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Transfers in coarse homology

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(Communicated by Peter Schneider)

Dedicated to Christopher Deninger on the occasion of his 60th birthday

Abstract. We enlarge the category of bornological coarse spaces by adding transfer morphisms and introduce the notion of an equivariant coarse homology theory with transfers. We then show that equivariant coarse algebraic K-homology and equivariant coarse ordinary homology can be extended to equivariant coarse homology theories with transfers. In the case of a finite group, we observe that equivariant coarse homology theories with transfers provide Mackey functors. We express standard constructions with Mackey functors in terms of coarse geometry, and we demonstrate the usage of transfers in order to prove injectivity results about assembly maps.

Contents

1. Introduction

In order to capture the large scale and the local finiteness behavior of metric spaces, groups, and other geometric objects, the category **BornCoarse** of bornological coarse spaces with proper controlled maps as morphisms was introduced in [3]. In the present paper we will work in the equivariant situation. So let G be a group and G**BornCoarse** denote the category of G-bornological coarse spaces [4, Section 2]. Further, let $\mathbf{C}$ be a cocomplete stable ∞-category. Following [4, Section 3], an equivariant $\mathbf{C}$-valued coarse homology theory is a functor

$$E\colon G\mathbf{BornCoarse} \to \mathbf{C}$$

which satisfies four axioms:

(i) coarse invariance,
(ii) excision,
(iii) vanishing on flasques,
(iv) and u-continuity.

In [4, Def. 4.9], we construct a universal G-equivariant coarse homology theory

$$\mathrm{Yo}^s\colon G\mathbf{BornCoarse} \to G\mathbf{Sp}\mathcal{X}$$

whose target is the presentable stable ∞-category of equivariant coarse motivic spectra. Any other equivariant coarse homology theory factorizes in an essentially unique way over the universal example Yo^s. More precisely, precomposition with Yo^s induces an equivalence from the ∞-category

$$\mathbf{Fun}^{\mathrm{colim}}(G\mathbf{Sp}\mathcal{X}, \mathbf{C})$$

of colimit-preserving functors to the ∞-category of $\mathbf{C}$-valued equivariant coarse homology theories [4, Cor. 4.10].

The main goal of the present paper is to add transfers as a new type of morphisms between bornological coarse spaces and to show that important examples of coarse homology theories admit transfers. We will furthermore construct the universal equivariant coarse homology theory with transfers. To this end, we enlarge the category $G\mathbf{BornCoarse}$ of G-bornological coarse spaces to the category $G\mathbf{BornCoarse}_{\mathrm{tr}}$ of G-bornological coarse spaces with transfers (see Section 2.23).

Given a G-set I and a G-bornological coarse space, we can form the G-bornological coarse space $I_{\mathrm{min,min}} \otimes X$ (see [4, Ex. 2.17]), or equivalently, the bounded union $\coprod_{i\in I}^{\mathrm{bd}} X$ of I copies of X (Definition 2.51). If i is a G-fixed point in I, then $j_i\colon X \to \coprod_{i\in I}^{\mathrm{bd}} X$ denotes the inclusion of the component with index i which is a morphism in $G\mathbf{BornCoarse}$. In general, if i is not fixed by G, then we can consider this morphism after forgetting the G-action.

By design (see Definition 2.27), $G\mathbf{BornCoarse}_{\mathrm{tr}}$ contains a transfer morphism

$$\mathrm{tr}_{X,I}\colon X \to I_{\mathrm{min,min}} \otimes X,$$

which morally is the sum $\sum_{i\in I} j_i$ of the inclusion morphisms. It will actually turn out that $G\mathbf{BornCoarse}_{\mathrm{tr}}$ is semi-additive and therefore enriched in commutative monoids. If I is finite and has the trivial G-action, then

$$\mathrm{tr}_{X,I} = \sum_{i\in I} j_i$$

is a literally true identity in $G\mathbf{BornCoarse}_{\mathrm{tr}}$. But transfers are most interesting in the case of infinite sets I.

If E is an equivariant coarse homology theory, then the construction of an extension of E to $G\mathbf{BornCoarse}_{\mathrm{tr}}$ should be guided by the idea that the morphism

$$E(\mathrm{tr}_{X,I})\colon E(X) \to E(I_{\min,\min} \otimes X)$$

places identical copies of a cycle for $E(X)$ on each component of the bounded union.

The projection $I_{\min,\min} \otimes X \to X$ is a controlled and bornological map, but not a proper map and therefore not a morphism of bornological coarse spaces. It is an example of a bounded covering, a notion which we will introduce in the present paper. The category $G\mathbf{BornCoarse}_{\mathrm{tr}}$ will be defined by adding wrong-way maps for all bounded coverings.

On the technical level, we use spans to construct $G\mathbf{BornCoarse}_{\mathrm{tr}}$ as a ∞-category (see Section 2.23). We further construct an embedding

$$\iota\colon G\mathbf{BornCoarse} \to G\mathbf{BornCoarse}_{\mathrm{tr}}$$

(see Definitions 2.25 and 2.35).

Let $\mathbf{C}$ be a stable cocomplete ∞-category.

Definition 1.1. A $\mathbf{C}$-valued *equivariant coarse homology theory with transfers* is a functor

$$E\colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{C}$$

such that

$$E \circ \iota\colon G\mathbf{BornCoarse} \to \mathbf{C}$$

is a $\mathbf{C}$-valued equivariant coarse homology theory.

By excision, an equivariant coarse homology theory with transfers preserves coproducts and is therefore an additive functor from $G\mathbf{BornCoarse}_{\mathrm{tr}}$ to $\mathbf{C}$.

Definition 1.2. We will say that a $\mathbf{C}$-valued equivariant coarse homology theory E *admits transfers* if there exists a functor

$$E_{\mathrm{tr}}\colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{C}$$

such that $E_{\mathrm{tr}} \circ \iota \simeq E$.

The condition that a coarse homology theory E admits transfers is used in order to show a version of the coarse Baum–Connes conjecture for E and scalable spaces [2, Section 10.3]. Furthermore, the existence of transfers is an important ingredient in [7] where we show that G-equivariant finite decomposition complexity of X implies that a certain forget-control map $E(\beta_X)$ is an equivalence.

In analogy with the universal equivariant coarse homology theory, we will construct the universal equivariant coarse homology theory with transfers

$$\mathrm{Yo}^s_{\mathrm{tr}}\colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}}.$$

Let $\mathbf{C}$ be a stable, cocomplete ∞-category. The next proposition is true by design of $\mathrm{Yo}^s_{\mathrm{tr}}$.

Proposition 1.3 (Proposition 2.59)**.** *Precomposition with* $\mathrm{Yo}^s_{\mathrm{tr}}$ *induces an equivalence from the* ∞*-category*

$$\mathbf{Fun}^{\mathrm{colim}}(G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}}, \mathbf{C})$$

to the ∞*-category of* $\mathbf{C}$*-valued equivariant coarse homology theories with transfers.*

In the present paper we consider the following examples of equivariant coarse homology theories:

(i) equivariant coarse ordinary homology $H\mathcal{X}^G$;
(ii) equivariant coarse algebraic K-homology $K\mathbf{A}\mathcal{X}^G$ of an additive category $\mathbf{A}$ with a strict action of G.

Their construction is given in [4, Section 7 and 8]. In this paper we are interested in the existence of transfers.

Theorem 1.4. *The equivariant coarse homology theories* $H\mathcal{X}^G$ *and* $K\mathbf{A}\mathcal{X}^G$ *admit transfers.*

The assertions of the theorem are shown in Section 3. The case of algebraic K-theory is actually quite involved and relies on the preparations in Section 3.1.

In the final Section 4, we show that for a finite group G, a $\mathbf{C}$-valued equivariant coarse homology theory E admitting transfers gives rise to a $\mathbf{C}$-valued Mackey functor which will be denoted by EM.

If V is a finite-dimensional orthogonal representation of G, then we can express the delooping of EM along the representation sphere $S(V)_\infty$ in terms of the equivariant coarse homology E_V obtained from E by twisting with V, where V is considered as a G-bornological coarse space. More precisely, we show the following.

Proposition 1.5 (Propostion 4.15)**.** *We have a canonical equivalence of* $\mathbf{C}$*-valued Mackey functors*

$$S(V)_\infty \wedge EM \simeq E_V M.$$

In [6] we use transfers in order to prove injectivity results for assembly maps. In the present paper we demonstrate this method in the simple case of a finite group G. Consider, for example, the family of solvable subgroups **Sol**. Let $G\mathbf{Orb}$ be the orbit category and let $G_{\mathbf{Sol}}\mathbf{Orb}$ be its subcategory of orbits with stabilizers in **Sol**. We consider a cocomplete and complete stable ∞-category $\mathbf{C}$ and a functor $E\colon G\mathbf{Orb} \to \mathbf{C}$. The following theorem is a special case of Theorem 4.22.

Theorem 1.6. *If* E *extends to a* $\mathbf{C}$*-valued Mackey functor, then the assembly map*

$$\operatorname*{colim}_{T \in G_{\mathbf{Sol}}\mathbf{Orb}} E(T) \to E(*) \tag{1}$$

is split injective.

2. Equivariant coarse motives with transfers

For an introduction to G-bornological coarse spaces and the associated motives, we refer to [3, Sections 2–4] and to [4, Section 2]. In the present section we will discuss the new aspects related to transfers.

In order to incorporate transfers for equivariant coarse homology theories, we introduce the ∞-category $G\mathbf{BornCoarse}_{\mathrm{tr}}$ of G-bornological coarse spaces with transfers in Section 2.23. To this end, we introduce in Section 2.1 the notion of a bounded covering which appears in the definition of the morphisms in $G\mathbf{BornCoarse}_{\mathrm{tr}}$. In Section 2.37 we introduce the corresponding ∞-category of coarse motivic spectra with transfers, and in Section 2.56 we discuss equivariant coarse homology theories with transfers.

2.1. **Bounded coverings and admissible squares.** In particular, the ∞-category $G\mathbf{BornCoarse}_{\mathrm{tr}}$ contains for all G-sets I transfer morphisms

$$\mathrm{tr}_{X,I}\colon X \to I_{\min,\min} \otimes X.$$

The projection to the second factor from $I_{\min,\min} \otimes X$ to X is a morphism of the underlying G-coarse spaces, but it is in general not proper and therefore not a morphism in $G\mathbf{BornCoarse}$. The transfer is a kind of wrong-way map for this projection.

In this section we will introduce for G-bornological coarse spaces W and X the notion of a bounded covering from W to X, which generalizes the projection onto the second factor discussed above. By construction, the homotopy category of $G\mathbf{BornCoarse}_{\mathrm{tr}}$ will have transfer maps $\mathrm{tr}_w\colon X \to W$ for all bounded coverings w from W to X, see the Definition 2.26.

We start with recalling some basic definitions from coarse geometry.

Definition 2.2.

(i) A *G-coarse space* is a pair $(W, \mathcal{C}_W)$ of a G-set W and a coarse structure $\mathcal{C}_W$ such that $\mathcal{C}_W$ is G-invariant and the set of invariant entourages $\mathcal{C}_W^G$ is cofinal in $\mathcal{C}_W$.

(ii) If $(W, \mathcal{C}_W)$ and $(W', \mathcal{C}_{W'})$ are G-coarse spaces and $f\colon W \to W'$ is an equivariant map between the underlying G-sets, then f is called *controlled* if for every U in $\mathcal{C}_W$, we have $(f \times f)(U) \in \mathcal{C}_{W'}$.

(iii) By $G\mathbf{Coarse}$ we denote the category of G-coarse spaces and G-equivariant controlled maps.

The category $G\mathbf{Coarse}$ is complete and cocomplete by [4, Prop. 2.18 and 2.21]. Moreover, we have a forgetful functor $G\mathbf{BornCoarse} \to G\mathbf{Coarse}$ which preserves coproducts and which sends the symmetric monoidal structure $\otimes$ on $G\mathbf{BornCoarse}$ to the product in $G\mathbf{Coarse}$.

Let W be a set, U be a subset of $W \times W$, and A be a subset of W.

Definition 2.3. The *U-thickening* of A is defined by

$$U[A] := \{w \in W \mid \text{there exists } a \in A : (w,a) \in U\}.$$

Let W be a coarse space with coarse structure $\mathcal{C}_W$. Then the union

$$R_W := \bigcup_{U \in \mathcal{C}_W} U$$

of all coarse entourages of W is an equivalence relation on W. Using this equivalence relation, we introduce the following notions.

Let A and B be subsets of W.

Definition 2.4.

(i) The *coarse closure* $[A]$ of A is the closure of A with respect to the equivalence relation R_W.
(ii) If $A = [A]$, then A is said to be *coarsely closed.*
(iii) A and B are *coarsely disjoint* if $[A] \cap [B] = \varnothing$.

If A is a subset of W, then using Definition 2.3, we have

$$[A] = \bigcup_{U \in \mathcal{C}_W} U[A].$$

Let W be a coarse space with coarse structure $\mathcal{C}_W$.

Definition 2.5.

(i) The equivalence classes of W with respect to the equivalence relation R_W are called the *coarse components* of W.
(ii) The G-set of coarse components of W will be denoted by $\pi_0(W)$.

In the following we discuss various ways to construct G-coarse spaces.

Let W be a set and Q be a subset of $\mathcal{P}(W \times W)$.

Definition 2.6. The *coarse structure* $\mathcal{C}\langle Q\rangle$ *generated by* Q is the smallest coarse structure on W containing the set Q.

If W is a G-set and Q consists of G-invariant subsets, then $\mathcal{C}\langle Q\rangle$ is a G-coarse structure.

Let U be a G-invariant entourage on a G-set W.

Definition 2.7. We let W_U denote the G-coarse space $(W, \mathcal{C}\langle\{U\}\rangle)$.

Let W be a G-set. An *equivariant partition* of W is a partition $(W_i)_{i\in I}$ such that I is a G-set and $gW_i = W_{gi}$ for all i in I and g in G.

Let W be a G-set and let $\mathcal{W} := (W_i)_{i\in I}$ be an equivariant partition. Then we consider the invariant entourage

$$U(\mathcal{W}) := \bigsqcup_{i\in I} W_i \times W_i \tag{2}$$

on W. Note that we have a canonical equivariant bijection

$$\pi_0(W_{U(\mathcal{W})}) \cong I.$$

Assume now that W is a G-coarse space with coarse structure $\mathcal{C}$ and with an equivariant partition $\mathcal{W} := (W_i)_{i\in I}$.

Definition 2.8. We define the G-coarse structure $\mathcal{C}(\mathcal{W})$ on W by

$$\mathcal{C}(\mathcal{W}) := \mathcal{C}\langle\{U \cap U(\mathcal{W}) \mid U \in \mathcal{C}\}\rangle.$$

Finally, let W be a G-set, let U be a G-coarse space with coarse structure $\mathcal{C}_U$, and let $w\colon W \to U$ be an equivariant map of sets.

Definition 2.9. The *induced coarse structure* $w^{-1}\mathcal{C}_U$ on W is the maximal coarse structure on W such that the map w is controlled.

Note that $w^{-1}\mathcal{C}_U$ is a G-coarse structure and explicitly given by

$$w^{-1}\mathcal{C}_U = \mathcal{C}\langle\{(w^{-1} \times w^{-1})(E) \mid E \in \mathcal{C}_U\}\rangle.$$

We now turn to the definition of the notion of a bounded coarse covering. Let $w\colon W \to U$ be a morphism of G-coarse spaces with coarse structures $\mathcal{C}_W$ and $\mathcal{C}_U$, respectively. Let $\mathcal{W} := \pi_0(W)$ be the partition of W into coarse components.

Definition 2.10. We say that w is a *bounded coarse covering* if the following conditions are satisfied:

(i) $(w^{-1}\mathcal{C}_U)(\mathcal{W}) = \mathcal{C}_W$ (see Definition 2.9 and Definition 2.8).

(ii) For every W_0 in $\pi_0(W)$, the map $w_{|W_0}\colon W_0 \to w(W_0)$ is an isomorphism between coarse components (see Definition 2.5).

Let $w\colon W \to U$ and $u\colon U \to V$ be bounded coarse coverings between G-coarse spaces.

Lemma 2.11. *The composition $u \circ w\colon W \to V$ is a bounded coarse covering.*

Proof. We let $\mathcal{U}$ be the partition of U into coarse components. Then we have the following equalities:

$$\begin{aligned} ((u\circ w)^{-1}\mathcal{C}_V)(\mathcal{W}) &= (w^{-1}(u^{-1}\mathcal{C}_V))(\mathcal{W}) \\ &= (w^{-1}(u^{-1}\mathcal{C}_V)(\mathcal{U}))(\mathcal{W}) = (w^{-1}\mathcal{C}_U)(\mathcal{W}) = \mathcal{C}_W. \end{aligned} \tag{3}$$

Here we use that the decomposition $w^{-1}\mathcal{U}$ of W is coarser than the decomposition $\mathcal{W}$ for the second equality, and the assumption that u and w are bounded coarse coverings for the third and the last equalities.

If W_0 is a coarse component in W, then w maps it isomorphically to a coarse component U_0 of U, and u maps U_0 isomorphically to a coarse component in V. Hence $u \circ w$ maps W_0 isomorphically to a coarse component in V. □

We consider a cartesian diagram

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "w"'] & U \arrow[d, "u"] \\ V \arrow[r, "g"] & Z \end{tikzcd}$$

in $G\mathbf{Coarse}$.

Lemma 2.12. *If u is a bounded coarse covering, then w is a bounded coarse covering.*

Proof. Recall that the coarse structure $\mathcal{C}_W$ of the space W is generated by entourages of the form $w^{-1}(A) \cap f^{-1}(B)$ for entourages A in $\mathcal{C}_V$ and B in $\mathcal{C}_U$, and that the coarse structure $w^{-1}(\mathcal{C}_V)(\mathcal{W})$ is generated by entourages of the form $w^{-1}(A) \cap U(\mathcal{W})$ for entourages A in $\mathcal{C}_V$. Here $\mathcal{W} := \pi_0(W)$ is the partition of W into coarse components.

Let $\mathcal{U} := \pi_0(U)$. Given an entourage A in $\mathcal{C}_V$, define $B := u^{-1}(g(A)) \cap U(\mathcal{U})$, which is an entourage in $\mathcal{C}_U$. Then we get $f^{-1}(B) = w^{-1}(A) \cap f^{-1}(U(\mathcal{U}))$. Because $U(\mathcal{W})$ is contained in $f^{-1}(U(\mathcal{U}))$, we have

$$w^{-1}(A) \cap U(\mathcal{W}) \subseteq w^{-1}(A) \cap f^{-1}(U(\mathcal{U})) = f^{-1}(B),$$

and hence $w^{-1}(\mathcal{C}_V)(\mathcal{W})$ is contained in $\mathcal{C}_W$.

On the other hand, the inclusion $\mathcal{C}_W \subseteq w^{-1}(\mathcal{C}_V)(\mathcal{W})$ is clear.

Let W_0 be a coarse component in W. We first show that $w(W_0)$ is a coarse component of V. There exists a coarse component U_0 in U such that $f(W_0) \subseteq U_0$. We consider a point a in $[w(W_0)]$, and we must argue that $a \in w(W_0)$. Since $g(a)$ and $g(w(W_0))$ are in the same coarse component of Z, we have $g(a) \in [u(U_0)]$. Since $u_{|U_0} \colon U_0 \to u(U_0)$ is an isomorphism of coarse components, there exists b in U_0 with $g(a) = u(b)$. The pair (a, b) uniquely determines a point c in W. By the choice of a, there exists a point c_0 in W_0 such that $\{(a, w(c_0))\}$ is an entourage of V and

$$(c, c_0) \in w^{-1}(\{(a, w(c_0))\}).$$

Since $f(c_0) \in U_0$ and U_0 is a coarse component, $\{(b, f(c_0))\}$ is an entourage of U. Then

$$(c, c_0) \in f^{-1}(\{(b, f(c_0))\}).$$

Since the square is cartesian, the coarse structure $\mathcal{C}_W$ of the space W is generated by entourages of the form $w^{-1}(A) \cap f^{-1}(B)$ for entourages A in $\mathcal{C}_V$ and B in $\mathcal{C}_U$, and therefore $\{(c, c_0)\}$ is an entourage of W. Since W_0 is a coarse component and $c_0 \in W_0$, we see that $c \in W_0$. Hence $a = w(c) \in w(W_0)$. This finishes the verification that $w(W_0)$ is a coarse component.

We show that for every coarse component W_0 of W, the map $w_{|W_0} \colon W_0 \to w(W_0)$ is an isomorphism of coarse components. We first show that $w_{|W_0}$ is injective. Consider two points w_0 and w_1 in W_0 with $w(w_0) = w(w_1)$. Since $u_{|[f(W_0)]} \colon [f(W_0)] \to u([f(W_0)])$ is an isomorphism, we get $f(w_0) = f(w_1)$. Since the square is cartesian, this implies $w_0 = w_1$.

We already know that $\mathcal{C}_W = w^{-1}(\mathcal{C}_V)(\mathcal{W})$. Because W_0 is a coarse component of W, this implies $\mathcal{C}_W \cap (W_0 \times W_0) = w^{-1}(\mathcal{C}_V)$ showing that $w_{|W_0}$ is an isomorphism of coarse components. $\square$

We consider a map between sets equipped with bornological structures.

Definition 2.13. (i) The map is called *bornological* if it sends bounded subsets to bounded subsets.

(ii) The map is called *proper* if preimages of bounded subsets are bounded.

Definition 2.14. Let $G\widetilde{\mathbf{BornCoarse}}$ be the category whose objects are G-bornological coarse spaces, and morphisms are morphisms between the underlying G-coarse spaces.

The forgetful functor $G\widetilde{\mathbf{BornCoarse}} \to G\mathbf{Coarse}$ is an equivalence of categories. Therefore $G\widetilde{\mathbf{BornCoarse}}$ has all small limits and colimits.

For spaces X and Y in $G\mathbf{BornCoarse}$ it makes sense to require that a morphism $X \to Y$ in $G\widetilde{\mathbf{BornCoarse}}$ is proper or bornological, or both, as an additional property.

We consider two G-bornological coarse spaces X and Y and a morphism $u\colon X \to Y$ in $G\widetilde{\mathbf{BornCoarse}}$.

Definition 2.15. We say that u is a *bounded covering* if the following conditions are satisfied:

(i) u is a bounded coarse covering (see Definition 2.10).
(ii) u is a bornological map (see Definition 2.13 (i)).
(iii) For every bounded subset B of X, there exists a finite, coarsely disjoint partition $(B_a)_{a\in A}$ of B such that $u_{|[B_a]}\colon [B_a] \to [u(B_a)]$ is an isomorphism of coarse spaces (see Definitions 2.2 and 2.4).

Condition 2.15 (iii) gives that we have isomorphisms of coarse spaces

$$u_{|U[B_a]}\colon U[B_a] \to u(U[B_a])$$

for all coarse entourages U of X, see Definition 2.3. If X has the property that a bounded set meets at most finitely many coarse components, then Condition 2.15 (iii) is automatically satisfied. But it becomes relevant if bounded sets can meet more than finitely many coarse components.

Let X, Y, W and U be G-bornological coarse spaces and let $w\colon X \to Y$ and $u\colon W \to U$ be bounded coverings. Note that the coproduct in $G\mathbf{BornCoarse}$ is also the coproduct in $G\widetilde{\mathbf{BornCoarse}}$ and therefore we can form the morphism $w \sqcup u\colon X \sqcup W \to Y \sqcup U$ in $G\widetilde{\mathbf{BornCoarse}}$, where the coproduct of the spaces is understood in $G\mathbf{BornCoarse}$. Similarly, the underlying G-coarse space of the tensor product in $G\mathbf{BornCoarse}$ is the product of the underlying G-coarse spaces. Hence we have a map $w \times u\colon X \otimes W \to Y \otimes U$ in $G\widetilde{\mathbf{BornCoarse}}$. The following lemma follows directly from the definitions.

Lemma 2.16. *The maps* $w \sqcup u\colon X \sqcup W \to Y \sqcup U$ *and* $w \times u\colon X \otimes W \to Y \otimes U$ *are bounded coverings.*

Proof. The case of $w \sqcup u$ is obvious.

We consider the case of $w \times u$. Let us first verify Condition 2.15 (i), i.e., that $w \times u$ is a bounded coarse covering. Indeed, we have the following chain of equalities:

$$\begin{aligned}((w \times u)^{-1}\mathcal{C}_{Y\otimes U})(\pi_0(X \otimes W)) &= ((w \times u)^{-1}\langle \mathcal{C}_Y \times \mathcal{C}_U\rangle)(\pi_0(X \otimes W)) \\ &= \langle w^{-1}(\mathcal{C}_Y) \times u^{-1}(\mathcal{C}_U)\rangle(\pi_0(X \otimes W)) \\ &= \langle w^{-1}(\mathcal{C}_Y) \times u^{-1}(\mathcal{C}_U)\rangle(\pi_0(X) \times \pi_0(W))\end{aligned}$$

$$
\begin{aligned}
&= \langle w^{-1}(\mathcal{C}_Y)(\pi_0(X)) \times u^{-1}(\mathcal{C}_U)(\pi_0(W))\rangle \\
&= \langle \mathcal{C}_X \times \mathcal{C}_W\rangle \\
&= \mathcal{C}_{X\otimes W}.
\end{aligned}
$$

Moreover, every coarse component Z_0 of $X \otimes W$ is of the form $X_0 \times W_0$ for coarse components X_0 of X and W_0 of W and both $w_{|X_0}$ and $u_{|W_0}$ are isomorphisms between coarse components by assumption. Hence $(w \times u)_{|Z_0}$ is an isomorphism of coarse components.

Conditions 2.15 (ii) and 2.15 (iii) easily follow from the fact that the bornology on $X \otimes W$ is generated by $\mathcal{B}_X \times \mathcal{B}_W$. □

Example 2.17. Let X be a G-coarse space and I a G-set. Then we can form the product $I_{\min} \times X$ in G-coarse spaces, where $I_{\min}$ is the G-coarse space with underlying G-set I and the minimal coarse structure. The projection onto the second factor $\mathrm{pr}_2 \colon I_{\min} \times X \to X$ is a bounded coarse covering. If X is a G-bornological coarse space, then pr_2 is a bounded covering of G-bornological coarse spaces from $I_{\min,\min} \otimes X$ to X, where $I_{\min,\min}$ carries the minimal coarse and bornological structures.

More generally, assume that X is a G-coarse space and $I \to I'$ a map of G-sets. Then the induced map $I_{\min} \times X \to I'_{\min} \times X$ is a bounded coarse covering. If X is a G-bornological coarse space, then $I_{\min,\min} \otimes X \to I'_{\min,\min} \otimes X$ is a bounded covering.

Example 2.18. Let X be a G-bornological coarse space with bornology $\mathcal{B}$ and assume that $\mathcal{B}'$ is a compatible G-bornological structure such that $\mathcal{B}' \subseteq \mathcal{B}$. Then we consider the G-bornological coarse space X' obtained from X by replacing $\mathcal{B}$ by $\mathcal{B}'$. Then the identity map of the underlying sets is a bounded covering $X' \to X$. Indeed, the identity is clearly a bounded coarse covering. Condition 2.15 (iii) is also satisfied (even for arbitrary subsets in place of B and for the trivial partition). Finally, the identity is bornological, since $\mathcal{B}' \subseteq \mathcal{B}$.

We consider G-bornological coarse spaces X, Y and Z, and bounded coverings $u \colon X \to Y$ and $v \colon Y \to Z$.

Lemma 2.19. *The composition $v \circ u \colon X \to Z$ is a bounded covering.*

Proof. By Lemma 2.11, we know that $v \circ u$ is a bounded coarse covering. Furthermore, as a composition of bornological maps it is bornological.

Let B be a bounded subset of X and let $(B_a)_{a\in A}$ be a finite, coarsely disjoint partition such that $u_{|B_a} \colon [B_a] \to [u(B_a)]$ is an isomorphism of coarse spaces. For every a in A, let $(C_{a,i})_{i\in I_a}$ be a finite, coarsely disjoint partition of $u(B_a)$ such that $v_{|[C_{a,i}]} \colon [C_{a,i}] \to [v(C_{a,i})]$ is an isomorphism of coarse spaces. Note that this partition exists since $u(B_a)$ is bounded in Y. Then we set $B_{a,i} := u^{-1}(C_{a,i}) \cap B_a$ and observe that $((B_{a,i})_{i\in I_a})_{a\in A}$ is a finite, coarsely disjoint partition of B such that $(v \circ u)_{|[B_{a,i}]} \colon [B_{a,i}] \to [(v \circ u)(B_{a,i})]$ is an isomorphism of coarse spaces. □

We consider G-bornological coarse spaces W, U, V and Z, and a diagram

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "w"'] & U \arrow[d, "u"] \\ V \arrow[r, "g"] & Z \end{tikzcd} \tag{4}$$

in $G\widetilde{\mathbf{BornCoarse}}$.

Definition 2.20. The square (4) is called *admissible* if the following conditions are satisfied:

(i) The square (4) is cartesian.
(ii) g is proper and bornological.
(iii) f is proper and bornological.
(iv) u is a bounded covering.

Note that Condition 2.20 (i) is equivalent to the condition that the underlying square of (4) in $G\mathbf{Coarse}$ is cartesian.

Lemma 2.21. *If the square* (4) *is admissible, then w is a bounded covering.*

Proof. The map w is a bounded coarse covering by Lemma 2.12.

Moreover, w is bornological. Indeed, let B be a bounded subset of W. Then we have

$$w(B) \subseteq g^{-1}(u(f(B))).$$

Since f and u are bornological and g is proper, we see that $g^{-1}(u(f(B)))$ and hence $w(B)$ are bounded.

We finally verify Condition 2.15 (iii). Let B be a bounded subset of W. Then $f(B)$ is bounded in U since f is bornological. Let $(C_a)_{a\in A}$ be a finite, coarsely disjoint partition of $f(B)$ such that $u_{|[C_a]}\colon [C_a] \to [u(C_a)]$ is an isomorphism of coarse spaces for every a in A. We define $B_a := f^{-1}(C_a) \cap B$. Then $(B_a)_{a\in A}$ is a finite, coarsely disjoint partition of B. It suffices to show that for every a in A, the map $w_{|[B_a]}\colon [B_a] \to [w(B_a)]$ is injective since w is a bounded coarse covering and therefore an isomorphism on each coarse component of W. Let b, b' be points in $[B_a]$ and assume that $w(b) = w(b')$. Then $u(f(b)) = u(f(b'))$. Since $f(b), f(b') \in [C_a]$ and $u_{|[C_a]}$ is injective, we conclude that $f(b) = f(b')$. Since the square (4) is a pullback of sets, this implies $b = b'$. □

We consider G-bornological coarse spaces U, V, Z and a diagram

$$\begin{tikzcd} & U \arrow[d, "u"] \\ V \arrow[r, "g"] & Z \end{tikzcd} \tag{5}$$

in $G\widetilde{\mathbf{BornCoarse}}$ such that g is proper and bornological and u is a bounded covering.

Lemma 2.22. *There exists an extension* (W, w, f) *of* (5) *to an admissible square* (4).

If (W', w', f') *is a second admissible extension, then there exists a unique isomorphism of* G*-bornological coarse spaces* $\phi\colon W \to W'$ *such that*

$$\begin{tikzcd} V \arrow[d, equal] & W \arrow[l, "w"'] \arrow[r, "f"] \arrow[d, "\phi"] & U \arrow[d, equal] \\ V & W' \arrow[l, "w'"'] \arrow[r, "f'"] & U \end{tikzcd}$$

commutes.

Proof. We choose an object W which represents the pullback $V \times_Z U$ in $G\widetilde{\mathbf{BornCoarse}}$, and we can assume that W has the bornology $\mathcal{B}_W := f^{-1}\mathcal{B}_U$. This is an extension (W, w, f) of (5) to an admissible square.

Because W is a pullback in $G\widetilde{\mathbf{BornCoarse}}$, it is unique up to unique isomorphism in $G\widetilde{\mathbf{BornCoarse}}$. This provides us the map ϕ which is an isomorphism in $G\widetilde{\mathbf{BornCoarse}}$. Since the maps $f\colon W \to U$ and $f'\colon W' \to U$ are proper and bornological, the map ϕ is an isomorphism of G-bornological coarse spaces. $\square$

2.23. **The category $G\mathbf{BornCoarse}_{\mathrm{tr}}$.** In this section we first introduce the category $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ of G-bornological coarse spaces with transfers. It contains the category $G\mathbf{BornCoarse}$ of G-bornological coarse spaces as a subcategory such that the inclusion

$$\iota\colon G\mathbf{BornCoarse} \hookrightarrow \mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}}) \tag{6}$$

is a bijection on objects. We then define the ∞-category $G\mathbf{BornCoarse}_{\mathrm{tr}}$ which models the ordinary category $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ as its homotopy category as indicated by the notation. Finally, we discuss some basic properties of these categories.

Let X and Y be a G-bornological coarse spaces.

Definition 2.24. A *span* (W, w, f) from X to Y is a diagram

$$\begin{tikzcd} & W \arrow[dl, two heads, "w"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd}$$

in $G\widetilde{\mathbf{BornCoarse}}$ (see Definition 2.14) subject to the following conditions:

(i) f is a morphism in $G\mathbf{BornCoarse}$ which is, in addition, bornological (see Definition 2.13).

(ii) $w\colon W \to X$ is a bounded covering (see Definition 2.15).

We use double-headed arrows in order to indicate which map is a bounded covering.

An *isomorphism* between spans (W, w, f) and (W', w', f') is defined to be an isomorphism of G-bornological coarse spaces $\phi\colon W \to W'$ such that the diagram

$$(7)\qquad \begin{tikzcd} X \arrow[d, equal] & W \arrow[l, two heads, "w"'] \arrow[r, "f"] \arrow[d, "\phi", "\cong"'] & Y \arrow[d, equal] \\ X & W' \arrow[l, two heads, "w'"'] \arrow[r, "f'"] & Y \end{tikzcd}$$

in $G\widetilde{\mathbf{BornCoarse}}$ commutes.

We define[1] $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ as the category whose objects are G-bornological coarse spaces and whose morphisms are isomorphism classes of spans. Morphisms in the category $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ are called *generalized morphisms* of G-bornological coarse spaces.

The *composition* $(U, w \circ u, g \circ h)$ of the spans (W, w, f) from X to Y and (V, v, g) from Y to Z is determined by the choice of a span (U, u, h) such that the square in the diagram

$$(8)\qquad \begin{tikzcd} & & U \arrow[dl, two heads, "u"'] \arrow[dr, "h"] & & \\ & W \arrow[dl, two heads, "w"'] \arrow[dr, "f"] & & V \arrow[dl, two heads, "v"'] \arrow[dr, "g"] & \\ X & & Y & & Z \end{tikzcd}$$

is admissible (Definition 2.20).

Compositions in the category $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ always exist and are well-defined by Lemmas 2.21 and 2.22.

Definition 2.25. We define the embedding

$$\iota\colon G\mathbf{BornCoarse} \to \mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$$

as follows:

(i) It is given by the identity on objects.
(ii) It sends the morphism $f\colon X \to Y$ to the generalized morphism represented by the span

$$\begin{tikzcd} & \hat{X} \arrow[dl, two heads, "\mathrm{id}"'] \arrow[dr, "\hat{f}"] & \\ X & & Y, \end{tikzcd}$$

where $\hat{X}$ is the G-bornological coarse space obtained from the space X by replacing its bornology by the bornology $f^{-1}\mathcal{B}_Y$, the right leg is induced by f, and the left leg is induced by the identity of underlying coarse spaces.

[1] Later we define a ∞-category $G\mathbf{BornCoarse}_{\mathrm{tr}}$ whose homotopy category is $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ justifying this notation, see Lemma 2.32.

Note that $\hat{f}$ in Definition 2.25 is proper and bornological by construction. The bornology $f^{-1}\mathcal{B}_Y$ on $\hat{X}$ is compatible with the coarse structure of $\hat{X}$, because f is controlled. Since f is proper, the left leg is bornological. The left leg is a bounded covering by Example 2.18. It is easy to see that the inclusion $\iota\colon G\mathbf{BornCoarse} \to \mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ is a functor.

We will denote the generalized morphism represented by the span (W, w, f) by $[W, w, f]$. For a G-bornological coarse space X in $G\mathbf{BornCoarse}$, we will use the symbol X also to denote the object $\iota(X)$ of $G\mathbf{BornCoarse}_{\mathrm{tr}}$. Furthermore, for a morphism f in $G\mathbf{BornCoarse}$, we will keep the short notation f for the generalized morphism $\iota(f) = [\hat{X}, \mathrm{id}, \hat{f}]$.

Let W and X be G-bornological coarse spaces and let $w\colon W \to X$ be a bounded covering.

Definition 2.26. The morphism

$$\mathrm{tr}_w := [W, w, \mathrm{id}_W]\colon X \to W$$

in $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ is called the *transfer* for w.

We will, in particular, need the following special case. Let X be a G-bornological coarse space and let I be a G-set. By Example 2.17, the projection onto the second factor

$$u\colon I_{\min,\min} \otimes X \to X$$

is a bounded covering.

Definition 2.27. The generalized morphism

$$\mathrm{tr}_{X,I} := [I_{\min,\min} \otimes X, u, \mathrm{id}_{I_{\min,\min}\otimes X}]\colon X \to I_{\min,\min} \otimes X$$

is called the *transfer* for I.

We define now a ∞-category $G\mathbf{BornCoarse}_{\mathrm{tr}}$, which models the ordinary category $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ introduced in Definition 2.24 as its homotopy category.

Recall Definition 2.14 of the category $G\widetilde{\mathbf{BornCoarse}}$. We will describe $G\mathbf{BornCoarse}_{\mathrm{tr}}$ as a simplicial subset of $\mathrm{Hom}_{\mathbf{Cat}}(\mathbf{Tw}, G\widetilde{\mathbf{BornCoarse}})$, where $\mathbf{Tw}\colon \Delta \to \mathbf{Cat}$ denotes the cosimplicial category with $\mathbf{Tw}[n]$ the twisted arrow category of the poset $[n]$. Our approach is similar to the construction of the effective Burnside category of a disjunctive triple in [1], but it is formally not a special case.

Remark 2.28. In this remark we recall the definition of $\mathbf{Tw}$, see also [10, Section 2] or [1, Section 2], and provide an explicit description of $\mathbf{Fun}(\mathbf{Tw}, \mathbf{C})$ for a small category $\mathbf{C}$.

First of all $\mathbf{Tw}$ is the functor (compare [10, Ex. 2.4])

$$\mathbf{Tw}\colon \Delta \to \mathbf{Cat}, \quad [n] \mapsto \mathbf{Tw}[n],$$

where $\mathbf{Tw}[n]$ is the poset of pairs of integers (i, j) with $0 \le i \le j \le n$ such that $(i, j) \le (i', j')$ if and only if $i \le i' \le j' \le j$. If $\sigma\colon [n] \to [m]$ is a morphism in Δ, then we define the morphism

$$\mathbf{Tw}(\sigma)\colon \mathbf{Tw}[n] \to \mathbf{Tw}[m], \quad (i,j) \mapsto (\sigma(i), \sigma(j)).$$

For a category $\mathbf{C}$, we now obtain the simplicial set

$$\mathrm{Hom}_{\mathbf{Cat}}(\mathbf{Tw}, \mathbf{C})\colon \Delta^{\mathrm{op}} \to \mathbf{Set}.$$

In the following, we will use the following notation for the data of an n-simplex X in $\mathrm{Hom}_{\mathbf{Cat}}(\mathbf{Tw}, \mathbf{C})$. We write $X_{i,j}$ for the image under X of the pair (i,j) in $\mathbf{Tw}[n]$, and we use the shorthand X_i instead of $X_{i,i}$. We will, furthermore, only depict the morphisms $X_{i,j} \to X_{i',j'}$ if (i,j) and (i',j') are adjacent, i.e., if $i = i'$ and $j' + 1 = j$ or $i' = i + 1$ and $j = j'$. Note that these morphisms $(i,j) \to (i',j')$ generate all morphisms in $\mathbf{Tw}[n]$.

Definition 2.29. The simplicial set $G\mathbf{BornCoarse}_{\mathrm{tr}}$ is defined to be the subset of $\mathrm{Hom}_{\mathbf{Cat}}(\mathbf{Tw}, G\widetilde{\mathbf{BornCoarse}})$, whose n-simplices X satisfy the following:

(i) For every object (i,j) in $\mathbf{Tw}[n]$ with $j \geq 1$, the morphism $X_{i,j} \to X_{i,j-1}$ is a bounded covering.

(ii) For every object (i,j) in $\mathbf{Tw}[n]$ with $i \leq n-1$, the morphism $X_{i,j} \to X_{i+1,j}$ is proper and bornological.

(iii) For every object (i,j) with $1 \leq i \leq j \leq n-1$, the square

$$\begin{tikzcd} X_{i-1,j+1} \arrow[r] \arrow[d, two heads] & X_{i,j+1} \arrow[d, two heads] \\ X_{i-1,j} \arrow[r] & X_{i,j} \end{tikzcd}$$

is admissible.

Here we use double-headed arrows in order to indicate which maps are bounded coverings.

In the following we describe the 3-skeleton of $G\mathbf{BornCoarse}_{\mathrm{tr}}$ in terms of pictures. These pictures are very helpful in order to see the verification of the horn-filling conditions in the proof of Lemma 2.31, but also for understanding the proof of Lemma 3.2.

(i) The 0-simplices of $G\mathbf{BornCoarse}_{\mathrm{tr}}$ are the objects of $G\mathbf{BornCoarse}$.

(ii) 1-simplices of $G\mathbf{BornCoarse}_{\mathrm{tr}}$ are spans (see Definition 2.24)

$$\begin{tikzcd} & X_{0,1} \arrow[dl, two heads] \arrow[dr] & \\ X_0 & & X_1. \end{tikzcd}$$

The two faces of this one-simplex are X_0 and X_1.

(iii) 2-simplices are diagrams

$$\begin{tikzcd}[column sep=small]
 & & X_{0,2} \arrow[dl, two heads] \arrow[dr] & & \\
 & X_{0,1} \arrow[dl, two heads] \arrow[dr] & & X_{1,2} \arrow[dl, two heads] \arrow[dr] & \\
X_0 & & X_1 & & X_2,
\end{tikzcd}$$

where the square is admissible. The three faces are

$$\begin{tikzcd}[column sep=small]
 & X_{1,2} \arrow[dl, two heads] \arrow[dr] & & & X_{0,2} \arrow[dl, two heads] \arrow[dr] & & & X_{0,1} \arrow[dl, two heads] \arrow[dr] & \\
X_1 & & X_2 & X_0 & & X_2 & X_0 & & X_1.
\end{tikzcd}$$

(iv) 3-simplices are diagrams

$$\begin{tikzcd}[column sep=small]
 & & & X_{0,3} \arrow[dl, two heads] \arrow[dr] & & & \\
 & & X_{0,2} \arrow[dl, two heads] \arrow[dr] & & X_{1,3} \arrow[dl, two heads] \arrow[dr] & & \\
 & X_{0,1} \arrow[dl, two heads] \arrow[dr] & & X_{1,2} \arrow[dl, two heads] \arrow[dr] & & X_{2,3} \arrow[dl, two heads] \arrow[dr] & \\
X_0 & & X_1 & & X_2 & & X_3,
\end{tikzcd}$$

where again all squares are admissible. Its faces are

$$\begin{tikzcd}[column sep=small]
 & & X_{1,3} \arrow[dl, two heads] \arrow[dr] & & \\
 & X_{1,2} \arrow[dl, two heads] \arrow[dr] & & X_{2,3} \arrow[dl, two heads] \arrow[dr] & \\
X_1 & & X_2 & & X_3,
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X_{0,3} \arrow[dl, two heads] \arrow[dr] & & \\
 & X_{0,2} \arrow[dl, two heads] \arrow[dr] & & X_{2,3} \arrow[dl, two heads] \arrow[dr] & \\
X_0 & & X_2 & & X_3,
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X_{0,3} \arrow[dl, two heads] \arrow[dr] & & \\
 & X_{0,1} \arrow[dl, two heads] \arrow[dr] & & X_{1,3} \arrow[dl, two heads] \arrow[dr] & \\
X_0 & & X_1 & & X_3,
\end{tikzcd}$$

$$\begin{tikzcd}
 & & X_{0,2} \arrow[dl, two heads] \arrow[dr] & & \\
 & X_{0,1} \arrow[dl, two heads] \arrow[dr] & & X_{1,2} \arrow[dl, two heads] \arrow[dr] & \\
X_0 & & X_1 & & X_2.
\end{tikzcd}$$

Lemma 2.30. *The simplicial set* $G\mathbf{BornCoarse}_{\mathrm{tr}}$ *is* 2*-coskeletal.*

Proof. We observe that the data of an n-simplex is given by the collection of data of all 2-simplices in the n-simplex. Hence the restriction map

$$\mathrm{Hom}_{\mathbf{sSet}}(\Delta^n, G\mathbf{BornCoarse}_{\mathrm{tr}}) \to \mathrm{Hom}_{\mathbf{sSet}}(\Delta^n_{\leq 2}, G\mathbf{BornCoarse}_{\mathrm{tr}})$$

is an isomorphism for all $n \geq 3$, where $\Delta^n_{\leq 2}$ denotes the 2-skeleton of Δ^n. $\square$

Lemma 2.31. *The simplicial set* $G\mathbf{BornCoarse}_{\mathrm{tr}}$ *is a* ∞*-category.*

Proof. We must check the inner horn filling condition.

(i) The image of Λ^2_1 in Δ^2 has the form

$$\begin{tikzcd}
 & X_{0,1} \arrow[dl, two heads] \arrow[dr] & & X_{1,2} \arrow[dl, two heads] \arrow[dr] & \\
X_0 & & X_1 & & X_2.
\end{tikzcd}$$

By Lemma 2.22, it has a filling.

(ii) The image of Λ^3_2 in Δ^3 is the bold part of the following diagram:

$$\begin{tikzcd}
 & & & X_{0,3} \arrow[dl, twoheadrightarrow] \arrow[dr, dotted] \arrow[ddrr, bend left] & & & \\
 & & X_{0,2} \arrow[dl, twoheadrightarrow] \arrow[dr] \arrow[rr, phantom, "S_{12}"] & & X_{1,3} \arrow[dl, twoheadrightarrow] \arrow[dr] & & \\
 & X_{0,1} \arrow[dl, twoheadrightarrow] \arrow[dr] \arrow[rr, phantom, "S_1"] & & X_{1,2} \arrow[dl, twoheadrightarrow] \arrow[dr] \arrow[rr, phantom, "S_2"] & & X_{2,3} \arrow[dl, twoheadrightarrow] \arrow[dr] & \\
X_0 & & X_1 & & X_2 & & X_3.
\end{tikzcd}$$

We first get the dotted arrow using the cartesian property of the square S_2. We further know that the squares S_1, $S_2 + S_{12}$ and S_2 are admissible. We must show that S_{12} is admissible. Since $S_2 + S_{12}$ and S_2 are cartesian in G**BornCoarse**, we conclude that S_{12} is cartesian in G**BornCoarse**. Since the maps $X_{1,3} \to X_{2,3}$ and $X_{0,3} \to X_{2,3}$ are bornological and proper, also the map $X_{0,3} \to X_{1,3}$ is bornological and proper. This implies that S_{12} is admissible.

A similar argument applies to the inclusion of Λ^3_1 into Δ^3.

(iii) Since every inner horn Λ^n_k for $n \geq 4$ already contains the full 2-skeleton, it is fillable by Lemma 2.30. □

The following lemma justifies the choice of notation $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ for the category introduced in Definition 2.24.

Lemma 2.32. *The category* $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ *is canonically equivalent to the homotopy category of* $G\mathbf{BornCoarse}_{\mathrm{tr}}$.

Proof. The equivalence is given by the functor described as follows:

(i) The functor is the obvious bijection on objects.

(ii) The functor sends the class $[W, w, f]$ of spans from X to Y to the class of (W, w, f) in the homotopy category of $G\mathbf{BornCoarse}_{\mathrm{tr}}$.

We first argue that the functor is well defined on morphisms. If $\phi\colon (W, w, f) \to (W', w', f')$ is an isomorphism between spans, then we can consider the diagram

$$\begin{tikzcd}
 & & W' \arrow[dl, twoheadrightarrow, "\phi"'] \arrow[dr, "f'"] & & \\
 & W \arrow[dl, twoheadrightarrow, "w"'] \arrow[dr] & & Y \arrow[dl, twoheadrightarrow, "\mathrm{id}_Y"'] \arrow[dr, "\mathrm{id}_Y"] & \\
X & & Y & & Y,
\end{tikzcd}$$

which provides a homotopy between the morphisms (W, w, f) and (W', w', f') in the left mapping space $\mathrm{Hom}^L_{G\mathbf{BornCoarse}_{\mathrm{tr}}}(X, Y)$.

One easily checks the compatibility with composition, so that we have a well-defined functor. It is, furthermore, obvious that the functor is full.

On the other hand, homotopies of spans in the left mapping space

$$\mathrm{Hom}^L_{G\mathbf{BornCoarse}_{\mathrm{tr}}}(X,Y)$$

are precisely of the above form. Because ϕ is a pullback of an isomorphism, ϕ defines an isomorphism of spans. This shows that the functor is also faithful. This completes the proof. $\square$

Remark 2.33. A higher categorical refinement of $G\mathbf{BornCoarse}_{\mathrm{tr}}$ can also be obtained in the form of a bi-category $G\mathbf{BornCoarse}^{\mathrm{bi}}_{\mathrm{tr}}$. Since $G\widetilde{\mathbf{BornCoarse}}$ admits fibre products, we can form the bi-category $\mathbf{Span}(G\widetilde{\mathbf{BornCoarse}})$ of spans in $G\widetilde{\mathbf{BornCoarse}}$ [8]. We obtain the bi-category $G\mathbf{BornCoarse}^{\mathrm{bi}}_{\mathrm{tr}}$ from $\mathbf{Span}(G\widetilde{\mathbf{BornCoarse}})$ by the following steps, which all yield bi-categories:

(i) In a first step we take a subcategory by requiring the left legs of the spans to be bounded coverings and the right legs to be proper and bornological. Compositions still exist by Lemma 2.22 in connection with Lemmas 2.11, 2.12 and 2.21.

Identity morphisms belong to our category. All relations involving 2-isomorphisms are automatically implemented by morphisms between G-bornological coarse spaces.

(ii) The bi-category $G\mathbf{BornCoarse}^{\mathrm{bi}}_{\mathrm{tr}}$ is defined to be the subcategory whose 2-morphisms between spans are implemented by morphisms of G-bornological coarse spaces.

According to [13, Def. 6.1.6.13], an ∞-category is called semi-additive if it is pointed, and finite coproducts and products exist and are equivalent.

Lemma 2.34. $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ *and* $G\mathbf{BornCoarse}_{\mathrm{tr}}$ *are semi-additive.*

Proof. We shall show that the empty space $\varnothing$ is both initial and final in $G\mathbf{BornCoarse}_{\mathrm{tr}}$. Let X be a G-bornological coarse space. We will use the simplicial set of right morphisms $\mathrm{Hom}^R_{G\mathbf{BornCoarse}_{\mathrm{tr}}}(\varnothing,X)$ (see [14, Section 1.2.2] for details). $\mathrm{Hom}^R_{G\mathbf{BornCoarse}_{\mathrm{tr}}}(\varnothing,X)$ is the one-point space. To see this, note that, e.g., the unique 2-simplex in this simplicial set is given by

$$\begin{tikzcd}
 & & & \varnothing \arrow[dl, twoheadrightarrow] \arrow[dr] & & & \\
 & & \varnothing \arrow[dl, twoheadrightarrow] \arrow[dr] & & \varnothing \arrow[dl, twoheadrightarrow] \arrow[dr] & & \\
 & \varnothing \arrow[dl, twoheadrightarrow] \arrow[dr] & & \varnothing \arrow[dl, twoheadrightarrow] \arrow[dr] & & \varnothing \arrow[dl, twoheadrightarrow] \arrow[dr] & \\
\varnothing & & \varnothing & & \varnothing & & X.
\end{tikzcd}$$

This shows that $\varnothing$ is an initial object.

To see that $\varnothing$ is also final, we use the simplicial set $\mathrm{Hom}^L_{G\mathbf{BornCoarse}_{\mathrm{tr}}}(X,\varnothing)$ of left morphisms and again observe that it is a one-point space.

Thus also the homotopy category $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ of $G\mathbf{BornCoarse}_{\mathrm{tr}}$ is pointed.

Since semi-additivity can be checked on the level of homotopy categories by [13, Rem. 6.1.6.15], it remains to check that $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ is semi-additive.

We show that $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ admits finite products and coproducts, and that they are naturally isomorphic.

We first claim that the inclusion $\iota\colon \mathbf{BornCoarse} \to \mathbf{Ho}(\mathbf{BornCoarse}_{\mathrm{tr}})$ preserves finite coproducts. Let X and Y be G-bornological spaces. Then we have a coproduct $X \sqcup Y$ in $G\mathbf{BornCoarse}$ together with canonical morphisms

$$i\colon X \to X \sqcup Y \quad \text{and} \quad j\colon Y \to X \sqcup Y.$$

Let now Z be a G-bornological coarse space and let

$$[W,w,f]\colon X \to Z \quad \text{and} \quad [V,v,g]\colon Y \to Z$$

be generalized morphisms. They extend uniquely to a generalized morphism

$$[W \sqcup V, w \sqcup v, f+g]\colon X \sqcup Y \to Z.$$

Note that $w \sqcup v$ is a bounded covering by Lemma 2.16. Then

$$[W \sqcup V, w \sqcup v, f+g] \circ i = [W,w,f] \quad \text{and} \quad [W \sqcup V, w \sqcup v, f+g] \circ j = [V,v,g].$$

We have generalized morphisms

$$p := [X,i,\mathrm{id}_X]\colon X \sqcup Y \to X \quad \text{and} \quad q := [Y,j,\mathrm{id}_Y]\colon X \sqcup Y \to Y.$$

We claim that the morphisms p and q exhibit $X \sqcup Y$ as the product of X and Y in $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$. Let

$$[A,a,s]\colon Q \to X \quad \text{and} \quad [B,b,t]\colon Q \to Y$$

be generalized morphisms. There is a unique generalized morphism

$$[A \sqcup B, a \sqcup b, s+t]\colon Q \to X \sqcup Y.$$

Then

$$p \circ [A \sqcup B, a \sqcup b, s+t] = [A,a,s] \quad \text{and} \quad q \circ [A \sqcup B, a \sqcup b, s+t] = [B,b,t].$$

This completes the proof. $\square$

Lemma 2.34 implies that the embedding

$$G\mathbf{BornCoarse} \to \mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$$

does not preserve products.

Let i be an element of I which is fixed by G and set $I' := I \setminus \{i\}$. Then we have the equality

$$\mathrm{tr}_{X,I} = \mathrm{tr}_{X,I'} + j_i \tag{9}$$

in $\mathrm{Hom}_{\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})}(X, I_{\mathrm{min,min}} \otimes X)$, where the embedding $j_i \colon X \to I_{\mathrm{min,min}} \otimes X$ is induced by the inclusion $\{i\} \to I$. We furthermore have a generalized morphism

$$p_i := [X, j_i, \mathrm{id}_X] \colon I_{\mathrm{min,min}} \otimes X \to X, \tag{10}$$

called the projection onto the i-th component of $I_{\mathrm{min,min}} \otimes X$ such that $p_i \circ j_i = \mathrm{id}_X$.

Definition 2.35. We define the *canonical embedding*

$$\iota \colon \mathtt{N}(G\mathbf{BornCoarse}) \to G\mathbf{BornCoarse}_{\mathrm{tr}}. \tag{11}$$

as the natural refinement of Definition 2.25.

This canonical embedding sends, e.g., the 3-simplex

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} U$$

in $\mathtt{N}(G\mathbf{BornCoarse})$ to the 3-simplex

$$\begin{tikzcd}
 & & & \hat{X} \arrow[dl, equal] \arrow[dr, "\hat{f}"] & & & \\
 & & \hat{X} \arrow[dl, equal] \arrow[dr, "\hat{f}"] & & \hat{Y} \arrow[dl, equal] \arrow[dr, "\hat{g}"] & & \\
 & \hat{X} \arrow[dl, twoheadrightarrow] \arrow[dr, "\hat{f}"] & & \hat{Y} \arrow[dl, twoheadrightarrow] \arrow[dr, "\hat{g}"] & & \hat{Z} \arrow[dl, twoheadrightarrow] \arrow[dr, "\hat{h}"] & \\
X & & Y & & Z & & U
\end{tikzcd}$$

in $G\mathbf{BornCoarse}_{\mathrm{tr}}$, where we use the notation introduced in Definition 2.25.

Example 2.36. Let Q be a G-bornological coarse space. If

$$w \colon W \to X$$

is a bounded covering between G-bornological coarse spaces, then

$$w \times \mathrm{id}_Q \colon W \otimes Q \to X \otimes Q$$

is again a bounded covering between G-bornological coarse spaces by Lemma 2.16. Furthermore, if the diagram

$$\begin{tikzcd}
W \arrow[r, "f"] \arrow[d, twoheadrightarrow, "w"'] & U \arrow[d, twoheadrightarrow, "u"] \\
V \arrow[r, "g"] & Z
\end{tikzcd} \tag{12}$$

is an admissible square of G-bornological coarse spaces, then the square

$$\begin{tikzcd}
W \otimes Q \arrow[r, "f \times \mathrm{id}_Q"] \arrow[d, two heads, "w \times \mathrm{id}_Q"'] & U \otimes Q \arrow[d, two heads, "u \times \mathrm{id}_Q"] \\
V \otimes Q \arrow[r, "g \times \mathrm{id}_Q"] & Z \otimes Q
\end{tikzcd}$$

is admissible, too. We therefore get a functor

$$- \otimes Q \colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to G\mathbf{BornCoarse}_{\mathrm{tr}}.$$

This construction actually produces a bifunctor

$$G\mathbf{BornCoarse}_{\mathrm{tr}} \times G\mathbf{BornCoarse} \to G\mathbf{BornCoarse}_{\mathrm{tr}}. \tag{13}$$

To illustrate this, we show what this functor does on 2-simplices. Given a 2-simplex

$$\begin{tikzcd}
 & & X_{0,2} \arrow[dl, two heads, "f_{01}"'] \arrow[dr, "g_{12}"] & & \\
 & X_{0,1} \arrow[dl, two heads, "f_0"'] \arrow[dr, "g_1"] & & X_{1,2} \arrow[dl, two heads, "f_1"'] \arrow[dr, "g_2"] & \\
X_0 & & X_1 & & X_2
\end{tikzcd}$$

in $G\mathbf{BornCoarse}_{\mathrm{tr}}$ and a composition $Q_0 \xrightarrow{a_1} Q_1 \xrightarrow{a_2} Q_2$ in $G\mathbf{BornCoarse}$, we obtain a new 2-simplex

$$\begin{tikzcd}
 & & X_{0,2} \otimes Q_0'' \arrow[dl, two heads, "f_{01} \otimes \mathrm{id}"'] \arrow[dr, "g_{12} \otimes a_1"] & & \\
 & X_{0,1} \otimes Q_0' \arrow[dl, two heads, "f_0 \otimes \mathrm{id}"'] \arrow[dr, "g_1 \otimes a_1"] & & X_{1,2} \otimes Q_1' \arrow[dl, two heads, "f_1 \otimes \mathrm{id}"'] \arrow[dr, "g_2 \otimes a_2"] & \\
X_0 \otimes Q_0 & & X_1 \otimes Q_1 & & X_2 \otimes Q_2,
\end{tikzcd}$$

where Q_0', Q_0'' and denote Q_0 with the bornology changed to $a_1^{-1}(\mathcal{B}_{Q_1})$ and $(a_2 \circ a_1)^{-1}(\mathcal{B}_{Q_2})$, respectively, and Q_1' denotes Q_1 with the bornology changed to $a_2^{-1}(\mathcal{B}_{Q_2})$. That all arrows pointing to the left are bounded coverings follows from Example 2.18 and Lemma 2.16.

2.37. **Coarse motivic spectra with transfers.** In this section we define the category $G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}}$ of coarse motivic spectra with transfers. We closely follow [3, Sections 3 and 4] and [4, Section 4.1].

We start with the category

$$\mathbf{PSh}(G\mathbf{BornCoarse}_{\mathrm{tr}}) := \mathbf{Fun}(G\mathbf{BornCoarse}_{\mathrm{tr}}^{\mathrm{op}}, \mathbf{Spc})$$

of space-valued presheaves on $G\mathbf{BornCoarse}_{\mathrm{tr}}$.

Remark 2.38. The canonical embedding

$$\iota\colon \mathtt{N}(G\mathbf{BornCoarse}) \to G\mathbf{BornCoarse}_{\mathrm{tr}}$$

(see Definition 2.35) induces a restriction

$$\mathbf{PSh}(G\mathbf{BornCoarse}_{\mathrm{tr}}) \to \mathbf{PSh}(G\mathbf{BornCoarse}).$$

Note that this restriction does not preserve representables.

We let

$$\mathrm{yo}_{\mathrm{tr}}\colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{PSh}(G\mathbf{BornCoarse}_{\mathrm{tr}})$$

denote the Yoneda embedding. In the following we will omit the canonical embedding ι defined in Definition 2.35 from the notation.

For an equivariant big family $\mathcal{Y} := (Y_i)_{i\in I}$ [4, Def. 3.5] on a G-bornological coarse space X, we set

$$\mathrm{yo}_{\mathrm{tr}}(\mathcal{Y}) := \operatorname*{colim}_{i\in I} \mathrm{yo}_{\mathrm{tr}}(Y_i).$$

If X is a G-bornological coarse space and $(Z,\mathcal{Y})$ is an equivariant complementary pair [4, Def. 3.7] on X, then we consider the map

$$\mathrm{yo}_{\mathrm{tr}}(\mathcal{Y}) \sqcup_{\mathrm{yo}_{\mathrm{tr}}(Z\cap\mathcal{Y})} \mathrm{yo}_{\mathrm{tr}}(Z) \to \mathrm{yo}_{\mathrm{tr}}(X). \tag{14}$$

By [14, Thm. 5.1.5.6] for any small ∞-category $\mathbf{D}$ the restriction along the Yoneda embedding induces an equivalence

$$\mathbf{PSh}(\mathbf{D}) \simeq \mathbf{Fun}^{\lim}(\mathbf{PSh}(\mathbf{D})^{\mathrm{op}}, \mathbf{Spc}).$$

Consequently, if E is an object of $\mathbf{PSh}(G\mathbf{BornCoarse}_{\mathrm{tr}})$, then we can evaluate E on presheaves (essentially via right Kan extension). For a big family $\mathcal{Y}$ on X, we abbreviate

$$E(\mathcal{Y}) := E(\mathrm{yo}_{\mathrm{tr}}(\mathcal{Y})).$$

Then the evaluation satisfies

$$E(\mathrm{yo}_{\mathrm{tr}}(X)) \simeq E(X) \quad \text{and} \quad E(\mathcal{Y}) \simeq \lim_{i\in I} E(Y_i).$$

Definition 2.39. We say that E satisfies *excision* if

(i) $E(\varnothing) \simeq \varnothing$,
(ii) E is local with respect to the morphisms (14) for every G-bornological coarse space X with an equivariant complementary pair $(Z,\mathcal{Y})$.

Remark 2.40. Condition 2.39 (ii) is equivalent to the condition that for every G-bornological coarse space X with an equivariant complementary pair $(Z,\mathcal{Y})$, the square

$$\begin{tikzcd} E(X) \arrow[r] \arrow[d] & E(Z) \arrow[d] \\ E(\mathcal{Y}) \arrow[r] & E(Z\cap Y) \end{tikzcd}$$

is cartesian.

Let us define

$$E(X,\mathcal{Y}) := \mathrm{Fib}(E(X) \to E(\mathcal{Y})).$$

Then descent is also equivalent to the condition that the natural morphism

$$E(X,\mathcal{Y}) \to E(Z, Z \cap \mathcal{Y})$$

is an equivalence for every G-bornological coarse space X with an equivariant complementary pair $(Z,\mathcal{Y})$.

The presheaves which satisfy descent are called sheaves.

We denote the full subcategory of presheaves satisfying excision (called sheaves in the following) by

$$\mathbf{Sh}(G\mathbf{BornCoarse}_{\mathrm{tr}}) \subseteq \mathbf{PSh}(G\mathbf{BornCoarse}_{\mathrm{tr}}).$$

Then we have a localization

$$L\colon \mathbf{PSh}(G\mathbf{BornCoarse}_{\mathrm{tr}}) \leftrightarrows \mathbf{Sh}(G\mathbf{BornCoarse}_{\mathrm{tr}}) : \text{inclusion}.$$

For the following definition, recall the definition of a flasque G-bornological coarse space [4, Def. 3.8].

Moreover, $\{0,1\}_{\max,\max}$ denotes the G-bornological coarse space given by the two-element set $\{0,1\}$ with trivial G-action and equipped with the maximal bornological coarse structure. The projection

$$\{0,1\}_{\max,\max} \to * \tag{15}$$

is a morphism.

Finally, if X is a G-bornological coarse space with coarse structure $\mathcal{C}_X$ and if U in $\mathcal{C}_X$ is G-invariant, then X_U denotes the G-bornological coarse space obtained from X by replacing the coarse structure $\mathcal{C}_X$ by the coarse structure $\mathcal{C}\langle\{U\}\rangle$ (see Definition 2.7). If U' in $\mathcal{C}_X^G$ is such that $U \subseteq U'$, then we have morphisms $X_U \to X_{U'} \to X$ of G-bornological coarse spaces, all induced by the identity of the underlying set.

Let E be an object of $\mathbf{Sh}(G\mathbf{BornCoarse}_{\mathrm{tr}})$.

Definition 2.41.

(i) E is *coarsely invariant* if it is local with respect to the morphism

$$\mathrm{yo}_{\mathrm{tr}}(\{0,1\}_{\max,\max} \otimes X) \to \mathrm{yo}_{\mathrm{tr}}(X)$$

induced by (15) for all G-bornological coarse spaces X.

(ii) E *vanishes on flasques* if it is local for the morphisms

$$\varnothing \to \mathrm{yo}_{\mathrm{tr}}(X)$$

for all flasque G-bornological coarse spaces X.

(iii) E is *u-continuous* if E is local for the morphisms

$$\operatorname*{colim}_{U \in \mathcal{C}_X^G} \mathrm{yo}_{\mathrm{tr}}(X_U) \to \mathrm{yo}_{\mathrm{tr}}(X)$$

for all G-bornological coarse spaces X.

Definition 2.42. The category of G-equivariant motivic coarse spaces with transfers $G\mathbf{Spc}\mathcal{X}_{\mathrm{tr}}$ is defined to be the full subcategory of $\mathbf{Sh}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ which are coarsely invariant, vanish on flasques, and which are u-continuous.

We have a localization

$$\mathcal{L}_{\mathrm{tr}} \colon \mathbf{Sh}(G\mathbf{BornCoarse}_{\mathrm{tr}}) \leftrightarrows G\mathbf{Spc}\mathcal{X}_{\mathrm{tr}} : \text{inclusion}.$$

We furthermore have a functor

$$\mathrm{Yo}_{\mathrm{tr}} := \mathcal{L}_{\mathrm{tr}} \circ \mathrm{yo}_{\mathrm{tr}} \colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to G\mathbf{Spc}\mathcal{X}_{\mathrm{tr}}. \tag{16}$$

Remark 2.43. For a G-bornological coarse space X, the representable presheaf $\mathrm{yo}_{\mathrm{tr}}(X)$ is a compact object. Moreover, the category $\mathbf{PSh}(G\mathbf{BornCoarse}_{\mathrm{tr}})$ is compactly generated by representables.

To make the construction of the category of motivic coarse spaces precise, we assume that there is a regular cardinal κ which bounds the size of all coarse structures of spaces appearing in $G\mathbf{BornCoarse}_{\mathrm{tr}}$ (i.e., we consider a suitable subcategory which is large enough to contain all spaces of interest), and which also bounds the size of the index sets of big families involved in the descent condition. Then the locality conditions are generated by a small set of morphisms between κ-compact objects. It follows that $G\mathbf{Spc}\mathcal{X}_{\mathrm{tr}}$ is κ-compactly generated and closed under κ-filtered colimits. For a bornological coarse space X, the object $\mathrm{Yo}_{\mathrm{tr}}(X)$ is κ-compact. See also [14, Cor. 5.5.7.3].

By construction, $\mathbf{Spc}\mathcal{X}_{\mathrm{tr}}$ is a presentable ∞-category. Let $\mathbf{Pr}^L$ be the large ∞-category of presentable ∞-categories and left-exact functors. The inclusion $\mathbf{Pr}^L_{\mathrm{stab}} \to \mathbf{Pr}^L$ of presentable stable ∞-categories in all presentable ∞-categories fits into an adjunction

$$\mathrm{Stab} \colon \mathbf{Pr}^L \leftrightarrows \mathbf{Pr}^L_{\mathrm{stab}} : \text{inclusion}.$$

Definition 2.44. We define the category $G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}}$ of *coarse motivic spectra with transfers* as the stabilization $\mathrm{Stab}(G\mathbf{Spc}\mathcal{X}_{\mathrm{tr}})$.

By construction, it fits into the adjunction

$$\Sigma^\infty_+ \colon G\mathbf{Spc}\mathcal{X}_{\mathrm{tr}} \leftrightarrows G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}} : \Omega^\infty.$$

We define the Yoneda functor

$$\mathrm{Yo}^s_{\mathrm{tr}} := \Sigma^\infty_+ \circ \mathrm{Yo}_{\mathrm{tr}} \colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}}. \tag{17}$$

Recall that $G\mathbf{BornCoarse}_{\mathrm{tr}}$ is semi-additive, by Lemma 2.34, and that $G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}}$ is additive since it is a stable ∞-category.

Lemma 2.45. *The functor* $\mathrm{Yo}^s_{\mathrm{tr}}$ *is additive.*

Proof. It suffices to show that $\mathrm{Yo}^s_{\mathrm{tr}}$ preserves zero objects and coproducts. Both properties are consequences of excision.

The zero object in $G\mathbf{BornCoarse}_{\mathrm{tr}}$ is given by the empty space $\varnothing$. By excision, we have $\mathrm{Yo}^s_{\mathrm{tr}}(\varnothing) \simeq 0$.

Let X and Y be two G-bornological coarse spaces. Their coproduct in $G\mathbf{BornCoarse}_{\mathrm{tr}}$ is represented by the coproduct $X \sqcup Y$ in $G\mathbf{BornCoarse}$.

We let $i\colon X \to X \sqcup Y$ and $j\colon Y \to X \sqcup Y$ denote the inclusions, and we let (Y) denote the equivariant big family on $X \sqcup Y$ consisting just of Y. The pair $(X,(Y))$ is a complementary pair on $X \sqcup Y$. Since the subsets X and Y are disjoint, by excision the map

$$\mathrm{Yo}^s_{\mathrm{tr}}(X) \oplus \mathrm{Yo}^s_{\mathrm{tr}}(Y) \xrightarrow{\mathrm{Yo}^s_{\mathrm{tr}}(i)+\mathrm{Yo}^s_{\mathrm{tr}}(j)} \mathrm{Yo}^s_{\mathrm{tr}}(X \sqcup Y)$$

is an equivalence. □

Let X be a G-bornological coarse space, let I be a G-set, and let i be a G-fixed element of I. We set $I' := I \setminus \{i\}$. Then $(\{i\} \times X, I' \times X)$ is an invariant complementary pair on the space $I_{\min,\min} \otimes X$. By excision, we have a decomposition

$$\mathrm{Yo}^s_{\mathrm{tr}}(I_{\min,\min} \otimes X) \simeq \mathrm{Yo}^s_{\mathrm{tr}}(X) \oplus \mathrm{Yo}^s_{\mathrm{tr}}(I'_{\min,\min} \otimes X).$$

If we compose the motivic transfer map $\mathrm{Yo}^s_{\mathrm{tr}}(\mathrm{tr}_{X,I})$ with the projections to the respective summands, we get a decomposition

$$\mathrm{Yo}^s_{\mathrm{tr}}(\mathrm{tr}_{X,I}) \simeq a \oplus b,$$

where

$$a\colon \mathrm{Yo}^s_{\mathrm{tr}}(X) \to \mathrm{Yo}^s_{\mathrm{tr}}(X), \quad b\colon \mathrm{Yo}^s_{\mathrm{tr}}(X) \to \mathrm{Yo}^s_{\mathrm{tr}}(I'_{\min,\min} \otimes X).$$

Lemma 2.46. *We have equivalences*

$$a \simeq \mathrm{id}_{\mathrm{Yo}^s_{\mathrm{tr}}(X)}, \quad b \simeq \mathrm{Yo}^s_{\mathrm{tr}}(\mathrm{tr}_{X,I'}).$$

Proof. Let $j_i\colon X \to I_{\min,\min} \otimes X$ be the inclusion given by $x \mapsto (i,x)$. In $G\mathbf{BornCoarse}_{\mathrm{tr}}$, we have relation (9),

$$j_i + \mathrm{tr}_{X,I'} = \mathrm{tr}_{X,I}\,.$$

This implies, by Lemma 2.45, that

$$\mathrm{Yo}^s_{\mathrm{tr}}(j_i) + \mathrm{Yo}^s_{\mathrm{tr}}(\mathrm{tr}_{X,I'}) \simeq \mathrm{Yo}^s_{\mathrm{tr}}(\mathrm{tr}_{X,I}).$$

Using the projection (10), we now have

$$a \simeq \mathrm{Yo}^s_{\mathrm{tr}}(p_i) \circ \mathrm{Yo}^s_{\mathrm{tr}}(\mathrm{tr}_{X,I}) \simeq \mathrm{Yo}^s_{\mathrm{tr}}(p_i \circ \mathrm{tr}_{X,I}) \simeq \mathrm{id}_{\mathrm{Yo}^s_{\mathrm{tr}}(X)}$$

and

$$b \simeq \mathrm{Yo}^s_{\mathrm{tr}}(\mathrm{tr}_{X,I}) - \mathrm{Yo}^s_{\mathrm{tr}}(j_i) \circ a \simeq \mathrm{Yo}^s_{\mathrm{tr}}(\mathrm{tr}_{X,I}) - \mathrm{Yo}^s_{\mathrm{tr}}(j_i) \simeq \mathrm{Yo}^s_{\mathrm{tr}}(\mathrm{tr}_{X,I'}).$$ □

If $\mathcal{Y} := (Y_i)_{i \in I}$ is an equivariant big family on a G-bornological coarse space X, then we set

$$\mathrm{Yo}^s_{\mathrm{tr}}(\mathcal{Y}) := \operatorname*{colim}_{i \in I} \mathrm{Yo}^s_{\mathrm{tr}}(Y_i).$$

The following properties of the functor $\mathrm{Yo}^s_{\mathrm{tr}}$ are shown by the same arguments as given for [4, Cor. 4.12, 4.14 and 4.15].

Lemma 2.47. (i) *If X is a G-bornological coarse space and A is a nice invariant subset of X* [4, Def. 3.3]*, then*

$$\mathrm{Yo}^s_{\mathrm{tr}}(A) \to \mathrm{Yo}^s_{\mathrm{tr}}(\{A\})$$

is an equivalence.

(ii) *If (Y, Z) is an equivariant coarsely excisive pair on a G-bornological coarse space X, then we have a push-out:*

$$\begin{tikzcd} \mathrm{Yo}^s_{\mathrm{tr}}(Z\cap Y) \arrow[r] \arrow[d] & \mathrm{Yo}^s_{\mathrm{tr}}(Z) \arrow[d] \\ \mathrm{Yo}^s_{\mathrm{tr}}(Y) \arrow[r] & \mathrm{Yo}^s_{\mathrm{tr}}(X). \end{tikzcd}$$

(iii) *If I_pX is a coarse cylinder* [3, Section 4.3] *over a G-bornological coarse space X, then the projection $I_pX \to X$ induces an equivalence*

$$\mathrm{Yo}^s_{\mathrm{tr}}(I_pX) \to \mathrm{Yo}^s_{\mathrm{tr}}(X).$$

Remark 2.48. Let A, B be objects in a stable ∞-category. Then we have an action

$$\mathbb{N} \times \mathrm{Map}(A, B) \to \mathrm{Map}(A, B), \quad (n, f) \mapsto nf.$$

It sends a morphism $f\colon A \to B$ to the composition

$$A \xrightarrow{\mathrm{diag}} \bigoplus_{i=1}^{n} A \xrightarrow{\oplus f} \bigoplus_{i=1}^{n} B \xrightarrow{+} B.$$

Here the diagonal map uses the interpretation of the sum as a product, while the last map is induced by the projections to the summands and interprets the sum as a coproduct.

Let X be a G-bornological coarse space and let I be a set. We consider I as a G-set with the trivial G-action. If I is finite, then $I_{\min,\min} \to *$ is a morphism of G-bornological coarse spaces. Hence we get a morphism $\rho\colon I_{\min,\min} \otimes X \to X$.

Lemma 2.49. *If I is finite, then*

$$\mathrm{Yo}^s_{\mathrm{tr}}(\rho) \circ \mathrm{Yo}^s_{\mathrm{tr}}(\mathrm{tr}_{X,I}) \simeq |I| \cdot \mathrm{id}_{\mathrm{Yo}^s_{\mathrm{tr}}(X)}\,.$$

Proof. We have a commuting diagram

$$\begin{tikzcd} \mathrm{Yo}^s_{\mathrm{tr}}(X) \arrow[r, "\mathrm{Yo}^s_{\mathrm{tr}}(\mathrm{tr}_{X,I})"] \arrow[d, equal] & \mathrm{Yo}^s_{\mathrm{tr}}(I_{\min,\min} \otimes X) \arrow[r, "\mathrm{Yo}^s_{\mathrm{tr}}(\rho)"] \arrow[d, "\cong"] & \mathrm{Yo}^s_{\mathrm{tr}}(X) \arrow[d, equal] \\ \mathrm{Yo}^s_{\mathrm{tr}}(X) \arrow[r, "\mathrm{diag}_{\mathrm{Yo}^s_{\mathrm{tr}}(X)}"] & \bigoplus_{i\in I} \mathrm{Yo}^s_{\mathrm{tr}}(X) \arrow[r, "+"] & \mathrm{Yo}^s_{\mathrm{tr}}(X), \end{tikzcd}$$

where the middle vertical isomorphism is induced by excision. Lemma 2.46 ensures that the first square commutes. The second square commutes in view of Lemma 2.45. □

2.50. **Bounded and free unions.** Let X be a G-bornological coarse space and let I be a G-set.

Definition 2.51. The *bounded union* $\coprod_{i\in I}^{\mathrm{bd}} X$ in $G\mathbf{BornCoarse}$ is defined as follows:

(i) The underlying G-set of $\coprod_{i\in I}^{\mathrm{bd}} X$ is the product $I \times X$ with the diagonal G-action.

(ii) The bornology of $\coprod_{i\in I}^{\mathrm{bd}} X$ is given by the subsets B satisfying the following two conditions:
   (a) The image of B under the projection $I \times X \to I$ is finite.
   (b) The image of B under the projection $I \times X \to X$ is bounded.

(iii) The coarse structure of $\coprod_{i\in I}^{\mathrm{bd}} X$ is generated by the entourages $\mathrm{diag}_I \times U$ for all entourages U of X.

Remark 2.52. We can consider the G-set I as the G-bornological coarse space $I_{\mathrm{min,min}}$ with the minimal bornology and the discrete coarse structure. Then we have an isomorphism of G-bornological coarse spaces

$$\coprod_{i\in I}^{\mathrm{bd}} X \cong I_{\mathrm{min,min}} \otimes X,$$

where $\otimes$ is the symmetric monoidal structure on G**BornCoarse**, see [4, Section 4.3].

We say that a G-set I has finite orbits if for every i in I, the orbit Gi is finite.

Assume that I is a G-set with finite orbits. Let X be a G-bornological coarse space.

Definition 2.53. We define the *free union* $\coprod_{i\in I}^{\mathrm{free}} X$ in G**BornCoarse** as follows:

(i) The underlying G-bornological space of $\coprod_{i\in I}^{\mathrm{free}} X$ coincides with the one of $\coprod_{i\in I}^{\mathrm{bd}} X$.

(ii) The coarse structure of $\coprod_{i\in I}^{\mathrm{free}} X$ is generated by the entourages $\bigsqcup_{i\in I} U_i$ for all families $(U_i)_{i\in I}$ of coarse entourages of X.

Remark 2.54. The restriction on the G-action on I is necessary in order to ensure that the coarse structure described in Definition 2.53 (ii) is a G-coarse structure.

If I is more general, we could modify Point (ii) of Definition 2.53 and instead take the induced G-coarse structure. But then we may lose the compatibility with the bornology described in Point (i) of Definition 2.53.

Remark 2.55. If I is a G-set with finite orbits and X is a G-bornological coarse space, then we have a canonical morphism

$$\coprod_{i\in I}^{\mathrm{bd}} X \to \coprod_{i\in I}^{\mathrm{free}} X$$

induced by the identity of the underlying set.

In particular, if we assume that I has the trivial G-action and X is a G-bornological coarse space, then we have morphisms

$$\coprod_{i\in I} X \to \coprod_{i\in I}^{\mathrm{bd}} X \to \coprod_{i\in I}^{\mathrm{free}} X,$$

all induced by the identity map of the underlying set.

2.56. **Equivariant coarse homology theories with transfers.** We recall the definition of an equivariant coarse homology theory [4, Def. 3.10]. Let $\mathbf{C}$ be a cocomplete stable ∞-category and $E\colon G\mathbf{BornCoarse} \to \mathbf{C}$ be a functor.

Definition 2.57. E is an *equivariant* $\mathbf{C}$*-valued coarse homology theory* if it satisfies the following:

(i) E is excisive for equivariant complementary pairs.
(ii) E is coarsely invariant.
(iii) E vanishes on flasque G-bornological coarse spaces.
(iv) E is u-continuous.

We refer to [4] for details on the notions appearing in the above definition.

Recall the embedding $\iota\colon \mathtt{N}(G\mathbf{BornCoarse}) \to G\mathbf{BornCoarse}_{\mathrm{tr}}$ given in Definition 2.35.

Definition 2.58. An *equivariant* $\mathbf{C}$*-valued coarse homology theory with transfers* is a functor

$$E\colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{C}$$

such that $E \circ \iota$ is an equivariant $\mathbf{C}$-valued coarse homology theory.

The conditions listed in Definition 2.57 determine the full sub-∞-category

$$G\mathbf{CoarseHomology}^{\mathbf{C}}_{\mathrm{tr}} \subseteq \mathbf{Fun}(G\mathbf{BornCoarse}_{\mathrm{tr}}, \mathbf{C}).$$

of $\mathbf{C}$-valued equivariant coarse homology theories with transfer.

By the construction of $G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}}$, we have the following proposition:

Proposition 2.59. *The pre-composition with* $\mathrm{Yo}^s_{\mathrm{tr}}$ (*see* (17)) *induces an equivalence*

$$\mathbf{Fun}^{\mathrm{colim}}(G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}}, \mathbf{C}) \xrightarrow{\simeq} G\mathbf{CoarseHomology}^{\mathbf{C}}_{\mathrm{tr}}$$

of the ∞*-category of equivariant* $\mathbf{C}$*-valued coarse homology theories with the* ∞*-category* $\mathbf{Fun}^{\mathrm{colim}}(G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}}, \mathbf{C})$ *of colimit-preserving functors from* $G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}}$ *to* $\mathbf{C}$.

The argument is completely analogous to the one for [3, Cor. 4.6].

Let $E\colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{C}$ be an equivariant coarse homology theory with transfers.

Corollary 2.60. *The functor* $E\colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{C}$ *is additive.*

Proof. This follows from Lemma 2.45. □

Pullback along the inclusion $\iota\colon \mathtt{N}(G\mathbf{BornCoarse}) \to G\mathbf{BornCoarse}_{\mathrm{tr}}$ sends equivariant coarse homology theories with transfers to equivariant coarse homology theories in the sense considered in [4]. Applied to $\mathrm{Yo}^s_{\mathrm{tr}} \circ \iota$, we get a colimit-preserving functor

$$\iota^{Mot}\colon G\mathbf{Sp}\mathcal{X} \to G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}}$$

such that

$$\mathrm{Yo}^s_{\mathrm{tr}} \circ \iota \simeq \iota^{Mot} \circ \mathrm{Yo}^s\,.$$

Remark 2.61. For every G-set I with finite G-orbits and every G-bornological coarse space X, we have a version of the transfer

$$\mathrm{tr}^{\mathrm{free}}_{X,I}\colon X \xrightarrow{\mathrm{tr}_{X,I}} \coprod_{i\in I}^{\mathrm{bd}} X \to \coprod_{i\in I}^{\mathrm{free}} X \tag{18}$$

for the free union in $G\mathbf{BornCoarse}_{\mathrm{tr}}$. Furthermore, for every G-fixed point j in I, we have the generalized morphism

$$p_j^{\mathrm{free}}\colon \coprod_{i\in I}^{\mathrm{free}} X \to X, \tag{19}$$

represented by the span

$$\begin{tikzcd} & X \arrow[dl, two heads] \arrow[dr, "\mathrm{id}_X"] & \\ \coprod_{i\in I}^{\mathrm{free}} X & & X, \end{tikzcd}$$

whose left leg is the inclusion of the jth component.

If E is now an equivariant coarse homology theory with transfers, then we have induced morphisms

$$E(\mathrm{tr}^{\mathrm{free}}_{X,I})\colon E(X) \to E\biggl(\coprod_{i\in I}^{\mathrm{free}} X\biggr), \quad E(p_j^{\mathrm{free}})\colon E\biggl(\coprod_{i\in I}^{\mathrm{free}} X\biggr) \to E(X).$$

Applying excision for the equivariant coarsely excisive decomposition

$$\biggl(X_j, \coprod_{i\in I\setminus\{j\}}^{\mathrm{free}} X\biggr)$$

of $\coprod_{i\in I}^{\mathrm{free}} X$, we get the right vertical arrow in the diagram

$$\begin{tikzcd} E(X) \arrow[rr, "E(\mathrm{tr}^{\mathrm{free}}_{X,I})"] \arrow[d, equal] & & E(\coprod_{i\in I}^{\mathrm{free}} X) \arrow[d, "\simeq"] \arrow[dr, "E(p_j^{\mathrm{free}})"] & \\ E(X) \arrow[rr, "\mathrm{id}_{E(X)} \oplus E(\mathrm{tr}^{\mathrm{free}}_{X,I\setminus\{j\}})"] & & E(X) \oplus E(\coprod_{i\in I\setminus\{j\}}^{\mathrm{free}} X) \arrow[r, "\mathrm{pr}_1"'] & E(X), \end{tikzcd} \tag{20}$$

which commutes in view of Lemma 2.46.

Example 2.62. Let E be an equivariant $\mathbf{C}$-valued coarse homology theory with transfers and let Q be any G-bornological coarse space. Then, in view of the Example 2.36 and by [4, Section 4.3], the twist of E by Q, which is defined as the composition

$$E(-\otimes Q)\colon G\mathbf{BornCoarse}_{\mathrm{tr}} \xrightarrow{-\otimes Q} G\mathbf{BornCoarse}_{\mathrm{tr}} \xrightarrow{E} \mathbf{C},$$

is again an equivariant $\mathbf{C}$-valued coarse homology theory with transfers. For fixed Q, we thus get a colimit-preserving functor

$$E(-\otimes Q)\colon G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}} \to \mathbf{C}.$$

Using the bifunctor (13), we see that this construction is also functorial in Q and satisfies the axioms of an equivariant coarse homology theory in this variable. In order to see the last assertion, note that a functor $G\mathbf{BornCoarse} \to \mathbf{Fun}^{\mathrm{colim}}(G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}}, \mathbf{C})$ is a coarse homology theory if and only if its evaluation at $\mathrm{Yo}^s_{\mathrm{tr}}(X)$ for each object X of $G\mathbf{BornCoarse}_{\mathrm{tr}}$ is a coarse homology theory. The objects of $G\mathbf{BornCoarse}_{\mathrm{tr}}$ are the objects of $G\mathbf{BornCoarse}$, and we already know that twisting with a G-bornological coarse space preserves equivariant coarse homology theories by [4, Section 4.3]. Consequently, we get a bifunctor

$$E(-\otimes -)\colon G\mathbf{Sp}\mathcal{X}_{\mathrm{tr}} \otimes G\mathbf{Sp}\mathcal{X} \to \mathbf{C}$$

which preserves colimits in each argument.

We will show that if an equivariant coarse homology theory E has transfers, then it has weak transfers [7, Def. 2.4].

We consider a family $(X_i)_{i\in I}$ of G-bornological coarse spaces and a G-fixed point j in I, and we set $I'_j := I \setminus \{j\}$. Then the pair of invariant subsets $(X_j, \coprod^{\mathrm{free}}_{i\in I'_j} X_i)$ of $\coprod^{\mathrm{free}}_{i\in I} X_i$ is an invariant coarsely excisive decomposition (see [4, Def. 4.13]). If E is an equivariant coarse homology theory, then E satisfies excision for invariant coarsely excisive decompositions [4, Cor. 4.14]. Therefore, we can define a projection

$$p^{\mathrm{ex}}_j\colon E\Big(\coprod_{i\in I}^{\mathrm{free}} X_i\Big) \simeq E(X_j) \oplus E\Big(\coprod_{i\in I'_j}^{\mathrm{free}} X_i\Big) \to E(X_j), \tag{21}$$

where the superscript ex is a reminder for the fact that the morphism uses excision for E.

Let I be a set with the trivial G-action and let $E\colon G\mathbf{BornCoarse} \to \mathbf{C}$ be an equivariant coarse homology theory. Then we define a functor

$$E^I\colon G\mathbf{BornCoarse} \to \mathbf{C}, \quad X \mapsto E\Big(\coprod_{i\in I}^{\mathrm{free}} X\Big).$$

For every j in I, the projection (21) provides a natural transformation of functors

$$p^{\mathrm{ex}}_j\colon E^I \to E.$$

Let E be an equivariant coarse homology theory.

Definition 2.63. E has *weak transfers* for I if there exists a natural transformation

$$\mathrm{tr}_I\colon E \to E^I$$

such that

$$p^{\mathrm{ex}}_j \circ \mathrm{tr}_I \simeq \mathrm{id}_E \tag{22}$$

for every j in I.

Lemma 2.64. *If E admits transfers (see Definition 1.2), then E has weak transfers.*

Proof. For every set I and G-bornological coarse space X, we have morphisms $\mathrm{tr}^{\mathrm{free}}_{X,I}$ and p^{free}_j, see (18) and (19), which satisfy the relation

$$p^{\mathrm{free}}_j \circ \mathrm{tr}^{\mathrm{free}}_{X,I} = \mathrm{id}_X \,. \tag{23}$$

If E admits transfers, then by the commutativity of the right triangle in (20) (where E is replaced by the extension E_{tr}, which exists by assumption), we have the equivalence

$$E_{\mathrm{tr}}(p^{\mathrm{free}}_j) \simeq p^{\mathrm{ex}}_j \tag{24}$$

for every j in I. Here and below we implicitly identify the values of E_{tr} and E on objects.

The morphism $\mathrm{tr}_{X,I}$ is natural in X. We can therefore form the natural transformation

$$\mathrm{tr}^{\mathrm{free}}_{-,I} \colon \mathrm{id}_{G\mathbf{BornCoarse}_{\mathrm{tr}}} \to \coprod_{i\in I}^{\mathrm{free}} - \colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to G\mathbf{BornCoarse}_{\mathrm{tr}}$$

of endofunctors of $G\mathbf{BornCoarse}_{\mathrm{tr}}$. We now define the natural transformation

$$\mathrm{tr}_I := E_{\mathrm{tr}}(\mathrm{tr}^{\mathrm{free}}_{-,I}) \colon E \to E^I .$$

The relation (22) is implied by (23) and (24). ☐

Theorem 1.4 in combination with Lemma 2.64 has the following corollary.

Corollary 2.65.
(i) *Equivariant coarse ordinary homology $H\mathcal{X}^G$ has weak transfers.*
(ii) *Equivariant coarse algebraic K-homology $K\mathbf{A}\mathcal{X}^G$ with coefficients in an additive category $\mathbf{A}$ with a strict G-action has weak transfers.*

For $K\mathbf{A}\mathcal{X}^G$, an alternative and independent argument is given in [7, Ex. 2.5].

Let $E\colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{C}$ be an equivariant $\mathbf{C}$-valued coarse homology theory with transfers.

Definition 2.66. E is called *strongly additive* if for every family $(X_i)_{i\in I}$ of G-bornological coarse spaces, the morphism

$$E\Big(\coprod_{i\in I}^{\mathrm{free}} X_i\Big) \to \prod_{j\in I} E(X_j) \tag{25}$$

induced by the family $(E(p^{\mathrm{free}}_j))_{j\in I}$, see (19), is an equivalence.

Remark 2.67. An equivariant coarse homology theory with transfers E is strongly additive if and only if the underlying equivariant coarse homology theory $E\circ\iota$ is strongly additive in the sense of [4, Def. 3.12]. This follows from the commutativity of the right triangle in (20), which compares the projection $E(p^{\mathrm{free}}_j)$ with the projection p_j defined by excision (the down-right composition in the triangle) used in the reference.

Example 2.68. Examples of strongly additive coarse homology theories with transfers are coarse algebraic K-homology and coarse ordinary homology, see Sections 3.3 and 3.11.

3. EXAMPLES

In this section we show that equivariant coarse algebraic K-homology and equivariant coarse ordinary homology extend to equivariant coarse homology theories with transfers.

3.1. Functors out of $G\mathbf{BornCoarse}_{\mathrm{tr}}$.

In order to construct coarse homology theories with transfers (see Definition 2.58), we must construct functors out of the ∞-category $G\mathbf{BornCoarse}_{\mathrm{tr}}$. Since this category is given in Section 2.23 explicitly as some simplicial set, there are essentially two options. The simpler option is to start with the canonical functor

$$G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$$

and then to construct ordinary functors out of $\mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}})$. This option works in the case of the construction of equivariant ordinary coarse homology with transfers in Section 3.11. The more complicated option is to describe directly a map of simplicial sets with domain $G\mathbf{BornCoarse}_{\mathrm{tr}}$. In the case of the construction of equivariant coarse algebraic K-homology with coefficients in a G-equivariant additive category in Section 3.3, the target of this map is the nerve of the strict $(2,1)$-category $\mathbf{Add}$ of additive categories.

The main goal of the present section is to prepare the construction of coarse algebraic K-homology with transfers by describing the data necessary to define a functor from $G\mathbf{BornCoarse}_{\mathrm{tr}}$ to the nerve of some strict $(2,1)$-category $\mathbf{C}$.

Applying the usual nerve functor $\mathtt{N}\colon \mathbf{Cat} \to \mathbf{sSet}$ to the morphism categories, we get a category $\mathtt{N}(\mathbf{C})$ which is enriched in Kan complexes. We can now further apply the homotopy coherent nerve functor $\mathcal{N}$. In this way we get an ∞-category which, following [10, Def. A.12], will be denoted by $\mathtt{N}_2(\mathbf{C})$. In the following, we describe sufficient data (justified by Lemma 3.2 below) for a functor

$$\mathbf{V}_{\mathrm{tr}}\colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathtt{N}_2(\mathbf{C}). \tag{26}$$

Suppose we are given the following data:

(i) a functor $\mathbf{V}\colon G\mathbf{BornCoarse} \to u(\mathbf{C})$, where $u(\mathbf{C})$ is the 1-category obtained from $\mathbf{C}$ by forgetting the rest of the 2-category structure;

(ii) for every bounded covering $w\colon W \to Z$ (see Definition 2.15), a 1-morphism
$$w^*\colon \mathbf{V}(Z) \to \mathbf{V}(W);$$

(iii) for every two composable bounded coverings $w\colon W \to Z$ and $v\colon V \to W$, a 2-isomorphism
$$a_{v,w}\colon (w \circ v)^* \Rightarrow v^* \circ w^*;$$

(iv) for every admissible square

$$\begin{tikzcd} & W \arrow[dl, twoheadrightarrow, "w"'] \arrow[dr, "f"] & \\ V \arrow[dr, "g"'] & & U \arrow[dl, twoheadrightarrow, "u"] \\ & Z & \end{tikzcd}$$

of G-bornological coarse spaces (see Definition 2.20), a 2-morphism

$$b_{g,u} \colon f_* \circ w^* \Rightarrow u^* \circ g_*,$$

where we write f_* and g_* for $\mathbf{V}(f)$ and $\mathbf{V}(g)$, respectively.

We assume that this data satisfies the following conditions:

(i) If the bounded covering $w \colon W \to Z$ is an isomorphism of the underlying G-coarse spaces, then we require that $w^* = (w^{-1})_*$. This is possible since the inverse of a bornological bijection is proper and hence $w^{-1} \colon Z \to W$ is a morphism of G-bornological coarse spaces.

(ii) If two composable bounded coverings $w \colon W \to Z$ and $v \colon V \to W$ are isomorphisms of the underlying G-coarse spaces, then $\alpha_{v,w}$ is the identity of $(v^{-1})_* \circ (w^{-1})_* = ((w \circ v)^{-1})_*$. Note that this is possible to require by Condition (i).

(iii) For every three composable bounded coverings $w \colon W \to Z$, $v \colon V \to W$ and $u \colon U \to V$, the square

(27)
$$\begin{tikzcd} (w \circ v \circ u)^* \arrow[r, Rightarrow, "a_{vu,w}"] \arrow[d, Rightarrow, "a_{u,wv}"'] & (v \circ u)^* \circ w^* \arrow[d, Rightarrow, "a_{u,v} \circ w^*"] \\ u^* \circ (w \circ v)^* \arrow[r, Rightarrow, "u^* \circ a_{v,w}"] & u^* \circ v^* \circ w^* \end{tikzcd}$$

commutes.

(iv) In the case of an admissible square with morphisms w, f, g, u, if u (and therefore also w) is an isomorphism of the underlying G-coarse spaces, then we require that $b_{g,u}$ is the identity of $f_* \circ (w^{-1})_* = (u^{-1})_* \circ g_*$.

(v) In the case of an admissible square with morphisms w, f, g, u, if f and g are identities and therefore $w = u$, then we require that $b_{g,u}$ is the identity of $w^* = u^*$.

(vi) For every diagram

$$\begin{tikzcd}[column sep=small]
 & & T \arrow[dl, two heads, "t"'] \arrow[dr, "m"] & & \\
 & U \arrow[dr, "h"] & & S \arrow[dl, two heads, "s"'] \arrow[dr, "n"] & \\
 & & V \arrow[dr, "g"] & & R \arrow[dl, two heads, "r"] \\
 & & & Z &
\end{tikzcd}$$

consisting of two admissible squares, we have the relation

$$b_{gh,r} = (b_{g,r} \circ h_*)(n_* \circ b_{h,s}). \tag{28}$$

(vii) For every diagram

$$\begin{tikzcd}[column sep=small]
 & & T \arrow[dl, two heads, "t"'] \arrow[dr, "m"] & \\
 & U \arrow[dl, two heads, "u"'] \arrow[dr, "h"] & & S \arrow[dl, two heads, "s"] \\
W \arrow[dr, "f"] & & V \arrow[dl, two heads, "v"] & \\
 & Y & &
\end{tikzcd}$$

consisting of two admissible squares, we have the relation

$$(a_{s,v} \circ f_*) b_{f,vs} = (s^* \circ b_{f,v})(b_{h,s} \circ u^*)(m_* \circ a_{t,u}). \tag{29}$$

Lemma 3.2. *The data as described above determines a functor*

$$\mathbf{V}_{\mathrm{tr}} \colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathtt{N}_2(\mathbf{C})$$

such that the diagram

$$\begin{tikzcd}
G\mathbf{BornCoarse}_{\mathrm{tr}} \arrow[r, "\mathbf{V}_{\mathrm{tr}}"] & \mathtt{N}_2(\mathbf{C}) \\
G\mathbf{BornCoarse} \arrow[u, "\iota"] \arrow[r, "\mathbf{V}"] & \mathtt{N}(u(\mathbf{C})) \arrow[u]
\end{tikzcd} \tag{30}$$

commutes.

Proof. It is known that the nerve $\mathtt{N}_2(\mathbf{C})$ for a strict $(2,1)$-category is 3-coskeletal [10, Prop. A.16]. Therefore it suffices to provide the map $\mathbf{V}_{\mathrm{tr}}$ on simplices of dimensions $0, 1, 2$ and 3. We need an explicit description of the 3-skeleton of the nerve $\mathtt{N}_2(\mathbf{C})$ (compare [10, Rem. A.18]).

For every n in $\mathbb{N}$, we consider the simplicially enriched category $\mathcal{C}[n]$ with objects $\{0,\dots,n\}$ and whose morphism space $\mathrm{Map}_{\mathcal{C}[n]}(i,j)$ is the nerve of the poset of subsets of $[i,j]$ containing i and j. Then, by definition,

$$\mathtt{N}_2(\mathbf{C})[n] = \mathrm{Hom}_{\mathbf{sCat}}(\mathcal{C}[n], \mathtt{N}(\mathbf{C})).$$

The following describes the n-simplices of $\mathtt{N}_2(\mathbf{C})$ for $n \le 3$:

(i) $\mathtt{N}_2(\mathbf{C})[0] = \mathrm{Ob}(\mathbf{C})$.

(ii) We have $\mathtt{N}_2(\mathbf{C})[1] = \mathbf{Fun}(\mathcal{C}[1], \mathbf{C})$. Note that $\mathrm{Map}_{\mathcal{C}[1]}(0,1) = \{*\}$. Therefore a one-simplex in $\mathtt{N}_2(\mathbf{C})$ is a morphism $X \to Y$ in $\mathbf{C}$ and its faces are X and Y.

(iii) A two-simplex in $\mathtt{N}_2(\mathbf{C})$ is given by a diagram

$$\begin{tikzcd} & Y \arrow[dr] & \\ X \arrow[ur] \arrow[rr] \arrow[r, phantom, ""] & \arrow[u, phantom, "\Uparrow"] & Z. \end{tikzcd}$$

(iv) The mapping spaces $\mathrm{Hom}_{\mathcal{C}[3]}(0,1)$, $\mathrm{Hom}_{\mathcal{C}[3]}(1,2)$ and $\mathrm{Hom}_{\mathcal{C}[3]}(2,3)$ are points. The mapping spaces $\mathrm{Hom}_{\mathcal{C}[3]}(0,2)$ and $\mathrm{Hom}_{\mathcal{C}[3]}(1,3)$ are isomorphic to Δ^1 and we call their one-simplexes α and β. The mapping space $\mathrm{Hom}_{\mathcal{C}[3]}(0,3)$ is the square

$$\begin{tikzcd} & \{0,1,3\} \arrow[dr, Rightarrow, "\beta\circ\{0,1\}"] & \\ \{0,3\} \arrow[ur, Rightarrow, "\delta"'] \arrow[rr, Rightarrow] \arrow[dr, Rightarrow, "\gamma"] & & \{0,1,2,3\}. \\ & \{0,2,3\} \arrow[ur, Rightarrow, "\{2,3\}\circ\alpha"'] & \end{tikzcd}$$

Hence, in order to provide a 3-simplex in $\mathtt{N}_2(\mathbf{C})$, we must provide the following data:

(i) four objects X_0, X_1, X_2, X_3;

(ii) six 1-morphisms $f_{ij}\colon X_i \to X_j$ for $i < j$;

(iii) four 2-morphisms

$$\alpha\colon f_{02} \Rightarrow f_{12} \circ f_{01}, \quad \beta\colon f_{13} \Rightarrow f_{23} \circ f_{12},$$
$$\gamma\colon f_{03} \Rightarrow f_{23} \circ f_{02}, \quad \delta\colon f_{03} \Rightarrow f_{13} \circ f_{01},$$

satisfying the relation

$$(\beta \circ f_{01})\delta = (f_{23} \circ \alpha)\gamma. \tag{31}$$

We can now construct $\mathbf{V}_{\mathrm{tr}}$ using the data described above.

(i) On zero simplices of $G\mathbf{BornCoarse}_{\mathrm{tr}}$, we define $\mathbf{V}_{\mathrm{tr}}(X) := \mathbf{V}(X)$.

(ii) On 1-simplices of $G\mathbf{BornCoarse}_{\mathrm{tr}}$, the functor $\mathbf{V}_{\mathrm{tr}}$ sends the span (W, w, f) to the morphism

$$f_* \circ w^*\colon \mathbf{V}(X) \to \mathbf{V}(Y).$$

Note that if (W, w, f) is in the image of ι, then (30) commutes on the level of 1-simplices by Condition (i).

(iii) The functor $\mathbf{V}_{\mathrm{tr}}$ sends a 2-simplex

$$\begin{tikzcd}
 & & U \arrow[dl, "u"'] \arrow[dr, "h"] & & \\
 & W \arrow[dl, "w"'] \arrow[dr, "f"] & & V \arrow[dl, "v"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd} \tag{32}$$

to the diagram

$$\begin{tikzcd}
 & \mathbf{V}(Y) \arrow[dr, "g_*\circ v^*"] & \\
\mathbf{V}(X) \arrow[ur, "f_*\circ w^*"] \arrow[rr, "(g\circ h)_*\circ(w\circ u)^*"'] \arrow[rr, phantom, shift left=4ex, "\Uparrow"] & & \mathbf{V}(Z)
\end{tikzcd}$$

filled by the 2-morphism

$$\begin{aligned}
(g\circ h)_*\circ(w\circ u)^* &\xRightarrow{(g\circ h)_*\circ a_{u,w}} (g\circ h)_*\circ u^*\circ w^* \\
&= g_*\circ(h_*\circ u^*)\circ w^* \xRightarrow{g_*\circ b_{f,v}\circ w^*} g_*\circ v^*\circ f_*\circ w^*.
\end{aligned}$$

If the 2-simplex is in the image of ι, then (30) commutes on the level of 2-simplices by Conditions (i), (ii) and (iv).

(iv) The functor $\mathbf{V}_{\mathrm{tr}}$ sends a 3-simplex

$$\begin{tikzcd}
 & & & T \arrow[dl, two heads, "t"'] \arrow[dr, "m"] & & & \\
 & & U \arrow[dl, two heads, "u"'] \arrow[dr, "h"] & & S \arrow[dl, two heads, "s"'] \arrow[dr, "n"] & & \\
 & W \arrow[dl, two heads, "w"'] \arrow[dr, "f"] & & V \arrow[dl, two heads, "v"'] \arrow[dr, "g"] & & R \arrow[dl, two heads, "r"'] \arrow[dr, "l"] & \\
X & & Y & & Z & & Q
\end{tikzcd}$$

to the 3-simplex of $\mathtt{N}_2(\mathbf{C})$ given by the following data:

(i) The objects of the 3-simplex are $\mathbf{V}(X)$, $\mathbf{V}(Y)$, $\mathbf{V}(Z)$, and $\mathbf{V}(Q)$.

(ii) The 1-morphisms are

(1) $f_{01} := f_*\circ w^*$,

(2) $f_{12} := g_*\circ v^*$,

(3) $f_{23} := l_* \circ r^*$,
(4) $f_{02} := (g \circ h)_* \circ (w \circ u)^*$,
(5) $f_{13} := (l \circ n)_* \circ (v \circ s)^*$,
(6) $f_{03} := (l \circ n \circ m)_* \circ (w \circ u \circ t)^*$.

(iii) The 2-morphisms are
(1) $\alpha := (g_* \circ b_{f,v} \circ w^*)((g \circ h)_* \circ a_{u,w})$,
(2) $\beta := (l_* \circ b_{g,r} \circ v^*)((l \circ n)_* \circ a_{s,v})$,
(3) $\gamma := (l_* \circ b_{gh,r} \circ (w \circ u)^*)((l \circ n \circ m)_* \circ a_{t,wu})$,
(4) $\delta := ((l \circ n)_* \circ b_{f,vs} \circ w^*)((l \circ n \circ m)_* \circ a_{ut,w})$.

We must check relation (31):

$$\begin{aligned}
(\beta \circ f_{01})\delta &= (l_* b_{g,r} v^* f_* w^*)(l_* n_* \boldsymbol{a_{s,v} f_*} w^*)(l_* n_* \boldsymbol{b_{f,vs}} w^*)(l_* n_* m_* a_{ut,w}) \\
&\overset{(29)}{=} (l_* b_{g,r} v^* f_* w^*)(l_* n_* s^* b_{f,v} w^*)(l_* n_* b_{h,s} u^* w^*) \\
&\qquad (l_* n_* m_* \boldsymbol{a_{t,u} w^*})(l_* n_* m_* \boldsymbol{a_{ut,w}}) \\
&\overset{(27)}{=} (l_* \boldsymbol{b_{g,r} v^* f_*} w^*)(l_* \boldsymbol{n_* s^* b_{f,v}} w^*)(l_* n_* b_{h,s} u^* w^*) \\
&\qquad (l_* n_* m_* t^* a_{u,w})(l_* n_* m_* a_{t,wu}) \\
&\overset{!}{=} (l_* r^* g_* b_{f,v} w^*)(l_* \boldsymbol{b_{g,r} h_*} u^* w^*)(l_* \boldsymbol{n_* b_{h,s}} u^* w^*) \\
&\qquad (l_* n_* m_* t^* a_{u,w})(l_* n_* m_* a_{t,wu}) \\
&\overset{(28)}{=} (l_* r^* g_* b_{f,v} w^*)(l_* \boldsymbol{b_{gh,r} u^* w^*})(l_* \boldsymbol{n_* m_* t^* a_{u,w}})(l_* n_* m_* a_{t,wu}) \\
&\overset{!}{=} (l_* r^* g_* b_{f,v} w^*)(l_* r^* g_* h_* a_{u,w})(l_* b_{gh,r} (wu)^*)(l_* n_* m_* a_{t,wu}) \\
&= (f_{23} \circ \alpha)\gamma.
\end{aligned}$$

For better legibility, we omitted the composition sign $\circ$ and marked boldface the part to which the respective relation is applied. The equations marked by ! hold in every $(2,1)$-category.

One again checks that the diagram (30) commutes on the level of 3-simplices because of Conditions (i), (ii) and (iv).

It is immediate from the definitions that our construction is compatible with the face maps. To verify the compatibility with the degeneracy maps we use Conditions (i), (ii), (iv) and (v) applied to identity maps in the appropriate places. □

3.3. **Coarse algebraic K-homology.** Let $\mathbf{A}$ be an additive category with a strict G-action. In this section we construct the extension of equivariant coarse algebraic K-homology $K\mathbf{A}\mathcal{X}^G \colon G\mathbf{BornCoarse} \to \mathbf{Sp}$ to an equivariant coarse homology theory with transfers $K\mathbf{A}\mathcal{X}^G_{\mathrm{tr}}$. For the construction of the functor $K\mathbf{A}\mathcal{X}^G$ (which will be recalled in detail below) and the verification of the axioms of an equivariant coarse homology theory, we refer to [4, Section 8].

We first explain how the algebraic K-theory functor for additive categories can be extended to a functor defined on the ∞-category $\mathtt{N}_2(\mathbf{Add})$, see the beginning of Section 3.1 for the notation $\mathtt{N}_2$. We start with a non-connective algebraic K-theory functor

$$K\colon \mathbf{Add} \to \mathbf{Sp}, \quad \mathbf{A} \mapsto K(\mathbf{A}),$$

for additive categories, see [18]. More precisely, we consider K a functor between ∞-categories

$$K\colon \mathtt{N}(\mathbf{Add}) \to \mathbf{Sp}.$$

Let W be the class of equivalences of additive categories in $\mathbf{Add}$. Since K sends equivalences between additive categories to equivalences of spectra, it has an essentially unique factorization over the localization $\mathtt{N}(\mathbf{Add}) \to \mathtt{N}(\mathbf{Add})[W^{-1}]$. Because the natural inclusion $\mathtt{N}(\mathbf{Add}) \to \mathtt{N}_2(\mathbf{Add})$ sends equivalences between additive categories to equivalences in the ∞-category $\mathtt{N}_2(\mathbf{Add})$ it induces a functor $\mathtt{N}(\mathbf{Add})[W^{-1}] \to \mathtt{N}_2(\mathbf{Add})$. The latter is an equivalence of ∞-categories [5, Section 3.1].

Hence we get a commuting diagram in $\mathbf{Cat}_\infty$

(33)
$$\begin{tikzcd}
\mathtt{N}(\mathbf{Add}) \arrow[rr, "K"] \arrow[d] & & \mathbf{Sp} \\
\mathtt{N}(\mathbf{Add})[W^{-1}] \arrow[urr, dotted] \arrow[d, "\simeq"] & & \\
\mathtt{N}_2(\mathbf{Add}). \arrow[uurr, "\mathbf{K}"'] & &
\end{tikzcd}$$

It provides an essentially unique extension of K to a functor

(34)
$$\mathbf{K}\colon \mathtt{N}_2(\mathbf{Add}) \to \mathbf{Sp}.$$

Let X be a G-bornological coarse space. The spectrum $K\mathbf{A}\mathcal{X}^G(X)$ is the non-connective algebraic K-theory spectrum of the additive category $\mathbf{V}^G_{\mathbf{A}}(X)$ of equivariant X-controlled objects of A and equivariant morphisms with controlled propagation [4, Section 8.2]. The functor $K\mathbf{A}\mathcal{X}^G$ is defined as the composition

$$K\mathbf{A}\mathcal{X}^G := K \circ \mathbf{V}^G_{\mathbf{A}}\colon G\mathbf{BornCoarse} \to \mathbf{Add} \to \mathbf{Sp}.$$

For the verification that $K\mathbf{A}\mathcal{X}^G$ satisfies the axioms of a strongly additive equivariant coarse homology theory, we refer to [4, Thm. 8.9 and Prop. 8.19].

In order to construct the extension $K\mathbf{A}\mathcal{X}^G_{\mathrm{tr}}$, we use the method described in Section 3.1 to construct an extension

$$\mathbf{V}^G_{\mathbf{A},\mathrm{tr}}\colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathtt{N}_2(\mathbf{Add})$$

of the functor $\mathbf{V}^G_{\mathbf{A}}$, and compose it then with the functor $\mathbf{K}$ in (34).

We start with recalling the details of the definition of the $\mathbf{Add}$-valued functor $\mathbf{V}^G_{\mathbf{A}}$ from [4, Section 8.2]. Let $\mathbf{A}$ be an additive category with a strict G-action and let X be a G-bornological coarse space. We consider the bornology $\mathcal{B}$ of X as a poset with a G-action and hence as a category with a G-action.

If $A\colon \mathcal{B} \to \mathbf{A}$ is a functor and g is an element of G, then $gA\colon \mathcal{B} \to \mathbf{A}$ denotes the functor which sends a bounded set B in $\mathcal{B}$ to the object $gA(g^{-1}(B))$ of $\mathbf{A}$.

If $\rho\colon A \to A'$ is a natural transformation between two such functors, then we let $g\rho\colon gA \to gA'$ denote the canonically induced natural transformation.

Definition 3.4. An *equivariant X-controlled $\mathbf{A}$-object* is a pair (A, ρ) consisting of a functor $A\colon \mathcal{B} \to \mathbf{A}$ and a family $\rho = (\rho(g))_{g\in G}$ of natural isomorphisms $\rho(g)\colon A \to gA$ satisfying the following conditions:

(i) $A(\varnothing) \cong 0$.

(ii) For all B, B' in $\mathcal{B}$, the commutative square

$$\begin{tikzcd} A(B\cap B') \arrow[r] \arrow[d] & A(B) \arrow[d] \\ A(B') \arrow[r] & A(B\cup B') \end{tikzcd}$$

is a pushout square.

(iii) For all B in $\mathcal{B}$, there exists some finite subset F of B such that the inclusion $F \to B$ induces an isomorphism $A(F) \xrightarrow{\cong} A(B)$.

(iv) For all pairs of elements g, g' of G, we have the relation $\rho(gg') = g\rho(g') \circ \rho(g)$.

If U is an invariant coarse entourage of X, i.e., an element of $\mathcal{C}^G$, then we get a G-equivariant functor

$$U[-]\colon \mathcal{B} \to \mathcal{B}$$

which sends a bounded subset B of X to its U-thickening

$$U[B] := \{x \in X \mid \text{there exists } b \in B : (x, b) \in U\}.$$

Note that $U[B]$ is again bounded by the compatibility of the coarse structure $\mathcal{C}$ and the bornology $\mathcal{B}$. For g in G, we have the equality $U[gB] = gU[B]$ by the G-invariance of U. Furthermore, note that for B' in $\mathcal{B}$ with $B \subseteq B'$, we have $U[B] \subseteq U[B']$.

Let $(A, \rho), (A', \rho')$ be equivariant X-controlled $\mathbf{A}$-objects and let U be an invariant coarse entourage of X.

Definition 3.5. An *equivariant U-controlled morphism* $\phi\colon (A, \rho) \to (A', \rho')$ is a natural transformation

$$\phi\colon A(-) \to A'(U[-])$$

such that $\rho'(g) \circ \phi = (g\phi) \circ \rho(g)$ for all elements g of G.

We let $\mathrm{Mor}_U((A, \rho), (A', \rho'))$ be the abelian group of equivariant U-controlled morphisms.

If U' is in $\mathcal{C}^G$ and such that $U \subseteq U'$, then for every B in $\mathcal{B}$, we have $U[B] \subseteq U'[B]$. These inclusions induce a transformation between functors $A'(U[-]) \to A'(U'[-])$ and therefore a map

$$\mathrm{Mor}_U((A, \rho), (A', \rho')) \to \mathrm{Mor}_{U'}((A, \rho), (A', \rho')),$$

by post-composition. Using these maps in the interpretation of the colimit, we define the abelian group of equivariant controlled morphisms from A to A' by

$$\mathrm{Hom}_{\mathbf{V}^G_{\mathbf{A}}(X)}((A,\rho),(A',\rho')) := \operatorname*{colim}_{U\in\mathcal{C}^G} \mathrm{Mor}_U((A,\rho),(A',\rho')).$$

We now consider a pair of morphisms in

$$\mathrm{Hom}_{\mathbf{V}^G_{\mathbf{A}}(X)}((A,\rho),(A',\rho')) \quad \text{and} \quad \mathrm{Hom}_{\mathbf{V}^G_{\mathbf{A}}(X)}((A',\rho'),(A'',\rho'')),$$

respectively, which are represented by

$$\phi\colon A(-)\to A'(U[-]) \quad \text{and} \quad \phi'\colon A'(-)\to A''(U'[-]).$$

We define the composition of the two morphisms to be represented by the morphism

$$U[-]^*\phi'\circ\phi\colon A\to A''((U'\circ U)[-]),$$

where $U[-]^*\phi'\colon A'(U[-])\to A''((U'\circ U)[-])$ is defined in the canonical manner.

We denote now the resulting additive category of equivariant X-controlled $\mathbf{A}$-objects and equivariant controlled morphisms by $\mathbf{V}^G_{\mathbf{A}}(X)$.

Let $f\colon (X,\mathcal{B},\mathcal{C})\to(X',\mathcal{B}',\mathcal{C}')$ be a morphism of G-bornological coarse spaces, and let (A,ρ) be an equivariant X-controlled $\mathbf{A}$-object. Since f is proper, it induces a functor $f^{-1}\colon \mathcal{B}'\to\mathcal{B}$, and we can define a functor $f_*A\colon \mathcal{B}'\to\mathbf{A}$ by

$$f_*A := A\circ f^{-1}.$$

Furthermore, we define

$$f_*\rho(g) := \rho(g)\circ f^{-1}.$$

Let U be in $\mathcal{C}^G$ and let $\phi\colon (A,\rho)\to(A',\rho')$ be an equivariant U-controlled morphism. Then $V := (f\times f)(U)$ belongs to $\mathcal{C}'^G$ and $U[f^{-1}(B')]\subseteq f^{-1}(V[B'])$ for all bounded subsets B' of X'. Therefore we obtain an induced V-controlled morphism

$$f_*\phi = \bigl\{f_*A(B') \xrightarrow{\phi_{f^{-1}(B')}} A(U[f^{-1}(B')]) \to f_*A(V[B'])\bigr\}_{B'\in\mathcal{B}'}.$$

One checks that this construction defines an additive functor

$$f_*\colon \mathbf{V}^G_{\mathbf{A}}(X)\to\mathbf{V}^G_{\mathbf{A}}(X').$$

This completes the construction of the functor

$$\mathbf{V}^G_{\mathbf{A}}\colon G\mathbf{BornCoarse}\to\mathbf{Add}.$$

We now start the construction of the functor $\mathbf{V}^G_{\mathbf{A},\mathrm{tr}}$.

Let $w\colon W\to Z$ be a bounded covering. Given a controlled object (A,ρ) in $\mathbf{V}^G_{\mathbf{A}}(Z)$, we define $w^*(A,\rho)=(w^*A,w^*\rho)$ as follows. Let $\mathcal{B}_W$ denote the category of bounded subsets of W and let

$$\mathcal{B}'_W\subseteq\mathcal{B}_W \tag{35}$$

be the full subcategory consisting of coarsely connected bounded subsets. Let

$$\hat{w}\colon \mathcal{B}'_W\to\mathcal{B}_Z$$

be the functor sending B in $\mathcal{B}'_W$ to $w(B)$ in $\mathcal{B}_Z$. We define $w^*A := \mathrm{Lan}(A\hat{w})$ to be a left Kan extension of $A\hat{w}$ along the inclusion $i\colon \mathcal{B}'_W \to \mathcal{B}_W$ as indicated in the following diagram:

$$\begin{tikzcd} \mathcal{B}'_W \arrow[r, "A\hat{w}"] \arrow[d, "i"'] \arrow[d, phantom, shift left=4, "\Swarrow \tau_{A,w}"] & \mathbf{A} \\ \mathcal{B}_W. \arrow[ur, "\mathrm{Lan}(A\hat{w})"'] & \end{tikzcd}$$

The definition of w^*A involves a choice. It is fixed uniquely up to unique isomorphism if we take into account the natural transformation

$$\tau_{A,w}\colon A\hat{w} \to \mathrm{Lan}(A\hat{w})i,$$

which is actually a natural isomorphism since i is fully faithful. If w is an isomorphism of the underlying coarse spaces, then $w^{-1}\colon Z \to B$ is proper, and we can choose the object $w^*A := w^{-1}_*A$ and let $\tau_{A,w}$ be the identity. This ensures Conditions (i) and (ii) formulated in Section 3.1. We suppress $\tau_{A,w}$ from notation unless we need to mention it explicitly.

For every g in G, we further define $w^*\rho(g)\colon w^*A \to gw^*A$ as the composition

$$\mathrm{Lan}(A\hat{w}) \xrightarrow{\mathrm{Lan}(\rho(g)\hat{w})} \mathrm{Lan}(gA\hat{w}) \xrightarrow{\iota} g\,\mathrm{Lan}(A\hat{w}),$$

where the morphisms are uniquely determined by the universal property of left Kan extensions and the relations

$$\tau_{gA,w}(\rho(g)\hat{w}) = (\mathrm{Lan}(\rho(g)\hat{w}) \circ i)\tau_{A,w}, \quad (\iota \circ i)\tau_{gA,w} = g\tau_{A,w}.$$

The morphism ι is an isomorphism since $(g\,\mathrm{Lan}(A\hat{w}), g\tau_{A,w})$ has the property of a left Kan extension of $gA\hat{w}$ along i.

Note that $\mathbf{A}$ admits finite sums but is in general not cocomplete.[2] Therefore we must check that the Kan extensions actually exist and land in the desired functor category.

Lemma 3.6. *The Kan extensions involved in the construction of $A\hat{w}$ exists and $(w^*A, w^*\rho)$ is an object of $\mathbf{V}^G_{\mathbf{A}}(W)$.*

Proof. By [15, Cor. X.3.4], the Kan extension exists if for every B of B_W, the colimit

$$\operatorname*{colim}_{(B'\subseteq B)\in\mathcal{B}'_W/B} A(w(B'))$$

exists. Fix B in B_W. Since w is a bounded covering (see Definition 2.15), there exists a finite, coarsely disjoint partition $(B_j)_{j\in J}$ of B such that $w_{|[B_j]}\colon [B_j] \to [w(B_j)]$ is an isomorphism of coarse spaces for every j in J. Since every element of $\mathcal{B}'_W$ is coarsely connected, we have a decomposition of categories

$$\mathcal{B}'_W/B \simeq \coprod_{j\in J} \mathcal{B}'_W/B_j.$$

[2] Such a condition would actually lead, by an Eilenberg swindle, to a very uninteresting K-theory.

For every j in J, the inclusion of the discrete subcategory

$$\{(B_j \cap W_0 \subseteq B_j)\}_{W_0 \in \pi_0(W)}$$

into the comma category $\mathcal{B}'_W/B_j$ is cofinal. Hence we have to show that the sum

$$\bigoplus_{j \in J} \bigoplus_{W_0 \in \pi_0(W)} A(w(W_0 \cap B_j)) \tag{36}$$

exists. Since $w(B_j)$ is a bounded subset of Z by Property 3.4 (iii) of A, it admits a finite subset F_j such that $A(F_j) \xrightarrow{\cong} A(w(B_j))$. We can choose a finite subset P_j of $\pi_0(W)$ such that $F_j \cap w(B_j \cap W_0) = \varnothing$ for all W_0 in $\pi_0(W) \setminus P_j$. In (36) we can therefore restrict the sum to the finite set P_j. Since $\mathbf{A}$ admits finite sums, this completes the proof of the existence of the Kan extension.

We use Properties 3.4 (ii) and 3.4 (iii) for A in order to calculate the sum in (36), and hence the value of the Kan extension at B, explicitly. We obtain an isomorphism

$$\mathrm{Lan}(A\hat{w})(B) \cong \bigoplus_{j \in J} A(w(B_j)). \tag{37}$$

It is now straight-forward to check that w^*A satisfies Conditions 3.4 (i)–(iii) for a W-controlled $\mathbf{A}$-object. The Relation 3.4 (iv) can be checked by a similar reasoning as in the construction of $w^*\rho(g)$ using the universal property of left Kan extensions. □

The following observation is stated here for later use. Let W be a G-bornological coarse space and let $i\colon \mathcal{B}'_W \to \mathcal{B}_W$ be the inclusion as in (35). Let (A, ρ) be an object of $\mathbf{V}^G_{\mathbf{A}}(W)$.

Lemma 3.7. *Then A is canonically isomorphic to* $\mathrm{Lan}(Ai)$*, the left Kan extension of $A \circ i$ along i.*

Proof. The argument is similar to the argument leading to (37) in the proof above. □

Lemma 3.8. *If w is an isomorphism on the underlying coarse spaces, then $\mathcal{B}_W$ is a subset of $\mathcal{B}_Z$ and, in this case w^*A, is isomorphic to the restriction of A to $\mathcal{B}_W$.*

Proof. The first statement follows by Definition 2.15 of a bounded covering. The second one from the pointwise formula for Kan extensions and Properties 3.4 (ii) and (iii) of A. □

This finishes the construction of w^* on objects. We now define w^* on morphism as follows. Let U be an invariant entourage of Z and let a natural transformation $\phi\colon A \to A' \circ U[-]$ be given. We set $U_W := (w \times w)^{-1}(U) \cap U(\pi_0(W))$, where $U(\pi_0(W))$ is defined as in (2). Then we consider the commutative diagram

$$\begin{tikzcd}
\mathcal{B}_W \arrow[r, "U_W[-]"] & \mathcal{B}_W \\
\mathcal{B}'_W \arrow[u, "i", "\subseteq"'] \arrow[r, "U_W[-]"] \arrow[d, "\hat{w}"'] & \mathcal{B}'_W \arrow[u, "i", "\subseteq"'] \arrow[d, "\hat{w}"'] \\
\mathcal{B}_Z \arrow[r, "U[-]"] & \mathcal{B}_Z .
\end{tikzcd}$$

We consider the composition

$$\begin{aligned}
&\mathrm{Nat}(A\hat{w}, A'U[-]\hat{w}) \\
&\quad = \mathrm{Nat}(A\hat{w}, A'\hat{w}U_W[-]) \xrightarrow[(\tau_{A',w})_*]{\cong} \mathrm{Nat}(A\hat{w}, \mathrm{Lan}(A'\hat{w})iU_W[-]) \\
&\quad = \mathrm{Nat}(A\hat{w}, \mathrm{Lan}(A'\hat{w})U_W[-]i) \xrightarrow[(\tau^*_{A,w})^{-1}]{\cong} \mathrm{Nat}(\mathrm{Lan}(A\hat{w}), \mathrm{Lan}(A'\hat{w})U_W[-]),
\end{aligned}$$

and we define the morphism $w^*\phi\colon w^*A \to (w^*A')U_W[-]$ to be the image of $\phi \circ \hat{w}$ under this map. In other words, the morphism $w^*\phi$ is uniquely determined by the equation

$$(w^*\phi \circ i)\tau_{A,w} = \tau_{A',w} \circ (\phi\hat{w}). \tag{38}$$

Using this equation, one checks easily that the construction of w^* is compatible with the composition.

Given two bounded coverings $V \xrightarrow{v} W \xrightarrow{w} Z$, we now have to define a natural isomorphism

$$a_{v,w}\colon (wv)^*A \to v^*w^*A.$$

Let $j\colon \mathcal{B}'_V \to \mathcal{B}_V$ be the inclusion analogous to the one in (35). We observe that $\hat{v}$ has a canonical factorization $\hat{v}\colon \mathcal{B}'_V \xrightarrow{\hat{v}'} \mathcal{B}'_W \xrightarrow{i} \mathcal{B}_W$ such that $\widehat{wv} = \hat{w}\hat{v}'$. Since we have a natural isomorphism

$$\begin{aligned}
A\widehat{wv} = A\hat{w}\hat{v}' &\xrightarrow[\tau_{A,w}\circ\hat{v}']{\cong} \mathrm{Lan}(A\hat{w})i\hat{v}' \\
&= \mathrm{Lan}(A\hat{w})\hat{v} \xrightarrow[\tau_{\mathrm{Lan}(A\hat{w}),v}]{\cong} \mathrm{Lan}(\mathrm{Lan}(A\hat{w})\hat{v})j,
\end{aligned} \tag{39}$$

the functor $\mathrm{Lan}(\mathrm{Lan}(A\hat{w})\hat{v})$ is a left Kan extension of $A\widehat{wv}$ along j, by Lemma 3.7. We define the natural isomorphism $a_{v,w}$ by

$$a_{v,w}\colon (wv)^*A = \mathrm{Lan}(A\widehat{wv}) \xrightarrow{(39)} \mathrm{Lan}(\mathrm{Lan}(A\hat{w})\hat{v}) = v^*w^*A.$$

In particular, $a_{v,w}$ is uniquely determined by the equality

$$(a_{v,w} \circ j)\tau_{A,wv} = \tau_{\mathrm{Lan}(A\hat{w}),v}(\tau_{A,w} \circ \hat{v}'). \tag{40}$$

Since $(wv)^*\rho(g)$ and $v^*w^*\rho(g)$ are the natural equivalences of the left Kan extensions induced by $\rho(g)$, they agree under the above natural isomorphism.

Given three bounded coverings $U \xrightarrow{u} V \xrightarrow{v} W \xrightarrow{w} Z$, we have a commutative diagram

$$\begin{tikzcd}
\mathcal{B}'_U \arrow[r, "\hat{u}'"] \arrow[d, "k"] \arrow[dr, "\hat{u}"] \arrow[rr, bend left, "\widehat{uv}'"] & \mathcal{B}'_V \arrow[r, "\hat{v}'"] \arrow[d, "j"] \arrow[dr, "\hat{v}"] & \mathcal{B}'_W \arrow[d, "i"] \arrow[dr, "\hat{w}"] & \\
\mathcal{B}_U & \mathcal{B}_V & \mathcal{B}_W & \mathcal{B}_Z .
\end{tikzcd}$$

We conclude that

$$\begin{aligned}
(((a_{u,v} \circ w^*)a_{vu,w}) \circ k)\tau_{A,wvu} &= (a_{u,v} \circ w^* \circ k)(\mathbf{a_{vu,w}} \circ \mathbf{k})\tau_{\mathbf{A,wvu}} \\
&\overset{(40)}{=} (\mathbf{a_{u,v}} \circ \mathbf{w}^* \circ \mathbf{k})\tau_{\mathrm{Lan}(\mathbf{A\hat{w}}),\mathbf{vu}}(\tau_{A,w} \circ \widehat{vu}') \\
&\overset{(40)}{=} \tau_{\mathrm{Lan}(\mathrm{Lan}(A\hat{w})\hat{v}),u}(\tau_{\mathrm{Lan}(\mathbf{A\hat{w}}),\mathbf{v}} \circ \hat{\mathbf{u}}')(\tau_{\mathbf{A,w}} \circ \widehat{\mathbf{vu}}') \\
&\overset{(40)}{=} \tau_{\mathrm{Lan}(\mathrm{Lan}(A\hat{w})\hat{v}),u}(((a_{v,w} \circ j)\tau_{A,wv}) \circ \hat{u}') \\
&= \tau_{\mathrm{Lan}(\mathrm{Lan}(\mathbf{A\hat{w}})\hat{\mathbf{v}}),\mathbf{u}}(\mathbf{a_{v,w}} \circ \hat{\mathbf{u}})(\tau_{A,wv} \circ \hat{u}') \\
&\overset{(38)}{=} (u^* a_{v,w} \circ k)\tau_{\mathrm{Lan}(\mathbf{A\widehat{wv}}),\mathbf{u}}(\tau_{\mathbf{A,wv}} \circ \hat{\mathbf{u}}') \\
&\overset{(40)}{=} (u^* a_{v,w} \circ k)(a_{u,wv} \circ k)\tau_{A,wvu} \\
&= (((u^* a_{v,w})a_{u,wv}) \circ k)\tau_{A,wvu},
\end{aligned}$$

which proves that relation (27) holds.

Given an admissible square

$$\begin{tikzcd}
& W \arrow[dl, "w"'] \arrow[dr, "f"] & \\
V \arrow[dr, "g"] & & U \arrow[dl, "u"] \\
& Z, &
\end{tikzcd}$$

we consider the diagram

$$\begin{tikzcd}
\mathcal{B}_U \arrow[r, "f^{-1}"] & \mathcal{B}_W \\
\mathcal{B}'_U \arrow[u, "i", "\subseteq"'] \arrow[r, "f^{-1}"] \arrow[d, "\hat{u}"] & \mathcal{B}'_W \arrow[u, "\subseteq"'] \arrow[d, "\hat{w}"] \\
\mathcal{B}_Z \arrow[r, "g^{-1}"] & \mathcal{B}_V ,
\end{tikzcd}$$

which is commutative since w is bornological and admissible squares are pullbacks of the underlying coarse spaces. We define

$$b'_{g,u} \colon \operatorname{Lan}((g_*A)\hat{u}) \to f_* \operatorname{Lan}(A\hat{w})$$

to be the natural isomorphism induced by the natural isomorphism

$$(g_*A)\hat{u} = Ag^{-1}\hat{u} = A\hat{w}f^{-1} \xrightarrow[\tau_{A,w}\circ f^{-1}]{\cong} \operatorname{Lan}(A\hat{w})if^{-1} = f_* \operatorname{Lan}(A\hat{w})i.$$

In particular, $b'_{g,u}$ is uniquely determined by the equation

$$(b'_{g,u} \circ i)\tau_{g_*A,u} = \tau_{A,w} \circ f^{-1}. \tag{41}$$

We finally define $b_{g,u}$ as the inverse of $b'_{g,u}$. As above this morphism is compatible with ρ.

We check the relations (28) and (29). Suppose that we have three admissible squares

$$\begin{tikzcd}
 & & T \arrow[dl, twoheadrightarrow, "t"'] \arrow[dr, "m"] & & \\
 & U \arrow[dl, twoheadrightarrow, "u"'] \arrow[dr, "h"] & & S \arrow[dl, twoheadrightarrow, "s"'] \arrow[dr, "n"] & \\
W \arrow[dr, "f"] & & V \arrow[dl, twoheadrightarrow, "v"'] \arrow[dr, "g"] & & R \arrow[dl, twoheadrightarrow, "r"'] \\
 & Y & & Z. &
\end{tikzcd}$$

We denote the inclusion $\mathcal{B}'_R \to \mathcal{B}_R$ (the analog of (35)) by i_R. By repeated application of (41), we then have

$$\begin{aligned}
(b'_{gh,r} \circ i_R)\tau_{g_*h_*A,t} &= \tau_{A,t} \circ (nm)^{-1} \\
&= (\tau_{A,t} \circ m^{-1}) \circ n^{-1} \\
&= ((b'_{h,s} \circ i_R)\tau_{h_*A,s}) \circ n^{-1} \\
&= ((b'_{h,s} \circ i_R) \circ n^{-1})(\tau_{h_*A,s} \circ n^{-1}) \\
&= ((n_* \circ b'_{h,s}) \circ i_R)((b'_{g,r} \circ h_* \circ i_R)\tau_{g_*h_*A,t} \\
&= (((n_* \circ b'_{h,s})(b'_{g,r} \circ h_*)) \circ i_R)\tau_{g_*h_*A,t}.
\end{aligned}$$

This proves that relation (28) holds.

Finally, we compute

$$
\begin{aligned}
&((b'_{h,s}\circ u^*)(s^*\circ b_{f,v})(\boldsymbol{a_{s,v}\circ f_*}))\circ \boldsymbol{i_S})\boldsymbol{\tau_{f_*A,vs}}\\
&\overset{(40)}{=} ((b'_{h,s}\circ u^*)\circ i_S)\boldsymbol{((s^*\circ b'_{f,v})\circ i_S)\tau_{\mathrm{Lan}((f_*A)\hat{v}),s}}(\tau_{f_*A,v}\circ \hat{s}')\\
&\overset{(38)}{=} ((b'_{h,s}\circ u^*)\circ i_S)\tau_{h_*\mathrm{Lan}(A\hat{u}),s}\boldsymbol{(b'_{f,v}\circ \hat{s})(\tau_{f_*A,v}\circ \hat{s}')}\\
&\overset{(41)}{=} \boldsymbol{((b'_{h,s}\circ u^*)\circ i_S)\tau_{h_*\mathrm{Lan}(A\hat{u}),s}}(\tau_{A,u}\circ h^{-1}\hat{s}')\\
&\overset{(41)}{=} (\tau_{\mathrm{Lan}(A\hat{u}),t}\circ m^{-1})(\tau_{A,u}\circ \hat{t}'m^{-1})\\
&= (\tau_{\mathrm{Lan}(A\hat{u}),t}(\tau_{A,u}\circ \hat{t}'))\circ m^{-1}\\
&\overset{(40)}{=} ((a_{t,u}\circ i_T)\tau_{A,ut})\circ m^{-1}\\
&= (m_*\circ a_{t,u}\circ i_S)(\tau_{A,ut}\circ m^{-1})\\
&\overset{(41)}{=} (m_*\circ a_{t,u}\circ i_S)(b'_{f,vs}\circ i_S)\tau_{f_*A,vs}\\
&= (((m_*\circ a_{t,u})b'_{f,vs})\circ i_S)\tau_{f_*A,vs}.
\end{aligned}
$$

This implies immediately that relation (29) holds as well.

By Lemma 3.2 the above data induces a functor

$$\mathbf{V}^G_{\mathbf{A},\mathrm{tr}}\colon G\mathbf{BornCoarse}_{\mathrm{tr}}\to \mathtt{N}_2(\mathbf{Add}).$$

Definition 3.9. We define the equivariant algebraic K-homology with transfers

$$K\mathcal{AX}^G_{\mathrm{tr}}\colon G\mathbf{BornCoarse}_{\mathrm{tr}}\to \mathbf{Sp}$$

as the composition

$$K\mathcal{AX}^G_{\mathrm{tr}} := \mathbf{K}\circ \mathbf{V}_{\mathrm{tr}}.$$

Proposition 3.10. *The functor $K\mathcal{AX}^G$ is equivalent to $K\mathcal{AX}^G_{\mathrm{tr}}\circ\iota$.*

Proof. This follows from the definition since the diagram

$$
\begin{tikzcd}
G\mathbf{BornCoarse}_{\mathrm{tr}} \arrow[r, "\mathbf{V}_{\mathrm{tr}}"] & \mathtt{N}_2(\mathbf{Add}) \arrow[r, "\mathbf{K}"] & \mathbf{Sp}\\
\mathtt{N}(G\mathbf{BornCoarse}) \arrow[u, "\iota"] \arrow[r, "\mathbf{V}"] & \mathtt{N}(\mathbf{Add}) \arrow[u] \arrow[ur, "K"'] &
\end{tikzcd}
$$

commutes by Lemma 3.2 and (33). ☐

3.11. **Coarse ordinary homology.** We first recall the construction of equivariant coarse ordinary homology

$$H\mathcal{X}^G\colon G\mathbf{BornCoarse}\to \mathbf{Sp}$$

from [4, Section 7]. One starts with a functor

$$C\mathcal{X}^G\colon G\mathbf{BornCoarse}\to \mathbf{Ch},$$

which associates to a G-bornological coarse space X the chain complex of G-invariant, locally finite and controlled chains (the definitions will be recalled below). We then use the Eilenberg–MacLane functor

$$\mathcal{EM}\colon \mathbf{Ch} \to \mathbf{Sp}$$

in order to define the equivariant coarse ordinary homology functor

$$H\mathcal{X}^G := \mathcal{EM} \circ C\mathcal{X}^G \colon G\mathbf{BornCoarse} \to \mathbf{Sp}.$$

In order to define equivariant coarse ordinary homology with transfers

$$H\mathcal{X}^G_{\mathrm{tr}} \colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{Sp},$$

we will define a functor

$$C\mathcal{X}^G_{\mathrm{tr}} \colon \mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}}) \to \mathbf{Ch}$$

such that $C\mathcal{X}^G_{\mathrm{tr}} \circ \iota = C\mathcal{X}^G$. It then induces the desired extension $H\mathcal{X}^G_{\mathrm{tr}}$ of $H\mathcal{X}^G$ as the composition

$$G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{Ho}(G\mathbf{BornCoarse}_{\mathrm{tr}}) \xrightarrow{C\mathcal{X}^G_{\mathrm{tr}}} \mathbf{Ch} \xrightarrow{\mathcal{EM}} \mathbf{Sp}, \tag{42}$$

where we omitted the nerve functor to consider ordinary categories as ∞-categories.

The construction of $H\mathcal{X}^G_{\mathrm{tr}}$ turns out to be considerably less involved than in the construction of K-homology $K\mathbf{A}\mathcal{X}^G_{\mathrm{tr}}$ given in Section 3.3, since we can stick to one-categorical considerations. We now explain the details.

Recall that the objects of $G\mathbf{BornCoarse}_{\mathrm{tr}}$ are G-bornological coarse spaces. Hence on objects we can define

$$C\mathcal{X}^G_{\mathrm{tr}}(X) := C\mathcal{X}^G(X).$$

To define $C\mathcal{X}^G_{\mathrm{tr}}$ as a functor, we must extend the functor $C\mathcal{X}^G$ to generalized morphisms, see Definition 2.24.

We now recall the definition of $C\mathcal{X}^G(X)$. For an n in $\mathbb{N}$ the group $C\mathcal{X}^G_n(X)$ consists of functions $c\colon X^{n+1} \to \mathbb{Z}$ which are G-invariant, and whose support is controlled and locally finite. Here the group G acts diagonally on the $(n+1)$-fold product X^{n+1} of X with itself. We say that a subset S of X^{n+1} is controlled if there exists an entourage U of X such that $(x_0, \dots, x_n) \in S$ implies that $(x_i, x_j) \in U$ for all i, j in $\{0, \dots, n\}$. Finally, a subset S of X^{n+1} is locally finite if for every bounded set B of X the set $\{s \in S \mid s \text{ meets } B\}$ is finite, where we say that $s = (x_0, \dots, x_n)$ meets B if there exists i in $\{0, \dots, n\}$ such that $x_i \in B$. The differential

$$\partial\colon C\mathcal{X}^G_n(X) \to C\mathcal{X}^G_{n-1}(X)$$

is defined by $\partial := \sum_{i=0}^n (-1)^i \partial_i$, where ∂_i is the linear extension of the map $X^{n+1} \to X^n$ which omits the ith entry.

We consider now a generalized morphism $[W, w, f]$ from X to Y, see the Definition 2.24. We consider the entourage $U(\pi_0(W))$ defined as in (2) for the partition $\pi_0(W)$ of W into coarse components. We let $\chi^n_{\pi_0(W)}$ in $(\mathbb{Z}^{W^{n+1}})^G$ denote the G-invariant characteristic function of the set

$$\{(w_0,\dots,w_n) \mid \text{for all } i,j \in \{0,\dots,n\}, (w_i,w_j) \in U(\pi_0(W))\},$$

i.e., the maximal $U(\pi_0(W))$-controlled subset of W^{n+1}. The map $w\colon W \to X$ induces a G-equivariant map $\hat{w}\colon W^{n+1} \to X^{n+1}$ which we can use to pullback a G-invariant function c on X^{n+1} to a G-invariant function $\hat{w}^*c$ on W^{n+1}. Then we can define a map

$$w^*\colon C\mathcal{X}^G(X) \to (\mathbb{Z}^{W^{n+1}})^G, \quad w^*c := \chi^n_{\pi_0(W)} \cdot \hat{w}^*c.$$

We now show that w^*c actually belongs to $C\mathcal{X}^G_n(W)$. Let B be a bounded subset of W. By Definition 2.15 of a bounded covering, there exists a finite partition $(B_\alpha)_{\alpha\in I}$ of B such that $w_{|[B_\alpha]}\colon [B_\alpha] \to [w(B_\alpha)]$ is an isomorphism of coarse spaces. Moreover, since w is bornological and hence $w(B_\alpha)$ is bounded for every α in I, only finitely many points of the support of w^*c meet B_α. Since I is finite only finitely many points of the support of w^*c meet B.

There exists an entourage U of X such that c is U-controlled. Then it is straight-forward to see that w^*c is $w^{-1}U \cap U(\pi_0(W))$ controlled. Since $w^{-1}U \cap U(\pi_0(W))$ is an entourage of W by the definition of a bounded coarse covering (Definition 2.10), we see that w^*c is controlled.

We have therefore defined a homomorphism

$$w^*\colon C\mathcal{X}^G_n(X) \to C\mathcal{X}^G_n(W).$$

We now consider the compatibility of w^* with the differential. For notational simplicity, we consider the case of ∂_n. We have

$$\begin{aligned}(\partial_n w^*c)&(w_0,\dots,w_{n-1})\\ &= \sum_{w_n\in W} \chi^n_{\pi_0(W)}(w_0,w_1,\dots,w_n)c(w(w_0),w(w_1),\dots,w(w_n)).\end{aligned}$$

We fix w_0 in W and let W_0 be the coarse component of w_0. Because of the $\chi^n_{\pi_0(W)}$-factor a summand on the right-hand side is nontrivial only if the points $w_1,\dots,w_n$ all belong to W_0. Since w is a bounded covering the restriction of w to W_0 is a bijection $w_{|W_0}\colon W_0 \to w(W_0)$ between coarse components. Since c is controlled, we see that $c(w(w_0),\dots,w(w_{n-1}),x_n) = 0$ if $x_n \notin w(W_0)$. We therefore get the equality

$$\begin{aligned}(\partial_n w^*c)&(w_0,\dots,w_{n-1})\\ &= \sum_{w_n\in W} \chi^n_{\pi_0(W)}(w_0,w_1,\dots,w_n)c(w(w_0),w(w_1),\dots,w(w_n))\\ &= \sum_{x_n\in X} \chi^{n-1}_{\pi_0(W)}(w_0,\dots,w_{n-1})c(w(w_0),\dots,w(w_{n-1}),x_n)\\ &= (w^*\partial_n c)(w_0,\dots,w_n).\end{aligned}$$

We thus have seen that w^* induces a morphism of complexes. We can now define

$$[W,w,f]_*\colon C\mathcal{X}^G(X)\to C\mathcal{X}^G(X), \quad [W,w,f]_* := f_*\circ w^*.$$

We must verify that $[W,w,f]_*$ is well-defined independently of the choice of the representative (W,w,f) of the generalized morphism, and that this definition is compatible with the composition.

Assume now that $\phi\colon W\to W'$ induces an isomorphism between the spans (W,w,f) and (W',w',f') from X to Y. Then the commutative diagram (7) induces a commutative diagram of chain complexes

$$\begin{tikzcd} C\mathcal{X}^G(X) \arrow[r, "w^*"] \arrow[d, equal] & C\mathcal{X}^G(W) \arrow[r, "f_*"] \arrow[d, bend right, "\cong"', "\phi^*"] & C\mathcal{X}^G(Y) \arrow[d, equal] \\ C\mathcal{X}^G(X) \arrow[r, "w'^{,*}"] & C\mathcal{X}^G(W') \arrow[r, "f'_*"] \arrow[u, bend right, "\cong"', "\phi_*"] & C\mathcal{X}^G(Y), \end{tikzcd}$$

where we use that ϕ_* is inverse to ϕ^*. We conclude that $f_*w^* = f'_*w'^{,*}$ and therefore that $[W,w,f]$ is well defined.

Let now $[V,v,g]$ be a generalized morphism from Y to Z. Then we consider a representative $[U,(wu),(gh)]$ of the composition fitting into the diagram of G-coarse spaces

(43)
$$\begin{tikzcd} & & U \arrow[dl, "u"'] \arrow[dr, "h"] & & \\ & W \arrow[dl, "w"'] \arrow[dr, "f"] & & V \arrow[dl, "v"'] \arrow[dr, "g"] & \\ X & & Y & & Z, \end{tikzcd}$$

where the square is admissible. We get a diagram

(44)
$$\begin{tikzcd} & & C\mathcal{X}^G(U) \arrow[dr, "h_*"] & & \\ & C\mathcal{X}^G(W) \arrow[ur, "u^*"] \arrow[dr, "f_*"] & & C\mathcal{X}^G(V) \arrow[dr, "g_*"] & \\ C\mathcal{X}^G(X) \arrow[ur, "w^*"] & & C\mathcal{X}^G(Y) \arrow[ur, "v^*"] & & C\mathcal{X}^G(Z) \end{tikzcd}$$

of chain complexes. The relation

$$[V,v,g]_*\circ[W,w,f]_* = [U,(wu),(gh)]_*$$

is now implied by the following two relations:

$$u^*w^* = (wu)^*, \quad h_*u^* = v^*f_*,$$

which we will verify in the following to paragraphs.

Since u is a morphism in $G\widetilde{\mathbf{BornCoarse}}$, we have the equality

$$u^{-1}U(\pi_0(W)) \cap U(\pi_0(U)) = U(\pi_0(U)).$$

This implies the relation $\chi^n_{\pi_0(U)}(\hat{u}^*\chi^n_{\pi_0(W)}) = \chi^n_{\pi_0(U)}$. Therefore for c in $C\mathcal{X}^G_n(X)$ we get the chain of equalities

$$\begin{aligned} u^*w^*c &= \chi^n_{\pi_0(U)}(\hat{u}^*(\chi^n_{\pi_0(W)}\hat{w}^*c)) = \chi^n_{\pi_0(U)}(\hat{u}^*\chi^n_{\pi_0(W)})(\hat{u}^*\hat{w}^*c) \\ &= \chi^n_{\pi_0(U)}(\hat{u}^*\hat{w}^*c) = \chi^n_{\pi_0(U)}\widehat{(wu)}^*c = (wu)^*c. \end{aligned}$$

Let now $(v_0,\dots,v_n)$ be a point in V^{n+1} and let c be in $C\mathcal{X}^G_n(W)$. Then we have the following chain of equalities:

$$\begin{aligned} &(h_*u^*c)(v_0,\dots,v_n) \\ &\qquad = \sum_{(u_0,\dots,u_n)\in h^{-1}(v_0,\dots,v_n)} (u^*c)(u_0,\dots,u_n) \\ &\qquad = \sum_{(u_0,\dots,u_n)\in h^{-1}(v_0,\dots,v_n)} \chi^n_{\pi_0(U)}(u_0,\dots,u_n)c(u(u_0),\dots,u(u_n)) \\ &\qquad \overset{!}{=} \sum_{(u_0,\dots,u_n)\in h^{-1}(v_0,\dots,v_n)} \chi^n_{\pi_0(V)}(v_0,\dots,v_n)c(u(u_0),\dots,u(u_n)) \\ &\qquad \overset{!!}{=} \sum_{(w_0,\dots,w_n)\in f^{-1}(v(v_0),\dots,v(v_n))} \chi^n_{\pi_0(V)}(v_0,\dots,v_n)c(w_0,\dots,w_n) \\ &\qquad = (v^*f_*c)(v_0,\dots,v_n), \end{aligned}$$

where for the equality marked with '!', we use the fact that (since c is controlled and the square is admissible) if $(u_0,\dots,u_n)$ in U^{n+1} is such that $c(u(u_0),\dots,u(u_n)) \neq 0$, then the conditions $\chi^n_{\pi_0(U)}(u_0,\dots,u_n) = 1$ and $\chi^n_{\pi_0(V)}(h(u_0),\dots,h(u_n)) = 1$ are equivalent.

For the equality marked with '!!', we use that an admissible square is a pullback square, and hence u induces a bijection

$$\{(u_0,\dots,u_n) \mid h(u_i) = v_i\} \to \{(w_0,\dots,w_n) \mid f(w_i) = v(v_i)\}.$$

Definition 3.12. We define

$$H\mathcal{X}^G_{\mathrm{tr}} \colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{Sp}.$$

as the composition (42).

Lemma 3.13. *$H\mathcal{X}^G_{\mathrm{tr}}$ is an equivariant strongly additive coarse homology theory with transfers.*

Proof. By construction, $H\mathcal{X}^G_{\mathrm{tr}} \circ \iota \simeq H\mathcal{X}^G$ is a strongly additive equivariant coarse homology theory by [4, Thm. 7.3 and Lem. 7.11]. □

4. Application: Mackey functors

In this final section we assume that G is a finite group. In Section 4.1 we show that any G-equivariant $\mathbf{C}$-valued coarse homology theory with transfers gives rise to a $\mathbf{C}$-valued Mackey functor. In the special case when $\mathbf{C}$ is the category of spectra, we obtain a spectral Mackey functor which is equivalent to the datum of a genuine G-equivariant spectrum, see Remark 4.4. Our main result is Propostion 4.15 which expresses the delooping along a representation sphere of a Mackey functor obtained from an equivariant coarse homology theory with transfers in terms of coarse geometry.

Our main application of transfers for equivariant coarse homology theories is the descent argument leading to injectivity results for assembly maps. We refer to [6] for more details. In Section 4.18 we explain the main principle of the descent argument in the case of finite groups. On the one hand, we can avoid all the difficulties connected with infinite groups, but on the other hand, even for finite groups, we obtain interesting consequences.

4.1. **Mackey functors from equivariant coarse homology theories with transfers.** We let $G\mathbf{Fin}$ denote the category of finite G-sets and equivariant maps. This category admits fibre products and we can form the bicategory $\mathbf{Span}(G\mathbf{Fin})$ of spans in $G\mathbf{Fin}$. Its homotopy category is called the effective Burnside category of G. The ∞-categorical version of the effective Burnside category is the subcategory $A^{\mathrm{eff}}(G)$ of $\mathbf{Fun}(\mathbf{Tw}, G\mathbf{Fin})$ (compare Remark 2.28) defined as follows.

Definition 4.2. For every n in $\mathbb{N}$, the set of n-simplices of the ∞-category $A^{\mathrm{eff}}(G)$ is the set of functors X in $\mathbf{Fun}(\mathbf{Tw}[n], G\mathbf{Fin})$ such that the squares

$$\begin{tikzcd} X_{i,j} \arrow[r] \arrow[d] & X_{i',j} \arrow[d] \\ X_{i,j'} \arrow[r] & X_{i',j'} \end{tikzcd}$$

for all $0 \le i \le i' \le j' \le j \le n$ are pull-backs.

Let $\mathbf{C}$ be some ∞-category.

Definition 4.3. We define the ∞-category $\mathbf{Mack}_{\mathbf{C}}(G)$ of $\mathbf{C}$-*valued Mackey functors* to be the full subcategory of $\mathbf{Fun}(A^{\mathrm{eff}}(G)^{\mathrm{op}}, \mathbf{C})$ of the coproduct preserving (or equivalently, additive) functors.

Remark 4.4. The stable ∞-category $\mathbf{Mack}_{\mathbf{Sp}}(G)$ is called the ∞-category of spectral Mackey functors, and it models the genuine stable homotopy category associated to the group G, see [11, 1]. Typical constructions in genuine equivariant stable homotopy theory are fixed points with respect to subgroups of G, deloopings along representation spheres, and geometric fixed points. In the present section we explain how these operations can be expressed in terms of coarse geometry provided the spectral Mackey functor is derived from an equivariant coarse homology theory with transfers, see Definition 4.7.

A G-set S naturally gives rise to a G-bornological coarse space $S_{\min,\min}$ obtained by equipping S with the minimal coarse and bornological structures. If $S \to T$ is a map between finite G-sets, then $S_{\min,\min} \to T_{\min,\min}$ is controlled and proper. We therefore have a functor

$$M\colon G\mathbf{Fin} \to G\mathbf{BornCoarse}, \quad S \mapsto S_{\min,\min}. \tag{45}$$

Lemma 4.5. (i) *The functor M preserves finite coproducts.*
(ii) *The functor M intertwines the cartesian product on $G\mathbf{Fin}$ with the symmetric monoidal structure $\otimes$ on $G\mathbf{BornCoarse}$.*
(iii) *The functor M sends every morphism to a bounded covering.*
(iv) *The functor M sends pullback squares to admissible squares.*

Proof. A finite coproduct in $G\mathbf{BornCoarse}$ of G-sets with the minimal structures is the coproduct of the underlying G-sets equipped with the minimal structures. This implies Assertion (i). The finiteness assumption is necessary because an infinite coproduct in $G\mathbf{BornCoarse}$ of nonempty G-sets with the minimal structures would not have the minimal bornology anymore.

To see Assertion (ii), note that the $\otimes$-product of two finite G-sets with minimal structures in $G\mathbf{BornCoarse}$ is the product of the underlying sets with the minimal structures.

It has been observed in Example 2.17 that a map between G-sets with minimal structures is a bounded covering. This implies Assertion (iii).

To see Assertion (iv), note that a cartesian square of finite G-sets becomes an admissible square (Definition 2.20) if one equips the G-sets in the square with the minimal structures. □

The following corollary is an immediate consequence of Lemma 4.5 (iv) and the fact (observed in the proof of Lemma 2.34) that the inclusion of $G\mathbf{BornCoarse}$ into $G\mathbf{BornCoarse}_{\mathrm{tr}}$ preserves finite coproducts.

Corollary 4.6. *The functor M naturally induces a coproduct preserving functor*

$$M\colon A^{\mathrm{eff}}(G) \to G\mathbf{BornCoarse}_{\mathrm{tr}}. \tag{46}$$

For a fixed S in $G\mathbf{Fin}$, we have a functor

$$P_S := S \otimes (-)\colon A^{\mathrm{eff}}(G) \to A^{\mathrm{eff}}(G) \tag{47}$$

given by the cartesian product of objects, spans, etc., with S. Recall that $G\mathbf{Orb}$ denotes the full subcategory of $G\mathbf{Fin}$ of transitive G-sets.

The effective Burnside category of G has a canonical duality

$$D\colon A^{\mathrm{eff}}(G)^{\mathrm{op}} \to A^{\mathrm{eff}}(G) \tag{48}$$

described in [9, Section 2.17] (note that $G\mathbf{Fin}$ admits a terminal object). It is the identity on objects. For the moment, we only need to understand the functorial equivalence of mapping spaces

$$\mathrm{Map}_{A^{\mathrm{eff}}(G)}(S \otimes R, T) \simeq \mathrm{Map}_{A^{\mathrm{eff}}(G)}(R, D(S) \otimes T) \tag{49}$$

for all R, T in $A^{\mathrm{eff}}(G)$ and S in $G\mathbf{Orb}$. Note that the right-hand side is defined by considering $D(S)$ as an object of $G\mathbf{Fin}$. The equivalence in (49) is induced by the evaluation and coevaluation spans

$$S \times S \xleftarrow{\operatorname{diag}(S)} S \to *, \quad * \leftarrow S \xrightarrow{\operatorname{diag}(S)} S \times S.$$

For details, we refer to [9].

Let now $\mathbf{C}$ be stable and let $E\colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{C}$ be an equivariant coarse homology theory with transfers.

Definition 4.7. We define the functor

$$EM := E \circ M \circ D\colon A^{\mathrm{eff}}(G)^{\mathrm{op}} \to \mathbf{C}.$$

The following corollary is an immediate consequence of Colorrary 2.60 and Corollary 4.6.

Corollary 4.8. *The functor EM preserves coproducts, i.e., it belongs to the subcategory $\mathbf{Mack}_{\mathbf{C}}(G)$ of $\mathbf{Fun}(A^{\mathrm{eff}}(G)^{\mathrm{op}}, \mathbf{C})$.*

The functor EM is the Mackey functor associated to the $\mathbf{C}$-valued equivariant coarse homology theory with transfers E.

For a subgroup H of G, we define the functor

$$(-)^H\colon \mathbf{Mack}_{\mathbf{C}}(G) \xrightarrow{\mathtt{ev}_{G/H}} \mathbf{C} \tag{50}$$

of evaluation at the G-set G/H.

Remark 4.9. The above notation is motivated by the notation for the H-fixed points F^H of a genuine G-equivariant spectrum F. Indeed, under the correspondence of spectral Mackey functors with genuine G-equivariant spectra (see Remark 4.4), the operation (50) corresponds to the operation of taking (categorical) H-fixed points.

Let $E\colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{C}$ be an equivariant coarse homology theory with transfers. The following corollary is an immediate consequence of the definitions.

Corollary 4.10. *We have an equivalence*

$$EM^H \simeq E((G/H)_{\min,\min}).$$

Remark 4.11. The cartesian product of $G\mathbf{Fin}$ induces a symmetric monoidal structure $\otimes$ on $A^{\mathrm{eff}}(G)$. The following constructions could be written more naturally using this symmetric monoidal structure. Since in the present section we do not want to discuss this symmetric monoidal structure in detail, we proceed in a more direct way.

The functor P_S defined in (47) preserves coproducts. Consequently, precomposition by the functor P_S preserves Mackey functors. Motivated by [9, Cor. 4.5.1], we define the power functor by the prescription

$$G\mathbf{Orb}^{\mathrm{op}} \times \mathbf{Mack}_{\mathbf{C}}(G) \to \mathbf{Mack}_{\mathbf{C}}(G), \quad (S, F) \mapsto F^S := P_S^* F. \tag{51}$$

Here we implicitly use the functorial dependence

$$G\mathbf{Orb} \ni S \mapsto P_S \in \mathbf{Fun}(A^{\mathrm{eff}}(G), A^{\mathrm{eff}}(G)).$$

Since Mackey functors are contravariant functors on $A^{\mathrm{eff}}(G)$, we eventually get the contravariant dependence on S in equation (51).

We now assume that $\mathbf{C}$ is both presentable and stable. By stability, finite coproducts and products in $\mathbf{C}$ coincide. Using this fact we can describe the category $\mathbf{Mack}_{\mathbf{C}}(G)$ as the full subcategory of sheaves in the ∞-category of $\mathbf{C}$-valued presheaves $\mathbf{Fun}(A^{\mathrm{eff}}(G)^{\mathrm{op}}, \mathbf{C})$ on $A^{\mathrm{eff}}(G)$ with respect to the Grothendieck topology given by finite disjoint decompositions into G-invariant subsets. It follows then that $\mathbf{Mack}_{\mathbf{C}}(G)$ is presentable as well.

Since limits in $\mathbf{Fun}(A^{\mathrm{eff}}(G)^{\mathrm{op}}, \mathbf{C})$ are defined objectwise, the functor of precomposition with P_S preserves small limits. Since also the inclusion $\mathbf{Mack}_{\mathbf{C}}(G) \to \mathbf{Fun}(A^{\mathrm{eff}}(G)^{\mathrm{op}}, \mathbf{C})$ detects and preserves limits, it follows that

$$(-)^S \colon \mathbf{Mack}_{\mathbf{C}}(G) \to \mathbf{Mack}_{\mathbf{C}}(G)$$

preserves small limits. We therefore have an adjunction

$$S \otimes (-) \colon \mathbf{Mack}_{\mathbf{C}}(G) \leftrightarrows \mathbf{Mack}_{\mathbf{C}}(G) : (-)^S$$

which determines the tensor structure

$$G\mathbf{Orb} \times \mathbf{Mack}_{\mathbf{C}}(G) \to \mathbf{Mack}_{\mathbf{C}}(G), \quad (S, F) \mapsto S \otimes F. \tag{52}$$

Recall that, by Elmendorf's theorem, the ∞-category $\mathbf{PSh}(G\mathbf{Orb})$ models the homotopy theory of G-spaces. We can left-Kan extend the tensor structure (52) (along the Yoneda embedding $G\mathbf{Orb} \to \mathbf{PSh}(G\mathbf{Orb})$) to a functor

$$\mathbf{PSh}(G\mathbf{Orb}) \times \mathbf{Mack}_{\mathbf{C}}(G) \to \mathbf{Mack}_{\mathbf{C}}(G), \quad (X, F) \mapsto X \otimes F,$$

preserving colimits in the first variable. Similarly, we can also right-Kan extend the power structure (51) to a functor

$$\mathbf{PSh}(G\mathbf{Orb})^{\mathrm{op}} \times \mathbf{Mack}_{\mathbf{C}}(G) \to \mathbf{Mack}_{\mathbf{C}}(G), \quad (X, F) \mapsto F^X,$$

preserving limits in the first variable.

Let $E \colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{C}$ be an equivariant coarse homology theory with transfers and recall Definition 4.7 of the $\mathbf{C}$-valued Mackey functor EM associated to E. Using (51), we define the functor

$$G\mathbf{Orb}^{\mathrm{op}} \to \mathbf{Mack}_{\mathbf{C}}(G), \quad S \mapsto EM^S.$$

For every equivariant coarse homology theory with transfers

$$E \colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{C}$$

and every G-bornological coarse space V, we can form a new equivariant coarse homology theory with transfers $E_V \colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{C}$, called the twist of E by V, see Example 2.62. Recall the definition of the functor M in equation (46). We can define the functor

$$G\mathbf{Orb}^{\mathrm{op}} \to \mathbf{Mack}_{\mathbf{C}}(G), \quad S \mapsto E_{M(D(S))}M.$$

To see the functorial dependence on S, note that, by Example 2.17, for every map $S \to T$ in $G\mathbf{Orb}$ and G-bornological coarse space V, the induced map $S_{\min,\min} \otimes V \to T_{\min,\min} \otimes V$ is a bounded covering, and therefore can serve as a left leg of a span from $T_{\min,\min} \otimes V$ to $S_{\min,\min} \otimes V$ whose right leg is the identity.

Proposition 4.12. *We have an equivalence of functors* $EM^{(-)} \simeq E_{M(D(-))}M$.

Proof. The equivalence is implemented by the following chain of equivalences which are natural in S:

$$\begin{aligned} EM^S(-) &\simeq (E \circ M \circ D)^S(-) \\ &\overset{(51)}{\simeq} E \circ M \circ D \circ (S \otimes (-)) \\ &\simeq E \circ M \circ (D(S) \otimes D(-)) \\ &\overset{!}{\simeq} E \circ (M(D(S)) \otimes M(D(-))) \\ &\simeq E_{M(D(S))}M(-), \end{aligned}$$

where for the marked equivalence we use Lemma 4.5 (ii). □

The equivalence (49) implies the natural equivalence of $\mathbf{Spc}$-valued Mackey functors

$$y_{A^{\mathrm{eff}}(G)}(D(S) \otimes T) \simeq y_{A^{\mathrm{eff}}(G)}(T)^S \tag{53}$$

for every T in $A^{\mathrm{eff}}(G)$ and S in $G\mathbf{Orb}$. Using now that the functors $D(S) \otimes -$ and $(-)^S$ on $\mathbf{Mack}_{\mathbf{C}}(G)$ preserve colimits and that we can write any $\mathbf{C}$-valued Mackey functor as a colimit of a diagram of functors of the form $y_{A^{\mathrm{eff}}(G)}(T) \otimes C$ for T in $A^{\mathrm{eff}}(G)$ and C in $\mathbf{C}$ (here $\otimes$ is the tensor structure of $\mathbf{C}$ over $\mathbf{Spc}$), the equivalence (53) extends to the Wirthmüller equivalence of functors

$$D(-) \otimes F \simeq F^{(-)} \colon G\mathbf{Orb}^{\mathrm{op}} \to \mathbf{Mack}_{\mathbf{C}}(G)$$

for every $\mathbf{C}$-valued Mackey functor F. If we combine the Wirthmüller equivalence with the equivalence shown in Proposition 4.12, we get the following consequence.

Let $\mathbf{C}$ be a presentable stable ∞-category and let $E \colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{C}$ be an equivariant coarse homology theory with transfers.

Corollary 4.13. *We have a natural equivalence of functors*

$$(S \mapsto S \otimes EM \simeq E_{S_{\min,\min}}M) \colon G\mathbf{Orb} \to \mathbf{Mack}_{\mathbf{C}}(G).$$

Let X be a pointed G-space, i.e., an object of $\mathbf{Fun}(G\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}_*)$, and let F be a $\mathbf{C}$-valued Mackey functor for a presentable and stable ∞-category $\mathbf{C}$.

Definition 4.14. We define the $\mathbf{C}$-valued Mackey functor

$$X \wedge F := \mathrm{Cofib}(* \otimes F \to X \otimes F).$$

We have a canonical equivalence

$$X \wedge F \simeq \mathrm{Fib}(X \otimes F \to * \otimes F). \tag{54}$$

A G-topological space A gives rise to an object (also denoted by A) of $\mathbf{PSh}(G\mathbf{Orb})$, which sends the G-orbit S to the space represented by the topological mapping space $\mathrm{Map}_{G\mathbf{Top}}(S_{\mathrm{disc}}, A)$. We use a similar notation convention for pointed G-topological spaces which yield objects of

$$\mathbf{Fun}(G\mathbf{Orb}^{\mathrm{op}}, \mathbf{Spc}_*).$$

Let V be a finite-dimensional Euclidean vector space with an orthogonal representation of G. We consider V as an object of $G\mathbf{BornCoarse}_{\mathrm{tr}}$ with the structures induced by the metric. Let $S^1(V)$ be the G-topological space given by the unit sphere in V. Furthermore, let $S(V)$ be the pointed G-topological space given by the one-point compactification of V by the point ∞. We will write $S(V)_\infty$ for the corresponding based space.

Let $\mathbf{C}$ be a presentable stable ∞-category and $E\colon G\mathbf{BornCoarse}_{\mathrm{tr}} \to \mathbf{C}$ be an equivariant coarse homology theory with transfers, let EM be the $\mathbf{C}$-valued Mackey functor associated to E (see Definition 4.7), and let V be a finite-dimensional Euclidean vector space with an orthogonal representation of G.

Proposition 4.15. *We have a canonical equivalence of* $\mathbf{C}$*-valued Mackey functors*

$$S(V)_\infty \wedge EM \simeq E_V M.$$

Proof. The cone $\mathcal{O}(A)$ (see [4, Section 9.4]) of a compact metrizable G-space A is a well-defined object of $G\mathbf{BornCoarse}$, and its underlying G-set is the product of G-sets $[0,\infty)\times A$. Its bornology is generated by the subsets $[0,n]\times A$ for all n in $\mathbb{N}$. Finally, its coarse structure is the hybrid coarse structure associated to the uniform structure for some choice of a metric d on A and the maximal coarse structure, and the exhaustion $([0,n] \times A)_{n\in\mathbb{N}}$. The notation $\mathcal{O}(A)$ abbreviates the longer symbol $\mathcal{O}(A_{d,\max,\max})$ used, e.g., in [6, Section 4]. The cone at infinity $\mathcal{O}^\infty(A)$ is then defined as $\mathrm{Yo}^s(\mathcal{O}(A), ([0,n] \times A)_{n\in\mathbb{N}})$, see [4, Section 9.5].

Let S be in $G\mathbf{Orb}$. We have an equivalence (see [4, Prop. 9.35])

$$\mathcal{O}^\infty(S_{\mathrm{disc},\max,\max}) \simeq \mathcal{O}^\infty(S_{\mathrm{disc},\min,\max}) \simeq \Sigma\,\mathrm{Yo}^s(S_{\min,\max}).$$

Since S is a finite set, the bornological coarse spaces $S_{\min,\max}$ and $S_{\min,\min}$ coincide. By Corollary 4.13, we therefore have the equivalence

$$S \otimes EM \simeq E_{S_{\min,\min}} M \simeq \Sigma^{-1} E_{\mathcal{O}^\infty(S_{\mathrm{disc},\max,\max})} M \tag{55}$$

of $\mathbf{C}$-valued Mackey functors which is natural in S. Note that the twist with an object of $G\mathbf{Sp}\mathcal{X}$ is well-defined by Example 2.62. Let

$$\mathcal{O}^\infty_{\mathrm{hlg}}\colon \mathbf{PSh}(G\mathbf{Orb}) \to G\mathbf{Sp}\mathcal{X}$$

be the left-Kan extension of the functor

$$G\mathbf{Orb} \to G\mathbf{Sp}\mathcal{X}, \quad S \mapsto \mathcal{O}^\infty(S_{\mathrm{disc},\max,\max}).$$

Note that this is consistent with the definition of $\mathcal{O}^\infty_{\mathrm{hlg}}$ from [6, Def. 8.16], see also [6, Rem. 8.17] which explains the difference with the definitions given in [4].

By left Kan-extension along the Yoneda embedding $G\mathbf{Orb} \to \mathbf{PSh}(G\mathbf{Orb})$, both sides of (55) can be extended to colimit-preserving functors $\mathbf{PSh}(G\mathbf{Orb}) \to \mathbf{Mack}_{\mathbf{C}}(G)$. Using also that E preserves colimits, we get the equivalence of $\mathbf{C}$-valued Mackey functors

$$X \otimes EM \simeq \Sigma^{-1} E_{\mathcal{O}^\infty_{\mathrm{hlg}}(X)} M, \tag{56}$$

which is natural for X in $\mathbf{PSh}(G\mathbf{Orb})$.

In the following, we want to rewrite the cone sequence (see [4, Cor. 9.30])

$$\begin{aligned}\mathrm{Yo}^s(S(V)_{\max,\max}) &\to \mathrm{Yo}^s(\mathcal{O}(S(V))) \\ &\to \mathcal{O}^\infty(S(V)) \to \Sigma\,\mathrm{Yo}^s(S(V)_{\max,\max})\end{aligned} \tag{57}$$

in $G\mathbf{Sp}\mathcal{X}$ in simpler terms.

Pulling back the G-bornological coarse structure of $V \oplus \mathbb{R}$ along the map

$$[0,\infty) \times S^1(V \oplus \mathbb{R}) \to V \oplus \mathbb{R}, \quad (t,\xi) \mapsto t\xi, \tag{58}$$

of G-sets induces a G-bornological coarse structure on $[0,\infty) \times S^1(V \oplus \mathbb{R})$ which we call the Euclidean cone structure. We let $\mathcal{O}_{eu}(S^1(V \oplus \mathbb{R}))$ denote the G-set $[0,\infty) \times S^1(V \oplus \mathbb{R})$ equipped with this structure. The identity of the underlying sets induces a morphism

$$\mathcal{O}_{eu}(S^1(V \oplus \mathbb{R})) \to \mathcal{O}(S^1(V \oplus \mathbb{R}))$$

of G-bornological coarse spaces. By arguments which are analogous to the ones given in [2, Section 8], one can show that this morphism induces an equivalence

$$\mathrm{Yo}^s(\mathcal{O}_{eu}(S^1(V \oplus \mathbb{R}))) \xrightarrow{\simeq} \mathrm{Yo}^s(\mathcal{O}(S^1(V \oplus \mathbb{R}))) \tag{59}$$

in $G\mathbf{Sp}\mathcal{X}$. The map (58) has a right-inverse

$$V \oplus \mathbb{R} \to [0,\infty) \times S^1(V \oplus \mathbb{R})$$

which sends the origin of $V \oplus \mathbb{R}$ to the point $(0,(0,1))$. One easily checks that these maps implement an equivalence of G-bornological coarse spaces between $\mathcal{O}_{eu}(S^1(V \oplus \mathbb{R}))$ and $V \oplus \mathbb{R}$. In particular, we get the third equivalence in the chain

$$\mathrm{Yo}^s(\mathcal{O}(S(V))) \simeq \mathrm{Yo}^s(\mathcal{O}(S^1(V \oplus \mathbb{R}))) \overset{(59)}{\simeq} \mathrm{Yo}^s(\mathcal{O}_{eu}(S^1(V \oplus \mathbb{R}))) \simeq \mathrm{Yo}^s(V \oplus \mathbb{R}).$$

For the first equivalence, we use the usual equivariant homeomorphism $S(V) \cong S^1(V \oplus \mathbb{R})$.

Since $S(V)$ has a G-fixed point, the projection map $S(V)_{\max,\max} \to *$ is an equivalence of G-bornological coarse spaces. Therefore we have an equivalence

$$\mathrm{Yo}^s(S(V)_{\max,\max}) \xrightarrow{\simeq} \mathrm{Yo}^s(*).$$

We finally note that $S(V)$ is homotopy equivalent to a finite G-CW complex. Since the functors $\mathcal{O}^\infty$ and $\mathcal{O}^\infty_{\mathrm{hlg}}$ behave as $G\mathbf{Sp}\mathcal{X}$-valued homology theories on the category of finite G-CW-complexes (see [6, Lem. 8.23]) and coincide on G-orbits, we have an equivalence

$$\mathcal{O}^\infty(S(V)) \simeq \mathcal{O}^\infty_{\mathrm{hlg}}(S(V)).$$

The cone sequence (57) is therefore equivalent to the fibre sequence

$$\mathrm{Yo}^s(*) \to \mathrm{Yo}^s(V \oplus \mathbb{R}) \to \mathcal{O}^\infty_{\mathrm{hlg}}(S(V)) \to \Sigma\,\mathrm{Yo}^s(*) \tag{60}$$

in $G\mathbf{Sp}\mathcal{X}$. Using the functoriality of $Q \mapsto E_Q$ (Example 2.62) and the obvious equivalence $E_{\mathrm{Yo}^s(*)} \simeq E$, we therefore get the fibre sequence of $\mathbf{C}$-valued Mackey functors

$$EM \to E_{V\oplus\mathbb{R}}M \to E_{\mathcal{O}^\infty_{\mathrm{hlg}}(S(V))}M \to \Sigma EM.$$

We now apply the equivalence (56) to the third term and obtain the sequence

$$EM \to E_{V\oplus\mathbb{R}}M \to \Sigma(S(V) \otimes EM) \to \Sigma EM. \tag{61}$$

We have a commuting diagram

$$\begin{tikzcd}
\Sigma(S(V)\otimes EM) \arrow[r, "(56)", "\simeq"'] \arrow[d] & E_{\mathcal{O}^\infty_{\mathrm{hlg}}(S(V))}M \arrow[r] \arrow[d] & \Sigma EM \arrow[d, equal] \\
\Sigma(*\otimes EM) \arrow[r, "(56)", "\simeq"'] & E_{\mathcal{O}^\infty_{\mathrm{hlg}}(*)}M \arrow[r, "\simeq"'] & \Sigma EM,
\end{tikzcd}$$

where the unnamed vertical maps are induced by the projection $S(V) \to *$. It follows that the last map in (61) is induced by the projection $S(V) \to *$.

In view of (54), the fibre sequence (61) gives an equivalence

$$E_{V\oplus\mathbb{R}}M \simeq \Sigma(S(V)_\infty \wedge EM). \tag{62}$$

Applied to the special case where V is the zero-dimensional representation, we get the equivalence

$$E_{\mathbb{R}}M \simeq \Sigma EM. \tag{63}$$

In the category $G\mathbf{BornCoarse}$, we have an equivalence $V \oplus \mathbb{R} \cong V \otimes \mathbb{R}$. This implies the second equivalence in

$$\Sigma(S(V)_\infty \wedge EM) \overset{(62)}{\simeq} E_{\mathbb{R}\oplus V}M \simeq (E_V)_{\mathbb{R}}M \simeq \Sigma E_V M,$$

where in the last equivalence we apply (63) to the twisted homology theory E_V in place of E. Since Σ is an equivalence, the proposition follows. $\square$

Finally, we discuss the geometric fixed point functor for a subgroup H of G. In genuine equivariant stable homotopy theory the geometric fixed points of a genuine G-equivariant spectrum F can be calculated by the formula

$$\Phi^H(F) \simeq \underset{V^H=\{0\}}{\mathrm{colim}}\,[S(V)\wedge F]^V,$$

where the colimit runs over all finite-dimensional representations of G with no nontrivial H-fixed vectors [16, Def. II.2.10]. In the following, we use this formula as a definition:

Definition 4.16. We define the *geometric fixed point functor*

$$\Phi^H : \mathbf{Mack_C}(G) \to \mathbf{C}$$

by

$$\Phi^H(F) := \operatorname*{colim}_{V^H=\{0\}} [S(V)_\infty \wedge F]^H.$$

Let $E\colon G\mathbf{BornCoarse}^Q_{\mathrm{tr}} \to \mathbf{C}$ be an equivariant coarse homology theory with transfers, let EM be the $\mathbf{C}$-valued Mackey functor associated to E, and let H be a subgroup of G.

Proposition 4.17. *In* $\mathbf{C}$*, we have an equivalence*

$$\Phi^H(EM) \simeq \operatorname*{colim}_{V^H=\{0\}} E((G/H)_{\min,\min} \otimes V),$$

where the colimit runs over orthogonal representations V *of* G *with no nontrivial* H*-fixed vectors.*

Proof. We use Corollary 4.10 and Propostion 4.15 in order to rewrite the formula from Definition 4.16 in the desired form. □

4.18. **A descent principle.** In this section we explain how transfers can be applied to show injectivity of the assembly map in the case of finite groups. In view of Remark 4.39, the main result itself (see Corollary 4.38) is not really new. Our main point is to give a self-contained proof using the descent principle, which avoids both the usage of the connection between spectral Mackey functors and genuine equivariant spectra and of results from genuine equivariant stable homotopy theory. Since we consider finite groups, we can drop all arguments involving coarse geometry.

Let G be a finite group and let $\mathcal{F}$ be a set of subgroups of G.

Definition 4.19. The set $\mathcal{F}$ is called a family of subgroups if it is nonempty and closed under conjugation in G and taking subgroups.

Let $\mathcal{F}$ be a family of subgroups. Then we consider the full subcategory $G_{\mathcal{F}}\mathbf{Orb}$ of $G\mathbf{Orb}$ of transitive G-sets with stabilizers in $\mathcal{F}$. For every cocomplete target category $\mathbf{C}$, the inclusion $G_{\mathcal{F}}\mathbf{Orb} \to G\mathbf{Orb}$ induces an adjunction

$$\mathrm{Ind}_{\mathcal{F}} \colon \mathbf{Fun}(G_{\mathcal{F}}\mathbf{Orb}, \mathbf{C}) \leftrightarrows \mathbf{Fun}(G\mathbf{Orb}, \mathbf{C}) : \mathrm{Res}_{\mathcal{F}}\,. \tag{64}$$

We have a similar adjunction for contravariant functors.

We consider a functor $E\colon G\mathbf{Orb} \to \mathbf{C}$ with a cocomplete target $\mathbf{C}$ and let $\mathcal{F}$ be a family of subgroups of G. Note that pt denotes the final object of $G\mathbf{Orb}$ given by the one-point G-set.

Definition 4.20. The morphism

$$\alpha_{\mathcal{F}} \colon (\mathrm{Ind}_{\mathcal{F}} \circ \mathrm{Res}_{\mathcal{F}}(E))(pt) \to E(pt), \tag{65}$$

given by the counit of the adjunction (64), is called the *assembly map*.

Let $*_{\mathcal{F}}$ denote the final object of $\mathbf{PSh}(G_{\mathcal{F}}\mathbf{Orb})$.

Definition 4.21. The object $E_{\mathcal{F}}G := \mathrm{Ind}_{\mathcal{F}}(*_{\mathcal{F}})$ of $\mathbf{PSh}(G\mathbf{Orb})$ is called the *classifying space* of the family $*_{\mathcal{F}}$.

One can check that

$$E_{\mathcal{F}}G(S) \simeq \begin{cases} * & \text{if } S \in G_{\mathcal{F}}\mathbf{Orb}, \\ \varnothing & \text{else.} \end{cases} \tag{66}$$

The main result of this section is the next theorem. Let $E\colon G\mathbf{Orb} \to \mathbf{C}$ be a functor.

Theorem 4.22. *Assume:*

(i) $\mathbf{C}$ *is stable, complete and cocomplete;*
(ii) E *extends to a Mackey functor;*
(iii) $E_{\mathcal{F}}G$ *is a compact object.*

Then the assembly map (65) *is split injective.*

Assumption (ii) will be explained in Definition 4.33 below.

Remark 4.23. Let $\mathbf{S}$ be an ∞-category and $\mathbf{C}$ a cocomplete ∞-category. Then pullback along the Yoneda embedding $\mathrm{yo}\colon \mathbf{S} \to \mathbf{PSh}(\mathbf{S})$ induces an equivalence of ∞-categories

$$\mathbf{Fun}^{\mathrm{colim}}(\mathbf{PSh}(\mathbf{S}), \mathbf{C}) \xrightarrow{\simeq} \mathbf{Fun}(\mathbf{S}, \mathbf{C}),$$

where the superscript indicates the full subcategory of colimit preserving functors. For a functor $F\colon \mathbf{S} \to \mathbf{C}$, we let $\widetilde{F}\colon \mathbf{PSh}(\mathbf{S}) \to \mathbf{C}$ denote the essentially uniquely determined colimit-preserving functor corresponding to F under this equivalence. Note that $\widetilde{F}$ (together with the identification of its restriction with F) is a left Kan extension of F along the Yoneda embedding.

Similarly, for a complete target $\mathbf{C}$, we have an equivalence

$$\mathbf{Fun}^{\mathrm{lim}}(\mathbf{PSh}(\mathbf{S})^{\mathrm{op}}, \mathbf{C}) \xrightarrow{\simeq} \mathbf{Fun}(\mathbf{S}^{\mathrm{op}}, \mathbf{C}).$$

Again, for a functor $F\colon \mathbf{S}^{\mathrm{op}} \to \mathbf{C}$, we let $\widetilde{F}\colon \mathbf{PSh}(\mathbf{S})^{\mathrm{op}} \to \mathbf{C}$ denote the essentially uniquely determined limit-preserving functor corresponding to F under the above equivalence. Note that $\widetilde{F}$ (together with the identification of its restriction with F) is a right Kan extension of the functor F along the Yoneda embedding. If we consider $\widetilde{F}$ as a contravariant functor from $\mathbf{PSh}(\mathbf{S})$ to $\mathbf{C}$, then it sends colimits to limits.

Let $\mathbf{S}$ and $\mathbf{T}$ be ∞-categories and assume now that we have a bifunctor

$$F\colon \mathbf{S}^{\mathrm{op}} \times \mathbf{T} \to \mathbf{C}$$

with a complete and cocomplete target $\mathbf{C}$. Then we can define a functor

$$\widetilde{F}\colon \mathbf{PSh}(\mathbf{S})^{\mathrm{op}} \times \mathbf{PSh}(\mathbf{T}) \to \mathbf{C}$$

by first right Kan extending F in the first variable, and then left Kan extending the result in the second variable. We consider F and $\widetilde{F}$ as contravariant functors in the first variable. The functor $\widetilde{F}$ is essentially uniquely determined by the property that it restricts to F along the product of Yoneda embeddings $\mathbf{S} \times \mathbf{T} \to \mathbf{PSh}(\mathbf{S}) \times \mathbf{PSh}(\mathbf{T})$ and satisfies

$$\widetilde{F}\big(\operatorname*{colim}_I X, \operatorname*{colim}_J \mathrm{yo}(Y)\big) \simeq \operatorname*{colim}_J \lim_I \widetilde{F}(X, \mathrm{yo}(Y)) \tag{67}$$

for all diagrams $X\colon I \to \mathbf{PSh}(\mathbf{S})$ and $Y\colon J \to \mathbf{T}$.

Similarly, switching the order of the left and right Kan extensions, we obtain a functor (contravariant in its first variable)

$$\widetilde{F}'\colon \mathbf{PSh}(\mathbf{S}) \times \mathbf{PSh}(\mathbf{T}) \to \mathbf{C}.$$

Again, the functor $\widetilde{F}'$ is essentially uniquely determined by the property that it restricts to F along the product of Yoneda embeddings and satisfies

$$\widetilde{F}'\big(\operatorname*{colim}_I \mathrm{yo}(X), \operatorname*{colim}_J Y\big) \simeq \lim_I \operatorname*{colim}_J \widetilde{F}'(\mathrm{yo}(X), Y) \tag{68}$$

for all diagrams $X\colon I \to \mathbf{S}$ and $Y\colon J \to \mathbf{PSh}(\mathbf{T})$.

Finally, note that the natural comparison morphism

$$\operatorname*{colim}_J \lim_I \to \lim_I \operatorname*{colim}_J$$

provides a comparison morphism

$$c\colon \widetilde{F} \to \widetilde{F}'.$$

Let $E\colon G\mathbf{Orb} \to \mathbf{C}$ be a functor. In the following statement, '$*$' denotes the final object of $\mathbf{PSh}(G\mathbf{Orb})$. Recall the notation introduced in Remark 4.23.

Lemma 4.24. *The assembly map* (65) *is equivalent to the morphism*

$$\widetilde{E}(E_{\mathcal{F}}G) \to \widetilde{E}(*)$$

induced by the morphism $E_{\mathcal{F}}G \to *$.

Proof. We have an equivalence $* \simeq \mathrm{yo}(pt)$. Moreover, $E_{\mathcal{F}}G = \mathrm{Ind}_{\mathcal{F}}(*_{\mathcal{F}})$ can be expressed in terms of a left Kan extension, and the pointwise formula gives

$$E_{\mathcal{F}}G \simeq \operatorname*{colim}_{S \in G_{\mathcal{F}}\mathbf{Orb}/pt} \mathrm{yo}(S).$$

Since $\widetilde{E}$ preserves colimits, the morphism $\widetilde{E}(E_{\mathcal{F}}G) \to \widetilde{E}(*)$ is equivalent to the morphism

$$\operatorname*{colim}_{S \in G_{\mathcal{F}}\mathbf{Orb}/pt} E(S) \to E(pt)$$

induced by the morphisms $S \to pt$ in $G\mathbf{Orb}$. But this is now exactly the formula for the assembly map if one expresses $\mathrm{Ind}_{\mathcal{F}} \circ \mathrm{Res}_{\mathcal{F}}(E)$ as a left Kan extension of $\mathrm{Res}_{\mathcal{F}}(E)$ and again applies the pointwise formula. $\square$

Recall Definition 4.2 of the ∞-category $A^{\mathrm{eff}}(G)$ modeling the effective Burnside category of G. We have a functor

$$m\colon G\mathbf{Fin} \times G\mathbf{Fin}^{\mathrm{op}} \to A^{\mathrm{eff}}(G), \tag{69}$$

which is characterized by the property that it sends a pair $(\psi\colon Q\to R, \phi\colon T\to S)$ of morphisms in $G\mathbf{Fin}\times G\mathbf{Fin}$ to the morphism

$$\begin{tikzcd} & Q\otimes T \arrow[dl, "\mathrm{id}\otimes\phi"'] \arrow[dr, "\psi\otimes\mathrm{id}"] & \\ Q\otimes S & & R\otimes T \end{tikzcd}$$

in $A^{\mathrm{eff}}(G)$. Note that we consider m as a contravariant functor in the second argument.

We now consider a functor $M\colon A^{\mathrm{eff}}(G)^{\mathrm{op}}\to \mathbf{C}$.

Definition 4.25. We define the functor

$$F := M\circ m^{\mathrm{op}}\colon G\mathbf{Fin}^{\mathrm{op}}\times G\mathbf{Fin}\to \mathbf{C}.$$

Note that the functor F depends on M, but this is not reflected in the notation.

The inclusion

$$r\colon G\mathbf{Orb}\to G\mathbf{Fin} \tag{70}$$

induces an adjunction

$$r_!\colon \mathbf{PSh}(G\mathbf{Orb}) \leftrightarrows \mathbf{PSh}(G\mathbf{Fin}) : r^*. \tag{71}$$

We consider a functor $M\colon A^{\mathrm{eff}}(G)^{\mathrm{op}}\to \mathbf{C}$ for a complete and cocomplete target $\mathbf{C}$, let F be as in Definition 4.25, and use the notation introduced in Remark 4.23. The counit

$$r_!r^*\to \mathrm{id} \tag{72}$$

of the adjunction (71) induces transformations

$$u\colon \widetilde{F}(-,-)\to \widetilde{F}(r_!r^*(-),-),\quad v\colon \widetilde{F}(-,r_!r^*(-))\to \widetilde{F}(-,-). \tag{73}$$

Recall Definition 4.3 of a Mackey functor.

Lemma 4.26. *If M is a Mackey functor, then the transformations u and v in* (73) *are equivalences.*

Proof. We show that v is an equivalence. The proof for u is similar. We first show that

$$\widetilde{F}(\mathrm{yo}(T), r_!r^*\,\mathrm{yo}(S))\to \widetilde{F}(\mathrm{yo}(T),\mathrm{yo}(S)) \tag{74}$$

is an equivalence for all $S,T\in G\mathbf{Fin}$.

We observe that

$$r_!r^*(\mathrm{yo}(S))\to \mathrm{yo}(S)$$

is equivalent to the morphism

$$\coprod_{R\in G\backslash S}\mathrm{yo}(r(R))\to \mathrm{yo}(S),$$

induced by the family of inclusions $(r(R) \to S)_{R\in G\backslash S}$ (see [6, Lem. 5.10] for more details). Using the fact that $\widetilde{F}$ preserves colimits in its second argument, we conclude that the morphism (74) is equivalent to the morphism

$$\coprod_{R\in G\backslash S} \widetilde{F}(\mathrm{yo}(T), \mathrm{yo}(r(R))) \to \widetilde{F}(\mathrm{yo}(T), \mathrm{yo}(S)). \tag{75}$$

In view of the defining relation between $\widetilde{F}$ and F (see Remark 4.23), the morphism (75) is in turn equivalent to the morphism

$$\coprod_{R\in G\backslash S} F(T, r(R)) \to F(T, S). \tag{76}$$

By Definition 4.25, the morphism (76) is equivalent to the morphism

$$\coprod_{R\in G\backslash S} M(T \times r(R)) \to M(T \times S) \tag{77}$$

obtained from the transfers along the inclusions of the orbits of S. We now use that these transfers also induce an equivalence

$$\coprod_{R\in G\backslash S} T \times r(R) \simeq T \times S$$

in $A^{\mathrm{eff}}(G)^{\mathrm{op}}$ and that M is a Mackey functor, i.e., coproduct preserving. This implies that (77) and hence (74) is an equivalence.

Finally, using (67) and the fact that $r_! r^*$ preserves colimits, we can extend the equivalence (74) to all objects of $\mathbf{PSh}(G\mathbf{Fin})^{\mathrm{op}} \times \mathbf{PSh}(G\mathbf{Fin})$. □

We consider a functor $M\colon A^{\mathrm{eff}}(G)^{\mathrm{op}} \to \mathbf{C}$ for a complete and cocomplete target $\mathbf{C}$ and we let S be an object S of $G\mathbf{Fin}$. Let F be as in Definition 4.25 and recall the notation introduced in Remark 4.23.

Lemma 4.27. *There is an equivalence*

$$s\colon \widetilde{F}(-, \mathrm{yo}(S)) \simeq \widetilde{F}(- \times \mathrm{yo}(S), *)$$

in $\mathbf{Fun}(\mathbf{PSh}(G\mathbf{Fin})^{\mathrm{op}}, \mathbf{C})$.

Proof. By definition of m (see (69)), we have an equivalence

$$m(-, S) \simeq m(- \times S, \mathrm{pt})$$

of functors $G\mathbf{Fin} \to A^{\mathrm{eff}}(G)$. Composing with M and using the Definition 4.25, we get an equivalence

$$F(-, S) \simeq F(- \times S, pt) \tag{78}$$

of functors $G\mathbf{Fin}^{\mathrm{op}} \to \mathbf{C}$. We abbreviate

$$F_1 := F(-, S), \quad F_2 := F(- \times S, pt).$$

By (78), we have an equivalence

$$\widetilde{F_1} \simeq \widetilde{F_2} \tag{79}$$

of contravariant functors from $\mathbf{PSh}(G\mathbf{Fin})$ to $\mathbf{C}$ which send colimits to limits. We now observe that, by the definitions given in Remark 4.23,

$$\widetilde{F_1}(-) \simeq \widetilde{F}(-, \mathrm{yo}(S)). \tag{80}$$

Furthermore, since yo preserves products, for T in $G\mathbf{Fin}$, we have an equivalence

$$\widetilde{F_2}(\mathrm{yo}(T)) \simeq F(T \times S, pt) \simeq \widetilde{F}(\mathrm{yo}(T \times S), *) \simeq \widetilde{F}(\mathrm{yo}(T) \times \mathrm{yo}(S), *). \tag{81}$$

We now use the general fact that for X in $\mathbf{PSh}(G\mathbf{Fin})$, the functor

$$- \times X \colon \mathbf{PSh}(G\mathbf{Fin}) \to \mathbf{PSh}(G\mathbf{Fin})$$

of taking the product with X preserves colimits.[3] This implies that the equivalence (81) extends to an equivalence

$$\widetilde{F_2}(-) \simeq \widetilde{F}(- \times \mathrm{yo}(S), *) \tag{82}$$

of contravariant functors from $\mathbf{PSh}(G\mathbf{Fin})$ to $\mathbf{C}$ sending colimits to limits. Combining now (82), (80) and (79), we get the equivalence asserted in the lemma. □

Let $M \colon A^{\mathrm{eff}}(G)^{\mathrm{op}} \to \mathbf{C}$ be a functor, let F be as in Definition 4.25, and recall the notation introduced in Remark 4.23 and (70). We consider an object A in $\mathbf{PSh}(G\mathbf{Fin})$ and a transitive G-set R in $G\mathbf{Orb}$. Let

$$p_R \colon \widetilde{F}(*, \mathrm{yo}(r(R))) \to \widetilde{F}(A, \mathrm{yo}(r(R))) \tag{83}$$

be the map induced by $A \to *$ (note that $\widetilde{F}$ is contravariant in the first variable).

Proposition 4.28. *Assume:*

(i) $R \in G_{\mathcal{F}}\mathbf{Orb}$;
(ii) M *is a Mackey functor;*
(iii) r^*A *in* $\mathbf{PSh}(G\mathbf{Orb})$ *is equivalent to* $E_{\mathcal{F}}G$.

Then (83) *is an equivalence.*

[3] It is a general property of ∞-topoi that colimits are universal, i.e., preserved by fibre products. We note that $\mathbf{PSh}(G\mathbf{Fin})$ is an ∞-topos.

Proof. Recall the equivalences u and s from Lemma 4.26 and Lemma 4.27 and consider the following commutative diagram:

$$\begin{tikzcd}
\widetilde{F}(*, \mathrm{yo}(r(R))) \arrow[r, "p_R"] \arrow[d, "s"', "\simeq"] & \widetilde{F}(A, \mathrm{yo}(r(R))) \arrow[d, "s"', "\simeq"] \\
\widetilde{F}(\mathrm{yo}(r(R)), *) \arrow[r] \arrow[d, "u"', "\simeq"] & \widetilde{F}(A \times \mathrm{yo}(r(R)), *) \arrow[d, "u"', "\simeq"] \\
\widetilde{F}(r_! r^*(\mathrm{yo}(r(R))), *) \arrow[r] \arrow[d, "!"', "\simeq"] & \widetilde{F}(r_! r^*(A \times \mathrm{yo}(r(R))), *) \arrow[d, "!"', "\simeq"] \\
\widetilde{F}(r_!(\mathrm{yo}(R))) \arrow[r] \arrow[d, equal] & \widetilde{F}(r_!(r^* A \times \mathrm{yo}(R))) \arrow[d, "\simeq"] \\
\widetilde{F}(r_!(\mathrm{yo}(R))) \arrow[r, "!!", "\simeq"'] & \widetilde{F}(r_!(E_{\mathcal{F}}G \times \mathrm{yo}(R))).
\end{tikzcd}$$

For the equivalences marked by '!', we use the canonical equivalence $r^* \mathrm{yo}(r(R)) \simeq \mathrm{yo}(R)$ and that r^* preserves limits.

Let S be in $G\mathbf{Orb}$. By Assumption (i), the relation $\mathrm{yo}(R)(S) \neq \varnothing$ implies that $S \in G_{\mathcal{F}}\mathbf{Orb}$. Hence, by (66),

$$E_{\mathcal{F}}G \times \mathrm{yo}(R) \simeq \mathrm{yo}(R)$$

and the map marked by '!!' is an equivalence as claimed. $\square$

In the situation of Proposition 4.28, we can consider the map

$$p_A \colon \widetilde{F}(*, A) \to \widetilde{F}(A, A) \tag{84}$$

induced by $A \to *$.

Corollary 4.29. *Assume:*

(i) *M is a Mackey functor;*
(ii) *$r^* A$ in $\mathbf{PSh}(G\mathbf{Orb})$ is equivalent to $E_{\mathcal{F}}G$.*

Then (84) *is an equivalence.*

Proof. Since $r^* A$ is equivalent to $E_{\mathcal{F}}G$, A is a colimit of objects of the form $\mathrm{yo}(S)$ with S in $G_{\mathcal{F}}\mathbf{Fin}$. Since $\widetilde{F}$ preserves colimits in its second argument, it suffices to show that

$$p_{\mathrm{yo}(S)} \colon \widetilde{F}(*, \mathrm{yo}(S)) \to \widetilde{F}(A, \mathrm{yo}(S)) \tag{85}$$

is an equivalence for all S in $G_{\mathcal{F}}\mathbf{Fin}$. By Lemma 4.26, in (85), we can replace $\mathrm{yo}(S)$ by $r_! r^* \mathrm{yo}(S)$. We have

$$r_! r^* \mathrm{yo}(S) \simeq r_!\Big(\coprod_{R \in G \backslash S} \mathrm{yo}(R)\Big) \simeq \coprod_{R \in G \backslash S} \mathrm{yo}(r(R)).$$

Since $\tilde{F}$ preserves colimits in its second argument, the map (85) is equivalent to the map

$$\coprod_{R \in G \backslash S} \widetilde{F}(*, \mathrm{yo}(r(R))) \to \coprod_{R \in G \backslash S} \widetilde{F}(A, \mathrm{yo}(r(R))).$$

Since $R \in G \setminus S$ implies $R \in G_{\mathcal{F}}\mathbf{Orb}$, this map is an equivalence by Proposition 4.28. □

We now consider the comparison morphism

$$c\colon \widetilde{F} \to \widetilde{F}'$$

introduced in Remark 4.23.

Let $M\colon A^{\mathrm{eff}}(G)^{\mathrm{op}} \to \mathbf{C}$ be a functor for a complete and cocomplete target $\mathbf{C}$, let F be as in Definition 4.25, and recall the notation introduced in Remark 4.23. Let A and B be in $\mathbf{PSh}(G\mathbf{Fin})$.

Lemma 4.30. *Assume:*

(i) $\mathbf{C}$ *is stable;*
(ii) A *or* B *is compact.*

Then the map

$$c\colon \widetilde{F}(A, B) \to \widetilde{F}'(A, B)$$

is an equivalence.

Proof. Any compact presheaf is a retract of a finite colimit of representable presheaves. Since a retract of an equivalence is an equivalence, it suffices to show the assertion under the assumption that A or B is a finite colimit of representables. To this end, we use the equivalences (67) and (68) and the fact that in a stable ∞-category, finite colimits commute with all limits and finite limits commute with all colimits. □

Let $M\colon A^{\mathrm{eff}}(G)^{\mathrm{op}} \to \mathbf{C}$ be a functor for a complete and cocomplete target $\mathbf{C}$, let F be as in Definition 4.25, and recall the notation introduced in the Remark 4.23. We consider an object A in $\mathbf{PSh}(G\mathbf{Fin})$. Let

$$p'_A\colon \widetilde{F}'(A, A) \to \widetilde{F}'(A, *)$$

be the map induced by $A \to *$

Analogously to Corollary 4.29, we obtain the following statement.

Corollary 4.31. *Assume:*

(i) M *is a Mackey functor;*
(ii) r^*A *in* $\mathbf{PSh}(G\mathbf{Orb})$ *is equivalent to* $E_{\mathcal{F}}G$.

Then p'_A *is an equivalence.*

One can even formally deduce this statement from Corollary 4.29 by going over to opposite categories in the appropriate way.

There is a canonical morphism

$$i\colon G\mathbf{Orb} \to A^{\mathrm{eff}}(G)^{\mathrm{op}}, \tag{86}$$

which is the obvious inclusion on objects and sends the morphism $f\colon S \to T$ to the span

$$\begin{tikzcd} & S \arrow[dl, "f"'] \arrow[dr, "\mathrm{id}_S"] & \\ T & & S. \end{tikzcd}$$

Let $M\colon A^{\mathrm{eff}}(G)^{\mathrm{op}} \to \mathbf{C}$ be a functor for a complete and cocomplete target $\mathbf{C}$, let F be as in Definition 4.25, and recall the notation introduced in Remark 4.23. We set $E := i^*M$.

Lemma 4.32. *The assembly map* (65) *is equivalent to the morphism*

$$\alpha\colon \widetilde{F}(*, r_! E_{\mathcal{F}}G) \to \widetilde{F}(*, *)$$

induced by the projection $r_! E_{\mathcal{F}}G \to *$.

Proof. By Lemma 4.24, the assembly map is equivalent to the morphism

$$\widetilde{E}(E_{\mathcal{F}}G) \to \widetilde{E}(*)$$

induced by the projection $E_{\mathcal{F}}G \to *$. The relations $i(-) \simeq m(*, r(-))$ and $E \simeq i^*M$ now imply that

$$E(-) \simeq F(pt, r(-)).$$

We therefore get an equivalence

$$\widetilde{E}(-) \simeq \widetilde{F}(*, r_!(-))$$

of colimit-preserving functors from $\mathbf{PSh}(G\mathbf{Orb})$ to $\mathbf{C}$. The assertion is now obvious. $\square$

Let $E\colon G\mathbf{Orb} \to \mathbf{C}$ be a functor and let i be as in (86).

Definition 4.33. We say that E *extends to a Mackey functor* if there exists a Mackey functor $M\colon A^{\mathrm{eff}}(G)^{\mathrm{op}} \to \mathbf{C}$ such that $i^*M \simeq E$.

Proof of Theorem 4.22. Let $M\colon A^{\mathrm{eff}}(G)^{\mathrm{op}} \to \mathbf{C}$ be a Mackey functor such that $E \simeq i^*M$. Let F be as in Definition 4.25 and recall the notation introduced in Remark 4.23.

We define the object $A := r_! E_{\mathcal{F}}G$ of $\mathbf{PSh}(G\mathbf{Fin})$. Because the functor $r\colon G\mathbf{Orb} \to G\mathbf{Fin}$ is fully faithful, we have an equivalence $r^* r_! \simeq \mathrm{id}$. In particular, we get the equivalence $r^*A \simeq r^* r_! E_{\mathcal{F}}G \simeq E_{\mathcal{F}}G$. Since $r_!$ is left adjoint to r^* and r^* preserves colimits, $r_!$ preserves compacts. Therefore A is a compact object.

We now consider the diagram

$$\begin{tikzcd}
\widetilde{F}(*,A) \arrow[r, "\alpha"] \arrow[d, "p_A"', "\simeq"] & \widetilde{F}(*,*) \arrow[dr, "c", "\simeq"'] & \\
\widetilde{F}(A,A) \arrow[dr, "\simeq", "c"'] & & \widetilde{F}'(*,*) \arrow[d] \\
& \widetilde{F}'(A,A) \arrow[r, "p'_A", "\simeq"'] & \widetilde{F}'(A,*).
\end{tikzcd}$$

The maps labeled with p_A, p'_A and c are equivalences by Corollary 4.29, Corollary 4.31 and Lemma 4.30. The diagram yields a left-inverse of α. Theorem 4.22 now follows from Lemma 4.32. $\square$

To apply Theorem 4.22, we have to verify the assumption on the compactness of $E_{\mathcal{F}}G$. Following [17], we introduce the following condition on the family $\mathcal{F}$.

Definition 4.34. We call $\mathcal{F}$ *separating* if for every two subgroups H and K of G such that H is normal in K and K/H is prime cyclic, either both K and H belong to $\mathcal{F}$, or both are not contained in $\mathcal{F}$.

Example 4.35. The family **Sol** of solvable subgroups of G is separating.

Theorem 4.36 ([17, Thm. 4])**.** *If $\mathcal{F}$ is a separating family for a finite group G, then there exists a finite G-CW-complex of the homotopy type of $E^{\mathrm{top}}_{\mathcal{F}}G$.*

Note that Oliver's theorem actually states that there exists a disc with a G-action with the correct homotopy types of fixed point spaces. By a theorem of Illman [12], one can then find a finite G-CW-complex in the same G-homotopy type.

Corollary 4.37. *If $\mathcal{F}$ is a separating family, then $E_{\mathcal{F}}G$ is compact.*

Proof. We let $G\mathbf{Top}[W^{-1}]$ denote the ∞-category obtained from the category of topological spaces by inverting G-weak homotopy equivalences, i.e., G-maps which induce weak equivalences on the fixed points spaces for all subgroups of G. By Elmendorf's theorem, we have the equivalence

$$G\mathbf{Top}[W^{-1}] \simeq \mathbf{PSh}(G\mathbf{Orb}).$$

Under this equivalence, a G-CW-complex of the homotopy type of $E^{\mathrm{top}}_{\mathcal{F}}G$ goes to a presheaf equivalent to $E_{\mathcal{F}}G$. We now note that a finite G-CW-complex represents a compact object in $G\mathbf{Top}[W^{-1}]$ and therefore in $\mathbf{PSh}(G\mathbf{Orb})$. $\square$

Therefore, Theorem 4.22 has the following corollary.

Let G be a finite group, let $\mathcal{F}$ be a family of subgroups of G, and let $E\colon G\mathbf{Orb} \to \mathbf{C}$ be a functor.

Corollary 4.38. *Assume:*

(i) $\mathbf{C}$ *is stable, complete and cocomplete;*
(ii) E *extends to a Mackey functor;*
(iii) $\mathcal{F}$ *is separating.*

Then the assembly map $\mathrm{Ind}_{\mathcal{F}} \circ \mathrm{Res}_{\mathcal{F}}(E)(pt) \to E(pt)$ *is split injective.*

By Example 4.35, this corollary applies to the family $\mathcal{F} = \mathbf{Sol}$.

Remark 4.39. In the case $\mathbf{C} = \mathbf{Sp}$, Corollary 4.38 is a consequence of known facts in equivariant stable homotopy theory under the identification of spectral Mackey functors with the category of equivariant spectra. Let $\mathbf{Sp}^G$ denote the stable ∞-category of equivariant spectra. Denote the equivariant sphere spectrum by $\mathbf{S}_G$. Following [19], the Burnside ring $A(G)$ is given by equivalence classes of finite G-CW-complexes subject to the relation that $[X] = [Y]$ if the Euler characteristics of all fixed points of X and Y agree. Addition in $A(G)$ is induced by the coproduct, and multiplication corresponds to the cartesian product. Then $A(G)$ is isomorphic to $\pi_0^G(\mathbf{S}_G) \simeq \pi_0 \mathrm{Map}_{\mathbf{Sp}^G}(\mathbf{S}_G, \mathbf{S}_G)$ via the map that sends a finite G-CW-complex to its trace. Moreover, the ghost map $A(G) \to \mathbf{Z}$ associated to any (conjugacy class of) subgroup H of G sends the class of a finite G-CW-complex X to the Euler characteristic of X^H. These maps fit into the following commutative diagram:

$$\begin{tikzcd} A(G) \arrow[r, "\cong"] \arrow[d] & \pi_0 \mathrm{Map}_{\mathbf{Sp}^G}(\mathbf{S}_G, \mathbf{S}_G) \arrow[d] \\ \prod_{(H)} \mathbf{Z} & \prod_{(H)} \pi_0 \mathrm{Map}_{\mathbf{Sp}}(\Phi^H(\mathbf{S}_G), \Phi^H(\mathbf{S}_G)). \arrow[l, "\cong"'] \end{tikzcd} \tag{87}$$

The products are indexed by conjugacy classes of subgroups of G. The right vertical map is induced by taking H-geometric fixed points of endomorphisms. The bottom horizontal isomorphism is given by taking mapping degrees. As indicated, the commutativity is therefore a consequence of the fact that geometric fixed points are monoidal and commute with the duality functor D, and that the mapping degree of the trace of a finite CW-complex is precisely its Euler characteristic.

Suppose now that there is a finite G-CW-complex $E_{\mathcal{F}}^{\mathrm{top}}G$ of the homotopy type of the classifying space for the family $\mathcal{F}$. Since $E_{\mathcal{F}}^{\mathrm{top}}G \times E_{\mathcal{F}}^{\mathrm{top}}G \simeq E_{\mathcal{F}}^{\mathrm{top}}G$, the class $[E_{\mathcal{F}}^{\mathrm{top}}G]$ is an idempotent in $A(G)$, and thus gives rise to an idempotent map $p\colon \mathbf{S}_G \to \mathbf{S}_G$. By the commutativity of (87),

$$\Phi^H(p) \simeq \begin{cases} 0 & \text{if } H \notin \mathcal{F}, \\ \mathrm{id}_{\mathbf{S}} & \text{if } H \in \mathcal{F}. \end{cases}$$

The idempotent p induces a splitting $\mathbf{S}_G \simeq \mathbf{S}_{G,\mathcal{F}} \oplus \mathbf{S}_G^{\mathcal{F}}$, where $\mathbf{S}_{G,\mathcal{F}}$ is given by the mapping telescope along p:

$$\mathbf{S}_{G,\mathcal{F}} \simeq \mathrm{colim}(\mathbf{S}_G \xrightarrow{p} \mathbf{S}_G \xrightarrow{p} \mathbf{S}_G \xrightarrow{p} \cdots).$$

Since Φ^H commutes with colimits, $\Phi^H(\mathbf{S}_{G,\mathcal{F}}) \simeq \mathbf{S}$ or $\Phi^H(\mathbf{S}_{G,\mathcal{F}}) \simeq 0$, depending on whether H lies in $\mathcal{F}$ or not.

Let $\Sigma^\infty_+ \colon G\mathbf{Top} \to \mathbf{Sp}^G$ denote the equivariant suspension spectrum. Consider the commutative diagram

$$\begin{tikzcd} \mathbf{S}_{G,\mathcal{F}} \arrow[d] & \mathbf{S}_{G,\mathcal{F}} \otimes \Sigma^\infty_+ E^{\mathrm{top}}_{\mathcal{F}} G \arrow[l] \arrow[r] \arrow[d] & \Sigma^\infty_+ E^{\mathrm{top}}_{\mathcal{F}} G \arrow[d] \\ \mathbf{S}_G & \mathbf{S}_G \otimes \Sigma^\infty_+ \mathrm{pt} \arrow[l] \arrow[r] & \Sigma^\infty_+ \mathrm{pt}, \end{tikzcd}$$

in which all arrows are induced by the inclusion $\mathbf{S}_{G,\mathcal{F}} \to \mathbf{S}_G$, the projection $E^{\mathrm{top}}_{\mathcal{F}} G \to \mathrm{pt}$ and the unit map of the monoidal structure. As the geometric fixed point functors are monoidal and jointly detect equivalences in $\mathbf{Sp}^G$, all horizontal arrows in this diagram are equivalences.

Therefore, in the induced splitting $M \simeq (M \otimes \mathbf{S}_{G,\mathcal{F}}) \oplus (M \otimes \mathbf{S}^{\mathcal{F}}_G)$ of an arbitrary equivariant spectrum M, the inclusion $M \otimes \mathbf{S}_{G,\mathcal{F}} \to M$ is equivalent to the map

$$M \otimes \Sigma^\infty_+ E^{\mathrm{top}}_{\mathcal{F}} G \to M$$

induced by the projection $E^{\mathrm{top}}_{\mathcal{F}} G \to \mathrm{pt}$. Finally, since

$$\begin{aligned} (M \otimes \Sigma^\infty_+ E^{\mathrm{top}}_{\mathcal{F}} G)^G &\simeq \Big(M \otimes \operatorname*{colim}_{G/H \in G_{\mathcal{F}}\mathbf{Orb}} \Sigma^\infty_+ G/H\Big)^G \\ &\simeq \operatorname*{colim}_{G/H \in G_{\mathcal{F}}\mathbf{Orb}} (M \otimes \Sigma^\infty_+ G/H)^G \\ &\simeq \operatorname*{colim}_{G/H \in G_{\mathcal{F}}\mathbf{Orb}} M^H, \end{aligned}$$

the split inclusion $(M \otimes \mathbf{S}_{G,\mathcal{F}})^G \to M^G$ is equivalent to the assembly map.

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