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Persistence approximation property and controlled operator K-theory

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Abstract. In this paper, we introduce and study the persistence approximation property for quantitative K-theory of filtered C^*-algebras. In the case of crossed product C^*-algebras, the persistence approximation property follows from the Baum–Connes conjecture with coefficients. We also discuss some applications of the quantitative K-theory to the Novikov conjecture.

Contents

1. Introduction

The idea of quantitative operator K-theory was first introduced in [15] to study the Novikov conjecture for groups with finite asymptotic dimension. In [9], we introduced a general quantitative K-theory for filtered C^*-algebras. Examples of filtered C^*-algebras include group C^*-algebras, crossed product C^*-algebras, Roe algebras, foliation C^*-algebras and finitely generated C^*-algebras. For a C^*-algebra A with a filtration, the K-theory of A can be approximated by the quantitative K-theory groups $K_*^{\varepsilon,r}(A)$ when r goes to

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infinity. The crucial point is that quantitative K-theory is often more computable using certain controlled exact sequences (see, e.g., [9, 15]). The study of K-theory for the Roe algebra can be reduced to that of quantitative K-theory for the Roe algebra associated to finite metric spaces, which in essence is a finite-dimensional linear algebra problem.

The main purpose of this paper is to introduce and study the persistence approximation property for quantitative K-theory of filtered C^*-algebras. Roughly speaking, the persistence approximation property means that the convergence of $K_*^{\varepsilon,r}(A)$ to $K_*(A)$ is uniform. More precisely, we say that the filtered C^*-algebra A has persistence approximation property if for each ε in $(0, \frac{1}{4})$ and $r > 0$, there exist $r' \geq r$ and ε' in $[\varepsilon, \frac{1}{4})$ such that an element from $K_*^{\varepsilon,r}(A)$ is zero in $K_*(A)$ if and only if it is zero in $K_*^{\varepsilon',r'}(A)$. The main motivation to study the persistence approximation property is that it provides an effective way of approximating K-theory with quantitative K-theory. In the case of crossed product C^*-algebras, the Baum–Connes conjecture with coefficients provides many examples that satisfy the persistence approximation property. It turns out that this property provides geometrical obstruction for the Baum–Connes conjecture. In order to study this obstruction in full generality, we consider the persistence approximation property for filtered C^*-algebra $A \otimes \mathscr{K}(\ell^2(\Sigma))$, where A is a C^*-algebra and Σ is a discrete metric space with bounded geometry. For this purpose,

- we introduce a family of quantitative local assembly maps valued in the quantitative K-theory for $A \otimes \mathscr{K}(\ell^2(\Sigma))$;
- proceeding as in [9, §6.2] for the quantitative Baum–Connes assembly maps, we set quantitative statements for these local quantitative Baum–Connes assembly maps.

These quantitative statements can be viewed as a geometric version of those stated in [9, §6.2]. We also show that if these statements hold uniformly for the family of finite subsets of a discrete metric space Σ with bounded geometry, the coarse Baum–Connes conjecture for Σ is satisfied. In particular, in the case of a finitely generated group Γ provided with the metric arising from any word length, these uniform statements for finite metric subsets of Γ imply the Novikov conjecture for Γ on homotopy invariance of higher signatures. We point out that in this case, these statements reduce to finite-dimensional problems in linear algebra and analysis.

The paper is organized as follows. In Section 2, we review the main results of [9] concerning quantitative K-theory. In Section 3, we introduce the persistence approximation property. We prove that if Γ is a finitely generated group that satisfies the Baum–Connes conjecture with coefficients and which admits a cocompact universal example for proper actions, then for any Γ-C^*-algebra A, the reduced crossed product $A \rtimes_r \Gamma$ satisfies the persistence approximation property. In the special case of the action of the group Γ on $C_0(\Gamma)$ by translation, we get a canonical identification between $C_0(\Gamma) \rtimes \Gamma$ and $\mathscr{K}(\ell^2(\Sigma))$ that preserves the filtration structure. Hence, the persistence approximation property can be stated in a completely geometrical way. This leads us to consider this property for the algebra $A \otimes \mathscr{K}(\ell^2(\Sigma))$, where A is a C^*-algebra and Σ is a proper discrete metric space, with filtration structure induced by the metric of Σ. In Section 4, following the idea of the Baum–Connes conjecture in order to compute the quantitative K-theory groups for $A \otimes \mathscr{K}(\ell^2(\Sigma))$, we construct a family of quantitative assembly maps $\nu^{\varepsilon,r,d}_{\Sigma,A,*}$. In view of the proof of the persistence approximation property in the crossed product algebras case, we introduce a geometrical assembly map $\nu^{\infty}_{\Sigma,A,*}$ (which plays the role of the Baum–Connes assembly map with relevant coefficients). Following the route of [11], we show that the target of these geometric assembly maps is indeed the K-theory of the crossed product algebra of an appropriate C^*-algebra $\mathcal{A}_{C_0(\Sigma)}$ by the groupoid G_Σ associated in [11] to the coarse structure of Σ. In Section 5, we study the Baum–Connes assembly map for the pair $(G_\Sigma, \mathcal{A}_{C_0(\Sigma)})$ and we show that the bijectivity of the geometric assembly maps $\nu^{\infty}_{\Sigma,A,*}$ is equivalent to the Baum–Connes conjecture for $(G_\Sigma, \mathcal{A}_{C_0(\Sigma)})$. We set in the geometric setting the analog of the quantitative statements of [9, §6.2] for the quantitative Baum–Connes assembly maps and we prove that these statements hold when Σ coarsely embeds into a Hilbert space. We then apply these results to the persistence approximation property for $A \otimes \mathscr{K}(\ell^2(\Sigma))$. In particular, we prove it when Σ coarsely embeds into a Hilbert space, under an assumption of coarse uniform contractibility. This condition is the analog of the existence of a cocompact universal example for proper actions in the geometric setting and is satisfied for instance for Gromov hyperbolic discrete metric spaces. In Section 6, we show that for a discrete metric space with bounded geometry, if the quantitative statements of Section 5 for $\nu^{\infty}_{F,A,*}$ hold uniformly when F runs through finite subsets of Σ, then Σ satisfies the coarse Baum–Connes conjecture.

2. Survey on quantitative K-theory

In this section, we collect the main results of [9] concerning quantitative K-theory and that we shall use throughout this paper. Quantitative K-theory was introduced to describe propagation phenomena in higher index theory for noncompact spaces. More generally, we use the framework of filtered C^*-algebras to model the concept of propagation.

Definition 2.1. A filtered C^*-algebra A is a C^*-algebra equipped with a family $(A_r)_{r>0}$ of closed linear subspaces indexed by positive numbers such that

- $A_r \subseteq A_{r'}$ if $r \leq r'$;
- A_r is stable under involution;
- $A_r \cdot A_{r'} \subseteq A_{r+r'}$;
- the subalgebra $\bigcup_{r>0} A_r$ is dense in A.

If A is unital, we also require that the identity 1 is an element of A_r for every positive number r. The elements of A_r are said to have *propagation* r.

Let A and A' be C^*-algebras filtered by $(A_r)_{r>0}$ and $(A'_r)_{r>0}$, respectively. A $*$-homomorphism of C^* -algebras $\phi : A \to A'$ is a *filtered homomorphism* (or a *homomorphism of filtered C^*-algebras*) if $\phi(A_r) \subseteq A'_r$ for any positive number r.

If A is not unital, let us denote by A^+ its unitarization, i.e.

$$A^+ = \{(x,\lambda) \mid x \in A,\, \lambda \in \mathbb{C}\}$$

with the product

$$(x,\lambda)(x',\lambda') = (xx' + \lambda x' + \lambda' x, \lambda\lambda')$$

for all (x,λ) and (x',λ') in A^+. Then A^+ is filtered by

$$A^+_r = \{(x,\lambda) \mid x \in A_r,\, \lambda \in \mathbb{C}\}.$$

We also define $\rho_A : A^+ \to \mathbb{C}$, $(x,\lambda) \mapsto \lambda$.

2.2. **Definition of quantitative K-theory.** Let A be a unital filtered C^*-algebra. For any positive numbers r and ε, we call

- an element u in A an ε-r-unitary if u belongs to A_r, $\|u^* \cdot u - 1\| < \varepsilon$ and $\|u \cdot u^* - 1\| < \varepsilon$. The set of ε-r-unitaries on A will be denoted by $\mathrm{U}^{\varepsilon,r}(A)$.
- an element p in A an ε-r-projection if p belongs to A_r, $p = p^*$ and $\|p^2 - p\| < \varepsilon$. The set of ε-r-projections on A will be denoted by $\mathrm{P}^{\varepsilon,r}(A)$.

Notice that an ε-r-unitary is invertible, and that if p is an ε-r-projection in A with $\varepsilon < \frac{1}{4}$, then it has a spectral gap around $\frac{1}{2}$ and then gives rise by functional calculus to a projection $\kappa_0(p)$ in A such that $\|p - \kappa_0(p)\| < 2\varepsilon$.

For any integer n, we set

$$\mathrm{U}^{\varepsilon,r}_n(A) = \mathrm{U}^{\varepsilon,r}(M_n(A)), \quad \mathrm{P}^{\varepsilon,r}_n(A) = \mathrm{P}^{\varepsilon,r}(M_n(A)).$$

For any unital filtered C^*-algebra A, any positive numbers ε and r and any positive integer n, we consider inclusions

$$\mathrm{P}^{\varepsilon,r}_n(A) \hookrightarrow \mathrm{P}^{\varepsilon,r}_{n+1}(A), \quad p \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$\mathrm{U}^{\varepsilon,r}_n(A) \hookrightarrow \mathrm{U}^{\varepsilon,r}_{n+1}(A), \quad u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

This allows us to define

$$\mathrm{U}^{\varepsilon,r}_\infty(A) = \bigcup_{n\in\mathbb{N}} \mathrm{U}^{\varepsilon,r}_n(A), \quad \mathrm{P}^{\varepsilon,r}_\infty(A) = \bigcup_{n\in\mathbb{N}} \mathrm{P}^{\varepsilon,r}_n(A).$$

For a unital filtered C^*-algebra A, we define the following equivalence relations on $\mathrm{P}^{\varepsilon,r}_\infty(A) \times \mathbb{N}$ and on $\mathrm{U}^{\varepsilon,r}_\infty(A)$:

- If p and q are elements of $\mathrm{P}^{\varepsilon,r}_\infty(A)$, l and l' are positive integers, then $(p,l) \sim (q,l')$ if there exist a positive integer k and an element h of $\mathrm{P}^{\varepsilon,r}_\infty(A[0,1])$ such that $h(0) = \mathrm{diag}(p, I_{k+l'})$ and $h(1) = \mathrm{diag}(q, I_{k+l})$.
- If u and v are elements of $\mathrm{U}^{\varepsilon,r}_\infty(A)$, then $u \sim v$ if there exists an element h of $\mathrm{U}^{3\varepsilon,2r}_\infty(A[0,1])$ such that $h(0) = u$ and $h(1) = v$.

If p is an element of $\mathrm{P}_\infty^{\varepsilon,r}(A)$ and l is an integer, we denote by $[p,l]_{\varepsilon,r}$ the equivalence class of (p,l) modulo $\sim$. If u is an element of $\mathrm{U}_\infty^{\varepsilon,r}(A)$ we denote by $[u]_{\varepsilon,r}$ its equivalence class modulo $\sim$.

Definition 2.3. Let r and ε be positive numbers with $\varepsilon < \frac{1}{4}$. We define:

(i) $K_0^{\varepsilon,r}(A) = \mathrm{P}_\infty^{\varepsilon,r}(A) \times \mathbb{N}/\sim$ for A unital and

$$K_0^{\varepsilon,r}(A) = \left\{[p,l]_{\varepsilon,r} \in \mathrm{P}^{\varepsilon,r}(A^+) \times \mathbb{N}/\sim \mid \operatorname{rank}\kappa_0(\rho_A(p)) = l\right\}$$

for A nonunital and $\kappa_0(\rho_A(p))$ being the spectral projection associated to $\rho_A(p)$;

(ii) $K_1^{\varepsilon,r}(A) = \mathrm{U}_\infty^{\varepsilon,r}(A^+)/\sim$, with $A = A^+$ if A is already unital.

Then $K_0^{\varepsilon,r}(A)$ turns to be an abelian group [9, Lem. 1.15] where

$$[p,l]_{\varepsilon,r} + [p',l']_{\varepsilon,r} = [\operatorname{diag}(p,p'), l+l']_{\varepsilon,r}$$

for any $[p,l]_{\varepsilon,r}$ and $[p',l']_{\varepsilon,r}$ in $K_0^{\varepsilon,r}(A)$. According to [9, Lem. 1.15], $K_1^{\varepsilon,r}(A)$ is equipped with a structure of abelian group such that

$$[u]_{\varepsilon,r} + [u']_{\varepsilon,r} = [\operatorname{diag}(u,v)]_{\varepsilon,r}$$

for any $[u]_{\varepsilon,r}$ and $[u']_{\varepsilon,r}$ in $K_1^{\varepsilon,r}(A)$.

Recall from [9, Cor. 1.19 and 1.21] that for any positive numbers r and ε with $\varepsilon < \frac{1}{4}$, we have that

$$K_0^{\varepsilon,r}(\mathbb{C}) \to \mathbb{Z}, \quad [p,l]_{\varepsilon,r} \mapsto \operatorname{rank}\kappa_0(p) - l$$

is an isomorphism and $K_1^{\varepsilon,r}(\mathbb{C}) = \{0\}$.

For any filtered C^*-algebra A and any positive numbers r, r', ε and ε' with $\varepsilon \le \varepsilon' < \frac{1}{4}$ and $r \le r'$, we have natural group homomorphisms

- $\iota_0^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \to K_0(A)$, $[p,l]_{\varepsilon,r} \mapsto [\kappa_0(p)] - [I_l]$ (where $\kappa_0(p)$ is the spectral projection associated to p);
- $\iota_1^{\varepsilon,r} : K_1^{\varepsilon,r}(A) \to K_1(A)$, $[u]_{\varepsilon,r} \mapsto [u]$;
- $\iota_*^{\varepsilon,r} = \iota_0^{\varepsilon,r} \oplus \iota_1^{\varepsilon,r}$;
- $\iota_0^{\varepsilon,\varepsilon',r,r'} : K_0^{\varepsilon,r}(A) \to K_0^{\varepsilon',r'}(A)$, $[p,l]_{\varepsilon,r} \mapsto [p,l]_{\varepsilon',r'}$;
- $\iota_1^{\varepsilon,\varepsilon',r,r'} : K_1^{\varepsilon,r}(A) \to K_1^{\varepsilon',r'}(A)$, $[u]_{\varepsilon,r} \mapsto [u]_{\varepsilon',r'}$;
- $\iota_*^{\varepsilon,\varepsilon',r,r'} = \iota_0^{\varepsilon,\varepsilon',r,r'} \oplus \iota_1^{\varepsilon,\varepsilon',r,r'}$.

If some of the indices r, r' or ε, ε' are equal, we shall not repeat it in $\iota_*^{\varepsilon,\varepsilon',r,r'}$. The following result is a consequence of [9, Rem. 1.17].

Proposition 2.4. *Let A be a C^*-algebra filtered by $(A_r)_{r>0}$.*

(i) *For any ε in $(0,\frac{1}{4})$ and any y in $K_*(A)$, there exist a positive number r and an element x in $K_*^{\varepsilon,r}(A)$ such that $\iota_*^{\varepsilon,r}(x) = y$.*

(ii) *There exists a positive number $\lambda > 1$ independent of A such that the following is satisfied: Let ε be in $(0,\frac{1}{4})$, let r be a positive number and let x be an element in $K_*^{\varepsilon,r}(A)$ such that $\iota_*^{\varepsilon,r}(x) = 0$ in $K_*(A)$. Then there exists a positive number r' with $r' > r$ such that*

$$\iota_*^{\varepsilon,\lambda\varepsilon,r,r'}(x) = 0 \textit{ in } K_*^{\lambda\varepsilon,r'}(A).$$

If $\phi : A \to B$ is a homomorphism of filtered C^*-algebras, then since ϕ preserves ε-r-projections and ε-r-unitaries, it obviously induces, for any positive number r and any $\varepsilon \in (0, \frac{1}{4})$, a group homomorphism

$$\phi_*^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \to K_*^{\varepsilon,r}(B).$$

Moreover, the quantitative K-theory is homotopy invariant with respect to homotopies that preserve propagation [9, Lem. 1.26]. There is also a quantitative version of Morita equivalence [9, Prop. 1.28].

Proposition 2.5. *If A is a filtered algebra and $\mathscr{H}$ is a separable Hilbert space, then the homomorphism*

$$A \to \mathscr{K}(\mathscr{H}) \otimes A, \quad a \mapsto \begin{pmatrix} a & & \\ & 0 & \\ & & \ddots \end{pmatrix}$$

induces a ($\mathbb{Z}_2$-graded) group isomorphism (the Morita equivalence)

$$\mathcal{M}_A^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \to K_*^{\varepsilon,r}(A \otimes \mathscr{K}(\mathscr{H}))$$

for any positive number r and any $\varepsilon \in (0, \frac{1}{4})$.

2.6. **Quantitative objects.** In order to study the functorial properties of quantitative K-theory, we introduce the concept of a quantitative object.

Definition 2.7. A quantitative object is a family $\mathcal{O} = (O^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}$ of abelian groups, together with a family of group homomorphisms

$$\iota_{\mathcal{O}}^{\varepsilon,\varepsilon',r,r'} : O^{\varepsilon,r} \to O^{\varepsilon',r'}$$

for $0 < \varepsilon \leq \varepsilon' < \frac{1}{4}$ and $0 < r \leq r'$ such that

$$\iota_{\mathcal{O}}^{\varepsilon,\varepsilon,r,r} = \mathrm{Id}_{O^{\varepsilon,r}}$$

for any $0 < \varepsilon < \frac{1}{4}$ and $r > 0$; and

$$\iota_{\mathcal{O}}^{\varepsilon',\varepsilon'',r',r''} \circ \iota_{\mathcal{O}}^{\varepsilon,\varepsilon',r,r'} = \iota_{\mathcal{O}}^{\varepsilon,\varepsilon'',r,r''}$$

for any $0 < \varepsilon \leq \varepsilon' \leq \varepsilon'' < \frac{1}{4}$ and $0 < r \leq r' \leq r''$.

Example 2.8. (i) Our prominent example will be of course quantitative K-theory $\mathcal{K}_*(A) = (K_*^{\varepsilon,r}(A))_{0<\varepsilon<\frac{1}{4},r>0}$ of a filtered C^*-algebra A with structure maps

$$\iota_*^{\varepsilon,\varepsilon',r,r'} : K_*^{\varepsilon,r}(A) \to K_*^{\varepsilon',r'}(A)$$

for $0 < \varepsilon \leq \varepsilon' < \frac{1}{4}$ and $0 < r \leq r'$.

(ii) If $(\mathcal{O}_i)_{i\in\mathbb{N}}$ is a family of quantitative objects with $\mathcal{O}_i = (O_i^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}$ for any integer i, we define

$$\prod_{i\in\mathbb{N}} \mathcal{O}_i = \Big(\prod_{i\in\mathbb{N}} O_i^{\varepsilon,r}\Big)_{0<\varepsilon<\frac{1}{4},r>0}.$$

Then $\prod_{i\in\mathbb{N}} \mathcal{O}_i$ is also a quantitative object. In the case of a constant family $(\mathcal{O}_i)_{i\in\mathbb{N}}$ with $\mathcal{O}_i = \mathcal{O}$ a quantitative object, we set $\mathcal{O}^{\mathbb{N}}$ for $\prod_{i\in\mathbb{N}} \mathcal{O}_i$.

2.9. **Controlled morphisms.** Obviously, the definition of controlled morphism [9, §2] can be then extended to quantitative objects.

Definition 2.10. A control pair is a pair (λ, h), where

- $\lambda \geq 1$;
- $h : (0, \frac{1}{4\lambda}) \to [1, +\infty)$, $\varepsilon \mapsto h_\varepsilon$ is a map such that there exists a non-increasing map $g : (0, \frac{1}{4\lambda}) \to [1, +\infty)$, with $h \leq g$.

The set of control pairs is equipped with a partial order: $(\lambda, h) \leq (\lambda', h')$ if $\lambda \leq \lambda'$ and $h_\varepsilon \leq h'_\varepsilon$ for all $\varepsilon \in (0, \frac{1}{4\lambda'})$.

Definition 2.11. Let (λ, h) be a control pair and let $\mathcal{O} = (O^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}$ and $\mathcal{O}' = (O'^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}$ be quantitative objects. A (λ, h)-controlled morphism $\mathcal{F} : \mathcal{O} \to \mathcal{O}'$ is a family $\mathcal{F} = (F^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\lambda},r>0}$ of group homomorphisms

$$F^{\varepsilon,r} : \mathcal{O}^{\varepsilon,r} \to \mathcal{O}'^{\lambda\varepsilon,h_\varepsilon r}$$

such that for any positive numbers ε, ε', r and r' with $0 < \varepsilon \leq \varepsilon' < \frac{1}{4\lambda}$, $r \leq r'$ and $h_\varepsilon r \leq h_{\varepsilon'} r'$, we have

$$F^{\varepsilon',r'} \circ \iota_{\mathcal{O}}^{\varepsilon,\varepsilon',r,r'} = \iota_{\mathcal{O}'}^{\lambda\varepsilon,\lambda\varepsilon',h_\varepsilon r,h_{\varepsilon'} r'} \circ F^{\varepsilon,r}.$$

When it is not necessary to specify the control pair, we will just say that $\mathcal{F}$ is a controlled morphism. If $\mathcal{O} = (O^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}$ is a quantitative object, let us define the identity $(1, 1)$-controlled morphism

$$\mathcal{I}d_{\mathcal{O}} = (\mathrm{Id}_{O^{\varepsilon,r}})_{0<\varepsilon<\frac{1}{4},r>0} : \mathcal{O} \to \mathcal{O}.$$

Recall that if A and B are filtered C^*-algebras and if $\mathcal{F} : \mathcal{K}_*(A) \to \mathcal{K}_*(B)$ is a (λ, h)-controlled morphism, then $\mathcal{F}$ induces a morphism $F : K_*(A) \to K_*(B)$ uniquely defined by $\iota_*^{\varepsilon,r} \circ F^{\varepsilon,r} = F \circ \iota_*^{\varepsilon,r}$.

If (λ, h) and (λ', h') are two control pairs, define

$$h * h' : \left(0, \frac{1}{4\lambda\lambda'}\right) \to (0, +\infty), \quad \varepsilon \mapsto h_{\lambda'\varepsilon} h'_\varepsilon.$$

Then $(\lambda\lambda', h * h')$ is again a control pair. Let

$$\mathcal{O} = (O^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}, \quad \mathcal{O}' = (O'^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}, \quad \mathcal{O}'' = (O''^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}$$

be quantitative objects, let

$$\mathcal{F} = (F^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{F}}},r>0} : \mathcal{O} \to \mathcal{O}'$$

be an $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism, and let

$$\mathcal{G} = (G^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{G}}},r>0} : \mathcal{O}' \to \mathcal{O}''$$

be an $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$-controlled morphism. Then the family of homomorphisms $\mathcal{G} \circ \mathcal{F} : \mathcal{O} \to \mathcal{O}''$ is the $(\alpha_{\mathcal{G}}\alpha_F, k_{\mathcal{G}} * k_{\mathcal{F}})$-controlled morphism defined by the family

$$(G^{\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\varepsilon} r} \circ F^{\varepsilon,r} : O^{\varepsilon,r} \to O''^{\alpha_{\mathcal{G}}\alpha_{\mathcal{F}}\varepsilon,k_{\mathcal{F},\varepsilon} k_{\mathcal{G},\alpha_{\mathcal{F}},\varepsilon} r})_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{F}}\alpha_{\mathcal{G}}},r>0}.$$

Notation 2.12. Let $\mathcal{O} = (O^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}$ and $\mathcal{O}' = (O'^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}$ be quantitative objects, let $\mathcal{F} = (F^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0} : \mathcal{O} \to \mathcal{O}'$ be an $(\alpha_\mathcal{F}, k_\mathcal{F})$-controlled morphism, let $\mathcal{G} = (G^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0} : \mathcal{O} \to \mathcal{O}'$ be an $(\alpha_\mathcal{G}, k_\mathcal{G})$-controlled morphism, and let (λ, h) be a control pair. We write

$$\mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{G}$$

if

- $(\alpha_\mathcal{F}, k_\mathcal{F}) \leq (\lambda, h)$ and $(\alpha_\mathcal{G}, k_\mathcal{G}) \leq (\lambda, h)$;
- for every ε in $(0, \frac{1}{4\lambda})$ and $r > 0$, we have

$$\iota_{\mathcal{O}'}^{\alpha_\mathcal{F}\varepsilon,\lambda\varepsilon,k_{\mathcal{F},\varepsilon}r,h_\varepsilon r} \circ F^{\varepsilon,r} = \iota_{\mathcal{O}'}^{\alpha_\mathcal{G}\varepsilon,\lambda\varepsilon,k_{\mathcal{G},\varepsilon}r,h_\varepsilon r} \circ G^{\varepsilon,r}.$$

Definition 2.13. Let (λ, h) be a control pair, and let $\mathcal{F} : \mathcal{O} \to \mathcal{O}'$ be an $(\alpha_\mathcal{F}, k_\mathcal{F})$-controlled morphism with $(\alpha_\mathcal{F}, k_\mathcal{F}) \leq (\lambda, h)$. $\mathcal{F}$ is called (λ, h)-invertible or a (λ, h)-isomorphism if there exists a controlled morphism $\mathcal{G} : \mathcal{O}' \to \mathcal{O}$ such that

$$\mathcal{G} \circ \mathcal{F} \overset{(\lambda,h)}{\sim} \mathcal{I}d_\mathcal{O} \quad \text{and} \quad \mathcal{F} \circ \mathcal{G} \overset{(\lambda,h)}{\sim} \mathcal{I}d_{\mathcal{O}'}.$$

The controlled morphism $\mathcal{G}$ is called a (λ, h)-inverse for $\mathcal{G}$.

In particular, if A and B are filtered C^*-algebras and if $\mathcal{G} : \mathcal{K}_*(A) \to \mathcal{K}_*(B)$ is a (λ, h)-isomorphism, then the induced morphism $G : K_*(A) \to K_*(B)$ is an isomorphism and its inverse is induced by a controlled morphism (indeed induced by any (λ, h)-inverse for $\mathcal{F}$).

If $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ is any family of filtered C^*-algebras and if $\mathscr{H}$ a separable Hilbert space, set

$$\mathcal{A}^\infty_{c,r} = \prod_{i\in\mathbb{N}} \mathscr{K}(\mathscr{H}) \otimes A_{i,r}$$

for any $r > 0$ and define the C^*-algebra $\mathcal{A}^\infty_c$ as the closure of $\bigcup_{r>0} A^\infty_{c,r}$ in $\prod_{i\in\mathbb{N}} \mathscr{K}(\mathscr{H}) \otimes A_i$.

Lemma 2.14. *Let $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ be a family of filtered C^*-algebras. With notations of Example 2.8* (ii)*, consider*

$$\mathcal{F}_{\mathcal{A},*} = (F^{\varepsilon,r}_\mathcal{A})_{0<\varepsilon,\frac{1}{4},r>0} : \mathcal{K}_*(\mathcal{A}^\infty_c) \to \prod \mathcal{K}_*(A_i),$$

where

$$F^{\varepsilon,r}_{\mathcal{A},*} : K^{\varepsilon,r}_*(\mathcal{A}^\infty_c) \to \prod_{i\in\mathbb{N}} K^{\varepsilon,r}_*(A_i)$$

is the map induced on the jth factor and up to the Morita equivalence by the restriction to $\mathcal{A}^\infty_c$ of the evaluation $\prod_{i\in\mathbb{N}} \mathscr{K}(\mathscr{H}) \otimes A_i \to \mathscr{K}(\mathscr{H}) \otimes A_j$ at $j \in \mathbb{N}$. Then, $\mathcal{F}_{\mathcal{A},*}$ is an (α, h)-controlled isomorphism for a control pair (α, h) independent of the family $\mathcal{A}$.

We postpone the proof of this lemma until the end of the next subsection.

2.15. **Control exact sequences.**

Definition 2.16. Let (λ, h) be a control pair. Let

$$\mathcal{O} = (O_{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}, \quad \mathcal{O}' = (O'_{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}, \quad \mathcal{O}'' = (O''_{\varepsilon,r})_{0<\varepsilon<\frac{1}{4},r>0}$$

be quantitative objects, let

$$\mathcal{F} = (F^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\lambda},r>0} : \mathcal{O} \to \mathcal{O}'$$

be an $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$-controlled morphism, and let

$$\mathcal{G} = (G^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{G}}},r>0} : \mathcal{O}' \to \mathcal{O}''$$

be an $(\alpha_{\mathcal{G}}, k_{\mathcal{G}})$-controlled morphism. Then the composition

$$\mathcal{O} \xrightarrow{\mathcal{F}} \mathcal{O}' \xrightarrow{\mathcal{G}} \mathcal{O}''$$

is said to be (λ, h)-exact at $\mathcal{O}'$ if $\mathcal{G} \circ \mathcal{F} = 0$ and if for any

$$0 < \varepsilon < \frac{1}{4\max\{\lambda\alpha_{\mathcal{F}}, \alpha_{\mathcal{G}}\}},$$

any $r > 0$ and any y in $O'^{\varepsilon,r}$ such that $G^{\varepsilon,r}(y) = 0$ in $O''_{\varepsilon,r}$, there exists an element x in $O^{\lambda\varepsilon,h_\varepsilon r}$ such that

$$F^{\lambda\varepsilon,h_\varepsilon r}(x) = \iota_{\mathcal{O}'}^{\varepsilon,\alpha_{\mathcal{F}}\lambda\varepsilon,r,k_{\mathcal{F},\lambda\varepsilon}h_\varepsilon r}(y) \text{ in } O'^{\alpha_{\mathcal{F}}\lambda\varepsilon,k_{\mathcal{F},\lambda\varepsilon}h_\varepsilon r}.$$

A sequence of controlled morphisms

$$\cdots \mathcal{O}_{k-1} \xrightarrow{\mathcal{F}_{k-1}} \mathcal{O}_k \xrightarrow{\mathcal{F}_k} \mathcal{O}_{k+1} \xrightarrow{\mathcal{F}_{k+1}} \mathcal{O}_{k+2} \cdots$$

is called (λ, h)-exact if for every k, the composition

$$\mathcal{O}_{k-1} \xrightarrow{\mathcal{F}_{k-1}} \mathcal{O}_k \xrightarrow{\mathcal{F}_k} \mathcal{O}_{k+1}$$

is (λ, h)-exact at $\mathcal{O}_k$.

Examples of controlled exact sequences in quantitative K-theory are provided by controlled six-term exact sequences associated to a completely filtered extensions of C^*-algebras [9, §3].

Definition 2.17. Let A be a C^*-algebra filtered by $(A_r)_{r>0}$, let J be an ideal of A, and let us set $J_r = J \cap A_r$. The extension of C^*-algebras

$$0 \to J \to A \to A/J \to 0$$

is called a completely filtered extension of C^*-algebras if the bijective continuous linear map

$$A_r/J_r \to (A_r + J)/J$$

induced by the inclusion $A_r \hookrightarrow A$ is a complete isometry, i.e.

$$\inf_{y \in M_n(J_r)} \|x + y\| = \inf_{y \in M_n(J)} \|x + y\|$$

for any integer n, any positive number r and any x in $M_n(A_r)$.

Notice that in this case, the ideal J is filtered by $(J_r)_{r>0}$ and A/J is filtered by $(A_r+J)_{r>0}$. A particular case of a completely filtered extension of C^*-algebra is the case of a filtered and semi-split extension of C^*-algebras [9, Lem. 3.3] (or a semi-split extension of filtered algebras), i.e. extension

$$0 \to J \to A \to A/J \to 0,$$

where

- A is filtered by $(A_r)_{r>0}$;
- there exists a completely positive and completely contractive (if A is not unital) cross-section $s : A/J \to A$ such that

$$s(A_r + J) \subseteq A_r$$

  for any number $r > 0$.

For any extension of C^*-algebras

$$0 \to J \to A \to A/J \to 0$$

we denote by $\partial_{J,A} : K_*(A/J) \to K_{*+1}(J)$ the associated boundary map.

Proposition 2.18. *There exists a control pair $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$ such that for any completely filtered extension of C^*-algebras*

$$0 \to J \to A \stackrel{q}{\to} A/J \to 0,$$

there exists an $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$-controlled morphism of odd degree

$$\mathcal{D}_{J,A} = (\partial^{\varepsilon,r}_{J,A})_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{D}}},r>0} : \mathcal{K}_*(A/J) \to \mathcal{K}_{*+1}(J)$$

which induces in K-theory $\partial_{J,A} : K_*(A/J) \to K_{*+1}(J)$.

Moreover, the controlled boundary map enjoys the usual natural properties with respect to extensions.

Theorem 2.19. *There exists a control pair (λ, h) such that for any completely filtered extension of C^*-algebras*

$$0 \to J \stackrel{\jmath}{\to} A \stackrel{q}{\to} A/J \to 0,$$

the following six-term sequence is (λ, h)-exact:

$$\begin{array}{ccccc} \mathcal{K}_0(J) & \xrightarrow{\jmath_*} & \mathcal{K}_0(A) & \xrightarrow{q_*} & \mathcal{K}_0(A/J) \\ {\scriptstyle \mathcal{D}_{J,A}}\uparrow & & & & \downarrow{\scriptstyle \mathcal{D}_{J,A}} \\ \mathcal{K}_1(A/J) & \xleftarrow[q_*]{} & \mathcal{K}_1(A) & \xleftarrow[\jmath_*]{} & \mathcal{K}_1(J). \end{array}$$

In the particular case of a filtered extension of C^*-algebras

$$0 \to J \stackrel{\jmath}{\to} A \stackrel{q}{\to} A/J \to 0$$

that splits by a filtered morphism, the following sequence is (λ, h)-exact:

$$0 \to \mathcal{K}_0(J) \stackrel{\jmath}{\to} \mathcal{K}_0(A) \stackrel{q}{\to} \mathcal{K}_0(A/J) \to 0.$$

Proof of Lemma 2.14. Assume first that all the A_i are unital. Then the result is a consequence of [9, Prop. 1.30]. If A_i is not unital for some i, then for every integer i, let us endow

$$\widetilde{A}_i = \big\{(x, \lambda) \mid x \in A_i,\, \lambda \in \mathbb{C}\big\}$$

with the product

$$(x, \lambda)(x', \lambda') = (xx' + \lambda x' + \lambda' x, \lambda\lambda')$$

for all (x, λ) and (x', λ') in A_i. Then $\widetilde{A}_i$ is filtered by

$$\widetilde{A}_{i,r} = \big\{(x, \lambda) \mid x \in A_{i,r},\, \lambda \in \mathbb{C}\big\}.$$

Set then $\widetilde{\mathcal{A}} = (\widetilde{A}_i)_{i\in\mathbb{N}}$. Let us denote by $\mathcal{C}$ the constant family of the C^*-algebra $\mathbb{C}$. Then

$$0 \to \mathcal{A}_c^\infty \to \widetilde{\mathcal{A}}_c^\infty \to \mathcal{C}_c^\infty \to 0$$

is a split extension of filtered C^*-algebras. Then we have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathcal{K}_*(\mathcal{A}_c^\infty) & \longrightarrow & \mathcal{K}_*(\widetilde{\mathcal{A}}_c^\infty) & \longrightarrow & \mathcal{K}_*(\mathcal{C}_c^\infty) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle \mathcal{F}_{\mathcal{A},*}} & & \big\downarrow{\scriptstyle \mathcal{F}_{\widetilde{\mathcal{A}},*}} & & \big\downarrow{\scriptstyle \mathcal{F}_{\mathcal{C},*}} & & \\
0 & \longrightarrow & \prod_{i\in\mathbb{N}} \mathcal{K}_*(A_i) & \longrightarrow & \prod_{i\in\mathbb{N}} \mathcal{K}_*(\widetilde{A}_i) & \longrightarrow & \mathcal{K}_*^{\mathbb{N}}(\mathbb{C}) & \longrightarrow & 0,
\end{array}$$

with (λ, h)-exact rows for the control pair (λ, h) of Theorem 2.19. The result is now a consequence of a five-lemma type argument. □

2.20. ***KK*-theory and controlled morphisms.** Let A be a C^*-algebra and let B be a filtered C^*-algebra filtered by $(B_r)_{r>0}$. Let us define $A \otimes B_r$ as the closure of the algebraic tensor product of A and B_r in the spatial tensor product $A \otimes B$. Then the C^*-algebra $A \otimes B$ is filtered by $(A \otimes B_r)_{r>0}$. If $f : A_1 \to A_2$ is a homomorphism of C^*-algebras, let us set

$$f_B : A_1 \otimes B \to A_2 \otimes B, \quad a \otimes b \mapsto f(a) \otimes b.$$

Recall from [3] that for C^*-algebras A_1, A_2 and B, Kasparov defined a tensorization map

$$\tau_B : KK_*(A_1, A_2) \to KK_*(A_1 \otimes B, A_2 \otimes B).$$

If B is a filtered C^*-algebra, then for any z in $KK_*(A_1, A_2)$ the morphism

$$K_*(A_1 \otimes B) \to K_*(A_2 \otimes B), \quad x \mapsto x \otimes_{A_1 \otimes B} \tau_B(z)$$

is induced by a control morphism [9, Thm. 4.4].

Theorem 2.21. *There exists a control pair $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$ such that*

- *for any filtered C^*-algebra B;*
- *for any C^*-algebras A_1 and A_2;*
- *for any element z in $KK_*(A_1, A_2)$.*

There exists an $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$-controlled morphism

$$\mathcal{T}_B(z) = (\tau_B^{\varepsilon,r})_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{T}}},r>0} : \mathcal{K}_*(A_1 \otimes B) \to \mathcal{K}_*(A_2 \otimes B)$$

with same degree as z that satisfies the following:

(i) $\mathcal{T}_B(z) : \mathcal{K}_*(A_1 \otimes B) \to \mathcal{K}_*(A_2 \otimes B)$ *induces the right multiplication by $\tau_B(z)$ in K-theory.*

(ii) *For any elements z and z' in $KK_*(A_1, A_2)$, we have*

$$\mathcal{T}_B(z + z') = \mathcal{T}_B(z) + \mathcal{T}_B(z').$$

(iii) *Let A_1' be a filtered C^*-algebra and let $f : A_1 \to A_1'$ be a homomorphism of C^*-algebras. For any z in $KK_*(A_1', A_2)$, we have*

$$\mathcal{T}_B(f^*(z)) = \mathcal{T}_B(z) \circ f_{B,*}.$$

(iv) *Let A_2' be a C^*-algebra and let $g : A_2' \to A_2$ be a homomorphism of C^*-algebras. For any z in $KK_*(A_1, A_2')$, we have*

$$\mathcal{T}_B(g_*(z)) = g_{B,*} \circ \mathcal{T}_B(z).$$

(v) *We have*

$$\mathcal{T}_B([\mathrm{Id}_{A_1}]) \overset{(\alpha_{\mathcal{T}},k_{\mathcal{T}})}{\sim} \mathcal{I}d_{\mathcal{K}_*(A_1\otimes B)}.$$

(vi) *For any C^*-algebra D and any element z in $KK_*(A_1, A_2)$, we have*

$$\mathcal{T}_B(\tau_D(z)) = \mathcal{T}_{B\otimes D}(z).$$

(vii) *For any semi-split extension of C^*-algebras $0 \to J \to A \to A/J \to 0$ with corresponding element $[\partial_{J,A}]$ of $KK_1(A/J, J)$ that implements the boundary map, we have*

$$\mathcal{T}_B([\partial_{J,A}]) = \mathcal{D}_{J\otimes B,A\otimes B}.$$

Moreover, $\mathcal{T}_B$ is compatible with Kasparov products [9, Thm. 4.5].

Theorem 2.22. *There exists a control pair (λ, h) such that the following holds: Let A_1, A_2 and A_3 be separable C^*-algebras and let B be a filtered C^*-algebra. Then for any z in $KK_*(A_1, A_2)$ and any z' in $KK_*(A_2, A_3)$, we have*

$$\mathcal{T}_B(z \otimes_{A_2} z') \overset{(\lambda,h)}{\sim} \mathcal{T}_B(z') \circ \mathcal{T}_B(z).$$

In the case of a finitely generated group, we also have a controlled version of the Kasparov transformation. Let Γ be a finitely generated group. Recall that a length on Γ is a map $\ell : \Gamma \to \mathbb{R}^+$ such that

- $\ell(\gamma) = 0$ if and only if γ is the identity element e of Γ;
- $\ell(\gamma\gamma') \leq \ell(\gamma) + \ell(\gamma')$ for all element γ and γ' of Γ;
- $\ell(\gamma) = \ell(\gamma^{-1})$.

In what follows, we will assume that ℓ is a word length arising from a finite generating symmetric set S, i.e.

$$\ell(\gamma) = \inf\{d \mid \gamma = \gamma_1 \cdots \gamma_d \text{ with } \gamma_1, \dots, \gamma_d \text{ in } S\}.$$

Let us denote by $B(e, r)$ the ball centered at the neutral element of Γ with radius r, i.e. $B(e, r) = \{\gamma \in \Gamma \mid \ell(\gamma) \leq r\}$. Let A be a separable Γ-C^*-algebra, i.e. a separable C^*-algebra provided with an action of Γ by automorphisms. For any positive number r, we set

$$(A \rtimes_{\mathrm{red}} \Gamma)_r := \big\{f \in C_c(\Gamma, A) \text{ with support in } B(e,r)\big\}.$$

Then the C^*-algebra $A \rtimes_{\mathrm{red}} \Gamma$ is filtered by $((A \rtimes_{\mathrm{red}} \Gamma)_r)_{r>0}$. Moreover, if $f : A \to B$ is a Γ-equivariant morphism of C^*-algebras, then the induced homomorphism $f_\Gamma : A \rtimes_{\mathrm{red}} \Gamma \to B \rtimes_{\mathrm{red}} \Gamma$ is a filtered homomorphism. Recall from [3] that for Γ-C^*-algebras A and B, Kasparov also defined a natural transformation

$$J_\Gamma^{\mathrm{red}} : KK_*^\Gamma(A,B) \to KK_*(A \rtimes_{\mathrm{red}} \Gamma, B \rtimes_{\mathrm{red}} \Gamma)$$

that preserves Kasparov products. Then for any z in $KK_*(A,B)$ the morphism

$$K_*(A \rtimes_{\mathrm{red}} \Gamma) \to K_*(B \rtimes_{\mathrm{red}} \Gamma), \quad x \mapsto x \otimes_{A \rtimes_{\mathrm{red}} \Gamma} J_\Gamma(z)$$

is induced by a control morphism [9, Thm. 5.3].

Theorem 2.23. *There exists a control pair $(\alpha_{\mathcal{J}}, k_{\mathcal{J}})$ such that*

- *for any separable Γ-C^*-algebras A and B,*
- *for any elements z and z' in $KK_*^\Gamma(A,B)$,*

there exists an $(\alpha_{\mathcal{J}}, k_{\mathcal{J}})$-controlled morphism

$$\mathcal{J}_\Gamma^{\mathrm{red}}(z) = (J_\Gamma^{\mathrm{red},\varepsilon,r}(z))_{0<\varepsilon<\frac{1}{4\alpha_{\mathcal{J}}}, r>0} : \mathcal{K}_*(A \rtimes_{\mathrm{red}} \Gamma) \to \mathcal{K}_*(B \rtimes_{\mathrm{red}} \Gamma)$$

of same degree as z that satisfies the following:

(i) *The controlled morphism $\mathcal{J}_\Gamma^{\mathrm{red}}(z) : \mathcal{K}_*(A \rtimes_{\mathrm{red}} \Gamma) \to \mathcal{K}_*(B \rtimes_{\mathrm{red}} \Gamma)$ induces right multiplication by $J_\Gamma^{\mathrm{red}}(z)$ in K-theory.*

(ii) *For any elements z and z' in $KK_*^\Gamma(A,B)$, we have*

$$\mathcal{J}_\Gamma^{\mathrm{red}}(z+z') = \mathcal{J}_\Gamma^{\mathrm{red}}(z) + \mathcal{J}_\Gamma^{\mathrm{red}}(z').$$

(iii) *For any Γ-C^*-algebra A', any homomorphism $f : A \to A'$ of Γ-C^*-algebras and any z in $KK_*^\Gamma(A',B)$, we have*

$$\mathcal{J}_\Gamma^{\mathrm{red}}(f^*(z)) = \mathcal{J}_\Gamma^{\mathrm{red}}(z) \circ f_{\Gamma,*}.$$

(iv) *For any Γ-C^*-algebra B', any homomorphism $g : B \to B'$ of Γ-C^*-algebras and any z in $KK_*^\Gamma(A,B)$, we have*

$$\mathcal{J}_\Gamma^{\mathrm{red}}(g_*(z)) = g_{\Gamma,*} \circ \mathcal{J}_\Gamma^{\mathrm{red}}(z).$$

(v) *If $0 \to J \to A \to A/J \to 0$ is a semi-split exact sequence of Γ-C^*-algebras and $[\partial_{J,A}]$ is the element of $KK_1^\Gamma(A/J, J)$ that implements the boundary map $\partial_{J,A}$, we have*

$$\mathcal{J}_\Gamma^{\mathrm{red}}([\partial_{J,A}]) = \mathcal{D}_{J \rtimes_{\mathrm{red}} \Gamma, A \rtimes_{\mathrm{red}} \Gamma}.$$

The controlled Kasparov transformation is compatible with Kasparov products [9, Thm. 5.4].

Theorem 2.24. *There exists a control pair (λ, h) such that the following holds: For any separable Γ-C^*-algebras A, B and D, any elements z in $KK_*^\Gamma(A,B)$ and z' in $KK_*^\Gamma(B,D)$, we have*

$$\mathcal{J}_\Gamma^{\mathrm{red}}(z \otimes_B z') \overset{(\lambda,h)}{\sim} \mathcal{J}_\Gamma^{\mathrm{red}}(z') \circ \mathcal{J}_\Gamma^{\mathrm{red}}(z).$$

We have a similar result for maximal crossed products [9, Thms. 5.5 and 5.6].

2.25. **Quantitative assembly maps.** Let Γ be a finitely generated group and let B be a Γ-C^*-algebra B. We equip Γ with any word metric. Recall that if d is a positive number, then the Rips complex of degree d is the set $P_d(\Gamma)$ of probability measures on Γ with support of diameter at most d. Then $P_d(\Gamma)$ is a locally finite simplicial complex and provided with the simplicial topology, $P_d(\Gamma)$ is endowed with a proper and cocompact action of Γ by left translation. Recall from [9] that for any Γ-C^*-algebra B, there exists a family of quantitative assembly maps

$$\mu_{\Gamma,B,*}^{\varepsilon,r,d} : KK_*^\Gamma(C_0(P_d(\Gamma)), B) \to K_*^{\varepsilon,r}(B \rtimes_{\mathrm{red}} \Gamma),$$

with $d > 0$, $\varepsilon \in (0, \frac{1}{4})$ and $r \geq r_{d,\varepsilon}$, for a function

$$[0, +\infty) \times (0, \tfrac{1}{4}) \to (0, +\infty) : \ (d, \varepsilon) \mapsto r_{d,\varepsilon}$$

independent of B and Γ, non-decreasing in d and non-increasing in ε. Moreover, the maps $\mu_{\Gamma,B,*}^{\varepsilon,r,d}$ induce the usual assembly maps

$$\mu_{\Gamma,B,*}^{d} : KK_*^\Gamma(C_0(P_d(\Gamma)), B) \to K_*(B \rtimes_{\mathrm{red}} \Gamma),$$

i.e. $\mu_{\Gamma,B,*}^{d} = \iota_*^{\varepsilon,r} \circ \mu_{\Gamma,B,*}^{\varepsilon,r,d}$. Let us recall now the definition of the quantitative assembly maps. Observe first that any x in $P_d(\Gamma)$ can be written down in a unique way as a finite convex combination

$$x = \sum_{\gamma \in \Gamma} \lambda_\gamma(x) \delta_\gamma,$$

where δ_γ is the Dirac probability measure at γ in Γ. The functions

$$\lambda_\gamma : P_d(\Gamma) \to [0,1]$$

are continuous and $\gamma(\lambda_{\gamma'}) = \lambda_{\gamma\gamma'}$ for all γ and γ' in Γ. The function

$$p_{\Gamma,d} : \Gamma \to C_0(P_d(\Gamma)), \quad \gamma \mapsto \sum_{\gamma \in \Gamma} \lambda_e^{1/2} \lambda_\gamma^{1/2}$$

is a projection of $C_0(P_d(\Gamma)) \rtimes_{\mathrm{red}} \Gamma$ with propagation less than $2d$. Let us set then $r_{d,\varepsilon} = 2k_{\mathcal{J},\varepsilon/\alpha_\mathcal{J}} d$. Recall that $k_\mathcal{J}$ can be chosen non-increasing and in this case, $r_{d,\varepsilon}$ is non-decreasing in d and non-increasing in ε.

Definition 2.26. For any Γ-C^*-algebra A and any positive numbers ε, r and d with $\varepsilon < \frac{1}{4}$ and $r \geq r_{d,\varepsilon}$, we define the quantitative assembly map

$$\mu_{\Gamma,A,*}^{\varepsilon,r,d} : KK_*^\Gamma(C_0(P_d(\Gamma)), A) \to K_*^{\varepsilon,r}(A \rtimes_{\mathrm{red}} \Gamma),$$
$$z \mapsto \big(J_\Gamma^{\mathrm{red}, \frac{\varepsilon}{\alpha_\mathcal{J}}, \frac{r}{k_{\mathcal{J},\varepsilon/\alpha_\mathcal{J}}}}(z)\big)\big([p_{\Gamma,d}, 0]_{\frac{\varepsilon}{\alpha_\mathcal{J}}, \frac{r}{k_{\mathcal{J},\varepsilon/\alpha_\mathcal{J}}}}\big).$$

Then according to Theorem 2.23, the map $\mu_{\Gamma,A,*}^{\varepsilon,r,d}$ is a homomorphism of groups. For any positive numbers d and d' such that $d \leq d'$, we denote by

$$q_{d,d'} : C_0(P_{d'}(\Gamma)) \to C_0(P_d(\Gamma))$$

the homomorphism induced by the restriction from $P_{d'}(\Gamma)$ to $P_d(\Gamma)$. It is straight-forward to check that if d, d' and r are positive numbers such that $d \leq d'$ and $r \geq r_{d',\varepsilon}$, then $\mu_{\Gamma,A,*}^{\varepsilon,r,d} = \mu_{\Gamma,A,*}^{\varepsilon,r,d'} \circ q_{d,d',*}$. Moreover, for every positive numbers ε, ε', d, r and r' such that $\varepsilon \leq \varepsilon' < \frac{1}{4}$, $r_{d,\varepsilon} \leq r$, $r_{d,\varepsilon'} \leq r'$, and $r \leq r'$, we get by definition of a controlled morphism that

$$\iota_*^{\varepsilon,\varepsilon',r,r'} \circ \mu_{\Gamma,A,*}^{\varepsilon,r,d} = \mu_{\Gamma,A,*}^{\varepsilon',r',d}.$$

3. Persistence approximation property

In this section, we introduce the persistence approximation property for filtered C^*-algebras. In the case of a crossed product C^*-algebra by a finitely generated group, we prove that the persistence approximation property follows from the Baum–Connes conjecture with coefficients.

Let B be a filtered C^*-algebra. As a consequence of Proposition 2.4, we see that there exists for every $\varepsilon \in (0, \frac{1}{4})$ a surjective map

$$\lim_{r>0} K_*^{\varepsilon,r}(B) \to K_*(B)$$

induced by $(\iota_*^{\varepsilon,r})_{r>0}$. Moreover, although this morphism is not a priori one-to-one, if ε is a positive and small enough number, then for every positive number r and any x in $K_*^{\varepsilon,r}(B)$, there exist positive numbers ε' in $[\varepsilon, \frac{1}{4})$ (indeed independent of x and B) and $r' > r$ such that

$$\iota_*^{\varepsilon,r}(x) = 0 \quad \Longrightarrow \quad \iota_*^{\varepsilon,\varepsilon',r,r'}(x) = 0 \text{ in } K_*^{\varepsilon',r'}(B).$$

It is of relevance to ask whether this r' depends on x, in other words whether the family $(K_*^{\varepsilon,r}(B))_{0<\varepsilon<\frac{1}{4},r>0}$ provides a persistence approximation for $K_*(B)$ in the following sense: for any ε in $(0, \frac{1}{4})$ small enough and for any $r > 0$, there exist ε' in $(\varepsilon, \frac{1}{4})$ and $r' \geq r$ such that for any x in $K_*^{\varepsilon,r}(B)$, we have

$$\iota_*^{\varepsilon,\varepsilon',r,r'}(x) \neq 0 \text{ in } K_*^{\varepsilon',r'}(B) \quad \Longrightarrow \quad \iota_*^{\varepsilon,r}(x) \neq 0 \text{ in } K_*(B).$$

Let us consider the following statement, for a filtered C^*-algebra B and positive numbers ε, ε' and r' such that $0 < \varepsilon \leq \varepsilon' < \frac{1}{4}$ and $0 < r \leq r'$:

$\mathcal{PA}_*(B, \varepsilon, \varepsilon', r, r')$: for any $x \in K_*^{\varepsilon,r}(B)$,

$$\iota_*^{\varepsilon,r}(x) = 0 \text{ in } K_*(B) \quad \Longrightarrow \quad \iota_*^{\varepsilon,\varepsilon',r,r'}(x) = 0 \text{ in } K_*^{\varepsilon',r'}(B).$$

Notice that $\mathcal{PA}_*(B, \varepsilon, \varepsilon', r, r')$ can be rephrased as follows: The restriction of $\iota_*^{\varepsilon',r'} : K_*^{\varepsilon',r'}(B) \to K_*(B)$ to $\iota_*^{\varepsilon,\varepsilon',r,r'}(K_*^{\varepsilon,r}(B))$ is one-to-one.

In this section, we investigate the following persistence approximation property: Given ε small enough and r positive numbers, do there exist positive numbers ε' and r' with $0 < \varepsilon \leq \varepsilon' < \frac{1}{4}$ and $r < r'$ such that $\mathcal{PA}_*(B, \varepsilon, \varepsilon', r, r')$ holds?

3.1. The case of crossed products.

Theorem 3.2. *Let Γ be a finitely generated group. Assume that*

- Γ *satisfies the Baum–Connes conjecture with coefficients;*
- Γ *admits a cocompact universal example for proper actions.*

Then there exists a universal constant $\lambda_{\mathrm{PA}} \geq 1$ such that for any ε in $(0, \frac{1}{4\lambda_{\mathrm{PA}}})$ and any $r > 0$, there exists $r' \geq r$ such that $\mathcal{PA}_*(A \rtimes_{\mathrm{red}} \Gamma, \varepsilon, \lambda_{\mathrm{PA}}\varepsilon, r, r')$ for any Γ-C^*-algebra A.

Proof. Notice first that since Γ satisfies the Baum–Connes conjecture with coefficients and admits a cocompact universal example for proper actions, there exist positive numbers d and d' with $d \leq d'$ such that for any Γ-C^*-algebra B, the following is satisfied:

- For any z in $K_*(B \rtimes_{\mathrm{red}} \Gamma)$, there exists x in $KK_*^{\Gamma}(C_0(P_d(\Gamma)), B)$ such that $\mu^d_{\Gamma,B,*}(x) = z$.
- For any x in $KK_*^{\Gamma}(C_0(P_d(\Gamma)), B)$ such that $\mu^d_{\Gamma,B,*}(x) = 0$, we have
$$q^*_{d,d'}(x) = 0 \text{ in } KK_*^{\Gamma}(C_0(P_{d'}(\Gamma)), B),$$
where $q^*_{d,d'} : KK_*^{\Gamma}(C_0(P_d(\Gamma)), B) \to KK_*^{\Gamma}(C_0(P_{d'}(\Gamma)), B)$ is induced by the inclusion $P_d(\Gamma) \hookrightarrow P_{d'}(\Gamma)$.

Let us fix such d and d', let λ be as in Proposition 2.4, pick (α, h) as in Lemma 2.14 and set $\lambda_{\mathrm{PA}} = \alpha\lambda$. Assume that this statement does not hold. Then there exist

- ε in $(0, \frac{1}{4\lambda_{\mathrm{PA}}})$ and $r > 0$,
- an unbounded increasing sequence $(r_i)_{i\in\mathbb{N}}$ bounded below by r,
- a sequence of Γ-C^*-algebras $(A_i)_{i\in\mathbb{N}}$,
- a sequence of elements $(x_i)_{i\in\mathbb{N}}$ with x_i in $K_*^{\varepsilon,r}(A_i \rtimes_{\mathrm{red}} \Gamma)$

such that, for every integer i,
$$\iota_*^{\varepsilon,r}(x_i) = 0 \text{ in } K_*(A_i \rtimes_{\mathrm{red}} \Gamma)$$
and
$$\iota_*^{\varepsilon,\lambda_{\mathrm{PA}}\varepsilon,r,r_i}(x_i) \neq 0 \text{ in } K_*^{\lambda_{\mathrm{PA}}\varepsilon,r_i}(A_i \rtimes_{\mathrm{red}} \Gamma).$$
We can assume without loss of generality that $r \geq r_{d',\varepsilon}$.

Since
$$\Big(\prod_{j\in\mathbb{N}} \mathscr{K}(\mathscr{H}) \otimes A_j\Big) \rtimes_{\mathrm{red}} \Gamma_{h_\varepsilon r} = \prod_{j\in\mathbb{N}} (\mathscr{K}(\mathscr{H}) \otimes A_j \rtimes_{\mathrm{red}} \Gamma_{h_\varepsilon r})$$
and according to Lemma 2.14, there exists an element
$$x \in K_*^{\alpha\varepsilon,h_\varepsilon r}\Big(\Big(\prod_{j\in\mathbb{N}} \mathscr{K}(\mathscr{H}) \otimes A_j\Big) \rtimes_{\mathrm{red}} \Gamma\Big)$$

that maps to $\iota_*^{\varepsilon,\alpha\varepsilon,r,h_\varepsilon r}(x_i)$ for all integers i under the composition

$$K_*^{\alpha\varepsilon,h_\varepsilon r}\Big(\Big(\prod_{j\in\mathbb{N}}\mathscr{K}(\mathscr{H})\otimes A_j\Big)\rtimes_{\mathrm{red}}\Gamma\Big)\to K_*^{\alpha\varepsilon,h_\varepsilon r}(\mathscr{K}(\mathscr{H})\otimes A_i\rtimes_{\mathrm{red}}\Gamma)$$
$$\xrightarrow{\mathcal{M}_{A_i}^{\alpha\varepsilon,h_\varepsilon r}} K_*^{\alpha\varepsilon,h_\varepsilon r}(\,A_i\rtimes_{\mathrm{red}}\Gamma),$$

where the first map is induced by the ith projection

$$\prod_{j\in\mathbb{N}}\mathscr{K}(\mathscr{H})\otimes A_j\to\mathscr{K}(\mathscr{H})\otimes A_i \tag{1}$$

and the map $\mathcal{M}_{A_i}^{\alpha\varepsilon,h_\varepsilon r}$ is the Morita equivalence of Proposition 2.5. Let

$$z\in KK_*^\Gamma\Big(C_0(P_d(\Gamma)),\prod_{j\in\mathbb{N}}\mathscr{K}(\mathscr{H})\otimes A_j\Big)$$

such that

$$\mu^d_{\Gamma,\prod_{j\in\mathbb{N}}\mathscr{K}(\mathscr{H})\otimes A_j,*}(z)=\iota_*^{\alpha\varepsilon,h_\varepsilon r}(x)\text{ in }K_*\Big(\Big(\prod_{j\in\mathbb{N}}\mathscr{K}(\mathscr{H})\otimes A_j\Big)\rtimes_{\mathrm{red}}\Gamma\Big).$$

Recall from [8, Prop. 3.4] that we have an isomorphism

$$KK_*^\Gamma\Big(C_0(P_d(\Gamma)),\prod_{j\in\mathbb{N}}\mathscr{K}(\mathscr{H})\otimes A_j\Big)\xrightarrow{\cong}\prod_{j\in\mathbb{N}}KK_*^\Gamma(C_0(P_d(\Gamma)),A_j) \tag{2}$$

induced on the ith factor and up to the Morita equivalence

$$KK_*^\Gamma(C_0(P_d(\Gamma)),A_j)\cong KK_*^\Gamma(C_0(P_d(\Gamma)),\mathscr{K}(\mathscr{H})\otimes A_j)$$

by the ith projection (1). Let $(z_j)_{j\in\mathbb{N}}$ be the element of

$$\prod_{j\in\mathbb{N}}KK_*^\Gamma(C_0(P_d(\Gamma)),A_j)$$

corresponding to z under this identification. Since the quantitative Baum–Connes assembly maps are compatible with the usual ones, we get that

$$\mu^d_{\Gamma,\prod_{j\in\mathbb{N}}\mathscr{K}(\mathscr{H})\otimes A_j,*}(z)=\iota_*^{\alpha\varepsilon,h_\varepsilon r}\circ\mu^{d,\alpha\varepsilon,h_\varepsilon r}_{\Gamma,\prod_{j\in\mathbb{N}}\mathscr{K}(\mathscr{H})\otimes A_j,*}(z).$$

But then, according to item (ii) of Proposition 2.4, there exists $R\geq h_\varepsilon r$ such that

$$\begin{aligned}\iota_*^{\alpha\varepsilon,\lambda_{\mathrm{PA}}\varepsilon,h_\varepsilon r,R}(x)&=\iota_*^{\alpha\varepsilon,\lambda_{\mathrm{PA}}\varepsilon,h_\varepsilon r,R}\circ\mu^{d,\alpha\varepsilon,h_\varepsilon r}_{\Gamma,\prod_{j\in\mathbb{N}}\mathscr{K}(H)\otimes A_j,*}(z)\\&=\mu^{d,\lambda_{\mathrm{PA}}\varepsilon,R}_{\Gamma,\prod_{j\in\mathbb{N}}\mathscr{K}(\mathscr{H})\otimes A_j,*}(z).\end{aligned}$$

Using once again the compatibility of the quantitative assembly maps with the usual ones, we obtain by naturality that $\mu^d_{\Gamma,A_i,*}(z_i)=0$ for every integer i and hence

$$q_{d,d',*}(z_i)=0\text{ in }KK_*^\Gamma(C_0(P_{d'}(\Gamma)),A_i).$$

Using once more equation (2), we deduce that

$$q_{d,d',*}(z)=0\text{ in }KK_*^\Gamma\Big(C_0(P_{d'}(\Gamma)),\prod_{j\in\mathbb{N}}\mathscr{K}(\mathscr{H})\otimes A_j\Big)$$

and since

$$\mu^{d,\lambda_{\mathrm{PA}}\varepsilon,R}_{\Gamma,\prod_{j\in\mathbb{N}}\mathscr{K}(\mathscr{H})\otimes A_j,*}(z)=\mu^{d',\lambda_{\mathrm{PA}}\varepsilon,R}_{\Gamma,\prod_{j\in\mathbb{N}}\mathscr{K}(\mathscr{H})\otimes A_j,*}\circ q_{d,d',*}(z)$$

that

$$\iota_*^{\alpha\varepsilon,\lambda_{\mathrm{PA}}\varepsilon,h_\varepsilon r,R}(x)=0 \text{ in } K_*^{\lambda_{\mathrm{PA}}\varepsilon,R}\Big(\Big(\prod_{j\in\mathbb{N}}\mathscr{K}(\mathscr{H})\otimes A_j\Big)\rtimes_{\mathrm{red}}\Gamma\Big).$$

By naturality, we see that $\iota_*^{\varepsilon,\lambda_{\mathrm{PA}}\varepsilon,r,R}(x_i)=0$ in $K_*^{\lambda_{\mathrm{PA}}\varepsilon,R}(A_i\rtimes_{\mathrm{red}}\Gamma)$ for every integer i. Picking an integer i such that $r_i\geq R$, we have

$$\iota_*^{\varepsilon,\lambda_{\mathrm{PA}}\varepsilon,r,r_i}(x_i)=\iota_*^{\lambda_{\mathrm{PA}}\varepsilon,R,r_i}\circ\iota_*^{\varepsilon,\lambda_{\mathrm{PA}}\varepsilon,r,R}(x_i)=0$$

which contradicts our assumption. □

Specifying the coefficients in the previous proof gives the next proposition.

Proposition 3.3. *Let Γ be a finitely generated group and let A be a Γ-C^*-algebra. Assume that*

- *Γ admits a cocompact universal example for proper actions;*
- *the Baum–Connes assembly map for Γ with coefficients in*

$$\ell^\infty(\mathbb{N},\mathscr{K}(\mathscr{H})\otimes A)$$

  is onto;
- *the Baum–Connes assembly map for Γ with coefficients in A is one to one.*

Then for some universal constant $\lambda_{\mathrm{PA}}\geq 1$, any ε in $(0,\frac{1}{4\lambda_{\mathrm{PA}}})$ and any $r>0$ there exists $r'\geq r$ such that $\mathcal{PA}(A\rtimes_{\mathrm{red}}\Gamma,\varepsilon,\lambda_{\mathrm{PA}}\varepsilon,r,r')$ is satisfied.

Since for any C^*-algebra B, the Baum–Connes assembly map for Γ with coefficients in $C_0(\Gamma,B)$ (B being provided with the trivial action) is an isomorphism and since $C_0(\Gamma,B)\rtimes_{\mathrm{red}}\Gamma\cong B\otimes\mathscr{K}(\ell^2(\Gamma))$, Proposition 3.3 leads to the following corollary.

Corollary 3.4. *Let Γ be a finitely generated group and let B be a C^*-algebra. Assume that*

- *Γ admits a cocompact universal example for proper actions;*
- *the Baum–Connes assembly map for Γ with coefficients in*

$$\ell^\infty(\mathbb{N},C_0(\Gamma,\mathscr{K}(\mathscr{H})\otimes B))$$

  is onto.

Then for some universal constant $\lambda_{\mathrm{PA}}\geq 1$, any ε in $(0,\frac{1}{4\lambda_{\mathrm{PA}}})$ and any $r>0$ there exists $r'\geq r$ such that $\mathcal{PA}(B\otimes\mathscr{K}(\ell^2(\Gamma)),\varepsilon,\lambda_{\mathrm{PA}}\varepsilon,r,r')$ is satisfied. Moreover, if Γ satisfies the Baum–Connes conjecture with coefficients, then r' does not depend on B.

If we take $B=\mathbb{C}$ in the previous corollary, we obtain the following linear algebra statement.

Proposition 3.5. *Let Γ be a finitely generated group and let H be a separable Hilbert space. Assume that*

- *Γ admits a cocompact universal example for proper actions;*
- *the Baum–Connes assembly map for Γ with coefficients in*

$$\ell^\infty(\mathbb{N}, C_0(\Gamma, \mathscr{K}(\mathscr{H})))$$

is onto.

Then for some universal constant $\lambda \geq 1$, any ε in $(0, \frac{1}{4\lambda})$ and any $r > 0$ there exists $R \geq r$ such that the following holds:

- *If u is an ε-r-unitary of $\mathscr{K}(\ell^2(\Gamma) \otimes \mathscr{H}) + \mathbb{C}\mathrm{Id}_{\ell^2(\Gamma)\otimes\mathscr{H}}$, then u is connected to $\mathrm{Id}_{\ell^2(\Gamma)\otimes\mathscr{H}}$ by a homotopy of $\lambda\varepsilon$-R-unitaries.*
- *If q_0 and q_1 are ε-r-projections of $\mathscr{K}(\ell^2(\Gamma) \otimes \mathscr{H})$ such that*

$$\operatorname{rank} \kappa_0(q_0) = \operatorname{rank} \kappa_0(q_1),$$

then q_0 and q_1 are connected by a homotopy of $\lambda\varepsilon$-R-projections.

3.6. **Induction and geometric setting.** The conclusions of Corollary 3.4 and of Proposition 3.5 concern only the metric properties of Γ (indeed as we shall see later up to quasi-isometries). For the purpose of having statements analogous to Corollary 3.4 in a metric setting, we need to have a completely geometric description of the quantitative assembly maps

$$\mu^{d,\varepsilon,r}_{\Gamma,\prod_{i\in\mathbb{N}} C_0(\Gamma,\mathscr{K}(\mathscr{H})\otimes A_i),*} : KK^\Gamma_*\Big(C_0(P_d(\Gamma)), \prod_{i\in\mathbb{N}} C_0(\Gamma, \mathscr{K}(\mathscr{H}) \otimes A_i)\Big)$$
$$\to K_*\Big(\Big(\prod_{i\in\mathbb{N}} C_0(\Gamma, \mathscr{K}(\mathscr{H}) \otimes A_i)\Big) \rtimes_{\mathrm{red}} \Gamma\Big)$$

(see the proof of Theorem 3.2). We study in this subsection a slight generalization of these maps to the case of induced algebras from the action of a finite subgroup of Γ.

Let Γ be a discrete group equipped with a proper length ℓ. Let F be a finite subgroup of Γ. For any F-C^*-algebra A, let us consider the induced Γ-algebra

$$\mathrm{I}^\Gamma_F(A) = \big\{f \in C_0(\Gamma, A) \mid f(\gamma) = kf(\gamma k) \text{ for every } k \text{ in } F\big\}.$$

Then left translation on $C_0(\Gamma, A)$ provides a Γ-C^*-algebra structure on $\mathrm{I}^\Gamma_F(A)$. Moreover, there is a covariant representation of $(\mathrm{I}^\Gamma_F(A), \Gamma)$ on the algebra of adjointable operators of the right Hilbert A-module $A \otimes \ell^2(\Gamma)$, where

- if f is in $\mathrm{I}^\Gamma_F(A)$, then f acts on $A \otimes \ell^2(\Gamma)$ by pointwise multiplication by $\gamma \mapsto \gamma^{-1}(f(\gamma))$;
- Γ acts by left translations.

The induced representation then provides an identification between the algebra $\mathrm{I}^\Gamma_F(A) \rtimes_{\mathrm{red}} \Gamma$ and the algebra of F-invariant elements of $A \otimes \mathscr{K}(\ell^2(\Gamma))$ for the diagonal action of F, the action on $\mathscr{K}(\ell^2(\Gamma))$ being by right translation. Let us denote by $A_{F,\Gamma}$ the algebra of F-invariant elements of $A \otimes \mathscr{K}(\ell^2(\Gamma))$ and by

$$\Phi_{A,F,\Gamma} : \mathrm{I}^\Gamma_F(A) \rtimes_{\mathrm{red}} \Gamma \to A_{F,\Gamma}$$

the isomorphism induced by the above covariant representation. The length ℓ gives rise to a filtration structure $(\mathrm{I}_F^\Gamma(A)\rtimes_{\mathrm{red}}\Gamma_r)_{r>0}$ on $\mathrm{I}_F^\Gamma(A)\rtimes_{\mathrm{red}}\Gamma$ (recall that $(\mathrm{I}_F^\Gamma(A)\rtimes_{\mathrm{red}}\Gamma_r)$ is the set of functions of $C_c(\Gamma,\mathrm{I}_F^\Gamma(A))$ with support in the ball of radius r centered at the neutral element). The right invariant metric associated to ℓ also provides a filtration structure on $\mathscr{K}(\ell^2(\Gamma))$ and hence on $A\otimes\mathscr{K}(\ell^2(\Gamma))$. This filtration is invariant under the action of F and moreover the isomorphism $\Phi_{A,F,\Gamma}:\mathrm{I}_F^\Gamma(A)\rtimes_{\mathrm{red}}\Gamma\to A_{F,\Gamma}$ preserves the filtrations. By using the induced algebra in the proof of Corollary 3.4, we get the following result.

Proposition 3.7. *Let F be a finite subgroup of a finitely generated group Γ and let A be an F-C^*-algebra. Assume that*

- *Γ admits a cocompact universal example for proper actions;*
- *the Baum–Connes assembly map for Γ with coefficients in*
$$\ell^\infty(\mathbb{N},C_0(\Gamma,\mathscr{K}(\mathscr{H})\otimes\mathrm{I}_F^\Gamma(A)))$$
is onto.

Then for some universal constant $\lambda_{\mathrm{PA}}\geq 1$, any ε in $(0,\frac{1}{4\lambda_{\mathrm{PA}}})$ and any $r>0$ there exists $r'\geq r$ such that $\mathcal{PA}_*(A_{F,\Gamma},\varepsilon,\lambda_{\mathrm{PA}}\varepsilon,r,r')$ is satisfied. Moreover, if Γ satisfies the Baum–Connes conjecture with coefficients, then r' does not depend on F and A.

In [7], an isomorphism

$$\mathrm{I}_F^\Gamma(P_s(\Gamma))_*:\lim_X KK_*^F(C(X),A)\xrightarrow{\cong}KK_*^\Gamma(C_0(P_s(\Gamma)),\mathrm{I}_F^\Gamma(A)) \tag{3}$$

was stated for any F-C^*-algebra, where X runs through F-invariant compact subsets of $P_s(\Gamma)$. In order to describe this isomorphism, let us first recall the definition of induction for equivariant KK-theory. Let A and B be F-C^*-algebras and let $(\mathcal{E},\rho,T)$ be a K-cycle for $KK_*^F(A,B)$, where

- $\mathcal{E}$ is a right B-Hilbert module provided with an equivariant action of F;
- $\rho:A\to\mathcal{L}_B(\mathcal{E})$ is an F-equivariant representation of A into the algebra $\mathcal{L}_B(\mathcal{E})$ of adjointable operators of $\mathcal{E}$;
- T is an F-equivariant operator of $\mathcal{L}_B(\mathcal{E})$ satisfying the K-cycle relations.

Let us define

$$\mathrm{I}_F^\Gamma(\mathcal{E})=\big\{f\in C_0(\Gamma,\mathcal{E})\mid f(\gamma)=kf(\gamma k)\text{ for every }k\text{ in }F\big\}.$$

Then $\mathrm{I}_F^\Gamma(\mathcal{E})$ is a right $\mathrm{I}_F^\Gamma(B)$-Hilbert module for the pointwise scalar product and multiplication, and the representation $\rho:A\to\mathcal{L}_B(\mathcal{E})$ gives rise in the same way to a representation

$$\mathrm{I}_F^\Gamma\rho:\mathrm{I}_F^\Gamma(A)\to\mathcal{L}_{\mathrm{I}_F^\Gamma(B)}(\mathrm{I}_F^\Gamma(\mathcal{E})).$$

Let $\mathrm{I}_F^\Gamma T$ be the operator of $\mathcal{L}_{\mathrm{I}_F^\Gamma(A)}(\mathrm{I}_F^\Gamma(\mathcal{E}))$ given by the pointwise multiplication by T. It is then plain to check that $(\mathrm{I}_F^\Gamma(\mathcal{E}),\mathrm{I}_F^\Gamma\rho,\mathrm{I}_F^\Gamma T)$ is a K-cycle for $KK_*^\Gamma(\mathrm{I}_F^\Gamma A,\mathrm{I}_F^\Gamma B)$ and that, moreover, $(\mathcal{E},\rho,T)\to(\mathrm{I}_F^\Gamma(\mathcal{E}),\mathrm{I}_F^\Gamma\rho,\mathrm{I}_F^\Gamma T)$ gives rise to a well-defined morphism $\mathrm{I}_F^\Gamma:KK_*^F(A,B)\to KK_*^\Gamma(\mathrm{I}_F^\Gamma(A),\mathrm{I}_F^\Gamma(B))$.

Back to the definition of the isomorphism of equation (3), let F be a finite subgroup of a discrete group Γ and let X be an F-invariant compact subset of $P_s(\Gamma)$ for $s > 0$. If we equip $\Gamma \times X$ with the diagonal action of F, where the action on Γ is by right multiplication, then there is a natural identification between $\mathrm{I}_F^\Gamma(C(X))$ and $C_0((\Gamma \times X)/F)$. The map

$$(\Gamma \times X)/F \to P_s(\Gamma), \quad [(\gamma, x)] \mapsto \gamma x$$

then gives rise to a Γ-equivariant homomorphism

$$\Upsilon_{F,X}^\Gamma : C_0(P_s(\Gamma)) \to \mathrm{I}_F^\Gamma(C(X)).$$

Then for any F-C^*-algebra A, the morphism

$$KK_*^F(C(X), A) \to KK_*^\Gamma(C_0(P_s(\Gamma)), \mathrm{I}_F^\Gamma(A)), \quad x \mapsto \Upsilon_{F,X}^{\Gamma,*}(\mathrm{I}_F^\Gamma(x))$$

is compatible with the inductive limit over F-invariant compact subsets of $P_s(\Gamma)$ and hence we eventually obtain a natural homomorphism

$$\mathrm{I}_F^\Gamma(P_s(\Gamma))_* : \lim_X KK_*^F(C(X), A) \to KK_*^\Gamma(C_0(P_s(\Gamma)), \mathrm{I}_F^\Gamma(A))$$

which turns out to be an isomorphism.

Let us consider now the composition

$$\Phi_{A,F,\Gamma,*} \circ \mu_{\Gamma,\mathrm{I}_F^\Gamma A,*}^{\varepsilon,r,s} \circ \mathrm{I}_F^\Gamma(P_s(\Gamma))_* : \lim_X KK_*^F(C(X), A) \to K_*^{\varepsilon,r}(A_{F,\Gamma}), \tag{4}$$

where X runs through F-invariant compact subsets of $P_s(\Gamma)$. Both sides of these maps only involve Γ as a metric space equipped with an isometric action of F. Our aim in the next section is to provide a geometric definition for this family of quantitative assembly maps.

4. Coarse geometry

Let Σ be a discrete proper metric space. For s a positive number, the Rips complex of degree s is the set $P_s(\Sigma)$ of probability measures on Σ with support of diameter less than s. If Σ is equipped with a free action of a finite group F by isometries and if A is an F-C^*-algebra, define then $A_{F,\Sigma}$ as the set of invariant elements of $A \otimes \mathscr{K}(\ell^2(\Sigma))$ for the diagonal action of F. For F trivial, we set $A_{\{e\},\Sigma} = A_\Sigma$. The filtration $(A \otimes \mathscr{K}(\ell^2(\Sigma))_r)_{r>0}$ on $A \otimes \mathscr{K}(\ell^2(\Sigma))$ is preserved by the action of the group F. Hence, if $A_{F,\Sigma,r}$ stands for the set of F-invariant elements of $A \otimes \mathscr{K}(\ell^2(\Sigma))_r$, then $(A_{F,\Sigma,r})_{r>0}$ provides $A_{F,\Sigma}$ with a structure of filtered C^*-algebra. Our aim in this section is to investigate the permanence approximation property for $A_{F,\Sigma}$. Let us set $\mathcal{PA}_{F,\Sigma,A,*}(\varepsilon, \varepsilon', r, r')$ for the property $\mathcal{PA}_*(A_{F,\Sigma}, \varepsilon, \varepsilon', r, r')$, i.e. the restriction of

$$\iota_*^{\varepsilon',r'} : K_*^{\varepsilon',r'}(A_{F,\Sigma}) \to K_*(A_{F,\Sigma})$$

to $\iota_*^{\varepsilon,\varepsilon',r,r'}(K_*^{\varepsilon,r}(A_{F,\Sigma}))$ is one-to-one.

Considering isometric actions of a finite group for the above persistence approximation property might have two interesting applications:

- Study the persistence approximation property for crossed product of a discrete proper group on a proper C^*-algebra.

- Using the previous point and some Poincaré duality for some examples of groups satisfying the Baum–Connes conjecture, try to compute explicitly r' in the persistence approximation property (see Theorem 3.2) in terms of ε and r.

Following the route of the proof of Theorem 3.2, and in view of equation (4), let us set

$$K_*^F(P_s(\Sigma), A) = \lim_X KK_*^F(C(X), A),$$

where in the inductive limit, X runs through F-invariant compact subsets of $P_s(\Sigma)$ for $s > 0$. Our purpose is to define a family of local quantitative coarse assembly maps

$$\nu_{F,\Sigma,A,*}^{\varepsilon,r,s} : K_*^F(P_s(\Sigma), A) \to K_*^{\varepsilon,r}(A_{F,\Sigma}),$$

for $s > 0$, $\varepsilon \in (0, \frac{1}{4})$, $r \geq r_{s,\varepsilon}$ and

$$[0, +\infty) \times (0, \tfrac{1}{4}) \to (0, +\infty) : (s, \varepsilon) \mapsto r_{s,\varepsilon}$$

a function independent on A, non-decreasing in s and non-increasing in ε such that, if F is a subgroup of a discrete group Γ equipped with right invariant metric arising from a proper length, then $\nu_{F,\Gamma,A,*}^{\varepsilon,r,s}$ coincides with the composition of equation (4).

4.1. **A local coarse assembly map.** Let Σ be a proper discrete metric space, with bounded geometry and equipped with a free action of a finite group F by isometries and let A be an F-algebra. Recall that $A_{F,\Sigma}$ is defined as the set of invariant elements of $A \otimes \mathscr{K}(\ell^2(\Sigma))$ for the diagonal action of F. Notice that since the action of F on Σ is free, the choice of an equivariant identification between $\Sigma/F \times F$ and Σ (i.e. the choice of a fundamental domain) gives rise to a Morita equivalence between $A_{F,\Sigma}$ and $A \rtimes F$. The aim of this section is to construct for $s > 0$ a family of local coarse assembly maps

$$\nu_{F,\Sigma,A,*}^{s} K_*^F(P_s(\Sigma), A) \to K_*(A_{F,\Sigma}).$$

Let us define first for any F-algebras A and B a map

$$\tau_{F,\Sigma} : KK_*^F(A, B) \to KK_*(A_{F,\Sigma}, B_{F,\Sigma})$$

analogous to the Kasparov transformation.

Let z be an element in $KK_*^F(A, B)$. Then z can be represented by an equivariant K-cycle $(\pi, T, \mathscr{H} \otimes \ell^2(F) \otimes B)$ where

- $\mathscr{H}$ is a separable Hilbert space;
- F acts diagonally on $\mathscr{H} \otimes \ell^2(F) \otimes B$, trivially on $\mathscr{H}$ and by the right regular representation on $\ell^2(F)$;
- π is an F-equivariant representation of A in the algebra $\mathcal{L}_B(\mathscr{H} \otimes \ell^2(F) \otimes B)$ of adjointable operators of $\mathscr{H} \otimes \ell^2(F) \otimes B$;
- T is an F-equivariant selfadjoint operator of $\mathcal{L}_B(\mathscr{H} \otimes \ell^2(F) \otimes B)$ satisfying the K-cycle conditions, i.e. $[T, \pi(a)]$ and $\pi(a)(T^2 - \mathcal{I}d_{\mathscr{H} \otimes \ell^2(F) \otimes B})$ belong to $\mathscr{K}(\mathscr{H} \otimes \ell^2(F)) \otimes B$, for every a in A.

Let $\mathscr{H}_{B,F,\Sigma}$ be the set of invariant elements in $\mathscr{H}\otimes\ell^2(F)\otimes B\otimes\mathscr{K}(\ell^2(\Sigma))$. Then $\mathscr{H}_{B,F,\Sigma}$ is obviously a right $B_{F,\Sigma}$-Hilbert module, and π induces a representation $\pi_{F,\Sigma}$ of $A_{F,\Sigma}$ on the algebra $\mathcal{L}_{B_{F,\Sigma}}(\mathscr{H}_{B,F,\Sigma})$ of adjointable operators of $\mathscr{H}_{B,F,\Sigma}$ and T gives rise also to a selfadjoint element $T_{B,F,\Sigma}$ of $\mathcal{L}_{B_{F,\Sigma}}(\mathscr{H}_{B,F,\Sigma})$. Moreover, by choosing an equivariant identification between $\Sigma/F\times F$ and Σ, we can check that the algebra of F-equivariant compact operators on $\mathscr{H}\otimes\ell^2(F)\otimes\ell^2(\Sigma)\otimes B$ coincides with the algebra of compact operators on the right $B_{F,\Sigma}$-Hilbert module $\mathscr{H}_{B,F,\Sigma}$. Hence, $(\pi_{F,\Sigma},T_{B,F,\Sigma},\mathscr{H}_{B,F,\Sigma})$ is a K-cycle for $KK_*(A_{F,\Sigma},B_{F,\Sigma})$. Furthermore, its class in $KK_*(A_{F,\Sigma},B_{F,\Sigma})$ only depends on z and thus we end up with a morphism

$$\tau_{F,\Sigma}:KK_*^F(A,B)\to KK_*(A_{F,\Sigma},B_{F,\Sigma}). \tag{5}$$

It is also quite easy to see that $\tau_{F,\Sigma}$ is functorial in both variables. Namely, for any F-equivariant homomorphism $f:A\to B$ of F-algebras, let us set $f_{F,\Sigma}:A_{F,\Sigma}\to B_{F,\Sigma}$ for the induced homomorphism. Then for any F-algebras A_1, A_2, B_1 and B_2 and any homomorphism of F-algebra $f:A_1\to A_2$ and $g:B_1\to B_2$, we have

$$\tau_{F,\Sigma}(f^*(z))=f^*_{F,\Sigma}(\tau_{F,\Sigma}(z))$$

and

$$\tau_{F,\Sigma}(g_*(z))=g_{F,\Sigma,*}(\tau_{F,\Sigma}(z))$$

for any z in $KK_*^F(A_2,B_1)$.

We are now in a position to define the index map. Observe that any x in $P_s(\Sigma)$ can be written as a finite convex combination

$$x=\sum_{\sigma\in\Sigma}\lambda_\sigma(x)\delta_\sigma,$$

where

- δ_σ is the Dirac probability measure at σ in Σ;
- for every σ in Σ, the coordinate function $\lambda_\sigma:P_s(\Sigma)\to[0,1]$ is continuous with support in the ball centered at σ and with radius 1 for the simplicial distance.

Moreover, for any σ in Σ and k in F, we have $\lambda_{k\sigma}(kx)=\lambda_\sigma(x)$. Let X be a compact F-invariant subset of $P_d(\Sigma)$. Let us define

$$P_X:C(X)\otimes\ell^2(\Sigma)\to C(X)\otimes\ell^2(\Sigma)$$

by

$$(P_X\cdot h)(x,\sigma)=\lambda_\sigma^{1/2}(x)\sum_{\sigma'\in\Sigma}h(x,\sigma')\lambda_{\sigma'}^{1/2}(x) \tag{6}$$

for any h in $C(X)\otimes\ell^2(\Sigma)$. Since $\sum_{\sigma\in\Sigma}\lambda_\sigma=1$, it is straight-forward to check that P_X is an F-equivariant projection in $C(X)\otimes\mathscr{K}(\ell^2(\Sigma))$ with propagation less than $2s$. Hence, P_X gives rise in particular to a class $[P_X]$ in $K_0(C(X)_{F,\Sigma})$. For any F-C^*-algebra A, the map

$$KK_*^F(C(X),A)\to K_*(A_{F,\Sigma}),\quad x\mapsto[P_X]\otimes_{C(X)_{F,\Sigma}}\tau_{F,\Sigma}(x)$$

is compatible with the inductive limit over F-invariant compact subsets of $P_s(\Sigma)$ and hence gives rise to a local coarse assembly map

$$\nu^s_{F,\Sigma,A,*} : K^F_*(P_s(\Sigma), A) \to K_*(A_{F,\Sigma}).$$

This local coarse assembly map is natural in the F-algebra. Furthermore, let us denote for any positive numbers s and s' such that $s \le s'$ by

$$q_{s,s',*} : K^F_*(P_s(\Sigma), A) \to K^F_*(P_{s'}(\Sigma), A)$$

the homomorphism induced by the inclusion $P_s(\Sigma) \hookrightarrow P_{s'}(\Sigma)$. Then it is straight-forward to check that

$$\nu^s_{F,\Sigma,A,*} = \nu^{s'}_{F,\Sigma,A,*} \circ q_{s,s',*}.$$

4.2. **Quantitative local coarse assembly maps.** With notation of Section 4.1, if Σ is a proper discrete metric space equipped with an action of a finite group F by isometries, then since the action of F preserves the filtration of $A\otimes\mathscr{K}(\ell^2(\Sigma))$, the C^*-algebra $A_{F,\Sigma}$ inherits a structure of a filtered C^*-algebra from $A \otimes \mathscr{K}(\ell^2(\Sigma))$. Our aim is to define a quantitative version of the local assembly map $\nu^s_{F,\Sigma,A,*}$. The argument of the proof of Theorem 2.21, can be easily adapted to prove the next theorem.

Theorem 4.3. *There exists a control pair $(\alpha_\mathcal{T}, k_\mathcal{T})$ such that*

- *for any proper discrete metric space Σ equipped with a free action of a finite group F by isometries,*
- *for any F-C^*-algebras A and B,*
- *for any z in $KK^F_*(A, B)$,*

there exists an $(\alpha_\mathcal{T}, k_\mathcal{T})$-controlled morphism

$$\mathcal{T}_{F,\Sigma}(z) = (\tau^{\varepsilon,r}_{F,\Sigma}(z))_{0<\varepsilon<\frac{1}{4\alpha_\mathcal{T}}} : \mathcal{K}_*(A_{F,\Sigma}) \to \mathcal{K}_*(B_{F,\Sigma})$$

that satisfies the following:

(i) *$\mathcal{T}_{F,\Sigma}(z) : \mathcal{K}_*(A_{F,\Sigma}) \to \mathcal{K}_*(B_{F,\Sigma})$ induces the right multiplication by the element $\tau_{F,\Sigma}(z) \in KK_*(A_{F,\Sigma}, B_{F,\Sigma})$, defined by equation (5), in K-theory.*

(ii) *For any elements z and z' in $KK^F_*(A, B)$, we have*

$$\mathcal{T}_{F,\Sigma}(z + z') = \mathcal{T}_{F,\Sigma}(z) + \mathcal{T}_{F,\Sigma}(z').$$

(iii) *Let A' be an F-C^*-algebra and let $f : A \to A'$ be an F-equivariant homomorphism of C^*-algebras. Then $\mathcal{T}_{F,\Sigma}(f^*(z)) = \mathcal{T}_{F,\Sigma}(z) \circ f_{F,\Sigma,*}$ for all z in $KK^F_*(A', B)$.*

(iv) *Let B' be an F-C^*-algebra and let $g : B' \to B$ be a homomorphism of C^*-algebras. Then $\mathcal{T}_{F,\Sigma}(g_*(z)) = g_{F,\Sigma,*} \circ \mathcal{T}_{F,\Sigma}(z)$ for any z in $KK^F_*(A, B')$.*

(v) *We have*

$$\mathcal{T}_{F,\Sigma}([\mathrm{Id}_A]) \overset{(\alpha_\mathcal{T},k_\mathcal{T})}{\sim} \mathcal{I}d_{\mathcal{K}_*(A_{F,\Sigma})}.$$

(vi) *For any semi-split extension of F-C^*-algebras $0 \to J \to A \to A/J \to 0$ with corresponding element $[\partial_{J,A}]$ of $KK_1(A/J, J)$ that implements the boundary map, we have*

$$\mathcal{T}_{F,\Sigma}([\partial_{J,A}]) = \mathcal{D}_{J_{F,\Sigma},A_{F,\Sigma}}.$$

We can proceed as in the proof of Theorem 2.22 to get the compatibility of $\mathcal{T}_{F,\Sigma}$ with the Kasparov product.

Theorem 4.4. *There exists a control pair (λ, h) such that the following holds: Let F be a finite group acting freely by isometries on a discrete metric space Σ and let A_1, A_2 and A_3 be F-C^*-algebras. Then for any z in $KK_*(A_1, A_2)$ and any z' in $KK_*(A_2, A_3)$, we have*

$$\mathcal{T}_{F,\Sigma}(z \otimes_{A_2} z') \overset{(\lambda,h)}{\sim} \mathcal{T}_{F,\Sigma}(z') \circ \mathcal{T}_{F,\Sigma}(z).$$

Let us set $r_{s,\varepsilon} = 2sk_{\mathcal{T},\varepsilon/\alpha_{\mathcal{T}}}$ for any ε in $(0, \frac{1}{4})$ and $s > 0$. Then for any F-C^*-algebra A and any $r \geq r_{s,\varepsilon}$, the map

$$\begin{aligned} &KK_*^F(C(X), A) \to K_*^{\varepsilon,r}(A_{F,\Sigma}),\\ &x \mapsto \big(\tau_{F,\Sigma}^{\varepsilon/\alpha_{\mathcal{T}},r/k_{\mathcal{T},\varepsilon/\alpha_{\mathcal{T}}}}(x)\big)([P_X, 0]_{\varepsilon/\alpha_{\mathcal{T}},r/k_{\mathcal{T},\varepsilon/\alpha_{\mathcal{T}}}}) \end{aligned}$$

is compatible with the inductive limit over F-invariant compact subsets of $P_s(\Sigma)$ and hence gives rise to a quantitative local coarse assembly map

$$\nu_{F,\Sigma,A,*}^{\varepsilon,r,s} : K_*^F(P_s(\Sigma), A) \to K_*^{\varepsilon,r}(A_{F,\Sigma}).$$

The quantitative local coarse assembly maps are natural in the F-algebras. It is straight-forward to check that

- $\iota_{\varepsilon,\varepsilon',r,r'}^* \circ \nu_{F,\Sigma,A,*}^{\varepsilon,r,s} = \nu_{F,\Sigma,A,*}^{\varepsilon',r',s}$ for any positive numbers ε, ε', r, r' and s such that $\varepsilon \leq \varepsilon' < \frac{1}{4}$, $r_{s,\varepsilon} \leq r$, $r_{s,\varepsilon'} \leq r'$ and $r \leq r'$;
- $\nu_{F,\Sigma,A,*}^{\varepsilon,r,s'} \circ q_{s,s',*} = \nu_{F,\Sigma,A,*}^{\varepsilon,r,s}$ for any positive numbers ε, r, s and s' such that $\varepsilon < \frac{1}{4}$, $s \leq s'$ and $r_{s',\varepsilon} \leq r$;
- $\nu_{F,\Sigma,A,*}^{s} = \iota_*^{\varepsilon,r} \circ \nu_{F,\Sigma,A,*}^{\varepsilon,r,s}$ for any positive numbers ε, r and s such that $\varepsilon < \frac{1}{4}$ and $r_{s,\varepsilon} \leq r$.

Let F be a finite subgroup of a finitely generated group Γ equipped with a right invariant metric. Let us show that

$$\nu_{F,\Gamma,A,*}^{\varepsilon,r,s} : K_*^F(P_s(\Gamma), A) \to K_*^{\varepsilon,r}(A_{F,\Gamma})$$

coincides with the composition of equation (4). Using the naturality of the map $\Phi_{\cdot,F,\Gamma} : I_F^\Gamma(\cdot) \rtimes_{\mathrm{red}} \Gamma \to \cdot_{F,\Gamma}$ and by construction of $\mathcal{T}_{F,\Gamma}$ and $\mathcal{J}_\Gamma^{\mathrm{red}}$ (see [9, §5.2]), we get the following.

Lemma 4.5. *Let F be a finite subgroup of a finitely generated discrete group Γ. Then for any F-algebras A and B and any x in $KK_*^F(A, B)$, we have*

$$\Phi_{B,F,\Gamma,*} \circ \mathcal{J}_\Gamma^{\mathrm{red}}(I_F^\Gamma(x)) = \mathcal{T}_{F,\Gamma}(x) \circ \Phi_{A,F,\Gamma,*}.$$

Proposition 4.6. *Let Γ be a finitely generated group, let F be a finite subgroup of Γ and let A be an F-C^*-algebra. Then for any positive numbers ε, r and s with $\varepsilon < \frac{1}{4}$ and $r \geq r_{s,\varepsilon}$, the following diagram is commutative:*

$$\begin{array}{ccc} KK_*^\Gamma(C_0(P_s(\Gamma)), \mathrm{I}_F^\Gamma(A)) & \xrightarrow{\mu^{s,\varepsilon,r}_{\Gamma,\mathrm{I}_F^\Gamma(A),*}} & K_*^{\varepsilon,r}(\mathrm{I}_F^\Gamma(A) \rtimes_{\mathrm{red}} \Gamma) \\ \uparrow {\scriptstyle \mathrm{I}_F^\Gamma(P_s(\Gamma))_*} & & \downarrow {\scriptstyle \Phi^{\varepsilon,r}_{A,F,\Gamma,*}} \\ K_*^F(P_s(\Gamma), A) & \xrightarrow{\nu^{\varepsilon,r,s}_{F,\Gamma,A,*}} & K_*^{\varepsilon,r}(A_{F,\Gamma}). \end{array}$$

Proof. Let us set $(\alpha, k) = (\alpha_{\mathcal{J}}, k_{\mathcal{J}}) = (\alpha_{\mathcal{T}}, k_{\mathcal{T}})$. Let X be an F-invariant compact subset of $P_s(\Gamma)$ and let x be an element of $KK_*^F(C(X), A)$. The definition of the quantitative assembly maps was recalled in Section 2.25. We have set

$$p_{\Gamma,s} : \Gamma \to C_0(P_s(\Gamma)), \quad \gamma \mapsto \lambda_e^{1/2}\lambda_\gamma^{1/2}.$$

Then

$$z_{\Gamma,s} = [p_{\Gamma,s}, 0]_{\frac{\varepsilon}{\alpha}, \frac{r}{k_{\varepsilon/\alpha}}}$$

defines an element in

$$K_0^{\frac{\varepsilon}{\alpha}, \frac{r}{k_{\varepsilon/\alpha}}}(C_0(P_s(\Gamma)) \rtimes \Gamma).$$

Moreover, we have the equalities

$$\begin{aligned} (7) \qquad & \Phi^{\varepsilon,r}_{A,F,\Gamma,*} \circ \mu^{s,\varepsilon,r}_{\Gamma,\mathrm{I}_F^\Gamma(A),*} \circ \mathrm{I}_F^\Gamma(P_s(\Gamma))_*(x) \\ &= \Phi^{\varepsilon,r}_{A,F,\Gamma,*} \circ \Big(J_\Gamma^{\mathrm{red},\frac{\varepsilon}{\alpha},\frac{r}{k_{\varepsilon/\alpha}}}(\Upsilon^{\Gamma,*}_{F,X} \circ \mathrm{I}_F^\Gamma(x))\Big)(z_{\Gamma,s}) \\ &= \Phi^{\varepsilon,r}_{A,F,\Gamma,*} \circ \Big(J_\Gamma^{\mathrm{red},\frac{\varepsilon}{\alpha},\frac{r}{k_{\varepsilon/\alpha}}}(\mathrm{I}_F^\Gamma(x))\Big) \circ \Upsilon^{\Gamma,\frac{\varepsilon}{\alpha},\frac{r}{k_{\varepsilon/\alpha}}}_{F,X,\Gamma,*}(z_{\Gamma,s}) \\ &= \mathcal{T}_{F,\Gamma}^{\frac{\varepsilon}{\alpha},\frac{r}{k_{\varepsilon/\alpha}}}(x) \circ \Phi^{\frac{\varepsilon}{\alpha},\frac{r}{k_{\varepsilon/\alpha}}}_{C(X),F,\Gamma,*} \circ \Upsilon^{\Gamma,\frac{\varepsilon}{\alpha},\frac{r}{k_{\varepsilon/\alpha}}}_{F,X,\Gamma,*}(z_{\Gamma,s}), \end{aligned}$$

where

- $\Upsilon^\Gamma_{F,X,\Gamma} : C_0(P_s) \rtimes \Gamma \to I_F^\Gamma(C(X)) \rtimes \Gamma$ is the morphism induced by $\Upsilon^\Gamma_{F,X}$;
- the second equality in (7) is a consequence of the naturality of $\mathcal{J}_\Gamma^{\mathrm{red}}$ (see Section 2);
- the third equality in (7) is a consequence of Lemma 4.5.

Since

$$\Phi^{\frac{\varepsilon}{\alpha},\frac{r}{k_{\varepsilon/\alpha}}}_{C(X),F,\Gamma,*} \circ \Upsilon^{\Gamma,\frac{\varepsilon}{\alpha},\frac{r}{k_{\varepsilon/\alpha}}}_{F,X,\Gamma,*}(z_{\Gamma,s}) = [\Phi_{C(X),F,\Gamma} \circ \Upsilon^\Gamma_{F,X,\Gamma}(p_{\Gamma,s}), 0]_{\frac{\varepsilon}{\alpha},\frac{r}{k_{\varepsilon/\alpha}}},$$

the proposition is then a consequence of the equality

$$\Phi_{C(X),F,\Gamma} \circ \Upsilon^\Gamma_{F,X,\Gamma}(p_{\Gamma,s}) = P_X,$$

where P_X is the projection of $C(X)_{F,\Sigma}$ defined by equation (6). □

4.7. **A geometric assembly map.** In order to generalize Proposition 3.7 to the setting of proper discrete metric spaces equipped with an isometric action of a finite group F, we need

- an analog of the algebra $\ell^\infty(\mathbb{N}, \mathscr{K}(\mathscr{H}) \otimes \mathrm{I}_F^\Gamma(A)) \rtimes_{\mathrm{red}} \Gamma$ for an action on a C^*-algebra A of a finite subgroup F of a finitely generated group Γ;
- an assembly map that computes its K-theory.

For a family $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ of F-C^*-algebras, let us define $\mathcal{A}_{F,\Sigma,r} = \prod_{i\in\mathbb{N}} A_{i,F,\Sigma,r}$ and let $\mathcal{A}_{F,\Sigma}$ be the closure of $\bigcup_{r>0} \mathcal{A}_{F,\Sigma,r}$ in $\prod_{i\in\mathbb{N}} A_{i,F,\Sigma}$. Then $\mathcal{A}_{F,\Sigma}$ is obviously a filtered C^*-algebra. We set for the trivial group $\mathcal{A}_{\{e\},\Sigma} = \mathcal{A}_\Sigma$ and thus, if Σ is acted upon by a finite group F by isometries, F acts on $\mathcal{A}_\Sigma$ and preserves the filtration. Clearly, $\mathcal{A}_{F,\Sigma}$ is the F-fixed points algebra of $\mathcal{A}_\Sigma$. If $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ is a family of F-C^*-algebras, we set $\mathcal{A}^\infty = (\mathscr{K}(\mathscr{H}) \otimes A_i)_{i\in\mathbb{N}}$, where $\mathscr{K}(\mathscr{H})$ is equipped with the trivial action of F. We can then define $\mathcal{A}^\infty_{F,\Sigma}$ and $\mathcal{A}^\infty_\Sigma$ from $\mathcal{A}^\infty$ as above. For an F-C^*-algebra A, we set $A^{\mathbb{N}} = (A_i)_{\in\in\mathbb{N}}$ for the constant family of F-C^*-algebras $A = A_i$ for all integers i and define from this $A^{\mathbb{N}}_{F,\Sigma}$ and $A^{\mathbb{N},\infty}_{F,\Sigma}$ as above. For any family $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ of F-C^*-algebras, let us consider the following controlled morphism:

$$\mathcal{G}_{F,\Sigma,\mathcal{A},*} = (G^{\varepsilon,r}_{F,\Sigma,\mathcal{A}})_{0<\varepsilon<\frac{1}{4},r>0} : \mathcal{K}_*(\mathcal{A}^\infty_{F,\Sigma}) \to \prod_{i\in\mathbb{N}} \mathcal{K}_*(A_{i,F,\Sigma}),$$

where

$$G^{\varepsilon,r}_{F,\Sigma,\mathcal{A},*} : K^{\varepsilon,r}_*(\mathcal{A}^\infty_{F,\Sigma}) \to \prod_{i\in\mathbb{N}} K^{\varepsilon,r}_*(A_{i,F,\Sigma})$$

is the map induced on the jth factor and up to the Morita equivalence by the restriction to $\mathcal{A}^\infty_{F,\Sigma}$ of the evaluation $\prod_{i\in\mathbb{N}} \mathscr{K}(\mathscr{H}) \otimes A_{i,F,\Sigma} \to \mathscr{K}(\mathscr{H}) \otimes A_{j,F,\Sigma}$ at $j \in \mathbb{N}$. As a consequence of Lemma 2.14, we have the following.

Lemma 4.8. *There exists a control pair (α, h) such that*

- *for any finite group F,*
- *for any proper discrete metric space Σ provided with an action of F by isometries,*
- *for any family $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ of F-algebras,*

the controlled morphism

$$\mathcal{G}_{F,\Sigma,\mathcal{A},*} : \mathcal{K}_*(\mathcal{A}^\infty_{F,\Sigma}) \to \prod_{i\in\mathbb{N}} \mathcal{K}_*(A_{i,F,\Sigma})$$

is an (α, h)-controlled isomorphism.

For any families of F-C^*-algebras $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ and $\mathcal{B} = (B_i)_{i\in\mathbb{N}}$ of F-C^*-algebras and any family $f = (f_i : A_i \to B_i)_{i\in\mathbb{N}}$ of F-equivariant homomorphisms, let us set

$$f_{F,\Sigma} = \prod_{i\in\mathbb{N}} f_{i,F,\Sigma} : \mathcal{A}_{F,\Sigma} \to \mathcal{B}_{F,\Sigma}$$

and

$$f^\infty_{F,\Sigma} = \prod_{i\in\mathbb{N}} \mathrm{Id}_{\mathscr{K}(\mathscr{H})} \otimes f_{i,F,\Sigma} : \mathcal{A}^\infty_{F,\Sigma} \to \mathcal{B}^\infty_{F,\Sigma}.$$

Then together with Theorem 4.3, Lemma 4.8 yields the following.

Corollary 4.9. *There exists a control pair* (α, h) *such that*

- *for any proper discrete metric space* Σ *equipped with a free action of a finite group* F *by isometries,*
- *for any families of* F*-*C^**-algebras* $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ *and* $\mathcal{B} = (B_i)_{i\in\mathbb{N}}$*,*
- *for any* $z = (z_i)_{i\in\mathbb{N}}$ *in* $\prod_{i\in\mathbb{N}} KK_*^F(A_i, B_i)$*,*

there exists an (α, h)*-controlled morphism*

$$\mathcal{T}_{F,\Sigma}^{\infty}(z) = (\tau_{F,\Sigma}^{\infty,\varepsilon,r}(z))_{0<\varepsilon<\frac{1}{4\alpha},r>0} : \mathcal{K}_*(\mathcal{A}_{F,\Sigma}^{\infty}) \to \mathcal{K}_*(\mathcal{B}_{F,\Sigma}^{\infty})$$

that satisfies the following:

(i) *For any elements* $z = (z_i)_{i\in\mathbb{N}}$ *and* $z' = (z'_i)_{i\in\mathbb{N}}$ *in* $\prod_{i\in\mathbb{N}} KK_*^F(A_i, B_i)$*, we have*

$$\mathcal{T}_{F,\Sigma}^{\infty}(z + z') = \mathcal{T}_{F,\Sigma}^{\infty}(z) + \mathcal{T}_{F,\Sigma}^{\infty}(z')$$

for $z + z' = (z_i + z'_i)_{i\in\mathbb{N}}$*.*

(ii) *For* $\mathcal{A}' = (A'_i)_{i\in\mathbb{N}}$ *a family of* F*-*C^**-algebras and* $f = (f_i : A'_i \to A_i)_{i\in\mathbb{N}}$ *a family of* F*-equivariant homomorphisms of* C^**-algebras, we have*

$$\mathcal{T}_{F,\Sigma}^{\infty}(f^*(z)) = \mathcal{T}_{F,\Sigma}^{\infty}(z) \circ f_{F,\Sigma,*}^{\infty},$$

where $f^*(z) = (f_i^*(z_i))_{i\in\mathbb{N}}$*.*

(iii) *For* $\mathcal{B}' = (B_i)_{i\in\mathbb{N}}$ *a family of* F*-*C^**-algebras and* $g = (g_i : B_i \to B'_i)_{i\in\mathbb{N}}$ *a family of* F*-equivariant homomorphism of* C^**-algebras, we have*

$$\mathcal{T}_{F,\Sigma}^{\infty}(g_*(z)) = g_{F,\Sigma,*}^{\infty} \circ \mathcal{T}_{F,\Sigma}^{\infty}(z),$$

where $g_*(z) = (g_{i,*}(z_i))_{i\in\mathbb{N}}$*.*

(iv) *If we set* $\mathrm{Id}_{\mathcal{A}} = (\mathrm{Id}_{A_i})_{i\in\mathbb{N}}$*, then*

$$\mathcal{T}_{F,\Sigma}^{\infty}([\mathrm{Id}_{\mathcal{A}}]) \overset{(\alpha,k)}{\sim} \mathcal{I}d_{\mathcal{K}_*(\mathcal{A}_{F,\Sigma}^{\infty})}.$$

(v) *For any family of semi-split extensions of* F*-*C^**-algebras*

$$0 \to J_i \to A_i \to A_i/J_i \to 0$$

with corresponding element $[\partial_{J_i,A_i}]$ *of* $KK_1(A_i/J_i, J_i)$ *that implements the boundary maps, let us set* $\mathcal{J} = (J_i)_{i\in\mathbb{N}}$*,* $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$*,* $\mathcal{A}/\mathcal{J} = (A_i/J_i)_{i\in\mathbb{N}}$ *and* $[\partial_{\mathcal{J},\mathcal{A}}] = ([\partial_{J_i,A_i}])_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} KK_1^{\Gamma}(A_i/J_i, J_i)$*. Then we have*

$$\mathcal{T}_{F,\Sigma}^{\infty}([\partial_{\mathcal{J},\mathcal{A}}]) = \mathcal{D}_{\mathcal{J}_{F,\Sigma}^{\infty},\mathcal{A}_{F,\Sigma}^{\infty}}.$$

The following proposition is a consequence of Theorem 4.4 and Lemma 4.8.

Proposition 4.10. *There exists a control pair* (λ, h) *such that the following holds: Let* F *be a finite group acting freely by isometries on a discrete metric space* Σ *and let* $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$*,* $\mathcal{B} = (B_i)_{i\in\mathbb{N}}$ *and* $\mathcal{B}' = (B'_i)_{i\in\mathbb{N}}$ *be families of* F*-*C^**-algebras. Let us set* $z \otimes_{\mathcal{B}} z' = (z_i \otimes_{B_i} z'_i)_{i\in\mathbb{N}}$ *for any* $z = (z_i)_{i\in\mathbb{N}}$ *in* $\prod_{i\in\mathbb{N}} KK_*^F(A_i, B_i)$ *and any* $z' = (z'_i)_{i\in\mathbb{N}}$ *in* $\prod_{i\in\mathbb{N}} KK_*^F(B_i, B'_i)$*. Then we have*

$$\mathcal{T}_{F,\Sigma}^{\infty}(z \otimes_{\mathcal{B}} z') \overset{(\lambda,h)}{\sim} \mathcal{T}_{F,\Sigma}^{\infty}(z') \circ \mathcal{T}_{F,\Sigma}^{\infty}(z).$$

If F is a finite group and if $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ is a family of F-C^*-algebras, let us consider the family $\mathcal{A}\otimes\mathscr{K}(\ell^2(F)) = (A_i\otimes\mathscr{K}(\ell^2(F)))_{i\in\mathbb{N}}$, provided by the diagonal action of F where the action on $\mathscr{K}(\ell^2(F))$ is induced with the right regular representation. If moreover F acts on Σ by isometries, $\mathcal{A}^\infty_\Sigma$ is indeed an F-C^*-algebra and we have a natural identification of filtered C^*-algebras

$$\mathcal{A}^\infty_\Sigma \rtimes F \cong (\mathcal{A}\otimes\mathscr{K}(\ell^2(F)))^\infty_{F,\Sigma}, \tag{8}$$

where $\mathcal{A}^\infty_\Sigma\rtimes F$ is filtered by $(C(F,\mathcal{A}^\infty_{\Sigma,r}))_{r>0}$. Applying Proposition 4.10 to the family $M_{\mathcal{A},F} = (M_{A_i,F})_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} KK^F_*(A_i, A_i\otimes\mathscr{K}(\ell^2(F)))$ of F-equivariant Morita equivalences, we get the following lemma.

Lemma 4.11. *There exists a control pair (α,h) such that for any finite group F, any family $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ of F-C^*-algebras and any discrete metric space Σ equipped with a free action of F by isometries, we have that, under the identification of equation* (8),

$$\mathcal{M}^\infty_{\mathcal{A},F} := \mathcal{T}^\infty_{F,\Sigma}(M_\mathcal{A}) : \mathcal{K}_*(\mathcal{A}^\infty_{F,\Sigma}) \to \mathcal{K}_*(\mathcal{A}^\infty_\Sigma\rtimes F)$$

is an (α,h)-controlled isomorphism.

Recall that to any F-invariant compact subset X of $P_s(\Sigma)$ a projection P_X of $C(X)_{F,\Sigma}$ is associated. Indeed $P_X(x)$ is for every x in X the matrix with almost all vanishing entries indexed by $\Sigma\times\Sigma$ defined by $P_X(x)_{\sigma,\sigma'} = \lambda_\sigma(x)^{1/2}\lambda_{\sigma'}(x)^{1/2}$ (recall that $(\lambda_\sigma)_{\sigma\in\Sigma}$ is the set of coordinate functions on $P_r(\Sigma)$). For any family $\mathcal{X} = (X_i)_{i\in\mathbb{N}}$ of compact F-invariant subsets of $P_s(\Sigma)$, let us set $\mathcal{C}_\mathcal{X} = (C(X_i))_{i\in\mathbb{N}}$ and consider the projection $P^\infty_\mathcal{X} = (P_{X_i}\otimes e)_{i\in\mathbb{N}}$ of $\mathcal{C}^\infty_{\mathcal{X},F,\Sigma}$, where e is a fixed rank-one projection of $\mathscr{K}(\mathscr{H})$. The propagation of $P^\infty_\mathcal{X}$ is less than $2s$. Hence for the control pair (α,h) of Corollary 4.9, any family $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ of F-C^*-algebras, any $\varepsilon\in(0,\frac{1}{4})$, any $s>0$ and any $r\geq r_{s,\varepsilon}$, the map

$$\prod_{i\in\mathbb{N}} KK^F_*(C(X_i),A_i) \to K^{\varepsilon,r}_*(\mathcal{A}^\infty_{F,\Sigma}),\quad z\mapsto \tau^{\infty,\varepsilon/\alpha,r/h_{\varepsilon/\alpha}}_{F,\Sigma}(z)(P^\infty_\mathcal{X})$$

is compatible with inductive limit of families $\mathcal{X} = (X_i)_{i\in\mathbb{N}}$ of compact F-invariant subsets of $P_s(\Sigma)$. By composition with the controlled isomorphism

$$\mathcal{T}^\infty_{F,\Sigma}(M_\mathcal{A}) : \mathcal{K}_*(\mathcal{A}^\infty_{F,\Sigma}) \to \mathcal{K}_*(\mathcal{A}^\infty_\Sigma\rtimes F),$$

we get for a function $(0,\frac{1}{4})\times(0,\infty)\to(0,\infty)$, $(\varepsilon,s)\mapsto r_{s,\varepsilon}$ non-decreasing in s, non-increasing in ε and independent of F, Σ and $\mathcal{A}$ and for any ε in $(0,\frac{1}{4})$, any positive numbers s and r such that $r\geq r_{s,\varepsilon}$ a quantitative geometric assembly map

$$\nu^{\infty,\varepsilon,r,s}_{F,\Sigma,\mathcal{A},*} : \prod_{i\in\mathbb{N}} K^F_*(P_s(\Sigma),A_i) \to K^{\varepsilon,r}_*(\mathcal{A}^\infty_\Sigma\rtimes F).$$

Therefore, for s a fixed positive number, the family of maps $(\nu^{\infty,\varepsilon,r,s}_{F,\Sigma,\mathcal{A},*})_{\varepsilon>0,r\geq r_{s,\varepsilon}}$ gives rise to a geometric assembly map

$$\nu^{\infty,s}_{F,\Sigma,\mathcal{A},*} : \prod_{i\in\mathbb{N}} K^F_*(P_s(\Sigma),A_i) \to K_*(\mathcal{A}^\infty_\Sigma\rtimes F)$$

uniquely defined by $\nu^{\infty,s}_{F,\Sigma,\mathcal{A},*} = \iota^{\varepsilon,r}_* \circ \nu^{\infty,\varepsilon,r,s}_{F,\Sigma,\mathcal{A},*}$ for any positive numbers ε, r and s such that $\varepsilon < \frac{1}{4}$ and $r \geq r_{s,\varepsilon}$.

The quantitative assembly maps $\nu^{\infty,\varepsilon,r,s}_{F,\Sigma,\mathcal{A},*}$ are compatible with inclusions of Rips complexes: let

$$q^{\infty}_{s,s',*} : \prod_{i\in\mathbb{N}} K^F_*(P_s(\Sigma), A_i) \to \prod_{i\in\mathbb{N}} K^F_*(P_{s'}(\Sigma), A_i)$$

be the map induced by the inclusion $P_s(\Sigma) \hookrightarrow P_{s'}(\Sigma)$, then we have

$$\nu^{\infty,\varepsilon,r,s'}_{F,\Sigma,\mathcal{A},*} \circ q^{\infty}_{s,s',*} = \nu^{\infty,\varepsilon,r,s}_{F,\Sigma,\mathcal{A},*}$$

for any positive numbers ε, s, s', and r such that $\varepsilon \in (0, \frac{1}{4})$, $s \leq s'$, $r \geq r_{s',\varepsilon}$, and thus

$$\nu^{\infty,s'}_{F,\Sigma,\mathcal{A},*} \circ q^{\infty}_{s,s',*} = \nu^{\infty,s}_{F,\Sigma,\mathcal{A},*}$$

for any positive numbers s and s' such that $s \leq s'$.

Eventually, we can take the inductive limit over the degree of the Rips complex and set

$$\begin{aligned} K^{\mathrm{top},\infty}_*(F, \Sigma, \mathcal{A}) &= \lim_{s>0,} \prod_{i\in\mathbb{N}} K^F_*(P_s(\Sigma), A_i) \\ &= \lim_{s>0,(X^s_i)_{i\in\mathbb{N}}} \prod_{i\in\mathbb{N}} KK^F_*(C(X^s_i), A_i), \end{aligned}$$

where in the inductive limit on the right-hand side, s runs through positive numbers and $(X^s_i)_{i\in\mathbb{N}}$ runs through families of F-invariant compact subset of $P_s(\Sigma)$. We get then an assembly map

$$\nu^{\infty}_{F,\Sigma,\mathcal{A},*} : K^{\mathrm{top},\infty}_*(F, \Sigma, \mathcal{A}) \to K_*(\mathcal{A}^{\infty}_{\Sigma} \rtimes F). \tag{9}$$

4.12. **The groupoid approach.** In order to generalize the proof of Proposition 3.7 in the setting of a discrete metric space, our purpose in this section is to follow the route of [11] and to show that if $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ is a family of C^*-algebras, then $\mathcal{A}^{\infty}_{\Sigma}$ is the reduced crossed product of the algebra $\prod_{i\in\mathbb{N}} C_0(\Sigma, A_i \otimes \mathcal{K}(H))$ by the diagonal action of the groupoid attached to the coarse structure of the discrete metric space Σ.

The coarse groupoid G_Σ associated to a discrete metric space Σ with bounded geometry was introduced in [11]. The groupoid G_Σ has the Stone–Čech compactification β_Σ of Σ as unit space, and the Roe algebra of Σ is the reduced crossed product of $\ell^\infty(\Sigma, \mathscr{K}(\mathscr{H}))$ by an action of G_Σ. Let us describe the construction of this groupoid. If (Σ, d) is a discrete metric space with bounded geometry. Then a subset E of $\Sigma \times \Sigma$ is called an entourage for Σ if there exists $r > 0$ such that

$$E \subseteq \big\{(x, y) \in \Sigma \times \Sigma \mid d(x, y) < r\big\}.$$

If E is an entourage for Σ, denote by $\bar{E}$ its closure in the Stone–Čech compactification $\beta_{\Sigma\times\Sigma}$ of $\Sigma \times \Sigma$. Then there is a unique structure of a groupoid on

$$G_\Sigma = \bigcup_{E \text{ entourage}} \bar{E} \subseteq \beta_{\Sigma\times\Sigma}$$

with the Stone–Čech compactification β_Σ of Σ being the unit space which extends the pair groupoid $\Sigma\times\Sigma$. Let $o: G_\Sigma\to\beta_\Sigma$ and $t: G_\Sigma\to\beta_\Sigma$ be respectively the source (origin) and range (target) map. For any $C(\beta_\Sigma)$-algebra B, let $o^*B=C_0(G_\Sigma)\otimes_o B$ and $t^*B=C(G_\Sigma)\otimes_t B$ be respectively the balanced product of $C_0(G_\Sigma)$ and B relatively to the $C(\beta_\Sigma)$-algebra structures on B induced by o and t. If g is a continuous function in $C_0(G_\Sigma)$ and b is an element in B, we denote respectively by $g\otimes_o b$ and $g\otimes_t b$ the elementary tensors of g and b in o^*B and t^*B. For a family $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ of C^*-algebras, let us set $\mathcal{A}_{C_0(\Sigma)}=\prod_{i\in\mathbb{N}}C_0(\Sigma,A_i)$. Then the diagonal action of $\ell^\infty(\Sigma)$ by multiplication clearly provides $\mathcal{A}_{C_0(\Sigma)}$ with a structure of $C(\beta_\Sigma)$-algebra. Our aim is to show that G_Σ acts diagonally on $\mathcal{A}_{C_0(\Sigma)}$ and that $\mathcal{A}_{C_0(\Sigma)}\rtimes_{\mathrm{red}}G_\Sigma$ is canonically isomorphic to $\mathcal{A}_\Sigma$.

Let $C_0(G_\Sigma,\mathcal{A})$ be the closure in $\prod_{i\in\mathbb{N}}C_0(\Sigma\times\Sigma,A_i)$ of

$$\big\{(f_i)_{i\in\mathbb{N}}\mid \exists r>0,\ \forall i\in\mathbb{N},\ \forall(\sigma,\sigma')\in\Sigma^2,\ d(\sigma,\sigma')>r\Rightarrow f_i(\sigma,\sigma')=0\big\}.$$

For an entourage E and an element $f=(f_i)_{i\in\mathbb{N}}$ of $\mathcal{A}_{C_0(\Sigma)}$, let us define

$$f_t^E=(f_{t,i}^E)_{i\in\mathbb{N}}\quad\text{and}\quad f_o^E=(f_{o,i}^E)_{i\in\mathbb{N}}$$

by

$$f_{t,i}^E(\sigma,\sigma')=\chi_E(\sigma,\sigma')f_i(\sigma)\quad\text{and}\quad f_{o,i}^E(\sigma,\sigma')=\chi_E(\sigma,\sigma')f_i(\sigma')$$

for any integer i and any σ and σ' in Σ.

Lemma 4.13. *Let $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ be a family of C^*-algebras. Then we have isomorphisms of $C(G_\Sigma)$-algebras*

$$\Psi_t: t^*\mathcal{A}_{C_0(\Sigma)}\to C_0(G_\Sigma,\mathcal{A})$$

and

$$\Psi_o: o^*\mathcal{A}_{C_0(\Sigma)}\to C_0(G_\Sigma,\mathcal{A})$$

only defined by $\Psi_t(\chi_E\otimes_t f)=f_r^E$ and $\Psi_o(\chi_E\otimes_o f)=f_o^E$ for any f in $\mathcal{A}_{C_0(\Sigma)}$ and any entourage E for Σ.

Proof. It is clear that the definitions of Ψ_t and Ψ_o by the formulas above are consistent. Moreover, Ψ_t and Ψ_o are isometries. Let us prove for instance that Ψ_t is an isomorphism. Surjectivity of Ψ_t amounts to proving that for any $(h_i)_{i\in\mathbb{N}}$ in $\prod_{i\in\mathbb{N}}C_0(\Sigma\times\Sigma,A_i)$ and any entourage E, the family of maps $h=(\chi_E h_i)_{i\in\mathbb{N}}$ is in the range of Ψ_t. According to [11, Lem. 2.7], we can assume that the restrictions $s:E\to\Sigma$ and $r:E\to\Sigma$ are one-to-one. For any integer i, then define $f_i:\Sigma\to A_i$ by

$$f_i(\sigma)=\begin{cases}h_i(\sigma,\sigma') & \text{if there exists } \sigma' \text{ such that } (\sigma,\sigma') \text{ is in } E,\\ 0 & \text{otherwise.}\end{cases}$$

Then f_i is in $C_0(\Sigma,A_i)$ for every integer i and if we set $f=(f_i)_{i\in\mathbb{N}}$, then $f_t^E=h$ and hence h is in the range of Ψ_t. □

Let us define $V_\Sigma = \Psi_t \circ \Psi_o^{-1}$. Then $V_\Sigma : o^*\mathcal{A}_{C_0(\Sigma)} \to s^*\mathcal{A}_{C_0(\Sigma)}$ is an isomorphism of $C(G_\Sigma)$-algebras that can be described on elementary tensors as follows. For an entourage E such that the restrictions $o : E \to \Sigma$ and $t : E \to \Sigma$ are one-to-one, for every σ in $t(E)$ there exists a unique σ' in $o(E)$ such that (σ, σ') is in E. For any $f = (f_i)_{i\in\mathbb{N}}$ in $\mathcal{A}_{C_0(\Sigma)}$, we define $E \circ f = (E \circ f_i)_{i\in\mathbb{N}}$ in $\mathcal{A}_{C_0(\Sigma)}$, where for any integer i,

$$E \circ f_i(\sigma) = \begin{cases} f_i(\sigma') & \text{if } \sigma \text{ is in } r(E) \text{ and } (\sigma,\sigma') \text{ is in } E, \\ 0 & \text{otherwise.} \end{cases}$$

Then under the above assumptions, we have $V_\Sigma(\chi_E \otimes_o f) = \chi_E \otimes_t E \circ f$.

Lemma 4.14. *For every family $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ of C^*-algebras,*

$$V_\Sigma : o^*\mathcal{A}_{C_0(\Sigma)} \to t^*\mathcal{A}_{C_0(\Sigma)}$$

is an action of the groupoid G_Σ on $\mathcal{A}_{C_0(\Sigma)}$.

Proof. For an element γ in G_Σ, let

$$V_{\Sigma,\gamma} : \mathcal{A}_{C_0(\Sigma)o(\gamma)} \to \mathcal{A}_{C_0(\Sigma)t(\gamma)}$$

be the map induced by V_Σ on the fiber of $\mathcal{A}_{C_0(\Sigma)}$ at $o(\gamma)$. Let γ and γ' be elements in G_Σ such that $o(\gamma) = t(\gamma')$. Let E and E' be entourages such that the restrictions of o and t to E and E' are one-to-one and such that $\gamma \in \bar{E}$ and $\gamma' \in \bar{E}'$. Let us set

$$E \circ E' = \big\{(\sigma, \sigma'') \in \Sigma \times \Sigma \mid \exists\, \sigma' \in \Sigma,\, (\sigma,\sigma') \in E \text{ and } (\sigma',\sigma'') \in E'\big\}.$$

Then $\gamma \cdot \gamma'$ is in $\overline{E \circ E'}$ and the restrictions of o and s to $E \circ E'$ are one-to-one. Moreover, we clearly have $(E \circ E') \circ f = E \circ (E' \circ f)$ for all f in $\mathcal{A}_{C_0(\Sigma)}$. Hence, we get

$$\begin{aligned} V_{\Sigma,\gamma\cdot\gamma'}(f_{o(\gamma')}) &= (E \circ E' \circ f)_{t(\gamma)} \\ &= V_{\Sigma,\gamma}((E' \circ f)_{o(\gamma)}) \\ &= V_{\Sigma,\gamma}((E' \circ f)_{t(\gamma')}) \\ &= V_{\Sigma,\gamma} \circ V_{\Sigma,\gamma'}(f_{o(\gamma)}). \end{aligned}$$

□

Proposition 4.15. *Let Σ be a discrete metric space with bounded geometry and let $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ be a family of C^*-algebras. Then we have a natural isomorphism*

$$\mathcal{I}_{\Sigma,\mathcal{A}} : \mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G_\Sigma \xrightarrow{\cong} \mathcal{A}_\Sigma.$$

Proof. Following the proof of [11, Lem. 4.4], we obviously have that $J_{\Sigma,A} = \bigoplus_{i\in\mathbb{N}} C_0(\Sigma, A_i)$ is a G_Σ-invariant ideal of $\mathcal{A}_{C_0(\Sigma)}$. For any σ' in Σ, we have at any element of Σ a canonical identification of the fiber of $J_{\Sigma,A}$ with $\bigoplus_{i\in\mathbb{N}} A_i$ and under this identification, the action of $\Sigma \times \Sigma \subseteq G_\Sigma$ on $J_{\Sigma,A}$ is trivial. According to [11, Lem. 4.3], the reduced crossed product $\mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G_\Sigma$ is faithfully represented in the right $J_{\Sigma,A}$-Hilbert module

$$L^2(G_\Sigma, J_{\Sigma,A}) \cong L^2(G_\Sigma, \mathcal{A}_{C_0(\Sigma)}) \otimes_{\mathcal{A}_{C_0(\Sigma)}} J_{\Sigma,A}.$$

But we have a natural identification of $J_{\Sigma,A}$-right Hilbert modules

$$L^2(G_\Sigma, J_{\Sigma,A}) \cong C_0\Big(\Sigma, \Big(\bigoplus_{i\in\mathbb{N}} A_i\Big) \otimes \ell^2(\Sigma)\Big).$$

Under this identification, the representation of $\mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G_\Sigma$ indeed arises from a pointwise action on $(\bigoplus_{i\in\mathbb{N}} A_i) \otimes \ell^2(\Sigma)$. As such, the underlying representation of $\mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G_\Sigma$ on $(\bigoplus_{i\in\mathbb{N}} A_i) \otimes \ell^2(\Sigma)$ is faithful. Let us describe this action.

- An element $f = (f_i)_{i\in\mathbb{N}}$ in $\mathcal{A}_{C_0(\Sigma)} \cong \prod_{i\in\mathbb{N}} A_i \otimes C_0(\Sigma)$ acts on $(\bigoplus_{i\in\mathbb{N}} A_i) \otimes \ell^2(\Sigma)$ in the obvious way.
- If E is an entourage, then the action of χ_E on $(\bigoplus_{i\in\mathbb{N}} A_i) \otimes \ell^2(\Sigma)$ is defined by pointwise multiplication by $\mathrm{Id}_{\bigoplus_{i\in\mathbb{N}} A_i} \otimes T_E$, where the operator T_E is defined by $T_{E,\sigma,\sigma'} = \chi_E(\sigma, \sigma')$ for any σ and σ' in Σ.

The algebra $\mathcal{A}_\Sigma$ acts also faithfully on $(\bigoplus_{i\in\mathbb{N}} A_i) \otimes \ell^2(\Sigma)$ by pointwise action at each integer i of $A_i \otimes \mathscr{K}(\ell^2(\Sigma))$ on $A_i \otimes \ell^2(\Sigma)$. It is then clear that if f is in $\mathcal{A}_{C_0(\Sigma)}$ and E is an entourage, then fT_E is in $\mathcal{A}_\Sigma$. Conversely, let us show any element in A_Σ acts on $(\bigoplus_{i\in\mathbb{N}} A_i) \otimes \ell^2(\Sigma)$ as an element of $\mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G_\Sigma$. Let $(T_i)_{i\in\mathbb{N}}$ be an element of $A_{\Sigma,r}$. We can assume that for every integer i, there exists a finite subset X_i of Σ such that $T_i = (T_{i,\sigma,\sigma'})_{(\sigma,\sigma')\in\Sigma^2}$ lies indeed in $A_i \otimes \mathscr{K}(\ell^2(X_i))$. Applying [11, Lem. 2.7] to the union of the support of the T_i when i runs through integers, we can actually assume without loss of generality that there exists an entourage E such that

- the restrictions of o and t to E are one-to-one;
- for any integer i and any σ and σ' in Σ, the inequality $T_{i,\sigma,\sigma'} \neq 0$ implies that (σ, σ') is in E.

Define then, for any integer i,

$$f_i(\sigma) = \begin{cases} T_{i,\sigma,\sigma'} & \text{if there exists } \sigma' \in X_i \text{ such that } (\sigma, \sigma') \in E \cap (X_i \times X_i), \\ 0 & \text{otherwise.} \end{cases}$$

Then f_i is in $C_0(\Sigma, A_i)$ for every integer i and if we set $f = (f_i)_{i\in\mathbb{N}}$, then fT_E acts on $(\bigoplus_{i\in\mathbb{N}} A_i) \otimes \ell^2(\Sigma)$ as $(T_i)_{i\in\mathbb{N}}$. □

If Σ is equipped with an action of a finite group F by isometries, then the diagonal action of F on Σ induces an action of F on G_Σ by automorphisms of groupoids. Moreover, for any family $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ of F-C^*-algebras, the action of G_Σ on $\mathcal{A}_{C_0(\Sigma)} = \prod_{i\in\mathbb{N}} C_0(\Sigma, A_i)$ is covariant with respect to the pointwise diagonal action of F. Hence, we end up in this way with an action of F on $\mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G_\Sigma$ by automorphisms. Namely, let us consider the semi-direct product groupoid

$$G_{F,\Sigma} = G_\Sigma \rtimes F = \{(\gamma, x) \in G_\Sigma \times F\}$$

provided with the source map

$$G_{F,\Sigma} \to \beta_\Sigma, \quad (\gamma, x) \mapsto o(x^{-1}(\gamma))$$

and range map

$$G_{F,\Sigma} \to \beta_\Sigma, \quad (\gamma, x) \mapsto t(\gamma)$$

and composition rule $(\gamma, x)\cdot(\gamma', x') = (\gamma \cdot x(\gamma'), xx')$ if $o(x^{-1}(\gamma)) = t(\gamma')$. Then $\mathcal{A}_{C_0(\Sigma)}$ is actually a $G_{F,\Sigma}$-C^*-algebra and we have a natural identification

$$(10) \qquad (\mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G_\Sigma) \rtimes F \cong \mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G_{F,\Sigma}.$$

On the other hand, F also acts for each integer i on $\mathscr{K}(\ell^2(\Sigma)) \otimes A_i$ and hence pointwise on $\mathcal{A}_\Sigma$. The isomorphism of Proposition 4.15 is then clearly F-equivariant and hence gives rise under the identification of equation (10) to an isomorphism

$$(11) \qquad \mathcal{I}_{F,\Sigma,\mathcal{A}} : \mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G_{F,\Sigma} \xrightarrow{\cong} \mathcal{A}_\Sigma \rtimes F.$$

Since $C(\beta_{\mathbb{N}\times\Sigma}) \cong \ell^\infty(\mathbb{N} \times \Sigma)$, the C^*-algebra $\mathcal{A}_{C_0(\Sigma)}$ is a $C(\beta_{\mathbb{N}\times\Sigma})$-algebra for any family $\mathcal{A}$. Let us show that $\beta_{\mathbb{N}\times\Sigma}$ is actually provided with an action of G_Σ on the right that makes $\mathcal{A}_{C_0(\Sigma)}$ into a $\beta_{\mathbb{N}\times\Sigma} \rtimes G_\Sigma$-algebra.

Let $p : \beta_{\mathbb{N}\times\Sigma} \to \beta_\Sigma$ be the (only) map extending the projection $\mathbb{N} \times \Sigma \to \Sigma$ by continuity. Let x be an element of β_Σ, let γ be an element of G_Σ such that $t(\gamma) = x$ and let $E \subseteq \Sigma \times \Sigma$ be an entourage such that

- γ belongs to $\bar{E}$;
- the restrictions of o and t to E are one-to-one.

Let $(n_k, \sigma_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{N} \times \Sigma$ converging to some z in $\beta_{\mathbb{N}\times\Sigma}$ and such that σ_k is in $t(E)$ for every integer k. For any integer k, let σ'_k be the unique element of $o(E)$ such that (σ_k, σ'_k) is in E. Then the sequence $(n_k, \sigma'_k)_{k\in\mathbb{N}}$ converges in $\beta_{\mathbb{N}\times\Sigma}$ to an element z' such that $p(z') = o(\gamma)$. This limit does not depend on the choice of E and $(n_k, \sigma_k)_{k\in\mathbb{N}}$ that satisfy the conditions above. If we set $z \cdot \gamma = z'$, we obtain an action of G_Σ on $\beta_{\mathbb{N}\times\Sigma}$ on the left. Obviously, the restriction of $\beta_{\mathbb{N}\times\Sigma} \rtimes G_\Sigma$ to the saturated open subset $\mathbb{N} \times \Sigma$ of $\beta_{\mathbb{N}\times\Sigma}$ is the union of pair groupoids on $\{n\} \times \Sigma$. If $\mathcal{A}$ is a family of C^*-algebras, the multiplier action of $C(\beta_{\mathbb{N}\times\Sigma})$ is G_Σ-equivariant and hence we end up with an action of $\beta_{\mathbb{N}\times\Sigma} \rtimes G_\Sigma$ on $\mathcal{A}_{C_0(\Sigma)}$.

If Σ is endowed with an action of a finite group F by isometries, then the diagonal action of F on $\mathbb{N} \times \Sigma$ (trivial on $\mathbb{N}$) gives rise to an action of F on $\beta_{\mathbb{N}\times\Sigma}$ by homeomorphisms which makes the action of G_Σ covariant. Hence $\beta_{\mathbb{N}\times\Sigma}$ is provided with an action of $G_{F,\Sigma} = G_\Sigma \rtimes F$. Moreover, if $\mathcal{A}$ is a family of F-C^*-algebras, then $\mathcal{A}_{C_0(\Sigma)}$ is a $\beta_{\mathbb{N}\times\Sigma} \rtimes G_{F,\Sigma}$-algebra.

Consider now the spectrum $\beta^0_{\mathbb{N}\times\Sigma}$ of the ideal $\ell^\infty(\mathbb{N}, C_0(\Sigma))$ of $C(\beta_{\mathbb{N}\times\Sigma}) \cong \ell^\infty(\mathbb{N} \times \Sigma)$. Then $\beta^0_{\mathbb{N}\times\Sigma}$ is a saturated open subset of $\beta_{\mathbb{N}\times\Sigma}$, the pointwise multiplication of $\ell^\infty(\mathbb{N}, C_0(\Sigma))$ on $\mathcal{A}_{C_0(\Sigma)} = \prod_{i\in\mathbb{N}} C_0(\Sigma, A_i)$ provides $\mathcal{A}_{C_0(\Sigma)}$ with a structure of $C(\beta^0_{\mathbb{N}\times\Sigma})$-algebra and thus we see that $\mathcal{A}_{C_0(\Sigma)}$ is indeed a $\beta^0_{\mathbb{N}\times\Sigma} \rtimes G_\Sigma$-algebra. The balanced products of $\mathcal{A}_{C_0(\Sigma)}$ respectively with $C_0(G_\Sigma)$, $C_0(\beta_{\mathbb{N}\times\Sigma} \rtimes G_\Sigma)$ and $C_0(\beta^0_{\mathbb{N}\times\Sigma} \rtimes G_\Sigma)$ over the origin map coincide (using the $C(\beta_{\mathbb{N}\times\Sigma})$-linearity). The same holds for the balanced products over the target map. Hence, by construction of the reduced groupoid, the three crossed products $\mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G_\Sigma$, $\mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} (\beta_{\mathbb{N}\times\Sigma} \rtimes G_\Sigma)$ and $\mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} (\beta^0_{\mathbb{N}\times\Sigma} \rtimes G_\Sigma)$

coincide. If Σ is equipped with an action of a finite group F by isometries, then $\beta^0_{\mathbb{N}\times\Sigma}$ is F-invariant and hence endowed with an action of $G_{F,\Sigma}$. Moreover, for any family $\mathcal{A}$ of F-C^*-algebras, $\mathcal{A}_{C_0(\Sigma)}$ is a $\beta^0_{\mathbb{N}\times\Sigma} \rtimes G_{F,\Sigma}$-algebra. Let us set then $G^{\mathbb{N}}_\Sigma$ (resp. $G^{0,\mathbb{N}}_\Sigma$) for the groupoid $\beta_{\mathbb{N}\times\Sigma} \rtimes G_\Sigma$ (resp. $\beta^0_{\mathbb{N}\times\Sigma} \rtimes G_\Sigma$), and if Σ is provided with an action of a finite group F by isometries, set then $G^{\mathbb{N}}_{F,\Sigma} = G^{\mathbb{N}}_\Sigma \rtimes F$.

Lemma 4.16. *Let E be a subset of $\mathbb{N}\times\Sigma\times\Sigma$ and assume that there exists $r > 0$ such that for all integers i and all σ and σ' in Σ, the element (i,σ,σ') in E implies that $d(\sigma,\sigma') < r$. Then there exist*

- *$f_1,\dots,f_k$ in $\ell^\infty(\mathbb{N}\times\Sigma)$;*
- *$E_1,\dots,E_k$ entourages of Σ included in $\bigcup_{i\in\mathbb{N}}\{(\sigma,\sigma')\in\Sigma^2 \mid (i,\sigma,\sigma')\in E\}$,*

such that $\chi_E(i,\sigma,\sigma') = \sum_{j=1}^k f_j(i,\sigma)\chi_{E_j}(\sigma,\sigma')$ for every integer i and all σ and σ' in Σ.

Proof. Let us set $E_1 = \bigcup_{i\in\mathbb{N}}\{(\sigma,\sigma')\in\Sigma^2 \mid (i,\sigma,\sigma')\in E\}$. Using [11, Lem. 2.7], we can assume without loss of generality that the restrictions of s and r to E_1 are one-to-one. Define then $f_1 : \mathbb{N}\times\Sigma\to\mathbb{C}$ by

$$f_1(i,\sigma) = \begin{cases} 1 & \text{if there exists } \sigma' \text{ in } \Sigma \text{ such that } (i,\sigma,\sigma') \text{ is in } E,\\ 0 & \text{otherwise.}\end{cases}$$

Then $\chi_E(i,\sigma,\sigma') = f_1(i,\sigma)\chi_{E_1}(\sigma,\sigma')$ for every integer i and all σ and σ' in Σ. □

5. THE BAUM–CONNES ASSEMBLY MAP FOR $(G_{F,\Sigma}, \mathcal{A}_{C_0(\Sigma)})$

Recall that the definition of the Baum–Connes assembly map has been extended to the setting of groupoids in [12]. Let G be a locally compact groupoid equipped with a Haar system and let B be a C^*-algebra acted upon by G. Then there is an assembly map

$$\mu_{G,B,*} : K^{\mathrm{top}}_*(G,B) \to K_*(B \rtimes_{\mathrm{red}} G),$$

where $K^{\mathrm{top}}_*(G,B)$ is the topological K-theory for the groupoid G with coefficients in B. Our aim in this section is to describe the left-hand side of this assembly map for the action of G_Σ on $\mathcal{A}^\infty_{C_0(\Sigma)}$ and then to show that the Baum–Connes conjecture is equivalent to the bijectivity of the geometric assembly map

$$\nu^\infty_{F,\Sigma,\mathcal{A},*} : K^{\mathrm{top},\infty}_*(F,\Sigma,\mathcal{A}) \to K_*(\mathcal{A}^\infty_\Sigma \rtimes F)$$

defined in Section 4.7. Using results of [11] on the Baum–Connes conjecture for groupoid affiliated to coarse structures, we get examples of coarse spaces that satisfy the permanence approximation property. Notice that $G^{\mathbb{N}}_{F,\Sigma}$ is clearly a σ-compact and étale groupoid and that according to [11, Lem. 4.1], the Baum–Connes conjectures for the action of $G_{F,\Sigma}$ on $\mathcal{A}^\infty_{C_0(\Sigma)}$ and for the action of $G^{\mathbb{N}}_{F,\Sigma}$ on $\mathcal{A}^\infty_{C_0(\Sigma)}$ are indeed equivalent.

5.1. **The classifying space for proper actions of the groupoid $G_\Sigma^{\mathbb{N}}$.** For a σ-compact and étale groupoid G, the following description for the left-hand side of the assembly map was given in [13, §3]. Let K be a compact subset of G and let us consider the space $P_K(G)$ of probability measures η on G such that for all γ and γ' in the support of η,

- γ and γ' have same target;
- $\gamma^{-1}\cdot\gamma'$ is in K.

We endow $P_K(G)$ with the weak-$*$ topology, and equip it with the natural left action of G. Then according to [13, Prop. 3.1], the action of G on $P_K(G)$ is proper and cocompact. If $K\subseteq K'$ is an inclusion of compact subsets of G, then for any G-algebra B, the inclusion $P_K(G)\hookrightarrow P_{K'}(G)$ induces a morphism $KK_*^G(C_0(P_K(G)),B)\to KK_*^G(C_0(P_{K'}(G)),B)$ and we have

$$K_*^{\mathrm{top}}(G,B)=\lim_K KK_*^G(C_0(P_K(G)),B),$$

where in the inductive limit, K runs through compact subsets of G. If the groupoid G is provided with an action of a finite group F by automorphisms, then for any F-invariant subset of G, the space $P_K(G)$ is F-invariant and for any $G\rtimes F$-algebra B, we get

$$K_*^{\mathrm{top}}(G\rtimes F,B)=\lim_K KK_*^{G\rtimes F}(C_0(P_K(G)),B),$$

where in the inductive limit, K runs through compact and F-invariant subsets of G. If Σ is a proper discrete metric space and if r is a nonnegative number, let us set

$$E_r=\big\{(\sigma,\sigma')\in\Sigma\times\Sigma\mid d(\sigma,\sigma')\leq r\big\},$$

and then consider the element $\chi_r=1\otimes_{C(\beta_\Sigma)}\chi_{E_r}$ of $C_c(G_\Sigma^{\mathbb{N}})$. Then we have $\chi_r^2=\chi_r$ for the pointwise multiplication as continuous functions in $C_0(G_\Sigma^{\mathbb{N}})$ and hence

$$\operatorname{supp}\chi_r=\big\{\gamma\in G_\Sigma^{\mathbb{N}}\mid \chi_r(\gamma)=1\big\}$$

is a compact subset of $G_\Sigma^{\mathbb{N}}$. Let us set then $P_r(G_\Sigma^{\mathbb{N}})=P_{\operatorname{supp}\chi_r}(G_\Sigma^{\mathbb{N}})$. If Σ is provided with an action of a finite group F by isometries, χ_r being F-invariant, we see that $P_r(G_\Sigma^{\mathbb{N}})$ is for any $r>0$ provided with an action of F by homeomorphisms.

For any ω in $\beta_{\mathbb{N}\times\Sigma}$ and any subset Y of some $P_r(G_\Sigma^{\mathbb{N}})$, let us set Y_ω for the fiber of Y at ω, i.e. the set of probability measures of Y supported in the set of elements of $G_\Sigma^{\mathbb{N}}$ with range ω. If W is a subset of $\beta_{\mathbb{N}\times\Sigma}$ then define $Y_{/W}=\bigcup_{\omega\in W}Y_\omega$. Let us define

$$P_r(G_\Sigma^{0,\mathbb{N}})=P_r(G_\Sigma^{\mathbb{N}})_{/\beta^0_{\mathbb{N}\times\Sigma}}.$$

Fix a positive number r and let (n,σ,x) be any element in $\mathbb{N}\times\Sigma\times P_r(\Sigma)$. Since (n,σ) is in $\beta^0_{\mathbb{N}\times\Sigma}$ and since the fiber of $G_\Sigma^{\mathbb{N}}$ at (n,σ) for the target map is Σ, we see that (n,σ,x) can be viewed as an element in $P_r(G_\Sigma^{0,\mathbb{N}})$. For any family $\mathcal{X}=(X_i)_{i\in\mathbb{N}}$ of compact subsets of $P_r(\Sigma)$, let us set $Z_{\mathcal{X}}$ for the closure of

$$\big\{(n,\sigma,x)\in\mathbb{N}\times\Sigma\times P_r(\Sigma)\mid n\in\mathbb{N},\,\sigma\in X_n,\,x\in X_n\big\}$$

in $P_r(G_\Sigma^{\mathbb{N}})$ (we view an element σ of Σ as an element of $P_r(\Sigma)$, the Dirac measure at σ).

Lemma 5.2. *Let r be a positive number and let $\mathcal{X} = (X_i)_{i\in\mathbb{N}}$ be a family of compact subsets of $P_r(\Sigma)$. Then $Z_\mathcal{X}$ is a compact subset of $P_r(G_\Sigma^{0,\mathbb{N}})$.*

Proof. Since X_i is a compact subset of the locally finite simplicial complex $P_r(\Sigma)$, there exists a finite set Y_i of Σ such that every element of X_i is supported in Y_i. Applying Lemma 4.16 to

$$E = \{(n,\sigma,\sigma') \in \mathbb{N}\times\Sigma\times\Sigma \mid \sigma \in Y_n,\ \sigma' \in Y_n,\ d(\sigma,\sigma') \leq r\},$$

we see that there exist $f_1,\dots,f_k$ in $\ell^\infty(\mathbb{N}\times\Sigma) \cong C(\beta_{\mathbb{N}\times\Sigma})$ and $E_1,\dots,E_k$ entourages of Σ of diameter less than r such that

$$\chi_E(i,\sigma,\sigma') = \sum_{j=1}^k f_j(i,\sigma)\chi_{E_j}(\sigma,\sigma')$$

for every integer j and all σ and σ' in Σ. Set then

$$\tilde{\chi}_E = \sum_{j=1}^k f_j \otimes_{C(\beta_\Sigma)} \chi_{E_j} \in C_c(G_\Sigma^{0,\mathbb{N}}).$$

Then $\tilde{\chi}_E$ is valued in $\{0,1\}$. The set of probability measures η such that $\eta(\tilde{\chi}_E) = 1$ is closed in the unit ball of the dual of $C_c(G_\Sigma^{\mathbb{N}})$ equipped with the weak topology and hence is compact. Since $\eta(\tilde{\chi}_E) = 1$ for any η in $Z_\mathcal{X}$, we get that $Z_\mathcal{X}$ is compact in $P_r(G_\Sigma^{\mathbb{N}})$. But since we also have $\eta(\tilde{\chi}_E f) = \eta(f)$ for any η in $Z_\mathcal{X}$ and any f in $C_c(G_\Sigma^{\mathbb{N}})$ and since $\tilde{\chi}_E$ is in $C_c(G_\Sigma^{0,\mathbb{N}})$, we deduce that $Z_\mathcal{X}$ is included in $P_r(G_\Sigma^{0,\mathbb{N}})$. □

Corollary 5.3. *Let r be a positive number and let $\mathcal{X} = (X_i)_{i\in\mathbb{N}}$ be a family of compact subsets of $P_r(\Sigma)$. Then the closure of $\bigcup_{i\in\mathbb{N}}\{i\}\times\Sigma\times X_i$ in $P_r(G_\Sigma^{\mathbb{N}})$ is a $G_\Sigma^{\mathbb{N}}$-invariant and $G_\Sigma^{\mathbb{N}}$-compact subset of $P_r(G_\Sigma^{0,\mathbb{N}})$.*

Proof. The closure of $\bigcup_{i\in\mathbb{N}}\{i\}\times\Sigma\times X_i$ in $P_r(G_\Sigma^{\mathbb{N}})$ is the $G_\Sigma^{\mathbb{N}}$-orbit of $Z_\mathcal{X}$ and hence is $G_\Sigma^{\mathbb{N}}$-invariant and $G_\Sigma^{\mathbb{N}}$-compact in $P_r(G_\Sigma^{0,\mathbb{N}})$. □

5.4. **Topological K-theory for the groupoid $G_{F,\Sigma}$ with coefficients in $\mathcal{A}_{C_0(\Sigma)}$.** The aim of this subsection is to show that for any free action of a finite group F by isometries on Σ and any family $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ of F-C^*-algebras, we have a natural identification between $K_*^{\mathrm{top}}(G_{F,\Sigma}^{\mathbb{N}}, \mathcal{A}_{C_0(\Sigma)}^\infty)$ and $K_*^{\mathrm{top},\infty}(F,\Sigma,\mathcal{A})$.

Let Σ be a discrete metric space with bounded geometry. For any $G_\Sigma^{\mathbb{N}}$-invariant and $G_\Sigma^{\mathbb{N}}$-compact subset Y of $P_r(G_\Sigma^{0,\mathbb{N}})$, the restriction $Y_{/\{i\}\times\Sigma\times P_r(\Sigma)}$ is an invariant and cocompact subset of $\{i\}\times\Sigma\times P_r(\Sigma)$ for the action of the groupoid of pairs $\Sigma\times\Sigma$. Hence, there exists a family $\mathcal{X}^Y = (X_i^Y)_{i\in\mathbb{N}}$ of compact subsets of $P_r(\Sigma)$ such that

$$Y_{/\{i\}\times\Sigma\times P_r(\Sigma)} = \{i\}\times\Sigma\times X_i^Y \tag{12}$$

for every integer i. Notice that if Σ is provided with an action of a finite group F by isometries and if Y as above is moreover F-invariant, then X_i^Y is F-invariant for every integer i. For any $G^{\mathbb{N}}_{F,\Sigma}$-invariant and $G^{\mathbb{N}}_{\Sigma}$-compact subset Y of $P_r(G_\Sigma^{0,\mathbb{N}})$ and any family $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ of F-C^*-algebras, consider the following composition (recall that $\mathcal{A}^\infty_{C_0(\Sigma)}=\prod_{i\in\mathbb{N}}C_0(\Sigma,A_i\otimes\mathscr{K}(\mathscr{H})))$:

$$(13)\qquad \begin{aligned} i_Y : KK_*^{G^{\mathbb{N}}_{F,\Sigma}}(C_0(Y),\mathcal{A}^\infty_{C_0(\Sigma)}) &\to \prod_{i\in\mathbb{N}} KK_*^F(C(X_i^Y),\mathscr{K}(\mathscr{H})\otimes A_i)\\ &\to \prod_{i\in\mathbb{N}} KK_*^F(C(X_i^Y),A_i), \end{aligned}$$

where

- X_i^Y is for any integer i defined by equation (12);
- the first map is induced by groupoid functoriality with respect to the family of groupoid morphisms
$$F\hookrightarrow(\mathbb{N}\times\Sigma\times\Sigma)\rtimes F,\quad x\mapsto(i,\sigma,x(\sigma),x),$$
where i runs through integers and σ is a fixed element of Σ (recall that $\mathbb{N}\times\Sigma\times\Sigma$ is an F-invariant subgroupoid of $G^{\mathbb{N}}_\Sigma$);
- the second map is given for every integer i by the Morita equivalence between $\mathscr{K}(\mathscr{H})\otimes A_i$ and A_i.

Let $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ and $\mathcal{B}=(B_i)_{i\in\mathbb{N}}$ be families of F-algebras and let $z=(z_i)_{i\in\mathbb{N}}$ be a family in $\prod_{i\in\mathbb{N}}KK_*^F(A_i,B_i)$. Without loss of generality, for every integer i, we can assume that z_i is represented by an F-equivariant K-cycle $(\ell^2(F)\otimes\mathscr{H}\otimes B_i,\pi_i,T_i)$ where

- $\mathscr{H}$ is a separable Hilbert space;
- F acts diagonally on $\ell^2(F)\otimes\mathscr{H}\otimes B_i$ by the right regular representation on $\ell^2(F)$ and trivially on $\mathscr{H}$;
- π_i is an F-equivariant representation of A_i in the algebra $\mathcal{L}_{B_i}(\ell^2(F)\otimes\mathscr{H}\otimes B_i)$ of adjointable operators of $\ell^2(F)\otimes\mathscr{H}\otimes B_i$;
- T_i is an F-equivariant selfadjoint operator of $\mathcal{L}_{B_i}(\ell^2(F)\otimes\mathscr{H}\otimes B_i)$ satisfying the K-cycle conditions, i.e. $[T_i,\pi_i(a)]$ and $\pi_i(a)(T_i^2-\mathcal{I}d_{\ell^2(F)\otimes\mathscr{H}\otimes B_i})$ belong to $\mathscr{K}(\ell^2(F)\otimes\mathscr{H})\otimes B_i$ for every a in A_i.

Note that we have an identification between the algebras $\mathcal{L}_{B_i}(\ell^2(F)\otimes\mathscr{H}\otimes B_i)$ and $\mathcal{L}_{\mathscr{K}(\mathscr{H})\otimes B_i}(\ell^2(F)\otimes\mathscr{K}(\mathscr{H})\otimes B_i)$. Indeed these two C^*-algebras can be viewed as the multiplier algebra of $\mathscr{K}(\ell^2(F)\otimes\mathscr{H})\otimes B_i$. We see that the pointwise diagonal multiplication by

$$\mathbb{N}\to\mathcal{L}_{\mathscr{K}(\mathscr{H})\otimes B_i}(\ell^2(F)\otimes\mathscr{K}(\mathscr{H})\otimes B_i),\quad i\mapsto T_i$$

gives rise to an F-equivariant adjointable operator $T^\infty_{C_0(\Sigma)}$ of the right $\mathcal{B}^\infty_{C_0(\Sigma)}$-Hilbert module

$$\prod_{i\in\mathbb{N}}C_0(\Sigma,\ell^2(F)\otimes\mathscr{K}(\mathscr{H})\otimes B_i)\cong\ell^2(F)\otimes\mathcal{B}^\infty_{C_0(\Sigma)}.$$

The family of representations $(\pi_i)_{i\in\mathbb{N}}$ gives rise to a representation $\pi^\infty_{C_0(\Sigma)}$ of $\mathcal{A}_{C_0(\Sigma)}$ on the algebra of adjointable operators of

$$\prod_{i\in\mathbb{N}} C_0(\Sigma,\ell^2(F)\otimes\mathscr{K}(\mathscr{H})\otimes B_i).$$

It is then straight-forward to check that $\pi^\infty_{C_0(\Sigma)}$ and $T^\infty_{C_0(\Sigma)}$ are indeed $G^{\mathbb{N}}_{F,\Sigma}$-equivariant and that $T^\infty_{C_0(\Sigma)}$ satisfies the K-cycle conditions. Therefore, we obtain in this way a K-cycle for $KK^{G^{\mathbb{N}}_{F,\Sigma}}_*(\mathcal{A}_{C_0(\Sigma)},\mathcal{B}^\infty_{C_0(\Sigma)})$ and we can define then a morphism

$$\tau^\infty_{C_0(\Sigma)}:\prod_{i\in\mathbb{N}} KK^F_*(A_i,B_i)\to KK^{G^{\mathbb{N}}_{F,\Sigma}}_*(\mathcal{A}_{C_0(\Sigma)},\mathcal{B}^\infty_{C_0(\Sigma)})$$

which is moreover bifunctorial, i.e. if $\mathcal{A}=(A_i)$ and $\mathcal{B}=(B_i)$ are families of F-algebras, then

- for any family $\mathcal{A}'=(A'_i)$ of F-algebras and any family $f=(f_i)_{i\in\mathbb{N}}$ of F-equivariant homomorphisms $f_i:A_i\to A'_i$, we have
$$\tau^\infty_{C_0(\Sigma)}(f^*(z))=f^*_{C_0(\Sigma)}(\tau^\infty_{C_0(\Sigma)}(z))$$
for any z in $\prod_{i\in\mathbb{N}} KK^F_*(A'_i,B_i)$, where $f_{C_0(\Sigma)}:\mathcal{A}_{C_0(\Sigma)}\to\mathcal{A}'_{C_0(\Sigma)}$ is induced by $f=(f_i)_{i\in\mathbb{N}}$;
- for any family $\mathcal{B}'=(B'_i)$ of F-algebras and any family $g=(g_i)_{i\in\mathbb{N}}$ of F-equivariant homomorphisms $g_i:B_i\to B'_i$, we have
$$\tau^\infty_{C_0(\Sigma)}(g_*(z))=g^\infty_{C_0(\Sigma),*}(\tau^\infty_{C_0(\Sigma)}(z))$$
for any z in $\prod_{i\in\mathbb{N}} KK^F_*(A_i,B_i)$, where $g^\infty_{C_0(\Sigma)}:\mathcal{B}^\infty_{C_0(\Sigma)}\to\mathcal{B}'^\infty_{C_0(\Sigma)}$ is induced by $g=(g_i)_{i\in\mathbb{N}}$.

Recall that for a family $\mathcal{X}=(X_i)_{i\in\mathbb{N}}$ of compact subsets in some $P_r(\Sigma)$, we defined $\mathcal{C}_\mathcal{X}=(C(X_i))_{i\in\mathbb{N}}$. If $\mathcal{X}'=(X'_i)_{i\in\mathbb{N}}$ is another such family satisfying $X_i\subseteq X'_i$ for any integer i (we say that $(\mathcal{X},\mathcal{X}')$ is a relative pair of families), then let us set $\mathcal{C}_{\mathcal{X},\mathcal{X}'}=(C_0(X'_i\setminus X_i))_{i\in\mathbb{N}}$. Let Z be a $G^{\mathbb{N}}_\Sigma$-compact subset of some $P_K(G^{\mathbb{N}}_\Sigma)$ for a given compact subset K in $G^{\mathbb{N}}_\Sigma$. Let us fix $r>0$ such that $P_K(G^{\mathbb{N}}_\Sigma)\subseteq P_r(G^{\mathbb{N}}_\Sigma)$ and let $\mathcal{X}=(X_i)_{i\in\mathbb{N}}$ be a family of compact subsets in $P_r(\Sigma)$ such that $Z_\mathcal{X}\subseteq Z$. Define then the $G^{\mathbb{N}}_\Sigma$-equivariant homomorphism

$$\Lambda^Z_\mathcal{X}:C_0(Z)\to\mathcal{C}_{\mathcal{X},C_0(\Sigma)},\quad f\mapsto(f_i)_{i\in\mathbb{N}},$$

with f_i in $C_0(\Sigma\times X_i)$ defined by $f_i(\sigma,x)=f(i,\sigma,x)$ for any integer i, any σ in Σ and any x in X_i. In the same way, if (Z,Z') is a relative pair of $G^{\mathbb{N}}_\Sigma$-compact subsets of $P_K(G^{\mathbb{N}}_\Sigma)$ and if $(\mathcal{X},\mathcal{X}')$ is a relative pair of families of compact subsets in $P_r(\Sigma)$ such that $Z_\mathcal{X}\subseteq Z$ and $Z_{\mathcal{X}'}\subseteq Z'$, the restriction of $\Lambda^{Z'}_{\mathcal{X}'}$ to $C_0(Z'\setminus Z)$ gives rise to a $G^{\mathbb{N}}_\Sigma$-equivariant homomorphism

$$\Lambda^{Z,Z'}_{\mathcal{X},\mathcal{X}'}:C_0(Z'\setminus Z)\to\mathcal{C}_{\mathcal{X},\mathcal{X}',C_0(\Sigma)}.$$

If (Z,Z') is a relative pair of $G^{\mathbb{N}}_\Sigma$-compact subsets of $P_K(G^{\mathbb{N}}_\Sigma)$ and if $\mathcal{X}'$ is a family of compact subsets in $P_r(\Sigma)$ such that $Z_{\mathcal{X}'}\subseteq Z'$, then there exists a unique family $\mathcal{X}'_{/Z}=(X'_{i,/Z})_{i\in\mathbb{N}}$ of compact subsets in $P_r(\Sigma)$ such that

$(\mathcal{X}'_{/Z}, \mathcal{X}')$ is a relative pair of families and $Z_{\mathcal{X}'_{/Z}} = Z_{\mathcal{X}'} \cap Z$. If the relative pair (Z, Z') is moreover F-invariant, then $(\mathcal{X}'_{/Z}, \mathcal{X}')$ is a relative pair of families of F-invariant compact spaces and the map

$$\prod_{i\in\mathbb{N}} KK^F_*(C_0(X'_i \setminus X'_{i/Z}), A_i) \to KK^{G^{\mathbb{N}}_{F,\Sigma}}_*(C_0(Z' \setminus Z), \mathcal{A}^\infty_{C_0(\Sigma)}),$$
$$z = (z_i)_{i\in\mathbb{N}} \mapsto \Lambda^{Z,Z',*}_{\mathcal{X},\mathcal{X}'}(\tau^\infty_{C_0(\Sigma)}(z))$$

is compatible with family of inclusions $(X_i \hookrightarrow X'_i)_{i\in\mathbb{N}}$ of F-invariant compact subsets. Hence, taking the inductive limit and setting

$$K^{F,\infty}_*(Z, Z', \mathcal{A}) = \lim_{\mathcal{X}'} \prod_{i\in\mathbb{N}} KK^F_*(C_0(X'_i \setminus X'_{i/Z}), A_i),$$

where $\mathcal{X}' = (X'_i)_{i\in\mathbb{N}}$ runs through the family of compact F-invariant subsets in $P_r(\Sigma)$ such that $Z_{\mathcal{X}'} \subseteq Z'$, we end up with a morphism

$$\upsilon^{Z,Z'}_{F,\Sigma,\mathcal{A},*} : K^{F,\infty}_*(Z, Z', \mathcal{A}) \to KK^{G^{\mathbb{N}}_{F,\Sigma}}_*(C_0(Z' \setminus Z), \mathcal{A}^\infty_{C_0(\Sigma)}). \tag{14}$$

We set $K^{F,\infty}_*(Z, \mathcal{A})$ for $K^{F,\infty}_*(\varnothing, Z, \mathcal{A})$ and $\upsilon^Z_{F,\Sigma,\mathcal{A},*}$ for $\upsilon^{\varnothing,Z}_{F,\Sigma,\varnothing,\mathcal{A},*}$.

Lemma 5.5. *Let Z be a $G^{\mathbb{N}}_{F,\Sigma}$-invariant closed subset of some $P_K(G^{\mathbb{N}}_\Sigma)$ for K a compact subset of $G^{\mathbb{N}}_\Sigma$. Assume that the restriction to Z of the anchor map for the action of $G^{\mathbb{N}}_\Sigma$ on $P_K(G^{\mathbb{N}}_\Sigma)$ is locally injective, i.e. there exists a covering of Z by open subsets for which the restriction of the anchor map is one-to-one. Then for any family $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ of F-algebras,*

$$\upsilon^Z_{F,\Sigma,\mathcal{A},*} : K^{F,\infty}_*(Z, \mathcal{A}) \to KK^{G^{\mathbb{N}}_{F,\Sigma}}_*(C_0(Z), \mathcal{A}^\infty_{C_0(\Sigma)})$$

is an isomorphism.

Proof. According to [13], since $\mathcal{A}^\infty_{C_0(\Sigma)}$ is indeed a $C(\beta^0_{\mathbb{N}\times\Sigma})$-algebra, there is an isomorphism

$$\lim_{Z'} : KK^{G^{\mathbb{N}}_{F,\Sigma}}_*(C_0(Z'), \mathcal{A}^\infty_{C_0(\Sigma)}) \to KK^{G^{\mathbb{N}}_{F,\Sigma}}_*(C_0(Z), \mathcal{A}^\infty_{C_0(\Sigma)}), \tag{15}$$

where

- in the inductive limit of the left-hand side, Z' runs through $G^{\mathbb{N}}_\Sigma$-compact and F-invariant subsets of $Z_{/\beta^0_{\mathbb{N}\times\Sigma}}$;
- the map is then induced by the inclusion $Z' \hookrightarrow Z$.

Under the identification of equation (15), the family of maps defined by equation (13),

$$i_{Z'} : KK^{G^{\mathbb{N}}_{F,\Sigma}}_*(C_0(Z'), \mathcal{A}^\infty_{C_0(\Sigma)}) \to \prod_{i\in\mathbb{N}} KK^F_*(C(X^{Z'}_i), A_i),$$

where Z' runs through $G^{\mathbb{N}}_\Sigma$-compact and F-invariant subsets of $Z_{/\beta^0_{\mathbb{N}\times\Sigma}}$, provides an inverse for $\upsilon^Z_{F,\Sigma,\mathcal{A},*}$. □

Since for any compact subset K of $G_\Sigma^{\mathbb{N}}$, there exists $r>0$ such that $P_K(G_\Sigma^{\mathbb{N}})\subseteq P_r(G_\Sigma^{\mathbb{N}})$, we get that

$$K_*^{\mathrm{top},\infty}(F,\Sigma,\mathcal{A})=\lim_K K_*^{F,\infty}(P_K(G_\Sigma^{\mathbb{N}}),\mathcal{A}),\tag{16}$$

where in the right-hand side, K runs through compact F-invariant subsets of G_Σ, and the inductive limit is taken under the maps induced by inclusions $P_K(G_\Sigma^{\mathbb{N}})\hookrightarrow P_{K'}(G_\Sigma^{\mathbb{N}})$ corresponding to relative pairs (K,K') of F-invariant compact subsets of $G_\Sigma^{\mathbb{N}}$. The maps

$$\upsilon_{F,\Sigma,\mathcal{A},*}^{P_K(G_\Sigma^{\mathbb{N}})}:K_*^{F,\infty}(P_K(G_\Sigma^{\mathbb{N}}),\mathcal{A})\to K_*^{G_{F,\Sigma}^{\mathbb{N}}}(C_0(P_K(G_\Sigma^{\mathbb{N}})),\mathcal{A}_{C_0(\Sigma)}^\infty)$$

are then obviously compatible with the inductive limit of equation (16) and hence give rise to a morphism

$$\upsilon_{F,\Sigma,\mathcal{A},*}:K_*^{\mathrm{top},\infty}(F,\Sigma,\mathcal{A})\to K_*^{\mathrm{top}}(G_{F,\Sigma}^{\mathbb{N}},\mathcal{A}_{C_0(\Sigma)}^\infty).$$

We end this subsection by proving that $\upsilon_{F,\Sigma,\mathcal{A},*}$ is an isomorphism. The idea is to use the simplicial structure of $P_K(G_\Sigma^{\mathbb{N}})$ to carry out a Mayer–Vietoris argument. In order to do that, we need first to reduce to the case of a second-countable and étale groupoid. Recall from [11, Lem. 4.1] that there exists a second countable étale groupoid G'_Σ with compact base space β'_Σ and an action of G'_Σ on β_Σ such that $G_\Sigma=\beta_\Sigma\rtimes G'_\Sigma$. The groupoid G'_Σ then acts on $\beta_{\mathbb{N}\times\Sigma}$ through the action of G_Σ and $\beta_{\mathbb{N}\times\Sigma}\rtimes G_\Sigma=\beta_{\mathbb{N}\times\Sigma}\rtimes G'_\Sigma$. For any subset X of a G'_Σ-space, let us set $X_\Sigma^{\mathbb{N}}=\beta_{\mathbb{N}\times\Sigma}\times_{\beta'_\Sigma}X$. If Σ is provided with an action by isometries of a finite group F, then G'_Σ can be chosen provided with an action of F by automorphisms that makes the action on β_Σ and hence on $\beta_{\mathbb{N}\times\Sigma}$ equivariant. If we set $G'_{F,\Sigma}=G'_\Sigma\rtimes F$, then for any family $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ of F-algebras, $\mathcal{A}_{C_0(\Sigma)}$ is a $G'_{F,\Sigma}$-algebra. Let Y be a locally compact space equipped with a proper and cocompact action of $G'_{F,\Sigma}$. Then the map

$$KK_*^{G_{F,\Sigma}^{\mathbb{N}}}(C_0(Y_\Sigma^{\mathbb{N}}),\mathcal{A}_{C_0(\Sigma)})\to KK_*^{G'_{F,\Sigma}}(C_0(Y),\mathcal{A}_{C_0(\Sigma)})$$

obtained by forgetting the $C(\beta_{\mathbb{N}\times\Sigma})$-action is an isomorphism. Moreover, up to the identification

$$\mathcal{A}_{C_0(\Sigma)}\rtimes_{\mathrm{red}}G_{F,\Sigma}^{\mathbb{N}}\cong\mathcal{A}_{C_0(\Sigma)}\rtimes_{\mathrm{red}}G'_{F,\Sigma},$$

the Baum–Connes conjecture for $G_{F,\Sigma}^{\mathbb{N}}$ and for $G'_{F,\Sigma}$ for the coefficient $\mathcal{A}_{C_0(\Sigma)}$ are equivalent [11]. For any compact subset K of G'_Σ, we have a natural identification

$$P_{K_\Sigma^{\mathbb{N}}}(G_\Sigma)\cong P_K(G'_\Sigma)_\Sigma^{\mathbb{N}}.$$

For a compact subset K of G'_Σ, fix a positive number r such that

$$P_{K_\Sigma^{\mathbb{N}}}(G_\Sigma)\subseteq P_r(G_\Sigma^{\mathbb{N}}).$$

Let Y be a $G'_{F,\Sigma}$-invariant closed subset of $P_K(G'_\Sigma)$ and let $\mathcal{X}=(X_i)_{i\in\mathbb{N}}$ be a family of F-invariant compact subsets of $P_r(G_\Sigma^{\mathbb{N}})$ such that $Z_{\mathcal{X}}\subseteq Y_\Sigma^{\mathbb{N}}$. Let us consider the composition

$$\Lambda_{\mathcal{X}}'^{Y}:C_0(Y)\to C_0(Y_\Sigma^{\mathbb{N}})\xrightarrow{\Lambda_{\mathcal{X}}^{Y_\Sigma^{\mathbb{N}}}}\mathcal{C}_{\mathcal{X},C_0(\Sigma)},$$

where the first map of the composition is induced by the projection $Y_\Sigma^{\mathbb{N}} \to Y$. Let us also consider the relative version: let (Y, Y') be a relative pair of $G'_{F,\Sigma}$-invariant closed subsets of $P_K(G'_\Sigma)$ and let $(\mathcal{X}, \mathcal{X}')$ be a relative family of compact F-invariant subsets of $P_r(G_\Sigma^{\mathbb{N}})$ such that $Z_\mathcal{X} \subseteq Y_\Sigma^{\mathbb{N}}$ and $Z'_\mathcal{X} \subseteq Y_\Sigma'^{\mathbb{N}}$. Define

$$\Lambda'^{Y,Y'}_{\mathcal{X},\mathcal{X}'} : C_0(Y' \setminus Y) \to \mathcal{C}_{\mathcal{X},\mathcal{X}',C_0(\Sigma)}$$

as the restriction of $\Lambda'^{Y}_{\mathcal{X}}$ to $C_0(Y' \setminus Y')$. Let us then proceed as we did to define $\upsilon^{Y,Y'}_{F,\Sigma,\mathcal{A},*}$ in equation (13).

If $\tau'^{\infty}_{C_0(\Sigma)}(\cdot)$ stands for the restriction of $\tau^{\infty}_{C_0(\Sigma)}(\cdot)$ to $KK_*^{G'_{F,\Sigma}}(\cdot,\cdot)$, then the map

$$\prod_{i\in\mathbb{N}} KK_*^F(C_0(X'_i \setminus X'_{i,/Y_\Sigma^{\mathbb{N}}}), A_i) \to KK_*^{G'_{F,\Sigma}}(C_0(Y' \setminus Y), \mathcal{A}^\infty_{C_0(\Sigma)}),$$

$$z = (z_i)_{i\in\mathbb{N}} \mapsto \Lambda'^{Y,Y',*}_{\mathcal{X},\mathcal{X}'}(\tau'^{\infty}_{C_0(\Sigma)}(z))$$

is compatible with the family of inclusions $(X_i \hookrightarrow X'_i)_{i\in\mathbb{N}}$ of F-invariant compact subset. Hence, proceeding as in the definition of $\upsilon^{\cdot,\cdot}_{F,\Sigma,\mathcal{A},*}$, we end up as in equation (14) with a morphism

$$\upsilon'^{Y,Y'}_{F,\Sigma,\mathcal{A},*} : K_*^{F,\infty}(Y_\Sigma^{\mathbb{N}}, Y_\Sigma'^{\mathbb{N}}, \mathcal{A}) \to KK_*^{G'_{F,\Sigma}}(C_0(Y' \setminus Y), \mathcal{A}^\infty_{C_0(\Sigma)}),$$

which is indeed the composition

$$K_*^{F,\infty}(Y_\Sigma^{\mathbb{N}}, Y_\Sigma'^{\mathbb{N}}, \mathcal{A}) \xrightarrow{\upsilon^{Y,Y'}_{F,\Sigma,\mathcal{A},*}} KK_*^{G^{\mathbb{N}}_{F,\Sigma}}(C_0(Y_\Sigma'^{\mathbb{N}} \setminus Y_\Sigma^{\mathbb{N}}), \mathcal{A}^\infty_{C_0(\Sigma)})$$
$$\xrightarrow{\cong} KK_*^{G'_{F,\Sigma}}(C_0(Y' \setminus Y), \mathcal{A}^\infty_{C_0(\Sigma)}),$$

where the second map is induced by the projection $Y_\Sigma'^{\mathbb{N}} \to Y'$. We set also $\upsilon'^{Y}_{F,\Sigma,\mathcal{A},*}$ for $\upsilon'^{\varnothing,Y}_{F,\Sigma,\mathcal{A},*}$.

Lemma 5.6. *Let Y be a $G'_{F,\Sigma}$-simplicial complex in the sense of* [13, Def. 3.7] *lying in some $P_K(G'_{F,\Sigma})$ for K a compact subset of $G'_{F,\Sigma}$. Then for any family $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ of F-algebras,*

$$\upsilon^{Y_\Sigma^{\mathbb{N}}}_{F,\Sigma,\mathcal{A},*} : K_*^{F,\infty}(Y_\Sigma^{\mathbb{N}}, \mathcal{A}) \to KK_*^{G^{\mathbb{N}}_{F,\Sigma}}(C_0(Y_\Sigma^{\mathbb{N}}), \mathcal{A}^\infty_{C_0(\Sigma)})$$

is an isomorphism.

Proof. Notice first that, as we have already mentioned, this is equivalent to proving that

$$\upsilon'^{Y}_{F,\Sigma,\mathcal{A},*} : K_*^{F,\infty}(Y_\Sigma^{\mathbb{N}}, \mathcal{A}) \to KK_*^{G'_{F,\Sigma}}(C_0(Y), \mathcal{A}^\infty_{C_0(\Sigma)})$$

is an isomorphism. Let us prove the result by induction on the dimension of the $G'_{F,\Sigma}$-simplicial complex Y. If Y has dimension 0, the anchor map for the action of $G^{\mathbb{N}}_{F,\Sigma}$ is locally injective and hence, the result is a consequence of Lemma 5.5. We can assume without loss of generality that Y is typed and that the action of $G'_{F,\Sigma}$ is typed preserving. Let $Y_0 \subseteq Y_1 \subseteq \cdots \subseteq Y_n = Y$ be the skeleton of Y, and assume that we have proved that

$$\upsilon_{F,\Sigma,\mathcal{A},*}^{Y^{\mathbb{N}}_{n-1,\Sigma}} : K_*^{F,\infty}(Y^{\mathbb{N}}_{n-1,\Sigma}, \mathcal{A}) \to K_*^{G^{\mathbb{N}}_{F,\Sigma}}(C_0(Y^{\mathbb{N}}_{n-1,\Sigma}), \mathcal{A}^\infty_{C_0(\Sigma)})$$

is an isomorphism. Since Y is second countable, the inclusion $Y_{n-1} \hookrightarrow Y_n$ gives rise to a long exact sequence

$$\begin{aligned}\cdots &\to KK_i^{G'_{F,\Sigma}}(C_0(Y_{n-1}), \mathcal{A}^\infty_{C_0(\Sigma)})\\ &\to KK_i^{G'_{F,\Sigma}}(C_0(Y_n), \mathcal{A}^\infty_{C_0(\Sigma)})\\ &\to KK_i^{G'_{F,\Sigma}}(C_0(Y_n \setminus Y_{n-1}), \mathcal{A}^\infty_{C_0(\Sigma)})\\ &\to KK_{i-1}^{G'_{F,\Sigma}}(C_0(Y_{n-1}), \mathcal{A}^\infty_{C_0(\Sigma)}) \to \cdots .\end{aligned}$$

In the same way, we have a long exact sequence

$$\begin{aligned}\cdots &\to K_i^{F,\infty}(Y^{\mathbb{N}}_{n-1,\Sigma}, \mathcal{A})\\ &\to K_i^{F,\infty}(Y^{\mathbb{N}}_{n,\Sigma}, \mathcal{A})\\ &\to K_i^{F,\infty}(Y^{\mathbb{N}}_{n-1,\Sigma}, Y^{\mathbb{N}}_{n,\Sigma}, \mathcal{A})\\ &\to K_{i-1}^{F,\infty}(Y^{\mathbb{N}}_{n-1,\Sigma}, \mathcal{A}) \to \cdots .\end{aligned}$$

By naturality of the morphisms $\Lambda'^{\cdot,\cdot}_{\cdot,\cdot}$ and $\tau'^{\infty}_{C_0(\Sigma)}(\cdot)$, these two long exact sequences are intertwined by the maps $\upsilon^{\cdot,\cdot}_{F,\Sigma,\mathcal{A},*}$. Using a five-lemma argument, the proof of the result amounts to showing that

$$\upsilon'^{Y^{\mathbb{N}}_{n-1,\Sigma}, Y^{\mathbb{N}}_{n,\Sigma}}_{F,\Sigma,\mathcal{A},*} : K_*^{F,\infty}(Y^{\mathbb{N}}_{n-1,\Sigma}, Y^{\mathbb{N}}_{n,\Sigma}, \mathcal{A}) \to KK_*^{G'_{F,\Sigma}}(C_0(Y_n \setminus Y_{n-1}), \mathcal{A}^\infty_{C_0(\Sigma)})$$

is an isomorphism. Let Y' be the set of centers of n simplices of Y. Since the action of $G'_{F,\Sigma}$ is type preserving, we have a $G'_{F,\Sigma}$-equivariant identification

$$Y_n \setminus Y_{n-1} \cong Y' \times \mathring{\Delta}, \tag{17}$$

where $\mathring{\Delta}$ is the interior of the standard simplex, and where the action of $G'_{F,\Sigma}$ on $Y' \times \mathring{\Delta}$ is diagonal through Y'. Let then $[\partial_{Y_{n-1},Y_n}]$ be the element of $KK_*^{G'_{F,\Sigma}}(C_0(Y'), C_0(Y_n \setminus Y_{n-1}))$ that implements up to the identification of equation (17) the Bott periodicity isomorphism. We can assume without loss of generality that in the definition of $KK_*^{F,\infty}(Y^{\mathbb{N}}_{n-1,\Sigma}, Y^{\mathbb{N}}_{n,\Sigma}, \mathcal{A})$, the inductive limit is taken over families $\mathcal{X} = (X_i)_{i\in\mathbb{N}}$ of F-invariant compact subsets of some $P_r(\Sigma)$ such that

- X_i is for every integer i a finite union of n-simplices with respect to the simplicial structure inherited from Y;
- $Z_{\mathcal{X}} \subseteq Y^{\mathbb{N}}_{n,\Sigma}$.

Let $\mathcal{X}$ be such a family and let X'_i be for every integer i the set of centers of n-simplices of X_i. Let us set then $\mathcal{X}' = (X'_i)_{i\in\mathbb{N}}$. Since the action of F is type preserving, we have an F-equivariant identification

$$X_i \setminus X_{i/Y^{\mathbb{N}}_{n-1,\Sigma}} \cong X'_i \times \mathring{\Delta}, \tag{18}$$

the action of F on $\mathring{\Delta}$ being trivial. Let $[\partial_i]$ be the element of

$$KK_*^F(C(X'_i), C_0(X_i \setminus X_{i/Y^{\mathbb{N}}_{n-1,\Sigma}}))$$

that implements up to this identification the Bott periodicity isomorphism. Set then

$$[\tilde{\partial}] = ([\partial_i])_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} KK_*^F(C(X'_i), C_0(X_i \setminus X_{i/Y^{\mathbb{N}}_{n-1,\Sigma}})).$$

The Bott generator of $KK_*(\mathbb{C}, C_0(\mathring{\Delta}))$ can be represented by a K-cycle $(C_0(\mathring{\Delta}, \mathbb{C}^l), \phi, T)$ for some integer l, where ϕ is the obvious representation of $\mathbb{C}$ on $C_0(\mathring{\Delta}, \mathbb{C}^l)$ by scalar multiplication, and T is an adjointable operator on $C_0(\mathring{\Delta}, \mathbb{C}^l)$ that satisfies the K-cycle conditions. Then for any integer i, the element $[\partial_i]$ of $KK_*^F(C(X'_i), C_0(X_i \setminus X_{i/Y^{\mathbb{N}}_{n-1,\Sigma}}))$ can be represented by the K-cycle

$$(C_0(X'_i \times \mathring{\Delta}, \mathbb{C}^l), \phi_i, \mathrm{Id}_{C(X'_i)} \otimes T),$$

where $C_0(X'_i \times \mathring{\Delta}, \mathbb{C}^l)$ is viewed as a right $C_0(X_i \setminus X_{i/Y^{\mathbb{N}}_{n-1,\Sigma}})$-Hilbert module by using the identification of equation (18) and ϕ_i is the obvious diagonal representation of $C_0(X'_i)$ on $C_0(X'_i \times \mathring{\Delta}, \mathbb{C}^l)$. Let $[\partial_{\mathcal{X},C_0(\Sigma)}]$ be the class of the K-cycle

$$\Big(\prod_{i\in\mathbb{N}} C_0(\Sigma \times X'_i \times \mathring{\Delta}, \mathbb{C}^l), \prod_{i\in\mathbb{N}} \mathrm{Id}_{C_0(\Sigma)} \otimes \phi_i, \prod_{i\in\mathbb{N}} \mathrm{Id}_{C_0(\Sigma\times X'_i)} \otimes T\Big)$$

in $KK_*^{G'_{F,\Sigma}}(\mathcal{C}_{\mathcal{X}',C_0(\Sigma)}, \mathcal{C}_{\mathcal{X}_{/Y^{\mathbb{N}}_{n-1,\Sigma}},\mathcal{X},C_0(\Sigma)})$. Then we have

$$\Lambda'^{Y',*}_{\mathcal{X}'}([\partial_{\mathcal{X},C_0(\Sigma)}]) = \Lambda'^{Y_{n-1},Y_n}_{\mathcal{X}_{/Y^{\mathbb{N}}_{n-1,\Sigma}},\mathcal{X},*}[\partial_{Y_{n-1},Y_n}]. \tag{19}$$

Let $z = (z_i)_{i\in\mathbb{N}}$ be a family in $\prod_{i\in\mathbb{N}} KK^F(C_0(X_i \setminus X_{i/Y^{\mathbb{N}}_{n-1,\Sigma}}), A_i)$. Then using the characterization of the Kasparov product (see [3] and [5] for the groupoid case), we get that

$$\Lambda'^{Y',*}_{\mathcal{X}'}([\partial_{\mathcal{X},C_0(\Sigma)}]) \otimes \tau^{\infty}_{C_0(\Sigma)}(z) = \Lambda'^{Y',*}_{\mathcal{X}'}(\tau^{\infty}_{C_0(\Sigma)}([\tilde{\partial}] \otimes z)).$$

This in turn implies that

$$\begin{aligned}
[\partial_{Y_{n-1},Y_n}] \otimes \Lambda'^{Y_{n-1},Y_n,*}_{\mathcal{X}_{/Y^{\mathbb{N}}_{n-1,\Sigma}},\mathcal{X}}(\tau^{\infty}_{C_0(\Sigma)}(z)) &= \Lambda'^{Y_{n-1},Y_n}_{\mathcal{X}_{/Y^{\mathbb{N}}_{n-1,\Sigma}},\mathcal{X},*}([\partial_{Y_{n-1},Y_n}]) \otimes \tau^{\infty}_{C_0(\Sigma)}(z) \\
&= \Lambda'^{Y',*}_{\mathcal{X}'}([\partial_{\mathcal{X},C_0(\Sigma)}]) \otimes \tau^{\infty}_{C_0(\Sigma)}(z) \\
&= \Lambda'^{Y',*}_{\mathcal{X}'}(\tau^{\infty}_{C_0(\Sigma)}([\tilde{\partial}] \otimes z)),
\end{aligned}$$

where the first equality is a consequence of bifunctoriality of the Kasparov product and the second equality holds by equation (19). From this, we get the existence of a commutative diagram

$$\begin{array}{ccc}
K_*^{F,\infty}(Y^{\mathbb{N}}_{n-1,\Sigma}, Y^{\mathbb{N}}_{n,\Sigma}, \mathcal{A}) & \xrightarrow{\quad\cong\quad} & K_*^{F,\infty}(Y'^{\mathbb{N}}_{\Sigma}, \mathcal{A}) \\
\Big\downarrow{\scriptstyle \upsilon'^{Y_{n-1},Y_n}_{F,\Sigma,\mathcal{A},*}} & & \Big\downarrow{\scriptstyle \upsilon'^{Y'}_{F,\Sigma,\mathcal{A},*}} \\
KK_*^{G'_{F,\Sigma}}(C_0(Y_n \setminus Y_{n-1}), \mathcal{A}^{\infty}_{C_0(\Sigma)}) & \xrightarrow{[\partial_{Y_{n-1},Y_n}]\otimes} & KK_*^{G'_{F,\Sigma}}(C_0(Y'), \mathcal{A}^{\infty}_{C_0(\Sigma)}),
\end{array}$$

where the top row is obtained by taking inductive limit over morphisms

$$\big([\partial_i]\otimes : KK_*^F(C_0(X_i \setminus X_{i/Y^{\mathbb{N}}_{n-1,\Sigma}}), A_i) \xrightarrow{\cong} KK_*^F(C(X'_i), A_i)\big)_{i\in\mathbb{N}}$$

relative to families $\mathcal{X} = (X_i)$ of F-invariant compact subsets of some $P_r(\Sigma)$ such that

- X_i is for every integer i a finite union of n-simplices;
- $Z_{\mathcal{X}} \subseteq Y^{\mathbb{N}}_{n,\Sigma}$.

Since Y' is a $G'_{F,\Sigma}$-simplicial complex of degree 0 and (as we have already seen) $\upsilon'^{Y'}_{F,\Sigma,\mathcal{A},*}$ is an isomorphism, we have that $\upsilon'^{Y_{n-1},Y_n}_{F,\Sigma,\mathcal{A},*}$ is an isomorphism. From this we deduce that $\upsilon'^{Y_n}_{F,\Sigma,\mathcal{A},*}$ is an isomorphism and hence that

$$\upsilon^{Y^{\mathbb{N}}_{n,\Sigma}}_{F,\Sigma,\mathcal{A},*}$$

is an isomorphism. ∎

Corollary 5.7. *Let Σ be a proper discrete metric space provided with an action of a finite group F by isometries. Let $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ be a family of F-algebras. Then,*

$$\upsilon_{F,\Sigma,\mathcal{A},*} : K_*^{\mathrm{top},\infty}(F,\Sigma,\mathcal{A}) \to K_*^{\mathrm{top}}(G^{\mathbb{N}}_{F,\Sigma}, \mathcal{A}^{\infty}_{C_0(\Sigma)})$$

is an isomorphism.

5.8. **The assembly map for the action of $G_{F,\Sigma}$ on $\mathcal{A}^{\infty}_{C_0(\Sigma)}$.** The aim of this subsection is to show that up to the identifications provided on the left-hand side by Corollary 5.7 and on the right-hand side by equation (11), the maps

$$\mu_{G^{\mathbb{N}}_{F,\Sigma},\mathcal{A}^{\infty}_{C_0(\Sigma)},*} : K_*^{\mathrm{top}}(G^{\mathbb{N}}_{F,\Sigma}, \mathcal{A}^{\infty}_{C_0(\Sigma)}) \to K_*(\mathcal{A}^{\infty}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G^{\mathbb{N}}_{F,\Sigma})$$

and

$$\nu^{\infty}_{F,\Sigma,\mathcal{A},*} : K_*^{\mathrm{top},\infty}(F,\Sigma,\mathcal{A}) \to K_*(\mathcal{A}^{\infty}_{\Sigma} \rtimes F)$$

coincide for any family $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ of F-algebras and any proper discrete metric space Σ equipped with a free action of F by isometries.

Fix a rank-one projection e in $\mathscr{K}(\mathscr{H})$ and define

$$\jmath : \mathbb{C} \to \mathscr{K}(H), \quad \lambda \mapsto \lambda e.$$

Let us consider the family of homomorphisms

$$(\jmath_{\mathcal{A}} = \jmath \otimes \mathrm{Id}_{A_i} : A_i \to A_i \otimes \mathscr{K}(\mathscr{H}))_{i\in\mathbb{N}}.$$

Proposition 5.9. *For any families of F-algebras $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ and $\mathcal{B} = (B_i)_{i\in\mathbb{N}}$ and any element $z = (z_i)_{i\in\mathbb{N}}$ in $\prod_{i\in\mathbb{N}} KK_*^F(A_i, B_i)$, we have for any proper discrete metric space Σ equipped with a free action of F by isometries a commutative diagram*

$$\begin{array}{ccc} K_*(\mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G^{\mathbb{N}}_{F,\Sigma}) & \xrightarrow{\otimes J_{G^{\mathbb{N}}_{F,\Sigma}}(\tau^{\infty}_{C_0(\Sigma)}(z))} & K_*(\mathcal{B}^{\infty}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G^{\mathbb{N}}_{F,\Sigma}) \\ \Big\downarrow{\scriptstyle \jmath_{\mathcal{A},F,\Sigma,*}\circ\mathcal{I}_{F,\Sigma,\mathcal{A},*}} & & \Big\downarrow{\scriptstyle \mathcal{I}_{F,\Sigma,\mathcal{B}^{\infty},*}V} \\ K_*(\mathcal{A}^{\infty}_{\Sigma} \rtimes_{\mathrm{red}} F) & \xrightarrow{\tau^{\infty}_{F,\Sigma}(z)} & K_*(\mathcal{B}^{\infty}_{\Sigma} \rtimes_{\mathrm{red}} F), \end{array}$$

where up to the identifications

$$K_*(\mathcal{A}^\infty_\Sigma \rtimes_{\mathrm{red}} F) \cong K_*(\mathcal{A}^\infty_{F,\Sigma}) \quad \textit{and} \quad K_*(\mathcal{B}^\infty_\Sigma \rtimes_{\mathrm{red}} F) \cong K_*(\mathcal{B}^\infty_{F,\Sigma}),$$

the morphism $\tau^\infty_{F,\Sigma}(z)$ is induced in K-theory by the controlled morphism $\mathcal{T}^\infty_{F,\Sigma}(z) : \mathcal{K}_*(\mathcal{A}^\infty_{F,\Sigma}) \to \mathcal{K}_*(\mathcal{B}^\infty_{F,\Sigma})$.

Proof. Assume first that the family $z = (z_i)_{i\in\mathbb{N}}$ is of even degree. According to [4, Lem. 1.6.9], there exist for any integer i

- an F-algebra A'_i;
- two F-equivariant homomorphisms $\alpha_i : A'_i \to B_i$ and $\beta_i : A'_i \to A_i$ such that the induced element $[\beta_i] \in KK^F_*(A'_i, A_i)$ is invertible and such that $z_i = \alpha_{i,*}([\beta_i]^{-1})$.

By naturality of $\jmath_{\cdot,F,\Sigma}$ and $\mathcal{I}_{F,\Sigma,\cdot,*}$ and by left functoriality of $\tau^\infty_{C_0(\Sigma)}$, $J_{G^{\mathbb{N}}_\Sigma}$ and $\tau^\infty_{F,\Sigma}$, we can actually assume that for any integer i, we have

$$z_i = [\beta_i]^{-1}$$

for a homomorphism $\beta_i : B_i \to A_i$ such that the induced element $[\beta_i] \in KK^F_*(B_i, A_i)$ is KK-invertible. Let us consider the family of homomorphisms $\beta = (\beta_i)_{i\in\mathbb{N}}$. Using the bifunctoriality of $\tau^\infty_{F,\Sigma}$, we see that $\tau^\infty_{F,\Sigma}(z)$ is an isomorphism with inverse $\beta^\infty_{F,\Sigma,*}$. But then, if we set $[\mathrm{Id}_\mathcal{A}] = ([\mathrm{Id}_{A_i}])_{i\in\mathbb{N}}$, using once again the naturality of $\jmath_{\cdot,F,\Sigma}$ and $\mathcal{I}_{F,\Sigma,\cdot,*}$ and right functoriality of $J_{G^{\mathbb{N}}_{F,\Sigma}}$ and $\tau^\infty_{F,\Sigma}$, we have

$$\begin{aligned}\beta^\infty_{F,\Sigma,*} \circ \mathcal{I}_{F,\Sigma,\mathcal{B}^\infty,*}(J_{G^{\mathbb{N}}_{F,\Sigma}}(\tau^\infty_{C_0(\Sigma)}(z))) &= \mathcal{I}_{F,\Sigma,\mathcal{A}^\infty,*}(J_{G^{\mathbb{N}}_{F,\Sigma}}(\tau^\infty_{C_0(\Sigma)}(\beta_*(z))))\\ &= \mathcal{I}_{F,\Sigma,\mathcal{A}^\infty,*}(J_{G^{\mathbb{N}}_{F,\Sigma}}(\tau^\infty_{C_0(\Sigma)}([\mathrm{Id}_\mathcal{A}]))).\end{aligned}$$

But up to the identifications provided by $\mathcal{I}_{F,\Sigma,\cdot}$, then $J_{G^{\mathbb{N}}_{F,\Sigma}}(\tau^\infty_{C_0(\Sigma)}([\mathrm{Id}_\mathcal{A}]))$ coincides with $\jmath_{\mathcal{A},F,\Sigma,*}$ and hence we get the result in the even case.

If $z = (z_i)_{i\in\mathbb{N}}$ is a family of odd degree, then, for every integer i, the element z_i of $KK^F_1(A_i, B_i)$ can be viewed up to Morita equivalence as implementing the boundary element of a semi-split extension of F-algebras

$$0 \to \mathscr{K}(\mathscr{H}) \otimes B_i \to E_i \to A_i \to 0.$$

If we set $\mathcal{E} = (E_i)_{i\in\mathbb{N}}$, then the induced extension

$$0 \to \mathcal{B}^\infty_{C_0(\Sigma)} \to \mathcal{E}_{C_0(\Sigma)} \to \mathcal{A}_{C_0(\Sigma)} \to 0$$

is a semisplit extension of the $G^{\mathbb{N}}_{F,\Sigma}$-algebra and hence gives rise to an extension of C^*-algebras

$$0 \to \mathcal{B}^\infty_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G^{\mathbb{N}}_{F,\Sigma} \to \mathcal{E}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G^{\mathbb{N}}_{F,\Sigma} \to \mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G^{\mathbb{N}}_{F,\Sigma} \to 0.$$

Moreover, by naturality of $\mathcal{I}_{F,\Sigma,\cdot}$, we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{B}^\infty_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G^{\mathbb{N}}_{F,\Sigma} & \longrightarrow & \mathcal{E}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G^{\mathbb{N}}_{F,\Sigma} & \longrightarrow & \mathcal{A}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G^{\mathbb{N}}_{F,\Sigma} & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathcal{I}_{F,\Sigma,\mathcal{B}^\infty}} & & \Big\downarrow{\scriptstyle \mathcal{I}_{F,\Sigma,\mathcal{E}}} & & \Big\downarrow{\scriptstyle \mathcal{I}_{F,\Sigma,\mathcal{A}}} & & \\ 0 & \longrightarrow & \mathcal{B}^\infty_\Sigma \rtimes F & \longrightarrow & \mathcal{E}_\Sigma \rtimes F & \longrightarrow & \mathcal{A}_\Sigma \rtimes F & \longrightarrow & 0. \end{array}$$

By using naturality of the boundary map in K-theory, the result in the odd case is a consequence of the two following observations:

- $J_{G^{\mathbb{N}}_{F,\Sigma}}(\tau^{\infty}_{C_0(\Sigma)}(z))$ implements the boundary map of the top extension.
- If $\partial_{\mathcal{B}^{\infty}_{\Sigma}\rtimes F,\mathcal{E}_\Sigma\rtimes F}$ stands for the boundary map of the bottom extension, then

$$\partial_{\mathcal{B}^{\infty}_{\Sigma}\rtimes F,\mathcal{E}_\Sigma\rtimes F}=\tau^{\infty}_{F,\Sigma}(z)\circ \jmath_{\mathcal{A},F,\Sigma,*}. \qquad \square$$

Proposition 5.10. *Let Σ be a discrete metric space provided with a free action of a finite group F by isometries. Let $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ be a family of F-algebras. Then we have a commutative diagram*

$$\begin{array}{ccc} K^{\mathrm{top},\infty}_*(F,\Sigma,\mathcal{A}) & \xrightarrow{\upsilon_{F,\Sigma,\mathcal{A},*}} & K^{\mathrm{top}}_*(G^{\mathbb{N}}_{F,\Sigma},\mathcal{A}^{\infty}_{C_0(\Sigma)}) \\ \Big\downarrow \nu^{\infty}_{F,\Sigma,\mathcal{A},*} & & \Big\downarrow \mu_{G^{\mathbb{N}}_{F,\Sigma},\mathcal{A}_{C_0(\Sigma)},*} \\ K_*(\mathcal{A}^{\infty}_{\Sigma}\rtimes F) & \xrightarrow{\mathcal{I}^{-1}_{F,\Sigma,\mathcal{A}^{\infty},*}} & K_*(\mathcal{A}^{\infty}_{C_0(\Sigma)}\rtimes_{\mathrm{red}} G^{\mathbb{N}}_{F,\Sigma}). \end{array}$$

Proof. Let $Z=P_K(G^{\mathbb{N}}_{\Sigma})$ for K an F-invariant subset in $G^{\mathbb{N}}_{\Sigma}$. Let us fix $r>0$ such that $Z\subseteq P_r(G^{\mathbb{N}}_{\Sigma})$. Let us define

$$\phi_Z: Z\to\mathbb{C},\quad \eta\mapsto\eta(\chi_0),$$

where χ_0 is the characteristic function of the diagonal of $\Sigma\times\Sigma$. Then ϕ_Z is a cut-off function for the proper action of $G^{\mathbb{N}}_{\Sigma}$ on Z. Let

$$P_{Z,G^{\mathbb{N}}_{\Sigma}}: Z\times_{\beta_{\mathbb{N}\times\Sigma}} G^{\mathbb{N}}_{\Sigma}\to\mathbb{C},\quad (\eta,\gamma)\mapsto\phi_Z(\eta)^{1/2}\phi_Z(\eta\cdot\gamma)^{1/2}$$

be the Mishchenko projection of $C_0(Z)\rtimes_{\mathrm{red}} G_\Sigma$ associated to ϕ_Z. For a family $\mathcal{X}=(X_i)_{i\in\mathbb{N}}$ of F-invariant compact subsets of $P_r(G^{\mathbb{N}}_{\Sigma})$ such that $Z_{\mathcal{X}}\subseteq Z$, let us consider $P_{\mathcal{X}}=(P_{X_i})_{i\in\mathbb{N}}$ in $\mathcal{C}_{\mathcal{X},F,\Sigma}$, where P_{X_i} is for each integer i the projection defined by equation (6) of Section 4.1. Recall that then, $P^{\infty}_{\mathcal{X}}=(P_{X_i}\otimes e)_{i\in\mathbb{N}}$ in $\mathcal{C}^{\infty}_{\mathcal{X},\Sigma}$ for e a fixed rank-one projection in $\mathscr{K}(H)$. Notice that $[P^{\infty}_{\mathcal{X}}]=\jmath^{\infty}_{\mathcal{C}_{\mathcal{X},F,\Sigma},*}[P_{\mathcal{X}}]$ in $K_0(\mathcal{C}^{\infty}_{\mathcal{X},F,\Sigma}))\cong K_0(\mathcal{C}^{\infty}_{\mathcal{X},\Sigma}\rtimes F)$. Thus the commutativity of the diagram amounts to showing that

$$\mathcal{I}_{F,\Sigma,\mathcal{A}^{\infty},*}\big([P_{Z,G^{\mathbb{N}}_{\Sigma}}]\otimes J_{G^{\mathbb{N}}_{\Sigma}}(\Lambda^{Z,*}_{\mathcal{X}}(\tau^{\infty}_{C_0(\Sigma)})(z))\big)=\tau^{\infty}_{F,\Sigma}(z)([\jmath_{\mathcal{C}_{\mathcal{X},F,\Sigma}}(P_{\mathcal{X}})])$$

for all z in $\prod_{i\in\mathbb{N}}KK^F_*(C_0(X_i),A_i)$ up to the identification $K_*(A^{\infty}_{\Sigma}\rtimes F)\cong K_*(A^{\infty}_{F,\Sigma})$. But it is straight-forward to check that

$$\Lambda^{Z}_{\mathcal{X}}(\phi_Z)=(\phi_{\Sigma,i})_{i\in\mathbb{N}}$$

with $\phi_{\Sigma,i}:\Sigma\times X_i\to\mathbb{C}$, $(\sigma,x)\mapsto\lambda_\sigma(x)$. Hence, if

$$\Lambda^{Z}_{\mathcal{X},G^{\mathbb{N}}_{F,\Sigma}}: C_0(Z)\rtimes G^{\mathbb{N}}_{F,\Sigma}\to\mathcal{C}_{\mathcal{X},C_0(\Sigma)}\rtimes_{\mathrm{red}} G^{\mathbb{N}}_{F,\Sigma}$$

stands for the map induced by $\Lambda^{Z}_{\mathcal{X}}$ on the reduced crossed-products, we have

$$\mathcal{I}_{F,\Sigma,\mathcal{C}_{\mathcal{X}}}\circ\Lambda^{Z}_{\mathcal{X},G^{\mathbb{N}}_{F,\Sigma}}(P_{Z,G^{\mathbb{N}}_{\Sigma}})=P_{\mathcal{X}}. \tag{20}$$

From this, we deduce

$$\begin{aligned}
&\mathcal{I}_{F,\Sigma,\mathcal{A}^\infty,*}([P_{Z,G^{\mathbb{N}}_\Sigma}]\otimes J_{G^{\mathbb{N}}_{F,\Sigma}}(\Lambda^Z_{\mathcal{X},*}(\tau^\infty_{C_0(\Sigma)})(z)))\\
&\quad=\mathcal{I}_{F,\Sigma,\mathcal{A}^\infty,*}([\Lambda^{Z,}_{\mathcal{X},G^{\mathbb{N}}_{F,\Sigma}}(P_{Z,G^{\mathbb{N}}_{F,\Sigma}})]\otimes J_{G^{\mathbb{N}}_{F,\Sigma}}(\tau^\infty_{C_0(\Sigma)})(z)))\\
&\quad=[\Lambda^Z_{\mathcal{X},G^{\mathbb{N}}_{F,\Sigma}}(P_{\mathcal{X},G^{\mathbb{N}}_\Sigma})]\otimes \mathcal{I}_{F,\Sigma,\mathcal{A}^\infty,*}(J_{G^{\mathbb{N}}_{F,\Sigma}}(\tau^\infty_{C_0(\Sigma)})(z))\\
&\quad=\tau^\infty_{F,\Sigma}(z)\circ \jmath^\infty_{\mathcal{C}_\mathcal{X},F,\Sigma,*}\circ \mathcal{I}_{F,\Sigma,\mathcal{C}_\mathcal{X},*}\circ \Lambda^Z_{\mathcal{X},G^{\mathbb{N}}_{F,\Sigma},*}([P_{Z,G^{\mathbb{N}}_\Sigma}])\\
&\quad=\tau^\infty_{F,\Sigma}(z)\circ \jmath^\infty_{\mathcal{C}_\mathcal{X},F,\Sigma,*}([P_\mathcal{X}]),
\end{aligned}$$

where the first equality holds by naturality of $J_{G^{\mathbb{N}}_{F,\Sigma}}$ and left functoriality of the Kasparov product, the second equality holds by right functoriality of the Kasparov product, the third equality is a consequence of Proposition 5.9, and the fourth equality holds by equation (20). □

As a consequence of Corollary 5.7 and Proposition 5.10 we obtain the following theorem.

Theorem 5.11. *Let F be a finite group acting freely on a discrete metric space Σ with bounded geometry. Let $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ be a family of C^*-algebras. Then the three following assertions are equivalent:*

(i) $\nu^\infty_{F,\Sigma,\mathcal{A},*}: K^{\mathrm{top},\infty}_*(F,\Sigma,\mathcal{A})\to K_*(\mathcal{A}^\infty_\Sigma\rtimes F)$ *is an isomorphism.*
(ii) *The groupoid $G_{F,\Sigma}$ satisfies the Baum–Connes conjecture with coefficients in $\mathcal{A}^\infty_{C_0(\Sigma)}$.*
(iii) *The groupoid $G^{\mathbb{N}}_{F,\Sigma}$ satisfies the Baum–Connes conjecture with coefficients in $\mathcal{A}^\infty_{C_0(\Sigma)}$.*

5.12. **Quantitative statements.** We are now in a position to state the analog of the quantitative statements of [9, §6.2] in the setting of discrete metric spaces with bounded geometry.

Let F be a finite group, let Σ a be discrete metric space with bounded geometry provided with an action of F by isometries, and let A be an F-algebra. Let us consider the following statements for d,d',r,r',ε and ε' positive numbers with $d\le d'$, $\varepsilon'\le\varepsilon<\frac14$, $r_{d,\varepsilon}\le r$ and $r'\le r$:

$QI_{F,\Sigma,A,*}(d,d',r,\varepsilon)$: For any element x in $K^F_*(P_d(\Sigma),A)$, the following holds:

$$\nu^{\varepsilon,r,d}_{F,\Sigma,A,*}(x)=0 \text{ in } K^{\varepsilon,r}_*(A_{F,\Sigma}) \quad\Longrightarrow\quad q^*_{d,d'}(x)=0 \text{ in } K^F_*(P_{d'}(\Sigma),A).$$

$QS_{F,\Sigma,A,*}(d,r',r,\varepsilon',\varepsilon)$: For every y in $K^{\varepsilon',r'}_*(A_{F,\Sigma})$, there exists an element x in $K^F_*(P_d(\Sigma),A)$ such that

$$\nu^{\varepsilon,r,d}_{F,\Sigma,A,*}(x)=\iota^{\varepsilon',\varepsilon,r',r}_*(y).$$

The following results provide numerous examples that satisfy these quantitative statements.

Theorem 5.13. *Let F be a finite group, let Σ be a discrete metric space with bounded geometry provided with a free action of F by isometries, and let A be an F-algebra. Then the following assertions are equivalent:*

(i) *For any positive numbers d, ε and r with $\varepsilon < \frac{1}{4}$ and $r \geq r_{d,\varepsilon}$, there exists a positive number d' with $d' \geq d$ for which $QI_{F,\Sigma,A,*}(d,d',r,\varepsilon)$ is satisfied.*

(ii) *The assembly map*

$$\nu^{\infty}_{F,\Sigma,A,*} : K^{\mathrm{top},\infty}_*(F,\Sigma,A^{\mathbb{N}}) \to K_*(A^{\mathbb{N},\infty}_{\Sigma} \rtimes F)$$

is one-to-one.

(iii) *The assembly map*

$$\mu^{\infty}_{G_{F,\Sigma},A^{\mathbb{N},\infty}_{C_0(\Sigma)},*} : K^{\mathrm{top}}_*(G_{F,\Sigma}, A^{\mathbb{N},\infty}_{C_0(\Sigma)}) \to K_*(A^{\mathbb{N},\infty}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G_{F,\Sigma})$$

is one-to-one.

Proof. Notice that injectivity for

$$\mu^{\infty}_{G_{F,\Sigma},A^{\mathbb{N},\infty}_{C_0(\Sigma)},*} \quad \text{and} \quad \mu^{\infty}_{G^{\mathbb{N}}_{F,\Sigma},A^{\mathbb{N},\infty}_{C_0(\Sigma)},*}$$

are equivalent. Thus the equivalence of (ii) and (iii) is a consequence of Corollary 5.7 and Proposition 5.10.

Let us prove that points (i) and (ii) are equivalent. Assume that condition (i) holds. Let $x = (x_i)_{i\in\mathbb{N}}$ be a family of elements in some $K^F_*(P_d(\Sigma),A)$ such that $\nu^{\infty,d}_{F,\Sigma,A,*}(x) = 0$. By definition of $\nu^{\infty,d}_{F,\Sigma,A,*}(x)$, we have

$$\iota^{\varepsilon',r'}_*(\nu^{\infty,\varepsilon',r',d}_{F,\Sigma,A,*}(x)) = 0$$

for any ε' in $(0,\frac{1}{4})$ and $r' \geq r_{d,\varepsilon'}$. Hence, by Proposition 2.4, we can find ε in $(0,\frac{1}{4})$ and $r \geq r_{d,\varepsilon}$ such that $\nu^{\infty,\varepsilon,r,d}_{F,\Sigma,A,*}(x) = 0$. But up to the controlled isomorphisms of Proposition 4.10 and of Lemma 4.11, $\nu^{\infty,\varepsilon,r,d}_{F,\Sigma,A,*}(x)$ coincides with $\prod_{i\in\mathbb{N}} \nu^{\varepsilon,r,d}_{F,\Sigma,A,*}(x_i)$, so up to rescaling ε and r by a (universal) control pair, we can assume that

$$\nu^{\varepsilon,r,d}_{F,\Sigma,A,*}(x_i) = 0$$

for every integer i. Let $d' \geq d$ be a number such that $QI_{F,\Sigma,A,*}(d,d',r,\varepsilon)$ is satisfied. Then we get that $q_{d,d',*}(x_i) = 0$ and hence $q_{d,d',*}(x) = 0$.

Let us prove the converse. Assume first that there exist positive numbers d, ε and r with $\varepsilon < \frac{1}{4}$ and $r \geq r_{d,\varepsilon}$ and such that for all $d' \geq d$, the condition $QI_{\Sigma,F,A,*}(d,d',r,\varepsilon)$ does not hold. Let us prove that $\nu^{\infty,d}_{F,\Sigma,A,*}$ is not one-to-one. Let $(d_i)_{i\in\mathbb{N}}$ be an increasing and unbounded sequence of positive numbers such that $d_i \geq d$ for every integer i. For every integer i, let x_i be an element in $K^F_*(P_d(\Sigma),A)$ such that

$$\nu^{\varepsilon,r,d}_{F,\Sigma,A,*}(x_i) = 0 \text{ in } K^{\varepsilon,r}_*(A_{F,\Sigma}) \quad \text{and} \quad q_{d,d_i,*}(x_i) \neq 0 \text{ in } K^F_*(P_{d_i}(\Sigma),A)$$

and set $x = (x_i)_{i\in\mathbb{N}}$. Then we have $\nu^{\infty,d}_{F,\Sigma,A,*}(x) = 0$ and $q_{d,d_i,*}(x) \neq 0$ for all i. Since the sequence $(d_i)_{i\in\mathbb{N}}$ is unbounded, we deduce that the kernel of $\nu^{\infty}_{F,\Sigma,A,*}$ is nontrivial. □

Theorem 5.14. *There exists $\lambda > 1$ such that for any finite group F, any discrete metric space Σ with bounded geometry, provided with a free action of F by isometries, and any F-algebra A, the following assertions are equivalent:*

(i) *For any positive numbers ε and r' with $\varepsilon < \frac{1}{4\lambda}$, there exist positive numbers d and r with $r_{d,\varepsilon} \leq r$ and $r' \leq r$ for which $QS_{F,\Sigma,A,*}(d, r', r, \varepsilon, \lambda\varepsilon)$ is satisfied.*

(ii) *The assembly map*

$$\nu^{\infty}_{F,\Sigma,A,*} : K^{\mathrm{top},\infty}_*(F, \Sigma, A^{\mathbb{N}}) \to K_*(A^{\mathbb{N},\infty}_{\Sigma} \rtimes F)$$

is onto.

(iii) *The assembly map*

$$\mu_{G_{F,\Sigma},A^{\mathbb{N},\infty}_{C_0(\Sigma)},*} : K^{\mathrm{top}}_*(G_{F,\Sigma}, A^{\mathbb{N},\infty}_{C_0(\Sigma)}) \to K_*(A^{\mathbb{N},\infty}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G_{F,\Sigma})$$

is onto.

Proof. Notice that surjectivity for

$$\mu^{\infty}_{G_{F,\Sigma},A^{\mathbb{N},\infty}_{C_0(\Sigma)},*} \quad \text{and} \quad \mu^{\infty}_{G^{\mathbb{N}}_{F,\Sigma},A^{\mathbb{N},\infty}_{C_0(\Sigma)},*}$$

are equivalent. Thus the equivalence of (ii) and (iii) is a consequence of Corollary 5.7 and Proposition 5.10.

Choose λ as in Proposition 2.4 and assume that condition (i) holds. Let z be an element in $K_*(A^{\mathbb{N},\infty}_{\Sigma} \rtimes F)$ and let y be an element in $K^{\varepsilon,r'}_*(A^{\mathbb{N},\infty}_{F,\Sigma})$ such that $\iota^{\varepsilon,r'}_*(y)$ corresponds to z up to the identification

$$K_*(A^{\mathbb{N},\infty}_{\Sigma} \rtimes F) \cong K_*(A^{\mathbb{N},\infty}_{F,\Sigma}).$$

Let y_i be the image of y under the composition

$$K^{\varepsilon,r'}_*(A^{\mathbb{N},\infty}_{F,\Sigma}) \to K^{\varepsilon,r'}_*(\mathscr{K}(\mathscr{H}) \otimes A_{F,\Sigma}) \xrightarrow{\cong} K^{\varepsilon,r'}_*(A_{F,\Sigma}), \tag{21}$$

where the first map is induced by the evaluation $A^{\mathbb{N},\infty}_{F,\Sigma} \to A_{F,\Sigma} \otimes \mathscr{K}(\mathscr{H})$ at the ith coordinate and the second map is the Morita equivalence. Let d and r be numbers with $r \geq r'$ and $r \geq r_{d,\varepsilon}$ and such that $QS_{F,\Sigma,A,*}(d, r', r, \varepsilon, \lambda\varepsilon)$ holds. Then for any integer i, there exists an x_i in $K^F_*(P_d(\Sigma), A)$ such that

$$\nu^{\lambda\varepsilon,r,d}_{F,\Sigma,A,*}(x_i) = \iota^{\varepsilon,\lambda\varepsilon,r',r}_*(y_i) \text{ in } K^{\lambda\varepsilon,r}_*(A_{F,\Sigma}).$$

Consider then $x = (x_i)_{i\in\mathbb{N}}$ in $K^{\mathrm{top},\infty}_*(F, \Sigma, A^{\mathbb{N}})$. By construction of the map $\nu^{\infty}_{F,\Sigma,A,*}$, we clearly have $\nu^{\infty}_{F,\Sigma,A,*}(x) = z$.

Conversely, assume that there exist positive numbers ε and r' with $\varepsilon < \frac{1}{4\lambda}$ such that for any positive numbers r and d with $r \geq r'$ and $r \geq r_{d,\varepsilon}$, statement $QS_{F,\Sigma,A,*}(d, r', r, \varepsilon, \lambda\varepsilon)$ does not hold. Let us prove then that $\nu^{\infty}_{F,\Sigma,A,*}$ is not onto. Assume first, for the sake of simplicity, that A is unital. Let $(d_i)_{i\in\mathbb{N}}$ and $(r_i)_{i\in\mathbb{N}}$ be increasing and unbounded sequences of positive numbers such that $r_i \geq r_{d_i,\lambda\varepsilon}$ and $r_i \geq r'$. Let y_i be an element in $K^{\varepsilon,r'}_*(A_{F,\Sigma})$ such that $\iota^{\varepsilon,\lambda\varepsilon,r',r_i}_*(y_i)$ is not in the range of $\nu^{\lambda\varepsilon,r_i,d_i}_{F,\Sigma,A,*}$. There exists an element y in $K^{\varepsilon,r'}_*(A^{\mathbb{N},\infty}_{F,\Sigma})$ such that for every integer i, the image of y under the composition of equation (21) is y_i. Assume that for some d' there is an x in $K_*^{\mathrm{top},\infty}(F,\Sigma,A^{\mathbb{N}})$ such that, up to the identification $K_*(A_\Sigma^{\mathbb{N},\infty}\rtimes F)\cong K_*(A_{F,\Sigma}^{\mathbb{N},\infty})$,

$$\iota_*^{\varepsilon,r'}(y)=\mu_{F,\Sigma,A,*}^{\infty,d'}(x).$$

Using Proposition 2.4, we see that there exists a positive number r with $r'\leq r$ and $r_{d',\lambda\varepsilon}\leq r$ such that

$$\nu_{F,\Sigma,A,*}^{\infty,\lambda\varepsilon,r,d'}(x)=\iota_*^{\varepsilon,\lambda\varepsilon,r',r}(y).$$

But then, if we choose i such that $r_i\geq r$ and $d_i\geq d'$, we get by using the definition of the geometric assembly map $\nu_{F,\Sigma,\cdot,*}^{\infty,\cdot,\cdot,\cdot}$ and by equation (21) that $\iota_*^{\varepsilon,\lambda\varepsilon,r',r_i}(y_i)$ belongs to the image of $\nu_{F,\Sigma,A,*}^{\lambda\varepsilon,r_i,d_i}$, which contradicts our assumption. If A is not unital, then we use the control pair of Lemma 2.14 to rescale λ. □

Replacing in the proof of "(ii) ⇒ (i)" of Theorems 5.13 and 5.14 the constant family $A^{\mathbb{N}}$ by a family $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ of F-algebras, we can prove indeed the following result.

Theorem 5.15. *Let Σ be a discrete metric space with bounded geometry provided with a free action of a finite group F by isometries.*

(i) *Assume that for any family $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ of F-algebras, the assembly map*

$$\mu_{G_{F,\Sigma},\mathcal{A}^\infty_{C_0(\Sigma)},*}:K_*^{\mathrm{top}}(G_{F,\Sigma},\mathcal{A}^\infty_{C_0(\Sigma)})\to K_*(\mathcal{A}^\infty_{C_0(\Sigma)}\rtimes_{\mathrm{red}}G_{F,\Sigma})$$

is one-to-one. Then for any positive numbers d, ε, r with $\varepsilon<\frac14$ and $r\geq r_{d,\varepsilon}$, there exists a positive number d' with $d'\geq d$ such that statement $QI_{\Sigma,F,A,*}(d,d',r,\varepsilon)$ is satisfied for every F-algebra A.

(ii) *Assume that for any family $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ of F-algebras, the assembly map*

$$\mu_{G_{F,\Sigma},\mathcal{A}^\infty_{C_0(\Sigma)},*}:K_*^{\mathrm{top}}(G_{F,\Sigma},\mathcal{A}^\infty_{C_0(\Sigma)})\to K_*(\mathcal{A}^\infty_{C_0(\Sigma)}\rtimes_{\mathrm{red}}G_{F,\Sigma})$$

is onto. Then for some $\lambda>1$ and for any positive numbers ε and r' with $\varepsilon<\frac{1}{4\lambda}$, there exist positive numbers d and r with $r_{d,\varepsilon}\leq r$ and $r'\leq r$ such that $QS_{\Sigma,F,A,*}(d,r',r,\varepsilon,\lambda\varepsilon)$ is satisfied for every F-algebra A.

Recall from [11, 16] that if Σ coarsely embeds in a Hilbert space, then the groupoid $G_{F,\Sigma}$ satisfies the Baum–Connes conjecture for any coefficients. In the case of space of finite asymptotic dimension and when the group F is trivial, following the idea of [15], more precise statements are given in [10] without using infinite dimension analysis. We shall briefly describe them here.

Recall first that for a metric set X and a positive number r, a cover $(U_i)_{i\in\mathbb{N}}$ has r-multiplicity n if any ball of radius r in X intersects at most n elements in $(U_i)_{i\in\mathbb{N}}$.

Definition 5.16. Let Σ be a proper discrete metric set. Then Σ has *finite asymptotic dimension* if there exists an integer m such that for any positive number r, there exists a uniformly bounded cover $(U_i)_{i\in\mathbb{N}}$ with finite r-multiplicity $m+1$. The smallest integer that satisfies this condition is called the *asymptotic dimension* of Σ.

Let Σ be a proper metric space with asymptotic dimension m. Then there exists a sequence of positive numbers $(R_k)_{k\in\mathbb{N}}$ and for any positive integer k a cover $(U_i^{(k)})_{i\in\mathbb{N}}$ of Σ such that

- $R_{k+1} > 4R_k$ for every positive integer k;
- $U_i^{(k)}$ has diameter less than R_k for all positive integers i and k;
- for any positive integer k, the R_k-multiplicity of $(U_i^{(k+1)})_{i\in\mathbb{N}}$ is $m+1$.

The sequence $(R_k)_{k\in\mathbb{N}}$ is called the *m-growth* of Σ.

From now on, if Σ is a discrete metric space, then

$$QI_{\Sigma,A,*}(d,d',r,\varepsilon) \quad \text{and} \quad QS_{\Sigma,A,*}(d,r,r',\varepsilon,\varepsilon')$$

respectively stand for $QI_{\{e\},\Sigma,A,*}(d,d',r,\varepsilon)$ and $QS_{\{e\},\Sigma,A,*}(d,r,r',\varepsilon,\varepsilon')$. The following results were proven in [10].

Theorem 5.17. *Let m be an integer and let $(R_k)_{k\in\mathbb{N}}$ be a sequence of positive numbers such that $R_{k+1} > 4R_k$ for every integer k. Then, for any positive numbers d, ε and r with $\varepsilon < \frac{1}{4}$ and $r \geq r_{d,\varepsilon}$, there exists a positive number d' with $d' \geq d$ for which $QI_{\Sigma,A,*}(d,d',r,\varepsilon)$ is satisfied for any discrete proper metric space Σ with bounded geometry, asymptotic dimension m and m-growth $(R_k)_{k\in\mathbb{N}}$ and any C^*-algebra A.*

Theorem 5.18. *There exists a positive number $\lambda > 1$ for which for any integer m and any sequence of positive numbers $(R_k)_{k\in\mathbb{N}}$ such that $R_{k+1} > 4R_k$ for every integer k, the following is satisfied: For any positive numbers ε and r' with $\varepsilon < \frac{1}{4\lambda}$, there exist positive numbers d and r with $r_{d,\varepsilon} \leq r$ and $r' \leq r$ for which $QS_{\Sigma,A,*}(d,r',r,\varepsilon,\lambda\varepsilon)$ is satisfied for any discrete proper metric space Σ with bounded geometry, asymptotic dimension m and m-growth $(R_k)_{k\in\mathbb{N}}$ and any C^*-algebra A.*

The proofs of these theorems extend without difficulties to the equivariant case with respect to isometric actions of a given finite group.

5.19. **Application to the persistence approximation property.** Let F be a finite group, let Σ be a discrete metric space with bounded geometry provided with a free action of F by isometries and let A be an F-algebra. We apply the results of the previous section to the persistence approximation for $A_{F,\Sigma}$: For any ε small enough and any $r > 0$ there exist ε' in $(\varepsilon, \frac{1}{4})$ and $r' \geq r$ such that $\mathcal{PA}_{\Sigma,F,A,*}(\varepsilon,\varepsilon',r,r')$ is satisfied.

Notice that the approximation property is coarse invariant. To apply quantitative statements of the last subsection to our persistence approximation property, we have to define the analog of the existence of a cocompact universal example for proper action of a discrete group in the setting of discrete proper metric space.

Definition 5.20. A discrete metric space Σ provided with a free action of a finite group is coarsely uniformly F-contractible if for every $d > 0$ there exists $d' > d$ such that any invariant compact subset of $P_d(\Sigma)$ lies in an F-equivariantly contractible invariant compact subset of $P_{d'}(\Sigma)$.

Example 5.21. Any (discrete) Gromov hyperbolic metric space provided with a free action of a finite group F by isometries is coarsely uniformly F-contractible [6].

Lemma 5.22. *Let Σ be a proper discrete metric space provided with a free action of a finite group F by isometries. Assume that Σ is coarsely uniformly F-contractible. Then for any positive number d, there exists a positive number d' such that the following is satisfied: For any x in $K_*^F(\mathrm{P}_d(\Sigma), A)$ such that $\nu^d_{F,\Sigma,A,*}(x) = 0$ in $K_*(A_{F,\Sigma})$, we have*

$$q_*^{d,d'}(x) = 0 \text{ in } K_*^F(\mathrm{P}_{d'}(\Sigma), A).$$

Proof. Let A be an F-algebra and let x be an element of $K_*^F(\mathrm{P}_d(\Sigma), A)$ such that $\nu^d_{F,\Sigma,A,*}(x) = 0$ in $K_*(A_{F,\Sigma})$. Let $d' \geq d$ be a positive number such that every invariant compact subset of $P_d(\Sigma)$ lies in an F-equivariantly contractible invariant compact subset of $P_{d'}(\Sigma)$. Then

$$q_*^{d,d'}(x) \in K_*(\mathrm{P}_{d'}(\Sigma), A)$$

comes indeed from an element of $KK_*^F(C(\{p\}), A) \cong KK_*^F(\mathbb{C}, A)$ for p an F-invariant element in $P_{d'}(\Sigma)$. But under the identification between $K_*(A_{F,\Sigma})$ and $K_*(A \rtimes F)$ given by Morita equivalence (see Section 4.1), the map

$$KK_*^F(\mathbb{C}, A) \to K_*(A_{F,\Sigma}), \quad x \mapsto [P_{\{p\}}] \otimes_{C(\{p\})_{F,\Sigma}} \tau_{F,\Sigma}(x)$$

is the Green–Julg duality isomorphism for finite groups [2]. Since

$$\nu^{d'}_{F,\Sigma,A,*} \circ q_*^{d,d'}(x) = \nu^d_{F,\Sigma,A,*}(x) = 0,$$

we deduce that $q_*^{d,d'}(x) = 0$. □

Theorem 5.23. *There exists $\lambda > 1$ such that for any finite group F and any F-algebra A the following holds: Let Σ be a discrete metric space with bounded geometry, provided with a free action of F by isometries. Assume that*

- *the assembly map*

$$\mu_{G_{F,\Sigma}, A^{\mathbb{N},\infty}_{C_0(\Sigma)},*} : K_*^{\mathrm{top}}(G_{F,\Sigma}, A^{\mathbb{N},\infty}_{C_0(\Sigma)}) \to K_*(A^{\mathbb{N},\infty}_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G_{F,\Sigma})$$

  is onto;
- *Σ is uniformly F-contractible.*

Then for any ε in $(0, \frac{1}{4\lambda})$ and any $r > 0$, there exists $r' > 0$ such that $\mathcal{PA}_{F,\Sigma,A,*}(\varepsilon, \lambda\varepsilon, r, r')$ holds.

Proof. In view of Corollary 5.7 and Proposition 5.10, we get that under the assumptions of the theorem,

$$\nu^{\infty}_{F,\Sigma,\mathcal{A},*} : K_*^{\mathrm{top},\infty}(F, \Sigma, \mathcal{A}) \to K_*(\mathcal{A}^{\infty}_{\Sigma} \rtimes F)$$

is onto for any F-algebra A. Consider λ as in Theorem 5.14. Let ε and r be positive numbers with $\varepsilon < \frac{1}{4\lambda}$ and let d and r' be positive numbers with $r' \geq r_{d,\varepsilon}$ such that $QS_{\Sigma,F,A,*}(d, r, r', \varepsilon, \lambda\varepsilon)$ is satisfied for every F-algebra A. Choose d' as in Lemma 5.22 with respect to d. We can assume without loss of generality that $r' \geq r_{d',\lambda\varepsilon}$. Let y be an element of $K_*^{\varepsilon,r}(A_{F,\Sigma})$ such that $\iota_*^{\varepsilon,r}(y) = 0$ in $K_*(A_{F,\Sigma})$. Then there exists x in $K_*(\mathrm{P}_d(\Sigma), A)$ such that $\nu_{F,\Sigma,A,*}^{\lambda\varepsilon,r',d}(x) = \iota_*^{\varepsilon,\lambda\varepsilon,r,r'}(y)$. Then we have

$$\begin{aligned}\nu_{F,\Sigma,A,*}^{d}(x) &= \iota_*^{\lambda\varepsilon,r'} \circ \iota_*^{\varepsilon,\lambda\varepsilon,r,r'}(y) \\ &= \iota_*^{\varepsilon,r}(y) \\ &= 0\end{aligned}$$

and hence, according to Lemma 5.22,

$$\begin{aligned}\iota_*^{\varepsilon,\lambda\varepsilon,r,r'}(y) &= \nu_{F,\Sigma,A,*}^{\lambda\varepsilon,r',d}(x) \\ &= \nu_{F,\Sigma,A,*}^{\lambda\varepsilon,r',d'} \circ q_*^{d,d'}(x) \\ &= 0.\end{aligned}$$

□

Similarly, using Theorem 5.15, we get:

Theorem 5.24. *There exists $\lambda > 1$ such that for any finite group F the following holds: Let Σ be a discrete metric space with bounded geometry, provided with a free action of F by isometries. Assume that*

- *for any family $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ of F-algebras, the assembly map*
$$\mu_{G_{F,\Sigma},\mathcal{A}^\infty_{C_0(\Sigma)},*} : K_*^{\mathrm{top}}(G_{F,\Sigma}, \mathcal{A}^\infty_{C_0(\Sigma)}) \to K_*(\mathcal{A}^\infty_{C_0(\Sigma)} \rtimes_{\mathrm{red}} G_{F,\Sigma})$$
is onto;
- *Σ is coarsely uniformly F-contractible.*

Then for any ε in $(0, \frac{1}{4\lambda})$ and any $r > 0$, there exists $r' > 0$ such that $\mathcal{PA}_{F,\Sigma,A,*}(\varepsilon, \lambda\varepsilon, r, r')$ holds for any F-algebra A.

Corollary 5.25. *There exists $\lambda > 1$ such that for any finite group F and any discrete Gromov hyperbolic metric space Σ provided with a free action of F by isometries, the following holds: For any ε in $(0, \frac{1}{4\lambda})$ and any $r > 0$, there exists $r' > 0$ such that $\mathcal{PA}_{*,F,\Sigma,A}(\varepsilon, \lambda\varepsilon, r, r')$ holds for any F-algebra A.*

6. Applications to Novikov conjecture

In this section, we investigate the connection between the quantitative statements of Section 5.12 and the Novikov conjecture. More precisely, we consider the uniform version of these statements for the family of all finite subsets of a discrete metric space Σ with bounded geometry and we apply them to prove the coarse Baum–Connes conjecture.

6.1. **The coarse Baum–Connes conjecture.** Let us first briefly recall the statement of the coarse Baum–Connes conjecture. Let Σ be a discrete metric space with bounded geometry and let $\mathscr{H}$ be a separable Hilbert space. Set $C[\Sigma]_r$ for the space of locally compact operators on $\ell^2(\Sigma)\otimes\mathscr{H}$ with propagation less than r, i.e. operators that can be written as blocks $T = (T_{x,y})_{(x,y)\in\Sigma^2}$ of compact operators of $\mathscr{H}$ such that $T_{x,y}=0$ if $d(x,y)>r$. The Roe algebra of Σ is then

$$C^*(\Sigma)=\overline{\bigcup_{r>0} C[\Sigma]_r}\subseteq \mathcal{L}(\ell^2(\Sigma)\otimes\mathscr{H})$$

and is by definition filtered by $(C[\Sigma]_r)_{r>0}$. The analog of the Baum–Connes assembly maps in the setting of coarse geometry was defined in [1] as a family of coarse assembly maps (we shall recall the definition of these maps later on)

$$\mu^s_{\Sigma,*}:K_*(P_s(\Sigma))\to K_*(C^*(\Sigma))$$

compatible with the maps $K_*(P_s(\Sigma))\to K_*(P_{s'}(\Sigma))$ induced by the inclusions of Rips complexes $P_s(\Sigma)\hookrightarrow P_{s'}(\Sigma)$ for $s\le s'$. Taking the inductive limit, we end up with the coarse Baum–Connes assembly map

$$\mu_{\Sigma,*}:\lim_{s>0}K_*(P_s(\Sigma))\to K_*(C^*(\Sigma)).$$

We say that Σ satisfies the coarse Baum–Connes conjecture if $\mu_{\Sigma,*}$ is an isomorphism.

The coarse Baum–Connes conjecture is related to the quantitative statements of Section 5.12 in the following way. If Σ is a discrete metric space, then

$$QI_{\Sigma,*}(d,d',r,\varepsilon)\quad\text{and}\quad QS_{\Sigma,*}(d,r,r',\varepsilon,\varepsilon')$$

respectively stand for $QI_{\{e\},\Sigma,\mathbb{C},*}(d,d',r,\varepsilon)$ and $QS_{\{e\},\Sigma,\mathbb{C},*}(d,r,r',\varepsilon,\varepsilon')$.

Theorem 6.2. *Let Σ be a discrete metric space with bounded geometry. Assume that there exists a positive number ε_0 with $\varepsilon_0<\frac14$ such that the following assertions hold:*

- (i) *For any positive numbers d, ε and r with $\varepsilon<\varepsilon_0$ and $r\ge r_{d,\varepsilon}$, there exists a positive number d' with $d'\ge d$ such that $QI_{X,*}(d,d',r,\varepsilon)$ holds for any finite subset X of Σ.*
- (ii) *For any positive numbers ε and r with $\varepsilon<\varepsilon_0$, there exist positive numbers d, ε' and r' with $r'\ge r_{d,\varepsilon'}$, $r'\ge r$ and ε' in $[\varepsilon,\frac14)$ such that $QS_{X,*}(d,r,r',\varepsilon,\varepsilon')$ holds for any finite subset X of Σ.*

Then Σ satisfies the coarse Baum–Connes conjecture.

This theorem will be proved in Section 6.8. Let us recall now the definition of the coarse Baum–Connes assembly maps given in [11, §2.3]. Indeed, the definition of the coarse Baum–Connes assembly map was extended to Roe algebras with coefficients in a C^*-algebra. Let $\mathscr{H}$ be a separable Hilbert space, let Σ be a proper discrete metric space with bounded geometry, and let B be a C^*-algebra. Define $C^*(\Sigma,B)$, the Roe algebra of Σ with coefficients in B, as the closure of locally compact with finite propagation operators in the C^*-algebra of adjointable operators on the right Hilbert B-module $\ell^2(\Sigma)\otimes\mathscr{H}\otimes B$. Then $C^*(\Sigma,B)$ is a sub-C^*-algebra of $\mathcal{L}_B(\ell^2(\Sigma)\otimes\mathscr{H}\otimes B)$. This construction is moreover functorial. Any morphism $f:A\to B$ induces in the obvious way a

C^*-algebra morphism $f_\Sigma : C^*(\Sigma, A) \to C^*(\Sigma, B)$. For any C^*-algebras A and B, an analog in the setting of coarse geometry of the Kasparov transformation has been defined in [11, §2.3] as a natural transformation

$$\sigma_\Sigma : KK_*(A, B) \to KK_*(C^*(\Sigma, A), C^*(\Sigma, B)).$$

Let $(\mathscr{H} \otimes B, \pi, T)$ be a non-degenerate K-cycle for $KK_*(A, B)$. Define then $\widetilde{T} = \mathrm{Id}_{\ell^2(\Sigma)\otimes\mathscr{H}} \otimes T$ acting on the Hilbertian right B-module $\ell^2(\Sigma)\otimes\mathscr{H}\otimes\mathscr{H}\otimes B$. The map

$$\mathcal{L}_A(\ell^2(\Sigma) \otimes \mathscr{H} \otimes A) \to \mathcal{L}_B(\ell^2(\Sigma) \otimes \mathscr{H} \otimes \mathscr{H} \otimes B), \quad T \mapsto T \otimes_\pi \mathrm{Id}_{\mathscr{H}\otimes B}$$

induces, by restriction and under the identification between $\mathcal{L}_B(\ell^2(\Sigma) \otimes \mathscr{H} \otimes \mathscr{H} \otimes B)$ and $\mathcal{L}_{\mathscr{K}(\mathscr{H})\otimes B}(\ell^2(\Sigma) \otimes \mathscr{H} \otimes \mathscr{K}(\mathscr{H}) \otimes B)$, a morphism

$$\widetilde{\pi} : C^*(\Sigma, A) \to M(C^*(\Sigma, B \otimes \mathscr{K}(\mathscr{H}))),$$

where $M(C^*(\Sigma, B \otimes \mathscr{K}(\mathscr{H})))$ stands for the multiplier algebra of $C^*(\Sigma, B \otimes \mathscr{K}(\mathscr{H}))$. Then

$$(M(C^*(\Sigma, B \otimes \mathscr{K}(\mathscr{H}))), \widetilde{\pi}, \widetilde{T})$$

is a K-cycle for $KK_*(C^*(\Sigma, A), C^*(\Sigma, B\otimes\mathscr{K}(\mathscr{H})))$ and hence, under the identification between $C^*(\Sigma, B\otimes\mathscr{K}(\mathscr{H}))$ and $C^*(\Sigma, B)$ we end up with an element in $KK_*(C^*(\Sigma, A), C^*(\Sigma, B))$. We obtain in this way a natural transformation

$$\sigma_\Sigma : KK_*(A, B) \to KK_*(C^*(\Sigma, A), C^*(\Sigma, B)).$$

This transformation is also bifunctorial, i.e. for any C^*-algebra morphisms $f : A_1 \to A_2$ and $g : B_1 \to B_2$ and any element z in $KK_*(A_2, B_1)$, we have

$$\sigma_\Sigma(f^*(z)) = f_\Sigma^*(\sigma_\Sigma(z)) \quad \text{and} \quad \sigma_\Sigma(g_*(z)) = g_{\Sigma,*}(\sigma_\Sigma(z)).$$

If z is an element of $KK_*(A, B)$, we define

$$\mathcal{S}_\Sigma(z) : K_*(C^*(\Sigma, A)) \to K_*(C^*(\Sigma, B)), \quad x \mapsto x \otimes_{C^*(\Sigma,A)} \sigma_\Sigma(z)$$

induced by right multiplication by $\sigma_\Sigma(z)$.

Notice that if

$$0 \to J \to A \to A/J \to 0 \tag{22}$$

is a semi-split extension of C^*-algebras, then $C^*(\Sigma, J)$ can be viewed as an ideal of $C^*(\Sigma, A)$ and we get then a semi-split extension of C^*-algebras

$$0 \to C^*(\Sigma, J) \to C^*(\Sigma, A) \to C^*(\Sigma, A/J) \to 0. \tag{23}$$

If z is the element of $KK_1(A/J, J)$ corresponding to the boundary element of the extension (22), then $\mathcal{S}_\Sigma(z) : K_*(C^*(\Sigma, A/J)) \to K_{*+1}(C^*(\Sigma, J))$ is the boundary morphism associated to the extension (23).

For a C^*-algebra A, let us denote by SA its suspension, i.e.

$$SA = C_0((0, 1), A),$$

by CA its cone, i.e.

$$CA = \{f \in C_0([0, 1], A) \mid f(1) = 0\},$$

and by $ev_0 : CA \to A$ the evaluation map at zero. Let us consider for any C^*-algebra A the Bott extension

$$0 \to SA \to CA \xrightarrow{ev_0} A \to 0,$$

with associated boundary map $\partial_A : K_*(A) \to K_{*+1}(SA)$. It is well known that the corresponding element $[\partial_A]$ of $KK_1(A, SA)$ is invertible with inverse up to Morita equivalence the element of $KK_1(A \otimes \mathscr{K}(\ell^2(\mathbb{N})), SA)$ corresponding to the Toeplitz extension

$$0 \to A \otimes \mathscr{K}(\ell^2(\mathbb{N})) \to \mathcal{T}_0 \otimes A \to SA \to 0.$$

Lemma 6.3. *For any C^*-algebra A, the morphism $\mathcal{S}_\Sigma([\partial_A]^{-1})$ is a left inverse for $\mathcal{S}_\Sigma([\partial_A])$.*

Proof. Consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & SC^*(\Sigma, A) & \longrightarrow & CC^*(\Sigma, A) & \longrightarrow & C^*(\Sigma, A) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \jmath} & & \downarrow & & \downarrow{\scriptstyle =} & & \\ 0 & \longrightarrow & C^*(\Sigma, SA) & \longrightarrow & C^*(\Sigma, CA) & \longrightarrow & C^*(\Sigma, A) & \longrightarrow & 0, \end{array}$$

where

- the top row is the Bott extension for $C^*(\Sigma, A)$ with boundary map
$$\partial_{C^*(\Sigma,A)} : K_*(C^*(\Sigma, A)) \to K_{*+1}(SC^*(\Sigma, A));$$
- the bottom row is the extension induced for Roe algebras by the Bott extension for A with boundary map
$$\mathcal{S}_\Sigma([\partial_A]) : K_*(C^*(\Sigma, A)) \to K_{*+1}(C^*(\Sigma, SA));$$
- the left and the middle vertical arrows are the obvious inclusions.

Consider similarly the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C^*(\Sigma, A) \otimes \mathscr{K}(\ell^2(\mathbb{N})) & \longrightarrow & CC^*(\Sigma, A \otimes \mathcal{T}_0) & \longrightarrow & C^*(\Sigma, SA) & \longrightarrow & 0 \\ & & \uparrow & & \uparrow & & {\scriptstyle \jmath}\uparrow & & \\ 0 & \longrightarrow & C^*(\Sigma, A) \otimes \mathscr{K}(\ell^2(\mathbb{N})) & \longrightarrow & C^*(\Sigma, A) \otimes \mathcal{T}_0 & \longrightarrow & SC^*(\Sigma, A) & \longrightarrow & 0, \end{array}$$

where

- the bottom row is the Toeplitz extension for $C^*(\Sigma, A)$;
- the top row is the extension induced for Roe algebras by the Toeplitz extension for A;
- the left and the middle vertical arrows are the obvious inclusions.

By naturality of the boundary map in the first commutative diagram, we see that

$$\mathcal{S}_\Sigma([\partial_A]) = \jmath_* \circ \partial_{C^*(\Sigma,A)},$$

where $\jmath_* : K_*(SC^*(\Sigma, A)) \to K_*(C^*(\Sigma, SA))$ is the map induced in K-theory by the inclusion $\jmath : SC^*(\Sigma, A) \hookrightarrow C^*(\Sigma, SA)$. Using now the naturality of the boundary map in the second commutative diagram, we see that up to

Morita equivalence, $\mathcal{S}_\Sigma([\partial_A]^{-1})\circ \jmath_*$ is the boundary map for the Toeplitz extension associated to $C^*(\Sigma, A)$ and hence is an inverse for $\partial_{C^*(\Sigma,A)}$. Therefore, $\mathcal{S}_\Sigma([\partial_A]^{-1})$ is a left inverse for $\mathcal{S}_\Sigma([\partial_A])$. □

The transformation $\mathcal{S}_\Sigma$ is compatible with the Kasparov product in the following sense.

Proposition 6.4. *If A, B and D are separable C^*-algebras, let z be an element in $KK_*(A,B)$ and let z' be an element in $KK_*(B,D)$. Then we have*

$$\mathcal{S}_\Sigma(z\otimes_B z') = \mathcal{S}_\Sigma(z')\circ \mathcal{S}_\Sigma(z).$$

Proof. Assume first that z is even. Then according to [4, Thm. 1.6.11], there exist

- a C^*-algebra A_1,
- a morphism $\nu : A_1 \to B$,
- a morphism $\theta : A_1 \to A$ such that the associated element $[\theta]$ in $KK_*(A_1, A)$ is invertible,

such that $z = \nu_*([\theta]^{-1})$. By bifunctoriality of the Kasparov product, we have

$$z\otimes_B z' = \nu_*([\theta]^{-1})\otimes_B z' = [\theta]^{-1}\otimes_{A_1}\nu^*(z').$$

Since σ_Σ and hence $\mathcal{S}_\Sigma$ are natural, we see that $\mathcal{S}_\Sigma([\theta]^{-1})$ is invertible, with inverse induced by $\theta_\Sigma : C^*(\Sigma, A_1) \to C^*(\Sigma, A)$. Then using once again the naturality of $\mathcal{S}_\Sigma$, we have

$$\begin{aligned}
\mathcal{S}_\Sigma(z\otimes_B z')\circ\theta_{\Sigma,*} &= \mathcal{S}_\Sigma(\nu^*(z'))\\
&= \mathcal{S}_\Sigma(z')\circ\nu_{\Sigma,*}\\
&= \mathcal{S}_\Sigma(z')\circ\nu_{\Sigma,*}\circ\mathcal{S}_\Sigma([\theta]^{-1})\circ\theta_{\Sigma,*}\\
&= \mathcal{S}_\Sigma(z')\circ\mathcal{S}_\Sigma(\nu_*([\theta]^{-1}))\circ\theta_{\Sigma,*}\\
&= \mathcal{S}_\Sigma(z')\circ\mathcal{S}_\Sigma(z)\circ\theta_{\Sigma,*}.
\end{aligned}$$

Since $\theta_{\Sigma,*}$ is invertible, we deduce that $\mathcal{S}_\Sigma(z\otimes_B z') = \mathcal{S}_\Sigma(z')\circ\mathcal{S}_\Sigma(z)$. If z' is even, we proceed similarly.

If z and z' are both odd. Let $[\partial_B]$ be the element of $KK_1(B, SB)$ corresponding to the boundary morphism $\partial_B : K_*(B) \to K_{*+1}(SB)$ associated to the Bott extension $0 \to SB \to CB \to B \to 0$. Then

$$\begin{aligned}
\mathcal{S}_\Sigma(z\otimes_B z') &= \mathcal{S}_\Sigma(z\otimes_B [\partial_B]\otimes_{SB}[\partial_B]^{-1}\otimes_B z')\\
&= \mathcal{S}_\Sigma([\partial_B]^{-1}\otimes_B z')\circ\mathcal{S}_\Sigma(z\otimes_B[\partial_B])\\
&= \mathcal{S}_\Sigma([\partial_B]^{-1}\otimes_B z')\circ\mathcal{S}_\Sigma([\partial_B])\circ\mathcal{S}_\Sigma([\partial_B]^{-1})\circ\mathcal{S}_\Sigma(z\otimes_B[\partial_B])\\
&= \mathcal{S}_\Sigma(z')\circ\mathcal{S}_\Sigma(z),
\end{aligned}$$

where the second and the fourth equality hold by the even cases, and the third equality is a consequence of Lemma 6.3. □

Now let Σ be a discrete metric space with bounded geometry. Let $\mathscr{H}$ be a separable Hilbert space and fix a unit vector ξ_0 in $\mathscr{H}$. For any positive number s, let $Q_{s,\Sigma}$ be the operator of $\mathcal{L}_{C_0(P_s(\Sigma))}(C_0(P_s(\Sigma))\otimes\ell^2(\Sigma)\otimes\mathscr{H})$ defined by

$$(Q_{s,\Sigma}\cdot h)(x,\sigma)=\lambda_\sigma^{1/2}(x)\sum_{\sigma'\in\Sigma}\lambda_{\sigma'}^{1/2}(x)\langle h(x,\sigma'),\xi_0\rangle\xi_0,$$

where h in $C_0(P_s(\Sigma))\otimes\ell^2(\Sigma)\otimes\mathscr{H}$ is viewed as function on $P_s(\Sigma)\times\Sigma$ with values in $\mathscr{H}$ (recall that $(\lambda_\sigma)_{\sigma\in\Sigma}$ is the family of coordinate functions in $P_s(\Sigma)$). Then $Q_{s,\Sigma}$ is a projection of $C^*(\Sigma,C_0(P_s(\Sigma)))$. Let B be a C^*-algebra. Then the maps

$$\mu^s_{\Sigma,B,*}:KK_*(P_s(\Sigma),B)\to K_*(C^*(\Sigma,B)),$$
$$z\mapsto[Q_{s,\Sigma}]\otimes_{C^*(\Sigma,C_0(P_s(\Sigma)))}\sigma_\Sigma(z)$$

are compatible with the maps $K_*(P_s(\Sigma))\to K_*(P_{s'}(\Sigma))$ induced by the inclusion of Rips complexes $P_s(\Sigma)\hookrightarrow P_{s'}(\Sigma)$. Taking the inductive limit, we obtain the coarse Baum–Connes assembly map with coefficients in B,

$$\mu_{\Sigma,B,*}:\lim_s KK_*(P_s(\Sigma),B)\to K_*(C^*(\Sigma,B)).$$

If $\mu_{\Sigma,B,*}$ is an isomorphism, we say that Σ satisfies the coarse Baum–Connes conjecture with coefficients in B. When $B=\mathbb{C}$, we set $\mu^s_{\Sigma,*}$ for $\mu^s_{\Sigma,\mathbb{C},*}$, $\mu_{\Sigma,*}$ for $\mu_{\Sigma,\mathbb{C},*}$ and we say that Σ satisfies the coarse Baum–Connes conjecture if

$$\mu_{\Sigma,*}:\lim_s K_*(P_s(\Sigma))\to K_*(C^*(\Sigma))$$

is an isomorphism. Recall that if Γ is a finitely generated group, and if $|\Gamma|$ stands for the metric space arising from any word metric, then the coarse Baum–Connes conjecture for $|\Gamma|$ implies the Novikov conjecture on higher signatures for the group Γ.

6.5. **A geometric assembly map for families of finite metric spaces.** To prove Theorem 6.2, we will need a slight modification of the map $\nu^\infty_{F,\Sigma,\mathcal{A},*}$ defined by equation (9). Let $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ be a family of C^*-algebras and let $\mathcal{X}=(X_n)_{n\in\mathbb{N}}$ be a family of discrete proper metric spaces. Define $\mathcal{A}^\infty_{\mathcal{X}}$ as the closure of the set of

$$x=(x_n)_{n\in\mathbb{N}}\in\prod_{n\in\mathbb{N}}A_n\otimes\mathscr{K}(\ell^2(X_n)\otimes\mathscr{H})$$

such that, for some $r>0$, x_n has propagation less than r for every integer n. Then $\mathcal{A}^\infty_{\mathcal{X}}$ is obviously a filtered C^*-algebra. When $\mathcal{A}$ is the constant family $A_i=\mathbb{C}$, we set $C^*(\mathcal{X})$ for $\mathcal{A}^\infty_{\mathcal{X}}$. According to Lemma 2.14, there exists for a universal control pair (α,h), any family $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ of C^*-algebras and any family $\mathcal{X}=(X_n)_{n\in\mathbb{N}}$ of discrete proper metric spaces, an (α,h)-controlled isomorphism

$$\mathcal{K}_*(\mathcal{A}^\infty_{\mathcal{X}})\to\prod_{n\in\mathbb{N}}\mathcal{K}_*(A_n\otimes\mathscr{K}(\ell^2(X_n)))$$

induced on the j factor and up to Morita equivalence by the restriction to $\mathcal{A}^\infty_\mathcal{X}$ of the evaluation

$$\prod_{n\in\mathbb{N}} A_n \otimes \mathscr{K}(\ell^2(X_n)\otimes\mathscr{H}) \to A_j \otimes \mathscr{K}(\ell^2(X_j)\otimes\mathscr{H}).$$

Proceeding as in Corollary 4.9, we see that there exists a universal control pair (α, h) such that

- for any family $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$ of finite metric spaces,
- for any families of C^*-algebras $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ and $\mathcal{B} = (B_i)_{i\in\mathbb{N}}$,
- for any $z = (z_i)_{i\in\mathbb{N}}$ in $\prod_{i\in\mathbb{N}} KK_*(A_i, B_i)$,

there exists an (α, h)-controlled morphism

$$\mathcal{T}^\infty_\mathcal{X}(z) = (\tau^{\infty,\varepsilon,r}_\mathcal{X}(z))_{0<\varepsilon<\frac{1}{4\alpha},r>0} : \mathcal{K}_*(\mathcal{A}^\infty_\mathcal{X}) \to \mathcal{K}_*(\mathcal{B}^\infty_\mathcal{X})$$

that satisfies in this setting the analogous properties to those listed in Corollary 4.9 and Proposition 4.10 for $\mathcal{T}^\infty_{F,\Sigma}(\cdot)$. Let us denote by

$$\tau^\infty_\mathcal{X}(z)\, K_*(\mathcal{A}^\infty_\mathcal{X}) \to \mathcal{K}_*(\mathcal{B}^\infty_\mathcal{X})$$

the morphisms induced in K-theory by $\mathcal{T}^\infty_\mathcal{X}(z)$, i.e.

$$\tau^\infty_\mathcal{X}(z)\circ\iota^{\varepsilon,r}_*(x) = \iota^{\alpha\varepsilon,h_\varepsilon r}_*\circ\tau^{\infty,\varepsilon,r}_\mathcal{X}(z)(x)$$

for any positive numbers ε and r with $\varepsilon < \frac{1}{4\alpha}$ and any x in $K^{\varepsilon,r}(\mathcal{A}^\infty_\mathcal{X})$. Recall that for a finite metric space Z and a positive number s, the C^*-algebra $C(P_s(Z))\otimes\mathscr{K}(\ell^2(Z))$ inherits a structure of a filtered C^*-algebra from the one on $\mathscr{K}(\ell^2(Z))$ (arising from the metric on Z). The projection $Q_{s,Z}$ of $C(P_s(Z))\otimes\mathscr{K}(\ell^2(Z))$ is defined by

$$Q_{s,Z}(h)(y,z) = \lambda_z^{1/2}(y)\sum_{z'\in Z} h(y,z')\lambda_{z'}^{1/2}(y)$$

for any h in $C(P_s(Z))\otimes\ell^2(Z)\cong C(P_s(Z)\times Z)$ where $(\lambda_z)_{z\in Z}$ is the family of coordinate functions of $P_s(Z)$, i.e. $y = \sum_{z\in Z}\lambda_z(y)$ for any y in $P_s(Z)$. Then $Q_{s,Z}$ has propagation less than $2s$. If we fix any rank-one projection e in $\mathscr{K}(\mathscr{H})$, for any family $\mathcal{X} = (X_i)_{i\in\mathbb{N}}$ of finite metric spaces, then $Q^\infty_{s,\mathcal{X}} = (Q_{s,X_i}\otimes e)_{i\in\mathbb{N}}$ is a projection of propagation less than $2s$ in $\mathcal{A}^\infty_\mathcal{X}$, where $\mathcal{A}$ is the family $(C(P_s(X_i)))_{i\in\mathbb{N}}$.

Now we can proceed as in Section 4.7 to define a quantitative geometric assembly map valued in $C^*(\mathcal{X})$. For any ε in $(0,\frac{1}{4})$, any positive numbers s and r such that $r\geq r_{d,\varepsilon}$, define

$$\nu^{\infty,\varepsilon,r,s}_{\mathcal{X},*} : \prod_{i\in\mathbb{N}} K_*(P_s(X_i)) \to K^{\varepsilon,r}_*(C^*(\mathcal{X})),$$
$$z\mapsto \tau^{\infty,\varepsilon/\alpha,r/h_{\varepsilon/\alpha}}_\mathcal{X}(z)([Q^\infty_{s,\mathcal{X}},0]_{\varepsilon/\alpha,r/h_{\varepsilon/\alpha}}).$$

The family of maps $(\nu^{\infty,\varepsilon,r,s}_{\mathcal{X},*})_{0<\varepsilon<\frac{1}{4},r>r_{s,\varepsilon}}$ is obviously compatible with the structure maps of $\mathcal{K}_*(C^*(\mathcal{X}))$, i.e.

$$\iota^{\varepsilon,\varepsilon',r,r'}_*\circ\nu^{\infty,\varepsilon,r,s}_{\mathcal{X},*} = \nu^{\infty,\varepsilon',r',s}_{\mathcal{X},*}$$

for $0 < \varepsilon \leq \varepsilon' < \frac{1}{4}$ and $r_{s,\varepsilon} < r \leq r'$. This allows us to define

$$\nu_{\mathcal{X},*}^{\infty,s} : \prod_{i\in\mathbb{N}} K_*(P_s(\mathcal{X})) \to K_*(C^*(\mathcal{X}))$$

as $\nu_{\mathcal{X},*}^{\infty,s} = \iota_*^{\varepsilon,r} \circ \nu_{\mathcal{X},*}^{\infty,\varepsilon,r,s}$. The quantitative assembly maps are also compatible with inclusion of Rips complexes. Let

$$q_{s,s',*}^{\infty} : \prod_{i\in\mathbb{N}} K_*(P_s(X_i)) \to \prod_{i\in\mathbb{N}} K_*(P_{s'}(X_i))$$

be the map induced by the family of inclusions $P_s(X_i) \hookrightarrow P_{s'}(X_i)$. Then we have

$$\nu_{\mathcal{X},*}^{\infty,\varepsilon,r,s'} \circ q_{s,s',*}^{\infty} = \nu_{\mathcal{X},*}^{\infty,\varepsilon,r,s}$$

for any positive numbers ε, s, s', and r such that $\varepsilon \in (0, \frac{1}{4})$, $s \leq s'$, $r \geq r_{s',\varepsilon}$, and thus

$$\nu_{\mathcal{X},*}^{\infty,s'} \circ q_{s,s',*}^{\infty} = \nu_{\mathcal{X},*}^{\infty,s}$$

for any positive numbers s and s' such that $s \leq s'$.

Let Σ be a graph space in the sense of [14], i.e. $\Sigma = \bigsqcup_{i\in\mathbb{N}} X_i$, where $(X_i)_{i\in\mathbb{N}}$ is a family of finite metric spaces such that

- for any $r > 0$, there exists an integer N_r such that for any integer i, any ball of radius r in X_i has at most N_r element;
- the distance between X_i and X_j is at least $i + j$ for any distinct integers i and j.

If $\mathcal{X}_\Sigma$ stands for the family $(X_i)_{i\in\mathbb{N}}$, we obviously have an inclusion of filtered C^*-algebras $\jmath_{\mathcal{X}_\Sigma} : C^*(\mathcal{X}_\Sigma) \hookrightarrow C^*(\Sigma)$.

Proposition 6.6. *Let Σ be a graph space $\Sigma = \bigsqcup_{i\in\mathbb{N}} X_i$ as above and let s be a positive number such that $d(X_i, X_j) > s$ if $i \neq j$. Then we have a commutative diagram*

$$\begin{array}{ccc} \prod_{i\in\mathbb{N}} K_*(P_s(X_i)) & \xrightarrow{\nu_{\mathcal{X},*}^{\infty,s}} & K_*(C^*(\mathcal{X}_\Sigma)) \\ \downarrow \simeq & & \downarrow \jmath_{\mathcal{X}_\Sigma,*} \\ K_*(P_s(\Sigma)) & \xrightarrow{\mu_{\Sigma,*}^{s}} & K_*(C^*(\Sigma)), \end{array}$$

where in view of the equality $P_s(\Sigma) = \bigsqcup_{i\in\mathbb{N}} P_s(X_i)$, the left vertical map is the identification between $\prod_{i\in\mathbb{N}} K_*(P_s(X_i))$ and $K_*(\bigsqcup_{i\in\mathbb{N}} P_s(X_i))$.

The proof of this proposition will require some preliminary steps. If $\mathcal{A} = (A_i)_{i\in\mathbb{N}}$ is a family of C^*-algebras, we set $\mathcal{A}^\oplus = \bigoplus_{n\in\mathbb{N}} A_i$. The orthogonal family $(A_i \otimes \mathscr{K}(\ell^2(X_i) \otimes \mathscr{H}))_{i\in\mathbb{N}}$ of corners in $\mathcal{A}^\oplus \otimes \mathscr{K}(\ell^2(\Sigma) \otimes \mathscr{H})$ gives rise to a one-to-one morphism $\jmath_{\mathcal{A},\mathcal{X}_\Sigma} : \mathcal{A}_{\mathcal{X}_\Sigma}^\infty \to C^*(\Sigma, \mathcal{A}^\oplus)$. Let $z = (z_i)_{i\in\mathbb{N}}$ be a family in $\prod_{i\in\mathbb{N}} KK_*(A_i, \mathbb{C})$. Recall that we have a canonical identification between $\prod_{i\in\mathbb{N}} KK_*(A_i, \mathbb{C})$ and $KK_*(\mathcal{A}^\oplus, \mathbb{C})$. Let $\widetilde{z}$ be the element of $KK_*(\mathcal{A}^\oplus, \mathbb{C})$ corresponding to z under this identification.

Lemma 6.7. *For any family $\mathcal{A}=(A_i)_{i\in\mathbb{N}}$ of C^*-algebras, any graph space $\Sigma=\bigsqcup_{i\in\mathbb{N}}X_i$ and any z in $\prod_{i\in\mathbb{N}}KK_*(A_i,\mathbb{C})$, we have a commutative diagram*

$$\begin{array}{ccc} K_*(\mathcal{A}^\infty_{\mathcal{X}_\Sigma}) & \xrightarrow{\tau^\infty_{\mathcal{X}_\Sigma}(z)} & K_*(C^*(\mathcal{X}_\Sigma)) \\ \Big\downarrow{\scriptstyle \jmath_{\mathcal{A},\mathcal{X}_\Sigma,*}} & & \Big\downarrow{\scriptstyle \jmath_{\mathcal{X}_\Sigma,*}} \\ K_*(C^*(\Sigma,\mathcal{A}^\oplus)) & \xrightarrow{\mathcal{S}_\Sigma(\tilde{z})} & K_*(C^*(\Sigma)). \end{array}$$

Proof. Assume first that z is odd. Let us fix a separable Hilbert space $\mathscr{H}$. For each integer i, let $(\mathscr{H},\pi_i,T_i)$ be the K-cycle for $KK_*(A_i,\mathbb{C})$ representing z_i with $\pi_i:A_i\to\mathcal{L}(\mathscr{H})$ a representation and T_i in $\mathcal{L}(\mathscr{H})$ satisfying the K-cycle conditions. Let us set

$$P_i=\frac{T_i+\mathcal{I}d_{\mathscr{H}}}{2}$$

and

$$E_i=\big\{(x,T)\in A_i\oplus\mathcal{L}(\mathscr{H})\mid P_i\pi_i(x)P_i-T\in\mathscr{K}(\mathscr{H})\big\}.$$

We have an inclusion

$$\mathscr{K}(\mathscr{H})\hookrightarrow E_i,\quad T\mapsto(0,T)$$

as an ideal and a surjection

$$E_i\to A_i,\quad (x,T)\to x.$$

Up to Morita equivalence, z_i induces by left multiplication the boundary morphism of the semi-split extension

$$0\to\mathscr{K}(\mathscr{H})\to E_i\to A_i\to 0.$$

Let $\mathcal{E}$ be the family $(E_i)_{i\in N}$ and set $\mathcal{CH}$ for the constant family $(\mathscr{K}(\mathscr{H}))_{i\in\mathbb{N}}$. Then the extension

$$\begin{aligned}0\mapsto\prod_{i\in\mathbb{N}}\mathscr{K}(\mathscr{H})\otimes\mathscr{K}(\ell^2(X_i)\otimes\mathscr{H})\to\prod_{i\in\mathbb{N}}E_i\otimes\mathscr{K}(\ell^2(X_i)\otimes\mathscr{H})\\ \to\prod_{i\in\mathbb{N}}A_i\otimes\mathscr{K}(\ell^2(X_i)\otimes\mathscr{H})\to 0\end{aligned}$$

restricts to a semi-split extension of filtered C^*-algebras:

$$0\mapsto\mathcal{CH}^\infty_{\mathcal{X}_\Sigma}\to\mathcal{E}^\infty_{\mathcal{X}_\Sigma}\to\mathcal{A}^\infty_{\mathcal{X}_\Sigma}\to 0.$$

Up to the identification between $K_*(\mathcal{CH}^\infty_{\mathcal{X}_\Sigma})$ and $K_*(C^*(\mathcal{X}_\Sigma))$ arising from Morita equivalence between $\mathbb{C}$ and $\mathscr{K}(\mathscr{H})$, the boundary morphism associated to this extension is

$$\mathcal{T}^\infty_{\mathcal{X}_\Sigma}(z):K_*(\mathcal{A}^\infty_{\mathcal{X}_\Sigma})\to K_{*+1}(C^*(\mathcal{X}_\Sigma)).$$

In the same way, let

$$E=\Big\{((x_i)_{i\in\mathbb{N}},T)\in\Big(\bigoplus_{n\in\mathbb{N}}A_i\Big)\oplus\mathcal{L}(\ell^2(\mathbb{N},\mathscr{H}))\\ \mid\Big(\bigoplus_{i\in\mathbb{N}}p_i\pi_i(x_i)p_i\Big)-T\in\mathscr{K}(\ell^2(\mathbb{N},\mathscr{H}))\Big\}.$$

As before we have a semi-split extension

$$0\to\mathscr{K}(\ell^2(\mathbb{N})\otimes\mathscr{H})\to E\to\mathcal{A}^{\oplus}\to 0. \tag{24}$$

Moreover,

$$\mathcal{S}_\Sigma(\tilde{z}):K_*(C^*(\Sigma,\mathcal{A}^\oplus))\to K_{*+1}(C^*(\Sigma))$$

is, up to the identification between $K_*(C^*(\Sigma))$ and $K_*(C^*(\Sigma,\mathscr{K}(\ell^2(\mathbb{N})\otimes\mathscr{H})))$ arising from Morita equivalence, the boundary morphism for the extension

$$0\to C^*(\Sigma,\mathscr{K}(\ell^2(\mathbb{N})\otimes\mathscr{H}))\to C^*(\Sigma,E)\to C^*(\Sigma,\mathcal{A}^\oplus)\to 0$$

induced by the extension of equation (24). For every integer i, there is an obvious representation of $\mathscr{K}(\mathscr{H}\otimes\ell^2(X_i))\otimes E_i$ on the right E-Hilbert module $\mathscr{H}\otimes\ell^2(\Sigma)\otimes E$ as a corner which gives rise when i runs through integers to a C^*-morphism $\jmath'_{\mathcal{E},\mathcal{X}_\Sigma}:\mathcal{E}^\infty_{\mathcal{X}_\Sigma}\to C^*(\Sigma,E)$ such that

$$\jmath'_{\mathcal{E},\mathcal{X}_\Sigma}(\mathcal{CH}^\infty_{\mathcal{X}_\Sigma})\subseteq C^*(\Sigma,\mathscr{K}(\ell^2(\mathbb{N})\otimes\mathscr{H})).$$

We have then a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{CH}^\infty_{\mathcal{X}_\Sigma} & \longrightarrow & \mathcal{E}^\infty_{\mathcal{X}_\Sigma} & \longrightarrow & \mathcal{A}^\infty_{\mathcal{X}_\Sigma} & \longrightarrow & 0\\ & & \downarrow{\scriptstyle \jmath'_{\mathcal{E},\mathcal{X}_\Sigma}} & & \downarrow{\scriptstyle \jmath'_{\mathcal{E},\mathcal{X}_\Sigma}} & & \downarrow{\scriptstyle \jmath_{\mathcal{A},\mathcal{X}_\Sigma}} & & \\ 0 & \longrightarrow & C^*(\Sigma,\mathscr{K}(\ell^2(\mathbb{N})\otimes\mathscr{H})) & \longrightarrow & C^*(\Sigma,E) & \longrightarrow & C^*(\Sigma,\mathcal{A}^\oplus) & \longrightarrow & 0. \end{array} \tag{25}$$

The restriction morphism

$$\mathcal{CH}^\infty_{\mathcal{X}_\Sigma}\xrightarrow{\jmath'_{\mathcal{E},\mathcal{X}_\Sigma}}C^*(\Sigma,\mathscr{K}(\ell^2(\mathbb{N})\otimes\mathscr{H}))$$

is homotopic to the composition

$$\mathcal{C}\mathscr{H}^\infty_{\mathcal{X}_\Sigma}\to C^*(\Sigma,\mathscr{K}(\mathscr{H}))\to C^*(\Sigma,\mathscr{K}(\ell^2(\mathbb{N})\otimes\mathscr{H})), \tag{26}$$

where the first map is induced by the obvious representation of

$$\mathcal{C}\mathscr{H}^\infty_{\mathcal{X}_\Sigma}=\prod_{i=1}^\infty\mathscr{K}(\mathscr{H}\otimes\ell^2(X_i))\otimes\mathscr{K}(\mathscr{H})$$

on the $\mathscr{K}(\mathscr{H})$-right Hilbert module $\mathscr{H}\otimes\ell^2(\Sigma)\otimes\mathscr{K}(\mathscr{H})$ (each factor acting as a corner); the second map is induced by the morphism

$$\mathscr{K}(\mathscr{H})\to\mathscr{K}(\ell^2(\mathbb{N})\otimes\mathscr{H}),\quad x\mapsto x\otimes e,$$

where e is any rank-one projection in $\mathscr{K}(\ell^2(\mathbb{N}))$. But up to the identification on one hand between $K_*(\mathcal{C}\mathscr{H}^\infty_{\mathcal{X}_\Sigma})$ and $K_*(C^*(\mathcal{X}_\Sigma))$, and on the other hand between

$K_*(C^*(\Sigma, \mathscr{K}(\ell^2(\mathbb{N}) \otimes \mathscr{H})))$ and $K_*(C^*(\Sigma))$, the morphism of equation (26) induces in K-theory

$$\jmath_{\mathcal{X}_\Sigma,*} : K_*(C^*(\mathcal{X}_\Sigma)) \to K_*(C^*(\Sigma)).$$

Since in the commutative diagram (25), $\mathcal{S}_\Sigma(\widetilde{z})$ is the boundary morphism associated to the top row and $\mathcal{T}^\infty_{\mathcal{X}_\Sigma}(z)$ is the boundary morphism associated to the bottom row, the lemma in the odd case is then a consequence of the naturality of the boundary morphisms.

If z is even, set

$$[\partial_\mathcal{A}] = ([\partial_{A_i}])_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} KK_*(A_i, SA_i)$$

and

$$[\partial_\mathcal{A}]^{-1} = ([\partial_{A_i}]^{-1})_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} KK_*(SA_i, A_i).$$

Let us also define the families $\mathcal{SA} = (SA_i)_{i\in\mathbb{N}}$ and $\mathcal{CA} = (CA_i)_{i\in\mathbb{N}}$ and set

$$z' = ([\partial_{A_i}]^{-1} \otimes_{A_i} z_i)_{i\in\mathbb{N}} \in \prod_{i\in\mathbb{N}} KK_*(SA_i, \mathbb{C}).$$

Using the odd case and the compatibility of the transformation $\mathcal{T}^\infty_{\mathcal{X}_\Sigma}(\cdot)$ with Kasparov products, we get that

$$\begin{aligned} \jmath_{\mathcal{X}_\Sigma,*} \circ \mathcal{T}^\infty_{\mathcal{X}_\Sigma}(z) &= \jmath_{\mathcal{X}_\Sigma,*} \circ \mathcal{T}^\infty_{\mathcal{X}_\Sigma}(z') \circ \mathcal{T}^\infty_{\mathcal{X}_\Sigma}([\partial_\mathcal{A}]) \\ &= \mathcal{S}_\Sigma(\widetilde{z'}) \circ \jmath_{\mathcal{SA},\mathcal{X}_\Sigma,*} \circ \mathcal{T}^\infty_{\mathcal{X}_\Sigma}([\partial_\mathcal{A}]). \end{aligned} \tag{27}$$

Under the canonical identifications $(\mathcal{SA})^\oplus \simeq S\mathcal{A}^\oplus$ and $(\mathcal{CA})^\oplus \simeq C\mathcal{A}^\oplus$, we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{SA}^\infty_{\mathcal{X}_\Sigma} & \longrightarrow & \mathcal{CA}^\infty_{\mathcal{X}_\Sigma} & \longrightarrow & \mathcal{A}^\infty_{\mathcal{X}_\Sigma} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \jmath_{\mathcal{SA},\mathcal{X}_\Sigma}} & & \downarrow{\scriptstyle \jmath_{\mathcal{CA},\mathcal{X}_\Sigma}} & & \downarrow{\scriptstyle \jmath_{\mathcal{A},\mathcal{X}_\Sigma}} & & \\ 0 & \longrightarrow & C^*(\Sigma, S\mathcal{A}^\oplus) & \longrightarrow & C^*(\Sigma, C\mathcal{A}^\oplus) & \longrightarrow & C^*(\Sigma, \mathcal{A}^\oplus) & \longrightarrow & 0, \end{array}$$

where the rows both arise from the family of Bott extensions

$$(0 \to SA_i \to CA_i \to A_i \to 0)_{i\in\mathbb{N}}.$$

Then

$$\mathcal{T}^\infty_{\mathcal{X}_\Sigma}([\partial_\mathcal{A}]) : K_*(\mathcal{A}^\infty_{\mathcal{X}_\Sigma}) \to K_{*+1}(\mathcal{SA}^\infty_{\mathcal{X}_\Sigma})$$

is the boundary morphism for the top row, and

$$\mathcal{S}_\Sigma([\partial_{\mathcal{A}^\oplus}]) : K_*(C^*(\Sigma, \mathcal{A}^\oplus)) \to K_{*+1}(C^*(\Sigma, S\mathcal{A}^\oplus))$$

is the boundary morphism for the bottom row.

By naturality of the boundary extension, we get that

$$\jmath_{\mathcal{SA},\mathcal{X}_\Sigma,*} \circ \mathcal{T}^\infty_{\mathcal{X}_\Sigma}([\partial_\mathcal{A}]) = \mathcal{S}_\Sigma([\partial_{\mathcal{A}^\oplus}]) \circ \jmath_{\mathcal{A},\mathcal{X}_\Sigma,*}.$$

Hence, using Proposition 6.4, we deduce from equation (27) that

$$\jmath_{\mathcal{X}_\Sigma,*} \circ \mathcal{T}^\infty_{\mathcal{X}_\Sigma}(z) = \mathcal{S}_\Sigma([\partial_{\mathcal{A}^\oplus}] \otimes_{\mathcal{A}^\oplus} \widetilde{z'}) \circ \jmath_{\mathcal{A},\mathcal{X}_\Sigma,*}.$$

But using the Connes–Skandalis characterization of Kasparov products, we get that $[\partial_{\mathcal{A}^{\oplus}}] \otimes_{\mathcal{A}^{\oplus}} \widetilde{z'} = \widetilde{z}$ and hence

$$\jmath_{\mathcal{X}_\Sigma,*} \circ \mathcal{T}^{\infty}_{\mathcal{X}_\Sigma}(z) = \mathcal{S}_\Sigma(\widetilde{z}) \circ \jmath_{\mathcal{A},\mathcal{X}_\Sigma,*}. \qquad \square$$

Proof of Proposition 6.6. Let $z = (z_i)_{i\in\mathbb{N}}$ be a family in

$$\prod_{i\in\mathbb{N}} K_*(P_s(X_i)) = \prod_{i\in\mathbb{N}} KK_*(C(P_s(X_i)), \mathbb{C}).$$

Then under the identification between the groups $\prod_{i\in\mathbb{N}} KK_*(C(P_s(X_i)), \mathbb{C})$ and $KK_*(C_0(P_s(\Sigma)), \mathbb{C})$ given by the equality

$$C_0(P_s(\Sigma)) = \bigoplus_{i\in\mathbb{N}} C(P_s(X_i)),$$

we have a correspondence between z and $\widetilde{z}$ and hence the commutativity of the diagram amounts to proving the equality

$$\mathcal{S}_\Sigma(\widetilde{z})([Q_{s,\Sigma}, 0]) = \jmath_{\mathcal{X}_\Sigma,*} \circ \mathcal{T}^{\infty}_{\mathcal{X}_\Sigma}(z)([Q^{\infty}_{s,\mathcal{X}_\Sigma}, 0]). \tag{28}$$

Let us consider the family $\mathcal{A} = (C(P_s(X_i)))_{n\in\mathbb{N}}$. Since $d(X_i, X_j) \geq s$ if $i \neq j$, we see that $\jmath_{\mathcal{A},\mathcal{X}_\Sigma}(Q^{\infty}_{s,\mathcal{X}_\Sigma}) = Q_{s,\Sigma}$ and hence

$$\mathcal{S}_\Sigma(\widetilde{z})([Q_{s,\Sigma}, 0]) = \mathcal{S}_\Sigma(\widetilde{z}) \circ \jmath_{\mathcal{A},\mathcal{X}_\Sigma,*}([Q^{\infty}_{s,\mathcal{X}_\Sigma}, 0]).$$

Equality (28) is then a consequence of Lemma 6.7. $\square$

6.8. **Proof of Theorem 6.2.** Let Σ be a discrete metric space with bounded geometry that satisfies the assumptions of Lemma 6.7. According to [14], we can assume by using a coarse Mayer–Vietoris argument that Σ is a graph space $\Sigma = \bigsqcup_{i\in\mathbb{N}} X_i$.

Let us show that $\mu_{\Sigma,*}$ is one-to-one. Let d be a positive number and let x be an element in $K_*(P_d(\Sigma))$ such that $\mu_{\Sigma,*}(x) = 0$. Fix $\varepsilon > 0$ small enough and choose a positive number λ as in the second point of Proposition 2.4. We can assume without loss of generality that $d(X_i, X_j) \geq d$ if $i \neq j$. Then $P_d(\Sigma) = \bigsqcup_{i\in\mathbb{N}} P_d(X_i)$ and up to the corresponding identification between $K_*(P_d(\Sigma))$ and $\prod_{i\in\mathbb{N}} K_*(P_d(X_i))$, we can view x as a family $(x_i)_{i\in\mathbb{N}}$ in $\prod_{i\in\mathbb{N}} K_*(P_d(X_i))$. According to Proposition 6.6, we get that

$$\jmath_{\mathcal{X}_\Sigma,*} \circ \nu^{\infty,d}_{\mathcal{X}_\Sigma,*}(x) = 0.$$

If we fix $r \geq r_{d,\varepsilon}$, then we have

$$\begin{aligned}\jmath_{\mathcal{X}_\Sigma,*} \circ \nu^{\infty,d}_{\mathcal{X}_\Sigma,*} &= \jmath_{\mathcal{X}_\Sigma,*} \circ \iota^{\varepsilon,r}_{*} \circ \nu^{\infty,\varepsilon,r,d}_{\mathcal{X}_\Sigma,*} \\ &= \iota^{\varepsilon,r}_{*} \circ \jmath^{\varepsilon,r}_{\mathcal{X}_\Sigma,*} \circ \nu^{\infty,\varepsilon,r,d}_{\mathcal{X}_\Sigma,*}.\end{aligned}$$

Hence according to the second point of Proposition 2.4, there exists $r' \geq r$ such that

$$\jmath^{\lambda\varepsilon,r'}_{\mathcal{X}_\Sigma,*} \circ \nu^{\infty,\lambda\varepsilon,r',d}_{\mathcal{X}_\Sigma,*}(x) = 0.$$

Therefore, replacing $\lambda\varepsilon$ by ε and r' by r, we see that there exist ε in $(0, \frac{1}{4})$ and a positive number r such that

$$\jmath^{\varepsilon,r}_{\mathcal{X}_\Sigma,*} \circ \nu^{\infty,\varepsilon,r,d}_{\mathcal{X}_\Sigma,*}(x) = 0.$$

We can also assume without loss of generality that $d(X_i, X_j) \geq r$ if $i \neq j$ and hence

$$\nu^{\infty,\varepsilon,r}_{\mathcal{X}_\Sigma,*}(x) = 0 \text{ in } K^{\varepsilon,r}_*(C^*(\mathcal{X}_\Sigma)).$$

Using the controlled isomorphism between the groups $\mathcal{K}_*(C^*(\mathcal{X}_\Sigma))$ and $\prod_{i\in\mathbb{N}} \mathcal{K}_*(\mathscr{K}(\ell^2(X_i)))$, we see that up to rescaling ε and r, we can assume that

$$\nu^{\varepsilon,r,d}_{X_i,*}(x_i) = 0 \text{ in } K^{\varepsilon,r}_*(\mathscr{K}(\ell^2(X_i)))$$

for every integer i. Let then $d' \geq d$ be such that $QI_{X,*}(d, d', r, \varepsilon)$ is satisfied for every finite subset X of Σ. We have then $q_{d,d',*}(x_i) = 0$ in $K_*(P_{d'}(X_i))$ for every integer i and therefore $q_{s,s',*}(x) = 0$ in $K_*(P_{d'}(\Sigma))$. Hence $\mu_{\Sigma,*}$ is one-to-one.

Let us prove that $\mu_{\Sigma,*}$ is onto. Let z be an element in $K_*(C^*(\Sigma))$ and fix ε' small enough. Then for some positive number r', there exists y' in $K^{\varepsilon',r'}_*(C^*(\Sigma))$ such that $z = \iota^{\varepsilon',r'}_*(y')$. Pick ε in $[\varepsilon', \frac{1}{4})$, d a positive number and $r \geq r'$ such that $QS_{X,*}(d, r', r, \varepsilon', \varepsilon)$ holds for any finite subset X of Σ. We can assume without loss of generality that $d(X_i, X_j) > r$ and $d(X_i, X_j) > d$ if $i \neq j$. Then there exists an element y in $K^{\varepsilon',r'}_*(C^*(\mathcal{X}_\Sigma))$ such that

$$\jmath^{\varepsilon',r'}_{\mathcal{X}_\Sigma,*}(y) = y'.$$

For every integer i, let y_i be the image of y under the composition

$$K^{\varepsilon',r'}_*(C^*(\mathcal{X}_\Sigma)) \to K^{\varepsilon',r'}_*(\mathscr{K}(\ell^2(X_i) \otimes \mathscr{H})) \to K^{\varepsilon',r'}_*(\mathscr{K}(\ell^2(X_i))),$$

where

- the first morphism is induced by the restriction to $C^*(\mathcal{X}_\Sigma)$ of the ith projection $\prod_{n\in\mathbb{N}} \mathscr{K}(\ell^2(X_n) \otimes \mathscr{H}) \to \mathscr{K}(\ell^2(X_i) \otimes \mathscr{H})$;
- the second morphism is the Morita equivalence.

For every integer i, there exists x_i in $K_*(P_d(X_i))$ such that

$$\nu^{\varepsilon,r,d}_{X_i,*}(x_i) = \iota^{\varepsilon',\varepsilon,r',r}_*(y_i).$$

Set then $x = (x_i)_{i\in\mathbb{N}}$ in $\prod_{i\in\mathbb{N}} K_*(P_d(X_i))$. Then $\nu^{\infty,d}_{\mathcal{X}_\Sigma,*}(x) = \iota^{\varepsilon,r}_*(y)$ and hence according to Proposition 6.6 and under the identification between $K_*(P_d(\Sigma))$ and $\prod_{i\in\mathbb{N}} K_*(P_d(X_i))$, we get that

$$\mu^d_{\Sigma,*}(x) = \jmath_{\mathcal{X}_\Sigma,*}(\iota^{\varepsilon,r}_*(y)) = \iota^{\varepsilon,r}_*(y') = z.$$

Hence $\mu_{\Sigma,*}$ is onto.

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