An algebraic characterization of the Weyl curvature of $S^m \times S^m$

Florian Jäger

2016

Mathematik

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Inaugural-Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften im Fachbereich
Mathematik und Informatik
der Mathematisch-Naturwissenschaftlichen Fakultät
der Westfälischen Wilhelms-Universität Münster

vorgelegt von
Florian Jäger
aus Werne
– 2016 –

Dekan:	Prof. Dr. Martin Stein
Erster Gutachter:	Prof. Dr. Christoph Böhm
Zweiter Gutachter:	Prof. Dr. Burkhard Wilking
Tag der mündlichen Prüfung:	18.07.2016
Tag der Promotion:	18.07.2016

Abstract

In the present work, the eigenvalue equation $W^2 + W^\# = \theta W$, which is closely related to the evolution equation of a curvature operator under the Ricci flow, is analyzed for Weyl curvature operators W. A proof that under certain conditions θ is maximal if and only if W is the Weyl curvature of $S^m \times S^m$ is given. Moreover, infinite series of new solutions to this eigenvalue equation are constructed.

Contents

Introduction

Hamilton introduced the Ricci flow in 1982 in [Ha1] to prove that compact three-manifolds which admit Riemannian metrics of positive Ricci curvature are spherical space forms. In the subsequent years, the Ricci flow evolved to a powerful tool in differential geometry. It was used in many proofs of important results such as Perelman's proof of the Poincaré conjecture in [P1] and [P2], the proof of the differentiable sphere theorem by Brendle and Schoen in [BS], and Böhm and Wilking's proof that compact Riemannian manifolds with positive curvature operators are space forms in [BW], to name but a few.
Hamilton showed in [Ha2] that the evolution equation of the Riemannian curvature operator is given by the partial differential equation

$$\frac{\partial R}{\partial t} - \Delta R = 2(R^2 + R^\#).$$

Hamilton's maximum principle asserts that an open, convex, $O(n)$-invariant subset of the space $S^2_B(\mathfrak{so}(n))$ of algebraic curvature operators which is invariant under the associated ordinary differential equation $\frac{dR}{dt} = R^2 + R^\#$ defines a Ricci flow invariant curvature condition. Even more importantly, pinching sets of such curvature conditions are constructed to show that the Ricci flow evolves to metrics with constant sectional curvature. Notice, if such an invariant curvature condition C contains an operator $R \notin \langle I \rangle$ with

$$R^2 + R^\# = \theta R \tag{1}$$

for $\theta \in \mathbb{R}$, no such pinching family can exist. Here, θ is called the eigenvalue of R. This is why a classification of solutions to (1) would be very desirable. The curvature operators of compact, irreducible symmetric spaces and also their Weyl curvature operators provide such solutions. For example, for even integers $n \geq 4$ the normalized Weyl curvature operator of $S^{\frac{n}{2}} \times S^{\frac{n}{2}}$ satisfies equation (1) with $\theta = \frac{\sqrt{2(n-1)(n-2)}}{n} =: \theta_0(n)$.
In the present work, the following result will be proven.

Theorem A. Let $n \geq 16$ be an even integer, and let $W \in \langle \mathbf{W} \rangle_d$ with $\|W\| = 1$ be given such that $W^2 + W^\# = \theta W$ for a $\theta \geq \theta_0(n)$. Then $\theta = \theta_0(n)$, and W is the normalized Weyl curvature of $S^{\frac{n}{2}} \times S^{\frac{n}{2}}$.

The conjecture is that this is also true for all even integers $n \geq 12$ and all operators $W \in \langle \mathbf{W} \rangle$.
Here, $\langle \mathbf{W} \rangle \subset S^2_B(\mathfrak{so}(n))$ denotes the space of Wely curvature operators and $\langle \mathbf{W} \rangle_d$ its subspace of operators which are diagonal with respect to the standard basis $e_1 \wedge e_2, \dots, e_{n-1} \wedge e_n$ of $\mathfrak{so}(n)$. Those operators $W \in \langle \mathbf{W} \rangle_d$ can be expressed as a symmetric $n \times n$-matrices $(w_{ij})_{1 \leq i,j \leq n}$ with 0 on the diagonal via $W(e_i \wedge e_j) = w_{ij}\, e_i \wedge e_j$ for all $1 \leq i < j \leq n$. The condition that W is Weyl then reduces to $\sum_{i=1}^n w_{ij} = 0$ for all $1 \leq j \leq n$. Further, if $W \in \langle \mathbf{W} \rangle_d$, the same is also true for $W^2 + W^\#$. Here, the Lie theoretic quadratic $\#\colon S^2(\mathfrak{so}(n)) \to S^2(\mathfrak{so}(n)); R \mapsto R^\#$ is defined by

$$\langle R^\#(v), w \rangle = -\frac{1}{2} \mathrm{tr}\,(\mathrm{ad}_w \circ R \circ \mathrm{ad}_v \circ R)$$

for $v, w \in \mathfrak{so}(n)$, where $\mathrm{ad}\colon \Lambda^2(\mathfrak{so}(n)) \to \mathfrak{so}(n)$ denotes the adjoint representation of $\mathfrak{so}(n)$.
Moreover, $SO(n)$ acts on $S^2_B(\mathfrak{so}(n))$ by means of the adjoint representation $\mathrm{Ad}\colon SO(n) \to \mathrm{Aut}(\mathfrak{so}(n))$ of $SO(n)$ via $g.R = \mathrm{Ad}_g R\, \mathrm{Ad}_g^T$. For curvature operators R of compact, irreducible symmetric spaces of dimension at least 3 the isotropy group $SO(n)_R$ is infinite.
In the present work, the following result will be proven.

Theorem B. There exist solutions in $\langle \mathbf{W} \rangle_d$ to equation (1) with at most finite isotropy groups in dimension p and p^2 for a prime number p such that 4 divides $p-1$ with eigenvalue 0 and in dimension mn for integers $3 \leq n < m$ with positive eigenvalue.

Clearly, these new examples do not emerge from symmetric spaces since their isotropy groups are not infinite. The solutions in dimension p and p^2 will be constructed by means of the Legendre Symbol $\left(\frac{a}{p}\right) \in \{\pm 1\}$, a number theoretic quantity that measures whether a given integer a is a square modulo a given odd prime number p or not. The solution to (1) in dimension p is of the form

$$\begin{pmatrix} 0 & \left(\frac{1}{p}\right) & \cdots & \left(\frac{p-1}{p}\right) \\ \left(\frac{p-1}{p}\right) & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \left(\frac{1}{p}\right) \\ \left(\frac{1}{p}\right) & \cdots & \left(\frac{p-1}{p}\right) & 0 \end{pmatrix},$$

and it gives rise to the constructions in dimension p^2. The series in dimension mn will be constructed without using number theory.
Since all of the new solutions are purely algebraic and do not arise from symmetric spaces, a general classification of solutions to equation (1) seems to be complicated.

The proof of theorem A will be separated into several steps. First of all, it will be shown in section 3.1 that the normalized Weyl curvature operator of $S^{\frac{n}{2}} \times S^{\frac{n}{2}}$, which will be called W_0, is a solution to (1) with eigenvalue $\theta_0(n)$. Then for W as in the statement the related operator $|W|$ will be defined by

$$\langle |W|(e_i \wedge e_j), e_i \wedge e_j\rangle = |\langle W(e_i \wedge e_j), e_i \wedge e_j\rangle|$$

for all $1 \leq i < j \leq n$. Since $|W_0|$ is somehow close to the normalized identity, its scalar curvature is almost maximal under all normalized curvature operators. The idea is now to show that the same is also true for $|W|$. The first step of this will be to prove in section 3.2 that the scalar curvature of $|W|$ cannot be contained in a certain interval with upper bound close to the scalar curvature of the normalized identity. It will then be proven via contradiction in the sections 3.3, 3.4, and 3.5 that actually the scalar curvature of $|W|$ has to be greater than the lower bound and therefore also greater than the upper bound of this interval. In order to do this, some estimates for related quantities will be proven. Afterwards, this will be used in section 3.6 to show that W has only two distinct eigenvalues which in fact agree with those of W_0 and that the dimensions of their eigenspaces are the same as those of the eigenvalues of W_0. It will then be deduced via the classification of symmetric spaces that the eigenspace of the positive eigenvalue of W is isomorphic to the Lie subalgebra $\mathfrak{so}(\frac{n}{2}) \oplus \mathfrak{so}(\frac{n}{2})$ of $\mathfrak{so}(n)$, from which finally follows that W is equal to W_0.
All solutions W to (1) constructed for theorem B will be expressed as generalized circulant matrices. The part for the solutions in dimension p and p^2 will then be proven via the basic properties of the Legendre symbol. By the definition of Legendre symbols it can be seen that W^2 is the identity on $\mathfrak{so}(p)$ and $\mathfrak{so}(p^2)$, respectively, and further properties of the Legendre symbol will finally be used to show that $W^{\#}$ is equal to -1 times the identity. The part in dimension mn will be proven via a straight forward computation. For all solutions it will be shown that $\ker(W^2 \# I - W^{\#}) = \{0\}$, which implies that their isotropy groups are at most finite.
The present work is structured into three parts. In section 1.1, a brief introduction to the theory of Ricci flows with a focus on the evolution of curvature operators will be given. Moreover, the space of algebraic curvature operators and the #-map will be discussed in the sections 1.2 and 1.3. Finally, the set of algebraic curvature operators which are diagonal with respect to the standard basis of $\mathfrak{so}(n)$ will be analyzed more explicitly in section 1.4.
In chapter 2, some results about Legendre symbols will be given in section 2.1 as well as a generalized version of the definition of a circulant matrix. Theorem B will then be proven in the sections 2.2 and 2.3.
In chapter 3, the proof of theorem A will be given. It will be divided into six sections.

Acknowledgement

I would like to express my sincere gratitude to my advisor Prof. Dr. Christoph Böhm for his constant and competent support during my PhD study. His suggestions and guidance helped me during my whole research. I also would like to thank all former and actual members of the Münster Differential Geometry Research Group for helpful discussions and all kind of scientific support.

Chapter 1

Preliminaries

1.1 The Ricci flow and the evolution of curvature operators

Introduced by Hamilton in [Ha1], a Ricci flow on a smooth manifold M is a family of Riemannian metrics $(g(t))_{t\in I}$ on M for an interval $I \subset \mathbb{R}$ such that $g(t)$ satisfies the Ricci flow equation

$$\frac{\partial g(t)}{\partial t} = -2\,\mathrm{Ric}_{g(t)}.$$

Clearly, if the metric changes with t, also all related geometric quantities do so as well. The Riemannian curvature operator $R(t)$ of $(M, g(t))$ for $t \in I$ carries all information of the different curvature quantities, and the understanding of its behaviour under the Ricci flow is hence a major concern. Using Uhlenbeck's Trick (cf. [CK], chapter 6), the following evolution equation for the curvature operator $R(t)$ of $g(t)$ can be obtained:

$$\frac{\partial R}{\partial t} = \Delta R + 2\left(R^2 + R^{\#}\right). \tag{1.1}$$

The term $R^{\#}$ will be explained more explicitly in section 1.3. There is the following related ordinary differential equation to this partial differential equation:

$$\frac{dR}{dt} = R^2 + R^{\#}. \tag{1.2}$$

Due to Hamilton's maximum principle (cf. [C$^+$], chapter 10.2.2), in many cases it suffices to analyze equation (1.2) instead of the much more complex equation (1.1). More specifically, let $(g(t))_{t\in[0,T)}$ be a Ricci flow on a smooth manifold M such that for every point $p \in M$ the curvature operator $R_p(0)$ in p at the initial time satisfies a certain curvature condition, that is $R_p(0) \in C$ for all $p \in M$ and an $O(n)$-invariant subset C of $S^2_B(\mathfrak{so}(n))$. Here, $S^2_B(\mathfrak{so}(n))$

is the set of algebraic curvature operators, which will be discussed in the next section, and $\Lambda^2(T_pM)$ is identified with $\mathfrak{so}(n)$. Moreover, let C be invariant under the ordinary differential equation (1.2), that is all solutions $S\colon [t_0,t_1) \to S^2_B(\mathfrak{so}(n))$ of (1.2) with $S(t_0) \in C$ have the property that also $S(t) \in C$ for all $t \in (t_0,t_1)$. Then also $R_p(t)$ satisfies these curvature condition for all $p \in M$ and all $t \in (0,T)$.

1.2 Algebraic curvature operators

If V is a Euclidean vector space, the exterior product $\Lambda^2 V$ will be endowed with the natural inner product given by

$$\langle v \wedge w, x \wedge y\rangle = \langle v,x\rangle\langle w,y\rangle - \langle v,y\rangle\langle w,x\rangle$$

for all $v,w,x,y \in V$. Moreover, two endomorphisms A and B on V induce a linear map on $\Lambda^2 V$ by

$$A \wedge B\colon \Lambda^2 V \to \Lambda^2 V;\ v \wedge w \mapsto \frac{1}{2}\left(A(v)\wedge B(w) + B(v)\wedge A(w)\right).$$

Clearly, $A \wedge B = B \wedge A$. In the following, $S^2(V)$ will always denote the set of selfadjoint endomorphisms on a given vector space V, and it will be endowed with the inner product given by $\langle A,B\rangle = \mathrm{tr}\,(AB)$ for $A,B \in S^2(V)$. It is easy to see that $A \wedge B \in S^2(\Lambda^2 V)$ for all $A,B \in S^2(V)$. Furthermore, $\Lambda^2\mathbb{R}^n$ will be identified with $\mathfrak{so}(n)$ as follows:
Let $e_1,\dots,e_n$ be the standard basis of $\mathbb{R}^n$. Then for $1 \le i < j \le n$ $e_i \wedge e_j$ will be identified with E_{ij}, where

$$(E_{ij})_{kl} = \begin{cases} 1 & \text{if } k = i, l = j,\\ -1 & \text{if } k = j, l = i,\\ 0 & \text{otherwise} \end{cases}$$

for all $1 \le k,l \le n$. Under this identification the scalar product on $\mathfrak{so}(n)$ corresponds to

$$\langle A,B\rangle = -\frac{1}{2}\mathrm{tr}\,(AB).$$

An element $R \in S^2(\mathfrak{so}(n))$ that satisfies the first Bianchi identity

$$\langle R(x \wedge y), z \wedge w\rangle + \langle R(y \wedge z), x \wedge w\rangle + \langle R(z \wedge x), y \wedge w\rangle = 0$$

for all $x,y,z,w \in \mathbb{R}^n$ is called an *algebraic curvature operator*, and the set of all algebraic curvature operators will be denoted with $S^2_B(\mathfrak{so}(n))$. A straight forward calculation shows that $A \wedge B \in S^2_B(\mathfrak{so}(n))$ for all $A,B \in S^2(\mathbb{R}^n)$. *The Ricci curvature* $\mathrm{Ric}(R)\colon \mathbb{R}^n \to \mathbb{R}^n$ and *the scalar curvature* $\mathrm{scal}(R)$ of an algebraic curvature operator R can be defined analogously to those of a Riemannian curvature operator by

$$\langle \mathrm{Ric}(R)(e_i), e_j \rangle = \sum_{k=1}^{n} \langle R(e_i \wedge e_k), e_j \wedge e_k \rangle = \sum_{k=1}^{n} R_{kijk} \quad \text{and}$$
$$\mathrm{scal}(R) = \mathrm{tr}\,\mathrm{Ric}(R).$$

Moreover, the *traceless Ricci tensor* of R can be defined as

$$\mathrm{Ric}_0(R) = \mathrm{Ric}(R) - \frac{\mathrm{scal}(R)}{n}\mathrm{id}.$$

For $n \geq 4$ there is the following natural decomposition of $S^2(\mathfrak{so}(n))$ into $O(n)$-invariant, irreducible, and orthogonal subspaces (cf. [C+], proposition 11.8):

$$S^2(\mathfrak{so}(n)) = \langle I \rangle \oplus \langle \mathrm{Ric}_0 \rangle \oplus \langle \mathbf{W} \rangle \oplus \Lambda^4\mathbb{R}^n.$$

Here, $\langle I \rangle$ is the space of multiples of the identity $I = \mathrm{id} \wedge \mathrm{id}$ on $\mathfrak{so}(n)$, $\langle \mathbf{W} \rangle$ is defined to be the kernel of the map $\mathrm{Ric}\colon S^2_B(\mathfrak{so}(n)) \to S^2(\mathbb{R}^n); R \mapsto \mathrm{Ric}(R)$ and is called *the set of Weyl curvature operators*, and $\langle \mathrm{Ric}_0 \rangle$ is the subspace

$$\langle \mathrm{Ric}_0 \rangle = \left\{ A \wedge \mathrm{id} \in S^2(\mathfrak{so}(n)) \,\middle|\, A \in S^2(\mathbb{R}^n) \text{ and } \mathrm{tr}\, A = 0 \right\}.$$

It is known that $R \in S^2(\mathfrak{so}(n))$ is an algebraic curvature operator if and only if the $\Lambda^4\mathbb{R}^n$-part of R vanishes (cf [C+], chapter 11.2.2). Thus,

$$S^2_B(\mathfrak{so}(n)) = \langle I \rangle \oplus \langle \mathrm{Ric}_0 \rangle \oplus \langle \mathbf{W} \rangle.$$

In the following, the projections of a curvature operator $R \in S^2_B(\mathfrak{so}(n))$ onto $\langle I \rangle$, $\langle \mathrm{Ric}_0 \rangle$, and $\langle \mathbf{W} \rangle$ will be denoted with R_I, R_{Ric_0}, and $R_{\mathbf{W}}$, respectively. Then

$$R_I = \frac{\mathrm{scal}(R)}{n(n-1)} I \quad \text{and} \quad R_{\mathrm{Ric}_0} = \frac{2}{n-2} \mathrm{Ric}_0 \wedge \mathrm{id} \tag{1.3}$$

(cf. [C+], chapter 11.2.2.3). Moreover, set $R_{\mathrm{Ric}} := R_I + R_{\mathrm{Ric}_0}$.

1.3 The #-map and its properties

Let $N = \frac{n(n-1)}{2} = \dim(\mathfrak{so}(n))$. Following Hamilton (cf. [Ha2]), # will be defined as the map

$$\#\colon S^2(\mathfrak{so}(n)) \times S^2(\mathfrak{so}(n)) \to S^2(\mathfrak{so}(n)); (R, S) \mapsto R\#S$$

given by

$$\langle (R\#S)(v), w\rangle = \frac{1}{2}\sum_{\alpha,\beta=1}^{N} \langle [R(b_\alpha), S(b_\beta)], v\rangle \, \langle [b_\alpha, b_\beta], w\rangle$$

for $v, w \in \mathfrak{so}(n)$ and an orthonormal basis $b_1, \dots, b_N$ of $\mathfrak{so}(n)$. Again following Hamilton, the notation

$$R^{\#} := R\#R$$

will be used. # has the following properties (cf [Ha2] or [C+], chapter 11.1.2 and 11.4.2):

- # is an $O(n)$-equivariant bilinear map.
- # is symmetric in R and S.
- Let $R \in S^2_B(\mathfrak{so}(n))$ be given. Then

$$R + R\#I = (n-1)R_I + \frac{n-2}{2} R_{\mathrm{Ric}_0}. \tag{1.4}$$

  In particular, $I + I^{\#} = (n-1)I$ and $W + W\#I = 0$ for every $W \in \langle \mathbf{W} \rangle$.

Moreover, $R\#S$ can be described invariantly via the following.

Lemma 1.3.1. Let $\mathrm{ad}\colon \Lambda^2\mathfrak{so}(n) \to \mathfrak{so}(n); v \wedge w \mapsto [v, w]$ denote the adjoint representation of $\mathfrak{so}(n)$, and let $\mathrm{ad}_v\colon \mathfrak{so}(n) \to \mathfrak{so}(n); w \mapsto [v, w]$ for $v \in \mathfrak{so}(n)$. Then

$$\langle (R\#S)(v), w\rangle = -\frac{1}{2}\mathrm{tr}\,(\mathrm{ad}_w R\,\mathrm{ad}_v S)$$

for all $R, S \in S^2_B(\mathfrak{so}(n))$ and all $v, w \in \mathfrak{so}(n)$.

Proof. Let $b_1, \dots, b_N$ be an orthonormal basis of $\mathfrak{so}(n)$, and let $v, w \in \mathfrak{so}(n)$ and $R, S \in S^2_B(\mathfrak{so}(n))$ be fixed. Then

$$\begin{aligned}
\langle (R\#S)(v), w\rangle &= \frac{1}{2}\sum_{\alpha,\beta=1}^{N} \langle [R(b_\alpha), S(b_\beta)], v\rangle \, \langle [b_\alpha, b_\beta], w\rangle \\
&= \frac{1}{2}\sum_{\alpha,\beta=1}^{N} \langle \mathrm{ad}_{S(b_\beta)}(R(b_\alpha)), v\rangle \, \langle \mathrm{ad}_{b_\beta}(b_\alpha), w\rangle \\
&= \frac{1}{2}\sum_{\alpha,\beta=1}^{N} \langle b_\alpha, (R\,\mathrm{ad}_v S)\,(b_\beta)\rangle \, \langle b_\alpha, \mathrm{ad}_w(b_\beta)\rangle
\end{aligned}$$

because $\mathrm{ad}_{\tilde{v}}^T = -\mathrm{ad}_{\tilde{v}}$ and $\mathrm{ad}_{\tilde{v}}(\tilde{w}) = [\tilde{v}, \tilde{w}] = -[\tilde{w}, \tilde{v}] = -\mathrm{ad}_{\tilde{w}}(\tilde{v})$ for all $\tilde{v}, \tilde{w} \in \mathfrak{so}(n)$ and because R is selfadjoint. Since $b_1, \dots b_N$ is an orthonormal basis of $\mathfrak{so}(n)$, it follows that

$$\begin{aligned}\langle (R\#S)(v), w\rangle &= \frac{1}{2}\sum_{\beta=1}^{N} \langle (R\,\mathrm{ad}_v S)\,(b_\beta), \mathrm{ad}_w(b_\beta)\rangle \\ &= \frac{1}{2}\sum_{\beta=1}^{N} \langle (-\mathrm{ad}_w R\,\mathrm{ad}_v S)\,(b_\beta), b_\beta\rangle \\ &= -\frac{1}{2}\mathrm{tr}\,\left(\mathrm{ad}_w R\,\mathrm{ad}_v S\right),\end{aligned}$$

which was to be proven. □

The quadratic $R^2 + R^\#$ for $R \in S_B^2(\mathfrak{so}(n))$ is of major interest in the present work. First of all, $R^2 + R^\#$ satisfies the first Bianchi identity for all $R \in S_B^2(\mathfrak{so}(n))$ (cf. [C$^+$], chapter 11.2.3). Moreover, the following properties are known for this expression and will be used later (cf. [C$^+$], corollary 11.18):

$$\mathrm{Ric}(R^2 + R^\#)_{ij} = \sum_{k,l=1}^{n} \mathrm{Ric}(R)_{kl} R_{kijl} \quad \text{for all } 1 \le i, j \le n \text{ and} \tag{1.5}$$

$$\mathrm{scal}(R^2 + R^\#) = \sum_{i,j=1}^{n} (\mathrm{Ric}(R)_{ij})^2. \tag{1.6}$$

In particular,

$$W^2 + W^\# \in \langle \mathbf{W} \rangle \tag{1.7}$$

for all $W \in \langle \mathbf{W} \rangle$.

1.4 Diagonal algebraic curvature operators

In the present work, the focus will be on those algebraic curvature operators which are diagonal with respect to the standard basis $e_1 \wedge e_2, \dots, e_{n-1} \wedge e_n$ of $\mathfrak{so}(n)$. Referring to this basis, such an operator $R \in S_B^2(\mathfrak{so}(n))$ can be expressed as a symmetric $n \times n$-matrix $(r_{ij})_{1 \le i,j \le n}$ instead of being viewed as a diagonal $N \times N$-matrix via

$$\begin{aligned}&R(e_i \wedge e_j) = r_{ij}\, e_i \wedge e_j && \text{for all } 1 \le i < j \le n \text{ and} \\ &r_{ii} = 0 && \text{for all } 1 \le i \le n.\end{aligned}$$

In the following, the sets of all n-dimensional diagonal algebraic curvature operators and of all n-dimensional diagonal Weyl curvature operators will be denoted with $S_{B,d}^2(\mathfrak{so}(n))$ and $\langle \mathbf{W} \rangle_d^n$, respectively. Moreover, in the rest of the present work, operators in $S^2_{B,d}(\mathfrak{so}(n))$ and $\langle \mathbf{W}\rangle^n_d$ will be identified with their associated matrices with respect to the standard basis. With this notation, it is easy to see that for $R = (r_{ij})_{1\leq i,j\leq n}, S = (s_{ij})_{1\leq i,j\leq n} \in S^2_{B,d}(\mathfrak{so}(n))$ the following properties hold:

- $\langle R, S\rangle = \operatorname{tr}(RS) = \sum\limits_{1\leq i<j\leq n} r_{ij}s_{ij}$.

- $\operatorname{Ric}(R)_{ij} = \begin{cases} \sum\limits_{k=1}^{n} r_{ik} & \text{if } i = j, \\ 0 & \text{otherwise} \end{cases}$ for all $1 \leq i, j \leq n$ and

  $\operatorname{scal}(R) = \sum\limits_{i,j=1}^{n} r_{ij} = 2\langle R, I\rangle$.

  Therefore, $W \in S^2_{B,d}(\mathfrak{so}(n))$ is Weyl if and only if $\sum_{j=1}^{n} w_{ij} = 0$ for all $1 \leq i \leq n$.

- The operator R^2 is also contained in $S^2_{B,d}(\mathfrak{so}(n))$ and is given by $(R^2)_{ij} = r^2_{ij}$ for all $1 \leq i < j \leq n$.

- The operator $R\#S$ is also contained in $S^2_{B,d}(\mathfrak{so}(n))$ and is given by the formula $(R\#S)_{ij} = \frac{1}{2}\sum\limits_{k=1}^{n}(r_{ik}s_{jk} + r_{jk}s_{ik})$ for all $1 \leq i < j \leq n$ since

$$
\begin{aligned}
&\langle (R\#S)(e_i \wedge e_j), e_k \wedge e_l\rangle \\
&= \frac{1}{2}\sum_{1\leq a<b\leq n}\sum_{1\leq c<d\leq n} r_{ab}s_{cd}\langle [e_a \wedge e_b, e_c \wedge e_d], e_i \wedge e_j\rangle \\
&\quad \langle [e_a \wedge e_b, e_c \wedge e_d], e_k \wedge e_l\rangle \\
&= \frac{1}{2}\delta_{ik}\delta_{jl}\sum_{1\leq a<b\leq n}\sum_{1\leq c<d\leq n} r_{ab}s_{cd}\langle [e_a \wedge e_b, e_c \wedge e_d], e_i \wedge e_j\rangle^2 \\
&= \frac{1}{2}\delta_{ik}\delta_{jl}\sum_{1\leq a<b\leq n}\sum_{1\leq c<d\leq n} r_{ab}s_{cd} \\
&\quad \langle -\delta_{ac}\, e_b \wedge e_d + \delta_{bc}\, e_a \wedge e_d + \delta_{ad}\, e_b \wedge e_c - \delta_{bd}\, e_a \wedge e_c, e_i \wedge e_j\rangle^2 \\
&= \frac{1}{2}\delta_{ik}\delta_{jl}\sum_{1\leq a<b\leq n}\sum_{1\leq c<d\leq n} r_{ab}s_{cd}\Big(-\delta_{ac}\,(\delta_{bi}\delta_{dj} - \delta_{bj}\delta_{di}) \\
&\quad + \delta_{bc}\,(\delta_{ai}\delta_{dj} - \delta_{aj}\delta_{di}) + \delta_{ad}\,(\delta_{bi}\delta_{cj} - \delta_{bj}\delta_{ci}) - \delta_{bd}\,(\delta_{ai}\delta_{cj} - \delta_{aj}\delta_{ci})\Big)^2 \\
&= \frac{1}{2}\delta_{ik}\delta_{jl}\sum_{1\leq a<b\leq n}\sum_{1\leq c<d\leq n} r_{ab}s_{cd}\Big(\delta_{ac}\,(\delta_{bi}\delta_{dj} + \delta_{bj}\delta_{di}) \\
&\quad + \delta_{bc}\,(\delta_{ai}\delta_{dj} + \delta_{aj}\delta_{di}) + \delta_{ad}\,(\delta_{bi}\delta_{cj} + \delta_{bj}\delta_{ci}) + \delta_{bd}\,(\delta_{ai}\delta_{cj} + \delta_{aj}\delta_{ci})\Big)
\end{aligned}
$$

$$= \frac{1}{2}\delta_{ik}\delta_{jl}\left(\sum_{1\le a<i}(r_{ai}s_{aj}+r_{aj}s_{ai})+\sum_{i<b<j}r_{ib}s_{bj}+\sum_{i<a<j}r_{aj}s_{ia}\right.$$
$$\left.+\sum_{j<b\le n}(r_{ib}s_{jb}+r_{jb}s_{ib})\right)$$
$$= \frac{1}{2}\delta_{ik}\delta_{jl}\sum_{a=1}^{n}(r_{ai}s_{aj}+r_{aj}s_{ai})$$

for all $1 \le i < j \le n$ and all $1 \le k < l \le n$.

Chapter 2

Some new examples

In order to find sets of algebraic curvature operators which are invariant under the evolution equation of curvature operators under the Ricci flow, a classification of solutions to the eigenvalue equation

$$R^2 + R^{\#} = \theta R \tag{2.1}$$

for $\theta \in \mathbb{R}$ and $R \in S^2_B(\mathfrak{so}(n))$ would be very desirable. The curvature operators of irreducible, compact symmetric spaces and also their Weyl curvatures are such solutions as the theorem below will show. In addition, infinite series of new solutions to equation (2.1) in $\langle \mathbf{W} \rangle^n_d$ that do not arise from symmetric spaces will be constructed in certain dimensions n in section 2.2. Since they are purely algebraic, a general classification of solutions to (2.1) seems to be complicated.

Theorem 2.0.1. Let R be a curvature operator of an irreducible, compact symmetric space and let W denote its Weyl curvature. Then there exists $\theta \in \mathbb{R}$ with

$$R^2 + R^{\#} = \theta R \quad \text{and} \quad W^2 + W^{\#} = \theta W.$$

Proof. Let (M, g) be an irreducible, compact symmetric space with curvature operator R. Then g is an Einstein metric with an Einstein constant $\theta > 0$, that is $\mathrm{Ric}_g = \theta g$. Moreover, $g(t) := (1 - 2\theta t)g$ for $t \in [0, \frac{1}{2\theta}) =: J$ defines a Ricci flow on M with $g(0) = g$ since

$$\frac{\partial}{\partial t} g(t) = -2\theta g = -2\mathrm{Ric}_g = -2\mathrm{Ric}_{g(t)}$$

for all $t \in J$. The last step is true because Ric is invariant under scaling. Let $R_{g(t)}$ denote the curvature operator of $(M, g(t))$. Then $R_{g(t)} = \frac{1}{1-2\theta t} R$ for all $t \in J$, and thus, $\frac{\partial}{\partial t} R_{g(t)} = \frac{2\theta}{(1-2\theta t)^2} R$ for all $t \in J$. Moreover $\nabla R \equiv 0$ because (M, g) is symmetric, and thus, $\Delta R \equiv 0$. In total this shows together with the evolution equation for curvature operators under the Ricci flow

$$R^2 + R^{\#} = \frac{1}{2}\frac{\partial}{\partial t}_{|t=0} R_{g(t)} = \theta R.$$

Consider now the decomposition $R = R_I + R_{\mathrm{Ric}_0} + R_{\mathbf{W}}$ of R as mentioned in section 1.2. Since g is an Einstein metric, $R_{\mathrm{Ric}_0} = 0$. Thus,

$$\begin{aligned}\theta R =& (R_I + R_{\mathbf{W}})^2 + (R_I + R_{\mathbf{W}})^{\#} \\ =& R_I^2 + R_I^{\#} + R_{\mathbf{W}}^2 + R_{\mathbf{W}}^{\#} + 2\left(R_I R_{\mathbf{W}} + R_I \# R_{\mathbf{W}}\right).\end{aligned}$$

As seen in (1.4) and (1.7) in section 1.3, it is true that $R_I R_{\mathbf{W}} + R_I \# R_{\mathbf{W}} = 0$, $R_I^2 + R_I^{\#} \in \langle I \rangle$, and $R_{\mathbf{W}}^2 + R_{\mathbf{W}}^{\#} \in \langle \mathbf{W} \rangle$. Therefore,

$$\begin{aligned}\theta R_{\mathbf{W}} =& \left((R_I + R_{\mathbf{W}})^2 + (R_I + R_{\mathbf{W}})^{\#}\right)_{\mathbf{W}} \\ =& R_{\mathbf{W}}^2 + R_{\mathbf{W}}^{\#},\end{aligned}$$

which concludes the proof. □

2.1 Legendre symbols and circulant matrices

The most of the new solutions to equation (2.1) that will be constructed in this chapter will use a number theoretic function that measures whether a given integer is a quadratic modulo a given prime number or not. This map is called Legendre symbol and is defined as follows.

Definition 2.1.1. Let p be an odd prime number, and let a be an integer. *The Legendre symbol* $\left(\frac{a}{p}\right)$ *of* a *and* p is defined as

$$\left(\frac{a}{p}\right) := \begin{cases} 1 & \text{if } x^2 \equiv a \pmod{p} \text{ has a solution and } a \not\equiv 0 \pmod{p}, \\ -1 & \text{if } x^2 \equiv a \pmod{p} \text{ has no solution}, \\ 0 & \text{if } a \equiv 0 \pmod{p}. \end{cases}$$

The following properties hold for the Legendre symbol (cf. [S], chapter I.3.2).

Theorem 2.1.2. Let p be an odd prime number, and let a and b be integers. Then

1. $a \equiv b \pmod{p}$ implies $\left(\frac{a}{p}\right) = \left(\frac{b}{p}\right)$,
2. $\left(\frac{a}{p}\right) \equiv a^{\frac{p-1}{2}} \pmod{p}$ (Euler),
3. $\left(\frac{-1}{p}\right) = (-1)^{\frac{p-1}{2}} = \begin{cases} 1 & \text{if } p \equiv 1 \pmod{4}, \\ -1 & \text{if } p \equiv 3 \pmod{4}, \end{cases}$

4. $\left(\frac{ab}{p}\right) = \left(\frac{a}{p}\right)\left(\frac{b}{p}\right)$ (multiplicity), and

5. $\sum_{i=0}^{p-1}\left(\frac{i}{p}\right) = 0.$

For more information about Legendre symbols, their properties, and ways for explicit calculations see for example [S], chapter I, or [SF], chapter V.3. Another number theoretic result that will be used for the construction of new solutions to equation (2.1) is the lemma below (cf. [S], chapter I.2.1).

Lemma 2.1.3. Let p be an odd prime number, and let a be a nonnegative integer. Then

$$\sum_{i=0}^{p-1} i^a \equiv \begin{cases} -1 \pmod{p} & \text{if } a \geq 1 \text{ and } p-1|a, \\ 0 \pmod{p} & \text{otherwise.} \end{cases}$$

By means of theorem 2.1.2 and lemma 2.1.3, the next lemma can be proven.

Lemma 2.1.4. Let p be an odd prime number, and let $k \notin p\mathbb{Z}$ be an integer. Then

$$\sum_{i=0}^{p-1}\left(\frac{i(i+k)}{p}\right) = -1.$$

Proof. Let $0 \leq i \leq p-1$. Then $\left(\frac{i(i+k)}{p}\right) = 0$ if and only if $i = 0$ or $i + k \equiv 0 \pmod{p}$, which is true for exactly one $1 \leq i \leq p-1$, since $k \notin p\mathbb{Z}$. Therefore,

$$\sum_{i=0}^{p-1}\left(\frac{i(i+k)}{p}\right) \in \{-(p-2), \ldots, p-2\}. \tag{2.2}$$

Set $q := \frac{p-1}{2}$. Property 2 of theorem 2.1.2 shows

$$\sum_{i=0}^{p-1}\left(\frac{i(i+k)}{p}\right) \equiv \sum_{i=0}^{p-1}\left(i^2 + ik\right)^q \pmod{p}. \tag{2.3}$$

Using binomial expansion leads to

$$\begin{aligned} \sum_{i=0}^{p-1}\left(i^2 + ik\right)^q &= \sum_{i=0}^{p-1}\sum_{j=0}^{q}\binom{q}{j} i^{2q-2j}(ik)^j \\ &= \sum_{j=0}^{q}\binom{q}{j} k^j \sum_{i=0}^{p-1} i^{p-1-j} \\ &\equiv (-1) \pmod{p} \end{aligned} \tag{2.4}$$

since for $0 \leq j \leq q$ lemma 2.1.3 shows that

$$\sum_{i=0}^{p-1} i^{p-1-j} \equiv \begin{cases} -1 \pmod p & \text{if } j = 0, \\ 0 \pmod p & \text{otherwise.} \end{cases}$$

The claim is now an immediate result from (2.2), (2.3), and (2.4). □

The solutions to equation (2.1) which will be constructed in this section are of a very special form. This is a generalization of a so called *circulant matrix*, that is a matrix which is fully determined by its first row, and each other row can be received from the previous one by shifting each entry by one (cf. [D], chapter 3). Even though not every solution to (2.1) constructed in the present work is a circulant matrix in the sense of [D], this terminology will be adopted as follows.

Definition 2.1.5. Let m and n be positive integers with $n \geq 2$. A matrix $A \in \mathbb{R}^{mn \times mn}$ is said to be an (m, n)-*circulant matrix* if there exist $A^{(0)}, \dots, A^{(n-1)} \in \mathbb{R}^{m \times m}$ such that

$$A = \begin{pmatrix} A^{(0)} & A^{(1)} & A^{(2)} & \dots & A^{(n-1)} \\ A^{(n-1)} & A^{(0)} & A^{(1)} & \dots & A^{(n-2)} \\ A^{(n-2)} & A^{(n-1)} & A^{(0)} & \dots & A^{(n-3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ A^{(1)} & A^{(2)} & A^{(3)} & \dots & A^{(0)} \end{pmatrix}.$$

Then A will shortly be written as $A = \operatorname{circ}\left(A^{(0)}, \dots, A^{(n-1)}\right)$.

Remark.

1. Only $(1, n)$-circulant matrices as in the defininitition above are also circulant matrices in the sense of [D].
2. Suppose that $R = \operatorname{circ}\left(A^{(0)}, \dots, A^{(n-1)}\right)$ is an (m, n)-circulant matrix with $A^{(1)} = (A^{(n-1)})^T, \dots, A^{(\frac{n}{2}-1)} = (A^{(\frac{n}{2}+1)})^T, A^{(\frac{n}{2})} = (A^{(\frac{n}{2})})^T$ if n is even and $A^{(1)} = (A^{(n-1)})^T, \dots, A^{(\frac{n-1}{2})} = (A^{(\frac{n+1}{2})})^T$ if n is odd, respectively, and that $A^{(0)}$ is symmetric with $(A^{(0)})_{ii} = 0$ for all $1 \leq i \leq n$. Then R is symmetric and all entries on its diagonal are equal to 0. Thus, R can be understood as an operator in $S^2_{B,d}(\mathfrak{so}(mn))$ as pointed out in section 1.4.

Notation 2.1.6. For integers n and a let $[a]_n$ denote the unique element in $a + n\mathbb{Z} \cap \{0, \dots, n-1\}$.

With this notation, the following is obvious.

Lemma 2.1.7. Let m and n be positive integers with $n \geq 2$, and let $A \in R^{mn \times mn}$. The following are equivalent:

1. A is an (m,n)-circulant matrix.

2. $a_{im+k,jm+l} = a_{[i-z]_n m+k,[j-z]_n m+l}$ for all $0 \le i,j \le n-1$, all $1 \le k,l \le n$, and all $z \in \mathbb{Z}$.

3. $a_{im+k,jm+l} = a_{k,[j-i]_n m+l}$ for all $0 \le i,j \le n-1$ and all $1 \le k,l \le m$.

Circulant matrices have some useful properties as the following lemma will show.

Lemma 2.1.8. Let m and n be positive integers with $n \ge 2$. If R and S are (m,n)-circulant matrices, the same is also true for the matrices A and B given by $a_{ij} = r_{ij}s_{ij}$ and $b_{ij} = \frac{1}{2}\sum_{k=1}^{mn}(r_{ik}s_{jk} + r_{jk}s_{ik})$ for $1 \le i,j \le mn$. In particular, if $R, S \in S^2_{B,d}(\mathfrak{so}(mn))$ are (m,n)-circulant matrices, the same is also true for the operators RS and $R\#S$.

Proof. Obviously, A is an (m,n)-circulant matrix. Furthermore, by lemma 2.1.7,

$$\begin{aligned}
&b_{im+k,jm+l}\\
&= \frac{1}{2}\sum_{x=0}^{n-1}\sum_{y=1}^{m}\left(r_{im+k,xm+y}\, s_{jm+l,xm+y} + s_{im+k,xm+y}\, r_{jm+l,xm+y}\right)\\
&= \frac{1}{2}\sum_{x=0}^{n-1}\sum_{y=1}^{m}\Big(r_{k,[x-i]_n m+y}\, s_{jm+l,xm+y} + s_{k,[x-i]_n m+y}\, r_{jm+l,xm+y}\Big)\\
&= \frac{1}{2}\sum_{x=-i}^{n-1-i}\sum_{y=1}^{m}\Big(r_{k,[x]_n m+y}\, s_{jm+l,(x+i)m+y} + s_{k,[x]_n m+y}\, r_{jm+l,(x+i)m+y}\Big)\\
&= \frac{1}{2}\sum_{x=-i}^{n-1-i}\sum_{y=1}^{m}\Big(r_{k,[x]_n m+y}\, s_{[j-i]_n m+l,[x]_n m+y}\\
&\quad + s_{k,[x]_n m+y}\, r_{[j-i]_n m+l,[x]_n m+y}\Big)\\
&= \frac{1}{2}\sum_{x=0}^{n-1}\sum_{y=1}^{m}\Big(r_{k,xm+y}\, s_{[j-i]_n m+l,xm+y} + s_{k,xm+y}\, r_{[j-i]_n m+l,xm+y}\Big)\\
&= b_{k,[j-i]_n m+l}
\end{aligned}$$

for all $0 \le i,j \le n-1$ and all $1 \le k,l \le m$. Applying again lemma 2.1.7 proves that B is an (m,n)-circulant matrix. If now $R, S \in S^2_{B,d}(\mathfrak{so}(mn))$, then

$$(R\#S)_{ij} = \frac{1}{2}\sum_{k=1}^{mn}(r_{ik}s_{jk} + r_{jk}s_{ik})$$

for all $1 \le i < j \le mn$ as shown in section 1.4. This concludes the proof. □

2.2 Infinite series of new solutions to $W^2 + W^\# = \theta W$

Now, new solutions to equation (2.1) can be given in the dimensions p, p^2, and $2p$ for a prime number p such that 4 divides $p-1$. Moreover, another infinite series of new solutions to equation (2.1) in dimension mn will be constructed for integers $m, n \geq 2$.

Theorem 2.2.1. Let p be a prime number such that 4 divides $p-1$, and let $W := \operatorname{circ}\left(\left(\frac{0}{p}\right), \left(\frac{1}{p}\right), \ldots, \left(\frac{p-1}{p}\right)\right) \in \mathbb{R}^{p \times p}$. Then W defines an operator in $\langle \mathbf{W} \rangle_d^p$ with $W^2 + W^\# = 0$.

Proof. Since 4 divides $p-1$, it follows from 1, 4, and 3 of theorem 2.1.2 that

$$\left(\frac{i}{p}\right) = \left(\frac{i-p}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{p-i}{p}\right) = \left(\frac{p-i}{p}\right). \tag{2.5}$$

Thus, W is symmetric, and since every entry on the diagonal of W is equal to $\left(\frac{0}{p}\right) = 0$, W defines an operator in $S^2_{B,d}(\mathfrak{so}(p))$. W is Weyl because of 5 of theorem 2.1.2, and $W^2 = I = \mathrm{id}_{\mathfrak{so}(p)}$ is obvious. Thus, it remains to show that $W^\# = -I$. Note that $W_{ij} = \left(\frac{j-i}{p}\right)$ for all $1 \leq i, j \leq n$. Because of the multiplicity of the Legendre symbol and of lemma 2.1.4,

$$W^\#_{1j} = \sum_{k=0}^{p-1} \left(\frac{k}{p}\right)\left(\frac{k+1-j}{p}\right) = \sum_{k=0}^{p-1} \left(\frac{k(k-(j-1))}{p}\right) = -1$$

for all $2 \leq j \leq p$. Since $W^\#$ is a $(1,p)$-circulant matrix by lemma 2.1.8, $W^\# = \operatorname{circ}(0, -1 \ldots, -1) = -I$, which concludes the proof. □

Example 2.2.2.

- The solution to $W^2 + W^\# = 0$ for $p = 5$ presented in the previous theorem is given by

$$W = \begin{pmatrix} 0 & + & - & - & + \\ & 0 & + & - & - \\ & & 0 & + & - \\ & & & 0 & + \\ & & & & 0 \end{pmatrix}$$

  where $+$ stands for 1 and $-$ for -1.

- The solution to $W^2 + W^\# = 0$ for $p = 13$ presented in the previous theorem is given by

$$
W = \begin{pmatrix}
0 & + & - & + & + & - & - & - & - & + & + & - & + \\
 & 0 & + & - & + & + & - & - & - & - & + & + & - \\
 & & 0 & + & - & + & + & - & - & - & - & + & + \\
 & & & 0 & + & - & + & + & - & - & - & - & + \\
 & & & & 0 & + & - & + & + & - & - & - & - \\
 & & & & & 0 & + & - & + & + & - & - & - \\
 & & & & & & 0 & + & - & + & + & - & - \\
 & & & & & & & 0 & + & - & + & + & - \\
 & & & & & & & & 0 & + & - & + & + \\
 & & & & & & & & & 0 & + & - & + \\
 & & & & & & & & & & 0 & + & - \\
 & & & & & & & & & & & 0 & + \\
 & & & & & & & & & & & & 0
\end{pmatrix},
$$

where again $+$ stands for 1 and $-$ for -1.

Theorem 2.2.3. Let p be a prime number such that 4 divides $p-1$. Set

$$
\begin{aligned}
&A^{(0)} := \operatorname{circ}(0, 1, \dots, 1) \in \mathbb{R}^{p\times p} \quad \text{and} \\
&A^{(i)} := \operatorname{circ}\left(-1, \left(\frac{i\cdot 1}{p}\right), \dots, \left(\frac{i\cdot(p-1)}{p}\right)\right) \in \mathbb{R}^{p\times p}
\end{aligned}
$$

for $1 \le i \le p-1$, and let $W := \operatorname{circ}\left(A^{(0)}, A^{(1)}, \dots, A^{(p-1)}\right)$. Then W defines an operator in $\langle \mathbf{W} \rangle_d^{p^2}$ with $W^2 + W^{\#} = 0$.

Proof. Due to equation (2.5) and the multiplicity of the Legendre symbol, $A^{(i)}$ is symmetric for all $0 \le i \le p-1$. The same arguments show $A^{(i)} = A^{(p-i)}$ for all $1 \le i \le p-1$. Therefore, W is symmetric, and also every entry on the diagonal of W is equal to 0. Hence, $W \in S^2_{B,d}(\mathfrak{so}(p^2))$. W is Weyl because

$$
\begin{aligned}
\sum_{j=1}^{p^2} W_{ij} &= \sum_{k=0}^{p-1}\sum_{j=1}^{p} \left(A^{(k)}\right)_{ij} \\
&= p - 1 + \sum_{k=1}^{p-1}\left(-1 + \left(\frac{k}{p}\right)\sum_{j=1}^{p}\left(\frac{j}{p}\right)\right) \\
&= 0
\end{aligned}
$$

for all $1 \le i \le p^2$ by 5 of theorem 2.1.2. Since $W^2 = I = \mathrm{id}_{\mathfrak{so}(p^2)}$, it remains to prove that $W^{\#} = -I$. Due to lemma 2.1.8, it is enough to show that $(W^{\#})_{1j} = -1$ for all $2 \le j \le p^2$ since all $A^{(i)}$ are circulant matrices.
Let $A_j^{(i)}$ for $0 \le i \le p-1$ and $1 \le j \le p$ denote the j^{th} row of $A^{(i)}$, that is

$$
A_j^{(i)} = \left(\left(\frac{i(p+1-j)}{p}\right), \dots, \left(\frac{i(p-1)}{p}\right), -1, \left(\frac{i}{p}\right), \dots, \left(\frac{i(p-j)}{p}\right)\right)
$$

for $1 \leq i \leq p-1$. Then $\langle A_1^{(0)}, A_j^{(0)} \rangle = p-2$ for all $2 \leq j \leq p$. Let now $1 \leq i \leq p-1$ and $2 \leq j \leq p$ be fixed. The application of the properties of the Legendre symbol and lemma 2.1.4 shows now

$$\begin{aligned}\langle A_1^{(i)}, A_j^{(i)} \rangle &= -2\left(\frac{i(1-j)}{p}\right) + \left(\frac{i}{p}\right)^2 \sum_{k=1}^{p-1} \left(\frac{k(k+1-j)}{p}\right) \\ &= -2\left(\frac{i(1-j)}{p}\right) - 1.\end{aligned}$$

Therefore,

$$\begin{aligned}(W^{\#})_{1j} &= \sum_{i=0}^{p-1} \langle A_1^{(i)}, A_j^{(i)} \rangle \\ &= p-2-2\left(\frac{1-j}{p}\right) \sum_{i=1}^{p-1} \left(\frac{i}{p}\right) - (p-1) \\ &= -1\end{aligned}$$

again by 5 of theorem 2.1.2.
Fix now $1 \leq i, k \leq p-1$ with $i \neq p-k$ and $2 \leq j \leq p$. Then

$$\begin{aligned}\langle A_1^{(i)}, A_1^{([i+k]_p)} \rangle &= 1 + \left(\frac{i}{p}\right)\left(\frac{i+k}{p}\right) \sum_{l=1}^{p-1} \left(\frac{l}{p}\right)^2 \\ &= 1 + (p-1)\left(\frac{i(i+k)}{p}\right)\end{aligned}$$

and

$$\begin{aligned}&\langle A_1^{(i)}, A_j^{([i+k]_p)} \rangle \\ &= -\left(\frac{(i+k)(1-j)}{p}\right) + \left(\frac{i(i+k)}{p}\right) \sum_{\substack{l=1 \\ l \neq j-1}}^{p-1} \left(\frac{l}{p}\right)\left(\frac{l+1-j}{p}\right) \\ &\quad - \left(\frac{i(j-1)}{p}\right) \\ &= -\left(\frac{i(j-1)}{p}\right) - \left(\frac{(i+k)(j-1)}{p}\right) - \left(\frac{i(i+k)}{p}\right).\end{aligned}$$

The last step follows again from lemma 2.1.4. If now $i = 0$ or $i = p-k$, then

$$\langle A_1^{(0)}, A_1^{(k)} \rangle = \left(\frac{k}{p}\right) \sum_{l=1}^{p-1} \left(\frac{l}{p}\right) = 0 = \langle A_1^{(p-k)}, A_1^{(0)} \rangle$$

and

$$\begin{aligned}\langle A_1^{(0)}, A_j^{(k)}\rangle &= -1 + \left(\frac{k}{p}\right) \sum_{\substack{l=1\\ l\neq p-j+1}}^{p-1} \left(\frac{l}{p}\right) &= -1 - \left(\frac{k(1-j)}{p}\right)\\ \langle A_1^{(p-k)}, A_j^{(0)}\rangle &= -1 + \left(\frac{p-k}{p}\right) \sum_{\substack{l=1\\ l\neq j-1}}^{p-1} \left(\frac{l}{p}\right) &= -1 - \left(\frac{k(1-j)}{p}\right).\end{aligned}$$

Using again lemma 2.1.4, all together leads to

$$\begin{aligned}(W^{\#})_{1,kp+1} &= \sum_{i=0}^{p-1} \langle A_1^{(i)}, A_1^{([i+k]_p)}\rangle\\ &= p - 2 + (p-1) \sum_{i=1}^{p-1} \left(\frac{i(i+k)}{p}\right)\\ &= -1\end{aligned}$$

and

$$\begin{aligned}(W^{\#})_{1,kp+j} &= \sum_{i=0}^{p-1} \langle A_1^{(i)}, A_j^{([i+k]_p)}\rangle\\ &= -2 - 2\left(\frac{k(1-j)}{p}\right) - \sum_{\substack{i=1\\ i\neq p-k}}^{p-1} \left(\left(\frac{i(j-1)}{p}\right)\right.\\ &\quad \left. + \left(\frac{(i+k)(j-1)}{p}\right) + \left(\frac{i(i+k)}{p}\right)\right)\\ &= -2 - \sum_{i=0}^{p-1} \left(\left(\frac{i(j-1)}{p}\right) + \left(\frac{(i+k)(j-1)}{p}\right) + \left(\frac{i(i+k)}{p}\right)\right)\\ &= -1 - 2\left(\frac{j-1}{p}\right) \sum_{i=0}^{p-1} \left(\frac{i}{p}\right)\\ &= -1\end{aligned}$$

for all $1 \leq k \leq p-1$ and all $2 \leq j \leq p$. The penultimate step follows again from lemma 2.1.4, the multiplicity of the Legendre symbol and a rearrange of the indices in the second sum. The last step is again 5 of theorem 2.1.2. Thus, $(W^{\#})_{1j} = -1$ for every $2 \leq j \leq p^2$, which concludes the proof. □

Theorem 2.2.4. Let p be a prime number such that 4 divides $p-1$, and let $a_i := -\left(1 + \left(\frac{i}{p}\right)\sqrt{2p-1}\right)$ for $1 \leq i \leq p-1$. Moreover, set

$$A^{(0)} := \begin{pmatrix} 0 & 2(p-1) \\ 2(p-1) & 0 \end{pmatrix} \quad \text{and} \quad A^{(i)} := \begin{pmatrix} a_i & a_i \\ a_i & a_i \end{pmatrix}$$

for $1 \leq i \leq p-1$, and let $W := \operatorname{circ}\left(A^{(0)}, \ldots, A^{(p-1)}\right)$. Then W defines an operator in $\langle \mathbf{W} \rangle_d^{2p}$ with $W^2 + W^{\#} = 2(2p-1)W$.

Proof. Set $\theta := 2(2p-1)$. Clearly, $a_i = a_{p-i}$ for all $1 \leq i \leq p-1$. Therefore, W is symmetric by equation (2.5). In addition, all entries on its diagonal are equal to 0. Since

$$\sum_{j=1}^{2p} W_{ij} = 2(p-1) - 2\sum_{j=1}^{p-1}\left(1 + \left(\frac{j}{p}\right)\sqrt{2p-1}\right) = 0$$

for all $1 \leq i \leq 2p$ by 5 of theorem 2.1.2, it follows that $W \in \langle \mathbf{W} \rangle_d^{2p}$. Consider now

$$a_i^2 = 2p + 2\left(\frac{i}{p}\right)\sqrt{2p-1} = 2(p-1-a_i)$$

for all $1 \leq i \leq p-1$. Therefore,

$$\left(W^{\#}\right)_{12} = 2\sum_{i=1}^{p-1} a_i^2 = 4\sum_{i=1}^{p-1}\left(p + \left(\frac{i}{p}\right)\sqrt{2p-1}\right) = 4p(p-1), \tag{2.6}$$

and thus,

$$\left(W^2 + W^{\#}\right)_{12} = 4(p-1)^2 + 4p(p-1) = 4(p-1)(2p-1) = \theta W_{12}.$$

Furthermore,

$$\begin{aligned}
&\left(W^{\#}\right)_{j,2i+k} \\
&= 2\sum_{\substack{l=1 \\ l\neq i}}^{p-1} a_l a_{[l-i]_p} + 4(p-1)a_i \\
&= 2\sum_{\substack{l=1 \\ l\neq i}}^{p-1}\left(1 + \left(\frac{l(l-i)}{p}\right)(2p-1) + \left(\left(\frac{l}{p}\right) + \left(\frac{l-i}{p}\right)\right)\sqrt{2p-1}\right) \\
&\quad - 4(p-1)\left(1 + \left(\frac{i}{p}\right)\sqrt{2p-1}\right) \\
&= 2(p-2) - 2(2p-1) - 4(p-1) - 4p\left(\frac{i}{p}\right)\sqrt{2p-1} \\
&= 4pa_i - 2(p-1)
\end{aligned} \tag{2.7}$$

for all $1 \leq i \leq p-1$ and all $1 \leq j, k \leq 2$ by theorem 2.1.2 and lemma 2.1.4, and hence,

$$\begin{aligned}
\left(W^2 + W^{\#}\right)_{j,2i+k} &= a_i^2 + 4pa_i - 2(p-1) \\
&= 2(p-1-a_i) + 4pa_i - 2(p-1) \\
&= 2(2p-1)a_i \\
&= \theta W_{j,2i+k}
\end{aligned}$$

for all $1 \leq i \leq p-1$ and all $1 \leq j, k \leq 2$. This proves the claim by means of lemma 2.1.8. $\square$

Remark. Let W be as in theorem 2.2.4. Then

$$\begin{aligned}\|W\|^2 &= 4p(p-1)^2 + 2p\sum_{i=1}^{p-1} a_i^2 \\ &= 4p(p-1)^2 + 4p^2(p-1) \\ &= 4p(p-1)(2p-1)\end{aligned}$$

by equation (2.6). Let $\tilde{W} := \frac{W}{\|W\|}$. Then $\tilde{W}^2 + \tilde{W}^{\#} = \tilde{\theta}\tilde{W}$ with

$$\tilde{\theta} = \frac{2(2p-1)}{2\sqrt{p(p-1)(2p-1)}} = \sqrt{\frac{2p-1}{p(p-1)}},$$

which tends to 0 as $p \to \infty$.

In addition to the solutions to the eigenvalue equation (2.1) which are all constructed above by means of Legendre symbols, there is also the following series of new solutions.

Theorem 2.2.5. Let $m, n \geq 2$ be integers, and let $x := \frac{m-1}{n-1}$ and $y := -\frac{2}{n-1}$. Moreover, set $A := \operatorname{circ}(0, 1, \cdots, 1)$, $B := \operatorname{circ}(x, y, \cdots, y) \in \mathbb{R}^{m \times m}$, and let $W := \operatorname{circ}(A, B, \ldots, B) \in \mathbb{R}^{mn \times mn}$. Then W defines an operator in $\langle \mathbf{W} \rangle_d^{mn}$ with $W^2 + W^{\#} = \left(m - 1 - \frac{4}{n-1}\right) W$.

Proof. Set $\theta := m - 1 - \frac{4}{n-1}$. Clearly, W is symmetric and all entries on its diagonal are equal to 0. Moreover,

$$\begin{aligned}\sum_{j=1}^{mn} W_{ij} &= m - 1 + (n-1)x + (m-1)(n-1)y \\ &= m - 1 + m - 1 - 2(m-1) \\ &= 0\end{aligned}$$

for all $1 \leq i \leq mn$. Therefore, $W \in \langle \mathbf{W} \rangle_d^{mn}$. Consider now

$$\begin{aligned}\left(W^{\#}\right)_{1j} &= m - 2 + 2(n-1)xy + (n-1)(m-2)y^2 \\ &= m - 2 - 4\frac{m-1}{n-1} + 4\frac{m-2}{n-1} \\ &= m - 2 - \frac{4}{n-1} \qquad (2.8)\end{aligned}$$

for $2 \leq j \leq m$. Therefore, $\left(W^2 + W^{\#}\right)_{1j} = \theta W_{1j}$ for $2 \leq j \leq m$. Moreover,

$$\begin{aligned}
\left(W^{\#}\right)_{1,im+1} &= 2(m-1)y + (n-2)\left(x^2 + (m-1)y^2\right) \\
&= -\frac{4(m-1)}{n-1} + \frac{(n-2)(m-1)^2}{(n-1)^2} + \frac{4(n-2)(m-1)}{(n-1)^2} \\
&= \frac{m-1}{n-1}\frac{(m-1)(n-2)-4}{n-1} \qquad (2.9)
\end{aligned}$$

for all $1 \leq i \leq n-1$, and thus,

$$\begin{aligned}
\left(W^2 + W^{\#}\right)_{1,im+1} &= x^2 + \frac{m-1}{n-1}\frac{(m-1)(n-2)-4}{n-1} \\
&= \frac{m-1}{n-1}\left(m-1-\frac{4}{n-1}\right) \\
&= \theta W_{1,im+1}
\end{aligned}$$

for all $1 \leq i \leq n-1$. Finally,

$$\begin{aligned}
&\left(W^{\#}\right)_{1,im+j} \\
&= 2(x + (m-2)y) + (n-2)\left(2xy + (m-2)y^2\right) \\
&= 2\frac{m-1}{n-1} - 4\frac{m-2}{n-1} - 4\frac{(n-2)(m-1)}{(n-1)^2} + 4\frac{(m-2)(n-2)}{(n-1)^2} \\
&= -2\frac{m-3}{n-1} - 4\frac{n-2}{(n-1)^2} \\
&= -\frac{2}{n-1}\frac{(m-3)(n-1)+2(n-2)}{n-1} \\
&= -\frac{2}{n-1}\left(m-1-\frac{2}{n-1}\right) \qquad (2.10)
\end{aligned}$$

for all $1 \leq i \leq n-1$ and $2 \leq j \leq m$, and hence,

$$\left(W^2 + W^{\#}\right)_{1,im+j} = y^2 - \frac{2}{n-1}\left(m-1-\frac{2}{n-1}\right) = \theta W_{1,im+j}$$

for all $1 \leq i \leq n-1$ and $2 \leq j \leq m$. This proves the claim by means of lemma 2.1.8. □

Remark. Let W be as in theorem 2.2.5. Then

$$\begin{aligned}
\|W\|^2 &= n\frac{m(m-1)}{2} + \frac{n(n-1)}{2}\left(mx^2 + m(m-1)y^2\right) \\
&= \frac{mn(m-1)(n+m+2)}{2(n-1)}.
\end{aligned}$$

Let $\tilde{W} := \frac{W}{\|W\|}$. Then $\tilde{W}^2 + \tilde{W}^\# = \tilde{\theta}\tilde{W}$ with

$$\tilde{\theta} = \frac{m-1-\frac{4}{n-1}}{\sqrt{\frac{mn(m-1)(n+m+2)}{2(n-1)}}} = \sqrt{2}\frac{(m-1)(n-1)-4}{\sqrt{mn(m-1)(n-1)(n+m+2)}},$$

which tends to 0 as $m \to \infty$ or $n \to \infty$.
Moreover, it is easy to see that $m - 1 - \frac{4}{n-1} = 0$ if and only if $(m, n) \in \{(3, 3), (2, 5), (5, 2)\}$

2.3 The isotropy groups of the new solutions

The adjoint representation $\mathrm{Ad}\colon SO(n) \to \mathrm{Aut}(\mathfrak{so}(n))$ of $SO(n)$ defines an action of $SO(n)$ on $S^2_B(\mathfrak{so}(n))$ via $g.R = \mathrm{Ad}_g R \,\mathrm{Ad}_g^T$ for $g \in SO(n)$ and $R \in S^2_B(\mathfrak{so}(n))$. In this part, the isotropy group

$$SO(n)_R = \{g \in SO(n) | \mathrm{Ad}_g R \,\mathrm{Ad}_g^T = R\}$$

of a curvature operator $R \in S^2_B(\mathfrak{so}(n))$ under this action will be analyzed for solutions to the eigenvalue equation $R^2 + R^\# = \theta R$. This will first be done for curvature operators of symmetric spaces and afterwards for the solutions constructed in the previous section.

Proposition 2.3.1. Let $M = G/H$ be a symmetric space with H connected. Let further $p = eH \in M$, where e denotes the unit element in G, and let R_p be the curvature operator of M at p. Then $dh_p \in SO(n)_{R_p}$ for all $h \in H$.

Proof. Ad acts under the identification $\Lambda^2(T_pM) = \Lambda^2(\mathbb{R}^n) = \mathfrak{so}(n)$ via $\mathrm{Ad}_g(x \wedge y) = gx \wedge gy$ for all $g \in SO(n)$ and all $x, y \in T_pM$. Let $h \in H$, that is h is an isometry on M with $h(p) = p$ and $dh_p \in SO(n)$. Since the $(3, 1)$-curvature tensor $R_p(\cdot, \cdot)\cdot$ is invariant under isometries, it follows that

$$\begin{aligned}
\langle R_p(x \wedge y), z \wedge w\rangle_p &= \langle R_p(x, y)w, z\rangle_p \\
&= \langle dh_p R_p(x, y)w, dh_p z\rangle_p \\
&= \langle R_p(dh_p x, dh_p y)dh_p w, dh_p z\rangle_p \\
&= \langle R_p(dh_p x \wedge dh_p y), dh_p z \wedge dh_p w\rangle_p \\
&= \langle (R_p \,\mathrm{Ad}_{dh_p})(x \wedge y), \mathrm{Ad}_{dh_p}(z \wedge w)\rangle_p \\
&= \langle (\mathrm{Ad}_{dh_p}^T R_p \,\mathrm{Ad}_{dh_p})(x \wedge y), z \wedge w\rangle_p
\end{aligned}$$

for all $x, y, z, w \in T_pM$. Thus, $\mathrm{Ad}_{dh_p}^T R_p \,\mathrm{Ad}_{dh_p} = R_p$ for all $h \in H$, which was to be proven. □

In particular, if there exists an $h \in H$ with dh_p not equal to the identity in G, then the isotropy group of R_p is infinite.

Let now R be an arbitrary algebraic curvature operator, and suppose that $v \in \mathfrak{so}(n)$ satisfies $g(t) := \exp(tv) \in SO(n)_R$ for all $t \in (-\epsilon, \epsilon)$ for an $\epsilon > 0$, that is

$$\mathrm{Ad}_{g(t)} R\, \mathrm{Ad}^T_{g(t)} \equiv R \tag{2.11}$$

for all $t \in (-\epsilon, \epsilon)$. Since $\frac{d}{dt}_{|t=0} \mathrm{Ad}_{g(t)} = \mathrm{ad}_{g'(0)} = \mathrm{ad}_v$, taking the derivative of (2.11) at $t = 0$ yields

$$\begin{aligned} 0 &= \frac{d}{dt}_{|t=0} \left(\mathrm{Ad}_{g(t)} R\, \mathrm{Ad}^T_{g(t)} \right) \\ &= \mathrm{ad}_v R - R\, \mathrm{ad}_v \\ &= [\mathrm{ad}_v, R]. \end{aligned}$$

If now the other way around $v \in \mathfrak{so}(n)$ satisfies $[\mathrm{ad}_v, R] = 0$, it follows that $\mathrm{Ad}_{g(t)} R\, \mathrm{Ad}^T_{g(t)}$ is constant. Since $g(0) = e$ is the unit element of $SO(n)$, $g(t)$ satisfies (2.11). In total this shows that the isotropy group of R is in one to one correspondence to

$$\mathfrak{so}(n)_R := \{v \in \mathfrak{so}(n) | [\mathrm{ad}_v, R] = 0\}.$$

Note that the standard scalar product on the vector space of endomorphisms on $\mathfrak{so}(n)$ is given by $\langle \cdot, \cdot \rangle \colon \mathrm{End}(\mathfrak{so}(n)) \times \mathrm{End}(\mathfrak{so}(n)) \to \mathbb{R}; (A, B) \mapsto \mathrm{tr}\,(AB^T)$ and that

$$\begin{aligned} [\mathrm{ad}_v, R]^T &= (\mathrm{ad}_v R - R\, \mathrm{ad}_v)^T = R\, \mathrm{ad}_v^T - \mathrm{ad}_v^T R \\ &= -R\, \mathrm{ad}_v + \mathrm{ad}_v R = [\mathrm{ad}_v, R]. \end{aligned}$$

for all $v \in \mathfrak{so}(n)$ and all $R \in S^2_B(\mathfrak{so}(n))$. Therefore, in order to compute the isotropy groups of the solutions to the eigenvalue equation (2.1) constructed in the previous section, the following lemma will be useful.

Lemma 2.3.2. Let $R \in S^2_B(\mathfrak{so}(n))$, and let $v, w \in \mathfrak{so}(n)$. Then

$$\mathrm{tr}\,([\mathrm{ad}_v, R]\,[\mathrm{ad}_w R]) = 4\langle (R^2 \# I - R^{\#})(v), w\rangle.$$

Proof. Let $R \in S^2_B(\mathfrak{so}(n))$ and $v, w \in \mathfrak{so}(n)$ be given. Then

$$\mathrm{tr}\,([\mathrm{ad}_v, R]\,[\mathrm{ad}_w R]) = \mathrm{tr}\,((\mathrm{ad}_v R - R\, \mathrm{ad}_v)\,(\mathrm{ad}_w R - R\, \mathrm{ad}_w)).$$

Since

$$\begin{aligned} \mathrm{tr}\,(R\, \mathrm{ad}_v R\, \mathrm{ad}_w) &= \mathrm{tr}\,(\mathrm{ad}_v R\, \mathrm{ad}_w R) \quad \text{and} \\ \mathrm{tr}\,(R\, \mathrm{ad}_v \mathrm{ad}_w R) &= \mathrm{tr}\left(\mathrm{ad}_w R^2\, \mathrm{ad}_v\right) \\ &= \mathrm{tr}\left((\mathrm{ad}_w R^2\, \mathrm{ad}_v)^T\right) \\ &= \mathrm{tr}\left((-\mathrm{ad}_v) R^2 (-\mathrm{ad}_w)\right) \\ &= \mathrm{tr}\left(\mathrm{ad}_v R^2 \mathrm{ad}_w\right), \end{aligned}$$

it follows that

$$\begin{aligned}\operatorname{tr}\left([\operatorname{ad}_v, R]\,[\operatorname{ad}_w R]\right) &= 2\operatorname{tr}\left(\operatorname{ad}_v R\operatorname{ad}_w R - \operatorname{ad}_v R^2\operatorname{ad}_w\right)\\ &= 4\langle (R^2\#I - R^{\#})(v), w\rangle\end{aligned}$$

by lemma 1.3.1, which was to be proven. □

Thus, the following is true.

Corollary 2.3.3. Let $R \in S^2_B(\mathfrak{so}(n))$. Then there is a one to one correspondence between $SO(n)_R$ and $\ker(R^2\#I - R^{\#})$.

Proof. An element $v \in \mathfrak{so}(n)$ is contained in $\ker(R^2\#I - R^{\#})$ if and only if $\operatorname{tr}\left([\operatorname{ad}_v, R][\operatorname{ad}_w, R]\right) = 0$ for all $w \in \mathfrak{so}(n)$ by the previous lemma. Since this trace defines an inner product on $\{[ad_w, R] | w \in \mathfrak{so}(n)\}$, it follows that $v \in \ker(R^2\#I - R^{\#})$ if and only if $v \in \mathfrak{so}(n)_R$, which concludes the proof. □

Clearly, it follows that the isotropy group of a given curvature operator R is at most finite if and only if $R^2\#I - R^{\#}$ has no kernel.
Now, the isotropy groups of the solutions to equation (2.1) constructed in the previous section can be analyzed.

Proposition 2.3.4. Let W be as in theorem 2.2.1 or 2.2.3. Then $SO(n)_W$ is finite.

Proof. Because of $W^2 = I$, $W^{\#} = -I$, and $I^{\#} = (n-2)I$, it follows that $W^2\#I - W^{\#} = (n-1)I$. Thus, $\ker\left(W^2\#I - W^{\#}\right) = \{0\}$, which concludes the proof. □

Proposition 2.3.5. Let W be as in theorem 2.2.4. Then

$$\ker(W^2\#I - W^{\#}) = \langle e_1 \wedge e_2\rangle \oplus \langle e_3 \wedge e_4\rangle \oplus \cdots \oplus \langle e_{2p-1} \wedge e_{2p}\rangle.$$

Proof. Consider first

$$\left(W^2\#I\right)_{12} = 2\sum_{i=1}^{p-1} a_i^2 = \left(W^{\#}\right)_{12}.$$

Therefore, $\left(W^2\#I - W^{\#}\right)_{12} = 0$. Moreover,

$$\begin{aligned}\left(W^2\#I\right)_{j,2i+k} &= 4(p-1)^2 + 2\sum_{l=1}^{p-1} a_l^2 - a_i^2\\ &= 4(p-1)^2 + 4p(p-1) - 2p - 2\left(\frac{i}{p}\right)\sqrt{2p-1}\\ &= 2\left(4p^2 - 7p + 2 - \left(\frac{i}{p}\right)\sqrt{2p-1}\right)\end{aligned}$$

for all $1 \le i \le p-1$ and all $1 \le j, k \le 2$ by equation (2.6), and thus,

$$\begin{aligned}\left(W^2\#I - W^\#\right)_{j,2i+k} =& 2\left(4p^2 - 7p + 2 - \left(\frac{i}{p}\right)\sqrt{2p-1}\right)\\ &+ 2(3p-1) + 4p\left(\frac{i}{p}\right)\sqrt{2p-1}\\ =& 2\left(2p-1\right)^2 + 2(2p-1)^{\frac{3}{2}}\left(\frac{i}{p}\right)\\ >& 0\end{aligned}$$

for all $1 \le i \le p-1$ and all $1 \le j, k \le 2$ by equation (2.7). Since W is a $(2,p)$-circulant matrix, the same is true for $W^2\#I - W^\#$ by lemma 2.1.8. This proves the statement. □

Proposition 2.3.6. Let W be as in theorem 2.2.5. Then

1. $SO(n)_W$ is finite if $m \ge 3$ or $n \ge 3$, and
2. $\ker\left(W^2\#I - W^\#\right) = \langle e_1 \wedge e_4\rangle \oplus \langle e_2 \wedge e_3\rangle$ if $m = n = 2$.

Proof. Consider first

$$\begin{aligned}\left(W^2\#I\right)_{1j} &= m - 2 + (n-1)\left(x^2 + (m-1)y^2\right)\\ &= m - 2 + \frac{(m-1)(m+3)}{n-1}\end{aligned}$$

for all $2 \le j \le m$. Therefore,

$$\left(W^2\#I - W^\#\right)_{1j} = \frac{(m-1)(m+3)}{n-1} + \frac{4}{n-1} = \frac{(m+1)^2}{n-1} > 0$$

for all $2 \le j \le m$ by equation (2.8). Furthermore,

$$\begin{aligned}\left(W^2\#I\right)_{1,im+1} &= m - 1 + (n-2)x^2 + (n-1)(m-1)y^2\\ &= \frac{m-1}{n-1}\,\frac{(n-1)^2 + (n-2)(m-1) + 4(n-1)}{n-1}\end{aligned}$$

for all $1 \le i \le n-1$, which shows together with equation (2.9)

$$\begin{aligned}\left(W^2\#I - W^\#\right)_{1,im+1} &= \frac{m-1}{(n-1)^2}\left((n-1)^2 + 4(n-1) + 4\right)\\ &= \frac{(m-1)(n+1)^2}{(n-1)^2}\\ &> 0\end{aligned}$$

for all $1 \leq i \leq n-1$. Finally

$$\left(W^2 \# I\right)_{1,im+j} = m-1+(n-1)x^2+((n-1)(m-1)-1)\,y^2$$
$$= \frac{(m-1)(n-1)(m+n+2)-4}{(n-1)^2}$$

for all $1 \leq i \leq n-1$ and all $2 \leq j \leq m$. This shows together with equation (2.10)

$$\left(W^2 \# I - W^{\#}\right)_{1,im+j} = \frac{(m-1)(n-1)(m+n+4)-8}{(n-1)^2}$$

for all $1 \leq i \leq n-1$ and all $2 \leq j \leq m$. Since $m, n \geq 2$, this is equal to 0 if and only if $m = n = 2$, and it is positive otherwise. By means of lemma 2.1.8, this shows all together $\ker\left(W^2 \# I - W^{\#}\right) = \{0\}$ if $m \geq 3$ or $n \geq 3$ and $\ker\left(W^2 \# I - W^{\#}\right) = \langle e_1 \wedge e_4 \rangle \oplus \langle e_2 \wedge e_3 \rangle$ if $m = n = 2$, which was to be proven. □

Thus, it is now clear that the solutions constructed in the previous section in the dimensions p, p^2, and mn for $m \geq 3$ or $n \geq 3$ do not emerge from symmetric spaces, which concludes the proof of theorem B.

Chapter 3

Proof of theorem A

Let m be a positive integer, and let $n = 2m$. In this part, it will be proven that for solutions (W, θ) to the eigenvalue equation

$$W^2 + W^{\#} = \theta W \tag{3.1}$$

such that $\theta \in \mathbb{R}$ and $W \in \langle \mathbf{W} \rangle_d^n$ with $\|W\| = 1$ the eigenvalue θ is maximal if and only if W is the normalized Weyl curvature operator of $S^m \times S^m$. Note that the condition $\|W\| = 1$ is not a restriction but a necessary assumption since also $(sW, s\theta)$ satisfies (3.1) for every solution (W, θ) of (3.1) and every $s \in \mathbb{R}$. This can easily be seen by the computation

$$(sW)^2 + (sW)^{\#} = s^2 \left(W^2 + W^{\#} \right) = s^2 \theta W.$$

The following lemma will show that the Weyl curvature of $S^m \times S^m$ actually is a solution to (3.1) and will also specify its eigenvalue.

Lemma 3.0.1. Let $m \geq 2$ be an integer, let $n = 2m$, and define

$$\theta_0(n) := \sqrt{2} \frac{\sqrt{(n-1)(n-2)}}{n}.$$

Let further denote W_0 the normalized Weyl curvature of $S^m \times S^m$. Then

$$W_0^2 + W_0^{\#} = \theta_0(n) W_0.$$

The proof of lemma 3.0.1 will be given in section 3.1. Now, the main theorem of the present work can be stated.

Theorem 3.0.2. Let $m \geq 8$ be an integer, and let $n = 2m$. Let further $\theta \geq \theta_0(n)$ be given such that there exists $W \in \langle \mathbf{W} \rangle_d^n$ with $\|W\| = 1$ and $W^2 + W^{\#} = \theta W$. Then $\theta = \theta_0(n)$, and W is the normalized Weyl curvature of $S^m \times S^m$.

The following section will prove lemma 3.0.1, while the rest of the present work is dedicated to the proof of theorem 3.0.2 and will roughly be divided into three parts. Instead of working directly with W, a related operator called $|W|$ will be analyzed. In section 3.2 a gap phenomenon for the scalar curvature of $|W|$ will be proven, that is the scalar curvature of $|W|$ cannot lie in a certain interval. The second part needs the most work. There will be shown that the scalar curvature cannot be smaller than the lower bound of this interval, which will be done in the sections 3.3, 3.4, and 3.5. Finally, in the last section, it will be deduced that W is in fact the normalized Weyl curvature of $S^m \times S^m$

Remark. Let $\tilde{W}_0$ be the normalized Weyl curvature operator of $S^m \times S^{m+1}$ for $m \geq 2$, and let $n = m(m+1)$. Then

$$\tilde{W}_0^2 + \tilde{W}_0^{\#} = \tilde{\theta}_0(n)\tilde{W}_0$$

for $\tilde{\theta}_0(n) = \sqrt{2}\sqrt{\frac{(n-1)(n-3)}{(n+1)(n-2)}}$ can be proven. The conjecture is now that there exists an analogue version of theorem 3.0.2 for $\tilde{W}_0$ and $\tilde{\theta}_0(n)$.
Moreover, $\tilde{\theta}_0(n) \leq \theta_0(n)$ for all $n \geq 3$, and most of the results will be proven for $\tilde{\theta}_0(n)$ and will therefore work for both, the even and the odd dimensional case. Only at the end of the whole proof it will be needed that $\theta \geq \theta_0(n)$ instead of only $\theta \geq \tilde{\theta}_0(n)$. Thus, if the odd dimensional version of the theorem is also true, then better estimates will be needed.

The following will be supposed to be true for the rest of the present work.

Assumption 1. Let $m \geq 8$ be an integer, and let $n = 2m$. Moreover, let $\theta \geq \theta_0(n)$ and $W \in \langle \mathbf{W} \rangle_d^n$ with $|W| = 1$ be given such that

$$W^2 + W^{\#} = \theta W.$$

Furthermore, the following notation will be used in the rest of the present work.

Notation 3.0.3. Let m, n, θ, W be as in assumption 1.

- If not stated differently, all components of elements in $S^2(\mathbb{R}^n)$ and in $S_B^2(\mathfrak{so}(n))$ will be understood with respect to the standard basis $e_1, \ldots, e_n$ and $e_1 \wedge e_2, \ldots, e_{n-1} \wedge e_n$, respectively.

- Define algebraic curvature operators $W_+, W_- \geq 0$ by

$$(W_+)_{ij} := \begin{cases} w_{ij} & \text{if } w_{ij} \geq 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (W_-)_{ij} := \begin{cases} -w_{ij} & \text{if } w_{ij} < 0, \\ 0 & \text{otherwise} \end{cases}$$

  for all $1 \leq i < j \leq n$. Then of course $W = W_+ - W_-$.

- If $R \in S^2_{B,d}(\mathfrak{so}(n))$ is any other diagonal curvature operator, define R_+, R_- by

$$(R_+)_{ij} := \begin{cases} r_{ij} & \text{if } w_{ij} \geq 0, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad (R_-)_{ij} := \begin{cases} r_{ij} & \text{if } w_{ij} < 0, \\ 0 & \text{otherwise} \end{cases}$$

  for all $1 \leq i < j \leq n$, and thus, $R = R_+ + R_-$.

- Set

$$\begin{aligned} |W| &:= W_+ + W_-, & \mathrm{Ric} &:= \mathrm{Ric}(|W|), \\ \mathrm{Ric}_0 &:= \mathrm{Ric}_0(|W|), & \mathrm{scal} &:= \mathrm{scal}(|W|), \\ r &:= \|\mathrm{Ric}\|, & r_0 &:= \|\mathrm{Ric}_0\|, \\ \bar{\lambda} &:= \frac{\mathrm{scal}}{n}, & \lambda_i &:= \mathrm{Ric}_{ii}, \quad \text{and} \\ \mu_i &:= (\mathrm{Ric}_0)_{ii} \end{aligned}$$

  for $1 \leq i \leq n$.

- Define $E \in S^2_{B,d}(\mathfrak{so}(n))$ with $\|E\| =: e$ such that $E \perp \mathrm{Ric} \wedge \mathrm{Ric}$ and

$$|W| = \frac{\sqrt{1-e^2}}{\|\mathrm{Ric} \wedge \mathrm{Ric}\|} \mathrm{Ric} \wedge \mathrm{Ric} + E.$$

- In many cases it will be more convenient to work with the normalized Ricci curvature instead of working with Ric. Thus, set

$$S := \frac{\mathrm{Ric}}{r}.$$

- Finally, recall that

$$\theta_0(n) = \sqrt{2}\frac{\sqrt{(n-1)(n-2)}}{n} \quad \text{and} \quad \tilde{\theta}_0(n) = \sqrt{2}\sqrt{\frac{(n-1)(n-3)}{(n+1)(n-2)}},$$

  and define

$$\hat{\theta}_0(n) := \frac{n-2}{n-3}\tilde{\theta}_0(n) = \sqrt{2}\sqrt{\frac{(n-1)(n-2)}{(n+1)(n-3)}}.$$

3.1 The Weyl curvature of $S^m \times S^m$

In this section, a proof for lemma 3.0.1 will be given. Let $m \geq 2$, and let $n = 2m$. The curvature operator of $S^m \times S^m$ is given by

$$R_{S^m \times S^m} = \begin{pmatrix} \mathrm{id}_{\mathfrak{so}(m)} & 0 \\ 0 & \mathrm{id}_{\mathfrak{so}(m)} \end{pmatrix}.$$

Let $W_{S^m\times S^m}$ denote the Weyl curvature of $S^m\times S^m$. Since

$$\begin{aligned}\mathrm{Ric}(R_{S^m\times S^m}) &= (m-1)\mathrm{id},\\ \mathrm{scal}(R_{S^m\times S^m}) &= 2m(m-1) = \frac{n(n-2)}{2}, \quad \text{and}\\ \mathrm{Ric}_0(R_{S^m\times S^m}) &= 0,\end{aligned}$$

it follows by the decomposition of curvature operators given in section 1.2 that

$$\begin{aligned}W_{S^m\times S^m} =& R_{S^m\times S^m} - \frac{n-2}{2(n-1)}I\\ =& \frac{n}{2(n-1)}\begin{pmatrix} 0 & 1 & \dots & 1 & \frac{2-n}{n} & \frac{2-n}{n} & \dots & \frac{2-n}{n}\\ & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots\\ & & \ddots & 1 & \frac{2-n}{n} & \frac{2-n}{n} & \dots & \frac{2-n}{n}\\ & & & 0 & \frac{2-n}{n} & \frac{2-n}{n} & \dots & \frac{2-n}{n}\\ & & & & 0 & 1 & \dots & 1\\ & & & & & \ddots & \ddots & \vdots\\ & & & & & & \ddots & 1\\ & & & & & & & \end{pmatrix}.\end{aligned}$$

Set now $W_0 := \frac{W_{S^m\times S^m}}{\|W_{S^m\times S^m}\|}$ and consider

$$\begin{aligned}\|W_{S^m\times S^m}\|^2 &= \left(\frac{n}{2(n-1)}\right)^2\left(\frac{n(n-2)}{4} + \frac{n^2}{4}\frac{(n-2)^2}{n^2}\right)\\ &= \left(\frac{n}{2(n-1)}\right)^2\frac{(n-1)(n-2)}{2}.\end{aligned}$$

Therefore,

$$W_0 = \sqrt{\frac{2}{(n-1)(n-2)}}\begin{pmatrix} 0 & 1 & \dots & 1 & \frac{2-n}{n} & \frac{2-n}{n} & \dots & \frac{2-n}{n}\\ & \ddots & \ddots & \vdots & \vdots & \vdots & & \vdots\\ & & \ddots & 1 & \frac{2-n}{n} & \frac{2-n}{n} & \dots & \frac{2-n}{n}\\ & & & 0 & \frac{2-n}{n} & \frac{2-n}{n} & \dots & \frac{2-n}{n}\\ & & & & 0 & 1 & \dots & 1\\ & & & & & \ddots & \ddots & \vdots\\ & & & & & & \ddots & 1\\ & & & & & & & \end{pmatrix}.$$

Therefore, by the formula for $R\#S$ for operators $R, S \in S^2_{B,d}(\mathfrak{so}(n))$ which was given in section 1.4, $W_0^2 + W_0^{\#}$ can be computed as follows.

Let first $1 \leq i < j \leq m$ or $m+1 \leq i < j \leq 2m$. Then

$$\begin{aligned}
\left(W_0^2 + W_0^{\#}\right)_{ij} =& \frac{2}{(n-1)(n-2)} \left(1 + \frac{n}{2} - 2 + \frac{n}{2}\left(\frac{n-2}{n}\right)^2\right) \\
=& \frac{2}{(n-1)(n-2)} \frac{n-2}{2} \left(1 + \frac{n-2}{n}\right) \\
=& \frac{2}{n} \\
=& \sqrt{2}\frac{\sqrt{(n-1)(n-2)}}{n} \sqrt{\frac{2}{(n-1)(n-2)}} \\
=& \sqrt{2}\frac{\sqrt{(n-1)(n-2)}}{n} (W_0)_{ij}.
\end{aligned}$$

If now $1 \leq i \leq m$ and $m+1 \leq j \leq 2m$, then

$$\begin{aligned}
\left(W_0^2 + W_0^{\#}\right)_{ij} =& \frac{2}{(n-1)(n-2)} \left(\left(\frac{n-2}{n}\right)^2 - (n-2)\frac{n-2}{n}\right) \\
=& - \frac{2}{(n-1)(n-2)} \frac{(n-2)^2}{n^2}(n-1) \\
=& \sqrt{2}\frac{\sqrt{(n-1)(n-2)}}{n} \sqrt{\frac{2}{(n-1)(n-2)}} \frac{2-n}{n} \\
=& \sqrt{2}\frac{\sqrt{(n-1)(n-2)}}{n} (W_0)_{ij}.
\end{aligned}$$

Hence, $W_0^2 + W_0^{\#} = \theta_0(n) W_0$ for $\theta_0(n) := \sqrt{2}\frac{\sqrt{(n-1)(n-2)}}{n}$.

This proves lemma 3.0.1.

3.2 A gap phenomenon for the scalar curvature

The normalized Weyl curvature operator W_0 of $S^m \times S^m$ has the property that up to a sign all entries agree almost with each other for large m. The operator which can be obtained by taking the absolute value of each entry of W_0 is therefore almost equal to the normalized identity, and its scalar curvature is almost maximal under all normalized curvature operators. The idea is now to show that the same is also true for $|W|$. In fact, it will be proven that the scalar curvature of $|W|$ has to be greater than or equal to $(n+1)\tilde{\theta}_0(n)$ with $|W|$ and $\tilde{\theta}_0(n)$ as in notation 3.0.3, which is close to $\mathrm{scal}(\frac{I}{\|I\|}) = \sqrt{2}\sqrt{n(n-1)}$. In order to show this, the existence of a gap for the scalar curvature of $|W|$ will be proven in this section, that is it will be shown that scal cannot be contained in a certain interval. In the whole section notation 3.0.3 will be used, and assumption 1 will be supposed to hold.

By assumption 1, (W,θ) satisfies (3.1). Obviously, the same is not necessarily the case for $|W|$, but at least the following result is true.

Lemma 3.2.1. Let assumption 1 hold, and set

$$X(W) := 4\left(W_+ \# W_-\right)_+ + 2\left((W_+)^\# + (W_-)^\#\right)_- .$$

Then

$$\theta|W| = |W|^2 + |W|^\# - 2W_-^2 - X(W).$$

In particular, $X(W) \geq 0$, and thus,

$$\theta|W| \leq |W|^2 + |W|^\# - 2W_-^2.$$

Proof. Define $W_\pm^\# := (W_\pm)^\#$ and $W_\pm^2 := (W_\pm)^2 = (W^2)_\pm$. Because of $W = W_+ - W_-$ and $\theta W = W^2 + W^\#$, it is clear that

$$\theta W_+ = W_+^2 + \left(W^\#\right)_+ \quad \text{and} \quad \theta W_- = -W_-^2 - \left(W^\#\right)_- .$$

Moreover,

$$W^\# = (W_+ - W_-)^\# = W_+^\# + W_-^\# - 2W_+ \# W_-,$$

and therefore,

$$\begin{aligned} \left(W^\#\right)_+ &= \left(W_+^\# + W_-^\#\right)_+ - 2\left(W_+ \# W_-\right)_+ \quad \text{and} \\ -\left(W^\#\right)_- &= -\left(W_+^\# + W_-^\#\right)_- + 2\left(W_+ \# W_-\right)_- . \end{aligned}$$

Furthermore,

$$|W|^\# = (W_+ + W_-)^\# = W_+^\# + W_-^\# + 2W_+ \# W_-.$$

Combined, this shows

$$\begin{aligned} \theta|W| =& W_+^2 - W_-^2 + \left(W^\#\right)_+ - \left(W^\#\right)_- \\ =& W^2 - 2W_-^2 + \left(W_+^\# + W_-^\#\right)_+ - 2\left(W_+ \# W_-\right)_+ \\ & - \left(W_+^\# + W_-^\#\right)_- + 2\left(W_+ \# W_-\right)_- \\ =& W^2 - 2W_-^2 + |W|^\# - 4\left(W_+ \# W_-\right)_+ - 2\left(W_+^\# + W_-^\#\right)_- , \end{aligned}$$

which concludes the proof since $X(W) \geq 0$ is obvious by the definitions of W_+ and W_-. ☐

The following lemma gives an inequality for the scalar curvature of $|W|$.

Lemma 3.2.2. Let assumption 1 hold, and set $C_{\#}(W) := \operatorname{scal}(X(W))$, where $X(W)$ is defined as in lemma 3.2.1. Then

$$\theta n\bar{\lambda} = r^2 - 4\|W_-\|^2 - C_{\#}(W).$$

In particular, $C_{\#}(W) \geq 0$, and thus,

$$\theta n\bar{\lambda} \leq r^2 - 4\|W_-\|^2.$$

Proof. As seen in equation (1.6), it is known that

$$\operatorname{scal}(R^2 + R^{\#}) = \sum_{i,j=1}^{n} (\operatorname{Ric}(R)_{ij})^2$$

for all operators $R \in S^2_B(\mathfrak{so}(n))$. This shows together with lemma 3.2.1

$$\begin{aligned}\theta \operatorname{scal} &= \operatorname{scal}(|W|^2 + |W|^{\#}) - 2\operatorname{scal}\left(W_-^2\right) - C_{\#}(W) \\ &= \sum_{i=1}^{n}(\operatorname{Ric}_{ii})^2 - 4\sum_{1\leq i<j\leq n}\left(W_-^2\right)_{ij} - C_{\#}(W) \\ &= r^2 - 4\|W_-\|^2 - C_{\#}(W).\end{aligned}$$

Moreover, $C_{\#}(W) \geq 0$ because $X(W) \geq 0$. This concludes the proof. $\square$

In order to apply lemma 3.2.2, the following upper bound for $\|W_-\|$ will be useful.

Lemma 3.2.3. The following is true:

$$2\|W_-\|^2 \geq 1 - \sqrt{1 - \||W|_{\operatorname{Ric}}\|^2}.$$

Proof. Set $a := \|W_+\|^2 - \|W_-\|^2 = \langle W_+ + W_-, W_+ - W_-\rangle = \langle |W|, W\rangle$. Since W and $|W|$ are both operators of norm 1,

$$a^2 \leq \||W|_{\langle \mathbf{W}\rangle}\|^2 = 1 - \||W|_{\operatorname{Ric}}\|^2.$$

By construction of W_+ and W_-, $\|W_+\|^2 + \|W_-\|^2 = 1$. Therefore,

$$2\|W_-\|^2 = 1 - \|W_+\|^2 - a + \|W_+\|^2 \geq 1 - \sqrt{1 - \||W|_{\operatorname{Ric}}\|^2},$$

which was to be shown. $\square$

The following lemma will give a better understanding of $\||W|_{\operatorname{Ric}}\|$.

Lemma 3.2.4. Let $R \in S^2_B(\mathfrak{so}(n))$. Then

$$\|R_{\operatorname{Ric}}\|^2 = \frac{1}{n-2}\|\operatorname{Ric}(R)\|^2 - \frac{\operatorname{scal}(R)^2}{2(n-1)(n-2)}.$$

Proof. As pointed out in equation (1.3),

$$R_{\mathrm{Ric}} = \frac{\mathrm{scal}(R)}{n(n-1)} I + \frac{2}{n-2} \mathrm{Ric}_0(R) \wedge \mathrm{id}.$$

Since $\langle I, \mathrm{Ric}_0(R) \wedge \mathrm{id} \rangle = 0$ and since $\|I\|^2 = \frac{n(n-1)}{2}$,

$$\|R_{\mathrm{Ric}}\|^2 = \frac{\mathrm{scal}(R)^2}{2n(n-1)} + \frac{4}{(n-2)^2} \|\mathrm{Ric}_0(R) \wedge \mathrm{id}\|^2.$$

Let $\nu_1, \ldots, \nu_n$ denote the eigenvalues of $\mathrm{Ric}_0(R)$, and let $b_1, \ldots, b_n$ be the corresponding orthonormal eigenbasis. Then

$$\langle (\mathrm{Ric}_0(R) \wedge \mathrm{id})(b_i \wedge b_j), b_i \wedge b_j \rangle = \frac{1}{2} (\nu_i + \nu_j)$$

for all $1 \leq i < j \leq n$. Therefore, since $\sum_{i=1}^n \nu_i = \mathrm{tr}\left(\mathrm{Ric}_0(R)\right) = 0$,

$$\begin{aligned} 4\|\mathrm{Ric}_0(R) \wedge \mathrm{id}\|^2 &= \sum_{1 \leq i < j \leq n} (\nu_i + \nu_j)^2 \\ &= (n-1) \sum_{i=1}^n \nu_i^2 + 2 \sum_{1 \leq i < j \leq n} \nu_i \, \nu_j \\ &= (n-2) \sum_{i=1}^n \nu_i^2 + \sum_{i,j=1}^n \nu_i \, \nu_j \\ &= (n-2) \, \|\mathrm{Ric}_0(R)\|^2. \end{aligned}$$

Furthermore,

$$\begin{aligned} \|\mathrm{Ric}_0(R)\|^2 &= \left\| \mathrm{Ric}(R) - \frac{\mathrm{scal}(R)}{n} \, \mathrm{id} \right\|^2 \\ &= \|\mathrm{Ric}(R)\|^2 + \frac{\mathrm{scal}(R)^2}{n} - 2 \frac{\mathrm{scal}(R)}{n} \langle \mathrm{Ric}(R), \mathrm{id} \rangle \\ &= \|\mathrm{Ric}(R)\|^2 - \frac{\mathrm{scal}(R)^2}{n} \end{aligned}$$

since $\langle \mathrm{Ric}(R), \mathrm{id} \rangle = \mathrm{scal}(R)$. Therefore, all together shows

$$\begin{aligned} \|R_{\mathrm{Ric}}\|^2 &= \frac{\mathrm{scal}(R)^2}{2n(n-1)} + \frac{1}{n-2} \left(\|\mathrm{Ric}(R)\|^2 - \frac{\mathrm{scal}(R)^2}{n} \right) \\ &= \frac{1}{n-2} \|\mathrm{Ric}(R)\|^2 - \frac{\mathrm{scal}(R)^2}{2(n-1)(n-2)}, \end{aligned}$$

which was to be proven. □

The following result will be the last step in order to prove the gap phenomenon.

Lemma 3.2.5. Let assumption 1 hold. Then there exists a $\theta' \geq \theta$ such that

$$\left(\theta' n\bar{\lambda} - r^2 + 2\frac{n-3}{n-2}\right)^2 = \frac{2(n\bar{\lambda})^2}{(n-1)(n-2)} + 4\left(\frac{n-3}{n-2}\right)^2 - \frac{4\theta' n\bar{\lambda}}{n-2}.$$

Proof. Because of the lemmata 3.2.2 and 3.2.3,

$$\theta n\bar{\lambda} \leq r^2 - 4\|W_-\|^2 \leq r^2 - 2\left(1 - \sqrt{1 - \||W|_{\mathrm{Ric}}\|^2}\right).$$

Now, choose $\theta' \geq \theta$ with equality. Then lemma 3.2.4 shows

$$\begin{aligned}\left(\theta' n\bar{\lambda} - r^2 + 2\right)^2 &= 4\left(1 - \||W|_{\mathrm{Ric}}\|^2\right)\\ &= 4\left(1 - \frac{1}{n-2}r^2 + \frac{(n\bar{\lambda})^2}{2(n-1)(n-2)}\right),\end{aligned}$$

which is equivalent to

$$\begin{aligned}\frac{2(n\bar{\lambda})^2}{(n-1)(n-2)} &= \left(\theta' n\bar{\lambda} - r^2\right)^2 + 4\theta' n\bar{\lambda} - 4\frac{n-3}{n-2}r^2\\ &= \left(\theta' n\bar{\lambda} - r^2 + 2\frac{n-3}{n-2}\right)^2 - 4\left(\frac{n-3}{n-2}\right)^2 + \frac{4\theta' n\bar{\lambda}}{n-2}.\end{aligned}$$

This proves the claim. □

The mentioned gap phenomenon for the scalar curvature of $|W|$ can now be obtained from the previous result as follows.

Corollary 3.2.6. Let assumption 1 hold. Then

$$\mathrm{scal} \notin \left((n-3)\tilde{\theta}_0(n), (n+1)\tilde{\theta}_0(n)\right).$$

Proof. In lemma 3.2.5 the existence of a $\theta' \geq \theta$ with

$$0 \leq 2\left(\frac{n-3}{n-2}\right)^2 - \frac{2\theta' n\bar{\lambda}}{n-2} + \frac{(n\bar{\lambda})^2}{(n-1)(n-2)}$$

was shown. Since $\theta \geq \theta_0(n) > \tilde{\theta}_0(n)$ by assumption 1,

$$\begin{aligned}0 &< \frac{(n-3)(n+1)}{(n-1)(n-2)}\tilde{\theta}_0(n)^2 - \tilde{\theta}_0(n)n\bar{\lambda}\frac{2}{n-2} + \frac{(n\bar{\lambda})^2}{(n-1)(n-2)}\\ &= \frac{(n-3)(n+1)\tilde{\theta}_0(n)^2 - 2(n-1)\tilde{\theta}_0(n)n\bar{\lambda} + (n\bar{\lambda})^2}{(n-1)(n-2)}\\ &= \frac{\left(n\bar{\lambda} - (n-3)\tilde{\theta}_0(n)\right)\left(n\bar{\lambda} - (n+1)\tilde{\theta}_0(n)\right)}{(n-1)(n-2)},\end{aligned}$$

which proves the claim. □

It will be shown in the following three sections that actually scal has to be greater than or equal to $(n+1)\tilde{\theta}_0(n)$.

3.3 Estimates for r^2, scal, $\operatorname{tr}\operatorname{Ric}^3$, and $\operatorname{tr}\operatorname{Ric}^4$

The goal for the next three sections will be to prove that the scalar curvature of $|W|$ has to be strictly greater than $(n-3)\tilde{\theta}_0(n)$, from which scal $\geq (n+1)\tilde{\theta}_0(n)$ would follow by corollary 3.2.6. In this part, estimates for and relations between r^2, scal, $\operatorname{tr}\operatorname{Ric}^3$, and $\operatorname{tr}\operatorname{Ric}^4$ will be proven, where all the abbreviations and definitions of notation 3.0.3 will be used. Also it will be supposed that assumption 1 holds.
The proof of the lower bound for the scalar curvature that will be given here will work via contradiction. Therefore, the following will be assumed for the whole section.

Assumption 2. Assume scal $\leq (n-3)\tilde{\theta}_0(n)$.

Lemma 3.3.1. Let assumptions 1 and 2 hold. Then

$$r^2 \geq \frac{n-2}{n-3}\tilde{\theta}_0(n)\mathrm{scal} = \hat{\theta}_0(n)\mathrm{scal}.$$

Proof. Let θ' be as in lemma 3.2.5. It was shown that

$$r^2 = \theta' n\bar{\lambda} + 2\frac{n-3}{n-2} - \sqrt{4\frac{(n-3)^2}{(n-2)^2} - \frac{4\theta' n\bar{\lambda}}{n-2} + \frac{2(n\bar{\lambda})^2}{(n-1)(n-2)}}.$$

The right hand side of this equation is obviously increasing in θ', and since $\theta' \geq \theta \geq \tilde{\theta}_0(n)$,

$$r^2 \geq \tilde{\theta}_0(n)n\bar{\lambda} + 2\frac{n-3}{n-2} - \sqrt{4\frac{(n-3)^2}{(n-2)^2} - \frac{4\tilde{\theta}_0(n)n\bar{\lambda}}{n-2} + \frac{2(n\bar{\lambda})^2}{(n-1)(n-2)}}.$$

The radicand can be written as

$$\begin{aligned}
&4\frac{(n-3)^2}{(n-2)^2} - \frac{4\tilde{\theta}_0(n)n\bar{\lambda}}{n-2} + \frac{2(n\bar{\lambda})^2}{(n-1)(n-2)} \\
&\quad= \left(2\frac{n-3}{n-2} - \frac{\tilde{\theta}_0(n)n\bar{\lambda}}{n-3}\right)^2 - \frac{\tilde{\theta}_0(n)^2(n\bar{\lambda})^2}{(n-3)^2} + \frac{2(n\bar{\lambda})^2}{(n-1)(n-2)} \\
&\quad= \left(2\frac{n-3}{n-2} - \frac{\tilde{\theta}_0(n)n\bar{\lambda}}{n-3}\right)^2 - \frac{2(n\bar{\lambda})^2}{n-2}\left(\frac{n-1}{(n+1)(n-3)} - \frac{1}{n-1}\right) \\
&\quad= \left(2\frac{n-3}{n-2} - \frac{\tilde{\theta}_0(n)n\bar{\lambda}}{n-3}\right)^2 - \frac{2(n\bar{\lambda})^2\left((n-1)^2-(n+1)(n-3)\right)}{(n+1)(n-1)(n-2)(n-3)} \\
&\quad= \left(2\frac{n-3}{n-2} - \frac{\tilde{\theta}_0(n)n\bar{\lambda}}{n-3}\right)^2 - \frac{8(n\bar{\lambda})^2}{(n+1)(n-1)(n-2)(n-3)}.
\end{aligned}$$

Since $n\bar{\lambda} \leq (n-3)\tilde{\theta}_0(n)$ by assumption 2,

$$\frac{\tilde{\theta}_0(n)n\bar{\lambda}}{n-3} \leq \tilde{\theta}_0(n)^2 = 2\frac{(n-1)(n-3)}{(n-2)(n+1)} \leq 2\frac{n-3}{n-2}.$$

Therefore,

$$\sqrt{4\frac{(n-3)^2}{(n-2)^2} - \frac{4\tilde{\theta}_0(n)n\bar{\lambda}}{n-2} + \frac{2(n\bar{\lambda})^2}{(n-1)(n-2)}} \leq 2\frac{n-3}{n-2} - \frac{\tilde{\theta}_0(n)n\bar{\lambda}}{n-3}.$$

In summary, this proves

$$r^2 \geq \left(1 + \frac{1}{n-3}\right)\tilde{\theta}_0(n)n\bar{\lambda} = \frac{n-2}{n-3}\tilde{\theta}_0(n)n\bar{\lambda} = \hat{\theta}_0(n)n\bar{\lambda},$$

which was claimed. □

The quantities $\frac{r}{\mathrm{scal}}$, $\operatorname{tr} S^3$, and $\operatorname{tr} S^4$ are bounded below and related as follows.

Lemma 3.3.2. Let assumptions 1 and 2 hold. Then

1. $\operatorname{tr} S^4 \leq \operatorname{tr} S^3$ and
2. $\dfrac{n-2}{(n-3)^2} \leq \dfrac{r^2}{\mathrm{scal}^2} \leq \left(\operatorname{tr} S^3\right)^2 \leq \operatorname{tr} S^4.$

Proof. Recall that $\lambda_i = \mathrm{Ric}_{ii}$ for all $1 \leq i \leq n$. Since $S \geq 0$ and $\|S\| = 1$, the eigenvalues $\frac{\lambda_1}{r}, \ldots, \frac{\lambda_n}{r}$ of S are for all $1 \leq i \leq n$ bounded above by 1 and below by 0. Thus, inequality 1 is obviously true.
The last inequality of 2 can be shown by the following computation:

$$\left(\operatorname{tr} S^3\right)^2 = \frac{1}{r^6}\left(\sum_{i=1}^n \lambda_i^3\right)^2 \leq \frac{1}{r^6}\sum_{i=1}^n \lambda_i^2 \sum_{i=1}^n \lambda_i^4 = \operatorname{tr} S^4,$$

where just the Cauchy-Schwarz inequality and the fact that $\sum_{i=1}^n \frac{\lambda_i^2}{r^2} = \|S\|^2 = 1$ were used.
For the second inequality of 2 consider

$$\begin{aligned}
\mathrm{scal}\operatorname{tr}\mathrm{Ric}^3 - r^4 &= \sum_{i,j=1}^n \lambda_i\lambda_j^3 - \sum_{i,j=1}^n \lambda_i^2\lambda_j^2 \\
&= \frac{1}{2}\sum_{i,j=1}^n \left(\lambda_i\lambda_j^3 + \lambda_i^3\lambda_j - 2\lambda_i^2\lambda_j^2\right) \\
&= \frac{1}{2}\sum_{i,j=1}^n \lambda_i\lambda_j\left(\lambda_i - \lambda_j\right)^2 \\
&\geq 0,
\end{aligned}$$

since $\lambda_i \geq 0$ for all $1 \leq i \leq n$, which proves $\frac{r}{\mathrm{scal}} \leq \operatorname{tr} S^3$.
Lemma 3.3.1 and assumption 2 show now the first inequality of 2:

$$\frac{r^2}{\mathrm{scal}^2} \geq \frac{n-2}{n-3}\frac{\tilde{\theta}_0(n)}{\mathrm{scal}} \geq \frac{n-2}{(n-3)^2}.$$

This concludes the proof. □

Furthermore, $\|\mathrm{Ric} \wedge \mathrm{Ric}\| = r^2 \|S \wedge S\|$ can be computed via the following.

Lemma 3.3.3. If $A \in S^2(\mathbb{R}^n)$, then $\|A \wedge A\| = \frac{1}{\sqrt{2}}\sqrt{\|A\|^4 - \operatorname{tr} A^4}$. In particular, $\|S \wedge S\| = \frac{1}{\sqrt{2}}\sqrt{1 - \operatorname{tr} S^4}$.

Proof. Let $a_1, \ldots, a_n$ denote the eigenvalues of A, and let $b_1, \ldots, b_n$ be the corresponding orthonormal eigenbasis. Then $\langle (A \wedge A)(b_i \wedge b_j), b_i \wedge b_j \rangle = a_i a_j$ for all $1 \leq i < j \leq n$. The following computation proves the claim:

$$\begin{aligned} \|A \wedge A\|^2 &= \sum_{1 \leq i < j \leq n} (a_i a_j)^2 \\ &= \frac{1}{2}\left(\sum_{i,j=1}^{n} a_i^2 a_j^2 - \sum_{i=1}^{n} a_i^4 \right) \\ &= \frac{1}{2}\left(\|A\|^4 - \operatorname{tr} A^4 \right). \end{aligned}$$

□

Recall that $|W|$ decomposes as

$$|W| = \frac{\sqrt{1-e^2}}{\|\mathrm{Ric} \wedge \mathrm{Ric}\|} \mathrm{Ric} \wedge \mathrm{Ric} + E, \tag{3.2}$$

where $E \in S^2_{B,d}(\mathfrak{so}(n))$ with $E \perp \mathrm{Ric} \wedge \mathrm{Ric}$ and $e = \|E\|$. With this, the following estimate for r^2 can be obtained.

Lemma 3.3.4. Let assumption 1 hold. Then

$$\theta r^2 \leq \sqrt{2} r^2 \sqrt{1 - \operatorname{tr} S^4}\sqrt{1-e^2} - 2\langle \mathrm{Ric}(W_-^2), \mathrm{Ric} \rangle.$$

Proof. The application of lemma 3.2.1 shows

$$\theta \mathrm{Ric} \leq \mathrm{Ric}\left(|W|^2 + |W|^{\#} \right) - 2\mathrm{Ric}\left(W_-^2 \right).$$

Therefore,

$$\theta r^2 \leq \langle \mathrm{Ric}\left(|W|^2 + |W|^{\#} \right), \mathrm{Ric} \rangle - 2\langle \mathrm{Ric}\left(W_-^2 \right), \mathrm{Ric} \rangle. \tag{3.3}$$

Because of $(\mathrm{Ric} \wedge \mathrm{Ric})_{ij} = \lambda_i \lambda_j$ for all $1 \leq i < j \leq n$ and

$$\mathrm{Ric}\left(|W|^2 + |W|^{\#}\right)_{ii} = \sum_{j=1}^{n} |W|_{ij} \lambda_j$$

for all $1 \leq i \leq n$ by equation (1.5) in chapter 1.3,

$$\begin{aligned}
\langle \mathrm{Ric}\left(|W|^2 + |W|^{\#}\right), \mathrm{Ric}\rangle &= \sum_{i,j=1}^{n} |W|_{ij} \lambda_j \lambda_i \\
&= 2 \sum_{1 \leq i < j \leq n} |W|_{ij} \lambda_j \lambda_i \\
&= 2\langle |W|, \mathrm{Ric} \wedge \mathrm{Ric}\rangle \\
&= 2\|\mathrm{Ric} \wedge \mathrm{Ric}\| \sqrt{1 - e^2} \\
&= r^2 \sqrt{2} \sqrt{1 - \mathrm{tr}\, S^4} \sqrt{1 - e^2},
\end{aligned}$$

where in the penultimate step (3.2) and $E \perp \mathrm{Ric} \wedge \mathrm{Ric}$ were used. The last equality follows from lemma 3.3.3.
Together with inequality (3.3) this concludes the proof. □

The inequality in lemma 3.3.4 will be one of the main tools for the construction of a contradiction to assumption 2, but the term $-2\langle \mathrm{Ric}(W_-^2), \mathrm{Ric}\rangle$ is not easy to work with, and without that term the inequality is not strong enough. Thus, it is necessary to find an estimate for that term which is on the one hand easy enough to work with and on the other strong enough for a contradiction. This will be done in the next chapter.
However, some preparations will be needed first for which lemma 3.3.4 will also be useful. The next lemma will prove a first estimate for $\langle \mathrm{Ric}(W_-^2), \mathrm{Ric}\rangle$ which will again not be strong enough for contradicting assumption 2, but it will be sufficient to apply lemma 3.3.4 for getting useful bounds for $\mathrm{tr}\, S^4$, scal, r^2, and e.

Lemma 3.3.5. The following is true:

$$\langle \mathrm{Ric}(W_-^2), \mathrm{Ric}\rangle \geq \frac{1}{4(n-2)} \mathrm{tr}\, \mathrm{Ric}^3.$$

Proof. Set $\lambda_i^- := \mathrm{Ric}(W_-)_{ii}$ for $1 \leq i \leq n$. Then

$$\langle \mathrm{Ric}\left(W_-^2\right), \mathrm{Ric}\left(W_-\right)\rangle = \sum_{i=1}^{n} \lambda_i^- \sum_{j=1}^{n} (W_-)_{ij}^2 \quad \text{and} \tag{3.4}$$

$$\mathrm{tr}\, \mathrm{Ric}(W_-)^3 = \sum_{i=1}^{n} {\lambda_i^-}^3 = \sum_{i=1}^{n} \lambda_i^- \left(\sum_{j=1}^{n} (W_-)_{ij}\right)^2. \tag{3.5}$$

Clearly, $(W_-)_{ii} = 0$ for all $1 \leq i \leq n$. Since $W \in \langle \mathbf{W} \rangle_d^n$, for every $1 \leq i \leq n$ either $(W_-)_{ij} = 0$ for at least one $1 \leq j \leq n$ with $j \neq i$ or $\lambda_i^- = 0$. Consider the function

$$f\colon \mathbb{R}^{n-2} \to \mathbb{R}; (x_1, \ldots, x_{n-2}) \mapsto \left(\sum_{i=1}^{n-2} x_i \right)^2 .$$

It is easy to see that f attains its maximum under the side conditions $\sum_{i=1}^{n-2} x_i^2 = 1$ and $x_i \geq 0$ for all $1 \leq i \leq n-2$ in $x_1 = \cdots = x_{n-2} = \frac{1}{\sqrt{n-2}}$. Since $f\left(\frac{1}{\sqrt{n-2}}, \ldots, \frac{1}{\sqrt{n-2}}\right) = n-2$, it follows that

$$\left(\sum_{i=1}^{n-2} x_i \right)^2 \leq (n-2) \sum_{i=1}^{n-2} x_i^2,$$

which is scale invariant. This shows together with (3.4) and (3.5)

$$\langle \mathrm{Ric}\left(W_-^2\right), \mathrm{Ric}\left(W_-\right) \rangle \geq \frac{1}{n-2} \operatorname{tr} \mathrm{Ric}(W_-)^3.$$

Because of $\mathrm{Ric} = 2\,\mathrm{Ric}\left(W_-\right)$, the claim is proven. □

Now, an upper bound for $\operatorname{tr} S^4$ can be given.

Lemma 3.3.6. Let assumptions 1 and 2 hold. Then $\operatorname{tr} S^4 \leq \dfrac{16}{8n+9}$.

Proof. From the lemmata 3.3.1, 3.3.2, and 3.3.5 it is known that

$$\langle \mathrm{Ric}\left(W_-^2\right), \mathrm{Ric} \rangle \geq \frac{1}{4(n-2)} \operatorname{tr} \mathrm{Ric}^3 \geq \frac{r^4}{4(n-2)\mathrm{scal}} \geq \frac{\tilde{\theta}_0(n)}{4(n-3)} r^2.$$

Since $\theta \geq \tilde{\theta}_0(n)$, lemma 3.3.4 shows now

$$\tilde{\theta}_0(n) \leq \sqrt{2}\sqrt{1 - \operatorname{tr} S^4} - \frac{\tilde{\theta}_0(n)}{2(n-3)},$$

which leads to

$$\begin{aligned}
\operatorname{tr} S^4 \leq & 1 - \frac{\tilde{\theta}_0(n)^2}{2} \left(1 + \frac{1}{2(n-3)}\right)^2 \\
= & 1 - \frac{(n-1)(n-3)}{(n+1)(n-2)} \left(\frac{2n-5}{2(n-3)}\right)^2 \\
= & \frac{8n^2 - 41n + 49}{4(n+1)(n-2)(n-3)} \\
\leq & \frac{16}{8n+9},
\end{aligned}$$

where the last inequality follows from

$$\left(8n^2 - 41n + 49\right)(8n+9) - 64(n+1)(n-2)(n-3)$$
$$= -41n + 57 \leq 0$$

for all $n \geq 2$. Therefore, the statement is proven. □

The following lower bounds for the scalar curvature and the norm of the Ricci curvature of $|W|$ can be obtained from the previous results.

Corollary 3.3.7. Let assumptions 1 and 2 hold. Then

1. $\operatorname{scal} \geq \dfrac{n-2}{n-3}\dfrac{8n+9}{16}\tilde{\theta}_0(n) = \dfrac{8n+9}{16}\hat{\theta}_0(n)$ and
2. $r^2 \geq \dfrac{(n-2)^2}{(n-3)^2}\dfrac{8n+9}{16}\tilde{\theta}_0(n)^2 = \dfrac{8n+9}{16}\hat{\theta}_0(n)^2.$

Proof. Since $\frac{\operatorname{scal}^2}{r^2} \geq \frac{8n+9}{16}$ by the lemmata 3.3.2 and 3.3.6, estimate 1 follows immediately from lemma 3.3.1.
Estimate 2 is just a direct consequence of 1 and the repeated application of lemma 3.3.1. □

Consider $e = \|E\|$ as defined in notation 3.0.3. Since $\|W\| = 1$, it is clear that $e \leq 1$. However, a better bound for e can be obtained from the previous results as the following lemma will show.

Lemma 3.3.8. Let assumptions 1 and 2 hold. Then $e^2 \leq \frac{3}{2n}$.

Proof. As proven in the lemmata 3.3.4 and 3.3.5,

$$\tilde{\theta}_0(n) r^2 \leq r^2 \sqrt{2}\sqrt{1 - \operatorname{tr} S^4}\sqrt{1-e^2} - \frac{1}{2(n-2)}\operatorname{tr}\operatorname{Ric}^3.$$

Dividing this by r^2 shows that this is equivalent to

$$\tilde{\theta}_0(n) \leq \sqrt{2}\sqrt{1 - \operatorname{tr} S^4}\sqrt{1-e^2} - \frac{r}{2(n-2)}\operatorname{tr} S^3.$$

Therefore,

$$e^2 \leq 1 - \frac{\left(\tilde{\theta}_0(n) + \frac{r}{2(n-2)}\operatorname{tr} S^3\right)^2}{2(1 - \operatorname{tr} S^4)}.$$

Since $\frac{1}{n} \leq \left(\operatorname{tr} S^3\right)^2 \leq \operatorname{tr} S^4$ and $r \geq \frac{n-2}{n-3}\tilde{\theta}_0(n)\sqrt{\frac{n}{2}}$ by lemma 3.3.2 and corollary 3.3.7, respectively,

$$\begin{aligned}
e^2 &\leq 1 - \frac{\tilde{\theta}_0(n)^2}{2}\frac{n}{n-1}\left(1 + \frac{1}{2\sqrt{2}(n-3)}\right)^2 \\
&= 1 - \frac{n(n-3)}{(n+1)(n-2)}\frac{8n^2 - (48 - 4\sqrt{2})n + 73 - 12\sqrt{2}}{8(n-3)^2} \\
&= \frac{(16 - 4\sqrt{2})n^2 - (65 - 12\sqrt{2})n + 48}{8(n+1)(n-2)(n-3)}.
\end{aligned}$$

Consider now

$$\begin{aligned}
&24(n+1)(n-2)(n-3) - 2n\left((16 - 4\sqrt{2})n^2 - (65 - 12\sqrt{2})n + 48\right) \\
&\quad = 8(\sqrt{2} - 1)n^3 + (34 - 24\sqrt{2})n^2 - 72n + 144 \\
&\quad =: f(n).
\end{aligned}$$

Then $f(3) = 18$,

$$\begin{aligned}
f'(n) &= 24(\sqrt{2} - 1)n^2 + (68 - 48\sqrt{2})n - 72, \quad \text{and} \\
f''(n) &= 48(\sqrt{2} - 1)n + 68 - 48\sqrt{2}.
\end{aligned}$$

Thus, $f'(3) = 72\sqrt{2} - 84 > 0$ and $f''(n) \geq f''(3) = 96\sqrt{2} - 76 > 0$ for all $n \geq 3$. Therefore, f is nonnegative on $[3, \infty)$, and hence,

$$e^2 \leq \frac{3}{2n},$$

which was to be shown. □

3.4 An estimate for $-2\langle \mathbf{Ric}(W_-^2), \mathbf{Ric}\rangle$

As mentioned before, the goal of this chapter will be to find an upper bound for $-2\langle \mathrm{Ric}(W_-^2), \mathrm{Ric}\rangle$ which could help to use lemma 3.3.4 for the construction of a contradiction for assumption 2. For the whole section it will be supposed that assumptions 1 and 2 hold, and notation 3.0.3 will be used. Recall that there exists an operator $E \in S^2_{B,d}(\mathfrak{so}(n))$ with $e := \|E\|$ such that

- $E \perp \mathrm{Ric} \wedge \mathrm{Ric}$ and
- $|W| = \dfrac{\sqrt{1 - e^2}}{\|\mathrm{Ric} \wedge \mathrm{Ric}\|}\mathrm{Ric} \wedge \mathrm{Ric} + E.$

The following result will be useful and is well known (cf. [C+], lemma 11.15). Note that in the present work the scalar product on $S^2(\mathbb{R}^n)$ is given by

$$\langle A, B\rangle = \operatorname{tr}(AB),$$

whereas in [C+] it differs by the factor $\frac{1}{2}$.

Lemma 3.4.1. Let $R \in S^2_B(\mathfrak{so}(n))$ and $A \in S^2(\mathbb{R}^n)$ be given. Then

$$\langle R, A \wedge 2\,\mathrm{id}\rangle = \langle \mathrm{Ric}(R), A\rangle.$$

With this, $-2\langle \mathrm{Ric}(W_-^2), \mathrm{Ric}\rangle$ can be decomposed as follows.

Lemma 3.4.2. Define

$$T_1 := 2\langle W_- (\mathrm{Ric} \wedge \mathrm{Ric}), \mathrm{Ric} \wedge 2\,\mathrm{id}\rangle \quad \text{and}$$
$$T_2 := -2\langle W_- E, \mathrm{Ric} \wedge 2\,\mathrm{id}\rangle.$$

Then

$$-2\langle \mathrm{Ric}(W_-^2), \mathrm{Ric}\rangle = -\frac{\sqrt{1-e^2}}{\|\mathrm{Ric} \wedge \mathrm{Ric}\|} T_1 + T_2.$$

Proof. Because of lemma 3.4.1 and $W_-^2 = W_-|W|$,

$$\begin{aligned}
\langle \mathrm{Ric}(W_-^2), \mathrm{Ric}\rangle &= \langle W_-^2, \mathrm{Ric} \wedge 2\,\mathrm{id}\rangle \\
&= \langle W_-|W|, \mathrm{Ric} \wedge 2\,\mathrm{id}\rangle \\
&= \frac{\sqrt{1-e^2}}{\|\mathrm{Ric} \wedge \mathrm{Ric}\|}\langle W_- (\mathrm{Ric} \wedge \mathrm{Ric}), \mathrm{Ric} \wedge 2\,\mathrm{id}\rangle \\
&\quad + \langle W_- E, \mathrm{Ric} \wedge 2\,\mathrm{id}\rangle,
\end{aligned}$$

which was to be proven. □

In the rest of this chapter, estimates for the two terms T_1 and T_2 will be calculated.

3.4.1 An estimate for T_1

In this part, the following estimate for T_1 will be proven.

Lemma 3.4.3. Let $\epsilon \geq \frac{\sqrt{10}-3}{2}$. Then

$$T_1 \geq n\bar{\lambda}^4 + \bar{\lambda}^2(2-\epsilon)r_0^2 - \bar{\lambda}(1+\epsilon)\mathrm{tr}\,\mathrm{Ric}_0^3.$$

In order to prove lemma 3.4.3, the next lemmata will be needed.

Lemma 3.4.4. Let $R \in S^2_B(\mathfrak{so}(n))$, and let $\nu_1, \dots, \nu_n$ be the eigenvalues of $\mathrm{Ric}_0(R)$. Then

1. $\displaystyle\sum_{\substack{i,j=1\\ i\neq j}}^n (\nu_i + \nu_j) = 0,$
2. $\displaystyle\sum_{\substack{i,j=1\\ i\neq j}}^n \nu_i\nu_j = -\|\mathrm{Ric}_0(R)\|^2,$
3. $\displaystyle\sum_{\substack{i,j=1\\ i\neq j}}^n (\nu_i^2 + \nu_j^2) = 2(n-1)\|\mathrm{Ric}_0(R)\|^2,$
4. $\displaystyle\sum_{\substack{i,j=1\\ i\neq j}}^n \nu_i^2\nu_j = -\mathrm{tr}\,\mathrm{Ric}_0(R)^3,$
5. $\displaystyle\sum_{\substack{i,j=1\\ i\neq j}}^n (\nu_i^3 + \nu_j^3) = 2(n-1)\mathrm{tr}\,\mathrm{Ric}_0(R)^3.$

Proof. Since $\sum_{i=1}^{n} \nu_i = \operatorname{tr}\operatorname{Ric}_0(R) = 0$, the following calculations prove the claims:

$$\sum_{\substack{i,j=1\\ i\neq j}}^{n} (\nu_i + \nu_j) = \sum_{i=1}^{n} \left((n-2)\nu_i + \sum_{j=1}^{n} \nu_j \right) = 0,$$

$$\sum_{\substack{i,j=1\\ i\neq j}}^{n} \nu_i\nu_j = \sum_{i=1}^{n} \nu_i \sum_{\substack{j=1\\ j\neq i}}^{n} \nu_j = -\sum_{i=1}^{n} \nu_i^2 = -\|\operatorname{Ric}_0(R)\|^2,$$

$$\sum_{\substack{i,j=1\\ i\neq j}}^{n} (\nu_i^2 + \nu_j^2) = \sum_{i=1}^{n} \left((n-2)\nu_i^2 + \sum_{j=1}^{n} \nu_j^2 \right) = 2(n-1)\|\operatorname{Ric}_0(R)\|^2,$$

$$\sum_{\substack{i,j=1\\ i\neq j}}^{n} \nu_i^2\nu_j = \sum_{i=1}^{n} \nu_i^2 \sum_{\substack{j=1\\ j\neq i}}^{n} \nu_j = -\sum_{i=1}^{n} \nu_i^3 = -\operatorname{tr}\operatorname{Ric}_0(R)^3,$$

$$\sum_{\substack{i,j=1\\ i\neq j}}^{n} (\nu_i^3 + \nu_j^3) = \sum_{i=1}^{n} \left((n-2)\nu_i^3 + \sum_{j=1}^{n} \nu_j^3 \right) = 2(n-1)\operatorname{tr}\operatorname{Ric}_0(R)^3.$$

□

Lemma 3.4.5. Let $x, y \geq -1$ and $\epsilon \geq \frac{\sqrt{10}-3}{2}$ be given. Then

$$0 \leq (2+\epsilon)\left(x^2 + y^2\right) + xy(4 + x + y).$$

Proof. Let $F\colon [-1,\infty) \times [-1,\infty) \to \mathbb{R}$ be defined by

$$(x, y) \mapsto (2+\epsilon)\left(x^2 + y^2\right) + xy(4 + x + y)$$

for $\epsilon \geq \frac{\sqrt{10}-3}{2}$, and let f be the restriction of F to $(-1,\infty) \times (-1,\infty)$. Clearly, f is differentiable with

$$\nabla f(x, y) = \begin{pmatrix} (4+2\epsilon)x + 4y + 2xy + y^2 \\ (4+2\epsilon)y + 4x + 2xy + x^2 \end{pmatrix}$$

for every $(x, y) \in (-1,\infty) \times (-1,\infty)$. Let now $(x, y) \in (-1,\infty) \times (-1,\infty)$ be a critical point of f. The subtraction of the second equation from the first of $\nabla f(x, y) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ leads to

$$0 = (4+2\epsilon)(x-y) + 4(y-x) + y^2 - x^2 = (x + y - 2\epsilon)(y - x).$$

If $x = y$, then $0 = (8+2\epsilon)x + 3x^2$. Therefore, $x = y = 0$ since $x > -1$.
If $x + y = 2\epsilon$, then

$$\begin{aligned} 0 &= (4+2\epsilon)(2\epsilon - y) + 4y + 2y(2\epsilon - y) + y^2 \\ &= -y^2 + 2\epsilon y + 8\epsilon + 4\epsilon^2. \end{aligned}$$

Hence, $y = \epsilon \pm \sqrt{5\epsilon^2 + 8\epsilon}$ and $x = \epsilon \mp \sqrt{5\epsilon^2 + 8\epsilon}$.
Consider now the function

$$\tilde{f} \colon [-1, \infty) \to \mathbb{R}, x \mapsto F(x, -1) = (1+\epsilon)x^2 - 3x + 2 + \epsilon.$$

It is easy to see that $\tilde{f}$ has a global minimum in $\frac{3}{2(1+\epsilon)}$. Further, F is obviously nonnegative on $[0, \infty) \times [0, \infty)$. Moreover,

$$\lim_{x \to \infty} F(x, y) = \infty$$

for every $y \in [-1, 0)$ because the map $x \mapsto F(x, y)$ is for every $y \in [-1, 0)$ a parabola that opens upward. Since $F(0, 0) = 0$ and since F is symmetric in x and y, all of this shows that the global minimum of F has to be

$$\min \left\{ 0, F\left(\epsilon + \sqrt{5\epsilon^2 + 8\epsilon}, \epsilon - \sqrt{5\epsilon^2 + 8\epsilon}\right), F\left(\frac{3}{2(1+\epsilon)}, -1\right) \right\}$$

if $\epsilon - \sqrt{5\epsilon^2 + 8\epsilon} > -1$, or else it is

$$\min \left\{ 0, F\left(\frac{3}{2(1+\epsilon)}, -1\right) \right\}.$$

However,

$$\begin{aligned} &F\left(\epsilon + \sqrt{5\epsilon^2 + 8\epsilon}, \epsilon - \sqrt{5\epsilon^2 + 8\epsilon}\right) \\ &\quad = (2+\epsilon)\left(12\epsilon^2 + 16\epsilon\right) - \left(4\epsilon^2 + 8\epsilon\right)(4 + 2\epsilon) \\ &\quad = 4\epsilon^2(2+\epsilon) \\ &\quad > 0 \end{aligned}$$

for all $\epsilon \geq \frac{\sqrt{3}-10}{2}$ such that $F\left(\epsilon + \sqrt{5\epsilon^2 + 8\epsilon}, \epsilon - \sqrt{5\epsilon^2 + 8\epsilon}\right)$ is defined, and

$$\begin{aligned} F\left(\frac{3}{2(1+\epsilon)}, -1\right) &= (2+\epsilon)\left(\frac{9}{4(1+\epsilon)^2} + 1\right) - \frac{9}{2(1+\epsilon)}\left(1 + \frac{1}{2(1+\epsilon)}\right) \\ &= -\frac{9}{4(1+\epsilon)} + 2 + \epsilon \\ &= \frac{4\epsilon^2 + 12\epsilon - 1}{4(1+\epsilon)}, \end{aligned}$$

which is nonnegative for all $\epsilon \geq \frac{\sqrt{10}-3}{2}$. This proves the claim. □

This will now be helpful to show the following.

Lemma 3.4.6. Let $\epsilon \geq \frac{\sqrt{10}-3}{2}$ and $R \in S^2_B(\mathfrak{so}(n))$ with $R \geq 0$ be given. Then

$$\begin{aligned} &\langle R\left(\operatorname{Ric}(R) \wedge \operatorname{Ric}(R)\right), \operatorname{Ric}(R) \wedge 2\,\mathrm{id} \rangle \\ &\quad \geq \frac{\operatorname{scal}(R)^4}{n^3} + \frac{\operatorname{scal}(R)^2}{n^2}(2-\epsilon)\|\operatorname{Ric}_0(R)\|^2 \\ &\quad\quad - \frac{\operatorname{scal}(R)}{n}(1+\epsilon)\operatorname{tr}\operatorname{Ric}_0(R)^3. \end{aligned}$$

Proof. Let $\bar{\lambda}_R := \frac{\mathrm{scal}(R)}{n}$, and let $\eta_1, \dots, \eta_n$ and $\nu_1, \dots, \nu_n$ be the eigenvalues of $\mathrm{Ric}(R)$ and $\mathrm{Ric}_0(R)$, respectively. Without loss of generality, it can be assumed that $\eta_1 \geq \dots \geq \eta_n$ as well as $\nu_1 \geq \dots \geq \nu_n$. Further, let $b_1, \dots, b_n$ denote the corresponding orthonormal basis of eigenvectors of $\mathrm{Ric}(R)$. All components of algebraic curvature operators in this proof will be understood with respect to this basis.
Since $\mathrm{Ric}(R) = \bar{\lambda}_R \,\mathrm{id} + \mathrm{Ric}_0(R)$, $\eta_i = \bar{\lambda}_R + \nu_i$ for every $1 \leq i \leq n$. Therefore,

$$\begin{aligned}
&((\mathrm{Ric}(R) \wedge \mathrm{Ric}(R)) (\mathrm{Ric}(R) \wedge 2\,\mathrm{id}))_{ijji} \\
&\quad = \langle (\mathrm{Ric}(R) \wedge \mathrm{Ric}(R)) (\mathrm{Ric}(R) \wedge 2\,\mathrm{id}) (b_i \wedge b_j), b_i \wedge b_j \rangle \\
&\quad = \eta_i \eta_j (\eta_i + \eta_j) \\
&\quad = \left(\bar{\lambda}_R + \nu_i\right) \left(\bar{\lambda}_R + \nu_j\right) \left(2\bar{\lambda}_R + \nu_i + \nu_j\right) \\
&\quad = 2\bar{\lambda}_R^3 + 3\bar{\lambda}_R^2 (\nu_i + \nu_j) + \bar{\lambda}_R \left(4\nu_i\nu_j + \nu_i^2 + \nu_j^2\right) + \nu_i\nu_j (\nu_i + \nu_j)
\end{aligned} \tag{3.6}$$

for every $1 \leq i, j \leq n$ with $i \neq j$.
Suppose first that there exists $1 \leq i \leq n$ with $\nu_i \neq 0$. In particular, $\nu_n < 0$, and thus, $\nu_n = -1$ can be assumed since the inequality in the statement of the lemma is scale invariant. Therefore, lemma 3.4.5 can be applied. Hence,

$$\begin{aligned}
0 &\leq (2 + \epsilon) \left(\nu_i^2 + \nu_j^2\right) + 4\nu_i\nu_j + \nu_i\nu_j (\nu_i + \nu_j) \\
&\leq \bar{\lambda}_R \left((2 + \epsilon) \left(\nu_i^2 + \nu_j^2\right) + 4\nu_i\nu_j\right) + \nu_i\nu_j (\nu_i + \nu_j)
\end{aligned}$$

for all $\epsilon \geq \frac{\sqrt{10}-3}{2}$ and all $1 \leq i, j \leq n$. Here, the second inequality results from $\bar{\lambda}_R = \eta_n - \nu_n \geq 1$ and

$$(2 + \epsilon) \left(\nu_i^2 + \nu_j^2\right) + 4\nu_i\nu_j \geq 2 (\nu_i + \nu_j)^2 \geq 0.$$

This shows together with equation (3.6) that

$$\begin{aligned}
&((\mathrm{Ric}(R) \wedge \mathrm{Ric}(R)) (\mathrm{Ric}(R) \wedge 2\,\mathrm{id}))_{ijji} \\
&\quad \geq 2\bar{\lambda}_R^3 + 3\bar{\lambda}_R^2 (\nu_i + \nu_j) - \bar{\lambda}_R (1 + \epsilon) \left(\nu_i^2 + \nu_j^2\right)
\end{aligned}$$

for all $\epsilon \geq \frac{\sqrt{10}-3}{2}$ and all $1 \leq i, j \leq n$ with $i \neq j$.
The right hand side of this inequality defines now an algebraic curvature operator of Ricci type, that is the Weyl part vanishes. This can be seen as follows: Let

$$\tilde{R}_{ijji} := 2\bar{\lambda}_R^3 + 3\bar{\lambda}_R^2 (\nu_i + \nu_j) - \bar{\lambda}_R (1 + \epsilon) \left(\nu_i^2 + \nu_j^2\right).$$

Then

$$\begin{aligned}
\mathrm{Ric}(\tilde{R})_{ii} = &2(n-1)\bar{\lambda}_R^3 + 3(n-2)\bar{\lambda}_R^2 \nu_i \\
&- \bar{\lambda}_R (1 + \epsilon) \left((n-2)\nu_i^2 + \|\mathrm{Ric}_0(R)\|^2\right)
\end{aligned}$$

for $1 \leq i \leq n$,

$$\operatorname{scal}(\tilde{R}) = 2n(n-1)\bar{\lambda}_R^3 - \bar{\lambda}_R(1+\epsilon)2(n-1)\|\operatorname{Ric}_0(R)\|^2,$$

and

$$\begin{aligned}\operatorname{Ric}_0(\tilde{R})_{ii} =& \operatorname{Ric}(\tilde{R})_{ii} - \frac{\operatorname{scal}(\tilde{R})}{n}\\ =& 3(n-2)\bar{\lambda}_R^2\nu_i - \bar{\lambda}_R(1+\epsilon)\left((n-2)\nu_i^2 - \frac{n-2}{n}\|\operatorname{Ric}_0(R)\|^2\right)\end{aligned}$$

for $1 \leq i \leq n$. Thus, since $(\operatorname{Ric}_0(\tilde{R}) \wedge \operatorname{id})_{ijji} = \frac{1}{2}\left(\operatorname{Ric}_0(\tilde{R})_{ii} + \operatorname{Ric}_0(\tilde{R})_{jj}\right)$ for all $1 \leq i, j \leq n$ with $i \neq j$,

$$\begin{aligned}&\left(\tilde{R}_I + \tilde{R}_{\operatorname{Ric}_0}\right)_{ijji}\\ &= \frac{\operatorname{scal}(\tilde{R})}{n(n-1)} + \frac{1}{n-2}\left(\operatorname{Ric}_0(\tilde{R})_{ii} + \operatorname{Ric}_0(\tilde{R})_{jj}\right)\\ &= 2\bar{\lambda}_R^3 - \frac{2(1+\epsilon)}{n}\bar{\lambda}_R\|\operatorname{Ric}_0(R)\|^2 + \frac{1}{n-2}\Bigg(3(n-2)\bar{\lambda}_R^2(\nu_i+\nu_j)\\ &\quad - \bar{\lambda}_R(1+\epsilon)\Big((n-2)(\nu_i^2+\nu_j^2) - 2\frac{n-2}{n}\|\operatorname{Ric}_0(R)\|^2\Big)\Bigg)\\ &= \tilde{R}_{ijji}\end{aligned}$$

for all $1 \leq i, j \leq n$ with $i \neq j$. Thus, $\tilde{R}_W = 0$ by the decomposition of curvature operators as seen in chapter 1.2.
If now $\nu_1 = \cdots = \nu_n = 0$ is assumed, then

$$\left((\operatorname{Ric}(R) \wedge \operatorname{Ric}(R))\,(\operatorname{Ric}(R) \wedge 2\operatorname{id})\right)_{ijji} = 2\bar{\lambda}_R^3$$

by equation (3.6), and thus, $(\operatorname{Ric}(R) \wedge \operatorname{Ric}(R))\,(\operatorname{Ric}(R) \wedge 2\operatorname{id}) \in \langle I \rangle$.
Since R decomposes as $R = \frac{\bar{\lambda}_R}{n-1}I + \frac{2}{n-2}\operatorname{Ric}_0(R) \wedge \operatorname{id} + R_W$, in both cases lemma 3.4.4 shows

$$
\begin{aligned}
&\langle R\left(\operatorname{Ric}(R)\wedge\operatorname{Ric}(R)\right),\operatorname{Ric}(R)\wedge 2\,\mathrm{id}\rangle\\
&\quad\geq \sum_{1\leq i<j\leq n}\left(\frac{\bar\lambda_R}{n-1}+\frac{\nu_i+\nu_j}{n-2}\right)\left(2\bar\lambda_R^3+3\bar\lambda_R^2\left(\nu_i+\nu_j\right)\right.\\
&\qquad\left.-\bar\lambda_R(1+\epsilon)\left(\nu_i^2+\nu_j^2\right)\right)\\
&\quad= n\bar\lambda_R^4+\frac{1}{2}\sum_{\substack{i,j=1\\ i\neq j}}^{n}\left(\bar\lambda_R^3\left(\frac{3}{n-1}+\frac{2}{n-2}\right)\left(\nu_i+\nu_j\right)\right.\\
&\qquad+\bar\lambda_R^2\left(\left(\frac{3}{n-2}-\frac{1+\epsilon}{n-1}\right)\left(\nu_i^2+\nu_j^2\right)+\frac{6}{n-2}\nu_i\nu_j\right)\\
&\qquad\left.-\bar\lambda_R\frac{1+\epsilon}{n-2}\left(\nu_i^3+\nu_j^3+\nu_i^2\nu_j+\nu_i\nu_j^2\right)\right)\\
&\quad= n\bar\lambda_R^4+\frac{\bar\lambda_R^2}{2}\left(2(n-1)\left(\frac{3}{n-2}-\frac{1+\epsilon}{n-1}\right)-\frac{6}{n-2}\right)\|\operatorname{Ric}_0(R)\|^2\\
&\qquad-\frac{\bar\lambda_R}{2}\frac{1+\epsilon}{n-2}(2(n-1)-2)\operatorname{tr}\operatorname{Ric}_0(R)^3\\
&\quad= n\bar\lambda_R^4+\bar\lambda_R^2(2-\epsilon)\|\operatorname{Ric}_0(R)\|^2-\bar\lambda_R(1+\epsilon)\operatorname{tr}\operatorname{Ric}_0(R)^3.
\end{aligned}
$$

This concludes the proof. □

Now, the proof of lemma 3.4.3 is just a direct application of lemma 3.4.6.

Proof of lemma 3.4.3. With $R := 2W_-$, the lemma follows directly from lemma 3.4.6 and the fact that $\operatorname{Ric}(2W_-) = \operatorname{Ric}$. □

3.4.2 An estimate for T_2

In this section, the following estimate for T_2 will be proven.

Lemma 3.4.7. Let assumptions 1 and 2 hold. Then

$$
\begin{aligned}
T_2 &\leq \tilde\theta_0(n)r^2\left(1-\sqrt{1-e^2}\right)\\
&\quad+\frac{2\sqrt{1-e^2}}{\tilde\theta_0(n)}\left(\operatorname{tr}S^4+\sqrt{\frac{\left(\operatorname{tr}S^4\right)^2-\left(\operatorname{tr}S^3\right)^4}{1-\left(\operatorname{tr}S^3\right)^2}}\right).
\end{aligned}
$$

The proof of lemma 3.4.7 will be separated into several steps. The first of these will be to introduce the quantity X in the following lemma and to show that T_2 and X are related as follows.

Lemma 3.4.8. Let

$$X := e\|W(\mathrm{Ric}\wedge 2\,\mathrm{id})\| - \frac{\sqrt{1-e^2}}{\|\mathrm{Ric}\wedge\mathrm{Ric}\|}\langle(\mathrm{Ric}\wedge\mathrm{Ric})(\mathrm{Ric}\wedge 2\,\mathrm{id}), E\rangle.$$

Then

$$T_2 \le X - \langle \mathrm{Ric}\wedge 2\,\mathrm{id}, E^2\rangle.$$

Proof. Since $W = W_+ - W_-$, $|W| = W_+ + W_-$, and

$$|W| = \frac{\sqrt{1-e^2}}{\|\mathrm{Ric}\wedge\mathrm{Ric}\|}\mathrm{Ric}\wedge\mathrm{Ric} + E,$$

the Cauchy-Schwarz inequality shows

$$\begin{aligned}-2\langle W_-E, \mathrm{Ric}\wedge 2\,\mathrm{id}\rangle &= -2\langle W_-(\mathrm{Ric}\wedge 2\,\mathrm{id}), E\rangle \\ &= \langle W(\mathrm{Ric}\wedge 2\,\mathrm{id}), E\rangle - \langle |W|(\mathrm{Ric}\wedge 2\,\mathrm{id}), E\rangle \\ &\le X - \langle \mathrm{Ric}\wedge 2\,\mathrm{id}, E^2\rangle,\end{aligned}$$

which was to be proven. □

The main work will now be to estimate the terms of X. To do this, the next two lemmata will be needed.

Lemma 3.4.9. Let $A \in S^2(\mathbb{R}^n)$ and a positive integer k be given. Then $\left(\mathrm{tr}\,A^k\right)^2 \le \mathrm{tr}\,A^{2(k-1)}\mathrm{tr}\,A^2$. Furthermore, if k is odd, then in addition $\left(\mathrm{tr}\,A^k\right)^2 \le \mathrm{tr}\,A^{k-1}\,\mathrm{tr}\,A^{k+1}$.

Proof. The following computation proves the first inequality by means of the Cauchy-Schwarz inequality:

$$\left(\mathrm{tr}\,A^k\right)^2 = \langle A^{k-1}, A\rangle^2 \le \|A^{k-1}\|^2 \cdot \|A\|^2 = \mathrm{tr}\,A^{2(k-1)}\mathrm{tr}\,A^2.$$

Let k now be odd. Then

$$\left(\mathrm{tr}\,A^k\right)^2 = \langle A^{\frac{k-1}{2}}, A^{\frac{k+1}{2}}\rangle^2 \le \|A^{\frac{k-1}{2}}\|^2 \cdot \|A^{\frac{k+1}{2}}\|^2 = \mathrm{tr}\,A^{k-1}\,\mathrm{tr}\,A^{k+1}$$

again by the Cauchy-Schwarz inequality. This concludes the proof. □

Lemma 3.4.10. Let $A \in S^2(\mathbb{R}^n)$ with $A \ge 0$ be given. Then

$$\begin{aligned}\|(A\wedge A)(A\wedge 2\,\mathrm{id})\|^2 \le &\frac{1}{\|A\wedge A\|^2}\langle (A\wedge A)(A\wedge 2\,\mathrm{id}), A\wedge A\rangle^2 \\ &+ \mathrm{tr}\,A^2\mathrm{tr}\,A^4 - \left(\mathrm{tr}\,A^3\right)^2.\end{aligned}$$

If in addition $\|A\| = 1$, then

$$\|(A\wedge A)(A\wedge 2\,\mathrm{id})\|^2 \le \left(\mathrm{tr}\,A^4 + \left(\mathrm{tr}\,A^3\right)^2\right)\frac{\left(1-\mathrm{tr}\,A^4\right)^2}{1-\left(\mathrm{tr}\,A^3\right)^2}.$$

Proof. Let $a_1, \ldots, a_n$ denote the eigenvalues of A. Then

$$\begin{aligned}
\|(A \wedge A)(A \wedge 2\,\mathrm{id})\|^2 &= \sum_{1 \le i < j \le n} (a_i a_j (a_i + a_j))^2 \\
&= \sum_{1 \le i < j \le n} \left(a_i^4 a_j^2 + 2 a_i^3 a_j^3 + a_i^2 a_j^4\right) \\
&= \sum_{i,j=1}^{n} \left(a_i^4 a_j^2 + a_i^3 a_j^3\right) - 2 \sum_{i=1}^{n} a_i^6 \\
&= \operatorname{tr} A^2 \operatorname{tr} A^4 + \left(\operatorname{tr} A^3\right)^2 - 2 \operatorname{tr} A^6. \qquad (3.7)
\end{aligned}$$

For the first inequality consider the following:

$$\begin{aligned}
\langle (A \wedge A)(A \wedge 2\,\mathrm{id}), A \wedge A \rangle &= \sum_{1 \le i < j \le n} a_i^2 a_j^2 (a_i + a_j) \\
&= \sum_{i,j=1}^{n} a_i^3 a_j^2 - \sum_{i=1}^{n} a_i^5 \\
&= \operatorname{tr} A^2 \operatorname{tr} A^3 - \operatorname{tr} A^5.
\end{aligned}$$

This shows together with lemma 3.3.3 and equation (3.7)

$$\begin{aligned}
T :=& \left(\operatorname{tr} A^2 \operatorname{tr} A^4 - \left(\operatorname{tr} A^3\right)^2 - \|(A \wedge A)(A \wedge 2\,\mathrm{id})\|^2\right) \|A \wedge A\|^2 \\
&+ \langle (A \wedge A)(A \wedge 2\,\mathrm{id}), A \wedge A \rangle^2 \\
=& \left(\operatorname{tr} A^6 - \left(\operatorname{tr} A^3\right)^2\right) \left(\left(\operatorname{tr} A^2\right)^2 - \operatorname{tr} A^4\right) + \left(\operatorname{tr} A^2 \operatorname{tr} A^3 - \operatorname{tr} A^5\right)^2 \\
=& \left(\operatorname{tr} A^2\right)^2 \operatorname{tr} A^6 - \operatorname{tr} A^4 \operatorname{tr} A^6 + \left(\operatorname{tr} A^3\right)^2 \operatorname{tr} A^4 \\
&- 2 \operatorname{tr} A^2 \operatorname{tr} A^3 \operatorname{tr} A^5 + \left(\operatorname{tr} A^5\right)^2 \\
=& \left(\sqrt{\operatorname{tr} A^4 \operatorname{tr} A^6} - \operatorname{tr} A^5\right) \left(2 \operatorname{tr} A^2 \operatorname{tr} A^3 - \sqrt{\operatorname{tr} A^4 \operatorname{tr} A^6} - \operatorname{tr} A^5\right) \\
&+ \left(\operatorname{tr} A^3 \sqrt{\operatorname{tr} A^4} - \operatorname{tr} A^2 \sqrt{\operatorname{tr} A^6}\right)^2.
\end{aligned}$$

The application of lemma 3.4.9 proves

$$\sqrt{\operatorname{tr} A^4 \operatorname{tr} A^6} - \operatorname{tr} A^5 \ge 0$$

and

$$2 \operatorname{tr} A^2 \operatorname{tr} A^3 - \sqrt{\operatorname{tr} A^4 \operatorname{tr} A^6} - \operatorname{tr} A^5 \ge 2 \left(\operatorname{tr} A^2 \operatorname{tr} A^3 - \sqrt{\operatorname{tr} A^4 \operatorname{tr} A^6}\right).$$

Since the eigenvalues $a_1, \ldots, a_n$ of A are all bounded below by 0,

$$\operatorname{tr} A^2 \operatorname{tr} A^3 = \sum_{i,j=1}^{n} a_i^2 a_j^3 \ge \sqrt{\sum_{i,j=1}^{n} a_i^4 a_j^6} = \sqrt{\operatorname{tr} A^4 \operatorname{tr} A^6}.$$

Thus, $T \geq 0$, which proves the first inequality.
Let now $\|A\| = 1$. Consider

$$
\begin{aligned}
&\left(1 - \operatorname{tr} A^4\right)^2 - \left(1 - 2\operatorname{tr} A^4 + \left(\operatorname{tr} A^3\right)^2\right)\left(1 - \left(\operatorname{tr} A^3\right)^2\right) \\
&\quad = \left(\operatorname{tr} A^4\right)^2 - 2\operatorname{tr} A^4 \left(\operatorname{tr} A^3\right)^2 + \left(\operatorname{tr} A^3\right)^4 \\
&\quad = \left(\operatorname{tr} A^4 - \left(\operatorname{tr} A^3\right)^2\right)^2 \\
&\quad \geq 0,
\end{aligned}
$$

which is equivalent to

$$
1 - 2\operatorname{tr} A^4 + \left(\operatorname{tr} A^3\right)^2 \leq \frac{(1 - \operatorname{tr} A^4)^2}{1 - (\operatorname{tr} A^3)^2}. \tag{3.8}
$$

Equation (3.7), inequality (3.8), and lemma 3.4.9 can be combined to

$$
\begin{aligned}
&\left(\operatorname{tr} A^4 + \left(\operatorname{tr} A^3\right)^2\right) \frac{(1 - \operatorname{tr} A^4)^2}{1 - (\operatorname{tr} A^3)^2} - \|(A \wedge A)(A \wedge 2\,\mathrm{id})\|^2 \\
&\geq \left(\operatorname{tr} A^4 + \left(\operatorname{tr} A^3\right)^2\right)\left(1 - 2\operatorname{tr} A^4 + \left(\operatorname{tr} A^3\right)^2\right) \\
&\quad - \operatorname{tr} A^4 - \left(\operatorname{tr} A^3\right)^2 + 2\operatorname{tr} A^6 \\
&= -2\left(\operatorname{tr} A^4\right)^2 - \operatorname{tr} A^4 \left(\operatorname{tr} A^3\right)^2 + \left(\operatorname{tr} A^3\right)^4 + 2\operatorname{tr} A^6 \\
&= 2\operatorname{tr} A^6 - 3\left(\operatorname{tr} A^4\right)^2 + \operatorname{tr} A^4 \left(\operatorname{tr} A^3\right)^2 + \left(\operatorname{tr} A^4 - \left(\operatorname{tr} A^3\right)^2\right)^2 \\
&\geq \operatorname{tr} A^6 - 2\operatorname{tr} A^3 \operatorname{tr} A^5 + \operatorname{tr} A^4 \left(\operatorname{tr} A^3\right)^2 + \left(\operatorname{tr} A^4 - \left(\operatorname{tr} A^3\right)^2\right)^2 \\
&\geq \operatorname{tr} A^6 - 2\operatorname{tr} A^3 \sqrt{\operatorname{tr} A^4 \operatorname{tr} A^6} + \operatorname{tr} A^4 \left(\operatorname{tr} A^3\right)^2 + \left(\operatorname{tr} A^4 - \left(\operatorname{tr} A^3\right)^2\right)^2 \\
&= \left(\sqrt{\operatorname{tr} A^6} - \sqrt{\operatorname{tr} A^4}\operatorname{tr} A^3\right)^2 + \left(\operatorname{tr} A^4 - \left(\operatorname{tr} A^3\right)^2\right)^2 \\
&\geq 0,
\end{aligned}
$$

which proves inequality 2. □

Now, the first term of X can be estimated.

Lemma 3.4.11. Let

$$
\begin{aligned}
Y := \Bigg(& 2\left(1 - e^2\right) r^2 \frac{(1 - \operatorname{tr} S^4)\left(\operatorname{tr} S^4 + (\operatorname{tr} S^3)^2\right)}{1 - (\operatorname{tr} S^3)^2} \\
&+ \operatorname{tr}\left(E^2 \left(\mathrm{Ric} \wedge 2\,\mathrm{id}\right)^2\right)\Bigg)^{1/2}.
\end{aligned}
$$

Then

$$\|W(\mathrm{Ric}\wedge 2\,\mathrm{id})\| \leq Y + \frac{\sqrt{1-e^2}}{Y\|\mathrm{Ric}\wedge\mathrm{Ric}\|}\langle(\mathrm{Ric}\wedge\mathrm{Ric})(\mathrm{Ric}\wedge 2\,\mathrm{id})^2, E\rangle.$$

Proof. Since $\|W(\mathrm{Ric}\wedge 2\,\mathrm{id})\| = \||W|(\mathrm{Ric}\wedge 2\,\mathrm{id})\|$ and

$$|W| = \frac{\sqrt{1-e^2}}{\|\mathrm{Ric}\wedge\mathrm{Ric}\|}\mathrm{Ric}\wedge\mathrm{Ric} + E = \sqrt{2}\frac{\sqrt{1-e^2}}{\sqrt{1-\mathrm{tr}\,S^4}}S\wedge S + E$$

by lemma 3.3.3, it follows that

$$\begin{aligned}\|W(\mathrm{Ric}\wedge 2\,\mathrm{id})\|^2 = {} & \frac{2r^2\left(1-e^2\right)}{1-\mathrm{tr}\,S^4}\|(S\wedge S)(S\wedge 2\,\mathrm{id})\|^2 \\ & + \|E(\mathrm{Ric}\wedge 2\,\mathrm{id})\|^2 \\ & + 2\frac{\sqrt{1-e^2}}{\|\mathrm{Ric}\wedge\mathrm{Ric}\|}\langle E, (\mathrm{Ric}\wedge\mathrm{Ric})(\mathrm{Ric}\wedge 2\,\mathrm{id})^2\rangle.\end{aligned}$$

Since $\|E(\mathrm{Ric}\wedge 2\,\mathrm{id})\|^2 = \mathrm{tr}\,\left(E^2(\mathrm{Ric}\wedge 2\,\mathrm{id})^2\right)$, this shows with lemma 3.4.10

$$\|W(\mathrm{Ric}\wedge 2\,\mathrm{id})\|^2 \leq Y^2 + 2\frac{\sqrt{1-e^2}}{\|\mathrm{Ric}\wedge\mathrm{Ric}\|}\langle E, (\mathrm{Ric}\wedge\mathrm{Ric})(\mathrm{Ric}\wedge 2\,\mathrm{id})^2\rangle.$$

In general it is true that every $a, b \in \mathbb{R}$ with $a \neq 0$ satisfy $a^2 + b \leq \left(a + \frac{b}{2a}\right)^2$, which concludes the proof since obviously $Y \neq 0$. □

To avoid or reduce long and too complex terms the following abbreviation will be used.

Notation 3.4.12. Let $c \in \mathbb{R}$ and $A \in S^2(\mathbb{R}^n)$ with $A \geq 0$ be given . Then $R_c(A)$ will be defined as

$$R_c(A) := (A\wedge A)(A\wedge 2\,\mathrm{id}) - c(A\wedge A)(A\wedge 2\,\mathrm{id})^2.$$

Corollary 3.4.13. Let $h := \frac{e}{Y}$. Then $X \leq eY - \frac{\sqrt{1-e^2}}{\|\mathrm{Ric}\wedge\mathrm{Ric}\|}\langle E, R_h(\mathrm{Ric})\rangle$.

Proof. This follows directly from lemma 3.4.11 and the definitions of X and $R_h(\mathrm{Ric})$. □

The next step will be to analyze and to bound the term $\langle E, R_h(\mathrm{Ric})\rangle$. For this the following lemma will be useful.

Lemma 3.4.14. Let $A \in S^2(\mathbb{R}^n)$ with eigenvalues $a_1, \ldots, a_n \geq 0$, and let $c > 0$ with $c\,(2a_{i_1} + a_{i_2} + a_{i_3}) \leq 2$ for all distinct $1 \leq i_1, i_2, i_3 \leq n$ be given. Then

$$\|R_c(A)\|^2 - \left\langle R_c(A), \frac{A\wedge A}{\|A\wedge A\|}\right\rangle^2 \leq \mathrm{tr}\,A^4\,\mathrm{tr}\,A^2 - \left(\mathrm{tr}\,A^3\right)^2.$$

Proof. Without loss of generality, let $a_1 \geq \cdots \geq a_n$. Using the first estimate of lemma 3.4.10 shows

$$\begin{aligned}
&\|R_c(A)\|^2 - \operatorname{tr} A^4 \operatorname{tr} A^2 + \left(\operatorname{tr} A^3\right)^2 \\
&\quad \leq c^2 \left\langle (A \wedge A)^2, (A \wedge 2\,\mathrm{id})^4 \right\rangle - 2c \left\langle (A \wedge A)^2, (A \wedge 2\,\mathrm{id})^3 \right\rangle \\
&\qquad + \frac{1}{\|A \wedge A\|^2} \left\langle (A \wedge A)^2, A \wedge 2\,\mathrm{id} \right\rangle^2 .
\end{aligned} \tag{3.9}$$

Furthermore,

$$\begin{aligned}
\langle R_c(A), A \wedge A\rangle^2 &= \left\langle (A \wedge A)^2, A \wedge 2\,\mathrm{id} \right\rangle^2 + c^2 \left\langle (A \wedge A)^2, (A \wedge 2\,\mathrm{id})^2 \right\rangle^2 \\
&\quad - 2c \left\langle (A \wedge A)^2, A \wedge 2\,\mathrm{id} \right\rangle \left\langle (A \wedge A)^2, (A \wedge 2\,\mathrm{id})^2 \right\rangle .
\end{aligned} \tag{3.10}$$

To deduce the proof of the lemma from (3.9) and (3.10) the occurring terms will now be compared separately. Consider first

$$\begin{aligned}
&\left\langle (A \wedge A)^2, (A \wedge 2\,\mathrm{id})^3 \right\rangle \|A \wedge A\|^2 \\
&\quad - \left\langle (A \wedge A)^2, A \wedge 2\,\mathrm{id} \right\rangle \left\langle (A \wedge A)^2, (A \wedge 2\,\mathrm{id})^2 \right\rangle \\
&= \sum_{1 \leq i < j \leq n} \sum_{1 \leq k < l \leq n} a_i^2 a_j^2 a_k^2 a_l^2 \left((a_i + a_j)^3 - (a_i + a_j)(a_k + a_l)^2 \right) \\
&= \frac{1}{2} \sum_{1 \leq i < j \leq n} \sum_{1 \leq k < l \leq n} a_i^2 a_j^2 a_k^2 a_l^2 \Big((a_i + a_j)^3 + (a_k + a_l)^3 \\
&\quad - (a_i + a_j)(a_k + a_l)^2 - (a_i + a_j)^2 (a_k + a_l) \Big) \\
&= \frac{1}{2} \sum_{1 \leq i < j \leq n} \sum_{1 \leq k < l \leq n} a_i^2 a_j^2 a_k^2 a_l^2 (a_i + a_j + a_k + a_l)(a_i + a_j - a_k - a_l)^2 .
\end{aligned}$$

The last step is true since in general $x^3 + y^3 - xy^2 - x^2 y = (x + y)(x - y)^2$ for all $x, y \in \mathbb{R}$. Moreover,

$$\begin{aligned}
&\left\langle (A \wedge A)^2, (A \wedge 2\,\mathrm{id})^4 \right\rangle \|A \wedge A\|^2 - \left\langle (A \wedge A)^2, (A \wedge 2\,\mathrm{id})^2 \right\rangle^2 \\
&= \sum_{1 \leq i < j \leq n} \sum_{1 \leq k < l \leq n} a_i^2 a_j^2 a_k^2 a_l^2 (a_i + a_j)^4 - \left(\sum_{1 \leq i < j \leq n} a_i^2 a_j^2 (a_i + a_j)^2 \right)^2 \\
&= \frac{1}{2} \sum_{1 \leq i < j \leq n} \sum_{1 \leq k < l \leq n} a_i^2 a_j^2 a_k^2 a_l^2 \Big((a_i + a_j)^4 + (a_k + a_l)^4 \\
&\quad - 2(a_i + a_j)^2 (a_k + a_l)^2 \Big) \\
&= \frac{1}{2} \sum_{1 \leq i < j \leq n} \sum_{1 \leq k < l \leq n} a_i^2 a_j^2 a_k^2 a_l^2 (a_i + a_j + a_k + a_l)^2 (a_i + a_j - a_k - a_l)^2
\end{aligned}$$

since in general $x^4+y^4-2x^2y^2=(x+y)^2(x-y)^2$ for all $x,y\in\mathbb{R}$. Therefore,

$$\begin{aligned}&\frac{\|A\wedge A\|^2}{c}\left(\|R_c(A)\|^2-\left\langle R_c(A),\frac{A\wedge A}{\|A\wedge A\|^2}\right\rangle^2-\operatorname{tr}A^4\operatorname{tr}A^2+\left(\operatorname{tr}A^3\right)^2\right)\\&\leq\frac{1}{2}\sum_{1\leq i<j\leq n}\sum_{1\leq k<l\leq n}a_i^2a_j^2a_k^2a_l^2(a_i+a_j+a_k+a_l)(a_i+a_j-a_k-a_l)^2\\&\quad(-2+c(a_i+a_j+a_k+a_l))\\&\leq 0.\end{aligned}$$

The last step is true because of the following: $a_i\geq 0$ for all $1\leq i\leq n$. Further, by assumption,

$$-2+c(a_i+a_j+a_k+a_l)\leq -2+c(2a_1+a_2+a_3)\leq 0$$

for all $1\leq i<j\leq n$ and all $1\leq k<l\leq n$ if not $i=k=1$ and $j=l=2$ since $a_1\geq\cdots\geq a_n$ for all $1\leq i\leq n$. However, if $i=k$ and $j=l$, then $a_i+a_j-a_k-a_l=0$. Hence, the statement is proven. □

The term $\langle E,R_h(\mathrm{Ric})\rangle$ can now be estimated by means of the above lemma as follows.

Corollary 3.4.15. Let assumptions 1 and 2 hold. Then

$$-\langle E,R_h(\mathrm{Ric})\rangle\leq r^3e\sqrt{\operatorname{tr}S^4-(\operatorname{tr}S^3)^2}.$$

Proof. Without loss of generality, $\lambda_1\geq\cdots\geq\lambda_n$ can be assumed. The goal is now to show $h(2\lambda_1+\lambda_2+\lambda_3)\leq 2$ to apply lemma 3.4.14.
By definition of h,

$$h\leq\frac{e\sqrt{1-(\operatorname{tr}S^3)^2}}{r\sqrt{2\left(1-e^2\right)\left(1-\operatorname{tr}S^4\right)\left(\operatorname{tr}S^4+(\operatorname{tr}S^3)^2\right)}}.$$

Moreover, $\frac{1}{n}\leq\left(\operatorname{tr}S^3\right)^2\leq\operatorname{tr}S^4\leq\frac{2}{n}$ by the lemmata 3.3.2 and 3.3.6 and $e^2\leq\frac{3}{2n}$ by lemma 3.3.8. Therefore,

$$\begin{aligned}1-\left(\operatorname{tr}S^3\right)^2&\leq\frac{n-1}{n}, & e&\leq\sqrt{\frac{3}{2n}},\\ 1-\operatorname{tr}S^4&\geq\frac{n-2}{n}, & 1-e^2&\geq\frac{2n-3}{2n},\quad\text{and}\\ \operatorname{tr}S^4+\left(\operatorname{tr}S^3\right)^2&\geq\frac{2}{n}.\end{aligned}$$

Thus, $h \leq \frac{\sqrt{3}}{2r}\sqrt{\frac{n(n-1)}{(2n-3)(n-2)}}$. Furthermore,

$$\begin{aligned}6r^2 &\geq 6\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)\\ &= (2\lambda_1+\lambda_2+\lambda_3)^2 + 2\lambda_1^2 + 5\lambda_2^2 + 5\lambda_3^2 - 4\lambda_1\lambda_2 - 4\lambda_1\lambda_3 - 2\lambda_2\lambda_3\\ &= (2\lambda_1+\lambda_2+\lambda_3)^2 + (\lambda_1-2\lambda_2)^2 + (\lambda_1-2\lambda_3)^2 + (\lambda_2-\lambda_3)^2,\end{aligned}$$

and hence, $2\lambda_1+\lambda_2+\lambda_3 \leq \sqrt{6}r$. Therefore,

$$h\left(2\lambda_1+\lambda_2+\lambda_3\right) \leq \frac{3}{\sqrt{2}}\sqrt{\frac{n(n-1)}{(2n-3)(n-2)}}.$$

Because of

$$8(2n-3)(n-2) - 9n(n-1) = 7n^2 - 47n + 48 \geq 0$$

for all $n \geq 6$, $h\left(2\lambda_1+\lambda_2+\lambda_3\right) \leq 2$ follows.
Let $R_h(\mathrm{Ric})_{\langle S\wedge S\rangle^\perp}$ be the projection of $R_h(\mathrm{Ric})$ to the orthogonal complement of $S\wedge S$. Then, by lemma 3.4.14 and since $E \perp S\wedge S$, $e = \|E\|$, and $\frac{S\wedge S}{\|S\wedge S\|} = \frac{\mathrm{Ric}\wedge\mathrm{Ric}}{\|\mathrm{Ric}\wedge\mathrm{Ric}\|}$,

$$\begin{aligned}-\left\langle \frac{E}{e}, R_h(\mathrm{Ric})\right\rangle &\leq \left\langle \frac{R_h(\mathrm{Ric})_{\langle S\wedge S\rangle^\perp}}{\|R_h(\mathrm{Ric})_{\langle S\wedge S\rangle^\perp}\|}, R_h(\mathrm{Ric})\right\rangle\\ &= \sqrt{\|R_h(\mathrm{Ric})\|^2 - \left\langle R_h(\mathrm{Ric}), \frac{\mathrm{Ric}\wedge\mathrm{Ric}}{\|\mathrm{Ric}\wedge\mathrm{Ric}\|}\right\rangle^2}\\ &\leq \sqrt{r^2 \mathrm{tr}\,\mathrm{Ric}^4 - \left(\mathrm{tr}\,\mathrm{Ric}^3\right)^2}\\ &= r^3\sqrt{\mathrm{tr}\,S^4 - (\mathrm{tr}\,S^3)^2}.\end{aligned}$$

This concludes the proof. □

Now, a proof of lemma 3.4.7 can be given.

Proof of lemma 3.4.7. The combination of the definition of Y, lemma 3.3.3, and the corollaries 3.4.13 and 3.4.15 shows

$$\begin{aligned}\frac{X}{\sqrt{1-e^2}} \leq& -\frac{1}{\|\mathrm{Ric}\wedge\mathrm{Ric}\|}\langle E, R_h(\mathrm{Ric})\rangle + \frac{eY}{\sqrt{1-e^2}}\\ \leq&\sqrt{2}re\left(\sqrt{\frac{\mathrm{tr}\,S^4 - (\mathrm{tr}\,S^3)^2}{1-\mathrm{tr}\,S^4}}\right.\\ &\left.+\sqrt{\frac{(1-\mathrm{tr}\,S^4)\left(\mathrm{tr}\,S^4 + (\mathrm{tr}\,S^3)^2\right)}{1-(\mathrm{tr}\,S^3)^2} + \frac{\mathrm{tr}\left(E^2(\mathrm{Ric}\wedge 2\,\mathrm{id})^2\right)}{2r^2\left(1-e^2\right)}}\right).\end{aligned}$$

Moreover, it is easy to see that $2xy \leq ax^2 + \frac{y^2}{a}$ for all $a, x, y \in \mathbb{R}$ with $a > 0$. This will be applied to

$$\begin{aligned} a :=& \frac{\tilde{\theta}_0(n)}{\sqrt{1-e^2}}, \\ x :=& \frac{er}{\sqrt{2}}, \qquad \text{and} \\ y :=& \sqrt{\frac{(1-\operatorname{tr} S^4)\left(\operatorname{tr} S^4 + (\operatorname{tr} S^3)^2\right)}{1-(\operatorname{tr} S^3)^2} + \frac{\operatorname{tr}\,(E^2(\operatorname{Ric}\wedge 2\operatorname{id})^2)}{2r^2(1-e^2)}} \\ &+ \sqrt{\frac{\operatorname{tr} S^4 - (\operatorname{tr} S^3)^2}{1-\operatorname{tr} S^4}}. \end{aligned}$$

However, before doing so, it will be useful to show that

$$\frac{(1-\operatorname{tr} S^4)\left(\operatorname{tr} S^4 + (\operatorname{tr} S^3)^2\right)}{1-(\operatorname{tr} S^3)^2} + \frac{\operatorname{tr} S^4 - (\operatorname{tr} S^3)^2}{1-\operatorname{tr} S^4} \leq 2\operatorname{tr} S^4.$$

Therefore, consider

$$\begin{aligned} &\left(1-\operatorname{tr} S^4\right)^2\left(\operatorname{tr} S^4 + \left(\operatorname{tr} S^3\right)^2\right) + \left(\operatorname{tr} S^4 - \left(\operatorname{tr} S^3\right)^2\right)\left(1-\left(\operatorname{tr} S^3\right)^2\right) \\ &\qquad - 2\operatorname{tr} S^4\left(1-\left(\operatorname{tr} S^3\right)^2\right)\left(1-\operatorname{tr} S^4\right) \\ &\quad = -\operatorname{tr} S^4\left(\operatorname{tr} S^3\right)^2 + \left(\operatorname{tr} S^3\right)^4 + \left(\operatorname{tr} S^4\right)^3 - \left(\operatorname{tr} S^4\right)^2\left(\operatorname{tr} S^3\right)^2 \\ &\quad = \left(\left(\operatorname{tr} S^4\right)^2 - \left(\operatorname{tr} S^3\right)^2\right)\left(\operatorname{tr} S^4 - \left(\operatorname{tr} S^3\right)^2\right) \\ &\quad \leq 0, \end{aligned}$$

where the last step is a consequence of lemma 3.3.2. Combining everything shows

$$
\begin{aligned}
&\frac{X}{\sqrt{1-e^2}} \\
&\leq \frac{e^2 r^2 \tilde{\theta}_0(n)}{2\sqrt{1-e^2}} + \frac{\operatorname{tr}\left(E^2(\operatorname{Ric}\wedge 2\,\mathrm{id})^2\right)}{2\tilde{\theta}_0(n) r^2 \sqrt{1-e^2}} + \frac{2\sqrt{1-e^2}}{\tilde{\theta}_0(n)} \left(\operatorname{tr} S^4 \right. \\
&\quad \left. + \sqrt{\operatorname{tr} S^4 - (\operatorname{tr} S^3)^2} \sqrt{\frac{\operatorname{tr} S^4 + (\operatorname{tr} S^3)^2}{1-(\operatorname{tr} S^3)^2} + \frac{\operatorname{tr}\left(E^2(\operatorname{Ric}\wedge 2\,\mathrm{id})^2\right)}{2r^2\left(1-e^2\right)\left(1-\operatorname{tr} S^4\right)}} \right) \\
&\leq \frac{\operatorname{tr}\left(E^2(\operatorname{Ric}\wedge 2\,\mathrm{id})^2\right)}{2\tilde{\theta}_0(n) r^2 \sqrt{1-e^2}} \left(1 + \sqrt{\frac{\left(\operatorname{tr} S^4 - (\operatorname{tr} S^3)^2\right)\left(1-(\operatorname{tr} S^3)^2\right)}{(1-\operatorname{tr} S^4)^2 \left(\operatorname{tr} S^4 + (\operatorname{tr} S^3)^2\right)}} \right) \\
&\quad + \frac{e^2 r^2 \tilde{\theta}_0(n)}{2\sqrt{1-e^2}} + \frac{2\sqrt{1-e^2}}{\tilde{\theta}_0(n)} \left(\operatorname{tr} S^4 + \sqrt{\frac{(\operatorname{tr} S^4)^2 - (\operatorname{tr} S^3)^4}{1-(\operatorname{tr} S^3)^2}} \right).
\end{aligned}
$$

The last inequality is true since $\sqrt{x+y} \leq \sqrt{x} + \frac{y}{2\sqrt{x}}$ for arbitrary $x, y \in \mathbb{R}$ with $x, y > 0$. Furthermore,

$$
\begin{aligned}
\frac{\left(\operatorname{tr} S^4 - (\operatorname{tr} S^3)^2\right)\left(1-(\operatorname{tr} S^3)^2\right)}{(1-\operatorname{tr} S^4)^2 \left(\operatorname{tr} S^4 + (\operatorname{tr} S^3)^2\right)} &\leq \frac{\left(\frac{2}{n} - \frac{n-2}{(n-3)^2}\right)\left(1 - \frac{n-2}{(n-3)^2}\right)}{2\left(1-\frac{2}{n}\right)^2 \frac{n-2}{(n-3)^2}} \\
&= \frac{n\left(n^2-10n+18\right)\left((n-3)^2-(n-2)\right)}{2(n-2)^3(n-3)^2} \\
&\leq \frac{1}{2}
\end{aligned}
$$

by the lemmata 3.3.2 and 3.3.6 and because $n(n^2-10n+18) \geq 0$ for $n \geq 8$ as well as $(n-2)^3 - n(n^2-10n+18) = 4n^2 - 6n - 8 \geq 0$ for $n \geq 3$. Since $\lambda_1 \geq \cdots \geq \lambda_n$, $\operatorname{Ric}\wedge 2\,\mathrm{id} \leq (\lambda_1+\lambda_2)I \leq \operatorname{scal} I$. Therefore,

$$
\begin{aligned}
&\frac{\operatorname{tr}\left(E^2(\operatorname{Ric}\wedge 2\,\mathrm{id})^2\right)}{2\tilde{\theta}_0(n) r^2} \left(1 + \sqrt{\frac{\left(\operatorname{tr} S^4 - (\operatorname{tr} S^3)^2\right)\left(1-(\operatorname{tr} S^3)^2\right)}{(1-\operatorname{tr} S^4)^2 \left(\operatorname{tr} S^4 + (\operatorname{tr} S^3)^2\right)}} \right) \\
&\leq \frac{\operatorname{tr}\ \left(E^2(\operatorname{Ric}\wedge 2\,\mathrm{id})\right)\operatorname{scal}}{2r^2\tilde{\theta}_0(n)} \left(1 + \frac{\sqrt{2}}{2}\right) \\
&\leq \frac{1}{2}\frac{n+1}{n-1} \operatorname{tr}\left(E^2(\operatorname{Ric}\wedge 2\,\mathrm{id})\right) \\
&\leq \operatorname{tr}\bigl(E^2(\operatorname{Ric}\wedge 2\,\mathrm{id})\bigr).
\end{aligned}
$$

Here, the penultimate inequality follows from

$$
\frac{\operatorname{scal}}{r^2 \tilde{\theta}_0(n)} \leq \frac{n-3}{(n-2)\tilde{\theta}_0(n)^2} = \frac{n+1}{2(n-1)}
$$

by lemma 3.3.1. The last inequality is true since $\frac{1}{2}\frac{n+1}{n-1} \leq 1$ for all $n \geq 3$. Everything together shows now

$$X \leq \frac{e^2 r^2 \tilde{\theta}_0(n)}{2} + \frac{2\left(1-e^2\right)}{\tilde{\theta}_0(n)} \left(\operatorname{tr} S^4 + \sqrt{\frac{\left(\operatorname{tr} S^4\right)^2 - \left(\operatorname{tr} S^3\right)^4}{1-\left(\operatorname{tr} S^3\right)^2}}\right)$$
$$+ \operatorname{tr}\left(E^2(\operatorname{Ric} \wedge 2\operatorname{id})\right).$$

Since $\operatorname{tr}\left(E^2(\operatorname{Ric} \wedge 2\operatorname{id})\right) = \langle \operatorname{Ric} \wedge 2\operatorname{id}, E^2 \rangle$, $\frac{e^2}{2} \leq 1 - \sqrt{1-e^2}$, and $(1-e^2) \leq \sqrt{1-e^2}$, the claim follows from lemma 3.4.8. □

3.5 The scalar curvature cannot be too small

In this part, the contradiction for assumption 2 will be constructed. The main tool for this step in the proof of theorem 3.0.2 will be the following result, which in fact was proven in the three previous sections. In the whole chapter notation 3.0.3 will be used, and assumption 1 will be supposed. Of course, assumption 2 will only assumed to be true until the contradiction is constructed.

Corollary 3.5.1. Let assumptions 1 and 2 hold. Then

$$\tilde{\theta}_0(n) r^2 - \sqrt{2} r^2 \sqrt{1 - \operatorname{tr} S^4} + \frac{n\bar{\lambda}^4 + \frac{23}{12}\bar{\lambda}^2 r_0^2 - \frac{13}{12}\bar{\lambda} \operatorname{tr} \operatorname{Ric}_0^3}{\|\operatorname{Ric} \wedge \operatorname{Ric}\|}$$
$$\leq \frac{2}{\tilde{\theta}_0(n)} \left(\operatorname{tr} S^4 + \sqrt{\frac{\left(\operatorname{tr} S^4\right)^2 - \left(\operatorname{tr} S^3\right)^4}{1-\left(\operatorname{tr} S^3\right)^2}}\right).$$

Proof. Since $\frac{1}{12} \geq \frac{\sqrt{10}-3}{2}$, lemma 3.4.3 can be applied with $\epsilon = \frac{1}{12}$. The combination of the lemmata 3.4.2, 3.4.3, and 3.4.7 shows now

$$- 2\langle \operatorname{Ric}(W_-^2), \operatorname{Ric} \rangle$$
$$\leq -\frac{\sqrt{1-e^2}}{\|\operatorname{Ric} \wedge \operatorname{Ric}\|} \left(n\bar{\lambda}^4 + \frac{23}{12}\bar{\lambda}^2 r_0^2 - \frac{13}{12}\bar{\lambda} \operatorname{tr} \operatorname{Ric}_0^3\right)$$
$$+ \frac{2\sqrt{1-e^2}}{\tilde{\theta}_0(n)} \left(\operatorname{tr} S^4 + \sqrt{\frac{\left(\operatorname{tr} S^4\right)^2 - \left(\operatorname{tr} S^3\right)^4}{1-\left(\operatorname{tr} S^3\right)^2}}\right)$$
$$+ \tilde{\theta}_0(n) r^2 \left(1 - \sqrt{1-e^2}\right).$$

Moreover, in lemma 3.3.4 it was shown that

$$\tilde{\theta}_0(n) r^2 \leq \sqrt{2} r^2 \sqrt{1 - \operatorname{tr} S^4} \sqrt{1-e^2} - 2\langle \operatorname{Ric}(W_-^2), \operatorname{Ric} \rangle$$

since $\tilde{\theta}_0(n) \leq \theta$. The combination of both inequalities divided by $\sqrt{1-e^2}$ yields what was claimed. □

The rough idea for the construction of the contradiction to assumption 2 will be to use corollary 3.5.1 for deducing a convex function on a compact interval with endpoints of the interval equal to the bounds of $n\bar{\lambda}$ given by corollary 3.3.7 and assumption 2 which is nonnegative in $n\bar{\lambda}$. It will then be shown that this function is negative at the boundary points, which is a contradiction. In order to prepare the construction of this function, the following results will be needed.

Lemma 3.5.2. The global minimum of the function

$$g\colon (0,1]\times[-1,0]\to\mathbb{R}; (x,y)\mapsto \frac{6x^2-6x+1}{2x^2}y^2+\frac{11}{12}\frac{2x-1}{x}y+\frac{23}{12}$$

is $\frac{263}{288}$.

Proof. Assume that g has a critical point $(x,y)\in(0,1)\times(-1,0)$ in the interior. Then

$$\nabla g(x,y)=\begin{pmatrix}(3x-1)\frac{y^2}{x^3}+\frac{11}{12}\frac{y}{x^2}\\ (6x^2-6x+1)\frac{y}{x^2}+\frac{11}{12x}(2x-1)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix}(3x-1)y+\frac{11}{12}x\\ (6x^2-6x+1)y+\frac{11}{12}(2x^2-x)\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}.$$

Since $x\neq\frac{1}{3}$, because otherwise $\frac{11}{36}=0$ by the first equation, $y=\frac{11x}{12(1-3x)}$. Using the second equation shows

$$0=6x^2-6x+1+(2x-1)(1-3x)=-x,$$

which is a contradiction. Thus, g has no critical point in the interior. Consider now the boundary functions

$$G_1\colon (0,1]\to\mathbb{R}; x\mapsto g(x,-1)=\frac{37x^2-25x+6}{12x^2}\quad\text{and}$$
$$G_2\colon [-1,0]\to\mathbb{R}; y\mapsto g(1,y)=\frac{1}{2}y^2+\frac{11}{12}y+\frac{23}{12}.$$

Then $G_1'(x)=\frac{25}{12x^2}-\frac{1}{x^3}$ for all $x\in(0,1)$, which vanishes if and only if $x=\frac{12}{25}$. Further $G_1''\left(\frac{12}{25}\right)=\left(\frac{25}{12}\right)^4>0$. Since $\lim_{x\searrow 0}G_1(x)=\infty$ and $G_1(1)=\frac{3}{2}$, G_1 attains its global minimum in $\frac{12}{25}$ with a value of

$$G_1\left(\frac{12}{25}\right)=\frac{37}{12}-\frac{1}{2}\left(\frac{25}{12}\right)^2=\frac{263}{288}.$$

Moreover, $G_2(y) = \frac{1}{2}\left(y + \frac{11}{12}\right)^2 + \frac{431}{288} \geq \frac{431}{288}$ for all $y \in [-1, 0]$.
Finally, let $(x_k, y_k)_{k\in\mathbb{N}}$ be a convergent sequence in $(0, 1] \times [-1, 0]$ with

$$\lim_{k\to\infty} x_k = 0 \quad \text{and} \quad \lim_{k\to\infty} y_k =: y_\infty \in [-1, 0].$$

If $y_\infty = 0$, then $\frac{y_k}{x_k} \leq 0$ for all $k \in \mathbb{N}$ and $-3y_k - \frac{11}{12} \leq 0$ for k large enough. Hence,

$$g(x_k, y_k) = 3y_k^2 + \frac{11}{6}y_k + \frac{23}{12} + \frac{y_k}{x_k}\left(\frac{y_k}{2x_k} - 3y_k - \frac{11}{12}\right) \geq \frac{23}{12}$$

for k large enough. Otherwise $\lim_{k\to\infty} \frac{y_k}{x_k} = -\infty$, and thus,

$$\lim_{k\to\infty} g(x_k, y_k) = \lim_{k\to\infty} 3y_k^2 + \frac{11}{6}y_k + \frac{23}{12} + \frac{y_k}{x_k}\left(\frac{y_k}{2x_k} - 3y_k - \frac{11}{12}\right) = \infty.$$

Therefore, the absolute minimum of g is $\frac{263}{288}$. □

By means of the lemma above the following result can be proven.

Lemma 3.5.3. Let assumptions 1 and 2 hold, and let $d \in \mathbb{R}$ with $d \leq \frac{263}{288}$ be given. Then

$$\frac{1}{2}\operatorname{tr}\operatorname{Ric}^4 + n\bar{\lambda}^4 + \left(\frac{23}{12} - d\right) r_0^2\bar{\lambda}^2 - \frac{13}{12}\bar{\lambda}\operatorname{tr}\operatorname{Ric}_0^3 - \frac{3}{2n}r^4 \geq 0.$$

Proof. Without loss of generality, let $\mu_1 < \cdots < \mu_l$ for $1 \leq l \leq n$ be all different eigenvalues of Ric_0, and let $\hat{x}_i$ for all $1 \leq i \leq l$ be the dimension of the eigenspace of μ_i. Since the statement of the lemma is invariant under scaling, $\bar{\lambda} = 1$ can be assumed. Then $\mu_i = \lambda_i - \bar{\lambda} \geq -1$ for every $1 \leq i \leq n$, and therefore, $-1 \leq \mu_1 < \cdots < \mu_l \leq n - 1$. Moreover,

$$\begin{aligned} r^4 &= r_0^4 + 2nr_0^2\bar{\lambda}^2 + n^2\bar{\lambda}^4 \quad \text{and} \\ \operatorname{tr}\operatorname{Ric}^4 &= \operatorname{tr}\operatorname{Ric}_0^4 + 4\bar{\lambda}\operatorname{tr}\operatorname{Ric}_0^3 + 6\bar{\lambda}^2 r_0^2 + n\bar{\lambda}^4. \end{aligned}$$

Therefore,

$$\begin{aligned} &\frac{1}{n}\left(\frac{1}{2}\operatorname{tr}\operatorname{Ric}^4 + n\bar{\lambda}^4 + \left(\frac{23}{12} - d\right) r_0^2\bar{\lambda}^2 - \frac{13}{12}\bar{\lambda}\operatorname{tr}\operatorname{Ric}_0^3 - \frac{3}{2n}r^4\right) \\ &\quad = \frac{1}{n}\left(\frac{1}{2}\operatorname{tr}\operatorname{Ric}_0^4 + \frac{11}{12}\operatorname{tr}\operatorname{Ric}_0^3 + \left(\frac{23}{12} - d\right) r_0^2 - \frac{3}{2n}r_0^4\right) \\ &\quad = \frac{1}{2}\sum_{i=1}^{l}\frac{\hat{x}_i}{n}\mu_i^4 + \frac{11}{12}\sum_{i=1}^{l}\frac{\hat{x}_i}{n}\mu_i^3 + \left(\frac{23}{12} - d\right)\sum_{i=1}^{l}\frac{\hat{x}_i}{n}\mu_i^2 - \frac{3}{2}\left(\sum_{i=1}^{l}\frac{\hat{x}_i}{n}\mu_i^2\right)^2. \end{aligned}$$

Consider now the set

$$K := \bigcup_{k=1}^{n} \left\{ (x_1, \ldots, x_k, y_1, \ldots, y_k) \in \mathbb{R}^{2k} \,\middle|\, x_1, \ldots, x_k \in [0,1], \right.$$
$$\left. -1 \leq y_1 \leq \cdots \leq y_k \leq n-1, \sum_{i=1}^{k} x_i = 1, \sum_{i=1}^{k} x_i y_i = 0 \right\}.$$

Since K is compact, the continuous function $f\colon K \to \mathbb{R}$ given by

$$f(x_1, \ldots, x_k, y_1, \ldots, y_k) := \frac{1}{2}\sum_{i=1}^{k} x_i y_i^4 + \frac{11}{12}\sum_{i=1}^{k} x_i y_i^3$$
$$+ \left(\frac{23}{12} - d\right)\sum_{i=1}^{k} x_i y_i^2 - \frac{3}{2}\left(\sum_{i=1}^{k} x_i y_i^2\right)^2$$

has a global minimum. Let M be this minimum. In order to prove the lemma, it is sufficient to show that $M \geq 0$ since $\left(\frac{\hat{x}_1}{n}, \ldots, \frac{\hat{x}_l}{n}, \mu_1, \ldots, \mu_l\right) \in K$. Assume $M < 0$. Let $1 \leq k \leq n$ and $(x_1, \ldots, x_k, y_1, \ldots, y_k) \in K$ be chosen such that $f(x_1, \ldots, x_k, y_1, \ldots, y_k) = M$ and such that if $1 \leq k' < k$ and $(\tilde{x}_1, \ldots, \tilde{x}_{k'}, \tilde{y}_1, \ldots, \tilde{y}_{k'}) \in K$, then $f(\tilde{x}_1, \ldots, \tilde{x}_{k'}, \tilde{y}_1, \ldots, \tilde{y}_{k'}) > M$.
In particular, it is easy to see that if $k > 1$, then $x_i \neq 0$ and $y_i \neq 0$ for all $1 \leq i \leq k$ and $y_1 < \cdots < y_k$.
Moreover, $k = 1$ is impossible since then $x_1 y_1 = 0$ by the definition of K and $f(x_1, y_1) = 0$, which would contradict the assumption $M < 0$.
For now, assume $k \geq 3$. Then there exist $a_1, \ldots, a_k$ with $a_i \neq 0$ for at least one $1 \leq i \leq k$ such that $\sum_{i=1}^{k} a_i = 0$ and $\sum_{i=1}^{k} a_i y_i = 0$. Set

$$t_{\min} := \max_{1 \leq i \leq k \,|\, a_i > 0} -\frac{x_i}{a_i} < 0 \quad \text{and} \quad t_{\max} := \min_{1 \leq i \leq k \,|\, a_i < 0} -\frac{x_i}{a_i} > 0,$$

and define $x_i(t) := x_i + t a_i$ for $t \in [t_{\min}, t_{\max}]$. Then

$$x_i(t) \geq 0 \text{ for all } 1 \leq i \leq k,$$
$$\sum_{i=1}^{k} x_i(t) = \sum_{i=1}^{k} x_i + t\sum_{i=1}^{k} a_i = 1, \text{ and hence, } x_i(t) \leq 1, \text{ and}$$
$$\sum_{i=1}^{k} x_i(t) y_i = \sum_{i=1}^{k} x_i y_i + t\sum_{i=1}^{k} a_i y_i = 0$$

for all $t \in [t_{\min}, t_{\max}]$. Thus, $(x_1(t), \ldots, x_k(t), y_1, \ldots, y_k) \in K$ for all $t \in [t_{\min}, t_{\max}]$. Define now

$$\tilde{f}\colon [t_{\min}, t_{\max}] \to \mathbb{R}; t \mapsto f(x_1(t), \ldots, x_k(t), y_1, \ldots, y_k).$$

$\tilde{f}$ attains its minimum in $t = 0$ because f does so in $(x_1, \dots, x_k, y_1, \dots, y_k)$. Since

$$\tilde{f}''(t) = -3 \left(\sum_{i=1}^{k} x_i y_i^2 \right)^2 \leq 0$$

for all $t \in (t_{\min}, t_{\max})$, $\tilde{f}$ is concave. Hence, $\tilde{f}$ has to be constant, which contradicts the choice of k because $x_i(t_{\min}) = 0$ for at least one $1 \leq i \leq k$. Thus, $k = 2$. Since $(x_1, x_2, y_1, y_2) \in K$ and $x_1, x_2 \in (0, 1)$, it follows that $x_1 = 1 - x_2$ and $y_2 = -\frac{1-x_2}{x_2} y_1$. Hence, by assumption,

$$\begin{aligned}
0 >& f(x_1, x_2, y_1, y_2) \\
=& (1 - x_2) y_1^2 \left(\frac{1}{2} \left(1 + \frac{(1-x_2)^3}{x_2^3} \right) y_1^2 + \frac{11}{12} \left(1 - \frac{(1-x_2)^2}{x_2^2} \right) y_1 \right. \\
&\left. + \left(\frac{23}{12} - d \right) \left(1 + \frac{1-x_2}{x_2} \right) - \frac{3}{2} (1 - x_2) \left(1 + \frac{1-x_2}{x_2} \right)^2 y_1^2 \right) \\
=& \frac{1-x_2}{x_2} y_1^2 \left(\frac{6x_2^2 - 6x_2 + 1}{2x_2^2} y_1^2 + \frac{11}{12} \frac{2x_2 - 1}{x_2} y_1 + \frac{23}{12} - d \right),
\end{aligned}$$

which is equivalent to

$$d > \frac{6x_2^2 - 6x_2 + 1}{2x_2^2} y_1^2 + \frac{11}{12} \frac{2x_2 - 1}{x_2} y_1 + \frac{23}{12}.$$

This contradicts the precondition on d since $d > \frac{263}{288}$ by lemma 3.5.2. Thus, $M \geq 0$, and the lemma is proven. □

For the preparation of the next step in the construction of the contradiction the following lemma will be useful.

Lemma 3.5.4. Let $k \geq 12$, $Y \geq -1 + \frac{3}{k}$, and $x \in [0, \frac{k-2}{k}]$ with

$$\frac{k^2 - 7k + 8}{k^2(k+1)(k-2)} + \frac{k-4}{k^2(k-2)} x - \frac{1}{4k^2} x^2 \leq \frac{2(k-2) - x}{k} Y + Y^2$$

be given. Then

$$Y \geq \frac{2k - 9}{4k^3} + \frac{3x}{8k^2}.$$

Proof. Let $Y_0 := \frac{2k-9}{4k^3} + \frac{3x}{8k^2}$. Then

$$\begin{aligned}
&\frac{2(k-2) - x}{k} Y_0 + Y_0^2 - \frac{k^2 - 7k + 8}{k^2(k+1)(k-2)} - \frac{k-4}{k^2(k-2)} x + \frac{1}{4k^2} x^2 \\
&= \frac{16k^2 - 24k + 9}{64k^4} x^2 - \frac{4k^4 - 8k^3 - 106k^2 + 111k - 54}{16k^5(k-2)} x \\
&\quad - \frac{8k^5 - 92k^4 - 24k^3 + 179k^2 + 9k + 162}{16k^6(k+1)(k-2)}.
\end{aligned}$$

Consider the function

$$f\colon \mathbb{R} \to \mathbb{R}; k \mapsto -\Big(8k^5 - 92k^4 - 24k^3 + 179k^2 + 9k + 162\Big).$$

Since $f(-1) = -256$, $f(0) = -162$, $f(1) = -242$, $f(2) = 512$, and $f(12) = -67518$, all critical points of f are contained in $(-\infty, 12)$. Thus, f restricted to $[12, \infty)$ has to be monotonically decreasing. Since $f(12) < 0$,

$$-\frac{8k^5 - 92k^4 - 24k^3 + 179k^2 + 9k + 162}{16k^6(k+1)(k-2)} \leq 0$$

for every $k \geq 12$.
Because of $\frac{16k^2-24k+9}{64k^4} = \frac{(4k-3)^2}{64k^4} \geq 0$ for all $k \neq 0$ and $0 \leq x \leq \frac{k-2}{k}$,

$$\begin{aligned}
&\frac{16k^2 - 24k + 9}{64k^4}x - \frac{4k^4 - 8k^3 - 106k^2 + 111k - 54}{16k^5(k-2)} \\
&\quad \leq \frac{16k^2 - 24k + 9}{64k^4}\frac{k-2}{k} - \frac{4k^4 - 8k^3 - 106k^2 + 111k - 54}{16k^5(k-2)} \\
&\quad = -\frac{56k^3 - 593k^2 + 576k - 252}{64k^5(k-2)}.
\end{aligned}$$

Define now

$$g\colon \mathbb{R} \to \mathbb{R}; k \mapsto -\Big(56k^3 - 593k^2 + 576k - 252\Big).$$

Then $g(10) = -2208$ and $g'(k) = -168k^2 + 1186k - 576 \leq g'(10) = -5516$ for all $k \geq 10$, which shows that g restricted to $[10, \infty)$ is nonpositve. Therefore,

$$\frac{16k^2 - 24k + 9}{64k^4}x^2 - \frac{4k^4 - 8k^3 - 106k^2 + 111k - 54}{16k^5(k-2)}x \leq 0$$

for all $k \geq 12$ and $x \in [0, \frac{k-2}{k}]$.
Further, it is easy to see that the map $Y \mapsto \frac{2(k-2)-x}{k}Y + Y^2$ is strictly monotonically increasing on $\Big[-1 + \frac{2}{k} + \frac{x}{2k}, \infty\Big)$ since its vertex lies at $\frac{x-2(k-2)}{2k} = -1 + \frac{2}{k} + \frac{x}{2k}$. Therefore,

$$\begin{aligned}
\frac{2(k-2)-x}{k}Y + Y^2 &< \frac{2(k-2)-x}{k}Y_0 + Y_0^2 \\
&\leq \frac{k^2 - 7k + 8}{k^2(k+1)(k-2)} + \frac{k-4}{k^2(k-2)}x - \frac{1}{4k^2}x^2
\end{aligned}$$

for all $Y \in \Big[-1 + \frac{3}{k}, Y_0\Big)$ since $\frac{x}{2k} \leq \frac{1}{k}$, which concludes the proof. □

With the help of lemma 3.5.4, the following result can be proven. The inequality stated there will be the main ingredient for the convex function mentioned above.

Lemma 3.5.5. Let assumptions 1 and 2 hold, and let $x := n\,\mathrm{tr}\,S^4 - 1$. Then

$$
\begin{aligned}
0 \leq & -\frac{263}{144}\frac{n-3}{n^2(n-2)}\left((n\bar{\lambda})^2 - \frac{(n\bar{\lambda})^3}{n\hat{\theta}_0(n)}\right) - 2\tilde{\theta}_0(n)\left(\frac{2n-9}{4n^3} + \frac{3x}{8n^2}\right)n\bar{\lambda} \\
& + 2\left(\mathrm{tr}\,S^4 + \sqrt{\frac{(\mathrm{tr}\,S^4)^2 - (\mathrm{tr}\,S^3)^4}{1-(\mathrm{tr}\,S^3)^2}}\right).
\end{aligned}
$$

Proof. Recall that $\|\mathrm{Ric}\wedge\mathrm{Ric}\| = \frac{r^2}{\sqrt{2}}\sqrt{1-\mathrm{tr}\,S^4}$. The idea is to use corollary 3.5.1. For that define

$$
\begin{aligned}
X :=& \|\mathrm{Ric}\wedge\mathrm{Ric}\|\left(\tilde{\theta}_0(n)r^2 - \sqrt{2}\sqrt{1-\mathrm{tr}\,S^4}r^2\right) + n\bar{\lambda}^4 + \frac{23}{12}r_0^2\bar{\lambda}^2 \\
& - \frac{13}{12}\bar{\lambda}\,\mathrm{tr}\,\mathrm{Ric}_0^3 \\
=& \frac{\tilde{\theta}_0(n)}{\sqrt{2}}r^4\sqrt{1-\mathrm{tr}\,S^4} - r^4 + \mathrm{tr}\,\mathrm{Ric}^4 + n\bar{\lambda}^4 + \frac{23}{12}r_0^2\bar{\lambda}^2 - \frac{13}{12}\bar{\lambda}\,\mathrm{tr}\,\mathrm{Ric}_0^3
\end{aligned}
$$

and

$$
Y := \frac{1}{r^4}\left(X - \frac{263}{288}r_0^2\bar{\lambda}^2\right).
$$

Lemma 3.5.3 proves now

$$
r^4 - \frac{1}{2}\mathrm{tr}\,\mathrm{Ric}^4 - \frac{3}{2n}r^4 + r^4 Y \geq \frac{\tilde{\theta}_0(n)}{\sqrt{2}}r^4\sqrt{1-\mathrm{tr}\,S^4}, \tag{3.11}
$$

which shows on the one hand with the known fact $\frac{1}{n} \leq \mathrm{tr}\,S^4 \leq \frac{2}{n}$ that

$$
\begin{aligned}
Y \geq& -1 + \frac{1}{2}\mathrm{tr}\,S^4 + \frac{3}{2n} + \frac{\tilde{\theta}_0(n)}{\sqrt{2}}\sqrt{1-\mathrm{tr}\,S^4} \\
\geq& -1 + \frac{2}{n} + \frac{\tilde{\theta}_0(n)}{\sqrt{2}}\sqrt{\frac{n-2}{n}} \\
=& -1 + \frac{2}{n} + \sqrt{\frac{(n-1)(n-3)}{n(n+1)}} \\
\geq& -1 + \frac{3}{n}.
\end{aligned} \tag{3.12}
$$

Here, the last step is true because of the following: Consider

$$
n(n-1)(n-3) - (n+1) = n^3 - 4n^2 + 2n - 1 =: f(n).
$$

Since $f(4) = 7$ and $f'(n) = 3n^2 - 8n + 2 \geq f'(4) = 18$ for all $n \geq 4$, f restricted to $[4,\infty)$ is nonnegative, which shows $\sqrt{\frac{(n-1)(n-3)}{n(n+1)}} \geq \frac{1}{n}$.

On the other hand, it follows with $\frac{\tilde{\theta}_0(n)^2}{2} = \frac{(n-1)(n-3)}{(n+1)(n-2)} = 1 - \frac{3n-5}{(n+1)(n-2)}$ and $\operatorname{tr} S^4 = \frac{1+x}{n}$ from (3.11) that

$$\left(1 - \frac{1+x}{2n} - \frac{3}{2n} + Y\right)^2 \geq \left(1 - \frac{3n-5}{(n+1)(n-2)}\right)\left(1 - \frac{1+x}{n}\right).$$

This is equivalent to

$$\frac{(1+x)^2}{4n^2} + \left(\frac{3}{2n} - Y\right)^2 - \left(2 - \frac{1+x}{n}\right)\left(\frac{3}{2n} - Y\right) \geq -\frac{3n-5}{(n+1)(n-2)}\left(1 - \frac{1+x}{n}\right).$$

Since

$$-\frac{(3n-5)(n-1)}{n(n+1)(n-2)} - \frac{5}{2n^2} + \frac{3(2n-1)}{2n^2} = \frac{n^2-7n+8}{n^2(n+1)(n-2)} \qquad \text{and}$$

$$\frac{3n-5}{n(n+1)(n-2)} - \frac{2}{n^2} = \frac{n^2-3n+4}{n^2(n+1)(n-2)} \geq \frac{n-4}{n^2(n-2)},$$

it follows that

$$\frac{n^2-7n+8}{n^2(n+1)(n-2)} + \frac{n-4}{n^2(n-2)}x - \frac{x^2}{4n^2} \leq \frac{2(n-2)-x}{n}Y + Y^2.$$

By (3.12) and lemma 3.5.4, this can only be true if

$$Y \geq \frac{2n-9}{4n^3} + \frac{3x}{8n^2}.$$

Note that indeed $x \in \left[0, \frac{n-2}{n}\right]$ since $\operatorname{tr} S^4 \leq \frac{16}{8n+9} \leq \frac{2}{n} - \frac{2}{n^2}$, which follows from lemma 3.3.6 and from

$$16n^2 - 2(8n+9)(n-1) = -2n + 18 \leq 0 \tag{3.13}$$

for all $n \geq 9$.

The application of corollary 3.5.1 and the definitions of X and Y show now

$$\begin{aligned}\frac{263}{288} r_0^2 \bar{\lambda}^2 + \left(\frac{2n-9}{4n^3} + \frac{3x}{8n^2}\right) r^4 &\leq X \\ &\leq \frac{2}{\tilde{\theta}_0(n)} \|\mathrm{Ric} \wedge \mathrm{Ric}\| \left(\operatorname{tr} S^4 + \sqrt{\frac{(\operatorname{tr} S^4)^2 - (\operatorname{tr} S^3)^4}{1 - (\operatorname{tr} S^3)^2}}\right).\end{aligned} \tag{3.14}$$

Since $r^2 \geq n\bar{\lambda}\hat{\theta}_0(n)$ by lemma 3.3.1,

$$\frac{r_0^2\bar{\lambda}^2}{r^2} = \left(1 - \frac{n\bar{\lambda}^2}{r^2}\right)\bar{\lambda}^2 = \frac{1}{n^2}\left(\left(n\bar{\lambda}\right)^2 - \frac{\left(n\bar{\lambda}\right)^4}{nr^2}\right)$$
$$\geq \frac{1}{n^2}\left(\left(n\bar{\lambda}\right)^2 - \frac{\left(n\bar{\lambda}\right)^3}{n\hat{\theta}_0(n)}\right). \tag{3.15}$$

The multiplication of (3.14) with $\frac{2}{r^2}\frac{n-3}{n-2}$ leads with the applications of (3.15), $\|\mathrm{Ric}\wedge\mathrm{Ric}\| = \frac{r^2}{\sqrt{2}}\sqrt{1-\mathrm{tr}\, S^4}$, and lemma 3.3.1 to

$$0 \leq -\frac{263}{144}\frac{n-3}{n^2(n-2)}\left(\left(n\bar{\lambda}\right)^2 - \frac{\left(n\bar{\lambda}\right)^3}{n\hat{\theta}_0(n)}\right) - 2\tilde{\theta}_0(n)\left(\frac{2n-9}{4n^3} + \frac{3x}{8n^2}\right)n\bar{\lambda}$$
$$+ \frac{4}{\hat{\theta}_0(n)}\left(\mathrm{tr}\, S^4 + \sqrt{\frac{\left(\mathrm{tr}\, S^4\right)^2 - \left(\mathrm{tr}\, S^3\right)^4}{1-\left(\mathrm{tr}\, S^3\right)^2}}\right)\frac{1}{\sqrt{2}}\sqrt{1-\mathrm{tr}\, S^4}.$$

This concludes the proof because

$$1 - \frac{\hat{\theta}_0(n)^2}{2} = 1 - \frac{(n-1)(n-2)}{(n+1)(n-3)} = \frac{n-5}{(n+1)(n-3)} \leq \frac{1}{n} \leq \mathrm{tr}\, S^4,$$

which shows $\frac{\sqrt{2}}{\hat{\theta}_0(n)}\sqrt{1-\mathrm{tr}\, S^4} \leq 1$. □

Finally, all the work that was done before will be needed in the following result to prove assumption 2 to be wrong.

Lemma 3.5.6. Let assumption 1 hold. Then

$$\mathrm{scal} \geq (n+1)\tilde{\theta}_0(n).$$

Proof. Suppose that assumption 2 is true, that is $\mathrm{scal} \leq (n-3)\tilde{\theta}_0(n)$. Then, by corollary 3.3.7, the scalar curvature lies in $\left[\frac{8n+9}{16}\hat{\theta}_0(n), (n-3)\tilde{\theta}_0(n)\right]$. By lemma 3.5.5 it is known that

$$0 \leq -\frac{263}{144}\frac{n-3}{n^2(n-2)}\left(\left(n\bar{\lambda}\right)^2 - \frac{\left(n\bar{\lambda}\right)^3}{n\hat{\theta}_0(n)}\right) - 2\tilde{\theta}_0(n)\left(\frac{2n-9}{4n^3} + \frac{3x}{8n^2}\right)n\bar{\lambda}$$
$$+ 2\left(\mathrm{tr}\, S^4 + \sqrt{\frac{\left(\mathrm{tr}\, S^4\right)^2 - \left(\mathrm{tr}\, S^3\right)^4}{1-\left(\mathrm{tr}\, S^3\right)^2}}\right)$$

with $x = n\,\mathrm{tr}\, S^4 - 1$ as in lemma 3.5.5. The right hand side of this inequality can be viewed as a function

$$
\begin{aligned}
RHS\colon \ & \left[\frac{8n+9}{16}\hat{\theta}_0(n), (n-3)\tilde{\theta}_0(n)\right] \to \mathbb{R};\\
t \mapsto & -\frac{263}{144}\frac{n-3}{n^2(n-2)}\left(t^2 - \frac{t^3}{n\hat{\theta}_0(n)}\right) - 2\tilde{\theta}_0(n)\left(\frac{2n-9}{4n^3} + \frac{3x}{8n^2}\right)t\\
& + 2\left(\operatorname{tr} S^4 + \sqrt{\frac{\left(\operatorname{tr} S^4\right)^2 - \left(\operatorname{tr} S^3\right)^4}{1-\left(\operatorname{tr} S^3\right)^2}}\right)
\end{aligned}
$$

with nonnegative maximum since $RHS(n\bar{\lambda}) \geq 0$. The second derivative

$$
RHS''(t) = -\frac{263}{72}\frac{n-3}{n^2(n-2)}\left(1 - \frac{3t}{n\hat{\theta}_0(n)}\right)
$$

is nonnegative since $\frac{3t}{n\hat{\theta}_0(n)} \geq 3\frac{8n+9}{16n} \geq 1$ for all $t \in \left[\frac{8n+9}{16}\hat{\theta}_0(n), (n-3)\tilde{\theta}_0(n)\right]$. Thus, RHS is convex, and hence, it attains its maximum on the boundary. The idea is now to estimate in both boundary points the terms of RHS to create a contradiction to the fact that the maximum of RHS is nonnegative. Recall that $\frac{1}{n} \leq \left(\operatorname{tr} S^3\right)^2 \leq \operatorname{tr} S^4 \leq \frac{16}{8n+9} \leq \frac{2}{n} - \frac{2}{n^2}$, where the last inequality was shown in (3.13) in the proof of lemma 3.5.5. Further, consider the function

$$
f\colon \left[\frac{1}{n}, \frac{2}{n} - \frac{2}{n^2}\right] \to \mathbb{R}; y \mapsto \frac{\left(\operatorname{tr} S^4\right)^2 - y^2}{1-y}.
$$

Then $f'(y) = \frac{y^2 - 2y + \left(\operatorname{tr} S^4\right)^2}{(1-y)^2}$. The numerator satisfies

$$
y^2 - 2y + \left(\operatorname{tr} S^4\right)^2 \leq \frac{8}{n^4} - \frac{16}{n^3} + \frac{8}{n^2} - \frac{2}{n} \leq 0 \tag{3.16}
$$

for all $n \geq 1$. To see this, consider the function

$$
\tilde{f}\colon \mathbb{R} \to \mathbb{R}; n \mapsto -n^3 + 4n^2 - 8n + 4.
$$

Then $\tilde{f}(1) = -1$ and $\tilde{f}'(n) = -3n^2 + 8n - 8 < 0$ for all $n \in \mathbb{R}$, and hence, (3.16) is true. Thus, f is monotonically decreasing in y. Therefore,

$$
\sqrt{\frac{\left(\operatorname{tr} S^4\right)^2 - \left(\operatorname{tr} S^3\right)^4}{1-\left(\operatorname{tr} S^3\right)^2}} \leq \sqrt{\frac{\frac{4}{n^2} - \frac{1}{n^2}}{1 - \frac{1}{n}}} = \frac{\sqrt{3}}{\sqrt{n(n-1)}} \leq \frac{\sqrt{3}}{n-1}.
$$

Case 1. Assume that the maximum is attained in $\hat{\theta}_0(n)\frac{8n+9}{16}$. Then

$$
\begin{aligned}
& -\frac{n-3}{n^2(n-2)}\hat{\theta}_0(n)^2\left(\frac{8n+9}{16}\right)^2\left(1 - \frac{8n+9}{16n}\right)\\
& \quad = -\frac{2}{n^2}\frac{n-1}{n+1}\left(\frac{8n+9}{16}\right)^2\frac{8n-9}{16n}\\
& \quad \leq -\frac{(n-1)(n+1)(8n-9)}{32n^3}
\end{aligned}
$$

and

$$
\begin{aligned}
&-2\tilde{\theta}_0(n)\left(\frac{2n-9}{4n^3}+\frac{3x}{8n^2}\right)\hat{\theta}_0(n)\frac{8n+9}{16}\\
&\quad=-2(n-1)\frac{8n+9}{8(n+1)}\left(\frac{2n-9}{4n^3}+\frac{3x}{8n^2}\right)\\
&\quad\le -2(n-1)\left(\frac{2n-9}{4n^2(n-2)}+\frac{3}{8n^2}\right)x\\
&\quad=-\frac{(n-1)(7n-24)}{4n^2(n-2)}x.
\end{aligned}
$$

In total this shows

$$
\begin{aligned}
0\le&-\frac{263}{144}\frac{(n-1)(n+1)(8n-9)}{32n^3}-\frac{(n-1)(7n-24)}{4n^2(n-2)}x\\
&+2\frac{1+x}{n}+\frac{2\sqrt{3}}{n-1}\\
\le&-\frac{20}{11}\frac{(n-1)(n+1)(8n-9)}{32n^3}+\frac{n^2+15n-24}{4n^2(n-2)}x+\frac{2}{n}+\frac{2\sqrt{3}}{n-1}\\
\le&-\frac{5(n-1)(n+1)(8n-9)}{88n^3}+\frac{n^2+15n-24}{4n^3}+\frac{2}{n}+\frac{305}{88(n-1)}
\end{aligned}
$$

since $\frac{263}{144}\ge\frac{20}{11}$, $2\sqrt{3}\le\frac{305}{88}$, $n^2+15n-24>0$ for $n\ge 2$, and $x\le\frac{n-2}{n}$. By multiplication with $88n^3(n-1)$ this is equivalent to

$$
\begin{aligned}
0\le&-5(n-1)^2(n+1)(8n-9)+22(n-1)\left(n^2+15n-24\right)\\
&+176n^2(n-1)+305n^3\\
=&-40n^4+588n^3+127n^2-943n+573\\
=:&\,g(n).
\end{aligned}
$$

However, $g(15)=-25497$, $g'(15)=-140233$, and $g''(n)=-480n^2+3528n+254\le 0$ for $n\ge 8$, which implies that g is monotonically decreasing on $[15,\infty)$. Thus, $g(n)<0$ for all $n\ge 15$, which is a contradiction.

Case 2. Assume now that the maximum is attained in $(n-3)\tilde{\theta}_0(n)$. Consider

$$
\begin{aligned}
&-\frac{n-3}{n^2(n-2)}\tilde{\theta}_0(n)^2(n-3)^2\left(1-\frac{(n-3)^2}{n(n-2)}\right)\\
&\quad=-2\frac{(n-3)^4(n-1)(4n-9)}{n^3(n-2)^3(n+1)}\\
&\quad\le -2\frac{(n-3)^3(4n-9)}{n^3(n+1)(n-1)}.
\end{aligned}
$$

The last step is true since $(n-1)^2(n-3)-(n-2)^3=n^2-5n+5\ge 0$ for all $n\ge 4$. Moreover,

$$
-2\tilde{\theta}_0(n)^2(n-3)\left(\frac{2n-9}{4n^3}+\frac{3x}{8n^2}\right)\le -\frac{3}{2}\frac{(n-1)(n-3)^2}{n^2(n+1)(n-2)}x.
$$

Therefore,

$$0 \leq -\frac{273(n-3)^3(4n-9)}{72n^3(n+1)(n-1)} - \frac{3}{2}\frac{(n-1)(n-3)^2}{n^2(n+1)(n-2)}x$$
$$+ 2\frac{1+x}{n} + \frac{2\sqrt{3}}{n-1}.$$

To see that this is again increasing in x, consider

$$-\frac{3}{2}\frac{(n-1)(n-3)^2}{n^2(n+1)(n-2)} + \frac{2}{n} = \frac{n^3+17n^2-53n+27}{2n^2(n+1)(n-2)}.$$

This is positive for $n \geq 3$ since the numerator at $n = 3$ is equal to 48 and its derivative is equal to $3n^2 + 34n - 53$, which is positive for $n \geq 2$. Because $2\sqrt{3} \leq \frac{250}{72}$ and $x \leq \frac{n-2}{n}$,

$$0 \leq -\frac{273(n-3)^3(4n-9)}{72n^3(n+1)(n-1)} + \frac{n^3+17n^2-53n+27}{2n^3(n+1)} + \frac{2}{n} + \frac{250}{72(n-1)},$$

which is by multiplication with $72n^3(n+1)(n-1)$ equivalent to

$$\begin{aligned}
0 \leq & -273(n-3)^3(4n-9) + 36(n-1)\left(n^3+17n^2-53n+27\right) \\
& + 144n^2\left(n^2-1\right) + 250n^3(n+1) \\
= & -662n^4 + 13111n^3 - 54261n^2 + 98703n - 67311 \\
=: & \, h(n).
\end{aligned}$$

However, $h(15) = -59616$, $h'(15) = -1616202$, and $h''(n) = -7944n^2 + 78666n - 108522 \leq 0$ for $n \geq 9$, which implies that h is monotonically decreasing on $[15, \infty)$. Thus, $h(n) < 0$ for all $n \geq 15$, which is a contradiction. Hence, scal $> (n-3)\tilde{\theta}_0(n)$, which proves lemma 3.5.6 by means of corollary 3.2.6. □

3.6 Conclusion: W is the normalized Wely curvature of $S^m \times S^m$

As seen in the previous chapter, the scalar curvature of $|W|$ has to be greater than or equal to $(n+1)\tilde{\theta}_0(n)$. This result will be used in this section to conclude that W is in fact the normalized Weyl curvature of $S^m \times S^m$. In the whole section assumption 1 will be supposed to hold, and in addition to notation 3.0.3 the following notation will be used.

Notation 3.6.1. In the following, let $a \in (-1, 1)$ and $b \in [0, 1]$ be chosen such that

$$|W| = \sqrt{1-a^2-b^2}\frac{I}{\|I\|} + aW + bX \tag{3.17}$$

for an $X \in S^2_{B,d}(\mathfrak{so}(n))$ with $\|X\| = 1$ and $X \perp W$ as well as $X \perp I$. Clearly, $a^2 + b^2 \leq 1$.
Further, let I_+, I_-, X_+, X_- be defined as in notation 3.0.3, and set

$$d_+ := \operatorname{tr} I_+ = \dim\left(\operatorname{Im}(I_+)\right) \quad \text{and} \quad d_- := \operatorname{tr} I_- = \dim\left(\operatorname{Im}(I_-)\right),$$

where $\operatorname{Im}(I_+)$ and $\operatorname{Im}(I_-)$ denotes the image of I_+ and I_-, respectively, in $\mathfrak{so}(n)$.

Lemma 3.6.2. The following is true:

1. $2 \operatorname{tr} W_- = \sqrt{1 - a^2 - b^2}\, \|I\|$,
2. $2\|W_-\|^2 = 1 - a$, and
3. $r^2 = 2(n-1)\left(1 - a^2 - b^2\right) + (n-2) b^2 \|X_{\mathrm{Ric}_0}\|^2$.

Proof. Since $0 = \operatorname{tr} W = \operatorname{tr} W_+ - \operatorname{tr} W_-$, $\operatorname{tr} I = \|I\|^2$, and $\operatorname{tr} X = \langle X, I \rangle = 0$, the first equation can be obtained by taking the trace from (3.17).
For the second equation subtract W from (3.17). Then

$$2W_- = \sqrt{1 - a^2 - b^2} \frac{I}{\|I\|} + (a-1)W + bX.$$

Taking the square of the norm of this shows

$$4\|W_-\|^2 = 1 - a^2 - b^2 + (a-1)^2 + b^2 = 2(1-a),$$

which proves equation 2.
For equation 3 apply Ric to both sides of (3.17) and take the square of the norm of the result. Since $\|\mathrm{Ric}(I)\|^2 = n(n-1)^2 = 2(n-1)\|I\|^2$ and $\mathrm{Ric}(W) = 0$, this yields

$$r^2 = 2(n-1)\left(1 - a^2 - b^2\right) + b^2 \|\mathrm{Ric}\, X\|^2.$$

Thus, it remains to show that $\|\mathrm{Ric}\, X\|^2 = (n-2)\|X_{\mathrm{Ric}_0}\|^2$. This can be proven as follows:
Recall from chapter 1.2 that $X_{\mathrm{Ric}_0} = \frac{2}{n-2} \mathrm{id} \wedge \mathrm{Ric}_0(X)$, and let $\nu_1, \ldots, \nu_n$ denote the eigenvalues of $\mathrm{Ric}_0(X)$. Then

$$\begin{aligned}
\|X_{\mathrm{Ric}_0}\|^2 =& \frac{4}{(n-2)^2} \|\mathrm{id} \wedge \mathrm{Ric}_0(X)\|^2 \\
=& \frac{4}{(n-2)^2} \sum_{1 \leq i < j \leq n} \frac{1}{4} (\nu_i + \nu_j)^2 \\
=& \frac{1}{2(n-2)^2} \left(\sum_{i,j=1}^{n} \left(\nu_i^2 + \nu_j^2 + 2\nu_i \nu_j\right) - \sum_{i=1}^{n} 4\nu_i^2 \right) \\
=& \frac{1}{2(n-2)^2} \left(2(n-2)\|\mathrm{Ric}_0(X)\|^2 + 2\left(\operatorname{tr}\left(\mathrm{Ric}_0(X)\right)\right)^2 \right) \\
=& \frac{1}{n-2} \|\mathrm{Ric}(X)\|^2,
\end{aligned}$$

where the last equality results from $\operatorname{tr}(\operatorname{Ric}_0(X)) = 0$ and $\operatorname{Ric}(X) = \operatorname{Ric}_0(X)$ since $\operatorname{scal}(X) = \langle X, I \rangle = 0$. This concludes the proof. $\square$

The quantities a, b, d_+, and d_- will in the following be analyzed more precisely. As a consequence of the lower bound of the scalar curvature of $|W|$ which was calculated in the previous sections, $a^2 + b^2$ cannot be too big. This will be shown in the following lemma.

Lemma 3.6.3. Let assumption 1 hold. Then

$$a^2 + b^2 \leq \frac{3}{n(n-2)}.$$

Proof. Since $\operatorname{scal}(W) = \operatorname{scal}(X) = 0$,

$$(n+1)\tilde{\theta}_0(n) \leq \operatorname{scal} = \frac{\sqrt{1 - a^2 - b^2}}{\|I\|} \operatorname{scal}(I) = \sqrt{1 - a^2 - b^2}\sqrt{2n(n-1)}$$

by lemma 3.5.6. Taking the square of this inequality and using the definition of $\tilde{\theta}_0(n)$ show

$$2\frac{(n+1)(n-1)(n-3)}{n-2} \leq 2n(n-1)\left(1 - a^2 - b^2\right),$$

which is equivalent to

$$a^2 + b^2 \leq 1 - \frac{(n+1)(n-3)}{n(n-2)} = \frac{3}{n(n-2)}.$$

This was to be proven. $\square$

The next step will be to show that $|W|$ has only components in the direction of I and W, that is b has to vanish, and to compute concrete values for a, d_+, and d_-. The first part in order to find these values will be to improve the bounds of a and b which are already known from the definition of a and b and from lemma 3.6.3.

Lemma 3.6.4. Let assumption 1 hold. Then

$$a \in \left[\frac{1}{n-1} - \frac{3}{n^2}, \frac{1}{n-1}\right] \quad \text{and} \quad b \in \left[0, \sqrt{\frac{3}{n(n-1)(n-2)}}\right].$$

Proof. By lemma 3.2.2 it is known that $\theta_0(n)\operatorname{scal} \leq r^2 - 4\|W_-\|^2$. Therefore,

$$2\theta_0(n) \operatorname{tr} W_- \leq \frac{r^2}{2} - 2\|W_-\|^2 \tag{3.18}$$

since $4\operatorname{tr} W_- = 2\operatorname{tr}|W| = \mathrm{scal}$. The left hand side of (3.18) satisfies

$$\begin{aligned}
2\theta_0(n)\operatorname{tr} W_- &= \sqrt{2}\frac{\sqrt{(n-1)(n-2)}}{n}\sqrt{1-a^2-b^2}\|I\| \\
&= (n-1)\sqrt{\frac{n-2}{n}}\sqrt{1-a^2-b^2},
\end{aligned} \tag{3.19}$$

which was proven in lemma 3.6.2. Multiplying (3.18) with $(n-1)^{-1}$ and applying (3.19) as well as 2 and 3 of lemma 3.6.2 show

$$\begin{aligned}
\sqrt{\frac{n-2}{n}}\sqrt{1-a^2-b^2} &\leq 1-a^2-b^2+\frac{n-2}{2(n-1)}b^2\|X_{\mathrm{Ric}_0}\|^2+\frac{a-1}{n-1} \\
&\leq 1-a^2+\frac{a-1}{n-1}-\frac{n}{2(n-1)}b^2
\end{aligned}$$

since $\|X_{\mathrm{Ric}_0}\| \leq \|X\| = 1$. Taking the square of this inequality results in

$$\begin{aligned}
\frac{n-2}{n}\left(1-a^2-b^2\right) \leq &\left(1-a^2+\frac{a-1}{n-1}\right)^2+\frac{n^2}{4(n-1)^2}b^4 \\
&-\frac{n}{n-1}\left(1-a^2+\frac{a-1}{n-1}\right)b^2.
\end{aligned}$$

This is equivalent to

$$\begin{aligned}
&b^2\left(\frac{n}{n-1}\left(1-a^2+\frac{a-1}{n-1}\right)-\frac{n-2}{n}-\frac{n^2}{4(n-1)^2}b^2\right) \\
&\quad\leq \left(1-a^2+\frac{a-1}{n-1}\right)^2-\frac{n-2}{n}\left(1-a^2\right) \\
&\quad= (1-a)\left((1-a)\left(1+a-\frac{1}{n-1}\right)^2-\frac{n-2}{n}(1+a)\right).
\end{aligned} \tag{3.20}$$

Define now $G\colon \left[-\sqrt{\frac{3}{n(n-2)}}, \sqrt{\frac{3}{n(n-2)}}\right] \times \left[0, \sqrt{\frac{3}{n(n-2)}}\right] \to \mathbb{R}$ by

$$G(x,y) := \frac{n}{n-1}\left(1-x^2+\frac{x-1}{n-1}\right)-\frac{n-2}{n}-\frac{n^2}{4(n-1)^2}y^2,$$

and set $z := a-\frac{1}{n-1}$. Note that $a \in \left[-\sqrt{\frac{3}{n(n-2)}}, \sqrt{\frac{3}{n(n-2)}}\right]$ and $b \in \left[0, \sqrt{\frac{3}{n(n-2)}}\right]$ by the definition of a and b and by lemma 3.6.3. Thus, by (3.20),

$$\begin{aligned}
b^2G(a,b) \leq &(1-a)\left(\left(\frac{n-2}{n-1}-z\right)(1+z)^2-\frac{n-2}{n}\left(z+\frac{n}{n-1}\right)\right) \\
= &-(1-a)z\left(z^2+\frac{n}{n-1}z+\frac{2}{n(n-1)}\right).
\end{aligned} \tag{3.21}$$

Considering the definition of G shows

$$
\begin{aligned}
G(x,y) \geq& \frac{n}{n-1}\left(\frac{n-2}{n-1} - \frac{3}{n(n-2)} - \frac{1}{n-1}\sqrt{\frac{3}{n(n-2)}}\right) - \frac{n-2}{n} \\
&- \frac{3n}{4(n-1)^2(n-2)} \\
=& \frac{8n^3 - 51n^2 + 60n - 16 - 4n\sqrt{3n(n-2)}}{4n(n-1)^2(n-2)} \\
\geq& \frac{8n^3 - 59n^2 + 60n - 16}{4n(n-1)^2(n-2)}
\end{aligned}
$$

for all $x \in \left[-\sqrt{\frac{3}{n(n-2)}}, \sqrt{\frac{3}{n(n-2)}}\right]$ and $y \in \left[0, \sqrt{\frac{3}{n(n-2)}}\right]$. The numerator at $n = 7$ is equal to 257, and its derivative is equal to $24n^2 - 118n + 60$, which is positive for all $n \geq 5$. Therefore, G is nonnegative, and hence,

$$
0 \geq z\left(z^2 + \frac{n}{n-1}z + \frac{2}{n(n-1)}\right)
$$

by (3.21), which can only be true if

$$
\begin{aligned}
z \in& \left(-\infty, -\frac{n}{2(n-1)} - \sqrt{\frac{n^3-8n+8}{4n(n-1)^2}}\right] \\
&\cup \left[-\frac{n}{2(n-1)} + \sqrt{\frac{n^3-8n+8}{4n(n-1)^2}}, 0\right].
\end{aligned}
$$

However,

$$
\begin{aligned}
z = a - \frac{1}{n-1} \geq& -\sqrt{\frac{3}{n(n-2)}} - \frac{1}{n-1} \\
>& -\frac{n}{2(n-1)} - \sqrt{\frac{n^3-8n+8}{4n(n-1)^2}}
\end{aligned}
$$

since $\frac{n}{2(n-1)} - \frac{1}{n-1} = \frac{n-2}{2(n-1)} > 0$ and

$$
\frac{n^3-8n+8}{4n(n-1)^2} - \frac{3}{n(n-2)} = \frac{n^4 - 2n^3 - 20n^2 + 48n - 28}{4n(n-1)^2(n-2)} > 0.
$$

The last step is true because at $n = 5$ the numerator is equal to 87 and its first derivative is equal to 198 and because the second derivative is given by $12n^2 - 12n - 40$, which is positive for all $n \geq 3$. Therefore, z has to be contained in the second interval. Moreover, consider

$$
\begin{aligned}
&n^4(n-1)\left(\frac{n^3-8n+8}{4n(n-1)^2} - \left(\frac{n}{2(n-1)} - \frac{3}{n^2}\right)^2\right) \\
&= n^3 - 9n + 9 \\
&=: f(n).
\end{aligned}
$$

Then $f(3) = 9$ and $f'(n) = 3n^2 - 9$, which is positive for all $n \geq 2$. Thus, f restricted to $[3, \infty)$ is nonnegative, and hence,

$$-\frac{n}{2(n-1)} + \sqrt{\frac{n^3 - 8n + 8}{4n(n-1)^2}} \geq -\frac{3}{n^2}.$$

Therefore, $z \in [-\frac{3}{n^2}, 0]$, which implies $a \in \left[\frac{1}{n-1} - \frac{3}{n^2}, \frac{1}{n-1}\right]$.
$G(a,b)$ can now be estimated as follows:

$$\begin{aligned} G(a,b) &\geq \frac{n}{n-1}\left(\frac{n-2}{n-1} - \frac{1}{(n-1)^2}\right) - \frac{n-2}{n} - \frac{3n}{4(n-1)^2(n-2)} \\ &= \frac{8n^4 - 51n^3 + 95n^2 - 64n + 16}{4n(n-1)^3(n-2)}. \end{aligned}$$

Consider

$$\begin{aligned} n&\left(8n^4 - 51n^3 + 95n^2 - 64n + 16\right) - 8(n-1)^3(n-2)^2 \\ &= 5n^4 - 57n^3 + 136n^2 - 112n + 32 \\ &=: g(n). \end{aligned}$$

Since $g(9) = 1292$, $g'(9) = 3065$, and $g''(n) = 60n^2 - 342n + 272$, which is positive for all $n \geq 5$, g restricted to $[9, \infty)$ is nonnegative. Therefore,

$$G(a,b) \geq \frac{2(n-2)}{n^2}.$$

Thus,

$$b^2 \leq -(1-a)z\left(z^2 + \frac{n}{n-1}z + \frac{2}{n(n-1)}\right)\frac{n^2}{2(n-2)}$$

by inequality (3.21). Furthermore, $-z\left(z^2 + \frac{n}{n-1}z\right) \leq 0$ and $-(1-a)z \leq -z$ since $z \in \left[-\frac{3}{n^2}, 0\right]$ and $a \geq 0$. Therefore,

$$\begin{aligned} b^2 &\leq -\frac{zn}{(n-1)(n-2)} \qquad (3.22) \\ &\leq \frac{3}{n(n-1)(n-2)}, \end{aligned}$$

which concludes the proof. □

In order to compute concrete values for a, b, d_+, and d_-, the following to results will be useful.

Lemma 3.6.5. The following is true:

$$\left(1 - a^2 - b^2\right)\left(\frac{d_+ - d_-}{\|I\|^2} + a\right) = b^2\left(\|X_+\|^2 - \|X_-\|^2 + a\right).$$

Proof. Since $W = W_+ - W_-$ and $|W| = W_+ + W_-$, it follows with (3.17) on the one hand that

$$\begin{aligned} W_+ &= \frac{\sqrt{1-a^2-b^2}}{(1-a)\|I\|} I_+ + \frac{b}{1-a} X_+ \quad \text{and} \\ W_- &= \frac{\sqrt{1-a^2-b^2}}{(1+a)\|I\|} I_- + \frac{b}{1+a} X_- . \end{aligned}$$

Subtracting the second equation from the first and taking the trace show

$$0 = \frac{\sqrt{1-a^2-b^2}}{\|I\|} \left(\frac{d_+}{1-a} - \frac{d_-}{1+a} \right) + b \left(\frac{\operatorname{tr} X_+}{1-a} - \frac{\operatorname{tr} X_-}{1+a} \right).$$

This is via multiplication with $1 - a^2$ equivalent to

$$b \left(\operatorname{tr} X_+ - \operatorname{tr} X_- \right) = - \frac{\sqrt{1-a^2-b^2}}{\|I\|} \left(d_+ - d_- + a\|I\|^2 \right) \tag{3.23}$$

since $\operatorname{tr} X_+ + \operatorname{tr} X_- = \operatorname{tr} X = 0$ and $d_+ + d_- = \|I\|^2$.
On the other hand, W_+ and W_- can also be written as

$$\begin{aligned} W_+ &= \frac{\sqrt{1-a^2-b^2}}{\|I\|} I_+ + aW_+ + bX_+ \quad \text{and} \\ W_- &= \frac{\sqrt{1-a^2-b^2}}{\|I\|} I_- - aW_- + bX_- . \end{aligned}$$

Taking the scalar product of the difference of both equations with bX shows

$$0 = \frac{\sqrt{1-a^2-b^2}}{\|I\|} b \left(\operatorname{tr} X_+ - \operatorname{tr} X_- \right) + ab^2 + b^2 \left(\|X_+\|^2 - \|X_-\|^2 \right)$$

since $X \perp W$ and $\langle |W|, X \rangle = b$. With (3.23) follows now

$$0 = - \left(1 - a^2 - b^2 \right) \left(\frac{d_+ - d_-}{\|I\|^2} + a \right) + b^2 \left(\|X_+\|^2 - \|X_-\|^2 + a \right),$$

which concludes the proof. □

From the lemma above the following result can be obtained.

Corollary 3.6.6. The following is true:

$$\left| \frac{d_+ - d_-}{\|I\|^2} + a \right| \leq b^2 \frac{1+a}{1-a^2-b^2}.$$

Proof. Since $\|X\| = 1$, $-1 \leq \|X_+\|^2 - \|X_-\|^2 \leq 1$. Therefore, lemma 3.6.5 shows on the one hand

$$\frac{d_+ - d_-}{\|I\|^2} + a \leq b^2 \frac{1+a}{1-a^2-b^2}$$

and on the other

$$-\left(\frac{d_+ - d_-}{\|I\|^2} + a\right) \le b^2 \frac{1-a}{1-a^2-b^2} \le b^2 \frac{1+a}{1-a^2-b^2}.$$

This proves the statement. □

Now, concrete values for d_+, d_-, a, and b can be calculated.

Lemma 3.6.7. Let assumption 1 hold. Then $d_+ = \frac{n(n-2)}{4}$, $d_- = \frac{n^2}{4}$, $a = \frac{1}{n-1}$, and $b = 0$.

Proof. Assume first that $d_+ \neq \frac{n(n-2)}{4}$. Since $d_+ = \operatorname{tr} I_+$, there exists a $k \in \left\{-\frac{n(n-2)}{4}, \dots, -1, 1, \dots, \frac{n^2}{4}\right\}$ with $d_+ = \frac{n(n-2)}{4} + k$. Thus, $d_- = \frac{n^2}{4} - k$. Let $z = a - \frac{1}{n-1}$ be defined as in lemma 3.6.4. Then

$$\left|\frac{2(d_+ - d_-)}{n(n-1)} + a\right| = \left|\frac{4k}{n(n-1)} + z\right| \ge \frac{4}{n(n-1)} - \frac{3}{n^2} \ge \frac{1}{n^2}.$$

Therefore,

$$\begin{aligned} \frac{1}{n^2} &\le b^2 \frac{1+a}{1-a^2-b^2} \\ &\le \frac{3}{n(n-1)(n-2)} \frac{\frac{n}{n-1}}{1 - \frac{1}{(n-1)^2} - \frac{3}{n(n-1)(n-2)}} \\ &= \frac{3n}{n^4 - 4n^3 + 4n^2 - 3n + 3} \end{aligned} \tag{3.24}$$

by corollary 3.6.6 and lemma 3.6.4. Define $f(n) := n^4 - 7n^3 + 4n^2 - 3n + 3$. Then $f(7) = 178$, $f'(7) = 396$, and $f''(n) = 12n^2 - 42n + 8 \ge f''(4) = 32$ for all $n \ge 4$. Hence, f restricted to $[7, \infty)$ is positive, which contradicts (3.24). Therefore, $d_+ = \frac{n(n-2)}{4}$ and $d_- = \frac{n^2}{4}$.

As a consequence, $\left|\frac{2(d_+ - d_-)}{n(n-1)} + a\right| = |z| = -z$. Using again corollary 3.6.6 and $a \le \frac{1}{n-1}$ as well as $b^2 \le -\frac{zn}{(n-1)(n-2)}$, which was shown in (3.22), a related calculation to that in (3.24) proves

$$-z \le -\frac{zn^3}{n^4 - 4n^3 + 4n^2 - 3n + 3}.$$

A similar argument as above shows that $g(n) := n^4 - 5n^3 + 4n^2 - 3n + 3$ is positive for $n \ge 5$. Thus, z has to vanish, which proves $a = \frac{1}{n-1}$ and $b = 0$. □

Corollary 3.6.8. Let assumption 1 hold, and define $\alpha_+ := \sqrt{\frac{2}{(n-1)(n-2)}}$ and $\alpha_- := \alpha_+ \frac{n-2}{n}$. Then

$$W_+ = \alpha_+ I_+ \quad \text{and} \quad W_- = \alpha_- I_-.$$

Proof. Since $|W| = \sqrt{1-a^2}\frac{I}{\|I\|} + aW$ with $a = \frac{1}{n-1}$ by lemma 3.6.7,

$$W_+ = \frac{\sqrt{1-a^2}}{(1-a)\|I\|} I_+ = \sqrt{\frac{2}{(n-1)(n-2)}}\, I_+$$

and

$$W_- = \frac{\sqrt{1-a^2}}{(1+a)\|I\|} I_- = \sqrt{\frac{2}{(n-1)(n-2)}} \frac{n-2}{n} I_-.$$

This was to be proven. □

It is now known that the eigenvalues of W are the same as those of W_0, and also the dimensions of their eigenspaces agree, but it is still not clear if W is in fact equal to the normalized Weyl curvature operator of $S^m \times S^m$.

Remark 3.6.9. Usually,in the present work $S^2_{B,d}(\mathfrak{so}(n))$ is understood to be the set

$$\{R = (r_{ij})_{1\le i,j\le n} \in \mathbb{R}^{n\times n} | R^T = R \text{ and } r_{ii} = 0 \text{ for all } 1 \le i \le n\}.$$

The permutation group S_n acts on this set by $(\sigma.R)_{ij} = r_{\sigma^{-1}(i)\sigma^{-1}(j)}$ for all $1 \le i < j \le n$ and all $\sigma \in S_n$. $\sigma.R$ and R may be different as matrices, but they define the same operator with respect to different bases, namely to $s_{12}e_{\sigma^{-1}(1)} \wedge e_{\sigma^{-1}(2)}, \dots, s_{n-1,n}e_{\sigma^{-1}(n-1)} \wedge e_{\sigma^{-1}(n)}$ for $s_{ij} := \operatorname{sgn}(\sigma^{-1}(j) - \sigma^{-1}(i))$ and $e_1 \wedge e_2, \dots, e_{n-1} \wedge e_n$, respectively. Therefore, in order to prove theorem 3.0.2, it is enough to show that there exists a $\sigma \in S_n$ with $W = \sigma.W_0$.

To use this remark, the operators I_+ and I_- as well as their images in $\mathfrak{so}(n)$ will now be analyzed more precisely.

Lemma 3.6.10. Let assumption 1 hold. Then

$$\operatorname{tr}_- I_+^{\#} = 0 \quad \text{and} \quad \operatorname{tr}_- I_-^{\#} = 0,$$

where $\operatorname{tr}_- R := \langle R, I_- \rangle$ for operators $R \in S^2_B(\mathfrak{so}(n))$.

Proof. First of all, since $a = \frac{1}{n-1}$, observe that

$$\begin{aligned} r^2 - 4\|W_-\|^2 =& 2(n-1)\left(1-a^2\right) - 2(1-a) \\ =& 2\frac{n(n-2)}{n-1} - 2\frac{n-2}{n-1} = 2(n-2) \end{aligned}$$

by lemma 3.6.2. Furthermore,

$$\begin{aligned} \theta_0(n)\operatorname{tr} W_- =& \theta_0(n) d_- \alpha_- \\ =& \sqrt{2}\frac{\sqrt{(n-1)(n-2)}}{n}\frac{n^2}{4}\sqrt{\frac{2}{(n-1)(n-2)}}\frac{n-2}{n} \\ =& \frac{n-2}{2}. \end{aligned}$$

Therefore,

$$\theta n\bar{\lambda} = \theta \mathrm{scal} = 2\theta \mathrm{tr}\, |W| = 4\theta \mathrm{tr}\, W_- \geq 2(n-2).$$

Recall now the definitions

$$X(W) = 4\left(W_+ \# W_-\right)_+ + 2\left(W_+^\# + W_-^\#\right)_- \quad \text{and}$$
$$C_\#(W) = \mathrm{scal}(X(W))$$

that were made in lemma 3.2.1 and 3.2.2, respectively. Combined with lemma 3.2.2, everything together shows

$$2(n-2) \leq \theta n\bar{\lambda} = r^2 - 4\|W_-\|^2 - C_\#(W) = 2(n-2) - C_\#(W),$$

which implies $C_\# W = 0$ since $C_\#(W) \geq 0$. Because of $X(W) \geq 0$, this also proves $X(W) = 0$. In particular, $\left(W_+^\#\right)_- = 0$ and $\left(W_-^\#\right)_- = 0$. By corollary 3.6.8, this shows

$$\mathrm{tr}_-\, I_+^\# = \frac{1}{\alpha_+^2} \mathrm{tr}_-\, W_+^\# = 0 \quad \text{and} \quad \mathrm{tr}_-\, I_-^\# = \frac{1}{\alpha_-^2} \mathrm{tr}_-\, W_-^\# = 0,$$

which concludes the proof. □

In the following, it will be shown that $\mathrm{Im}(I_+)$ is a Lie subalgebra of $\mathfrak{so}(n)$ that is isomorphic to $\mathfrak{so}(m) \oplus \mathfrak{so}(m)$, which will be helpful to conclude that W is a permutation of W_0. In order to prove this, the following definition together with the proposition thereafter will be needed (both cf. [He], chapter IV.3).

Definition 3.6.11. Let $\mathfrak{g}$ be a real Lie algebra with center $\mathfrak{z}$, and let s be an involutive automorphism of $\mathfrak{g}$ with set of fixed points $\mathfrak{k}$ such that $\mathfrak{k}$ is a compactly embedded subalgebra of $\mathfrak{g}$ and such that $\mathfrak{k} \cap \mathfrak{z} = \{0\}$. Then the pair $(\mathfrak{g}, s)$ is called an *effective orthogonal symmetric Lie algebra*. Is further G a connected Lie group with Lie algebra $\mathfrak{g}$ and K a Lie subgroup of G with Lie algebra $\mathfrak{k}$, then the pair (G, K) is said to be *associated* with the effective orthogonal symmetric Lie Algebra $(\mathfrak{g}, s)$.

For effective orthogonal symmetric Lie algebras the following is true.

Proposition 3.6.12. Let $(\mathfrak{g}, s)$ be an effective orthogonal symmetric Lie algebra with set of fixed points $\mathfrak{k}$ and with center of $\mathfrak{g}$ equal to $\{0\}$. Moreover, let (G, K) be associated with $(\mathfrak{g}, s)$ such that G is simply connected and K is connected. Then G/K is a Riemannian symmetric space.

Together with the classification of symmetric spaces (for example cf. [Be] or [He]) this will help to prove the following.

Lemma 3.6.13. Let assumption 1 hold, and let $\mathfrak{k} = \mathrm{Im}(I_+)$. Then

$$\mathfrak{k} \simeq \mathfrak{so}(m) \oplus \mathfrak{so}(m).$$

Proof. Set $\mathfrak{p} := \mathrm{Im}(I_-)$. Then $\mathfrak{so}(n) = \mathfrak{k} \oplus \mathfrak{p}$. Consider further the map $s\colon \mathfrak{so}(n) \to \mathfrak{so}(n)$ given by $s(x+y) = x - y$ for $x \in \mathfrak{k}$ and $y \in \mathfrak{p}$. It will be shown that $(\mathfrak{so}(n), s)$ is an effective orthogonal symmetric Lie algebra.
Clearly, s is involutive with set of fixed points equal to $\mathfrak{k}$. To prove that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{so}(n)$, it is enough to show that $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$.
Let $x := \sum\limits_{1\le i<j\le n} x_{ij}\, e_i \wedge e_j, y := \sum\limits_{1\le i<j\le n} y_{ij}\, e_i \wedge e_j \in \mathfrak{so}(n)$ and define $x_{ji} := x_{ij}$, $y_{ji} := y_{ij}$ for all $1 \le i < j \le n$ and $x_{ii} := 0$, $y_{ii} := 0$ for all $1 \le i \le n$. Then

$$
\begin{aligned}
[x,y] &= \sum_{1\le i<j\le n} \sum_{1\le k<l\le n} x_{ij} y_{kl} [e_i \wedge e_j, e_k \wedge e_l] \\
&= \sum_{1\le i<j\le n} \sum_{1\le k<l\le n} x_{ij} y_{kl} \left(-\delta_{ik} e_j \wedge e_l + \delta_{jk}\ e_i \wedge e_l + \delta_{il}\ e_j \wedge e_k \right.\\
&\quad \left. -\delta_{jl}\ e_i \wedge e_k\right) \\
&= -\sum_{1\le i<j<l\le n} (x_{ij} y_{il} - x_{il} y_{ij})\, e_j \wedge e_l + \sum_{1\le i<j<l\le n} x_{ij} y_{jl}\, e_i \wedge e_l \\
&\quad + \sum_{1\le k<i<j\le n} x_{ij} y_{ki}\, e_j \wedge e_k - \sum_{1\le i<k<j\le n} (x_{ij} y_{kj} - x_{kj} y_{ij})\, e_i \wedge e_k \\
&= \sum_{1\le i<j<k\le n} \Big((x_{ik} y_{ij} - x_{ij} y_{ik})\, e_j \wedge e_k + (x_{ij} y_{jk} - x_{jk} y_{ij})\, e_i \wedge e_k \\
&\quad + (x_{jk} y_{ik} - x_{ik} y_{jk})\, e_i \wedge e_j \Big).
\end{aligned}
\tag{3.25}
$$

Let now $x, y \in \mathfrak{k}$. Since all entries of I_+ are in $\{0,1\}$ and since

$$
0 = \mathrm{tr}_-\, I_+^{\#} = \sum_{\substack{1\le a<b\le n \\ I_{+ab}=0}} \sum_{c=1}^{n} I_{+ac} I_{+bc}
$$

by lemma 3.6.10, it follows that if $I_{+ab} = 0$ for some $1 \le a < b \le n$, then for all $1 \le c \le n$ $I_{+ac} = 0$ or $I_{+bc} = 0$. Thus, for all $1 \le c \le n$ $x_{ac} = y_{ac} = 0$ or $x_{bc} = y_{bc} = 0$. Hence, $[x, y] \in \mathfrak{k}$ by (3.25), which proves that $\mathfrak{k}$ is a Lie subalgebra of $\mathfrak{so}(n)$.
Suppose now $x, y \in \mathfrak{p}$. Since also all entries of I_- are in $\{0,1\}$ and since

$$
0 = \mathrm{tr}_-\, I_-^{\#} = \sum_{\substack{1\le a<b\le n \\ I_{+ab}=0}} \sum_{c=1}^{n} I_{-ac} I_{-bc}
$$

by lemma 3.6.10, it follows analogously that if $I_{+ab} = 0$ for some $1 \le a < b \le n$, then for all $1 \le c \le n$ $x_{ac} = y_{ac} = 0$ or $x_{bc} = y_{bc} = 0$. With (3.25), this proves $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$.
Let now $x \in \mathfrak{k}$ and $y \in \mathfrak{p}$. Choose $1 \le a < b \le n$ with $I_{-ab} = 0$ and let $1 \le c \le n$. If $I_{+ac} = 0$, then also $I_{+bc} = 0$ as seen above. Thus,

$x_{ac}y_{bc} - x_{bc}y_{ac} = 0$. An analogue argument shows $x_{ac}y_{bc} - x_{bc}y_{ac} = 0$ if $I_{+bc} = 0$. Let now $I_{+ac}, I_{+bc} \neq 0$. Then $I_{-ac} = I_{-bc} = 0$, which implies that $y_{ac} = y_{bc} = 0$. Therefore, $x_{ac}y_{bc} - x_{bc}y_{ac} = 0$. In total this shows with (3.25) that $[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$.
Let now $x_1, x_2 \in \mathfrak{k}$ and $y_1, y_2 \in \mathfrak{p}$ be given. Then $[x_1, x_2], [y_1, y_2] \in \mathfrak{k}$ and $[x_1, y_2], [y_1, x_2] \in \mathfrak{p}$ as seen above. Therefore,

$$\begin{aligned}
[s(x_1 + y_1), s(x_2 + y_2)] &= [x_1 - y_1, x_2 - y_2] \\
&= [x_1, x_2] + [y_1, y_2] - [x_1, y_2] - [y_1, x_2] \\
&= s\left([x_1, x_2] + [y_1, y_2] + [x_1, y_2] + [y_1, x_2]\right) \\
&= s\left([x_1 + y_1, x_2 + y_2]\right).
\end{aligned}$$

Hence, s is an automorphism. Since the center of $\mathfrak{so}(n)$ is equal to $\{0\}$, $(\mathfrak{so}(n), s)$ is an effective orthogonal symmetric Lie algebra, and proposition 3.6.12 can be applied:
Spin(n) is the universal cover of $SO(n)$, is hence simply connected and has $\mathfrak{so}(n)$ as Lie algebra. Let K be a connected Lie subalgebra of Spin(n) with Lie algebra $\mathfrak{k}$. Since $\dim(\mathrm{Spin}(n)/K) = \dim \mathfrak{p} = \frac{n^2}{4} = m^2$ by lemma 3.6.7, the classification of symmetric spaces (cf. [Be], 7.102 Table 1) shows now that $\mathfrak{k}$ has to be isomorphic to $\mathfrak{so}(m) \oplus \mathfrak{so}(m)$. This was to be proven. □

Finally, the proof of theorem 3.0.2 can be given.

Proof of theorem 3.0.2. Since $\mathfrak{k} = \mathrm{Im}(I_+)$, there exists $A \subset \{(1,2), (1,3)\ldots, (n-1, n)\}$ with

$$\mathfrak{k} = \bigoplus_{(i,j) \in A} \langle e_i \wedge e_j \rangle \tag{3.26}$$

Define $A_k := \{j \in \{1, \ldots, n\} | e_k \wedge e_j \in \mathfrak{k}\}$ for $1 \leq k \leq n$. Then $A_l = A_k$ for all $l \in A_k$, since $e_l \wedge e_j = [e_l \wedge e_k, e_k \wedge e_j] \in \mathfrak{k}$ if $j \in A_k$, and conversely $e_k \wedge e_j = [e_k \wedge e_l, e_l \wedge e_j] \in \mathfrak{k}$ if $j \in A_l$. Further, set $\mathfrak{i}_k := \bigoplus_{a,b \in A_k, a<b} \langle e_a \wedge e_b \rangle$. Clearly $\mathfrak{i}_k = \mathfrak{i}_l$ for every $l \in A_k$. The following will show that $\mathfrak{i}_k$ is an ideal in $\mathfrak{k}$ for all $1 \leq k \leq n$:
If $a, b \in A_k$, then $e_a \wedge e_b = [e_a \wedge e_k, e_k \wedge e_b] \in \mathfrak{k}$. Moreover,

$$[e_a \wedge e_b, e_c \wedge e_d] = \delta_{bc} e_a \wedge e_d - \delta_{ac} e_b \wedge e_d + \delta_{ad} e_b \wedge e_c - \delta_{bd} e_a \wedge e_c \in \mathfrak{i}_k$$

for all $a, b, c, d \in A_k$. Therefore, $\mathfrak{i}_k$ is a Lie subalgebra of $\mathfrak{k}$. Assume that there exists $a \notin A_k$ and $b \in A_k$ such that $e_a \wedge e_b \in \mathfrak{k}$. Then $e_a \wedge e_k = [e_a \wedge e_b, e_b \wedge e_k] \in \mathfrak{k}$ by the choice of b, which is a contradiction to the choice of a. Let now $a, b \in A_k$ and $c, d \notin A_k$ be given. Then $\delta_{ac} = \delta_{ad} = \delta_{bc} = \delta_{bd} = 0$. Thus, $[e_a \wedge e_b, e_c \wedge e_d] = 0 \in \mathfrak{i}_k$. This proves that $\mathfrak{i}_k$ is an ideal for every $1 \leq k \leq n$. Moreover, $\mathfrak{i}_k$ is equal to the ideal $\mathfrak{i}(e_k \wedge e_j)$ generated by $e_k \wedge e_j$ for every $j \in A_k \setminus \{k\}$.

Since $\mathfrak{k} \simeq \mathfrak{so}(m) \oplus \mathfrak{so}(m)$ by lemma 3.6.13, $\mathfrak{k}$ has only two nontrivial ideals, which both are isomorphic to $\mathfrak{so}(m)$. Assume now that there exists $1 \leq k \leq n$ with $\mathfrak{i}_k = \mathfrak{k}$. In particular, $\{(a,b)|a,b \in A_k \text{ and } a < b\} = A$, which implies that for every $1 \leq j \leq n$ either $\mathfrak{i}_j = \mathfrak{k}$ or $\mathfrak{i}_j = \{0\}$.
Let $\mathfrak{i} \neq \{0\}$ now be any ideal in $\mathfrak{k}$ and suppose that there exists a linear combination $x = x_1\, e_a \wedge e_b + x_2\, e_c \wedge e_d$ in $\mathfrak{i}$ with $(a,b),(c,d) \in A$. Without loss of generality, $a \neq c$ and $a \neq d$ can be assumed. Further, choose $l \in A_k$ with $l \notin \{a,b,c,d\}$ which is possible because $m \geq 8$ and $\#A = m(m-1)$ by lemma 3.6.7. Since $\mathfrak{i}$ is an ideal, $x_1\, e_l \wedge e_b = [e_l \wedge e_a, x] \in \mathfrak{i}$, which implies that $\mathfrak{i} = \mathfrak{k}$. Analogously, if any ideal $\mathfrak{i}$ in $\mathfrak{k}$ contains a linear combination of s different basis vectors of the form $e_a \wedge e_b$ for $(a,b) \in A$ and $1 \leq s \leq n$, it can be shown that $\mathfrak{i}$ also contains a linear combination of $s-1$ vectors of that form. Via induction it can be seen that every ideal in $\mathfrak{k}$ not equal to $\{0\}$ contains an element of the form $e_a \wedge e_b$ for some $(a,b) \in A$. Therefore, $\mathfrak{k}$ is a simple lie algebra, which is of course a contradiction to $\mathfrak{k} \simeq \mathfrak{so}(m) \oplus \mathfrak{so}(m)$. Thus, every ideal that is generated by a vector of the form $e_a \wedge e_b$ has to be isomorphic to $\mathfrak{so}(m)$. Clearly, not every $e_a \wedge e_b$ can generate the same ideal since otherwise $k \simeq \mathfrak{so}(m)$. Thus, there exist $(a,b),(c,d) \in A$ such that $\mathfrak{i}_a = \mathfrak{i}(e_a \wedge e_b) \simeq \mathfrak{so}(m)$, $\mathfrak{i}_c = \mathfrak{i}(e_c \wedge e_d) \simeq \mathfrak{so}(m)$, and $\mathfrak{k} = \mathfrak{i}_a \oplus \mathfrak{i}_c$. In particular, there exists a permutation $\sigma \in S_n$ with

$$\mathfrak{i}_a = \langle e_{\sigma(1)} \wedge e_{\sigma(2)} \rangle \oplus \cdots \oplus \langle e_{\sigma(m-1)} \wedge e_{\sigma(m)} \rangle \quad \text{and}$$
$$\mathfrak{i}_c = \langle e_{\sigma(m+1)} \wedge e_{\sigma(m+2)} \rangle \oplus \cdots \oplus \langle e_{\sigma(2m-1)} \wedge e_{\sigma(2m)} \rangle.$$

This is true since $\dim(\mathfrak{i}_a) = \dim(\mathfrak{i}_c) = \frac{m(m-1)}{2}$, which implies $\#A_a = \#A_c = m$, and $A_a \cap A_c = \emptyset$. This shows together with corollary 3.6.8 that $W = \sigma^{-1}.W_0$, which concludes the proof of theorem 3.0.2 by means of remark 3.6.9. □

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