].

The category of finite sets over Y has a symmetric monoidal structure determined by the disjoint union in the category of finite sets and the fact that $Y^{S \coprod T} \approx Y^S \times Y^T$ if the spaces are given the compactly generated topology.

Precisely, $(S, \varphi) \coprod (T, \psi) = (S \coprod T, \varphi \times \psi)$. Associated to this structure there is a Γ-space $M_{\mathcal{C}_Y}$ such that $\Omega \mathrm{B} M_{\mathcal{C}_X} \simeq \Omega^\infty \Sigma^\infty Y_*$. If Y is the singleton space, then the construction above gives the first step of a proof of the Barrat-Priddy-Quillen-Segal theorem. We remit the reader to the article [Sch07] for a further , more recent reference of these facts. We point out that altough the construction does not take pointed spaces as input, the topological category of sets over Y does have a basis point: the empty function.

As it is a leitmotiv in this work, we make this constructions equivariant by considering categories of functors out of the transport category.

Definition 28. Let Y be a G-CW complex. The category of G-sets over Y is the category of functors from the transport category to the category of sets over Y. The Γ-space of G-sets over Y is the Γ-space associated to the symmetric monoidal structure of $\mathcal{C}_Y$, as in remark 1. In symbols

$$M_{\mathcal{C}_Y}(\bar{n}) = \mid \mathrm{Fun}(\mathcal{E}(G), \mathcal{C}_Y(\bar{n})) \mid$$

there is an action of G on this space determined by the action on the category of functors which is described by $g \cdot f(h) = (S, g \cdot \varphi)$, where $g \cdot \varphi : S \mapsto gf(hg^{-1})$. Note that this Γ-space also admits integer deloopings, denoted by B and that $\Omega \mathrm{B} M_{\mathcal{C}_Y}(\mathbf{1})$ is an infinite loop space with an action of G.

Let $M^0(X, Y)$ be the set of homotopy clases of G maps from X info the group completion of $M_{\mathcal{C}_Y}$. Note that M is a contravariant, abelian group valued functor of X which has the universal property of extendig transformations into monoid valued functors (fundamental property of group completion, proposition 4.1 in [Seg74]). Hence, our strategy to proof that $M^*(\quad,\quad)$ extends $\omega_{\underline{\mathrm{E}}G}(\quad,\quad)$. if to give a natural transformation S of monoid-valued, contravariant functors of X as in the following diagram, where the dotted map is given by the universal property of the group completion.

$$\begin{tikzcd}
{[X, M_{\mathcal{C}(Y)}(\bar{1})]} \arrow[r, "S"] \arrow[d] & \omega_{\underline{E}G}(X,Y) \\
M^0(X,Y) \arrow[ur, dashed] &
\end{tikzcd}$$

we then prove that these transformation restrict to isomorphisms on finite proper G-CW complexes Y, X by examining the behaviour on equivariant cells of X. Our construction is a generalization of the parametrized Segal map constructed before. The first step in this direction is to give a geometric interpretation of maps into the corresponding Γ-space.

Definition 29. Let $\varphi : S \to Y$ be a finite set over Y. A φ-covering map over X consists of

1. An $\mid S \mid$-sheeted, G-covering map $\tilde{X} \xrightarrow{p} X$.
2. A map $\tilde{\varphi} : \tilde{X} \to \mathrm{SP}^k Y$ which is fibrewise the image of φ. that is, such that the following diagram commutes for every $x \in X$,

$$\begin{tikzcd}
S \arrow[r, "\varphi"] & \mathrm{SP}^k Y \\
p^{-1}(\{x\}) \arrow[u, "\approx"'] \arrow[ur, "\tilde{\varphi}|_{p^{-1}(\{x\})}"'] &
\end{tikzcd}$$

and $\mathrm{Sp}^k = Y^k/\Sigma_k$ denotes the $k-$th symmetric product of Y

We shall call the set of φ-covering maps with varying φ, a covering map labeled on Y.

φ, $\varphi^{'}$-covering maps are called equivalent if there is a covering map isomorphism over X such that the following diagram commutes

$$\begin{tikzcd}
 & \mathrm{SP}^k(Y) \arrow[r, "\mathrm{id}"] & \mathrm{SP}^k(Y) \\
\tilde{X} \arrow[ur, "\varphi"] \arrow[r] \arrow[d] & \tilde{X}^{'} \arrow[ur, "\varphi^{'}"'] \arrow[dl] & \\
X & &
\end{tikzcd}$$

Let X, Y be proper G-CW complexes and $r_X : X \to \underline{E}G$, $r_Y : Y \to \underline{E}G$ be representatives of the unique G-homotopy classes. A φ-covering is said to be admisible if for every $s \in p^{-1}(\{x\})$, $\varphi(s)$ lies in the same connected component of G/Y as $r_Y^{-1}(\{r_X(x)\})$.

The sum (fiberwise disjoint union) of covering maps and the symmetric monoidal structure in $\mathcal{C}_Y$ determine an abelian monoid structure on the set of admisible coverings labelled in Y. We shall denote this by $\mathrm{Cov}_Y(X)$

Proposition 30. Let X and Y be proper finite G-CW complexes. The space $M_{\mathcal{C}_Y}(1)$ classifies admissible covering maps labeled on Y. That is, there is an in X natural isomorphism of abelian monoids

$$[X, M_{\mathcal{C}}(Y)(\bar{1})] \cong \mathrm{Cov}_Y(X)$$

Proof. Let f be a G-map $X \to M_{\mathcal{C}_Y}(\bar{1})$. We construct out of f a covering map labelled on Y. Consider the functor $C : \mathcal{C}_Y \to \mathcal{C}_{\{*\}}$ induced by the collapse map into the singleton space. Note that there is an equivalence of categories $\mathcal{C}_{\{*\}} \cong \mathrm{Set}$ into the category of finite sets and bijections. Hence, there is an equivariant map $M_{\mathcal{C}_Y}(1) \to \underline{G\text{−Set}}$.

On the other hand, by pullback of a covering of the form $\underline{G\text{−Frame}} \underset{\Sigma_{|S|}}{\times} S \to \underline{G\text{−Set}}$, we constructed previously a G-covering over X. Now note that φ determines a map $\tilde{\varphi}$ as in the following diagram

$$\begin{tikzcd}
\tilde{X} \arrow[r] \arrow[d] & \underline{G\text{−Frame}} \underset{\Sigma_{|S|}}{\times} S \arrow[r, "\tilde{\varphi}"] \arrow[d] & \mathrm{Sp}^k Y \\
X \arrow[r] & \underline{G\text{−Set}} &
\end{tikzcd}$$

The covering map $\tilde{X}$ equipped with the map $\tilde{X} \to \mathrm{SP}^k Y$ described above determines a covering map labelled in Y which only depends on the G-homotopy class of f. ☐

Proposition 31. Let X, Y be proper, cocompact G-CW complexes. There is an in X natural morphism of abelian monoids $\mathrm{Cov}_Y(X) \to \omega^0_{\underline{E}G}(X; Y)$

Proof. Consider an admissible φ-covering over X with total space $\tilde{X}$. Let us recall after lemma 3 the existence of a vector bundle ξ over $\underline{E}G$ such that $\tilde{X} \to E$ covers injectively the classifying map r_X (Lemma 3 gives a vector bundle over X and, one can push it forward into a summand of a vector bundle over $\underline{E}G$ by means of lemma 3.7 of[LO01b]. Denote by $\{y_1, \dots, y_k\}$ the image of φ. The *i-stratum* $\tilde{X}_i$ of $\tilde{X}$ consists of the points which are fibrewise mapped onto y_i. Note that $\tilde{X}$ is fibrewise decomposed over its strata. By conveniently choosing the embedding into the vector bundle, one can choose vector bundles E_i such that $\tilde{X}_i$ injectively lies on E_i with disjoint images. Let $s : \tilde{X} \to \xi = \bigoplus_i E_i$ be a stratified embedding as described. Choose a G-invariant riemannian metric on ξ and a tubular neighborhood $U_\epsilon(\tilde{X})$ of the image of the embedding adequate for considering a collapse situation.

Consider now the map $S^{r_X^*\xi} X \overset{\wedge^i_{\underline{E}G}\theta_i}{\longrightarrow} S^{r_Y^*\xi} Y$ which is given on the stratum E_i as the map $(x, v) \mapsto (y_i, \theta_{S_i}(v))$ for the collapse map θ_{S_i} of the embedding of $\tilde{X}_i \to E_i$.

This determines a stable map $\Sigma^\xi X \to \Sigma^\xi Y$, which is well defined after considering homotopy classes of maps over $\underline{E}G$. It is additive by a careful fibrewise examination which runs parallel to lemma 5. ☐

We now examine the behaviour on equivariant cells. Of course, this is a special case covered by the following slightly more general

Proposition 32. Let H be a finite subgroup of G. Let X be space with a trivial action of H. Then, for every G-CW complex Y, there is an isomorphism

$$[X \times G/H, \Omega \mathrm{B} M_{\mathcal{C}_Y}]_G \cong [X, Q(Y)]_H$$

where $Q(Y)$ denotes the H-space obtained as the colimit over a cofinal set of representations of the H-equivariant mapping spaces $\mathrm{Map}_*(S^V, S^V \wedge Y_+)$, and Y has the action given by the restriction to H.

Proof. We first recall that from our remarks on functors going out of transport and grupoid categories, lemma 1, the left hand side admits a decription as $[X, |\operatorname{Fun}(\mathcal{B}H, \mathcal{C}_Y)\,||]_H$. We note that giving a functor out of the grupoid category amounts to give an object of $\mathcal{C}_Y$ with a conveniently defined action of the finite group H. That is, we have $H \to \mathrm{Auto}(S, \varphi)$. It follows that we can identify such a functor with a finite H-set and a map to Y -defined on the finite set- which is compatible with the action. That is to say, the map takes values into the H-fixed point set. We shall calll such objects a finite H-set over Y. The geometrical realization has the homotopy type of

$$\coprod_{S,\varphi} \mathrm{BAuto}_{S,\varphi}$$

where S, φ runs over the isomorphism classes of finite H-sets over Y. As before, given an H-set S, we look at its decomposition into orbits of the type H/K_i and write $S = \coprod n_i H/K_i$ in what we expect is a natural notation. We consider now an H universe, V^∞, as definied in the proof of proposition 4. For the same reason there -the equivariant version of the Whitney embedding theorems-, the space of equivariant embeddings $\mathrm{Emb}_H(H/K, V^\infty)$ is a model for the classifying space of the Weyl group of K in H, in symbols

$$\mathrm{EW_{K,H} \simeq_H Emb_H(H/K, V^\infty)}$$

On the other hand, for every map $\varphi : S \to Y$, evaluation at the identity coset gives a map

$$\mathrm{Map}(n_i H/K_i, V^\infty) \approx \prod_i \mathrm{Map}(H/K_i, V^\infty) \xrightarrow{\varphi} \prod_i Y^{K_i}$$

which we denote by φ_i. By means of this we can identify

$$\coprod_{S,\varphi} \mathrm{BAuto}_{S,\varphi} \simeq \coprod_n \coprod_K \mathrm{Emb}(nH/K, V^\infty) \underset{(W_{K,H})^n \wr \Sigma_n}{\times} (Y^K)^n$$

Now, up to homotopy, given an embedding of $\coprod n_i H/K_i$ in V^∞, we can choose finite dimensional, $W_{K_i,H}$-invariant subspaces V_i such that $n_i H/K_i \subset U_i$ for a neighborhood which is suitable for constructing a collapse map $S_i : S^{\mathrm{V_i}} \to S^{V_i}$. We now form the $W_{K,H}$ -equivariant map $\tau_i : S^{V_i} \to S^{V_i} \wedge Y_+^{K_i}$ defined by

$$x \mapsto \begin{cases} \infty & v \in V^i - \bar{U} \text{ or } v = \infty \\ S(v) \wedge \varphi_i(eH) & v \in U \end{cases}$$

This factors trough the quotient determined by $\mathrm{Emb}_H(H/K, V^\infty) \times Y^K \to \mathrm{Emb}(H/K, V^\infty) \underset{W_{K,H}}{\times} Y^K$ and thus gives a map into the equivariant stable $W_{K,H}$-maps as follows

$$\mid \mathrm{Fun}(\mathcal{B}H, \mathcal{C}(Y) \mid \rightarrow \prod_{K} \{\{*\}, \mathrm{EW_{K,H}} \underset{\mathrm{W_{H,K}}}{\times} \mathrm{Y^K}\}_{\mathrm{W_{K,H}}}$$

To resume, we get a map

$$[X, \mid \mathrm{Fun}(\mathcal{B}H, \mathcal{C}(Y)) \mid]_H \xrightarrow{\alpha} \prod_{K} [X, Q_{W_{H,K}}(\mathrm{EW_{K,H}} \underset{\mathrm{W_{K,H}}}{\times} \mathrm{Y^K})]$$

The map α determines a transformation of contravariant functors between a monoid valued and a group valued functor, thus factoring out trough our group completion. Hence, from the universal property of the group completion, the map $\tilde{\alpha}$ in the following diagram is an isomorphism

$$\begin{tikzcd}
{[X, \mid \mathrm{Fun}(\mathcal{B}H, \mathcal{C}(Y)) \mid]_H} \arrow[r] \arrow[d] & {\prod_K [X, Q_{W_{K,H}}(\mathrm{EW_{K.H}} \underset{\mathrm{W_{K,H}}}{\times} \mathrm{Y^K})] \cong [\mathrm{X}, \mathrm{Q_H Y}]_{\mathrm{H}}} \\
{[X, \Omega \mathrm{B} M_{\mathcal{C}(Y)}]_H} \arrow[ur, "\tilde{\alpha}"'] &
\end{tikzcd}$$

Where the right upper isomorphism is obtained by a splitting result for finite groups, for instance Theorem 2.1 p. 206 in [May96]. ☐

Since both theories agree on cells, and every finite G-CW complexes is exhausted after a finite number of cell attachments, we have the following

Theorem 14. The natural transformation

$$[X, \Omega \mathrm{B} M_{\mathcal{C}(Y)}(\bar{1})]_G \rightarrow \omega^0_{\underline{\mathrm{E}}G}(X, Y)$$

consists of isomorphisms in the category of finite G-CW complexes.

We examinate now the behaviour of the functor $M_{\mathcal{C}(X)}$ in subspaces . The following technical result is crucial in order to verify that this construction defines an equivariant homology theory being dual to equivariant stable cohomotopy.

Lemma 16.

1. Let (X, A) be a G-CW pair. The identification map $X \xrightarrow{p} X/A$ induces a continuous functor $M_{\mathcal{C}}(A) \rightarrow M_{\mathcal{C}}(X) \xrightarrow{p} M_{\mathcal{C}}(X/A)$. For any subgroup $H \subset G$, the functor restricts to a quasifibration on the H-fixed point space of objects $M^H_{\mathcal{C}(X)} \rightarrow M^H_{\mathcal{C}_{X/A}}$ with fiber $M^H_{\mathcal{C}_A}$.
2. The functor p_* induces a quasifibration at the leves of fixed points of classifying spaces. $\mid M_{\mathcal{C}(A)} \mid^H \rightarrow \mid M_{\mathcal{C}(X)} \mid^H \rightarrow \mid M_{\mathcal{C}(X/A)} \mid^H$
3. The functor p_* induces a long exact sequence of fixed point sets after performing group completion $\Omega\mathrm{B}$ of the Γ-space structure associated with the category of sets over X.

Proof. 1. Using a well known chriterion due to Waner [Wan80](Proposition 2.7), It amounts to find a G- filtration of *distinguished* sets U_i of the space $M_{\mathcal{C}_X}$, in the sense that The map $p_* : \pi_*(p^{-1}(U_i), p^{-1}(\{x\}), y) \rightarrow \pi_*(U_i, x)$ is an isomorphism for any H. and for any i, there is a G-invariant open subset U of U_i containing U_{i-1} together with G- homotopies $h : U \times I \rightarrow U$, $H : p^{-1}(U) \times I \rightarrow p^{-1}(U)$ for which :

- $h_0 = \mathrm{id}$, $h_t(U_{i-1}) \subset U_{i-1}$, $h_1(U) \subset U_{i-1}$
- $H_0 = \mathrm{id}$ and H covers h.
- H_1 restricts to a weak G_b-equivalence $p^{-1}(b) \to p^{-}1(h_1(b)$ for any b.

Recall [Lüc89], lemma 1.10 that A is an equivariant G- deformation retract of some G-invariant open neighborhood $\tilde{U}$ of A. For any natural number or zero, Let $M^i_{\mathcal{C}_X}$ be the collection of finite sets labbelled over X, for which the image of $x : S \to X$ has cardinality less or equal to i. define $V = M^i_{\mathcal{C}X/A} - M^{i-1}_{\mathcal{C}_{X/A}}$ the map $p^{-1}(V) \to V \times M_{\mathcal{C}_A}$ is an equivariant homeomorphism, as it is the case for any open, G-invariant subset. Let V' be the subset of $M^i_{\mathcal{C}_X}$ for which the map p maps at least one point into $p(\tilde{U})$. $V^{'}$ is a G- neighborhood deformation retract in $M^n_{\mathcal{C}_{X/A}}$. The set $V^{'}$ is also distinguished using a similar argument. As V, $V^{'}$, $V^{'} \cap V$ are distinguished, so is $V^{'} \cup V = M^i_{\mathcal{C}_X}$. for any i. The sets U are constructed using a Neighborhood deformation retract of A, as well as the homotopies h and H, which satisfy the three requirements above..

2. We use for this theorem B of Daniel Quillen, see [Qui73]. We recall that this result constructs a long exact sequence of homotopy groups of classifying spaces of categories (as desired here) out of certain assumptions on a functor.

   We quickly indroduce the notation needed in this setting. If $f : C \to C^{'}$ is a functor, and c is an object in $c^{'}$, then the category of objects over c, denoted by c/f has as objects pairs (d, e), where d is an object in c and $e : c \to f(e)$ is a morphism. A morphism between (d, e) and $(d^{'}, e^{'})$ is a morphism in C, $w : d \to d'$ such that $f(w)(e) = e^{'}$.

   We have to verify that for every morphism $(T, \varphi) \xrightarrow{\theta} (T^{'}, \varphi^{'})$ in $W_{\mathcal{C}}(X/A)$, the functor $p^* : (T^{'}, \varphi^{'})/p_* \to (T, \varphi)/p_*$ induces a homotopy equivalence of classifying spaces. This is achieved by an application of theorem A [Qui73], which is in fact a particular case of theorem B. That is, for every object (S, ψ) in the category $(T, \varphi)/p_*$, and any morphism θ^* in this category, the category $(S, \psi)/\theta^*$ is filtering. We begin with this fact. Let (T_1, η_1, θ_1) and (T_2, η_2, θ_2) be elements in $(S, \psi)/p_*$ and assume that $\theta_{1,2} : (T_1, \eta_1, \theta_1) \to (T_2, \eta_2, \theta_2)$ is a morphism. Note that, by definition, $\theta_i^* : p_*(\varphi) \to \varphi$ is an isomorphism induced by the isomorphism of sets T_1, T_2, and the same occurs for $\theta_{1,2}$. The object $(S, \varphi, \mathrm{id}) \in (S, \varphi)/p_*$ has maps induced by isomorphisms of sets $(T_i, \eta_i, \theta_i) \to (S, \varphi, id)$. If $\theta_i : (T, \eta, \tau) \to (U, \delta, \rho)$ are two morphisms with common domain and range, then by definition of the category there exists an isomorphism of sets $\xi : U \to S$ such that $\xi\theta_1 = \xi\theta_2$. This finishes the proof that the category is directed, and hence filtering, Hence, theorem B applies and gives the result.

3. This is now easier in view of part 2., since the delooping $BM_{\mathcal{C}(X)}(1)$ is defined to be the composite of

   - Shift of the Γ-space structure: $S \mapsto M_{\mathcal{C}(X)}(1 \times S)$.
   - Thickening τ of the simplicial set $M.$.

- Geometric realization of the thickened and shifted Γ-space $\mid \tau M_{\mathcal{C}(X)} \mid$.

The first part is clear to be true, second part follows from examination of the simplicial structure, the third one is Consequence of part 2. Finally, this commutes with fixed point sets because the fixed point sets are also Γ-spaces, and with the loop functor by obvious reasons(the long exact sequence only gets shifted).

□

Definition 30 (Equivariant Stable homotopy groups)**.** Let (X, A) be a proper G-CW pair. Define the equivariant stable homotopy groups as $\pi_n(\Omega \mathrm{B} M_{\mathcal{C}(X \cup \mathrm{Cone} A)}(1))$ if $n \geq 0$ and $\pi_1(\Omega B^{n+1} M_{\mathcal{C}(X \cup \mathrm{Cone} A)}$

We recall the axiomatic description of an equivariant homology theory as given in [LR05].

Definition 31. Let G be a group annd fix an associative ring with unit R. A G-homology theory with values in R-modules is a collection of covariant functors $\mathcal{H}^G_n$ indexed by the integer numers $\mathcal{Z}$ from the category of G-CW pairs together with natural transformations $\partial n^G : \mathcal{H}_n(X, A) \to \mathcal{H}^G_{n-1}(A) := \mathcal{H}_{n-1}(A, \phi)$, such that the followinaxioms are satisfied:

1. If f_0 and f_1 are G-homotopic maps $(X, A) \to (Y, B)$ of G-CW pairs, then $\mathcal{H}^G_n(f_0) = \mathcal{H}^G_n(f_1)$ for all n.
2. Given a pair (X, A) of G-CW complexes, there is a long exact sequence
$$\ldots \overset{\mathcal{H}^G_{n+1}(j)}{\rightarrow} \mathcal{H}^G_{n+1}(X, A) \overset{\partial^G_{n+1}}{\rightarrow} \mathcal{H}^G_n(A) \overset{\mathcal{H}^G_n(i)}{\rightarrow} \mathcal{H}^G_n(X)$$
$$\overset{\mathcal{H}_n(j)}{\rightarrow} \mathcal{H}^G_n(X, A) \overset{\partial^G_n}{\rightarrow} \mathcal{H}^G_{n-1}(A) \overset{\mathcal{H}^G_{n-1}(i)}{\rightarrow} \ldots$$
where $i : A \to X$ and $j : X \to (X, A)$ are the inclusions.
3. Let (X, A) be a G-CW pair and $f : A \to B$ be a cellular map. The canonical mal $(F, f) : (X, A) \to (X \cup_f B, B)$ induces an isomorphism
$$\mathcal{H}^G_n(X, A) \overset{\cong}{\rightarrow} \mathcal{H}^G_n(X \cup_f B, B)$$
.
4. Let $\{X_i \mid i \in \mathcal{I}\}$ be a family of G-CW-complexes and denote by $j_i : X_i \to \coprod_{i \in \mathcal{I}} X_i$ the inclusion map. Then the map $\oplus_{i \in \mathcal{I}} \mathcal{H}^G_n(j_i)$
$$\oplus_{i \in \mathcal{I}} \mathcal{H}^G_n(X_i) \overset{\cong}{\rightarrow} \mathcal{H}^G_n(\coprod_i X_i)$$
is bijective for each $n \in \mathbb{Z}$.

An equivariant homology theory consists of a family of G-homology theories $\mathcal{H}^G_*$ together with an induction structure

$$H^H_n(X, A) \cong \mathcal{H}^G_n(\mathrm{ind}_\alpha(X, A))$$

for group homomorphisms $\alpha : H \to G$ whose kernel acts freely on X satisfying the following conditions:

1. For any n, $\partial_n^G \circ \mathrm{ind}_\alpha = \mathrm{ind}_\alpha \circ \partial_n^G$.
2. For any group homomorphism $\beta : G \to K$ such that $\ker \beta \circ \alpha$ acts freely on X, one has

   $$\mathrm{ind}_{\alpha\circ\beta} = \mathcal{H}_n^K(f_1 \circ \mathrm{ind}_\beta \circ \mathrm{ind}_\alpha) : \mathcal{H}_n^H(X, A) \to \mathcal{H}_n^K(\mathrm{ind}_{\beta\circ\alpha}(X, A))$$

   where $f_1 : \mathrm{ind}_\beta \mathrm{ind}_\alpha \to \mathrm{ind}_{\beta\circ\alpha}$ is the canonical G-homeomorphism.
3. For any $n \in \mathbb{Z}$, any $g \in G$, the homomorphism

   $$\mathrm{ind}_{c(g):G\to G}\mathcal{H}_n^G(X, A) \to \mathcal{H}_n^G(\mathrm{ind})_{\mathrm{c(g):G\to G}}(\mathrm{X, A}))$$

   agrees with the map $\mathcal{H}_n^G(f_2)$, where $f_2 : (X, A) \to \mathrm{ind}_{c(g):G\to G}$ sends x to $(1, g^{-1}x)$ and $c(g)$ is the conjugation isomorphism in G.

Theorem 15. $\pi_*^G(X, A)$ defines an equivariant homology theory in the sense of [LR05]

Remark 13 (An Approach via OR(G)-spectra.)**.** As stated in [LR05], in order to construct a G-Homology theory, it is necessary to construct a covariant functor from the orbit category to the category of spectra. Due to theorem 7.4 of [DL98], This is consistent with the approach used here, by considering the functor

$$G/H \mapsto \Omega \mathrm{B} M_{\mathcal{C}_{G/H}}$$

Remark 14. Our definition of equivariant homology theories is crucially different from others in the literature, especially from that one in [May96]. We make no assuptions of our theory being $RO(G)$- graded(which amounts to finer equivariant deloopings of the Γ-space with an action M_{C_X} which we construct here). The analogon of proposition 32 is not expected to be true for more general topological groups (even Lie). See [Blu06] for all these issues.

Chapter 6

Noncommutative Geometry

Noncommutative geometry arose as an ambitious generalization of the methods and crucial questions of topology. This discipline, which was developed in the last decades by Alain Connes, Israil Gelf́and, Gennadi Kasparov, Paul Baum and several other mathematicians deals with the interaction of functional analysis, topology, differential geometry and K-theory. The author of this work admires this ideas and being from his education a topologist, would like to do a modest contribution to this vaste and diverse field of research.

As this work which does not deal until here with the methods of operator theory, we present some account of basic methods and techniques of noncommutative topology

Definition 32. 1. A complex Banach algebra is a $\mathbb{C}$-algebra which (with the same $\mathbb{C}$-linear structure is a complete normed vector space such that the norm satisfies

$$\|xy\| \leq \|x\| \|y\|$$

2. A C^*-algebra is a complex Banach Algebra A with an involutive antiautomorphism $*$ such that the following is true:

$$\|xx^*\| = \|x\| \|x\|$$

We are mainly interested in operator theory methods as a non trivial extension of topology. This is certainly explained in the following classical and today basic result:

Theorem 16 (Gel'fand- Naimark)**.** Let A be an abelian, σ-unital C^*-algebra and let $a \in A$. Denote by $\hat{} : A \to C_0(\hat{A})$ the function representation of A, characterized by the fact that $\hat{a}(\rho) = \rho(a)$.

Then, there exists an isometric $*$-homomorphism inverse to $\hat{}$

$$C : C_0(\hat{A}) \longrightarrow A$$

called the functional calculus with respect to a. It is unique with the property that $C(i) = a$, where $i : \mathrm{Spec}(a) \to C_0(\hat{A})$ is the map which maps identically an element of the spectrum to itself. Both the function representation and the functional calculus are natural and give rise to an equivalence of categories between the category of locally compact hausdorff spaces and the category of commutative C^* -algebras. .

6.1 Preliminaries on proper C^*-algebras

The notion of a proper group action is at the same time a leitmotiv and a motivation in this work. We give an account of different notions of properness for C^*-algebras with an action of a locally compact group.

A good reference in generalized notions of proper actions on C^* algebras is [Mey01].

Extending directly the notion for spaces as in [Pal61] via the Gel'fand-Naimark Theorem leads to the notion of a spectrally proper C^*-algebra. This concept is too rigid for our purposes. On the other extreme, the notion of square integrable action on a C^*-algebra is the laxest generalization allowing the use of techniques from the theory of Hilbert modules. We do not deal with this generalization in this work and remit to [Mey01] for the definition and the comparison with other notions, as well as examples showing that they do not agree in general.

The notion of properness we deal with arose in connection with the developement of equivariant KK-theory. It can be morally thought as a precise formulation of the intuition that a proper C^*-algebra is pretty much like the algebra of functions in a proper G-space.

Definition 33.

1. Let X be a locally compact, Hausdorff G-space. X is called proper if the map $X \times G \to X \times X$ $x \mapsto (gx, x)$ is proper.
2. Let A be a C^*-algebra with a strongly continuous action of G (briefly: a G-C^* algebra). A is spectrally proper if the states space $\hat{A}$ is proper as a G-space.
3. Let A be a G-C^* algebra. Recall that the multiplier algebra of A, [Bla98], p.103 is defined as the smallest unital algebra containing A as an essential ideal. A G-C^* algebra A is proper if there is a proper space X and an essential $*$-homomorphism into the center of the multiplier algebra of A.

$$C_0(X) \to Z(\mathcal{M}(A))$$

Proposition 33. Let A be a proper G-C^* algebra. There is a canonical $*$-homomorphism $\phi : C_0(\underline{E}G) \to \mathcal{L}(A)$, which allows to form the tensor product $A \otimes_\phi C_0(X)$ for any space X with a proper G-map $X \to C_0(\underline{E}G)$.

Proof. Suppose $A \subset \mathcal{L}(H)$ for some Hilbert space. then a concrete model for the multiplicator algebra is given as

$$\mathcal{M}(A) = \{m \in \mathcal{L}(H) \mid ma, am \in A \forall a \in A\}$$

Of course, the map $C_0(\underline{E}G) \ni x \mapsto \mathcal{Z}(\mathcal{M}(A)) \subset \mathcal{L}(H)$ restricts to a $*$-homo morphism into $\mathcal{L}(A)$. We now briefly recall the construction of the spatial tensor product. For more details on this, see [Bla98], [Lan95]. Consider a faithful representation $\pi^{'} : C_0(X) \to \mathbb{L}(H^{'})$ and denote by $\pi : A \to \mathcal{L}(H)$ the a above mentioned faithful representation. Consider the tensor product of representations $\pi \odot_\varphi \pi^{'} : A \odot C_0(X) \to \mathcal{L}(H \otimes H^{'})$. and define the tensor product to be the completion of $A \odot C_0(X)$ with respect to the spatial norm $t \mapsto \|\pi \odot \pi^{'}(t)\|$. In fact, under additional choices there is a structure of Hilbert $C_0(\underline{E}G)$- module on this C^* algebra. We warn, however, that the topology is not determined by this structure and will not explote this fact. □

6.2 Asymptotic homomorphisms

Mapping cones and reduced suspensions of topological spaces are elementary though fundamental tools of homotopy theory. Their convenient interaction in a long exact sequence of homotopy sets can be considered as the birth of homotopy theory and is at the same time the fundamental notion which is developed in further broad-reach generalizations like triangulated categories.

Unexpected problems arise when trying to extend to the noncommutative setting the long exact sequence involving cones and suspensions. This was first noticed in research around the properties of the Kasparov bivariant functor [Ska91], and is a purely operator theoretical phenomenon, in the sense that we do not know an analogous complication in the topology of CW-complexes.

One of the theoretical solutions to this problem was the definition of E-theory. Imprecisely, it could be seen as a homotopy theoretical refinement of Kasparov theory. The fundamental difference of E-theory in comparison with other homotopy theoretical-approaches to bivariant theories is the substitution of $*$-homomorphisms by a coarser notion, the so-called assymptotic homomorphisms. A good intuition to this point is to think about "$*$-homomorphisms at infinity". For the reader coming from operator theory, the author would like to say that assymptotic homomorphism are a convenient way of allowing generalized morphisms which appear, for instance, after the use of approximate units.

Definition 34. [GHT00] Let A, B be G-C^*- algebras. Denote by $\mathfrak{T}B$ the C^*-algebra of continuous,G-continuous bounded functions from the locally compact space $T = [1, \infty)$ to B. Let $\mathfrak{T}_0 B$ be the ideal in $\mathfrak{T}B$ consisting of the funcions from T to B which vanish in norm at infinity. The assymptotic algebra of B, $\mathfrak{A}(B)$ is defined to be the quotient $\mathfrak{T}B/\mathfrak{T}_0B$. An equivariant assymptotic homomorphism from A to B is an equivariant $*$-homomorphism from A to the assymptotic algebra $\mathfrak{A}B$.

Notice that there is a $*$-homomorphism $\alpha_B : B \to \mathfrak{A}B$ which assigns to $b \in B$ the class in $\mathfrak{A}B$ of the constant function $t \mapsto b$. Of course, the construction $\mathfrak{A}$ is functorial.

An assymptotic homomorphism $\varphi : A \to \mathfrak{A}B$ determines after choice of a section to $C_b([1, \infty), B) \to \mathfrak{A}B$ a family of continuous, bounded functions $\{\varphi_t : A \to B\}_{t \in [1,\infty)}$, satisfying that for any $a, a^{'} \in A$ and $\lambda \in \mathbb{C}$ the limit at infinity of the expressions

$$\varphi_t(a)^* - \varphi_t(a^*)$$

$$\varphi_t(a) + \lambda\varphi_t(a^{'}) - \varphi_t(a + \lambda a^{'})$$

$$\varphi_t(a)\varphi_t(a^{'}) - \varphi_t(aa^{'})$$

is zero in norm. When needed, we shall call such a family an associated family to the assymptotic homomorphism φ.

Composition and homotopy notions for assymptotic homomorphisms is a tecnically delicate question. There exist very simple examples showing that the composition of assymptotic homomorphisms is not an assymptotic homomorphism. One approach to this problem -the original followed by Connes and

Higson in their foundational work [CH90]- is an operator theoretical alteration of the morphisms. We rather follow an alternative solution, due to Guentner, Higson and Trout [GHT00], which can be thought of a categorical, rather than operator theoretical substitution. We point out that we only deal with this notions in connection with homotopy.

Definition 35. Let A, B be G-C^* algebras. The interval over B, IB is the C^*-algebra which consists of the continuous functions $f : [0,1] \to B$. Denote by $\mathfrak{A}^n$ the n-fold composition of the functor $\mathfrak{A}$ with itself. Two equivariant $*$-homomorphisms $\varphi_0, \varphi_1 : A \to \mathfrak{A}^n B$ are called n-homotopic if there is an equivariant $*$-homomorphism $\varphi : A \to \mathfrak{A}^n IB$ whose evaluation at the endpoints gives the φ_i. We shall denote by $[A,B]_n$ the homotopy classes of equivariant $*$-homomorphisms from A to $\mathfrak{A}^n B$.

Let $\alpha_B : B \to \mathfrak{A}B$ be the homomorphism described in definition 34. Note that given an equivariant $*$-homomorphism $A \xrightarrow{\varphi} \mathfrak{A}^n B$ one can form the composition $A \xrightarrow{\varphi} \mathfrak{A}^n \xrightarrow{\mathfrak{A}^n(\alpha_B)} \mathfrak{A}^{n+1}B$ We obtain by this mean a map $[A,B]_n \to [A,B]_{n+1}$

Definition 36. Let A, B be G-C^* algebras. The homotopy set of equivariant assymptotic homomorphisms from A to B, denoted by $[A,B]$ is defined as the colimit of the following diagram of sets

$$[A,B]_0 \to [A,B]_1 \to [A,B]_2 \to \dots$$

We summarize in the following result the advantages of considering homotopy classes of assymptotic homomorphisms. We remark that the proofs can be found in chapters 2 to 5 of the excellent exposition [GHT00].

Proposition 34. 1. Let A, B, C be G-C^* algebras. Given an equivariant $*$-homomorphisms $\varphi : A : \mathfrak{A}^j B$, $\psi : B \to \mathfrak{A}^k C$, the construction

$$A \xrightarrow{\varphi} \mathfrak{A}^j B \xrightarrow{\mathfrak{A}^j(\psi)} \mathfrak{A}^{j+k} C$$

defines an associative product

$$[A,B] \times [B,C] \to [A,C]$$

2. Denote by $D \otimes A$ the maximal tensor product of D and A. There is a functor from the category of G-C^* algebras which assignes $D \otimes A$ to the G-C^*-algebra D. and which assigns to a morphism $A \to \mathfrak{A}^j B$ the composition

$$D \otimes A \xrightarrow{1\otimes\psi} D \otimes \mathfrak{A}^j B \xrightarrow{i_j} \mathfrak{A}^j(D \otimes B)$$

where i_j denotes the morphisms constructed inductively as the composition:

$$D \otimes \mathfrak{A}^j B \xrightarrow{i} \mathfrak{A}(A \otimes \mathfrak{A}^{i_{j-1}}) \xrightarrow{\mathfrak{A}^{(i_{j-1})}} \mathfrak{A}^j(D \otimes B)$$

and $i : D \otimes \mathfrak{A}C \to \mathfrak{A}(D \otimes C)$ is the map determined from the fact that the maximal tensor product is an exact functor.

3. The corresponding statement for 2. is true if one considers the maximal tensor product $A \otimes D$. This construction is associative, in the sense that

for every pair of morphisms $\varphi : A_1 \to A_2$, $\psi : B_1 \to B_2$ in the homotopy category of assymptotic morphisms, the compositions

$$A_1 \otimes B_1 \overset{\varphi\otimes\mathrm{id}}{\to} A_2 \otimes B_1 \overset{\mathrm{id}\otimes\psi}{\to} A_2 \otimes B_2$$

and

$$A_1 \otimes B_1 \overset{\mathrm{id}\otimes\psi}{\to} A_1 \otimes B_2 \overset{\varphi\otimes\mathrm{id}}{\to} A_2 \otimes B_2$$

are equal in the homotopy category of assymptotic homomorphisms.

As we mentioned, the fundamental contributions of assymptotic homomorphisms are focused around the construction of exact sequences involving mapping cones and suspensions. We recall briefly the corresponding definitions

Definition 37. 1. Let B be a G-C^* algebra. The suspension of B is the C^* algebra

$$\Sigma B = \{f : [0,1] \to B \mid f \text{ is continuous and } f(0) = 0 = f(1)\}$$

endorsed with the obious action of the group G. Of course, the operation of suspension is a functor, and slightly more generally, an equivariant $*$-homomorphism $A \to \mathfrak{A}^n B$ induces an equivariant $*$-homomorphism $\Sigma A \to \Sigma\mathfrak{A}^n B$.

2. Let $\theta : B \to A$ be an equivariant $*$-homomorphism between G-C^* algebras. The mapping cone C_θ of θ is the G-C^* algebra

$$C_\theta = \{b \oplus f \in C_0([0,1), B) \oplus A \mid \theta(b) = f(0)\}$$

furnished with the obvious G-action. There exist $*$-homomorphisms

$$\alpha : C_\theta \to B \qquad \beta : \Sigma A \to C_\theta$$

defined by $\alpha(b \oplus f) = b$ and $\beta(f) = 0 \oplus f$.

The following is a fundamental though technical result:

Proposition 35. Given a short exact sequence of separable G-C^*-algebras

$$0 \to J \to B \overset{\pi}{\to} A \to 0$$

and an approximate unit $\{u_t\}_t$ for $J \subset B$, there is an assymptotic morphism $\sigma : A \to \mathfrak{A}J$ such that for any associated family $\{\sigma_t\}$, any set theoretical section (not necesarilly equivariant) $s : A \to B$ of the quotient map and any $f \in \Sigma$ and $a \in A$, the limit when t tends to infinity of the difference $\sigma_t(f \otimes x) - f(u_t)s(a)$ is zero .

We use it mainly in the form of the following consecuences, stated in [GHT00], proposition 5.14. and 5.16.

Proposition 36. Let $\pi : B \to A$ be a surjective $*$-homomorphism and denote by J its kernel. Let $\pi_1.CB \to C_\pi$ be the morphism given by $f \mapsto (f(0), \pi(f))$. Let $\sigma : \Sigma C_\pi \to \mathfrak{A}\Sigma J$ be the assymptotic homomorphism obtained from applying proposition 35 to the sequence $0 \to \Sigma J \to CB \overset{\pi_1}{\to} C_\pi \to 0$. Then, σ is an inverse in the homotopy categoy of assymptotic homomorphisms to the inclusion $*$-homomorphism $\Sigma\tau : \Sigma J \to \Sigma C_\pi$

Corollary 2. Let $\theta : B \to A$ be an equivariant $*$-homomorphism and let D be a G-C^*-algebra. The sequence of pointed sets

$$[D, C_\theta] \overset{\alpha_*}{\to} [D, B] \overset{\theta_*}{\to} [D, A]$$

is exact

The following result summarizes some technical facts concerning the relationship of homotopy classes of assymptotic homomorphisms and the rigid, classical notion of homotopy.

Proposition 37. Let A, B G-C^* algebras.

1. If A is a separable C^* algebra, then the map
$$[A, B]_1 \to \mathrm{colim}[A, B]_n$$
is an isomorphism.

2. Suppose that A is a separable nuclear G-C^* algebra such that the space of states $\hat{A}$ is a G-absolute neighborhood retract (G-ANR). Then, the canonical map
$$[A, C_0(Y)]_0 \to [A, C_0(Y)]_1$$
is an isomorphism for any locally compact metrizable G-space Y. In particular, the notions of equivariant homotopy for $*$-homomorphisms and assymptotic homomorphisms agree.

Proof. 1. This is the content of Theorem 2.16 in page 15 of [GHT00].

2. We adapt some arguments due to Marius Dădărlat from proposition 16 of [Dăd94]. Our aim is to show that the map $\alpha_{C_0(Y)} : [A, C_0(Y)]_0 \to [A, C_0(Y)]_1$ is an isomorphism. The equivalence of an arbitrary equivariant assymptotic homomorphism $\varphi : A \to \mathfrak{A}B$ with an equivariant $*$-homomorphism is reached in a number of steps:

   - Choose an associated assymptotic family $\{\varphi_t\}_{t\in[1,\infty)}$. Since A is nuclear, we can use a completely positive, contractive approximation of id_A to produce a completely positive , linear and contractive map $\tilde{\varphi} : C_b([1, \infty), C_0(Y))$, where b stays for boundedness. Construct a map with values on completely positive, linear G-invariant maps $f : (1, \infty) \times Y \to CP(A, \mathbb{C})$ by $(t, y) \mapsto \tilde{\varphi}(a)(t)(y)$.
   - For any G-invariant neighborhood V of $\hat{A}$ in $CP(A, \mathbb{C})$, there exists t_0 such that $f(t, y) \in V$ for $t \geq t_0$. This follows from the fact that $CP(A, \mathbb{C}) - U$ is compact (since $CP(A, \mathbb{C})$ is a weak*-compact subset of the dual Banach space A^*). Suppose that there is a sequence (t_n, y_n) with $t_n \to \infty$ such that $f(t_n, y_n) \in CP(A, \mathbb{C}) - U$. By compactness, one can assume convergence to an element $h \in CP(A, \mathbb{C}) - \hat{A}$. But on the other side, h must be an equivariant $*$-homomorphism, since φ is an equivariant assymptotic homomorphism.

- Since $\hat{A}$ is a G-ANR, there is an equivariant retraction $r : U \to \hat{A}$ from some invariant neighborhood in the space of completely positive elements. The composition $r \circ f$ defines fon any $t \geq t_0$ a $*$-homomorphism $\psi : A \to C_b([1,\infty), Y)$. In particular, $\psi \mid_{t_0}: A \to C_0(Y)$ is a *-homomorphism. This is well defined at the level of homotopy classes, for if $t \geq t_0$, the map $H_t(a)(s) = \psi_{(1-s)+ts}$ is a homotopy from $\psi \mid_{t_0}$ to $\psi \mid_t$.
- We now show that $\alpha_{C_0(Y)}\psi \mid_{t_0}$ determines the same assymptotic morphism as $\alpha_{C_0(Y)}(\varphi)$. Since A is separable, we can assume the existence of a G-invariant metric d on $CP(A,\mathbb{C})$. So we are done if we are able to show that

  $$\lim_{t\to\infty} \sup_{y\in Y} d(f(t,y)\psi(t,y)) = 0$$

  In order to do this, we find a sequence of invariant neighborhoods U_i of $\hat{A}$ in $CP(A,\mathbb{C})$ with $d(\alpha, r\alpha) < \frac{1}{i}$ for all $\alpha \in U_i$. We can construct a sequence t_i of real numbers diverging to infinity such that $d(f(t,y),\psi(t,y)) \leq \frac{1}{i}$ for $t \geq t_i$, from where

  $$\lim_{t\to\infty} \sup_{y} d(f(t,y),\psi(t,y)) = 0$$

  follows.
- If φ and ψ are $*$-homomorphisms which are homotopic as assymptotic homomorphisms and H is a homotopy between them, we can form as previously a map T such that for $s \mapsto T(t)(s)$ is a G-homotopy of $*$-homomorphisms for t big enough.

$\square$

6.3 A bivariant homotopy theory for proper algebras

We restrict now ourselves in this section to discrete groups which have a cocompact model for the classifying space of proper actions $\underline{E}G$. As above mentioned, mapping class groups, word hyperbolic groups, one relator groups and Cat(0)-groups are among possible examples. The point of this restriction is to ensure the existence of finite dimensional vector bundles, which was the starting point of Lück's notions and our bivariant modification of it.

Let $\xi : E \to \underline{E}G$ be a finite dimensional *complex* G-vector bundle. Let S^ξ be the total space of the locally trivial bundle which is fiberwise the one point compactification. Notice that the projection $S^\xi \to \underline{E}G$ is a proper map, since a locally trivial fibration of compact fiber. Hence, there exists a $*$-homomorphism $C_0(\underline{E}G) \to C_0(S^\xi)$.

Let now A and B be proper G-C^*-algebras. As a consequence of lemma 33, it makes sense to form the tensor products of G-Hilbert modules $A \underset{C_0(\underline{E}G)}{\otimes} C_0(S^\xi)$ and $B \underset{C_0(\underline{E}G)}{\otimes} C_0(S^\xi)$, denoted $\Sigma^\xi A$, respectively $\Sigma^\xi B$. Notice that we still have G-C^* algebras, so we can form the set of homotopy classes of equivariant assymptotic homomorphisms

$$[\Sigma^\xi A, \Sigma^\xi B]$$

Given a finite dimensional G-vector bundle ζ, we consider the zero inclusion $i : \xi \to \xi \oplus \zeta$ and the canonical projection $p : \xi \oplus \zeta \to \xi$, which determine a map as follows:

$$\mathrm{Mor}_{\mathfrak{A}}[\mathrm{A} \underset{\mathrm{C_0(\underline{E}G)}}{\otimes} \mathrm{C_0(S^\xi),)B} \underset{\mathrm{C_0(\underline{E}G)}}{\otimes} \mathrm{C_0(S^\xi)}]$$

$$\xrightarrow{[\mathrm{id}\otimes_{C_0(\underline{E}G)} id, \mathfrak{A}(\mathrm{id}\otimes C_0(i))]}$$

$$\mathrm{Mor}_{\mathfrak{A}}[A \underset{C_0(\underline{E}G)}{\otimes} S^{(\xi\oplus\zeta)}, B \underset{C_0(\underline{E}G)}{\otimes} S^{(\xi\oplus\zeta)}]$$

which we denote at the level of homotopy sets as $\wedge$id, in remembrance of its classical origin:

$$[\Sigma^\xi A, \Sigma^\xi B] \overset{\wedge \mathrm{id}}{\to} [\Sigma^{\xi\oplus\zeta} A, \Sigma^{\xi\oplus\zeta} B]$$

Two assymptotic homomorphisms $\Sigma^{\xi_i} A \to \Sigma^{\xi_i} B$ for $i = 0, 1$ are said to be stably equivalent if there exist finite dimensional vector bundles μ_i and an isomorphism $\nu : \xi_0 \oplus \mu_1 \to \xi_1 \oplus \mu_1$ such that they are mapped up to homotopy of assymptotic homomorphisms as in the following diagram:

$$\begin{tikzcd}
{[\Sigma^{\xi_0} A, \Sigma^{\xi_0} B]} \arrow[r, "\wedge \mathrm{id}"] & {[\Sigma^{\xi_0\oplus\mu_0} A, \Sigma^{\xi_0\oplus\mu_0} B]} \arrow[d, "\cong \nu^*"] \\
{[\Sigma^{\xi_1} A, \Sigma^{\xi_1} B]} \arrow[r, "\wedge \mathrm{id}"] & {[\Sigma^{\xi_1\oplus\mu_1} A, \Sigma^{\xi_1\oplus\mu_1} B]}
\end{tikzcd}$$

Definition 38. Let A, B be proper G-C^* algebras with an action of a discrete group G with a finite model for $\underline{E}G$. let $n \in \mathbb{Z}$. The bivariant homotopy groups in degree n, $F^n_G(A, B)$ are defined to be the set of equivalence classes of assymptotic homomorphisms of virtual difference n. In symbols

$$F^n_G(A, B) = [\Sigma^{\xi\oplus\mathbb{R}^k} A, \Sigma^{\xi\oplus\mathbb{R}^{k+n}} B] \diagup \sim$$

Let us justify now the assumption that F^n defines a group. Denote by $C(\nabla) : C_0(\mathbb{R}) \oplus C_0(\mathbb{R}) \to C_0(\mathbb{R})$ the pinching map. The sum of the homotopy classes of two assymptotic homomorphisms f_1, f_2 from $\Sigma^\xi A$ to $\Sigma^\mu B$ is defined as follows. We begin with the element

$$f_0 \otimes \mathrm{id} \bigoplus f_1 \otimes \mathrm{id} \in$$

$$\mathrm{Mor}(A \underset{C_0(\underline{E}G)}{\otimes} C_0(S^\xi) \otimes C_0(\mathbb{R}),$$

$$\mathfrak{A}(B \underset{C_0(\underline{E}G)}{\otimes} C_0(S^\mu) \otimes C_0(\mathbb{R})) \bigoplus \mathfrak{A}(B \underset{C_0(\underline{E}G)}{\otimes} C_0(S^\mu) \otimes C_0(\mathbb{R}))$$

We define their sum to be the element which is determined by its image under the following secuence of maps:

$$\begin{gathered}
[A \underset{C_0(\underline{E}G)}{\otimes} C_0(S^\xi) \otimes C_0(\mathbb{R}), \\
\mathfrak{A}(B \underset{C_0(\underline{E}G)}{\otimes} C_0(S^\mu) \otimes C_0(\mathbb{R})) \bigoplus \mathfrak{A}(B \underset{C_0(\underline{E}G)}{\otimes} C_0(S^\mu) \otimes C_0(\mathbb{R}))] \xrightarrow{\cong} \\
[A \underset{C_0(\underline{E}G)}{\otimes} C_0(S^\xi) \otimes C_0(\mathbb{R}), \\
\mathfrak{A}(B \underset{C_0(\underline{E}G)}{\otimes} C_0(S^\mu) \otimes C_0(\mathbb{R}) \bigoplus B \underset{C_0(\underline{E}G)}{\otimes} S^\mu \otimes C_0(\mathbb{R}))] \xrightarrow{\cong} \\
[A \underset{C_0(\underline{E}G)}{\otimes} C_0(S^\xi) \otimes C_0(\mathbb{R}), \mathfrak{A}(B \underset{C_0(\underline{E}G)}{\otimes} S^\mu \otimes (C_0(\mathbb{R}) \bigoplus C_0(\mathbb{R}))] \xrightarrow{\mathfrak{A}(\mathrm{id}\otimes C(\nabla)\otimes \mathrm{id})} \\
[A \underset{C_0(\underline{E}G)}{\otimes} C_0(S^\xi) \otimes C_0(\mathbb{R}), \mathfrak{A}(B \underset{C_0(\underline{E}G)}{\otimes} C_0(S^\mu) \otimes C_0(\mathbb{R}))]
\end{gathered}$$

Here are some explanations. The first isomorphism is given by the isomorphism $\mathfrak{A}(E) \oplus \mathfrak{A}(F) \cong \mathfrak{A}(E \oplus F)$, cf. lemma 2.5 in [GHT00], while the second one is given by the distributivity of the tensor product with respect to sums. .

Theorem 17. Let X and Y be finite, proper G-CW complexes for a discrete group G with a cocompact model for the classifying space of proper actions. The canonical Gel'fand-Naimark correspondence induces an isomorphism

$$C : \omega^n_{\underline{E}G}(X, Y) \longrightarrow F^n(C_0(Y), C_0(X))$$

Proof. Let $f : \Sigma^\xi X \to \Sigma^\mu(Y)$ be a representative of an element in $\omega^0_{\underline{E}G}(X, Y)$. Since f is a fibred continuous map, the naturality of the Gelfand transformation, theorem 16 gives a $*$-homomorphism $C(f) : C_0(\Sigma^\mu Y) \to C_0(\Sigma^\xi X)$. We consider the composition with the canonical map $\alpha : C_0((\Sigma^\xi X) \to \mathfrak{A}(C_0(\Sigma^\xi X))$ and obtain by this means a well defined map

$$[\Sigma^\xi X, \Sigma^\mu Y]_{\underline{E}G} \to [C_0(S^\mu \wedge_{\underline{E}G} Y), C_0(S^\xi \wedge_{\underline{E}G} X)]_1$$

where the 1 stays for homotopy of assymptotic homomorphisms. Since the spaces involved are in particular ANR's, and the algebras involved are commutative, part 1. and 2. of proposition 37 allows the substitution of this by homotopy classes of equivariant $*$-homomorphisms $[C_0(S^\mu \wedge_{\underline{E}G} Y), C_0(S^\xi \wedge_{\underline{E}G} X)]$, which can canonically be identifyied with $[\Sigma^\mu C_0(Y), \Sigma^\xi C_0(X)]$ as defined for proper C^*-algebras. □

Bibliography

[AHJM88] J. F. Adams, J.-P. Haeberly, S. Jackowski, and J. P. May. A generalization of the Segal conjecture. *Topology*, 27(1):7–21, 1988.

[AM69] M. F. Atiyah and I. G. Macdonald. *Introduction to commutative algebra.* Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., 1969.

[Ant85] S. A. Antonyan. Equivariant generalization of Dugundji's theorem. *Mat. Zametki*, 38(4):608–616, 636, 1985.

[AS69] M. F. Atiyah and I. M. Singer. Index theory for skew-adjoint Fredholm operators. *Inst. Hautes Études Sci. Publ. Math.*, (37):5–26, 1969.

[Bau89] Stefan Bauer. On the Segal conjecture for compact Lie groups. *J. Reine Angew. Math.*, 400:134–145, 1989.

[Ber77] Melvin S. Berger. *Nonlinearity and functional analysis.* Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1977. Lectures on nonlinear problems in mathematical analysis, Pure and Applied Mathematics.

[BF04] Stefan Bauer and Mikio Furuta. A stable cohomotopy refinement of Seiberg-Witten invariants. I. *Invent. Math.*, 155(1):1–19, 2004.

[Bla98] Bruce Blackadar. *K-theory for operator algebras*, volume 5 of *Mathematical Sciences Research Institute Publications.* Cambridge University Press, Cambridge, second edition, 1998.

[Blu06] Andrew J. Blumberg. Continuous functors as a model for the equivariant stable homotopy category. *Algebr. Geom. Topol.*, 6:2257–2295, 2006.

[Car84] Gunnar Carlsson. Equivariant stable homotopy and Segal's Burnside ring conjecture. *Ann. of Math. (2)*, 120(2):189–224, 1984.

[CDD] Gunnar Carlsson, Christopher Douglas, and Bjorn Dundas. Higher topological cyclic homology and the segal conjecture for tori. ArXiv:0803.2745v1.

[CH90] Alain Connes and Nigel Higson. Déformations, morphismes asymptotiques et K-théorie bivariante. *C. R. Acad. Sci. Paris Sér. I Math.*, 311(2):101–106, 1990.

[Cra07] M. C. Crabb. The fibrewise Leray-Schauder index. *J. Fixed Point Theory Appl.*, 1(1):3–30, 2007.

[CW85] J. Caruso and S. Waner. An approximation theorem for equivariant loop spaces in the compact Lie case. *Pacific J. Math.*, 117(1):27–49, 1985.

[Dăd94] Marius Dădărlat. A note on asymptotic homomorphisms. *K-Theory*, 8(5):465–482, 1994.

[Dei85] Klaus Deimling. *Nonlinear functional analysis.* Springer-Verlag, Berlin, 1985.

[DL98] James F. Davis and Wolfgang Lück. Spaces over a category and assembly maps in isomorphism conjectures in K- and L-theory. *K-Theory*, 15(3):201–252, 1998.

[Dol74] Albrecht Dold. The fixed point index of fibre-preserving maps. *Invent. Math.*, 25:281–297, 1974.

[Dre69] Andreas Dress. A characterisation of solvable groups. *Math. Z.*, 110:213–217, 1969.

[dSp04] Clément de Seguins pazzis. K-theorie équivariante pur des actions propres de groupes de lie non compacts. Unpublished, Université Paris 13, 2004.

[Dwy96] W. G. Dwyer. Transfer maps for fibrations. *Math. Proc. Cambridge Philos. Soc.*, 120(2):221–235, 1996.

[Fes79] Mark Feshbach. The transfer and compact Lie groups. *Trans. Amer. Math. Soc.*, 251:139–169, 1979.

[Fes87] Mark Feshbach. The Segal conjecture for compact Lie groups. *Topology*, 26(1):1–20, 1987.

[GHT00] Erik Guentner, Nigel Higson, and Jody Trout. Equivariant E-theory for C^*-algebras. *Mem. Amer. Math. Soc.*, 148(703):viii+86, 2000.

[Hau77] H. Hauschild. Zerspaltung äquivarianter Homotopiemengen. *Math. Ann.*, 230(3):279–292, 1977.

[Hau80] H. Hauschild. Äquivariante Konfigurationsräume und Abbildungsräume. In *Topology Symposium, Siegen 1979 (Proc. Sympos., Univ. Siegen, Siegen, 1979)*, volume 788 of *Lecture Notes in Math.*, pages 281–315. Springer, Berlin, 1980.

[Hoc65] G. Hochschild. *The structure of Lie groups.* Holden-Day Inc., San Francisco, 1965.

[HST] Henning Hohnhold, Stephan Stolz, and Peter Teichner. K-theory: From minimal geodesics tu susy field theories.

[Ize05] Jorge Ize. Equivariant degree. In *Handbook of topological fixed point theory*, pages 301–337. Springer, Dordrecht, 2005.

[Kit09] Nitu Kitchloo. Dominant K-theory and integrable highest weight representations of Kac-Moody groups. *Adv. Math.*, 221(4):1191–1226, 2009.

[Lan95] E. C. Lance. *Hilbert C^*-modules*, volume 210 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1995. A toolkit for operator algebraists.

[LO01a] Wolfgang Lück and Bob Oliver. Chern characters for the equivariant K-theory of proper G-CW-complexes. In *Cohomological methods in homotopy theory (Bellaterra, 1998)*, volume 196 of *Progr. Math.*, pages 217–247. Birkhäuser, Basel, 2001.

[LO01b] Wolfgang Lück and Bob Oliver. The completion theorem in K-theory for proper actions of a discrete group. *Topology*, 40(3):585–616, 2001.

[LR05] Wolfgang Lück and Holger Reich. The Baum-Connes and the Farrell-Jones conjectures in K- and L-theory. In *Handbook of K-theory. Vol. 1, 2*, pages 703–842. Springer, Berlin, 2005.

[Lüc89] Wolfgang Lück. *Transformation groups and algebraic K-theory*, volume 1408 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1989. Mathematica Gottingensis.

[Lüc05a] Wolfgang Lück. The Burnside ring and equivariant stable cohomotopy for infinite groups. *Pure Appl. Math. Q.*, 1(3):479–541, 2005.

[Lüc05b] Wolfgang Lück. Equivariant cohomological Chern characters. *Internat. J. Algebra Comput.*, 15(5-6):1025–1052, 2005.

[Lüc05c] Wolfgang Lück. Survey on classifying spaces for families of subgroups. In *Infinite groups: geometric, combinatorial and dynamical aspects*, volume 248 of *Progr. Math.*, pages 269–322. Birkhäuser, Basel, 2005.

[Lüc08] Wolfgang Lück. The segal conjecture for infinite groups. Preprint, 2008.

[May92] J. Peter May. *Simplicial objects in algebraic topology.* Chicago Lectures in Mathematics. University of Chicago Press, Chicago, IL, 1992. Reprint of the 1967 original.

[May96] J. P. May. *Equivariant homotopy and cohomology theory*, volume 91 of *CBMS Regional Conference Series in Mathematics.* Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996. With contributions by M. Cole, G. Comezana, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner.

[Mey01] Ralf Meyer. Generalized fixed point algebras and square-integrable groups actions. *J. Funct. Anal.*, 186(1):167–195, 2001.

[Mic56] Ernest Michael. Continuous selections. I. *Ann. of Math. (2)*, 63:361–382, 1956.

[Moo01] John Douglas Moore. *Lectures on Seiberg-Witten invariants*, volume 1629 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, second edition, 2001.

[MS76] D. McDuff and G. Segal. Homology fibrations and the "group-completion" theorem. *Invent. Math.*, 31(3):279–284, 1975/76.

[Nic00] Liviu I. Nicolaescu. *Notes on Seiberg-Witten theory*, volume 28 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2000.

[Pal61] Richard S. Palais. On the existence of slices for actions of non-compact Lie groups. *Ann. of Math. (2)*, 73:295–323, 1961.

[Phi89] N. Christopher Phillips. *Equivariant K-theory for proper actions*, volume 178 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1989.

[PU91] Carlos Prieto and Hanno Ulrich. Equivariant fixed point index and fixed point transfer in nonzero dimensions. *Trans. Amer. Math. Soc.*, 328(2):731–745, 1991.

[Qui73] Daniel Quillen. Higher algebraic K-theory. I. In *Algebraic K-theory, I: Higher K-theories (Proc. Conf., Battelle Memorial Inst., Seattle, Wash., 1972)*, pages 85–147. Lecture Notes in Math., Vol. 341. Springer, Berlin, 1973.

[RS00] Colin Rourke and Brian Sanderson. Equivariant configuration spaces. *J. London Math. Soc. (2)*, 62(2):544–552, 2000.

[Sch93] Herbert Schröder. *K-theory for real C^*-algebras and applications*, volume 290 of *Pitman Research Notes in Mathematics Series*. Longman Scientific & Technical, Harlow, 1993.

[Sch07] Christian Schlichtkrull. The homotopy infinite symmetric product represents stable homotopy. *Algebr. Geom. Topol.*, 7:1963–1977, 2007.

[Seg68] Graeme Segal. Classifying spaces and spectral sequences. *Inst. Hautes Études Sci. Publ. Math.*, (34):105–112, 1968.

[Seg71] G. B. Segal. Equivariant stable homotopy theory. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 2*, pages 59–63. Gauthier-Villars, Paris, 1971.

[Seg73] Graeme Segal. Configuration-spaces and iterated loop-spaces. *Invent. Math.*, 21:213–221, 1973.

[Seg74] Graeme Segal. Categories and cohomology theories. *Topology*, 13:293–312, 1974.

[Ska91] Georges Skandalis. Le bifoncteur de Kasparov n'est pas exact. *C. R. Acad. Sci. Paris Sér. I Math.*, 313(13):939–941, 1991.

[Sma65] S. Smale. An infinite dimensional version of Sard's theorem. *Amer. J. Math.*, 87:861–866, 1965.

[Ste67] N. E. Steenrod. A convenient category of topological spaces. *Michigan Math. J.*, 14:133–152, 1967.

[Šva64] A. S. Švarc. On the homotopic topology of Banach spaces. *Dokl. Akad. Nauk SSSR*, 154:61–63, 1964.

[tD75] Tammo tom Dieck. The Burnside ring of a compact Lie group. I. *Math. Ann.*, 215:235–250, 1975.

[tD79] Tammo tom Dieck. *Transformation groups and representation theory*, volume 766 of *Lecture Notes in Mathematics*. Springer, Berlin, 1979.

[tD87] Tammo tom Dieck. *Transformation groups*, volume 8 of *de Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 1987.

[Tel] Andrei Teleman. Introduction à la théorie de jauge.

[Ulr88] Hanno Ulrich. *Fixed point theory of parametrized equivariant maps*, volume 1343 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.

[Wan80] Stefan Waner. Equivariant classifying spaces and fibrations. *Trans. Amer. Math. Soc.*, 258(2):385–405, 1980.

[Was69] Arthur G. Wasserman. Equivariant differential topology. *Topology*, 8:127–150, 1969.

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