# A Flow Space for a Relatively Hyperbolic Group

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# A Flow Space for a Relatively Hyperbolic Group

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# 1 Introduction

## 1.1 The Farrell-Jones Conjecture

The Farrell-Jones Conjecture, if true for a group $G$ and a ring $R$, provides a strategy for computing the $K$-theory of the group ring $R[G]$ in terms of the $K$-theory of group rings of the form $R[V]$ where $V$ is a virtually cyclic subgroup of $G$. (There is also a version of the Farrell-Jones Conjecture for $L$-theory but we are not interested in that version here.)

In the particular case $R = \mathbb{Z}$, the Farrell-Jones Conjecture would allow us to more easily compute $K_*(\mathbb{Z}[G])$. This case is interesting in topology because the Whitehead group, which for example appears in the $s$-cobordism thoerem, is a quotient of $K_1(\mathbb{Z}[G])$.

The Farrell-Jones Conjecture also has direct applications to other conjectures, such as the Kaplansky Conjecture about idempotents in the group ring $R[G]$. See [BLR08b] or [LR05] for more information about the Farrell-Jones Conjecture and its applications.

In [BLR08c] Bartels-Lück-Reich proved that the Farrell-Jones Conjecture holds for a hyperbolic group $G$. Their proof made use of special covers of $G \times \overline{X}$ where $X$ is the Rips complex associated to $G$ and $\overline{X}$ is the compactification using the Gromov boundary. These covers were constructed in [BLR08a], using a flow space, first defined by Mineyev in [Min05]. Special covers were constructed over this flow space and then pulled-back to $G \times \overline{X}$ along a suitable embedding.

In this thesis we construct a flow space for a relatively hyperbolic group, i.e. a group $G$ that is hyperbolic relative to a set of peripheral subgroups. The hope is to use this flow space to prove that if the Farrell-Jones Conjecture holds for all the peripheral subgroups then the Farrell-Jones Conjecture also holds for the group $G$.

## 1.2 Relatively hyperbolic groups

The concept of a hyperbolic group was first introduced by Gromov in [Gro87] and since then there has been considerable research on hyperbolic groups.

Relative hyperbolicity generalises this concept by allowing the group to be non-hyperbolic so long as it only behaves in a non-hyperbolic manner within certain subgroups. So, informally, a group $G$ is hyperbolic relative to a set $\mathcal{P}$ of subgroups if $G$ is hyperbolic outside the subgroups $P \in \mathcal{P}$ and their conjugates (see section 2.4 for an exact definition). Hence, if the subgroups $P \in \mathcal{P}$ are themselves hyperbolic then $G$ is hyperbolic everywhere, and thus is a hyperbolic group (see [Osi06, Corollary 2.41] for a proof of this fact).

Since we have refrained from giving an exact definition here we instead give some examples of groups and subgroups to which they are relatively hyperbolic.

- A group is hyperbolic if and only if it is hyperbolic relative to the trivial subgroup, hence relative hyperbolicity is indeed a generalisation of hyperbolicity.

- If $G = A * B$ is a free product then $G$ is hyperbolic relative to $\{A, B\}$. More generally, if $G = A *_H B$ is a free product amalgamated over a finite subgroup $H$ then $G$ is hyperbolic relative to $\{A, B\}$.

- If $X$ is a systolic complex with isolated flats and $G$ acts cocompactly and properly discontinuously on $X$ then $G$ is hyperbolic relative to its maximal virtually abelian subgroups of rank 2 (see [Els, Corollary 5.14]).

- A limit group is hyperbolic relative to its maximal non-cyclic abelian subgroups (proven independently by F. Dahmani as [Dah03a, Theorem 4.5] and E. Alibegović as [Ali05, Theorem 3.4]).

- More geometrically, the fundamental group of a hyperbolic manifold with finite volume is hyperbolic relative to the cusp subgroups.

Many facts about hyperbolic groups have been proven to hold for relatively hyperbolic groups, assuming that the fact holds for all of the peripheral subgroups. For example, Osin proved that a relatively hyperbolic group $G$ has a solvable word problem if all the peripheral subgroups have a solvable word problem (see [Osi06, Theorem 5.1]) and that if none of the peripheral subgroups contain a copy of a Baumslag-Solitar group then neither does $G$ (see [Osi06, Corollary 4.22]).

Therefore it is reasonable to ask if a group $G$ is hyperbolic relative to $\mathcal{P}$ and the Farrell-Jones Conjecture holds for all the peripheral subgroups then does it also hold for $G$?

The Farrell-Jones Conjecture can be generalised to allow an arbitray family of subgroups instead of using virtually cyclic subgroups. With this more general version there is a transitivity principle, (see [LR05, Theorem 2.9]) that says;

Suppose $\mathcal{E}, \mathcal{F}$ are two families of subgroups of $G$ with $\mathcal{E} \subseteq \mathcal{F}$ and for $F \in \mathcal{F}$ let $\mathcal{E} \cap F = \{E \cap F \mid E \in \mathcal{E}\}$. If the Farrell-Jones Conjecture with the family $\mathcal{F}$ holds for $G$ and the Farrell-Jones Conjecture with the family $\mathcal{E} \cap F$ holds for all $F \in \mathcal{F}$ then the Farrell-Jones Conjecture with the family $\mathcal{E}$ holds for $G$.

Then given a group $G$ that is hyperbolic relative to $\mathcal{P}$ we can define the family $\mathcal{F}[\mathcal{P}]$ to consist of all virtually cyclic subgroups and all peripheral subgroups, and then ask whether the Farrell-Jones Conjecture with the family $\mathcal{F}[\mathcal{P}]$ holds for $G$. The transitivity principle would then imply that if the Farrell-Jones Conjecture (with the family of virtually cyclic subgroups) is true for all the peripheral subgroups then it is also true for $G$.

## 1.3 The flow space

If a group $G$ is hyperbolic relative to a set $\mathcal{P}$ of subgroups then Mineyev and Yaman in [MY06] constructed a simplicial complex $X$ associated to $G$, which can be thought of as an analogue of the Rips complex for a hyperbolic group. In particular, the space $X$ is contractible, has finite dimension, and its 1-skeleton is a Gromov hyperbolic metric space. Furthermore, we can define a Gromov boundary for $X$ and hence get a topological space $\overline{X} = X \cup \partial X$ (see section 2.5).

In [Min05] Mineyev also defined a flow space associated to what he calls a 'hyperbolic complex', which is a simplicial complex whose 1-skeleton is a uniformly locally finite, Gromov-hyperbolic graph.

If $(Y, d_Y)$ is a uniquely geodesic metric space then we can take the flow space $FS(Y)$ to be the space of all geodesics in $Y$. These geodesics may be finite or infinite in length, so we say a map $c\colon \mathbb{R} \to Y$ is a *generalised geodesic* if there are real numbers $t_0 \leq t_1 \in [-\infty, \infty]$ such that $c|_{(t_0,t_1)}$ is a geodesic and $c$ is constant on $(-\infty, t_0]$ and $[t_1, \infty)$. Then

$$FS(Y) := \{c\colon \mathbb{R} \to Y \mid c \text{ is a generalised geodesic}\}$$

with the topology defined by the metric

$$d_{FS}(c,c') := \int_{\mathbb{R}} e^{-\frac{1}{2}|t|} d_Y\big(c(t),c'(t)\big)\,dt.$$

The flow on $FS(Y)$ is given by translation on $\mathbb{R}$, i.e. $\Phi_\tau(c)(t) := c(t+\tau)$. Then the flow space constructed by Mineyev in [Min05] is a space that formally replicates what happens in the case of a uniquely geodesic space.

Unfortunately, the space $X$ constructed in [MY06] does not have a uniformly locally finite 1-skeleton (in fact the 1-skeleton is not locally finite at all) and so the construction in [Min05] is not applicable.

Therefore, the construction of the flow space has to be adapted to use the properties we do have about the 1-skeleton of $X$, in particular we need to use the *uniform fineness* of the 1-skeleton.

Fineness is a property about graphs that is weaker than local finiteness, and can be thought of as a kind of local finiteness for edges. Fineness was introduced by Bowditch in [Bow97] and in this thesis will be a fundamental ingredient in the construction of the flow space. In section 2.3 we will give a definition of fineness and state the facts we will need later on.

Chapter 7 is about the properties of this flow space. In section 7.1 we examine the topology of the flow space induced by the metric, and in particular prove theorem 7.3 that says there is a homeomorphism

$$\big(\overline{X}\times\overline{X}\setminus\Delta(\overline{X})\big)\times\mathbb{R}\longrightarrow\Big(FS(X)\setminus FS(X)^{\mathbb{R}},\tilde{d}\Big)$$

where the domain of the map has the standard product topology.

The important properties are found in section 7.4, where we look at what happens when two formal geodesics have a common end-point. We prove that under the action of the flow two such formal geodesics become arbitrarily close, as stated in the following theorem.

**Theorem** (7.5)**.** *Let $X$ be a simplicial complex whose 1-skeleton is a uniformly fine, Gromov hyperbolic graph. For any $x,x'\in X, y\in\overline{X}$ and all $t,t'\in\mathbb{R}$ there exists a constant $t_0\in\mathbb{R}$ such that*

$$d_{FS}\big(\Phi_\tau(x,y,t),\Phi_\tau(x',y,t'+t_0)\big)\longrightarrow 0 \quad \text{as } \tau\to\infty.$$

Since we are specifically interested in when we start with a relatively hyperbolic group we can say even more;

**Theorem** (7.6)**.** *Let $G$ be a group that is hyperbolic relative to a finite collection $\mathcal{P}$ of subgroups and let $X$ be the associated simplicial complex from section 3.1. Fix a base-point $x_0\in X$ and let $G$ act on $G\times\overline{X}$ via the diagonal action.*

*There is a continuous $G$-equivariant map $j\colon G\times\overline{X}\to FS(X)$ that satisfies the following property;*

*For any $\alpha > 0$ there is a function $f_\alpha\colon \mathbb{R} \to [0, \infty)$ with $f_\alpha(\tau) \to 0$ as $\tau \to \infty$, and there is a $\beta = \beta(\alpha) > 0$ such that for all $g, h \in G$ and $y \in \overline{X}$, if $d_G(g, h) \le \alpha$ then there exists some $t_0 \in [-\beta, \beta]$ such that for all $\tau \in \mathbb{R}$*

$$d_{FS}\left(\Phi_\tau j(g, y), \Phi_{\tau+t_0} j(h, y)\right) \le f_\alpha(\tau).$$

An informal explanation as to what this theorem is actually saying can be found after its appearance in section 7.4.

## 1.4 Outline of the Construction

The goal of this thesis is the construction of a flow space associated to a simplicial complex $X$ whose 1-skeleton $\mathcal{G}$ is a uniformly fine, Gromov hyperbolic graph (such as the simplicial complex $X$ associated to a relatively hyperbolic group in [MY06]).

The set underlying the flow space $FS(X)$ is defined in section 3.3 and uses the idea of Mineyev in [Min05]. The definition of the metric on $FS(X)$ is the complicated part. In section 3.4 we start the construction of this metric, but it will not be completed until section 6.3. The construction requires that the *double difference*

$$(x, x'|y, y') := d(x, y) - d(x, y') - d(x', y) + d(x', y')$$

of four points $x, x', y, y' \in X$ can be extended to allow $x, x'$ to be points in the boundary $\partial X$. However, this is not necessarily possible if we start with the standard metric on $\mathcal{G}$ and so we need to create a new metric on $\mathcal{G}$ for which the double difference can be extended. In the original hyperbolic case covered by Mineyev, the construction of this metric was done in [MY02] building on work in [Min01].

A possible way to adapt the work from [Min01] was suggested Mineyev and Yaman in [MY06] and is done here explicitly in chapter 4, although we do not use precisely their suggested method.

The idea is to consider the vertex set $V = V(\mathcal{G})$ and edge set $E = E(\mathcal{G})$ as bases of $\mathbb{Q}$-vector spaces $\mathbb{Q}V$ and $\mathbb{Q}E$ respectively, and then construct a map $g\colon \mathbb{Q}V \oplus \mathbb{Q}V \to \mathbb{Q}E$ such that $g(a, b)$ is the end result of some kind of projection from $b$ to $a$. The projection is built from many small steps in an iterative fashion. Each step is a projection by a distance $\mu$, where $\mu$ is a fixed constant.

We start with a map $f_\aleph\colon \mathbb{Q}V \oplus \mathbb{Q}V \to \mathbb{Q}E$ in section 4.1 that depends on a constant $\aleph$. This map is the first step in the projection towards the sphere of radius $\aleph$ around $a$. The element $f_\aleph(a,b) \in \mathbb{Q}E$ is in essence an average over the intersection of a sphere around $a$ with a neighbourhood of the geodesics from $a$ to $b$. This averaging procedure is why we need to work over edges instead of vertices, because we can use fineness to ensure we are averaging over finite sets, whereas if we worked with vertices then the analogous sets could be infinite. Hyperbolicity tells us that if $b$ and $c$ are close then the geodesics from $a$ to $c$ are close to the geodesics from $a$ to $b$, in particular we get the following;

**Lemma** (4.6)**.** *There exists a constant $\eta_0 \in (0,1)$ such that for $a,b,c \in V$ if $d(b,c) \leq 2\mu$ and*

$$d(a,b) = (m+1)\mu + \aleph = d(a,c)$$

*for some $m \in \mathbb{N}$ then*

$$|f_\aleph(a,b) - f_\aleph(a,c)|_1 \leq 2\eta_0.$$

Section 4.2 is concerned with what happens when we iterate $f_\aleph$. Then in section 4.3 we define the map $g_\aleph \colon \mathbb{Q}V \oplus \mathbb{Q}V \to \mathbb{Q}E$, which is given by repeatedly applying the small step $f_\aleph$ until we reach the sphere of radius $\aleph$. A crucial property of this map is the following proposition,

**Proposition** (4.18)**.** *If $2\mu \geq 7\delta + 4$ then there are constants $L > 0$ and $\lambda \in (0,1)$ such that for all $a,b,b' \in V$*

$$|g_\aleph(a,b) - g_\aleph(a,b')|_1 \leq L\lambda^{(b|b')_a}$$

*where $(b|b')_a = \frac{1}{2}\big(d(a,b) + d(a,b') - d(b,b')\big)$ is the Gromov product.*

Then in section 4.4 we take an average over such $g_\aleph$ using initial spheres of differing radii to get the map $g$.

The element $g(a,b) \in \mathbb{Q}E$ was the result of a projection from $b$ to $a$, but turning around our point of view we can also think of it as the first step in moving $a$ towards $b$. By counting how many times we have to apply $g$ to move $a$ to $b$ we get a function $r \colon \mathbb{Q}V \oplus \mathbb{Q}V \to [0,\infty)$ which is our first attempt at a new metric on $V = V(\mathcal{G})$. This function is defined and studied in section 5.1. Although this work is based on ideas from [MY02] the results in chapter 5 are completely new.

The main results about the function $r$ are summarised below.

**Theorem.** *The function $r \colon \mathbb{Q}V \oplus \mathbb{Q}V \to [0,\infty)$ satisfies*

(*i*) *There exists a constant $K \geq 1$ such that for all $a,b \in V$*

$$\frac{1}{K} d(a,b) \leq r(a,b) \leq d(a,b).$$

(*ii*) *There exists a constant* $N \geq 0$ *such that for all* $a, a' \in V$

$$|r(a,b) - r(a',b)| \leq d(a,a') + N.$$

(*iii*) *There exists a constant* $M \geq 1$ *such that for all* $a, b, b' \in V$

$$|r(a,b) - r(a,b')| \leq M\, d(b,b').$$

(*iv*) *There exist constants* $C > 0, \omega \in (0,1)$ *such that for all* $a, a', b, b' \in V$ *if* $d(a,a') \leq 1$ *and* $d(b,b') \leq 1$ *then*

$$|r(a,b) - r(a,b') - r(a',b) + r(a',b')| \leq C\omega^{d(a,b)}.$$

This theorem is a combination of proposition 5.2, lemma 5.5, proposition 5.8, and proposition 5.10. The hardest of these is proposition 5.10 whose proof takes up all of section 5.2. From $r$ we can define a function $\hat{d}\colon V \times V \to [0,\infty)$ which is a metric on $V$. This is done in section 5.3.

Before we can use this to define a metric on the flow space though we need to extend $\hat{d}$ to all of $X$. An attempt at an extension was given by Mineyev in [Min05] and this extension was later used by Bartels-Lück-Reich in [BLR08a], but this extension is not a metric so in section 6.1 we give a framework for extending a metric from the vertex set of a simplicial complex to the whole simplicial complex. In particular this fixes the problem in [Min05] which carried over to [BLR08a].

Then in section 6.2 we show what happens in our case when we use $\hat{d}$ as an input for this framework to get a metric $\tilde{d}$ on $X$. The main results about this metric $\tilde{d}$ are proposition 6.12 and theorem 6.13, which tell us the following two facts about the double difference

$$\langle a, a' \wr b, b' \rangle := \tilde{d}(a,b) - \tilde{d}(a,b') - \tilde{d}(a',b) + \tilde{d}(a',b')$$

with respect to $\tilde{d}$;

**Theorem.** *There are constants* $C > 0, \omega \in (0,1)$ *such that for all* $a, a', b, b' \in V$ *if* $d(a,a') \leq 1$ *and* $d(b,b') \leq 1$ *then*

$$\left|\langle a, a' \wr b, b' \rangle\right| \leq \frac{1}{2} C\omega^{d(a,b)}.$$

*Moreover, the double difference extends continuously to* $\overline{X} \times \overline{X} \times X \times X$.

Using this extension we can construct a metric on the flow space in terms of $\tilde{d}$, which is done in section 6.3.

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# 2 Prerequisites

We begin the thesis with some definitions and facts that are (mostly) not new but it will be helpful to have these results stated somewhere and so we do that here.

However, section 2.6 may not be a new idea but I have not seen the argument written up in such generality elsewhere.

## 2.1 Graphs

First of all we recall some definitions and fix some notation related to graphs.

**Definition 2.1.** A *graph* $\Gamma$ is a set $V = V(\Gamma)$ of vertices together with a set $E = E(\Gamma)$ of edges and two maps $\varphi_\pm \colon E \to V$ such that $\varphi_\pm(e)$ are the end-points of the edge $e$. Two vertices are *adjacent* if there is an edge between them, an edge is *incident* to a vertex if the vertex is one of the end-points of the edge, and two edges are *coincident* if they share an end-point.

If $v$ is a vertex of a graph $\Gamma$ then define the graph $\Gamma - v$ to have vertex set $V(\Gamma - v) := V(\Gamma) \backslash \{v\}$ and whose edge set consists of all edges of $\Gamma$ that do not have $v$ as an end-point.

A *path* between two vertices $u, v$ in a graph $\Gamma$ is a sequence $e_1, \ldots, e_k$ of edges such that for every

- $u$ is an end-point of $e_1$,
- $v$ is an end-point of $e_2$,
- for all $i = 1, \ldots, k-1$ the edges $e_i$ and $e_{i+1}$ are coincident.

A path that starts and ends at the same vertex is called a *loop*, and a loop that does not hit any vertex more than once is called a *circuit* (sometimes such a loop is also called a *cycle*).

A graph is *connected* if for any pair of vertices $u, v$ there is a path from $u$ to $v$. A connected graph $\Gamma$ is 2-*vertex-connected* if for every vertex $v \in V(\Gamma)$ the graph $\Gamma - v$ is connected.

We will always assume that a graph is connected.

The *length* of a path is the number of edges in the path. Then we can define the distance $d_\Gamma(u, v)$ between two vertices $u, v$ of $\Gamma$ to be the length of the shortest path between $u$ and $v$. This gives every edge length 1, and the metric $d_\Gamma$ on $V$ can naturally be extended to a path metric on all of $\Gamma$. We call $d_\Gamma$ the *word metric* on $\Gamma$, where the terminology comes from geometric group theory and the concept of Cayley graphs (see definition 2.2).

A *geodesic* in $\Gamma$ is a map $\alpha\colon [t_0, t_1] \to \Gamma$ with $t_0, t_1 \in [0, \infty)$ such that for all $t, t' \in [t_0, t_1]$

$$d_\Gamma\big(\alpha(t), \alpha(t')\big) = \left|t - t'\right|.$$

If $\alpha$ is a geodesic in $\Gamma$ and $x_0 = \alpha(s_0), x_1 = \alpha(s_1)$ are points on the image of $\alpha$ with $s_0 \leq s_1$ then let $\alpha_{[x_0,x_1]}$ denote the part of the geodesic $\alpha$ between $x_0$ and $x_1$, i.e. $\alpha_{[x_0,x_1]} \equiv \alpha|_{[s_0,s_1]}$.

For $x, x' \in \Gamma$ we will use $[x, x']$ to denote a geodesic that starts at $x$ and ends at $x'$. For any pair $x, x' \in V$ of vertices in $\Gamma$ let $\mathrm{Geod}[x, x']$ be the set of geodesics in $\Gamma$ from $x$ to $x'$, let $V[x, x']$ be the set of all vertices of $\Gamma$ that lie on some geodesic from $\mathrm{Geod}[x, x']$, and for $k \in \mathbb{N}$ let $V[x, x'; k]$ be the vertices from $V[x, x']$ whose distance from $x$ is $k$.

Similarly, for any pair $x, x' \in V$ of vertices, let $E[x, x']$ be the set of all edges of $\Gamma$ that form part of some geodesic from $\mathrm{Geod}[x, x']$, and for $k \in \mathbb{N}$ let $E[x, x'; k]$ be the edges from $E[x, x']$ whose distance from $x$ is $k$.

We are interested in groups, in particular we want to consider a group as a metric space. We do this via Cayley graphs.

**Definition 2.2.** Let $G$ be a group and let $S \subseteq G \backslash \{e\}$ be a generating set of $G$. The *Cayley graph of $G$ with respect to $S$* is the graph $\mathrm{Cay}(G; S)$ whose vertex set is $G$ and for every $s \in S$ and every $g \in G$ we add an edge from $g$ to $gs$.

**Remark 2.3.** I have defined a Cayley graph such that the Cayley graph of $\mathbb{Z}/2\mathbb{Z}$ is a circle; two vertices with a pair of edges between them.

Any graph has a canonical metric as in definition 2.1, and so we get a metric on a Cayley graph of a group. By restricting this metric to the vertex set we get a metric on the group itself, which we call the *word metric on $G$ with respect to $S$* and denote by $d_S$ since it depends on the choice of generating set.

If we use a different generating set then we get a different graph (and thus a different metric on $G$), but there is some notion of equivalence between the two graphs.

**Definition 2.4.** Let $X, Y$ be metric spaces. Suppose $A \geq 1$ and $B \geq 0$ are fixed constants. A (not necessarily continuous) map $f\colon X \to Y$ is an $(A, B)$*-quasi-isometric embedding* if for all $x_1, x_2 \in X$,

$$\frac{1}{A}\, d_X(x_1, x_2) - B \leq d_Y\left(f(x_1), f(x_2)\right) \leq A\, d_X(x_1, x_2) + B.$$

A (not necessarily continuous) map $f\colon X \to Y$ is *quasi-dense* if there exists a constant $C \geq 0$ such that $Y$ is contained in the $C$-neighbourhood of the image of $f$, i.e. for any $y \in Y$ there is some $x \in X$ with $d_Y\left(y, f(x)\right) \leq C$. A (not necessarily continuous) map $f$ is an $(A, B)$*-quasi-isometry* if it is a quasi-dense $(A, B)$-quasi-isometric-embedding.

The particular constants $A, B$ are not always important and so are often omitted from the notation; a map $f$ is a *quasi-isometry* if there are constants $A \geq 1$ and $B \geq 0$ such that $f$ is an $(A, B)$-quasi-isometry.

Two metric spaces $X, Y$ are *quasi-isometric* if there exists a quasi-isometry between them.

**Remark 2.5.** The property of being quasi-isometric is an equivalence relation on the class of metric spaces. In particular, the composition of two quasi-isometries is a quasi-isometry and if there exists a quasi-isometry $X \to Y$ then there exists a quasi-isometry $Y \to X$.

**Proposition 2.6.** *Let $G$ be a finitely generated group. For any finite generating set $S$ the inclusion $(G, d_S) \hookrightarrow \mathrm{Cay}(G; S)$ is a quasi-isometry.*

*Moreover, for any two finite generating sets $S_1, S_2$, the graphs $\mathrm{Cay}(G; S_1)$ and $\mathrm{Cay}(G; S_2)$ are quasi-isometric.*

*Proof.* The canonical inclusion $G \hookrightarrow \mathrm{Cay}(G, S)$ is an isometric embedding and is also $\frac{1}{2}$-dense, hence it is a quasi-isometry.

Given two finite generating sets $S_1, S_2$ of $G$ we can write all the elements of $S_2$ as words over $S_1$, and vice versa. With this we can write a word in the generators from $S_1$ as a word in the generators from $S_2$, and so we can compare $d_{S_1}$ with $d_{S_2}$. See [BH99, Example I.8.17(3)] for full details. □

In particular, the metric spaces $(G, d_{S_1})$ and $(G, d_{S_2})$ are quasi-isometric. Therefore we will often think of $G$ as a metric space without explicitly stating which generating set is used.

For our purposes we will not always want to work with a Cayley graph associated to a group, instead we will want to use a graph on which the group acts with certain properties. A group acts freely, isometrically, and cocompactly on its Cayley graph, but we consider a slightly weaker type of group action.

Recall that a group action $G \curvearrowright X$ is *free* if the stabiliser of every point is trivial, is *proper* if for every compact subset $K \subseteq X$ the number of elements $g \in G$ such that $g \cdot K \cap K \neq \emptyset$ is finite, and is *cocompact* if there is a compact subset $K \subset X$ such that $G \cdot K = X$. A metric space is a *length space* if the distance between any two points is equal to the infimum of lengths of paths between them.

**Lemma 2.7** (Švarc-Milnor Lemma)**.** *If a group $G$ acts properly, isometrically, and cocompactly on a length space $X$ then $G$ is finitely generated and quasi-isometric to $X$.*

*Proof.* See [BH99, Proposition I.8.19]. □

## 2.2 Hyperbolicity

The concept of Gromov hyperbolicity has been quite extensively studied, and there exist a few books on the topic, see [GdLH90] for example. Hence here we only give a brief overview of facts we will need, omitting many details and proofs.

**Definition 2.8.** Let $(X, d)$ be a geodesic metric space, i.e. for any pair of points there is a geodesic between them. Define the *Gromov product* of three points $x, x', y \in X$ to be

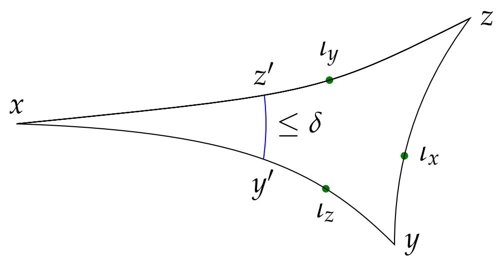
$$(x|x')_y = \frac{1}{2}\left(d(x,y) + d(x',y) - d(x,x')\right).$$

If $\Delta$ is a geodesic triangle in $X$ with corners $x, y, z$ then we can find a point $\iota_z$ on $[x, y]$ such that $d(x, \iota_z) = (y|z)_x$ and $d(\iota_z, y) = (x|z)_y$. Similarly we can define a point $\iota_x$ on $[y, z]$ and a point $\iota_y$ on $[x, z]$.

Given $\delta \geq 0$, a geodesic triangle $\Delta = \Delta(x, y, z)$ is *$\delta$-thin* if whenever $y'$ is a point on $[x, y]$ and $z'$ is a point on $[x, z]$ such that

$$d(x, y') = d(x, z') \leq (y|z)_x$$

then $d(y', z') \leq \delta$.



A geodesic metric space $X$ is *$\delta$-hyperbolic* if every geodesic triangle is $\delta$-thin, and a geodesic metric space is called *Gromov hyperbolic* if it is $\delta$-hyperbolic for some $\delta \geq 0$.

Note that if a metric space is $\delta$-hyperbolic then it is also $\delta'$-hyperbolic for all $\delta' \geq \delta$, so we may always assume $\delta \geq 1$.

For alternative definitions see [BH99, Proposition III.H.1.17].

In the degenerate case that $y = z$ we have $(y|y)_x = d(x, y)$, which immediately gives the following lemma.

**Lemma 2.9.** *Suppose $X$ is a $\delta$-hyperbolic space. Fix $x, y \in X$. For all $k \leq d(x, y)$ and all $z, z' \in V[x, y; k]$ we have $d(z, z') \leq \delta$.*

One useful alternative formulation of Gromov hyperbolicity is given by the next proposition.

**Proposition 2.10.** *A geodesic metric space $X$ is Gromov hyperbolic if and only if there exists some $\delta \geq 0$ such that for any four points $x, y, z, w \in X$,*

$$(x|z)_w \geq \min\{(x|y)_w, (y|z)_w\} - \delta. \tag{2.1}$$

*Proof.* See [BH99, Proposition III.H.1.22]. □

Since we defined the Gromov product in terms of distances, we do not need $X$ to be a geodesic space in order to make sense of the Gromov product $(x|z)_w$ and so we can use inequality (2.1) to define Gromov hyperbolicity for non-geodesic metric spaces.

**Definition 2.11.** An arbitrary metric space $X$ is *Gromov hyperbolic* if there exists some $\delta \geq 0$ such that inequality (2.1) holds for all $x, y, z, w \in X$.

**Remark 2.12.** The condition that equation (2.1) holds for all $x, y, z, w \in X$ is equivalent to the condition that

$$d(x, y) + d(z, w) \leq \max\{d(x, z) + d(y, w), d(x, w) + d(y, z)\} + 2\delta$$

holds for all $x, y, z, w \in X$.

We are interested in groups as metric spaces (via Cayley graphs as in definition 2.2). We want to say that a group is hyperbolic if its Cayley graph is Gromov hyperbolic, but the Cayley graph of a group is only defined up-to quasi-isometry, so we need that the property of being Gromov hyperbolic is preserved under quasi-isometries.

**Proposition 2.13.** *Let $X, Y$ be geodesic metric spaces. Suppose $X$ and $Y$ are quasi-isometric. If $X$ is $\delta$-hyperbolic then there exists some $\delta' \geq 0$ such that $Y$ is $\delta'$-hyperbolic.*

*Proof.* See [BH99, Theorem III.H.1.9]. □

Proposition 2.13 needs the metric spaces to be geodesic metric spaces, so we cannot apply this to the inclusion $(G, d_S) \hookrightarrow \mathrm{Cay}(G; S)$. However, every graph with its canonical metric is a geodesic metric space, which leads to the following corollary of proposition 2.13.

**Corollary 2.14.** *Let $G$ be a finitely generated group. Suppose $S_1, S_2 \subseteq G\backslash\{e\}$ are two finite generating sets of $G$. The graph $\mathrm{Cay}(G; S_1)$ is Gromov hyperbolic if and only if the graph $\mathrm{Cay}(G; S_2)$ is Gromov hyperbolic.*

Thus the following definition makes sense.

**Definition 2.15.** A finitely generated Group $G$ is *hyperbolic* if for one (and hence for all) finite generating sets $S \subseteq G\backslash\{e\}$ the Cayley graph $\mathrm{Cay}(G; S)$ is Gromov hyperbolic.

**Remark 2.16.** We could use definition 2.11 to define Gromov hyperbolicity for a group $G$ with generating set $S$, but then we would have to worry about whether or not this depended on the choice of generating set.

We can formulate hyperbolicity for a group in terms of actions of the group on Gromov hyperbolic metric spaces, using the Švarc-Milnor lemma (lemma 2.7).

**Proposition 2.17.** *A finitely generated group $G$ is hyperbolic if and only if there exists a Gromov hyperbolic graph on which $G$ acts properly, isometrically, and cocompactly.*

*Proof.* If a group is hyperbolic then its Cayley graph (with respect to any finite generating set) is Gromov hyperbolic, and $G$ acts on it properly, isometrically, and cocompactly.

Conversely, if there exists such a graph $\Gamma$ then by the Švarc-Milnor lemma (lemma 2.7) the group $G$ is quasi-isometric to $\Gamma$. Hence $\Gamma$ is quasi-isometric to the Cayley graph of $G$ (with respect to some finite generating set) by proposition 2.6. Then proposition 2.13 tells us that the Cayley graph of $G$ is Gromov hyperbolic. □

It is this formulation of a hyperbolic group that we will generalise to define a relatively hyperbolic group in section 2.4.

## 2.3 Fine graphs

Although proposition 2.17 only requires the graph to be Gromov hyperbolic, it follows from the existence of a proper, cocompact group action (by isometries) that the graph is locally finite.

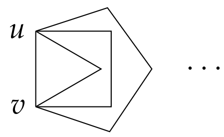
Being locally finite is too strong of a condition for our purposes. We need to use a weaker notion for defining a relatively hyperbolic group.

**Definition 2.18.** A graph $\Gamma$ is *fine* if for all $n \in \mathbb{N}$ and all edges $e \in E$ there are only finitely many circuits containing $e$ and whose length is $\leq n$.

A graph $\Gamma$ is *uniformly fine* if for all $n \in \mathbb{N}$ there is a constant $K \in \mathbb{N}$ such that for any edge $e \in E$ there are at most $K$ circuits containing $e$ and whose length is $\leq n$.

A locally finite graph is automatically fine, but a fine graph is not necessarily locally finite, as in the example below.

Construct a graph by starting with two vertices $u, v$ and then for every positive integer $n \in \mathbb{N}_{>0}$ add a path of length $n$ from $u$ to $v$.




This graph is not locally finite, but it is fine.

I have given the definition of a fine graph that I have seen used the most often. However the definition of a fine graph in [MY06] is a uniformly fine graph in our terminology. When this difference is important in the statement or proof of something then it will be made clear what changes have to be made. The difference will not pose a problem for our main results since we will only consider fine graphs with a group acting cocompactly on them, and such graphs are automatically uniformly fine, due to the following lemma.

**Lemma 2.19.** *Let $G$ be a group that acts on a graph $\Gamma$ such that there are only finitely many orbits of edges. If $\Gamma$ is fine then $\Gamma$ is uniformly fine.*

*Proof.* Fix $n \in \mathbb{N}$. We need to bound the number of circuits of length $\leq n$ through an arbitrary edge $e$ of $\Gamma$.

Pick a representative $e_i$ of every $G$-orbit of edges of $\Gamma$. Let $k_i$ be the number of circuits through $e_i$ of length $\leq n$, which is finite since the graph is fine. Now set $K = \max_i k_i$, which is finite since there are only finitely many $G$-orbits of edges.

Given an arbitrary edge $e$ there is an $e_i$ and $g \in G$ such that $ge = e_i$. Then for every circuit $c$ of length $\leq n$ containing $e$, the loop $gc$ is a circuit of length $\leq n$ containing $e_i$. Thus the number of circuits of length $\leq n$ containing $e$ is $\leq k_i \leq K$. Therefore the graph is uniformly fine. $\square$

There are alternative formulations of the fineness condition, which give alternative ways to think of a fine graph and these will be useful for later proofs.

To state these alternative formulations we need some more terminology. An *arc* in a graph is a path that hits every vertex at most once, i.e. a path that never returns to a vertex that it has already visited. An *inner vertex of an arc* is a vertex hit by the arc but is not an end-point of the arc. Two arcs are *independent* if the they do not have any inner vertices in common. (So we allow independent arcs to share end-points.)

**Lemma 2.20** ([Bow97, Proposition 2.1])**.** *Let* $\Gamma$ *be a graph. The following are equivalent;*

$(i)$ $\Gamma$ *is fine.*

$(ii)$ *For all* $x, x' \in V$ *and* $n \in \mathbb{N}$ *the set of arcs of length* $n$ *connecting* $x$ *to* $x'$ *is finite.*

$(iii)$ *For all* $x, x' \in V$ *and* $n \in \mathbb{N}$ *there does not exist an infinite collection of pairwise independent arcs of length* $n$ *connecting* $x$ *to* $x'$.

In particular, from lemma 2.20$(ii)$ by setting $n = d(x, x')$ we immediately obtain the following corollary;

**Corollary 2.21.** *If* $\Gamma$ *is a fine graph then for any pair* $x, x'$ *of vertices of* $\Gamma$ *there are only finitely many geodesics from* $x$ *to* $x'$.

## 2.4 Relatively hyperbolic groups

As with hyperbolic groups, there are many different definitions of a relatively hyperbolic group. Each definition has its own advantages and disadvantages. The first definition we give comes from [Bow97] (cf. proposition 2.17).

**Definition 2.22.** A finitely generated group $G$ is *hyperbolic relative* to a set $\mathcal{P}$ of infinite subgroups of $G$ if there exists a graph $\Gamma$ with a $G$-action such that

- the graph $\Gamma$ is connected, Gromov-hyperbolic, and fine,
- there are only finitely many $G$-orbits of edges of $\Gamma$,
- the stabiliser of any edge is finite,

- the stabiliser of any vertex is finite or a conjugate of an element of $\mathcal{P}$,
- for all $P \in \mathcal{P}$ and $g \in G$ there is precisely one vertex whose stabiliser is $gPg^{-1}$,
- the set $\mathcal{P}$ is finite,
- every element of $\mathcal{P}$ is finitely generated.

We call the elements of $\mathcal{P}$ (and their conjugates) the *peripheral subgroups*.

This definition is combinatoric and neatly states all the properties required of the graph $\Gamma$. However, it doesn't give any clue as to how to find such a graph, nor is it easy to show that a group is *not* relatively hyperbolic to a set of subgroups. There is an alternative definition that is more explicit, which comes from [GM08].

**Definition 2.23.** Let $G$ be a finitely generated group and let $\mathcal{P}$ be a finite set of finitely generated subgroups of $G$. Let $S$ be a finite generating set of $G$ that is compatible with $\mathcal{P}$, i.e. for all $P \in \mathcal{P}$ the set $S \cap P$ generates $P$. Let $\mathrm{Cay}(G, S)$ be the Cayley graph of $G$ with respect to $S$. For every coset $gP$ of every peripheral subgroup $P \in \mathcal{P}$ add a new vertex $v_{gP}$ to $\mathrm{Cay}(G; S)$ and join every element of $gP$ to this new vertex with an edge. Call the resulting graph the *coned-off Cayley graph of $G$ with respect to $S$ and $\mathcal{P}$*. Denote this graph by $\widehat{\mathrm{Cay}}(G, S; \mathcal{P})$.

Then we say that $G$ is *hyperbolic relative to $\mathcal{P}$* if the coned-off Cayley graph $\widehat{\mathrm{Cay}}(G, S; \mathcal{P})$ is Gromov hyperbolic and fine.

**Remark 2.24.** Farb introduced the coned-off Cayley graph in [Far98], in which a group $G$ was defined to be hyperbolic relative to a finite set $\mathcal{P}$ of subgroups if and only if the coned-off Cayley graph is Gromov hyperbolic. This is a strictly weaker condition.

For example consider the group $\mathbb{Z}^2 = \langle a, b \mid [a, b] \rangle$ relative to the subgroup $\langle a \rangle$. The coned-off Cayley graph is Gromov hyperbolic but is not fine.

However, Farb often uses a property he calls *bounded coset penetration*, and the coned-off Cayley graph is Gromov hyperbolic and fine if and only if it is Gromov hyperbolic and satisfies the bounded coset penetration property (see proposition 1 and lemma 5 in the appendix of [Dah03b]).

The action of the group $G$ on the Cayley graph $\mathrm{Cay}(G, S)$ extends to an action of $G$ on the coned-off Cayley graph $\widehat{\mathrm{Cay}}(G, S; \mathcal{P})$, specifically $G$ acts by left multiplication on non-cone vertices and $h \cdot v_{gP} := v_{hgP}$. Then the action of $G$ on the coned-off Cayley graph satisfies all the conditions from definition 2.22. Thus if $G$ is hyperbolic relative to $\mathcal{P}$ in the sense of definition 2.23 then $G$ is also hyperbolic relative to $\mathcal{P}$ in the sense of definition 2.22.

The converse is not so easy to show. In [Bow97] Bowditch proved that definition 2.22 is equivalent to the original definition given by Gromov in [Gro87]. Szczepański showed in [Szc98] that the Gromov definition implies the coned-off Cayley graph is Gromov hyperbolic, and then Dahmani in the appendix of his thesis ([Dah03b]) proved that the coned-off Cayley graph is also fine.

## 2.5 The boundary of a fine, Gromov hyperbolic graph

The ultimate aim of this thesis is to define a flow space, and this will be done using geodesics. We will often refer to the end-points of a geodesic but we want to allow geodesics of infinite length, so we need some concept of a boundary.

**Definition 2.25.** Let $\Gamma$ be a fine, $\delta$-hyperbolic graph. A *geodesic ray* is a geodesic $\alpha\colon [0,\infty) \to \Gamma$. Two geodesic rays $\alpha, \alpha'\colon [0,\infty) \to \Gamma$ are *asymptotic* if there is a constant $C \geq 0$ such that $d\left(\alpha(t), \alpha'(t)\right) \leq C$ for all $t \in [0,\infty)$.

Given a fixed base-point $x_0 \in \Gamma$, we can define the *(visual) boundary of* $\Gamma$ to be the set of geodesic rays starting at $x_0$ modulo the equivalence relation of being asymptotic, i.e.

$$\partial\Gamma := \{\alpha\colon [0,\infty) \to \Gamma \mid \alpha \text{ is a geodesic ray with } \alpha(0) = x_0\} / \sim$$

where $\alpha \sim \alpha'$ if and only if they are asymptotic. We write $\alpha(\infty)$ for the point of $\partial\Gamma$ represented by the geodesic ray $\alpha$.

Set $\overline{\Gamma} = \Gamma \cup \partial\Gamma$. We want a topology on $\overline{\Gamma}$, but it should not change the topology on $\Gamma$, i.e. we want the inclusion $\Gamma \hookrightarrow \overline{\Gamma}$ to be an embedding. Moreover, if we insist that $\Gamma \subseteq \overline{\Gamma}$ is an open subset then to define a topology on $\overline{\Gamma}$ we only need to define neighbourhoods of points on the boundary. For this, note that every point of $\Gamma$ can be represented by a geodesic (of finite length) starting at $x_0$. Thus we define a *generalised geodesic ray* to be a map $\alpha\colon [0,\infty) \to \Gamma$ that is either a geodesic ray or there exists some $l \geq 0$ such that $\alpha|_{[0,l]}$ is a geodesic and $\alpha|_{[l,\infty)}$ is constant. Denote by $\alpha(\infty)$ the end-point in $\overline{\Gamma}$ of $\alpha$.

Then given a fixed constant $r > \delta$ and a point $\xi \in \partial\Gamma$ represented by a geodesic ray $\alpha_0$ with $\alpha_0(0) = x_0$, we can define a base of neighbourhoods of $\xi$ to be sets of the form

$$U_{x_0}(\alpha_0, n, r) := \left\{ \alpha(\infty) \,\middle|\, \begin{array}{c} \alpha \text{ is a generalised geodesic with } \alpha(0) = x_0 \\ \text{and } d(\alpha_0(n), \alpha(n)) \leq r \end{array} \right\}$$

for $n \in \mathbb{N}$. We need $r > \delta$ so that we can always find a neighbourhood in the intersection of two such sets. It doesn't matter if $r > \delta$ is fixed or not, since for any $n, r, r'$ hyperbolicity gives

$$U_{x_0}(\alpha_0, n + r', r') \subseteq U_{x_0}(\alpha_0, n, \delta) \subseteq U_{x_0}(\alpha_0, n, r).$$

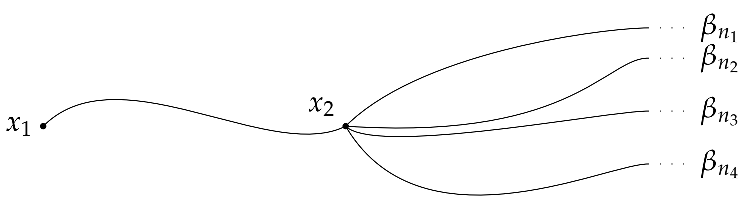
The boundary of a $\delta$-hyperbolic geodesic metric space $X$ has been oft studied, but many facts about the boundary use the assumption that $X$ is a proper metric space, see for example [BH99, Section III.H.3]. A fine graph is not necessarily a proper metric space. However, properness is only used to construct the limit of a sequence of geodesic rays, by invoking Arzelà-Ascoli. So if we have an appropriate method for constructing geodesic rays then facts about proper Gromov hyperbolic spaces will also hold for fine graphs.

The corollary of Arzelà-Ascoli we need is the following.

**Lemma 2.26.** *Let $\Gamma$ be a fine, $\delta$-hyperbolic graph. Given any point $x_1 \in \Gamma$ and any sequence $(y_n)_{n\in\mathbb{N}}$ of points in $\Gamma$, let $\beta_n$ be a geodesic from $x_1$ to $y_n$. If $(y_n|y_m)_{x_1} \to \infty$ as $n, m \to \infty$ then there is a subsequence of $(\beta_n)_{n\in\mathbb{N}}$ that converges to a geodesic ray $\alpha_1$.*

*Proof.* We construct the geodesic ray $\alpha_1$ one edge at a time, showing that at every step there will be a next edge that is contained in infinitely many of the $\beta_n$. Formally we use a kind of inductive proof by contradiction.

Suppose $(\beta_{n_k})_{k\in\mathbb{N}}$ is a subsequence of $(\beta_n)_{n\in\mathbb{N}}$ such that the first $l$ edges of all the $\beta_{n_k}$ coincide but that the $(l+1)$-th edges are distinct, i.e. for all $k \neq k'$ the $(l+1)$-th edge of $\beta_{n_k}$ is not the same as the $(l+1)$-th edge of $\beta_{n_{k'}}$. Let $x_2$ be the vertex where all the $\beta_{n_k}$ split.




We may assume that all the geodesics $\beta_{n_k}$ start at $x_2$.

Since $(y_n|y_m)_{x_2} \geq (y_n|y_m)_{x_1} - d(x_1, x_2) \to \infty$ we can pick $N \in \mathbb{N}$ large enough such that for all $k, l \geq N$ we have $(y_{n_k}|y_{n_l})_{x_2} \geq [\delta + 1] =: d$, where $[\delta + 1]$ denotes the integer part of $\delta + 1$. For $k \geq N$ set $z_k = \beta_{n_k}(d)$. Then $d(z_N, z_k) \leq \delta$ by hyperbolicity and the choice of $N$. Let $\gamma_k$ be a geodesic from $z_k$ to $z_N$. The concatenation of $\gamma_k$ with $\beta_{n_k}|_{[0,d]}$ is a path from $x_2$ to $z_N$ whose length is $\leq d + \delta$, from which we get an arc $a_k$ from $x_2$ to $z_N$ whose length is $\leq d + \delta$.

Since $d > \delta$ we know that each geodesic $\gamma_k$ cannot pass through $x_2$. Therefore the first edge of $a_k$ is the same as the first edge of $\beta_{n_k}$. Hence we have a sequence $(a_k)_{k\in\mathbb{N}}$ of distinct arcs from $x_2$ to $z_N$, all of which have length $\leq d+\delta$. This contradicts the fineness of the graph, using lemma 2.20.

This means there cannot exist such a subsequence $(\beta_{n_k})_{k\in\mathbb{N}}$ that coincide on the first $l$ edges but all take a different $(l+1)$-th edge.

So we can pass to a subsequence of $(\beta_n)_{n\in\mathbb{N}}$ consisting of geodesics that all agree on the first edge. Then we can pass to a further subsequence of geodesics that agree on the first two edges. Iterating this gives a sequence of subsequences $((\beta_{k,n})_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ such that the geodesics $\{\beta_{k,n} \mid n \in \mathbb{N}\}$ coincide on the first $k$ edges. Then the diagonal sequence $(\beta_{n,n})_{n\in\mathbb{N}}$ converges to a geodesic ray $\alpha_1$. □

Now that we have this lemma, we can use arguments as in the proper case to deduce facts about the boundary of a fine graph. In particular, the boundary $\partial\Gamma$ and the topology on $\overline{\Gamma}$ are independent of the choice of base-point. (See [BH99, Proposition III.H 3.7(1) on page 429].) Note however, that the space $\overline{\Gamma}$ need not be compact, unlike for a proper Gromov hyperbolic metric space.

The Cayley graph of a group is only defined up to quasi-isometry; using different generating sets of the group yields quasi-isometric graphs. Hence it is important to look at what a quasi-isometry does to the boundary. For this, it is helpful to give an alternative definition of the boundary in terms of quasi-geodesic rays. But first we recall the definition of the Hausdorff distance; Let $X$ be a metric space. The *Hausdorff distance* of two subsets $A, B \subseteq X$ is given by

$$d_H(A,B) = \max\left\{\sup_{a\in A}\inf_{b\in B} d_X(a,b)\,,\ \sup_{b\in B}\inf_{a\in A} d_X(a,b)\right\}.$$

**Definition 2.27.** Let $\Gamma$ be a $\delta$-hyperbolic graph. A *quasi-geodesic* in $\Gamma$ is a quasi-isometric embedding of the form $\alpha\colon [0,l] \to \Gamma$. A *quasi-geodesic ray* in $\Gamma$ is a quasi-isometric embedding of the form $\alpha\colon [0,\infty) \to \Gamma$.

Two quasi-geodesic rays $\alpha, \alpha'\colon [0,\infty) \to \Gamma$ are *asymptotic* if the Hausdorff distance between their images is finite.

Then given any choice of base-point $x_0 \in \Gamma$ we can define the boundary $\partial_q\Gamma$ of $\Gamma$ to be the set of quasi-geodesic rays starting at $x_0$ up to asymptoty.

**Proposition 2.28.** *If $\Gamma$ is a fine $\delta$-hyperbolic graph with base-point $x_0$, then there is a natural bijection $\partial\Gamma \to \partial_q\Gamma$.*

*Proof.* Any geodesic ray is a quasi-geodesic ray. Two geodesic rays are asymptotic as geodesic rays if and only if they are asymptotic as quasi-geodesic rays, so there is a natural map $\partial\Gamma \to \partial_q\Gamma$, which is injective.

For surjectivity; Given a quasi-geodesic ray $\alpha$ we can use lemma 2.26 to construct a geodesic ray $\beta$ that is asymptotic to $\alpha$. □

Using quasi-geodesic rays we can show that a quasi-isometry $f\colon \Gamma \to \Gamma'$ induces a homeomorphism $f_\partial\colon \partial\Gamma \to \partial\Gamma$. (See [BH99, Theorem III.H 3.9].)

The definition of the boundary in terms of geodesic rays may be intuitive but it is not always the most convenient to work with. There is an alternative formulation in terms of sequences of points and the Gromov product.

**Definition 2.29.** Let $\Gamma$ be a fine, $\delta$-hyperbolic graph and let $x_0$ be a fixed base-point. Define the *boundary* of $\Gamma$ to be

$$\partial_s\Gamma := \{(x_n)_{n\in\mathbb{N}} \mid (x_n|x_m)_{x_0} \to \infty \text{ as } n, m \to \infty\}/\sim$$

where $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$ if and only if $(x_n|y_n)_{x_0} \to \infty$ as $n \to \infty$. To see that $\sim$ is indeed an equivalence relation on the set of such sequences we need to use proposition 2.10. If a sequence $(x_n)_{n\in\mathbb{N}}$ in $\Gamma$ represents a point $\xi \in \partial_s\Gamma$ then we write $\xi = \lim x_n$ and $x_n \to \xi$.

This does not depend on the choice of base-point $x_0$ since for any other choice $y_0$

$$\begin{aligned}(x_n|x_m)_{x_0} &= \frac{1}{2}\big(d(x_n, x_0) + d(x_m, x_0) - d(x_n, x_m)\big)\\ &\leq \frac{1}{2}\big(d(x_n, y_0) + d(y_0, x_0) + d(x_m, y_0) + d(y_0, x_0) - d(x_m, x_n)\big)\\ &= (x_n|x_m)_{y_0} + d(x_0, y_0)\end{aligned}$$

and similarly $(x_n|x_m)_{y_0} \leq (x_n|x_m)_{x_0} + d(x_0, y_0)$.

**Remark 2.30.** Given any geodesic ray $\alpha\colon [0,\infty) \to \Gamma$ with $\alpha(0) = x_0$ we can set $x_n = \alpha(n)$ to get a point in $\partial_s\Gamma$. Asymptotic rays yield the same point in $\partial_s\Gamma$ so we have a map $\partial\Gamma \to \partial_s\Gamma$.

Conversely, given any sequence of points $(x_n)_{n\in\mathbb{N}}$ representing a point in $\partial_s\Gamma$ we can use lemma 2.26 to construct a geodesic ray $\alpha$ such that

$$\lim x_n = \lim \alpha(n) \in \partial_s\Gamma$$

and so we have an inverse $\partial_s\Gamma \to \partial\Gamma$.

Therefore we may choose to represent a point in the boundary by a geodesic ray or a sequence of points whose Gromov product tends to infinity.

The concept of a Gromov product can be extended to the boundary using the $\partial_s\Gamma$ definition.

**Definition 2.31.** Let $\Gamma$ be a fine, Gromov hyperbolic graph. For any two points $x, y \in \Gamma \cup \partial_s\Gamma$ and any point $z \in \Gamma$ define the *Gromov product* of $x, y$ at $z$ to be

$$(x|y)_z := \sup \liminf_{n\to\infty} (x_n|y_n)_z$$

where the supremum is taken over all sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ in $\Gamma$ such that $x_n \to x$ and $y_n \to y$.

**Remark 2.32.** Since the Gromov product is continuous on $\Gamma$, if $x$ lies in $\Gamma$ then we may take the sequence $(x_n)_{n\in\mathbb{N}}$ to be constantly $x$, i.e.

$$(x|y)_z = \sup \liminf_{n\to\infty} (x|y_n)_z$$

where the supremum is taken over all sequences $(y_n)_{n\in\mathbb{N}}$ such that $y_n \to y$ as $n \to \infty$. In particular, if $x$ and $y$ both lie in $\Gamma$ then this supremum just gives back the original Gromov product.

Furthermore, for arbitrary $x, y \in \overline{\Gamma}$ and $z \in \Gamma$ we can use a diagonal argument to find two sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ converging to $x$ and $y$ respectively such that

$$(x|y)_z = \lim_{n\to\infty} (x_n|y_n)_z.$$

**Proposition 2.33.** *Let $\Gamma$ is a fine, $\delta$-hyperbolic graph. For any $x, y, z \in \Gamma \cup \partial_s\Gamma$ and $w \in \Gamma$ the Gromov product $(x|y)_w$ has the following properties.*

*(i)* $(x|y)_w = \infty$ *if and only if* $x = y \in \partial_s\Gamma$;

*(ii)* $(x|z)_w \geq \min\{(x|y)_w, (y|z)_w\} - 2\delta$;

*(iii) Suppose* $x, y \in \partial_s\Gamma$. *If* $(x_n)_{n\in\mathbb{N}}$ *and* $(y_n)_{n\in\mathbb{N}}$ *are sequences in* $\Gamma$ *that tend to* $x$ *and* $y$ *respectively then*

$$(x|y)_w - 2\delta \leq \liminf_{n\to\infty} (x_n|y_n)_w \leq (x|y)_w.$$

*Proof.* Let $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}, (y'_n)_{n\in\mathbb{N}}, (z_n)_{n\in\mathbb{N}}$ be sequences in $\Gamma$ that converge to $x, x, y, y$ and $z$ respectively.

*(i)* If $x \in \partial_s\Gamma$ then $(x_n|x'_n)_w \to \infty$ by definition, and so $(x|x)_w = \infty$. So we just need to show the converse.

If $x$ or $y$ are in $\Gamma$ then the Gromov product is finite, since it is bounded by the distance to $w$. So assume $x, y \in \partial_s\Gamma$. If $x \neq y$ then by definition of the equivalence relation there is some constant $K$ such that $(x_n|y_n)_w \leq K$ for infinitely many $n$. Hence $\liminf_{n\to\infty}(x_n|y_n)_w \leq K$. Then by proposition 2.10

$$\begin{aligned}\liminf_{n\to\infty} (x_n|y_n)_w &\geq \liminf_{n\to\infty} \left(\min\{(x_n|x'_n)_w, (x'_n|y'_n)_w, (y'_n|y_n)_w\} - 2\delta\right) \\ &= \liminf_{n\to\infty} (x'_n|y'_n)_w - 2\delta \qquad (2.2)\end{aligned}$$

since $(x_n|x'_n)_w$ and $(y_n|y'_n)_w$ both tend to $\infty$. Therefore $(x|y)_w \leq K + 2\delta$ which is finite.

*(ii)* The argument again uses proposition 2.10. If $y \in \Gamma$ then we may assume $y_n = y = y'_n$ and use inequality (2.1) for $x_n, y, z_n$. If $y \in \partial_s\Gamma$ then $(y_n|y'_n)_w \to \infty$ and

$$\begin{aligned}\liminf_{n\to\infty}(x_n|z_n)_w &\geq \liminf_{n\to\infty}\left(\min\{(x_n|y_n)_w, (y_n|y'_n)_w, (y'_n|z_n)_w\} - 2\delta\right)\\ &= \min\left\{\liminf_{n\to\infty}(x_n|y_n)_w, \liminf_{n\to\infty}(y'_n|z_n)_w\right\} - 2\delta.\end{aligned}$$

This holds for arbitrary such sequences so it must also hold when taking the supremum.

*(iii)* The upper bound is immediate since $(x|y)_w$ is defined to be the supremum over all such values. The lower bound follows from inequality (2.2). □

**Remark 2.34.** We can use the (extended) Gromov product to define a topology on $\Gamma \cup \partial_s\Gamma$, by saying a base for the neighbourhoods of a point $x \in \Gamma$ is given by metric balls $B_r(x)$, and a base for the neighbourhoods of a point $\xi \in \partial_s\Gamma$ is given by the sets

$$W_R(\xi) := \{x \in \Gamma \cup \partial_s\Gamma \mid (x|\xi)_{x_0} \geq R\}. \tag{2.3}$$

with $R \geq 0$.

The advantage of this method is that it allows us to define the boundary $\partial_s X$ of an arbitrary Gromov hyperbolic metric space $X$, where we use definition 2.11 to define hyperbolicity for a metric space when it is not a geodesic metric space. We can still extend the Gromov product as in definition 2.31, and the statements in proposition 2.33 still hold. Hence we can define a topology on $\overline{X} := X \cup \partial_s X$.

**Proposition 2.35.** *Let $\Gamma$ be a fine, Gromov hyperbolic graph. The natural map $\Gamma \cup \partial\Gamma \to \Gamma \cup \partial_s\Gamma$, using the identity on $\Gamma$ and the natural bijection $\partial\Gamma \to \partial_s\Gamma$ from remark 2.30, is a homeomorphism.*

*Proof.* It is automatically a bijection, so we only need to show it is continuous and its inverse is continuous. The topology on $\Gamma$ is unchanged, and it is an open subset, hence we only need to look at neighbourhoods of points at infinity.

Fix a base-point $x_0 \in \Gamma$ and a point $\xi \in \partial\Gamma$. By definition there exists a geodesic ray $\alpha_0 \colon [0,\infty) \to \Gamma$ with $\alpha_0(0) = x_0$ and $\alpha_0(\infty) = \xi$. Then a base of neighbourhoods for $\xi$ in $\Gamma \cup \partial\Gamma$ is comprised of sets of the form

$$U_{x_0}(\alpha_0, n, r) := \left\{\alpha(\infty) \,\middle|\, \begin{array}{c}\alpha \text{ is a generalised geodesic with } \alpha(0) = x_0\\ \text{and } d(\alpha_0(n), \alpha(n)) \leq r\end{array}\right\}$$

for $n \in \mathbb{N}$, where $r > \delta$ is some fixed constant.

Furthermore, a base of neighbourhoods for $\xi$ in $\Gamma \cup \partial_s \Gamma$ is comprised of sets of the form

$$W_R(\xi) := \{x \in \Gamma \cup \partial_s \Gamma \mid (x|\xi)_{x_0} \geq R\}$$

for $R > 0$. We need to show that for every $n \in \mathbb{N}$ we can find some $R' > 0$ such that $W_{R'}(\xi) \subseteq U_{x_0}(\alpha_0, n, r)$, and conversely for every $R > 0$ there is some $n' \in \mathbb{N}$ such that $U_{x_0}(\alpha_0, n', r) \subseteq W_R(\xi)$.

Given $n \in \mathbb{N}$ take $R' > n + 2\delta$. For any $x \in W_{R'}(\xi)$ we can find a generalised geodesic ray $\alpha$ from $x_0$ to $x$. Proposition 2.33(*iii*) tells us that

$$n < (x|\xi)_{x_0} - 2\delta \leq \liminf_{m\to\infty} (\alpha(m)|\alpha_0(m))_{x_0} \leq (x|\xi)_{x_0}$$

and in particular, for some large $m \in \mathbb{N}$ we have $n \leq (\alpha(m)|\alpha_0(m))_{x_0}$. Then hyperbolicity gives $d(\alpha(n), \alpha_0(n)) \leq \delta$, and therefore $x \in U_{x_0}(\alpha_0, n, r)$. Since $x$ was arbitrary, we conclude $W_{R'}(\xi) \subseteq U_{x_0}(\alpha_0, n, r)$.

Conversely, suppose we are given $R > 0$, so that we need to find $n' \in \mathbb{N}$ such that $U_{x_0}(\alpha_0, n', r) \subseteq W_R(\xi)$. If $x \in U_{x_0}(\alpha_0, n', r)$ then there is a generalised geodesic ray $\alpha$ from $x_0$ to $x$ such that $d(\alpha(n'), \alpha_0(n')) \leq r$. Hence

$$\begin{aligned}(\alpha(n')|\alpha_0(n'))_{x_0} &= \frac{1}{2}\left(d(\alpha(n'), x_0) + d(\alpha_0(n'), x_0) - d(\alpha(n'), \alpha_0(n'))\right) \\ &\geq n' - \frac{1}{2}r. \end{aligned} \tag{2.4}$$

Now we aim to show that $(x|\xi)_{x_0} \geq (\alpha(n')|\alpha_0(n'))_{x_0}$. To see this, note that the Gromov product doesn't decrease as we travel along the geodesic rays $\alpha, \alpha_0$, i.e. for all $m \geq n'$;

$$\begin{aligned}(\alpha(m)|\alpha_0(m))_{x_0} &= \frac{1}{2}\left(d(\alpha(m), x_0) + d(\alpha_0(m), x_0) - d(\alpha(m), \alpha_0(m))\right) \\ &\geq m - \frac{1}{2}\left(d(\alpha(m), \alpha(n')) + d(\alpha(n'), \alpha_0(n')) + d(\alpha_0(n'), \alpha_0(m))\right) \\ &= n' - \frac{1}{2}d(\alpha(n'), \alpha_0(n')) \\ &= (\alpha(n')|\alpha_0(n'))_{x_0}.\end{aligned}$$

So we can use proposition 2.33(*iii*) to get

$$(\alpha(n')|\alpha_0(n'))_{x_0} \leq \liminf_{m\to\infty} (\alpha(m)|\alpha_0(m))_{x_0} \leq (x|\xi)_{x_0}.$$

Combining this with inequality (2.4) gives $(x|\xi)_{x_0} \geq n' - \frac{1}{2}r$. Therefore, if we take $n' \geq R + \frac{1}{2}r$ then $U_{x_0}(\alpha_0, n', r) \subseteq W_R(\xi)$.

This proves that we get the same topology around $\xi$ regardless of which base we use, and so the map $\Gamma \cup \partial\Gamma \to \Gamma \cup \partial_s\Gamma$ is a homeomorphism. □

This allows us to talk about the space $\overline{\Gamma}$ without needing to specify if we are using $\partial\Gamma$ or $\partial_s\Gamma$ to define the boundary and the topology.

**Remark 2.36.** If $X$ is a proper geodesic Gromov hyperbolic metric space then the space $\overline{X}$ is compact. However, it is not necessarily true that $\overline{\Gamma}$ is compact for an arbitrary fine, $\delta$-hyperbolic graph $\Gamma$. There is a way to define $\overline{\Gamma}$ to make it compact (see [Bow97, Proposition 8.6], where as sets his $\Delta(K)$ is equivalent to our $\overline{\Gamma}$) but this changes the topology on $\Gamma$, which we want to avoid doing here.

## 2.6 Metrising $\overline{X}$

In section 2.5 we defined a boundary of a fine, Gromov hyperbolic graph $\Gamma$, and used this to define a space $\overline{\Gamma}$. It is known that if $\Gamma$ is a locally finite, Gromov hyperbolic graph then the space $\overline{\Gamma}$ is metrisable. (See for example, [BH99, Exercise 3.18(4)].)

We want to use a similar argument to show that the space $\overline{\Gamma}$ is still metrisable, even under the weaker assumptions that the graph $\Gamma$ is fine and Gromov hyperbolic. We can actually do this more generally for any Gromov hyperbolic metric space $X$, where we use definition 2.11 to define hyperbolicity when the space is not a geodesic metric space, and remark 2.34 to define the topological space $\overline{X}$.

To show $\overline{X}$ is metrisable we use uniform structures;

**Definition 2.37.** Let $S$ be a set. A family $\mathcal{B}$ of subsets of $S \times S$ is a *base of a uniform structure* on $S$ if

$(i)$ Every $B \in \mathcal{B}$ is symmetric;

$(ii)$ The diagonal $\Delta(S)$ is contained in every $B \in \mathcal{B}$;

$(iii)$ For any $B_1, B_2 \in \mathcal{B}$ there exists a $B \in \mathcal{B}$ such that $B \subseteq B_1 \cap B_2$;

$(iv)$ For all $B \in \mathcal{B}$ there exists a $B' \in \mathcal{B}$ such that for all $x, y, z \in S$

$$(x, y), (y, z) \in B' \Rightarrow (x, z) \in B.$$

Moreover, a uniform structure is *separated* if for all $x \neq y \in S$ there exists some $B \in \mathcal{B}$ such that $(x, y) \notin B$.

A metric on a set $Y$ determines a separated uniform structure whose base consists of elements of the form $B_r := \{(x, y) \mid d_Y(x, y) < r\}$.

Conversely, a base $\mathcal{B}$ of a uniform structure on a set $S$ induces a topology on $S$ where for any point $x \in S$ the sets $B[x] := \{y \in S \mid (x,y) \in B\}$ form a base for the neighbourhoods of $x$ (see [Kel55, Theorem 6.5]). We will refer to the set $S$ with this induced topology as a *uniform space*. We can use the following theorem to say when a uniform space is metrisable.

**Theorem 2.38.** *A uniform space $S$ is metrisable if and only if the uniform structure is separated and has a countable base.*

*Proof.* See [Kel55, Theorem 6.13], and note that a uniform space is Hausdorff if and only if the uniform structure is separated. □

So to prove a topological space is metrisable, it suffices to find a countable base for a separated uniform structure that induces that topology on the set. This is how we will show that $\overline{X}$ is metrisable when $X$ is a Gromov hyperbolic metric space.

The topology induced on $X$ by the uniform structure should coincide with the original topology induced by the metric $d_X$, so for any $\epsilon \in \mathbb{Q}_+$ set $U_\epsilon = \{(x,y) \in X \mid d_X(x,y) < \epsilon\}$. For neighbourhoods at infinity, fix a base-pt $x_0 \in X$ and consider the sets $W_r := \{(x,y) \in \overline{X} \mid (x|y)_{x_0} \geq r\}$ with $r \in \mathbb{Q}_+$. Then set $B_{\epsilon,r} = U_\epsilon \cup W_r$ and $\mathcal{B} = \{B_{\epsilon,r} \mid \epsilon, r \in \mathbb{Q}_+\}$.

The goal is to show that $\mathcal{B}$ is the base of a separated uniform structure on $\overline{X}$ and that the topology induced by $\mathcal{B}$ coincides with the topology given in remark 2.34.

**Proposition 2.39.** *Let $X$ be a $\delta$-hyperbolic metric space. The family $\mathcal{B}$ is a base of a separated uniform structure on $\overline{X}$.*

*Proof.* We need to show that the family $\mathcal{B}$ satisfies properties $(i)$-$(iv)$ from definition 2.37, and that $\mathcal{B}$ is separated.

$(i)$ All of the sets $U_\epsilon$ and $W_r$ are symmetric, hence every $B_{\epsilon,r}$ is symmetric.

$(ii)$ For any $\epsilon > 0$ we know $\Delta(X) \subseteq U_\epsilon$, and by proposition 2.33$(i)$ we know $(x|x)_{x_0} = \infty$ for all $x \in \partial_s X$, thus $\Delta(\partial_s X) \subseteq W_r$ for any $r \in \mathbb{Q}_+$. Combining the two tells us that $\Delta(\overline{X}) \subseteq B_{\epsilon,r}$.

$(iii)$ Given $B_{\epsilon_1,r_1}$ and $B_{\epsilon_2,r_2}$ take $\epsilon = \min\{\epsilon_1, \epsilon_2\}$ and $r = \max\{r_1, r_2\}$. Then $B_{\epsilon,r} \subseteq B_{\epsilon_1,r_1} \cap B_{\epsilon_2,r_2}$.

$(iv)$ Given $B_{\epsilon,r} \in \mathcal{B}$ we know $B_{\epsilon,r} = U_\epsilon \cup W_r$. We need to find $\epsilon', r' \in \mathbb{Q}_+$ such that

$$(x,y),(y,z) \in B_{\epsilon',r'} \Rightarrow (x,z) \in B_{\epsilon,r}. \tag{2.5}$$

We have different cases depending on whether the points are in $X$ or $\partial_s X$. If $x,y,z \in X$ then we must have $(x,y),(y,z) \in U_{\epsilon'}$ and then $(x,z) \in U_{2\epsilon'}$ by the triangle inequality. If $y \in \partial_s X$ or $x,z \in \partial_s X$ then $(x,y),(y,z) \in W_{r'}$ and so $(x,z) \in W_{r'-2\delta}$ by proposition 2.33$(ii)$. This leaves the case $y \in X$ and exactly one of $x, z$ is in the boundary. Without loss of generality $z \in \partial_s X$. Hence $(x, y) \in U_{\epsilon'}$ and $(y, z) \in W_{r'}$ and applying proposition 2.33(*ii*) gives

$$\begin{aligned}(x|z)_{x_0} &\geq \min\{(x|y)_{x_0}, (y|z)_{x_0}\} - 2\delta \\ &\geq \min\{d(y, x_0) - d(x, y), (y|z)_{x_0}\} - 2\delta \\ &\geq \min\{(y|z)_{x_0} - \epsilon', (y|z)_{x_0}\} - 2\delta \\ &\geq r' - \epsilon' - 2\delta\end{aligned}$$

and so $(x, z) \in W_{r'-\epsilon'-2\delta}$. Therefore if we take $\epsilon' = \frac{1}{2}\epsilon$ and $r' = r + \frac{1}{2}\epsilon + 2\delta$ we know that (2.5) holds.

So we have proven that $\mathcal{B}$ is a base of a uniform structure on $\overline{X}$. It remains to prove that the base is separated.

Given $x, y \in \overline{X}$ with $x \neq y$, if $x, y \notin \partial_s X$ then $d_X(x, y)$ is finite and for $\epsilon < d_X(x, y)$ we know that $(x, y) \notin U_\epsilon$. Otherwise, at least one of the points $x, y$ lies in the boundary. Since $x \neq y$ we know $(x|y)_{x_0} < \infty$ (see proposition 2.33(*i*)) and so for $r > (x|y)_{x_0}$ we must have $(x, y) \notin W_r$. Therefore the base is separated. □

The family $\mathcal{B}$ is also countable, since we took the parameters $\epsilon, r$ to be in $\mathbb{Q}_+$, and thus the topology induced by this uniform structure is metrisable by theorem 2.38. However, we have to check that this induced topology is the topology we want on $\overline{X}$.

**Proposition 2.40.** *Let $X$ be a Gromov hyperbolic metric space. The topology on $\overline{X}$ induced by the separated uniform structure $\mathcal{B}$ coincides with the topology given in remark 2.34.*

*Proof.* Fix $x \in \overline{X}$ and consider neighbourhoods of the point $x$ in the two topologies. Let $\tau$ denote the topology given in remark 2.34 and let $\tau_{\mathcal{B}}$ denote the topology induced by the uniform structure.

We know that the sets $B_{\epsilon,r}[x] := \{y \in \overline{X} \mid (x, y) \in B_{\epsilon,r}\}$ define a base for the neighbourhoods of $x$ in $\tau_{\mathcal{B}}$.

If $x \in X$ then the sets $U_\epsilon[x]$ form a base for the neighbourhoods of $x$ in $\tau$, and for any $R > d_X(x, x_0)$ the set $W_R[x]$ is empty. Hence for any $\epsilon, r \in \mathbb{Q}_+$ we have $B_{\epsilon,R}[x] = U_\epsilon[x] \subseteq B_{\epsilon,r}[x]$ and so $\tau_{\mathcal{B}}$ coincides with $\tau$ on $X$.

It remains to consider neighbourhoods of $x \in \partial_s X$. By definition the sets $W_r[x]$ form a base of open neighbourhoods of $x$ in $\tau$, and we also know that the sets $B_{\epsilon,r}[x]$ form a base of neighbourhoods of $x$ in $\tau_{\mathcal{B}}$. But $U_\epsilon[x] = \emptyset$ for any $\epsilon \in \mathbb{Q}_+$ and so $W_r[x] = B_{\epsilon,r}[x]$.

Therefore the topologies $\tau_{\mathcal{B}}$ and $\tau$ coincide on all of $\overline{X}$. □

Combining the results of this section leads to the following theorem.

**Theorem 2.41.** *If $X$ is a Gromov hyperbolic metric space then the space $\overline{X}$ as defined in remark 2.34 is metrisable.*

*Proof.* The family $\mathcal{B} = \{U_\epsilon \cup W_r | \epsilon, r \in \mathbb{Q}_+\}$ is a countable base of a separated uniform strucutre, using proposition 2.39, and by proposition 2.40 the topology induced on $\overline{X}$ by $\mathcal{B}$ coincides with the topology defined in remark 2.34. Then theorem 2.38 says the space is metrisable. □

## 2.7 Angles in a graph

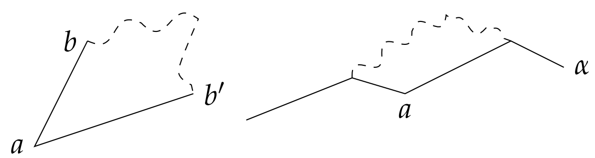
The concept of angles in a graph was used by Dahmani in his Ph.D. thesis ([Dah03b]) and expanded upon by Mineyev and Yaman in [MY06]. In this section we give definitions and state the results we will need from that article.

**Definition 2.42.** Let $\Gamma$ be a graph. Given two coincident edges $e = (a, b)$ and $e' = (a, b')$ that are both incident to a vertex $a$ in $\Gamma$ define the *angle between e and e′ at a* to be the distance between $b$ and $b'$ in the graph $\Gamma - a$, i.e.

$$\text{ang}_a(e, e') := d_{\Gamma - a}(b, b')$$

where we define $\text{ang}_a(e, e') = \infty$ if $b$ and $b'$ are in different components of $\Gamma - a$.

Given any geodesic $\alpha$ in $\Gamma$ and an internal vertex $a$ of $\alpha$ we define the *angle of $\alpha$ at $a$* to be the angle between the pair of edges in $\alpha$ that are incident to $a$, and we denote this by $\text{ang}_a(\alpha)$. Set $\text{maxang}(\alpha)$ to be the maximum of all the angles of $\alpha$.




In order to try to motivate why angles in a graph may be useful, recall the coned-off Cayley graph from definition 2.23, and here we use the degenerate case $\mathcal{P} = \{G\}$, where $G$ is a finitely generated group. If $S$ is a finite generating set of $G$ then the coned-off Cayley graph $\widehat{\text{Cay}}(G, S; \{G\})$ is formed by taking the standard Cayley graph $\text{Cay}(G, S)$ and adding a new vertex $v_G$ to $\text{Cay}(G, S)$ with edges joining it to every element of $G = V(\text{Cay}(G, S))$.

If $g$ and $g'$ are two non-adjacent elements in the Cayley graph $\text{Cay}(G, S)$ then in the coned-off Cayley graph the path given by the edges $(g, v_G)$ and $(v_G, g')$ is a geodesic (of length 2). So by attaching the cone-vertex we have shortened all distances to at most 2, which loses the information given by the word metric $d_S$ on $G$ with respect to the finite generating set $S$. However, this information is retained via angles, namely for any $g, g' \in G$

$$\mathrm{ang}_{v_G}\big((g, v_G), (v_G, g')\big) = d_S(g, g').$$

Thus we can think of angles in the coned-off Cayley graph $\widehat{\mathrm{Cay}}(G, S; \mathcal{P})$ as a way of extracting information about the geometry of the original Cayley graph $\mathrm{Cay}(G, S)$.

The following lemma shows what angles in a Gromov hyperbolic graph can tell us about the behaviour of geodesics and geodesic triangles.

**Lemma 2.43** ([MY06, Lemma 3]). *Let $\mathcal{G}$ be a graph with the path metric $d$ having $\delta$-thin triangles. There exists a constant $\kappa$ depending only on $\delta$ such that given vertices $a, b, c$, and geodesics $\alpha \in Geod[b, c]$, $\beta \in Geod[a, c]$ and $\gamma \in Geod[a, b]$ we have the following:*

*$(i)$ If $ang_z(\alpha) > \kappa$ for some $z \in \alpha$ distinct from $b$ and $c$, then $z \in \beta$, or $z \in \gamma$.*

*$(ii)$ If $z \in \alpha$, $d(c, z) < (a|b)_c$ and $ang_z(\alpha) > \kappa$, then $z \in \beta$.*

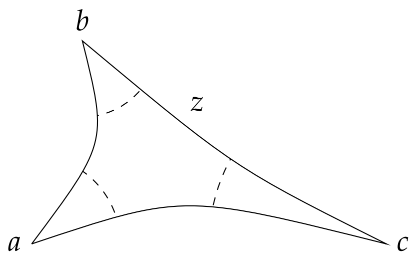
*$(iii)$ If $ang_c(\alpha, \beta) > \kappa$, then $c \in \gamma$.*

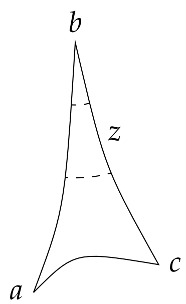
*$(iv)$ If $b = a$ i.e. $\gamma$ is a null geodesic, then $ang_c(\alpha, \beta) \leq \kappa$.*

*Sketch Proof.* We give a quick sketch of why the first part is true. The other parts can be proven analogously.

Let $z_\pm$ be the two vertices of $\alpha$ that are adjacent to $z$. If we assume that the point $z$ does not lie on either $\beta$ or $\gamma$ then we can use the triangle to find a path from $z_-$ to $z_+$ that does not pass through $z$, and this is how we can bound the angle of $\alpha$ at $z$.

There are two possibilities; maybe we have to use all sides of the triangle to create the path $z_-$ to $z_+$, or perhaps we only need to jump to one other side. The two pictures below indicate the idea behind how to construct a path $z_-$ to $z_+$ that bypasses $z$.




For details see [MY02, Lemma 3]. □

For working with angles it is helpful to consider a metric on edges defined in terms of angles, but first we show that the angle at a vertex $a \in V$ defines a metric on the edges incident to $a$, which follows from the next lemma.

**Lemma 2.44** ([MY06, Proposition 1]). *Given three coincident edges $e_1, e_2, e_3$ that are all incident to a vertex a in a graph $\mathcal{G}$ one has*

- $ang_a(e_1, e_2) = ang_a(e_2, e_1)$.
- $ang_a(e_1, e_3) \leq ang_a(e_1, e_2) + ang_a(e_2, e_3)$.

**Definition 2.45.** Let $\Gamma$ be a graph and let $e, e'$ be a pair of arbitrary edges of $\Gamma$. An *admissable sequence of edges from from e to e'* is a sequence $e_0, e_1, \ldots, e_k$ of edges of $\Gamma$ such that $e_0 = e$, $e_k = e'$, and for every $i = 1, \ldots, k$ the edges $e_{i-1}$ and $e_i$ are coincident.

Then the *snake distance $d^\zeta$ from e to e'* is defined by

$$d^\zeta(e, e') := \inf \sum_{i=1}^{k} \text{ang}(e_{i-1}, e_i)$$

where the infimum is taken over all admissable sequences of edges from $e$ to $e'$.

**Remark 2.46.** If $\Gamma$ is a connected graph then the function $d^\zeta \colon E \times E \to [0, \infty]$ is an extended metric on the set of edges of $\Gamma$. Recall that an extended metric is a function satisfying all the properties of being a metric except it is allowed to take the value $\infty$.

Furthermore, if $\Gamma$ is 2-vertex-connected then $d^\zeta \colon E \times E \to [0, \infty)$ is a metric on the set of edges, and we call this metric the *snake metric*.

This metric on $E$ is invariant under the action of $\text{Isom}(\Gamma)$ on $E$, meaning for any isometry $\psi$ of $\Gamma$ and any edges $e, e' \in E$

$$d^\zeta(\psi(e), \psi(e')) = d^\zeta(e, e').$$

As noted earlier, the fineness condition for a graph is weaker than being locally finite. In particular, the balls around an arbitrary vertex in a fine graph need not be finite. However, we do have a kind of local finiteness for edges;

**Lemma 2.47.** *Let $\Gamma$ be a graph. If $\Gamma$ is fine then any ball in the metric space $(E, d^\zeta)$ is finite, i.e.*

$$\forall R > 0, \forall e \in E, \exists k \in \mathbb{N}, \quad \left| B_R^\zeta(e) \right| \leq k.$$

*where $B_R^\zeta(e) := \{e' \in E \mid d^\zeta(e, e') \leq R\}$.*

This is a version of [MY06, Lemma 11] using our terminology. In that paper their definition of a fine graph coincides with our definition of a uniformly fine graph. With this stronger condition they can show that the balls in $(E, d^\zeta)$ are uniformly finite, so the parameter $k$ would not depend on $e$ (although it would still depend on $R$).

We can think of an edge as a subset of a graph, and so we can set

$$d(e, e') = \inf\{d(e, e') \mid x \in e,\ x' \in e'\}.$$

The snake metric is defined using admisssable sequences of edges, and such a sequence determines a path in the graph, so we can compare $d(e, e')$ to $d^\zeta(e, e')$.

**Lemma 2.48** ([MY06, Lemma 6])**.** *Let $\Gamma$ be a hyperbolic graph. For any arbitrary edges $e, e' \in E$,*

$$d(e, e') \leq d^\zeta(e, e').$$

**Corollary 2.49.** *Let $\Gamma$ be a hyperbolic graph. Let $e, e' \in E$ be arbitray edges. Let $x \in V$ be an end-point of $e$ and let $x' \in V$ be an end-point of $e'$. Then*

$$d(x, x') \leq d^\zeta(e, e') + 2.$$

We have already explained how fineness of a graph tells us something about $d^\zeta$, but for a relatively hyperbolic group as in definition 2.22 we also require the graph to be Gromov hyperbolic, and then we can use the following lemma to get a version of hyperbolicity for $d^\zeta$ (cf. definition 2.8).

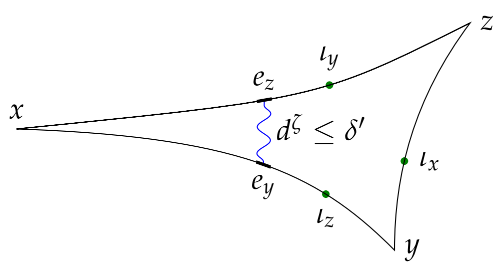
**Lemma 2.50** ([MY02, Proposition 10])**.** *A connected graph $\Gamma$ is $\delta$-hyperbolic if and only if there exists a constant $\delta' \geq 0$ such that for every geodesic triangle $\Delta(x, y, z)$ in $\Gamma$, if $e_y$ is an edge on $[x, y]$ and $e_z$ is an edge on $[x, z]$ with*

$$d(x, e_y) = d(x, e_z) \leq (y|z)_x$$

*then*

$$d^\zeta(e_y, e_z) \leq \delta'.$$

Sometimes a picture can be very illustrative;




A corollary to lemma 2.50 is the degenerate case where $y = z$;

**Corollary 2.51.** *Let* $\Gamma$ *be a* $\delta$*-hyperbolic graph. Fix two distinct points* $x, y \in \Gamma$*. For any* $k \leq d(x,y) - 1$ *and for all* $e, e' \in E[x,y;k]$ *we have* $d^{\zeta}(e,e') \leq \delta$*.*

Note that lemma 2.50 is *not* claiming that the space $(E, d^{\zeta})$ is Gromov hyperbolic (in the sense of definition 2.11).

**Remark 2.52.** Although the edge-hyperbolicity uses a different constant $\delta'$ we may always assume $\delta$ is large enough for edge-hyperbolicity, i.e. we take our new $\delta$ to be the maximum of the old $\delta$ and the $\delta'$.

# 3 Construction of the Flow Space

The purpose of this thesis is to construct a flow space associated to a relatively hyperbolic group $G$. We already have an associated graph $\Gamma$ (as per definition 2.22), but for a flow space we would like to start with a simplicial complex associated to the group, so in section 3.1 we show how to turn the graph $\Gamma$ into a simplicial complex which has nice properties, eg is contractible.

Section 3.2 is a short detour into what a flow space over a tree would look like in order to motivate the definition of the flow space more generally in section 3.3. Then in section 3.4 we begin to outline how to construct a metric on the flow space.

## 3.1 The MY-space

Associated to a hyperbolic group $G$ we have a Cayley graph $\Gamma$ and over this we can build a Rips complex $P_r(\Gamma)$ with parameter $r$ by adding an edge $(u, v)$ to $\Gamma$ whenever $d_\Gamma(u, v) \leq r$ and then taking the flag complex over the resulting graph, i.e. we add as many simplices as the resulting graph allows.

In this case if the parameter $r$ is sufficiently large then the Rips complex $P_r(\Gamma)$ is finite-dimensional and contractible simplicial complex (see [BH99, Theorem III.Γ.3.21]). Moreover, it is a universal space for proper actions of the group $G$ (see [MS02]).

If the group $G$ is hyperbolic relative to a finite set $\mathcal{P}$ of subgroups then we use the coned-off Cayley graph instead, but we want to retain some of the information about distances in the original Cayley graph so we do not take the Rips complex over the coned-off Cayley graph but perform a similar construction that also takes angles into account (since angles are a way of retaining information about distances in the original Cayley graph, see the comments after definition 2.42).

**Definition 3.1.** Let $\Gamma$ be a graph and fix a parameter $\eta \geq 1$. For vertices $v, v'$ of $\Gamma$, an *$\eta$-geodesic* is a geodesic $\alpha$ in $\Gamma$ from $v$ to $v'$ of length $\leq \eta$ such that $\operatorname{maxang}(\alpha) \leq \eta$.

The *MY-space over $\Gamma$ with parameter $\eta$* is the simplicial complex $X_\eta(\Gamma)$ with vertex set $V(X_\eta(\Gamma)) = V(\Gamma)$ such that $\{v_0, \dots, v_k\}$ spans a simplex in $X_\eta(\Gamma)$ if and only if for every pair $v_i, v_j$ there is an $\eta$-geodesic in $\Gamma$ from $v_i$ to $v_j$.

I call this simplicial complex the "MY-space" to reflect who first defined it, namely Mineyev and Yaman in [MY06].

Note that an edge has no inner vertices, so its maximum angle is 0. Therefore, if $\eta \geq 1$ then the graph $\Gamma$ embeds into the 1-skeleton of $X_\eta(\Gamma)$.

The MY-space will be our replacement of the Rips complex. To help motivate why the MY-space is a good replacement we quote two properties about it in the next two propositions.

**Proposition 3.2** ([MY06, Corollary 17])**.** *Let $\Gamma$ be a uniformly fine, $\delta$-hyperbolic graph. If $\eta \geq 1$ then the MY-space $X_\eta(\Gamma)$ is finite-dimensional.*

For proposition 3.2 it is important that the graph is uniformly fine. To see this, let $K_n$ denote the complete graph on $n$ vertices and consider the graph $\Gamma := \bigvee_{n \in \mathbb{N}} K_n$, where the base-point of each $K_n$ is a vertex. Then for any $\eta \geq 1$ the space $X_\eta(\Gamma)$ is $\bigvee_{n \in \mathbb{N}} \Delta^n$, a wedge of simplices. The graph $\Gamma$ is fine and 2-hyperbolic but the space $X_\eta(\Gamma)$ is infinite-dimensional.

**Proposition 3.3** ([MY06, Theorem 19])**.** *Let $\Gamma$ be a fine, $\delta$-hyperbolic graph. Let $\kappa$ be the constant from in lemma 2.43. Then for $\eta \geq 3\kappa$ the MY-space $X_\eta(\Gamma)$ is contractible.*

By definition if $G$ is a group that is hyperbolic relative to $\mathcal{P}$ then there is a fine, $\delta$-hyperbolic graph $\Gamma$ on which $G$ acts with finitely many orbits of vertices and edges and such that the stabiliser of any vertex is finite or an element of $\mathcal{P}$. So for $\kappa$ as in lemma 2.43 and $\eta \geq 3\kappa$ we know that $X_\eta(\Gamma)$ is contractible. Moreover, the graph $\Gamma$ is uniformly fine by lemma 2.19 so we also know that $X_\eta(\Gamma)$ is finite-dimensional.

Ergo, associated to a relatively hyperbolic group $G$ we have a finite-dimensional, contractible simplicial complex $X_\eta(\Gamma)$ on which the group acts. It would be nice if this action were cocompact. To prove this we need another lemma from [MY06].

**Lemma 3.4** ([MY06, Lemma 15])**.** *Let $\Gamma$ be a fine, $\delta$-hyperbolic graph. Fix $\eta \geq 1$. For any edge $e = (a, b)$ of $X_\eta(\Gamma)$ there are only finitely many vertices $c$ of $X_\eta(\Gamma)$ that are connected to both $a$ and $b$ by edges in $X_\eta(\Gamma)$.*

**Corollary 3.5.** *If $\Gamma$ is a fine, $\delta$-hyperbolic graph and $\eta \geq 1$ then every edge of $X_\eta(\Gamma)$ is contained in only finitely many simplices.*

Now we prove that the action of $G$ on $X_\eta(\Gamma)$ is cocompact.

**Proposition 3.6.** *Let $G$ be a group and let $\Gamma$ be a fine graph on which $G$ acts cocompactly. Fix a parameter $\eta \geq 1$. For all $k \in \mathbb{N}$ there are only finitely many $G$-orbits of $k$-simplices in the simplicial complex $X_\eta(\Gamma)$.*

*Proof.* It suffices to prove that there are only finitely many $G$-orbits of edges in $X_\eta(\Gamma)$ because then corollary 3.5 says there are only finitely many $G$-orbits of $k$-simplices.

To show that there are only finitely many $G$-orbits of edges in $X_\eta(\Gamma)$ it is enough to show that there are only finitely many $G$-orbits of $\eta$-geodesics in $\Gamma$. We prove this by showing that for any edge $e$ in $\Gamma$ there are only finitely many $\eta$-geodesics whose first edge is $e$, and then use the fact that there are only finitely many $G$-orbits of edges in $\Gamma$. So it is enough to show that for a fixed edge $e_1$ of $\Gamma$ there are only finitely many $\eta$-geodesics in $\Gamma$ whose first edge is $e_1$.

Let $\alpha$ be an $\eta$-geodesic in $\Gamma$ whose first edge is $e_1$ and let $e_k$ be the $k$-th edge of $\alpha$. The geodesic $\alpha$ gives an admissable sequence of edges from $e_1$ to $e_k$ (recall from definition 2.45 that a sequence of edges is admissable if consecutive edges are coincident). The angle between consecutive edges of $\alpha$ is $\leq \eta$, hence

$$d^\zeta(e_1, e_k) \leq \sum_{i=2}^{k} \operatorname{ang}(e_{i-1}, e_i) \leq (k-1)\eta.$$

So for any edge $e$ of $\alpha$, we have $d^\zeta(e_1, e) \leq \eta(\eta - 1)$, since the length of $\alpha$ is bounded by $\eta$. By lemma 2.47 there are only finitely many such edges, therefore there are only finitely many $\eta$-geodesics whose first edge is $e_1$. □

The 1-skeleton of any simplicial complex is a graph. We will often work with the 1-skeleton of $X_\eta(\Gamma)$, which we denote by $\mathcal{G}_\eta(\Gamma)$. Hence we want to know how graph properties of $\Gamma$ can be translated to $\mathcal{G}_\eta(\Gamma)$.

**Lemma 3.7.** *Let $\Gamma$ be a graph and let $d_\Gamma$ denote the canonical metric on $\Gamma$. Fix the parameter $\eta \geq 1$ and let $\mathcal{G} = \mathcal{G}_\eta(\Gamma)$ denote the 1-skeleton of $X_\eta(\Gamma)$. If $d_\mathcal{G}$ denotes the canonical metric on the graph $\mathcal{G}$ then the inclusion map $(\Gamma, d_\Gamma) \hookrightarrow (\mathcal{G}, d_\mathcal{G})$ is a quasi-isometry.*

*Proof.* The vertex set of a graph is quasi-dense and $V(\Gamma) = V(\mathcal{G}) =: V$, hence it is enough to show that the identity map $\mathrm{id}_V \colon (V, d_\Gamma) \to (V, d_\mathcal{G})$ is a quasi-isometry. It follows from the definition of $\mathcal{G}$ that for any $x, y \in V$

$$d_\mathcal{G}(x, y) \leq d_\Gamma(x, y) \leq \eta\, d_\mathcal{G}(x, y)$$

and therefore the identity map is a quasi-isometry. □

Gromov hyperbolicity is preserved under quasi-isometries (see proposition 2.13) so lemma 3.7 yields the following corollary.

**Corollary 3.8.** *If $\Gamma$ is a Gromov hyperbolic graph then the graph $\mathcal{G}_\eta(\Gamma)$ is Gromov hyperbolic (for any $\eta \geq 1$).*

The graph property fineness is not preserved by quasi-isometries, but we can use still say something due to a lemma by Bowditch;

**Lemma 3.9** ([Bow97, Lemma 4.5])**.** *Let $G$ be a group. Let $V$ be a set on which $G$ acts with finite pair stabilisers, i.e. for all $u, v \in V$ there are only finitely many elements of $G$ that fix both $u$ and $v$. Let $K, L$ be connected $G$-invariant graphs with vertex set $V$ and with finite quotient under $G$, i.e. there are only finitely many $G$-orbits of vertices and finitely many $G$-orbits of edges.*

*If $K$ is fine then $L$ is fine.*

**Remark 3.10.** If we start with a relatively hyperbolic group $G$ and have an associated graph $\Gamma$, as in definition 2.22, then for any $\eta \geq 1$ the 1-skeleton of $X_\eta(\Gamma)$ is a uniformly fine, Gromov hyperbolic graph, by corollary 3.8 and lemma 3.9, where the uniformity of the fineness uses lemma 2.19 and proposition 3.6.

The group $G$ acts on $X$ via the action of $G$ on $V = V(\Gamma) = V(X)$. There are only finitely many orbits of simplices, by proposition 3.6.

We end this section by saying something about the stabilisers of the $G$-action on $X$.

Before we can do that though, we need a result from Osin about the intersection of peripheral subgroups.

**Lemma 3.11** ([Osi06, Proposition 2.36])**.** *If a group $G$ is hyperbolic relative to a finite set $\mathcal{P}$ of subgroups then for all $P, P' \in \mathcal{P}$ and for all $g, g' \in G$ either*

$$gPg^{-1} = g'P'g'^{-1}$$

*or*

$$\left| gPg^{-1} \cap g'P'g'^{-1} \right| < \infty.$$

Osin uses yet another definition of relatively hyperbolicity in terms of (finite) relative presentations and relative Dehn functions, but Theorem 6.10 in the appendix of [Osi06] proves the equivalence of his definition to the definition given in section 2.4 here.

**Proposition 3.12.** *Let $G$ be a group that is hyperbolic relative to a finite set $\mathcal{P}$ of subgroups and let $\Gamma$ be an associated graph (as in definition 2.22).*

*For any $x \in X := X_\eta(\Gamma)$ either the stabiliser of $x$ is finite or $x \in V$.*

*Proof.* Suppose $x \in X$ has an infinite stabiliser $H$. Write $x = \sum_{v \in V} x_v v$ in barycentric coordinates. The group $H$ must permute the support of $x$, but the support of $x$ is finite thus the stabiliser of every $v \in \operatorname{supp}(x)$ is infinite.

So for every $v \in \operatorname{supp}(x)$ we can find some $g_v \in G$ and $P_v \in \mathcal{P}$ such that the stabiliser of $v$ is $g_v P_v g_v^{-1}$.

Furthermore, by considering the diagonal action of $H$ on $(\operatorname{supp}(x))^2$ we see that the pair stabilisers are infinite, i.e. the intersection of the stabilisers of any two vertices in the support of $x$ is infinite. But by lemma 3.11 the intersection of two distinct conjugates of elements of $\mathcal{P}$ is always finite. Therefore there cannot be any pair stabilisers, and the support of $x$ is a single vertex $v$, in which case $x = v$. □

## 3.2 Motivation

The motivation for the flow space comes from considering the a tree $T$, since a tree is the special case of a 0-hyperbolic graph. Consider the space $\overline{T}$ as defined in section 2.5. Between any pair of points in $\overline{T}$ there is a unique geodesic. Then we say that a map $c \colon \mathbb{R} \to T$ is a *generalised geodesic* if there exists $t_-, t_+ \in \overline{\mathbb{R}} := \mathbb{R} \cup \{-\infty, \infty\}$ such that

- $c|_{(-\infty, t_-]}$ is constant;
- $c|_{[t_-, t_+]}$ is a geodesic;
- $c|_{[t_+, \infty)]}$ is constant.

So intuitively, we have a geodesic $c$ which we extend to a map $\mathbb{R} \to T$ by making it constant at either end (if it is not already an infinite geodesic).

Then the *flow space* is defined to be the set of generalised geodesics in $T$. We make this a metric space via

$$d(c, c') := \int_{t \in \mathbb{R}} e^{-\frac{1}{2}|t|^2} d_T(c(t), c'(t))\, dt.$$

Finally, we can define a flow $\Phi$ on this space via $\Phi_\tau(c)(t) := c(t + \tau)$.

This doesn't work for a general Gromov hyperbolic space because the geodesics are not necessarily unique. Hence we need to formally replicate the idea.

## 3.3 The flow space

Let $X$ be a simplicial complex whose 1-skeleton is a fine, Gromov hyperbolic graph. In our motivating example of a tree $T$, the image of a generalised geodesic $c\colon \mathbb{R} \to T$ is determined by its end-points, hence our first attempt at a flow space is $\overline{X} \times \overline{X}$. It is possible for our generalised geodesic to be stationary (i.e. $c(-\infty) = c(\infty)$) but if one end-point lies in the boundary then the generalised geodesic cannot be stationary. Thus we consider the set $(\overline{X} \times \overline{X})\backslash\Delta(\partial X)$, where $\Delta(\partial X)$ is the diagonal of $\partial X \times \partial X$.

This determines the 'image' of a generalised geodesic but we also need a way of encoding the parametrisation. Going back to our motivating example, fix a base-point $x_0 \in T$. For any geodesic $c$ in $T$ there is a unique point on $c$ that is closest to $x_0$. Then we can determine the parametrisation of $c$ by what time it hits this point (or leaves the point if this unique point is the starting point). This gives us a value $t \in \mathbb{R}$. Therefore, we consider the space $((\overline{X} \times \overline{X})\backslash\Delta(\partial X)) \times \mathbb{R}$.

However, there are stationary geodesics, for which the time coordinate plays no role. So we quotient out by the relation $(x, x, t) \sim (x, x, t')$.

We are not quite finished. In a tree $T$, if a generalised geodesic $c$ starts at a point $x \in T$ then the flow $\Phi_\tau(c)$ should converge to the stationary geodesic at $x$ as $\tau \to -\infty$. To replicate this we use $\overline{\mathbb{R}}$ instead of $\mathbb{R}$ in our definition of the flow space, i.e. we consider $((\overline{X} \times \overline{X})\backslash\Delta(\partial X)) \times \overline{\mathbb{R}}$, where we want the point $(x, y, -\infty)$ to correspond to the stationary geodesic at $x$. This is only possible if $x \notin \partial X$ so we need to remove such points beforehand. Then we know that the stationary geodesic at $x$ is given by the point $(x, x, t)$ (for any $t \in \overline{\mathbb{R}}$) so we need to identify $(x, y, -\infty)$ with $(x, x, -\infty)$. Similarly we need to identify $(x, y, \infty)$ with $(y, y, \infty)$.

Putting everything together, we obtain the following definition.

**Definition 3.13.** Let $X$ be a simplicial complex whose 1-skeleton is a fine, Gromov hyperbolic graph. First set

$$X' := \left\{ (x, y, t) \in \overline{X} \times \overline{X} \times \overline{\mathbb{R}} \;\middle|\; \begin{array}{l} x, y \in \partial X \Rightarrow x \neq y, \\ t = -\infty \Rightarrow x \notin \partial X, \\ t = \infty \Rightarrow y \notin \partial X \end{array} \right\} \tag{3.1}$$

and then define the *flow space* as the quotient $FS(X) := X'/\sim$ where

$$\begin{array}{ll} \forall x \in \overline{X}, \forall t, t' \in \overline{\mathbb{R}}, & (x, x, t) \sim (x, x, t'); \\ \forall x, y, y' \in \overline{X}, & (x, y, -\infty) \sim (x, y', -\infty); \\ \forall x, x', y \in \overline{X}, & (x, y, \infty) \sim (x', y, \infty). \end{array}$$

The topology on $FS(X)$ is defined to be the quotient topology, where we give $\overline{X} \times \overline{X} \times \overline{\mathbb{R}}$ the product topology and $X'$ the subspace topology. The flow on $FS(X)$ is given by

$$\Phi_\tau(x,y,t) := (x,y,t+\tau). \tag{3.2}$$

Note that we do have $\Phi_\tau(x,y,t) \to (x,x,-\infty)$ as $\tau \to -\infty$ with this definition, and analogously for $\tau \to \infty$.

This formulation of a flow space was first given by Mineyev in [Min05], where he used the notation $\circledast\overline{X}$ for the flow space $FS(X)$.

A quick remark about this definition. In the motivating example of a tree $T$ we thought of the flow space as the generalised geodesics $c\colon \mathbb{R} \to T$ and then said that given a base-point $x_0 \in T$ we could always get a parameter $t$. This gives an identification of the space of generalised geodesics with the abstract flow space as in definition 3.13, but this identification depends on the choice of base-point. Using a different base-point would still yield an identification of the two versions of a flow space but it would not necessarily give a generalised geodesic $c$ the same parameter $t \in \overline{\mathbb{R}}$.

Therefore, although we do not use a choice of base-point in this definition we will often think of one as being implicit. Sometimes we will want to explicitly use a base-point, in which case we will use the following definition.

**Definition 3.14.** Let $X$ be a finite-dimensional simplicial complex whose 1-skeleton $\mathcal{G}$ is a fine, Gromov hyperbolic graph. Suppose $\tilde{d}$ is a metric on $X$ whose restriction to $X^{(0)}$ coincides with $d_{\mathcal{G}}$. Given any fixed base-point $x_0 \in X$, for any $x, y \in \overline{X}$ define the map $\theta^{x_0}_{x,y}\colon \overline{\mathbb{R}} \to [-(y|x_0)_x, (x|x_0)_y]$ by

$$\theta^{x_0}_{x,y}(t) := \begin{cases} -(y|x_0)_x & \text{if } t \leq -(y|x_0)_x \\ t & \text{if } -(y|x_0)_x \leq t \leq (x|x_0)_y \\ (x|x_0)_y & \text{if } (x|x_0)_y \leq t \end{cases}$$

and observe that for the case $x = y$ the map $\theta^{x_0}_{x,x}$ is identically zero.

The idea is to replicate what happens in a tree. If we have a fixed base-point $x_0 \in X$ then we can think of $(x,y,0)$ as the formal generalised geodesic whose 'closest' point to $x_0$ is at time 0. Then for arbitrary $t$ we just use the flow relation $(x,y,t) = \Phi_t(x,y,0)$.

So the map $\theta^{x_0}_{x,y}$ can be thought of as evaluation at time 0; starting at the point on $[x,y]$ that is 'closest' to $x_0$ we move a distance $|\theta^{x_0}_{x,y}(t)|$ along $[x,y]$ and the sign says which direction, namely towards $x$ if negative and towards $y$ if positive.

## 3.4 Constructing a metric

If $X$ is a Gromov hyperbolic metric space then we would like to define a metric on $FS(X)$. In our motivating example, we defined the distance between $c, c' \in FS(T)$ by integrating over the image of the generalised geodesics $c, c'$. Note that $c(s) = \Phi_s(c)(0)$ for all $s \in \mathbb{R}$ so it is enough to know distances between the points $c(0)$ and $c'(0)$ for all $c, c' \in FS(X)$.

Back in the general case the idea is to think of a point $(x, y, t) \in FS(X)$ as a formal generalised geodesic. Then there should be a point $(x, y, t)_0 \in X$ that corresponds to the formal generalised geodesic evaluated at time $t = 0$. However, we have no such evaluation map $FS(X) \to X$ so we need a way of calculating the distance without using an evaluation map.

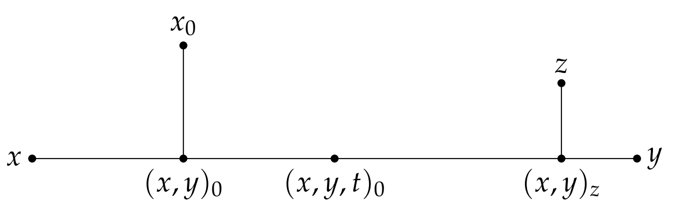
We start by defining a function $X \times FS(X) \to \mathbb{R}$ that in a sense gives a distance from a point $z \in X$ to the 'point' $(x, y, t)_0$.

Then for any two points $(x, y, t), (x', y', t') \in FS(X)$ we can compare how close $(x, y, t)_0$ and $(x', y', t')_0$ are to a point $z \in X$. If we do this for all points $z \in X$ then we should be able to say something about how close the 'points' $(x, y, t)_0$ and $(x', y', t')_0$ are.

To define a distance from $(x, y, t)_0$ to a point $z \in X$ we want to find a point $(x, y)_z$ on the formal generalised geodesic $[x, y]$ that is 'closest' to $z$ and then the distance from $(x, y, t)_0$ to $z$ is the distance along the formal generalised geodesic from $(x, y, t)_0$ to $(x, y)_z$ plus the distance from $(x, y)_z$ to $z$. The distance from $z$ to the point $(x, y)_z$ is the distance from $z$ to the formal generalised geodesic $[x, y]$, which is given by the Gromov product $(x|y)_z$.

To make sense of $(x, y, t)_0$ we need a base-point $x_0 \in X$ (see definition 3.14). We need to know what the distance between $(x, y)_z$ and $(x, y, t)_0$ is in terms of $x_0, x, y, t$ and $z$. If $(x, y)_0$ denotes the point of $[x, y]$ that is 'closest' to $x_0$ then by definition the distance between $(x, y)_0$ and $(x, y, t)_0$ is $|\theta^{x_0}_{x,y}(t)|$, where $\theta^{x_0}_{x,y}(t)$ is positive if $(x, y)_0$ lies between $(x, y, t)_0$ and $x$, and is negative if $(x, y)_0$ lies between $(x, y, t)_0$ and $y$. So for $x, y \in X$ we can write the distance between $(x, y, t)_0$ and $(x, y)_z$ as the following expression in terms of Gromov products;

$$\theta^{x_0}_{x,y}(t) - ((y|z)_x - (y|x_0)_x) = \theta^{x_0}_{x,y}(t) - \big((x|x_0)_y - (x|z)_y\big)$$



However, if $x$ and $y$ are in the boundary then all of the Gromov products are infinite and so the expression does not make sense. We get around this problem by looking specifically at the value $(y|z)_x - (y|x_0)_x$;

**Definition 3.15.** Let $X$ be an arbitrary metric space. The *double difference* of four points $x, y, z, w \in X$ is

$$(x, y|z, w) := (y|z)_x - (y|w)_x. \tag{3.3}$$

Before returning to the problem of defining a metric on our flow space, we list some properties of the double difference.

**Proposition 3.16.** *Let $X$ be an arbitrary metric space. The double difference satisfies the following properties for any points in $X$;*

(0) $(x, x'|y, y') = \frac{1}{2}\left(d_X(x, y) - d_X(x, y') - d_X(x', y) + d_X(x', y')\right)$;

(1) $(x, x'|y, y') = (y, y'|x, x')$;

(2) $(x, x'|y, y') = -(x', x|y, y') = -(x, x'|y', y)$;

(3) $(x, x|y, y') = 0 = (x, x'|y, y)$;

(4) $(x, x'|y, y') + (x', x''|y, y') = (x, x''|y, y')$;

(5) $(x, y|z, w) + (z, x|y, w) + (y, z|x, w) = 0$.

Now we want to define the distance from a point $z \in X$ to $(x, y, t)_0$ for $(x, y, t) \in FS(X)$ to be $(x|y)_z + |\theta^{x_0}_{x,y}(t) - (x, y|z, x_0)|$ but for this we need to know that the double difference is defined for $x, y \in \partial X$.

This is not necessarily true for an arbitrary metric space $X$. Moreover, it is not true for an arbitrary Gromov hyperbolic graph. To solve this we construct a new metric on a uniformly fine, $\delta$-hyperbolic graph such that the double difference with respect to this new metric can be extended to the boundary. Then we can use this new metric to define a metric on $FS(X)$ using the argument laid out above, which is done formally in section 6.3.

# 4 Projecting Along Geodesics

In section 3.4 we started to sketch how to define a metric on the flow space $FS(X)$ for a Gromov hyperbolic metric space $X$, but we encountered a problem that the double difference (as in definition 3.15) is not well-defined for points on the boundary of $X$.

In this section we will use the work of [Min01] together with the modification suggested in [MY06] to define a projection map $g$ that will be used in chapter 5 to construct the metric $\hat{d}$ on the vertex set of $X$.

Our projection map $g$ is essentially the map $\bar{f}$ as in [MY06, Proposition 45], where several facts about the map $\bar{f}$ are stated but not explicitly proven. Instead it is said that the proofs are analogous to proofs given in [Min01]. In this section we will define the projection map $g$, dependent upon three constants $I$, $J$, and $R$, and prove the following theorem.

**Theorem 4.1.** *Let $\mathcal{G}$ be a uniformly fine, Gromov hyperbolic graph. There is a map $g\colon \mathbb{Q}V \oplus \mathbb{Q}V \to \mathbb{Q}E$ that satisfies the following properties;*

(i) *It is invariant under isometries of $\mathcal{G}$ (where $\operatorname{Isom}(\mathcal{G})$ acts on $\mathcal{G} \times \mathcal{G}$ diagonally).*

(ii) *For all $a, b \in V$, for every integer $\aleph$ with $I \leq \aleph \leq J$ the support of $g(a,b)$ contains $E[a,b;\aleph]$.*

(iii) *For all $a, b \in V$ and all $x \in supp(\varphi g(a,b))$, $d(x,b) \leq d(a,b) - 1$.*

(iv) *For all $a, b \in V$ and all $x \in supp\big(\varphi g(a,b)\big)$ there is an $\aleph$ with $I \leq \aleph \leq J$ such that for all $u_\aleph \in V[a,b;\aleph]$ we have $d(x,u_\aleph) \leq 3\delta + R + 2 =: C_0$.*

(v) *For all $a, b \in V$, $diam\big(supp(\varphi g(a,b))\big) \leq 2C_0 + J - I$.*

(vi) *There exist constants $L > 0$ and $\lambda \in (0,1)$ such that for all $a, b, b' \in V$*

$$|g(a,b) - g(a,b')|_1 \leq L\lambda^{(b|b')_a}.$$

(*vii*) *There exists a constant* $\nu \in (0,1)$ *such that for all* $a, a', b \in V$ *if*

- $d(a,b) \geq J+1$ *and* $d(a',b) \geq J+1$
- $|d(a,b) - d(a',b)| \leq J - I$
- $(a|b)_{a'} \leq J$ *and* $(a'|b)_a \leq J$ *then*

$$|g(a,b) - g(a',b)|_1 \leq 2\nu.$$

*where the map* $\varphi \colon \mathbb{Q}E \to \mathbb{Q}V$ *picks out the end-points of every edge.*

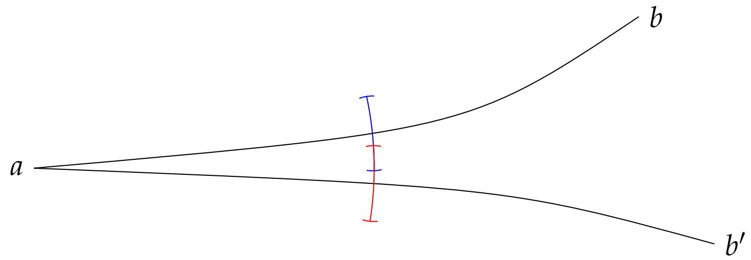
Many of the objects in the statement of theorem 4.1 are yet to be defined but all will be explained in this section. At the very end of chapter 4 we will give the proof of the theorem, which will just show where in the section to look for the proof of the relevant facts.

## 4.1 The first projection

On our uniformly fine $\delta$-hyperbolic graph we want to define a new metric that has better convergence properties than the word metric. To do this we modify the ideas used in [Min01].

In the following picture we consider two points $b, b'$ and a third point $a$ that is comparatively far away. The upper arc perpendicular to $[a,b]$ (blue if you have colour) is the intersection of a sphere around $a$ with the $\delta$-neighbourhood of the geodesic $[a,b]$, whereas the lower arc perpendicular to $[a,b']$ (red) is the intersection of the sphere around $a$ with the $\delta$-neighbourhood of the geodesic $[a,b']$.

*Fig. 4.1:* Overlapping Neighbourhoods




The geodesics may not overlap but by hyperbolicity their $\delta$-neighbourhoods will overlap close to $a$. The idea behind the new metric therefore is to average over the $\delta$-neighbourhoods of the geodesics between the two points. We will need to use the fineness of the graph to ensure we are averaging over something finite. So we average over edges.

For this average to make sense, we work in $\mathbb{Q}E$, the $\mathbb{Q}$-vector space with basis $E$. For any finite subset $S \subseteq E$ define the *average of $S$* to be

$$\mathrm{av}(S) := \frac{1}{|S|} \sum_{s \in S} s \in \mathbb{Q}E.$$

Then in $\mathbb{Q}E$ we can measure the size of the overlap using the $l^1$-norm, which we denote by $|..|_1$, i.e.

$$\left| \sum_{s \in S} q_s s \right|_1 := \sum_{s \in S} |q_s|.$$

In lemma 2.50 we needed $d(a, e) = d(a, e')$ so we want to keep the distance to $a$ constant. Therefore, given $e \in E[a, V]$ we define the *flower of radius $r$ around $e$ with centre $a$* to be the set

$$\mathrm{Fl}[a, e; r] := E[a, V; d(a, e)] \cap B_r^{\zeta}(e).$$

Similarly, for a vertex $b \in V[a, V]$ we can define the *flower of radius $r$ around $b$ with centre $a$* to be $\mathrm{Fl}[a, b; r] := V[a, V; d(a, b)] \cap B_r(b)$.

By taking the average we can turn the flower of radius $r$ around $e$ with centre $a$ into an element of $\mathbb{Q}E[a, V]$, and so for a fixed vertex $a$ and flower radius $r$ we get a map $\mathrm{av}_{\mathrm{Fl}(a;r)} \colon E[a, V] \to \mathbb{Q}E[a, V]$, which we can extend $\mathbb{Q}$-linearly to get a map $\mathrm{av}_{\mathrm{Fl}(a;r)} \colon \mathbb{Q}E[a, V] \to \mathbb{Q}E[a, V]$.

We do not want to project all the way to the vertex $a$ so we need a constant $\aleph > 0$ such that inside the ball of radius $\aleph$ around $a$ we should not change $b$ much. We will only spread out at certain distances away from $a$, so we need a second constant $\mu > 0$ such that we only spread out when $d(a, b)$ is the initial constant $\aleph$ plus a multiple of the moving constant $\mu$. First, we define a function $f_\aleph \colon \mathbb{Q}V \oplus \mathbb{Q}V \to \mathbb{Q}E$ which is moving one step before spreading out.

**Definition 4.2.** Let $\mathcal{G}$ be a fine, $\delta$-hyperbolic graph. Define a $\mathbb{Q}$-bilinear map $f_\aleph \colon \mathbb{Q}V \oplus \mathbb{Q}V \to \mathbb{Q}E$ on vertices as follows; Given two vertices $a, b$ of $\mathcal{G}$ either $d(a, b) \leq \aleph$ or there is an $m \in \mathbb{N}$ such that

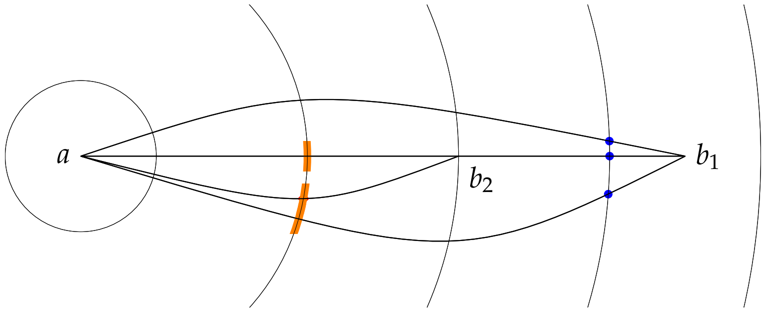
$$m\mu + \aleph < d(a, b) \leq (m+1)\mu + \aleph$$

and then $f_\aleph$ is given by

$$f_\aleph(a, b) := \begin{cases} 0 & \text{if } a = b \\ \mathrm{av}(E[a, b; d(a, b) - 1]) & \text{if } 1 \leq d(a, b) \leq \aleph \\ \mathrm{av}(E[a, b; m\mu + \aleph]) & \text{if } m\mu + \aleph < d(a, b) < (m+1)\mu + \aleph \\ \mathrm{av}_{\mathrm{Fl}(a;\delta)}(\mathrm{av}(E[a, b; m\mu + \aleph])) & \text{if } d(a, b) = (m+1)\mu + \aleph \end{cases}$$

The function $f_{\aleph}$ depends on both $\aleph$ and $\mu$. We will keep $\mu$ fixed so it is suppressed in the notation. In section 4.4 we will allow $\aleph$ to vary so we want to keep track of it in the notation.

For understanding what the map does it is more intuitive to consider the end-points closest to $a$, namely let $\iota_a\colon \mathbb{Q}E[a,V] \to \mathbb{Q}V[a,V]$ be the $\mathbb{Q}$-linear extension of the map that given an edge $e \in E[a,V]$ picks out the end-point of $e$ that is closest to $a$ and consider $\iota_a f_{\aleph}(a,b) \in \mathbb{Q}V$.




If the vertex $b$ does not lie on one of the spheres then we just project to the next sphere by averaging along all of the geodesics from $b$ to $a$. So $b_1$ in the picture is sent to the (blue) dots.

The interesting part happens when $b$ lies on one of the spheres, in which case we take the projection as in the previous case but then we also spread out when we reach the next sphere. This is represented in the picture by $b_2$ being moved to the (orange) bars on the next-smallest sphere.

**Remark 4.3.** If $a \neq b$ then the element $f_{\aleph}(a,b) \in \mathbb{Q}E$ is a convex combination, i.e. all coefficients are non-negative and their sum is 1. Furthermore, the map $f_{\aleph}$ is invariant under the diagonal $\mathrm{Isom}(\mathcal{G})$-action on $\mathbb{Q}V \times \mathbb{Q}V$.

The map $f_{\aleph}$ is only interesting when $d(a,b) > \aleph$, and the next lemma gives some properties of the support of $f_{\aleph}(a,b)$ in this case.

**Lemma 4.4.** *Let $\mathcal{G}$ be a fine, $\delta$-hyperbolic graph. Suppose $a,b \in V$ have distance $d(a,b) > \aleph$. Pick $m \in \mathbb{N}$ such that*

$$m\mu + \aleph < d(a,b) \leq (m+1)\mu + \aleph.$$

*Then the following statements hold;*

*(i)* $E[a,b;m\mu+\aleph] \subseteq \mathit{supp}(f_{\aleph}(a,b))$;

*(ii)* *For any* $e \in E[a,b;m\mu+\aleph]$, $\mathit{supp}(f_{\aleph}(a,b)) \subseteq \mathit{Fl}[a,e;2\delta]$.

*Proof.* *(i)* Follows immediately from the definition.

*(ii)* Corollary 2.51 gives $d^{\zeta}(e,e') \leq \delta$ for all $e,e' \in E[a,b;m\mu+\aleph]$. In particular, the triangle inequality shows $\mathrm{Fl}[a,e';\delta] \subseteq \mathrm{Fl}[a,e;2\delta]$.

If $d(a,b) < (m+1)\mu + \aleph$ then $f_\aleph(a,b) = \mathrm{av}(E[a,b;m\mu+\aleph])$ and so

$$\mathrm{supp}(f_\aleph(a,b)) = E[a,b;m\mu+\aleph] \subseteq \mathrm{Fl}[a,e;2\delta].$$

If $d(a,b) = (m+1)\mu + \aleph$ then $f_\aleph(a,b) = \mathrm{av}_{\mathrm{Fl}(a;\delta)}(\mathrm{av}(E[a,b;m\mu+\aleph]))$. So

$$\mathrm{supp}(f_\aleph(a,b)) = \bigcup_{\bar{e}\in E[a,b;m\mu+\aleph]} \mathrm{Fl}[a,\bar{e};\delta] \subseteq \mathrm{Fl}[a,e;2\delta]. \qquad \square$$

We will iterate this projection via $\iota_a$ so it will be useful to also know what the support of $\iota_a f_\aleph(a,b)$ is like.

**Corollary 4.5.** *Suppose we have the same set-up as in lemma 4.4. Then*

(i) $V[a,b;m\mu+\aleph] \subseteq \mathit{supp}(\iota_a f_\aleph(a,b))$;

(ii) *For any* $v \in V[a,b;m\mu+\aleph]$, $\mathit{supp}(\iota_a f_\aleph(a,b)) \subseteq \mathit{Fl}[a,v;2\delta+2]$.

*Proof.* Use lemma 4.4 and corollary 2.49. $\square$

So we know that the support is never too far away from any geodesic from $a$ to $b$.

The spreading out in the fourth case in the definition is done to create an overlap, as in figure 4.1. If two vertices $b, c$ are close then we want the supports of $\iota_a f_\aleph(a,b)$ and $\iota_a f_\aleph(a,c)$ to remain close, and furthermore we want the intersection of the supports to be non-empty.

For there to be a global bound we assume $\mathcal{G}$ is uniformly fine, in which case the balls in the edge-metric $d^\zeta$ are uniformly finite, by lemma 2.47, and we set

$$B_\delta^\zeta = \max\left\{\left|B_\delta^\zeta(e)\right| : e \in E\right\}.$$

**Lemma 4.6.** *Let $\mathcal{G}$ be a fine, $\delta$-hyperbolic graph. Let $a,b,c \in V$ be three vertices satisfying*

- $\exists m \in \mathbb{N}$ *such that* $d(a,b) = (m+1)\mu + \aleph = d(a,c)$;
- $d(b,c) \leq 2\mu$.

*Then for all* $e_b \in \mathit{supp}(f_\aleph(a,b))$ *and for all* $e_c \in \mathit{supp}(f_\aleph(a,c))$,

$$d^\zeta(e_b, e_c) \leq 3\delta.$$

*In particular* $d(\iota_a e_b, \iota_a e_c) \leq 3\delta + 2$. *Moreover, if $\mathcal{G}$ is uniformly fine then there exists a constant* $\eta_0 \in (0,1)$ *such that*

$$|f_\aleph(a,b) - f_\aleph(a,c)|_1 \leq 2\eta_0.$$

*Proof.* By the definition of $f_{\aleph}$, there is an edge $\bar{e}_b \in E[a,b;m\mu+\aleph]$ such that $e_b \in \mathrm{Fl}[a,\bar{e}_b;\delta]$. Pick $\bar{e}_c$ analogously. Under our assumptions on the distances between $a,b,c$ we know that $(b|c)_a \geq m\mu+\aleph$. Hence lemma 2.50 tells us that $d^{\zeta}(\bar{e}_b,\bar{e}_c) \leq \delta$. Thus

$$d^{\zeta}(e_b,e_c) \leq d^{\zeta}(e_b,\bar{e}_b) + d^{\zeta}(\bar{e}_b,\bar{e}_c) + d^{\zeta}(\bar{e}_c,e_c) \leq 3\delta.$$

In particular corollary 2.49 gives $d(\iota_a e_b, \iota_a e_c) \leq 3\delta + 2$.

Now suppose that $\mathcal{G}$ is uniformly fine (hence $B^{\zeta}_{\delta}$ is finite). To simplify the notation a bit, set $E_{ab} = E[a,b;m\mu+\aleph]$, and for $e \in E_{ab}$ set $\mathrm{Fl}_e = \mathrm{Fl}[a,e;\delta]$. Similarly we can define $E_{ac}$ and $\mathrm{Fl}_{\bar{e}}$ for $\bar{e} \in E_{ac}$.

Then by definition we have

$$f_{\aleph}(a,b) = \frac{1}{|E_{ab}|}\sum_{e\in E_{ab}} \frac{1}{|\mathrm{Fl}_e|}\sum_{e'\in \mathrm{Fl}_e} e' \quad , \quad f_{\aleph}(a,c) = \frac{1}{|E_{ac}|}\sum_{\bar{e}\in E_{ac}} \frac{1}{|\mathrm{Fl}_{\bar{e}}|}\sum_{\bar{e}'\in \mathrm{Fl}_{\bar{e}}} \bar{e}'.$$

For any $e \in E_{ab}$ and $\bar{e} \in E_{ac}$ we have $d^{\zeta}(e,\bar{e}) \leq \delta$ since $(b|c)_a \geq m\mu+\aleph$. So $\mathrm{Fl}_e \cap \mathrm{Fl}_{\bar{e}} \neq \emptyset$ and therefore

$$\begin{aligned}
&|f_{\aleph}(a,b) - f_{\aleph}(a,c)|_1 \\
&\quad \leq \frac{1}{|E_{ab}|}\sum_{e\in E_{ab}} \frac{1}{|\mathrm{Fl}_e|}\sum_{e'\in \mathrm{Fl}_e} \frac{1}{|E_{ac}|}\sum_{\bar{e}\in E_{ac}} \frac{1}{|\mathrm{Fl}_{\bar{e}}|}\sum_{\bar{e}'\in \mathrm{Fl}_{\bar{e}}} |e'-\bar{e}'|_1 \qquad (4.1)\\
&\quad = \frac{1}{|E_{ab}|}\sum_{e\in E_{ab}} \frac{1}{|E_{ac}|}\sum_{\bar{e}\in E_{ac}} 2\frac{|\mathrm{Fl}_e||\mathrm{Fl}_{\bar{e}}| - |\mathrm{Fl}_e \cap \mathrm{Fl}_{\bar{e}}|}{|\mathrm{Fl}_e||\mathrm{Fl}_{\bar{e}}|}\\
&\quad \leq \frac{1}{|E_{ab}|}\sum_{e\in E_{ab}} \frac{1}{|E_{ac}|}\sum_{\bar{e}\in E_{ac}} 2\left(1 - \frac{1}{(B^{\zeta}_{\delta})^2}\right)\\
&\quad = 2\left(1 - \frac{1}{(B^{\zeta}_{\delta})^2}\right)
\end{aligned}$$

and the proof is finished by setting $\eta_0 = 1 - \frac{1}{(B^{\zeta}_{\delta})^2} \in (0,1)$. □

## 4.2 Iterated projections

Using $f_{\aleph}$ we can project to the next sphere of radius $m\mu+\aleph$ around $a$. We can iterate these projections in the natural way.

**Definition 4.7.** Let $\mathcal{G}$ be a fine, $\delta$-hyperbolic graph. Given $k \in \mathbb{N}$ define $f^{(k)}_{\aleph}\colon \mathbb{Q}V \oplus \mathbb{Q}V \to \mathbb{Q}E$ to be the $\mathbb{Q}$-bilinear function that is defined inductively on vertices by

- $f_{\aleph}^{(0)}(a,b) := f_{\aleph}(a,b)$;
- $f_{\aleph}^{(k+1)}(a,b) := f_{\aleph}^{(k)}(a, \iota_a f_{\aleph}(a,b))$.

**Remark 4.8.** As with $f_{\aleph}(a,b)$, for $a \neq b$ the element $f_{\aleph}^{(k)}(a,b) \in \mathbb{Q}E$ is a convex combination, in fact it is a convex combination in $\mathbb{Q}E[a,V]$ so we can always consider $\iota_a f_{\aleph}^{(k)}(a,b)$. Furthermore, the map $f_{\aleph}^{(k)}$ is invariant under the action of $\mathrm{Isom}(\mathcal{G})$.

This recursive definition has an equivalent form, given in the following lemma.

**Lemma 4.9.** *Let $\mathcal{G}$ be a fine, $\delta$-hyperbolic graph. For all $a,b \in V$ and for all $k \in \mathbb{N}$*

$$f_{\aleph}^{(k+1)}(a,b) = f_{\aleph}(a, \iota_a f_{\aleph}^{(k)}(a,b)).$$

*Proof.* Induct on $k$.

The case $k = 0$ follows straight from the definition of the map $f_{\aleph}^{(0)}$.

For the inductive step;

$$\begin{aligned} f_{\aleph}^{(k+1)}(a,b) &= f_{\aleph}^{(k)}(a, \iota_a f_{\aleph}(a,b)) \\ &= f_{\aleph}(a, \iota_a f_{\aleph}^{(k-1)}(a, \iota_a f_{\aleph}(a,b))) \\ &= f_{\aleph}(a, \iota_a f_{\aleph}^{(k)}(a,b)) \end{aligned}$$

where the first equality uses the definition of $f_{\aleph}^{(k+1)}$, the second equality is the inductive hypothesis and the third equality is the definition of $f_{\aleph}^{(k)}$. □

From now on we will have to make an assumption about the constant $\mu$, which is how far we move in each step of the recursive function $f_{\aleph}^{(k)}$. Since we will later introduce more constants and more conditions we will start a list of all the assumptions we are making which we can add to later on.

**Assumptions 4.10.** We assume that the constant $\mu$ always satisfies the inequality

$$2\mu \geq 3\delta + 2. \tag{4.2}$$

The right-hand side of inequality (4.2) is chosen so that we can iterate lemma 4.6. We know that $3\delta + 2$ bounds $d(\iota_a e_b, \iota_a e_c)$ with $e_b$ in the support of $f_{\aleph}(a,b)$ and $e_c$ in the support of $f_{\aleph}(a,c)$ (for suitable vertices $a,b,c \in V$), but in lemma 4.6 we assume $d(b,c) \leq 2\mu$, so to iterate the lemma we have to assume $\mu$ satisfies inequality (4.2).

As with $f_{\aleph}$ it is helpful to have an idea what the support of $f_{\aleph}^{(k)}(a,b)$ is like. This leads to the following lemma (cf lemma 4.4).

**Lemma 4.11.** *Let $\mathcal{G}$ be a fine, $\delta$-hyperbolic graph. Suppose $a, b \in V$ have distance $d(a,b) > \aleph$. Pick $m \in \mathbb{N}$ such that*

$$m\mu + \aleph < d(a,b) \leq (m+1)\mu + \aleph.$$

*Then for all $k \in \{0, \ldots, m\}$;*

*(i) $E[a,b;(m-k)\mu + \aleph] \subseteq supp(f_\aleph^{(k)}(a,b))$;*

*(ii) For any $e \in E[a,b;(m-k)\mu + \aleph]$, $supp(f_\aleph^{(k)}(a,b)) \subseteq Fl[a,e;3\delta]$.*

*Proof.* We induct on $k$.

The case $k = 0$ is lemma 4.4.

Now for the inductive step.

$(i)$ Assume $E[a,b;(m-k)\mu + \aleph] \subseteq \text{supp}(f_\aleph^{(k)}(a,b))$. We want to look at the support of $f_\aleph^{(k+1)}(a,b)$. For any $e \in E[a,b;(m-k)\mu + \aleph]$ we know that any geodesic from $a$ to $b$ restricts to a geodesic from $a$ to $\iota_a e$. In particular

$$E[a,b;(m-(k+1))\mu + \aleph] \subseteq E[a,\iota_a e;(m-(k+1))\mu + \aleph].$$

Lemma 4.4 shows

$$E[a,\iota_a e;(m-(k+1))\mu + \aleph] \subseteq \text{supp}(f_\aleph(a,\iota_a e))$$

but by the alternative inductive definition of $f_\aleph^{(k+1)}$ given in lemma 4.9 we know that $\text{supp}(f_\aleph(a,\iota_a e)) \subseteq \text{supp}(f_\aleph^{(k+1)}(a,b))$ and bringing this altogether we obtain

$$E[a,\iota_a e;(m-(k+1))\mu + \aleph] \subseteq \text{supp}(f_\aleph^{(k+1)}(a,b)).$$

$(ii)$ Fix $e \in E[a,b;(m-(k+1))\mu + \aleph]$. We need to show that for any edge $e_1 \in \text{supp}(f_\aleph^{(k+1)}(a,b))$ we have $d^\zeta(e,e_1) \leq 3\delta$. By the definition of $f_\aleph^{(k+1)}$ there is some $e_1' \in \text{supp}(f_\aleph^{(k)}(a,b))$ with $e_1 \in \text{supp}(f_\aleph(a,\iota_a e_1'))$. Pick $e' \in E[a,b;(m-k)\mu + \aleph]$ such that there is a geodesic from $a$ to $b$ that contains both $e$ and $e'$. Then by part $(i)$ we know $e' \in \text{supp}(f_\aleph^{(k)}(a,b))$.

The inductive hypothesis says $d^\zeta(e',e_1') \leq 3\delta$, and then corollary 2.49 gives $d(\iota_a e', \iota_a e_1') \leq 3\delta + 2 \leq 2\mu$. So lemma 4.6 tells us that $d^\zeta(e,e_1) \leq 3\delta$, as desired. □

If we start with two vertices $b, c$ then we already know that $f_\aleph$ creates an overlap if $d(b,c)$ is small, and we hope that by iterating this overlap becomes larger, in other words we want $|f_\aleph^{(k)}(a,b) - f_\aleph^{(k)}(a,c)|_1$ to get smaller.

This is formalised in the following lemma (cf lemma 4.6).

**Lemma 4.12.** *Let $\mathcal{G}$ be a fine, $\delta$-hyperbolic graph. Let $a, b, c \in V$ be three vertices satisfying*

- *$\exists m \in \mathbb{N}$ such that $d(a,b) = (m+1)\mu + \aleph = d(a,c)$;*
- *$d(b,c) \leq 2\mu$.*

*For any $k \in \{0,\dots,m\}$, $x \in$ supp $\left(\iota_a f_\aleph^{(k)}(a,b)\right)$, and $y \in$ supp $\left(\iota_a f_\aleph^{(k)}(a,c)\right)$,*

$$d(x,y) \leq 3\delta + 2.$$

*Moreover, if $\mathcal{G}$ is uniformly fine then*

$$|f_\aleph^{(k)}(a,b) - f_\aleph^{(k)}(a,c)|_1 \leq 2\eta_0^k$$

*where $\eta_0 \in (0,1)$ is the constant from lemma 4.6.*

*Proof.* We induct on $k$.

The case $k=0$ is exactly lemma 4.6.

For the inductive step, use the alternative recursive definition of $f_\aleph^{(k+1)}$ given in lemma 4.9, which says $f_\aleph^{(k+1)}(a,b) = f_\aleph(a,\iota_a f_\aleph^{(k)}(a,b))$. So for any $x \in \operatorname{supp}(\iota_a f_\aleph^{(k+1)}(a,b))$ there is some $b_x \in \operatorname{supp}(\iota_a f_\aleph^{(k)}(a,b))$ such that $x \in \operatorname{supp}(\iota_a f_\aleph(a,b_x))$. Similarly for $y \in \operatorname{supp}(\iota_a f_\aleph^{(k+1)}(a,c))$ there is a corresponding $c_y \in \operatorname{supp}(\iota_a f_\aleph^{(k)}(a,c))$.

By the inductive hypothesis $d(b_x,c_y) \leq 3\delta+2 \leq 2\mu$. Thus we can apply lemma 4.6 to deduce that $d(x,y) \leq 3\delta+2$.

Moreover, suppose $\mathcal{G}$ is uniformly fine. We keep the notation as in the proof of lemma 4.6. By definition $f_\aleph^{(k+1)}(a,b) = f_\aleph^{(k)}(a,\iota_a f_\aleph(a,b))$. Using the $\mathbb{Q}$-linearity of $f_\aleph^{(k)}$ we can proceed analogously to the proof of lemma 4.6 and we get

$$\begin{aligned}&|f_\aleph^{(k+1)}(a,b) - f_\aleph^{(k+1)}(a,c)|_1\\ &\leq \frac{1}{|E_{ab}|}\sum_{e\in E_{ab}}\frac{1}{|\mathrm{Fl}_e|}\sum_{e'\in \mathrm{Fl}_e}\frac{1}{|E_{ac}|}\sum_{\bar{e}\in E_{ac}}\frac{1}{|\mathrm{Fl}_{\bar{e}}|}\sum_{\bar{e}'\in \mathrm{Fl}_{\bar{e}}}|f_\aleph^{(k)}(a,\iota_a e') - f_\aleph^{(k)}(a,\iota_a \bar{e}')|_1.\end{aligned}$$

Lemma 4.6 also tells us that $d(\iota_a e', \iota_a \bar{e}') \leq 3\delta+2 \leq 2\mu$, hence the inductive hypothesis gives

$$|f_\aleph^{(k)}(a,\iota_a e') - f_\aleph^{(k)}(a,\iota_a \bar{e}')|_1 \leq \eta_0^k$$

and therefore

$$\begin{aligned}&|f_\aleph^{(k+1)}(a,b) - f_\aleph^{(k+1)}(a,c)|_1\\ &\qquad\leq \frac{1}{|E_{ab}|}\sum_{e\in E_{ab}}\frac{1}{|\mathrm{Fl}_e|}\sum_{e'\in \mathrm{Fl}_e}\frac{1}{|E_{ac}|}\sum_{\bar{e}\in E_{ac}}\frac{1}{|\mathrm{Fl}_{\bar{e}}|}\sum_{\bar{e}'\in \mathrm{Fl}_{\bar{e}}}|e' - \bar{e}'|_1\eta_0^k\end{aligned}$$

but this is just the sum (4.1) multiplied by the constant $\eta_0^k$. Hence we conclude

$$|f_\aleph^{(k+1)}(a,b) - f_\aleph^{(k+1)}(a,c)|_1 \leq 2\eta_0^{k+1}$$

which finishes the induction. □

## 4.3 The total projection

Now we consider what happens when we repeatedly project until we reach the ball of radius $\aleph$ around $a$.

**Definition 4.13.** Let $\mathcal{G}$ be a fine, $\delta$-hyperbolic graph. Define a $\mathbb{Q}$-linear map $g_\aleph \colon \mathbb{Q}V \oplus \mathbb{Q}V \to \mathbb{Q}E$ on vertices as follows;

For all $a, b \in V$, if $d(a,b) > \aleph$ then there exists an $m \in \mathbb{N}$ such that

$$m\mu + \aleph < d(a,b) \leq (m+1)\mu + \aleph$$

and then

$$g_\aleph(a,b) := \begin{cases} f_\aleph(a,b) & \text{if } d(a,b) \leq \aleph \\ f_\aleph^{(m)}(a,b) & \text{if } d(a,b) > \aleph \end{cases}.$$

**Remark 4.14.** The element $g_\aleph(a,b) \in \mathbb{Q}E$ is a convex combination if $a \neq b$, and the map $g_\aleph$ is invariant under the action of $\mathrm{Isom}(\mathcal{G})$.

Previous results about $f_\aleph^{(k)}(a,b)$ immediately carry over to $g_\aleph(a,b)$. In particular lemma 4.11 gives;

**Lemma 4.15.** *Let $\mathcal{G}$ be a fine, $\delta$-hyperbolic graph. Suppose $a, b \in V$ have distance $d(a,b) > \aleph$. Pick $m \in \mathbb{N}$ such that*

$$m\mu + \aleph < d(a,b) \leq (m+1)\mu + \aleph.$$

*Then the following statements hold;*

*(i)* $E[a,b;\aleph] \subseteq supp(g_\aleph(a,b))$;

*(ii)* *For any* $e \in E[a,b;\aleph]$, $supp(g_\aleph(a,b)) \subseteq Fl[a,e;3\delta]$. □

Furthermore, lemma 4.12 yields the following lemma;

**Lemma 4.16.** *Let $\mathcal{G}$ be a fine, $\delta$-hyperbolic graph. Let $a, b, c \in V$ be three vertices satisfying*

- $\exists m \in \mathbb{N}$ *such that* $d(a,b) = (m+1)\mu + \aleph = d(a,c)$;
- $d(b,c) \leq 2\mu$.

*Then for all $x \in supp(\iota_a g_\aleph(a,b))$, and all $y \in supp(\iota_a g_\aleph(a,c))$,*

$$d(x,y) \leq 3\delta + 2.$$

*Moreover, if $\mathcal{G}$ is uniformly fine then*

$$|g_\aleph(a,b) - g_\aleph(a,c)|_1 \leq 2\eta_0^m$$

*where $\eta_0 \in (0,1)$ is the constant from lemma 4.6.*

Up to now we have been assuming that $d(b,c) \leq 2\mu$, and that they both lie on a sphere of radius $(m+1)\mu + \aleph$ around $a$. We want to drop these conditions on the vertices $b, c$ to get a general result about what happens when the second variable of $g_\aleph$ is changed.

However, we will have to assume a bit more about the constant $\mu$. So we have to update the assumption 4.10 to the following;

**Assumptions 4.17.** We assume that the constant $\mu$ always satisfies the inequality

$$2\mu \geq 7\delta + 4. \tag{4.3}$$

If we consider lemma 4.12 with $b = c$ then we know that the diameter of the support of $\iota_a f_\aleph^{(k)}(a,b)$ is bounded by $3\delta + 2$. Hence if we have three vertices $a, b, b' \in V$ and know that there is some vertex $x_0$ in the support of $\iota_a f_\aleph^{(k)}(a,b)$ and some $x_0'$ in the support of $\iota_a f_\aleph^{(k)}(a,b')$ with $d(x_0, x_0') \leq \delta$ then we can bound the distance between an arbitrary element in the support of $\iota_a f_\aleph^{(k)}(a,b)$ and an arbitrary element in the support of $\iota_a f_\aleph^{(k)}(a,b')$ by $7\delta + 4$. This is why we now assume that inequality (4.3) holds.

**Proposition 4.18.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. There are constants $L > 0$, $\lambda \in (0,1)$ such that for all $a, b, b' \in V$,*

$$|g_\aleph(a,b) - g_\aleph(a,b')|_1 \leq L\lambda^{(b|b')_a}.$$

*Proof.* Set $L = \max\left\{2\eta_0^{-\frac{\aleph}{\mu}}, \eta_0^{-1-\frac{\aleph}{\mu}}\right\}$ and $\lambda = \eta_0^{\frac{1}{\mu}} \in (0,1)$ where $\eta_0$ is the constant from lemma 4.6.

First consider the case $(b|b')_a \leq \aleph$. Here we have

$$\begin{aligned} |g_\aleph(a,b) - g_\aleph(a,b')|_1 &\leq |g_\aleph(a,b)|_1 + |g_\aleph(a,b')|_1 \\ &\leq 2 \\ &= 2\eta_0^{-\frac{\aleph}{\mu}}\eta_0^{\frac{\aleph}{\mu}} \\ &\leq 2\eta_0^{-\frac{\aleph}{\mu}}\eta_0^{\frac{(b|b')_a}{\mu}} \\ &\leq L\lambda^{(b|b')_a} \end{aligned}$$

and so the proposition is true in this special case.

Now consider the case $(b|b')_a > \aleph$. Then there is some $n \in \mathbb{N}$ such that $n\mu + \aleph < (b|b')_a \leq (n+1)\mu + \aleph$.

There is also some $m \in \mathbb{N}$ such that $m\mu + \aleph < d(a,b) \leq (m+1)\mu + \aleph$. So $m \geq n$ since $d(a,b) \geq (b|b')_a$. Set $k = m - n$ and $S = \operatorname{supp}(\iota_a f_\aleph^{(k)}(a,b))$. Write $\iota_a f_\aleph^{(k)}(a,b) = \sum_{x \in S} \alpha_x x$ and then $g_\aleph(a,b) = \sum_{x \in S} \alpha_x g_\aleph(a,x)$. Observe that for $x \in S$ we have $d(a,x) = n\mu + \aleph$.

Similarly for $b'$ we find $m' \geq n$, and analogously define $k', S'$, and $\alpha'_{x'}$ for $x' \in S'$. Again, observe that $d(a, x') = n\mu + \aleph$ for all $x' \in S'$.

We wish to apply lemma 4.16 to $a, x, x'$ with $x \in S$ and $x' \in S'$ but we need a bound on the distance between $x$ and $x'$.

Fix a vertex $x_0 \in V[a, b; n\mu + \aleph]$ and a vertex $x'_0 \in V[a, b'; n\mu + \aleph]$. Then $x_0 \in S$ and $x'_0 \in S'$ using lemma 4.11$(i)$. Moreover, $d(x_0, x'_0) \leq \delta$ by considering a geodesic triangle with corners $a, b, b'$ and that contains the two vertices $x_0, x'_0$ (this uses the fact $d(a, x) = d(a, x') \leq (b|b')_a$). So for any $x \in S$ and any $x' \in S'$, using the triangle-inequality together with lemma 4.12 gives

$$\begin{aligned} d(x, x') &= d(x, x_0) + d(x_0, x'_0) + d(x'_0, x') \\ &\leq 3\delta + 2 + \quad \delta \quad + 3\delta + 2 \\ &= 7\delta + 4 \\ &\leq 2\mu. \end{aligned}$$

Thus we can apply lemma 4.16 to get

$$|g_\aleph(a, x) - g_\aleph(a, x')|_1 \leq \eta_0^n.$$

Therefore

$$\begin{aligned} |g_\aleph(a, b) - g_\aleph(a, b')|_1 &= \left| \sum_{x \in S} \alpha_x g_\aleph(a, x) - \sum_{x' \in S'} \alpha'_{x'} g_\aleph(a, x') \right|_1 \\ &\leq \sum_{x \in S} \sum_{x' \in S'} \alpha_x \alpha'_{x'} |g_\aleph(a, x) - g_\aleph(a, x')|_1 \\ &\leq \sum_{x \in S} \sum_{x' \in S'} \alpha_x \alpha'_{x'} \eta_0^n \\ &= \eta_0^n \\ &\leq \eta_0^{-1 - \frac{\aleph}{\mu}} \eta_0^{\frac{(b|b')_a}{\mu}} \\ &\leq L \lambda^{(b|b')_a}. \end{aligned}$$

□

## 4.4 Varying the initial sphere

Proposition 4.18 gives us some control over what happens when we change the second variable of $g_\aleph$, but we still have no control over what happens when we change the first variable. The support of $g_\aleph(a, b)$ (for $d(a, b)$ large) is contained in the sphere of radius $\aleph$ around $a$. Even if $a'$ is close to $a$ there need not be a large overlap between $E[a, V; \aleph]$ and $E[a', V; \aleph]$ To compensate for this, we take an average over many different initial spheres.

So we introduce two new constants to designate the start and end of the range over which we will average. We will use $I$ for the radius of the smallest initial sphere to be considered, and $J$ for the largest. We also need a third constant $R$ that will be the radius of the balls in the edge metric over which we average each $g_\aleph(a, b)$. This is made formal in the following definition.

**Definition 4.19.** Let $\mathcal{G}$ be a fine, $\delta$-hyperbolic graph. Define the $\mathbb{Q}$-linear map $g\colon \mathbb{Q}V \oplus \mathbb{Q}V \to \mathbb{Q}E$ on vertices $a, b \in V$ by

$$g(a,b) := \frac{1}{1+J-I} \sum_{\aleph=I}^{J} \mathrm{av}_{B_R^\zeta}(g_\aleph(a,b)).$$

Some facts about $g_\aleph(a, b)$ carry straight to $g(a, b)$. For example, $g(a, b)$ is a convex combination if $a \neq b$, the map $g$ is invariant under the action of $\mathrm{Isom}(\mathcal{G})$, and lemma 4.15 gives the following lemma about $g(a, b)$.

**Lemma 4.20.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. Let $a, b \in V$ be two distinct vertices of $\mathcal{G}$ with $d(a, b) > J$. For every $\aleph$ with $I \leq \aleph \leq J$ pick an edge $e_\aleph \in E[a, b; \aleph]$. Then*

*(i) Every $e_\aleph$ is contained in the support of $g(a, b)$;*

*(ii) For any $e \in supp(g(a, b))$ there exists some $\aleph$ between $I$ and $J$ such that $d^\zeta(e, e_\aleph) \leq 3\delta + R$.* □

The support of $g(a, b)$ is not necessarily contained in $E[a, V]$ so we cannot apply $\iota_a$ to it. Recall that in the definition of a graph (see definition 2.1) we included the existence of two maps $\varphi_\pm\colon E \to V$ that pick out the end-points of an edge. These maps can be $\mathbb{Q}$-linearly extended and then we can take the average of the two end-points of an edge, to wit; let $\varphi\colon \mathbb{Q}E \to \mathbb{Q}V$ be the $\mathbb{Q}$-linear map defined by

$$\varphi(e) := \frac{1}{2}\varphi_+(e) + \frac{1}{2}\varphi_-(e).$$

Then the composition $\varphi g$ is a $\mathbb{Q}$-bilinear map $\mathbb{Q}V \oplus \mathbb{Q}V \to \mathbb{Q}V$.

**Corollary 4.21.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. Let $a, b \in V$ be two distinct vertices of $\mathcal{G}$ with $d(a, b) > J$. For every $\aleph$ with $I \leq \aleph \leq J$ pick a vertex $u_\aleph \in V[a, b; \aleph]$. Then*

*(i) Every $u_\aleph$ is in the support of $\varphi g(a, b)$;*

*(ii) For every $x \in supp(\varphi g(a, b))$ there exists some $\aleph$ between $I$ and $J$ such that $d(x, u_\aleph) \leq 3\delta + R + 2 =: C_0$;*

(iii) $diam(supp(\varphi g(a,b))) \leq 2C_0 + J - I$.

*Proof.* The first two statements follow from lemma 4.20. The third statement follows from the second statement. □

And since $\mathrm{av}_{B_R^\zeta}$ is $\mathbb{Q}$-linear we can use proposition 4.18 to bound how much $g(a,b)$ changes with $b$.

**Proposition 4.22.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. There are constants $L > 0$, $\lambda \in (0,1)$ such that for all $a, b, b' \in V$,*

$$|g(a,b) - g(a,b')|_1 \leq L\lambda^{(b|b')_a}.$$

*Proof.* Proposition 4.18 tell us that $|g_\aleph(a,b) - g_\aleph(a,b')|_1 \leq L\lambda^{(b|b')_a}$ for any $\aleph$ and then, using the $\mathbb{Q}$-linearity of $\mathrm{av}_{B_R^\zeta}$ we get

$$\begin{aligned}\left|\mathrm{av}_{B_R^\zeta} g_\aleph(a,b) - \mathrm{av}_{B_R^\zeta} g_\aleph(a,b')\right|_1 &= \left|\mathrm{av}_{B_R^\zeta}(g_\aleph(a,b) - g_\aleph(a,b'))\right|_1 \\ &= |g_\aleph(a,b) - g_\aleph(a,b')|_1 \\ &\leq L\lambda^{(b|b')_a}\end{aligned}$$

Then $g(a,b) - g(a,b')$ is an average of the $\mathrm{av}_{B_R^\zeta} g_\aleph(a,b) - \mathrm{av}_{B_R^\zeta} g_\aleph(a,b')$ so its $l^1$-norm is also bounded by $L\lambda^{(b|b')_a}$. □

For what follows we will need assumptions on the constants $R$ and $I$. Hence we add to the assumption 4.17 to now assume the following.

**Assumptions 4.23.** We assume the constants $\mu$, $R$, and $I$ are chosen to satisfy the following inequalities;

$$\begin{aligned}2\mu &\geq 7\delta + 4 \\ R &\geq 4\delta \\ I &\geq C_0 + 1\end{aligned}$$

where $C_0 = 3\delta + R + 2$ is the constant that appears in corollary 4.21(*ii*). We will use the lower bound on $R$ to show that there is an overlap in proposition 4.25. The proof of that proposition will be via induction, for which we need to know that the distance from $b$ to any element in the support of $\varphi g(a,b)$ is strictly less than $d(a,b)$. This fact is the focus of the next lemma, and needs the lower bound on $I$.

**Lemma 4.24.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. If $a, b \in V$ satisfy $d(a,b) > J + 1$ then for all $x \in supp(\varphi g(a,b))$,*

$$d(x,b) \leq d(a,b) - 1.$$

*Proof.* Given $x \in \operatorname{supp}(\varphi g(a,b))$ for any $u_\aleph \in V[a,b;\aleph]$ corollary 4.21 tells us that there is some $\aleph$ with $d(x,u_\aleph) \leq C_0$ where $C_0 = 3\delta + R + 2$ as in corollary 4.21(*ii*). Then

$$\begin{aligned} d(x,b) &\leq d(x,u_\aleph) + d(u_\aleph,b) \\ &\leq C_0 + d(a,b) - d(a,u_\aleph) \\ &\leq C_0 + d(a,b) - I \\ &\leq d(a,b) - 1. \end{aligned}$$ □

Next we turn to look at what happens when we change the first variable of $g$. We consider a special case (as we did when first considering the effect of changing the second variable). We will need to use hyperbolicity of the triangle $\Delta(a,a',b)$ around the corner $b$, so we assume that the vertices $a,a',b \in V$ satisfy $|d(a,b) - d(a',b)| \leq J - I$ to ensure that we can find some $\aleph,\aleph'$ with $I \leq \aleph,\aleph' \leq J$ such that $d(a,b) - \aleph = d(a',b) - \aleph'$ and we also assume that both $(a|b)_{a'}$ and $(a'|b)_a$ are bounded by $J$ so that we may apply hyperbolicity to move from the geodesic $[a,b]$ to the geodesic $[a',b]$.

**Proposition 4.25.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. There is a constant $\nu \in (0,1)$ such that for all $a,a',b \in V$, if*

- $d(a,b) \geq J+1$ *and* $d(a',b) \geq J+1$
- $|d(a,b) - d(a',b)| \leq J - I$
- $(a|b)_{a'} \leq J$ *and* $(a'|b)_a \leq J$

*then*

$$|g(a,b) - g(a',b)|_1 \leq 2\nu.$$

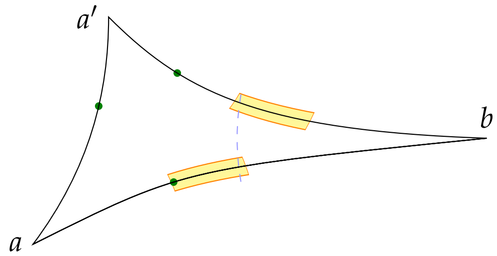
*Proof.* Without loss of generality $d(a',b) \leq d(a,b)$. So the first assumption becomes $0 \leq d(a,b) - d(a',b) \leq J - I$. Pick an edge $e_0 \in E[a,b;J]$. Then $\operatorname{supp}(g_J(a,b)) \subseteq \mathrm{Fl}[a,e_0;3\delta]$ by lemma 4.15. The idea behind the proof is to show that there is a corresponding edge $e_0' \in E[a',b]$ such that $d(e_0',b) = d(e_0,b) \leq (a|a')_b$ and $I \leq d(a',e_0') \leq J$. The first condition allows us to bound $d^\zeta(e_0,e_0')$ and the second condition ensures that $e_0'$ contributes to $g(a',b)$. These two facts lead to a bound on the size of the overlap between $g(a,b)$ and $g(a',b)$.

So pick an edge $e_0' \in E[b,a';d(b,e_0)]$. Then $d' := d(a',e_0') \leq J$ and from the first assumption we also obtain

$$\begin{aligned} d' &= d(a',b) - 1 - d(b,e_0') \\ &= d(a',b) - 1 - d(b,e_0) \\ &= d(a',b) - d(a,b) + d(a,e_0) \\ &\geq -(J-I) + J \\ &= I \end{aligned}$$

Thus $I \leq d' \leq J$ and $\mathrm{av}_{B_R^\zeta}(g_{d'}(a',b))$ contributes to $g(a',b)$.

The picture below shows a geodesic triangle with vertices $a, a', b$.




The (yellow) bars represent the union of the supports of the $g_\aleph(a,b)$ and $g_\aleph(a',b)$ for $I \leq \aleph \leq J$. The dashed (blue) line is part of a sphere around $b$ that intersects both (yellow) bars. The thick (green) dots are the inner points of the triangle.

We know that $(a'|b)_a \leq J$ and so $d(e_0,b) \leq (a|a')_b$, thus we can apply hyperbolicity (see lemma 2.50) to get $d^\zeta(e_0,e_0') \leq \delta$.

Also, by lemma 4.15, the support of $g_J(a,b)$ is contained in $\mathrm{Fl}[a,e_0;3\delta]$. Hence for all $e \in \mathrm{supp}(g_J(a,b))$

$$\begin{aligned} d^\zeta(e,e_0') &\leq d^\zeta(e,e_0) + d^\zeta(e_0,e_0') \\ &\leq \quad 3\delta \quad + \quad \delta \\ &= \quad 4\delta. \end{aligned}$$

Then for any $e' \in \mathrm{supp}(g_{d'}(a',b))$ we have $e' \in \mathrm{Fl}[a',e_0';3\delta]$ (by lemma 4.15) and so $e_0' \in B_R^\zeta(e) \cap B_R^\zeta(e')$. Thus there is an overlap between $\mathrm{av}_{B_R^\zeta}(g_J(a,b))$ and $\mathrm{av}_{B_R^\zeta}(g_{d'}(a',b))$. We bound this analogously to the proof of lemma 4.6.

$$\begin{aligned} &\left|\mathrm{av}_{B_R^\zeta}(g_J(a,b)) - \mathrm{av}_{B_R^\zeta}(g_{d'}(a',b))\right|_1 \\ &\qquad \leq \sum_{e\in S_J}\sum_{e'\in S'_{d'}} \frac{\alpha_e}{|B_R^\zeta(e)|}\frac{\alpha'_{e'}}{|B_R^\zeta(e')|}\sum_{\bar{e}\in B_R^\zeta(e)}\sum_{\bar{e}'\in B_R^\zeta(e')}|\bar{e}-\bar{e}'|_1 \\ &\qquad \leq 2\left(1-\frac{1}{(B_R^\zeta)^2}\right) \end{aligned} \tag{4.4}$$

where $S_J := \mathrm{supp}(g_J(a,b))$, $S'_{d'} := \mathrm{supp}(g_{d'}(a',b))$ and we express $g_J(a,b)$ and $g_{d'}(a',b)$ as the convex combinations

$$g_J(a,b) = \sum_{e\in S_J}\alpha_e e\,, \quad g_{d'}(a',b) = \sum_{e'\in S'_{d'}}\alpha'_{e'}e'.$$

So

$$\begin{aligned}
&\left|g(a,b)-g(a',b)\right|_1 \\
&\quad = \left|\frac{1}{1+J-I}\sum_{\aleph=I}^{J}\operatorname{av}_{B_R^\zeta}(g_\aleph(a,b)) - \frac{1}{1+J-I}\sum_{\aleph'=I}^{J}\operatorname{av}_{B_R^\zeta}(g_{\aleph'}(a',b))\right|_1 \\
&\quad \le \frac{1}{(1+J-I)^2}\sum_{\aleph,\aleph'=I}^{J}\left|\operatorname{av}_{B_R^\zeta} g_\aleph(a,b) - \operatorname{av}_{B_R^\zeta}(g_{\aleph'}(a',b))\right|_1 .
\end{aligned}$$

All these summands are bounded by 2, but the particular summand with $\aleph = J$ and $\aleph' = d'$ can be bounded as in inequality (4.4). Thus

$$\begin{aligned}
\left|g(a,b)-g(a',b)\right|_1 &\le \frac{1}{(1+J-I)^2}\left(2\big((1+J-I)^2-1\big)+2\left(1-\frac{1}{(B_R^\zeta)^2}\right)\right) \\
&= 2\left(1-\frac{1}{(1+J-I)^2(B_R^\zeta)^2}\right).
\end{aligned}$$

Therefore we set

$$\nu = 1-\frac{1}{(1+J-I)^2(B_R^\zeta)^2} \in (0,1)$$

to conclude $|g(a,b)-g(a',b)|_1 \le 2\nu$. □

We end the section by proving theorem 4.1, which summarises what has been done in this section.

*Proof of Theorem 4.1.* *(i)* The map $g$ is invariant by construction, this was explicitly stated after the definition of $g$ in definition 4.19.

*(ii)* The support of $g(a,b)$ contains the sets $E[a,b;\aleph]$ for $I \le \aleph \le J$ as stated in lemma 4.20, which followed immediately from lemma 4.15, which itself followed straight from lemma 4.11.

*(iii)* The support of $\varphi g(a,b)$ has distance strictly less than $d(a,b)$ to $b$ by lemma 4.24.

*(v)* The diameter of $\operatorname{supp}(\varphi g(a,b))$ was bounded in corollary 4.21*(iii)*, which is an easy corollary of lemma 4.20.

*(vi)* The bound when varying the second coordinate of $g$ was obtained in proposition 4.22, but the bulk of the work behind the proposition came in proposition 4.18 where the bound was obtained for the map $g_\aleph$ with arbitrary initial constant $\aleph$.

*(vii)* The bound when varying the first coordinate of $g$ was obtained in proposition 4.25. □

# 5 A New Metric on the Vertex Set

Given a uniformly fine, Gromov hyperbolic graph $\mathcal{G}$ we need to use the map $g\colon \mathbb{Q}V \oplus \mathbb{Q}V \to \mathbb{Q}E$ from theorem 4.1 to construct a metric on $V$. In section 5.1 we will define a function $r\colon \mathbb{Q}V \oplus \mathbb{Q}V \to [0, \infty)$, and prove some properties about this function, which will help for the triangle inequality. Section 5.2 is about the double difference of this function $r$. Then in section 5.3 we can define the metric $\hat{d}$ on the vertex set $V$ of the graph $\mathcal{G}$, and state some properties that this metric satisfies.

## 5.1 A first attempt at a metric

In chapter 4 we constructed the element $\varphi g(a, b) \in \mathbb{Q}V$ for vertices $a, b$ in a fine, Gromov hyperbolic graph $\mathcal{G}$, by moving $b$ in small steps towards $a$. This can also be thought of as moving $a$ one step closer to $b$, and then we can count how many steps it takes to move $a$ to $b$.

**Definition 5.1.** Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. As in chapter 4 we need to fix the constants $\mu, R, I$, and $J$ satisfying the inequalities given as assumptions 4.23. The map $r\colon \mathbb{Q}V \oplus \mathbb{Q}V \to [0, \infty)$ is defined recursively, given on vertices $a, b \in V$ by

$$r(a,b) := \begin{cases} 0 & \text{if } a = b \\ 1 & \text{if } 0 < d(a,b) \leq J+1 \\ 1 + r\left(\varphi g(a,b), b\right) & \text{if } d(a,b) > J+1 \end{cases}$$

and then $\mathbb{Q}$-linearly extending. To see that the map is well-defined we need to use theorem 4.1*(iii)* to say that the distance from $b$ to any vertex in the support of $\varphi g(a,b)$ is strictly less than $d(a,b)$, thus the recursion must stop in $\leq d(a,b)$ steps.

The map $r\colon \mathbb{Q}V \oplus \mathbb{Q}V \to [0,\infty)$ is our first attempt at a metric. It is positive-definite but not symmetric. However we can symmetrise it later. We want something resembling the triangle-inequality. Since $r$ is not symmetric, we will consider $|r(a,b) - r(a',b)|$ and $|r(a,b) - r(a,b')|$ separately. But before that, we compare the function $r$ to the original metric $d$ on $V$.

**Proposition 5.2.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. For all vertices $a,b \in V$ we have*

$$\frac{1}{J+1} d(a,b) \leq r(a,b) \leq d(a,b).$$

*Proof.* Induct on $d(a,b)$.

If $a = b$ then by definition $r(a,b) = 0$.

If $0 < d(a,b) \leq J+1$ then $r(a,b) = 1 \leq d(a,b)$ but also

$$\frac{1}{J+1} d(a,b) \leq \frac{1}{J+1}(J+1) = 1.$$

If $d(a,b) > J+1$ then $r(a,b) = 1 + r(\varphi g(a,b), b)$. For any $x$ in the support of $\varphi g(a,b)$ theorem 4.1(*iii*) tells us that $d(x,b) \leq d(a,b) - 1$, so the inductive hypothesis gives

$$\frac{1}{J+1} d(x,b) \leq r(x,b) \leq d(x,b).$$

If we write $\varphi g(a,b) = \sum_{x \in V} \alpha_x x$ where $\alpha_x \in [0,1]$ and $\sum_{x\in V} \alpha_x = 1$, which can be done since $\varphi g(a,b)$ is a convex combination, then

$$\begin{aligned} r(a,b) &= 1 + \sum_{x\in V} \alpha_x r(x,b) \\ &\leq 1 + \sum_{x\in V} \alpha_x d(x,b) \\ &\leq 1 + \sum_{x\in V} \alpha_x \big(d(a,b) - 1\big) \\ &= d(a,b). \end{aligned}$$

Moreover, for any $x \in \operatorname{supp}(\varphi g(a,b))$,

$$d(x,b) \geq d(a,b) - d(a,x) \geq d(a,b) - J$$

and so

$$\begin{aligned} r(a,b) &= 1 + \sum_{x \in V} \alpha_x r(x,b) \\ &\geq 1 + \sum_{x \in V} \frac{\alpha_x}{J+1} d(x,b) \\ &\geq 1 + \sum_{x \in V} \frac{\alpha_x}{J+1} \big(d(a,b) - J\big) \\ &\geq \frac{1}{J+1} d(a,b). \end{aligned}$$

□

**Remark 5.3.** The function $r\colon \mathbb{Q}V \oplus \mathbb{Q}V \to [0,\infty)$ is invariant under the action of $\mathrm{Isom}(\mathcal{G})$, i.e. if $\psi$ is an isometry of $\mathcal{G}$ (with respect to the word metric $d$) then for all $a, b \in V$

$$r\big(\psi(a), \psi(b)\big) = r(a,b).$$

For what follows we need to make more assumptions about the constants $I$ and $J$, so we add to assumptions 4.23 to get;

**Assumptions 5.4.** We assume the constants $\mu, R, I,$ and $J$ are chosen to satisfy the following inequalities;

$$\begin{aligned} 2\mu &\geq 7\delta + 4 \\ R &\geq 4\delta \\ I &\geq C_0 + \delta + 1 \\ J - I &\geq C_0 + \delta + 1 + I \end{aligned}$$

where $C_0$ is the constant from theorem 4.1$(iv)$. We have increased the lower bound on $I$ and introduced a lower bound on $J$. These new assumptions are there to allow us to show that the distance from $a'$ to an element in the support of $\varphi g(a,b)$ is strictly less than $d(a,a')$, under certain conditions on the vertices $a, a', b$. This is for the induction in the proof of lemma 5.5.

Now we turn to the triangle inequality. The next lemma gives a bound for $|r(a,b) - r(a',b)|$ in terms of $d(a,a')$.

**Lemma 5.5.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. There exists a constant $N \geq 0$ such that for all vertices $a, a', b \in V$ we have*

$$|r(a,b) - r(a',b)| \leq d(a,a') + N.$$

*Proof.* Without loss of generality, $d(a',b) \leq d(a,b)$. Pick $N \geq J + 2$ large enough to satisfy the equation

$$(1-\nu)N \geq 2\nu(C_0 + J) \tag{5.1}$$

where $\nu \in (0,1)$ is the constant from theorem 4.1$(vii)$ and $C_0$ is the constant from theorem 4.1$(iv)$.

The idea behind the proof is to move at least one of $a, a'$ closer to $b$ and use the inductive definition of $r$, hence we induct over $d(a,b) + d(a',b)$. But first we need a start point for the induction.

If $d(a',b) \leq J+1$ then $r(a',b) \in \{0,1\}$ by definition. Thus

$$\begin{aligned} |r(a,b) - r(a',b)| &\leq r(a,b) + r(a',b) \\ &\leq d(a,b) + 1 \\ &\leq d(a,a') + d(a',b) + 1 \\ &\leq d(a,a') + (J+1) + 1 \\ &\leq d(a,a') + N. \end{aligned}$$

Furthermore, if $d(a,b) + d(a',b) \leq 2(J+1)$ then $d(a',b) \leq J+1$, so for the inductive step we may assume $J+1 \leq d(a',b) \leq d(a,b)$. Then by the inductive definition $r(a,b) = 1 + r(\varphi g(a,b), b)$ and similarly for $r(a',b)$.

The idea is to move $a$ to $\varphi g(a,b)$, which is closer to $b$, by theorem 4.1$(iii)$. So moving $a$ to $\varphi g(a,b)$ reduces $d(a,b) + d(a',b)$ and we can apply the inductive hypothesis. However, $\varphi g(a,b)$ need not be closer to $a'$, and may even be further away. Hence we split the proof into three cases;

*Case* 1 When $a$ and $a'$ are about the same distance away from $b$ and this distance is large in comparison to $d(a,a')$.

*Case* 2 When $a$ and $a'$ are about the same distance away from $b$ but $d(a,a')$ is comparatively large.

*Case* 3 When $a$ and $a'$ are not roughly the same distance away from $b$.

In the first case we will move $a$ to $\varphi g(a,b)$ and $a'$ to $\varphi g(a',b)$ to get closer to $b$. This may increase $d(a,a')$ but we can use theorem 4.1$(vii)$ to compensate for this possibility.

In the second case we will move only $a$ to $\varphi g(a,b)$, but using the hyperbolic nature of the graph we can show that we have moved closer to $a'$ too.

In the third case, we move only $a$, and this moves us closer to $a'$ too because $a$ was much further away from $b$ than $a'$ is.

Before tackling these three cases we fix some notation.

Let $\Delta = \Delta([a,b],[a',b],[a,a'])$ be a geodesic triangle with vertices $a, a', b$. For $I \leq \aleph \leq J$ let $u_\aleph$ be the vertex of $[a,b]$ whose distance from $a$ is $\aleph$, and let $u'_\aleph$ be the vertex of $[a',b]$ whose distance from $a'$ is $\aleph$.

**Case 1:** Assume the following;

- $d(a,b) - d(a',b) \leq J - I$

- $(a|b)_{a'} \leq J$ and $(a'|b)_a \leq J$

i.e., assume that $a$ and $a'$ are about the same distance away from $b$ and that this distance is large in comparison to $d(a,a')$. We are interested in the expression

$$\begin{aligned}|r(a,b) - r(a',b)| &= |r(\varphi g(a,b),b) - r(\varphi g(a',b),b)| \\ &= |r(\varphi(g(a,b) - g(a',b)),b)|.\end{aligned}$$

There is some cancellation between $g(a,b)$ and $g(a',b)$, which is forumlated in theorem 4.1(*vii*). To use this cancellation we define two new elements $g_+, g_- \in \mathbb{Q}V$ by:

- $g_+ - g_- = g(a,b) - g(a',b)$
- all coefficients of $g_\pm$ are positive
- $\mathrm{supp}(g_+) \cap \mathrm{supp}(g_-) = \emptyset$

These three conditions determine $g_+$ and $g_-$. All we have done is cancel out as much as possible from $g(a,b) - g(a',b)$. In particular, since the support of $g_+$ is disjoint from the support of $g_-$ we have

$$\begin{aligned}|g_+|_1 + |g_-|_1 &= |g_+ - g_-|_1 \\ &= |g(a,b) - g(a',b)|_1 \\ &\leq 2\nu.\end{aligned}$$

Moreover, if we let $\epsilon\colon \mathbb{Q}V \to \mathbb{Q}$ be the augmentation homomorphism then the non-negativity of the coefficients of $g_\pm$ yields $|g_\pm|_1 = \epsilon(g_\pm)$. Then

$$\begin{aligned}|g_+|_1 - |g_-|_1 &= \epsilon(g_+) - \epsilon(g_-) \\ &= \epsilon(g_+ - g_-) \\ &= \epsilon(g(a,b) - g(a',b)) \\ &= \epsilon(g(a,b)) - \epsilon(g(a',b)) \\ &= |g(a,b)|_1 - |g(a',b)|_1 \\ &= 0\end{aligned}$$

Therefore $|g_+|_1 = |g_-|_1$.

Combining the previous two facts gives

$$|g_\pm|_1 = \frac{1}{2}(|g_+|_1 + |g_-|_1) \leq \nu.$$

Write $\varphi g_+ = \sum_{x \in V} \alpha_x x$ and $g_- = \sum_{x' \in V} \alpha'_{x'} x'$.

Observe that the support of $\varphi g_+$ is contained in the support of $\varphi g(a,b)$ and the support of $\varphi g_-$ is contained in the support of $\varphi g(a',b)$ and thus theorem 4.1(*iii*) tells us that for all $x \in \operatorname{supp}(\varphi g_+)$ and all $x' \in \operatorname{supp}(\varphi g_-)$ we have

$$d(x,b) + d(x',b) \leq d(a,b) - 1 + d(a',b) - 1.$$

Hence we can apply the inductive hypothesis to $x, x', b$ to obtain

$$|r(x,b) - r(x',b)| \leq d(x,x') + N.$$

Moreover, we know from theorem 4.1(*iv*) that there is some $\aleph$ and some $\aleph'$ between $I$ and $J$ such that $d^{\zeta}(x, u_{\aleph}) \leq C_0$ and $d^{\zeta}(x', u'_{\aleph'}) \leq C_0$. Therefore

$$\begin{aligned} |r(x,b) - r(x',b)| &\leq d(x,u_{\aleph}) + d(u_{\aleph}, u'_{\aleph'}) + d(u'_{\aleph'}, x') + N \\ &\leq 2C_0 + (J + d(a,a') + J) + N. \end{aligned}$$

Now we have

$$\begin{aligned} |r(a,b) - r(a',b)| &= |r(\varphi g_+, b) - r(\varphi g_-, b)| \\ &= \left| \sum_{x \in V} \alpha_x r(x,b) - \sum_{x' \in V} \alpha'_{x'} r(x',b) \right| \\ &\leq \sum_{x,x' \in V} \frac{\alpha_x \alpha'_{x'}}{|g_{\pm}|_1} \left| r(x,b) - r(x',b) \right| \\ &\leq \sum_{x,x' \in V} \frac{\alpha_x \alpha'_{x'}}{|g_{\pm}|_1} (2C_0 + 2J + d(a,a') + N) \\ &= |g_{\pm}|_1 (2C_0 + 2J + d(a,a') + N) \\ &\leq \nu(2C_0 + 2J + d(a,a') + N) \\ &\leq d(a,a') + N \end{aligned}$$

since we chose $N$ large enough to satisfy equation (5.1). This finishes case 1.

**Case 2:** Assume that $J < (a'|b)_a$.

Here the Gromov product is large and so any element in the support of $\varphi g(a,b)$ should be closer to $a'$ than $a$ is, and so we only move $a$.

Write $\varphi g(a,b) = \sum_{x \in V} \alpha_x x$ with $\sum_{x \in V} \alpha_x = 1$. For any $x$ in the support of $\varphi g(a,b)$ we can find an $\aleph$ such that $d(x, u_{\aleph}) \leq C_0$. Let $v_{\aleph}$ be the point on the geodesic $[a,a']$ with $d(a, v_{\aleph}) = \aleph = d(a, u_{\aleph})$. Since $\aleph \leq J < (a'|b)_a$ hyperbolicity gives $d(u_{\aleph}, v_{\aleph}) \leq \delta$. Then the distance from $x$ to $a'$ is;

$$\begin{aligned} d(x,a') &\leq d(x,u_{\aleph}) + d(u_{\aleph}, v_{\aleph}) + d(v_{\aleph}, a') \\ &\leq C_0 + \delta + (d(a,a') - d(a, v_{\aleph})) \\ &= C_0 + \delta + d(a,a') - \aleph \\ &\leq C_0 + \delta + d(a,a') - I \\ &\leq d(a,a') - 1. \end{aligned}$$

Using theorem 4.1$(iii)$ we can apply the inductive hypothesis to $x, a', b$. Hence;

$$\begin{aligned} |r(a,b) - r(a',b)| &= |1 + r(\varphi g(a,b), b) - r(a',b)| \\ &\leq 1 + \sum_{x \in V} \alpha_x |r(x,b) - r(a',b)| \\ &\leq 1 + \sum_{x \in V} \alpha_x \left(d(x,a') + N\right)\right) \\ &\leq 1 + N + \sum_{x \in V} \alpha_x (d(a,a') - 1) \\ &= d(a,a') + N. \end{aligned}$$

This finishes case 2.

**Case 3:** Assume that $|d(a,b) - d(a',b)| > J - I$, i.e. assume that $a$ is much further away from $b$ than $a'$ is. Since we have already covered the possibility $J < (a'|b)_a$ in case 2, we may also assume that $(a'|b)_a \leq J$.

Write $\varphi g(a,b) = \sum_{x \in V} \alpha_x x$ with $\sum_{x \in V} \alpha_x = 1$. For any $x$ in the support of $\varphi g(a,b)$ take $\aleph$ with $d(x, u_\aleph) \leq C_0$. We aim to show that $d(x,a')$ is smaller than $d(a,a')$.

If $\aleph = d(u_\aleph, a) \leq (a'|b)_a$ then we can repeat the argument in case 2 to get $d(x,a') \leq d(a,a') - 1$.

If $d(u_\aleph, a) > (a'|b)_a$ then $d(b, u_\aleph) \leq (a|a')_b$ and we use hyperbolicity to move to $[a', b]$ and bound how far we need to travel along $[a', b]$ to $a'$. Let $w_\aleph \in [a', b]$ satisfy $d(b, w_\aleph) = d(b, u_\aleph)$. Then hyperbolicity tells us that $d(w_\aleph, u_\aleph) \leq \delta$. Thus

$$\begin{aligned} d(u_\aleph, a') &\leq d(u_\aleph, w_\aleph) + d(w_\aleph, a') \\ &\leq \delta + (d(a',b) - d(b, w_\aleph)) \\ &= \delta + d(a',b) - d(b, u_\aleph) \\ &= \delta + d(a',b) - (d(a,b) - d(a, u_\aleph)) \\ &\leq \delta + d(a',b) - d(a,b) + J \\ &\leq \delta - (J - I) + J \\ &= \delta + I \end{aligned}$$

Moreover we know

$$d(a,a') \geq |d(a,b) - d(a',b)| > J - I$$

and so

$$\begin{aligned} d(x,a') &\leq d(x, u_\aleph) + d(u_\aleph, a') \\ &\leq C_0 + (\delta + I) \\ &\leq (d(a,a') - (J - I)) + C_0 + \delta + I \\ &\leq d(a,a') - 1. \end{aligned}$$

Then we proceed as in case 2 and conclude

$$|r(a,b) - r(a',b)| = d(a,a') + N.$$

This finishes case 3.

That covers all possible cases and therefore we have finished the inductive proof. $\square$

The following corollary can be thought of as a means for bounding the size of error terms in later proofs.

**Corollary 5.6.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. Let $\epsilon\colon \mathbb{Q}V \to \mathbb{Q}$ be the augmentation homomorphism. There is a constant $D > 0$ such that for any $b \in V$ and any $z \in \mathbb{Q}V$ if $\epsilon(z) = 0$ then*

$$|r(z,b)| \leq D|z|_1 diam(supp(z)).$$

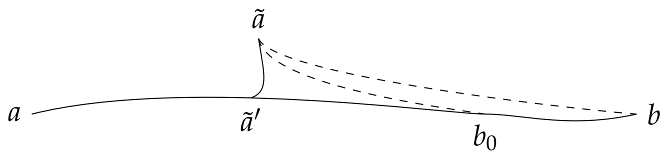
*Proof.* Let $z = \sum_{v\in V} \alpha_v v$. Then $\sum_{v\in V} \alpha_v = 0$ since $z$ is a cycle. So we can write $z$ in the form $z = \sum_{i\in I} \beta_i(u_i - w_i)$ where $\beta_i > 0$ and $\sum_{i\in I} \beta_i = \frac{1}{2}|z|_1$. Set $D = \frac{1}{2}(1+N)$, where $N$ is the constant from lemma 5.5. Applying lemma 5.5 gives

$$\begin{aligned} |r(z,b)| &\leq \sum_{i\in I} \beta_i |r(u_i,b) - r(w_i,b)| \\ &\leq \sum_{i\in I} \beta_i \big(d(u_i,w_i) + N\big) \\ &\leq \frac{1}{2}|z|_1\big(\mathrm{diam}(\mathrm{supp}(z)) + N\big) \\ &\leq D|z|_1 \mathrm{diam}(\mathrm{supp}(z)). \end{aligned}$$ $\square$

The next lemma goes towards proving that geodesics with respect to $d$ will not be not far away from being geodesics with respect to the new metric (which is defined in section 5.3).

**Lemma 5.7.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. There exists a constant $C_1 \geq 0$ such that if $\alpha$ is a geodesic in $\mathcal{G}$ from $a \in V$ to $b \in V$ then for all vertices $b_0$ on $\alpha$ and all vertices $\tilde{a} \in N(\alpha_{[a,b_0]}, C_0)$*

$$|r(\tilde{a},b) - r(\tilde{a},b_0) - r(b_0,b)| \leq C_1.$$



*Proof.* We induct on $d(\tilde{a}, b_0)$. Let $L > 0$ and $\lambda \in (0,1)$ be the constants from proposition 4.18, $N > 0$ be the constant from lemma 5.5, and $D > 0$ be the constant from corollary 5.6. In the induction we will acquire error terms but these error terms become exponentially small as $d(\tilde{a}, b)$ increases, so we take as our inductive hypothesis that

$$|r(\tilde{a}, b) - r(\tilde{a}, b_0) - r(b_0, b)| \le C_1' \sum_{k=0}^{d(\tilde{a}, b_0)} \lambda^k$$

where $C_1' := \max\{2(J+1) + 4C_0 + N, DL\lambda^{-2C_0}(2C_0 + 2J)\}$. If we can prove this then setting $C_1 = C_1' \sum_{k \in \mathbb{N}} \lambda^k = C_1'/1-\lambda$ finishes the proof.

To start the induction, assume $d(\tilde{a}, b_0) \le J + 1 + 2C_0$. Then

$$\begin{aligned}
|r(\tilde{a}, b) - r(\tilde{a}, b_0) - r(b_0, b)| &\le |r(\tilde{a}, b) - r(b_0, b)| + r(\tilde{a}, b_0) \\
&\le (d(\tilde{a}, b_0) + N) + d(\tilde{a}, b_0) \\
&\le 2(J + 1 + 2C_0) + N \\
&\le C_1' \sum_{k=0}^{d(\tilde{a}, b_0)} \lambda^k
\end{aligned}$$

using lemma 5.5, proposition 5.2 and the definition of $C_1'$.

For the inductive step assume $d(\tilde{a}, b_0) > J + 1 + 2C_0$. By the inductive definition $r(\tilde{a}, b_0) = 1 + r(\varphi g(\tilde{a}, b_0), b_0)$. Also, there is some vertex $\tilde{a}'$ on $\alpha_{[a,b_0]}$ such that $d(\tilde{a}, \tilde{a}') \le C_0$. Then $d(\tilde{a}', b) = d(\tilde{a}', b_0) + d(b_0, b)$ and

$$\begin{aligned}
d(\tilde{a}, b) &\ge d(\tilde{a}', b) - d(\tilde{a}, \tilde{a}') \ge d(\tilde{a}', b_0) - d(\tilde{a}, \tilde{a}') \ge d(\tilde{a}, b_0) - 2d(\tilde{a}, \tilde{a}') \\
&> J + 1 + 2C_0 - 2C_0 = J + 1
\end{aligned}$$

so $r(\tilde{a}, b) = 1 + r(\varphi g(\tilde{a}, b), b)$. Thus

$$\begin{aligned}
|r(\tilde{a}, b) - r(\tilde{a}, b_0) - r(b_0, b)| &= |r(\varphi g(\tilde{a}, b), b) - r(\varphi g(\tilde{a}, b_0), b_0) - r(b_0, b)| \\
&\le |r(\varphi g(\tilde{a}, b_0), b) - r(\varphi g(\tilde{a}, b_0), b_0) - r(b_0, b)| \\
&\quad + |r(\varphi g(\tilde{a}, b) - \varphi g(\tilde{a}, b_0), b)|.
\end{aligned}$$

We will bound these two terms separately, using the inductive hypothesis for the first term (the inductive term) and corollary 5.6 for the second term (the error term). We start by considering the inductive term. Write $\varphi g(\tilde{a}, b_0) = \sum_{y \in V} \alpha_y y$ with $\sum_{y \in V} \alpha_y = 1$. Then

$$\begin{aligned}
&|r(\varphi g(\tilde{a}, b_0), b) - r(\varphi g(\tilde{a}, b_0), b_0) - r(b_0, b)| \\
&\qquad \le \sum_{y \in V} \alpha_y |r(y, b) - r(y, b_0) - r(b_0, b)|
\end{aligned}$$

Theorem 4.1(*iii*) gives $d(y, b_0) \leq d(\tilde{a}, b_0) - 1$ for any $y \in \mathrm{supp}(\varphi g(\tilde{a}, b_0))$ and so we can apply the inductive hypothesis to get

$$\begin{aligned} |r(y,b) - r(y,b_0) - r(b_0,b)| &\leq C_1' \sum_{k=0}^{d(y,b_0)} \lambda^k \\ &\leq C_1' \sum_{k=0}^{d(\tilde{a},b_0)-1} \lambda^k. \end{aligned}$$

We still need to bound the error term $|r(\varphi g(\tilde{a}, b) - \varphi g(\tilde{a}, b_0), b)|$. Applying corollary 5.6 gives

$$\begin{aligned} &|r(\varphi g(\tilde{a},b) - \varphi g(\tilde{a},b_0),b)| \\ &\qquad \leq D|g(\tilde{a},b) - g(\tilde{a},b_0)|_1 \mathrm{diam}(\mathrm{supp}(\varphi g(\tilde{a},b) - \varphi g(\tilde{a},b_0))). \end{aligned}$$

and by theorem 4.1(*vi*) we have

$$|g(\tilde{a},b) - g(\tilde{a},b_0)|_1 \leq L\lambda^{(b|b_0)_{\tilde{a}}}.$$

Hence we need a lower bound for $(b|b_0)_{\tilde{a}}$ and an upper bound for the diameter of the support of $\varphi g(\tilde{a}, b) - \varphi g(\tilde{a}, b_0)$. We consider the Gromov product first.

$$\begin{aligned} (b|b_0)_{\tilde{a}} &= \frac{1}{2}\left(d(\tilde{a},b) + d(\tilde{a},b_0) - d(b,b_0)\right) \\ &\geq \frac{1}{2}\left(d(\tilde{a}',b) - C_0 + d(\tilde{a}',b_0) - C_0 - d(b,b_0)\right) \\ &= d(\tilde{a}',b_0) - C_0 \\ &\geq d(\tilde{a},b_0) - 2C_0. \end{aligned}$$

So $\lambda^{(b|b_0)_{\tilde{a}}} \leq \lambda^{d(\tilde{a},b_0)-2C_0}$. We still need an upper bound for the diameter of the support. Theorem 4.1(*v*) tells us that the diameter of the support of $\varphi g(\tilde{a}, b)$ is bounded by $2C_0 + J - I$, as is the diameter of the support of $\varphi g(\tilde{a}, b_0)$. We can bound the distance from an element in the support of $\varphi g(\tilde{a}, b)$ to an element in the support of $\varphi g(\tilde{a}, b_0)$ by going via the vertex $\tilde{a}$, hence we get a bound of $2C_0 + 2J$. Therefore

$$|r(\varphi g(\tilde{a},b) - \varphi g(\tilde{a},b_0),b)| \leq DL\lambda^{d(\tilde{a},b_0)-2C_0}(2C_0 + 2J).$$

Bringing everything together gives

$$\begin{aligned} |r(\tilde{a},b) - r(\tilde{a},b_0) - r(b_0,b)| &\leq C_1' \sum_{k=0}^{d(\tilde{a},b_0)-1} \lambda^k + DL\lambda^{d(\tilde{a},b_0)-2C_0}(2C_0 + 2J) \\ &\leq C_1' \sum_{k=0}^{d(\tilde{a},b_0)} \lambda^k. \end{aligned}$$

This finishes the inductive proof and the lemma follows as given at the start of this proof. □

For the triangle inequality we also need to look at $|r(a,b) - r(a,b')|$.

**Proposition 5.8.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. There exists a constant $M > 0$ such that for all $a, b, b' \in V$*

$$|r(a,b) - r(a,b')| \leq M\, d(b,b').$$

The proof will be by induction but in this induction there will be error terms so it is easier to use a different inductive hypothesis, which is given in the next lemma.

**Lemma 5.9.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. For any $a, b, b' \in V$ we have*

$$|r(a,b) - r(a,b')| \leq d(b,b') + 2(J+1) + \frac{1}{2}L \sum_{j=1}^{(b|b')_a} \lambda^j (2j + d(b,b')).$$

The term $2(J+1)$ is for the start of the induction, and the sum bounds the size of the error terms that we pick up. Before proving lemma 5.9 we will show how it implies proposition 5.8.

*Proof of Proposition 5.8.* If $d(b,b') = 0$ then $b = b'$ and the inequality is trivially satisfied (both sides equal zero). So assume that $d(b,b') > 0$. In particular $d(b,b') \geq 1$. Set $M = 1 + 2(J+1) + \frac{1}{2}L \sum_{j=1}^{\infty} \lambda^j (2j+1)$, where the infinte sum $\sum_{j=1}^{\infty} \lambda^j (2j+1)$ converges by using the ratio test. Then by lemma 5.9;

$$\begin{aligned} |r(a,b) - r(a,b')| &\leq d(b,b') + 2(J+1) + \frac{1}{2}L \sum_{j=1}^{(b|b')_a} \lambda^j (2j + d(b,b')) \\ &\leq \left(1 + 2(J+1) + \frac{1}{2}L \sum_{j=1}^{\infty} \lambda^j (2j+1)\right) d(b,b') \\ &= M\, d(b,b'). \end{aligned}$$

□

We still need to prove lemma 5.9.

*Proof of Lemma 5.9.* Without loss of generality $d(a,b) \geq d(a,b')$. The proof is by induction on $(b|b')_a$.

If $(b|b')_a \leq J+1$ then we use proposition 5.2;

$$\begin{aligned} |r(a,b) - r(a,b')| &\leq r(a,b) + r(a,b') \\ &\leq d(a,b) + d(a,b') \\ &= d(b,b') + 2(b|b')_a \\ &\leq d(b,b') + 2(J+1). \end{aligned}$$

This is the starting point for our induction.

So assume that $(b|b')_a > J+1$. Then $d(a,b) \geq d(a,b') \geq (b|b')_a > J+1$ and so

$$\begin{aligned} |r(a,b) - r(a,b')| &= |1 + r(\varphi g(a,b), b) - 1 - r(\varphi g(a,b'), b')| \\ &= |r(\varphi g(a,b), b) - r(\varphi g(a,b'), b')|. \end{aligned}$$

The idea is to say that since the Gromov product $(b|b')_a$ is large there is a large overlap between $g(a,b)$ and $g(a,b')$, and for any element in the support of both $g(a,b)$ and $g(a,b')$ we can apply the inductive hypothesis. We still need to worry about the elements in the support of one but not both, however these elements should not contribute much to the final total, and will be swallowed in the sum.

Formally, define new chains $g_+, g_-, g_0$ by saying that all their coefficients are positive, and that they satisfy;

- $\mathrm{supp}(g_+) \cap \mathrm{supp}(g_-) = \emptyset$
- $g(a,b) - g(a,b') = g_+ - g_-$
- $g(a,b) = g_+ + g_0$
- $g(a,b') = g_- + g_0$.

Then $g_0$ represents the overlap and $g_\pm$ represent the parts outside the overlap. Using the linearity of $r$ we have

$$\begin{aligned} &|r(\varphi g(a,b), b) - r(\varphi g(a,b'), b')| \\ &\qquad = |r(\varphi g_+, b) + r(\varphi g_0, b) - r(\varphi g_0, b') - r(\varphi g_-, b')| \\ &\qquad \leq |r(\varphi g_+, b)| + |r(\varphi g_0, b) - r(\varphi g_0, b')| + |r(\varphi g_-, b')|. \end{aligned}$$

We want to bound these three terms. The middle term is the inductive term. The first and third terms are error terms.

We start with the error terms $|r(\varphi g_+, b)|$ and $|r(\varphi g_-, b')|$. The contribution from these terms should be small because $|g_+|_1$ and $|g_-|_1$ are small. More precisely, by considering the augmentation homomorphism (as in case 1 of the proof of lemma 5.5) we know

$$|g_+|_1 - |g_-|_1 = |g(a,b)|_1 - |g(a,b')|_1 = 0$$

and thus $|g_+|_1 = |g_-|_1$. Moreover it follows from theorem 4.1$(vi)$ that

$$2|g_\pm|_1 = |g_+|_1 + |g_-|_1 = |g_+ - g_-|_1 = |g(a,b) - g(a,b')|_1 \leq L\lambda^{(b|b')_a}.$$

Write $\varphi g_+ = \sum_{x\in V} \alpha_x^+ x$. Theorem 4.1(*iii*) tells us that $d(x,b) \leq d(a,b) - 1$ for all $x \in \operatorname{supp}(\varphi g_+) \subseteq \operatorname{supp}(\varphi g(a,b))$. Hence proposition 5.2 gives;

$$\begin{aligned} |r(\varphi g_+, b)| &\leq \sum_{x\in V} \alpha_x^+ |r(x,b)| \\ &\leq \sum_{x\in V} \alpha_x^+ d(x,b) \\ &\leq \sum_{x\in V} \alpha_x^+ d(a,b) \\ &= |g_+|_1 d(a,b) \end{aligned}$$

Similarly $|r(\varphi g_-, b')| \leq |g_-|_1 d(a,b')$. Therefore

$$\begin{aligned} |r(\varphi g_+, b)| + |r(\varphi g_-, b')| &\leq |g_+|_1 d(a,b) + |g_-|_1 d(a,b') \\ &\leq \frac{1}{2} L\lambda^{(b|b')_a} \big(d(a,b) + d(a,b')\big) \\ &= \frac{1}{2} L\lambda^{(b|b')_a} \big(d(b,b') + 2(b|b')_a\big). \end{aligned}$$

Now consider the inductive term $|r(\varphi g_0, b) - r(\varphi g_0, b')|$. In barycentric coordinate write $\varphi g_0 = \sum_{y\in V} \alpha_y^0 y$ so

$$|r(\varphi g_0, b) - r(\varphi g_0, b')| \leq \sum_{y\in V} \alpha_y^0 |r(y,b) - r(y,b')|.$$

We want to apply the inductive hypothesis to $|r(y,b) - r(y,b')|$, so we need to know that $(b|b')_y < (b|b')_a$. The support of $\varphi g_0$ is contained in both the support of $\varphi g(a,b)$ and the support of $\varphi g(a,b')$ then theorem 4.1(*iii*) tells us that $d(y,b) \leq d(a,b) - 1$ and $d(y,b') \leq d(a,b') - 1$. Thus

$$\begin{aligned} (b|b')_y &= \frac{1}{2}\big(d(y,b) + d(y,b') - d(b,b')\big) \\ &\leq \frac{1}{2}\big(d(a,b) - 1 + d(a,b') - 1 - d(b,b')\big) \\ &= (b|b')_a - 1. \end{aligned}$$

Hence I can apply the inductive hypothesis to get

$$\begin{aligned} |r(y,b) - r(y,b')| &\leq d(b,b') + 2(J+1) + \frac{1}{2} L \sum_{j=1}^{(b|b')_y} \lambda^j \big(2j + d(b,b')\big) \\ &\leq d(b,b') + 2(J+1) + \frac{1}{2} L \sum_{j=1}^{(b|b')_a - 1} \lambda^j \big(2j + d(b,b')\big) \end{aligned}$$

and so

$$\begin{aligned} &|r(\varphi g_0, b) - r(\varphi g_0, b')| \\ &\qquad \leq |g_0|_1 \left( d(b,b') + 2(J+1) + \frac{1}{2} L \sum_{j=1}^{(b|b')_a - 1} \lambda^j \big(2j + d(b,b')\big) \right). \end{aligned}$$

Combining this with the bound on the other two terms together with the fact $|g_0|_1 \leq 1$ gives

$$\begin{aligned}|r(a,b) - r(a,b')| &\leq |r(g_+,b)| + |r(g_0,b) - r(g_0,b')| + |r(g_-,b')| \\ &\leq \frac{1}{2}L\lambda^{(b|b')_a}\left(d(b,b') + 2(b|b')_a\right) \\ &\qquad + d(b,b') + 2(J+1) + \frac{1}{2}L\sum_{j=1}^{(b|b')_a - 1}\lambda^j(2j + d(b,b')) \\ &= d(b,b') + 2(J+1) + \frac{1}{2}L\sum_{j=1}^{(b|b')_a}\lambda^j(2j + d(b,b'))\end{aligned}$$

as claimed. □

## 5.2 Convergence of the double difference

The reason for altering the metric on $\mathcal{G}$ was to be have better convergence properties at the boundary, namely we wanted to extend the double difference to the boundary (see section 3.4). Recall from proposition 3.16 that the double difference can be written as

$$(a,a'|b,b') = \frac{1}{2}\left(d(a,b) - d(a,b') - d(a',b) + d(a',b')\right).$$

Hence we look at $R(a,a',b,b') := r(a,b) - r(a,b') - r(a',b) + r(a',b')$. Our goal for this section is the following proposition about the behaviour of $R$.

**Proposition 5.10.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. There exists constants $C \geq 0$ and $\omega \in (0,1)$ such that for all $a,a',b,b' \in V$, if $d(a,a') \leq 1$ and $d(b,b') \leq 1$ then*

$$|R(a,a',b,b')| \leq C\omega^{d(a,b)}.$$

We will use induction on $d(a,b) + d(a',b)$ to prove proposition 5.10, for which we will move $a$ and/or $a'$ towards $b$ and $b'$. However, when we do this we will not always be able to assume $d(x,x') \leq 1$ to apply the inductive hypothesis to $R(x,x',b,b')$. Therefore we have to allow $d(a,a')$ to be larger in the inductive hypothesis. This forces us to take account of $d(a,a')$ in the right-hand side of the inequality.

To make life easier, we fix some notation:

$$\begin{aligned}D(a,a',b) &:= d(a,b) + d(a',b) \\ \zeta &:= 2C_0 + 2(J-I) + \delta.\end{aligned}$$

Then the inductive hypothesis we will actually use is the following.

**Proposition 5.11.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. There are constants $A, B > 0$ and $\rho \in (0,1)$ such that for any $a, a', b, b' \in V$, if $d(a,a') \leq \zeta$ and $d(b,b') \leq 1$ then*

$$|R(a,a',b,b')| \leq (A\, d(a,a') + B)\rho^{D(a,a',b)}.$$

Before proving the more induction-friendly proposition 5.11, we show how it implies proposition 5.10

*Proof of Proposition 5.10.* In proposition 5.10 we assume that $d(a,a') \leq 1$, so proposition 5.11 tells us

$$|R(a,a',b,b')| \leq (A+B)\rho^{D(a,a',b)}.$$

Moreover $D(a,a',b) \geq 2d(a,b) - 1$ since $d(a',b) \geq d(a,b) - 1$ by the triangle inequality. Hence

$$|R(a,a',b,b')| \leq (A+B)\rho^{2d(a,b)-1}$$

and proposition 5.10 follows from setting the constants $C = (A+B)\rho^{-1}$ and $\omega = \rho^2$. □

Throughout the rest of this section we keep the notation and assumptions as in proposition 5.11, namely we assume the following;

**Assumptions.** Suppose $\mathcal{G}$ is a uniformly fine, $\delta$-hyperbolic graph and assume $a, a', b, b' \in V$ are vertices such that $d(a,a') \leq \zeta$ and $d(b,b') \leq 1$. Moreover assume, without loss of generality, that $d(a,b) \geq d(a',b)$.

We are also still assuming that the constants $\mu, R, I$ and $J$ satisfy the inequalities as laid out as assumptions 5.4.

The proof of proposition 5.11 will be an induction on $D(a,a',b)$. We begin with a lemma which will start the induction.

**Lemma 5.12.** *For any $\xi > 0$, if $D(a,a',b) \leq \xi$ then*

$$|R(a,a',b,b')| \leq 2\xi + 2.$$

*Proof.* We use proposition 5.2 as follows:

$$\begin{aligned}|R(a,a',b,b')| &\leq r(a,b) + r(a,b') + r(a',b) + r(a',b') \\ &\leq d(a,b) + d(a,b') + d(a',b) + d(a',b') \\ &\leq 2d(a,b) + d(b,b') + 2d(a',b) + d(b,b') \\ &\leq 2\xi + 2.\end{aligned}$$

□

Now we need to prove the inductive step.

We shall split into two cases and get an inequality in each case.

The first case is when the distance from $a$ to $b$ is much larger than the distance from $a'$ to $b$. This is covered in lemma 5.13.

The second case is when $a$ and $a'$ have roughly the same distance to $b$. This is covered in lemma 5.14.

After this has been done we will need to show that the constants $A, B, \rho$ can be chosen such that these inequalities are always satisfied, which is done in lemma 5.24.

Note that $(a'|b)_a \geq (a|b)_{a'}$ since $d(a,b) \geq d(a',b)$ (by assumption). Also, by the definition of $\zeta$ and the assumptions 5.4

$$\begin{aligned} d(a,a') &\leq \zeta \\ &= 2C_0 + 2(J-I) + \delta \\ &\leq 2J - \delta - 2 \\ &< J \end{aligned} \tag{5.2}$$

In particular, both $(a|b)_{a'} < J$ and $(a'|b)_a < J$.

First we consider the case where $d(a,b)$ is much larger than $d(a',b)$.

**Lemma 5.13.** *If $D(a,a',b) > 2(J+2)$ and $d(a,b) - d(a',b) > J - I$ then*

$$\begin{aligned} |R(a,a',b,b')| \leq (A\, d(a,a') - A + B)\rho^{D(a,a',b)-J-C_0} \\ + 2DL\lambda^{\frac{1}{2}(D(a,a',b)-\zeta-2)}(2C_0 + J - I). \end{aligned}$$

*Proof.* The idea is use $\varphi g$ to move $a$ towards $b$ and $b'$, and by doing so we will also have moved closer to $a'$. There will be a large overlap between $\varphi g(a,b)$ and $\varphi g(a,b')$, on which we can apply the inductive hypothesis and this will give the first term on the right-hand side of the inequality. Then there are two small error terms given by the part of $\varphi g(a,b)$ that is not also in $\varphi g(a,b')$ and vice versa. These two error terms will be bounded using corollary 5.6 and yield the second term of the inequality.

That is the rough idea. Now we have to go through the argument formally. We are assuming that $d(a,b) \geq d(a',b)$. Hence

$$2d(a,b) \geq d(a,b) + d(a',b) = D(a,a',b) > 2(J+2)$$

and so $d(a,b) > J+2$. Then, by definition, $r(a,b) = 1 + r(\varphi g(a,b), b)$. Since $d(b,b') \leq 1$ the triangle inequality yields

$$d(a,b') \geq d(a,b) - d(b,b') > J+1$$

and so $r(a,b') = 1 + r(\varphi g(a,b'), b')$. Therefore

$$|R(a,a',b,b')| = |r(\varphi g(a,b),b) - r(\varphi g(a,b'),b') - r(a',b) + r(a',b')|.$$

As it is, the right-hand side cannot be written in terms of summands of the form $R(x, a', b, b')$. However, there will be an overlap in the support of $g(a, b)$ and the support of $g(a, b')$ and on this overlap we could try to use the inductive hypothesis. We make this formal, as in the proof of lemma 5.9, by defining new 0-chains $g_+, g_-, g_0$ to have the following conditions;

- all coefficients of $g_+, g_-, g_0$ are positive
- $g(a, b) - g(a, b') = g_+ - g_-$
- $g(a, b) - g_+ = g_0 = g(a, b') - g_-$
- $\operatorname{supp}(g_+) \cap \operatorname{supp}(g_-) = \emptyset$.

It follows, using a similar argument as the one in the proof of lemma 5.9, that

$$\alpha := |g_+|_1 = |g_-|_1 \leq \frac{1}{2} L \lambda^{(b|b')_a}.$$

Here the chain $g_0$ represents the overlap. Now

$$\begin{aligned}
&|R(a, a', b, b')| \\
&\quad = |r(\varphi g(a, b), b) - r(\varphi g(a, b'), b') - r(a', b) + r(a', b')| \\
&\quad = |r(\varphi g_+, b) + r(\varphi g_0, b) - r(\varphi g_-, b') - r(\varphi g_0, b') - r(a', b) + r(a', b')| \\
&\quad = |R(\varphi g_0, a', b, b') + r(\varphi g_+, b) - r(\varphi g_-, b')|.
\end{aligned}$$

The function $R$ is not linear so for any arbitrary element $\sum_{x \in V} \alpha_x x \in \mathbb{Q}V$ the equality $R(\sum_{x \in V} \alpha_x x, a', b, b') = \sum_{x \in V} \alpha_x R(x, a', b, b')$ does not necessarily hold but it does hold if $\sum_{x \in V} \alpha_x = 1$. Here $|g_0|_1 = 1 - \alpha$. To get around this problem, we fix a vertex $x_0 \in V[a, b; I]$ and consider $\varphi g_0 + \alpha x_0$;

$$\begin{aligned}
&|R(a, a', b, b')| \\
&\quad = |R(\varphi g_0, a', b, b') + r(\varphi g_+, b) - r(\varphi g_-, b')| \\
&\quad = |R(\varphi g_0 + \alpha x_0, a', b, b') + r(\varphi g_+ - \alpha x_0, b) - r(\varphi g_- - \alpha x_0, b')| \\
&\quad \leq |R(\varphi g_0 + \alpha x_0, a', b, b')| + |r(\varphi g_+ - \alpha x_0, b)| + |r(\varphi g_- - \alpha x_0, b')|.
\end{aligned}$$

Now we have an inductive term and two error terms, which can be bounded using corollary 5.6. We know $|\varphi g_\pm - \alpha x_0|_1 \leq 2\alpha$ but we need to bound the diameter of the support of $\varphi g_\pm - \alpha x_0$. For this we need to bound the diameter of the support of $\varphi g_\pm$ as well as their maximal distances to $x_0$. Observe that $\operatorname{supp}(g_+) \subseteq \operatorname{supp}(g(a, b))$ and $x_0 \in \operatorname{supp}(\varphi g(a, b))$, which follows from theorem 4.1$(ii)$. Then theorem 4.1$(v)$ shows that the diameter of the support of $\varphi g_+ - \alpha x_0$ is bounded by $2C_0 + J - I$.

Similarly $\operatorname{supp}(\varphi g_-) \subseteq \operatorname{supp}(\varphi g(a, b'))$ whose diameter is bounded by $2C_0 + J - I$. However we do not necessarily have $x_0 \in \operatorname{supp}(\varphi g(a, b'))$ so it remains to bound the maximal distance from $x_0$ to an element in the support of $\varphi g_-$.

Fix a geodesic triangle $\Delta$ with corners $a, b, b'$ such that $\Delta$ contains the point $x_0$, and let $x_0'$ be the vertex of $\Delta$ between $a$ and $b'$ whose distance to $a$ is $I$. Then $x_0' \in \mathrm{supp}(\varphi g(a, b'))$ and $d(x_0', y) \leq C_0 + J - I$ for any element $y \in \mathrm{supp}(\varphi g_-)$ using theorem 4.1(*ii*).

We want to use hyperbolicity to get $d(x_0, x_0') \leq \delta$ but for this we need to know that $(b|b')_a \geq I$. Recall that $d(a, b) \geq d(a', b)$ and then;

$$\begin{aligned}(b|b')_a &= \frac{1}{2}(d(a,b) + d(a,b') - d(b,b')) \\ &\geq \frac{1}{2}(d(a,b) + d(a,b) - 2d(b,b')) \\ &\geq \frac{1}{2}(D(a,a',b) - 2) \qquad (5.3) \\ &\geq I.\end{aligned}$$

So hyperbolicity gives $d(x_0, x_0') \leq \delta$ and since $C_0 \geq \delta$ we conclude that

$$\mathrm{diam}(\mathrm{supp}(\varphi g_\pm - \alpha x_0)) \leq 2C_0 + J - I.$$

Therefore corollary 5.6 gives

$$\begin{aligned}|r(\varphi g_\pm - \alpha x_0, b)| &\leq D|\varphi g_\pm - \alpha x_0|_1 \mathrm{diam}(\mathrm{supp}(\varphi g_\pm - \alpha x_0)) \\ &\leq DL\lambda^{(b|b')_a}(2C_0 + J - I).\end{aligned}$$

We want the power of $\lambda$ in terms of $D(a, a', b)$ and not $(b|b')_a$. However, inequality (5.3) gives $(b|b')_a \geq \frac{1}{2}(D(a,a',b) - 2) \geq \frac{1}{2}(D(a,a',b) - \zeta - 2)$. Thus

$$|r(\varphi g_\pm - \alpha x_0, b)| \leq DL\lambda^{\frac{1}{2}(D(a,a',b) - \zeta - 2)}(2C_0 + J - I).$$

Now consider the inductive term $|R(\varphi g_0 + \alpha x_0, a, b, b')|$. Using barycentric coordinates write $\varphi g_0 + \alpha x_0 = \sum_{x \in V} \alpha_x x$. Then

$$\sum_{x \in V} \alpha_x = |\varphi g_0 + \alpha x_0|_1 = |\varphi g_0|_1 + \alpha = 1.$$

Thus $R(\varphi g_0 + \alpha x_0, a, b, b') = \sum_{x \in V} \alpha_x R(x, a, b, b')$.

Before we can apply the inductive hypothesis to $R(x, a, b, b')$ we need to know that $d(x, a) \leq \zeta$ and $D(x, a', b) < D(a, a', b)$.

Observe that $\mathrm{supp}(\varphi g_0) \subseteq \mathrm{supp}(\varphi g(a, b))$. Moreover it follows from lemma 4.15 that $x_0 \in \mathrm{supp}(\varphi g_I(a, b))$. Then for all $x \in \mathrm{supp}(\varphi g_0 + \alpha x_0)$ theorem 4.1(*iii*) says $d(x, b) \leq d(a, b) - 1$ and so

$$D(x, a', b) = d(x, b) + d(a', b) \leq D(a, a', b) - 1.$$

We still need to show $d(x, a') \leq \zeta$ before we can use the inductive hypothesis. Consider a geodesic triangle $\Delta$ with corners $a, a', b$ such that $\Delta$ contains $x_0$. For $I \leq \aleph \leq J$ let $u_\aleph \in [a, b]$ satisfy $d(a, u_\aleph) = \aleph$. In particular $u_I = x_0$. To bound $d(x,a)$ we need to consider the possibilities $\aleph \leq (a'|b)_a$ and $\aleph > (a'|b)_a$ separately.

Suppose $\aleph \leq (a'|b)_a$. Let $v_\aleph \in [a,a']$ be the vertex with $d(v_\aleph, a) = \aleph$. By hyperbolicity $d(u_\aleph, v_\aleph) \leq \delta$. Then using theorem 4.1(*iv*)

$$\begin{aligned} d(x,a') &\leq d(x,u_\aleph) + d(u_\aleph, v_\aleph) + d(v_\aleph, a') \\ &\leq \quad C_0 \quad + \quad \delta \quad + d(a,a') - I \\ &\leq d(a,a') - 1. \end{aligned}$$

In particular, $d(x,a') \leq d(a,a') \leq \zeta$.

So suppose $\aleph > (a'|b)_a$ and let $w_\aleph$ be the vertex of the $[a',b]$ in the triangle $\Delta$ with $d(w_\aleph, b) = d(u_\aleph, b)$. Hyperbolicity tells us $d(u_\aleph, w_\aleph) \leq \delta$. Therefore

$$\begin{aligned} d(x,a') &\leq d(x,u_\aleph) + d(u_\aleph, w_\aleph) + d(w_\aleph, a') \\ &\leq C_0 + \delta + d(a',b) - d(w_\aleph, b) \\ &= C_0 + \delta + d(a',b) - d(u_\aleph, b) \\ &= C_0 + \delta + d(a',b) - d(a,b) + d(a, u_\aleph) \\ &\leq C_0 + \delta - (J - I) + J \\ &\leq J - I - 1 \\ &\leq d(a,b) - d(a',b) - 1 \\ &\leq d(a,a') - 1 \end{aligned}$$

and in particular, $d(x,a') \leq d(a,a') \leq \zeta$.

Now the inductive hypothesis holds and

$$\begin{aligned} |R(x,a',b,b')| &\leq (A\, d(x,a') + B)\rho^{D(x,a',b)} \\ &\leq (A\, d(a,a') - A + B)\rho^{D(x,a',b)} \end{aligned}$$

We need to remove the dependence on $x$ so we need an upper bound for $\rho^{D(x,a',b)}$. Since $\rho \in (0,1)$, we need a lower bound on $D(x,a',b)$;

$$\begin{aligned} D(x,a',b) &= d(x,b) + d(a',b) \\ &\geq d(a,b) - d(a,u_\aleph) - d(u_\aleph, x) + d(a',b) \\ &\geq d(a,b) - J - C_0 + d(a',b) \\ &= D(a,a',b) - (J + C_0). \end{aligned}$$

Therefore

$$|R(x,a',b,b')| \leq (A\, d(a,a') - A + B)\rho^{D(a,a',b)-(J-C_0)}.$$

and so

$$\begin{aligned}|R(\varphi g_0+\alpha x_0,a',b,b')| &\leq \sum_{x\in V}\alpha_x|R(x,a',b,b')| \\ &\leq \sum_{x\in V}\alpha_x(A\,d(a,a')-A+B)\rho^{D(a,a',b)-(J-C_0)} \\ &= (A\,d(a,a')-A+B)\rho^{D(a,a',b)-(J-C_0)}.\end{aligned}$$

So to conclude the proof, we put these terms together and obtain

$$\begin{aligned}|R(a,a',b,b')| \leq\ & |R(\varphi g_0+\alpha x_0,a',b,b')| \\ & +|r(\varphi g_+-\alpha x_0,b)|+|r(\varphi g_- -\alpha x_0,b')| \\ \leq\ & (A\,d(a,a')-A+B)\rho^{D(a,a',b)-(J-C_0)} \\ & +2DL\lambda^{\frac{1}{2}(D(a,a',b)-\zeta-2)}(2C_0+J-I)\end{aligned}$$

which is exactly the inequality that the lemma claims to be true. □

Next consider the case where $d(a,b)$ is roughly the same as $d(a',b)$. Set

$$\xi = 2\max\left\{J+1,\frac{\ln 2-\ln L}{\ln\lambda}\right\}+2\zeta+2. \tag{5.4}$$

We aim for the following lemma;

**Lemma 5.14.** *Suppose $D(a,a',b)\geq \xi+1$. If $d:=d(a,b)-d(a',b)\leq J-I$ then there exists some constant $\eta_2\in(0,1)$ such that*

$$\begin{aligned}|R(a,a',b,b')| \leq\ & \eta_2(A\zeta+B)\rho^{D(a,a',b)-2(C_0+J)} \\ & +4DL\lambda^{\frac{1}{2}(D(a,a',b)-\zeta-2)}(2C_0+J-I).\end{aligned}$$

The idea here is to move both $a$ and $a'$. As in the proof of lemma 5.13 we consider the overlap between $\varphi g(a,b)$ and $\varphi g(a,b')$. But we also consider the overlap between $\varphi g(a',b)$ and $\varphi g(a',b')$. This gives us four error terms and something to which we can apply the inductive hypothesis.

However, there is an overlap of the overlaps, and on this double overlap the double difference $R$ will vanish. This is where the constant $\eta_2$ will come from.

Before we can do this we need to know that $d(a,b)>J+1$ so that we do use the recursive definition of $r$. We need this for all four pairings of vertices.

**Lemma 5.15.** *Suppose $D(a,a',b)>\xi$. Then*

$$d(a,b)>J+1,\quad d(a,b')>J+1,\quad d(a',b)>J+1,\quad d(a',b')>J+1.$$

*Proof.* These four inequalities follow from the triangle inequality and the assumption $d(a,b) \geq d(a',b)$.

$$\begin{aligned} d(a,b) &\geq \frac{1}{2}\left(d(a,b)+d(a',b)\right) > \frac{1}{2}\zeta && \geq J+1+\zeta+1. \\ d(a,b') &\geq \quad d(a,b)-d(b,b') && > J+1. \\ d(a',b) &\geq \quad d(a,b)-d(a,a') && > J+1+1. \\ d(a',b') &\geq \quad d(a',b)-d(b,b') && > J+1. \end{aligned}$$

□

This tells us that $r(a,b) = 1 + r(\varphi g(a,b),b)$ and so on for the other pairings. Hence

$$\begin{aligned} R(a,a',b,b') = r(\varphi g(a,b),b) &- r(\varphi g(a,b'),b') \\ &- r(\varphi g(a',b),b) + r(\varphi g(a',b'),b'). \end{aligned}$$

Recall that $g(a,b) = \frac{1}{1+J-I}\sum_{\aleph=I}^{J} \mathrm{av}_{B_R^\zeta} g_\aleph(a,b)$. The condition $d \leq J-I$ means that $J-d \geq I$. The idea is to use an argument similar to the proof of lemma 5.13 to split $r(\varphi g(a,b),b) - r(\varphi g(a,b'),b')$ into four terms,

$$\begin{aligned} r(\varphi g(a,b),b) - r(\varphi g(a,b'),b') = r(\varphi g_0 + \alpha y_0, b) &- r(\varphi g_0 + \alpha y_0, b') \\ + r(\varphi g_+ - \alpha y_0, b) &- r(\varphi g_- - \alpha y_0, b') \end{aligned}$$

for some well-chosen $y_0 \in V$. The third and fourth terms are error terms and can be bounded using corollary 5.6. We can also do this for $a'$ in place of $a$, although we want to use the same $y_0$. This leaves the expression

$$\begin{aligned} r(\varphi g_0 + \alpha y_0, b) - r(\varphi g_0 + \alpha y_0, b') - r(\varphi g_0' + \alpha' y_0, b) + r(\varphi g_0' + \alpha' y_0, b') \\ = R(\varphi g_0 + \alpha y_0, \varphi g_0' + \alpha' y_0, b, b') \end{aligned}$$

and here we want to use an argument similar to the one used in case 1 of the proof of lemma 5.5; we want to use the cancellation between $\varphi g_0 + \alpha y_0$ and $\varphi g_0' + \alpha' y_0$.

To summarise, we need to pick a suitable $y_0$, split up once, bound the terms involving cycles, split up a second time, and apply the inductive hypothesis.

For $I \leq \aleph \leq J$ define new elements $g_{\aleph,+}, g_{\aleph,-}, g_{\aleph,0} \in \mathbb{Q}E$ by

- all coefficients are positive
- $g(a,b) - g_{\aleph,+} = g_{\aleph,0} = g(a,b') - g_{\aleph,-}$
- $g_\aleph(a,b) - g_\aleph(a,b') = g_{\aleph,+} - g_{\aleph,-}$
- $\mathrm{supp}(g_{\aleph,+}) \cap \mathrm{supp}(g_{\aleph,-}) = \emptyset$.

Then set $g_+ = \frac{1}{1+J-I}\sum_{\aleph=I}^{J} \mathrm{av}_{B_R^\zeta}(g_{\aleph,+})$. Similarly we can define $g_-$ and $g_0$. Observe that $g(a,b) = g_+ + g_0$ and $g(a,b') = g_- + g_0$ but the supports of $g_+$ and $g_-$ are not necessarily disjoint.

Later on it will be important that we split prior to taking the averages, which is why we have deviated slightly from the method of proof in lemmas 5.9 and 5.13.

For bounding the error terms we need to first bound the size of $|g_{\aleph,\pm}|_1$ and $|g'_{\aleph,\pm}|_1$.

**Lemma 5.16.** *For all* $\aleph$*,*

$$\alpha_\aleph := |g_{\aleph,+}|_1 = |g_{\aleph,-}|_1 \leq \frac{1}{2}L\lambda^{\frac{1}{2}(D(a,a',b)-\zeta-2)}.$$

*Proof.* Using the argument in the proof of lemma 5.9 we get that

$$|g_{\aleph,+}|_1 = |g_{\aleph,-}|_1 \leq \frac{1}{2}L\lambda^{(b|b')_a}.$$

(It was shown for $g$ instead of $g_\aleph$ but the proof is identical.) So we need to find a lower bound for the Gromov product $(b|b')_a$. We are still assuming $d(a,b) \geq d(a',b)$ so inequality (5.3) in the proof of lemma 5.13 still holds, i.e. $(b|b')_a \geq \frac{1}{2}(D(a,a',b)-2)$. Combining this with the inequality for $|g_{\aleph,\pm}|_1$ finishes the proof of the lemma. □

Analogously, we can define $g'_{\aleph,+}, g'_{\aleph,-}, g_{\aleph,0}$ for $a'$ in place of $a$. There is a version of lemma 5.16 here;

**Lemma 5.17.** *For all* $\aleph$*,*

$$\alpha'_\aleph := |g'_{\aleph,+}|_1 = |g'_{\aleph,-}|_1 \leq \frac{1}{2}L\lambda^{\frac{1}{2}(D(a,a',b)-\zeta-2)}.$$

*Proof.* Using the same argument as the proof of lemma 5.16 gives

$$|g'_{\aleph,+}|_1 = |g'_{\aleph,-}|_1 \leq \frac{1}{2}L\lambda^{(b|b')_{a'}}$$

and so we need a lower bound of $(b|b')_{a'}$;

$$\begin{aligned}(b|b')_{a'} &= \frac{1}{2}(d(a',b)+d(a',b')-d(b,b')) \\ &\geq \frac{1}{2}(d(a,b)-d(a,a')+d(a',b)-2d(b,b')) \\ &\geq \frac{1}{2}(D(a,a',b)-\zeta-2) \qquad (5.5)\end{aligned}$$

thus we get the inequality as claimed. □

Before we get to the inductive part we quickly deal with the error terms.

**Lemma 5.18.** *Suppose $D(a,a',b) \geq \xi + 1$. Given any vertex $y_0 \in V[a,b;J]$ all four terms*

- $|r(\varphi g_+ - \alpha y_0, b)|$
- $|r(\varphi g_- - \alpha y_0, b')|$
- $|r(\varphi g'_+ - \alpha' y_0, b)|$
- $|r(\varphi g'_- - \alpha' y_0, b')|$

*are bounded from above by the expression*

$$DL(2C_0 + J - I)\lambda^{\frac{1}{2}(D(a,a',b)-\zeta-2)}. \tag{5.6}$$

*Proof.* We want to use corollary 5.6.

We already know that $\varphi g_+ - \alpha y_0$ is a cycle in $\mathbb{Q}V$ but we need to bound the diameter of its support. The support of $\varphi g_+$ is contained in the support of $\varphi g(a,b)$ and it follows from theorem 4.1*(ii)* that $y_0 \in \operatorname{supp}(\varphi g(a,b))$. Then theorem 4.1*(v)* tells us that the diameter of the support of $\varphi g_+ - \alpha y_0$ is bounded by $2C_0 + J - I$.

Therefore, using lemma 5.16 with corollary 5.6 gives

$$|r(\varphi g_+ - \alpha y_0, b)| \leq DL(2C_0 + J - I)\lambda^{\frac{1}{2}(D(a,a',b)-\zeta-2)}.$$

We want to do a similar thing for the other terms.

The support of $\varphi g_-$ is contained in the support of $\varphi g(a,b')$ so using the argument as above all we need to do is bound the distance from an element in the support of $\varphi g(a,b')$ to $y_0$. Consider a geodesic triangle with corners $a, b, b'$ that contains the point $y_0$. Inequality (5.3) in the proof of lemma 5.13 states $(b|b')_a \geq \frac{1}{2}(D(a,a',b) - 2)$. But $D(a,a',b) \geq \xi$ by assumption and then from the definition of $\xi$ (see equation (5.4)) we obtain $(b|b')_a \geq J$. Hence we can use hyperbolicity to go from $y_0$ to the geodesic $[a,b']$. Using this, together with theorem 4.1*(v)*, for every $y$ in the support of $\varphi g(a,b')$;

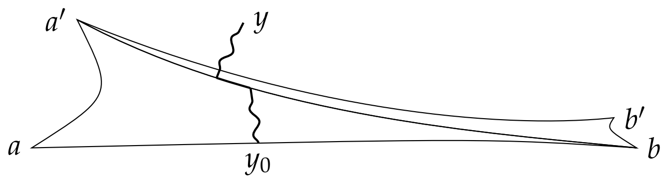
$$d(y,y_0) \leq C_0 + J - I + \delta \leq 2C_0 + J - I.$$

Therefore we can bound $|r(\varphi g_- - \alpha y_0, b')|_1$.

We want to use a similar argument to bound the remaining two terms. We need to use lemma 5.17 instead of lemma 5.16 but the rest of the argument is very similar. Observe that the support of $\varphi g'_+, \varphi g'_-$ is contained in the support of $\varphi g(a',b), \varphi g(a',b')$ respectively. Hence the only thing we need to do is bound the maximal distance from $y_0$ to the an element in the support of $\varphi g(a',b)$ or $\varphi g(a',b')$.

For $y \in \operatorname{supp}(\varphi g(a',b))$ we can consider a geodesic triangle $\Delta$ whose corners are $a, a', b$ and that contains the point $y_0$, and then the assumption $(a'|b)_a$ allows us to use hyperbolicity around the vertex $b$. This together with theorem 4.1(*iv*) gives $d(y, y_0) \leq C_0 + J - I + \delta$. This leads to the bound for $|r(\varphi g'_+ - \alpha' y_0, b)|$.

For $y' \in \operatorname{supp}(\varphi g(a',b'))$ we need to go further and consider a geodesic triangle $\Delta'$ whose corners are $a', b, b'$ such that it shares the side $[a', b]$ with the triangle $\Delta$, as in the picture below.




Inequality (5.5) states $(b|b')_{a'} \geq \frac{1}{2}(D(a,a',b) - \zeta - 2)$, and by assumption this leads to $(b|b')_{a'} \geq J$. Hence we can consider hyperbolicity in $\Delta'$ around the vertex $a'$ and conclude that $d(y', y_0) \leq C_0 + J - I + 2\delta$, where again we also need to use theorem 4.1(*iv*). Thus we get the bound on the term $|r(\varphi g'_- - \alpha' y_0, b')|$. □

Now we can move on to the term $|R(\varphi g_0 + \alpha y_0, \varphi g'_0 + \alpha' y_0, b, b')|$. The idea is to show that there is an overlap between $\varphi g_0 + \alpha y_0$ and $\varphi g'_0 + \alpha' y_0$ on which $R = 0$ and use an argument similar to that in the proof of lemma 5.9.

Once again we define new elements in $\mathbb{Q}V$; fix an element $y_0 \in V[a,b;J]$ and define $h_+, h_-, h_0 \in \mathbb{Q}V$ by the following properties

- all coefficients are positive
- $h_+ - h_- = \varphi g_0 + \alpha y_0 - (\varphi g'_0 + \alpha' y_0)$
- $\varphi g_0 + \alpha y_0 - h_+ = h_0 = \varphi g'_0 + \alpha' y_0$
- $\operatorname{supp}(h_+) \cap \operatorname{supp}(h_-) = \emptyset$.

So $h_0$ represents the overlap. These chains depend on the choice of $y_0$ in $V[a,b;J]$ but from now on we keep $y_0$ fixed so it is suppressed in the notation.

For any $x \in V$, $R(x, x, b, b') = 0$, so $h_0$ does not contribute anything and we need only consider $R(h_+, h_-, b, b')$. Hence we try to find a lower bound for $|h_0|_1$, i.e. a lower bound for how much cancels out.

The idea is to show that any edge $e_0 \in E[a,b;J]$ is $R$-close to every element in the support of $g_{J,0}$ and to every element in the support of $g'_{J-d,0}$.

Then the coefficient of $e_0$ in $\mathrm{av}_{B_R^\zeta} g_{J,0}$ is at least $|g_{J,0}|_1/B_R^\zeta$, where $B_R^\zeta$ is the maximum size of a ball of radius $R$ in the edge metric $d^\zeta$, which is finite by lemma 2.47 and the comments following it. This allows us to bound the coefficient of $y_0$ in $\varphi\mathrm{av}_{B_R^\zeta} g_{J,0}$ from below by $|g_{J,0}|_1/2B_R^\zeta$. Similarly we will get a lower bound for the coefficient of $y_0$ in $\varphi\mathrm{av}_{B_R^\zeta} g'_{J-d,0}$ of $|g'_{J-d,0}|_1/2B_R^\zeta$. From this we get a lower bound on $|h_0|_1$.

First we look for a lower bound of $|g_{J,0}|_1$ and $|g'_{J-d,0}|_1$.

**Lemma 5.19.** *If $D(a,a',b) \geq \xi + 1$ then $|g_{J,0}|_1 \geq 1-\lambda$ and $|g'_{J-d,0}|_1 \geq 1-\lambda$.*

*Proof.* Note that $|g_{J,0}|_1 = 1-\alpha_J \geq 1 - \frac{1}{2}L\lambda^{\frac{1}{2}(D(a,a',b)-\zeta-2)}$ using lemma 5.16. Moreover, we have

$$\frac{1}{2}(D(a,a',b)-\zeta-2) \geq \frac{\ln 2 - \ln L}{\ln \lambda} + 1$$

by the definition of $\xi$. Hence $\lambda^{\frac{1}{2}(D(a,a',b)-\zeta-2)} \leq \frac{2}{L}\lambda$ and so $|g_{J,0}|_1 \geq 1-\lambda$.

Similarly $|g'_{J-d,0}|_1 \geq 1-\lambda$, except we need to use lemma 5.17 instead of lemma 5.16. □

We can use the lower bounds in lemma 5.19 to find lower bounds for the coefficient of $y_0$ in $\varphi\mathrm{av}_{B_R^\zeta}(g_{J,0})$ and in $\varphi\mathrm{av}_{B_R^\zeta}(g'_{J-d,0})$, which will allow us to get the bound in the following lemma.

**Lemma 5.20.** *Suppose $D(a,a',b) \geq \xi+1$. If $d := d(a,b) - d(a',b) \leq J - I$ then there is a constant $\eta_1 \in (0,1)$ such that for all $I \leq \aleph, \aleph' \leq J$;*

(i) $\left|\varphi av_{B_R^\zeta}(g_{\aleph,0}) + \alpha_\aleph y_0\right|_1 = 1 = \left|\varphi av_{B_R^\zeta}(g'_{\aleph',0}) + \alpha'_{\aleph'} y_0\right|_1;$

(ii) $\left|\varphi av_{B_R^\zeta}(g_{J,0}) + \alpha_J y_0 - \left(\varphi av_{B_R^\zeta}(g'_{J-d,0}) + \alpha'_{J-d} y_0\right)\right|_1 \leq 2\eta_1.$

*Proof.* For the first part;

$$|\varphi\mathrm{av}_{B_R^\zeta}(g_{\aleph,0}) + \alpha_\aleph y_0|_1 = |g_{\aleph,0}|_1 + \alpha_\aleph = 1$$

and similarly for $|\varphi\mathrm{av}_{B_R^\zeta}(g'_{\aleph',0}) + \alpha'_{\aleph'} y_0|_1$.

For the second part, we look at the contribution from $y_0$ to find a bound on how much of the coefficient of $y_0$ gets cancelled. Let $\beta_J$ be the coefficient of $y_0$ in $\varphi\mathrm{av}_{B_R^\zeta}(g_{J,0})$ and let $\beta'_{J-d}$ be the coefficient of $y_0$ in $\varphi\mathrm{av}_{B_R^\zeta}(g'_{J-d,0})$. Then

$$\begin{aligned}
&|\varphi\mathrm{av}_{B_R^\zeta}(g_{J,0}) + \alpha_J y_0 - (\varphi\mathrm{av}_{B_R^\zeta}(g'_{J-d,0}) + \alpha'_{J-d} y_0)|_1 \\
&\quad \leq |\varphi\mathrm{av}_{B_R^\zeta}(g_{J,0}) - \beta_J y_0|_1 + \alpha_J \\
&\qquad + |\varphi\mathrm{av}_{B_R^\zeta}(g'_{J-d,0}) - \beta'_{J-d} y_0|_1 + \alpha'_{J-d} + |\beta_J - \beta'_{J-d}| \\
&\quad = |g_{J,0}|_1 - \beta_J + \alpha_J + |g'_{J-d,0}|_1 - \beta'_{J-d} + \alpha'_{J-d} + |\beta_J - \beta'_{J-d}| \\
&\quad = 2 - 2\min\{\beta_J, \beta'_{J-d}\}
\end{aligned}$$

and so we need a lower bound for $\beta_J$ and $\beta'_{J-d}$.

The vertex $y_0 \in V[a,b;J]$ must be the end-point of an edge $e_0 \in E[a,b;J]$ and by lemma 4.15 this edge is contained in the support of $g_J(a,b)$. The support of $g_{J,0}$ is also contained in the support of $g_J(a,b)$ so for any edge $e$ in the support of $g_{J,0}$ it follows from lemma 4.15 that $d^\zeta(e_0,e) \le 3\delta \le R$.

So $e_0$ is contained in $B_R^\zeta(e)$ for any edge $e$ in the support of $g_{J,0}$ and thus we know that $\beta_J \ge |g_{J,0}|_1/2B_R^\zeta$. Then we can apply lemma 5.19 to deduce that $\beta_J \ge (1-\lambda)/2B_R^\zeta$.

Analogously, if we can show that $e_0$ is $R$-close to any edge in the support of $g'_{J-d,0}$ then we would know that $\beta'_{J-d} \ge (1-\lambda)/2B_R^\zeta$.

Fix an edge $e'_0 \in E[a',b;J-d]$. The support of $g'_{J-d,0}$ is contained in the support of $g(a',b)$, hence lemma 4.15 tells us that the support of $g'_{J-d,0}$ is contained in $\mathrm{Fl}[a',e'_0;3\delta]$. We picked $e'_0$ such that $d(b,e'_0) = d(b,e_0)$, and the fact $(a'|b)_a < J$ (see inequality (5.2)) gives $d(b,e_0) \le (a|a')_b$ so by the edge-hyperbolicity of $\mathcal{G}$ (given in lemma 2.50) $d^\zeta(e_0,e'_0) \le \delta$. Therefore, for any edge $e$ in the support of $g'_{J-d,0}$ we deduce $d^\zeta(e,e_0) \le 4\delta \le R$.

Thus $\beta'_{J-d} \ge (1-\lambda)/2B_R^\zeta$.

So both $\beta_J$ and $\beta'_{J-d}$ are bounded from below by $(1-\lambda)/2B_R^\zeta$. Therefore

$$\begin{aligned} |\varphi \mathrm{av}_{B_R^\zeta}(g_{J,0}) + \alpha_J y_0 - (\varphi \mathrm{av}_{B_R^\zeta}(g'_{J-d,0}) + \alpha'_{J-d} y_0)|_1 &\le 2 - 2\min\{\beta_J, \beta'_{J-d}\} \\ &\le 2 - 2\left(\frac{1-\lambda}{2B_R^\zeta}\right) \\ &= 2\eta_1 \end{aligned}$$

where $\eta_1 := 1 - \frac{1-\lambda}{2B_R^\zeta}$. □

Now we need to use this to give a lower bound on $|h_0|_1$, which equates to an upper bound on $|h_+|_1$ and $|h_-|_1$.

**Lemma 5.21.** *Suppose $D(a,a',b) \ge \xi + 1$. If $d := d(a,b) - d(a',b) \le J - I$ then there exists $\eta_2 \in (0,1)$ such that;*

(*i*) $|h_+|_1 = |h_-|_1$

(*ii*) $|h_\pm|_1 \le \eta_2$.

*Proof.* The first fact can be proven using the augmentation homomorphism, see case 1 of the proof of lemma 5.5 for an example of how the argument works.

So we just need to bound $|h_\pm|_1$. Since all the coefficients are positive and the supports are disjoint we know $|h_+|_1 + |h_-|_1 = |h_+ - h_-|_1$. Then using the definitions gives

$$
\begin{aligned}
|h_\pm|_1 &= \frac{1}{2}|\varphi g_0 + \alpha y_0 - (\varphi g_0' + \alpha' y_0)|_1 \\
&\le \frac{1}{2(1+J-I)^2} \sum_{\aleph,\aleph'=I}^{J} \left| \varphi \mathrm{av}_{B_R^\zeta}(g_{\aleph,0}) + \alpha_\aleph y_0 - (\varphi \mathrm{av}_{B_R^\zeta}(g'_{\aleph',0}) + \alpha'_{\aleph'} y_0) \right|_1
\end{aligned}
$$

Every summand is bounded by 2, and furthermore lemma 5.20 tells us that the summand with $\aleph = J$ and $\aleph' = J - d$ is bounded by $2\eta_1$ with $\eta_1 \in (0,1)$. So this gives

$$
\begin{aligned}
|h_\pm|_1 &\le \frac{1}{2(1+J-I)^2}\Big(2\big((1+J-I)^2 - 1\big) + 2\eta_1\Big) \\
&= 1 - \frac{1-\eta_1}{(1+J-I)^2}
\end{aligned}
$$

and setting $\eta_2 = 1 - \frac{1-\eta_1}{(1+J-I)^2} \in (0,1)$ completes the proof of the lemma. □

This means that if we write $h_+ = \sum_{x\in V} \alpha_x^+ x$ and $h_- = \sum_{x'\in V} \alpha_{x'}^- x'$ then

$$
R(h_+, h_-, b, b') = \frac{1}{|h_\pm|_1} \sum_{x,x'\in V} \alpha_x^+ \alpha_{x'}^- R(x, x', b, b').
$$

Now we want to apply our inductive hypothesis to $R(x, x', b, b')$. For this, we need to know that the conditions for the inductive hypothesis are satisfied, namely we need $d(x, x') \le \zeta$ and $D(x, x', b) < D(a, a', b)$. Moreover, after we have applied the inductive hypothesis we will want to translate back from being in terms of $d(x, x')$ and $D(x, x', b)$ to being in terms of $d(a, a')$ and $D(a, a', b)$.

**Lemma 5.22.** *Suppose $D(a, a', b) \ge \zeta + 1$. If $d := d(a, b) - d(a', b) \le J - I$ then for any $x \in supp(h_+)$ and any $x' \in supp(h_-)$*

*(i)* $D(a, a', b) - 2(C_0 + J) \le D(x, x', b) \le D(a, a', b) - 2$;

*(ii)* $d(x, x') \le \zeta$.

*Proof.* *(i)* The support of $h_+$ is contained in the support of $\varphi g_0 + \alpha y_0$, and the support of $g_0$ is contained in both the support of $g(a, b)$ and the support of $g(a, b')$, so it is enough to prove the lemma for $x \in \mathrm{supp}(\varphi g(a, b))$. For such $x$ we have $d(x, b) \le d(a, b) - 1$ by theorem 4.1*(iii)*. Moreover,

$$
d(x, b) \ge d(a, b) - d(a, x) \ge d(a, b) - (C_0 + J)
$$

using theorem 4.1*(iv)*.

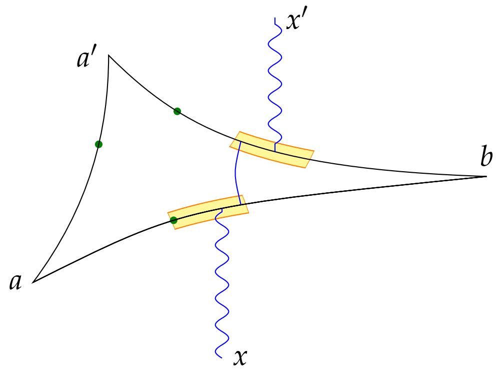

Similarly for any $x' \in \operatorname{supp}(\varphi g_0') \subseteq \operatorname{supp}(\varphi g(a',b))$ we have

$$d(a',b) - (C_0 - J) \leq d(x',b) \leq d(a',b) - 1.$$

We still need to find bounds for the possibility that $x$ or $x'$ is $y_0$. By definition $y_0 \in V[a,b;J]$ so $d(y_0,b) = d(a,b) - J$.

*(ii)* It follows from theorem 4.1*(ii)* that $y_0 \in \operatorname{supp}(\varphi g(a,b))$ so we fix an arbitrary $x \in \operatorname{supp}(\varphi g(a,b))$ and consider two possibilities depending on whether $x' = y_0$ or $x'$ is in the support of $\varphi g(a',b)$. .

If $x' = y_0$ then $d(x,x') \leq \operatorname{diam}(\operatorname{supp}(\varphi g(a,b))) \leq 2C_0 + J - I \leq \zeta$, using theorem 4.1*(v)*.




Recall that $(a'|b)_a \leq J$, which follows from inequality (5.2). So we can use theorem 4.1*(iv)* together with the assumption $d(a,b) - d(a',b) \leq J - I$ to show that if $x' \in \operatorname{supp}(\varphi g(a',b))$ then

$$d(x,x') \leq 2C_0 + \delta + 2(J - I) \leq \zeta.$$

(See the picture above.)

Therefore for any $x \in \operatorname{supp}(\varphi g_0 + \alpha y_0)$ and $x' \in \operatorname{supp}(\varphi g_0' + \alpha' y_0)$ we have $d(x,x') \leq \zeta$. □

With lemma 5.22 we can apply the inductive hypothesis to $R(x,x',b,b')$ for $x \in \operatorname{supp}(h_+)$ and $x' \in \operatorname{supp}(h_-)$.

**Corollary 5.23.** *Suppose $D(a,a',b) \geq \xi + 1$. If $d := d(a,b) - d(a',b) \leq J - I$ then*

$$|R(h_+,h_-,b,b')| \leq \eta_2(A\zeta + B)\rho^{D(a,a',b)-2(C_0+J)}$$

*where $\eta_2$ is the constant from lemma 5.21.*

*Proof.* Write $h_+ = \sum_{x \in V} \alpha_x^+ x$ and $h_- = \sum_{x' \in V} \alpha_{x'}^- x'$ so that

$$R(h_+, h_-, b, b') = \frac{1}{|h_\pm|_1} \sum_{x,x' \in V} \alpha_x^+ \alpha_{x'}^- R(x, x', b, b').$$

Lemma 5.22(*ii*) shows $d(x, x') \leq \zeta$ and $D(x, x', b) \leq D(a, a', b) - 2$ so we can apply the inductive hypothesis to $R(x, x', b, b')$ to get

$$|R(x, x', b, b')| \leq (A\, d(x, x') + B)\rho^{D(x,x',b)}.$$

Moreover lemma 5.22(*i*) tells us $D(x, x', b) \geq D(a, a', b) - 2(C_0 + J)$. Therefore

$$\begin{aligned} |R(h_+, h_-, b, b')| &\leq \frac{1}{|h_\pm|_1} \sum_{x,x' \in V} \alpha_x^+ \alpha_{x'}^- (A\zeta + B)\rho^{D(a,a',b) - 2(C_0+J)} \\ &= |h_\pm|_1 (A\zeta + B)\rho^{D(a,a',b) - 2(C_0+J)} \end{aligned}$$

and $|h_\pm|_1$ was bounded by $\eta_2$ in lemma 5.21. □

And so we can finish the inductive step in proving proposition 5.11 for this case, i.e. we can give the proof of lemma 5.14;

*Proof of Lemma 5.14.* Split up $R(a, a', b, b')$ into the five terms

$$\begin{aligned} R(a, a', b, b') = R(&\varphi g_0 + \alpha y_0, \varphi g_0' + \alpha' y_0, b, b') \\ &+ r(\varphi g_+ - \alpha y_0, b) \\ &- r(\varphi g_- - \alpha y_0, b') \\ &- r(\varphi g_+' - \alpha' y_0, b) \\ &+ r(\varphi g_-' - \alpha' y_0, b'). \end{aligned}$$

The inductive term $R(\varphi g_0 + \alpha y_0, \varphi g_0' + \alpha' y_0, b, b') = R(h_+, h_-, b, b')$ using corollary 5.23 and lemma 5.18 yields a bound for the four error terms, and combining these bounds gives

$$\begin{aligned} |R(a, a', b, b')| \leq\ & 4DL\lambda^{\frac{1}{2}(D(a,a',b) - \zeta - 2)}(2C_0 + J - I) \\ & + \eta_2(A\zeta + B)\rho^{D(a,a',b) - 2(C_0+J)}. \end{aligned}$$ □

To finish the proof of proposition 5.11 it remains to show that we can choose the constants $A, B$ and $\rho$ appropriately.

**Lemma 5.24.** *For any $\xi > 0$ there exists constants $A, B > 0$ and $\rho \in (0, 1)$ such that for any $a, a'$ with $1 \leq d(a, a') \leq \zeta$ and all $b, b'$ with $d(b, b') \leq 1$ the following three inequalities hold;*

$$2\xi + 2 \leq A\lambda^{\frac{1}{2}\xi} \tag{5.7}$$

$$(A\,d(a,a') - A + B)\rho^{-J-C_0} + K\left(\frac{\lambda^{\frac{1}{2}}}{\rho}\right)^{D(a,a',b)} \leq A\,d(a,a') + B \tag{5.8}$$

$$\eta_2(A\zeta + B)\rho^{-2(C_0+J)} + 2K\left(\frac{\lambda^{\frac{1}{2}}}{\rho}\right)^{D(a,a',b)} \leq A\,d(a,a') + B \tag{5.9}$$

*where* $K := 2DL(2C_0 + J - I)\lambda^{-\frac{1}{2}(\zeta+2)}$.

*Proof.* First pick $A > K$ large enough such that $2\xi + 2 \leq A\lambda^{\frac{1}{2}\xi}$. Then pick $\rho_1 \in (\lambda^{\frac{1}{2}}, 1)$ such that $\eta_2 < \rho_1^{2(C_0+J)}$, which is possible since $\eta_2 < 1$ and $\rho_1^{2(C_0+J)} \nearrow 1$ as $\rho_1 \nearrow 1$.

Next choose $B > A$ large enough such that

$$B \geq \frac{\eta_2 A\zeta\rho_1^{-2(C_0+J)} + 2K}{1 - \eta_2\,\rho_1^{-2(C_0+J)}}$$

and finally pick $\rho \in (\rho_1, 1)$ such that for any $l = 1, \ldots, \zeta$

$$\rho^{-(C_0+J)} \leq \frac{Al + B - K}{A(l-1) + B}$$

which is possible since $B > A > K$ and $\rho^{-(C_0+J)} \searrow 1$ as $\rho \nearrow 1$.

It remains to show that the three inequalities (5.7), (5.8), and (5.9) are satisfied. The constant $A$ was chosen to satisfy inequality (5.7) but we need to check the other two. For inequality (5.8);

$$\begin{aligned}(Ad(a,a') - A + B)\rho^{-(C_0+J)} + K\left(\frac{\lambda^{\frac{1}{2}}}{\rho}\right)^{D(a,a',b)} & \\ \leq (Ad(a,a') - A + B)\rho^{-(C_0+J)} + K & \\ \leq A\,d(a,a') + B &\end{aligned}$$

where the first inequality uses $\rho \geq \rho_1 \geq \lambda^{\frac{1}{2}}$ and the second inequality follows from the choice of $\rho$, since $d(a,a') \leq \zeta$. For inequality (5.9);

$$\begin{aligned}\eta_2(A\zeta + B)\rho^{-2(C_0+J)} + 2K\left(\frac{\lambda^{\frac{1}{2}}}{\rho}\right)^{D(a,a',b)} &\leq \eta_2(A\zeta + B)\rho_1^{-2(C_0+J)} + 2K \\ &\leq B \\ &\leq A\,d(a,a') + B\end{aligned}$$

where the first inequality uses $\rho \geq \rho_1 \geq \lambda^{\frac{1}{2}}$, and the second inequality uses the choice of $B$. □

Now we can finally prove proposition 5.11;

*Proof of Proposition 5.11.* Note that if $a = a'$ or $b = b'$ then $R(a, a', b, b') = 0$ so we may assume they are distinct. In particular $d(a, a') \geq 1$. Set

$$\xi = 2 \max \left\{ J + 1, \frac{\ln 2 - \ln L}{\ln \lambda} \right\} + 2\zeta + 2.$$

Then use lemma 5.24 to get constants $A, B > 0$ and $\rho \in (0, 1)$ satisfying the inequalities (5.7), (5.8), and (5.9).

We induct on $D(a, a', b)$.

For $D(a, a', b) \leq \xi$ lemma 5.12 says $|R(a, a', b, b')| \leq 2\xi + 2$ but then by the choice of $A$ we know $|R(a, a', b, b')| \leq A\lambda^{\frac{1}{2}\xi}$. We picked $\rho \geq \lambda^{\frac{1}{2}}$ hence

$$|R(a, a', b, b')| \leq A\rho^{\xi} \leq A\rho^{D(a,a',b)} \leq (A\, d(a, a') + B)\rho^{D(a,a',b)}.$$

For the inductive step assume $D(a, a', b) \geq \xi + 1$.

If $|d(a, b) - d(a', b)| > J - I$ then lemma 5.13 combined with inequality (5.8) gives

$$|R(a, a', b, b')| \leq (A\, d(a, a') + B)\rho^{D(a,a',b)}$$

as desired. Otherwise $|d(a, b) - d(a', b)| \leq J - I$ and then use lemma 5.14 and inequality (5.9). □

Thus we have finished the proof of proposition 5.11 and with that we have also proven proposition 5.10.

## 5.3 A metric on vertices

With the results we have already established we can now follow the arguments in [MY02] to define a metric. First we must symmetrise the function.

**Definition 5.25.** Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. Define a function $s\colon V \times V \to \mathbb{R}_{\geq 0}$ by

$$s(x, y) := \frac{1}{2}\big(r(x, y) + r(y, x)\big).$$

Some properties of $r$ immediately carry over to $s$ and we summarise these properties in the following lemma.

**Lemma 5.26.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph.*

(*i*) *The function* $s\colon V \times V \to \mathbb{R}_{\geq 0}$ *is invariant under the diagonal action of* $\mathrm{Isom}(\mathcal{G})$ *on* $V \times V$.

(*ii*) *There exists a constant* $J + 1 \geq 1$ *such that for all* $a, b \in V$ *we have*

$$\frac{1}{J+1} d(a,b) \leq s(a,b) \leq d(a,b).$$

(*iii*) *There exist constants* $N, M \geq 1$ *such that for all* $a, a', b \in V$

$$|s(a,b) - s(a',b)| \leq \frac{1}{2}(1+M)d(a,a') + \frac{1}{2}N.$$

(*iv*) *There exists a constant* $C_1 > 0$ *such that for* $a, b \in V$ *and* $x \in V[a,b]$

$$|s(a,b) - s(a.x) - s(x,b)| \leq C_1.$$

(*v*) *There exists constants* $C > 0$ *and* $\omega \in (0,1)$ *such that for* $a, a', b, b' \in V$ *if* $d(a,a') \leq 1$ *and* $d(b,b') \leq 1$ *then*

$$|s(a,b) - s(a',b) - s(a,b') + s(a',b')| \leq C\omega^{d(a,b)}.$$

*Proof.* The function is $\mathrm{Isom}(\mathcal{G})$-invariant since $r$ is $\mathrm{Isom}(\mathcal{G})$-invariant. The second part comparing $s$ to $d$ follows from proposition 5.2. The third part combines lemma 5.5 and proposition 5.8. The fifth part, which is a double difference type result, comes from proposition 5.10. The fourth part, which says geodesics with respect to $d$ are almost geodesic with respect to the wannabe metric $s$, uses a degenerate form of lemma 5.7, where $b_0 = x$ and $\tilde{a} = a$, to bound

$$|r(a,b) - r(a,x) + r(x,b)| \leq C_1.$$

We have to apply lemma 5.7 a second time, swapping the roles of $a$ and $b$ to get

$$|r(b,a) - r(b,x) - r(x,a)| \leq C_1$$

and then taking the average gives the inequality in $s$. □

For a metric we need the triangle inequality to hold. We don't have it yet but we do have the following lemma.

**Lemma 5.27.** *There exists a constant* $C_2 > 0$ *such that for all* $x, y, z \in V$,

$$s(x,z) \leq s(x,y) + s(y,z) + C_2.$$

*Proof.* The idea for the proof comes from [MY02, Proposition 11]. Let $\Delta$ be a geodesic triangle with corners $x, y, z$, and let $\iota_x, \iota_y, \iota_z$ be the inner points of $\Delta$, as in definition 2.8. Then

$$\begin{aligned} s(x,z) &\leq s(x,\iota_y) + s(\iota_y, z) + C_1 \\ &\leq \left(s(x,\iota_z) + \frac{1}{2}(1+M)d(\iota_z,\iota_y) + \frac{1}{2}N\right) \\ &\qquad + \left(\frac{1}{2}(1+M)d(\iota_y,\iota_x) + \frac{1}{2}N + s(\iota_x, z)\right) + C_1 \\ &\leq s(x,\iota_z) + s(\iota_x, z) + (1+M)\delta + N + C_1 \\ &\leq (s(x,y) + C_1) + (s(y,z) + C_1) + (1+M)\delta + N + C_1 \end{aligned}$$

where the first and fourth inequalities use lemma 5.26($iv$), the second inequality uses lemma 5.26($iii$), and the third inequality uses $\delta$-hyperbolicity. Then the lemma is finished by setting $C_2 = (1+M)\delta + N + 3C_1$. □

This is not quite the triangle inequality, but we can now define our metric.

**Definition 5.28.** Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. Define a function $\hat{d} \colon V \times V \to \mathbb{R}$ by

$$\hat{d}(x,y) = \begin{cases} s(x,y) + C_2 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

where $C_2$ is the constant from lemma 5.27.

**Theorem 5.29.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. The function $\hat{d}$ is a metric on $V$. Moreover it satisfies the following properties;*

($i$) *It is invariant under the action of* $\mathrm{Isom}(\mathcal{G})$ *on $V$.*

($ii$) *Let $C_2 \geq 0$ be the constant from lemma 5.27. For all $a, b \in V$*

$$\frac{1}{J+1} d(a,b) \leq \hat{d}(a,b) \leq d(a,b) + C_2.$$

($iii$) *There exists a constant $C_3 > 0$ such that for $a, b \in V$ and $x \in V[a,b]$*

$$0 \leq \hat{d}(a,x) + \hat{d}(x,b) - \hat{d}(a,b) \leq C_3.$$

($iv$) *There exist constants $C > 0, \omega \in (0,1)$ such that for all $a, a', b, b' \in V$ if $d(a,a') \leq 1$ and $d(b,b') \leq 1$ then*

$$\left|\hat{d}(a,b) - \hat{d}(a',b) - \hat{d}(a,b') + \hat{d}(a',b')\right| \leq C\omega^{d(a,b)}. \tag{5.10}$$

*Proof.* It is symmetric since $s$ is defined to be symmetric, and it satisfies the triangle inequality because of lemma 5.27. If $x \neq y \in V$ then by definition $\hat{d}(x,y) \geq C_2 > 0$ so it is positive-definite. Therefore $\hat{d}$ is a metric on $V$.

Moreover it is invariant under graph automorphisms of $\mathcal{G}$ because $s$ is invariant.

For any $a \neq b \in V$, lemma 5.26(*ii*) says $\frac{1}{J+1}d(a,b) \leq s(a,b) \leq d(a,b)$ and thus we obtain part (*ii*).

Part (*iii*) follows from lemma 5.26(*iv*), if we set $C_3 = C_1 + 2C_2$. (Note that $\hat{d}(a,x) + \hat{d}(x,b) - \hat{d}(a,b)$ is always non-negative by the triangle inequality.)

For part (*iv*) we can use lemma 5.26(*v*) to get

$$|s(a,b) - s(a',b) - s(a,b') + s(a',b')| \leq C\omega^{d(a,b)}.$$

If all four vertices are distinct then the four copies of $C_2$ cancel out and so we get inequality (5.10). If $a = a'$ or $b = b'$ then the left-hand side of inequality (5.10) is zero. Finally if one of $\{a,a'\}$ is equal to one of $\{b,b'\}$ then $\text{diam}\{a,a',b,b'\} \leq 2$ (with respect to the metric $d$) so the left-hand side of inequality (5.10) is bounded by $8 + 4C_2$ (using part (*ii*)). If we increase $C$ (if necessary) such that $8 + 4C_2 \leq C\omega^2$ then inequality (5.10) holds here too. □

**Remark 5.30.** In part (*iii*) of theorem 5.29 the notation $V[a,b]$ still refers to geodesics with respect to the original metric $d$ on $\mathcal{G}$. When we talk about actualy geodesics we will only ever consider geodesics in $\mathcal{G}$ with respect to the word metric.

We can now define Gromov products and the double difference in terms of this new metric.

**Definition 5.31.** Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph. Let $\hat{d}$ be the metric on $V$ from theorem 5.29. Define the *Gromov product* (with respect to $\hat{d}$) of three vertices $a, a', b \in V$ to be

$$\langle a|a'\rangle_b = \frac{1}{2}\big(\hat{d}(a,b) + \hat{d}(a',b) - \hat{d}(a,a')\big).$$

Define the *double difference* (with respect to $\hat{d}$) of four vertices $a, a', b, b' \in V$ by

$$\langle a,a'|b,b'\rangle = \frac{1}{2}\left(\hat{d}(a,b) - \hat{d}(a',b) - \hat{d}(a,b') + \hat{d}(a',b')\right).$$

Here we are using angular brackets to distinguish from the Gromov product and double difference defined using the word metric on the graph. We could define the double difference in terms of the Gromov product, as in definition 3.15.

The generic facts about the double difference in proposition 3.16 still hold. Moreover theorem 5.29 gives us a couple of facts for this new Gromov product and new double difference.

**Theorem 5.32.** *Let $\mathcal{G}$ be a uniformly fine, $\delta$-hyperbolic graph.*

$(i)$ *There exists a constant $C_3 > 0$ such that for $a, b \in V$ and $x \in V[a, b]$*

$$0 \leq \langle a|b\rangle_x \leq \frac{1}{2}C_3.$$

$(ii)$ *There exist constants $C > 0, \omega \in (0,1)$ such that for all $a, a', b, b' \in V$ if $d(a, a') \leq 1$ and $d(b, b') \leq 1$ then*

$$\left|\langle a, a'|b, b'\rangle\right| \leq \frac{1}{2}C\omega^{d(a,b)}. \tag{5.11}$$

Note that if $x$ lies on a geodesic (with respect to the word metric $d$ on $\mathcal{G}$) between $a$ and $b$ then $(a|b)_x = 0$. Part $(i)$ of theorem 5.29 says that $\langle a|b\rangle_x$ is bounded by some global constant.

Before showing how to extend this metric to the boundary of the graph, we need to extend the metric from the vertex set to all of the graph. More generally, in section 6.1 we will show how to extend a metric on the vertex set of a simplicial complex to the whole simplicial complex, and then show in section 6.2 that if we start with the metric $\hat{d}$ constructed here (see theorem 5.29) then we can extend the double difference to the boundary (theorem 6.13).

# 6 The Metric on the Flow Space

In chapter 5 we defined a new metric $\hat{d}$ on the vertex set of our space $X$ but to proceed with the construction from section 3.4 of a metric on the flow space of $X$ we need to extend the metric $\hat{d}$ to the whole simplicial complex $X$.

In section 6.1 we will work more generally, with an arbitrary simplicial complex $X$ and a metric $\hat{d}$ defined on the vertices of $X$, such that $\hat{d}$ is $(A, B)$-quasi-isometric to the word metric $d_{\mathcal{G}}$, where $\mathcal{G}$ is the 1-skeleton of $X$. Then in section 6.2 we will prove some properties of this metric on $X$ when we start with the $\hat{d}$ defined in section 5.3.

## 6.1 Extending a metric

In [Min05] Mineyev defined an extension of a metric $\hat{d}$ by taking a bilinear expansion with respect to the barycentric coordinates, see [Min05, Equation (31) on page 436]. If $x = \sum_{u \in V} x_u u$ and $y = \sum_{v \in V} y_v v$ then define

$$\hat{d}(x, y) = \sum_{u,v \in V} x_u y_v \, \hat{d}(u, v). \tag{6.1}$$

This extension was also used by Bartels-Lück-Reich in their proof of The Farrell-Jones Conjecture for hyperbolic groups (see [BLR08a, Section 6]). However, this extension does not define a metric, since it does not always satisfy the condition $\hat{d}(x, x) = 0$.

For example, suppose $x \in X$ is not a vertex. Then we can find two vertices $u, v \in V$ such that in barycentric coordinates the values $x_u, x_v$ are both non-zero. Thus

$$\hat{d}(x,x) \geq x_u x_v \, \hat{d}(u,v) > 0.$$

In this section we show how to define a metric $\tilde{d}$ on $X$ whose restriction to the vertex set coincides with $\hat{d}$ and moreover if $x, y \in X$ are sufficiently far apart then $\tilde{d}(x,y) = \hat{d}(x,y)$, i.e. the distance is given by the bilinear expansion of the barycentric coordinates (see theorem 6.9).

Since the applications of the metric in [Min05] and [BLR08a] only make use of the large-scale geometry of the metric, their arguments work with $\tilde{d}$ instead of $\hat{d}$, requiring only minor tweaking. For example [Min05, Proposition 38], which was quoted by Bartels-Lück-Reich as [BLR08a, Proposition 6.4], can be tweaked as demonstrated in proposition 7.2, which explains how to modify the proof given by Mineyev to use the extension $\tilde{d}$ instead of $\hat{d}$. Therefore the results obtained in those papers are valid.

Even though the bilinear extension is not a metric, it nevertheless satisfies the triangle inequality;

**Lemma 6.1.** *Let $X$ be a simplicial complex and let $\hat{d}$ be a metric on the vertex set of $X$. Extend $\hat{d}$ to a function on $X \times X$ by bilinearly extending, as in equation* (6.1)*. Then for all $x, y, z \in X$,*

$$\hat{d}(x,z) \leq \hat{d}(x,y) + \hat{d}(y,z).$$

*Proof.* Write $x, y, z$ in barycentric coordinates, and then

$$\begin{aligned}
\hat{d}(x,z) &= \sum_{u,w \in X^{(0)}} x_u z_w \, \hat{d}(u,w) \\
&= \sum_{u,v,w \in X^{(0)}} x_u y_v z_w \, \hat{d}(u,w) \\
&\leq \sum_{u,w \in X^{(0)}} x_u y_v z_w (\hat{d}(u,v) + \hat{d}(v,w)) \\
&= \sum_{u,v,w \in X^{(0)}} x_u y_v z_w \, \hat{d}(u,v) + \sum_{u,v,w \in X^{(0)}} x_u y_v z_w \, \hat{d}(v,w) \\
&= \sum_{u,v \in X^{(0)}} x_u y_v \, \hat{d}(u,v) + \sum_{v,w \in X^{(0)}} y_v z_w \, \hat{d}(v,w) \\
&= \hat{d}(x,y) + \hat{d}(y,z).
\end{aligned}$$

□

So this bilinear extension is *almost* a metric. It only fails to be a metric locally, with the problem that $\hat{d}(x,x)$ need not be zero. To construct an actual metric we will take the minimum of this bilinear extension and a genuine metric on $X$. We want this to extend the metric on the vertex set so we want the genuine metric on $X$ to be greater than $\hat{d}$ on the vertex set.

The metric we will use is a length metric induced by the $l^1$-metric on each simplex.

**Definition 6.2.** Let $X$ be a simplicial complex. For any simplex $\sigma$ of $X$ let $d^1_\sigma$ be the $l^1$-metric on $\sigma$, i.e. for all points $x, y \in \sigma$,

$$d^1_\sigma(x,y) := \frac{1}{2} \sum_{u \in \sigma^{(0)}} |x_u - y_u|. \tag{6.2}$$

The factor $\frac{1}{2}$ is to ensure that the distance between adjacent vertices is 1.

Then for $x, y \in X$ define a *path in $X$ from $x$ to $y$* to be a sequence of points $x = a_0, a_1, \ldots, a_r = y$ in $X$ such that for every $i = 1, \ldots, r$ there is a simplex $\sigma_i$ of $X$ that contains both $a_{i-1}$ and $a_i$. The *length* of such a path is the sum $\sum_{i=1}^r d^1_{\sigma_i}(a_{i-1}, a_i)$.

The length of a path does not depend on how we choose the simplices $\sigma_i$ because if a simplex $\tau$ is a subsimplex of $\sigma$ then $d^1_\tau \equiv d^1_\sigma|_\tau$.

Then we define a metric on $X$ by setting $d^1_X(x,y)$ to be the infimum of lengths of all such paths from $x$ to $y$.

We start by giving some properties of this metric.

**Proposition 6.3.** *Let $X$ be a simplicial complex.*

$(i)$ *For any simplex $\sigma$ in $X$, the restriction of $d^1_X$ to $\sigma$ coincides with $d^1_\sigma$.*

$(ii)$ *The diameter of any simplex is* 1*, unless the simplex is a vertex.*

$(iii)$ *If $x, y \in X$ have disjoint supports then $d^1_X(x,y) \geq 1$.*

*Proof.* $(i)$ Given any two points $x, y$ contained in one simplex $\sigma$, by considering the trivial path $x = a_0, a_1 = y$ we get $d^1_X(x,y) \leq d^1_\sigma(x,y)$ so we only need to show the inequality in the other direction.

Fix a simplex $\sigma$ and two points $x, y \in \sigma$. Consider a path $a_0, a_1, \ldots, a_r$ from $x$ to $y$ in $X$. Since it is a path we can find simplices $\sigma_i$ for $i = 1, \ldots, r$ such that $\sigma_i$ contains both $a_{i-1}$ and $a_i$. Let $Y \subseteq X$ be the union of all the $\sigma_i$ (which is a subcomplex of $X$) and denote the vertex set of $Y$ by $Y_0$ and let $\Delta = \Delta^{Y_0 - 1}$ be the standard simplex of dimension $Y_0 - 1$, where we identify the vertex set of $\Delta$ with $Y_0$

There is a canonical inclusion $Y \to \Delta$, and so we can think of $Y$ as a subcomplex of $\Delta$. Thus for each $i$ we have $d^1_{\sigma_i}(a_{i-1}, a_i) = d^1_\Delta(a_{i-1}, a_i)$ and then the triangle inequality tells us that the length of the path $a_0, \ldots, a_r$ is at least $d^1_\Delta(a_0, a_r) = d^1_\sigma(x,y)$.

We can do this for any path from $x$ to $y$ so $d^1_X(x,y) \geq d^1_\sigma(x,y)$. Therefore we can conclude that $d^1_X|_\sigma \equiv d^1_\sigma$.

$(ii)$ It follows from equation (6.2) that if two points $x, y$ lie in a common simplex $\sigma$ then the distance between them is $d^1_\sigma(x,y) \leq 1$. Taking $x, y$ to be distinct vertices of $\sigma$ shows that the bound is strict.

$(iii)$ If $v$ is a vertex in the support of $x$ but not in the support of $y$ then any path from $x$ to $y$ must eventually take the $v$-coordinate down from $x_v$

to zero. Formally, if $a_0, \ldots, a_r$ is a path from $x$ to $y$ let $a_{i,v}$ denote the $v$-coordinate of $a_i$. Then the $v$-coordinate contributes

$$\begin{aligned}\sum_{i=1}^{r} \frac{1}{2} \left|a_{i-1,v} - a_{i,v}\right| &\geq \frac{1}{2} \left|\sum_{i=1}^{r} (a_{i-1,v} - a_{i,v})\right| \\ &= \frac{1}{2} \left|a_{0,v} - a_{r,v}\right| = \frac{1}{2} x_v\end{aligned}$$

and thus the $v$-coordinate contributes at least $\frac{1}{2}x_v$ to $d^1_X(x,y)$.

Similarly if $w$ is in the support of $y$ but not the support of $x$ then any path must take the $w$-coordinate from zero up to $y_w$, hence the $w$-coordinate contributes at least $\frac{1}{2}y_w$ to $d^1_X(x,y)$.

Therefore if the support of $x$ is disjoint from the support of $y$ then we get at least $\frac{1}{2}$ from the coordinates in the support of $x$ and at least $\frac{1}{2}$ from the coordinates in the support of $y$. Thus $d^1_X(x,y) \geq 1$. □

If take the minimum of the bilinear extension (6.1) and this metric $d^1_X$ then we could encounter two problems, namely the triangle inequality might not be satisfied and this minimum might not restrict to $\hat{d}$ on the vertex set.

If the metric $\hat{d}$ is quasi-isometric to the word metric on the 1-skeleton of $X$ then we can fix both problems by re-scaling the metric $d^1_X$.

**Definition 6.4.** Let $X$ be a simplicial complex and let $\hat{d}$ be a metric on the vertices of $X$ that is $(A,B)$-quasi-isometric to the word metric induced by the 1-skeleton of $X$. Set $\hat{d}\colon X \times X \to \mathbb{R}$ to be the bilinear extension of $\hat{d}$ as in equation (6.1). Define a function $\tilde{d}\colon X \times X \to \mathbb{R}$ by

$$\tilde{d}(x,y) := \min\left\{\hat{d}(x,y)\,,\, 3(A+B)d^1_X(x,y)\right\}. \tag{6.3}$$

To prove that this function satisfies the triangle inequality we will use the following lemma.

**Lemma 6.5.** *Let $X$ be a simplicial complex and let $\hat{d}$ be a metric on the vertices of $X$ that is $(A,B)$-quasi-isometric to the word metric induced by the 1-skeleton of $X$. If $\hat{d}\colon X \times X \to \mathbb{R}$ is the bilinear extension of $\hat{d}$ as in equation (6.1) then for all $x,y,z \in X$,*

$$\hat{d}(x,z) \leq \hat{d}(x,y) + 2(A+B)d^1_X(y,z). \tag{6.4}$$

*Proof.* First we will consider the special case where $y$ and $z$ lie in a common simplex $\sigma$. By definition, we can expand bilinearly and obtain

$$\begin{aligned}\hat{d}(x,z) &= \sum_{u,w\in X^{(0)}} x_u z_w\, \hat{d}(u,w) \\ &= \sum_{u,w\in X^{(0)}} x_u y_w\, \hat{d}(u,w) + \sum_{u,w\in X^{(0)}} x_u (z_w - y_w)\, \hat{d}(u,w) \\ &= \hat{d}(x,y) + \sum_{u,w\in X^{(0)}} x_u (z_w - y_w)\, \hat{d}(u,w).\end{aligned}$$

Fix a vertex $v_0$ of $\sigma$. Then $\hat{d}(u,w) \leq \hat{d}(u,v_0) + \hat{d}(v_0,w)$ for any vertices $u, w$ of $X$. The distance $\hat{d}(u,v_0)$ is independent of the vertex $w$ so we have $\sum_{u,w\in X^{(0)}} x_u(z_w - y_w)\,\hat{d}(u,v_0) = 0$. Moreover, since $v_0$ is joined to any vertex of $\sigma$ by an edge, we know $\hat{d}(v_0,w) \leq A+B$. Therefore

$$\begin{aligned}\sum_{u,w\in X^{(0)}} x_u(z_w - y_w)\,\hat{d}(u,w) &\leq \sum_{u,w\in X^{(0)}} x_u|z_w - y_w|(A+B)\\ &= 2(A+B)d^1_\sigma(y,z).\end{aligned}$$

But then from proposition 6.3(*i*) we get $d^1_\sigma(y,z) = d^1_X(y,z)$.

Putting all this together gives the desired inequality in this special case.

For the general case, consider a path $y = a_0, a_1, \ldots, a_r = z$ from $y$ to $z$ (in the sense of definition 6.2). Then

$$\hat{d}(x,z) = \hat{d}(x,y) + \sum_{i=1}^{r}\left(\hat{d}(x,a_i) - \hat{d}(x,a_{i-1})\right)$$

We know that $a_{i-1}$ and $a_i$ lie in a common simplex so by the special case we have $\hat{d}(x,a_i) - \hat{d}(x,a_{i-1}) \leq 2(A+B)d^1_X(a_i,a_{i-1})$. Thus

$$\begin{aligned}\hat{d}(x,z) &\leq \hat{d}(x,y) + \sum_{i=1}^{r} 2(A+B)d^1_X(a_i,a_{i-1})\\ &= \hat{d}(x,y) + 2(A+B)l(\underline{a})\end{aligned}$$

where $l(\underline{a})$ is the length of the path $a_0, \ldots, a_r$.

This holds for any path from $y$ to $z$ so we conclude

$$\hat{d}(x,z) \leq \hat{d}(x,y) + 2(A+B)d^1_X(y,z). \qquad \square$$

With this lemma, we can prove that the function $\tilde{d}$ is a metric on the simplicial complex.

**Proposition 6.6.** *Let $X$ be a simplicial complex and let $\hat{d}$ be a metric on the vertices of $X$ that is $(A,B)$-quasi-isometric to the word metric induced by the* 1*-skeleton of $X$. Set $\hat{d}\colon X \times X \to \mathbb{R}$ to be the bilinear extension of $\hat{d}$ as in equation* (6.1)*. Then $\tilde{d}$ defined by equation* (6.3) *is a metric on $X$.*

*Proof.* It is symmetric non-negative since it is the minimum of two functions that are both symmetric non-negative. For all $x \in X$,

$$0 \leq \tilde{d}(x,x) \leq 3(A+B)d^1_X(x,x) = 0$$

so $\tilde{d}(x,x) = 0$. For any two distinct points $x \neq y \in X$ then $d^1_X(x,y) > 0$ since $d^1_X$ is a metric on $X$. Moreover we can find vertices $u \neq v$ with $x_u \neq 0$ and $y_v \neq 0$, thus $\hat{d}(x,y) \geq x_u y_v\,\hat{d}(u,v) > 0$. Therefore $\tilde{d}(x,y) > 0$.

The last thing to check for $\tilde{d}$ to be a metric is the triangle inequality. Given three points $x, y, z \in X$ we need to show

$$\tilde{d}(x,z) \leq \tilde{d}(x,y) + \tilde{d}(y,z).$$

Both $d^1_X$ and $\hat{d}$ satisfy the triangle inequality on $X$, where we use lemma 6.1 for $\hat{d}$, and in the mixed case we can use lemma 6.5. □

We want the metric $\tilde{d}$ to restrict to $\hat{d}$ on the vertex set. Hence we need to look at the metric $d^1_X$ on the vertex set.

**Lemma 6.7.** *Let $X$ be a simplicial complex, and let $\mathcal{G}$ be the 1-skeleton of $X$. For any $u, v \in V$ we have*

$$d_{\mathcal{G}}(u,v) = d^1_X(u,v). \tag{6.5}$$

*Proof.* Any path in the 1-skeleton is also a path in $X$, so $d_{\mathcal{G}}(u,v) \geq d^1_X(u,v)$. It remains to prove that the length of any path in $X$ from $u$ to $v$ is at least $d_{\mathcal{G}}(u,v)$.

For $k = 0, \dots, d := d_{\mathcal{G}}(u,v)$ let $S_k = \{z \in V \mid d_{\mathcal{G}}(u,z) = k\}$. The idea is that the support of any path from $u$ to $v$ must meet every $S_k$ and moreover the weight must come from $u$ and through each $S_k$ before arriving at $v$ (the path could be longer but we are only looking for a lower bound).

Let $u = a_0, a_1, \dots, a_r = v$ be a path in $X$ from $u$ to $v$. So for any $i$ there is always a simplex $\sigma_i$ that contains both $a_{i-1}$ and $a_i$. Write each $a_i = \sum_{z \in V} a_i^z z$ in barycentric coordinates and for all $k$ set

$$w_k(a_i) = \sum_{z \in S_k} a_i^z \in [0,1]$$

which is the weight of the point $a_i$ in the sphere $S_k$.

The length of this path $\underline{a}$ is

$$\begin{aligned}
\text{length}(\underline{a}) &= \sum_{i=1}^{r} d^1_{\sigma_i}(a_{i-1}, a_i) \\
&= \sum_{i=1}^{r} \sum_{z \in V} \frac{1}{2} \left| a_{i-1}^z - a_i^z \right| \\
&\geq \frac{1}{2} \sum_{i=1}^{r} \sum_{k=0}^{d} \sum_{z \in S_k} \left| a_{i-1}^z - a_i^z \right| \\
&\geq \frac{1}{2} \sum_{i=1}^{r} \sum_{k=0}^{d} \left| \sum_{z \in S_k} a_{i-1}^z - \sum_{z \in S_k} a_i^z \right| \\
&= \frac{1}{2} \sum_{k=0}^{d} \sum_{i=1}^{r} \left| w_k(a_{i-1}) - w_k(a_i) \right|.
\end{aligned}$$

Now we try to find a lower bound for the sums $\sum_{i=1}^{r} |w_k(a_{i-1}) - w_k(a_i)|$.

If $k = 0$ then

$$\begin{aligned}\sum_{i=1}^{r} |w_0(a_{i-1}) - w_0(a_i)| &\geq \left|\sum_{i=1}^{r} \big(w_0(a_{i-1}) - w_0(a_i)\big)\right| \\ &= |w_0(a_0) - w_0(a_r)| \\ &= 1\end{aligned}$$

since $a_0 = u \in S_0$ and $a_r = v \in S_d$. Similarly for $k = d$ we get

$$\sum_{i=1}^{r} |w_d(a_{i-1}) - w_d(a_i)| \geq |w_d(a_0) - w_d(a_r)| = 1.$$

So we are left with $1 \leq k < d$. We claim that for such a $k$ we can find some $i_k \in \{1, \ldots, d-1\}$ such that $w_k(a_{i_k}) = 1$.

If for now we assume the claim is true, then

$$\begin{aligned}&\sum_{i=1}^{r} |w_k(a_{i-1}) - w_k(a_i)| \\ &\quad= \sum_{i=1}^{i_k} |w_k(a_{i-1}) - w_k(a_i)| + \sum_{i=i_k+1}^{r} |w_k(a_{i-1}) - w_k(a_i)| \\ &\quad\geq \left|\sum_{i=1}^{i_k} \big(w_k(a_{i-1}) - w_k(a_i)\big)\right| + \left|\sum_{i=i_k+1}^{i_k} \big(w_k(a_{i-1}) - w_k(a_i)\big)\right| \\ &\quad= \big|w_k(a_0) - w_k(a_{i_k})\big| + \big|w_k(a_{i_k}) - w_k(a_r)\big| \\ &\quad= 2\end{aligned}$$

and we would get

$$\begin{aligned}\text{length}(\underline{a}) &\geq \frac{1}{2} \sum_{k=0}^{d} \sum_{i=1}^{r} |w_k(a_{i-1}) - w_k(a_i)| \\ &\geq \frac{1}{2}\big(1 + 2(d-2) + 1\big) \\ &= d = d_{\mathcal{G}}(u, v).\end{aligned}$$

Therefore, to finish the proof of the lemma we need to prove the claim that for all $k = 1, \ldots, d-1$ there is some $i_k$ with $w_k(a_{i_k}) = 1$.

For all $i$ there is a simplex containing both $a_i$ and $a_{i-1}$ hence every vertex in the support of $a_i$ is joined to any vertex in the support of $a_{i-1}$ by an edge. In particular, if $w_{k-1}(a_{i-1}) \neq 0$ then $w_{k+1}(a_i) = 0$ because we may move at most one further away from $u$ with successive $a_i$'s. So if we pick $i_k$ minimal with $w_{k+1}(a_{i_k+1}) \neq 0$ then $w_{k-1}(a_{i_k}) = 0$ and $w_{k+1}(a_{i_k}) = 0$. So all the weight of $a_{i_k}$ has to be at the sphere $S_k$, but $\sum_k w_k(a_{i_k}) = 1$. Therefore $w_k(a_{i_k}) = 1$.

This proves the claim, and with it also finishes the proof of the lemma by the earlier argument. □

Now we can prove that $\tilde{d}$ restricts to $\hat{d}$ on vertices. In fact we can prove a stronger result.

**Lemma 6.8.** *Let $X$ be a simplicial complex, and let $\mathcal{G}$ denote its 1-skeleton. Let $\hat{d}$ be a metric on the vertices of $X$ that is $(A, B)$-quasi-isometric to the word metric $d_{\mathcal{G}}$. For all $x, y \in X$ if the support of $x$ is disjoint from the support of $y$ then*

$$\tilde{d}(x,y) = \hat{d}(x,y).$$

*In particular, $\tilde{d}$ coincides with $\hat{d}$ on the vertex set.*

*Proof.* Start by considering vertices $u, v \in V$. If $u \neq v$ then $d_{\mathcal{G}}(u,v) \geq 1$. Moreover, by assumption we have $\hat{d}(u,v) \leq A\, d_{\mathcal{G}}(u,v) + B$. Therefore

$$\begin{aligned} \hat{d}(u,v) &\leq A\, d_{\mathcal{G}}(u,v) + B \\ &\leq (A+B)\, d_{\mathcal{G}}(u,v) \end{aligned}$$

since $B \geq 0$ and then applying lemma 6.7 yields

$$\hat{d}(u,v) \leq (A+B)\, d^1_X(u,v). \tag{$\star$}$$

Note that if $u = v$ then ($\star$) is trivially true, so inequality ($\star$) holds for all vertices $u, v \in V$.

Now consider the general case $x, y \in X$ with disjoint supports. Since the support of $x$ is disjoint from the support of $y$, we must have $d^1_X(x,y) \geq 1$ by proposition 6.3(*iii*). Moreover, for any $u \in \text{supp}(x)$ proposition 6.3(*ii*) tells us that $d^1_X(u,x) \leq 1$, and similarly for $v \in \text{supp}(y)$. Therefore

$$\begin{aligned} \hat{d}(x,y) &= \sum_{u,v \in V} x_u y_v\, \hat{d}(u,v) \\ &\leq \sum_{u,v \in V} x_u y_v (A+B) d^1_X(u,v) \\ &\leq \sum_{u,v \in V} x_u y_v (A+B) \big(d^1_X(u,x) + d^1_X(x,y) + d^1_X(y,v)\big) \\ &\leq \sum_{u,v \in V} x_u y_v (A+B) 3 d^1_X(x,y) \\ &= 3(A+B) d^1_X(x,y) \end{aligned}$$

and then $\tilde{d}(x,y) = \hat{d}(x,y)$ since $\tilde{d}$ is defined to be the minimum of these two expressions. □

To summarise this section, we have the following theorem;

**Theorem 6.9.** *Let $X$ be a simplicial complex, and let $\mathcal{G}$ denote its 1-skeleton. Let $\hat{d}$ be a metric on the vertex set $V$ of $X$. If $\hat{d}$ is quasi-isometric to $d_{\mathcal{G}}$ then there exists a metric $\tilde{d}$ on all of $X$ such that for any $x, y \in X$ if the support of $x$ is disjoint from the support of $y$ then $\tilde{d}(x,y) = \hat{d}(x,y)$. In particular, $\tilde{d} \equiv \hat{d}$ on vertices.*

*Moreover, if $\hat{d}$ is preserved under simplicial automorphisms of $X$ then so is $\tilde{d}$.*

We have a metric on all of $X$, and we want to know that it doesn't change the large-scale geometry from $\hat{d}$.

**Proposition 6.10.** *Let $X$ be a simplicial complex, and let $\mathcal{G}$ denote its 1-skeleton. Let $\hat{d}$ be a metric on the vertex set $V$ of $X$ that is $(A, B)$-quasi-isometric to $d_{\mathcal{G}}$. The metric $\tilde{d}$ from theorem 6.9 is $(1, 2(A + B))$-quasi-isometric to $\hat{d}$.*

*Proof.* Fix $x, y \in X$. If the support of $x$ is disjoint from the support of $y$ then $\tilde{d}(x, y) = \hat{d}(x, y)$, so suppose there exists some $v \in \operatorname{supp}(x) \cap \operatorname{supp}(y)$. Then

$$\hat{d}(x, v) = \sum_{u \in V} x_u \hat{d}(u, v) \leq \sum_{u \in V} x_u (A + B) = (A + B)$$

and similarly $\hat{d}(v, y) \leq (A + B)$. Therefore

$$\begin{aligned} \tilde{d}(x, y) &\leq \tilde{d}(x, v) + \tilde{d}(v, y) \\ &\leq \hat{d}(x, v) + \hat{d}(v, y) \leq 2(A + B) \leq \hat{d}(x, y) + 2(A + B). \end{aligned}$$

Moreover,

$$\hat{d}(x, y) \leq \hat{d}(x, v) + \hat{d}(v, y) \leq 2(A + B) \leq \tilde{d}(x, y) + 2(A + B).$$

So

$$\hat{d}(x, y) - 2(A + B) \leq \tilde{d}(x, y) \leq \hat{d}(x, y) + 2(A + B).$$

□

## 6.2 Extending our metric

If $X$ is a simplicial complex whose 1-skeleton $\mathcal{G}$ is a uniformly fine, Gromov hyperbolic graph then the metric $\hat{d}$ defined in section 5.3 is quasi-isometric to the word metric (theorem 5.29(*ii*)) so theorem 6.9 provides an extension $\tilde{d}$, which is defined on all of $X$.

This extension is preserved under simplicial automorphisms of $X$ since $\hat{d}$ is preserved under simplicial automorphisms of $X$.

We constructed the metric $\hat{d}$ to have better convergence properties towards the boundary, which was formulated in terms of the double difference in theorem 5.32(*ii*), and we want the metric $\tilde{d}$ to also satisfy these better convergence properties.

As with the metrics $d$ and $\hat{d}$ we can define the *Gromov product with respect to $\tilde{d}$* of three points $x, y, z \in X$ to be

$$\langle x \wr y \rangle_z = \frac{1}{2}\big(\tilde{d}(x, z) + \tilde{d}(y, z) - \tilde{d}(x, y)\big)$$

and the *double difference with respect to* $\tilde{d}$ of four points $x, x', y, y' \in X$ to be

$$\langle x, x' \wr y, y' \rangle = \frac{1}{2}\left(\tilde{d}(x,y) - \tilde{d}(x,y') - \tilde{d}(x',y) + \tilde{d}(x',y')\right).$$

**Remark 6.11.** We can relate the double difference with respect to $\tilde{d}$ to the double difference with respect to $\hat{d}$: Theorem 5.29$(ii)$ tells us that $\hat{d}$ is quasi-isometric to $d$, so proposition 6.10 says that there is some constant $B' \geq 0$ such that $\tilde{d}$ is $(1, B')$-quasi-isometric to $\hat{d}$. Hence for any $x, x', y, y' \in X$;

$$\langle x, x' | y, y' \rangle - 2B' \leq \langle x, x' \wr y, y' \rangle \leq \langle x, x' | y, y' \rangle + 2B'. \tag{6.6}$$

The inequality from theorem 5.32$(ii)$ holds for the double difference with respect to $\tilde{d}$, since the metric $\tilde{d}$ agrees with $\hat{d}$ on the vertex set, i.e. we have the following;

**Proposition 6.12.** *Let $X$ be a simplicial complex whose 1-skeleton is a uniformly fine, Gromov hyperbolic graph. Let $\hat{d}$ be the metric from section 5.3 and let $\tilde{d}$ be the extension given by theorem 6.9. There exist constants $C > 0$ and $\omega \in (0,1)$ such that for all $a, a', b, b' \in V$ if $d(a,a') \leq 1$ and $d(b,b') \leq 1$ then*

$$\left|\langle a, a' \wr b, b' \rangle\right| \leq \frac{1}{2} C\omega^{d(a,b)}. \tag{6.7}$$

If we apply theorem 6.9 to the word metric of the 1-skeleton $\mathcal{G}$ of $X$ then we get a metric $\tilde{d}_{\mathcal{G}}$, and the metric space $(X, \tilde{d}_{\mathcal{G}})$ is Gromov hyperbolic, in the sense of definition 2.11. Therefore we can define the boundary $\partial_s X$ using sequences as in definition 2.29 define a topology on $\overline{X} := X \cup \partial_s X$ using remark 2.34. Now we can finally extend the double difference with respect to $\tilde{d}$ from the vertex set to all of $\overline{X}$.

**Theorem 6.13.** *Let $X$ be a simplicial complex whose 1-skeleton $\mathcal{G}$ is a uniformly fine, Gromov hyperbolic graph. Set $S$ to be the subset of $\overline{X}^4 := \overline{X} \times \overline{X} \times \overline{X} \times \overline{X}$ consisting of points $(x, x', y, y')$ such that*

- $x, y \in \partial X \Rightarrow x \neq y$;
- $x, y' \in \partial X \Rightarrow x \neq y'$;
- $x', y \in \partial X \Rightarrow x' \neq y$;
- $x', y' \in \partial X \Rightarrow x' \neq y'$.

*Then the double difference with respect to $\tilde{d}$ extends continuously to $S$ and satisfies*

*(a)* $\langle a, a' \wr b, b' \rangle = \langle b, b' \wr a, a' \rangle$;

*(b)* $\langle a, a' \wr b, b' \rangle = -\langle a', a \wr b, b' \rangle = -\langle a, a' \wr b', b \rangle$;

(c) $\langle a, a \wr b, b' \rangle = 0 = \langle a, a' \wr b, b \rangle$;

(d) $\langle a, a' \wr b, b' \rangle + \langle a', a'' \wr b, b' \rangle = \langle a, a'' \wr b, b' \rangle$;

(e) $\langle a, b \wr c, x \rangle + \langle c, a \wr b, x \rangle + \langle b, c \wr a, x \rangle = 0$;

(f) *There are constants* $\alpha \geq$ *and* $\beta \geq 0$ *such that for all vertices* $a, a', b, b' \in V$

$$\frac{1}{\alpha}(a, a'|b, b') - \beta \leq \langle a, a' \wr b, b' \rangle \leq \alpha(a, a'|b, b') + \beta.$$

*Proof.* This is essentially [Min05, Theorem 35]. There the double difference is extended further to a larger subset of $\overline{X}^4$, but the proof works by first extending to the set $S$. To extend it further requires allowing the double difference to take values $\pm\infty$, but here we want the double difference to only take values in $\mathbb{R}$ so we only extend to the smaller set $S$.

A couple of modifications must be made to the proof given by Mineyev in [Min05]. Firstly, his definition of a hyperbolic complex requires the 1-skeleton to be uniformly locally finite, but since we have theorem 5.29 the only property required of the graph is Gromov hyperbolicity.

Secondly, his extension is given by taking the bilinear extension of the metric $\hat{d}$ on the vertex set, but as noted earlier this does not give a metric. Hence we have used a different extension, namely $\tilde{d}$ defined by equation (6.3). However, this extension $\tilde{d}$ coincides with $\hat{d}$ on the vertex set (see lemma 6.8). So we can use the argument in the proof of [Min05, Theorem 35] to extend the double difference to the boundary using sequences of vertices.

When considering sequences of arbitrary points (i.e. not necessarily vertices) we know from lemma 6.8 that $\tilde{d}(x, y) = \hat{d}(x, y)$ whenever the support of $x$ is disjoint from the support of $y$. So if $(x_n)_{n \in \mathbb{N}}$ is a sequence of points that converges to $x \in \partial X$ and $(y_n)_{n \in \mathbb{N}}$ is a sequence of points that converges to $y \in \overline{X} \backslash \{x\}$ then for sufficiently large $n$, the support of $x_n$ will be disjoint from the support of $y_n$ and thus we can use the bilinear extension formula (6.1).

For part (f), Mineyev in [Min05, Theorem 35(h)] shows an equivalence between $\langle -, - | -, - \rangle$ and $(-, - | -, -)$ but we can go to $\langle -, - \wr -, - \rangle$ using remark 6.11. □

We know that for points $x, y, z, w \in X$, the double difference relates to the Gromov product via the equation

$$\langle x, y \wr z, w \rangle = \langle y \wr z \rangle_x - \langle y \wr w \rangle_x \tag{6.8}$$

and, in particular $\langle x, y \wr z, x \rangle = \langle y \wr z \rangle_x$. Hence we can use theorem 6.13 to extend the Gromov product to the boundary.

**Theorem 6.14.** *Let $X$ be a simplicial complex whose 1-skeleton $\mathcal{G}$ is a uniformly fine, Gromov hyperbolic graph. Set $T$ to be the subset of $\overline{X}^3 := \overline{X} \times \overline{X} \times \overline{X}$ consisting of points $(x, y, z)$ such that*

- $x \notin \partial X$;
- $y, z \in \partial X \Rightarrow y \neq z$.

*Then the Gromov product $\langle y \wr z \rangle_x$ with respect to $\tilde{d}$ extends continuously to $T$ and satisfies*

$$\langle a, a' \wr b, b' \rangle = \langle a' \wr b \rangle_a - \langle a' \wr b' \rangle_a \tag{6.9}$$

*whenever $(a, a', b, b') \in S$ (where $S$ is defined as in theorem 6.13) and $a \notin \partial X$.*

*Proof.* If $(x, y, z) \in T$ then $(x, y, z, x) \in S$ and we can define the Gromov product in terms of the double difference: $\langle y \wr z \rangle_x := \langle x, y \wr z, x \rangle$.

Equation (6.9) follows from theorem 6.13(d), where the properties (a) and (b) are also used implicitly. □

This extension of the Gromov product coincides with the sup lim inf extension given in definition 2.31, since for any $x \in X$ and $y, z \in \overline{X}$ with $y \neq z$, if $(y_n)$ and $(z_n)$ are sequences in $X$ converging to $y$ and $z$ respectively then the continuity of the double difference given by theorem 6.13 gives

$$\begin{aligned}\liminf_{n\to\infty} \langle y_n \wr z_n \rangle_x &= \liminf_{n\to\infty} \langle x, y_n \wr z_n, x \rangle \\ &= \langle x, y \wr z, x \rangle \\ &= \langle y \wr z \rangle_x.\end{aligned}$$

In particular, this means the facts from proposition 2.33 still hold.

We cannot extend $\langle -, - \wr -, - \rangle$ to all of $\overline{X}^4$ because it would need to be $\pm\infty$ in places, as the next lemma demonstrates.

**Lemma 6.15.** *Let $X$ be a simplicial complex whose 1-skeleton $\mathcal{G}$ is a uniformly fine, Gromov hyperbolic graph. If $(x_n)_{n\in\mathbb{N}}$ is a sequence in $X$ that converges to a point $x \in \partial X$ then for any $y, z \in \overline{X} \backslash \{x\}$ if $y \neq z$ then*

$$\langle x, y \wr x_n, z \rangle \longrightarrow -\infty \qquad \text{as } n \to \infty.$$

*Proof.* Pick a sequence $(z_k)_{k\in\mathbb{N}}$ in $X$ that converges to $z$. Then by continuity, as $k \to \infty$ we get $\langle z, z_k \wr y, x \rangle \longrightarrow \langle z, z \wr y, x \rangle = 0$. Therefore we can pick $k$ large enough such that $|\langle z, z_k \wr y, x \rangle| \leq 1$.

Combining theorems 6.13 and 6.14 gives

$$\begin{aligned}\langle x, y \wr x_n, z \rangle &= \langle z, x_n \wr y, x \rangle \\ &= \langle z, z_k \wr y, x \rangle + \langle z_k, x_n \wr y, x \rangle \\ &\leq 1 + \langle x_n \wr y \rangle_{z_k} - \langle x_n \wr x \rangle_{z_k}.\end{aligned}$$

Since the sequence $(x_n)_{n\in\mathbb{N}}$ converges to a point $x \in \partial_s X$ the Gromov product $\langle x_i \wr x_j \rangle_{z_k}$ tends up to $\infty$ as $i, j \to \infty$. Hence

$$\langle x_n \wr x \rangle_{z_k} \geq \liminf_{m\to\infty} \langle x_n \wr x_m \rangle_{z_k} \longrightarrow \infty \quad \text{as } n \to \infty.$$

By remark 2.32 we can pick sequences $(x'_n)$ and $(y_n)$ in $X$ with $x'_n \to x$ and $y_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \langle x'_n \wr y_n \rangle_{z_k} = \langle x \wr y \rangle_{z_k}$. Then

$$\langle x'_n \wr y_n \rangle_{z_k} \geq \min\{\langle x'_n \wr x_n \rangle_{z_k}, \langle x_n \wr y \rangle_{z_k}, \langle y \wr y_n \rangle_{z_k}\} - 4\delta$$

by proposition 2.33($ii$). Since the sequences $(x_n)$ and $(x'_n)$ both converge to $x \in \partial_s X$ proposition 2.33($i$) says $\langle x'_n \wr x_n \rangle_{z_k} \to \infty$ as $n \to \infty$. Also, the Gromov product $\langle y \wr y_n \rangle_{z_k}$ tends to infinity, repeating the argument for $\langle x_n \wr x \rangle_{z_k}$ above. Therefore

$$\begin{aligned} \lim_{n\to\infty} \langle x_n \wr y \rangle_{z_k} &\leq \lim_{n\to\infty} \langle x'_n \wr y \rangle_{z_k} + 4\delta \\ &= \langle x \wr y \rangle_{z_k} + 4\delta \end{aligned}$$

which is finite since $x \neq y$.

Bringing this all together,

$$\begin{aligned} \lim_{n\to\infty} \langle x, y \wr x_n, z \rangle &\leq 1 + \langle x \wr y \rangle_{z_k} + 2\delta - \lim_{n\to\infty} \langle x_n \wr x \rangle_{z_k} \\ &= -\infty. \end{aligned}$$

□

## 6.3 A metric on the flow space

In section 3.4 we outlined a plan to construct a metric on $FS(X)$ for a metric space $X$. We needed to extend the double difference to points on the boundary and to do this we constructed a new metric $\tilde{d}$ in the case that $X$ is a simplicial complex whose 1-skeleton is a uniformly fine, $\delta$-hyperbolic graph (see theorem 6.13).

For the rest of this section, we assume that $X$ is such a simplicial complex, and that $x_0 \in X$ is a fixed base-point, and we try to apply the construction from section 3.4 to the metric space $(X, \tilde{d})$.

**Definition 6.16.** We start by using the metric to interpret the parameter $t$ of a point in the flow space (cf definition 3.14). For any point $(x, y, t) \in FS(X)$ define a map $\tilde{\theta}^{x_0}_{x,y} \colon \overline{\mathbb{R}} \to [-\langle y \wr x_0 \rangle_x, \langle x \wr x_0 \rangle_y]$ by

$$\tilde{\theta}^{x_0}_{x,y}(t) = \begin{cases} -\langle y \wr x_0 \rangle_x & \text{if } t \leq -\langle y \wr x_0 \rangle_x \\ t & \text{if } -\langle y \wr x_0 \rangle_x \leq t \leq \langle x \wr x_0 \rangle_y \\ \langle x \wr x_0 \rangle_y & \text{if } \langle x \wr x_0 \rangle_y \leq t \end{cases}.$$

With this, we have an idea of how far $(x, y, t)_0$ (the fictitious point at time zero) is from the point of $[x, y]$ that is 'closest' to the base-point. Using the family of maps $(\tilde{\theta}^{x_0}_{x,y} | x, y \in \overline{X})$ we can define a distance from $(x, y, t)_0$ to an arbitrary point $z \in X$.

**Definition 6.17.** Define a map $l_{x_0} \colon X \times FS(X) \to [0, \infty)$ by

$$l_{x_0}(z, (x, y, t)) = \langle x \wr y \rangle_z + \left| \tilde{\theta}^{x_0}_{x,y}(t) - \langle x, y \wr z, x_0 \rangle \right|.$$

This function is well-defined by theorems 6.13 and 6.14, since $z$ and $x_0$ are both points in $X$, and not the boundary.

**Remark 6.18.** If $\underline{x} := [(x, x, 0)] \in FS(X)$ is a stationary point then for any $z \in X$ we have $l_{x_0}(z, \underline{x}) = \tilde{d}(z, x)$.

Thus we can say what the distance is from $(x, y, t)_0$ to $z$ but we are interested in the distance from $(x, y, t)_0$ to $(x', y', t')_0$. We do this by comparing their distances to $z$.

**Definition 6.19.** Define a function $\beta_{x_0} \colon X \times FS(X) \times FS(X) \to \mathbb{R}$ by

$$\beta_{x_0}(z, c, c') = l_{x_0}(z, c) - l_{x_0}(z, c'). \tag{6.10}$$

The next lemma states a couple of simple properties about the function $\beta_{x_0}$. The proofs are straight-forward and hence omitted.

**Lemma 6.20.** *The function $\beta_{x_0}$ satisfies the following properties;*

*(i)* *$\beta_{x_0}$ is invariant under the induced diagonal action of* $\mathrm{Isom}(X)$ *on $FS(X)$;*

*(ii)* *For any $x, x', z \in X$,* $\quad \beta_{x_0}(z, \underline{x}, \underline{x'}) = \tilde{d}(z, x) - \tilde{d}(z, x')$;

*(iii)* *$\beta_{x_0}$ is anti-symmetric with respect to the $FS(X) \times FS(X)$ factors, i.e. for all $z \in X$ and for all $c, c' \in FS(X)$, $\beta_{x_0}(z, c, c') = -\beta_{x_0}(z, c', c)$;*

*(iv)* *For all $c_1, c_2, c_3 \in FS(X)$ and all $z \in X$,*

$$\beta_{x_0}(z, c_1, c_3) = \beta_{x_0}(z, c_1, c_2) + \beta_{x_0}(z, c_2, c_3).$$

Some more interesting properties;

**Lemma 6.21.** *For any $x, y \in \overline{X}$, if $c_1 = (x, y, t)$ and $c_2 = (x, y, t_2)$ in $FS(X)$ for some $t_1, t_2 \in \mathbb{R}$ then for any $z \in X$,*

$$|\beta_{x_0}(z, c_1, c_2)| \le \left| \tilde{\theta}^{x_0}_{x,y}(t_1) - \tilde{\theta}^{x_0}_{x,y}(t_2) \right|.$$

*Moreover, if $x \in X$ then*

$$\beta_{x_0}(x, c_1, c_2) = \tilde{\theta}^{x_0}_{x,y}(t_1) - \tilde{\theta}^{x_0}_{x,y}(t_2).$$

*Proof.* If $c_1 = (x, y, t_1)$ and $c_2 = (x, y, t_2)$ then the Gromov product $\langle x \wr y \rangle_z$ cancels itself out. The special case with $x \in X$ uses $\langle x, y \wr x, x_0 \rangle = -\langle y \wr x_0 \rangle_x$ (by equation (6.9)) and $\tilde{\theta}^{x_0}_{x,y}(t) + \langle y \wr x_0 \rangle_x$ is always positive (by the definition of $\tilde{\theta}^{x_0}_{x,y}$).

The general case of $x \in \overline{X}$ can be shown by considering the various possibilities in turn. If both $\tilde{\theta}^{x_0}_{x,y}(t_1)$ and $\tilde{\theta}^{x_0}_{x,y}(t_2)$ are greater than $\langle x, y \wr z, x_0 \rangle$ then $\beta_{x_0}(z, c_1, c_2) = \tilde{\theta}^{x_0}_{x,y}(t_1) - \tilde{\theta}^{x_0}_{x,y}(t_2)$. If they are both less than $\langle x, y \wr z, x_0 \rangle$ then $\beta_{x_0}(z, c_1, c_2) = \tilde{\theta}^{x_0}_{x,y}(t_2) - \tilde{\theta}^{x_0}_{x,y}(t_1)$. This only leaves the possibility that one is greater and one is lesser. Without loss of generality, we assume that $\tilde{\theta}^{x_0}_{x,y}(t_1) < \langle x, y \wr z, x_0 \rangle < \tilde{\theta}^{x_0}_{x,y}(t_2)$. Hence

$$\begin{aligned} |\beta_{x_0}(z, c_1, c_2)| &= \left| \left| \langle x, y \wr z, x_0 \rangle - \tilde{\theta}^{x_0}_{x,y}(t_1) \right| - \left| \langle x, y \wr z, x_0 \rangle - \tilde{\theta}^{x_0}_{x,y}(t_2) \right| \right| \\ &= \left| 2\langle x, y \wr z, x_0 \rangle - \tilde{\theta}^{x_0}_{x,y}(t_1) - \tilde{\theta}^{x_0}_{x,y}(t_2) \right| \end{aligned}$$

and if $2\langle x, y \wr z, x_0 \rangle \leq \tilde{\theta}^{x_0}_{x,y}(t_1) + \tilde{\theta}^{x_0}_{x,y}(t_2)$ then

$$\begin{aligned} |\beta_{x_0}(z, c_1, c_2)| &= \tilde{\theta}^{x_0}_{x,y}(t_1) + \tilde{\theta}^{x_0}_{x,y}(t_2) - 2\langle x, y \wr z, x_0 \rangle \\ &\leq \tilde{\theta}^{x_0}_{x,y}(t_2) - \tilde{\theta}^{x_0}_{x,y}(t_1) \end{aligned}$$

since $\langle x, y \wr z, x_0 \rangle \geq \tilde{\theta}^{x_0}_{x,y}(t_1)$ by assumption.

Similarly the assumption $\langle x, y \wr z, x_0 \rangle \leq \tilde{\theta}^{x_0}_{x,y}(t_2)$ can be used to get the inequality if $2\langle x, y \wr z, x_0 \rangle \geq \tilde{\theta}^{x_0}_{x,y}(t_1) + \tilde{\theta}^{x_0}_{x,y}(t_2)$.

We have gone through all the possibilities so the inequality in the statement of the lemma must always hold. □

**Definition 6.22.** Define the function $L_{x_0} : FS(X) \times FS(X) \to \mathbb{R}$ by

$$L_{x_0}(c, c') = \sup_{z \in X} \left| \beta_{x_0}(z, c, c') \right|. \tag{6.11}$$

Informally, the function $L_{x_0}$ should represent the distance between the points of $c, c'$ at time zero.

**Proposition 6.23.** *The function $L_{x_0}$ is a pseudometric on $FS(X)$. Moreover, it satisfies the following properties;*

(*i*) *For all $x, x' \in X$,* $\quad L_{x_0}(\underline{x}, \underline{x'}) = \tilde{d}(x, x')$;

(*ii*) *For any $x, y \in \overline{X}$, if $c_1 = (x, y, t_1)$ and $c_2 = (x, y, t_2)$ in $FS(X)$ then*

$$L_{x_0}(c_1, c_2) = \left| \tilde{\theta}^{x_0}_{x,y}(t_1) - \tilde{\theta}^{x_0}_{x,y}(t_2) \right|.$$

*Proof.* To show that $L_{x_0}$ is a pseudometric, we need to show it is symmetric, positive semi-definite, and satisfies the triangle inequality.

The function $\beta_{x_0}$ is anti-symmetric, so taking the absolute value makes it symmetric and non-negative. To show that $L_{x_0}(c,c) = 0$ for any $c \in FS(X)$, we use lemma 6.21.

Finally, the triangle inequality follows from lemma 6.20$(iv)$.

Therefore the function $L_{x_0}$ is indeed a pseudometric on $FS(X)$.

Property $(i)$ follows from the triangle inequality and lemma 6.20$(ii)$.

It remains to show $(ii)$. If $x \notin \partial X$ then it follows from lemma 6.21. So assume $x \in \partial X$. Take a sequence $(x_n)_{n\in\mathbb{N}}$ of points in $X$ that converge to $x$. Then

$$\begin{aligned}\beta_{x_0}(x_n, c_1, c_2) &= l_{x_0}(x_n, c_1) - l_{x_0}(x_n, c_2)\\ &= \left|\tilde{\theta}^{x_0}_{x,y}(t_1) - \langle x, y \wr x_n, x_0\rangle\right| - \left|\tilde{\theta}^{x_0}_{x,y}(t_2) - \langle x, y \wr x_n, x_0\rangle\right|\end{aligned}$$

but $\langle x, y \wr x_n, x_0\rangle \to -\infty$ as $n \to \infty$ by lemma 6.15, thus we can find some large $n$ such that both $\tilde{\theta}^{x_0}_{x,y}(t_1) - \langle x, y \wr x_n, x_0\rangle$ and $\tilde{\theta}^{x_0}_{x,y}(t_2) - \langle x, y \wr x_n, x_0\rangle$ are positive. Therefore

$$L_{x_0}(c_1, c_2) \geq \left|\tilde{\theta}^{x_0}_{x,y}(t_1) - \tilde{\theta}^{x_0}_{x,y}(t_2)\right|$$

and lemma 6.21 provides the other inequality, hence we get equality. □

To make $L_{x_0}$ into a metric we need to be able to distinguish between two points of $FS(X)$. We do this by not only considering the points at time zero, but along the entire formal generalised geodesic, with an appropriate weighting.

Recall that the flow $\Phi$ on $FS(X)$ is given by $\Phi_\tau(x, y, t) := (x, y, t + \tau)$.

**Definition 6.24.** Define the function $d_{FS}\colon FS(X) \times FS(X) \to \mathbb{R}$ by

$$d_{FS}(c_1, c_2) = \int_{\mathbb{R}} \frac{L_{x_0}\left(\Phi_\tau(c_1), \Phi_\tau(c_2)\right)}{2e^{|\tau|}} d\tau. \tag{6.12}$$

**Proposition 6.25.** *The function $d_{FS}$ is a metric on $FS(X)$.*

*Proof.* That $d_{FS}$ is a pseudometric follows immediately from the fact that $L_{x_0}$ is a pseudometric, thus it only remains to prove that if $c_1 \neq c_2 \in FS(X)$ then $d_{FS}(c_1, c_2) \neq 0$. Write $c_1 = (x_1, y_1, t_1)$ and $c_2 = (x_2, y_2, t_2)$. First we show that if $x_1 \neq x_2$ then $d_{FS}(c_1, c_2) > 0$.

If $x_i \notin \partial X$ then $\Phi_\tau(c_i) \to \underline{x_i}$ as $\tau \to -\infty$. But $\beta_{x_0}(x_1, \underline{x_1}, \underline{x_2}) = -\tilde{d}(x_1, x_2)$ by lemma 6.20$(ii)$. Hence

$$\left|\beta_{x_0}(x_1, \Phi_\tau(c_1), \Phi_\tau(c_2))\right| \longrightarrow \tilde{d}(x_1, x_2) \quad \text{as } \tau \to -\infty.$$

Therefore if $x_1, x_2 \notin \partial X$ and $x_1 \neq x_2$ then the integral for $d_{FS}(c_1, c_2)$ is non-zero.

Suppose $x_1 \in \partial X$ but $x_2 \notin \partial X$. For $\tau << 0$ we know that $l_{x_0}(x_0, \Phi_\tau(c_2))$ is bounded by $\langle x_2 \wr y_2 \rangle_{x_0} + \langle x_0 \wr y_2 \rangle_{x_2}$ but

$$l_{x_0}(x_0, \Phi_\tau(c_1)) = \langle x_1 \wr y_1 \rangle_{x_0} - (t_1 + \tau)$$

and so $|\beta_{x_0}(x_0, \Phi_\tau(c_1), \Phi_\tau(c_2))| \to \infty$ as $\tau \to -\infty$. Since $L_{x_0}$ is defined as a supremum over $X$ of $\beta_{x_0}$ we get that $L_{x_0}(\Phi_\tau(c_1), \Phi_\tau(c_2))$ is also unbounded, and then the integral defining $d_{FS}(c_1, c_2)$ must be non-zero.

So suppose $x_1, x_2 \in \partial X$ and $x_1 \neq x_2$. First assume $x_1 \neq y_2$. For any $z \in X$ the Gromov product $\langle x_1 \wr x_2 \rangle_z$ is finite (since $x_1 \neq x_2$). Thus

$$\begin{aligned} \langle x_1 \wr y_1 \rangle_z - \langle x_2 \wr y_2 \rangle_z &= \big(\langle x_1 \wr y_1 \rangle_z - \langle x_1 \wr x_2 \rangle_z\big) - \big(\langle x_1 \wr x_2 \rangle_z - \langle x_2 \wr y_2 \rangle_z\big) \\ &= \langle z, x_2 \wr y_2, x_1 \rangle - \langle z, x_1 \wr x_2, y_1 \rangle. \end{aligned}$$

As $z \to x_1$ we can make $\langle z, x_2 \wr y_2, x_1 \rangle$ arbitrarily large (by lemma 6.15) and $\langle z, x_1 \wr x_2, y_1 \rangle$ arbitrarily close to 0. In particular, we can pick $z \in X$ such that $\langle x_1 \wr y_1 \rangle_z > \langle x_2 \wr y_2 \rangle_z$. Next set $\tau = t_2 - \langle x_2, y_2 \wr z, x_0 \rangle$, i.e. choose $\tau$ to minimise $l_{x_0}(z, \Phi_\tau(c_2))$. Then

$$\begin{aligned} \beta_{x_0}(z, \Phi_\tau(c_1), \Phi_\tau(c_2)) &= \langle x_1 \wr y_1 \rangle_z + |\tilde{\theta}^{x_0}_{x_1,y_1}(t_1 + \tau) - \langle x_1, y_1 \wr z, x_0 \rangle| \\ &\qquad - \langle x_2 \wr y_2 \rangle_z - |\tilde{\theta}^{x_0}_{x_2,y_2}(t_2 + \tau) - \langle x_2, y_2 \wr z, x_0 \rangle| \\ &= \langle x_1 \wr y_1 \rangle_z - \langle x_2 \wr y_2 \rangle_z \\ &\qquad + |\tilde{\theta}^{x_0}_{x_1,y_1}(t_1 + \tau) - \langle x_1, y_1 \wr z, x_0 \rangle| \\ &\geq \langle x_1 \wr y_1 \rangle_z - \langle x_2 \wr y_2 \rangle_z \\ &> 0 \end{aligned}$$

and it follows that $L_{x_0}(\Phi_\tau(c_1), \Phi_\tau(c_2)) > 0$ and thus $d_{FS}(c_1, c_2) > 0$.

Similarly, if $x_1, x_2 \in \partial X$ with $x_1 \neq x_2$ and $x_2 \neq y_1$ we get $d_{FS}(c_1, c_2) > 0$.

However, if $x_1, x_2 \in \partial X$ with $x_1 \neq x_2$ but $x_1 = y_2$ and $x_2 = y_1$, which intuitively means that $c_2$ is the formal geodesic $c_1$ in reverse, then we need a different argument. Fix some $\tau$ such that $t_1 + \tau \neq -(t_2 + \tau)$, and pick $z \in X$ such that

$$\langle x_1, y_1 \wr z, x_0 \rangle \geq \max\{t_1 + \tau, -(t_2 + \tau)\}.$$

We can find such a $z \in X$ since if $(z_n)_{n \in \mathbb{N}}$ is a sequence in $X$ with $z_n \to y_1$ as $n \to \infty$ then $\langle x_1, y_1 \wr z_n, x_0 \rangle \to \infty$ as $n \to \infty$ by lemma 6.15. Then

$$\begin{aligned} \beta_{x_0}(z, \Phi_\tau(c_1), \Phi_\tau(c_2)) &= \langle x_1 \wr y_1 \rangle_z + |(t_1 + \tau) - \langle x_1, y_1 \wr z, x_0 \rangle| \\ &\qquad - \big(\langle x_2 \wr y_2 \rangle_z + |(t_2 + \tau) - \langle x_2, y_2 \wr z, x_0 \rangle|\big) \\ &= \langle x_1, y_1 \wr z, x_0 \rangle - (t_1 + \tau) \\ &\qquad - \big(\langle x_1, y_1 \wr z, x_0 \rangle + (t_2 + \tau)\big) \\ &= -(t_1 + \tau) - (t_2 + \tau) \\ &\neq 0 \end{aligned}$$

by the choice of $\tau$. Hence $L_{x_0}\big(\Phi_\tau(c_1),\Phi_\tau(c_2)\big) > 0$ and thus $d_{FS}(c_1,c_2) > 0$.

Thus if $x_1 \neq x_2$ then $d_{FS}(c_1,c_2) > 0$. Similarly if $y_1 \neq y_2$. So assume that $x_1 = x_2 =: x$ and $y_1 = y_2 =: y$. If $x = y$ then $c_1 = \underline{x} = c_2$ which contradicts the assumption $c_1 \neq c_2$, so assume $x \neq y$. Then

$$\begin{aligned}|\beta_{x_0}(x_0,\Phi_\tau(c_1),\Phi_\tau(c_2)| &= \Bigg|\Big|\tilde{\theta}^{x_0}_{x,y}(t_1+\tau) - \langle x,y \wr x_0,x_0\rangle\Big| \\ &\qquad - \Big|\tilde{\theta}^{x_0}_{x,y}(t_2+\tau) - \langle x,y \wr x_0,x_0\rangle\Big|\Bigg| \\ &= \Big|\tilde{\theta}^{x_0}_{x,y}(t_1+\tau) - \tilde{\theta}^{x_0}_{x,y}(t_2+\tau)\Big|\end{aligned}$$

and we can find some $\tau_0 \in \mathbb{R}$ such that $t_1 + \tau_0 \in (-\langle y \wr x_0\rangle_x, \langle x \wr x_0\rangle_y)$ and then $\tilde{\theta}^{x_0}_{x,y}(t_1+\tau_0) = t_1 + \tau_0$. If $t_1 \neq t_2$ then $\tilde{\theta}^{x_0}_{x,y}(t_2+\tau_0) \neq t_1 + \tau_0$ and so $L_{x_0}(\Phi_{\tau_0}(c_1),\Phi_{\tau_0}(c_2) > 0$.

Therefore $c_1 \neq c_2$ implies $d_{FS}(c_1,c_2) > 0$, so $d_{FS}$ is a metric on $FS(X)$. □

# 7 Properties of the Flow Space

In section 3.3 we defined the set underlying the flow space, and subsequently a metric on $FS(X)$ was constructed, culminating in section 6.3.

This section is dedicated to properties of the metric space $\left(FS(X), d_{FS}\right)$. Section 7.1 is about the topology of this metric space, section 7.2 considers what happens we change the base-point of $X$, and section 7.3 shows how an isometric group action on $X$ induces an isometric action on the flow space.

Then section 7.4 contains some nice results about the convergence of formal geodesics in the flow space.

## 7.1 The induced topology

Given a simplicial complex $X$ whose 1-skeleton is a uniformly fine, Gromov hyperbolic graph we can define the flow space as in section 3.3. This is a topological space, which for now we denote by $(FS(X), \tau)$ and use $FS(X)$ to denote the underlying set. In section 6.3 we defined a metric on the set $FS(X)$. We would like the topology induced by this metric to coincide with the original topology $\tau$ as given in definition 3.13.

It follows from the definition of the flow space that the diagonal map $X \to (FS(X), \tau)$ is an embedding. We can also consider the diagonal map $X \to (FS(X), d_{FS})$.

**Proposition 7.1.** *Let $X$ be a simplicial complex whose 1-skeleton is a uniformly fine, Gromov hyperbolic graph. The map $(X, \tilde{d}) \to (FS(X), d_{FS})$ given by sending $x \in X$ to the class $\underline{x} := [(x, x, 0)] \in FS(X)$ is an isometric embedding.*

*Proof.* For any $x, x' \in X$ it follows from lemma 6.20$(ii)$ and the triangle inequality that $L_{x_0}(\underline{x}, \underline{x'}) = \tilde{d}(x, x')$. The geodesics are stationary with respect to the flow so in equation (6.12) we get

$$d_{FS}(\underline{x}, \underline{x'}) = \int_{\mathbb{R}} \frac{\tilde{d}(x, x')}{2e^{|\tau|}}\, d\tau = \tilde{d}(x, x'). \qquad \square$$

Next we consider what happens away from the stationary geodesics. For this, we would like to quote [Min05, Proposition 48]. However, the proof of that proposition uses [Min05, Proposition 38], whose proof used the bilinear extension of the metric $\hat{d}$ from the vertex set to all of $X$ (see equation (6.1)). Hence we state the latter proposition in our case so that we can fix this small problem, as well as explain why the rest of the argument still holds.

**Proposition 7.2.** *Let $X$ be a simplicial complex whose* 1*-skeleton is a uniformly fine, Gromov hyperbolic graph. Let $S$ be as in theorem 6.13. There exists a constant $\lambda_0 \in [0, 1)$ such that for all $\lambda \in [\lambda_0, 1)$ there is a $T \geq 0$ such that for all $a, b, c, u \in \overline{X}$ if*

- $(a, c, u, b) \in S$*;*
- $(b, c, u, a) \in S$*;*
- $T \leq \max\{\langle a, c \wr u, b\rangle, \langle b, c \wr u, a\rangle\} =: m$

*then $(a, b, u, c) \in S$ and*

$$|\langle a, b \wr u, c\rangle| \leq \lambda^m.$$

*Proof.* If $a, b, c, u$ are vertices or boundary points then we can use the proof of [Min05, Proposition 38], which only uses Gromov hyperbolicity of the 1-skeleton as well as facts established in our case in theorem 5.29.

In order to extend to arbitrary points in $X$ we need to do a bit more work, since we need to show that under the assumptions given we can use the bilinear extension of $\hat{d}$, i.e. we want

$$\langle a, b \wr u, c\rangle = \langle a, b | u, c\rangle$$

so that we can write $\langle a, b \wr u, c\rangle$ in terms of the double difference of vertices in the support of the points $a, b, u, c$.

Let $A, B$ be the constants appearing in a quasi-isometry between $\hat{d}$ and the word metric on the vertex set of $X$, so that

$$\tilde{d}(x, y) = \min\{\hat{d}(x, y), 3(A + B)d^1_X(x, y)\}.$$

The diameter (with respect to the metric $d^1_X$) of any simplex is $\leq 1$ by proposition 6.3$(ii)$. Moreover, if the supports of two points $x, y \in X$ are disjoint then $\tilde{d}(x, y) = \hat{d}(x, y)$, by lemma 6.8. Therefore, if $\tilde{d}(x, y) > 6(A + B)$ then the supports of $x, y$ are disjoint and $\tilde{d}(x,y) = \hat{d}(x,y)$. So we are interested in finding a lower bound for the distances used to define $\langle a, b \wr u, c\rangle$.

Without loss of generality suppose $m = \langle b, c \wr u, a\rangle$. Theorem 6.14 gives

$$\begin{aligned}\langle b, c \wr u, a\rangle &= \langle c \wr u\rangle_b - \langle c \wr a\rangle_b \\ &\leq \min\left\{\tilde{d}(c,b), \tilde{d}(u,b)\right\}\end{aligned}$$

and then the assumption $T \leq m$ gives $T \leq \tilde{d}(c,b), \tilde{d}(u,b)$. Moreover,

$$\begin{aligned}\langle b, c \wr u, a\rangle = -\langle a, u \wr b, c\rangle &= -(\langle u \wr b\rangle_a - \langle u \wr c\rangle_a) \\ &\leq \min\left\{\tilde{d}(u,a), \tilde{d}(c,a)\right\}\end{aligned}$$

and so $T \leq \tilde{d}(c,a), \tilde{d}(u,a)$ as well.

Thus if we take $T \geq 6(A+B)$ then for all four of these pairings $\tilde{d} \equiv \hat{d}$, and we get $\langle a, b \wr u, c\rangle = \langle a, b | u, c\rangle$. Since we are allowed to increase $T$ as we wish, the rest of the proof works as in [Min05, Proposition 38]. □

**Theorem 7.3.** *Let $X$ be a simplicial complex whose 1-skeleton is a uniformly fine, Gromov hyperbolic graph. Let $\Delta(\overline{X})$ be the diagonal subset of $\overline{X} \times \overline{X}$. Let $FS(X)^{\mathbb{R}}$ denote the stationary orbits, i.e. the orbits fixed by the flow. The map*

$$(\overline{X} \times \overline{X} \backslash \Delta(\overline{X})) \times \mathbb{R} \to \left(FS(X) \backslash FS(X)^{\mathbb{R}}, \tilde{d}\right) \;;\quad (x,y,t) \mapsto [x,y,t]$$

*is a homeomorphism.*

*Proof.* The proof is as for [Min05, Proposition 48]. It works in our case using the results we have already proven, for example our theorem 5.32 covers both [Min05, Theorem 32] and [Min05, Proposition 33]. □

## 7.2 Change of base-point

The metric $d_{FS}$ on $FS(X)$ constructed in section 6.3 depends on the choice of base-point $x_0 \in X$.

However, we do not want the topology of $FS(X)$ to depend on this base-point so we must look at what would happen if we chose a different base-point.

Let $d_{FS}^{x_0}$ denote the metric on the flow space using $x_0$ as base-point.

**Proposition 7.4.** *Let $X$ be a simplicial complex whose 1-skeleton is a fine, Gromov hyperbolic graph. Let $x_0, y_0 \in X$ be two choices of base-point. The map*

$$\left(FS(X), d_{FS}^{x_0}\right) \to \left(FS(X), d_{FS}^{y_0}\right) \;;\quad (x,y,t) \mapsto (x, y, t - \langle x, y \wr x_0, y_0\rangle)$$

*is an isometry.*

*Proof.* The maps $\tilde{\theta}^{x_0}_{x,y}$ and $\tilde{\theta}^{y_0}_{x,y}$, which allow us to interpret the parameter $t$, are related by the equation

$$\tilde{\theta}^{x_0}_{x,y}(t) - \langle x, y \wr x_0, y_0 \rangle = \tilde{\theta}^{y_0}_{x,y}(t - \langle x, y \wr x_0, y_0 \rangle)$$

and so for any $z \in X$

$$l_{x_0}\big(z, (x, y, t)\big) = l_{y_0}\big(z, (x, y, t - \langle x, y \wr x_0, y_0 \rangle)\big).$$

From this it follows that

$$\begin{aligned} &d^{x_0}_{FS}\big((x, y, t), (x', y', t')\big) \\ &\qquad = d^{y_0}_{FS}\big((x, y, t - \langle x, y \wr x_0, y_0 \rangle), (x', y', t' - \langle x', y' \wr x_0, y_0 \rangle)\big) \end{aligned}$$

i.e. the map $(x, y, t) \mapsto (x, y, t - \langle x, y \wr x_0, y_0 \rangle)$ is an isometry. □

Therefore, although the metric on the flow space depends on the choice of base-point, the topology of the flow space does not.

## 7.3 The group action on $FS(X)$

If $X$ is a simplicial complex with a uniformly fine, Gromov hyperbolic 1-skeleton then we have constructed a flow space $FS(X)$. If a group $G$ acts simplicially on $X$ then the metric $\tilde{d}$ is invariant under $G$ and we could consider the action of $G$ on $FS(X)$ given by

$$g \cdot (x, y, t) := (gx, gy, t).$$

However, the metric $d_{FS}$ is not necessarily invariant under this group action. The problem arises from the dependence of the metric $d_{FS}$ on the base-point $x_0$. Hence we define the action of $G$ on $FS(X)$ via

$$g \cdot (x, y, t) := \Big(gx, gy, t + \langle x, y \wr x_0, g^{-1}x_0 \rangle\Big).$$

With this definition, we can show

$$l_{x_0}\big(gz, g \cdot (x, y, t)\big) = l_{x_0}\big(z, (x, y, t)\big)$$

for any $z \in X$. From which it follows that for any $c, c' \in FS(X)$ we have $L_{x_0}(g \cdot c, g \cdot c') = L_{x_0}(c, c')$ and thus $d_{FS}$ is invariant under $G$.

So now the $G$-action on $X$ induces a $G$-action on $FS(X)$ that preserves the metric $d_{FS}$.

## 7.4 Convergence of formal geodesics

If a space is uniquely geodesic, i.e. between any pair of points there is precisely one geodesic between them, then the flow space is just the space of all generalised geodesics. More generally, the flow space was defined to formally replicate the space of generalised geodesics, even if we start with a Gromov hyperbolic metric space that is not necessarily a geodesic space.

We already know that if $y \in X$ then $(x, y, t) \to \underline{y}$ as $t \to \infty$, where $\underline{y}$ denotes the point of the flow space represented by $(y, y, 0)$. Recall that the flow is defined by $\Phi_\tau(x, y, t) = (x, y, t + \tau)$ so more generally we can say that $\Phi_\tau(x, y, t) \to \underline{y}$ as $\tau \to \infty$. In particular, for any $x, x' \in \overline{X}$ and $t, t' \in \mathbb{R}$

$$d_{FS}\left(\Phi_\tau(x, y, t), \Phi_\tau(x', y, t')\right) \longrightarrow 0 \quad \text{as } \tau \to \infty. \tag{7.1}$$

If $y \in \partial X$ then there is no such point $\underline{y}$ in the flow space, but we could still ask for (7.1) to hold. However, since a geodesic ray will never reach its end-point, it is possible that one geodesic is always running behind the other. For example, consider $\mathbb{R}$ as a graph with vertex set $\mathbb{Z}$ and base-point 0. Then for all $\tau \in [0, \infty)$

$$d_{FS}\left(\Phi_\tau(0, \infty, 0), \Phi_\tau(0, \infty, 1)\right) = 1$$

by proposition 6.23(*ii*). Hence we have to add the quantifier "there exists a $t_0 \in \mathbb{R}$" where this $t_0$ will bring the parametrisations of the geodesics in line. It turns out, that this is the only change we need.

**Theorem 7.5.** *Let $X$ be a simplicial complex whose 1-skeleton is a uniformly fine, Gromov hyperbolic graph. For any $x, x' \in X, y \in \overline{X}$ and all $t, t' \in \mathbb{R}$ there exists a constant $t_0 \in \mathbb{R}$ such that*

$$d_{FS}\left(\Phi_\tau(x, y, t), \Phi_\tau(x', y, t' + t_0)\right) \longrightarrow 0 \quad \text{as } \tau \to \infty.$$

*Proof.* This follows from [BLR08a, Theorem 8.9]. That article builds on the work in [Min05] and as such it assumes the 1-skeleton of the simplicial complex is uniformly locally finite, but this assumption is only necessary for quoting the results from [Min05], and these results have been proven for a uniformly fine 1-skeleton in this thesis.

Given the existence of the metric $\tilde{d}$ that satisfies nice properties (e.g. theorem 6.13 and proposition 7.2) the proof of the theorem is mainly computational. □

Recall that if a group $G$ is hyperbolic relative to a finite collection $\mathcal{P}$ of subgroups then, by definition, we can find a fine, Gromov hyperbolic graph $\Gamma$ on which $G$ acts such that there are only finitely many $G$-orbits of vertices and edges. The construction from section 3.1 applied to this graph yields a simplicial complex $X = X_\eta(\Gamma)$ whose 1-skeleton is a uniformly fine, Gromov hyperbolic graph. We can define a boundary of $X$ as in section 2.5, and then we also have the space $\overline{X}$, on which $G$ acts. In this case we can use [BLR08a, Theorem 1.5] to say something stronger.

(Recall that the action of $G$ on the flow space is via equation (7.3).)

**Theorem 7.6.** *Let $G$ be a group that is hyperbolic relative to a finite collection $\mathcal{P}$ of subgroups and let $X$ be the associated simplicial complex from section 3.1. Fix a base-point $x_0 \in X$ and let $G$ act on $G \times \overline{X}$ via the diagonal action.*

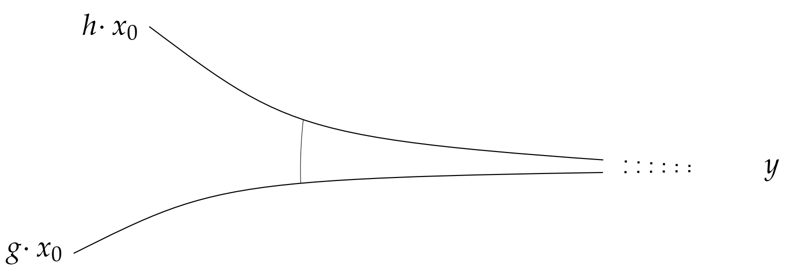
*There is a continuous $G$-equivariant map $j\colon G \times \overline{X} \to FS(X)$ that satisfies the following property;*

*For any $\alpha > 0$ there is a function $f_\alpha\colon \mathbb{R} \to [0, \infty)$ with $f_\alpha(\tau) \to 0$ as $\tau \to \infty$, and there is a $\beta = \beta(\alpha) > 0$ such that for all $g, h \in G$ and $y \in \overline{X}$, if $d_G(g, h) \leq \alpha$ then there exists some $t_0 \in [-\beta, \beta]$ such that for all $\tau \in \mathbb{R}$*

$$d_{FS}\left(\Phi_\tau j(g, y), \Phi_{\tau+t_0} j(h, y)\right) \leq f_\alpha(\tau). \tag{7.2}$$

The theorem has a lot of quantifiers so we shall try to give an informal explanation of its statement. The map $j\colon G \times \overline{X} \to FS(X)$ sends the pair $(g, y)$ to some point along the flow orbit with ends $g\cdot x_0$ and $y$, i.e. there is some $t_{(g,y)}$ such that $j(g, y) = (g\cdot x_0, y, t_{(g,y)})$. Although we haven't explained how to choose $t_{(g,y)}$ it should vary uniformly continuously with $g$.

The number $t_0$ represents how far the formal geodesic $(g\cdot x_0, y, t_{(g,y)})$ is ahead or behind the formal geodesic $(h\cdot x_0, y, t_{(h,y)})$ and so $|t_0|$ should be bounded by some expression involving $\tilde{d}(g\cdot x_0, h\cdot x_0)$ and $|t_{(g,y)} - t_{(h,y)}|$, and this expression would vary uniformly continuously with $g, h$. Therefore, the bound $\beta$ on $|t_0|$ should depend only on $\alpha$.




Then the theorem states that $\Phi_\tau j(g, y)$ and $\Phi_{\tau+t_0} j(h, y)$ converge uniformly as $\tau \to \infty$.

*Proof of Theorem 7.6.* The proof is as for the proof of [BLR08a, Theorem 1.5], but we need to make one remark here.

To prove the map $j$ is continuous they use the fact that $\overline{X}$ is metrisable, so we need to use section 2.6 to show $\overline{X}$ is metrisable for our choice of $X$. □

# 8 A Flow Space for a Relatively Hyperbolic Group

Now we want to summarise what happens in the case of a relatively hyperbolic group.

If a group $G$ is hyperbolic relative to a finite set $\mathcal{P}$ of subgroups of $G$ then, by definition, we can find a fine, Gromov hyperbolic graph $\Gamma$ on which $G$ acts with certain properties (see definition 2.22 for the full list of properties). To such a graph there is an associated simplicial complex $X = X_\eta(\Gamma)$ (which also depends upon a choice of parameter $\eta \geq 1$), and this simplicial complex is contractible and finite-dimensional. Furthermore, the 1-skeleton $\mathcal{G} = \mathcal{G}_\eta(\Gamma)$ of $X$ is a fine, Gromov hyperbolic graph. This was all done in section 3.1.

The group $G$ acts on $X$ such that there are only finitely many orbits of simplices, and in particular we can say that $\mathcal{G}$ is uniformly fine using lemma 2.19.

Therefore there is a metric space $(FS(X), d_{FS})$ associated to $X$, which has a flow $\Phi\colon FS(X) \times \mathbb{R} \to FS(X)$ such that two points $c_1, c_2$ with the same end-point in $X$ converge under this flow (see theorem 7.5). Moreover, there is an embedding $j\colon G \times \overline{X} \to FS(X)$ such that for any $y \in \overline{X}$ and $g, h \in G$ the elements $j(g, y)$ and $j(h, y)$ in $FS(X)$ converge uniformly under the flow (see theorem 7.6).

We hope that we will be able to use this flow space to help prove a version of the Farrell-Jones Conjecture for relatively hyperbolic groups, as explained at the very start of this thesis.

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