Helma C. Klüver

Representation theory of unipotent linear algebraic groups and a generalized Kirillov theory for a class of nilpotent groups

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Helma Klüver
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Dekan:	Prof. Dr. Dr. h.c. Joachim Cuntz
Erster Gutachter:	Prof. Dr. Siegfried Echterhoff
Zweiter Gutachter:	Prof. Dr. Markus Reineke
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Abstract

This thesis is concerned with two different subjects in the field of representation theory of nilpotent locally compact groups. The first part deals with the structure theory of abelian unipotent linear algebraic groups over a local field K of characteristic p, which are treated as abstract topological groups. The finite-dimensional Witt groups over K form a large class of these groups.

The first main result is that every such Witt group, $W_n(K)$, can be decomposed into a discrete and a compact part, each being the dual of the other. This implies that the topological group $W_n(K)$ is isomorphic to its dual group, i.e., to its group of characters. As a consequence we obtain that every abelian unipotent K-split group is topologically isomorphic to its dual group.

In the second chapter we develop a version of Kirillov theory that can be applied to a wide class of locally compact nilpotent groups, including nilpotent groups of characteristic 0, such as connected, simply connected nilpotent Lie groups and p-adic unipotent linear algebraic groups, and a large class of unipotent linear algebraic groups over local fields of characteristic p. The key concept is the notion of a nilpotent k-Lie pair $(G, \mathfrak{g})$, $k \in \mathbb{N}$. This serves as a substitute for the Lie algebra $\mathfrak{g}$ of a Lie group G and it contains suitable additional structure to obtain a reasonable definition of a dual space $\mathfrak{g}^*$ of $\mathfrak{g}$. Given such a structure, the Kirillov-orbit map associates with a quasi-orbit of the coadjoint action of G on the dual $\mathfrak{g}^*$ a primitive ideal of the group C^*-algebra of G.

The second main result is that this Kirillov-orbit map establishes a homeomorphism between the quasi-orbit space and $\mathrm{Prim}(C^*(G))$ for every k-Lie pair $(G, \mathfrak{g})$ satisfying some additional property.

Zusammenfassung

In dieser Arbeit werden zwei verschiedene Themen aus der Darstellungstheorie nilpotenter lokalkompakter Gruppen behandelt. Im ersten Kapitel wird die Struktur von abelschen unipotenten linearen algebraischen Gruppen über einem lokalen Körper K der Charakteristik p untersucht. Eine große Klasse dieser Gruppen bilden die endlich-dimensionalen Witt-Gruppen über K.
Das erste Hauptergebnis ist, dass sich jede solche Witt-Gruppe $W_n(K)$ in einen diskreten und einen kompakten Teil zerlegen läßt, die zueinander dual sind. Dieses Resultat wird dann verwendet, um zu zeigen, dass jede abelsche unipotente K-split Gruppe topologisch isomorph zu ihrem Dual ist.

Im zweiten Kapitel entwickeln wir eine Version der Kirillov-Theorie und zeigen, dass diese auf eine große Klasse lokalkompakter nilpotenter Gruppen angewendet werden kann; dazu gehören nicht nur nilpotente Gruppen der Charakteristik 0, wie zusammenhängende, einfach zusammenhängende nilpotente Lie Gruppen und p-adische unipotente linear algebraische Gruppen, sondern auch eine große Klasse von unipotenten linear algebraischen Gruppen über lokalen Körpern der Charakteristik p. Ein wichtiges Konzept ist der Begriff eines nilpotenten k-Lie-Paares $(G, \mathfrak{g})$, $k \in \mathbb{N}$. Dieses dient als Ersatz für die Liealgebra $\mathfrak{g}$ einer Liegruppe G und erlaubt uns insbesondere, einen Dualraum $\mathfrak{g}^*$ von $\mathfrak{g}$ zu definieren. Für jedes nilpotente k-Lie-Paar $(G, \mathfrak{g})$ erhalten wir dann eine Kirillov-Abbildung, die jeder Quasi-Bahn der koadjungierten Wirkung von G auf $\mathfrak{g}^*$ ein primitives Ideal der Gruppen-C^*-Algebra von G zuordnet.
Das zweite Hauptergebnis dieser Arbeit ist, dass für jedes nilpotente k-Lie-Paar, welches eine zusätzliche Bedingung erfüllt, diese Kirillov-Abbildung ein Homöomorphismus ist.

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Introduction

This thesis covers two different subjects in the field of representation theory of nilpotent locally compact groups. The first chapter is devoted to the structure theory of abelian unipotent linear algebraic groups over local fields of characteristic p. We consider these groups as abstract topological groups and study their dual group, i.e., the group of characters. In the second chapter we develop a Kirillov theory that can be applied not only to a large class of unipotent linear algebraic groups over local fields of characteristic p, but also to the classical case of connected, simply connected nilpotent Lie groups.

There are two important classes of unipotent linear algebraic groups over a local field K, the so-called K-split groups and K-wound groups. If K is a perfect field, in particular if K is a field of characteristic 0, then every unipotent linear algebraic group over K is K-split [3]. Generally, every unipotent linear algebraic group over K admits a largest algebraic subgroup N which is K-split and the quotient group G/N is K-wound [26]. But every K-wound group is a compact topological group [26] and thus every abelian K-wound group has a discrete dual group. This indicates that the structure of abelian K-split groups and their dual groups is more interesting, and therefore we will concentrate in the first chapter on the study of these groups.

If the characteristic of the field K is equal to 0 then every abelian K-split group is isomorphic to a product of copies of the additive group of K. In particular, every such group is isomorphic to its dual group. If the characteristic of K is different from 0, we will see that the abelian K-split groups are closely related to finite-dimensional Witt groups. We give the definition of $W_n(K)$, the $n+1$-dimensional Witt group of a local field K of characteristic p, and we derive a structure theorem which shows that such a group can be decomposed into a discrete and a compact part, each being the dual of the other. As a consequence we obtain that the topological group $W_n(K)$ is isomorphic to its dual group for every $n \in \mathbb{N}_0$ and every local field K of characteristic p. Since every abelian K-split group over a local field of characteristic p is isogenous to a finite product of Witt groups, we can use the results obtained for the latter to prove that every abelian K-split group is topologically isomorphic to its dual group.

For connected, simply connected nilpotent Lie groups G, A.A. Kirillov [23] provided an elegant geometric description of the dual space $\hat{G}$, i.e., of the equivalence classes of irreducible unitary representations of G. His method associates to each element of $\hat{G}$ an orbit of the coadjoint action of G on the linear dual $\mathfrak{g}^*$, where $\mathfrak{g}$ denotes the Lie algebra of G. Given an element of $\mathfrak{g}^*$, there exists a simple method of describing the irreducible representations corresponding to it.

This method has been extended in many different directions (see, for example, [6], [16], [17]) and the precise boundary between groups which admit a Kirillov theory and those which do not, is not well defined. Howe [16] constructed a version of Kirillov theory for nilpotent separable locally compact quasi-p groups. Every unipotent linear algebraic group over a local field of characteristic p is nilpotent and can be considered as a locally compact quasi-p group. In this second chapter we build on some ideas of Howe to obtain a modified Kirillov theory that extends the theory for unipotent linear algebraic groups and can be applied not only to the "classical groups", such as simply connected nilpotent Lie groups, but also to p-adic unipotent linear algebraic groups.

One problem of extending the Kirillov theory to nilpotent groups G which are not Lie groups is that the dual space $\hat{G}$ does not need to satisfy the T_0-axiom. So instead of studying the space $\hat{G}$, we will describe the primitive ideal space $\mathrm{Prim}(C^*(G))$ of the group C^*-algebra of G. Equipped with the hull-kernel topology, $\mathrm{Prim}(C^*(G))$ is a T_0-space and in the case of type I groups, $\hat{G}$ may be identified with $\mathrm{Prim}(C^*(G))$.

Another problem of extending the Kirillov theory to non-Lie groups is that of finding objects which can serve as a suitable substitute for the canonically attached Lie algebra and its dual space.

For every unipotent linear algebraic group G over a local field K of characteristic $p > 0$ there is a well-known algebraic construction [3] which attaches to it a Lie algebra $\mathfrak{g}$. This Lie algebra is a vector space over the field K. However, this Lie algebra is not the right object for our purpose. Indeed, every nilpotent Lie algebra of dimension two is abelian, but, unlike the case of Lie groups, there exist two-dimensional, two-step nilpotent linear algebraic groups which are not abelian. So if we attach to such groups an abelian Lie algebra, then the linear dual space of this Lie algebra can never reflect the structure of the irreducible representations of these non-abelian groups.

We introduce in Section 2.2 the notion of a nilpotent k-Lie pair $(G, \mathfrak{g})$, $k \in \mathbb{N}$, that is a locally compact, separable, l-step nilpotent group G to which we can attach a Lie algebra $\mathfrak{g}$ over the ring $\mathbb{Z}[\frac{1}{k!}]$ for some $k \geq l$, such that there exists a homeomorphism $\exp : \mathfrak{g} \to G$ with inverse map log satisfying the Campbell-Hausdorff formula. Note that such a Lie algebra does not necessarily carry the structure of a vector space and we can not make use of the linear dual of $\mathfrak{g}$ as in the classical case. To deal with this, we require additionally the existence of a locally compact abelian group $\mathfrak{w}$ which is also a $\mathbb{Z}[\frac{1}{k!}]$-module and we require the existence of a character $\epsilon \in \hat{\mathfrak{w}}$ such that the group of continuous, $\mathbb{Z}[\frac{1}{k!}]$-linear homomorphisms from $\mathfrak{g}$ to $\mathfrak{w}$, denoted by

$\mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$, is isomorphic to the Pontrjagin dual of the additive group $\mathfrak{g}$ via the map $f \mapsto \epsilon \circ f$. We will then use the group $\mathfrak{g}^* := \mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$ instead of the linear dual space of $\mathfrak{g}$.

Many statements for nilpotent k-Lie pairs $(G, \mathfrak{g})$ are proven by induction on the nilpotence class of G. Therefore, it is of interest to know what kind of subgroups H of G have the property that the pair $(H, \log(H))$ defines a nilpotent k-Lie pair. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$, then it follows directly from the Campbell-Hausdorff formula that $\exp(\mathfrak{h}) =: H$ is a subgroup of G. In Section 2.3 we develop two Inversion Formulas of the Campbell-Hausdorff formula, which will be a useful tool to decide whether or not a subgroup H of G has the property that $\log(H)$ defines a subalgebra of $\mathfrak{g}$. Furthermore, we prove that if $\log(H)$ is a closed subalgebra of $\mathfrak{g}$, then $(H, \mathfrak{h})$ is a nilpotent k-Lie pair.

In Section 2.6 we introduce for a homomorphism $f \in \mathfrak{g}^*$ the notion of a polarizing subalgebra $\mathfrak{r}$ for f and we explain how such a map f defines a character φ_f on the subgroup $R := \exp(\mathfrak{r})$ of G. Furthermore, we show in Section 2.8 that the induced representation $\mathrm{ind}_R^G \varphi_f$ is an irreducible representation of G for every homomorphism $f \in \mathfrak{g}^*$ and every polarizing subalgebra $\mathfrak{r}$ for f.

The main result of this chapter can be summarized in the following theorem.

Theorem. *(Corollary 2.10.32) Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. The Kirillov map*

$$\kappa : \mathfrak{g}^* \longrightarrow \mathrm{Prim}(C^*(G)),\ f \mapsto \ker(\mathrm{ind}_R^G \varphi_f)$$

is a well-defined, surjective map. If the group G satisfies some additional property (see Corollary 2.10.32), then we obtain a bijective Kirillov-orbit map

$$\tilde{\kappa} : \mathfrak{g}^* /_\sim \longrightarrow \mathrm{Prim}(C^*(G)),\ \mathcal{O} \mapsto \ker(\mathrm{ind}_R^G \varphi_f),$$

where $\mathfrak{g}^* /_\sim$ denotes the space of quasi-orbits of $\mathfrak{g}^*$ by the coadjoint action of G on $\mathfrak{g}^*$ and f denotes any representative of $\mathcal{O}$. Moreover, this map is a homeomorphism with respect to the natural quotient topology on the set of quasi-orbits and the hull-kernel topology on $\mathrm{Prim}(C^*(G))$.

Finally, we prove in Section 2.11 that every simply connected, connected nilpotent real Lie group G with Lie algebra $\mathfrak{g}$, as well as every p-adic unipotent linear algebraic group G with Lie algebra $\mathfrak{g}$, forms a nilpotent k-Lie pair. Moreover, we will show that we can find for a large class of unipotent linear algebraic groups G over a local field of characteristic p a natural number k and an algebra $\mathfrak{g}$ such that the pair $(G, \mathfrak{g})$ defines a nilpotent k-Lie pair. Since all these classes of groups satisfy the additional property of Corollary 2.10.32, the theorem above can be applied to them.

Chapter 1

Unipotent linear algebraic groups

In this chapter we analyze the structure of abelian unipotent linear algebraic groups over local fields of characteristic p. In particular we study their dual groups, i.e., the group of characters.

In Section 1.1 and Section 1.2 we recall some basic facts about the structure of local fields of characteristic p. The additive group of such a field can be decomposed into a discrete and a compact part, each being the dual of the other. Therefore, such topological groups are isomorphic to their dual groups.

Section 1.3 is devoted to the structure theory of unipotent linear algebraic groups over arbitrary local fields. We consider these groups as abstract groups, equipped with the locally compact topology from the additive group of the underlying field K, which is commonly denoted by G_a. A large class of unipotent linear algebraic groups over local fields admit a finite composition series consisting of characteristic subgroups such that all consecutive quotient groups are isomorphic to the additive group G_a. Such groups are called K-split.

If the characteristic of the field K is equal to 0 then every unipotent linear algebraic group over K is K-split. In this case, every abelian K-split group is isomorphic to a product of copies of the group G_a and thus every such abelian group is isomorphic to its dual group. If the characteristic of K is different from 0, we will see that the abelian K-split groups are closely related to finite-dimensional Witt groups, which we introduce in Section 1.4. We give in Subsection 1.4.1 the definition of $W_n(K)$, the nth Witt group of a local field K of characteristic p and we derive in Subsection 1.4.2 a structure theorem (Theorem 1.4.16) which shows that such groups can be decomposed into a discrete and a compact part, each being the dual of the other. This is the main result of this chapter and we use it to prove in Subsection 1.4.3 that the topological group $W_n(K)$ is isomorphic to its dual group for every $n \in \mathbb{N}_0$ and every local field K of characteristic p. Moreover, we give in Subsection 1.4.4 an explicit description of the characters of the first Witt group $W_1(K)$, where K is a local field of the form $\mathbb{F}_p((t))$ for some prime number p.

We see in Section 1.5 that every abelian K-split group over a local field of characteristic p is isogenous to a finite product of Witt groups and we can use the results obtained for the latter to prove in Section 1.6 that every abelian K-split group is topologically isomorphic to its dual group.

1.1 Local fields of characteristic p

Since we want to study linear algebraic groups over local fields of characteristic p we recall in this section some basic facts about the structure of these fields.

A local field is a locally compact, non-discrete field. Up to isomorphism, the local fields of characteristic 0 are $\mathbb{R}$, $\mathbb{C}$, and finite algebraic extensions of the field $\mathbb{Q}_p$ of p-adic numbers for all prime numbers p ([30], §3 Theorem 5). Furthermore, every local field of characteristic p is isomorphic to a field of formal Laurent series in one variable, denoted by $\mathbb{F}_q((t))$, where q is some power of the prime p ([30], §4 Theorem 8). A field of this form can be constructed as follows.

Let $\mathbb{F}_q$ be the finite field with q elements, where q is a power of some prime $p > 0$, and let $\mathbb{F}_q[[t]]$ be the ring of formal power series in one variable over $\mathbb{F}_q$. The invertible elements of this ring are power series $x = \sum_{n=0}^{\infty} x_n t^n$, where $x_0 \neq 0$. Put

$$\mathbb{F}_q((t)) := \mathbb{F}_q[[t]](t^{-1}).$$

The elements of $\mathbb{F}_q((t))$ are formal Laurent series of the form $\sum_{n=n_0}^{\infty} a_n t^n$, where $n_0 \in \mathbb{Z}$ and $a_n \in \mathbb{F}_q$. Since every element $x \in \mathbb{F}_q((t))$ can be written uniquely as $x = t^l y$ for some $l \in \mathbb{Z}$ and some $y \in \mathbb{F}_q[[t]]^\times$, every element of $\mathbb{F}_q((t))$ has a multiplicative inverse. Thus $\mathbb{F}_q((t))$ is a field, it is the quotient field of the ring $\mathbb{F}_q[[t]]$.

One can define a norm on $\mathbb{F}_q((t))$ in the following way. If $x \in \mathbb{F}_q((t))$ with $x = t^l y$ for some $l \in \mathbb{Z}$ and some $y \in \mathbb{F}_q[[t]]^\times$, then the map $\|.\| : K \to \mathbb{R}_{\geq 0},\ \|x\| := q^{-l}$ defines a norm on $\mathbb{F}_q((t))$. Equipped with this norm, the additive group of the field $\mathbb{F}_q((t))$ becomes a locally compact group.

Observe that the map

$$\phi : \mathbb{F}_q[[t]] \to \prod_{i=0}^{\infty} \mathbb{F}_q,\ \sum_{n=0}^{\infty} a_n t^n \mapsto (a_n)_{n \in \mathbb{Z}_{\geq 0}}$$

defines an isomorphism of groups, which is bi-continuous with respect to the product topology on the compact group $\prod_{i=0}^{\infty} \mathbb{F}_q$. Furthermore, the map

$$\Phi : \mathbb{F}_q((t)) \to \bigoplus_{i=-1}^{-\infty} \mathbb{F}_q \times \prod_{i=0}^{\infty} \mathbb{F}_q,\ \ \sum_{n=-m}^{\infty} a_n t^n \mapsto (a_n)_{n \in \mathbb{Z}_{\geq -m}},$$

is an isomorphism of additive groups and Φ is bi-continuous with respect to the discrete topology on the direct sum $\bigoplus_{i=-1}^{-\infty} \mathbb{F}_q$.

By means of this map, we obtain a natural decomposition of the local field $\mathbb{F}_q((t))$ into a discrete and a compact part:

$$\mathbb{F}_q((t)) \cong \bigoplus_{i=1}^{\infty} \mathbb{F}_q \times \prod_{i=0}^{\infty} \mathbb{F}_q. \tag{1.1}$$

1.2 Duality of local fields

Definition 1.2.1. Let G be a locally compact abelian group and $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$. The set

$$\hat{G} := \{\chi : G \to \mathbb{T} \mid \chi \text{ is a continuous homomorphism}\},$$

equipped with pointwise multiplication is a group, called the dual group of G. The elements of $\hat{G}$ are called characters of G. Given the topology of compact convergence on G, $\hat{G}$ is a locally compact group.

If G is the additive group of a local field then G is isomorphic to its dual group.

Proposition 1.2.2. *([30], Theorem 3) Let K be a non-discrete locally compact field and let χ be a non-trivial character of the additive group of K. Then the map $y \mapsto \chi_y$ from K to $\hat{K}$, where $\chi_y(x) := \chi(xy)$, is an isomorphism of topological groups.*

It follows directly from the proposition above that the field $\mathbb{F}_q((t))$, viewed as an additive locally compact group, is selfdual. We want to observe that there is also a different way of exhibiting the selfduality of $\mathbb{F}_q((t))$, using the structure of its additive group. For this, we recall the following facts about the dual group of a locally compact abelian group G which can all be found in [13], Chapter 4.

(1) If $G_1, \dots, G_n$ are locally compact abelian groups then

$$(G_1 \times \cdots \times G_n)\hat{} \cong \hat{G}_1 \times \cdots \times \hat{G}_n$$

and every finite abelian group is selfdual.

(2) If G is discrete then $\hat{G}$ is compact and if G is compact then $\hat{G}$ is discrete.

(3) If $G = \prod_{i \in I} G_i$, where each G_i is a compact abelian group then

$$\hat{G} \cong \bigoplus_{i \in I} \hat{G}_i.$$

(4) The map $\Phi : G \to \hat{\hat{G}}$ defined by $\Phi(x)(\chi) = \chi(x)$ is an isomorphism of topological groups.

Let G be the additive group of the local field $\mathbb{F}_q((t))$. Then we have by (1.1)

$$G \cong \bigoplus_{i=1}^{\infty} \mathbb{F}_q \times \prod_{i=0}^{\infty} \mathbb{F}_q$$

and using the facts listed above we obtain

$$\hat{G} \cong (\bigoplus_{i=1}^{\infty} \mathbb{F}_q \times \prod_{i=0}^{\infty} \mathbb{F}_q)\hat{} \cong (\bigoplus_{i=1}^{\infty} \mathbb{F}_q)\hat{} \times (\prod_{i=0}^{\infty} \mathbb{F}_q)\hat{} \cong \prod_{i=0}^{\infty} \mathbb{F}_q \times \bigoplus_{i=1}^{\infty} \mathbb{F}_q \cong G.$$

1.3 The structure of unipotent linear algebraic groups

Throughout this section, K denotes a local field. Recall the following definition.

Definition 1.3.1. Let V be a finite dimensional vector space over K and denote by $\mathrm{End}(V)$ the algebra of endomorphisms of V.

(a) An element $a \in \mathrm{End}(V)$ is called nilpotent if $a^m = 0$ for some $m \in \mathbb{N}$.

(b) An element $a \in \mathrm{End}(V)$ is called unipotent if $a - 1$ is nilpotent.

(c) A group $G \subseteq \mathrm{End}(V)$ is said to be unipotent if all its elements are unipotent.

Remark 1.3.2. In this section we want to study the structure of unipotent linear algebraic groups. By a linear algebraic group G over K we will always understand the K-rational points, $G(K)$, of such a group with the locally compact topology from K.

The most basic example is the additive group of the underlying field K which we denote in the following by G_a. The group G_a is unipotent since it is isomorphic to the group $G = \left\{ \left(\begin{smallmatrix} 1 & x \\ 0 & 1 \end{smallmatrix}\right), x \in K \right\} \subseteq \mathrm{GL}_2(K)$.

Proposition 1.3.3. *([3], Proposition 1.10) Let G be a linear algebraic group over K. Then G is isomorphic to a closed subgroup of $\mathrm{GL}_n(K)$ for some $n \in \mathbb{N}$.*

Definition 1.3.4. Let $n \in \mathbb{N}$. We define

$$Tr_1(n, K) := \{A \in GL_n(K) \mid A_{ij} = 0 \text{ for } j < i \text{ and } A_{ii} = 1 \text{ for all } 1 \leq i \leq n\}$$

to be the algebra of upper triangular $n \times n$-matrices over K with each diagonal entry equal to 1.

Theorem 1.3.5. *([3], Theorem 4.8) A linear algebraic group defined over K is unipotent if it is isomorphic to a closed algebraic subgroup of the upper triangular unipotent group $Tr_1(n, K)$ for some $n \in \mathbb{N}$.*

We will see in the following that every unipotent linear algebraic group is in particular nilpotent. For this, recall the definition of a nilpotent group.

Definition 1.3.6. Let G be a group.

(i) The descending central series of G is defined inductively by

$$Z_0(G) := G \quad \text{and} \quad Z_{i+1}(G) := (Z_i(G), G) \ \forall i \in \mathbb{N},$$

where $(Z_i(G), G)$ denotes the group commutator of $Z_i(G)$ and G.

(ii) The ascending central series of G is defined by

$$Z^0(G) := \{e\} \quad \text{and} \quad Z^{i+1}(G) := q^{-1}(Z(G/Z^i(G))),$$

where $Z(G/Z^i(G))$ denotes the center of $G/Z^i(G)$ and $q : G \to G/Z^i(G)$ the canonical quotient map.

(iii) The group G is nilpotent if the descending central series of G terminates within finitely many steps, i.e., if there exists $m \in \mathbb{N}$ such that

$$G = Z_0(G) \trianglerighteq Z_1(G) \trianglerighteq \cdots \trianglerighteq Z_m(G) = \{e\},$$

or equivalently, if the ascending central series of G terminates within finitely many steps, i.e., if there exists $k \in \mathbb{N}$ such that

$$\{e\} \trianglelefteq Z^1(G) \trianglelefteq \cdots \trianglelefteq Z^k(G) = G.$$

If m and k are minimal with these properties, then one can show that $m = k$. In this case G is said to be k-step nilpotent.

Remark 1.3.7. Now, let $Tr_1(n, K)$ be the algebra of upper triangular $n \times n$-matrices over K with each diagonal entry equal to 1, and let $Tr_0(n, K)$ be the algebra of upper triangular $n \times n$-matrices over K with each diagonal entry equal to 0. We have $Tr_0(n, K)^n = 0$ and thus $Tr_1(n, K) = E_n + Tr_0(n, K)$ is a unipotent group, i.e., a group consisting of unipotent elements. It is then easy to verify that each group $E_n + Tr_0(n, K)^i$, $1 \leq i \leq n$, is a normal subgroup of $Tr_1(n, K)$ satisfying the commutator formula $(E_n + Tr_0(n, K)^i, E_n + Tr_0(n, K)^j) \subseteq E_n + Tr_0(n, K)^{i+j}$. In particular, taking $i = 1$ and varying j, we see that $Tr_1(n, K)$ is a nilpotent group. Since every subgroup of a nilpotent group is again nilpotent, it follows that every unipotent linear algebraic group over K is nilpotent.

An important class of unipotent linear algebraic groups is the class of K-split groups, as follows.

Definition 1.3.8. ([3], Definition 15.1) A unipotent linear algebraic group G defined over K is called K-split if it admits a composition series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \cdots \trianglerighteq G_k = \{e\}$$

consisting of closed, normal linear algebraic subgroups of G such that G_i/G_{i+1} is isomorphic to the additive group G_a. In particular such a group is connected with respect to the Zariski topology ([26], V, 2.1).

Remark/Example 1.3.9. ([3], Corollary 15.5.) If the field K is perfect, in particular if the characteristic of K is equal to 0, then every unipotent linear algebraic group G over K is K-split.

Let G be a K-split group and let

$$G := Z_0(G) \trianglerighteq Z_1(G) \trianglerighteq \cdots \trianglerighteq Z_m(G) = \{e\} \tag{1.2}$$

be its descending central series. Notice that in this case $Z_{m-1}(G)$ is equal to the center of G. Since $(Z_i(G), Z_i(G)) \subseteq (Z_i(G), G)$ for all i, it follows that the canonical map

$$\varphi : Z_i(G)/(Z_i(G), Z_i(G)) \longrightarrow Z_i(G)/(Z_i(G), G)$$

is surjective. But we have $Z_i(G)/(Z_i(G), G) = Z_i(G)/Z_{i+1}(G)$ and since the quotient group $Z_i(G)/(Z_i(G), Z_i(G))$ is abelian it follows that all consecutive quotients $Z_i(G)/Z_{i+1}(G)$ appearing in the central series (1.2) are abelian unipotent K-split groups. Thus we can refine the central series (1.2) so that we obtain a new composition series in which all consecutive quotients are isomorphic to the additive group G_a.

But not every unipotent linear algebraic group is K-split.

Definition 1.3.10. ([26], V §3.1) Let G be a unipotent linear algebraic group over K. The group G is said to be K-wound if G does not admit a subgroup which is isomorphic to the additive group G_a.

Example 1.3.11. ([26], V §3.4) Let K be a local field of characteristic p and suppose that K is not perfect (i.e. $K^p \neq K$). Let t be an element of $K \setminus K^p$. One can show that the algebraic subgroup $H := \{(x, y) \mid x^p - ty^p = x\}$ of $G_a \times G_a$ is K-wound.

In [26], chapter V and VI, the author proves some interesting facts about the structure of K-wound groups.

Theorem 1.3.12. *([26], V §5) Let G be a unipotent linear algebraic group over a local field K of characteristic $p > 0$. Then the following are equivalent.*

(i) G is K-wound.

(ii) The topological group $G(K)$ is compact.

Theorem 1.3.13. *([26], VI §1) Every unipotent linear algebraic group G over K admits a largest algebraic subgroup N which is K-split. The quotient G/N is K-wound and it is the largest quotient of G with this property.*

It follows from Theorem 1.3.12 that every abelian K-wound group has a discrete dual group. This indicates that the structure of abelian K-split groups and their dual groups is more interesting and we will concentrate in the following on a detailed study of these groups.

As we have seen above, every unipotent K-split group is a multiple extension of groups of the type G_a. So in order to understand the structure of these groups more precisely, we first need to study the algebraic extensions of the additive group G_a with itself. For this we recall the following definitions.

An algebraic 2-cocycle f is a polynomial in two variables satisfying the equations

$$\begin{aligned} &f(x,0) = f(0,x) = 0 \quad \forall x \in G_a \quad \text{and} \\ &f(y,z) - f(x+y,z) + f(x,y+z) - f(x,y) = 0 \quad \forall x,y,z \in G_a. \end{aligned} \tag{1.3}$$

If $g : G_a \to G_a$ is any polynomial map, the function $h : G_a \times G_a \to G_a$ defined by

$$h(x,y) = g(x+y) - g(x) - g(y)$$

is an algebraic 2-cocycle and such a 2-cocycle is called trivial. The group of classes of algebraic 2-cocycles modulo the trivial 2-cocycles is denoted by $H^2(G_a, G_a)$. Every algebraic extension of the additive group G_a with itself is completely determined by an algebraic 2-cocycle $f : G_a \times G_a \to G_a$, and we will identify in the following such extensions with their 2-cocycles.

A 2-cocycle f is called symmetric if $f(x,y) = f(y,x)$ for all $x, y \in G_a$, and we denote the group of classes of symmetric, algebraic 2-cocycles modulo the trivial 2-cocycles by $\mathrm{Ext}(G_a, G_a)$. In characteristic $p > 0$, a non-trivial example of such a polynomial is

$$\omega(x,y) = \frac{1}{p}(x^p + y^p - (x+y)^p), \tag{1.4}$$

the 2-cocycle which defines the first Witt group over K. Finite-dimensional Witt groups will be introduced in Section 1.4 and we will see in the Section 1.5 that there is in fact a close connection between abelian K-split groups and Witt groups.

But first we want to remark that the class of polynomial 2-cocycles of G_a is much greater then the class of symmetric polynomial 2-cocycles. It is easy to see that, for example, every bi-additive polynomial $f : G_a \times G_a \to G_a$ satisfies Equation (1.3). In characteristic p, the bi-additive polynomials of G_a are of the form

$$f(x,y) = \sum_{m,n} a_{m,n} x^{p^m} y^{p^n},$$

where all but finitely many coefficients $a_{m,n}$ are equal to zero. A simple example of an asymmetric bi-additive polynomial is given by

$$f(x,y) = x^p y.$$

The non-abelian group G corresponding to this asymmetric cocycle can be realized as

$$G = \left\{ \left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & x^p \\ 0 & 0 & 1 \end{smallmatrix} \right), x, y \in G_a \right\}.$$

In fact, we can show the following result.

Proposition 1.3.14. *Every algebraic* 2*-cocycle of* G_a *is the sum of a symmetric* 2*-cocycle and a bi-additive* 2*-cocycle.*

Proof. Notice first that if $f(x,y)$ is a 2-cocycle of G_a then the polynomials $\bar{f}(x,y) := f(y,x)$ and $g(x,y) := f(x,y) - f(y,x)$ are also 2-cocycles. Furthermore, we have

$$g(x, y+z) - g(x,y) - g(x,z) = 0,$$

for all $x, y, z \in G_a$, and since $g(x,y) = -g(y,x)$ it follows that the polynomial g is bi-additive. We distinguish between two cases.

If the characteristic of K is anything except 2, then we can write

$$f(x,y) = \frac{1}{2}\big(f(x,y) + f(y,x)\big) + \frac{1}{2} g(x,y)$$

and thus we can write f as a sum of a symmetric and a bi-additive 2-cocycle.

If the characteristic of K is equal to 2, then $g(x,y) = f(x,y) + f(y,x)$ is bi-additive and symmetric. If f is of the form $f(x,y) = \sum a_{ij} x^i y^j$ then we have

$$g(x,y) = \sum (a_{ij} + a_{ji}) x^i y^j = \sum b_{ij} x^i y^j.$$

Furthermore we have $b_{ij} = b_{ji}$ and $b_{ii} = 2a_{ii} = 0$ for all i. Since $g(x,y)$ is bi-additive it follows that $b_{ij} = 0$ except for the case that i and j are both powers of 2. Now, let $h(x,y) = \sum_{i<j} b_{ij} x^i y^j$, then $h(x,y)$ is bi-additive and we have $h(x,y) + h(y,x) = g(x,y)$ for all $x, y \in G_a$. Since

$$f(x,y) + f(y,x) + h(x,y) + h(y,x) = 2g(x,y) = 0,$$

it follows that the sum $f(x,y) + h(x,y)$ is symmetric and hence $f(x,y)$ is the sum of a symmetric and a bi-additive 2-cocycle. $\square$

1.4 Witt groups

Finite-dimensional Witt groups over local fields K of characteristic p define an important class of abelian K-split groups. This section is divided into four subsections. In Subsection 1.4.1 we give, for every $n \in \mathbb{N}_0$, the definition of $W_n(K)$, the nth Witt group of a local field K of characteristic p. We prove in Subsection 1.4.2 that finite-dimensional Witt groups of any local field K of characteristic p can be decomposed into a discrete and a compact part, each being the dual of the other (Proposition 1.4.20). With this result we prove in Subsection 1.4.3 that, for every $n \in \mathbb{N}_0$ and every local field K of characteristic p, the topological group $W_n(K)$ is isomorphic to its dual group. Finally, we give in Subsection 1.4.4 an explicit description of the characters of the first Witt group of a local field K of the form $K = \mathbb{F}_p((t))$ for some prime number p.

1.4.1 Definition of finite-dimensional Witt groups

We first introduce the definition of the nth Witt ring $W_n(R)$ of a commutative ring R with unity. The approach we are following is given in [22].

Let $p \in \mathbb{N}$ be a fixed prime, let $n \in \mathbb{N}$, and consider the polynomial ring $A = \mathbb{Q}[X_0, Y_0, \ldots, X_n, Y_n]$. We will define a new ring structure on the set A^{n+1} via the following procedure. Let $x = (x_0, x_1, \ldots, x_n) \in A^{n+1}$ and define a map $\phi : A^{n+1} \to A^{n+1}$ by

$$\phi((x_0, x_1, \ldots, x_n)) = (x^{(0)}, x^{(1)}, \ldots, x^{(n)}),$$

where

$$x^{(0)} := x_0 \text{ and } x^{(i)} := x_0^{p^i} + p \cdot x_1^{p^{i-1}} + \cdots + p^{i-1} \cdot x_{i-1}^p + p^i \cdot x_i \text{ for } i \geq 1. \qquad (1.5)$$

Conversely, given an arbitrary vector $z = (x^{(0)}, x^{(1)}, \ldots, x^{(n)}) \in A^{n+1}$, define a map $\psi : A^{n+1} \to A^{n+1}$ by

$$\psi(z) = (x_0, x_1, \ldots, x_n),$$

where

$$x_0 := x^{(0)} \text{ and } x_i := \frac{1}{p^i}[x^{(i)} - x_0^{p^i} - p \cdot x_1^{p^{i-1}} - \cdots - p^{i-1} \cdot x_{i-1}^p] \text{ for } i \geq 1. \qquad (1.6)$$

The maps ϕ and ψ are inverse bijections. Using these maps we can introduce new binary operations of addition, denoted by $\oplus$, and multiplication, denoted by $\otimes$, on the set A^{n+1}. For this let $x, y \in A^{n+1}$ and define

$$x \oplus y := \phi^{-1}(\phi(x) + \phi(y)) \text{ and } x \otimes y := \phi^{-1}(\phi(x) \cdot \phi(y)). \qquad (1.7)$$

That is, $(x \oplus y)^{(i)} = x^{(i)} + y^{(i)}$ and $(x \otimes y)^{(i)} = x^{(i)} \cdot y^{(i)}$ for $i \geq 0$. Note that, in general, $x \oplus y \neq x + y$ and $x \otimes y \neq x \cdot y$. We write $W_n(A)$ for the set A^{n+1} endowed with the operations $\oplus$ and $\otimes$ as given above. One can show that $W_n(A)$ is a commutative ring of characteristic 0, isomorphic to the ring A^{n+1} under $\phi : W_n(A) \to A^{n+1}$. Observe that by (1.5), (1.6) and (1.7), we obtain for example:

$$\begin{aligned}
(x \oplus y)_0 &= x_0 + y_0, \\
(x \oplus y)_1 &= x_1 + y_1 + \tfrac{1}{p}(x_0^p + y_0^p - (x_0 + y_0)^p), \\
(x \otimes y)_0 &= x_0 y_0, \\
(x \otimes y)_1 &= x_0^p y_1 + x_1 y_0^p + p \cdot x_1 y_1.
\end{aligned}$$

Theorem 1.4.1. *([18], Theorem 8.25) Let $n \in \mathbb{N}$ and let $x = (x_0, x_1, \ldots, x_n)$ and $y = (y_0, y_1, \ldots, y_n)$ be elements of $W_n(A)$. Let $x \circ y$ denote $(x \oplus y)$, $x \otimes y$, or $x \ominus y$. Then $(x \circ y)_i \in \mathbb{Z}[x_0, y_0, \ldots x_i, y_i]$ for all $i \in \{0, \ldots, n\}$.*

Using this result we can now define for all $0 \leq i \leq n$:

$$\begin{aligned}
(x \oplus y)_i &=: A_i(x_0, y_0, \ldots, x_i, y_i) \in \mathbb{Z}[x_0, y_0, \ldots, x_i, y_i], \\
(x \otimes y)_i &=: M_i(x_0, y_0, \ldots, x_i, y_i) \in \mathbb{Z}[x_0, y_0, \ldots, x_i, y_i], \\
(x \ominus y)_i &=: S_i(x_0, y_0, \ldots, x_i, y_i) \in \mathbb{Z}[x_0, y_0, \ldots, x_i, y_i].
\end{aligned}$$

With these properties we can pass from the ring $A = \mathbb{Q}[X_0, Y_0, \ldots, X_n, Y_n]$ to any commutative ring R with identity. Let $\varphi : \mathbb{Z} \to R$ denote the natural homomorphism defined by $\varphi(c) = 1 \cdot c =: \bar{c}$ for $c \in \mathbb{Z}$. Let $\bar{A}_i(x_0, y_0, \ldots, x_i, y_i)$ and $\bar{M}_i(x_0, y_0, \ldots, x_i, y_i)$ be the polynomials in $R[x_0, y_0, \ldots, x_i, y_i]$ obtained from $A_i(x_0, y_0, \ldots, x_i, y_i)$ and $M_i[x_0, y_0, \ldots, x_i, y_i]$, respectively, by replacing each coefficient $c \in \mathbb{Z}$ by $\bar{c} \in R$. We can now give the definition of the nth Witt ring of a commutative ring with unity.

Definition 1.4.2. Let R be any commutative ring with unity and let $n \geq 0$. The nth Witt ring $W_n(R)$ of R is defined to be the set of all $(n+1)$-tuples $x = (x_0, \ldots, x_n)$, where $x_i \in R$ for every $i \in \{0, \ldots, n\}$, with equality defined as usual, and addition and multiplication defined as

$$\begin{aligned}
(x \oplus x')_i &=: \bar{A}_i(x_0, x'_0, \ldots, x_i, x'_i) \quad \text{and} \\
(x \otimes x')_i &=: \bar{M}_i(x_0, x'_0, \ldots, x_i, x'_i).
\end{aligned}$$

By [18], Theorem 8.26, $W_n(R)$ is a commutative ring. The zero and identity elements of $W_n(R)$ are $0 = (0, 0, \ldots, 0)$ and $(1, 0, \ldots, 0)$, respectively.

Remark 1.4.3. Let R be any commutative ring with unity and let $n \geq 0$. Let $x = (x_0, \ldots, x_n)$ and $x' = (x'_0, \ldots, x'_n)$ be elements of $W_n(R)$, and consider $B := \mathbb{Z}[Y_0, Y'_0, \ldots, Y_n, Y'_n]$. There exists a ring homomorphism θ from B to R, satisfying $\theta(c) = \bar{c}$ for $c \in \mathbb{Z}$, $\theta(Y_i) = x_i$, and $\theta(Y'_i) = x'_i$ for all $0 \leq i \leq n$. This map induces a ring homomorphism $\tilde{\theta} : W_n(B) \to W_n(R)$, where $\tilde{\theta}((a_0, \ldots, a_n)) = (\theta(a_0), \ldots, \theta(a_n))$, $a_i \in B$, satisfying $\tilde{\theta}((Y_0, \ldots, Y_n)) = (x_0, \ldots, x_n)$ and $\tilde{\theta}((Y'_0, \ldots, Y'_n)) = (x'_0, \ldots, x'_n)$.

In this way we obtain a functor W_n from the category of commutative rings of characteristic p into the category of commutative rings. In particular, if S is a subring of R, then $W_n(S)$ is a subring of $W_n(R)$.

Remark 1.4.4. Let R be any commutative ring with unity and let $n \geq 0$. Let $x = (x_0, \ldots, x_n)$ and $y = (y_0, \ldots, y_n)$ be elements of $W_n(R)$. Then $(x \oplus y)_i = \bar{A}_i(x_0, y_0, \ldots, x_i, y_i)$ for all $0 \leq i \leq n$ and it follows from Equation (1.5), (1.6), and (1.7) that $\bar{A}_i$ is of the form

$$\bar{A}_i(x_0, y_0, \ldots, x_i, y_i) = x_i + y_i + \omega_i(x_0, y_0, \ldots, x_{i-1}, y_{i-1}), \tag{1.8}$$

where ω_i, $i \in \{1, \ldots, n\}$, is a polynomial in the variables $x_0, y_0, \ldots, x_{i-1}, y_{i-1}$ with coefficients in R. Only ω_0 is equal to the zero polynomial and every ω_i, $i \in \mathbb{N}_0$, defines in fact a cocycle.

1.4.2 The structure of finite-dimensional Witt groups

In the following, let K be a local field of characteristic $p > 0$. The $(n+1)$-dimensional Witt ring $W_n(K)$, $n \in \mathbb{N}_{\geq 0}$, has the natural structure of an abelian, unipotent algebraic group. The elements of $W_n(K)$ are $(n+1)$-tuples $(x_0, \ldots, x_n)$, where $x_i \in K$ and hence, as a set, $W_n(K) = K^{n+1}$. It follows from Definition 1.4.2 and the remarks following it that the maps $\pi : W_n \times W_n \to W_n$, where $\pi(x, y) = x \oplus y$ and $i : W_n \to W_n$, where $i(x) = -x$ are morphisms of affine varieties $K^{2(n+1)} \to K^{n+1}$ and $K^{n+1} \to K^{n+1}$, respectively. Therefore, $W_n(K)$ is a (n+1)-dimensional abelian (affine) algebraic group. In order to see that the group $W_n(K)$ is unipotent we introduce the following maps.

(1) The Shift map

$$S : W_n(K) \to W_n(K), (x_0, x_1, \ldots, x_n) \mapsto (0, x_0, \ldots, x_{n-1}) \text{ and}$$

(2) the Frobenius map

$$F : W_n(K) \to W_n(K), (x_0, x_1, \ldots, x_n) \mapsto (x_0^p, x_1^p, \ldots, x_n^p).$$

These maps have the following important properties.

Lemma 1.4.5. *([22], Theorem 13.6.)*

(i) The Shift map S and the Frobenius map F are ring homomorphisms.

(ii) All elements $x = (x_0, \ldots, x_n) \in W_n(K)$ satisfy the following equation

$$\begin{aligned} p^k x &= S^k(F^k(x)) \quad \forall\, 0 \leq k \leq n \quad \textit{and hence} \\ p^k(x_0, \ldots, x_n) &= (0, 0, \ldots, 0, x_0^{p^k}, x_1^{p^k}, \ldots, x_{n-k}^{p^k}). \end{aligned} \tag{1.9}$$

As a direct consequence of this lemma we obtain the following result.

Corollary 1.4.6. *If* 1 *denotes the vector* $(1, 0 \dots, 0)$ *in the* n*th Witt ring* $W_n(K)$*, then* $p^{n+1} \cdot 1 = 0$ *and* $p^m \cdot 1 \neq 0$ *whenever* $m < n+1$*. Thus each element of* $W_n(K)$ *has additive order a power of* p*, and hence* $W_n(K)$ *is a unipotent group. Moreover,* p^{n+1} *is the smallest power of* p *satisfying the condition* $p^{n+1}x = 0$ *for all* $x \in W_n(K)$*.*

For the rest of this section, we endow $W_n(K)$ with the locally compact topology of the field K and consider in this way $W_n(K)$ as an abelian locally compact group. In the following we will write W_n instead of $W_n(K)$.

As we have seen in Section 1.1, we can decompose the additive group of the field K into a product of a discrete and a compact part, each being the dual of the other. Our aim in this section is to derive a structure theorem for all finite-dimensional Witt groups over K, which shows that these groups consist, like the field K, of a discrete and a compact part, each being the dual of the other.

Recall that every non-discrete locally compact field of characteristic $p > 0$ is isomorphic to a field of formal Laurent series in one variable, $\mathbb{F}_q((t))$, where q is some power of the prime p. We consider first the case that $q = p$, i.e., the case that the field K is isomorphic to $\mathbb{F}_p((t))$. We will use the following **notations**.

Notation 1.4.7. Let

- $k := \mathbb{Z}/p\mathbb{Z}$ be the finite field with p elements,
- $K := k((t))$ the field of Laurent-series over k,
- $K^+ := k[[t]] \subseteq K$ the power series ring over k, and
- $K^- := \{a_1 t^{-1} + a_2 t^{-2} + \ldots + a_n t^{-n} \mid n \in \mathbb{N}, a_i \in k\}$.

The set K^- is an additive subgroup of K which is also closed under multiplication. Clearly, every element $a \in K$ can be written uniquely as $a = a^+ + a^-$, where $a^+ \in K^+$ and $a^- \in K^-$.

Furthermore, if A is a finite abelian group with the discrete topology we define

- $A^\infty := \prod_{i=0}^\infty A = \{(a_0, a_1, \ldots) \mid a_i \in A\}$

to be the infinite direct product of A, which is a compact group with respect to the product topology. And we define

- $A^{(\infty)} := \bigoplus_{i=1}^\infty A = \{(a_1, a_2, \ldots) \mid a_i \in A, \ \ a_i = 0 \text{ for almost all } i\}$

to be the infinite direct sum, which is a discrete group with the usual direct-sum-topology. Moreover, let for every $n \in \mathbb{N}_0$

- $C_n := \mathbb{Z}/n\mathbb{Z}$ be the cyclic group with n elements,

- $W_n^+ := \{(x_0, \ldots,\ x_n) \in W_n \mid x_i \in K^+ \ \forall\ 0 \leq i \leq n\}$, and
- $W_n^- := \{(x_0, \ldots, x_n) \in W_n \mid x_i \in K^- \ \forall\ 0 \leq i \leq n\}$.

We recall that the field $K = \mathbb{F}_p((t))$ is isomorphic to the direct product of the discrete subgroup K^- and the compact subgroup K^+ and thus

$$K \cong K^- \times K^+ \cong \bigoplus_{i=1}^{\infty} k \times \prod_{i=0}^{\infty} k \cong C_p^{(\infty)} \times C_p^{\infty}.$$

The aim of this section is to show that the nth Witt group of K can be decomposed in a similar way. In fact we will prove

$$W_n \cong W_n^- \times W_n^+ \cong (C_{p^{n+1}}^{(\infty)} \times C_{p^n}^{(\infty)} \times \cdots \times C_p^{(\infty)}) \times (C_{p^{n+1}}^{\infty} \times C_{p^n}^{\infty} \times \cdots \times C_p^{\infty}).$$

The proof consists of several steps. At first we show that the nth Witt group W_n can be written as the direct product of its subgroups W_n^- and W_n^+.

Lemma 1.4.8. *(i) The sets W_n^+ and W_n^-, defined as above, are subgroups of W_n and we have $W_n^+ \cap W_n^- = \{0\}$.*

(ii) The map $\mu : W_n^+ \times W_n^- \to W_n,\ (x, y) \mapsto x \oplus y$ is a bi-continuous isomorphism.

Proof. (i): In order to show that W_n^+ is a subgroup of W_n notice first that the neutral element $(0, \ldots, 0)$ is an element of W_n^+. Moreover, if $(x_0, \ldots, x_n) \in W_n^+$ then also $-(x_0, \ldots, x_n) \in W_n^+$, since it follows from (1.5), (1.6), and (1.7), that $-(x_0, \ldots, x_n) = (-x_0, \ldots, -x_n)$. Now, let $x = (x_0, \ldots, x_n)$ and $x' = (x'_0, \ldots, x'_n)$ be two arbitrary elements of W_n^+. Then we have $x_i, x'_i \in K^+$ for all $0 \leq i \leq n$ and it follows from Definition 1.4.2 that $(x \oplus x')_i = \bar{A}_i(x_0, x'_0, \ldots, x_i, x'_i)$, where $\bar{A}_i(x_0, x'_0, \ldots, x_i, x'_i)$ denotes a polynomial in $x_0, \ldots, x_i$ and $x'_0, \ldots, x'_i$. Thus $\bar{A}_i(x_0, x'_0, \ldots, x_i, x'_i)$ is itself an element of K^+ and we obtain $(x \oplus x')_i \in K^+$ for all $0 \leq i \leq n$. This proves that W_n^+ is closed under addition. We can use the same arguments to show that W_n^- is a subgroup of W_n and obviously we have $W_n^+ \cap W_n^- = \{0\}$.

(ii): We show first that the map μ is a homomorphism. For this, let (x, y) and (v, z) be two elements of $W_n^+ \times W_n^-$. Since the group W_n is commutative we obtain

$$\begin{aligned} \mu((x, y) + (v, z)) &= \mu((x \oplus v, y \oplus z)) = (x \oplus v) \oplus (y \oplus z) \\ &= (x \oplus y) \oplus (v \oplus z) = \mu((x, y)) \oplus \mu((v, z)). \end{aligned}$$

In order to prove the injectivity of the map μ, let $x = (x_0, \ldots, x_n) \in W_n^+$ and let $y = (y_0, \ldots, y_n) \in W_n^-$ with $\mu((x, y)) = (0, \ldots, 0)$. Then

$$(0, 0, \ldots, 0) = (\bar{A}_0(x_0, y_0), \bar{A}_1(x_0, x_1, y_0, y_1), \ldots, \bar{A}_n(x_0, \ldots, x_n, y_0, \ldots, y_n))$$

and by comparing the components of the vectors above we obtain for all $0 \leq i \leq n$:

$$0 = \bar{A}_i(x_0, y_0, \ldots, x_i, y_i).$$

Rewriting the expression $\bar{A}_i(x_0, y_0, \dots, x_i, y_i)$ by means of (1.8) of Remark 1.4.4, we obtain for all $0 \le i \le n$:

$$0 = x_i + y_i + \omega_i(x_0, y_0, \dots, x_{i-1}, y_{i-1}). \tag{1.10}$$

The proof proceeds by induction on $i \in \{0, \dots, n\}$.

If $i = 0$, Equation (1.10) yields $0 = x_0 + y_0$ and since $x_0 \in K^+$ and $y_0 \in K^-$, it follows that $x_0 = 0$ and $y_0 = 0$, proving the base case.

So let $i \in \{0, \dots, n\}$ be fixed and suppose that $x_j = y_j = 0$ for all $0 \le j \le i$. Then we have $\omega_{i+1}(x_0, y_0, \dots, x_i, y_i) = 0$ and it follows from (1.10) that $0 = x_{i+1} + y_{i+1}$. But since $x_{i+1} \in K^+$ and $y_{i+1} \in K^-$, it we obtain $x_{i+1} = y_{i+1} = 0$.

In order to prove that the map μ is surjective let $x = (x_0, \dots, x_n)$ be an arbitrary element of W_n. We need to show that there exist two elements $y^- \in W_n^-$ and $y^+ \in W_n^+$ with $\mu((y^-, y^+)) = x$. We define these elements $y^- = (y_0^-, \dots, y_n^-)$ and $y^+ = (y_0^+, \dots, y_n^+)$ via the following procedure. For each $a \in K$ let $a^- \in K^-$ and $a^+ \in K^+$ be those elements, such that $a = a^- + a^+$ and define

$$\begin{aligned}
y_0^- &:= x_0^-,\ y_0^+ := x_0^+, \\
y_k^- &:= x_k^- - \omega_k((y_0^-, \dots, y_{k-1}^-), (y_0^+, \dots, y_{k-1}^+))^-, \text{ and} \\
y_k^+ &:= x_k^+ - \omega_k((y_0^-, \dots, y_{k-1}^-), (y_0^+, \dots, y_{k-1}^+))^+ \text{ for all } 1 \le k \le n.
\end{aligned}$$

It follows directly from this definition that $y^- \in W_n^-$, $y^+ \in W_n^+$ and clearly we have

$$\mu((y^-, y^+)) = (y_0^-, \dots, y_n^-) \oplus (y_0^+, \dots, y_n^+) = (x_0, \dots, x_n).$$

It only remains to show that μ is bi-continuous. But μ is the addition map in a topological group and thus continuous by definition. Since every continuous, bijective homomorphism between σ-compact locally compact groups is open, it follows that μ is bi-continuous. $\square$

In a second step we construct an isomorphism Λ between the subgroup W_n^+ of W_n and the compact group $(C_{p^{n+1}}^\infty \times C_{p^n}^\infty \times \dots \times C_p^\infty)$ and an isomorphism Ψ between the subgroup W_n^- of W_n and the discrete group $(C_{p^{n+1}}^{(\infty)} \times C_{p^n}^{(\infty)} \times \dots \times C_p^{(\infty)})$. For this we define maps Λ_k and Ψ_k, $k = 0, \dots, n$, which will be the "componentwise building blocks" for the maps Λ and Ψ, respectively. Since the definition of these maps is not canonical we explain the idea of the construction by means of the following special case.

Example 1.4.9. Let $p = 2$ and consider the local field $K = \mathbb{F}_2((t))$. The first Witt group $W_1(K)$ of K consists of the set of pairs $\{(x_0, x_1) \mid x_0, x_1 \in K\}$, where addition is defined as

$$(x_0, x_1) \oplus (y_0, y_1) := (x_0 + y_0, x_1 + y_1 + x_0 y_0).$$

We can view $W_1(K)$ as the group $G := \left\{ \left(\begin{smallmatrix} 1 & x_0 & x_1 \\ 0 & 1 & x_0 \\ 0 & 0 & 1 \end{smallmatrix} \right), x_0, x_1 \in K \right\}$, since the map

$$\Phi : W_1(K) \to G,\ (x_0, x_1) \mapsto \left(\begin{smallmatrix} 1 & x_0 & x_1 \\ 0 & 1 & x_0 \\ 0 & 0 & 1 \end{smallmatrix} \right)$$

defines an isomorphism of topological groups.

In order to define an isomorphism Λ between the subgroup $W_1^+(K)$ and the compact group $C_{p^2}^\infty \times C_p^\infty$, we introduce first two homomorphisms, $\Lambda_1 : C_{p^2}^\infty \to W_1^+(K)$ and $\Lambda_0 : C_p^\infty \to W_1^+(K)$. Notice that $C_{p^2} = C_4 = \mathbb{Z}/4\,\mathbb{Z}$ and $C_p = C_2 = \mathbb{Z}/2\,\mathbb{Z}$ and we identify elements of $\mathbb{Z}/4\,\mathbb{Z}$ and of $\mathbb{Z}/2\,\mathbb{Z}$ with numbers in $\{0,1,2,3\}$ and in $\{0,1\}$, respectively, in the canonical way. Thus we can multiply elements $a_m \in C_4$ with pairs $(x_0, x_1) \in W_1^+(K)$, where we understand the product as the a_m-fold sum of the pair (x_0, x_1) in $W_1^+(K)$. We can now define

$$\Lambda_1 : C_4^\infty \to W_1^+(K),\ (a_m)_{m\in\mathbb{N}_0} \mapsto \sum_{m\in\mathbb{N}_0} a_m(t^m, 0)$$

and we will prove in Lemma 1.4.12 that Λ_1 is a well-defined group homomorphism. Notice that

$$\begin{aligned} 0 \cdot (t^m, 0) &= (0,0), \\ 1 \cdot (t^m, 0) &= (t^m, 0), \\ 2 \cdot (t^m, 0) &= (t^m, 0) \oplus (t^m, 0) = (0, t^{2m}), \text{ and} \\ 3 \cdot (t^m, 0) &= (0, t^{2m}) \oplus (t^m, 0) = (t^m, t^{2m}). \end{aligned}$$

This computation shows that the image of Λ_1 is equal to the subgroup $K^+ \times (K^+)^2$ of $W_1^+(K)$ and since we want to obtain an isomorphism between $C_4^\infty \times C_2^\infty$ and $W_1^+(K)$, we define a second map Λ_0 as follows:

$$\Lambda_0 : C_2^\infty \to W_1^+(K),\ (a_m)_{m\in\mathbb{N}_0} \mapsto \sum_{\substack{m\in\mathbb{N}_0 \\ m\notin 2\,\mathbb{N}_0}} a_m(0, t^m).$$

But we do not want to use only every second term of the sequence $(a_m)_{m\in\mathbb{N}_0}$ (although we want to multiply a_m, $m \in \mathbb{N}_0$, only with the even powers of t), and thus we define the function f to be the unique monotone bijective function from $\mathbb{N}_0 \setminus 2\,\mathbb{N}_0$ to $\mathbb{N}_0$ and modify the definition of Λ_0 as follows:

$$\Lambda_0 : C_2^\infty \to W_1^+(K),\ (a_m)_{m\in\mathbb{N}_0} \mapsto \sum_{\substack{m\in\mathbb{N}_0 \\ m\notin 2\,\mathbb{N}_0}} a_{f(m)}(0, t^m).$$

We will show in Lemma 1.4.12 that Λ_0 is a well-defined group homomorphism and we will prove in Proposition 1.4.14 that the map

$$\Lambda : C_4^\infty \times C_2^\infty \to W_1^+(K), (a^{(1)}, a^{(0)}) \mapsto \Lambda_1(a^{(1)}) \oplus \Lambda_0(a^{(0)})$$

defines an isomorphism of topological groups.

In a similar way we will define an isomorphism Ψ between the subgroup $W_1^-(K)$ and the discrete group $C_{p^2}^{(\infty)} \times C_p^{(\infty)}$.

Recall that W_n denotes the nth Witt group of the field of Laurent series $\mathbb{F}_p((t))$. Generalizing the idea above, we introduce the following notation.

Definition 1.4.10. Let p be any prime number and $J := \mathbb{N}_0 \setminus p\,\mathbb{N}_0$. Define f to be the unique monotone bijective function from J to $\mathbb{N}_0$.

Definition 1.4.11. Let $n \in \mathbb{N}_0$ be fixed and define

$$\Lambda_n : C_{p^{n+1}}^{\infty} \to W_n^+,\ (a_m^{(n)})_{m\in\mathbb{N}_0} \mapsto \sum_{m\in\mathbb{N}_0} a_m^{(n)}(t^m, 0, \dots, 0). \tag{1.11}$$

We view $a_m^{(n)} \in C_{p^{n+1}} = \mathbb{Z}/p^{n+1}\mathbb{Z}$ as an integer between 0 and $p^{n+1}-1$ in the canonical way and understand the product $a_m^{(n)}(t^m, 0, \dots, 0)$ as the $a_m^{(n)}$-fold sum of the $(n+1)$-tuple $(t^m, 0, \dots, 0)$ in W_n^+. Furthermore, we define for every $0 \le k \le n-1$:

$$\Lambda_k : C_{p^{k+1}}^{\infty} \to W_n^+,\ (a_m^{(k)})_{m\in\mathbb{N}_0} \mapsto \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} a_{f(m)}^{(k)}(0, \cdots, 0, t^m, 0, \cdots, 0), \tag{1.12}$$

where the term t^m is at the $(n+1-k)$th position in the $(n+1)$-tuple $(0, \cdots, 0, t^m, 0, \cdots, 0)$. Again, we view $a_{f(m)}^{(k)} \in C_{p^{k+1}} = \mathbb{Z}/p^{k+1}\mathbb{Z}$ as an integer between 0 and $p^{k+1}-1$ in the canonical way and understand the product $a_{f(m)}^{(k)}(0, \cdots, 0, t^m, 0, \cdots, 0)$ as the $a_{f(m)}^{(k)}$-fold sum in W_n^+ of the $(n+1)$-tuple $(0, \cdots, 0, t^m, 0, \cdots, 0)$.

Furthermore, we define

$$\Psi_n : C_{p^{n+1}}^{(\infty)} \to W_n^+,\ (b_m^{(n)})_{m\in\mathbb{N}} \mapsto \sum_{m\in\mathbb{N}} b_m^{(n)}(t^{-m}, 0, \cdots, 0) \tag{1.13}$$

and for every $0 \le k \le n-1$:

$$\Psi_k : C_{p^{k+1}}^{(\infty)} \to W_n^-,\ (b_m^{(k)})_{m\in\mathbb{N}} \mapsto \sum_{\substack{m\in\mathbb{N}\\ m\notin p\,\mathbb{N}}} b_{f(m)}^{(k)}(0, \cdots, 0, t^{-m}, 0, \cdots, 0), \tag{1.14}$$

where the term t^{-m} stands at the $(n+1-k)$th position in the $(n+1)$-tuple $(0, \cdots, 0, t^{-m}, 0, \cdots, 0)$. Notice that the sums, appearing in (1.13) and (1.14), are finite.

Lemma 1.4.12. *The map Λ_k is a continuous group homomorphism for every $k \in \{0, \cdots, n\}$.*

Proof. In order to prove that Λ_k is a well-defined map for every $k \in \{0, \cdots, n\}$, we observe that if $(G, +)$ is any abelian group and $g_0, \cdots, g_i \in G$, $i \in \mathbb{N}_0$, then the map $\varphi^i : \mathbb{Z}^{i+1} \to G, (z_0, \cdots, z_i) \mapsto \sum_{m=0}^{i} z_m g_m$ is a group homomorphism. So if $G = W_n^+$ and if $g_i = (0, \cdots, 0, t^i, 0, \cdots, 0)$, $i \in \mathbb{N}_0$, is a vector in W_n^+, where the term t^i stands at the $(n+1-k)$th position in this $(n+1)$-tuple, we obtain for every $0 \leq k \leq n-1$ and every $i \in \mathbb{N}_0$ a group homomorphism

$$\varphi_k^i : \mathbb{Z}^{i+1} \to W_n^+, (z_0, \ldots, z_i) \mapsto \sum_{m=0}^{i} z_m (0, \ldots, 0, t^m, 0, \cdots, 0).$$

By the same argument we obtain a group homomorphism

$$\varphi_n^i : \mathbb{Z}^{i+1} \to W_n^+, (z_0, \ldots, z_i) \mapsto \sum_{m=0}^{i} z_m (t^m, \ldots, 0, \cdots, 0).$$

Moreover, if we denote by $V : W_k \to W_{k+1}, \ (x_0, \ldots, x_k) \mapsto (0, x_0, \ldots, x_k)$ the Shift homomorphism, then one can show (see for example [22]) that for every $k \in \{0, \ldots, n-1\}$, the image $V^{n-k}(W_k) \subseteq W_n$ is isomorphic to W_k. Since $p^{k+1}(x_0, \cdots, x_k) = (0, \cdots, 0)$ for every vector $(x_0, \cdots, x_k) \in W_{k+1}$ (Corollary 1.4.6), we obtain for every $i \in \mathbb{N}_0$ and every $0 \leq k \leq n-1$ a well-defined group homomorphism

$$\Lambda_k^i : (\mathbb{Z} / p^{k+1} \mathbb{Z})^{i+1} \to W_n^+, \ (a_0^{(k)}, \cdots, a_i^{(k)}) \mapsto \sum_{\substack{m=0 \\ m \notin p \mathbb{N}_0}}^{i} a_{f(m)}^{(k)} (0, \cdots, 0, t^m, 0, \cdots, 0).$$

Furthermore we obtain a well-defined group homomorphism

$$\Lambda_n^i : (\mathbb{Z} / p^{n+1} \mathbb{Z})^{i+1} \to W_n^+, \ (a_0^{(n)}, \cdots, a_i^{(n)}) \mapsto \sum_{m=0}^{i} a_m^{(n)} (t^m, 0, \cdots, 0).$$

Using the definition of addition in the nth Witt group W_n we can rewrite for every $k \in \{0, \ldots, n-1\}$, the $a_{f(m)}^{(k)}$-fold sum of the vector $(0, \cdots, 0, t^m, 0, \cdots, 0)$ in the following way:

$$a_{f(m)}^{(k)} (0, \cdots, 0, t^m, 0, \cdots, 0) = (0, \cdots, 0, a_{f(m)}^{(k)} t^m, c_{n+1-k}^{(k)}(m), \cdots, c_n^{(k)}(m)), \quad (1.15)$$

where every term $c_j^{(k)}(m)$, $j \in \{n+1-k, \ldots, n\}$ is a polynomial in t, whose smallest exponent of t is greater than or equal to m. (If $k = 0$, then the terms $c_j^{(0)}(m)$ do not occur.) Consequently, the sequence $(\Lambda_k^i(a_0^{(k)}, \cdots, a_i^{(k)}))_{i \in \mathbb{N}_0}$ converges in W_n to the element

$$\sum_{\substack{m=0 \\ m \notin p \mathbb{N}_0}}^{\infty} a_{f(m)}^{(k)} (0, \cdots, t^m, \cdots, 0) = \Lambda_k((a_m^{(k)})_{m \in \mathbb{N}_0})$$

for every $k = 0, \ldots, n-1$. Moreover, the sequence $\Lambda_n^i((a_0^{(n)}, \ldots a_i^{(n)})_{i\in\mathbb{N}_0}$ converges in W_n to the element

$$\sum_{m=0}^{\infty} a_m^{(n)}(t^m, 0, \cdots, 0) = \Lambda_n((a_m^{(n)})_{m\in\mathbb{N}_0}).$$

Now, let $k \in \{0, \ldots, n-1\}$ be fixed. Since the map Λ_k^i is an additive group homomorphism for every $i \in \mathbb{N}_0$, we obtain for all sequences $a^{(k)}$ and $b^{(k)}$ in $C_{p^{k+1}}^{\infty}$:

$$\begin{aligned}
\Lambda_k(a^{(k)} + b^{(k)}) &= \lim_{i\to\infty} \Lambda_k^i((a_0^{(k)}, \ldots, a_i^{(k)}) + (b_0^{(k)}, \ldots, b_i^{(k)})) \\
&= \lim_{i\to\infty} (\Lambda_k^i((a_0^{(k)}, \ldots, a_i^{(k)})) \oplus \Lambda_k^i((b_0^{(k)}, \ldots, b_i^{(k)}))) \\
&= \lim_{i\to\infty} \Lambda_k^i((a_0^{(k)}, \ldots, a_i^{(k)})) \oplus \lim_{i\to\infty} \Lambda_k^i((b_0^{(k)}, \ldots, b_i^{(k)}))) \\
&= \Lambda_k(a^{(k)}) \oplus \Lambda_k(b^{(k)}).
\end{aligned}$$

Hence Λ_k is an additive group homomorphism and it follows by the same argument that Λ_n is an additive group homomorphism.

It remains to show that each of the maps Λ_k, $k = 0, \ldots, n$, is continuous. For this, we observe that the infinite direct product $C_{p^{k+1}}^{\infty}$ is clearly a compact group with respect to the product topology. Furthermore, the group W_n^+ is, as a topological space, isomorphic to $(K^+)^n$, the n-fold direct product of compact groups and thus itself compact. So in order to show that the map Λ_k is continuous, it suffices to show that Λ_k is componentwise continuous. We will show that $\pi_j \circ \Lambda_k$ is continuous for every $j \in \{1, \ldots, n+1\}$, where $\pi_j : W_n^+ \to K^+, (x_0, \ldots, x_n) \mapsto x_{j+1}$ denotes the projection onto the jth component.

Let $k \in \{0, \ldots, n\}$. We have

$$\Lambda_k((a_m^{(k)})_{m\in\mathbb{N}_0}) = \sum_{\substack{m\in\mathbb{N}_0 \\ m\notin p\,\mathbb{N}_0}} a_{f(m)}^{(k)}(0, \cdots, 0, t^m, 0, \cdots, 0), \tag{1.16}$$

where the term t^m is at the $(n+1-k)$th position in the $(n+1)$-tuple $(0, \cdots, 0, t^m, 0, \cdots, 0)$. Hence $\pi_j \circ \Lambda_k((a_m)_{m\in\mathbb{N}_0}) = 0$ for every $j \in \{1, \ldots, n-k\}$ and in particular $\pi_j \circ \Lambda_k$ is continuous for every $j \in \{1, \ldots, n-k\}$. To see that $\pi_j \circ \Lambda_k$ is also continuous for every $j \in \{n+1-k, \ldots, n+1\}$ we observe that the locally compact topology on K is constructed in a way that open sets of the form $U_r :=< t^r >$ with $r \in \mathbb{N}_0$ form a neighborhoodbasis of $0 \in K$. Let U_r, $r \in \mathbb{N}_0$, be such a neighborhood in K. We show that there exists a neighborhood V_{s_r} of $0 \in C_{p^{k+1}}^{\infty}$ such that $(\pi_j \circ \Lambda_k)(V_{s_r}) \subseteq U_r$ for all $j \in \{n+1-k, \ldots, n+1\}$. For this, put $s_r := f^{-1}(r)$ and define

$$V_{s_r} := \{(a_m^{(k)})_{m\in\mathbb{N}_0} \in C_{p^{k+1}}^{\infty} \mid a_i^{(k)} = 0 \; \forall \; i = 0, \ldots, s_r - 1\}.$$

This is a neighborhood of 0 in the product topology of $C^{\infty}_{p^{k+1}}$.
Let $(0,\dots,0,a^{(k)}_{s_r},a^{(k)}_{s_r+1},\dots)$ be an arbitrary element of V_{s_r}, then

$$(\pi_j \circ \Lambda_k)(0,\dots,0,a^{(k)}_{s_r},a^{(k)}_{s_r+1},\dots) \in U_r$$

for all $j \in \{n+1-k,\dots,n+1\}$, since we have seen in (1.15) that the smallest exponent of t appearing in each nonzero entry of

$$\sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} a^{(k)}_{f(m)}(0,\cdots,0,t^m,0,\cdots,0)$$

is at least $f(s_r)=r$. Hence the map $\pi_j \circ \Lambda_k$ is continuous for every $j \in \{n+1-k,\dots,n+1\}$, which proves that Λ_k is continuous. The continuity of Λ_n can be obtained in the same way. □

Next, we obtain the same result for the maps $\Psi_k, k=0,\cdots,n$.

Lemma 1.4.13. *The map Ψ_k, as defined in (1.14), is a continuous group homomorphism for every $k \in \{0,\dots n-1\}$. Furthermore, the map Ψ_n, as defined in (1.13), is a continuous group homomorphism.*

Proof. We can apply the same arguments as in the proof of Lemma 1.4.12. In fact, the proof is even simpler since all the sums appearing in the definition of the maps $\Psi_k, k\in\{0,\dots n-1\}$ and Ψ_n are finite. □

We will now state and prove the key result of this section, namely that the sum of all the continuous group homomorphisms Λ_k, $k\in\{0,\dots,n\}$, yields an isomorphism Λ between the group W_n^+ and the compact group $C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p}$. Furthermore, we prove that the sum of all the continuous group homomorphisms Ψ_k, $k\in\{0,\dots,n\}$, yields an isomorphism Ψ between the group W_n^- and the discrete group $C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p}$.

Proposition 1.4.14. *(i) The map*

$$\begin{aligned}\Lambda : C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p} &\to W_n^+,\\ (a^{(n)},a^{(n-1)},\dots,a^{(0)}) &\mapsto \Lambda_n(a^{(n)}) \oplus \Lambda_{n-1}(a^{(n-1)}) \oplus \cdots \oplus \Lambda_0(a^{(0)})\end{aligned}$$

is an isomorphism of topological groups.

(ii) The map

$$\begin{aligned}\Psi : C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p} &\to W_n^-,\\ (b^{(n)},b^{(n-1)},\dots,b^{(0)}) &\mapsto \Psi_n(b^{(n)}) \oplus \Psi_{n-1}(b^{(n-1)}) \oplus \cdots \oplus \Psi_0(b^{(0)})\end{aligned}$$

is an isomorphism of topological groups.

Before we give a proof of this proposition we rewrite the sum appearing in the definition of the map Λ in the following way. Recall that we have for all vectors $(a^{(n)}, a^{(n-1)}, \ldots, a^{(0)}) \in C^\infty_{p^{n+1}} \times C^\infty_{p^n} \times \cdots \times C^\infty_p$,

$$\begin{aligned}\Lambda((a^{(n)}, a^{(n-1)}, \ldots, a^{(0)})) &= \Lambda_n(a^{(n)}) \oplus \Lambda_{n-1}(a^{(n-1)}) \oplus \cdots \oplus \Lambda_0(a^{(0)}) \\ &= \sum_{m\in\mathbb{N}_0} a^{(n)}_m (t^m, 0, \ldots, 0) \oplus \sum_{\substack{m\in\mathbb{N}_0 \\ m\notin p\,\mathbb{N}_0}} a^{(n-1)}_{f(m)} (0, t^m, 0, \ldots, 0) \oplus \cdots \\ &\quad \oplus \sum_{\substack{m\in\mathbb{N}_0 \\ m\notin p\,\mathbb{N}_0}} a^{(0)}_{f(m)} (0, \ldots, 0, t^m).\end{aligned}$$

We may write every coefficient $a^{(k)}_{f(m)} \in C_{p^{k+1}} = \mathbb{Z}/p^{k+1}\,\mathbb{Z}$, $0 \leq k < n$, $m \in \mathbb{N}_0$, with respect to its p-adic expansion, i.e., we can find uniquely determined numbers $0 \leq a^{(k)}_{f(m)_j} \leq p-1$, $j = 0, \ldots, k$, such that

$$a^{(k)}_{f(m)} = a^{(k)}_{f(m)_0} + a^{(k)}_{f(m)_1} p + a^{(k)}_{f(m)_2} p^2 + \cdots + a^{(k)}_{f(m)_k} p^k. \tag{1.17}$$

In the same way we can write every coefficient $a^{(n)}_m \in C_{p^{n+1}} = \mathbb{Z}/p^{n+1}\,\mathbb{Z}$, $m \in \mathbb{N}_0$, with respect to its p-adic expansion, i.e., we can find uniquely determined numbers $0 \leq a^{(n)}_{m_j} \leq p-1$, $j = 0, \ldots, n$, such that

$$a^{(n)}_{(m)} = a^{(n)}_{m_0} + a^{(n)}_{m_1} p + a^{(n)}_{m_2} p^2 + \cdots + a^{(n)}_{m_n} p^n. \tag{1.18}$$

Using (1.17) and (1.18) we obtain for all $0 \leq k \leq n-1$:

$$\begin{aligned}a^{(k)}_{f(m)}(0, \ldots, t^m, \ldots, 0) &= (a^{(k)}_{f(m)_0} + a^{(k)}_{f(m)_1} p + \cdots + a^{(k)}_{f(m)_k} p^k)(0, \ldots, 0, t^m, 0, \ldots, 0) \\ &= a^{(k)}_{f(m)_0}(0, \ldots, 0, t^m, 0, \ldots, 0) \oplus a^{(k)}_{f(m)_1} p\ (0, \ldots, 0, t^m, 0, \ldots, 0) \\ &\quad \oplus \cdots \oplus a^{(k)}_{f(m)_k} p^k\ (0, \ldots, 0, t^m, 0, \ldots, 0).\end{aligned}$$

If we apply part (ii) of Lemma 1.4.5 to the last expression we obtain

$$\begin{aligned}a^{(k)}_{f(m)}(0, \ldots, 0, t^m, 0, \ldots, 0) &= a^{(k)}_{f(m)_0}(0, \ldots, 0, t^m, 0, \ldots, 0) \\ &\quad \oplus a^{(k)}_{f(m)_1}(0, \ldots, 0, t^{mp}, 0, \ldots, 0) \oplus \cdots \oplus a^{(k)}_{f(m)_k}(0, \ldots, 0, t^{mp^k}).\end{aligned}$$

Using the definition of addition in the nth Witt group we obtain, as in (1.15), for every $0 \leq k \leq n-1$ and $0 \leq j \leq k$:

$$a^{(k)}_{f(m)_j}(0, \ldots, 0, t^{mp^j}, 0, \ldots, 0) = (0, \ldots, 0, a^{(k)}_{f(m)_j} t^{mp^j}, c^{(j,k)}_{n+1-k+j}(m), \ldots, c^{(j,k)}_n(m)),$$

where all the terms $c^{(j,k)}_{n-k+1+j}(m), \ldots, c^{(j,k)}_n(m)$ are polynomials in t, whose smallest exponent of t is greater than or equal to m and whose coefficients are uniquely determined by the coefficients $a^{(k)}_{f(m)_j}$. Notice that if $k=0$, then the terms $c^{(j,0)}$ do not occur. Furthermore, we have for all $0 \le j \le n$:

$$a^{(n)}_{m_j}(0,\dots,0,t^{mp^j},0,\dots,0) = (0,\dots,0,a^{(n)}_{m_j}t^{mp^j}, c^{(j,n)}_{j+1}(m),\dots,c^{(j,n)}_{n}(m)),$$

where all the terms $c^{(j,n)}_{j+1}(m),\dots,c^{(j,n)}_{n}(m)$ are polynomials in t, whose smallest exponent of t is at least m and whose coefficients are uniquely determined by the coefficients $a^{(n)}_{f(m)_j}$.

With this notation we obtain, for every $0 \le k \le n-1$,

$$\begin{aligned} a^{(k)}_{f(m)}(0,\dots,0,t^m,0,\dots,0) &= (0,\dots,0,a^{(k)}_{f(m)_0}t^m, c^{(0,k)}_{n+1-k},\dots,c^{(0,k)}_{n}) \\ &\oplus(0,\dots,0,0,a^{(k)}_{f(m)_1}t^{mp}, c^{(1,k)}_{n+1-k+1},\dots,c^{(1,k)}_{n}) \oplus \dots \oplus (0,0,\dots,0,a^{(k)}_{f(m)_k}t^{mp^k}). \end{aligned} \tag{1.19}$$

Furthermore, we have

$$\begin{aligned} a^{(k)}_{m}(t^m 0,\dots,0) &= (a^{(n)}_{m_0}t^m, c^{(0,n)}_{1},\dots,c^{(0,n)}_{n}) \\ &\oplus(0,a^{(n)}_{m_1}t^{mp}, c^{(1,n)}_{2},\dots,c^{(1,n)}_{n}) \oplus \dots \oplus (0,0,\dots,0,a^{(n)}_{m_n}t^{mp^n}). \end{aligned} \tag{1.20}$$

If we use (1.19) for every $k \in \{0,\dots,n-1\}$ and (1.20) for $k=n$, then we obtain

$$\begin{aligned} &\Lambda((a^{(n)},a^{(n-1)},\dots,a^{(0)})) \\ &= \sum_{m\in\mathbb{N}_0} a^{(n)}_{m}(t^m,0,\dots,0) \oplus \sum_{\substack{m\in\mathbb{N}_0 \\ m\notin p\mathbb{N}_0}} a^{(n-1)}_{f(m)}(0,t^m,0,\dots,0) \\ &\qquad \oplus \dots \oplus \sum_{\substack{m\in\mathbb{N}_0 \\ m\notin p\mathbb{N}_0}} a^{(0)}_{f(m)}(0,\dots,0,t^m) \\ &= \sum_{m\in\mathbb{N}_0} [(a^{(n)}_{m_0}t^m, c^{(0,n)}_{1}(m),\dots,c^{(0,n)}_{n}(m)) \\ &\qquad \oplus(0,a^{(n)}_{m_1}t^{mp}, c^{(1,n)}_{2}(m),\dots,c^{(1,n)}_{n}(m)) \oplus \dots \oplus (0,0,\dots,0,a^{(n)}_{m_n}t^{mp^n})] \\ &\quad \oplus \sum_{\substack{m\in\mathbb{N}_0 \\ m\notin p\mathbb{N}_0}} [(0,a^{(n-1)}_{f(m)_0}t^m, c^{(0,n-1)}_{2}(m),\dots,c^{(0,n-1)}_{n}(m)) \oplus \\ &\qquad (0,0,a^{(n-1)}_{f(m)_1}t^{mp}, c^{(1,n-1)}_{3}(m),\dots,c^{(1,n-1)}_{n}(m)) \\ &\qquad\quad \oplus \dots \oplus (0,0,\dots,0,a^{(n-1)}_{f(m)_{n-1}}t^{mp^{n-1}})] \\ &\quad \oplus \dots \\ &\quad \oplus \sum_{\substack{m\in\mathbb{N}_0 \\ m\notin p\mathbb{N}_0}} (0,0,\dots,0,a^{(0)}_{f(m)_0}t^m). \end{aligned}$$

Changing the order of summation in a suitable way yields

$$
\begin{aligned}
&\Lambda((a^{(n)}, a^{(n-1)}, \ldots, a^{(0)})) \qquad (1.21)\\
= &\sum_{m\in\mathbb{N}_0} (a^{(n)}_{m_0} t^m, c_1^{(0,n)}(m), \ldots, c_n^{(0,n)}(m))\\
\oplus &\sum_{m\in\mathbb{N}_0} (0, a^{(n)}_{m_1} t^{mp}, c_2^{(1,n)}(m), \ldots, c_n^{(1,n)}(m))\\
&\quad \oplus \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} (0, a^{(n-1)}_{f(m)_0} t^m, c_2^{(0,n-1)}(m), \ldots, c_n^{(0,n-1)}(m))\\
\oplus &\sum_{m\in\mathbb{N}_0} (0, 0, a^{(n)}_{m_2} t^{mp^2}, c_3^{(2,n)}(m), \ldots, c_n^{(2,n)}(m))\\
&\quad \oplus \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} [(0, 0, a^{(n-1)}_{f(m)_1} t^{mp}, c_3^{(1,n-1)}(m), \ldots, c_n^{(1,n-1)}(m))\\
&\qquad\qquad \oplus (0, 0, a^{(n-2)}_{f(m)_0} t^m, c_3^{(0,n-2)}(m), \ldots, c_n^{(0,n-2)}(m))]\\
\oplus &\ \ldots\\
\oplus &\sum_{m\in\mathbb{N}_0} (0, \ldots, 0, a^{(n)}_{m_n} t^{mp^n})\\
&\quad \oplus \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} [(0, \ldots, 0, a^{(n-1)}_{f(m)_{n-1}} t^{mp^{n-1}})\\
&\qquad\qquad \oplus (0, \ldots, 0, a^{(n-2)}_{f(m)_{n-2}} t^{mp^{n-2}}) \oplus \cdots \oplus (0, \ldots, 0, a^{(0)}_{f(m)_0} t^m)].
\end{aligned}
$$

To avoid lengthy descriptions we will use the following notation.

Notation 1.4.15. Let $t, s \in \mathbb{Z}$ with $t \mid s$, and let $a, b \in \mathbb{Z}/s\,\mathbb{Z} = C_s$. We say that $a = b \bmod t$ if $a + t \cdot C_s = b + t \cdot C_s$.

We now prove Proposition 1.4.14.

Proof. To (i): Observe first that, as a direct consequence of Lemma 1.4.12, the map Λ is a well-defined group homomorphism. So it remains to show that Λ is onto, one-to-one, and bi-continuous.

In order to prove the surjectivity of Λ, let $(x^{(n)}, x^{(n-1)}, \ldots, x^{(0)})$ be an arbitrary vector of W_n^+. Each entry $x^{(k)}$, $k = 0, \ldots, n$, of this vector is of the form

$$x^{(k)} = \sum_{i=0}^{\infty} x_i^{(k)} t^i \quad \text{with } x_i^{(k)} \in C_p = \mathbb{Z}/p\,\mathbb{Z}\,.$$

We need to show that there exists a vector

$$(a^{(n)}, a^{(n-1)}, \ldots, a^{(0)}) \in C^\infty_{p^{n+1}} \times C^\infty_{p^n} \times \ldots \times C^\infty_p$$

with the property that

$$\Lambda((a^{(n)}, a^{(n-1)}, \dots, a^{(0)})) = (x^{(n)}, x^{(n-1)}, \dots, x^{(0)}). \tag{1.22}$$

We prove by induction on k, $k \in \{0, \dots, n\}$, the statement
$I(k)$: We can find

(1.) a series $a^{(n)} \in C^{\infty}_{p^{n+1}}$, which is uniquely determined mod p^{k+1}, i.e., each element $a^{(n)}_m \in C_{p^{n+1}}$, $m \in \mathbb{N}_0$, of the series $a^{(n)}$ is uniquely determined mod p^{k+1}, and

(2.) for every $1 \leq j \leq k$, a series $a^{(n-j)} \in C^{\infty}_{p^{n-j+1}}$, which is uniquely determined mod p^{k+1-j}, i.e., each element $a^{(n-j)}_{f(m)} \in C_{p^{n-j+1}}$, $m \in \mathbb{N}_0$, of the series $a^{(n-j)}$ is uniquely determined mod p^{k+1-j},

such that the vector $(a^{(n)}, a^{(n-1)}, \dots, a^{(0)})$ satisfies Equation (1.22).

Notice that this proves then the surjectivity of Λ since by $I(n)$ we can find series

$$a^{(n)} \in C^{\infty}_{p^{n+1}}, a^{(n-1)} \in C^{\infty}_{p^{n}}, \dots, \text{ and } a^{(0)} \in C^{\infty}_{p}$$

such that the vector $(a^{(n)}, a^{(n-1)}, \dots, a^{(0)})$ satisfies (1.22).

If $k = 0$, we use the summation formula (1.21) to compare the first component of the vector $\Lambda((a^{(n)}, a^{(n-1)}, \dots, a^{(0)}))$ with the first component of the vector $(x^{(n)}, x^{(n-1)}, \dots, x^{(0)})$. This yields the following conditions for the series $a^{(n)} \in C^{\infty}_{p^{n+1}}$:

$$\sum_{m \in \mathbb{N}_0} a^{(n)}_{m_0} t^m = \sum_{m \in \mathbb{N}_0} x^{(n)}_m t^m.$$

By comparing the coefficients of these sums we obtain the defining equation:

$$a^{(n)}_{m_0} := x^{(n)}_m \quad \forall m \in \mathbb{N}_0\,. \tag{1.23}$$

But this means that the coefficients $a^{(n)}_m$, $m \in \mathbb{N}_0$, of the series $a^{(n)} \in C^{\infty}_{p^{n+1}}$ are determined mod p and we have proven the base case $I(0)$.

If $k = 1$, we use again formula (1.21) to compare the second component of the vector $\Lambda((a^{(n)}, a^{(n-1)}, \dots, a^{(0)}))$ with the second component of the vector $(x^{(n)}, x^{(n-1)}, \dots, x^{(0)})$. This leads to the following conditions for the series $a^{(n)} \in C^{\infty}_{p^{n+1}}$ and $a^{(n-1)} \in C^{\infty}_{p^{n}}$:

$$\sum_{m \in \mathbb{N}_0} a^{(n)}_{m_1} t^{mp} + \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} (a^{(n-1)}_{f(m)_0} t^m + c^{(0,n)}_1(m)) = \sum_{m \in \mathbb{N}_0} x^{(n-1)}_m t^m. \tag{1.24}$$

Notice that there do not appear the same exponents of t in the expressions

$$\sum_{m \in \mathbb{N}_0} a^{(n)}_{m_1} t^{mp} \quad \text{and} \quad \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} a^{(n-1)}_{f(m)_0} t^m.$$

Furthermore, all coefficients appearing in the polynomials $c_1^{(0,n)}(m)$, $m \in \mathbb{N}_0$, depend only on the numbers $a_{m_0}^{(n)}$, which are already uniquely defined by Equation (1.23). Thus if we compare coefficients in (1.24) we obtain the following defining equations:

$$\begin{aligned} a_{m_1}^{(n)} &:= x_{mp}^{(n-1)} - F(mp) \quad \forall m \in \mathbb{N}_0 \text{ and} \\ a_{f(m)_0}^{(n-1)} &:= x_m^{(n-1)} - F(m) \quad \forall m \in \mathbb{N}_0 \setminus p\,\mathbb{N}_0, \end{aligned} \tag{1.25}$$

where $F(m)$ and $F(mp)$ denote some numbers, which depend only on the coefficients of the term $c_1^{(0,n)}(m)$ and hence on the numbers $a_{m_0}^{(n)}$, $m \in \mathbb{N}_0$. Thus the series, $a^{(n)}$ and $a^{(n-1)}$, are determined mod p^2 and mod p, respectively, and we have proven the statement $I(1)$.

Let $k \in \{0, \dots, n-1\}$ be fixed and assume that $I(j)$ holds for every $0 \leq j \leq k$. In order to prove the statement $I(k+1)$ we use again formula (1.21) and compare the $(k+2)$nd component of the vector $\Lambda((a^{(n)}, a^{(n-1)}, \dots, a^{(0)}))$ with the $(k+2)$nd component of the vector $(x^{(n)}, x^{(n-1)}, \dots, x^{(0)})$. This yields the following condition for the series $a^{(n)} \in C_{p^{n+1}}^{\infty}, a^{(n-1)} \in C_{p^n}^{\infty}, \dots,$ and $a^{n-(k+1)} \in C_{p^{n-k}}^{\infty}$:

$$\sum_{m \in \mathbb{N}_0} a_{m_{k+1}}^{(n)} t^{mp^{k+1}} + \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} [a_{f(m)_k}^{(n-1)} t^{mp^k} + \cdots + a_{f(m)_0}^{(n-(k+1))} t^m] + X(m) = \sum_{m \in \mathbb{N}_0} x_m^{(n-(k+1))} t^m, \tag{1.26}$$

where $X(m)$ is a polynomial in t, whose coefficients consist of linear combinations in the numbers

$$a_{m_0}^{(n)}, a_{m_1}^{(n)}, \dots, a_{m_k}^{(n)}; a_{f(m)_0}^{(n-1)}, \dots, a_{f(m)_{k-1}}^{(n-1)}; a_{f(m)_0}^{(n-2)}, \dots, a_{f(m)_{k-2}}^{(n-2)}; \dots; a_{f(m)_0}^{(n-k)}.$$

But it follows from the induction hypothesis that these numbers are already uniquely determined by the elements of the series $x^{(n)}, x^{(n-1)}, \dots,$ and $x^{(n-k)}$. Furthermore, there do not appear the same exponents of t in the sums

$$\sum_{m \in \mathbb{N}_0} a_{m_{k+1}}^{(n)} t^{mp^{k+1}}, \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} a_{f(m)_k}^{(n-1)} t^{mp^k}, \cdots, \text{and} \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} a_{f(m)_0}^{(n-(k+1))} t^m,$$

so that comparing coefficients in (1.26) leads to the following defining equations for elements of the series $a^{(n)}, a^{(n-1)}, a^{(n-2)}, \dots,$ and $a^{(n-(k+1))}$:

$$\begin{array}{lcllll} a_{m_{k+1}}^{(n)} & := & x_{mp^{k+1}}^{(n-(k+1))} & -F(mp^{k+1}) & \forall m \in \mathbb{N}_0 \\ a_{f(m)_k}^{(n-1)} & := & x_{mp^k}^{(n-(k+1))} & -F(mp^k) & \forall m \in \mathbb{N}_0 \setminus p\,\mathbb{N}_0 \\ a_{f(m)_{k-1}}^{(n-2)} & := & x_{mp^{k-1}}^{(n-(k+1))} & -F(mp^{k-1}) & \forall m \in \mathbb{N}_0 \setminus p\,\mathbb{N}_0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{f(m)_0}^{(n-(k+1))} & := & x_m^{(n-(k+1))} & -F(m) & \forall m \in \mathbb{N}_0 \setminus p\,\mathbb{N}_0\,. \end{array}$$

Therefore, the series $a^{(n)}$, $a^{(n-1)}$, $a^{(n-2)}$, …, $a^{(n-(k+1))}$ are determined modulo p^{k+2}, p^{k+1}, p^k …, p, respectively, and so we have proven $I(k+1)$.

To prove the injectivity of the map Λ, let

$$(a^{(n)}, a^{(n-1)}, \dots, a^{(0)}) \in C^\infty_{p^{n+1}} \times C^\infty_{p^n} \times \dots \times C^\infty_p$$

with

$$\Lambda((a^{(n)}, a^{(n-1)}, \dots, a^{(0)})) = (0, \dots, 0) \in W_n^+.$$

We prove by induction on k, $k \in \{0, \dots, n\}$, the statement
$I(k)$: The series $a^{(n)}, a^{(n-1)}, \dots, a^{(n-k)}$ satisfy

(1.) $a^{(n)} = 0 \mod p^{k+1}$, i.e., $a^{(n)}_m = 0 \mod p^{k+1}$ for all $m \in \mathbb{N}_0$, and

(2.) $a^{(n-j)} = 0 \mod p^{k+1-j}$ for all $j \in \{1, \dots, k\}$, i.e., $a^{(n-j)}_{f(m)} = 0 \mod p^{k+1-j}$ for all $m \in \mathbb{N}_0$.

If $k = 0$, we obtain with formula (1.21)

$$\sum_{m \in \mathbb{N}_0} a^{(n)}_{m_0} t^m = 0, \tag{1.27}$$

and hence

$$a^{(n)}_{m_0} = 0 \ \forall m \in \mathbb{N}_0,$$

which proves the base case $I(0)$.

So let $k \in \{0, \dots, n-1\}$ be fixed and assume that $I(j)$ holds for every $j \in \{0, \dots, k\}$. Then we have, for every $m \in \mathbb{N}_0$,

(1.) $a^{(n)}_m = 0 \mod p^{k+1}$ and hence $a^{(n)}_{m_0} = a^{(n)}_{m_1} = \dots = a^{(n)}_{m_k} = 0$, and

(2.) $a^{(n-j)}_{f(m)} = 0 \mod p^{k+1-j}$ for all $j \in \{1, \dots, k\}$ and hence

$$a^{(n-j)}_{f(m)_0} = a^{(n-j)}_{f(m)_1} = \dots = a^{(n-j)}_{f(m)_{k-j}} = 0.$$

Thus, if we set the $(k+2)$nd component of the vector $\Lambda((a^{(n)}, a^{(n-1)}, \dots, a^{(0)}))$ equal to zero, we obtain, from formula (1.21), the following equation

$$\sum_{m \in \mathbb{N}_0} a^{(n)}_{m_{k+1}} t^{mp^{k+1}} + \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\mathbb{N}_0}} [a^{(n-1)}_{f(m)_k} t^{mp^k} + \dots + a^{(n-(k+1))}_{f(m)_0} t^m] + X(m) = 0, \tag{1.28}$$

where $X(m)$ is a polynomial in t, whose coefficients consist of linear combinations in the numbers

$$a^{(n)}_{m_0}, a^{(n)}_{m_1}, \dots, a^{(n)}_{m_k}, a^{(n-1)}_{f(m)_0}, \dots, a^{(n-1)}_{f(m)_{k-1}}, a^{(n-2)}_{f(m)_0}, \dots, a^{(n-2)}_{f(m)_{k-2}}, \dots, a^{(n-k)}_{f(m)_0}.$$

But it follows from the induction hypothesis that these numbers are all equal to zero and hence $X(m) = 0$. Therefore, (1.28) turns into

$$\sum_{m \in \mathbb{N}_0} a^{(n)}_{m_{k+1}} t^{mp^{k+1}} + \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} [a^{(n-1)}_{f(m)_k} t^{mp^k} + \cdots + a^{(n-(k+1))}_{f(m)_0} t^m] = 0. \tag{1.29}$$

Since there do not appear the same exponents of t in the sums

$$\sum_{m \in \mathbb{N}_0} a^{(n)}_{m_{k+1}} t^{mp^{k+1}}, \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} a^{(n-1)}_{f(m)_k} t^{mp^k}, \cdots, \text{and} \sum_{\substack{m \in \mathbb{N}_0 \\ m \notin p\,\mathbb{N}_0}} a^{(n-(k+1))}_{f(m)_0} t^m,$$

we may easily compare coefficients in (1.29) and obtain

$$\begin{array}{lcl} a^{(n)}_{m_{k+1}} & = & 0 \quad \forall m \in \mathbb{N}_0, \\ a^{(n-1)}_{f(m)_k} & = & 0 \quad \forall m \in \mathbb{N}_0 \setminus p\,\mathbb{N}_0, \\ a^{(n-2)}_{f(m)_{k-1}} & = & 0 \quad \forall m \in \mathbb{N}_0 \setminus p\,\mathbb{N}_0, \\ \vdots & \vdots & \vdots \quad \vdots \\ a^{(n-(k+1))}_{f(m)_0} & = & 0 \quad \forall m \in \mathbb{N}_0 \setminus p\,\mathbb{N}_0 . \end{array}$$

This proves the statement $I(k+1)$ and hence the injectivity of the map Λ.

It remains to prove that the map Λ is bi-continuous. But we have seen already in Lemma 1.4.12 that each map $\Lambda_k : C^\infty_{p^{k+1}} \to W^+_n$, $k \in \{0, \ldots, n\}$ is bi-continuous. Hence Λ is, as the sum of bi-continuous maps, itself bi-continuous.

The proof of part (ii) is similar. □

We now establish the main theorem of this section. It summarizes the results obtained so far and gives us precise information about the structure of the nth Witt group, $W_n(K)$, of the field $K = \mathbb{F}_p((t))$.

Theorem 1.4.16. *Let $K = \mathbb{F}_p((t))$ for some prime p, let $n \in \mathbb{N}_0$, and let W_n be the nth Witt group of K. The map*

$$\begin{aligned} \Theta :& (C^\infty_{p^{n+1}} \times C^\infty_{p^n} \times \cdots \times C^\infty_p) \times (C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_p) \longrightarrow W_n, \\ & ((a^{(n)}, a^{(n-1)}, \ldots, a^{(0)}), (b^{(n)}, b^{(n-1)}, \ldots, b^{(0)})) \mapsto \\ & \quad \Lambda((a^{(n)}, a^{(n-1)}, \ldots, a^{(0)})) \oplus \Psi((b^{(n)}, b^{(n-1)}, \ldots, b^{(0)})), \end{aligned}$$

where Λ and Ψ are defined as in Proposition 1.4.14, is an isomorphism of topological groups.

Proof. The map

$$\mu : W_n^+ \times W_n^- \to W_n, (x, y) \mapsto x \oplus y$$

is an isomorphism of topological groups (Lemma 1.4.8) and we have

$$\Theta((a, b)) = \mu(\Lambda(a), \Psi(b))$$

for all $a \in C_{p^{n+1}}^{\infty} \times C_{p^n}^{\infty} \times \cdots \times C_p^{\infty}$ and $b \in C_{p^{n+1}}^{(\infty)} \times C_{p^n}^{(\infty)} \times \cdots \times C_p^{(\infty)}$. Since, by Proposition 1.4.14, both maps Λ and Ψ are isomorphisms of topological groups, it follows that Θ is, as the composition of isomorphisms, itself an isomorphism of topological groups. □

We may also obtain a more general version of Theorem 1.4.16, i.e., a similar decomposition of the nth Witt group of every local field of characteristic p. For this, let p be a prime, let $k = \mathbb{F}_{p^r}$ for some fixed $r \in \mathbb{N}$, and let $K := \mathbb{F}_{p^r}((t))$ be the field of formal Laurent series over k. As for the field $\mathbb{F}_p((t))$ (see Notation 1.4.7), we introduce the following notations. We define

- $K^+ := k[[t]] \subseteq K$ to be the power series ring over k,
- $K^- := \{a_1 t^{-1} + a_2 t^{-2} + \ldots + a_n t^{-n} \mid n \in \mathbb{N},\ a_i \in k\}$,
- $W_n^+(K) := \{(x_0, \ldots,\ x_n) \in W_n \mid x_i \in K^+ \ \forall\ 0 \leq i \leq n\}$, and
- $W_n^-(K) := \{(x_0, \ldots, x_n) \in W_n \mid x_i \in K^- \ \forall\ 0 \leq i \leq n\}$.

The set K^- is an additive subgroup of K which is also closed under multiplication. Clearly, every element $a \in K$ can be written uniquely as $a = a^+ + a^-$, where $a^+ \in K^+$ and $a^- \in K^-$. Furthermore, we can decompose the nth Witt group $W_n(K)$ into a direct product of its subgroups $W_n^+(K)$ and $W_n^-(K)$.

Lemma 1.4.17. *(i) The sets $W_n^+(K)$ and $W_n^-(K)$, defined as above, are subgroups of $W_n(K)$ and we have $W_n^+(K) \cap W_n^-(K) = \{0\}$.*

(ii) The map $\mu : W_n^+(K) \times W_n^-(K) \to W_n(K),\ (x, y) \mapsto x \oplus y$ is a bi-continuous isomorphism.

Proof. The proof of Lemma 1.4.8 goes through without any modifications. □

We will show in the remaining part of this section that, for every $n \in \mathbb{N}_0$, the nth Witt group $W_n(K)$ can be decomposed as follows.

$$\begin{aligned} W_n(K) &\cong W_n^-(K) \times W_n^+(K) \\ &\cong (C_{p^{n+1}}^{(\infty)})^r \times (C_{p^n}^{(\infty)})^r \times \cdots \times (C_p^{(\infty)})^r \times (C_{p^{n+1}}^{\infty})^r \times (C_{p^n}^{\infty})^r \times \cdots \times (C_p^{\infty})^r. \end{aligned}$$

We observe that the finite field $k = \mathbb{F}_{p^r}$ is a vector space over the finite field $\mathbb{F}_p$, and we can choose elements $\omega_0, \omega_1, \ldots, \omega_{r-1} \in \mathbb{F}_{p^r}$ such that the set $\{\omega_0, \omega_1, \ldots, \omega_{r-1}\}$

is a basis of $\mathbb{F}_{p^r}$ over $\mathbb{F}_p$, i.e., for every element $x \in \mathbb{F}_{p^r}$ there exist unique elements ${}^0a, {}^1a, \dots, {}^{r-1}a \in \mathbb{F}_p$ such that $x = \sum_{i=0}^{r-1} {}^ia\omega_i$.

In the same way as in Definition 1.4.11, we can now define maps Λ_k and Ψ_k, $k \in \{0, 1, \dots, n\}$, between the r-fold direct product of $C_{p^{k+1}}^{\infty}$ and $W_n^+(K)$, and between the r-fold direct product of $C_{p^{k+1}}^{(\infty)}$ and $W_n^-(K)$. Recall that $f : \mathbb{N}_0 \setminus p\,\mathbb{N}_0 \to \mathbb{N}_0$ denotes the unique monotone bijective function from $\mathbb{N}_0 \setminus p\,\mathbb{N}_0$ to $\mathbb{N}_0$.

Definition 1.4.18. Let $n \in \mathbb{N}_0$ be fixed and define

$$\Lambda_n : (C_{p^{n+1}}^{\infty})^r \longrightarrow W_n^+(K), \quad \big(({}^0a_m^{(n)})_{m\in\mathbb{N}_0}, ({}^1a_m^{(n)})_{m\in\mathbb{N}_0}, \dots, ({}^{r-1}a_m^{(n)})_{m\in\mathbb{N}_0}\big) \mapsto$$
$$\sum_{m\in\mathbb{N}_0} \big({}^0a_m^{(n)}(\omega_0 t^m, 0, \dots, 0) \oplus {}^1a_m^{(n)}(\omega_1 t^m, 0, \dots, 0)$$
$$\oplus \dots \oplus {}^{r-1}a_m^{(n)}(\omega_{r-1} t^m, 0, \dots, 0)\big).$$

For every $i = 0, \dots, r-1$, we view ${}^ia_m^{(n)} \in C_{p^{n+1}} = \mathbb{Z}/p^{n+1}\mathbb{Z}$ as an integer between 0 and $p^{n+1}-1$ in the canonical way and understand the product ${}^ia_m^{(n)}(\omega_i t^m, 0, \dots, 0)$ as the ${}^ia_m^{(n)}$-fold sum of the $(n+1)$-tuple $(\omega_i t^m, 0, \dots, 0)$ in $W_n^+(K)$.

Furthermore, we define for every $0 \le k \le n-1$:

$$\Lambda_k : (C_{p^{k+1}}^{\infty})^r \longrightarrow W_n^+(K), \quad \big(({}^0a_m^{(k)})_{m\in\mathbb{N}_0}, ({}^1a_m^{(k)})_{m\in\mathbb{N}_0}, \dots, ({}^{r-1}a_m^{(k)})_{m\in\mathbb{N}_0}\big) \mapsto$$
$$\sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} \big({}^0a_{f(m)}^{(k)}(0, \dots, 0, \omega_0^{p^{n-k}} t^m, \dots, 0) \oplus {}^1a_{f(m)}^{(k)}(0, \dots, 0, \omega_1^{p^{n-k}} t^m, 0, \dots, 0)$$
$$\oplus \dots \oplus {}^{r-1}a_{f(m)}^{(k)}(0, \dots, 0, \omega_{r-1}^{p^{n-k}} t^m, 0, \dots, 0)\big),$$

where, for every $i = 0, \dots, r-1$, the term $\omega_i^{p^{n-k}} t^m$ is at the $(n+1-k)$th position in the $(n+1)$-tuple $(0, \cdots, 0, \omega_i^{p^{n-k}} t^m, 0, \cdots, 0)$. Again, we view $a_{f(m)}^{(k)} \in C_{p^{k+1}} = \mathbb{Z}/p^{k+1}\mathbb{Z}$ as an integer between 0 and $p^{k+1}-1$ in the canonical way and understand the product ${}^ia_{f(m)}^{(k)}(0, \cdots, 0, \omega_i^{p^{n-k}} t^m, 0, \cdots, 0)$ as the ${}^ia_{f(m)}^{(k)}$-fold sum in $W_n^+(K)$ of the $(n+1)$-tuple $(0, \cdots, 0, \omega_i^{p^{n-k}} t^m, 0, \cdots, 0)$.

Similarly, we define

$$\Psi_n : (C_{p^{n+1}}^{(\infty)})^r \longrightarrow W_n^+(K), \quad \big(({}^0b_m^{(n)})_{m\in\mathbb{N}_0}, ({}^1b_m^{(n)})_{m\in\mathbb{N}_0}, \dots, ({}^{r-1}b_m^{(n)})_{m\in\mathbb{N}_0}\big) \mapsto$$
$$\sum_{m\in\mathbb{N}_0} \big({}^0b_m^{(n)}(\omega_0 t^{-m}, 0, \dots, 0) \oplus {}^1b_m^{(n)}(\omega_1 t^{-m}, 0, \dots, 0)$$
$$\oplus \dots \oplus {}^{r-1}b_m^{(n)}(\omega_{r-1} t^{-m}, 0, \dots, 0)\big)$$

and for every $0 \le k \le n-1$:

$$\Psi_k : (C^{(\infty)}_{p^{k+1}})^r \to W^-_n(K), \quad \big(({}^0b^{(k)}_m)_{m\in\mathbb{N}_0}, ({}^1b^{(k)}_m)_{m\in\mathbb{N}_0}, \dots, ({}^{r-1}b^{(k)}_m)_{m\in\mathbb{N}_0}\big) \mapsto$$
$$\sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} \big({}^0b^{(k)}_{f(m)}(0,\dots,0,\omega_0^{p^{n-k}}t^{-m},\dots,0) \oplus {}^1b^{(k)}_{f(m)}(0,\dots,0,\omega_1^{p^{n-k}}t^{-m},0,\dots,0)$$
$$\oplus \dots \oplus {}^{r-1}b^{(k)}_{f(m)}(0,\dots,0,\omega_{r-1}^{p^{n-k}}t^{-m},0,\dots,0)\big),$$

where, for every $i = 0,\dots,r-1$, the term $\omega_i^{p^{n-k}}t^{-m}$ is at the $(n+1-k)$th position in the $(n+1)$-tuple $(0,\cdots,0,\omega_i^{p^{n-k}}t^{-m},0,\cdots,0)$. Again, we view $b^{(k)}_{f(m)} \in C_{p^{k+1}} = \mathbb{Z}/p^{k+1}\,\mathbb{Z}$ as an integer between 0 and $p^{k+1}-1$ in the canonical way and understand the product ${}^ib^{(k)}_{f(m)}(0,\cdots,0,\omega_i^{p^{n-k}}t^{-m},0,\cdots,0)$ as the ${}^ib^{(k)}_{f(m)}$-fold sum in $W^-_n(K)$ of the $(n+1)$-tuple $(0,\cdots,0,\omega_i^{p^{n-k}}t^{-m},0,\cdots,0)$.

Lemma 1.4.19. *The maps Λ_k and Ψ_k, as defined in Definition 1.4.18, are continuous group homomorphisms for every $k \in \{0,\dots,n\}$.*

Proof. Let $k \in \{0,\dots,n\}$. For every $i \in \{0,\dots,r-1\}$, we obtain by the same arguments as in the proof of Lemma 1.4.12 a continuous group homomorphism

$$\Lambda^i_k : C^\infty_{p^{k+1}} \longrightarrow W^+_n(K),\ ({}^ia^{(k)}_m)_{m\in\mathbb{N}_0} \mapsto \sum_{\substack{m\in\mathbb{N}_0\\ m\notin p\,\mathbb{N}_0}} {}^ia^{(k)}_{f(m)}(0,\dots,0,\omega_i^{p^{n-k}}t^m,\dots,0).$$

Since

$$\Lambda_k\big(({}^0a^{(k)},{}^1a^{(k)},\dots,{}^{r-1}a^{(k)})\big) = \Lambda^0_k({}^0a^{(k)}) \oplus \Lambda^1_k({}^1a^{(k)}) \oplus \dots \oplus \Lambda^{r-1}_k({}^{r-1}a^{(k)}),$$

it follows that the map Λ_k is, as the sum of the continuous group homomorphisms Λ^i_k, $i = 0,\dots,r-1$, itself a well-defined, continuous group homomorphism. $\square$

Proposition 1.4.20. *Let $K = \mathbb{F}_q((t))$, where $q = p^r$ for some prime $p > 0$ and some $r \in \mathbb{N}$, let $n \in \mathbb{N}_0$, and let $W_n(K)$ be the nth Witt group of K.*

(i) The map

$$\begin{aligned}\Lambda : (C^\infty_{p^{n+1}})^r \times (C^\infty_{p^n})^r \times \dots \times (C^\infty_p)^r &\longrightarrow W^+_n(K),\\ \big(({}^0a^{(n)},{}^1a^{(n)},\dots,{}^{r-1}a^{(n)}), ({}^0a^{(n-1)},{}^1a^{(n-1)},\dots,{}^{r-1}a^{(n-1)}),\dots,&\\ ({}^0a^{(0)},{}^1a^{(0)},\dots,{}^{r-1}a^{(0)})\big) &\mapsto\\ \Lambda_n\big(({}^0a^{(n)},{}^1a^{(n)},\dots,{}^{r-1}a^{(n)})\big) \oplus \Lambda_{n-1}\big(({}^0a^{(n-1)},{}^1a^{(n-1)},\dots,{}^{r-1}a^{(n-1)})\big)&\\ \oplus \dots \oplus \Lambda_0\big(({}^0a^{(0)},{}^1a^{(0)},\dots,{}^{r-1}a^{(0)})\big)&\end{aligned}$$

is an isomorphism of topological groups.

(ii) The map

$$\begin{aligned}\Psi : (C^{(\infty)}_{p^{n+1}})^r \times (C^{(\infty)}_{p^n})^r \times \cdots \times (C^{(\infty)}_{p})^r &\longrightarrow W_n^-(K),\\ \big(({}^0b^{(n)}, {}^1b^{(n)}, \ldots, {}^{r-1}b^{(n)}), ({}^0b^{(n-1)}, {}^1b^{(n-1)}, \ldots, {}^{r-1}b^{(n-1)}), \ldots,\\ ({}^0b^{(0)}, {}^1b^{(0)}, \ldots, {}^{r-1}b^{(0)})\big) \mapsto\\ \Psi_n\big(({}^0b^{(n)}, {}^1b^{(n)}, \ldots, {}^{r-1}b^{(n)})\big) \oplus \Psi_{n-1}\big(({}^0b^{(n-1)}, {}^1b^{(n-1)}, \ldots, {}^{r-1}b^{(n-1)})\big)\\ \oplus \cdots \oplus \Psi_0\big(({}^0b^{(0)}, {}^1b^{(0)}, \ldots, {}^{r-1}b^{(0)})\big)\end{aligned}$$

is an isomorphism of topological groups.

Proof. We observe first that if the set $\{\omega_0, \omega_1, \ldots, \omega_{r-1}\}$ is a basis of the finite field $\mathbb{F}_{p^r}$ over the finite field $\mathbb{F}_p$, then also the set $\{\omega_0^{p^i}, \omega_1^{p^i}, \ldots, \omega_{r-1}^{p^i}\}$, for every $i \in \mathbb{N}$. With this fact, the proof of the bijectivity of the maps Λ and Ψ is a straightforward application of the proof of Proposition 1.4.14. $\square$

In the same way as in Theorem 1.4.16 we may now obtain the following decomposition of the nth Witt group of any local field K into a discrete and a compact part.

Proposition 1.4.21. *Let $K = \mathbb{F}_q((t))$, where $q = p^r$ for some prime $p > 0$ and some $r \in \mathbb{N}$, let $n \in \mathbb{N}_0$, and let $W_n(K)$ be the nth Witt group of K. Then*

$$W_n(K) \cong (C^{\infty}_{p^{n+1}})^r \times \cdots \times (C^{\infty}_{p})^r \times (C^{(\infty)}_{p^{n+1}})^r \times \cdots \times (C^{(\infty)}_{p})^r. \tag{1.30}$$

Proof. The map

$$\mu : W_n^+(K) \times W_n^-(K) \to W_n(K),\ (x, y) \mapsto x \oplus y$$

is an isomorphism of topological groups (Lemma 1.4.17) and we have

$$\Theta((a, b)) = \mu(\Lambda(a), \Psi(b))$$

for all $a \in (C^{\infty}_{p^{n+1}})^r \times (C^{\infty}_{p^n})^r \times \cdots \times (C^{\infty}_{p})^r$ and $b \in (C^{(\infty)}_{p^{n+1}})^r \times (C^{(\infty)}_{p^n})^r \times \cdots \times (C^{(\infty)}_{p})^r$. Since, by Proposition 1.4.20, both maps Λ and Ψ are isomorphisms of topological groups, it follows that Θ is, as the composition of isomorphisms, itself an isomorphism of topological groups. $\square$

1.4.3 Duality of Witt groups

With the detailed information about the structure of finite-dimensional Witt groups over local fields of characteristic $p > 0$, it is now easy to see that such groups are topologically isomorphic to their dual groups.

Proposition 1.4.22. *Let $K = \mathbb{F}_p((t))$ for some prime p. The nth Witt group of K is, as a topological group, selfdual for every $n \in \mathbb{N}_0$, i.e., $\widehat{W_n(K)} \cong W_n(K)$.*

Proof. The proof of this proposition follows directly from Theorem 1.4.16 and the facts (1)-(4) about the dual group of locally compact abelian groups listed in Section 1.2. Theorem 1.4.16 yields

$$W_n(K) \cong (C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p}) \times (C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p}).$$

Since C_{p^k} is a finite cyclic group we have $(C_{p^k})\widehat{\ } \cong C_{p^k}$ for every $k \in \{1, \ldots, n+1\}$. Additionally, we have for every $k \in \{1, \ldots, n+1\}$,

$$(C^{\infty}_{p^k})\widehat{\ } \cong \big(\prod_{i=0}^{\infty} C_{p^k}\big)\widehat{\ } \cong \bigoplus_{i=0}^{\infty} (C_{p^k})\widehat{\ } \cong \bigoplus_{i=0}^{\infty} C_{p^k} = C^{(\infty)}_{p^k}$$

and

$$(C^{(\infty)}_{p^k})\widehat{\ } \cong \big(\bigoplus_{i=1}^{\infty} C_{p^k}\big)\widehat{\ } \cong \prod_{i=1}^{\infty} (C_{p^k})\widehat{\ } \cong \prod_{i=1}^{\infty} C_{p^k} = C^{\infty}_{p^k}.$$

With these results we obtain

$$\begin{aligned}
\widehat{W_n(K)} &\cong (C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p} \times C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p})\widehat{\ } \\
&\cong C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p} \times C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p} \\
&\cong W_n(K).
\end{aligned}$$

□

Corollary 1.4.23. *For any local field K of characteristic p and every $n \in \mathbb{N}_0$, the nth Witt group $W_n(K)$ is isomorphic to its dual group, as a topological group.*

Proof. Let K be any local field of characteristic p. Then K is isomorphic to a field of formal Laurent series in one indeterminate with coefficients in a finite field of characteristic p, i.e., $K \cong \mathbb{F}_q((t))$, where $q = p^r$ for some $r \in \mathbb{N}$. But by Proposition 1.4.21 we have

$$\begin{aligned}
W_n(K) &\cong \big((C^{\infty}_{p^{n+1}})^r \times \cdots \times (C^{\infty}_{p})^r\big) \times \big((C^{(\infty)}_{p^{n+1}})^r \times \cdots \times (C^{(\infty)}_{p})^r\big) \\
&\cong \big((C^{\infty}_{p^{n+1}} \times \cdots \times C^{\infty}_{p}) \times (C^{(\infty)}_{p^{n+1}} \times \cdots \times C^{(\infty)}_{p})\big)^r,
\end{aligned}$$

and thus, we obtain by the same arguments as in the proof of Proposition 1.4.22

$$\begin{aligned}
\widehat{W_n(K)} &\cong \big((C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p} \times C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p})^r\big)\widehat{\ } \\
&\cong \big((C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p} \times C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p})\widehat{\ }\big)^r \\
&\cong \big(C^{(\infty)}_{p^{n+1}} \times C^{(\infty)}_{p^n} \times \cdots \times C^{(\infty)}_{p} \times C^{\infty}_{p^{n+1}} \times C^{\infty}_{p^n} \times \cdots \times C^{\infty}_{p}\big)^r \\
&\cong W_n(K).
\end{aligned}$$

□

1.4.4 Characters of the first Witt group

In this subsection, we give an explicit description of the characters of the first Witt group $W_1(K)$, where $K = \mathbb{F}_p((t))$ for some prime p.

By Theorem 1.4.16 we have

$$W_1(K) \cong (\bigoplus_{i=1}^{\infty} C_{p^2} \times \prod_{i=0}^{\infty} C_{p^2}) \times (\bigoplus_{i=1}^{\infty} C_p \times \prod_{i=0}^{\infty} C_p) \cong \bigoplus_{i=1}^{\infty}(C_{p^2} \times C_p) \times \prod_{i=0}^{\infty}(C_{p^2} \times C_p).$$

So, in order to describe the characters of $W_1(K)$ we can use the isomorphism of Theorem 1.4.16 and describe instead the characters of the group

$$H := \bigoplus_{i=1}^{\infty}(C_{p^2} \times C_p) \times \prod_{i=0}^{\infty}(C_{p^2} \times C_p).$$

Since C_{p^j}, $j = 1, 2$, is a finite cyclic group, every character $\chi \in \widehat{C_{p^j}}$ is of the form

$$\chi = \chi_{v_j} : C_{p^j} \to \mathbb{T},\ s_j \mapsto \exp(\frac{2\pi i s_j v_j}{p^j}),$$

for some $v_j \in C_{p^j}$. Thus every character $\chi \in \widehat{C_{p^2} \times C_p}$ is of the form $\chi = \chi_v$, where $v = (v_1, v_2) \in C_{p^2} \times C_p$ and we have

$$\chi_v : C_{p^2} \times C_p \to \mathbb{T},\ \chi_v(s_1, s_2) = \chi_{v_1}(s_1) \cdot \chi_{v_2}(s_2),$$

where χ_{v_1} is a character of C_{p^2} and χ_{v_2} is a character of C_p, as above.

We define a "duality bracket" in the following way:

$$\langle v, s\rangle_{C^{p^2} \times C^p} := \langle v_1, s_1\rangle_{C^{p^2}} \cdot \langle v_2, s_2\rangle_{C^p} := \chi_{(v_1,v_2)}(s_1, s_2) = \chi_v(s). \tag{1.31}$$

Observe that

$$(\bigoplus_{i=1}^{\infty}(C_{p^2} \times C_p))\widehat{\ } \cong \prod_{i=0}^{\infty}(C_{p^2} \times C_p) \text{ and } (\prod_{i=0}^{\infty}(C_{p^2} \times C_p))\widehat{\ } \cong \bigoplus_{i=1}^{\infty}(C_{p^2} \times C_p).$$

Thus we can define a character $\chi_x \in \widehat{H}$, $x = (x_m)_{m\in\mathbb{Z}} \in H$ by defining it first on every component of the sequence $s = (s_m)_{m\in\mathbb{Z}} \in H$:

$$\chi_x(s_m) := \langle x_{-m}, s_m\rangle_{C^{p^2} \times C^p}.$$

The character $\chi_x \in \hat{H}$, $x = (x_m)_{m\in\mathbb{Z}} \in H$, is then of the form

$$\chi_x(s) = \prod_{m\in\mathbb{Z}} \langle x_{-m}, s_m\rangle_{C^{p^2} \times C^p} \tag{1.32}$$

and it is clear that every character of H is of such a form. Notice that since only finitely many components with negative subscript of x and s are nonzero, the product in (1.32) is well-defined.

1.5 The structure of abelian K-split groups

In the following, let K be a local field. Recall that we denote by G_a the additive group of the field K. In this section we give a complete characterization of abelian K-split groups. As we have seen in Section 1.3, the basic building-blocks for these groups are the abelian, algebraic extensions of the additive group G_a with itself. Recall that we denote by $\operatorname{Ext}(G_a, G_a)$ the set of all group extensions given by symmetric algebraic 2-cocycles $f : G_a \times G_a \to G_a$ and we will identify such group extensions with the corresponding 2-cocycle. During this section we will follow an approach of Serre [29], chapter VII to the structure of commutative unipotent groups, state the most important results, and prove some additional facts, which will be needed in the next section.

Remark 1.5.1. A general assumption made in [29], chapter VII, is that the base field K is algebraically closed. But studying the relevant proofs in that chapter, one can show that this assumption can be removed. In fact, all the results cited in this section hold for any local field K.

Proposition 1.5.2. *([29], Proposition 8) In characteristic* 0, $\operatorname{Ext}(G_a, G_a) = 0$. *In characteristic $p > 0$, the K-vector space* $\operatorname{Ext}(G_a, G_a)$ *admits for a basis the p^nth powers ($n \in \mathbb{N}_0$) of the 2-cocycle f which defines the first Witt group $W_1(K)$:*

$$f(x, y) = \frac{1}{p}(x^p + y^p - (x + y)^p).$$

Note that the right hand side of the equation above should be considered as a formal sum.

We sketch briefly the idea of the proof. One writes the polynomial $g(x, y)$, which determines the group extension, in the form $\sum a_{ij} x^i y^j$. Then formula (1.3) translates into identities for the coefficients a_{ij} which allow one to determine explicitly all symmetric 2-cocycles. For the details of the computation see [24], §III.

Corollary 1.5.3. *([29], Corollary of Proposition 8) In characteristic* 0*, every commutative connected unipotent group is isomorphic to a product of copies of the additive group G_a.*

Proposition 1.5.2 indicates the relevance of finite-dimensional Witt groups in the field of abelian K-split groups. In the following we recall and state some facts concerning these groups, see also [29], chapter VII. The definition of the nth Witt group $W_n(K) =: W_n$ of a field K is given in Section 1.4.1. There exist two maps which are very useful in this context:

(1) the Shift homomorphism $S : W_n \to W_{n+1}, (x_0, \ldots, x_n) \mapsto (0, x_0, \ldots, x_n)$ and

(2) the Restriction homomorphism $R : W_{n+1} \to W_n, (x_0, \ldots, x_{n+1}) \mapsto (x_0, \ldots, x_n)$.

These homomorphisms commute with each other and we obtain, for all $m, n \in \mathbb{N}_0$, an exact sequence:

$$0 \longrightarrow W_m \xrightarrow{S^{n+1}} W_{n+m+1} \xrightarrow{R^{m+1}} W_n \longrightarrow 0. \tag{1.33}$$

We denote the corresponding element of $\mathrm{Ext}(W_n, W_m)$ by V_n^m. The following commutative diagram shows the effect of the restriction homomorphism R on these extensions

$$\begin{tikzcd}
0 \arrow[r] & W_m \arrow[r] \arrow[d, "R"] & W_{n+m+1} \arrow[r] \arrow[d, "R"] & W_n \arrow[r] \arrow[d, equal, "id"] & 0 \\
0 \arrow[r] & W_{m-1} \arrow[r] & W_{n+m} \arrow[r] & W_n \arrow[r] & 0.
\end{tikzcd}$$

Thus we obtain the formula

$$R_*(V_n^m) = V_n^{m-1},$$

where $R_*(V_n^m)$ denotes the pushout of V_n^m by the map R as indicated in the above diagram. Analogously, we have the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & W_m \arrow[r] & W_{n+m+1} \arrow[r] & W_n \arrow[r] & 0 \\
0 \arrow[r] & W_m \arrow[r] \arrow[u, equal, "id"] & W_{n+m} \arrow[r] \arrow[u, "S"] & W_{n-1} \arrow[r] \arrow[u, "S"] & 0.
\end{tikzcd}$$

And thus we obtain the formula

$$S^*(V_n^m) = V_{n-1}^m, \tag{1.34}$$

where $S^*(V_n^m)$ denotes the pullback of V_n^m by the map S. In the same way one can show

$$S_*(V_n^m) = R^*(V_{n-1}^{m+1}).$$

We denote by A_n the ring of endomorphisms of the algebraic group W_n, $n \in \mathbb{N}_0$. The pushout operation $\varphi_*(V_n^m)$ and the pullback operation $\varphi^*(V_n^m)$ give the group $\mathrm{Ext}(W_n, W_m)$ the structure of a left module over A_m and a right module over A_n, respectively, and these two structures are compatible in the above sense.

Remark 1.5.4. The group W_0 is just the additive group G_a and the exact sequence (1.33) shows that the nth Witt group W_n, $n \in \mathbb{N}_0$, is a multiple extension of the additive group G_a. For $m \leq n$, we can identify W_m with a subgroup of W_n by means of S^{n-m} and we have $W_m = p^{n-m}W_n$ (see also Lemma 1.4.5 in Section 1.4.2). Furthermore, the mth Witt groups W_m, $m \leq n$, are the only connected subgroups of W_n ([29], VII, Section 8).

The following definition is a useful instrument in algebraic geometry.

Definition 1.5.5. (i) A homomorphism between two algebraic groups is called an isogeny if it is surjective with finite kernel.

(ii) We say that two algebraic groups G and H are isogenous if there exist isogenies $f : G \to H$ and $g : H \to G$.

Remark 1.5.6. ([29], chapter VII) Let $n \in \mathbb{N}_0$ and let G be an abelian unipotent linear algebraic group. The following are equivalent:

(i) There exists an isogeny $f : G \to W_n$.

(ii) There exists an isogeny $g : W_n \to G$.

Lemma 1.5.7. *([29], VII, §2, Lemma 3) Every element $H \in \mathrm{Ext}(G_a, G_a)$ can be written uniquely as $H = \varphi^*(V_0^0)$ (or $\psi_*(V_0^0)$), where φ and ψ are elements of A_0. Furthermore one has $\varphi^*(V_0^0)$ is the trivial extension if and only if φ is not an isogeny.*

Proof. The existence and uniqueness of φ works as follows. The element $V_0^0 \in \mathrm{Ext}(G_a, G_a)$ corresponds to a symmetric 2-cocycle $\omega : G_a \times G_a \to G_a$ which determines the first Witt group:

$$V_0^0 : \ 0 \longrightarrow G_a \longrightarrow W_1 \longrightarrow G_a \longrightarrow 0.$$

Let $H \in \mathrm{Ext}(G_a, G_a)$ be an abelian algebraic group extension of G_a. According to Proposition 1.5.2, the element H corresponds to a symmetric 2-cocycle of the form

$$f(x, y) = \sum_i a_i \, \omega(x, y)^{p^i} \text{ with } a_i \in K.$$

On the other hand, every endomorphisms φ of G_a can be written uniquely as

$$\varphi(x) = \sum_i b_i \, x^{p^i}.$$

Hence we have $H = \varphi^*(V_0^0)$ if and only if $b_i = a_i$ for all i, which proves the existence and uniqueness of φ. The other parts are similar. □

With Lemma 1.5.7 we can obtain a useful characterization of the elements of $\mathrm{Ext}(G_a, G_a)$.

Corollary 1.5.8. *Let H be an element of $\mathrm{Ext}(G_a, G_a)$. Then H is either isomorphic (as an algebraic group) to $G_a \times G_a$ or isogenous to the 2-dimensional Witt group $W_1(K)$.*

Proof. By Lemma 1.5.7 we can find a map $\varphi \in \mathrm{End}(G_a, G_a)$ such that $H = \varphi^*(V_0^0)$. If $H = \varphi^*(V_0^0) = 0$ then H splits, which means that H is isomorphic to $G_a \times G_a$. Otherwise the map φ is an isogeny, and since the corresponding pullback diagram

$$\begin{tikzcd} 0 \arrow[r] & G_a \arrow[r] & W_1 \arrow[r] & G_a \arrow[r] & 0 \\ 0 \arrow[r] & G_a \arrow[r] \arrow[u, equal, "id"'] & H \arrow[r] \arrow[u, "\phi"'] & G_a \arrow[r] \arrow[u, "\varphi"'] & 0 \end{tikzcd}$$

is commutative, it follows as an application of the Snake-Lemma that the map $\phi : H \to W_1$ is an isogeny. $\square$

There are also similar results for higher dimensional Witt groups.

Lemma 1.5.9. *([29], VII, §2, Lemma 6) Every element H of $\mathrm{Ext}(W_n, G_a)$ can be written as $H = \varphi^*(V_n^0)$ for some $\varphi \in A_n$. One has $\varphi^*(V_n^0) = 0$ if and only if φ is not an isogeny.*

One can also reverse the roles of W_n and G_a.

Lemma 1.5.10. *([29], VII, §2, Lemma 6') Every element H of $\mathrm{Ext}(G_a, W_n)$ can be written as $H = \varphi_*(V_0^n)$ for some $\varphi \in A_n$. One has $\varphi_*(V_0^n) = 0$ if and only if φ is not an isogeny.*

As in the case $n = 0$, we obtain a characterization of the elements of $\mathrm{Ext}(G_a, W_n)$ and $\mathrm{Ext}(W_n, G_a)$.

Corollary 1.5.11. *Let H be an element of either $\mathrm{Ext}(G_a, W_n)$ or $\mathrm{Ext}(W_n, G_a)$. Then H (i.e., the linear algebraic group defined by the exact sequence H) is either isomorphic to $W_n \times G_a$ or isogenous to W_{n+1}.*

Proof. We will prove the corollary for $\mathrm{Ext}(G_a, W_n)$, the case of $\mathrm{Ext}(W_n, G_a)$ is similar. As in the two-dimensional case we have either $H = (\varphi)^* V_0^n = 0$ for some $\varphi \in A_n$ and thus H splits and is isomorphic to $W_n \times G_a$, or there exists an isogeny φ from G_a to G_a such that H is the pullback of W_{n+1} and G_a under φ. Since the diagram

$$\begin{tikzcd} 0 \arrow[r] & W_n \arrow[r] & W_{n+1} \arrow[r] & G_a \arrow[r] & 0 \\ 0 \arrow[r] & W_n \arrow[r] \arrow[u, equal, "id"'] & H \arrow[r] \arrow[u, "\phi"'] & G_a \arrow[r] \arrow[u, "\varphi"'] & 0 \end{tikzcd}$$

is commutative, it follows as an application of the Snake-Lemma that the map $\phi : H \to W_n$ is an isogeny. $\square$

Lemma 1.5.12. *([29], VII, §2, Lemma 7) If $m \geq n$, every element $H \in \mathrm{Ext}(W_n, G_a)$ can be written as $H = f_*(V_m^0)$ with $f \in \mathrm{Hom}(W_n, W_m)$.*

The next theorem demonstrates the exact connection between abelian unipotent K-split groups and Witt groups.

Theorem 1.5.13. *([29], VII, §2, Theorem 1) Every commutative unipotent K-split group is isogenous to a finite product of Witt groups.*

In order to get a better understanding of this theorem, we give a sketch of the proof.

Proof. Let G be a commutative unipotent K-split group of dimension $n \in \mathbb{N}$. We argue by induction on n.

If $n = 1$ then $G = G_a = W_0(K)$ and there is nothing to prove.

So let $n \in \mathbb{N}$ and suppose that the theorem is shown for all abelian K-split groups of dimension less than n. The group G is an extension of a group H of dimension $n-1$ by the group G_a. Applying the induction hypothesis to the group H yields an isogeny

$$f : \prod_{i=1}^{k} W_{n_i} \to H.$$

Put $W := \prod_{i=1}^{k} W_{n_i}$. The pullback $f^*(G)$ is an extension of W by G_a and this pullback is isogenous to G:

$$\begin{tikzcd}
0 \arrow[r] & G_a \arrow[r] & G \arrow[r] & H \arrow[r] & 0 \\
0 \arrow[r] & G_a \arrow[r] \arrow[u, equal, "id"'] & f^*(G) \arrow[r] \arrow[u, "F"'] & W \arrow[r] \arrow[u, "f"'] & 0.
\end{tikzcd}$$

Thus it suffices to show that $f^*(G)$ is isogenous to a product of Witt groups. In other words we are reduced to the case where $H = W$. Replacing $f^*(G)$ by G, let us denote the extension in question by $\gamma \in \mathrm{Ext}(W, G_a)$.

The extension γ is defined by a family of elements $\gamma_i \in \mathrm{Ext}(W_{n_i}, G_a)$. Suppose that $n_1 \geq n_i$ for all i and let $V = \prod_{i=2}^{k} W_{n_i}$. We are going to distinguish two cases.

1.) $\gamma_1 = 0$. The group G is then the product of W_{n_1} and the extension of V by G_a, defined by the system $(\gamma_i)_{i \geq 2}$. By the induction hypothesis, this extension of V by G_a is isogenous to a product of Witt groups and hence G is isogenous to a product of Witt groups.

2.) $\gamma_1 \neq 0$. Let $\beta = (\beta_i) \in \mathrm{Ext}(W, G_a)$ be the element defined by $\beta_1 = V_{n_1}^0$ and $\beta_i = 0$ for $i \geq 2$. The extension G' corresponding to β is the product $W_{n_1+1} \times V$. We are going to show the existence of an isogeny $\varphi : W \to W$ such that $\varphi^*(G')$ is isomorphic to G:

$$\begin{tikzcd}
0 \arrow[r] & G_a \arrow[r] & G' \arrow[r] & W \arrow[r] & 0 \\
0 \arrow[r] & G_a \arrow[r] \arrow[u, equal, "id"'] & \varphi^*(G') \arrow[r] \arrow[u, "\phi"'] & W \arrow[r] \arrow[u, "\varphi"'] & 0.
\end{tikzcd}$$

It will follow from this that G is isogenous to G', which is a product of Witt groups.

Applying Lemma 1.5.12 to every $\gamma_i \in \mathrm{Ext}(W_{n_i}, G_a)$ yields homomorphism $f_i \in \mathrm{Hom}(W_{n_i}, W_{n_1})$ such that $\gamma_i = f_{i_*} V^0_{n_1}$. Define the map $\varphi : W \to W$ by

$$\varphi(w_1, w_2, \ldots, w_k) = (f_1(w_1) + f_2(w_2) + \cdots + f_k(w_k), w_2, \ldots, w_k).$$

Then $\varphi^*(\beta) = \gamma$. Since f_1 is surjective (see Lemma 1.5.9), it follows immediately that φ is surjective and every surjective homomorphism between two groups of the same dimension has a finite kernel. Thus the map φ defines the desired isogeny.

□

1.6 Duality of abelian K-split groups

In the following, let K be a local field of characteristic $p > 0$. The aim of this section is to show that every abelian K-split group is, as a topological group, selfdual. Certainly not all abelian unipotent groups over K are selfdual since, for example, the K-wound groups are compact topological groups and thus have discrete dual groups. But we have seen in Section 1.4.3 that every finite dimensional Witt group over K is isomorphic to its dual group. Since every abelian K-split group is isogenous to a finite product of Witt groups, the question arises if every abelian K-split group is isomorphic to its topological dual group. The main step to see that this is indeed true is provided by the following result.

Proposition 1.6.1. *Let H be a unipotent linear algebraic group and suppose H is isogenous to $G = W_n(K)$, the nth Witt group of the field $K = \mathbb{F}_p((t))$. Then H is topologically isomorphic to G.*

Proof. Since H is isogenous to G, we can find a finite subgroup F of G such that $H \cong G/F$. So in order to show that H is isomorphic to G, it suffices to prove that $G \cong G/F$, where F is an arbitrary finite subgroup of G. But every finite subgroup F is of the form $F = \langle x_1, \ldots, x_k \rangle$ for some $x_1, \ldots, x_k \in G$ and we will prove that $G/\langle x \rangle \cong G$ for every $x \in G$, where $\langle x \rangle$ denotes the additive subgroup in G generated by x. (Notice that by Corollary 1.4.6 of Section 1.4.2 the group $G = W_n(K)$ is of exponent p^{n+1}, so in particular every element $x \in G$ has finite order and thus $\langle x \rangle$ is finite for every $x \in G$.) It then follows by an induction argument that $G/F \cong (G/\langle x_1 \rangle)/\langle x_2 \ldots, x_k \rangle \cong G/\langle x_2, \ldots x_k \rangle \cong G$.

Recall that we denote by $C_n := \mathbb{Z}/n\mathbb{Z}$ the cyclic group with n elements. Furthermore we write A^∞ for the infinite direct product $\prod_{i=0}^{\infty} A$ of a finite abelian group A and $A^{(\infty)}$ for the infinite direct sum $\bigoplus_{i=0}^{\infty} A$.

By Theorem 1.4.16 of Section 1.4.2 we know that the topological group $G = W_n(K)$ is of the form

$$G \cong C_p^\infty \times C_{p^2}^\infty \times \cdots \times C_{p^{n+1}}^\infty \times C_p^{(\infty)} \times C_{p^2}^{(\infty)} \times \cdots \times C_{p^{n+1}}^{(\infty)}.$$

So if we define $H_{p^i} := C_{p^i}^\infty \times C_{p^i}^{(\infty)}$, then

$$G \cong H_p \times H_{p^2} \times \cdots \times H_{p^{n+1}}$$

and every $x \in G$ is of the form $x = (x^1, \ldots, x^{n+1})$ with $x^i \in H_{p^i}$ for every $i \in \{1, \ldots, n+1\}$.

Let $x \in G$ and suppose $x \neq 0$. The finite group $\langle x \rangle$ is a subgroup of $C_{p^{n+1}}$, and thus $\langle x \rangle \cong C_{p^m}$ for some $m \in \{1, \ldots, n+1\}$. But every cyclic group C_{p^m}, $m \geq 1$, has a subgroup which is isomorphic to C_p. Thus, by replacing x by x^{p^k} for a suitable power k, we can assume without loss of generality that $\langle x \rangle \cong C_p$.

We consider two different cases:

1.) The intersection of the element x and the group H_p is not trivial, i.e., $x^1 \neq 0$. Then $\langle x^1 \rangle \cong C_p$ and $x^1 \in H_p$ is a Laurent series of the form $x^1 = (x^1_m)_{m \in \mathbb{Z}}$, where $x^1_m \in C_p$ for every $m \in \mathbb{Z}$. But the series x^1 generates the cyclic group C_p, and thus we can find an integer k such that $\langle x^1_k \rangle \cong C_p$. Observe that for every $y = (y^1, \ldots, y^{n+1}) \in G$ we can find a unique element of the span $\langle x^1_k \rangle$ which is equal to y^1_k. We denote by $\bar{x}^1$ the element in H_p defined by $\bar{x}^1_k = x^1_k$ and $\bar{x}^1_m = 0$ for all $m \in \mathbb{Z} \setminus \{k\}$. We will show

(a) $G/\langle x \rangle \cong H_p/\langle \bar{x}^1 \rangle \times H_{p^2} \times \cdots \times H_{p^{n+1}}$ and

(b) $H_p/\langle \bar{x}^1 \rangle \cong H_p$.

It follows directly from (a) and (b) that $G/\langle x \rangle \cong G$.

In order to show part (a) we define the map

$$\Phi : G \longrightarrow G,\ y \mapsto y - \varphi(y),$$

where $\varphi(y) = x' \in \langle x \rangle$ with ${x'}^1_k = y^1_k$. We conclude from the above observation that Φ is well-defined and clearly, Φ is a group homomorphism. Furthermore, we have $y - \varphi(y) = 0$ if and only if $y = \varphi(y)$ if and only if $y \in \langle x \rangle$, which shows that $\ker(\Phi) = \langle x \rangle$. Hence $G/\langle x \rangle$ is isomorphic to the image of Φ, which is isomorphic to the direct product $H_p/\langle \bar{x}^1 \rangle \times H_{p^2} \times \cdots \times H_{p^{n+1}}$.

In order to prove part (b), we recall that

$$H_p \cong \bigoplus_{i=1}^{\infty} C_p \times \prod_{i=0}^{\infty} C_p \quad \text{and} \quad \langle \bar{x}^1 \rangle \cong \langle \bar{x}^1_k \rangle \cong C_p.$$

Without loss of generality we assume $k = 0$. Notice that if $y^1, z^1 \in [y^1] \in H_p/\langle \bar{x}^1 \rangle$ are two elements of the same coset, then $y^1 - z^1 \in \langle \bar{x}^1 \rangle$ which means that there exists a number $\lambda \in C_p$ such that $y^1_l - z^1_l = \lambda \bar{x}^1_l$ for all $l \in \mathbb{Z}$. In particular, if $y^1, z^1 \in [y^1]$ with $y^1_0 = z^1_0 = 0$ then we obtain $y^1_l = z^1_l = 0$ for all $l \in \mathbb{Z}$ and thus $y^1 = z^1$, since $\langle x^1_0 \rangle \neq 0$. This means that in every coset $[y^1] \in H_p/\langle \bar{x}^1 \rangle$ there exists a unique element z^1 with $z^1_0 = 0$. We now define the map

$$\Psi : \bigoplus_{i=1}^{\infty} C_p \times \{0\} \times \prod_{i=1}^{\infty} C_p \longrightarrow H_p/\langle \bar{x}^1 \rangle, \quad y^1 \mapsto [y^1].$$

It follows directly from the above that Ψ is well-defined and it is not hard to see that Ψ is a group isomorphism. But the group $\bigoplus_{i=1}^{\infty} C_p \times \{0\} \times \prod_{i=1}^{\infty} C_p$ is obviously isomorphic to H_p, which completes the proof of part (b).

2.) The intersection of the element x and the group H_p is trivial, i.e., $x^1 = 0$. Let $i \in \{2, \dots, n+1\}$ be minimal with respect to the property that $x^i \neq 0$. Since $\langle x \rangle \cong C_p$, we have $\langle x^i \rangle \cong C_p$. As in the case $i = 1$, we know that x^i is a Laurent series of the form $x^i = (x^i_m)_{m \in \mathbb{Z}}$ with $x^i_m \in C_{p^i}$ for every $m \in \mathbb{Z}$. Since $\langle x^i \rangle \cong C_p$, there exists $k \in \mathbb{Z}$ such that $\langle x^i_k \rangle \cong C_p$. So for every $y = (y^i, \dots, y^{n+1}) \in H_{p^i} \times \cdots \times H_{p^{n+1}}$ we can find a unique element $z \in \langle x^i_k \rangle$ with $y^i_k + \big(C_{p^i}/C_p\big) = z + \big(C_{p^i}/C_p\big)$. (Or, if we view x^i_k as an element of $\{0, \dots, p^i - 1\} \cong C_{p^i}$, then $z \equiv y^i_k \mod p$.) Denote by $\bar{x}^i$ the element of H_{p^i} defined by $\bar{x}^i_k = x^i_k$ and $\bar{x}^i_m = 0$ for all $m \in \mathbb{Z} \backslash \{k\}$. We have

$$\big(H_p \times H_{p^2} \times \cdots \times H_{p^{n+1}}\big)/\langle x \rangle \cong H_p \times \cdots \times H_{p^{i-1}} \times \big(H_{p^i} \times \cdots \times H_{p^{n+1}}/\langle (x^i, \dots, x^{n+1}) \rangle\big)$$

and claim that it suffices to prove the statements

(a) $H_{p^i} \times \cdots \times H_{p^{n+1}}/\langle (x^i, \dots, x^{n+1}) \rangle \cong H_{p^i}/\langle \bar{x}^i \rangle \times H_{p^{i+1}} \times \cdots \times H_{p^{n+1}}$ and

(b) $H_{p^i}/\langle \bar{x}^i \rangle \cong \bigoplus_{i=1}^{\infty} C_{p^i} \times (C_{p^i}/C_p) \times \prod_{i=1}^{\infty} C_{p^i}$.

Indeed, using (a) and (b) and the fact that $(\prod_{i=0}^{\infty} C_{p^{i-1}}) \times C_{p^{i-1}} \cong \prod_{i=0}^{\infty} C_{p^{i-1}}$ and $\prod_{i=1}^{\infty} C_{p^i} \cong \prod_{i=0}^{\infty} C_{p^i}$, we obtain

$$\begin{aligned}
G/\langle x \rangle &\cong H_p \times \cdots \times H_{p^{i-1}} \times (H_{p^i}/\langle x^i \rangle) \times H_{p^{i+1}} \times \cdots \times H_{p^{n+1}} \\
&\cong \bigoplus_{i=1}^{\infty} C_p \times \prod_{i=0}^{\infty} C_p \times \cdots \times \bigoplus_{i=1}^{\infty} C_{p^{i-1}} \times \prod_{i=0}^{\infty} C_{p^{i-1}} \\
&\quad \times \Big(\bigoplus_{i=1}^{\infty} C_{p^i} \times C_{p^{i-1}} \times \prod_{i=1}^{\infty} C_{p^i}\Big) \times \cdots \times \bigoplus_{i=1}^{\infty} C_{p^{n+1}} \times \prod_{i=0}^{\infty} C_{p^{n+1}} \\
&\cong \bigoplus_{i=1}^{\infty} C_p \times \prod_{i=0}^{\infty} C_p \times \cdots \times \bigoplus_{i=1}^{\infty} C_{p^{i-1}} \times \prod_{i=0}^{\infty} C_{p^{i-1}} \\
&\quad \times \bigoplus_{i=1}^{\infty} C_{p^i} \times \prod_{i=0}^{\infty} C_{p^i} \times \cdots \times \bigoplus_{i=1}^{\infty} C_{p^{n+1}} \times \prod_{i=0}^{\infty} C_{p^{n+1}} \\
&\cong G.
\end{aligned}$$

In order to prove the statement (a), we may use exactly the same idea as in the first case. We define a map

$$\Phi : H_{p^i} \times \cdots \times H_{p^{n+1}} \longrightarrow H_{p^i} \times \cdots \times H_{p^{n+1}},\ y \mapsto y - \varphi(y),$$

where $\varphi(y) = x' \in \langle (x^i, \ldots, x^{n+1}) \rangle$ is defined so that $y^i_k + \big(C_{p^i}/C_p\big) = {x'}^i_k + \big(C_{p^i}/C_p\big)$. By the above remarks we know that Φ is a well-defined group homomorphism. The kernel of Φ is equal to $\langle (x^i, \ldots, x^{n+1}) \rangle$ and hence the quotient group $H_{p^i} \times \cdots \times H_{p^{n+1}} / \langle (x^i, \ldots, x^{n+1}) \rangle$ is isomorphic to the image of Φ, which is isomorphic to $H_{p^i}/\langle \bar{x}^i \rangle \times H_{p^{i+1}} \times \cdots \times H_{p^{n+1}}$.

For the proof of part (b), we assume without loss of generality that $k = 0$ and apply the same argument as above to the map

$$\Psi : \bigoplus_{i=1}^{\infty} C_{p^i} \times \prod_{i=0}^{\infty} C_{p^i} \longrightarrow \bigoplus_{i=1}^{\infty} C_{p^i} \times \prod_{i=0}^{\infty} C_{p^i},\ y \mapsto y - \psi(y),$$

where $\psi(y) = x' \in \langle x^i \rangle$ with $y^i_0 + \big(C_{p^i}/C_p\big) = {x'}^i_0 + \big(C_{p^i}/C_p\big)$. Since $\langle x^i_0 \rangle \cong C_p$, it follows that the image of Ψ is isomorphic to $\bigoplus_{i=1}^{\infty} C_{p^i} \times C_{p^i}/C_p \times \prod_{i=1}^{\infty} C_{p^i}$, which finishes the proof. $\square$

Lemma 1.6.2. *Let G be a finite product of Witt groups of the field $K = \mathbb{F}_q((t))$, where $q = p^r$ for some prime p and some $r \in \mathbb{N}$, i.e., $G = \prod_{i=1}^{k} W_{n_i}(K)$ for some $k \in \mathbb{N}$ and some $n_i \in \mathbb{N}_0$, $i = 1, \ldots, k$. Let n_j be the maximum of the set $\{n_i \mid i = 1, \ldots, k\}$. Then G is, as a topological group, isomorphic to $W_{n_j}\big(\mathbb{F}_p((t))\big)$.*

Proof. Using Proposition 1.4.21, the topological group G is of the form

$$\begin{aligned} G \ &\cong\ \prod_{i=1}^{k} (C_p^{\infty})^r \times (C_{p^2}^{\infty})^r \times \cdots \times (C_{p^{n_i+1}}^{\infty})^r \times (C_p^{(\infty)})^r \times (C_{p^2}^{(\infty)})^r \times \cdots \times (C_{p^{n_i+1}}^{(\infty)})^r \\ &\cong\ \prod_{i=1}^{k} (C_p^{(\infty)} \times C_p^{\infty})^r \times (C_{p^2}^{(\infty)} \times C_{p^2}^{\infty})^r \times \cdots \times (C_{p^{n_i+1}}^{(\infty)} \times C_{p^{n_i+1}}^{\infty})^r. \end{aligned}$$

But for all $j = 1, \ldots, n_i + 1$, we have

$$(C_{p^j}^{(\infty)} \times C_{p^j}^{\infty})^r \cong (C_{p^j}^{(\infty)} \times C_{p^j}^{\infty})$$

(as additive topological groups) and since the finite product $\prod_{i=1}^{k} (C_{p^j}^{(\infty)} \times C_{p^j}^{\infty})$ is topologically isomorphic to the group $C_{p^j}^{(\infty)} \times C_{p^j}^{\infty}$ for all $j = 1, \ldots, n_i + 1$, it follows that

$$G \cong (C_p^{(\infty)} \times C_p^{\infty}) \times (C_{p^2}^{(\infty)} \times C_{p^2}^{\infty}) \times \cdots \times (C_{p^{n_j+1}}^{(\infty)} \times C_{p^{n_j+1}}^{\infty}) \cong W_{n_j}\big(\mathbb{F}_p((t))\big).$$

$\square$

Corollary 1.6.3. *If K is any local field of characteristic p and G a commutative K-split group then G is, as a topological group, isomorphic to its dual group.*

Proof. Let K be any local field of characteristic p. Then K is isomorphic to a field of formal Laurent series in one indeterminate with coefficients in a finite field of characteristic p, i.e., $K \cong \mathbb{F}_q((t))$, where $q = p^r$ for some $r \in \mathbb{N}$. Let G be a commutative K-split group. Then G is isogenous to a finite product of Witt groups (Theorem 1.5.13), i.e., there exists $k \in \mathbb{N}$ and there exist $n_i \in \mathbb{N}$, $i = 1, \ldots, k$, such that G is isogenous to H, where

$$H = \prod_{i=1}^{k} (C_p^{(\infty)} \times C_p^{\infty})^r \times (C_{p^2}^{(\infty)} \times C_{p^2}^{\infty})^r \times \cdots \times (C_{p^{n_i+1}}^{(\infty)} \times C_{p^{n_i+1}}^{\infty})^r.$$

By Lemma 1.6.2, the group H is topologically isomorphic to $W_{n_j}\big(\mathbb{F}_p((t))\big)$ for some $n_j \in \{n_i \mid i = 1, \ldots, k\}$ and thus G is isogenous to the Witt group $W_{n_j}\big(\mathbb{F}_p((t))\big)$. It follows then from Proposition 1.6.1 that the topological group G is isomorphic to $W_{n_j}\big(\mathbb{F}_p((t))\big)$. Since every such finite dimensional Witt group is, as a topological group, self-dual (Proposition 1.4.22), it follows that G is self-dual. $\square$

Chapter 2

Kirillov Theory

2.1 Introduction to the Kirillov theory

For connected, simply connected nilpotent Lie groups G, A.A. Kirillov [23] provided a nice geometric description of the dual space $\hat{G}$, i.e., the equivalence classes of irreducible unitary representations of G that we now explain.

Suppose G is a nilpotent Lie group with Lie algebra $\mathfrak{g}$. There is a natural linear action of G on $\mathfrak{g}$ called the adjoint action: $\mathrm{Ad}(x)Y$ is the tangent vector of the curve $t \to x(\exp tY)x^{-1}$ at $t = 0$. This yields a linear action on the real dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ called the coadjoint action, given by $\mathrm{Ad}^*(x) = (\mathrm{Ad}(x^{-1}))^*$. Now, if $l \in \mathfrak{g}^*$ is a linear functional of $\mathfrak{g}$ and $\mathfrak{r}$ a subalgebra of $\mathfrak{g}$ such that $l \equiv 0$ on $[\mathfrak{r}, \mathfrak{r}]$, then $l|_{\mathfrak{r}}$ is a Lie algebra homomorphism from $\mathfrak{r}$ to $\mathbb{R}$. If R is a Lie subgroup of G with Lie algebra $\mathfrak{r}$, we shall call a one-dimensional representation φ_l of R such that $\varphi_l(\exp X) = e^{2\pi i l(X)}$ for $X \in \mathfrak{r}$ a character of R corresponding to l (one uses the Campbell-Hausdorff formula in order to prove that φ_l is indeed a character of R).

Theorem. *Let G be a connected, simply connected nilpotent Lie group. Given $l \in \mathfrak{g}^*$, let $\mathfrak{r}$ be a maximal subalgebra of $\mathfrak{g}$ such that $l \equiv 0$ on $[\mathfrak{r}, \mathfrak{r}]$, and let R and φ_l be as above. Then $\mathrm{ind}_R^G \varphi_l$ is an irreducible representation of G, and its equivalence class depends only on the orbit $\mathcal{O}_l$ of l under the coadjoint action. The map $\mathcal{O}_l \mapsto [\mathrm{ind}_R^G \varphi_l]$ is a bijection from the set of coadjoint orbits of $\mathfrak{g}^*$ to $\hat{G}$ which is a homeomorphism with respect to the natural quotient topology on the set of orbits and the Fell topology on $\hat{G}$.*

This theorem was proven by Kirillov [23] except for the fact that the inverse map $[\mathrm{ind}_R^G \varphi_l] \to \mathcal{O}_l$ is continuous, which is due to Brown [5].

In this chapter we develop a version of Kirillov theory which extends the theory for connected, simply connected nilpotent Lie groups and which can be applied to a large class of unipotent linear algebraic groups over local fields of characteristic p.

In Section 2.2 we introduce for a locally compact separable l-step nilpotent group G the notion of a nilpotent k-Lie pair $(G, \mathfrak{g})$, $k \geq l$. In this way we attach to the group G a Lie algebra $\mathfrak{g}$ over the ring $\mathbb{Z}[\frac{1}{k!}]$, including additional structure to obtain a reasonable definition of a dual space $\mathfrak{g}^*$ of $\mathfrak{g}$. The Lie algebra $\mathfrak{g}$ acts on itself by the adjoint action ad and we can exponentiate this action to obtain an action Ad of G on $\mathfrak{g}$. This action yields an action of G on the dual space $\mathfrak{g}^*$, called the coadjoint action, and we denote by $\mathfrak{g}^* /_\sim$ the space of quasi-orbits of $\mathfrak{g}^*$ with respect to this action.
In Section 2.6 we introduce the notion of a polarizing subalgebra $\mathfrak{r}$ for a homomorphism $f \in \mathfrak{g}^*$ and we will explain how such a map f defines a character φ_f on the subgroup $R := \exp(\mathfrak{r})$ of G. Furthermore, we show in Section 2.8 that the induced character $\operatorname{ind}_R^G \varphi_f$ is an irreducible representation of G for every homomorphism $f \in \mathfrak{g}^*$ and every chosen polarizing subalgebra $\mathfrak{r}$ for f. The aim is to define for every nilpotent k-Lie pair $(G, \mathfrak{g})$ a Kirillov map κ in the following way:

$$\kappa : \mathfrak{g}^* \longrightarrow \operatorname{Prim}(C^*(G)),\ f \mapsto \ker(\operatorname{ind}_R^G \varphi_f). \tag{2.1}$$

One difficulty is to prove that the kernel of the induced representation $\operatorname{ind}_R^G \varphi_f$ does not depend on the choice of the polarizing subalgebra $\mathfrak{r}$ for f.
In Section 2.7 we prove that the map κ is well-defined for every two-step nilpotent k-Lie pair, $k \geq 2$. We then develop in Section 2.8 some representational machinery, which we need in Section 2.9 to prove by induction on the nilpotence class of the group G that the map κ is well-defined and surjective for every nilpotent k-Lie pair $(G, \mathfrak{g})$.
Moreover, we show in Section 2.10 that, under certain additional assumptions on the group G, the "T_0-ization" analogue of the Kirillov-orbit map

$$\tilde{\kappa} : \mathfrak{g}^* /_\sim \longrightarrow \operatorname{Prim}(C^*(G)),\ \mathcal{O} \mapsto \ker(\operatorname{ind}_R^G \varphi_f), \tag{2.2}$$

where $f \in \mathfrak{g}^*$ is any chosen representative of the coadjoint quasi-orbit $\mathcal{O}$, is a homeomorphism with respect to the quotient topology on the quasi-orbit space and the hull-kernel topology on the primitive ideal space of $C^*(G)$.
Finally, we discuss in Section 2.11 three classes of nilpotent locally compact groups G, for which there exist a Lie algebra $\mathfrak{g}$ and a natural number k, such that the pair $(G, \mathfrak{g})$ defines a nilpotent k-Lie pair.

2.2 Nilpotent Lie pairs

Since the main objects of concern are nilpotent groups and nilpotent Lie algebras we recall the most important definitions.

Definition 2.2.1. Let G be a group.

(1) Define the following subgroups inductively

$$Z^0(G) := \{e\},\ Z^1(G) := Z(G),\ \text{and}\quad Z^{i+1}(G) := q_i^{-1}\big(Z(G/Z^i(G))\big),$$

where $Z(G/Z^i(G))$ denotes the center of $G/Z^i(G)$ and $q_i : G \to G/Z^i(G)$ the canonical quotient map. The chain of subgroups

$$\{e\} \trianglelefteq Z(G) \trianglelefteq Z^2(G) \trianglelefteq \cdots$$

is called the ascending central series of G.

(2) The group G is called nilpotent if its ascending central series terminates within finitely many steps, i.e., if $Z^l(G) = G$ for some $l \in \mathbb{N}$. The smallest such l is called the nilpotence class of G and G is said to be l-step nilpotent.

Definition 2.2.2. Let R be a commutative ring with unity. An algebra $\mathfrak{g}$ over R is called a Lie algebra over R if its multiplication, denoted by $(X, Y) \mapsto [X, Y]$, satisfies the following identities:

(1) $[X, X] = 0$ and

(2) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$. (Jacobi-identity)

A topological Lie algebra over R is a Lie algebra over R with a Hausdorff topology such that the Lie algebra operations $X \mapsto -X$, $(X, Y) \mapsto X + Y$, and $(X, Y) \mapsto [X, Y]$ are continuous with respect to this topology.

Remark 2.2.3. The product $[X, Y]$ is called the commutator of X and Y. If $\mathfrak{g}$ is a Lie algebra over R then $\mathfrak{g}$ is in particular an R-module and the commutator map $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is bi-additive.

Definition 2.2.4. Let R be a commutative ring with unity and let $\mathfrak{g}$ be a Lie algebra over R.

(1) The center of $\mathfrak{g}$ is defined as

$$\mathfrak{z}(\mathfrak{g}) := \{X \in \mathfrak{g} \mid [X, Y] = 0\ \forall\, Y \in \mathfrak{g}\}$$

and one defines the following subalgebras inductively

$$\mathfrak{z}^0(\mathfrak{g}) := \{e\},\ \mathfrak{z}^1(\mathfrak{g}) := \mathfrak{z}(\mathfrak{g}),\ \text{and}\quad \mathfrak{z}^{i+1}(\mathfrak{g}) := q_i^{-1}(\mathfrak{z}(\mathfrak{g}/\mathfrak{z}^i(\mathfrak{g}))),$$

where $\mathfrak{z}(\mathfrak{g}/\mathfrak{z}^i(\mathfrak{g}))$ denotes the center of $\mathfrak{g}/\mathfrak{z}^i(\mathfrak{g})$ and $q_i : \mathfrak{g} \to \mathfrak{g}/\mathfrak{z}^i(\mathfrak{g})$ the canonical quotient map. The chain of ideals

$$\{0\} \trianglelefteq \mathfrak{z}(\mathfrak{g}) \trianglelefteq \mathfrak{z}^2(\mathfrak{g}) \trianglelefteq \cdots$$

is called the ascending central series of $\mathfrak{g}$.

(2) The Lie algebra $\mathfrak{g}$ is called nilpotent if its ascending central series terminates within finitely many steps, i.e., if $\mathfrak{z}^l(\mathfrak{g}) = \mathfrak{g}$ for some $l \in \mathbb{N}$. The smallest such l is called the nilpotence class of $\mathfrak{g}$ and $\mathfrak{g}$ is said to be l-step nilpotent.

We introduce now the notion of a nilpotent k-Lie pair $(G, \mathfrak{g})$, $k \in \mathbb{N}$, which turns out to be a suitable object for our purpose.

Definition 2.2.5. Let $k \in \mathbb{N}$, let G be a locally compact, separable, nilpotent group of nilpotence class $l \leq k$ and let $\mathfrak{g}$ be a topological Lie algebra over $\mathbb{Z}$. We call the pair $(G, \mathfrak{g})$ a nilpotent k-Lie pair if the following properties are satisfied:

(i) The additive group $\mathfrak{g}$ is a Λ_k-module, extending the $\mathbb{Z}$-module structure of $\mathfrak{g}$, where $\Lambda_k := \mathbb{Z}[\frac{1}{k!}]$ denotes the ring in which every prime number $p \leq k$ is invertible.

(ii) There exists a homeomorphism $\exp : \mathfrak{g} \to G$, with inverse denoted by log, satisfying the Campbell-Hausdorff formula (see Remark 2.2.7).

(iii) There exists a locally compact abelian group $\mathfrak{w}$ and there exists a character $\epsilon : \mathfrak{w} \to \mathbb{T}$ such that the following properties hold:

(a) The group $\mathfrak{w}$ is a Λ_k-module.

(b) There does not exist a non-trivial Λ_k-submodule of $\mathfrak{w}$ inside the kernel of the character ϵ.

(c) The map
$$\Phi : \mathrm{Hom}(\mathfrak{g}, \mathfrak{w}) \to \hat{\mathfrak{g}},\ f \mapsto \epsilon \circ f,$$
is an isomorphism of groups, where $\mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$ denotes the group of continuous group homomorphisms from $\mathfrak{g}$ to $\mathfrak{w}$ and $\widehat{\mathfrak{g}}$ denotes the Pontrjagin dual of the abelian group $\mathfrak{g}$.

(d) For every closed Λ_k-subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and for any $f \in \mathrm{Hom}(\mathfrak{h}, \mathfrak{w})$ there exists a map $\tilde{f} \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$ such that $\tilde{f}|_{\mathfrak{h}} = f$.

To get an idea of this technical definition we briefly describe an example, which will be discussed in full detail in Section 2.11.

Example 2.2.6. Let K be a local field of characteristic p. Let $Tr_1(n, K)$ be the group of upper triangular $n \times n$-matrices over K with each diagonal entry equal to 1 and let $Tr_0(n, K)$ be the group of upper triangular $n \times n$-matrices over K with each diagonal entry equal to 0 and suppose that $p > n$.

Let $k \in \{n, \ldots, p-1\}$. Equipped with the usual commutator of matrices, $Tr_0(n, K)$ becomes a Lie algebra over the ring $\mathbb{Z}[\frac{1}{k!}]$. Furthermore, since $p > n$, the exponential map $\exp : Tr_0(n, K) \to Tr_1(n, K)$, given by the usual power series, is a well-defined homeomorphism satisfying the Campbell-Hausdorff formula.

Since the characteristic of K is equal to p, it follows that $\chi(X) \in \mathcal{U}_p$ for every character χ of the additive group $Tr_0(n,K)$ and for all $X \in Tr_0(n,K)$, where $\mathcal{U}_p$ denotes the group of primitive pth roots of unity. Put $\mathfrak{w} := \mathcal{U}_p \subseteq \mathbb{T}$. Then $\mathfrak{w}$ is, as a discrete group, locally compact and clearly a Λ_k-module. We define $\epsilon : \mathcal{U}_p \hookrightarrow \mathbb{T}, \epsilon = Id$. This map is a character of $\mathfrak{w}$ and there does not exist a non-trivial Λ_k-module inside the kernel of ϵ. Moreover, we have

$$\mathrm{Hom}(Tr_0(n,K), \mathfrak{w}) = \widehat{Tr_0(n,K)}.$$

So if $\mathfrak{h}$ is any closed subalgebra of $Tr_0(n,K)$, then the additive group $\mathfrak{h}$ is a closed subgroup of the locally compact abelian group $Tr_0(n,K)$, and it follows from general representation theory of locally compact abelian groups that for every character $f \in \mathrm{Hom}(\mathfrak{h}, \mathfrak{w})$ there exists an extension $\tilde{f} \in \mathrm{Hom}(Tr_0(n,K), \mathfrak{w})$ of f. Therefore, the pair $(Tr_1(n,K), Tr_0(n,K))$ satisfies all the properties of Definition 2.2.5, which means that $(Tr_1(n,K), Tr_0(n,K))$ is a nilpotent k-Lie pair for every $n \leq k < p$.

Remark 2.2.7. The Campbell-Hausdorff formula describes the multiplication inside the group G using the laws of the Lie algebra $\mathfrak{g}$. If $X, Y \in \mathfrak{g}$ then $\exp(X)\exp(Y) = \exp(Z)$, where the element $Z = \log(\exp(X)\exp(Y))$ is of the form

$$Z = \sum_{n=1}^{l} Z_n = \sum_{n=1}^{l} \Big(\frac{1}{n} \sum_{s+t=n} (Z'_{s,t} + Z''_{s,t})\Big),$$

where

$$Z'_{s,t} \quad = \quad \sum_{\substack{s_1+\dots+s_m=s \\ t_1+\dots+t_{m-1}=t-1 \\ s_i+t_i \geq 1\ \forall i \\ s_m \geq 1}} \frac{(-1)^{m+1}}{m} \frac{\mathrm{ad}(X)^{s_1}\,\mathrm{ad}(Y)^{t_1} \dots \mathrm{ad}(X)^{s_m}(Y)}{s_1! t_1! \dots s_m!} \tag{2.3}$$

and

$$Z''_{s,t} \quad = \quad \sum_{\substack{s_1+\dots+s_{m-1}=s \\ t_1+\dots+t_{m-1}=t \\ s_i+t_i \geq 1\ \forall i}} \frac{(-1)^{m+1}}{m} \frac{\mathrm{ad}(X)^{s_1}\,\mathrm{ad}(Y)^{t_1} \dots \mathrm{ad}(Y)^{t_{m-1}}(X)}{s_1! t_1! \dots t_{m-1}!}. \tag{2.4}$$

Notice that $\mathrm{ad}(X)(Y) := [X,Y]$ for all $X, Y \in \mathfrak{g}$ and thus we have for all $X_i \in \mathfrak{g}$:

$$\mathrm{ad}(X_1) \dots \mathrm{ad}(X_{n-1})(X_n) = [X_1, [X_2, \dots, [X_{n-1}, X_n]] \dots].$$

Explicitly, the values of the first three homogeneous components of Z are

$Z_1 = X + Y$,

$Z_2 = \frac{1}{2}[X,Y]$, and

$Z_3 = \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]])$.

Hence we have for all $X, Y \in \mathfrak{g}$:

$$(\exp X)(\exp Y) = \exp(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]] + \cdots).$$

A complete description of this formula can be found for example in [4], §6. Since the Lie algebra $\mathfrak{g}$ is nilpotent, the right hand side of the equation above consists of a finite sum.

Definition 2.2.8. Let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. By a subalgebra of $\mathfrak{g}$ we understand a Lie-subalgebra of $\mathfrak{g}$, which is also a Λ_k-module. An ideal of $\mathfrak{g}$ is a subalgebra $\mathfrak{j}$ of $\mathfrak{g}$ which has the additional property that $[X, Y] \in \mathfrak{j}$ for all $X \in \mathfrak{g}$ and for all $Y \in \mathfrak{j}$.

Remark 2.2.9. Let $k \in \mathbb{N}$, let $\mathfrak{g}$ be a Lie algebra over $\mathbb{Z}$, and suppose that the additive group $\mathfrak{g}$ is a Λ_k-module extending the $\mathbb{Z}$-module structure. Then $\mathfrak{g}$ is a Lie algebra over Λ_k and in particular, the commutator map $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ is Λ_k-bilinear.

Notice that $\mathfrak{g}$ extends the $\mathbb{Z}$-module structure uniquely. Indeed, if $X, Y \in \mathfrak{g}$ and $\lambda \in \Lambda_k$ with $\lambda X = \lambda Y$, then $\lambda(X - Y) = 0$ and since λ is invertible we obtain $X - Y = 0$ and hence $X = Y$. Now, let $\lambda = \frac{1}{m} \in \Lambda_k$, $m \in \mathbb{Z}$. Since the commutator is bi-additive, we obtain for all $X, Y \in \mathfrak{g}$:

$$[X, Y] = [m\frac{1}{m}X, Y] = m[\frac{1}{m}X, Y]$$

and hence

$$\frac{1}{m}[X, Y] = \frac{1}{m}(m[\frac{1}{m}X, Y]) = [\frac{1}{m}X, Y].$$

In the same way it follows that

$$\frac{1}{m}[X, Y] = [X, \frac{1}{m}Y].$$

Remark 2.2.10. Let $\mathfrak{g}$ be a Lie algebra over the ring Λ_k and let $\mathfrak{w}$ be a Λ_k-module. If $f \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$ is a continuous group homomorphism, then f is automatically Λ_k-linear. Indeed, we obtain for all $X \in \mathfrak{g}$ and for all $\lambda = \frac{n}{m} \in \Lambda_k = \mathbb{Z}[\frac{1}{k!}]$:

$$nf(X) = f(nX) = f(n\, m\frac{1}{m}X) = mf(\frac{n}{m}X)$$

and thus

$$\frac{n}{m}f(X) = f(\frac{n}{m}X).$$

Remark 2.2.11. We will show in the next lemma that if $(G, \mathfrak{g})$ is a nilpotent k-Lie pair with chosen Λ_k-module $\mathfrak{w}$ and character $\epsilon \in \widehat{\mathfrak{w}}$, then every closed subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ satisfies the desirable property

$$\operatorname{Hom}(\mathfrak{h}, \mathfrak{w}) \cong \widehat{\mathfrak{h}} \quad \text{via} \quad f \mapsto \epsilon \circ f$$

for the same Λ_k-module $\mathfrak{w}$ and the same character $\epsilon \in \widehat{\mathfrak{w}}$. We want to demonstrate in this remark that the property (iii)(b) of Definition 2.2.5 is necessary to obtain this property.

For this let $k = 1$. If we associate to the locally compact group $G = \mathbb{R}$ the abelian Lie algebra $\mathfrak{g} = (\mathbb{R}, +, [.,.])$ then $\mathfrak{g}$ is clearly a Lie algebra over $\mathbb{Z}$ satisfying the first two properties of Definition 2.2.5. Put $\mathfrak{w} := \mathbb{R}$ and let $\epsilon \in \hat{\mathbb{R}}$, defined by $\epsilon : \mathbb{R} \to \mathbb{T}, t \mapsto e^{2\pi i t}$. Then $\mathfrak{w}$ is a $\mathbb{Z}$-module and the map

$$\Phi : \operatorname{Hom}(\mathbb{R}, \mathbb{R}) \to \widehat{\mathbb{R}}, f \mapsto \epsilon \circ f$$

is an isomorphism of groups. Consider the subalgebra $\mathfrak{h} := \mathbb{Z}$ of $\mathbb{R}$. Then $\mathfrak{h}$ is a closed $\mathbb{Z}$-subalgebra of $\mathbb{R}$, but the map

$$\Phi_{\mathfrak{h}} : \operatorname{Hom}(\mathbb{Z}, \mathbb{R}) \to \widehat{\mathbb{Z}}, \ f \mapsto \epsilon \circ f$$

is not injective. Indeed, define $f \in \operatorname{Hom}(\mathbb{Z}, \mathbb{R})$ by $f(n) := n$. Then $f \neq 0$, but the character $\epsilon \circ f$ is equal to the trivial character.

Notice that if we choose $\mathfrak{w} = \mathbb{T}$ and $\epsilon = Id$, then property (iii)(b) of Definition 2.2.5 is satisfied and the pair $(\mathbb{R}, \mathbb{R})$ with $\mathbb{Z}$-module $\mathfrak{w}$ and character $\epsilon \in \hat{\mathfrak{w}}$ defines a nilpotent 1-Lie pair.

Lemma 2.2.12. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. Let $\mathfrak{w}$ be a locally compact abelian group and let $\epsilon \in \widehat{\mathfrak{w}}$ such that $\mathfrak{w}$ and ϵ satisfy property (iii) of Definition 2.2.5. Then for every closed subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ one has*

(i) $\operatorname{Hom}(\mathfrak{h}, \mathfrak{w}) \cong \widehat{\mathfrak{h}}$ *via* $f \mapsto \epsilon \circ f$ *and*

(ii) $\operatorname{Hom}(\mathfrak{g}/\mathfrak{h}, \mathfrak{w}) \cong \widehat{\mathfrak{g}/\mathfrak{h}}$ *via* $\tilde{f} \mapsto \epsilon \circ \tilde{f}$.

Proof. Since $(G, \mathfrak{g})$ is a nilpotent k-Lie pair, the map Φ, defined by

$$\Phi : \operatorname{Hom}(\mathfrak{g}, \mathfrak{w}) \to \widehat{\mathfrak{g}}, \ f \mapsto \epsilon \circ f, \tag{2.5}$$

is an isomorphism of groups and we obtain a well-defined, continuous group homomorphism

$$\Phi_{\mathfrak{h}} : \operatorname{Hom}(\mathfrak{h}, \mathfrak{w}) \to \widehat{\mathfrak{h}}, \ f \mapsto \epsilon \circ f. \tag{2.6}$$

We will prove in the following that $\Phi_{\mathfrak{h}}$ is a bijective map. For this, we observe first that it is a well-known fact (see for example [13], Theorem 4.39.) that the exact sequence of locally compact abelian groups

$$0 \longrightarrow \mathfrak{h} \longrightarrow \mathfrak{g} \longrightarrow \mathfrak{g}/\mathfrak{h} \longrightarrow 0$$

yields, by taking duals, the following exact sequence of locally compact abelian groups:

$$1 \longrightarrow \widehat{\mathfrak{g}/\mathfrak{h}} \longrightarrow \widehat{\mathfrak{g}} \longrightarrow \widehat{\mathfrak{h}} \longrightarrow 1. \tag{2.7}$$

Now, let $\psi \in \widehat{\mathfrak{h}}$. Then there exists a character $\tilde{\psi} \in \widehat{\mathfrak{g}}$ with $\tilde{\psi}|_{\mathfrak{h}} = \psi$. But since the map Φ, defined in (2.5), is surjective, we can find a homomorphism $f \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$ such that $\tilde{\psi} = \epsilon \circ f$. Thus we obtain

$$\psi = \tilde{\psi}|_{\mathfrak{h}} = \epsilon \circ f|_{\mathfrak{h}},$$

and since $f|_{\mathfrak{h}} \in \mathrm{Hom}(\mathfrak{h}, \mathfrak{w})$, we have shown that the map $\Phi_{\mathfrak{h}}$ is surjective.

In order to prove that the map $\Phi_{\mathfrak{h}}$ is injective, let $f, g \in \mathrm{Hom}(\mathfrak{h}, \mathfrak{w})$ with

$$\psi_f := \epsilon \circ f = \epsilon \circ g =: \psi_g.$$

By property $(iii)(d)$ of Definition 2.2.5 we can find an extension $\tilde{f} \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$ of f and an extension $\tilde{g} \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$ of g and the maps $\tilde{\psi}_f := \epsilon \circ \tilde{f} \in \widehat{\mathfrak{g}}$ and $\tilde{\psi}_g := \epsilon \circ \tilde{g} \in \widehat{\mathfrak{g}}$ define a lift of $\psi_f \in \widehat{\mathfrak{h}}$ and $\psi_g \in \widehat{\mathfrak{h}}$, respectively. But we have $\tilde{\psi}_f \cong \tilde{\psi}_g \cdot \chi$ for some character $\chi \in \widehat{\mathfrak{g}/\mathfrak{h}}$ and since

$$\widehat{\mathfrak{g}/\mathfrak{h}} \cong \mathfrak{h}^{\perp} := \{\chi \in \widehat{\mathfrak{g}} \mid \chi(X) = 1 \; \forall \; X \in \mathfrak{h}\},$$

we can identify the character $\chi \in \widehat{\mathfrak{g}/\mathfrak{h}}$ with a character $\chi \in \mathfrak{h}^{\perp}$. By the surjectivity of Φ we can find a map $\gamma \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$ such that $\chi = \epsilon \circ \gamma$. We have $\gamma|_{\mathfrak{h}} \equiv 1$, because if not then $\gamma(\mathfrak{h})$ would be a nontrivial Λ_k-submodule of $\mathfrak{w}$ inside the kernel of the character ϵ, contradicting property $(iii)(b)$ of Definition 2.2.5. Therefore, we obtain for all $X \in \mathfrak{g}$

$$\epsilon(\tilde{f}(X)) = \epsilon(\tilde{g}(X)) \cdot \epsilon(\gamma(X)) = \epsilon((\tilde{g} + \gamma)(X)),$$

and since the map Φ is injective, it follows that $\tilde{f} = \tilde{g} + \gamma$. But $\gamma|_{\mathfrak{h}} \equiv 1$ and thus we obtain for all $X \in \mathfrak{h}$

$$f(X) = \tilde{f}(X) = \tilde{g}(X) + \gamma(X) = \tilde{g}(X) = g(X).$$

This proves the injectivity of $\Phi_{\mathfrak{h}}$. Since every continuous, bijective homomorphism between σ-compact locally compact groups is open, it follows that the map $\Phi_{\mathfrak{h}}$ is an isomorphism of groups.

For the proof of part (ii), we observe that the map Φ of (2.5) is an isomorphism of groups and thus we obtain a well-defined, continuous group homomorphism

$$\widetilde{\Phi} : \mathrm{Hom}(\mathfrak{g}/\mathfrak{h}, \mathfrak{w}) \to \widehat{\mathfrak{g}/\mathfrak{h}}, \; \tilde{f} \mapsto \epsilon \circ \tilde{f}. \tag{2.8}$$

In order to prove that the map $\widetilde{\Phi}$ is surjective let $\widetilde{\psi}$ be any given character of $\widehat{\mathfrak{g}/\mathfrak{h}}$. Since the sequence in (2.7) is exact we can find a character $\psi \in \widehat{\mathfrak{g}}$ such that

$\widetilde{\psi}(q(X)) = \psi(X)$ for all $X \in \mathfrak{g}$, where $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$ denotes the canonical quotient map. Let $f \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$ with $\psi = \epsilon \circ f$ and define a map $\tilde{f} : \mathfrak{g}/\mathfrak{h} \to \mathfrak{w}$ by

$$\tilde{f}(q(X)) := f(X) \quad \forall\, X \in \mathfrak{g}\,.$$

Then $\tilde{f} \in \mathrm{Hom}(\mathfrak{g}/\mathfrak{h}, \mathfrak{w})$ and we obtain

$$\tilde{\psi}(q(X)) = \psi(X) = \epsilon(f(X)) = \epsilon(\tilde{f}(q(X))),$$

which proves that the map $\widetilde{\Phi}$ is onto.

In order to see that the map $\widetilde{\Phi}$ is injective, let $\tilde{f}, \tilde{g} \in \mathrm{Hom}(\mathfrak{g}/\mathfrak{h}, \mathfrak{w})$ and suppose that

$$\epsilon \circ \tilde{f} = \epsilon \circ \tilde{g}.$$

Then both maps, $f = \tilde{f} \circ q$ and $g = \tilde{g} \circ q$, are elements of $\mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$ and we have $\epsilon \circ f = \epsilon \circ g$. But since the map Φ of (2.5) is injective, it follows that $f = g$ and hence $\tilde{f} = \tilde{g}$.

Since every continuous, bijective homomorphism between σ-compact locally compact groups is open, it follows that the map $\widetilde{\Phi}$ is an isomorphism of groups.

□

Notation 2.2.13. If $(G, \mathfrak{g})$ is a nilpotent k-Lie pair then there exists a Λ_k-module $\mathfrak{w}$ such that $\mathrm{Hom}(\mathfrak{g}, \mathfrak{w}) \cong \widehat{\mathfrak{g}}$. We let the group

$$\mathfrak{g}^* := \mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$$

serve as a substitute for a linear dual of $\mathfrak{g}$. Since $\mathrm{Hom}(\mathfrak{g}, \mathfrak{w}) \cong \hat{\mathfrak{g}}$, up to isomorphism, the group $\mathfrak{g}^*$ does not depend on the choice of the Λ_k-module $\mathfrak{w}$.

Remark/Definition 2.2.14. Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. Let $x \in G$ and put $X := \log(x) \in \mathfrak{g}$. By the Campbell-Hausdorff formula we have for all $n \in \mathbb{Z}$

$$x^n = (\exp(X))^n = \exp(X)\exp(X)\cdots\exp(X) = \exp(X + X + \cdots + X) = \exp(nX),$$

and thus we obtain for all fractions $\frac{1}{m} \in \mathbb{Z}[\frac{1}{k!}]$:

$$x = \exp(X) = \exp(m\frac{1}{m}X) = (\exp(\frac{1}{m}X))^m. \tag{2.9}$$

Let $\frac{1}{m} \in \mathbb{Z}[\frac{1}{k!}]$ be fixed. According to (2.9) we define the "mth" root of x as follows

$$x^{\frac{1}{m}} := \exp(\frac{1}{m}X).$$

Let $y \in G$ with $y^m = x$ and put $Y := \log(y) \in \mathfrak{g}$. Then $y^m = \exp(mY)$ and thus $\exp(mY) = x = \exp(X)$. But since the exponential map was assumed to be injective, we obtain $mY = X$ and hence $y = \exp(\frac{1}{m}X)$. This proves that there exists a unique element $y \in G$ with $y^m = x$, namely $y = \exp(\frac{1}{m}X)$.

Let $\lambda \in \mathbb{Z}[\frac{1}{k!}]$. Then λ is of the form $\lambda = \frac{n}{m}$ for some $n, m \in \mathbb{Z}$ and we define

$$x^\lambda := (x^{\frac{1}{m}})^n.$$

We have

$$x^\lambda = (x^{\frac{1}{m}})^n = (\exp((\frac{1}{m}X)))^n = \exp(\frac{n}{m}X) = \exp(\lambda X) \tag{2.10}$$

and it follows by the same arguments as above, that the element $y = \exp(\lambda X)$ is the unique element of G with the property that $x^\lambda = y$.

Lemma 2.2.15. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. Then we have for all $x \in G$ and $\lambda \in \mathbb{Z}[\frac{1}{k!}]$:*

$$\log(x^\lambda) = \lambda \log(x).$$

Proof. The proof is obvious, apply the map exp to both sides of the equation and use (2.10). □

Definition 2.2.16. Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. A closed subgroup H of G is called exponentiable, if $\log(H)$ is a subalgebra of $\mathfrak{g}$.

Remark 2.2.17. Notice that if $(G, \mathfrak{g})$ is a nilpotent k-Lie pair and H an exponentiable subgroup of G, then it follows from Lemma 2.2.12 that $(H, \log(H))$ is also a nilpotent k-Lie pair.

2.3 Inversion of the Campbell-Hausdorff formula

If $(G, \mathfrak{g})$ is a nilpotent k-Lie pair for some $k \in \mathbb{N}$ and if $\mathfrak{h}$ is any subalgebra of $\mathfrak{g}$, then it follows directly from the Campbell-Hausdorff formula that $\exp(\mathfrak{h}) =: H$ is a subgroup of G. So the question arises, what kind of closed subgroups H of G are exponentiable, i.e., have the property that $\log(H)$ is a subalgebra of $\mathfrak{g}$? To analyze this question we establish the following definition.

Definition 2.3.1. Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. A closed subgroup H of G is said to be k-complete, if $x^\lambda \in H$ for all $x \in H$ and for all $\lambda \in \mathbb{Z}[\frac{1}{k!}]$. (Note that by Remark/Definition 2.2.14, x^λ is a well-defined element of G for all $x \in H$ and for all $\lambda \in \mathbb{Z}[\frac{1}{k!}]$.)

Theorem 2.3.2. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. A closed subgroup H of G is k-complete if and only if H is exponentiable.*

For the proof of this theorem we need to show some facts about commutators of length m, $m \geq 1$, with entries in $\log(H)$.

Definition 2.3.3. Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair.

(i) A commutator $[X_1, \ldots, X_m]$ of length $m \geq 1$ in $\mathfrak{g}$ is defined inductively by

$$[X_1] := X_1 \text{ and } [X_1, \ldots, X_m] := [[X_1, \ldots, X_{m-1}], X_m], \quad X_i \in \mathfrak{g},\ i = 1, \ldots, m.$$

(ii) A group commutator $(x_1, \ldots, x_m)$ of length $m \geq 1$ in G is defined inductively by

$$\begin{aligned} &(x_1) := x_1,\ (x_1, x_2) := x_1 x_2 x_1^{-1} x_2^{-2}, \quad \text{and} \\ &(x_1, \ldots, x_m) = ((x_1, \ldots, x_{m-1}), x_m), \quad x_i \in G,\ i = 1, \ldots, m. \end{aligned}$$

(iii) Let H be a subgroup of G. A linear combination, C_r, of commutators of length $r \geq 1$ with entries in $\log(H)$ is a finite sum $\sum_i \lambda_i C_i$, where $\lambda_i \in \mathbb{Z}[\frac{1}{k!}]$ for all i and each C_i is of the form

$$C_i = [\log(x_{1,i}), \ldots, \log(x_{r,i})]$$

for some $x_{1,i}, \ldots, x_{r,i} \in H$.

Lemma 2.3.4. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. If $x_1, \ldots, x_m \in G$ then*

$$\log\big((x_1, \ldots, x_m)\big) = [\log(x_1), \ldots, \log(x_m)] + \sum_j F_j, \tag{2.11}$$

where each F_j is a linear combination of commutators of length $j > m$ with entries in $\log(G)$.

The proof is by induction on m and can be found in [19], Theorem 6.1.6.

Remark 2.3.5. The exact form of every sum F_j appearing in Equation (2.11) is determined by the Campbell-Hausdorff formula. In fact, one can prove that each term F_j in (2.11) is a linear combination of commutators $[\log(x_{i_1}), \ldots, \log(x_{i_j})]$ of length $j > m$ and $i_l \in \{1, \ldots, m\}$ for $1 \leq l \leq j$, such that each of $1, \ldots, m$ occurs at least once among the i_l.

Lemma 2.3.6. *Let $k \in \mathbb{N}$, let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair, and let H be a k-complete subgroup of G. Let $h \in H$, let D be any commutator of length $r \geq 1$ with entries in $\log(H)$, and let $\mu \in \mathbb{Z}[\frac{1}{k!}]$. Then there exists an element $h' \in H$ and there exist linear combinations, G_t, of commutators of length $t \geq r+1$ with entries in $\log(H)$, such that*

$$\log(h) + \mu D = \log(h') + \sum_t G_t.$$

Proof. Let $r \geq 1$ be fixed. Since D is a commutator of length r with entries in $\log(H)$, the term μD is of the form

$$\mu D = \mu[\log(x_1), \ldots, \log(x_r)]$$

for some $x_i \in H$, $i = 1, \ldots, r$. By Lemma 2.2.15 and the fact that the commutator is $\mathbb{Z}[\frac{1}{k!}]$- bilinear, we obtain

$$\mu[\log(x_1), \ldots, \log(x_r)] = [\log(x_1^\mu), \ldots, \log(x_r)].$$

Notice that since H is k-complete, we have $x_1^\mu \in H$. Furthermore, we obtain by Lemma 2.3.4

$$[\log(x_1^\mu), \ldots, \log(x_r)] = \log\big((x_1^\mu, \ldots, x_r)\big) - \sum_s F_s, \tag{2.12}$$

where each F_s is a linear combination of commutators of length $s > r$ in $\log(x_1^\mu)$ and in $\log(x_j)$, $j \in \{2, \ldots, r\}$. Recall the Campbell-Hausdorff formula (Remark 2.2.7)

$$\log(xy) = \log(x) + \log(y) + \sum_{t \geq 2} G_t,$$

where each G_t is a linear combination of commutators of length $t \geq 2$ in $\log(x)$ and $\log(y)$. Applying this formula to the elements $x = h$ and $y = (x_1^\mu, \ldots, x_r)$ yields

$$\log\big(h \cdot (x_1^\mu, \ldots, x_r)\big) = \log(h) + \log\big((x_1^\mu, \ldots, x_r)\big) + \sum_t G_t, \tag{2.13}$$

where each G_t is a linear combination of commutators of length $t \geq 2$ in $\log(h)$ and $\log\big((x_1^\mu, \ldots, x_r)\big)$. But (2.12) yields

$$\log\big((x_1^\mu, \ldots, x_r)\big) = [\log(x_1^\mu), \ldots, \log(x_r)] + \sum_s F_s,$$

and thus every term G_t in (2.13) is in fact a commutator of length $t \geq r+1$ in $\log(h)$, $\log(x_1^\mu)$, and $\log(x_j)$, $j \in \{2, \ldots, r\}$ and we obtain

$$\log(h) + \log\big((x_1^\mu, \ldots, x_r)\big) = \log\big(h \cdot (x_1^\mu, \ldots, x_r)\big) + \sum_t G_t, \tag{2.14}$$

where each G_t is a linear combination of commutators of length $t \geq r+1$ in $\log(h)$, $\log(x_1^\mu)$, and $\log(x_j)$, $j \in \{2, \ldots, r\}$. Since $h' := h(x_1^\mu, \ldots, x_r) \in H$, Equation (2.14) yields the desired result. □

The next lemma provides the main tool for the proof of Theorem 2.3.2. A similar result can be found in [4], chapter II, Exercises, §6.

Lemma 2.3.7. *Let $k \in \mathbb{N}$, let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair, and let H be a k-complete subgroup of G. For every linear combination C_j of commutators of length $j \geq 1$ with entries in $\log(H)$ one has*

$$\sum_{j=1}^{k} C_j \in \log(H). \tag{2.15}$$

Proof. Let $r := \min\{j \mid C_j \neq 0\}$. We may decompose the sum in (2.15) as follows:

$$\sum_j C_j = \sum_{n=1}^{N_r} \mu_n D_n + \sum_{\ell=r+1}^{L_r} E_\ell, \tag{2.16}$$

where $\mu_n \in \mathbb{Z}[\frac{1}{k!}]$ for every n, each term D_n is a commutator of length r with entries in $\log(H)$, and each E_ℓ is a linear combination of commutators of length $\ell \geq r+1$ with entries in $\log(H)$.

Pick the first term, D_1, of the sum $\sum_{n=1}^{N_r} \mu_n D_n$ of commutators of length r. Using $h = 1$ we can write

$$D_1 = \log(h) + D_1,$$

and it follows from Lemma 2.3.6 that there exists an element $h_{r,1} \in H$ and there exist linear combinations, G_t, of commutators of length $t \geq r+1$ with entries in $\log(H)$, such that

$$\log(h) + D_1 = \log(h_{r,1}) + \sum_t G_t.$$

So we may replace the commutator D_1 in Equation (2.16) by an element of the form $\log(h_{r,1}) + \sum_t G_t$ as described above.

Pick the second commutator of length r of the sum $\sum_{n=1}^{N_r} \mu_n D_n$. Again by Lemma 2.3.6 we can find an element, say $h_{r,2} \in H$, and linear combinations, G'_t, of commutators of length $t \geq r+1$ with entries in $\log(H)$, such that

$$\log(h_{r_1}) + D_2 = \log(h_{r,2}) + \sum_t G'_t.$$

In this way we may remove the terms D_n in Equation (2.16) one by one and obtain a new equation of the form

$$\sum_j C_j = \log(h_r) + \sum_{m \geq r+1} R_m, \tag{2.17}$$

where $h_r \in H$ and each R_m is a linear combination of commutators of length $m \geq r+1$ in elements of $\log(H)$. Iterating this procedure we obtain

$$\sum_j C_j = \log(h') + \sum_m S_m,$$

where $h' \in H$ and each term S_m is a linear combination of commutators of length $m \geq k+1$. Since the nilpotence class of H is less than or equal to k, it follows that $S_m = 0$ for every m. This proves

$$\sum_j C_j \in \log(H).$$

□

The proof of Theorem 2.3.2 is now an easy application of Lemma 2.3.7.

Proof. Let H be a k-complete subgroup of G. We need to show that $\log(H)$ is a subalgebra of $\mathfrak{g}$, i.e. we need to prove

(1) $\forall\, x, y \in H\ \exists\, z \in H$ with $\log(x) + \log(y) = \log(z)$ and

(2) $\forall\, x, y \in H\ \exists\, v \in H$ with $[\log(x), \log(y)] = \log(v)$.

Let $x, y \in H$. Since $C_1 = \log(x) + \log(y)$ is a sum of commutators of length $r = 1$ it follows from Lemma 2.3.7 that

$$\log(x) + \log(y) = \log(z)$$

for some $z \in H$. Furthermore, $C_2 = [\log(x), \log(y)]$ is a commutator of length $r = 2$ and thus we obtain by Lemma 2.3.7 the desired result

$$[\log(x), \log(y)] = \log(v)$$

for some $v \in H$.

Conversely, suppose that H is an exponentiable subgroup of G. Then $\log(H)$ is a subalgebra of $\mathfrak{g}$ and since $\lambda X \in \log(H)$ for all $\lambda \in \mathbb{Z}[\frac{1}{k!}]$ and for all $X \in \log(H)$ we obtain by Lemma 2.2.15

$$x^\lambda = \exp(\lambda X) \in H$$

for all $x = \exp(X) \in H$ and for all $\lambda \in \mathbb{Z}[\frac{1}{k!}]$. □

Remark/Example 2.3.8. It follows from Lemma 2.3.7 and the proof of Lemma 2.3.6 that for every nilpotent k-Lie pair $(G, \mathfrak{g})$ there exist two Inversion Formulas of the Campbell-Hausdorff formula. Since the method of writing the sum (resp. the commutator) of two logarithms as the logarithm of some element of G is quite complicated in practice, we want to demonstrate how the procedure in the proof of Lemma 2.3.7 works for three-step nilpotent groups.

Let $k \in \mathbb{N}_{\geq 3}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair of nilpotence class 3. The Campbell-Hausdorff formula reduces for all $X, Y \in \mathfrak{g}$ to

$$\exp(X)\exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] + \frac{1}{12}[Y, [Y, X]])$$

and thus we have for all $x, y \in G$:

$$\begin{aligned}\log(x) + \log(y) &= \log(xy) - \frac{1}{2}[\log(x), \log(y)] \\ &\quad -\frac{1}{12}[\log(x), [\log(x), \log(y)]] - \frac{1}{12}[\log(y), [\log(y), \log(x)]].\end{aligned} \tag{2.18}$$

In the following, we demonstrate how we may replace the sum on the right hand side of Equation (2.18) step by step to end up with an element of the form $\log(z)$ for some $z \in G$. Furthermore, we will compute the exact form of this element z. We start this procedure by rewriting the first commutator appearing on the right hand side of (2.18), the expression $-\frac{1}{2}[\log(x), \log(y)]$, as the logarithm of some element of G plus some additional terms which consist of commutators of length three. For this we compute first with the Campbell-Hausdorff formula the group commutator of two elements in G. A lengthy computation yields for all $X, Y \in \log(G)$:

$$\exp(X)\exp(Y)\exp(-X)\exp(-Y) = \exp([X, Y] + \frac{1}{2}[X, [X, Y]] - \frac{1}{2}[Y, [Y, X]]).$$

Thus we obtain the following equation for the logarithm of the group commutator of two elements $x, y \in G$:

$$\log\big((x, y)\big) = [\log(x), \log(y)] + \frac{1}{2}[\log(x), [\log(x), \log(y)]] - \frac{1}{2}[\log(y), [\log(y), \log(x)]].$$

This yields the following formula for the commutator of the logarithm of two elements $\log(x)$ and $\log(y)$ in $\log(G)$:

$$[\log(x), \log(y)] = \log\big((x, y)\big) - \frac{1}{2}[\log(x), [\log(x), \log(y)]] + \frac{1}{2}[\log(y), [\log(y), \log(x)]]. \tag{2.19}$$

Using this formula and the fact that

$$-\frac{1}{2}[\log(x), \log(y)] = [\log(x^{-\frac{1}{2}}), \log(y)],$$

Equation (2.18) turns into the following equation

$$\begin{aligned}\log(x) + \log(y) &= \log(xy) + \log\big((x^{-\frac{1}{2}}, y)\big) \\ &\quad -\frac{1}{2}[\log(x^{-\frac{1}{2}}), [\log(x^{-\frac{1}{2}}), \log(y)]] + \frac{1}{2}[\log(y), [\log(y), \log(x^{-\frac{1}{2}})]] \\ &\quad -\frac{1}{12}[\log(x), [\log(x), \log(y)]] - \frac{1}{12}[\log(y), [\log(y), \log(x)]].\end{aligned} \tag{2.20}$$

If we apply (2.18) to the elements xy and $(x^{-\frac{1}{2}}, y)$ we obtain

$$\begin{aligned}\log(xy) + \log\big((x^{-\frac{1}{2}}, y)\big) &= \log\big(xy\,(x^{-\frac{1}{2}}, y)\big) - \frac{1}{2}[\log(xy), \log\big((x^{-\frac{1}{2}}, y)\big)] \\ &\quad -\frac{1}{12}[\log(xy), [\log(xy), \log\big((x^{-\frac{1}{2}}, y)\big)]] \\ &\quad -\frac{1}{12}[\log\big((x^{-\frac{1}{2}}, y)\big), [\log\big((x^{-\frac{1}{2}}, y)\big), \log(xy)]].\end{aligned} \tag{2.21}$$

If we use this computation in Equation (2.20), we obtain

$$\begin{aligned}
\log(x)+\log(y) &= \log\big(xy\ (x^{-\frac{1}{2}},y)\big) - \frac{1}{2}[\log(xy),\log\big((x^{-\frac{1}{2}},y)\big)] \\
&\quad -\frac{1}{12}[\log(xy),[\log(xy),\log\big((x^{-\frac{1}{2}},y)\big)]] \\
&\quad -\frac{1}{12}[\log\big((x^{-\frac{1}{2}},y)\big),[\log\big((x^{-\frac{1}{2}},y)\big),\log(xy)]] \\
&\quad -\frac{1}{2}[\log(x^{-\frac{1}{2}}),[\log(x^{-\frac{1}{2}}),\log(y)]] + \frac{1}{2}[\log(y),[\log(y),\log(x^{-\frac{1}{2}})]] \\
&\quad -\frac{1}{12}[\log(x),[\log(x),\log(y)]] - \frac{1}{12}[\log(y),[\log(y),\log(x)]] \\
&= \log\big(xy\ (x^{-\frac{1}{2}},y)\big) - \frac{1}{2}[\log(xy),\log\big((x^{-\frac{1}{2}},y)\big)] \\
&\quad -\frac{1}{12}[\log(xy),[\log(xy),\log\big((x^{-\frac{1}{2}},y)\big)]] \\
&\quad -\frac{1}{12}[\log\big((x^{-\frac{1}{2}},y)\big),[\log\big((x^{-\frac{1}{2}},y)\big),\log(xy)]] \\
&\quad -\frac{5}{24}[\log(x),[\log(x),\log(y)]] - \frac{1}{3}[\log(y),[\log(y),\log(x)]].
\end{aligned} \tag{2.22}$$

We observe that there is only one remaining commutator of length two, namely the expression $-\frac{1}{2}[\log(xy),\log\big((x^{-\frac{1}{2}},y)\big)]$. With (2.19) we are able to rewrite this commutator as the logarithm of some group commutator plus a sum of commutators of length three:

$$\begin{aligned}
-\frac{1}{2}[\log(xy),\log\big((x^{-\frac{1}{2}},y)\big)] &= \log\big(((xy)^{-\frac{1}{2}},(x^{-\frac{1}{2}},y))\big) \\
&\quad -\frac{1}{2}[\log((xy)^{-\frac{1}{2}}),[\log((xy)^{-\frac{1}{2}}),\log\big((x^{-\frac{1}{2}},y)\big)]] \\
&\quad +\frac{1}{2}[\log\big((x^{-\frac{1}{2}},y)\big),[\log\big((x^{-\frac{1}{2}},y)\big),\log\big((xy)^{-\frac{1}{2}}\big)]].
\end{aligned} \tag{2.23}$$

Using (2.23), Equation (2.22) turns into the following equation

$$\begin{aligned}
\log(x)+\log(y) &= \log\big(xy\ (x^{-\frac{1}{2}},y)\big) + \log\big(((xy)^{-\frac{1}{2}},(x^{-\frac{1}{2}},y))\big) \\
&\quad -\frac{1}{2}[\log((xy)^{-\frac{1}{2}}),[\log((xy)^{-\frac{1}{2}}),\log\big((x^{-\frac{1}{2}},y)\big)]] \\
&\quad +\frac{1}{2}[\log\big((x^{-\frac{1}{2}},y)\big),[\log\big((x^{-\frac{1}{2}},y)\big),\log((xy)^{-\frac{1}{2}})]] \\
&\quad -\frac{1}{12}[\log(xy),[\log(xy),\log\big((x^{-\frac{1}{2}},y)\big)]] \\
&\quad -\frac{1}{12}[\log\big((x^{-\frac{1}{2}},y)\big),[\log\big((x^{-\frac{1}{2}},y)\big),\log(xy)]] \\
&\quad -\frac{5}{24}[\log(x),[\log(x),\log(y)]] - \frac{1}{3}[\log(y),[\log(y),\log(x)]].
\end{aligned} \tag{2.24}$$

It turns out that this expression can be highly simplified. By Lemma 2.3.4 we have for all $u, v, w, z \in G$

$$[\log(u), [\log(v), \log((w, z))]] = \log\big((u, (v, (w, z)))\big) + \sum_j F_j,$$

where each term F_j is a linear combination of commutators in u, v, and (w, z), of length greater or equal to 4. Since G is three-step nilpotent, it follows that not only $\sum_j F_j = 0$, but also $\log\big((u, (v, (w, z)))\big) = 0$. Hence we obtain for all $u, v, w, z \in G$

$$[\log(u), [\log(v), \log\big((w, z)\big)]] = 0.$$

It follows directly from this fact that the $3rd$, $4th$, $5th$, and $6th$ terms on the right hand side of Equation (2.24) are all equal to zero, so that (2.24) turns into

$$\begin{aligned} \log(x) + \log(y) &= \log\big(xy\ (x^{-\frac{1}{2}}, y)\big) + \log\big(((xy)^{-\frac{1}{2}}, (x^{-\frac{1}{2}}, y))\big) \\ &\quad -\frac{5}{24}[\log(x), [\log(x), \log(y)]] - \frac{1}{3}[\log(y), [\log(y), \log(x)]]. \end{aligned} \tag{2.25}$$

But by Lemma 2.3.4 we may replace every commutator of length three appearing in Equation (2.25) by the logarithm of the corresponding group commutator and obtain

$$\begin{aligned} \log(x) + \log(y) &= \log\big(xy\ (x^{-\frac{1}{2}}, y)\big) + \log\big(((xy)^{-\frac{1}{2}}, (x^{-\frac{1}{2}}, y))\big) \\ &\quad + \log\big((x^{-\frac{5}{24}}, (x, y))\big) + \log\big((y^{-\frac{1}{3}}, (y, x))\big). \end{aligned} \tag{2.26}$$

So it remains to compute the sum of the four logarithms on the right hand side of the above equation. One can do this step by step by computing first the sum of the first two terms, then the sum with the third term and so on. If we use first (2.18) and then (2.19) we obtain

$$\begin{aligned} &\log\big(xy\ (x^{-\frac{1}{2}}, y)\big) + \log\big(((xy)^{-\frac{1}{2}}, (x^{-\frac{1}{2}}, y))\big) \\ = \ &\log\big(xy\ (x^{-\frac{1}{2}}, y)\ ((xy)^{-\frac{1}{2}}, (x^{-\frac{1}{2}}, y))\big) + \sum_j F_j, \end{aligned} \tag{2.27}$$

where F_j is a linear combination of commutators of length greater or equal to two in the expressions $\log\big(xy\ (x^{-\frac{1}{2}}, y)\big)$ and in $\log\big(((xy)^{-\frac{1}{2}}, (x^{-\frac{1}{2}}, y))\big)$. As we explained above, it follows from Lemma 2.3.4 that each term F_j is equal to zero. Therefore, Equation (2.26) turns into the following equation

$$\begin{aligned} \log(x) + \log(y) &= \log\big(xy\ (x^{-\frac{1}{2}}, y)\ ((xy)^{-\frac{1}{2}}, (x^{-\frac{1}{2}}, y))\big) \\ &\quad + \log\big((x^{-\frac{5}{24}}, (x, y)) + \log((y^{-\frac{1}{3}}, (y, x))\big). \end{aligned} \tag{2.28}$$

In order to compute the sum of the first two terms on the right hand side of (2.28) we can use the same arguments as above. After repeating this procedure one more time we obtain a formula for the sum of the elements $\log(x)$ and $\log(y)$:

$$\log(x)+\log(y) = \log\big(xy(x^{-\frac{1}{2}}, y)((xy)^{-\frac{1}{2}}, (x^{-\frac{1}{2}}, y))(x^{-\frac{5}{24}}, (x, y))(y^{-\frac{1}{3}}, (y, x))\big). \tag{2.29}$$

To obtain a formula for the commutator of two elements $\log(x)$ and $\log(y)$ in G, we recall Equation (2.19):

$$[\log(x),\log(y)] = \log\big((x,y)\big) - \frac{1}{2}[\log(x),[\log(x),\log(y)]] + \frac{1}{2}[\log(y),[\log(y),\log(x)]].$$

Since

$$-\frac{1}{2}[\log(x),[\log(x),\log(y)]] = [\log(x^{-\frac{1}{2}}),[\log(x),\log(y)]] = \log\big((x^{-\frac{1}{2}},(x,y))\big)$$

and

$$\frac{1}{2}[\log(y),[\log(y),\log(x)]] = \log\big((y^{\frac{1}{2}},(y,x))\big),$$

we obtain

$$[\log(x),\log(y)] = \log\big((x,y)\big) + \log\big((x^{-\frac{1}{2}},(x,y))\big) + \log\big((y^{\frac{1}{2}},(y,x))\big). \tag{2.30}$$

Using the same arguments as above, (2.30) turns into a formula for the commutator of $\log(x)$ and $\log(y)$:

$$[\log(x),\log(y)] = \log\big((x,y)\ (x^{-\frac{1}{2}},(x,y))\ (y^{\frac{1}{2}},(y,x))\ \big). \tag{2.31}$$

Remark 2.3.9. Let $k \in \mathbb{N}_{\geq 3}$ and let $(G,\mathfrak{g})$ be a nilpotent k-Lie pair of nilpotence class 3. We computed in Remark 2.3.8 two explicit Inversion Formulas of the Campbell-Hausdorff formula. One formula expresses the sum of two elements $\log(x), \log(y) \in \log(G)$ as an element of $\log(G)$:

$$\log(x)+\log(y) = \log\big(xy\ (x^{-\frac{1}{2}},y)\ ((xy)^{-\frac{1}{2}},(x^{-\frac{1}{2}},y)))\ (x^{-\frac{5}{24}},(x,y))\ (y^{-\frac{1}{3}},(y,x))\big).$$

The other formula expresses the commutator of two elements $\log(x), \log(y) \in \log(G)$ as an element of $\log(G)$:

$$[\log(x),\log(y)] = \log\big((x,y)\ (x^{-\frac{1}{2}},(x,y))\ (y^{\frac{1}{2}},(y,x))\big).$$

More generally, we obtain from the computations in Remark 2.3.8 and the proof of Lemma 2.3.7 the following result.

Corollary 2.3.10. *Let $k \in \mathbb{N}$. For every nilpotent k-Lie pair $(G,\mathfrak{g})$ there exist two different Inversion Formulas of the Campbell-Hausdorff formula. One formula expresses the sum of two elements of $\log(G)$ as an element of $\log(G)$:*

$$\log(x)+\log(y) = \log\big(\prod_{m=1}^{k} C_m(x,y)\big), \tag{2.32}$$

where each $C_m(x,y)$ is a product of commutators $(z_1,\dots,z_m)$ of length m and where each z_i is equal to some rational power $\lambda \in \mathbb{Z}[\frac{1}{k!}]$ of some product in x and y. The explicit form of the first three terms, C_1, C_2, and C_3, is given in Remark 2.3.9.

The other formula expresses the commutator of two elements of $\log(G)$ *as an element of* $\log(G)$*:*

$$[\log(x), \log(y)] = \log\Big(\prod_{m=2}^{k} D_m(x, y)\Big), \tag{2.33}$$

where each $D_m(x, y)$ *is a product of commutators* $(z_1, \ldots, z_m)$ *of length* m *and where each* z_i *is equal to some rational power* $\lambda_m \in \mathbb{Z}[\frac{1}{k!}]$ *of either* x *or* y*. The explicit form of the first two terms,* D_2 *and* D_3*, is given in Remark 2.3.9.*

Corollary 2.3.11. *Let* $k \in \mathbb{N}$*, let* $(G, \mathfrak{g})$ *be a nilpotent* k*-Lie pair, and let* H *be a* k*-complete, normal subgroup of* G*. Then* $\log(H)$ *is an ideal of* $\mathfrak{g}$*.*

Proof. Let H be a k-complete, normal subgroup of G. It follows from Theorem 2.3.2 that $\log(H)$ is a subalgebra of $\mathfrak{g}$. Let $x \in G$ and let $y \in H$. Since H is k-complete we have $y^\lambda \in H$ for every $\lambda \in \mathbb{Z}[\frac{1}{k!}]$, and since H is a normal subgroup of G it follows that every commutator of length $m \geq 2$ in x and y^λ is an element of H. By Formula (2.33) we then obtain $[\log(x), \log(y)] = \log(v)$ for some $v \in H$ and hence $[\log(x), \log(y)] \in \log(H)$. □

2.4 Consequences of the Inversion formulas

In many of the proofs of this chapter, we want to pass from a given k-Lie pair $(G, \mathfrak{g})$ to a pair of quotients $(G/J, \mathfrak{g}/\mathfrak{j})$. If $\mathfrak{j}$ is an ideal of $\mathfrak{g}$, then the pair $(G/\exp(\mathfrak{j}), \mathfrak{g}/\mathfrak{j})$, defines a nilpotent k-Lie pair, as we now show.

Lemma 2.4.1. *Let* $k \in \mathbb{N}$ *and let* $(G, \mathfrak{g})$ *be a nilpotent* k*-Lie pair. If* $\mathfrak{j}$ *is an ideal of* $\mathfrak{g}$*, then* $J := \exp(\mathfrak{j})$ *is a normal subgroup of* G *and the pair of quotients* $(G/J, \mathfrak{g}/\mathfrak{j})$ *is a nilpotent* k*-Lie pair.*

Proof. Let $\mathfrak{j}$ be any ideal of $\mathfrak{g}$. In order to see that $J := \exp(\mathfrak{j})$ is a normal subgroup of G, let $x \in G$ and let $y \in J$. Then $Y := \log(y) \in \mathfrak{j}$, $X := \log(x) \in \mathfrak{g}$, and we obtain by the Campbell-Hausdorff formula

$$xyx^{-1} = \exp(X)\exp(Y)\exp(-X) = \exp\big(Y + [X, Y] + \sum_{m\geq 3} S_m(X, Y)\big),$$

where each term $S_m(X, Y)$, $m \geq 3$, is a linear combination of commutators in X and Y of length at least three with coefficients in Λ_k. Since $\mathfrak{j}$ is an ideal of $\mathfrak{g}$, it follows that $Y + [X, Y] + \sum_{m\geq 3} S_m(X, Y) \in \mathfrak{j}$. Hence $xyx^{-1} \in \log(\mathfrak{j}) = J$, which proves that J is a normal subgroup of G.

Let $q : G \to G/J$ and $q' : \mathfrak{g} \to \mathfrak{g}/\mathfrak{j}$ denote the canonical quotient maps. Notice that if $X + \mathfrak{j}$ and $Y + \mathfrak{j}$ are elements of $\mathfrak{g}/\mathfrak{j}$ then we may define the commutator map on the quotient algebra by

$$[X + \mathfrak{j}\,, Y + \mathfrak{j}\,] := [X, Y] + \mathfrak{j}.$$

Clearly, $\mathfrak{g}/\mathfrak{j}$ becomes in this way a Lie algebra over the ring Λ_k.

In order to prove property *(ii)* of Definition 2.2.5, we show that there exists a homeomorphism $\widetilde{\log} : G/J \to \mathfrak{g}/\mathfrak{j}$ such that the following diagram is commutative

$$\begin{tikzcd} G \arrow[r, "\log"] \arrow[d, "q"] & \mathfrak{g} \arrow[d, "q'"] \\ G/J \arrow[r, "\widetilde{\log}"] & \mathfrak{g}/\mathfrak{j}. \end{tikzcd}$$

But by the Campbell-Hausdorff formula we obtain for all $x \in G$ and $y \in J$,

$$q'(\log(yx)) = q'(\log(y) + \log(x) + \frac{1}{2}[\log(y), \log(x)] + \cdots).$$

Since $\mathfrak{j}$ is an ideal of $\mathfrak{g}$ it follows that the sum $\log(y) + \frac{1}{2}[\log(y), \log(x)] + \cdots$ is an element of the ideal $\mathfrak{j}$ and thus

$$q'(\log(yx)) = q'(\log(y) + \log(x) + \frac{1}{2}[\log(y), \log(x)] + \cdots) = q'(\log(x)).$$

Hence the map $\widetilde{\log} : G/J \to \mathfrak{g}/\mathfrak{j}$, defined by

$$\widetilde{\log}(q(x)) := q'(\log(x))$$

is well-defined. It remains to show that $\widetilde{\log}$ is a homeomorphism. For this we define the map $\widetilde{\exp} : \mathfrak{g}/\mathfrak{j} \to G/J$ by

$$\widetilde{\exp}(q'(X)) := q(\exp(X)) \tag{2.34}$$

and claim that $\widetilde{\exp}$ is well-defined inverse map of $\widetilde{\log}$. Indeed, we obtain by the Inversion Formula (2.32) for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{j}$:

$$\exp(X + Y) = \exp(X)\exp(Y) \prod_{m\geq 2}^{k} C_m(\exp(X), \exp(Y)),$$

where each term $C_m(\exp(X), \exp(Y))$ is a product of commutators $(z_1, \ldots, z_m)$ of length m and where each entry z_i is equal to some rational power $\lambda \in \Lambda_k$ of some product in $\exp(X)$ and $\exp(Y)$. But $\exp(Y) \in \exp(\mathfrak{j}) = J$ and since J is a normal subgroup of G, it follows that $\exp(Y) \prod_{m\geq 2}^{k} C_m(\exp(X), \exp(Y)) \in J$. Thus, we obtain for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{j}$:

$$q(\exp(X + Y)) = q(\exp(X)).$$

This proves that the map $\widetilde{\exp}$ is well-defined.

Furthermore we have for all $X \in \mathfrak{g}$:

$$\widetilde{\log}\big(\widetilde{\exp}(q'(X))\big) = \widetilde{\log}\big(q(\exp(X))\big) = q'\big(\log(\exp(X))\big) = q'(X)$$

and we have for all $x \in G$:

$$\widetilde{\exp}\big(\widetilde{\log}(q(x))\big) = \widetilde{\exp}\big(q'(\log(x))\big) = q\big(\exp(\log(x))\big) = q(x).$$

Therefore, the map $\widetilde{\exp}$ is the inverse map of $\widetilde{\log}$ and it follows that both maps, $\widetilde{\exp}$ and $\widetilde{\log}$ are bijective.

Since both quotient maps, $q : G \to G/J$ and $q' : \mathfrak{g} \to \mathfrak{g}/\mathfrak{j}$, are continuous and both maps, log and exp, are continuous, it follows that both maps, $\widetilde{\log}$ and $\widetilde{\exp}$, are continuous (with respect to the quotient topologies). Indeed, we have

- $\widetilde{\log}$ is continuous if and only if $\widetilde{\log} \circ q$ is continuous if and only if $q' \circ \log$ is continuous and
- $\widetilde{\exp}$ is continuous if and only if $\widetilde{\exp} \circ q'$ is continuous if and only if $q \circ \exp$ is continuous.

Furthermore, we have for all $X, Y \in \mathfrak{g}$:

$$\begin{aligned}
\widetilde{\exp}(q'(X))\,\widetilde{\exp}(q'(Y)) &= q(\exp(X))\,q(\exp(Y)) = q(\exp(X)\exp(Y)) \\
&= q(\exp(X + Y + \frac{1}{2}[X, Y] + \dots)) \\
&= \widetilde{\exp}(q'(X + Y + \frac{1}{2}[X, Y] + \dots)) \\
&= \widetilde{\exp}(q'(X) + q'(Y) + \frac{1}{2}[q'(X), q'(Y)] + \dots)).
\end{aligned}$$

This proves that the homeomorphism $\widetilde{\exp}$ satisfies the Campbell-Hausdorff formula.

Since $(G, \mathfrak{g})$ is a nilpotent k-Lie pair, we can find a Λ_k-module $\mathfrak{w}$ and a character $\epsilon \in \widehat{\mathfrak{w}}$ such that $\mathfrak{w}$ and ϵ satisfy property (iii) of Definition 2.2.5. But we have proven in Lemma 2.2.12 that this Λ_k-module $\mathfrak{w}$ and this character ϵ satisfy the property

$$\mathrm{Hom}(\mathfrak{g}/\mathfrak{h}, \mathfrak{w}) \cong \widehat{\mathfrak{g}/\mathfrak{h}} \text{ via } \tilde{f} \mapsto \epsilon \circ \tilde{f}.$$

Therefore, the pair $(G/J, \mathfrak{g}/\mathfrak{j})$ satisfies all the properties of Definition 2.2.5 and is in fact a nilpotent k-Lie pair. $\square$

Moreover, we may use the Inversion Formulas, (2.32) and (2.33), of the Campbell-Hausdorff formula to show that the ascending central series of G corresponds to the ascending central series of $\mathfrak{g}$ for every nilpotent k-Lie pair $(G, \mathfrak{g})$.

Lemma 2.4.2. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair of nilpotence class l. Then*

$$\exp(\mathfrak{z}^i(\mathfrak{g})) = Z^i(G) \quad \forall\, i = 1, \dots, l, \tag{2.35}$$

where $\mathfrak{z}^i(\mathfrak{g})$ denotes the ith element of the ascending central series of $\mathfrak{g}$ and $Z^i(G)$ denotes the ith element of the ascending central series of G. In particular, $Z^i(G)$ is k-complete for every $i = 1, \dots, l$.

Proof. We will prove Equation (2.35) by induction on $i = 1, \dots, l$.

Let $i = 1$ and let $x \in \exp(\mathfrak{z}(\mathfrak{g}))$. By the Campbell-Hausdorff-formula and the fact that $X := \log(x) \in \mathfrak{z}(\mathfrak{g})$ we obtain for all $y = \exp(Y) \in G$:

$$(x, y) = \exp(X)\exp(Y)\exp(-X)\exp(-Y) = \exp([X, Y]) = \exp(0) = 1,$$

where (x, y) denotes the group commutator of x and y. Thus the element $x \in G$ commutes with every element $y \in G$ which means that $x \in Z(G)$.

On the other hand, let $x \in Z(G)$. By the Inversion Formula (2.33) we have

$$[\log(x), \log(y)] = \log\big((x, y)\prod_{m\geq 2} D_m(x, y)\big),$$

where each $D_m(x, y)$, $m \geq 2$, is a product of commutators $(z_1, \dots, z_m)$ of length m and where z_i is equal either to the commutator (x, y) or to some rational power $\lambda_m \in \mathbb{Z}[\frac{1}{k!}]$ of x or y. But $(x, y) = 1$ for all $y \in G$ and it follows from Lemma 2.2.15 and the fact that the map log is bijective that $1^\lambda = 1$ for all $\lambda \in \mathbb{Z}[\frac{1}{k!}]$. Thus $D_m(x, y) = 1$ for all $y \in G$ and for all $m \geq 2$ and hence $(x, y)\prod_{m\geq 2} D_m(x, y) = 1$ for all $y \in G$. Therefore we obtain $[\log(x), \log(y)] = \log(1) = 0$ for all $y \in G$ which proves that $\log(x) \in \mathfrak{z}(\mathfrak{g})$. Moreover, $Z(G)$ is k-complete. Indeed, if $x \in Z(G)$ then we obtain by Lemma 2.2.15

$$\log(x^\lambda) = \lambda \log(x) \quad \forall\, \lambda \in \mathbb{Z}[\frac{1}{k!}].$$

But we have $\log(x) \in \mathfrak{z}(\mathfrak{g})$ and since the commutator is $\mathbb{Z}[\frac{1}{k!}]$-bilinear (Remark 2.2.9) it follows that $\lambda \log(x) \in \mathfrak{z}(\mathfrak{g})$ for all $\lambda \in \mathbb{Z}[\frac{1}{k!}]$. Hence $\exp(\lambda \log(x)) = x^\lambda \in Z(G)$.

Let $i \in \{2, \dots, l\}$ and suppose that (2.35) holds for every $1 \leq j < i$. In particular, we have $\exp(\mathfrak{z}^{i-1}(\mathfrak{g})) = Z^{i-1}(G)$ and since $\mathfrak{z}^{i-1}(\mathfrak{g})$ is a closed ideal of $\mathfrak{g}$, it follows from Lemma 2.2.12 that the pair of quotients $(G/Z^{i-1}, \mathfrak{g}/\mathfrak{z}^{i-1}(\mathfrak{g}))$ is a nilpotent k-Lie pair. Thus we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{z}^{i-1}(\mathfrak{g}) \arrow[r] \arrow[d, "\exp"] & \mathfrak{g} \arrow[r, "q_i"] \arrow[d, "\exp"] & \mathfrak{g}/\mathfrak{z}^{i-1}(\mathfrak{g}) \arrow[r] \arrow[d, "\widetilde{\exp}"] & 0 \\
1 \arrow[r] & Z^{i-1}(G) \arrow[r] & G \arrow[r, "q_i'"] & G/Z^{i-1}(G) \arrow[r] & 1,
\end{tikzcd}$$

where both maps, exp and $\widetilde{\exp}$, are homeomorphisms of groups. Since the pair of quotients $(G/Z^{i-1}, \mathfrak{g}/\mathfrak{z}^{i-1}(\mathfrak{g}))$ is a nilpotent k-Lie pair it follows from our prior computation that

$$\widetilde{\exp}(\mathfrak{z}(\mathfrak{g}/\mathfrak{z}^{i-1})) = Z(G/Z^{i-1}(G)).$$

But since the diagram above is commutative, we obtain the desired result

$$\exp(\mathfrak{z}^{\mathfrak{i}}(\mathfrak{g})) = \exp\left(q_i^{-1}(\mathfrak{z}(\mathfrak{g}/\mathfrak{z}^{i-1}))\right) = q_i'^{-1}\big(Z(G/Z^{i-1}(G))\big) = Z^i(G).$$

Moreover, the subgroup $Z^i(G)$ is k-complete. Indeed, since $\log(x^\lambda) = \lambda\log(x)$ for all $x \in Z^i(G)$ and for all $\lambda \in \mathbb{Z}[\frac{1}{k!}]$, it follows from the fact that the commutator is $\mathbb{Z}[\frac{1}{k!}]$-bilinear that $\lambda\log(x) \in \mathfrak{z}^i(\mathfrak{g})$ and hence $x^\lambda \in Z^i(G)$ for all $\lambda \in \mathbb{Z}[\frac{1}{k!}]$. □

Lemma 2.4.3. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. Let H be an exponentiable subgroup of G and denote by $\mathfrak{h}$ the subalgebra of $\mathfrak{g}$ corresponding to H. The closure of the commutator subgroup of H is a normal, exponentiable subgroup of H and one has*

$$\overline{(H,H)} = \exp(\overline{[\mathfrak{h},\mathfrak{h}]}).$$

Proof. Let $x, y \in H$ and put $X := \log(x)$ and $Y := \log(y)$. Then $X, Y \in \mathfrak{h}$ and we obtain by the Campbell-Hausdorff formula

$$\begin{aligned}(x,y) &= \exp(X)\exp(Y)\exp(-X)\exp(-Y)\\ &= \exp([X,Y] + \frac{1}{2}[X,[X,Y]] - \frac{1}{2}[Y,[Y,X]] + \dots),\end{aligned}$$

which proves that $(x,y) \in \exp([\mathfrak{h},\mathfrak{h}])$. Since the inverse map of exp is continuous it follows that

$$\overline{(H,H)} \subseteq \exp(\overline{[\mathfrak{h},\mathfrak{h}]}).$$

On the other hand, let $X, Y \in \mathfrak{h}$ and put $x := \exp(X)$ and $y := \exp(Y)$. Then $x, y \in H$ and we obtain by the Inversion Formula (2.33)

$$[\log(x),\log(y)] = \log\big((x,y)(x^{-\frac{1}{2}},(x,y))(y^{\frac{1}{2}}(y,x))\dots\big).$$

Since H is k-complete (Theorem 2.3.2) it follows that $[\log(x),\log(y)] \in \log((H,H))$ and thus $\exp([\mathfrak{h},\mathfrak{h}]) \subseteq (H,H)$. Since the map exp is continuous it follows that

$$\exp(\overline{[\mathfrak{h},\mathfrak{h}]}) \subseteq \overline{(H,H)}.$$

Since $[\mathfrak{h},\mathfrak{h}]$ is an ideal of $\mathfrak{h}$, it follows that $\overline{[\mathfrak{h},\mathfrak{h}]}$ is a closed ideal of $\mathfrak{h}$ and thus $\exp(\overline{[\mathfrak{h},\mathfrak{h}]}) = \overline{(H,H)}$ is a normal exponentiable subgroup of H (Lemma 2.4.1). □

2.5 The inner hull-kernel topology and the space $\mathcal{K}(G)$

In this section we recall certain facts concerning the inner hull-kernel topology of representations of a C^*-algebra A or a locally compact group G and we will define the compact-open topology on the space $\mathcal{K}(G)$ of all closed subgroups of G. We follow the approach of Fell [10], §1 and §2. Furthermore we state some general results about the representation theory of locally compact groups which we will need several times in this chapter.

In the following let A be a C^*-algebra and let $\mathrm{Rep}(A)$ be the space of equivalence classes of $*$-representations of A with dimension bounded by a fixed cardinal $\aleph$. The restriction on the dimension has to be made in order that $\mathrm{Rep}(A)$ be a set. However, we will assume that $\aleph$ is big enough to guarantee that all representations we are interested in have dimension less than $\aleph$. If $\pi \in \mathrm{Rep}(A)$ and $\Sigma \subseteq \mathrm{Rep}(A)$, then π is weakly contained in Σ (and we often write $\pi \prec \Sigma$), if $\ker(\pi) \supseteq \cap_{\sigma\in\Sigma} \ker(\sigma)$. Two subsets Σ_1 and Σ_2 of $\mathrm{Rep}(A)$ are weakly equivalent if every $\sigma_1 \in \Sigma_1$ is weakly contained in Σ_2, and conversely. Restricted to $\hat{A}$, the set of equivalence classes of irreducible unitary representations of A, the relation of weak containment defines the closure operation for the hull-kernel topology.

Definition 2.5.1. Let $\pi \in \mathrm{Rep}(A)$. The spectrum of π is defined as

$$\mathrm{Sp}(\pi) := \{\rho \in \hat{A} \mid \rho \prec \pi\}.$$

Each $\pi \in \mathrm{Rep}(A)$ is weakly equivalent to the unique closed subset $\mathrm{Sp}(\pi)$ of $\hat{A}$. Let $\mathcal{F}$ be a finite family of nonempty open subsets of $\hat{A}$ and define

$$\mathcal{U}(\mathcal{F}) := \{\pi \in \mathrm{Rep}(A) \mid \mathrm{Sp}(\pi) \cap V \neq \emptyset \ \forall\, V \in \mathcal{F}\}.$$

The inner hull-kernel topology of $\mathrm{Rep}(A)$ is the topology in which the set of all $\mathcal{U}(\mathcal{F})$ forms a basis of open sets. Relativized to $\hat{A}$, this is again the hull-kernel topology. In the following we assume that $\mathrm{Rep}(A)$ is equipped with the inner hull-kernel topology and $\hat{A}$ is equipped with the hull-kernel topology. The following facts will be very useful.

Proposition 2.5.2. *([10], Proposition 1.1) If $\pi \in \mathrm{Rep}(A)$ and $\Sigma \subseteq \mathrm{Rep}(A)$, then the following are equivalent:*

(a) π is weakly contained in Σ.

(b) The closure of $\cup_{\sigma\in\Sigma} \mathrm{Sp}(\sigma)$ contains $\mathrm{Sp}(\pi)$.

Proposition 2.5.3. *([10], Proposition 1.2) A net π_i converges to π in $\mathrm{Rep}(A)$ if and only if every subnet of π_i weakly contains π.*

Proposition 2.5.4. *([10], Proposition 1.3) If π_i converges to π in* $\mathrm{Rep}(A)$ *and π weakly contains ρ, then π_i converges to ρ in* $\mathrm{Rep}(A)$.

Proposition 2.5.5. *([28], Theorem 2.2) Let $(\pi_i)_{i\in I}$ be a net in* $\mathrm{Rep}(A)$ *with $\pi_i \to \pi$ for some $\pi \in \hat{A}$. For every $i \in I$, let D_i be a dense subset of* $\mathrm{Sp}(\pi_i)$. *Then there exists a subnet $(\pi_\lambda)_{\lambda\in\Lambda}$ of $(\pi_i)_{i\in I}$ and a net $(\rho_\lambda)_{\lambda\in\Lambda}$ in $\hat{A}$ such that $\rho_\lambda \in D_\lambda$ for all $\lambda \in \Lambda$ and $\rho_\lambda \to \pi$ in $\hat{A}$.*

If G is a locally compact group, we define weak containment, the inner hull-kernel topology, and so forth, for $\mathrm{Rep}(G)$ by passing to the group C^*-algebra and identifying $\mathrm{Rep}(G)$ with the family of all representations of the latter.

If X is an arbitrary locally compact (not necessarily Hausdorff) space, then we equip the family $\mathcal{X}(X)$ of all closed subsets of X with the compact-open topology as follows. For each compact subset C of X and each finite family $\mathcal{F}$ of nonempty open subsets of X, let

$$\mathcal{U}(C, \mathcal{F}) := \{Y \in \mathcal{X}(X) \mid Y \cap C = \emptyset, Y \cap B \neq \emptyset \;\forall\; B \in \mathcal{F}\}.$$

A subset $\mathcal{Y}$ of $\mathcal{X}(X)$ is open in the compact-open topology if and only if it is a union of certain of the $\mathcal{U}(C, \mathcal{F})$. With this topology $\mathcal{X}(X)$ becomes a compact Hausdorff space [9].

If X is a second countable Hausdorff space, we may describe the convergence of closed subsets of X as follows.

Proposition 2.5.6. *Let X be a second countable Hausdorff space and let $(A_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{X}(X)$. Then $A_n \to A$ in $\mathcal{X}(X)$ if and only if the following properties are satisfied:*

(a) If $a_n \in A_n$ and $a_n \to a$ then $a \in A$.

(b) If $a \in A$ then there exists a subsequence (A_{n_k}) of (A_n) and there exist elements $a_{n_k} \in A_{n_k}$ for all $k \in \mathbb{N}$ such that $a_{n_k} \to a$ in X.

If G is a locally compact group and $\mathcal{K}(G)$ the family of all closed subgroups of G, then, as a subset of $\mathcal{X}(G)$, $\mathcal{K}(G)$ is closed in the compact-open topology. Thus we obtain the following result.

Proposition 2.5.7. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair.*

(i) If $(L_n)_{n\in\mathbb{N}}$ is a sequence of closed, normal subgroups in G such that $L_n \to L$ in $\mathcal{X}(G)$, then L is also a closed, normal subgroup of G.

(ii) If $(L_n)_{n\in\mathbb{N}}$ is a sequence of exponentiable subgroups of G such that $L_n \to L$ in $\mathcal{X}(G)$ for some subgroup L of G, then L is also exponentiable.

Proof. Part (i) follows immediately from the characterization of convergence in $\mathcal{X}(G)$ given in Proposition 2.5.6. For part (ii), let $(L_n)_{n\in\mathbb{N}}$ be a sequence of exponentiable subgroups of G with $L_n \to L$ in $\mathcal{X}(G)$ for some closed subgroup L of G. Then the sequence of closed groups $(\ell_n)_{n\in\mathbb{N}}$ converges in $\mathcal{X}(\mathfrak{g})$, where $\ell_n := \log(L_n)$ denotes the corresponding subalgebra of L_n in $\mathfrak{g}$, for every $n \in \mathbb{N}$. Let ℓ be its limit. It follows from part (i) that ℓ is a closed subgroup of $\mathfrak{g}$ and since the commutator is continuous, one can use the same argument to show that ℓ is a subalgebra of $\mathfrak{g}$. Since the maps $\exp : \mathfrak{g} \to G$ and $\log : G \to \mathfrak{g}$ are homeomorphisms, it follows that $\exp(\ell) = L$ is the exponentiable limit of the sequence $(L_n)_{n\in\mathbb{N}}$. □

Since the dual space $\hat{A}$ of every C^*-algebra A is locally compact, it follows from the above that $\mathcal{X}(\hat{A})$ is compact in the compact-open topology. If we identify each representation π with its spectrum, the inner hull-kernel topology of $\mathrm{Rep}(A)$ is contained in the compact-open topology of $\mathcal{X}(\hat{A})$. Thus we obtain the following fact.

Proposition 2.5.8. *([10], Proposition 1.7) If G is a locally compact group,* $\mathrm{Rep}(G)$ *is compact (in the inner hull-kernel topology).*

If G is a locally compact group we shall assume that $\mathcal{K}(G)$ carries the relativized compact-open topology. Thus $\mathcal{K}(G)$ is a compact Hausdorff space.

Suppose G is a locally compact group and H is a closed subgroup of G. Any unitary representation of G can be restricted to H, and any unitary representation of H can be induced up to G. In the following we recall some results about the relationship between these procedures. But first we recall briefly the definition of the inducing construction for unimodular groups.

Remark/Definition 2.5.9. Let G be a unimodular locally compact group, H a closed subgroup of G, $q : G \to G/H$ the canonical quotient map, and σ a unitary representation of H on the Hilbert space H_σ. We denote the inner product on H_σ by $\langle u, v\rangle_\sigma$, and we denote by $C(G, H_\sigma)$ the space of continuous functions from G to H_σ. Following the approach in [13], §6.1, we will now construct the unitary representation $\mathrm{ind}_H^G \sigma : G \to \mathcal{U}(H_{\mathrm{ind}\,\sigma})$. The main ingredient in the inducing construction is the following space of vector valued functions

$$\begin{aligned}\mathcal{F}_\sigma := \{f \in C(G, H_\sigma) \mid \quad & q(\mathrm{supp}(f)) \text{ is compact and} \\ & f(x\xi) = \sigma(\xi^{-1})f(x) \text{ for } x \in G, \xi \in H\}.\end{aligned}$$

Elements of $\mathcal{F}_\sigma$ are of the following form.

Propositon. *([13], Proposition 6.1) If $\alpha : G \to H_\sigma$ is continuous with compact support, then the function*

$$f_\alpha(x) = \int_H \sigma(\eta)\alpha(x\eta)\, d\eta$$

belongs to $\mathcal{F}_\sigma$ and is uniformly continuous on G. Moreover, every element of $\mathcal{F}_\sigma$ is of the form f_α for some $\alpha \in C_c(G, H_\sigma)$.

The group G acts on the space $\mathcal{F}_\sigma$ by left translations and thus we obtain a unitary representation of G if we can impose an inner product on $\mathcal{F}_\sigma$ with respect to which these translations are isometries. If $f, g \in \mathcal{F}_\sigma$ then we define

$$\langle f, g\rangle := \int_G \beta(x)\langle f(x), g(x)\rangle \, d\mu(x), \tag{2.36}$$

where $\beta : G \to [0, \infty)$ is a Bruhat-section for G, i.e., β is a continuous function satisfying

(i) $\operatorname{supp}(\beta) \cap CH$ is compact for all $C \subseteq G$ compact and

(ii) $\int_H \beta(xh)\, d\mu(h) = 1$ for all $x \in G$.

One can show that such a Bruhat-section always exists and that the pairing $\langle ., .\rangle$ in (2.36) defines an inner product on $\mathcal{F}_\sigma$, which is preserved by left translations. We denote by $H_{\operatorname{ind}\sigma}$ the Hilbert space completion of $\mathcal{F}_\sigma$ with respect to this scalar product and we put for $s, t \in G$ and $\xi \in \mathcal{F}_\sigma$

$$(\operatorname{ind}_H^G \sigma(s)(\xi))(t) := \xi(s^{-1}t).$$

This left translation operator extends to a strongly continuous unitary operator on the Hilbert space $H_{\operatorname{ind}\sigma}$ and thus

$$\operatorname{ind}_H^G \sigma : G \to \mathcal{U}(H_{\operatorname{ind}\sigma}),\ s \mapsto \operatorname{ind}_H^G \sigma(s)$$

defines a unitary representation of G, called the representation induced by σ.

Induced representations have many nice properties. One fundamental result is the following theorem of "induction in stages".

Theorem 2.5.10. *([13], Theorem 6.14) Suppose H is a closed subgroup of G, L is a closed subgroup of H, and σ is a unitary representation of L. Then the representations $\operatorname{ind}_L^G \sigma$ and $\operatorname{ind}_H^G(\operatorname{ind}_L^H \sigma)$ are unitarily equivalent.*

Moreover, we will often rely on the following results. A proof of both theorems can be found for example in [13].

Theorem 2.5.11. *Let G be a locally compact group and let H be a closed subgroup of G. If π is a unitary representation of G and ρ a unitary representation of H, then*

$$\operatorname{ind}_H^G(\pi|_H \otimes \rho) \cong \pi \otimes \operatorname{ind}_H^G \rho.$$

Theorem 2.5.12. *Let L be a closed, normal subgroup of the locally compact group G and let H be a closed subgroup of G with $L \subseteq H \subseteq G$. If π is a unitary representation of H/L, then*

$$\operatorname{ind}_H^G(\pi \circ q) \cong (\operatorname{ind}_{H/L}^{G/L} \pi) \circ q,$$

where $q : G \to G/L$ denotes the canonical quotient map.

Let G be a locally compact second countable group and let N be a closed normal subgroup of G. The group G acts by conjugation as a group of automorphisms of N, and this action determines in a natural manner an action of G on $C^*(N)$, $\hat{N}$, and $\mathrm{Prim}(N)$.

Theorem 2.5.13. *([14], Theorem 2.1.) For each $\pi \in \hat{G}$ there exists $J \in \mathrm{Prim}(N)$ such that*

$$\ker(\pi|_N) = \bigcap_{g \in G} g \cdot J.$$

Proposition 2.5.14. *([11], Theorem 1) Let G be a separable locally compact group and let H be a closed subgroup of G. Let ρ and σ be two unitary representations of H and suppose that $\rho \prec \sigma$. If π is any unitary representation of H then $\pi \otimes \rho \prec \pi \otimes \sigma$.*

Proposition 2.5.15. *([10], Proposition 5.3) Let G be a locally compact group and let H be a closed, normal subgroup of G. For every representation $\sigma \in \hat{H}$ one has*

$$\sigma \prec (\mathrm{ind}_H^G \sigma)|_H.$$

Since locally compact nilpotent groups are in particular amenable, we can also make use of the following well-known facts concerning the relationship between induced and restricted representations of amenable groups.

Theorem 2.5.16. *([15], Theorem 5.1) Let G be an amenable, locally compact group and let H be a closed subgroup of G. Then*

$$1_G \prec \mathrm{ind}_H^G 1_H,$$

where 1_G denotes the trivial representation of G and 1_H the trivial representation of H.

The following theorem is also well-known. We include a short proof.

Theorem 2.5.17. *Let G be a locally compact, amenable group and let H be a closed subgroup of G. If $\pi \in \mathrm{Rep}(G)$, then $\pi \prec \mathrm{ind}_H^G(\pi|_H)$.*

Proof. Let π be a unitary representation of G. Applying Theorem 2.5.11 to the representation $\rho = 1_H$ yields

$$\mathrm{ind}_H^G(\pi|_H \otimes 1_H) \cong \pi \otimes \mathrm{ind}_H^G 1_H.$$

But since $1_G \prec \mathrm{ind}_H^G 1_H$ (Theorem 2.5.16), we obtain by Proposition 2.5.14 the desired result

$$\pi \cong \pi \otimes 1_G \prec \pi \otimes \mathrm{ind}_H^G 1_H \cong \mathrm{ind}_H^G(\pi|_H).$$

$\square$

2.6 Definition of the Kirillov-orbit map

In the following, let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. Then there exists a Λ_k-module $\mathfrak{w}$ and there exists a character $\epsilon \in \widehat{\mathfrak{w}}$ such that $\mathrm{Hom}(\mathfrak{g}, \mathfrak{w}) \cong \widehat{\mathfrak{g}}$ via the map $f \mapsto \epsilon \circ f$. Recall that the group $\mathfrak{g}^* := \mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$ serves as a substitute for a linear dual space of the Lie algebra $\mathfrak{g}$, and this group is unique up to isomorphism. In order to define a Kirillov map from the “dual space” $\mathfrak{g}^*$ to the primitive ideal space of $C^*(G)$, we will introduce the notion of a polarizing subalgebra $\mathfrak{r}$ for a homomorphism $f \in \mathfrak{g}^*$. Moreover, we will explain how such a group homomorphism $f \in \mathfrak{g}^*$ defines a character φ_f on the subgroup $R := \exp(\mathfrak{r})$ of G.

Definition 2.6.1. Let $f \in \mathfrak{g}^*$ be a given group homomorphism. A closed subalgebra $\mathfrak{r}$ of $\mathfrak{g}$ is said to be f-subordinate if

$$f([\mathfrak{r}, \mathfrak{r}]) \equiv 0.$$

If $\mathfrak{r}$ is maximal with this property we say that $\mathfrak{r}$ is a polarizing subalgebra for f.

We observe that a homomorphism $f \in \mathfrak{g}^*$ may have different, non-isomorphic polarizing subalgebras as indicated in the following example.

Example 2.6.2. Let $\mathfrak{g} = Tr_0(4, \mathbb{R})$, the nilpotent Lie algebra of upper triangular 4×4-matrices with entries in $\mathbb{R}$ and diagonal entries equals 0. A typical element of $\mathfrak{g}$ is of the form

$$A = \begin{pmatrix} 0 & x_1 & y_1 & z \\ 0 & 0 & x_2 & y_2 \\ 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} = zZ + y_1Y_1 + y_2Y_2 + x_1X_1 + x_2X_2 + x_3X_3,$$

where $Z, Y_1, Y_2, X_1, X_2, X_3$ is a basis of $\mathfrak{g}$ corresponding to the matrix entries. The nontrivial commutators in $\mathfrak{g}$ are:

$$[X_1, X_2] = Y_1, \; [X_3, X_2] = -Y_2, \; [X_1, Y_2] = Z, \text{ and } [X_3, Y_1] = -Z. \tag{2.37}$$

Let $\mathfrak{g}^*$ be the linear dual of $\mathfrak{g}$ and let $\{Z^*, Y_1^*, Y_2^*, X_1^*, X_2^*, X_3^*\}$ be a dual basis in $\mathfrak{g}^*$. This means that if

$$f := \gamma Z^* + \sum_{i=1}^{3} \alpha_i X_i^* + \sum_{i=1}^{2} \beta_i Y_i^* \in \mathfrak{g}^* \text{ and } W := cZ + \sum_{i=1}^{3} a_i X_i + \sum_{i=1}^{2} b_i Y_i \in \mathfrak{g},$$

for some real numbers $\gamma, \alpha_i, \beta_i$ and for some real numbers c, a_i, b_i, then

$$f(W) = c\gamma + \sum_{i=1}^{3} a_i \, \alpha_i + \sum_{i=1}^{2} b_i \, \beta_i \in \mathbb{R}\,.$$

Let $f := \gamma Z^* + \alpha_2 X_2^* \in \mathfrak{g}^*$ with $\gamma \neq 0$ and $\alpha_2 \neq 0$. Using the commutator relations of (2.37), it is easy to see that two possible polarizing subalgebras for f are given by

$$\mathfrak{r}_1 := \mathbb{R}-\text{span of } \{X_1, X_2, Y_1, Z\}, \quad \text{and} \quad \mathfrak{r}_2 := \mathbb{R}-\text{span of } \{X_2, Y_1, Y_2, Z\}.$$

Since $\mathfrak{r}_2$ is abelian and $\mathfrak{r}_1$ is not abelian, these two polarizing subalgebras of $\mathfrak{g}$ are not isomorphic.

However, we will see in Proposition 2.8.16 that there exists a polarizing subalgebra $\mathfrak{r}$ for every homomorphism $f \in \mathfrak{g}^*$.

Remark/Notation 2.6.3. Let $f \in \mathfrak{g}^*$ and let $\mathfrak{r}$ be a polarizing subalgebra for f. Then the map φ_f, defined by

$$\varphi_f(\exp X) := \epsilon(f(X)) \quad \text{for all } X \in \mathfrak{r}, \tag{2.38}$$

is a character of the closed subgroup $R := \exp \mathfrak{r}$ of G. Indeed, using the Campbell-Hausdorff formula we obtain for all $X, Y \in \mathfrak{r}$:

$$\exp(X)\exp(Y) \;=\; \exp(X + Y + \sum_{m\geq 2}^{k-1} C_m(X,Y)),$$

where each term $C_m(X,Y)$ is a linear combination of Lie products $[Z_1, \ldots, Z_m]$ of length m and each Z_i is equal either to X or to Y. Since the commutator is $\mathbb{Z}[\frac{1}{k!}]$-bilinear and $\mathfrak{r}$ is a f-subordinate subalgebra of $\mathfrak{g}$, it follows that $f(C_m(X,Y)) = 1$ for all $X, Y \in \mathfrak{r}$ and for all $m = 2, \ldots, k-1$. Hence $f(\sum_m C_m(X,Y)) = 1$ and we obtain for all $X, Y \in \mathfrak{r}$:

$$\begin{aligned}
\varphi_f(\exp X \exp Y) &= \varphi_f(\exp(X + Y + \sum_{m\geq 2} C_m(X,Y))) \\
&= \epsilon(f(X + Y + \sum_{m\geq 2} C_m(X,Y))) = \epsilon(f(X+Y)) \\
&= \epsilon(f(X)) \cdot \epsilon(f(Y)) = \varphi_f(\exp X) \cdot \varphi_f(\exp Y).
\end{aligned}$$

If $f \in \mathfrak{g}^*$ and if $\mathfrak{r}$ is a polarizing subalgebra of f then we denote by φ_f the character of $\exp(\mathfrak{r}) = R$ as defined in (2.38). We will prove in Proposition 2.8.16 that the induced character $\mathrm{ind}_R^G \varphi_f$ is an irreducible representation of G.

Recall the notion of a Lie algebra homomorphism.

Definition 2.6.4. Let R be a commutative ring with unity, let V be a R-module and let $\mathfrak{g}$ be a Lie algebra over R. A map $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is called Lie algebra homomorphism if for all $X, Y \in \mathfrak{g}$ one has

(1) $\rho(X+Y) = \rho(X) + \rho(Y)$ and

(2) $\rho([X,Y]) = [\rho(X), \rho(Y)]$.

Remark 2.6.5. Recall that the adjoint map $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is defined by

$$\mathrm{ad}(X)(Y) = [X,Y].$$

We claim that the map ad is a Lie algebra homomorphism. Indeed, the map ad is a homomorphism of the additive group $\mathfrak{g}$, since the commutator map is bi-additive. Moreover, $\mathrm{End}(\mathfrak{g})$ is an associative Lie algebra and thus we obtain by the Jacobi-identity for all $X, Y, Z \in \mathfrak{g}$:

$$\mathrm{ad}([X,Y])(Z) = [[X,Y],Z] = [X,[Y,Z]] - [Y,[X,Z]] = [\mathrm{ad}(X), \mathrm{ad}(Y)](Z).$$

We will see in the following that we can exponentiate the action $\mathrm{ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ of $\mathfrak{g}$ to obtain an action Ad of G on $\mathfrak{g}$. For this, we recall the definition of a nilpotent endomorphism.

Definition 2.6.6. Let V be a Λ_k-module. An element $\sigma \in \mathrm{End}(V)$ is called nilpotent, if $\sigma^n = 0$ for some $n \in \mathbb{N}$.

Remark 2.6.7. For every $X \in \mathfrak{g}$, the map $\mathrm{ad}(X) \in \mathrm{End}(\mathfrak{g})$ is nilpotent.

Proposition 2.6.8. *Let V be a Λ_k-module. If $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is a Lie algebra homomorphism of $\mathfrak{g}$ such that $\rho(X) \in \mathrm{End}(V)$ is nilpotent for every $X \in \mathfrak{g}$, then the map*

$$\exp(\rho) : G \to \mathrm{GL}(V),\ \exp(\rho)(\exp(X)) := \exp(\rho(X)))$$

is a representation of the group G.

Proof. Let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be a Lie algebra homomorphism. Since, for every $X \in \mathfrak{g}$, the map $\rho(X) \in \mathrm{End}(V)$ is nilpotent, it follows that $\exp(\rho(X))$ is well-defined and we obtain by the Campbell-Hausdorff formula for all $X, Y \in \mathfrak{g}$:

$$\begin{aligned}
\exp(\rho)(\exp(X)\exp(Y)) &= \exp(\rho)(\exp(X + Y + \frac{1}{2}[X,Y] + \dots)) \\
&= \exp(\rho(X + Y + \frac{1}{2}[X,Y] + \dots)) \\
&= \sum_{n=0}^{k} \frac{1}{n!}(\rho(X + Y + \frac{1}{2}[X,Y] + \dots))^n \\
&= \sum_{n=0}^{k} \frac{1}{n!}(\rho(X) + \rho(Y) + \frac{1}{2}[\rho(X), \rho(Y)] + \dots)^n \\
&= \exp(\rho(X) + \rho(Y) + \frac{1}{2}[\rho(X), \rho(Y)] + \dots) \\
&= \exp(\rho(X))\exp(\rho(Y)).
\end{aligned}$$

□

Definition 2.6.9. We define the adjoint action $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ by

$$\mathrm{Ad} := \exp(\mathrm{ad}).$$

Lemma 2.6.10. *The map* $\mathrm{Ad}(x) : \mathfrak{g} \to \mathfrak{g}$ *is a Lie algebra homomorphism for all* $x \in G$.

Proof. Let $x \in G$ and let $Y, Z \in \mathfrak{g}$. We need to show

(1) $\mathrm{Ad}(x)(Y + Z) = \mathrm{Ad}(x)(Y) + \mathrm{Ad}(x)(Z)$ and

(2) $\mathrm{Ad}(x)([Y, Z]) = [\mathrm{Ad}(x)(Y), \mathrm{Ad}(x)(Z)]$.

Property (1) follows directly from Remark 2.6.5 and Proposition 2.6.8. A proof of property (2) can be found for example in [4]. $\square$

Remark 2.6.11. It is a well-known fact (see for example [4], II, Exercise, §6) that

$$\mathrm{Ad}(\exp(X))(Y) = \log(\exp(X)\exp(Y)\exp(-X)) \quad \forall\, X, Y \in \mathfrak{g}\,. \tag{2.39}$$

We may now define an action of G on the group $\mathfrak{g}^*$, which we refer to as the coadjoint action.

Definition 2.6.12. We define

$$\mathrm{Ad}^* : G \times \mathfrak{g}^* \to \mathfrak{g}^*,\ (x \cdot f)(Y) = f(\mathrm{Ad}(x^{-1})Y) \quad \forall\, Y \in \mathfrak{g}$$

to be the coadjoint action of G on the group $\mathfrak{g}^*$ and denote by $G(f)$ the G-orbit of $f \in \mathfrak{g}^*$ under this action.

If two homomorphisms are in the same G-orbit under the coadjoint action, one can choose suitable polarizing subalgebras such that the corresponding induced characters define equivalent representations of G.

Lemma 2.6.13. *Let* $f \in \mathfrak{g}^*$ *and let* $x \in G$. *Then*

(1) $\mathfrak{r}$ *is a polarizing subalgebra for* f $\Leftrightarrow$ $\mathrm{Ad}(x)\mathfrak{r}$ *is a polarizing subalgebra for* $\mathrm{Ad}^*(x)f$.

(2) The induced representations $\mathrm{ind}_R^G \varphi_f$ *and* $\mathrm{ind}_{\exp(\mathrm{Ad}(x)\mathfrak{r})}^G \varphi_{\mathrm{Ad}^*(x)f}$ *are equivalent representations.*

Proof. By Lemma 2.6.10 we obtain for all $Y, Y' \in \mathfrak{r}$:

$$\mathrm{Ad}^*(x)f([\mathrm{Ad}(x)Y, \mathrm{Ad}(x)Y']) = f(\mathrm{Ad}(x^{-1})[\mathrm{Ad}(x)Y, \mathrm{Ad}(x)Y']) = f([Y, Y']), \tag{2.40}$$

which proves that $\mathrm{Ad}(x)\mathfrak{r}$ is $\mathrm{Ad}^*(x)f$-subordinate. Since the map $\mathrm{Ad}(x) \in \mathrm{GL}(\mathfrak{g})$ is a bijective map it follows that $\mathrm{Ad}(x)\mathfrak{r}$ is a polarizing subalgebra for $\mathrm{Ad}^*(x)f$, proving statement (1).

To prove part (2), notice that

$$\exp(\mathrm{Ad}(x)\mathfrak{r}) = \{xyx^{-1} \mid y \in R\} = xRx^{-1}.$$

Moreover, we have for all $Y \in \mathfrak{r}$:

$$\varphi_{\mathrm{Ad}^*(x)f}(\exp(Y)) = \epsilon(\mathrm{Ad}^*(x)f(Y)) = \epsilon(f(\mathrm{Ad}(x^{-1})Y)) = \varphi_f(x^{-1}\exp(Y)x)$$

and it is well-known (see for example [12], XI, 16.19) that the induced representations $\pi_f := \mathrm{ind}_R^G \varphi_f$ and $\pi_{\mathrm{Ad}^*(x)f} := \mathrm{ind}_{R^x}^G \varphi_{\mathrm{Ad}^*(x)f}$ are unitarily equivalent. An intertwining operator T for these representations is given by

$$T : H_{\pi_f} \to H_{\pi_{\mathrm{Ad}^*(x)f}},\ T(\xi)(g) = \xi(gx).$$

□

We will prove in Proposition 2.9.1 that the kernel of the induced character φ_f does not depend on the choice of the polarizing subalgebra $\mathfrak{r}$ for f.

Since the space of irreducible unitary representation of G does not necessarily satisfy the T_0-axiom as a topological space, we consider instead the primitive ideal space, $\mathrm{Prim}(C^*(G))$, of the group C^*-algebra of G. This space, equipped with the hull-kernel topology, is always T_0. Our aim is to define for every nilpotent k-Lie pair $(G, \mathfrak{g})$ a Kirillov-map as follows:

$$\kappa : \mathfrak{g}^* \longrightarrow \mathrm{Prim}(C^*(G)),\ f \mapsto \ker(\mathrm{ind}_R^G \varphi_f). \tag{2.41}$$

As a consequence, we will obtain a Kirillov-orbit map

$$\tilde{\kappa} : \mathfrak{g}^* /_\sim \longrightarrow \mathrm{Prim}(C^*(G)),\ \mathcal{O} \mapsto \ker(\mathrm{ind}_R^G \varphi_f), \tag{2.42}$$

where $f \in \mathfrak{g}^*$ is any chosen representative of $\mathcal{O}$, and where $\mathfrak{g}^* /_\sim$ denotes the quasi-orbit space of $\mathfrak{g}^*$ with respect to the coadjoint action of G. That means, $\sim$ denotes the following equivalence relation on $\mathfrak{g}^*$:

$$f \sim f' :\Leftrightarrow f \in \overline{G(f')} \text{ and } f' \in \overline{G(f)}.$$

Observe that the quasi-orbit space $\mathfrak{g}^* /_\sim$ satisfies the T_0-axiom.

We will prove in Section 2.9 that the Kirillov map κ is well-defined (Proposition 2.9.1) and surjective (Proposition 2.9.3) and we will obtain a well-defined (Corollary 2.10.12) and surjective (Corollary 2.10.13) Kirillov-orbit map $\tilde{\kappa}$. Furthermore, we will prove in Section 2.10 that, under certain additional assumptions on the group G, the map $\tilde{\kappa}$ is injective (Corollary 2.10.31) and bi-continuous (Corollary 2.10.32).

2.7 The Kirillov map for two-step nilpotent groups

In the following, let $k \in \mathbb{N}_{\geq 2}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair of nilpotence class 2. Then there exists a Λ_k-module $\mathfrak{w}$ and there exists a character $\epsilon \in \widehat{\mathfrak{w}}$ such that $\mathfrak{g}^* := \mathrm{Hom}(\mathfrak{g}, \mathfrak{w}) \cong \widehat{\mathfrak{g}}$ via $f \mapsto \epsilon \circ f$.

Recall that given a homomorphism $f \in \mathfrak{g}^*$, a closed subalgebra $\mathfrak{r}$ of $\mathfrak{g}$ is called polarizing subalgebra for f, if $f([\mathfrak{r}, \mathfrak{r}]) \equiv 0$ and if $\mathfrak{r}$ is maximal with respect to this property. Furthermore, we have seen in Remark 2.6.3 that every $f \in \mathfrak{g}^*$ defines a character φ_f on $R := \exp(\mathfrak{r})$ by

$$\varphi_f(\exp(X)) := \epsilon(f(X)) \quad \forall X \in \mathfrak{r}. \tag{2.43}$$

Note that a homomorphism $f \in \mathfrak{g}^*$ may have different, non-isomorphic polarizing subalgebras (Example 2.6.2). Since the Lie algebra $\mathfrak{g}$ is two-step nilpotent we have $[X, Y] \in \mathfrak{z}(\mathfrak{g})$ for all $X, Y \in \mathfrak{g}$. This shows that if f and f' are two homomorphisms in $\mathfrak{g}^*$ with the property that $f|_{\mathfrak{z}(\mathfrak{g})} = f'|_{\mathfrak{z}(\mathfrak{g})}$, then a subalgebra $\mathfrak{r}$ of $\mathfrak{g}$ polarizes f if and only if $\mathfrak{r}$ polarizes f'.

Our aim in this section is to use certain known facts from the representation theory of two-step nilpotent, locally compact groups to prove the following facts.
If $f \in \mathfrak{g}^*$ and if $\mathfrak{r}$ is a polarizing subalgebra for f, then the induced character $\mathrm{ind}_R^G \varphi_f$ defines an irreducible representation of G and the kernel of the representation $\mathrm{ind}_R^G \varphi_f$ does not depend on the choice of the polarizing subalgebra $\mathfrak{r}$ for f, so that we obtain a well-defined Kirillov map

$$\kappa : \mathfrak{g}^* \longrightarrow \mathrm{Prim}(C^*(G)),\ f \mapsto \ker(\mathrm{ind}_R^G \varphi_f). \tag{2.44}$$

Many authors have studied the problem of describing the primitive ideal space of two-step nilpotent, locally compact groups (see for example [1] and [21]). We will make use of a theorem, stated and proven in [8], which summarizes all the information we need for our situation.

Theorem 2.7.1. *([8], Theorem 3.2.3) Let G be a separable locally compact, two-step nilpotent group.*

(i) There exists a map $\Delta : \widehat{Z(G)} \to \mathcal{K}(G), \phi \mapsto H_\phi$ such that each H_ϕ is a maximal closed subgroup of G with respect to the property that there exists a unitary extension of ϕ to H_ϕ.

(ii) Let $\Delta : \widehat{Z(G)} \to \mathcal{K}(G)$ be as in (i) and let

$$\mathcal{D} := \{\xi \in \widehat{H_\phi} \mid \phi \in \widehat{Z(G)},\ \xi|_{Z(G)} = \phi\}.$$

Then $\ker(\mathrm{ind}_{H_\phi}^G \xi)$ is a primitive ideal in $C^*(G)$ for any $\xi \in \mathcal{D}$ and every primitive ideal of $C^*(G)$ can be obtained in this way. Moreover, $\ker(\mathrm{ind}_{H_\phi}^G \xi) = \ker(\mathrm{ind}_{H_{\phi'}}^G \xi')$ if and only if $\xi|_{Z(G)} = \phi = \phi' = \xi'|_{Z(G)}$ and ξ and ξ' lie in the same quasi-orbit in $\widehat{H_\phi}$.

(iii) ([8], Lemma 3.1.12.) Let ϕ be a character of $Z(G)$ and define

$$\Sigma_\phi := q^{-1}(Z(G/\ker(\phi))),$$

where $q : G \to G/\ker(\phi)$ denotes the canonical quotient map. If H_ϕ and H'_ϕ are two distinct closed subgroups of G which are maximal with respect to the property that ϕ extends unitarily to some character $\xi \in \widehat{H_\phi}$ and $\xi' \in \widehat{H'_\phi}$ and if $\xi|_{\Sigma_\phi} = \xi'|_{\Sigma_\phi}$, then $\ker(\operatorname{ind}_{H_\phi}^G \xi) = \ker(\operatorname{ind}_{H'_\phi}^G \xi')$.

Remark 2.7.2. In part (i) and (ii) of Theorem 2.7.1, $\mathcal{K}(G)$ denotes the set of all closed subgroups of G equipped with the compact-open topology as explained in Section 2.5.

In the following, we define a particular map $\Delta : \widehat{Z(G)} \to \mathcal{K}(G)$ and show that this map can be used in Theorem 2.7.1. Recall that every homomorphism $f \in \mathfrak{g}^*$ defines a character φ_f on $Z(G)$ by the usual construction:

$$\varphi_f(\exp(X)) = \epsilon(f(X)) \quad \forall\, X \in \mathfrak{z}(\mathfrak{g}).$$

Choose a polarizing subalgebra $\mathfrak{r}_f$ for $f \in \mathfrak{g}^*$ and put $R_f := \exp(\mathfrak{r}_f)$ (the existence of a polarizing subalgebra is shown in Proposition 2.8.16). Define

$$\Delta : \widehat{Z(G)} \to \mathcal{K}(G),\ \varphi_f \mapsto R_f.$$

In order to use this map Δ in part (i) and part (ii) of Theorem 2.7.1, we need to prove the following facts.

(1) Every character of $Z(G)$ is of the form φ_f for some homomorphism $f \in \mathfrak{g}^*$.

(2) The closed subgroup R_f of G, $f \in \mathfrak{g}^*$, is maximal with respect to the property that the character $\varphi_f \in \widehat{Z(G)}$ extends to some character of R_f.

For the proof of statement (1), let ϕ be an arbitrary character of $Z(G)$. Since $\mathfrak{z}(\mathfrak{g}) \cong Z(G)$ via the map $\exp|_{\mathfrak{z}(\mathfrak{g})}$, it follows that the character ϕ defines a character ψ on $\mathfrak{z}(\mathfrak{g})$ by

$$\psi(X) := \phi(\exp(X)), \quad X \in \mathfrak{z}(\mathfrak{g}).$$

Choose an extension $\widetilde{\psi} \in \widehat{\mathfrak{g}}$ of $\psi \in \widehat{\mathfrak{z}(\mathfrak{g})}$. Since $\mathfrak{g}^* \cong \widehat{\mathfrak{g}}$, we can find a homomorphism $f \in \mathfrak{g}^*$ with $\epsilon \circ f = \widetilde{\psi}$. But then we obtain for all $X \in \mathfrak{z}(\mathfrak{g})$:

$$\phi(\exp(X)) = \epsilon(f(X))$$

and thus $\phi = \varphi_f$ on $Z(G)$. This proves the first part.

Now, let $f \in \mathfrak{g}^*$ and let $\mathfrak{r}_f$ be a polarizing subalgebra for f. In order to prove that the group $R_f = \exp(\mathfrak{r}_f)$ is a maximal subgroup of G with respect to the property that there exists an extension of $\varphi_f \in \widehat{Z(G)}$ to R_f, we observe the following. The symmetry group of the character $\varphi_f \in \widehat{Z(G)}$ is defined as

$$\Sigma_{\varphi_f} = q^{-1}(Z(G/\ker(\varphi_f))),$$

where $q : G \to G/\ker(\varphi_f)$ denotes the canonical quotient map and we have

$$\Sigma_{\varphi_f} = \{x \in G \mid \ \varphi_f((x,y)) = 1 \; \forall \; y \in G\}.$$

Define $S_f := \{X \in \mathfrak{g} \mid \ f([X,Y]) = 0 \; \forall \; Y \in \mathfrak{g}\}$ and put $\Sigma_f := \exp(S_f)$.

Lemma 2.7.3. *One has* $\Sigma_f = \Sigma_{\varphi_f}$.

Proof. Let $x \in \Sigma_f$. Then $f([\log(x), \log(y)]) = 0$ for all $y \in G$. But since G is two-step nilpotent, we have

$$[\log(x), \log(y)] = \log((x,y))$$

and thus we obtain for all $y \in G$:

$$1 = \epsilon(f([\log(x), \log(y)])) = \epsilon(f(\log((x,y)))) = \varphi_f((x,y)). \tag{2.45}$$

This proves that $x \in \Sigma_{\varphi_f}$.

Conversely, let $x \in \Sigma_{\varphi_f}$. We obtain by (2.45) for all $y \in G$:

$$\epsilon(f([\log(x), \log(y)])) = 1.$$

But this means that the character $\epsilon(f([\log(x), .])) \in \widehat{\mathfrak{g}}$ is equal to the trivial character. Since the map

$$\Phi : \mathrm{Hom}(\mathfrak{g}, \mathfrak{w}) \to \widehat{\mathfrak{g}}, \; g \mapsto \epsilon \circ g$$

is an isomorphism of groups, it follows that the homomorphism

$$\Phi^{-1}(\epsilon(f([\log(x), .]))) = f([\log(x), .])$$

is equal to the trivial map in $\mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$. Thus we obtain for all $y \in G$:

$$f([\log(x), \log(y)]) = 0,$$

which proves that $x \in \Sigma_f$.

□

Notice that since $S_f \subseteq \mathfrak{r}_f$, it follows that $\Sigma_f = \exp(S_f) \subseteq \exp(\mathfrak{r}_f) = R_f$.

We want to make use of the following fact.

Lemma 2.7.4. *Let G be a two-step nilpotent, locally compact group and let ϕ be a character of $Z(G)$. Let $H \supseteq Z(G)$ be a closed subgroup of G and suppose that the character $\phi \in \widehat{Z(G)}$ extends unitarily to some character of H. Then the following are equivalent:*

(i) *The group* H *is a maximal closed subgroup of* G *with respect to the property that there exists a unitary extension of* $\phi \in \widehat{Z(G)}$ *to* H*.*

(ii) *The map*

$$\Theta_\phi : G/H \to \widehat{H/\Sigma_\phi},\ \dot{x} \mapsto \phi((x, .)) \tag{2.46}$$

is an injective homomorphism and the image of Θ_ϕ *is dense in* $\widehat{H/\Sigma_\phi}$*.*

Proof. A proof of the implication $(i) \Rightarrow (ii)$ is given in [8], Lemma 3.1.2.

To prove the implication $(ii) \Rightarrow (i)$, we suppose that the map Θ_ϕ, defined in (ii), is an injective homomorphism with dense image. Assume that there exists a closed subgroup $L \supsetneq H$ of G, that there exists a unitary extension of $\phi \in \widehat{Z(G)}$ to L, and that the map

$$\Psi_\phi : G \to \widehat{L/\Sigma_\phi},\ x \mapsto \phi((x, .))$$

is a homomorphism with $\ker(\Psi_\phi) = L$ and such that the image of Ψ_ϕ is dense in $\widehat{L/\Sigma_\phi}$. Since the map

$$p : \widehat{L/\Sigma_\phi} \to \widehat{H/\Sigma_\phi},\ \chi \mapsto \chi|_{H/\Sigma_\phi}$$

is surjective, it follows that the map

$$p \circ \Psi_\phi : G \to \widehat{H/\Sigma_\phi},\ x \mapsto \phi((x, .))|_{H/\Sigma_\phi}$$

is a homomorphism with dense image and it is $\ker(p \circ \Psi_\phi) \supseteq L$. But we have $\Psi_\phi = \Theta_\phi \circ q$, where $q : G \to G/H$ denotes the canonical quotient map and since the map Θ_ϕ is injective, it follows that $\ker(p \circ \Psi_\phi) = H$. This contradicts the assumption that $H \subsetneq L$. Thus H is maximal with respect to the property that there exists a unitary extension of $\phi \in \widehat{Z(G)}$ to H. □

Observe that $Z(G) \subseteq R_f$ and that the character $\varphi_f \in \widehat{Z(G)}$ clearly extends unitarily to a character of R_f. So in order to prove the maximality property of the closed subgroup R_f of G, it suffices by Lemma 2.7.4 to prove that the map

$$\Theta_f : G/R_f \to \widehat{R_f/\Sigma_{\varphi_f}},\ \dot{x} \mapsto \varphi_f((x, .)) \tag{2.47}$$

is a well-defined, injective homomorphism with dense image.

Notice that if $y \in \Sigma_{\varphi_f}$, then $\varphi_f((x, y)) = 1$ for all $x \in G$. Moreover, if $x, y \in G$ with $xR_f = yR_f$, then $xy^{-1} \in R_f$ and we obtain for all $z \in R_f$:

$$\varphi_f((xy^{-1}, z)) \;=\; \epsilon(f(\log((xy^{-1}, z)))) = \epsilon(f([\log(xy^{-1}), \log(z)])) = 1,$$

since $\log(xy^{-1})$ and $\log(z)$ are both elements of $\log(R_f) = \mathfrak{r}_f$, the polarizing subalgebra for f. This shows that Θ_f is well-defined.

Furthermore, it is proven in [8] that if $\phi \in \widehat{Z(G)}$ and if $H \supseteq \Sigma_\phi$ is any closed subgroup of G with the property that the character ϕ extends unitarily to some character of H, then the image of the map Θ_ϕ, as defined in (2.46), is dense in $\widehat{H/\Sigma_\phi}$. In particular, the image of the map Θ_f is dense.

So it remains to prove that Θ_f is injective. For this, let $x \in G$ with $\Theta_f(x) = 1$. We need to show that $x \in R_f$. But we have

$$1 \equiv \Theta_f(x) = \varphi_f((x,.)) = \epsilon(f([\log(x), \log(.)]))$$

and thus $\epsilon(f([\log(x), \log(.)])) \in \widehat{R_f/\Sigma_{\varphi_f}}$ is equal to the trivial character. Put $X := \log(x)$, then $X \in \mathfrak{g}$ and it follows from the above that the map

$$\psi_f : \mathfrak{r}_f/S_f \to \mathbb{T},\ Y \mapsto \epsilon(f([X,Y]))$$

is equal to the trivial character of $\mathfrak{r}_f/S_f$. Since $\mathrm{Hom}(\mathfrak{r}_f/S_f, \mathfrak{w}) \cong \widehat{\mathfrak{r}_f/S_f}$ via $g \mapsto \epsilon \circ g$, it follows that the homomorphism $f([X,.]) \in \mathrm{Hom}(\mathfrak{r}_f/S_f, \mathfrak{w})$ is equal to the trivial map. But this means that $f([X,Y]) = 0$ for all $Y \in \mathfrak{r}_f$ and since $\mathfrak{r}_f$ was chosen to be a maximal subalgebra of $\mathfrak{g}$ with respect to the property that $f([Z,Y]) = 0$ for all $Z, Y \in \mathfrak{r}_f$, it follows that $X \in \mathfrak{r}_f$. Hence $x = \log(X) \in R_f$. This proves the injectivity of the map Θ_f.

Hence the group R_f is maximal with respect to the property that the character $\varphi_f \in \widehat{Z(G)}$ extends unitarily to some character of R_f and we have proven the second statement.

Therefore, we can use the map

$$\Delta : \widehat{Z(G)} \to \mathcal{K}(G),\ \varphi_f \mapsto R_f$$

in Theorem 2.7.1 and it follows as an application of part (i) and (ii) of this theorem that for every homomorphism $f \in \mathfrak{g}^*$ there exists a polarizing subalgebra $\mathfrak{r}$ for f such that $\ker(\mathrm{ind}_R^G \varphi_f)$ is a primitive ideal of $C^*(G)$, where $R := \exp(\mathfrak{r})$.

Observe that the group $\Sigma_{\varphi_f} = \Sigma_f$ is clearly contained in every closed subgroup H_f of G which is maximal with respect to the property that there exists a unitary extension of the character $\varphi_f \in \widehat{Z(G)}$ to H_f. But if $f \in \mathfrak{g}^*$ and if $\mathfrak{r}$ and $\mathfrak{r}'$ are two distinct polarizing subalgebras for f, then we have shown above that both subgroups, $R = \exp(\mathfrak{r})$ and $R' = \exp(\mathfrak{r}')$, are maximal with respect to the property that the character $\varphi_f \in \widehat{Z(G)}$ extends unitarily to a character $\varphi_f \in \widehat{R}$ and to a character $\varphi'_f \in \widehat{R}'$. Thus, the symmetry group Σ_f is contained in both subgroups, R and R', and it follows directly from the construction of the character φ_f that

$$\varphi_f|_{R \cap R'} = \varphi'_f|_{R \cap R'}.$$

We obtain then as an application of part (*iii*) of Theorem 2.7.1 that

$$\ker(\operatorname{ind}_R^G \varphi_f) = \ker(\operatorname{ind}_{R'}^G \varphi'_f).$$

Therefore, the kernel of the induced character φ_f does not depend on the particular choice of the polarizing subalgebra $\mathfrak{r}$ for f. This shows that the Kirillov map

$$\kappa : \mathfrak{g}^* \to \operatorname{Prim}(C^*(G)),\ f \mapsto \ker(\operatorname{ind}_R^G \varphi_f)$$

is a well-defined map for every nilpotent k-Lie pair $(G, \mathfrak{g})$ of nilpotence class 2.

Moreover, it follows from part (*ii*) of Theorem 2.7.1 that every primitive ideal of $C^*(G)$ can be obtained in this way, which means that the map κ is onto.

2.8 The tool box for the general case

In this section we develop some "representation theoretic tools" for nilpotent, locally compact separable groups. We will use these not only to prove that every homomorphism $f \in \mathfrak{g}^*$ of a nilpotent k-Lie pair $(G, \mathfrak{g})$ admits a polarizing subalgebra $\mathfrak{r}$, but also to prove that the induced character $\operatorname{ind}_R^G \varphi_f$ defines an irreducible representation of G.

In the case of a simply connected, connected nilpotent real Lie group G with Lie algebra $\mathfrak{g}$ these facts are proven by induction on the dimension of the Lie algebra $\mathfrak{g}$ as a vector space over $\mathbb{R}$ [23]. If this dimension is equal to one or two, the nilpotent Lie algebra is abelian and the Lie algebra $\mathfrak{g}$ itself polarizes every continuous linear functional $f \in \mathfrak{g}^*$. If the dimension is greater or equal to three, one can find with the Lemma of Kirillov an ideal $\mathfrak{g}_1$ of $\mathfrak{g}$ of codimension one and elements $X \in \mathfrak{g} \setminus \mathfrak{g}_1$ and $Y \in \mathfrak{z}(\mathfrak{g}_1)$, such that $0 \neq [X, Y] =: Z \in \mathfrak{z}(\mathfrak{g})$. The group generated by the elements $x = \exp(X), y = \exp(Y)$, and $z = \exp(Z)$ is isomorphic to the three-dimensional Heisenberg group, and one uses in the induction process the good understanding of the irreducible representations of the latter. But, unlike the case of Lie groups, there exist two-dimensional, two-step nilpotent groups which are not abelian. Take for example

$$G := \left\{ \left(\begin{smallmatrix} 1 & x & y \\ 0 & 1 & x^p \\ 0 & 0 & 1 \end{smallmatrix}\right), x, y \in K \right\},\ \text{where } K = F_p((t)) \ \text{ for some prime } p.$$

This group is neither abelian, nor does it contain a subgroup which is isomorphic to the three-dimensional Heisenberg group, but we will see in Section 2.11 that it is possible to find a natural number k and a Lie algebra $\mathfrak{g}$ over the ring Λ_k, such that the pair $(G, \mathfrak{g})$ defines a nilpotent k-Lie pair.

Following an idea of Howe [16], we prove most of the results in the following sections by induction on the nilpotence class of the given group G.

If $(G, \mathfrak{g})$ is a nilpotent k-Lie pair of nilpotence class $l \geq 2$ then we define A to be a maximal abelian subgroup of $Z^2(G)$ and N to be the centralizer of A. We will see that such a group N is a normal subgroup of G of nilpotence class at most $(l-1)$ and both subgroups, A and N, are exponentiable subgroups of G, so that both pairs, $(A, \log(A))$ and $(N, \log(N))$, define nilpotent k-Lie pairs. The group A is contained in the center of N and for every element $y \in A \setminus Z(G)$ and $x \in G \setminus N$ one has $1 \neq (y,x) \in Z(G)$. This indicates that such normal subgroups A and N of G are suitable substitutes for the objects found by the Lemma of Kirillov in the case of Lie groups.

Remark 2.8.1. Let G be a l-step nilpotent group with $l \geq 2$, i.e., G is not commutative. Then the center of G is a proper subgroup of every maximal abelian subgroup A of $Z^2(G)$, because we can always find an element $y \in Z^2(G) \setminus Z(G)$ such that the subgroup generated by y and $Z(G)$ defines an abelian subgroup of $Z^2(G)$. Observe that

$$Z^2(G) = \{x \in G \mid (x,y) \in Z(G) \ \forall\, y \in G\}.$$

We claim that every maximal abelian subgroup A of $Z^2(G)$ is normal in G. Indeed, we have for all $x \in G$ and $y \in A$,

$$xyx^{-1} = xyx^{-1}y^{-1}y = (x,y)y \in A,$$

since the commutator $(x,y) \in Z(G) \subseteq A$ for all $x \in G$ and for all $y \in A$.

Remark 2.8.2. Let G be a l-step nilpotent group with $l \geq 2$ and let A be an abelian subgroup of $Z^2(G)$. Then we have for all $x \in G$ and $y \in A$:

$$(x,y) = (x^{-1}, y^{-1}) = (y, x^{-1}).$$

Indeed, let $x \in G$ and let $y \in A$. Since $(x,y) \in Z(G)$, it remains the same under conjugating it with any element $z \in G$ and thus we obtain

$$(x,y) = y^{-1}x^{-1}(x,y)xy = (y^{-1}x^{-1}xy)x^{-1}y^{-1}xy = (x^{-1}, y^{-1}).$$

By the same argument it follows that

$$(x,y) = x^{-1}(x,y)x = yx^{-1}y^{-1}x = (y, x^{-1}).$$

Lemma 2.8.3. *Let G be a l-step nilpotent locally compact separable group for $l > 1$. Let A be a maximal abelian subgroup of $Z^2(G)$, the second element of the ascending central series of G. The centralizer N of A is a closed normal subgroup of G, the nilpotence class of N is smaller or equal to $(l-1)$, and the quotient group G/N is abelian.*

Proof. We show first that N is a normal subgroup of G. Notice that

$$N := \{x \in G \mid xy = yx \ \forall y \in A\}$$

and thus N is closed. Let $x \in G$ and $v \in N$. It suffices to show that the conjugated element xvx^{-1} commutes with every element $y \in A$. But we have $(y, x) \in Z(G)$ for all $y \in A$, and since N commutes elementwise with A we obtain for all $y \in A$:

$$\begin{aligned} xvx^{-1}y &= xv(x^{-1}yxy^{-1})yx^{-1} = xv(x^{-1}, y)yx^{-1} = x(x^{-1}, y)vyx^{-1} = yxy^{-1}vyx^{-1} \\ &= yxvx^{-1}. \end{aligned}$$

In order to show that the subgroup N is of nilpotence class smaller or equal to $(l-1)$, we prove by induction on $m = 2, \ldots, l$:

$$N \cap Z^m(G) \subseteq Z^{m-1}(N). \tag{2.48}$$

Note that this suffices, since for $m = l$ we obtain $N \subseteq Z^{l-1}(N)$ which means that the ascending central series of N terminates (at most) within $(l-1)$ steps. We claim that $N \cap Z^2(G) = A$. This proves Equation (2.48) in the case $m = 2$ since A is obviously contained in $Z(N)$. For this, notice that the subgroup A is contained in both, N and $Z^2(G)$. If $x \notin A$, but $x \in Z^2(G)$ then $x \notin N$, since by the maximality property of A there exists at least one element $y \in A$ with $xy \neq yx$. Using the induction hypothesis and the fact that N is a normal subgroup of G, we obtain

$$\begin{aligned} N \cap Z^m(G) &= \{v \in N \mid (x, v) \in Z^{m-1}(G)\ \forall x \in G\} \\ &= \{v \in N \mid (x, v) \in Z^{m-1}(G) \cap N\ \forall x \in G\} \\ &\subseteq \{v \in N \mid (x, v) \in Z^{m-2}(N)\ \forall x \in G\} \\ &\subseteq Z^{m-1}(N). \end{aligned}$$

One possible way to see that the quotient group G/N is abelian is the following. Define a map

$$\Phi : G/N \to \mathrm{Hom}(A/Z(G), Z(G)),\ \dot{x} \mapsto \phi_{\dot{x}},$$

where $\phi_{\dot{x}}(\dot{y}) = (x, y)$ is defined to be the group commutator of x and y. Since both groups, $A/Z(G)$ and $Z(G)$, are abelian it follows that the group $\mathrm{Hom}(A/Z(G), Z(G))$ is abelian. Thus it suffices to show that Φ is an injective homomorphism. Note first, that if $x_1, x_2 \in G$ with $\dot{x_1} = \dot{x_2}$, then $x_1x_2^{-1} \in N$ and we obtain for all $y \in A$:

$$1 = (x_1x_2^{-1}, y) = x_1x_2^{-1}yx_2x_1^{-1}y^{-1} = x_1(x_2^{-1}, y)yx_1^{-1}y^{-1} = (x_2^{-1}, y)(x_1, y).$$

Therefore, we have $(y, x_2^{-1}) = (x_1, y)$ for all $y \in A$, and since $(y, x_2^{-1}) = (x_2, y)$ (Remark 2.8.2), it follows that $(x_2, y) = (x_1, y)$. In the same way one can show that if $y_1, y_2 \in A$ with $y_1y_2^{-1} \in Z(G)$, then $(x, y_1) = (x, y_2)$ for all $x \in G$. This proves that Φ is well-defined.

Since $(x_1x_2, y) = (x_1, y)(x_2, y)$ for all $x_1, x_2 \in G$ and $y \in A$ it follows that the map Φ is a homomorphism.

So it remains to show that Φ is one-to-one. For this suppose $\phi_{\dot{x}} \equiv 1$. Then we have $(x, y) = 1$ for all $y \in A$ and hence $x \in N$. $\square$

It turns out that, not only every maximal abelian subgroup A of $Z^2(G)$, but also its centralizer N, is an exponentiable subgroup of G.

Lemma 2.8.4. *Let $k \geq 2$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair of nilpotence class $l \geq 2$. Let A be a maximal abelian subgroup of $Z^2(G)$. Then $\log(A)$ is a subalgebra of $\mathfrak{g}$.*

Proof. By Theorem 2.3.2 it suffices to prove that the subgroup A is k-complete.

Let $y \in A$ be fixed. We need to show that $y^\lambda \in A$ for every $\lambda \in \mathbb{Z}[\frac{1}{k!}]$. Since $A \subseteq Z^2(G)$ it follows that the Inversion Formula (2.33) of the Campbell-Hausdorff formula reduces, for all $x \in G$, to the following equation

$$[\log(x), \log(y)] = \log((x, y)). \tag{2.49}$$

Using the fact that $\log(z^\lambda) = \lambda \log(z)$ for all $\lambda \in \mathbb{Z}[\frac{1}{k!}]$ and all $z \in G$ (Lemma 2.2.15), we obtain for all $x \in A$ and $\lambda \in \mathbb{Z}[\frac{1}{k!}]$:

$$\log((x, y^\lambda)) = [\log(x), \log(y^\lambda)] = \lambda[\log(x), \log(y)] = 0,$$

and hence $(x, y^\lambda) = 1$. But A was chosen to be a maximal abelian subgroup of $Z^2(G)$ and since $Z^2(G)$ is k-complete (Lemma 2.4.2), it follows that $y^\lambda \in A$. □

Lemma 2.8.5. *Let $k \geq 2$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair of nilpotence class $l \geq 2$. Let A be a maximal abelian subgroup of $Z^2(G)$ and let N be the centralizer of A. Then $\log(N)$ is a subalgebra of $\mathfrak{g}$.*

Proof. We use the same idea as in the proof of Lemma 2.8.4. By Theorem 2.3.2 it suffices to prove that the subgroup N is k-complete.

Let $x \in N$ and let $\lambda \in \mathbb{Z}[\frac{1}{k!}]$. Since N is defined as the centralizer of A, it suffices to show that $(x^\lambda, y) = 1$ for all $y \in A$. Using the fact that $\log(x^\lambda) = \lambda \log(x)$ (Lemma 2.2.15) we obtain by the Inversion Formula (2.33) for all $y \in A$:

$$\log((x^\lambda, y)) = [\log(x^\lambda), \log(y)] = \lambda[\log(x), \log(y)] = 0.$$

Hence, $(x^\lambda, y) = 1$ for all $y \in A$, which proves that $x^\lambda \in N$. □

Remark 2.8.6. Let $k \in \mathbb{N}_{\geq 2}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair of nilpotence class $l \geq 2$. Let A be a maximal abelian subgroup of $Z^2(G)$ and let N be the centralizer of A. We have shown in Lemma 2.8.4 and Lemma 2.8.5 that $\log(A) := \mathfrak{a}$ and $\log(N) := \mathfrak{n}$ are subalgebras of $\mathfrak{g}$. Furthermore, we have proven in Lemma 2.8.3 that the subgroup N is at most $(l-1)$-step nilpotent, so that the pair $(N, \mathfrak{n})$ is a nilpotent k-Lie pair of nilpotence class at most $l - 1$. Since both subgroups, A and N, are normal subgroups of G, it follows from Corollary 2.3.11 that $\mathfrak{a}$ and $\mathfrak{n}$ are ideals of $\mathfrak{g}$. Moreover, we have for all $X \in \mathfrak{n}$ and $Y \in \mathfrak{a}$:

$$[X, Y] = \log((\exp(X), \exp(Y)) = \log(1) = 0.$$

We show now that if G is a nilpotent locally compact separable group of nilpotence class $l \geq 2$ and if $I \in \mathrm{Prim}(C^*(G))$ with $I = \ker(\pi)$ for some irreducible unitary representation π of G, which is faithful on the center of G and not one-dimensional, then I is induced from some primitive ideal J of the normal subgroup N of G, where N is defined as in Lemma 2.8.3. Proposition 2.8.8 and Proposition 2.8.13 are due to Howe; we present a detailed proof of the arguments given in [16], Proposition 5. For this, we will use the following facts, which can be found for example in [27], §3.3.

Remark 2.8.7. If A is a C^*-algebra, we denote by $\mathcal{I}(A)$ the space of (closed two-sided) ideals in A, where a subbasis for the topology is given by the sets

$$U(I) = \{J \in \mathcal{I}(A) \mid J \cap I \neq \emptyset\}.$$

If G is a locally compact group and H a closed subgroup of G then there is a continuous map

$$\mathrm{res}_H^G : \mathcal{I}(C^*(G)) \to \mathcal{I}(C^*(H)),$$

such that for all non-degenerate representations π of $C^*(G)$:

$$\mathrm{res}_H^G(\ker \pi) = \ker(\mathrm{res}_H^G \pi).$$

Furthermore, there is a containment preserving continuous map

$$\mathrm{ind}_H^G : \mathcal{I}(C^*(H)) \to \mathcal{I}(C^*(G)),$$

such that for all non-degenerate representations π of $C^*(H)$:

$$\mathrm{ind}_H^G \ker(\pi) = \ker(\mathrm{ind}_H^G \pi).$$

Proposition 2.8.8. *Let G be nilpotent locally compact separable group of nilpotence class $l \geq 2$. Let $I \in \mathrm{Prim}(C^*(G))$ and suppose $I = \ker(\pi)$ for some irreducible unitary representation π of G which is faithful on $Z(G)$ and not one-dimensional. Then there exists a closed normal subgroup N of G of nilpotence class at most $l-1$ such that $I = \ker(\mathrm{ind}_N^G \pi|_N)$.*

Proof. It follows from Schur's Lemma that there exists a character $\psi \in \widehat{Z(G)}$ such that $\pi(z) = \psi(z) \cdot Id_{H_\pi}$ for all $z \in Z(G)$. In the following, we will identify the restricted representation $\pi|_{Z(G)}$ with this character ψ. Since π was assumed to be faithful on $Z(G)$, the map ψ is a faithful character of $Z(G)$. Let N be the centralizer of a maximal abelian subgroup A of $Z^2(G)$. We have seen in Lemma 2.8.3 that N is a closed normal subgroup of G of nilpotence class at most $l-1$ and the quotient group G/N is abelian. By Theorem 2.5.11 we obtain

$$\mathrm{ind}_N^G \pi|_N \cong \mathrm{ind}_N^G(\pi|_N \otimes 1_N) \cong \pi \otimes \mathrm{ind}_N^G 1_N \cong \pi \otimes \lambda_{G/N},$$

where $\lambda_{G/N}$ denotes the left regular representation of G/N on the Hilbert space $L^2(G/N)$, and thus

$$\ker(\operatorname{ind}_N^G \pi|_N) = \ker(\pi \otimes \lambda_{G/N}).$$

But since the quotient group G/N is abelian, we have

$$\ker(\lambda_{G/N}) = \ker(\bigoplus_{\chi \in \widehat{G/N}} \chi).$$

The dual group $\widehat{G/N}$ acts on $C^*(G)$ by the dual action which yields, by passing to kernels, an action of $\widehat{G/N}$ on the primitive ideal space of G. We will show in the following paragraph that the primitive ideal $I = \ker(\pi)$ is invariant under the action of all characters $\chi \in \widehat{G/N}$, i.e., we will show

$$\chi \cdot I = \ker(\chi \otimes \pi) = \ker(\pi) = I \quad \forall \chi \in \widehat{G/N}. \tag{2.50}$$

This suffices, since the process of taking tensor products is "continuous" (Proposition 2.5.14), so we have $\ker(\bigoplus_{\chi \in \widehat{G/N}} \chi \otimes \pi) = \ker(\lambda_{G/N} \otimes \pi)$ and hence

$$\ker(\operatorname{ind}_N^G \pi|_N) = \ker(\pi \otimes \lambda_{G/N}) = \ker(\bigoplus_{\chi \in \widehat{G/N}} \chi \otimes \pi) = \bigcap_{\chi \in \widehat{G/N}} \chi \cdot I = I.$$

In order to prove that $\ker(\chi \otimes \pi) = \ker(\pi)$ for all $\chi \in \widehat{G/N}$, we make the following observation. The normal abelian subgroup A of G acts on $\hat{G}$ by conjugation and we have

$$\operatorname{Ad}(y^{-1})\pi(x) = \pi(yxy^{-1}) = \pi(yxy^{-1}x^{-1}x) = \psi((y,x)) \cdot \pi(x),$$

for all $y \in A$ and $x \in G$. Thus, $\operatorname{Ad}(y^{-1})\pi(x) = (\psi^y \otimes \pi)(x)$, where the character ψ^y is defined by $\psi^y(x) := \psi((y,x))$. (We have shown in the proof of Lemma 2.8.3 that the map ψ^y, $y \in A$ is indeed a character.) Therefore, the representation $\operatorname{Ad}(y^{-1})\pi$ is unitarily equivalent to the product $\psi^y \otimes \pi$ for all $y \in A$, and since the conjugated representation $\operatorname{Ad}(y^{-1})\pi$, $y \in A$, is clearly unitarily equivalent to π itself it follows that

$$\ker(\psi^y \otimes \pi) = \ker(\pi) = I \quad \forall y \in A.$$

We will prove now that every character χ of G/N is a limit of some sequence of characters of the form ψ^{y_n} for some $y_n \in A$, $n \in \mathbb{N}$. For this, we show that the map

$$\Phi : A/Z(G) \to \widehat{G/N}, y \mapsto \psi^y$$

is a continuous, injective homomorphism with dense image. Note, that it follows directly from the definition of ψ^y that Φ is well-defined and continuous. Since the commutator $(y,x) \in Z(G)$ for all $y \in A$ and $x \in G$ we get $(y_1y_2, x) = (y_1, x)(y_2, x)$

for all $y_1, y_2 \in A$ and $x \in G$, which shows that Φ is a homomorphism. Furthermore, if $\psi^y(x) = \psi((y,x)) = 1$ for all $x \in G$, then $(y,x) = 1$ for all $x \in G$ (since ψ is faithful on the center of G) and hence $y \in Z(G)$. This proves the injectivity of Φ. Using well-known results from the representation theory of locally compact abelian groups ([13], § 4.3) we obtain

$$\begin{aligned}(\operatorname{im}\Phi)^{\perp} &= \{x \in G \mid \psi((y,x)) = 1 \;\forall\, y \in A\} \\ &= \{x \in G \mid (y,x) = 1 \;\forall\, y \in A\} \\ &= N.\end{aligned}$$

This shows that $(\operatorname{im}\Phi)^{\perp}$ is trivial in G/N and thus we obtain $\overline{\operatorname{im}\Phi} = \widehat{G/N}$. Since the image of Φ is dense in $\widehat{G/N}$, we can find for every character $\chi \in \widehat{G/N}$, a sequence $(y_n)_{n\in\mathbb{N}} \in A^{\mathbb{N}}$, such that $\psi^{y_n} \to \chi$ as $n \to \infty$. Since $\ker(\psi^y \otimes \pi) = \ker(\pi)$ for all $y \in A$ we obtain by continuity the desired result: $\ker(\chi \otimes \pi) = \ker(\pi)$ for all $\chi \in \widehat{G/N}$. $\square$

We show now that Proposition 2.8.8 remains true under slightly different assumptions.

Corollary 2.8.9. *Let $k \geq 2$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair of nilpotence class $l \geq 2$. Let $I \in \operatorname{Prim}(C^*(G))$ and suppose that $I = \ker(\pi)$ for some irreducible unitary representation π of G which is not one-dimensional. Let $f \in \mathfrak{g}^*$ with*

$$\pi|_{Z(G)} = \varphi_f Id_{H_\pi},$$

where $\varphi_f \in \widehat{Z(G)}$ denotes the character corresponding to f. If f is faithful on $\mathfrak{z}(\mathfrak{g})$ then there exists a closed normal subgroup N of G of nilpotence class at most $l-1$ such that $I = \ker(\operatorname{ind}_N^G \pi|_N)$.

Proof. Suppose that the homomorphism $f \in \mathfrak{g}^*$ is faithful on the center of $\mathfrak{g}$ and note that

$$\pi|_{Z(G)} = \varphi_f = \epsilon \circ f \circ \log .$$

Let A be a maximal abelian subgroup of $Z^2(G)$ and let N be the centralizer of A. By the arguments given in the proof of Proposition 2.8.8 it suffices to show that every character χ of the abelian quotient group G/N is a limit of characters of the form $\varphi_f^{y_n} \in \widehat{G/N}$, where $y_n \in A$ and $\varphi_f^{y_n}(\dot{x}) := \varphi_f((y_n, x))$ for all $x \in G$ and for all $n \in \mathbb{N}$.

For this, define $\mathfrak{a} := \log(A)$ and $\mathfrak{n} := \log(N)$. Both pairs, $(A, \mathfrak{a})$ and $(N, \mathfrak{n})$, are nilpotent k-Lie pairs (Remark 2.8.6), and it follows from Lemma 2.4.1 that the pairs, $(A/Z(G), \mathfrak{a}/\mathfrak{z}(\mathfrak{g}))$ and $(G/N, \mathfrak{g}/\mathfrak{n})$, are also nilpotent k-Lie pairs. But both quotient groups, $A/Z(G)$ and G/N, are abelian and thus we have $A/Z(G) \cong \mathfrak{a}/\mathfrak{z}(\mathfrak{g})$ and $G/N \cong \mathfrak{g}/\mathfrak{n}$. Furthermore, we have $\widehat{G/N} \cong \widehat{\mathfrak{g}/\mathfrak{n}}$ and since $y_n \in A$ for every $n \in \mathbb{N}$ and $x \in G/N$ we obtain

$$\epsilon(f([\log(y_n), \log(x)])) = \epsilon(f(\log((y_n, x)))) = \varphi_f((y_n, x)).$$

Thus it suffices to show that the map

$$\Phi_f : \mathfrak{a}/\mathfrak{z}(\mathfrak{g}) \to \widehat{\mathfrak{g}/\mathfrak{n}},\ \dot{Y} \mapsto \epsilon \circ f^Y$$

is an injective homomorphism with dense image, where $f^Y(\dot{X}) := f([Y,X])$. Clearly, the map ϕ_f is well-defined, continuous homomorphism.

In order to prove the injectivity of Φ_f, let $Y \in \mathfrak{a}$ and suppose that

$$1 = \epsilon(f^Y(X)) = \epsilon(f([Y,X])) \quad \text{for all } X \in \mathfrak{g}\,.$$

This means that the set $\{f([Y,X]) \mid X \in \mathfrak{g}\}$ is a Λ_k-submodule of $\mathfrak{w}$ inside the kernel of ϵ and since this set must be trivial, we obtain $f([Y,X]) = 0$ for all $X \in \mathfrak{g}$. But we have $[Y,X] \in \mathfrak{z}(\mathfrak{g})$ for all $X \in \mathfrak{g}$ and since f was chosen to be faithful on the center of $\mathfrak{g}$, it follows that $[Y,X] = 0$ for all $X \in \mathfrak{g}$. Therefore, Y must be an element of the center of $\mathfrak{g}$, proving the injectivity of Φ_f.

Finally, we obtain by the same arguments as above

$$\begin{aligned}
(\operatorname{im}\Phi)^\perp &= \{X \in \mathfrak{g} \mid \epsilon(f^Y(X)) = 1\ \forall\, Y \in \mathfrak{a}\} \\
&= \{X \in \mathfrak{g} \mid f([Y,X]) = 0\ \forall\, Y \in \mathfrak{a}\} \\
&= \{X \in \mathfrak{g} \mid [Y,X] = 0\ \forall\, Y \in \mathfrak{a}\} \\
&= \mathfrak{n}.
\end{aligned}$$

This proves that $(\operatorname{im}\Phi_f)^\perp$ is trivial in $\mathfrak{g}/\mathfrak{n}$ and thus $\overline{\operatorname{im}\Phi_f} = \widehat{\mathfrak{g}/\mathfrak{n}}$. □

Remark 2.8.10. Let $k \geq 2$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair of nilpotence class $l \geq 2$. Let $\pi \in \widehat{G}$ and suppose that $f \in \mathfrak{g}^*$ with

$$\pi|_{Z(G)} = \varphi_f Id_{H_\pi},$$

where $\varphi_f \in \widehat{Z(G)}$ denotes the character associated to f. We have shown in the proof of Corollary 2.8.9 that if f is faithful on $\mathfrak{z}(\mathfrak{g})$ then the map

$$\Phi : A/Z(G) \to \widehat{G/N},\ \dot{y} \mapsto \varphi_f^y, \text{ where } \varphi_f^y(\dot{x}) := \varphi_f((x,y)),$$

as well as the map

$$\Phi_f : \mathfrak{a}/\mathfrak{z}(\mathfrak{g}) \to \widehat{\mathfrak{g}/\mathfrak{n}},\ \dot{Y} \mapsto \epsilon \circ f^Y, \text{ where } f^Y(\dot{X}) := f([Y,X]),$$

is an injective homomorphism with dense image.

Given a homomorphism $f \in \mathfrak{g}^*$, we distinguish in some proofs of this chapter between the case that f is faithful on the center of $\mathfrak{g}$ and the case that f is not faithful on the center of $\mathfrak{g}$. The following two lemmas turn out to be a useful tool to deal with the latter case. If $(G, \mathfrak{g})$ is a nilpotent k-Lie pair for some $k \in \mathbb{N}$, recall that a closed subgroup H of G is called exponentiable if $\log(H)$ is a subalgebra of $\mathfrak{g}$.

Lemma 2.8.11. *Let $k \in \mathbb{N}$, let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair, and let $\pi \in \widehat{G}$.*

(i) There exists a maximal exponentiable normal subgroup J of G such that $J \subseteq \ker(\pi)$.

(ii) Let J be as in (i), let $\widetilde{\pi}$ be the irreducible representation of G/J induced by π, and let $\mathfrak{j} := \log(J)$. If $\tilde{f}$ denotes a homomorphism in $(\mathfrak{g}/\mathfrak{j})^*$ with the property that

$$\widetilde{\pi}|_{Z(G/J)} = \varphi_{\tilde{f}},$$

then $\tilde{f}$ is faithful on $\mathfrak{z}(\mathfrak{g}/\mathfrak{j})$.

Proof. To prove part (i) we define

$$\mathcal{M} := \{I \mid I \text{ is a normal exponentiable subgroup of } G \text{ and } I \subseteq \ker(\pi)\}.$$

The set $\mathcal{M}$ is partially ordered by inclusion and since $\{1_G\} \in \mathcal{M}$, it follows that $\mathcal{M}$ is nonempty. Let $\mathcal{K}$ be a chain in $\mathcal{M}$. We claim that the set

$$J := \overline{\bigcup_{I \in \mathcal{K}} I}$$

is an upper bound for $\mathcal{K}$. Clearly, $I \subseteq J$ for all $I \in \mathcal{K}$ and J is a normal subgroup of G. Furthermore, $I \subseteq \ker(\pi)$ for every $I \in \mathcal{K}$ and since the kernel of π is closed it follows that $J \subseteq \ker(\pi)$. So it suffices to show that J defines an exponentiable subgroup of G. But we have

$$\log(\bigcup_{I \in \mathcal{K}} I) = \bigcup_{I \in \mathcal{K}} \log(I)$$

and since the map $\log : G \to \mathfrak{g}$ is a homeomorphism it follows that

$$\log(\overline{\bigcup_{I \in \mathcal{K}} I}) = \overline{\bigcup_{I \in \mathcal{K}} \log(I)}.$$

Since $\log(I)$ is an ideal of $\mathfrak{g}$ for every $I \in \mathcal{K}$, it follows that $\bigcup_{I \in \mathcal{K}} \log(I)$ is a closed ideal of $\mathfrak{g}$ and hence $\overline{\bigcup_{I \in \mathcal{K}} \log(I)}$ is an ideal of $\mathfrak{g}$. This proves that J is exponentiable.

In order to prove part (ii), let $\tilde{f} \in (\mathfrak{g}/\mathfrak{j})^*$ such that

$$\widetilde{\pi}|_{Z(G/J)} = \varphi_{\tilde{f}}.$$

Assume that $\tilde{f}$ is not faithful on $\mathfrak{z}(\mathfrak{g}/\mathfrak{j})$. Then there exists an element $\dot{Z} \in \mathfrak{z}(\mathfrak{g}/\mathfrak{j})$ with $\dot{Z} \neq 0$, but $\tilde{f}(\dot{Z}) = 0$. Let $Z \in \mathfrak{g}$ with $q'(Z) = \dot{Z}$, where $q' : \mathfrak{g} \to \mathfrak{g}/\mathfrak{j}$ denotes the canonical quotient map. Then $Z \notin \mathfrak{j}$, but since $\dot{Z} \in \mathfrak{z}(\mathfrak{g}/\mathfrak{j})$, it follows that $[X, Z] \in \mathfrak{j}$ for all $X \in \mathfrak{g}$. Thus $\mathfrak{h} := \Lambda_k \cdot Z + \mathfrak{j}$ defines an ideal of $\mathfrak{g}$, which is larger than $\mathfrak{j}$. But we have for all $\lambda \in \Lambda_k$:

$$1 = \epsilon(\tilde{f}(\lambda\dot{Z})) = \widetilde{\pi}(\widetilde{\exp}(\lambda\dot{Z})) = \widetilde{\pi}(q(\exp(\lambda Z)))) = \pi(\exp(\lambda Z))$$

and thus $H := \exp(\overline{\mathfrak{h}}) \supsetneq J$ is a normal exponentiable subgroup of G inside the kernel of π, contradicting the choice of the subgroup J. $\square$

Lemma 2.8.12. *Let $k \in \mathbb{N}$, let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair, and let $f \in \mathfrak{g}^*$. There exists a largest ideal $\mathfrak{j}$ inside the kernel of f such that the corresponding homomorphism $\tilde{f}$ of the quotient algebra $\mathfrak{g}/\mathfrak{j}$ is faithful on the center of $\mathfrak{g}/\mathfrak{j}$. Furthermore, $\mathfrak{r}$ is a polarizing subalgebra for f if and only if $\tilde{\mathfrak{r}} := q(\mathfrak{r})$ is a polarizing subalgebra for $\tilde{f}$, where $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{j}$ denotes the canonical quotient map.*

Proof. Since the kernel of the character f does not have to be an ideal of the algebra $\mathfrak{g}$, we can not just pass to the quotient $\mathfrak{g}/\ker(f)$. Instead, let $\mathfrak{j}$ be the largest ideal of $\mathfrak{g}$ inside $\ker(f)$. Zorn's Lemma assures the existence of a maximal ideal while uniqueness is guaranteed by the fact that if $\mathfrak{j}_1, \mathfrak{j}_2$ are two such, then their sum would also be one, contradicting maximality.

Put $\tilde{\mathfrak{g}} := \mathfrak{g}/\mathfrak{j}$ and denote by $\tilde{f}$ the homomorphism of the quotient algebra, defined by $\tilde{f}(q(X)) := f(X)$, where $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{j}$ denotes the canonical quotient map. Assume that $\tilde{f}$ is not faithful on the center of $\tilde{\mathfrak{g}}$. Then there exists an element $\dot{Z} \in \mathfrak{z}(\tilde{\mathfrak{g}})$ with $\dot{Z} \neq 0$, but $\tilde{f}(\dot{Z}) = 0$. In particular, we have $f(Z) = 0$ and thus $Z \in \ker(f)$, but $Z \notin \mathfrak{j}$. Since f is Λ_k-linear (Remark 2.2.10), the kernel of f is a Λ_k-submodule of $\mathfrak{g}$. Furthermore, it follows from the fact that $\dot{Z} \in \mathfrak{z}(\tilde{\mathfrak{g}})$, that $[\dot{Z}, \dot{Y}] = 0$ for all $\dot{Y} \in \tilde{\mathfrak{g}}$ and hence $[Z, Y] \in \mathfrak{j}$ for all $Y \in \mathfrak{g}$. Therefore, $\mathfrak{h} := \mathfrak{j} + \Lambda_k \cdot Z$, the ideal generated by $\mathfrak{j}$ and Z, defines an ideal inside the kernel of f and $\mathfrak{h}$ is larger than $\mathfrak{j}$. This contradicts the maximality property of the ideal $\mathfrak{j}$.

Using the definition of the commutator in the quotient algebra as in Lemma 2.4.1 and the fact that every polarizing subalgebra $\mathfrak{r}$ for f contains the ideal $\mathfrak{j}$, we obtain for all $X, Y \in \mathfrak{g}$:

$$\tilde{f}([q(X), q(Y)]) = \tilde{f}(q([X, Y])) = f([X, Y]).$$

□

Proposition 2.8.13. *([16], Proposition 5) Let G be a nilpotent locally compact separable group and let $I \in \mathrm{Prim}(C^*(G))$. There exists a closed subgroup H of G and a character χ of H, such that the representation $\mathrm{ind}_H^G \chi$ is irreducible and $\ker(\mathrm{ind}_H^G \chi) = I$.*

Proof. We will prove this proposition by induction on the nilpotence class l of G. Let $\pi \in \hat{G}$ with $\ker(\pi) = I$. Notice that if the irreducible representation π is one-dimensional, so in particular if $l = 1$, then we can choose $H = G$ and there is nothing to prove.

Let $l \geq 2$ and suppose that π is not one-dimensional. Assume that the proposition is proven for all nilpotent locally compact groups of nilpotence class less than l. We will prove first that we can assume without loss of generality that the representation π is faithful on G. For this observe that $\ker(\pi)$ is a normal subgroup of G and define $q : G \to G/\ker(\pi)$ to be the canonical quotient group. If we denote by $\tilde{\pi}$ the corresponding representation of $\tilde{G} = G/\ker(\pi)$, so $\tilde{\pi} \circ q = \pi$, then $\tilde{\pi}$ is clearly faithful on $\tilde{G}$. Using $L = \ker(\pi)$ in Theorem 2.5.12 yields for every closed subgroup $L \subseteq H \subseteq G$:

$$\operatorname{ind}_H^G(\tilde{\pi} \circ q) \cong (\operatorname{ind}_{H/L}^{G/L} \tilde{\pi}) \circ q.$$

Thus, proven the proposition for the quotient group $\tilde{G}$ with corresponding faithful representation $\tilde{\pi}$, we may find a closed subgroup $\tilde{H}$ of $\tilde{G}$ and a character $\tilde{\chi}$ of $\tilde{H}$ such that

$$\ker(\tilde{\pi}) = \ker(\operatorname{ind}_{\tilde{H}}^{\tilde{G}} \tilde{\chi}).$$

With this result we obtain for the closed subgroup $H = q^{-1}(\tilde{H})$ and the character $\chi \in \hat{H}$, defined by $\chi = \tilde{\chi} \circ q$, the desired equation:

$$\ker(\pi) = \ker(\tilde{\pi} \circ q) = \ker((\operatorname{ind}_{\tilde{H}}^{\tilde{G}} \tilde{\chi}) \circ q) = \ker(\operatorname{ind}_H^G(\tilde{\chi} \circ q)) = \ker(\operatorname{ind}_H^G \chi).$$

So we can assume in the following that the representation π is faithful on G.

Let A be a maximal abelian subgroup of $Z^2(G)$ and let N be the centralizer of A. We have seen in Lemma 2.8.3 that N is a closed normal subgroup of G of nilpotence class at most $l-1$. In order to apply the induction hypothesis to N, we need to find first a suitable ideal $J \in \operatorname{Prim}(C^*(N))$. But by Theorem 2.5.13 there exists a primitive ideal $J \in \operatorname{Prim}(C^*(N))$ such that $\ker(\pi|_N) = \bigcap_{g\in G} gJg^{-1}$. Since $\operatorname{ind}_N^G J = \operatorname{ind}_N^G gJg^{-1}$ for all $g \in G$, it follows from Proposition 2.8.8 that

$$I = \operatorname{ind}_N^G I|_N = \operatorname{ind}_N^G J.$$

If we apply now the induction hypothesis to the nilpotent group N and the primitive ideal $J \in \operatorname{Prim}(C^*(N))$, we can find a closed subgroup H of N and a character $\chi \in \hat{H}$ such that $\rho := \operatorname{ind}_H^N \chi$ defines an irreducible representation of N with $\ker(\rho) = J$. Put $\tilde{\pi} := \operatorname{ind}_N^G \rho$. Then $\tilde{\pi} \cong \operatorname{ind}_H^G \chi$ and we have

$$\ker(\tilde{\pi}) = \ker(\operatorname{ind}_N^G \rho) = \operatorname{ind}_N^G(\ker \rho) = \operatorname{ind}_N^G J = I.$$

Thus it only remains to show that the representation $\tilde{\pi}$ is in fact irreducible. For this, let $T \in \mathcal{L}(\mathcal{H}_{\tilde{\pi}})$ be an intertwining operator for $\tilde{\pi}$. We need to show that T is a multiple of the identity $Id_{\mathcal{H}_{\tilde{\pi}}}$. But $\tilde{\pi}$ is an induced representation, so there exists a system of imprimitivity $\Sigma = (\tilde{\pi}, P)$, where

$$P : C_0(G/N) \to \mathcal{L}(\mathcal{H}_{\tilde{\pi}}),\ P(\varphi)(\xi) = \varphi \cdot \xi$$

defines a nondegenerate $*$-representation on $C_0(G/N)$. Recall that the set of intertwining operators of the imprimitivity system Σ is isometrically isomorphic to the set of intertwining operators of the representation ρ ([13], Theorem 6.28). Since ρ is irreducible, it suffices to show that the operator T commutes with $P(\varphi)$ for all $\varphi \in C_0(G/N)$. But the quotient group G/N is an abelian locally compact group and thus we have $C_0(G/N) \cong C^*(\widehat{G/N})$.

Therefore, it is enough to prove the equality $TP(\chi) = P(\chi)T$ for all $\chi \in \widehat{G/N}$. Denote by $\psi \in \widehat{Z(G)}$ the character, given by $\psi \mathrel{\widehat{=}} \pi|_{Z(G)}$. We have seen in the proof of Proposition 2.8.8 that the map

$$\Phi : A/Z(G) \to \widehat{G/N}, y \mapsto \psi^y,$$

is a continuous, injective homomorphism with dense image, where $\psi^y(x) := \psi((y,x))$, $x \in G$, and thus it suffices to prove that T commutes with $P(\psi^y)$ for all $y \in A$.

We will see in the following that the imprimitivity map P and the representation $\tilde{\pi}$ are closely related. For every $y \in A$, the operator $\tilde{\pi}(y)$ is a multiplication operator which coincides, up to multiplication with a scalar, with the imprimitivity map P on characters of the form ψ^y, $y \in A$. Indeed, using that $\tilde{\pi} = \operatorname{ind}_N^G \rho$ is an induced representation we obtain

$$(\tilde{\pi}(y)\xi)(x) = \xi(y^{-1}x) = \xi(xx^{-1}y^{-1}x) = \rho(x^{-1}yx) \cdot \xi(x)$$

for all $y \in A$, $x \in G$, and $\xi \in \mathcal{H}_{\tilde{\pi}}$. But A is a normal abelian subgroup of G and hence $\rho|_A$ can be identified with a character. Thus we obtain for all $y \in A, x \in G$, and $\xi \in \mathcal{H}_{\tilde{\pi}}$:

$$\rho(y^{-1}) \cdot (\tilde{\pi}(y)\xi)(x) = \rho(y^{-1})\rho(x^{-1}yx)\xi(x) = \rho((y^{-1}, x^{-1}))\xi(x).$$

Recall that, for all $y \in A$ and $x \in G$, we have $(y^{-1}, x^{-1}) = (y, x)$ (Remark 2.8.2). Furthermore, we have

$$I|_{Z(G)} = \bigcap_{x \in G} xJ|_{Z(G)}x^{-1} = J|_{Z(G)}$$

and hence $\rho|_{Z(G)} \mathrel{\widehat{=}} \pi|_{Z(G)} \mathrel{\widehat{=}} \psi$. In particular, we have

$$\rho((y^{-1}, x^{-1})) = \rho((y, x)) = \psi((y, x)) = \psi^y(x)$$

for all $y \in A, x \in G$ and hence

$$\begin{aligned} T(P(\psi^y)\xi) &= T(\psi^y \cdot \xi) = T(\rho(y^{-1}) \cdot (\tilde{\pi}(y)\xi)) = \rho(y^{-1}) \cdot (T\tilde{\pi}(y)\xi) \\ &= \rho(y^{-1}) \cdot (\tilde{\pi}(y)T\xi) = \psi^y \cdot (T\xi) = P(\psi^y)(T\xi), \end{aligned}$$

where we denote by $\rho(y^{-1}) \cdot (\tilde{\pi}(y)\xi)$ the operator $x \mapsto \rho(y^{-1})(\tilde{\pi}(y)\xi)(x)$. □

Remark 2.8.14. The arguments used in the proof of the proposition above do neither depend on a particular form of the subgroup H, nor on the character $\chi \in \hat{H}$.

We may use similar arguments to prove a more specific result which turns out be very useful in our context.

Lemma 2.8.15. *Let $k \in \mathbb{N}$, let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair of nilpotence class $l \geq 2$, and let $f \in \mathfrak{g}^*$ such that f is faithful on $\mathfrak{z}(\mathfrak{g})$. Let N be the centralizer of some maximal abelian subgroup A of $Z^2(G)$ and let $\mathfrak{n} := \log(N)$ be its corresponding subalgebra. If $\rho := \operatorname{ind}_R^N \varphi_{f|_\mathfrak{n}}$ is an irreducible representation of N, where $\log(R) = \mathfrak{r} \subseteq \mathfrak{n}$ denotes a polarizing subalgebra for $f|_\mathfrak{n}$, then $\pi := \operatorname{ind}_N^G \rho$ is an irreducible representation of G.*

Proof. Suppose that $\rho = \operatorname{ind}_R^N \varphi_{f|_\mathfrak{n}}$ is an irreducible representation of N, where $\log(R) = \mathfrak{r} \subseteq \mathfrak{n}$ denotes a polarizing subalgebra for $f|_\mathfrak{n}$. We need to show that the representation

$$\pi = \operatorname{ind}_N^G \rho = \operatorname{ind}_N^G \operatorname{ind}_R^N \varphi_{f|_\mathfrak{n}} \cong \operatorname{ind}_R^G \varphi_{f|_\mathfrak{n}}$$

is irreducible. For this, we follow the outline of the last part of the proof of Proposition 2.8.13.

Let $T \in \mathcal{L}(\mathcal{H}_\pi)$ be an intertwining operator for π. We need to show that T is a multiple of the identity $Id_{\mathcal{H}_\pi}$. But π is an induced representation, so there exists a system of imprimitivity $\Sigma = (\pi, P)$, where

$$P : C_0(G/N) \to \mathcal{L}(\mathcal{H}_\pi),\ P(\varphi)(\xi) = \varphi \cdot \xi$$

defines a nondegenerate $*$-representation on $C_0(G/N)$. Recall that the set of intertwining operators of the imprimitivity system Σ is isometrically isomorphic to the set of intertwining operators of the representation ρ ([13], Theorem 6.28). Since ρ is irreducible by assumption, it suffices to show that the operator T commutes with $P(\varphi)$ for all $\varphi \in C_0(G/N)$. But the quotient group G/N is an abelian locally compact group and thus we have $C_0(G/N) \cong C^*(\widehat{G/N})$. Therefore, it is enough to prove the equality $TP(\chi) = P(\chi)T$ for all $\chi \in \widehat{G/N}$. But since $f \in \mathfrak{g}^*$ is faithful on the center of $\mathfrak{g}$, it follows from Remark 2.8.10 that the map

$$\Phi : A/Z(G) \to \widehat{G/N},\ y \mapsto \varphi_f^y,$$

is a continuous, injective homomorphism with dense image, where $\varphi_f^y(x) := \varphi_f((y, x))$ for all $x \in G$. Hence it suffices to prove that T commutes with $P(\varphi_f^y)$ for all $y \in A$.

For every $y \in A$, the operator $\pi(y)$ is a multiplication operator which coincides, up to multiplication with a scalar, with the imprimitivity map P on characters of the form φ_f^y, $y \in A$. Indeed, using that $\pi = \operatorname{ind}_N^G \rho$ is an induced representation we obtain

$$(\pi(y)\xi)(x) = \xi(y^{-1}x) = \xi(xx^{-1}y^{-1}x) = \rho(x^{-1}yx) \cdot \xi(x)$$

for all $y \in A$, $x \in G$, and $\xi \in \mathcal{H}_\pi$. But A is a normal abelian subgroup of G and hence $\rho|_A$ can be identified with a character. Thus we obtain for all $y \in A, x \in G$, and $\xi \in \mathcal{H}_\pi$,

$$\rho(y^{-1}) \cdot (\pi(y)\xi)(x) = \rho(y^{-1})\rho(x^{-1}yx)\xi(x) = \rho((y^{-1}, x^{-1}))\xi(x).$$

Notice that $(y^{-1}, x^{-1}) = (y, x)$ for all $y \in A$ and $x \in G$ (Remark 2.8.2), and that $\rho|_{Z(G)} \mathrel{\widehat{=}} \pi|_{Z(G)} \mathrel{\widehat{=}} \varphi_f$. In particular, we have

$$\rho((y^{-1}, x^{-1})) = \rho((y, x)) = \varphi_f((y, x)) = \varphi_f^y(x)$$

for all $y \in A, x \in G$ and thus we obtain

$$\begin{aligned} T(P(\varphi_f^y)\xi) &= T(\varphi_f^y \cdot \xi) = T(\rho(y^{-1}) \cdot (\pi(y)\xi)) = \rho(y^{-1}) \cdot (T\pi(y)\xi) \\ &= \rho(y^{-1}) \cdot (\pi(y)T\xi) = \varphi_f^y \cdot (T\xi) = P(\varphi_f^y)(T\xi), \end{aligned}$$

where we denote by $\rho(y^{-1}) \cdot (\pi(y)\xi)$ the operator $x \mapsto \rho(y^{-1}) \cdot (\pi(y)\xi)(x)$. □

We prove now that if $(G, \mathfrak{g})$ is a nilpotent k-Lie pair then every homomorphism $f \in \mathfrak{g}^*$ admits a polarizing subalgebra $\mathfrak{r}$. Furthermore, we show that the induced representation $\operatorname{ind}_R^G \varphi_f$ is irreducible. The idea of Proposition 2.8.16 is due to Howe ([16], Lemma 11).

Proposition 2.8.16. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. For every homomorphism $f \in \mathfrak{g}^*$ there exists a polarizing subalgebra $\mathfrak{r}$ and the induced character $\operatorname{ind}_R^G \varphi_f$ is an irreducible representation of G, where φ_f and R are defined as in Remark 2.6.3.*

Proof. We will prove this proposition by induction on the nilpotence class $l \leq k$ of the group G. Let $f \in \mathfrak{g}^*$.

To start the induction process we observe that if the group G is abelian, then the abelian algebra $\mathfrak{g}$ polarizes every homomorphism $f \in \mathfrak{g}^*$ and the map φ_f defines a character of G.

So suppose $l \geq 2$ and assume that the proposition is proven for all nilpotent k-Lie pairs $(H, \mathfrak{h})$, where the nilpotence class of H (and hence that of $\mathfrak{h}$) is less than l.

Step 1: We consider the case that $f \in \mathfrak{g}^*$ is faithful on $\mathfrak{z}(\mathfrak{g})$.

Let A be a maximal abelian subgroup of $Z^2(G)$, the second element of the ascending central series of G, and let N be the centralizer of A. Then N is a closed normal subgroup of G of nilpotence class at most $(l-1)$ (Lemma 2.8.3). Put $\mathfrak{a} := \log(A)$ and $\mathfrak{n} := \log(N)$. Not only $\mathfrak{a}$, but also $\mathfrak{n}$, is a subalgebra of $\mathfrak{g}$ (Lemma 2.8.4 and Lemma 2.8.5) and in particular, the pair $(N, \mathfrak{n})$ is a nilpotent k-Lie pair of nilpotence class at most $(l-1)$.
Applying the induction hypothesis to the nilpotent k-Lie pair $(N, \mathfrak{n})$ and the homomorphism $f|_{\mathfrak{n}}$ yields a polarizing subalgebra $\mathfrak{r} \subseteq \mathfrak{n}$ and the induced character $\operatorname{ind}_R^N \varphi_{f|_{\mathfrak{n}}}$ defines an irreducible representation of N.

We will show now that the subalgebra $\mathfrak{r} \subseteq \mathfrak{n}$, which polarizes the restricted map $f|_{\mathfrak{n}}$, is already a polarizing subalgebra for $f \in \mathfrak{g}^*$. For this, it suffices to prove the following implication: If $X \in \mathfrak{g}$ and $f([X, Y]) = 0$ for all $Y \in \mathfrak{r}$, then $X \in \mathfrak{r}$. So suppose $X \in \mathfrak{g}$ and $f([X, Y]) = 0$ for all $Y \in \mathfrak{r}$. Since $\mathfrak{a} \subseteq \mathfrak{z}(\mathfrak{n}) \subseteq \mathfrak{r}$ we have in particular $f([X, B]) = 0$ for all $B \in \mathfrak{a}$. But $[X, B] \in \mathfrak{z}(\mathfrak{g})$ for all $B \in \mathfrak{a}$ and since the homomorphism f was assumed to be faithful on the center of $\mathfrak{g}$, it follows that $[X, B] = 0$ for all $B \in \mathfrak{a}$ and hence $X \in \mathfrak{n}$. Since the subalgebra $\mathfrak{r} \subseteq \mathfrak{n}$ was chosen to be maximal with respect to the property that it is $f|_{\mathfrak{n}}$-subordinate, we obtain $X \in \mathfrak{r}$.

It follows then from Lemma 2.8.15 that inducing the irreducible representation $\operatorname{ind}_R^N \varphi_{f|_{\mathfrak{n}}}$ from N to G yields an irreducible representation

$$\pi := \operatorname{ind}_N^G \operatorname{ind}_R^N \varphi_{f|_{\mathfrak{n}}} \cong \operatorname{ind}_R^G \varphi_f.$$

Step 2: If the homomorphism $f \in \mathfrak{g}^*$ is not faithful on $\mathfrak{z}(\mathfrak{g})$, then we pass to the quotient algebra $\tilde{\mathfrak{g}} := \mathfrak{g}/\mathfrak{j}$, where $\mathfrak{j}$ is defined to be the largest ideal inside the kernel of f as explained in Lemma 2.8.12. We have seen that $\exp(\mathfrak{j}) := J$ is a normal subgroup of G (Lemma 2.4.1) and the pair of quotients, $(\tilde{G}, \tilde{\mathfrak{g}})$, is a nilpotent k-Lie pair, where $\tilde{G} := G/J$. Moreover, the homomorphism $\tilde{f} \in \tilde{\mathfrak{g}}^*$, corresponding to the homomorphism $f \in \mathfrak{g}^*$, is faithful on the center of $\tilde{\mathfrak{g}}$.

If we apply the first step to the nilpotent k-Lie pair $(\tilde{G}, \tilde{\mathfrak{g}})$ and the homomorphism $\tilde{f}$, then we can find a polarizing subalgebra $\tilde{\mathfrak{r}} \subseteq \tilde{\mathfrak{g}}$ for $\tilde{f}$ such that the induced representation $\operatorname{ind}_{\tilde{R}}^{\tilde{G}} \varphi_{\tilde{f}}$ is irreducible, where $\tilde{R} := \exp(\tilde{\mathfrak{r}})$. But we have seen in Lemma 2.8.12 that if $\tilde{\mathfrak{r}}$ is a polarizing subalgebra for $\tilde{f}$, then $\mathfrak{r} = q^{-1}(\tilde{\mathfrak{r}})$ is a polarizing subalgebra for f, where $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{j}$ denotes the canonical quotient map. Using $H = R$ and $L = J$ in Theorem 2.5.12 yields

$$\operatorname{ind}_R^G(\varphi_{\tilde{f}} \circ q') \cong (\operatorname{ind}_{R/J}^{G/J} \varphi_{\tilde{f}}) \circ q',$$

where $q' : G \to G/J$ denotes the canonical quotient map. But since

$$\varphi_{\tilde{f}}(q'(x)) = \varphi_f(x)$$

for all $x \in R$, we obtain

$$\operatorname{ind}_R^G \varphi_f \cong (\operatorname{ind}_{R/J}^{G/J} \varphi_{\tilde{f}}) \circ q'.$$

Since the representation $\operatorname{ind}_{R/J}^{G/J} \varphi_{\tilde{f}}$ is irreducible, it follows that the representation $\pi = \operatorname{ind}_R^G \varphi_f$ is irreducible. □

Corollary 2.8.17. *Let $k \geq 2$, let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair of nilpotence class $l \geq 2$, and let $f \in \mathfrak{g}^*$ such that f is faithful on $\mathfrak{z}(\mathfrak{g})$. Let N be the centralizer of some maximal abelian subgroup A of $Z^2(G)$ and let $\mathfrak{n} := \log(N)$ be its corresponding subalgebra. If $\mathfrak{r} \subseteq \mathfrak{n}$ denotes a polarizing subalgebra for $f|_{\mathfrak{n}}$ then $\mathfrak{r}$ is also a polarizing subalgebra for f.*

Proof. This is shown in step 1 of the proof of Proposition 2.8.16. □

2.9 The Kirillov-orbit map in the general case

In this section we will prove that the Kirillov map κ is a well-defined, surjective map for every k-Lie pair $(G, \mathfrak{g})$.

Proposition 2.9.1. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. The Kirillov map*

$$\kappa : \mathfrak{g}^* \to \mathrm{Prim}(C^*(G)),\ f \mapsto \ker(\mathrm{ind}_R^G \varphi_f),$$

as explained in Section 2.6, is well-defined.

Proof. We have already seen in Proposition 2.8.16 that every homomorphism $f \in \mathfrak{g}^*$ admits a polarizing subalgebra $\mathfrak{r}$ and the induced representation $\mathrm{ind}_R^G \varphi_f$ is an irreducible representation of G, where $R = \exp(\mathfrak{r})$. Thus $\ker(\mathrm{ind}_R^G \varphi_f)$ is a primitive ideal of $C^*(G)$.

But, as we have seen in Example 2.6.2, there may exist several non-isomorphic polarizing subalgebras for one given homomorphism $f \in \mathfrak{g}^*$, and we need to show that the kernel of the induced representation $\mathrm{ind}_R^G \varphi_f$ does not depend on the choice of the polarizing subalgebra $\mathfrak{r}$ for f.

For this, let $f \in \mathfrak{g}^*$ and suppose that there exist two distinct polarizing subalgebras, $\mathfrak{r}$ and $\mathfrak{r}'$, for f. As usual, the map f defines a character φ_f on $R := \exp(\mathfrak{r})$, by $\varphi_f := \epsilon \circ f \circ \log$ and f defines a character φ'_f on $R' := \exp(\mathfrak{r}')$ in the same way, $\varphi'_f := \epsilon \circ f \circ \log$. We will prove by induction on the nilpotence class l of G that

$$\ker(\mathrm{ind}_R^G \varphi_f) = \ker(\mathrm{ind}_{R'}^G \varphi'_f). \tag{2.51}$$

(Notice that we can not expect in general to get equivalent representations.)

If $l = 1$ and hence if G is an abelian group, then the Lie algebra $\mathfrak{g}$ itself polarizes every homomorphism of $\mathfrak{g}^*$. It follows then from the maximality property of every polarizing subalgebra that $\mathfrak{r} = \mathfrak{r}' = \mathfrak{g}$ and there is nothing to prove.

If G is a two-step nilpotent group, Equation (2.51) is proven in Section 2.7.

So suppose $l \geq 3$, and assume that (2.51) holds for all k-Lie pairs $(H, \mathfrak{h})$, where the nilpotence class of H is smaller than l. Let A be a maximal abelian subgroup of $Z^2(G)$ and let N be the centralizer of A. We have shown in Lemma 2.8.3 that N is a closed normal subgroup of G of nilpotence class at most $(l-1)$. Furthermore, we have proven that both sets, $\mathfrak{a} := \log(A)$ and $\mathfrak{n} := \log(N)$, are subalgebras of $\mathfrak{g}$ (Lemma 2.8.4 and Lemma 2.8.5) and in particular, we have $[X, Y] = 0$ for all $X \in \mathfrak{n}$ and $Y \in \mathfrak{a}$. So the pair $(N, \mathfrak{n})$ is a nilpotent k-Lie pair of nilpotence class at most $(l-1)$. We consider two different cases.

Case 1: $\mathfrak{r}, \mathfrak{r}' \subseteq \mathfrak{n}$.

Applying the induction hypothesis to the closed normal subgroup N of G yields

$$\ker(\mathrm{ind}_R^N \varphi_f) = \ker(\mathrm{ind}_{R'}^N \varphi'_f),$$

and hence the desired equation

$$\ker(\operatorname{ind}_R^G \varphi_f) = \operatorname{ind}_N^G \ker(\operatorname{ind}_R^N \varphi_f) = \operatorname{ind}_N^G \ker(\operatorname{ind}_{R'}^N \varphi'_f) = \ker(\operatorname{ind}_{R'}^G \varphi'_f).$$

Case 2: $\mathfrak{r} \nsubseteq \mathfrak{n}$ or $\mathfrak{r}' \nsubseteq \mathfrak{n}$.
Suppose that $\mathfrak{r} \nsubseteq \mathfrak{n}$. The idea we will pursue is to pass from $\mathfrak{r}$ to a different polarizing subalgebra for f, denoted by $\mathfrak{s}$, which is contained in the subalgebra $\mathfrak{n}$, and satisfies the equation $\ker(\operatorname{ind}_S^G \varphi_f) = \ker(\operatorname{ind}_R^G \varphi_f)$, where $S := \exp(\mathfrak{s})$.
If also $\mathfrak{r}' \nsubseteq \mathfrak{n}$, we will pass in the same way from $\mathfrak{r}'$ to a different polarizing subalgebra for f, denotes by $\mathfrak{s}'$, which is contained in the subalgebra $\mathfrak{n}$, and satisfies the equation $\ker(\operatorname{ind}_{S'}^G \varphi_f) = \ker(\operatorname{ind}_{R'}^G \varphi'_f)$, where $S' := \exp(\mathfrak{s}')$. Done that, we can assume (without loss of generality) that we are in the situation of the first case and obtain the desired result.

Step 1: We assume that the homomorphism $f \in \mathfrak{g}^*$ is faithful on $\mathfrak{z}(\mathfrak{g})$.

In order to construct a polarizing subalgebra $\mathfrak{s}$ as mentioned above, we observe first that if $\mathfrak{r} \nsubseteq \mathfrak{n}$, then $\mathfrak{a} \nsubseteq \mathfrak{r}$. Indeed, if $\mathfrak{a} \subseteq \mathfrak{r}$ then $f([X,Y]) = 0$ for all $X \in \mathfrak{r}$ and $Y \in \mathfrak{a}$. But $[X,Y] \in \mathfrak{z}(\mathfrak{g})$ for all $X \in \mathfrak{r}, Y \in \mathfrak{a}$ and since f was assumed to be faithful on $\mathfrak{z}(\mathfrak{g})$, it follows that $[X,Y] = 0$ for all $X \in \mathfrak{r}$ and $Y \in \mathfrak{a}$. Therefore, every element $Y \in \mathfrak{a}$ commutes with every element $X \in \mathfrak{r}$ and hence $\mathfrak{r} \subseteq \mathfrak{n}$. Notice that in this case it can not happen that $\mathfrak{r} = \mathfrak{g}$.

We define now $\mathfrak{s}$ to be the closure of the algebra generated by $\mathfrak{r} \cap \mathfrak{n}$ and $\mathfrak{a}$. Since $[V,Y] = 0$ for all $V \in \mathfrak{r} \cap \mathfrak{n}$ and all $Y \in \mathfrak{a}$, we have $\langle \mathfrak{r} \cap \mathfrak{n}, \mathfrak{a} \rangle = \mathfrak{r} \cap \mathfrak{n} + \mathfrak{a}$ and thus

$$\mathfrak{s} = \overline{\mathfrak{r} \cap \mathfrak{n} + \mathfrak{a}}.$$

So every element B of $\mathfrak{s}$ is of the form $B = \lim_{n\to\infty} B_n$, where $B_n = V_n + Y_n$ for some $V_n \in \mathfrak{r} \cap \mathfrak{n}$ and some $Y_n \in \mathfrak{a}$, $n \in \mathbb{N}$. (Note that neither the sequence $(V_n)_{n\in\mathbb{N}}$, nor the sequence $(Y_n)_{n\in\mathbb{N}}$ needs to converge.)

Claim. The subalgebra $\mathfrak{s}$ is f-subordinate.

Clearly, both subalgebras, the intersection $\mathfrak{r} \cap \mathfrak{n}$ and the abelian subalgebra $\mathfrak{a}$, are f-subordinate. Since $[V,Y] = 0$ for all $V \in \mathfrak{r} \cap \mathfrak{n}$ and $Y \in \mathfrak{a}$, we obtain for two elements $V + Y, V' + Y' \in \mathfrak{r} \cap \mathfrak{n} + \mathfrak{a}$, where $V, V' \in \mathfrak{r} \cap \mathfrak{n}$ and $Y, Y' \in \mathfrak{a}$:

$$[V + Y, V' + Y'] = [V, V'] + [Y, V'] + [V, Y'] + [Y, Y'] = [V, V'].$$

Since $\mathfrak{r}$ was chosen to be f-subordinate, it follows that

$$f([V + Y, V' + Y']) = f([V, V']) = 0,$$

which proves that $\mathfrak{r} \cap \mathfrak{n} + \mathfrak{a}$ is f-subordinate. Since the commutator, as well as the homomorphism f, is a continuous map, it follows that the closure $\overline{\mathfrak{r} \cap \mathfrak{n} + \mathfrak{a}} = \mathfrak{s}$ is f-subordinate as well. This proves the claim.

Put $S := \exp(\mathfrak{s})$, the closed subgroup of G corresponding to the subalgebra $\mathfrak{s}$.

Claim. We have $S = \overline{(R \cap N)A}$.

Notice that A is a normal subgroup of G (Remark 2.8.1), so every element $x \in \langle R \cap N, A \rangle$ is of the form $x = vy$ for some $v \in R \cap N$ and some $y \in A$.

As $[V, Y] = 0$ for every $V \in \log(R \cap N) = \mathfrak{r} \cap \mathfrak{n}$ and every $Y \in \log(A) = \mathfrak{a}$, we obtain by the Campbell-Hausdorff formula for all $v \in R \cap N$ and $y \in A$:

$$\exp(\log(v)) \exp(\log(y)) = \exp(\log(v) + \log(y))$$

and hence

$$\log(v \cdot y) = \log(v) + \log(y).$$

This yields $\log((R \cap N)A) = \log(R \cap N) + \log(A) = (\mathfrak{r} \cap \mathfrak{n}) + \mathfrak{a}$ and since log is a continuous map we obtain $\log(\overline{(R \cap N)A}) = \overline{(\mathfrak{r} \cap \mathfrak{n}) + \mathfrak{a}} = \mathfrak{s}$ and thus $\overline{(R \cap N)A} = S$. This proves the claim.

We observe that the subalgebra $\mathfrak{s}$ is not necessarily maximal with respect to the property that it is f-subordinate as a subalgebra of $\mathfrak{g}$. But it turns out that $\mathfrak{s}$ is maximal f-subordinate inside a certain subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, which is defined as follows.

Let $\mathfrak{h}$ be the closure of the subalgebra generated by $\mathfrak{r}$ and $\mathfrak{a}$. Since $[\mathfrak{r}, \mathfrak{a}] \subseteq \mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{a}$, we have $\langle \mathfrak{r}, \mathfrak{a} \rangle = \mathfrak{r} + \mathfrak{a}$ and thus $\mathfrak{h} = \overline{\mathfrak{r} + \mathfrak{a}}$. Note that the subalgebra $\mathfrak{h}$ is not f-subordinate, because $\mathfrak{a} \nsubseteq \mathfrak{r}$ and $\mathfrak{r}$ was chosen to be a maximal f-subordinate subalgebra of $\mathfrak{g}$.

Claim. The algebra $\mathfrak{s}$ is a maximal f-subordinate subalgebra of $\mathfrak{h}$.

Assume it is not, then we can find an element $W \in \mathfrak{h} \setminus \mathfrak{s}$ with $f([W, Z]) = 0$ for all $Z \in \mathfrak{s}$. But since $\mathfrak{r} \nsubseteq \mathfrak{s}$ (otherwise $\mathfrak{r}$ would be contained in $\mathfrak{n}$), we can assume without loss of generality that the element W is of the form $W = X + Y$ for some $X \in \mathfrak{r} \setminus (\mathfrak{r} \cap \mathfrak{n})$ and some $Y \in \mathfrak{a}$. In particular, we obtain for all $Y' \in \mathfrak{a}$:

$$0 = f([X + Y, Y']) = f([X, Y'] + [Y, Y']) = f([X, Y']).$$

But $[X, Y'] \in \mathfrak{z}(\mathfrak{g})$ for all $Y' \in \mathfrak{a}$ and since f was assumed to be faithful on $\mathfrak{z}(\mathfrak{g})$, we obtain $[X, Y'] = 0$ for all $Y' \in \mathfrak{a}$, which contradicts the fact that $X \notin \mathfrak{n}$. Hence $\mathfrak{s}$ is a polarizing subalgebra for f within $\mathfrak{h}$, proving the claim.

Put $H := \overline{\langle R, A \rangle}$, the closure of the subgroup of G generated by R and A. Since A is a normal subgroup of G, every element $w \in \overline{\langle R, A \rangle}$ can be written as $w = \lim_{n \to \infty} w_n$, where $w_n = x_n y_n$ for some $x_n \in R$ and some $y_n \in A$ and we have $H = \overline{RA}$.

Claim. We have $\exp(\mathfrak{h}) = H$.

On the one hand, the Campbell-Hausdorff formula reduces for all $X \in \mathfrak{r}$ and $Y \in \mathfrak{a}$ to the following equation

$$\exp(X) \exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y]). \tag{2.52}$$

Therefore, every element $w = xy \in RA$ is of the form $xy = \exp(X + Y + \frac{1}{2}[X, Y])$, where $X = \log(x) \in \mathfrak{r}$ and $Y = \log(y) \in \mathfrak{a}$. Since $[\mathfrak{r}, \mathfrak{a}] \subseteq \mathfrak{z}(\mathfrak{g})$, it follows that $xy \in \exp(\mathfrak{r} + \mathfrak{a}) = \exp(\mathfrak{h})$. But since the map exp is a homeomorphism, $\exp(\mathfrak{h})$ is a closed subgroup of G, and every element $w = \lim_{n\in\mathbb{N}} w_n \in H$, where $w_n = x_n y_n \in RA$, is in $\exp(\mathfrak{h})$. This proves that $H \subseteq \exp(\mathfrak{h})$.

On the other hand, the Inversion Formula (2.32) reduces for all $x \in R$ and $y \in A$ to the following equation

$$\log(x) + \log(y) = \log(xy(x^{-\frac{1}{2}}, y)). \tag{2.53}$$

So if $X = \log(x) \in \mathfrak{r}$ and $Y = \log(y) \in \mathfrak{a}$, then (2.53) yields $\exp(X + Y) \in RA \subseteq H$. Since the map exp is continuous, it follows that $\exp(W) \in \overline{RA} = H$ for every $W \in \mathfrak{h}$ and hence $\exp(\mathfrak{h}) \subseteq H$. This proves the claim.

Summarized, we have the following situation. We constructed from the polarizing subalgebra $\mathfrak{r}$ for $f \in \mathfrak{g}^*$ with corresponding subgroup $R = \exp(\mathfrak{r})$ of G, the closed subalgebra $\mathfrak{h} = \overline{\mathfrak{r} + \mathfrak{a}}$ of $\mathfrak{g}$ and the closed subgroup $H = \overline{RA}$ of G, and the pair $(H, \mathfrak{h})$ defines a nilpotent k-Lie pair. Furthermore, we have shown that the subalgebra $\mathfrak{s} = \overline{\mathfrak{r} \cap \mathfrak{n} + \mathfrak{a}}$ is f-subordinate and, as a subalgebra of $\mathfrak{h}$, $\mathfrak{s}$ is maximal with respect to this property. The homomorphism f defines in the usual way a character φ_f of the closed subgroup R and a character φ'_f of the closed subgroup $S = \exp(\mathfrak{s})$.

It turns out that we can reduce the situation to the case of two-step nilpotent k-Lie pairs. For this, put

$$\mathfrak{K} := \ker(f|_{\mathfrak{r}\cap\mathfrak{n}}).$$

Claim. The set $\mathfrak{K}$ is an ideal of $\mathfrak{h}$.

We show that $\mathfrak{k}$ is an ideal of $\mathfrak{r} + \mathfrak{a}$. It follows then from the continuity of the commutator map that $\mathfrak{k}$ is an ideal of $\overline{\mathfrak{r} + \mathfrak{a}} = \mathfrak{h}$. Since $\mathfrak{k}$ is defined as the kernel of a group homomorphisms of the subalgebra $\mathfrak{r} \cap \mathfrak{n}$ of $\mathfrak{g}$, it follows that $(\mathfrak{k}, +)$ is an additive subgroup of $\mathfrak{g}$. Since the map $f \in \mathfrak{g}^*$ is a homomorphism of Λ_k-modules (Remark 2.2.10), it follows that $\mathfrak{k}$ is also a Λ_k-module. Therefore, it suffices to prove that $[V, W] \in \mathfrak{k}$ for all $V \in \mathfrak{k}$ and for all $W \in \mathfrak{r} + \mathfrak{a}$. For this, let $V \in \mathfrak{k}$ and let $W \in \mathfrak{r} + \mathfrak{a}$. The element W is of the form $W = X + Y$ for some $X \in \mathfrak{r}$ and some $Y \in \mathfrak{a}$ and since $V \in \mathfrak{r} \cap \mathfrak{n}$, we obtain

$$[V, W] = [V, X] + [V, Y] = [V, X] \in [\mathfrak{r}, \mathfrak{r}],$$

and thus $f([V, W]) = 0$. But since $\mathfrak{r}/(\mathfrak{r} \cap \mathfrak{n})$ is abelian, it follows that $[\mathfrak{r}, \mathfrak{r}] \subseteq \mathfrak{r} \cap \mathfrak{n}$ and hence $[V, W] \in \mathfrak{k}$. This proves the claim.

Since $\mathfrak{k}$ is an ideal of $\mathfrak{h}$, it follows from Lemma 2.4.1 that $K := \exp(\mathfrak{k})$ is a normal subgroup of H and the pair of quotients $(H/K, \mathfrak{h}/\mathfrak{k})$ defines a nilpotent k-Lie pair.

Claim. The nilpotent k-Lie pair $(H/K, \mathfrak{h}/\mathfrak{k})$ is of nilpotence class 2.

We prove that the Lie algebra $\mathfrak{h}/\mathfrak{k}$ is two-step nilpotent. For this, we show

$$[\mathfrak{h}/\mathfrak{k}, \mathfrak{h}/\mathfrak{k}] \subseteq (\mathfrak{r} \cap \mathfrak{n})/\mathfrak{k} \subseteq \mathfrak{z}(\mathfrak{h}/\mathfrak{k}). \tag{2.54}$$

Let $W, W' \in \mathfrak{h}$. We consider first the case that the element W is of the form $W = X + Y$ for some $X \in \mathfrak{r}$ and some $Y \in \mathfrak{a}$ and the element W' is of the form $W' = X' + Y'$ for some $X' \in \mathfrak{r}$ and some $Y' \in \mathfrak{a}$. We have

$$[X + Y, X' + Y'] = [X, X'] + [X, Y'] + [Y, X'] + [Y, Y'],$$

and since both commutators, $[X, Y']$ and $[Y, X']$, are elements of $\mathfrak{z}(\mathfrak{g})$ and since $[Y, Y'] = 0$, it follows that $[W, W'] \in [\mathfrak{r}, \mathfrak{r}] \subseteq \mathfrak{r} \cap \mathfrak{n}$. Using the continuity of the commutator map and the fact that the subalgebra $\mathfrak{r} \cap \mathfrak{n}$ is closed, it follows that $[W, W'] \in \mathfrak{r} \cap \mathfrak{n}$ for all elements $W, W' \in \mathfrak{h}$. This proves the first inclusion of (2.54), $[\mathfrak{h}/\mathfrak{k}, \mathfrak{h}/\mathfrak{k}] \subseteq (\mathfrak{r} \cap \mathfrak{n})/\mathfrak{k}$.

In order to prove the second inclusion, $(\mathfrak{r} \cap \mathfrak{n})/\mathfrak{k} \subseteq \mathfrak{z}(\mathfrak{h}/\mathfrak{k})$, let $V \in \mathfrak{r} \cap \mathfrak{n}$. Since $f([W, V]) = f([X, V]) = 0$ for every $W = X+Y \in \mathfrak{r}+\mathfrak{a}$, it follows that $f([W, V]) = 0$ for every $W \in \mathfrak{h}$. But, as we have seen above, we have $[W, V] \in \mathfrak{r} \cap \mathfrak{n}$ for every $W \in \mathfrak{h}$ and thus it follows that $[W, V] \in \mathfrak{k}$ for every $W \in \mathfrak{h}$. This proves that if $\dot{W} \in \mathfrak{h}/\mathfrak{k}$ is any element of the quotient algebra, then $[\dot{W}, \dot{V}]$ is trivial in $\mathfrak{h}/\mathfrak{k}$ and thus $\dot{V} \in \mathfrak{z}(\mathfrak{h}/\mathfrak{k})$, proving the claim.

Now, both characters, $\varphi_f \in \widehat{R}$ and $\varphi'_f \in \widehat{S}$, yield characters $\tilde{\varphi}_f$ and $\tilde{\varphi}'_f$ of the quotient groups R/K and S/K, respectively. Furthermore, the homomorphism $f|_{\mathfrak{h}} \in \mathfrak{h}^*$ yields a homomorphism $\tilde{f} \in (\mathfrak{h}/\mathfrak{k})^*$ and it follows from the construction that

$$\widetilde{\varphi_f} = \varphi_{\tilde{f}}.$$

Indeed, we have for all $X \in \mathfrak{r}$:

$$\widetilde{\varphi_f}(\widetilde{\exp}(q'(X))) = \widetilde{\varphi_f}(q(\exp(X))) = \varphi_f(\exp(X)) = \epsilon(f(X)) = \epsilon(\tilde{f}(q'(X))),$$

where $q : H \to H/K$ and $q' : \mathfrak{h} \to \mathfrak{h}/\mathfrak{k}$ denote the canonical quotient maps. In the same way we obtain $\widetilde{\varphi'_f} = \varphi'_{\tilde{f}}$. Since both algebras, $\mathfrak{r}$ and $\mathfrak{s}$, are polarizing subalgebras for f within $\mathfrak{h}$, it follows that $\mathfrak{r}/\mathfrak{k}$ and $\mathfrak{s}/\mathfrak{k}$ are polarizing subalgebras for $\tilde{f}$ within $\mathfrak{h}/\mathfrak{k}$.

It follows now directly from the results of Section 2.7 (the Kirillov map for two-step nilpotent groups) that

$$\ker(\operatorname{ind}_{R/K}^{H/K} \tilde{\varphi}_f) = \ker(\operatorname{ind}_{S/K}^{H/K} \tilde{\varphi}'_f). \tag{2.55}$$

If we use $G = H$, $L = K$, and $H = R$ in Theorem 2.5.12, we obtain

$$(\operatorname{ind}_{R/K}^{H/K} \tilde{\varphi}_f) \circ q \cong \operatorname{ind}_R^H(\tilde{\varphi}_f \circ q),$$

and with $G = H$, $L = K$, and $H = S$ in Theorem 2.5.12, we obtain

$$(\operatorname{ind}_{S/K}^{H/K} \tilde{\varphi}'_f) \circ q \cong \operatorname{ind}_S^H(\tilde{\varphi}'_f \circ q).$$

These facts, together with (2.55), yield then

$$\ker(\operatorname{ind}_R^H \tilde{\varphi_f} \circ q) = \ker(\operatorname{ind}_S^H \tilde{\varphi'_f} \circ q).$$

Since $\tilde{\varphi_f} \circ q = \varphi_f$ and $\tilde{\varphi'_f} \circ q = \varphi'_f$, it follows that

$$\ker(\operatorname{ind}_R^H \varphi_f) = \ker(\operatorname{ind}_S^H \varphi'_f).$$

Inducing once again from the closed subgroup H to the group G yields the desired result

$$\ker(\operatorname{ind}_R^G \varphi_f) = \operatorname{ind}_H^G \ker(\operatorname{ind}_R^H \varphi_f) = \operatorname{ind}_H^G \ker(\operatorname{ind}_S^H \varphi'_f) = \ker(\operatorname{ind}_S^G \varphi'_f).$$

Step 2: The original chosen homomorphism $f \in \mathfrak{g}^*$ is not faithful on $\mathfrak{z}(\mathfrak{g})$.

We pass to the quotient algebra $\dot{\mathfrak{g}} := \mathfrak{g}/\mathfrak{j}$, where $\mathfrak{j}$ denotes the largest ideal in $\ker(f)$. We have shown in Lemma 2.8.12 that such an ideal exists and that the corresponding homomorphism $\dot{f} \in (\dot{\mathfrak{g}})^*$ is faithful on $\mathfrak{z}(\dot{\mathfrak{g}})$. Let $J := \exp(\mathfrak{j})$ be the normal subgroup of G corresponding to the ideal $\mathfrak{j}$. Clearly, we have $\mathfrak{j} \subseteq \mathfrak{r}$ and $J \subseteq R$.

Applying the results of the first step to the quotient groups $\dot{G} := G/J$, $\dot{R} := R/J$, and $\dot{S} := S/J$ and the homomorphism $\dot{f} \in (\dot{\mathfrak{g}})^*$ yields

$$\ker(\operatorname{ind}_{\dot{R}}^{\dot{G}} \varphi_{\dot{f}}) = \ker(\operatorname{ind}_{\dot{S}}^{\dot{G}} \varphi'_{\dot{f}}).$$

If we use $L = J$ in Theorem 2.5.12 and the same arguments as in the first step, we obtain the desired equation

$$\ker(\operatorname{ind}_R^G \varphi_f) = \ker(\operatorname{ind}_S^G \varphi'_f).$$

This concludes the proof of Equation (2.51). We have shown that the primitive ideal $\ker(\operatorname{ind}_R^G \varphi_f)$, where $R := \exp(\mathfrak{r})$, does not depend on the choice of the polarizing subalgebra $\mathfrak{r}$ for f. □

Remark 2.9.2. Let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair, let $f \in \mathfrak{g}^*$ such that f is faithful on the center of $\mathfrak{g}$, and let $\mathfrak{r}$ be any polarizing subalgebra for f. We have shown in step 1 of the proof of Proposition 2.9.1 that we can always pass from $\mathfrak{r}$ to a polarizing subalgebra $\mathfrak{s}$ for f, such that $\mathfrak{s} \subseteq \mathfrak{n}$, where $\mathfrak{n}$ is defined to be the centralizer of an abelian subgroup $\mathfrak{a}$ of $\mathfrak{z}^2(\mathfrak{g})$.

With the results of Section 2.8 we can now show that the Kirillov map κ, as defined in (2.41), is surjective.

Proposition 2.9.3. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. The Kirillov map*

$$\kappa : \mathfrak{g}^* \to \operatorname{Prim}(C^*(G)),\ f \mapsto \ker(\operatorname{ind}_R^G \varphi_f)$$

is surjective.

Proof. We will prove this proposition by induction on the nilpotence class $l \leq k$ of the group G.

Let $k = 1$. Let $I \in \mathrm{Prim}(C^*(G))$ and let $\pi \in \hat{G}$ with $\ker(\pi) = I$. Then π is a character of the abelian group G and since the map $\exp : \mathfrak{g} \to G$ is in this case an isomorphism of groups, it follows that the map $\psi := \pi \circ \exp$ is a character of $\mathfrak{g}$. But we have $\mathfrak{g}^* \cong \widehat{\mathfrak{g}}$ via the map $f \mapsto \epsilon \circ f$ and thus $\psi \in \widehat{\mathfrak{g}}$ is of the form $\psi = \epsilon \circ f$ for some homomorphism $f \in \mathfrak{g}^*$. Therefore, the character $\pi \in \hat{G}$ is of the form

$$\pi = \epsilon \circ f \circ \log = \varphi_f$$

for some $f \in \mathfrak{g}^*$ and we have proven the surjectivity of κ in the abelian case.

Let $l \geq 2$ and assume the proposition is proven for all nilpotent k-Lie pairs $(H, \mathfrak{h})$ of nilpotence class $m < l$. Let $I \in \mathrm{Prim}(C^*(G))$ and let $\pi \in \widehat{G}$ with $\ker(\pi) = I$.

Suppose first that π is a one-dimensional representation of G. We claim that the map $\psi : \mathfrak{g} \to \mathbb{T}$, defined by

$$\psi(X) = \pi(\exp(X)) \quad \forall\, X \in \mathfrak{g},$$

is a character of $\mathfrak{g}$. For this, recall the Inversion Formula (2.32) of the Campbell-Hausdorff formula. For all $x, y \in G$:

$$\log(x) + \log(y) = \log\bigl(xy \prod_{m=2}^{k} C_m(x, y)\bigr),$$

where each $C_m(x, y)$ is a product of commutators $(z_1, \ldots, z_m)$ of length $m \geq 2$ and where each z_i is equal to some rational power $\lambda \in \mathbb{Z}[\frac{1}{k!}]$ of some product in x and y.

Since π is a one-dimensional homomorphism of G, we have $\pi((v, w)) = 1$ for all $v, w \in G$ and thus we obtain

$$\pi(\prod_{m=2}^{k} C_m(x, y)) = 1$$

for all elements $x, y \in G$. Therefore, we obtain for all $X = \log(x), Y = \log(y) \in \mathfrak{g}$:

$$\begin{aligned} \psi(X + Y) &= \pi(\exp(X + Y)) = \pi\bigl(xy \prod_{m=2}^{k} C_m(x, y)\bigr) = \pi(xy) = \pi(x)\pi(y) \\ &= \psi(X)\psi(Y). \end{aligned}$$

Since $\mathfrak{g}^* \cong \widehat{\mathfrak{g}}$ via the map $f \mapsto \epsilon \circ f$, we can find a homomorphism $f \in \mathfrak{g}^*$ such that

$$\pi = \psi \circ \log = \epsilon \circ f \circ \log = \varphi_f.$$

This proves the surjectivity of κ in the case that π is one-dimensional.

So suppose that π is not one-dimensional. By Schur's Lemma we can find a character $\chi \in \widehat{Z(G)}$ such that $\pi(z) = \chi(z) Id_{H_\pi}$ for all $z \in Z(G)$ and we will identify in the following the restricted representation $\pi|_{Z(G)}$ with this character χ of $Z(G)$. Since the abelian group $Z(G)$ is isomorphic to the abelian group $\mathfrak{z}(\mathfrak{g})$ via the map log, it follows that the dual group of $Z(G)$ is isomorphic to the dual group of $\mathfrak{z}(\mathfrak{g})$. Moreover, we have $\mathfrak{g}^* \cong \widehat{\mathfrak{g}}$ via the map $f \mapsto \epsilon \circ f$ and thus we can find a homomorphism $f \in \mathrm{Hom}(\mathfrak{z}(\mathfrak{g}), \mathfrak{w})$ such that

$$\pi|_{Z(G)} = \chi = \epsilon \circ f \circ \log .$$

Step 1: We consider the case that $f \in \mathrm{Hom}(\mathfrak{z}(\mathfrak{g}), \mathfrak{w})$ is a faithful map.
Let A be a maximal abelian subgroup of $Z^2(G)$ and let N be the centralizer of A. Then N is a closed normal subgroup of G of nilpotence class at most $l-1$ (Lemma 2.8.3) and we have seen in Lemma 2.8.5 that the set $\log(N) =: \mathfrak{n}$ is a subalgebra of $\mathfrak{g}$. The pair $(N, \mathfrak{n})$ is a nilpotent k-Lie pair of nilpotence class at most $l-1$ and Corollary 2.8.9 yields

$$I = \ker(\mathrm{ind}_N^G \pi|_N).$$

Furthermore, we have seen in the proof of Proposition 2.8.13 that

$$\ker(\pi|_N) = \bigcap_{g \in G} gJg^{-1}$$

for some primitive ideal $J \in \mathrm{Prim}(C^*(N))$ and since $\mathrm{ind}_N^G J = \mathrm{ind}_N^G gJg^{-1}$ for all $g \in G$, it follows that

$$I = \mathrm{ind}_N^G(\mathrm{res}_N^G I) = \mathrm{ind}_N^G J.$$

If we apply the induction hypothesis to the nilpotent k-Lie pair $(N, \mathfrak{n})$ and the ideal $J \in \mathrm{Prim}(C^*(N))$, we can find a homomorphism $g \in \mathfrak{n}^*$ and a polarizing subalgebra $\mathfrak{r} \subseteq \mathfrak{n}$ for g such that $J = \ker(\mathrm{ind}_R^N \varphi_g)$, where R and φ_g are defined as usual.

Let $\tilde{g} \in \mathfrak{g}^*$ be an extension of $g \in \mathfrak{n}^*$; the existence of such a homomorphism $\tilde{g}$ is assured by Definition 2.2.5. Since $g|_{\mathfrak{z}(\mathfrak{g})} = f$, it follows that g is faithful on the center of $\mathfrak{g}$ and the polarizing subalgebra $\mathfrak{r}$ for g is at the same time a polarizing subalgebra for $\tilde{g}$ (Corollary 2.8.17). Therefore, we obtain

$$I = \mathrm{ind}_N^G J = \ker(\mathrm{ind}_N^G \mathrm{ind}_R^N \varphi_{\tilde{g}}) = \ker(\mathrm{ind}_R^G \varphi_{\tilde{g}}),$$

which shows that I is in the range of κ, if the homomorphism $f \in \mathfrak{z}(\mathfrak{g})^*$ is faithful.

Step 2: The homomorphism $f \in \mathrm{Hom}(\mathfrak{z}(\mathfrak{g}), \mathfrak{w})$ is not faithful.
By Lemma 2.8.11 we can find a maximal exponentiable normal subgroup J inside the kernel of π. Let $\mathfrak{j} := \log(J)$ be the ideal of $\mathfrak{g}$ corresponding to J. We define $\tilde{G} := G/J$ and $\tilde{\mathfrak{g}} := \mathfrak{g}/\mathfrak{j}$. Then $(\tilde{G}, \tilde{\mathfrak{g}})$ is a nilpotent k-Lie pair and we denote by $q : G \to G/J$ and $q' : \mathfrak{g} \to \mathfrak{g}/\mathfrak{j}$ the canonical quotient maps. Moreover, let $\widetilde{\pi}$ be the

irreducible representation of $\tilde{G}$ induced by π, and choose a homomorphism $\tilde{g} \in \tilde{\mathfrak{g}}^*$ such that

$$\widetilde{\pi}|_{Z(G/I)} = \varphi_{\tilde{g}}.$$

It follows from part (*ii*) of Lemma 2.8.11 that the homomorphism $\tilde{g}$ is faithful on $\mathfrak{z}(\tilde{\mathfrak{g}})$. If we apply the first step to the irreducible representation $\widetilde{\pi}$ of $\tilde{G}$ and the faithful homomorphism $\widetilde{g}|_{\mathfrak{z}(\tilde{\mathfrak{g}})}$, we can find a homomorphism $\tilde{h} \in (\widetilde{\mathfrak{g}})^*$ and a polarizing subalgebra $\widetilde{\mathfrak{r}}$ for $\tilde{h}$ such that

$$\ker(\tilde{\pi}) = \ker(\operatorname{ind}_{\tilde{R}}^{\tilde{G}} \varphi_{\tilde{h}}).$$

Put $h = \tilde{h} \circ q'$ and put $\mathfrak{r} = q'^{-1}(\widetilde{\mathfrak{r}})$. Since $\mathfrak{j} \subseteq \mathfrak{r}$, it follows that $\mathfrak{r}/\mathfrak{j} \cong \widetilde{\mathfrak{r}}$. Furthermore, we denote by $\tilde{R} := \exp(\widetilde{\mathfrak{r}})$ and by $R := \exp(\mathfrak{r})$. Since $R/J \cong \tilde{R}$ we obtain by Theorem 2.5.12,

$$\operatorname{ind}_R^G(\varphi_{\tilde{h}} \circ q) \cong (\operatorname{ind}_{\tilde{R}}^{\tilde{G}} \varphi_{\tilde{h}}) \circ q$$

and thus

$$\ker(\pi) = \ker(\tilde{\pi} \circ q) = \ker((\operatorname{ind}_{\tilde{R}}^{\tilde{G}} \varphi_{\tilde{h}}) \circ q) = \ker(\operatorname{ind}_R^G(\varphi_{\tilde{h}} \circ q)) = \ker(\operatorname{ind}_R^G \varphi_h).$$

This shows that the primitive ideal I is in the range of the map κ. $\square$

Corollary 2.9.4. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. Let H be an exponentiable subgroup of G and let $\pi \in \hat{H}$ be a one-dimensional representation of H. Then π is of the form $\pi = \varphi_f$ for some $f \in \mathfrak{h}^*$, where $\mathfrak{h}$ denotes the subalgebra of $\mathfrak{g}$ corresponding to the subgroup H.*

Proof. This is shown in the proof of Proposition 2.9.3. $\square$

2.10 The Kirillov homeomorphism

2.10.1 A topology for representations of subgroups and the space $\mathcal{S}(G)$

Following the approach of Fell [10], §2, we will define in this subsection the subgroup algebra $\mathcal{A}_s(G)$ of a locally compact group G and we will use this subgroup algebra to introduce a topology on the set of all subgroup representation pairs $\mathcal{S}(G)$.

In the following, let G be a locally compact group and let $\mathcal{K}(G)$ be the family of all closed subgroups of G equipped with the compact-open topology as explained in Section 2.5. A choice of Haar measures in $\mathcal{K}(G)$ is a mapping $K \mapsto \mu_K$ assigning to each K in $\mathcal{K}(G)$ a left Haar measure μ_K on K. Such a choice is called smooth if, for each f in $C_c(G)$, the function $K \mapsto \int_K f(x) d\mu_K(x)$ is continuous on $\mathcal{K}(G)$. It is known that such smooth choices of Haar measures exist [10]. Let Y be the set of all pairs (K, x), where $K \in \mathcal{K}(G)$ and $x \in K$. One can show that Y is a closed subset of $\mathcal{K}(G) \times G$, and hence is itself locally compact in the relative topology. Let $\{\mu_K\}$ be a fixed smooth choice of Haar measures on $\mathcal{K}(G)$. Then one can prove the following.

Lemma 2.10.1. *([10], Lemma 2.1) If $f \in C_c(Y)$, the function*

$$K \mapsto \int_K f(K,x)\, d\mu_K(x)$$

is continuous on $\mathcal{K}(G)$.

Let Δ_K be the modular function for the closed subgroup K of G. Then $(K,x) \mapsto \Delta_K(x)$ is a continuous function on Y. We make $C_c(Y)$ into a normed $*$-algebra with the following definitions of convolution, involution and norm. If $f, g \in C_c(Y)$ we define:

$$\begin{aligned} (f * g)(K,x) &= \textstyle\int_K f(K,y)\, g(K,y^{-1}x)\, d\mu_K(y), \\ f^*(K,x) &= \overline{f(K,x^{-1})}\Delta_K(x^{-1}), \text{ and} \\ \|f\| &= \sup_{K \in \mathcal{K}(G)} \textstyle\int_K |f(K,x)|\, d\mu_K(x). \end{aligned}$$

Each element of $C_c(Y)$ can be thought of as a function on $\mathcal{K}(G)$, whose value at K is in the group algebra of K. The operations are pointwise. The completion of the normed $*$-algebra $C_c(Y)$ with respect to this norm is a Banach $*$-algebra, called the subgroup algebra of G and denoted by $A_s(G)$.

For each K in $\mathcal{K}(G)$, the mapping $\Phi_K : f \mapsto f_K$, initially defined on $C_c(Y)$ by $f_K(x) = f(K,x)$, extends to a continuous $*$-homomorphism of $A_s(G)$ onto a dense subalgebra of $L_1(K, \mu_K)$.

Lemma 2.10.2. *([10], Lemma 2.2) For each f in $A_s(G)$, the map $K \mapsto \|\Phi_K(f)\|$ is continuous on $\mathcal{K}(G)$ and $\|f\| = \sup_K \|\Phi_K(f)\|$.*

Each unitary representation π of a closed subgroup K of G can be lifted to a $*$-representation $W^{K,\pi}$ of $A_s(G)$, namely $W^{K,\pi} = \pi \circ \Phi_K$. It follows from this and Lemma 2.10.2 that $A_s(G)$ is a reduced Banach $*$-algebra. Its C^*-completion, denoted by $C_s^*(G)$, is called the subgroup C^*-algebra of G. The norm of $C_s^*(G)$ will be denoted by $\|.\|_c$. Corresponding representations of $A_s(G)$ and $C_s^*(G)$ will be designated by the same letter and representations of the form $W^{K,\pi}$ will be said to be lifted from K.

Let $\mathcal{S}(G)$ be the set of all subgroup representations, that is, pairs (K,π), where $K \in \mathcal{K}(G)$ and $\pi \in \mathrm{Rep}(K)$. We identify all pairs (K,π) for which the representation π is identically zero; the resulting element being called the zero element of $\mathcal{S}(G)$. (The zero representation is admitted as a unitary, but not as an irreducible representation of a locally compact group.) By the inner hull-kernel topology of $\mathcal{S}(G)$ we mean that topology which makes the one-to-one mapping $(K,\pi) \mapsto W^{K,\pi}$ a homeomorphism with respect to the inner hull-kernel topology of $\mathrm{Rep}(C_s^*(G))$. This is the only topology of $\mathcal{S}(G)$ which will be used. Some properties are the following.

Lemma 2.10.3. *([10], Lemma 2.3) The topology of $\mathcal{S}(G)$ is independent of the particular smooth choice of Haar measures $\{\mu_K\}$.*

Lemma 2.10.4. *([10], Lemma 2.4) The set $\{W^{K,\pi} \mid (K,\pi) \in \mathcal{S}(G)\}$ is closed in* $\mathrm{Rep}(A_s(G))$*. Thus $\mathcal{S}(G)$ is compact.*

Lemma 2.10.5. *([10], Lemma 2.5) The mapping $F : (K,\pi) \mapsto K$ from $\mathcal{S}(G) \setminus \{0\}$ to $\mathcal{K}(G)$ is continuous.*

Lemma 2.10.6. *([10], Lemma 2.6) For each K in $\mathcal{K}(G)$, the mapping $\pi \mapsto (K,\pi)$ is a homeomorphism of* $\mathrm{Rep}(K)$ *into $\mathcal{S}(G)$.*

Lemma 2.10.7. *([10], Lemma 2.8) Every irreducible $*$-representation of $C_s^*(G)$ is of the form $W^{K,\pi}$ for some unique (K,π) in $\mathcal{S}(G)$, $\pi \in \hat{K}$.*

For each K in $\mathcal{K}(G)$, define $\hat{A}_K := \{W^{K,\pi} \mid \pi \in \hat{K}\}$. It follows from Lemma 2.10.5 and Lemma 2.10.7 that the sets $\hat{A}_K$, $K \in \mathcal{K}(G)$, are pairwise disjoint nonempty closed subsets of $\widehat{C_s^*(G)}$ whose union is $\widehat{C_s^*(G)}$.

In §3 of [10], the topology of $\mathcal{S}(G)$ is described in terms of functions of positive type on subgroups. As a consequence, one obtains the continuity of the restriction operation with varying subgroups.

Theorem 2.10.8. *([10], Theorem 3.2) Let*

$$\mathcal{W} := \{(H,K,\pi) \mid (K,\pi) \in \mathcal{S}(G),\ H \in \mathcal{K}(G),\ H \subseteq K\}$$

and suppose $\mathcal{W}$ has the topology relativized from the product $\mathcal{K}(G) \times \mathcal{S}(G)$. Then the map $(H,K,\pi) \mapsto (H,\pi|_H)$ from $\mathcal{W}$ to $\mathcal{S}(G)$ is continuous.

In §4 of [10], a similar result is proven concerning the continuity of inducing representations to larger groups.

Theorem 2.10.9. *([10], Theorem 4.2) Let*

$$\mathcal{W} := \{(H,K,\pi) \mid (K,\pi) \in \mathcal{S}(G),\ H \in \mathcal{K}(G),\ H \supseteq K\}$$

and suppose $\mathcal{W}$ has the topology relativized from the product $\mathcal{K}(G) \times \mathcal{S}(G)$. Then the map $(H,K,\pi) \mapsto (H, \mathrm{ind}_K^H \pi)$ from $\mathcal{W}$ to $\mathcal{S}(G)$ is continuous.

It is a well-known result (see for example [13], §4.1) that the weak*-topology of characters on a locally compact abelian group coincides with the topology of uniform convergence on compact sets. The following lemma is a consequence of the results of this section.

Lemma 2.10.10. *Let (H_n,χ_n) be a sequence in $\mathcal{S}(G)$, let $(H,\chi) \in \mathcal{S}(G)$, and suppose that χ, χ_n, $n \in \mathbb{N}$, are characters. Then the following are equivalent:*

(i) $(H_n,\chi_n) \to (H,\chi)$ *in $\mathcal{S}(G)$.*

(ii) $H_n \to H$ *in $\mathcal{K}(G)$ and for every subsequence (H_{n_k}) of (H_n) and every element $h_{n_k} \in H_{n_k}$ with $h_{n_k} \to h$ for some $h \in H$, one has $\chi_{n_k}(h_{n_k}) \to \chi(h)$ in $\mathbb{C}$.*

2.10.2 Continuity of the Kirillov-orbit map

Let G be a locally compact group. We will denote by $\mathcal{K}(G)$ the set of all closed subgroups of G equipped with the compact-open topology and by $\mathcal{S}(G)$ the set of all subgroup representation pairs of G equipped with the subgroup-representation topology of Fell, as explained in Section 2.5 and in Section 2.10.1, respectively.

Proposition 2.10.11. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. The Kirillov map*

$$\kappa : \mathfrak{g}^* \longrightarrow \operatorname{Prim}(C^*(G)),\ f \mapsto \ker(\operatorname{ind}_R^G \varphi_f), \tag{2.56}$$

is continuous.

Proof. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathfrak{g}^*$ and suppose that $f_n \to f$ in $\mathfrak{g}^*$ for some $f \in \mathfrak{g}^*$. For every $n \in \mathbb{N}$, let $\mathfrak{r}_n$ be a polarizing subalgebra for f_n and define $R_n := \exp(\mathfrak{r}_n)$. We will prove that the sequence of ideals $(\ker(\operatorname{ind}_{R_n}^G \varphi_{f_n}))_{n\in\mathbb{N}}$ converges in $\operatorname{Prim}(C^*(G))$ to an ideal of the form $\ker(\operatorname{ind}_S^G \varphi_f)$, where $S = \exp(\mathfrak{s})$ denotes the subgroup of G, corresponding to some polarizing subalgebra $\mathfrak{s}$ for f.

Since $\mathcal{X}(\mathfrak{g})$ is a compact space with respect to the compact-open topology, as explained in Section 2.10.1, we can assume that $\mathfrak{r}_n \to \mathfrak{r}$ in $\mathcal{X}(\mathfrak{g})$ for some subalgebra $\mathfrak{r}$ of $\mathfrak{g}$ (otherwise we pass to a suitable subsequence). We claim that $\mathfrak{r}$ is f-subordinate. For this, let X and Y be two arbitrary elements of $\mathfrak{r}$. By Proposition 2.5.6 and by passing to a suitable subsequence we can find for every $n \in \mathbb{N}$, elements $X_n, Y_n \in \mathfrak{r}_n$, such that $X = \lim_{n\to\infty} X_n$ and $Y = \lim_{n\to\infty} Y_n$. Since the commutator is continuous, we obtain $[X_n, Y_n] \to [X, Y]$ as $n \to \infty$ and therefore

$$0 = f_n([X_n, Y_n]) \to f([X, Y]).$$

Hence we have $f([X, Y]) = 0$, which proves that $\mathfrak{r}$ is f-subordinate.

It is not clear whether the subalgebra $\mathfrak{r}$ is maximal with respect to the property that it is f-subordinate, but using Zorn's Lemma we can find a maximal f-subordinate subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ with $\mathfrak{r} \subseteq \mathfrak{s}$. Put $S := \exp(\mathfrak{s})$. Then we have $R \subseteq S$ and $\varphi_{f|_\mathfrak{r}} = \varphi_f|_R$. Since $f_n \to f$ in $\mathfrak{g}^*$, we have $\epsilon \circ f_n \to \epsilon \circ f$ in $\widehat{\mathfrak{g}}$ and thus

$$\varphi_{f_n} = \epsilon \circ f_n \circ \log \to \epsilon \circ f \circ \log = \varphi_f|_R.$$

But as $\mathfrak{r}_n \to \mathfrak{r}$ in $\mathcal{X}(\mathfrak{g})$, it follows that $R_n \to R$ in $\mathcal{X}(G)$ and thus

$$(R_n, \varphi_{f_n}) \to (R, \varphi_f|_R) \quad \text{in } \mathcal{S}(G).$$

Since the process of inducing subgroup-representation pairs is continuous (see Theorem 2.10.9), it follows that

$$(G, \operatorname{ind}_{R_n}^G \varphi_{f_n}) \to (G, \operatorname{ind}_R^G \varphi_f|_R) \text{ in } \mathcal{S}(G) \text{ as } n \to \infty.$$

Notice that the representation $\pi_f := \operatorname{ind}_R^G \varphi_f|_R$ does not have to be irreducible. But by Theorem 2.5.17 we have $\varphi_f \prec \operatorname{ind}_R^S \varphi_f|_R$ and thus we obtain, by the continuity of the inducing process,

$$\widetilde{\pi_f} := \operatorname{ind}_S^G \varphi_f \prec \operatorname{ind}_S^G \operatorname{ind}_R^S \varphi_f|_R \cong \operatorname{ind}_R^G \varphi_f|_R = \pi_f.$$

Since $\pi_{f_n} \to \pi_f$ and $\widetilde{\pi_f} \prec \pi_f$, it follows from Proposition 2.5.4 that $\pi_{f_n} \to \widetilde{\pi_f}$. Note that the representation $\widetilde{\pi_f}$ is irreducible. Therefore, we have

$$(G, \operatorname{ind}_{R_n}^G \varphi_{f_n}) \to (G, \operatorname{ind}_S^G \varphi_f) \text{ in } \mathcal{S}(G)$$

and thus

$$\ker(\operatorname{ind}_{R_n}^G \varphi_{f_n}) \to \ker(\operatorname{ind}_S^G \varphi_f) \text{ in } \operatorname{Prim}(C^*(G)) \text{ as } n \to \infty.$$

$\square$

We may now show that the Kirillov-orbit map $\tilde{\kappa}$, as defined in (2.42), is in fact a well-defined map.

Corollary 2.10.12. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. The Kirillov-orbit map*

$$\tilde{\kappa} : \mathfrak{g}^* /_\sim \to \operatorname{Prim}(C^*(G)),\ \mathcal{O} \mapsto \ker(\operatorname{ind}_R^G \varphi_f),$$

where $f \in \mathfrak{g}^*$ is any chosen representative of the coadjoint quasi-orbit $\mathcal{O}$, is a well-defined map.

Proof. By Proposition 2.9.1 it suffices to prove that if f and f' are two homomorphisms of $\mathfrak{g}^*$, such that f and f' are in the same quasi-orbit in $\mathfrak{g}^*$ under the coadjoint action, then $\ker(\operatorname{ind}_R^G \varphi_f) = \ker(\operatorname{ind}_{R'}^G \varphi_{f'})$, where $\mathfrak{r} = \log(R)$ and $\mathfrak{r}' = \log(R')$ are any chosen polarizing subalgebras for f and f', respectively.

Let $f \in \mathfrak{g}^*$ and let $\mathfrak{r}$ be a polarizing subalgebra for f. Let $f' \in \mathfrak{g}^*$ be an element of the same quasi-orbit as f. We have shown in Lemma 2.6.13 that we can choose for every element $g \in G(f)$ (the G-orbit of f under the coadjoint action Ad^*) a suitable polarizing subalgebra $\mathfrak{s}$ for g such that $\operatorname{ind}_R^G \varphi_f \cong \operatorname{ind}_S^G \varphi_g$, where $S = \exp(\mathfrak{s})$. But we have $f' \in \overline{G(f)}$ and since the Kirillov map κ is continuous (Proposition 2.10.11), we can choose for the homomorphism $f' \in \mathfrak{g}^*$ a polarizing subalgebra $\mathfrak{r}'$ such that

$$\ker(\operatorname{ind}_R^G \varphi_f) = \ker(\operatorname{ind}_{R'}^G \varphi_{f'}),$$

where $R' := \exp(\mathfrak{r}')$. Since we have proven in Proposition 2.9.1 that the primitive ideal $\ker(\operatorname{ind}_{R'}^G \varphi_{f'})$ does not depend on the choice of the polarizing subalgebra $\mathfrak{r}'$ for f', we obtain

$$\ker(\operatorname{ind}_R^G \varphi_f) = \ker(\operatorname{ind}_{R'}^G \varphi_{f'})$$

for all polarizing subalgebras $\mathfrak{r}'$ for f'. This proves that $\tilde{\kappa}$ is well-defined. $\square$

We obtain now as a direct consequence of Proposition 2.10.13 that the Kirillov-orbit map is surjective.

Corollary 2.10.13. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. The Kirillov-orbit map*

$$\tilde{\kappa} : \mathfrak{g}^* /_{\sim} \to \mathrm{Prim}(C^*(G)), \ \mathcal{O} \mapsto \ker(\mathrm{ind}_R^G \varphi_f),$$

where $f \in \mathfrak{g}^*$ is any chosen representative of the quasi-orbit $\mathcal{O}$, is surjective.

In order to prove the continuity of the Kirillov-orbit map, we want to observe first some facts about topological spaces and the T_0-axiom.

Definition 2.10.14. If X is a topological space, then the "T_0-ization" of X is the quotient space $(X)^{\sim}$, where $\sim$ is the equivalence relation of X defined by $x \sim y$ if $\overline{\{x\}} = \overline{\{y\}}$. We give $(X)^{\sim}$ the quotient topology, which is the largest topology making the quotient map $q : X \to (X)^{\sim}$ continuous.

Lemma 2.10.15. *If X is a topological space, then $(X)^{\sim}$ is a topological space satisfying the T_0-axiom. If Y is any topological T_0 space and if $f : X \to Y$ is continuous, then there is a continuous map $f' : (X)^{\sim} \to Y$ such that the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "q"'] & Y \\ (X)^{\sim} \arrow[ur, "f'"'] & \end{tikzcd}$$

commutes.

Proof. Suppose that $q(x)$ and $q(y)$ are distinct elements in $(X)^{\sim}$. If $\overline{\{x\}} \subset \overline{\{y\}}$, then $U = X \setminus \overline{\{x\}}$ is a saturated open set containing y but not x. Since $q^{-1}(q(U)) = U$, $q(U)$ is an open set in $(X)^{\sim}$ containing $q(x)$ but not $q(y)$. If $\overline{\{x\}} \not\subseteq \overline{\{y\}}$, then $V = X \setminus \overline{\{y\}}$ is a saturated open set containing x but not y. Thus $q(V)$ is an open set containing $q(x)$ but not $q(y)$. Hence $(X)^{\sim}$ is a T_0 space.

Now suppose that Y is T_0, and that $f : X \to Y$ is continuous. If $f(x) \neq f(y)$, then, interchanging x and y if necessary, there is an open set in X containing x but not containing y. Thus $x \notin \overline{\{y\}}$ and $q(x) \neq q(y)$. It follows that there is a well-defined function $f' : (X)^{\sim} \to Y$, given by $f'(q(x)) = f(x)$. If U is open in Y, then $q^{-1}(f'^{-1}(U)) = f^{-1}(U)$. Thus $f'^{-1}(U)$ is open and f' is continuous. $\square$

If X is a topological G-space, then the quasi-orbit space, which we denote in the following by $X/_{\sim}$, is the "T_0-ization" of the orbit space $G \setminus X$. The orbit map $p : X \to G \setminus X$ is continuous and open, where the orbit space $G \setminus X$ is equipped with the quotient topology. Therefore, the quasi-orbit map $\tilde{p} : X \to X/_{\sim}$ is, as the composition of the orbit map and the quotient map, continuous as well.

Consider now our situation of a nilpotent k-Lie pair $(G, \mathfrak{g})$. The group G acts on the space $X = \mathfrak{g}^*$ by the coadjoint action Ad^*, as explained in Section 2.6, and we denote by $G \setminus \mathfrak{g}^*$ the orbit space of $\mathfrak{g}^*$ with respect to this action. The space $Y = \mathrm{Prim}(C^*(G))$ is a topological space satisfying the T_0-axiom. Let $\kappa' : G \setminus \mathfrak{g}^* \to \mathrm{Prim}(C^*(G))$ denote the map such that the diagram

$$\begin{tikzcd} \mathfrak{g}^* \arrow[r, "\kappa"] \arrow[d, "p"] & \mathrm{Prim}(C^*(G)) \\ G \setminus \mathfrak{g}^* \arrow[ur, "\kappa'"'] & \end{tikzcd}$$

is commutative. Since the orbit map p is open and since the Kirillov map κ is continuous (Proposition 2.10.11), it follows that κ' is a continuous map. Let $\tilde{\kappa}$ denote the Kirillov-orbit map. Then we have a commutative diagram

$$\begin{tikzcd} G \setminus \mathfrak{g}^* \arrow[r, "\kappa'"] \arrow[d, "q"] & \mathrm{Prim}(C^*(G)) \\ \mathfrak{g}^* /_{\sim} \arrow[ur, "\tilde{\kappa}"'] & \end{tikzcd}$$

and it follows from Lemma 2.10.15 that the Kirillov-orbit map $\tilde{\kappa}$ is continuous. Therefore, we have proven the following fact.

Corollary 2.10.16. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. The Kirillov-orbit map*

$$\tilde{\kappa} : \mathfrak{g}^* /_{\sim} \to \mathrm{Prim}(C^*(G)),\ \mathcal{O} \mapsto \ker(\mathrm{ind}_R^G \varphi_f),$$

where $f \in \mathfrak{g}^*$ is any chosen representative of the quasi-orbit $\mathcal{O}$, is continuous.

2.10.3 Injectivity of the Kirillov-orbit map and continuity of the inverse map

In the following, let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair of nilpotence class $l \geq 2$. In this subsection we study the convergence of sequences $\{\sigma_n\}$ in $\hat{G}$ by studying sequences of subgroup-representation pairs $\{(H_n, \pi_n)\}$, where H_n is a closed subgroup of G and $\pi_n \in \hat{H}_n$. Following the approach of Joy [20], we develop necessary and sufficient conditions for the convergence of sequences in $\hat{G}$, in terms of convergence of associated sequences of subgroup-representation pairs in the subgroup-representation topology of Fell [10], as explained in Subsection 2.10.1. Using these results we will prove that, under certain additional assumptions on the group G, the Kirillov-orbit map $\tilde{\kappa}$ is one-to-one and bi-continuous. We denote by $\mathcal{K}(G)$ the set of all closed subgroups of G equipped with the compact-open topology and by $\mathcal{S}(G)$ the set of all subgroup-representation pairs of G equipped with the subgroup-representation pair topology of Fell.

In several proofs of this subsection we will use instead of induction on the nilpotence class of the given group, a more applicable induction method, for which we introduce the degree of a pair $(H,\pi) \in \mathcal{S}(G)$. Recall that a closed subgroup H of G is called exponentiable, if $\log(H)$ is a subalgebra of $\mathfrak{g}$.

Definition 2.10.17. A pair $(H,\pi) \in \mathcal{S}(G)$, where H is an exponentiable subgroup of G and $\pi \in \hat{H}$, is said to be of degree $c \in \mathbb{N}$, if H/J_π is of nilpotence class c, where J_π is the maximal exponentiable normal subgroup of H inside the kernel of π as in Lemma 2.8.11.

Lemma 2.10.18. *Let $(H,\pi) \in \mathcal{S}(G)$, where H is an exponentiable subgroup of G and $\pi \in \hat{H}$. Then the following are equivalent:*

(i) (H,π) is of degree one.

(ii) The irreducible representation π is one-dimensional.

Proof. Suppose that (H,π) is of degree one. Then H/J_π is an abelian group, where J_π denotes the maximal exponentiable normal subgroup of H inside the kernel of π as in Lemma 2.8.11. But $J_\pi \subseteq \ker(\pi)$ and thus $H/\ker(\pi)$ is an abelian group. Since π is irreducible, it follows that π is one-dimensional.

Conversely, suppose that $\pi \in \hat{H}$ is one-dimensional and let $\mathfrak{h} := \log(H)$ be the subalgebra of $\mathfrak{g}$ corresponding to H. Since H is an exponentiable subgroup of G it follows from Lemma 2.4.3 that $\overline{(H,H)}$ is a normal exponentiable subgroup of H. Since π is one-dimensional, we have $\pi((x,y)) = 1$ for all $x, y \in H$, and since π is continuous it follows that $\pi(\overline{(H,H)}) \equiv 1$. Therefore, we obtain $\overline{(H,H)} \subseteq J_\pi$. But $H/\overline{(H,H)}$ is an abelian group and thus H/J_π is an abelian group, proving that (H,π) is of degree one. □

Definition 2.10.19. Let $(H,\pi) \in \mathcal{S}(G)$, where H is an exponentiable subgroup of G and $\pi \in \hat{H}$ is not a character. A pair $(L,\rho) \in \mathcal{S}(G)$ is called inducing for (H,π) if L is an exponentiable subgroup of H, $\rho \in \hat{L}$, $\ker(\pi) = \ker(\operatorname{ind}_L^H \rho)$, and the degree of (L,ρ) is less than the degree of (H,π).

Lemma 2.10.20. *For every pair $(H,\pi) \in \mathcal{S}(G)$, where H is an exponentiable subgroup of G and $\pi \in \hat{H}$ is not a character, there exists an inducing pair $(L,\rho) \in \mathcal{S}(G)$.*

Proof. Let $(H,\pi) \in \mathcal{S}(G)$, where H is an exponentiable subgroup of G and $\pi \in \hat{H}$ is not a character. Let J_π be the maximal exponentiable normal subgroup of H inside the kernel of π as in Lemma 2.8.11, and define $\mathfrak{j}_\pi := \log(J_\pi)$ to be the subalgebra of $\mathfrak{h} = \log(H)$ corresponding to J_π. Let $q : H \to H/J_\pi$ be the canonical quotient map and put $\tilde{H} := H/J_\pi$ and $\tilde{\mathfrak{h}} := \mathfrak{h}/\mathfrak{j}_\pi$. Then $(\tilde{H}, \tilde{\mathfrak{h}})$ is a nilpotent k-Lie pair (Lemma 2.4.1). Choose $\tilde{f} \in \tilde{\mathfrak{h}}^*$ with $\ker(\tilde{\pi}) = \ker(\operatorname{ind}_{\tilde{R}}^{\tilde{H}} \varphi_{\tilde{f}})$, where $\tilde{R} = \exp(\tilde{\mathfrak{r}})$ for some polarizing subalgebra $\tilde{\mathfrak{r}}$ for $\tilde{f}$. The existence of such a homomorphism $\tilde{f}$ is assured by Proposition 2.9.3, and since $\tilde{\pi}|_{Z(\tilde{G})} = \varphi_{\tilde{f}}$, it follows from Lemma 2.8.11 that $\tilde{f}$ is faithful on $\mathfrak{z}(\tilde{\mathfrak{h}})$. Let $\tilde{L}$ be the centralizer of some maximal abelian subgroup of $Z^2(\tilde{H})$ and put $\tilde{\ell} = \log(\tilde{L})$, the subalgebra of $\tilde{\mathfrak{h}}$ corresponding to $\tilde{L}$. Notice that we have shown in Lemma 2.8.3 that the nilpotence class of $\tilde{L}$ is less than the nilpotence class of $\tilde{H}$. Since $\tilde{f}$ is faithful on the center of $\tilde{\mathfrak{g}}$, we can assume without loss of generality that the polarizing subalgebra $\tilde{\mathfrak{r}}$ for $\tilde{f}$ is contained in $\tilde{\ell}$ (Remark 2.9.2), and it follows from Proposition 2.8.16 that the induced representation $\tilde{\rho} := \operatorname{ind}_{\tilde{R}}^{\tilde{L}} \varphi_{\tilde{f}|_\ell}$ is an irreducible representation of $\tilde{L}$. Furthermore, we have

$$\ker(\operatorname{ind}_{\tilde{L}}^{\tilde{H}} \tilde{\rho}) = \ker(\operatorname{ind}_{\tilde{L}}^{\tilde{H}} \operatorname{ind}_{\tilde{R}}^{\tilde{L}} \varphi_{\tilde{f}|_\ell}) = \ker(\operatorname{ind}_{\tilde{R}}^{\tilde{H}} \varphi_{\tilde{f}}) = \ker(\tilde{\pi}). \tag{2.57}$$

Define $L := q^{-1}(\tilde{L})$ and $\rho := \tilde{\rho} \circ q$. Clearly, L is an exponentiable normal subgroup of H and $\rho \in \hat{L}$. Furthermore, we obtain with (2.57) and Theorem 2.5.12:

$$\ker(\pi) = \ker(\tilde{\pi} \circ q) = \ker((\operatorname{ind}_{\tilde{L}}^{\tilde{H}} \tilde{\rho}) \circ q) = \ker(\operatorname{ind}_L^H (\tilde{\rho} \circ q)) = \ker(\operatorname{ind}_L^H \rho).$$

Since $J_\pi \subseteq \ker(\rho)$ it follows that $J_\pi \subseteq J_\rho$, where J_ρ denotes the maximal exponentiable subgroup inside the kernel of ρ. Therefore, the nilpotence class of L/J_ρ is less than or equal to the nilpotence class of $L/J_\pi = \tilde{L}$, which is less than the nilpotence class of $H/J_\pi = \tilde{H}$. This proves that the pair (L, ρ) is an inducing pair for (H, π). $\square$

Remark 2.10.21. Let $(H, \pi) \in \mathcal{S}(G)$, where H is an exponentiable subgroup of G and $\pi \in \hat{H}$ is not a character. Then there exists an inducing pair (L, ρ) for (H, π) and by the proof of Lemma 2.10.20 we can choose the exponentiable subgroup L in the following way. Let J_π denote the maximal exponentiable normal subgroup inside the kernel of π as in Lemma 2.8.11 and let $q : H \to H/J_\pi := \tilde{H}$ denote the canonical quotient map. Put $L = q^{-1}(\tilde{L})$, where $\tilde{L}$ is the centralizer of some maximal abelian subgroup of $\tilde{H}$. We observe that since $\tilde{H}/\tilde{L} \cong H/L$ and $\tilde{H}/\tilde{L}$ is abelian, it follows that H/L is abelian, but the nilpotence class of L is not necessarily less than the nilpotence class of H.

Furthermore, if $\ker(\pi) = \ker(\operatorname{ind}_R^H \varphi_f)$ for some homomorphism $f \in \mathfrak{h}^*$, where $\mathfrak{h}$ denotes the subalgebra of $\mathfrak{g}$ corresponding to H, then we can choose the irreducible representation $\rho \in \hat{L}$ to be of the form $\rho = \operatorname{ind}_R^L \varphi_{f|_\ell}$, where ℓ denotes the subalgebra of $\mathfrak{h}$ corresponding to L.

Lemma 2.10.22. *Let H be a normal, exponentiable subgroup of G of nilpotence class at most $l-1$ and suppose that G/H is abelian. Let $\pi \in \hat{G}$ and let $f \in \mathfrak{g}^*$ with $\ker(\pi) = \ker(\operatorname{ind}_R^G \varphi_f)$, where φ_f and R are defined as in Remark 2.6.3. Let $\mathfrak{h} := \log(H)$ be the subalgebra associated to H. If $\rho \in \hat{H}$ with $\ker(\rho) = \ker(\operatorname{ind}_{R'}^H \varphi_{f|_\mathfrak{h}})$ for some polarizing subalgebra $\mathfrak{r}' = \log(R')$ for $f|_\mathfrak{h}$, then $\pi \prec \operatorname{ind}_H^G \rho$.*

Moreover, if the product of the closed normal subgroup H and any exponentiable subgroup of G is closed, then $\rho \prec \pi|_H$, i.e., $\rho \in \operatorname{Sp}(\pi|_H)$.

Proof. Observe first that if $\mathfrak{r}'$ denotes a polarizing subalgebra for the homomorphism $f|_{\mathfrak{h}} \in \mathfrak{h}^*$, then $\mathfrak{r}'$ is of the form $\mathfrak{r}' = \mathfrak{s} \cap \mathfrak{h}$ for some polarizing subalgebra $\mathfrak{s}$ for $f \in \mathfrak{g}^*$. Since by Proposition 2.9.1 the kernel of the induced character $\operatorname{ind}_R^H \varphi_f$ does not depend on the choice of the polarizing subalgebra $\mathfrak{r}$ for f, we can assume without loss of generality that $\mathfrak{s} = \mathfrak{r}$, and hence $\mathfrak{r}' = \mathfrak{r} \cap \mathfrak{h}$.

Let $\rho \in \hat{H}$ be an irreducible representation of H with

$$\ker(\rho) = \ker(\operatorname{ind}_{R\cap H}^H \varphi_{f|_{\mathfrak{h}}}).$$

Applying Theorem 2.5.17 to the character $\varphi_f \in \hat{R}$ yields

$$\varphi_f \prec \operatorname{ind}_{R\cap H}^R(\varphi_f|_{R\cap H}) = \operatorname{ind}_{R\cap H}^R \varphi_{f|_{\mathfrak{r}\cap\mathfrak{h}}}$$

and since the inducing operation preserves weak containment we obtain

$$\operatorname{ind}_R^G \varphi_f \prec \operatorname{ind}_{R\cap H}^G \varphi_{f|_{\mathfrak{r}\cap\mathfrak{h}}}.$$

Therefore, we obtain

$$\begin{aligned}\ker(\pi) &= \ker(\operatorname{ind}_R^G \varphi_f) \supseteq \ker(\operatorname{ind}_{R\cap H}^G \varphi_{f|_{\mathfrak{r}\cap\mathfrak{h}}}) = \ker(\operatorname{ind}_H^G \operatorname{ind}_{R\cap H}^H \varphi_{f|_{\mathfrak{r}\cap\mathfrak{h}}}) \\ &= \operatorname{ind}_H^G \ker(\operatorname{ind}_{R\cap H}^H \varphi_{f|_{\mathfrak{r}\cap\mathfrak{h}}}) = \operatorname{ind}_H^G \ker(\rho) = \ker(\operatorname{ind}_H^G \rho),\end{aligned}$$

and hence $\pi \prec \operatorname{ind}_H^G \rho$.

Suppose now that the product of the normal, exponentiable subgroup H and any exponentiable subgroup of G is closed. Since $R = \exp(\mathfrak{r})$ is an exponentiable subgroup of G, it follows that the product HR is closed. Consider the irreducible representations

$$\rho' := \operatorname{ind}_{R\cap H}^H \varphi_{f|_{\mathfrak{h}}} \quad \text{and} \quad \sigma := \operatorname{ind}_R^{HR} \varphi_f.$$

We claim that $\ker(\rho') = \ker(\sigma|_H)$. For this, observe that both representations, ρ' and σ, are induced and we have seen in Remark/Definition 2.5.9 that σ is of the form

$$\sigma : HR \to \mathcal{U}(H_\sigma),\ \sigma(g)(\xi)(t) = \xi(g^{-1}t),$$

where the Hilbert space H_σ can be described as the closure (with respect to a certain scalar product, see Remark 2.5.9) of the space of complex-valued functions

$$\begin{aligned}C_c(HR, \varphi_f) := \{&\xi : HR \to \mathbb{C} \text{ continuous } \mid \xi(gr) = \varphi_f(r^{-1})\xi(g)\ \forall\, g \in HR, r \in R \\ &\text{and with } q(\operatorname{supp}(\xi)) \subseteq HR/R \text{ compact}\},\end{aligned}$$

where $q : HR \to HR/R$ denotes the canonical quotient map. In the same way we have

$$\rho' : H \to \mathcal{U}(H_{\rho'}),\ \rho'(h)(\xi)(s) = \xi(h^{-1}s),$$

where the Hilbert space $H_{\rho'}$ can be described as the closure of the space of complex-valued functions

$$\begin{aligned} C_c(H, \varphi_f) \;&:=\; \{\xi : H \to \mathbb{C} \text{ continuous } \mid \xi(hr) = \varphi_f(r^{-1})\xi(h) \; \forall\, h \in H, r \in R \cap H \\ &\qquad \text{and with } q'(\mathrm{supp}(\xi)) \subseteq H/(R \cap H) \text{ compact}\}, \end{aligned}$$

where $q' : H \to H/(R \cap H)$ denotes the canonical quotient map.

The irreducible representations σ and ρ' determine irreducible representations of the group C^*-algebras $C^*(HR)$ and $C^*(H)$, still denoted by σ and ρ', respectively, in the usual way. We will prove now that one can identify functions in $C_c(HR, \varphi_f)$ with functions in $C_c(H, \varphi_f)$ by restricting them to the closed subgroup H. For this, we observe that if $\xi \in C_c(HR, \varphi_f)$, then $\xi|_H : H \to \mathbb{C}$ is a continuous map satisfying $\xi|_H(hr) = \varphi_f(r^{-1})\xi(h)$ for all $h \in H$ and $r \in R \cap H$. So it remains to show that there exists a compact set $K \subseteq H/R \cap H$ with $q'(\mathrm{supp}(\xi|_H)) = K$.

Since H is a normal subgroup of G, we have $HR = RH$. As the subgroup HR was assumed to be closed, it is locally compact and the map

$$\varphi : H \to HR/R, \; h \mapsto hR$$

defines a surjective, continuous map, which annihilates $R \cap H$. This yields a bijective, continuous map

$$\Phi : H/H \cap R \to HR/R, \; h \mapsto hR, \tag{2.58}$$

and one can show that Φ is a homeomorphism (see for example [12], Corollary 3.13). So if $\xi \in C_c(HR, \varphi_f)$, then $q(\mathrm{supp}(\xi)) \subseteq HR/R$ is compact, and we can realize the image of $\mathrm{supp}(\xi|_H)$ under the quotient map $q' : H \to H/H \cap R$ as the compact set

$$K := \Phi^{-1}(q(\mathrm{supp}(\xi))) \subseteq H/H \cap R.$$

Therefore, we obtain a well-defined map

$$\iota : C_c(HR, \varphi_f) \to C_c(H, \varphi_f), \; \xi \mapsto \xi|_H,$$

and we claim that ι is a bijective isometry.

(1) In order to prove the injectivity of ι, let $\xi, \eta \in C_c(HR, \varphi_f)$ with $\xi|_H = \eta|_H$. Using the definition of the space $C_c(HR, \varphi_f)$, we obtain for all $h \in H$ and $r \in R$,

$$\xi(hr) = \varphi_f(r^{-1})\xi(h) = \varphi_f(r^{-1})\eta(h) = \eta(hr),$$

which proves that $\xi = \eta$ on HR.

(2) To show that ι is onto, we prove that

$$\tau : C_c(H, \varphi_f) \to C_c(HR, \varphi_f), \; \xi \mapsto \tilde{\xi}, \; \text{where } \tilde{\xi}(hr) = \varphi_f(r^{-1})\xi(h)$$

is a well-defined map, being an inverse map of ι. For this, we need to show first that for any $\xi \in C_c(H, \varphi_f)$, the value $\tilde{\xi}(g) = \tilde{\xi}(hr)$ does not depend on the choice of the decomposition of the element $g \in HR$ into the product hr. So let $g \in HR$ with $g = hr = h'r'$ for some $h, h' \in H$ and $r, r' \in R$. Then $h' = hrr'^{-1}$ and since $rr'^{-1} \in R \cap H$ we obtain

$$\begin{aligned} \tilde{\xi}(h'r') &= \varphi_f(r'^{-1})\xi(h') = \varphi_f(r'^{-1})\xi(hrr'^{-1}) = \varphi_f(r'^{-1})\varphi_f((rr'^{-1})^{-1})\xi(h) \\ &= \varphi_f(r^{-1})\xi(h) = \tilde{\xi}(hr). \end{aligned}$$

Furthermore, if $\xi \in C_c(H, \varphi_f)$, then $\tilde{\xi} : HR \to \mathbb{C}$ is a continuous map which satisfies, for all $g = hs \in HR$ and $r \in R$,

$$\begin{aligned} \tilde{\xi}(gr) &= \tilde{\xi}(hsr) = \varphi_f((sr)^{-1})\xi(h) = \varphi_f(r^{-1})\varphi_f(s^{-1})\xi(h) \\ &= \varphi_f(r^{-1})\xi(hs) = \varphi_f(r^{-1})\xi(g). \end{aligned}$$

Using the homeomorphism defined in (2.58), for every function $\xi \in C_c(H, \varphi_f)$, we can identify the set $q(\mathrm{supp}(\tilde{\xi})) \subseteq HR/R$ with the compact set $q'(\mathrm{supp}(\xi)) \subseteq H/H \cap R$. This proves that τ is a well-defined map. But we have $\tilde{\xi}|_H = \xi$ and thus ι is a surjective map.

(3) It remains to show that ι is an isometry, i.e. we need to prove

$$\langle \xi, \eta \rangle = \langle \iota(\xi), \iota(\eta) \rangle \quad \text{for all } \xi, \eta \in C_c(HR, \varphi_f). \tag{2.59}$$

Recall that if $\xi, \eta \in C_c(HR, \varphi_f)$, then

$$\langle \xi, \eta \rangle = \int_{HR} \beta(g) \langle \xi(g), \eta(g) \rangle \, dg,$$

where $\beta : HR \to [0, \infty)$ is a Bruhat-section for HR, i.e., β is a continuous function satisfying

(i) $\mathrm{supp}(\beta) \cap CR$ is compact for all $C \subseteq HR$ compact and
(ii) $\int_R \beta(gr)\, dr = 1$ for all $g \in HR$.

One can show that such a map always exists. So in order to prove Equation (2.59), we need to understand the connection between a Bruhat-section for the product HR and a Bruhat-section for H. Given a Bruhat-section β for H, one can construct a Bruhat-section $\tilde{\beta}$ for HR in the following way.

Since H is a normal subgroup of G, the group R acts on H by conjugation, and the map

$$\Lambda : H \rtimes R \to HR, (h, r) \mapsto hr$$

defines a continuous, surjective homomorphism. Indeed, we have

$$\Lambda((h, r), (h'r')) = \Lambda((hrh'r^{-1}, rr')) = hrh'r' = \Lambda((h, r))\Lambda((h', r'))$$

for all pairs $(h,r),(h',r') \in H \rtimes R$. As

$$\ker(\Lambda) = \{(h,r) \mid h = r^{-1}\} = \{(r,r^{-1}) \mid r \in H \cap R\} \cong R \cap H,$$

we obtain

$$HR \cong H \rtimes R/(H \cap R) \quad \text{(as topological groups).}$$

Let β be a Bruhat-section for H. Then $\beta : H \to [0,\infty)$ is a continuous map satisfying

(i) $\operatorname{supp}(\beta) \cap K(H \cap R)$ is compact for all $K \subseteq H$ compact and
(ii) $\int_{H\cap R} \beta(hr)\, dr = 1$ for all $h \in H$.

Choose a function $\varphi \in C_c(R)$ with $\int_R \varphi(r)dr = 1$ and define

$$\gamma : H \rtimes R \to [0,\infty),\ \gamma((h,r)) = \beta(h)\varphi(r).$$

We claim that the map $\tilde{\beta} : HR \to [0,\infty)$, defined by

$$\tilde{\beta}(hr) = \int_{H\cap R} \gamma((hl, l^{-1}r))\, dl = \int_{H\cap R} \beta(hl)\varphi(l^{-1}r)\, dl$$

is a Bruhat-section for HR.

Using the left invariance property of the Haar measure, we obtain for all $h \in H$;

$$\begin{aligned}\int_R \tilde{\beta}(hr)\, dr &= \int_R \int_{R\cap H} \beta(hl)\varphi(l^{-1}r)\, dl\, dr = \int_{R\cap H} \beta(hl) \int_R \varphi(l^{-1}r)\, dr\, dl \\ &= \int_{R\cap H} \beta(hl)\, dl = 1.\end{aligned}$$

So it remains to prove that $\operatorname{supp}(\tilde{\beta}) \cap CR$ is a compact subset of HR for all compact sets $C \subseteq HR$. For this, let C be an arbitrary compact subset of HR. Using the homeomorphism defined in Equation (2.58), one can show that $CR = KR$ for some compact set $K \subseteq H$. We need to analyze the support of $\tilde{\beta}$. Recall that $\tilde{\beta}(hr) = \int_{H\cap R} \beta(hl)\varphi(l^{-1}r)dl$. We define

$$L_K := \{l \in H \cap R \mid \exists\, h \in K \text{ s.t. } hl \in \operatorname{supp}(\beta)\} = (H \cap R) \cap K^{-1} \operatorname{supp}(\beta).$$

Since K (and hence K^{-1}) is compact, it follows that L_K is closed. But we have

$$(H \cap R) \cap K^{-1} \operatorname{supp}(\beta) \subseteq K^{-1}(K(H \cap R) \cap \operatorname{supp}(\beta))$$

and since the set $K(H \cap R) \cap \operatorname{supp}(\beta)$ is compact by assumption, it follows that $K^{-1}(K(H \cap R) \cap \operatorname{supp}(\beta))$ is, as a closed subset of a compact set, itself compact. Hence L_K is compact. Furthermore, the set

$$R_K := \{r \in R \mid \exists\, l \in L_K \text{ s.t. } l^{-1}r \in \operatorname{supp}(\varphi)\} = L_K \operatorname{supp}(\varphi)$$

is, as the product of compact sets, itself compact. Since

$$\operatorname{supp}(\tilde{\beta}) \cap CR = \operatorname{supp}(\tilde{\beta}) \cap KR \subseteq K(L_K \operatorname{supp}(\varphi)),$$

it follows that $\operatorname{supp}(\tilde{\beta}) \cap CR$ is compact, proving that $\tilde{\beta}$ is a Bruhat-section for HR.

Now, let $\xi, \eta \in C_c(RH, \varphi_f)$. We want to observe that for all $h \in H$ and $r \in R$, the scalar product $\langle \xi(hr), \eta(hr) \rangle$ is not only independent of r, i.e.

$$\langle \xi(hr), \eta(hr) \rangle = \langle \varphi_f(r^{-1})\xi(h), \varphi_f(r^{-1})\eta(h) \rangle = \langle \xi(h), \eta(h) \rangle,$$

but also invariant under multiplication by elements of the intersection $H \cap R$ in the following sense. For every $l \in R \cap H$, we have

$$\langle \xi(hr), \eta(hr) \rangle \;=\; \langle \tilde{\xi}((h,r)), \tilde{\eta}((h,r)) \rangle = \langle \tilde{\xi}((hl, l^{-1}r)), \tilde{\eta}((hl, l^{-1}r)) \rangle,$$

where $\tilde{\xi}((h,r)) := \xi(hr)$ and $\tilde{\eta}((h,r)) := \eta(hr)$. Applying these facts and the formula of Weil for unimodular groups;

$$\int_G \varphi(g)dg = \int_{G/N} \int_N \varphi(gn)dn \; d(gN),$$

to the group $G = H \rtimes R$ with normal subgroup $N = H \cap R$ yields

$$\begin{aligned}
\langle \xi, \eta \rangle &= \int_{HR} \tilde{\beta}(hr) \langle \xi(hr), \eta(hr) \rangle \; d(hr) \\
&= \int_{HR} \int_{H \cap R} \beta(hl)\varphi(l^{-1}r) \; dl \; \langle \xi(hr), \eta(hr) \rangle \; d(hr) \\
&= \int_{H \rtimes R} \beta(h)\varphi(r) \langle \xi(hr), \eta(hr) \rangle \; d(h,r) \\
&= \int_H \int_R \beta(h) \langle \xi(h), \eta(h) \rangle \varphi(r) \; dr \; dh = \int_H \beta(h) \langle \xi(h), \eta(h) \rangle \; dh \\
&= \langle \iota(\xi), \iota(\eta) \rangle.
\end{aligned}$$

Therefore, we have proven that the map $\iota : C_c(HR, \varphi_f) \to C_c(H, \varphi_f), \xi \mapsto \xi|_H$ is a bijective isometry. Since both spaces, $C_c(HR, \varphi_f)$ and $C_c(H, \varphi_f)$, are dense subspaces of the Hilbert spaces H_σ and $H_{\rho'}$, respectively, it follows that the map ι extends to a bijective unitary operator $\iota : H_\sigma \to H_{\rho'}$ satisfying the property

$$\iota\sigma|_H(\eta) = \rho'(\eta)\iota \quad \text{for all } \eta \in C^*(H).$$

But this means that the representations $\sigma|_H$ and ρ' are unitarily equivalent, and in particular we obtain

$$\ker(\sigma|_H) = \ker(\rho') = \ker(\rho).$$

Since H is a normal subgroup of G and since the quotient group G/H is abelian, it follows that the product HR is a normal subgroup of G. Applying Theorem 2.5.15 to our situation yields

$$\sigma \prec (\operatorname{ind}_{HR}^{G} \sigma)|_{HR}$$

and hence

$$\sigma|_H \prec (\operatorname{ind}_{HR}^{G} \sigma)|_H \cong (\operatorname{ind}_{HR}^{G} \operatorname{ind}_{R}^{HR} \varphi_f)|_H.$$

Since $\ker(\operatorname{ind}_{HR}^{G} \operatorname{ind}_{R}^{HR} \varphi_f) = \ker(\pi)$, it follows that $\sigma|_H \prec \pi|_H$. As $\sigma|_H \cong \rho$, we obtain the desired result, $\rho \prec \pi|_H$. □

Remark 2.10.23. A large class of nilpotent k-Lie pairs $(G, \mathfrak{g})$ satisfies the property that the product of any normal, exponentiable subgroup and any exponentiable subgroup is closed. Examples are given in Section 2.11.

In the following we require the locally compact separable nilpotent group G to have the additional property that the product of any normal exponentiable subgroup and any exponentiable subgroup is closed. We now develop necessary and sufficient conditions for a sequence of subgroup representation pairs to converge in $\mathcal{S}(G)$.

Theorem 2.10.24. *Let $\{(H_n, \pi_n)\}$ be a sequence in $\mathcal{S}(G)$ and suppose that H_n is an exponentiable subgroup of G for every $n \in \mathbb{N}$, and $\pi_n \in \hat{H}_n$ is not a character. Then the following are equivalent:*

(i) $(H_n, \pi_n) \to (H, \pi)$ *in* $\mathcal{S}(G)$.

(ii) For every subsequence of $\{(H_n, \pi_n)\}$ there exists a subsequence $\{(H_{n_k}, \pi_{n_k})\}$ such that

  (a) for every $k \in \mathbb{N}$ there exists an inducing pair (L_{n_k}, ρ_{n_k}) for (H_{n_k}, π_{n_k}), such that $(L_{n_k}, \rho_{n_k}) \to (L, \rho)$ in $\mathcal{S}(G)$ for some exponentiable subgroup L of H and some $\rho \in \hat{L}$, and

  (b) $\pi \prec \operatorname{ind}_L^H \rho$ *and* $\rho \prec \pi|_L$.

Proof. Suppose $(H_n, \pi_n) \to (H, \pi)$ in $\mathcal{S}(G)$ and let $\{(H_{n_m}, \pi_{n_m})\}$ be a subsequence of $\{(H_n, \pi_n)\}$. For every $m \in \mathbb{N}$, let (L_{n_m}, σ_{n_m}) be an inducing pair for (H_{n_m}, π_{n_m}) (the existence of inducing pairs is assured by Lemma 2.10.20). So $(L_{n_m})_{m \in \mathbb{N}}$ is a sequence of closed subgroups in the compact space $\mathcal{K}(G)$ and thus there exists a subsequence, which we also denote by $(L_{n_m})_{m \in \mathbb{N}}$, and there exists a closed subgroup L of G such that $L_{n_m} \to L$ in $\mathcal{K}(G)$. Since each such subgroup L_{n_m} is an exponentiable, normal subgroup of H_{n_m}, it follows from Proposition 2.5.7 that L is a normal, exponentiable subgroup of H. Define $\ell := \log(L)$. Since the process of restricting representations to subgroups is continuous (Theorem 2.10.8) we obtain

$$(L_{n_m}, \pi|_{L_{n_m}}) \to (L, \pi|_L) \text{ in } \mathcal{S}(G) \text{ as } m \to \infty.$$

Since the Kirillov map is surjective, we can choose a homomorphism $f \in \mathfrak{h}^*$ and a polarizing subalgebra $\mathfrak{r}$ for f, such that $\ker(\pi) = \ker(\operatorname{ind}_R^H \varphi_f)$, where $R = \exp(\mathfrak{r})$. Consider the homomorphism $f|_\ell \in \ell^*$ and choose an irreducible unitary representation ρ of L with $\ker(\rho) = \ker(\operatorname{ind}_{R'}^L \varphi_{f|_\ell})$, where $\log(R') = \mathfrak{r}'$ denotes a polarizing subalgebra for $f|_\ell$. By Lemma 2.10.22 we obtain

$$\pi \prec \operatorname{ind}_L^G \rho \quad \text{and} \quad \rho \prec \pi|_L.$$

We need to find a sequence $(\rho_m)_{m\in\mathbb{N}}$ of irreducible representations such that

- $\rho_m \in \widehat{L_{n_m}}$ for every $m \in \mathbb{N}$,
- (L_{n_m}, ρ_m) is an inducing pair for (H_{n_m}, π_m) for every $m \in \mathbb{N}$, and
- $(L_{n_m}, \rho_m) \to (L, \rho)$ in $\mathcal{S}(G)$ as $m \to \infty$.

Notice that $\rho \prec \pi|_L$, and since $(L_{n_m}, \pi_{n_m}|_{L_{n_m}}) \to (L, \pi|_L)$ in $\mathcal{S}(G)$ it follows from Proposition 2.5.4 that

$$(L_{n_m}, \pi_{n_m}|_{L_{n_m}}) \to (L, \rho) \text{ as } m \to \infty.$$

Since, for every $m \in \mathbb{N}$, the pair (L_{n_m}, σ_{m_n}) is an inducing pair for (H_{n_m}, π_{n_m}), we have

$$\ker(\pi_{n_m}) = \ker(\operatorname{ind}_{L_{n_m}}^{H_{n_m}} \sigma_{n_m}),$$

and since L_{n_m} is a normal subgroup of H_{n_m}, we obtain for every $m \in \mathbb{N}$;

$$\ker(\pi_{n_m}|_{L_{n_m}}) = \ker((\operatorname{ind}_{L_{n_m}}^{H_{n_m}} \sigma_{n_m})|_{L_{n_m}}) = \bigcap_{h_m \in H_{n_m}} \ker(h_m \cdot \sigma_{n_m}).$$

It follows directly from this equation that

$$\overline{H_{n_m}(\sigma_{n_m})} = \operatorname{Sp}(\pi_{n_m}|_{L_{n_m}}) \quad \text{for all } m \in \mathbb{N}, \tag{2.60}$$

where $H_{n_m}(\sigma_{n_m})$ denotes the H_{n_m}-orbit of σ_{m_n} in $\hat{L}_{n_m}$. As the orbit $H_{n_m}(\sigma_{n_m})$ is dense in $\operatorname{Sp}(\pi_{n_m}|_{L_{n_m}})$, for every $m \in \mathbb{N}$, we can choose by Proposition 2.5.5 a subsequence of $(\pi_{n_m}|_{L_{n_m}})_{m\in\mathbb{N}}$ (denoted with the same indices) and a sequence $(\rho_m)_{m\in\mathbb{N}}$ such that

(1) $\rho_m \in H_{n_m}(\sigma_{n_m}) \subseteq \operatorname{Sp}(\pi_{n_m}|_{L_{n_m}})$ for every $m \in \mathbb{N}$, and

(2) $(L_{n_m}, \rho_m) \to (L, \rho)$ as $m \to \infty$.

Since, under induction, the representation σ_{n_m} yields the same kernel as every element of its orbit, we obtain for every $m \in \mathbb{N}$;

$$\ker(\pi_{n_m}) = \ker(\operatorname{ind}_{L_{n_m}}^{H_{n_m}} \sigma_{n_m}) = \ker(\operatorname{ind}_{L_{n_m}}^{H_{n_m}} \rho_m).$$

Since $\rho_m \in H_{n_m}(\sigma_{n_m})$ it follows that $L_{n_m}/\ker(\rho_m) \cong L_{n_m}/\ker(\sigma_{n_m})$ for every $m \in \mathbb{N}$ and thus $L_{n_m}/J_{\rho_m} \cong L_{n_m}/J_{\sigma_{n_m}}$, where J_{ρ_m} and $J_{\sigma_{n_m}}$ denote the maximal exponentiable normal subgroups inside the kernel of ρ_m and σ_{n_m}, respectively. Therefore, the nilpotence class of L_{n_m}/J_{ρ_m} is equal to the nilpotence class of $L_{n_m}/J_{\sigma_{n_m}}$, which is less than the nilpotence class of H_{n_m}/π_{n_m}. This proves that (L_{n_m}, ρ_m) is an inducing pair for (H_{n_m}, π_{n_m}), for every $m \in \mathbb{N}$, and the sequence $(L_{n_m}, \rho_m)_{m\in\mathbb{N}}$ with limit (L, ρ) satisfies the desired properties (a) and (b) of (ii).

To prove the implication $(ii) \Rightarrow (i)$, suppose that for each subsequence of the sequence $\{(H_n, \pi_n)\}$ there exists a subsequence, say $\{(H_{n_m}, \pi_{n_m})\}$, satisfying Conditions (a) and (b). By the continuity of the inducing operation in all variables (Theorem 2.10.9) we obtain

$$(H_{n_m}, \operatorname{ind}_{L_{n_m}}^{H_{n_m}} \rho_{n_m}) \to (H, \operatorname{ind}_L^H \rho) \quad \text{in } \mathcal{S}(G).$$

Since (L_{n_m}, ρ_{n_m}) is an inducing pair for (H_{n_m}, π_{n_m}), we have $\ker(\operatorname{ind}_{L_{n_m}}^{H_{n_m}} \rho_{n_m}) = \ker(\pi_{n_m})$, for every $m \in \mathbb{N}$, and thus

$$(H_{n_m}, \pi_{n_m}) \to (H, \operatorname{ind}_L^H \rho) \text{ in } \mathcal{S}(G).$$

But $\pi \prec \operatorname{ind}_L^H \rho$, and hence we obtain by Proposition 2.5.4 the desired result,

$$(H_{n_m}, \pi_{n_m}) \to (H, \pi) \text{ in } \mathcal{S}(G).$$

□

Remark 2.10.25. Let $\{(H_n, \pi_n)\}$ be a sequence in $\mathcal{S}(G)$, where, for every $n \in \mathbb{N}$, H_n is an exponentiable subgroup of G and $\pi_n \in \hat{H}_n$ is not a character. Suppose that $(H_n, \pi_n) \to (H, \pi)$ in $\mathcal{S}(G)$. Then H is an exponentiable subgroup of G and suppose that $\ker(\pi) = \ker(\operatorname{ind}_R^H \varphi_f)$ for some homomorphism $f \in \mathfrak{h}^*$, where $\mathfrak{h}$ denotes the subalgebra of $\mathfrak{g}$ corresponding to H. For every $n \in \mathbb{N}$, let $\mathfrak{h}_n$ be the subalgebra of $\mathfrak{g}$ corresponding to H_n, and suppose that $\ker(\pi_n) = \ker(\operatorname{ind}_{R_n}^{H_n} \varphi_{f_n})$ for some homomorphism $f_n \in \mathfrak{h}_\mathfrak{n}{}^*$ with polarizing subalgebra $\mathfrak{r}_\mathfrak{n}$. By Remark 2.10.21 we can choose for every pair (H_n, π_n) an inducing pair (L_n, σ_n), such that $\sigma_n = \operatorname{ind}_{R_n}^{L_n} \varphi_{f_n|_{\ell_n}}$, where ℓ_n denotes the subalgebra of $\mathfrak{g}$ corresponding to L_n. By Theorem 2.10.24 there exists a subsequence $\{(H_{n_k}, \pi_{n_k})\}$ of $\{(H_n, \pi_n)\}$, such that for every $k \in \mathbb{N}$, there exists an inducing pair (L_{n_k}, ρ_{n_k}) for (H_{n_k}, π_{n_k}), such that $(L_{n_k}, \rho_{n_k}) \to (L, \rho)$ in $\mathcal{S}(G)$ for some exponentiable subgroup L of H and some $\rho \in \hat{L}$. We have shown in the proof of Theorem 2.10.24 that the irreducible representation $\rho \in \hat{L}$ can be chosen such that $\ker(\rho) = \ker(\operatorname{ind}_{R'}^L \varphi_{f|_\ell})$ and the irreducible representation $\rho_{n_k} \in \widehat{L_{n_k}}$, $k \in \mathbb{N}$, can be chosen such that $\rho_{n_k} \in H_{n_k}(\sigma_{n_k})$, the H_{n_k}-orbit of σ_{n_k} in $\widehat{L_{n_k}}$.

In the case of an abelian group, we obtain the following interesting fact about the convergence of subgroup-representation pairs.

Lemma 2.10.26. *Let $\mathfrak{g}$ be a locally compact abelian group and let $\mathfrak{h}$ be a closed subgroup of $\mathfrak{g}$. Let $(\mathfrak{h}_n)_{n\in\mathbb{N}}$ be a sequence of closed subgroups of $\mathfrak{g}$, and for every $n \in \mathbb{N}$, let ψ_n be a character of $\mathfrak{h}_n$ such that $(\mathfrak{h}_n, \psi_n) \to (\mathfrak{h}, \psi|_{\mathfrak{h}})$ in $\mathcal{S}(\mathfrak{g})$ for some character ψ of $\mathfrak{g}$. Then there exists a subsequence $(\psi_{n_k})_{k\in\mathbb{N}}$ of $(\psi_n)_{n\in\mathbb{N}}$ and lifts $\chi_{n_k} \in \widehat{\mathfrak{g}}$ of ψ_{n_k}, i.e., $\chi_{n_k}|_{\mathfrak{h}_{n_k}} = \psi_{n_k}$ for every $k \in \mathbb{N}$, such that $\chi_{n_k} \to \psi$ in $\widehat{\mathfrak{g}}$ as $k \to \infty$.*

Proof. Suppose $(\mathfrak{h}_n, \psi_n) \to (\mathfrak{h}, \psi|_{\mathfrak{h}})$ in $\mathcal{S}(\mathfrak{g})$ for some $\psi \in \hat{\mathfrak{g}}$. Since the inducing operation is continuous in every variable (Theorem 2.10.9), we obtain

$$(\mathfrak{g}, \operatorname{ind}_{\mathfrak{h}_n}^{\mathfrak{g}} \psi_n) \to (\mathfrak{g}, \operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}} \psi|_{\mathfrak{h}}) \text{ in } \mathcal{S}(\mathfrak{g}) \text{ as } k \to \infty.$$

Furthermore, we have $\operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}} \psi|_{\mathfrak{h}} \cong \psi \otimes \lambda_{\mathfrak{g}/\mathfrak{h}}$, where $\lambda_{\mathfrak{g}/\mathfrak{h}}$ denotes the left regular representation of $\mathfrak{g}/\mathfrak{h}$ on the Hilbert space $L^2(\mathfrak{g}/\mathfrak{h})$, and thus

$$\operatorname{Sp}(\operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}} \psi|_{\mathfrak{h}}) = \psi \cdot \operatorname{Sp}(\lambda_{\mathfrak{g}/\mathfrak{h}}) = \psi \cdot \widehat{\mathfrak{g}/\mathfrak{h}}.$$

In particular, we have $\psi \in \operatorname{Sp}(\operatorname{ind}_{\mathfrak{h}}^{\mathfrak{g}} \psi|_{\mathfrak{h}})$, and it follows from Proposition 2.5.4 that

$$\operatorname{ind}_{\mathfrak{h}_n}^{\mathfrak{g}} \psi_n \to \psi \text{ in } \operatorname{Rep}(\mathfrak{g}) \text{ as } n \to \infty.$$

By Proposition 2.5.5 we can find a subsequence of $(\operatorname{ind}_{\mathfrak{h}_n}^{\mathfrak{g}} \psi_n)_{n\in\mathbb{N}}$, say $(\operatorname{ind}_{\mathfrak{h}_{n_k}}^{\mathfrak{g}} \psi_{n_k})_{k\in\mathbb{N}}$, and we can find a sequence of characters, $(\chi_k)_{k\in\mathbb{N}}$, of $\mathfrak{g}$ such that $\chi_k \in \operatorname{Sp}(\operatorname{ind}_{\mathfrak{h}_{n_k}}^{\mathfrak{g}} \psi_{n_k})$, for every $k \in \mathbb{N}$, and $\chi_k \to \psi$ in $\widehat{\mathfrak{g}}$ as $k \to \infty$. But since $\chi_k \prec \operatorname{ind}_{\mathfrak{h}_{n_k}}^{\mathfrak{g}} \psi_{n_k}$ and $\mathfrak{g}$ is abelian, it follows that $\psi_{n_k} \prec \chi_{n_k}|_{\mathfrak{h}_{n_k}}$, for every $k \in \mathbb{N}$, and hence $\psi_{n_k} = \chi_{n_k}|_{\mathfrak{h}_{n_k}}$. ☐

We can now use the lemma above to prove the following lemma, which provides a useful tool for the proof of Proposition 2.10.30.

Lemma 2.10.27. *Let $k \geq 2$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair of nilpotence class $l \geq 2$. Let A be a maximal abelian subgroup of $Z^2(G)$ and let N be the centralizer of A. Then N is a normal subgroup of G and we have proven in Lemma 2.8.5 that $\mathfrak{n} := \log(N)$ is a subalgebra of $\mathfrak{g}$. The group G acts on $\mathfrak{n}^*$ by the coadjoint action, denoted by Ad^*. If f and g are two homomorphisms of $\mathfrak{g}^*$, such that $f|_{\mathfrak{n}}$ and $g|_{\mathfrak{n}}$ are in the same $\operatorname{Ad}^*(G)$-quasi-orbit in $\mathfrak{n}^*$, and if furthermore, both homomorphisms are faithful on the center of $\mathfrak{g}$, then f and g are in the same $\operatorname{Ad}^*(G)$-quasi-orbit in $\mathfrak{g}^*$.*

Proof. Let f and g be two homomorphisms of $\mathfrak{g}^*$ such that $f|_{\mathfrak{n}}$ and $g|_{\mathfrak{n}}$ are in the same $\operatorname{Ad}^*(G)$-quasi-orbit in $\mathfrak{n}^*$ and suppose that both homomorphisms, f and g, are faithful on $\mathfrak{z}(\mathfrak{g})$. Since $f \in \overline{G(g)}$, there exists a sequence $(x_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}}$ such that

$$\operatorname{Ad}^*(x_n) f|_{\mathfrak{n}} \to g|_{\mathfrak{n}} \quad \text{as } n \to \infty.$$

Clearly, we have $f|_{\mathfrak{z}(\mathfrak{g})} = g|_{\mathfrak{z}(\mathfrak{g})}$. Recall that there exists a Λ_k-module $\mathfrak{w}$ and a character $\epsilon \in \widehat{\mathfrak{w}}$ such that $\operatorname{Hom}(\mathfrak{g}, \mathfrak{w}) \cong \widehat{\mathfrak{g}}$ via the map $f \mapsto \epsilon \circ f$.

For every $n \in \mathbb{N}$, let $\psi_n := \epsilon \circ \mathrm{Ad}^*(x_n)f$ and let $\xi := \epsilon \circ g$. Then ψ_n, $n \in \mathbb{N}$, and ξ are characters of $\mathfrak{g}$ and since $\mathrm{Ad}^*(x_n)f|_{\mathfrak{n}} \to g|_{\mathfrak{n}}$ in $\mathrm{Hom}(\mathfrak{n}, \mathfrak{w})$, we have $\psi_n|_{\mathfrak{n}} \to \xi|_{\mathfrak{n}}$ in $\widehat{\mathfrak{n}}$.

Applying Lemma 2.10.26 to the sequence of pairs $((\mathfrak{n}, \psi_n|_{\mathfrak{n}}))_{n\in\mathbb{N}}$ with limit $(\mathfrak{n}, \xi|_{\mathfrak{n}})$ yields the existence of a subsequence of $(\psi_n|_{\mathfrak{n}})_{n\in\mathbb{N}}$, which we denote with the same indices, and, for every $n \in \mathbb{N}$, the existence of a lift $\chi_n \in \widehat{\mathfrak{g}}$ of $\psi_n|_{\mathfrak{n}}$, such that $\chi_n \to \xi$ in $\widehat{\mathfrak{g}}$ as $n \to \infty$. For every $n \in \mathbb{N}$, the character χ_n is of the from $\chi_n = \epsilon \circ \alpha_n$ for some homomorphism $\alpha_n \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$, and thus $\alpha_n \to g$ in $\mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$ as $n \to \infty$. Therefore, both characters, χ_n and ψ_n, are lifts of the character $\psi_n|_{\mathfrak{n}}$ for every $n \in \mathbb{N}$, and it is a well-known fact in the representation theory of locally compact abelian groups ([13], Theorem 4.39) that these two lifts differ by the multiplication with some character λ_n of the abelian quotient group $\mathfrak{g}/\mathfrak{n}$. Therefore, we obtain for all $n \in \mathbb{N}$ and $V \in \mathfrak{g}$,

$$\chi_n(V) = \psi_n(V) \cdot \lambda_n(\dot{V}),$$

where we denote by $\dot{V}$ the image of V under the canonical quotient map $q : \mathfrak{g} \to \mathfrak{g}/\mathfrak{n}$.

But the homomorphism $f \in \mathfrak{g}^*$ was chosen to be faithful on the center of $\mathfrak{g}$, and we have seen in Remark 2.8.10 that the map

$$\Phi_f : \mathfrak{a}/\mathfrak{z}(\mathfrak{g}) \to \widehat{\mathfrak{g}/\mathfrak{n}}, \dot{B} \mapsto \epsilon \circ f^{\dot{B}}$$

defines an injective, continuous homomorphism with dense image, where $f^{\dot{B}}(\dot{W}) := f([B, W])$ for all $W \in \mathfrak{g}$. Notice that $\mathfrak{a}$ denotes the algebra corresponding to the subgroup A. Thus, for every character $\lambda_n \in \widehat{\mathfrak{g}/\mathfrak{n}}$, $n \in \mathbb{N}$, there exists a sequence $(\dot{B}_{n_m})_{m\in\mathbb{N}}$ in $\mathfrak{a}/\mathfrak{z}(\mathfrak{g})$ such that $\epsilon \circ f^{\dot{B}_{n_m}} \to \lambda_n$ as $m \to \infty$. Since $B_{n_m} \in \mathfrak{a} \subseteq \mathfrak{z}^2(\mathfrak{g})$, for all $m \in \mathbb{N}$, and thus $[B_{n_m}, V] \in \mathfrak{z}(\mathfrak{g})$ for all $V \in \mathfrak{g}$, we obtain for all $n \in \mathbb{N}$ and $V \in \mathfrak{g}$,

$$\begin{aligned}
\chi_n(V) &= \psi_n(V) \cdot \lambda_n(\dot{V}) \\
&= \lim_{m\to\infty} \epsilon(\mathrm{Ad}^*(x_n)f(V))\, \epsilon(f^{\dot{B}_{n_m}}(\dot{V})) \\
&= \lim_{m\to\infty} \epsilon(f(\exp(\mathrm{ad}(X_n)))\, \epsilon(f([B_{n_m}, V])) \\
&= \lim_{m\to\infty} \epsilon(f(\exp(\mathrm{ad}(X_n + B_{n_m})(V)))) \\
&= \lim_{m\to\infty} \epsilon(\mathrm{Ad}^*(\exp(X_n + B_{n_m})f(V))),
\end{aligned}$$

where $X_n = \log(x_n)$ for all $n \in \mathbb{N}$. But, for every $n \in \mathbb{N}$, we have $\chi_n = \epsilon \circ \alpha_n$ so that we obtain from the computation above

$$\epsilon \circ \mathrm{Ad}^*(\exp(\log(x_n) + B_{n_m}))f \to \epsilon \circ \alpha_n \quad \text{as } m \to \infty$$

and thus

$$\mathrm{Ad}^*(\exp(\log(x_n) + B_{n_m}))f \to \alpha_n \quad \text{as } m \to \infty. \tag{2.61}$$

Since $\exp(\log(x_n) + B_{n_m}) \in G$ for all $n \in \mathbb{N}$ and for all $m \in \mathbb{N}$, it follows directly from (2.61) that the homomorphism α_n is in the closure of the $\mathrm{Ad}^*(G)$-orbit of f for every $n \in \mathbb{N}$. Hence the homomorphism $g = \lim_{n\to\infty} \alpha_n$ is an element of the closure of the $\mathrm{Ad}^*(G)$-orbit of f.

Switching the roles of the homomorphisms f and g in the above argument yields that the homomorphism f is in the closure of the $\mathrm{Ad}^*(G)$-orbit of g, which proves that f and g are in the same $\mathrm{Ad}^*(G)$-quasi-orbit. □

Corollary 2.10.28. *Let $k \geq 2$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair of nilpotence class $l \geq 2$. Let $(H, \pi) \in \mathcal{S}(G)$, where $\pi \in \hat{H}$ is not a character and H is an exponentiable subgroup of G with corresponding subalgebra $\mathfrak{h} = \log(H)$. Let (L, ρ) be an inducing pair for (H, π), where the exponentiable normal subgroup L of H is chosen as in Remark 2.10.21. We denote by ℓ the subalgebra of $\mathfrak{h}$ corresponding to the subgroup L of H. Suppose that $\ker(\pi) = \ker(\mathrm{ind}_R^H \varphi_{f|_\mathfrak{h}})$ for some $f \in \mathfrak{g}^*$ and let $g \in \mathfrak{g}^*$, such that $f|_\ell$ and $g|_\ell$ are in the same $\mathrm{Ad}^*(H)$-quasi-orbit in ℓ^*. Then $f|_\mathfrak{h}$ and $g|_\mathfrak{h}$ are in the same $\mathrm{Ad}^*(H)$-quasi-orbit in $\mathfrak{h}^*$.*

Proof. Let J_π be the maximal exponentiable normal subgroup of H inside the kernel of π as in Lemma 2.8.11. We consider two different cases.

Case 1: J_π is trivial. It follows from Lemma 2.8.11 that the homomorphism $f|_\mathfrak{h} \in \mathfrak{h}^*$ (and hence also $g|_\mathfrak{h}$) is faithful on $\mathfrak{z}(\mathfrak{h})$. Since L is in this case isomorphic to the centralizer of some maximal abelian subgroup of $Z^2(H)$, we can apply Lemma 2.10.27 to obtain the desired result.

Case 2: J_π is not trivial. We denote by $\tilde{H} := H/J_\pi$, $\mathfrak{j}_\pi := \log(J_\pi)$ and $\tilde{\mathfrak{h}} := \mathfrak{h}/\mathfrak{j}_\pi$. If $\tilde{f}$ denotes the homomorphism in $\tilde{\mathfrak{h}}^*$ induced by f, then $\tilde{f}$ is faithful on $\mathfrak{z}(\tilde{\mathfrak{h}})$ (Lemma 2.8.11) and we have $\tilde{f}|_{\mathfrak{z}(\tilde{\mathfrak{h}})} = \tilde{g}|_{\mathfrak{z}(\tilde{\mathfrak{h}})}$. The quotient group $q(L) = \tilde{L}$ is the centralizer of some maximal abelian subgroup of $Z^2(\tilde{H})$ and since $f|_\ell$ and $g|_\ell$ are in the same $\mathrm{Ad}^*(H)$-quasi-orbit in ℓ^*, it follows that $\widetilde{f|_\ell}$ and $\widetilde{g|_\ell}$ are in the same $\mathrm{Ad}^*(\tilde{H})$-quasi-orbit in $(\tilde{\ell})^*$, where $\tilde{\ell} = \ell/\mathfrak{j}_\pi$. Applying Lemma 2.10.27 to the quotient group $\tilde{H}$ with normal subgroup $\tilde{L}$ and homomorphisms $\widetilde{f|_\mathfrak{h}}$ and $\widetilde{g|_\mathfrak{h}}$ yields that $\widetilde{f|_\mathfrak{h}}$ and $\widetilde{g|_\mathfrak{h}}$ are in the same $\mathrm{Ad}^*(\tilde{H})$-quasi-orbit in $(\tilde{\mathfrak{h}})^*$. But this means that the homomorphisms $f|_\mathfrak{h}$ and $g|_\mathfrak{h}$ are in the $\mathrm{Ad}^*(H)$-quasi-orbit in $\mathfrak{h}^*$. □

We can now use Theorem 2.10.24 and the lemmas above to prove a proposition, which provides the main argument for the Kirillov-orbit map to be a homeomorphism. Since we do not know yet if the Kirillov-orbit map is one-to-one, we often need to choose a quasi-orbit corresponding to a primitive ideal under the Kirillov-orbit map. To simplify this, we make the following definition.

Definition 2.10.29. Let $k \in \mathbb{N}$, let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair, and let H be a closed exponentiable subgroup of G with associated subalgebra $\mathfrak{h} = \log(H)$. We have seen in Section 2.6 that H acts on $\mathfrak{h}^*$ by the coadjoint action:

$$\mathrm{Ad}^* : H \times \mathfrak{h}^* \to \mathfrak{h}^*,\ x \cdot f(Y) = f(\exp(\mathrm{ad}(\log(x)))(Y)).$$

As usual, we denote by $\mathfrak{h}^*/_\sim$ the quasi-orbit space of $\mathfrak{h}$ with respect to the coadjoint action of H. Let π be an irreducible representation of H. We say that $\mathcal{O} \in \mathfrak{h}^*/_\sim$ is a quasi-orbit of π, if

$$\ker(\operatorname{ind}_R^H \varphi_f) = \ker(\pi)$$

for some $f \in \mathcal{O}$ (and hence for all $f \in \mathcal{O}$), where R and φ_f are defined as usual.

Proposition 2.10.30. *Let $k \in \mathbb{N}$ and let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair. Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of exponentiable subgroups of G and, for every $n \in \mathbb{N}$, let $\pi_n \in \hat{H}_n$ such that $(H_n, \pi_n) \to (H, \pi)$ in $\mathcal{S}(G)$ as $n \to \infty$ for some exponentiable subgroup H of G and some $\pi \in \hat{H}$. Let $\mathfrak{h} = \log(H)$ be the subalgebra of $\mathfrak{g}$ corresponding to H and, for every $n \in \mathbb{N}$, let $\mathfrak{h}_n$ be the subalgebra of $\mathfrak{g}$ corresponding to H_n. Let $\mathcal{O}$ be a quasi-orbit of π and, for every $n \in \mathbb{N}$, let $\mathcal{O}_n$ be a quasi-orbit of π_n. If $f \in \mathfrak{g}^*$ with $f|_\mathfrak{h} \in \mathcal{O}$, then for every subsequence of $((H_n, \pi_n))_{n\in\mathbb{N}}$ there exists a subsequence, say $((H_{n_m}, \pi_{n_m}))_{m\in\mathbb{N}}$, such that for every $m \in \mathbb{N}$, there exist a homomorphism $f_m \in \mathfrak{g}^*$ with $f_m|_{\mathfrak{h}_{n_m}} \in \mathcal{O}_{n_m}$ and $f_m \to f$ in $\mathfrak{g}^*$ as $m \to \infty$.*

Proof. Let $f \in \mathfrak{g}^*$ with $f|_\mathfrak{h} \in \mathcal{O}$. Note that such a homomorphism always exists, because we have shown in Proposition 2.9.3 that the Kirillov-map is surjective and we have

$$\ker(\operatorname{ind}_{R'}^H \varphi_{f|_\mathfrak{h}}) = \ker(\pi),$$

where $\mathfrak{r}' = \log(R')$ denotes some polarizing subalgebra for $f|_\mathfrak{h}$. Choose a polarizing subalgebra $\mathfrak{r}$ for $f \in \mathfrak{g}^*$ with $\mathfrak{r}' = \mathfrak{r} \cap \mathfrak{h}$. Let $((H_{n_m}, \pi_{n_m}))_{m\in\mathbb{N}}$ be a subsequence of $((H_n, \pi_n))_{n\in\mathbb{N}}$, chosen so that we may assume that the degree of all pairs (H_{n_m}, π_{n_m}), $m \in \mathbb{N}$, is constant. Let this constant be $l \in \mathbb{N}_{\leq k}$.

We consider first the case, where we can find a subsequence $((H_{n_m}, \pi_{n_m}))_{m\in\mathbb{N}}$ (denoted by the same indices), such that, for every $m \in \mathbb{N}$, the representation π_{n_m} of H_{n_m} is one-dimensional. Since the set of all one-dimensional representations of a C^*-algebra is closed and since $\pi_{n_m} \to \pi$, it follows that $\pi \in \hat{H}$ is also a one-dimensional representation. Observe that, for every $m \in \mathbb{N}$, we can identify the one-dimensional representation π_{n_m} with a character of H_{n_m}, and by Corollary 2.9.4, this character is of the form φ_{g_m} for some $g_m \in \mathfrak{h}_{n_m}^*$. With this identification we obtain, for every $m \in \mathbb{N}$, $\pi_{n_m} \circ \exp = \epsilon \circ g_m$, and clearly $g_m \in \mathcal{O}_{n_m}$. Furthermore, we can identify the one-dimensional representation $\pi \in \hat{H}$ with a character of H, and since $\ker(\operatorname{ind}_{R'}^H \varphi_{f|_\mathfrak{h}}) = \ker(\pi)$, it follows that this character is equal to $\varphi_f|_H = \varphi_{f|_\mathfrak{h}}$. Thus we obtain $\pi \circ \exp = \epsilon \circ f|_\mathfrak{h}$. Since $\pi_{n_m} \to \pi$, and since the map $\exp : \mathfrak{g} \to G$ is a homeomorphism of groups, it follows that $\epsilon \circ g_m \to \epsilon \circ f|_\mathfrak{h}$ as $m \to \infty$. By Lemma 2.10.26 we can find a subsequence of $(\epsilon \circ g_m)_{m\in\mathbb{N}}$, say $(\epsilon \circ g_{m_j})_{j\in\mathbb{N}}$, and, for every $j \in \mathbb{N}$, we can find an extension $\chi_{m_j} \in \widehat{\mathfrak{g}}$ of $\epsilon \circ g_{m_j}$, such that $\chi_{m_j} \to \epsilon \circ f$ in $\widehat{\mathfrak{g}}$. But every character $\chi_{m_j} \in \widehat{\mathfrak{g}}$ is of the form $\chi_{m_j} = \epsilon \circ f_{m_j}$ for some extension $f_{m_j} \in \mathfrak{g}^*$ of $g_{m_j} \in \mathfrak{h}_{n_{m_j}}^*$ and since $\epsilon \circ f_{m_j} \to \epsilon \circ f \in \widehat{\mathfrak{g}}$, we have $f_{m_j} \to f \in \mathfrak{g}^*$. Furthermore, we have $f_{m_j}|_{\mathfrak{h}_{n_{m_j}}} = g_{m_j}$ and thus $f_{m_j}|_{\mathfrak{h}_{n_{m_j}}} \in \mathcal{O}_{n_{m_j}}$ for every $j \in \mathbb{N}$. This proves the proposition in this case.

The proof of the general case proceeds by induction on $l \in \mathbb{N}_{\leq k}$.

If $l = 1$, i.e., if the degree of every pair (H_{n_m}, π_{n_m}), $m \in \mathbb{N}$, is equal to one, then every irreducible representation $\pi_{n_m} \in \widehat{H_{n_m}}$ is one-dimensional (Lemma 2.10.18) and we are done with the above observation.

So suppose $l \geq 2$, and assume that the proposition has been shown for all convergent sequences $((L_n, \rho_n))_{n\in\mathbb{N}}$ in $\mathcal{S}(G)$ whose elements have degree less than l. By the above, and by passing to a suitable subsequence if necessary, we may assume that each irreducible representation π_{n_m}, $m \in \mathbb{N}$, is not one-dimensional.

For every $m \in \mathbb{N}$, choose a homomorphism $g_m \in \mathcal{O}_{n_m}$. By Theorem 2.10.24 there exists a subsequence of $((H_{n_m}, \pi_{n_m}))_{m\in\mathbb{N}}$, which we denote by the same indices, and there exists, for every $m \in \mathbb{N}$, an inducing pair (L_{n_m}, ρ_{n_m}) for (H_{n_m}, π_{n_m}), satisfying

(1) $(L_{n_m}, \rho_{n_m}) \to (L, \rho)$ in $\mathcal{S}(G)$ for some exponentiable subgroup L of G and some irreducible representation $\rho \in \hat{L}$, and

(2) $\pi \prec \operatorname{ind}_L^H \rho$.

Furthermore, we can choose, for every $m \in \mathbb{N}$, the subgroup L_{n_m} of H_{n_m} as in Remark 2.10.25, and we can choose the irreducible representation $\rho \in \hat{L}$ and the irreducible representation $\rho_{n_m} \in \widehat{L_{n_m}}$, $m \in \mathbb{N}$, such that

(3) $\ker(\rho) = \ker(\operatorname{ind}_{L\cap R}^L \varphi_{f|_\ell})$, where $\ell := \log(L)$, and $\rho_{n_m} \in H_{n_m}(\sigma_{n_m})$ (the H_{n_m}-orbit of σ_{n_m} in $\hat{L}_{n_m}$), where $\sigma_{n_m} = \operatorname{ind}_{R_{n_m}}^{L_{n_m}} \varphi_{g_m|_{\ell_{n_m}}}$.

Thus we can find, for every $m \in \mathbb{N}$, an element $h_m \in H_{n_m}$ such that

$$\ker(\rho_{n_m}) = \ker(\operatorname{ind}_{\exp(Ad(h_m)\mathfrak{r}_{n_m})}^{L_{n_m}} \varphi_{\mathrm{Ad}^*(h_m)g_m|_{\ell_{n_m}}}). \tag{2.62}$$

(We have seen in Lemma 2.6.13 that if g is a homomorphism with polarizing subalgebra $\mathfrak{r}$, then $\mathrm{Ad}(x)\mathfrak{r}$ is a polarizing subalgebra for the homomorphism $\mathrm{Ad}^*(x)g$.)

For every $m \in \mathbb{N}$, let $\mathcal{O}'_{n_m}$ be the quasi-orbit of ρ_{n_m} with $g_m|_{\ell_{n_m}} \in \mathcal{O}'_{n_m}$ and let $\mathcal{O}'$ be the quasi-orbit of ρ with $f|_\ell \in \mathcal{O}'$ (the existence of such quasi-orbits is assured by (2.62) and by property (3) above). We can now apply the induction hypothesis to the sequence of pairs $((L_{n_m}, \rho_{n_m}))_{m\in\mathbb{N}}$ with limit (L, ρ) and quasi-orbits $\mathcal{O}'_{n_m}$ of ρ_{n_m}, for every m, and the quasi-orbit $\mathcal{O}'$ of ρ. This yields us, for every subsequence of $(L_{n_m}, \rho_{n_m})_{m\in\mathbb{N}}$, a subsequence, which we denote with the same indices, and homomorphisms $f_m \in \mathfrak{g}^*$ with $f_m|_{\ell_{n_m}} \in \mathcal{O}'_{n_m}$, for every $m \in \mathbb{N}$, and $f_m \to f$ in $\mathfrak{g}^*$ as $m \to \infty$.

It remains to prove that $f_m|_{\mathfrak{h}_{n_m}} \in \mathcal{O}_{n_m}$ for every $m \in \mathbb{N}$. So let $m \in \mathbb{N}$ and notice that both homomorphisms, $g_m|_{\ell_{n_m}}$ and $f_m|_{\ell_{n_m}}$, are elements of the same $\mathrm{Ad}^*(H_{n_m})$-quasi-orbit in $\ell^*_{n_m}$, namely $\mathcal{O}'_{n_m}$. Since (L_{n_m}, ρ_{n_m}) is an inducing pair for (H_{n_m}, π_{n_m}) and the subgroup L_{n_m} of H_{n_m} is chosen as in Remark 2.10.25 it follows from Corollary 2.10.28 that the homomorphisms g_m and $f_m|_{\mathfrak{h}_{n_m}}$ are in the same H_{n_m}-quasi-orbit in $\mathfrak{h}^*_{n_m}$. Since g_m was chosen to be an element of the quasi-orbit $\mathcal{O}_{n_m}$, it follows that $f_m|_{\mathfrak{h}_{n_m}} \in \mathcal{O}_{n_m}$. □

Both desired properties, the injectivity of the Kirillov-orbit map and the continuity of its inverse map, are now an easy application of Proposition 2.10.30.

Corollary 2.10.31. *Let $k \in \mathbb{N}$, let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair, and suppose that the product of every normal exponentiable subgroup and every exponentiable subgroup of G is closed. The Kirillov-orbit map*

$$\tilde{\kappa} : \mathfrak{g}^* /_\sim \to \operatorname{Prim}(C^*(G)),\ \mathcal{O} \mapsto \ker(\operatorname{ind}_R^G \varphi_f),$$

where $f \in \mathfrak{g}^*$ is any chosen representative of the quasi-orbit $\mathcal{O}$, is an injective map.

Proof. Let $\mathcal{O}$ and $\mathcal{O}'$ be two quasi-orbits in $\mathfrak{g}^* /_\sim$ with $\tilde{\kappa}(\mathcal{O}) = \tilde{\kappa}(\mathcal{O}')$. Then we have

$$\ker(\operatorname{ind}_R^G \varphi_f) = \ker(\operatorname{ind}_{R'}^G \varphi_{f'}) \tag{2.63}$$

for all $f \in \mathcal{O}$ and for all $f' \in \mathcal{O}'$, where $\mathfrak{r} = \log(R)$ denotes some polarizing subalgebra for f and $\mathfrak{r}' = \log(R')$ denotes some polarizing subalgebra for f'. Fix a homomorphism $f \in \mathcal{O}$ and a homomorphism $f' \in \mathcal{O}'$. Since the constant sequence $(\ker(\operatorname{ind}_R^G \varphi_f))_{n\in\mathbb{N}}$ converges to $\ker(\operatorname{ind}_{R'}^G \varphi_{f'})$ in the space $\operatorname{Prim}(C^*(G))$ it follows that

$$(G, \operatorname{ind}_R^G \varphi_{f_n}) \to (G, \operatorname{ind}_{R'}^G \varphi_{f'}) \text{ in } \mathcal{S}(G) \text{ as } n \to \infty,$$

where $f_n = f$ for all $n \in \mathbb{N}$. By Proposition 2.10.30 we can find homomorphisms $g_n \in \mathfrak{g}^*$ such that $g_n \in \mathcal{O}$ for every $n \in \mathbb{N}$ and $g_n \to f'$ as $n \to \infty$. Since the closure of the quasi-orbit $\mathcal{O}$ is contained in the closure of the $\operatorname{Ad}^*(G)$-orbit of f, $G(f)$, it follows that $f' \in \overline{G(f)}$.

Switching the roles of f and f' in the argument above yields $f \in \overline{G(f')}$, and hence we obtain $\mathcal{O} = \mathcal{O}'$. □

We can now prove the main result of this section.

Corollary 2.10.32. *Let $k \in \mathbb{N}$, let $(G, \mathfrak{g})$ be a nilpotent k-Lie pair, and suppose that the product of every normal exponentiable subgroup and every exponentiable subgroup of G is closed. The Kirillov-orbit map*

$$\tilde{\kappa} : \mathfrak{g}^* /_\sim \to \operatorname{Prim}(C^*(G)),\ \mathcal{O} \mapsto \ker(\operatorname{ind}_R^G \varphi_f),$$

where $f \in \mathfrak{g}^*$ is any chosen representative of the quasi-orbit $\mathcal{O}$, is a homeomorphism.

Proof. We have already shown that the Kirillov-orbit map $\tilde{\kappa}$ is a well-defined, continuous bijection (Corollary 2.10.12, Corollary 2.10.13, Proposition 2.10.11, and Corollary 2.10.31). So it remains to prove that the inverse map of $\tilde{\kappa}$ is continuous.

For this, let $(\pi_n)_{n\in\mathbb{N}}$ be a sequence in $\widehat{G}$ and suppose that $\ker(\pi_n) \to \ker(\pi)$ in $\operatorname{Prim}(C^*(G))$ for some $\pi \in \widehat{G}$. Choose a homomorphism $f \in \mathfrak{g}^*$ with $\ker(\operatorname{ind}_R^G \varphi_f) = \ker(\pi)$. Since

$$(G, \pi_n) \to (G, \pi) \text{ in } \mathcal{S}(G) \text{ as } n \to \infty,$$

we can find by Proposition 2.10.30 for every subsequence of $\{(G,\pi_n)\}$, a subsequence, say $\{(G,\pi_{n_m})\}$, and we can find, for every $m\in\mathbb{N}$, a homomorphism $f_m\in\mathfrak{g}^*$, such that $\ker(\operatorname{ind}_{R_m}^G\varphi_{f_m})=\ker(\pi_{n_m})$ and $f_m\to f$ in $\mathfrak{g}^*$ as $m\to\infty$. Hence $\mathcal{O}_{f_m}\to\mathcal{O}_f$ in $\mathfrak{g}^*/_\sim$, where, for every m, $\mathcal{O}_{f_m}$ denotes the quasi-orbit of f_m and $\mathcal{O}$ denotes the quasi-orbit of f. Since this results holds for every subsequence of $\{(G,\pi_n)\}$ and since $\tilde{\kappa}$ is an injective map, it follows that if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{g}^*$ with $\ker(\operatorname{ind}_{R_n}^G\varphi_{f_n})=\ker(\pi_n)$ for every $n\in\mathbb{N}$, then $\mathcal{O}_{f_n}\to\mathcal{O}_f$ in $\mathfrak{g}^*/_\sim$. $\square$

2.11 Examples

In this section we show for three different kinds of locally compact, nilpotent groups G that there exist a natural number k and a Lie algebra $\mathfrak{g}$ over the ring Λ_k, such that the pair $(G,\mathfrak{g})$ defines a nilpotent k-Lie pair. But before we discuss some explicit examples, we want to observe some general facts.

Theorem 2.11.1. *([30], Theorem 3) Let K be any local field, let V be a finite dimensional vector space over K, and let χ be a non-trivial character of the additive group of K. The map*

$$\Phi_\chi : V^*\to\widehat{V},\ f\mapsto\chi\circ f,$$

defines an isomorphism (of vector spaces) from the space of continuous linear functionals of V onto the Pontrjagin dual of the additive group of V.

Theorem 2.11.2. *(Hahn-Banach) Let X be a normable vector space and let f be a continuous linear functional on a subspace Y of X. Then there exists a continuous functional $\tilde{f}\in X^*$ such that $\tilde{f}=f$ on Y.*

Example 2.11.3. Let G be a connected, simply connected, l-step nilpotent Lie group over $\mathbb{R}$ with Lie algebra $\mathfrak{g}$.
We will show in the following that we can find for every natural number $k\geq l$, a locally compact group $\mathfrak{w}$ and a character $\epsilon\in\widehat{\mathfrak{w}}$ such that the pair $(G,\mathfrak{g})$, together with this group $\mathfrak{w}$ and this character $\epsilon\in\widehat{\mathfrak{w}}$, satisfies all the properties of Definition 2.2.5, which means that $(G,\mathfrak{g})$ is a nilpotent k-Lie pair.

Since $\mathfrak{g}$ is a Lie algebra over $\mathbb{R}$, it clearly defines a Lie algebra over the ring $\mathbb{Z}[\frac{1}{k!}]$ for every $k\geq 1$ and property (i) of Definition 2.2.5 is satisfied for every $k\geq 1$. Furthermore, it follows from the following theorem that the pair $(G,\mathfrak{g})$ satisfies property (ii) of Definition 2.2.5. A proof can be found for example in [7].

Theorem. *Let G be a connected, simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$.*

(i) The map $\exp:\mathfrak{g}\to G$ is an analytic diffeomorphism.

(ii) The Campbell-Hausdorff formula holds for all $X,Y\in\mathfrak{g}$.

So it remains to show that, for every $k \in \mathbb{N}_{\geq l}$, there exists a locally compact abelian group $\mathfrak{w}$ and a character $\epsilon \in \widehat{\mathfrak{w}}$ such that the following properties hold.

(a) The group $\mathfrak{w}$ is a Λ_k-module.

(b) There does not exist a non-trivial Λ_k-submodule of $\mathfrak{w}$ inside the kernel of the character ϵ.

(c) The map
$$\Phi : \mathrm{Hom}(\mathfrak{g}, \mathfrak{w}) \to \hat{\mathfrak{g}},\ f \mapsto \epsilon \circ f,$$
is an isomorphism of groups, where $\mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$ denotes the group of continuous group homomorphisms from $\mathfrak{g}$ to $\mathfrak{w}$ and $\widehat{\mathfrak{g}}$ denotes the Pontrjagin dual of the abelian group $(\mathfrak{g}, +)$.

(d) For every closed Λ_k-subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and for any $f \in \mathrm{Hom}(\mathfrak{h}, \mathfrak{w})$, there exists a map $\tilde{f} \in \mathrm{Hom}(\mathfrak{g}, \mathfrak{w})$ such that $\tilde{f}|_{\mathfrak{h}} = f$.

If $l \geq 2$, then we put $\mathfrak{w} := \mathbb{R}$ and define
$$\epsilon : \mathbb{R} \to \mathbb{T},\ t \mapsto e^{2\pi i t}$$
to be the standard character of $\mathbb{R}$. We claim that the group $\mathfrak{w}$ and the character $\epsilon \in \widehat{\mathfrak{w}}$ satisfy properties $(a) - (d)$ above, for every $k \in \mathbb{N}_{\geq 2}$. Let $k \in \mathbb{N}_{\geq 2}$. Clearly, $\mathfrak{w} = \mathbb{R}$ is a Λ_k-module and since the abelian group $\mathfrak{g}$ is a vector space over $\mathbb{R}$, we have $\mathrm{Hom}(\mathfrak{g}, \mathbb{R}) = \mathfrak{g}^*$, and it follows from Theorem 2.11.1 that the map
$$\Phi_\epsilon : \mathfrak{g}^* \to \widehat{\mathfrak{g}},\ f \mapsto \epsilon \circ f$$
is an isomorphism of groups. Moreover, if $\mathfrak{h}$ is any closed subalgebra of $\mathfrak{g}$, then $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$ and if $f \in \mathrm{Hom}(\mathfrak{h}, \mathfrak{w}) = \mathfrak{h}^*$ is a continuous functional of $\mathfrak{h}$, then there exists by the Hahn-Banach Theorem a continuous functional $\tilde{f} \in \mathfrak{g}^*$ with $\tilde{f}|_{\mathfrak{h}} = f$.

It remains to prove that there does not exist a non-trivial Λ_k-submodule of $\mathbb{R}$ inside the kernel of the character ϵ. But we have $\ker(\epsilon) \subseteq \mathbb{Z}$ and since it is not possible for a subgroup of $\mathbb{Z}$ to be a $\mathbb{Z}[\frac{1}{k!}]$-submodule of $\mathbb{R}$ for $k \geq 2$, there does not exist a non-trivial Λ_k-submodule of $\mathbb{R}$ inside the kernel of ϵ.

Therefore, if G is any connected, simply connected l-step nilpotent Lie group over $\mathbb{R}$ with Lie algebra $\mathfrak{g}$ and if $l \geq 2$, then the pair $(G, \mathfrak{g})$ defines a nilpotent k-Lie pair for every $k \geq 2$.

If $l = 1$, then we can choose, for every $k \in \mathbb{N}_{\geq 2}$, the group $\mathfrak{w}$ and the character $\epsilon \in \widehat{\mathfrak{w}}$ as above. For $k = 1$, we put $\mathfrak{w} = \mathbb{T}$ and $\epsilon = Id$. The locally compact abelian group $\mathfrak{w}$ is clearly a $\mathbb{Z}$-module and $\ker(\epsilon) = \{1\}$. Since $\mathrm{Hom}(\mathfrak{g}, \mathfrak{w}) = \widehat{\mathfrak{g}}$ (not only isomorphic), property (d) above follows from the fact that every character of a closed subgroup $\mathfrak{h}$ of the locally compact abelian group $\mathfrak{g}$ can be lifted to a character of $\mathfrak{g}$.

Furthermore, every connected and simply connected nilpotent Lie groups satisfies the property that the product of any normal, exponentiable subgroup and any exponentiable subgroup is closed. (This property is needed in Subsection 2.10.3, where we prove that the Kirillov-orbit map is a homeomorphism for every nilpotent k-Lie pair with this additional property.) Indeed, since every exponentiable subgroup is in particular closed, this additional property follows directly from the following lemma.

Lemma. *([2],Lemma 1.1) Let G be a connected and simply connected nilpotent Lie group, and let H and N be closed subgroups of G. If N is connected and normal, then $H \cdot N$ is closed.*

Therefore, we obtain for every connected, simply connected nilpotent Lie group over $\mathbb{R}$ with Lie algebra $\mathfrak{g}$ a homeomorphism

$$\tilde{\kappa} : \mathfrak{g}^* /_{\sim} \longrightarrow \mathrm{Prim}(C^*(G)),\ \mathcal{O} \mapsto \ker(\mathrm{ind}_R^G \varphi_f),$$

where $f \in \mathfrak{g}^*$ is any chosen representative of the coadjoint quasi-orbit $\mathcal{O}$.

Example 2.11.4. Let K be a local field of characteristic p. Let $Tr_1(n, K)$ be the group of upper triangular $n \times n$-matrices over K with each diagonal entry equal to 1 and let $Tr_0(n, K)$ be the group of upper triangular $n \times n$-matrices over K with each diagonal entry equal to 0, and suppose that $p > n$.

We will show in the following that we can find a natural number $k \geq n$, a locally compact abelian group $\mathfrak{w}$, and a character $\epsilon \in \widehat{\mathfrak{w}}$ such that the pair $(G, \mathfrak{g})$ together with this group $\mathfrak{w}$ and this character ϵ satisfy all the properties of Definition 2.2.5, so that $(G, \mathfrak{g})$ is a nilpotent k-Lie pair.

Let $n \leq k < p$ be arbitrary. Equipped with the usual commutator of matrices, $Tr_0(n, K)$ becomes a Lie algebra over the ring $\mathbb{Z}[\frac{1}{k!}]$. Furthermore, since $p > n$, the exponential map

$$\exp : Tr_0(n, K) \to Tr_1(n, K),\ X \mapsto \sum_{m=0}^{n-1} \frac{X^m}{m!}$$

is a well-defined homeomorphism satisfying the Campbell-Hausdorff formula. Its inverse map log is also given by the usual power series,

$$\log : Tr_1(n, K) \to Tr_0(n, K),\ x = 1 + u \mapsto \sum_{m=1}^{n-1} (-1)^{m+1} \frac{u^m}{m}.$$

Since the characteristic of the field K is equal to p, we have $pX = 0$ for all $X \in Tr_0(n, K)$ and since

$$\exp(mX) = (\exp(X))^m$$

for all $X \in \mathfrak{g}$ and $m \in \mathbb{N}$, it follows that $x^p = 1$ for all $x \in Tr_1(n, K)$. Thus, if $\chi : Tr_0(n, K) \to \mathbb{T}$ denotes any character of the additive group $Tr_0(n, K)$ then we have for all $X \in Tr_0(n, K)$,

$$\chi(pX) = (\chi(X))^p = 1.$$

Therefore, $\chi(X) \in \mathcal{U}_p$ for all $X \in Tr_0(n, K)$, where $\mathcal{U}_p$ denotes the group of primitive pth roots of unity.

Put $\mathfrak{w} := \mathcal{U}_p \subseteq \mathbb{T}$. Then $\mathfrak{w}$ is, as a discrete group, locally compact and clearly a Λ_k-module. We define $\epsilon := Id$. The map $\epsilon : \mathcal{U}_p \hookrightarrow \mathbb{T}$ is a character of $\mathfrak{w}$ and there does not exist a non-trivial Λ_k-module inside the kernel of ϵ. Moreover, we have

$$\mathrm{Hom}(Tr_0(n, K), \mathfrak{w}) = \widehat{Tr_0(n, K)}.$$

So if $\mathfrak{h}$ is any closed subalgebra of $Tr_0(n, K)$, then every homomorphism $f \in \mathfrak{h}^*$ is a character of the additive group $\mathfrak{h}$, and can therefore be lifted to a character $\tilde{f} \in \widehat{Tr_0(n, K)}$. This proves that the pair $(Tr_1(n, K), Tr_0(n, K))$ with locally compact abelian group $\mathfrak{w} = \mathcal{U}_p$ and character $\epsilon = Id$ defines a nilpotent k-Lie pair for every $n \leq k < p$.

Proposition 2.11.5. *Let G be a unipotent, linear algebraic group defined over a local field K of characteristic p. Then G is isomorphic to an algebraic subgroup of the upper triangular unipotent group $Tr_1(n, K)$ for some $n \in \mathbb{N}$. If $p > n$, then $(G, \log(G))$ is a nilpotent k-Lie pair for every $n \leq k < p$.*

Proof. Every unipotent linear algebraic group G over K is isomorphic to an algebraic subgroup of the upper triangular unipotent group $Tr_1(n, K)$ for some $n \in \mathbb{N}$ (see Chapter 1, Section 1.3). Since $(Tr_1(n, K), Tr_0(n, K))$ is a n-step nilpotent k-Lie pair for every $n \leq k < p$, we need to prove that $\log(G)$ is a subalgebra of $Tr_0(n, K)$. But by Theorem 2.3.2 of Section 2.3, it suffices to show that G is a k-complete subgroup of $Tr_1(n, K)$.

So let $x \in G$ and let $\lambda \in \mathbb{Z}[\frac{1}{k!}]$. We need to prove that $x^\lambda \in G$. For this we write $\lambda = \frac{r}{s}$ for some $r, s \in \mathbb{Z}$ with $\gcd(r, s) = 1$. Since $\gcd(s, p) = 1$, we can find integers ℓ, m with $s\ell + pm = 1$. Put $y := x^\ell$. Then $y \in G$ and since $x^p = 1$, we obtain

$$y^s = (x^\ell)^s = x^{\ell s} = x^{1-pm} = x\ (x^p)^{-m} = x.$$

Hence $y = x^{\frac{1}{s}}$ and thus $x^{\frac{1}{s}} \in G$. It follows then that $y^r = x^\lambda \in G$. □

Furthermore, every unipotent linear algebraic group over a local field K of characteristic p satisfies the property that the product of any normal, exponentiable subgroup and any exponentiable subgroup is closed. In fact, every such group satisfies the more general property that the product of any two closed subgroups is closed.

To prove this, let G be a unipotent linear algebraic group over a local field K of characteristic p for some prime number $p > 0$. Then K is isomorphic to the field $\mathbb{F}_q((t))$, where q is some power of the prime p and, as mentioned above, G is isomorphic to an algebraic subgroup of the upper triangular unipotent group $Tr_1(k, K)$ for some $k \in \mathbb{N}$. Observe that there exists a sequence, $(U_n)_{n\in\mathbb{Z}}$, of compact open subgroups of G such that $U_{n+1} \subseteq U_n$ for all $n \in \mathbb{Z}$ and $\bigcup_{n\in\mathbb{Z}} U_n = G$. For example, one can choose, for every $n \in \mathbb{Z}$, the subgroup U_n to be the intersection of G and the subgroup

$$A_n : \;=\; \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1,k} \\ 0 & 1 & a_{23} & \cdots & a_{2,k} \\ 0 & 0 & \ddots & & \vdots \\ & & & 1 & a_{k-1,k} \\ & & & & 1 \end{pmatrix}, a_{ij} \in K, \|a_{12}, \|, \|a_{23}\|, \ldots, \|a_{k-1,k}\| \leq q^{-n}; \right.$$

$$\left. \|a_{13}\|, \|a_{24}\|, \ldots, \|a_{k-2,k}\| \leq (q^{-n})^2; \ldots; \|a_{1,k}\| \leq (q^{-n})^{k-1} \right\},$$

where $\|.\| : K \to \mathbb{R}_{\geq 0}$ denotes the norm function on K, which is defined as follows. Every element $x \in \mathbb{F}_q((t))$ can be written uniquely as $x = t^l u$ for some $u \in \mathbb{F}[[t]]^\times$ and some $l \in \mathbb{Z}$ and one puts $\|x\| = q^{-l}$. We claim that every subgroup L of G satisfies the property,

$$L \text{ is closed} \iff L \cap U_n \text{ is closed} \;\forall\, n \in \mathbb{Z}\,. \tag{2.64}$$

If L is closed, then also its intersection with every compact subgroup U_n, $n \in \mathbb{Z}$. In order to prove the other direction, suppose $(x_n)_{n\in\mathbb{N}}$ is a sequence in L which converges to some element $x \in G$. Then every entry of the matrix x is a Laurent series and the norms of these Laurent series are all bounded by some value, say q^{-l}. Thus we have $\{x, x_n, n \in \mathbb{N}\} \subseteq U_l$ and since $L \cap U_l$ is closed, it follows that $x \in L \cap U_l$. Hence $x \in L$.

Now, let H and N be two arbitrary closed subgroups of G. Then, for every $n \in \mathbb{Z}$, $H \cap U_n$ and $N \cap U_n$ are compact subgroups of U_n. As the product of compact subsets is compact, we have $(H \cap U_n) \cdot (N \cap U_n)$ is compact for every $n \in \mathbb{Z}$. But, for every $n \in \mathbb{Z}$, there exists an integer $m \leq n$, such that $(H \cap U_n) \cdot (N \cap U_n) \hookrightarrow HN \cap U_m$. Furthermore, we can find some integer $l \geq n$ such that $HN \cap U_l \subseteq (H \cap U_n) \cdot (N \cap U_n)$. Since the quotient $(HN \cap U_m / HN \cap U_l)$ is finite, it follows that $(H \cap U_n) \cdot (N \cap U_n)$ has finite index in $HN \cap U_m$, which proves that $HN \cap U_m$ is compact for every $m \in \mathbb{Z}$. It follows then from (2.64) that the group HN is closed.

Therefore, if G is a group which is isomorphic to an algebraic subgroup of the upper triangular unipotent group $Tr_1(n, K)$ for some $n \in \mathbb{N}$, where K is any local field K of characteristic $p > n$, then $(G, \log(G))$ is a nilpotent k-Lie pair for every $n \leq k < p$. In particular, the Kirillov-orbit map

$$\tilde{\kappa} : \mathfrak{g}^* /_\sim \longrightarrow \operatorname{Prim}(C^*(G)),\; \mathcal{O} \mapsto \ker(\operatorname{ind}_R^G \varphi_f),$$

where $f \in \mathfrak{g}^*$ is any chosen homomorphism of $\mathcal{O}$, is a homeomorphism.

Example 2.11.6. Let p be any prime number and let G be a unipotent, linear algebraic group defined over the local field $\mathbb{Q}_p$ of p-adic numbers. Then G is isomorphic to an algebraic subgroup of the upper triangular unipotent group $Tr_1(n, \mathbb{Q}_p)$ for some $n \in \mathbb{N}$. Let $\mathfrak{g}$ denote the Lie algebra of G and let $l := \max\{p, n\}$.

We will show in the following that we can find for every natural number $k \geq l$, a locally compact group $\mathfrak{w}$ and a character $\epsilon \in \widehat{\mathfrak{w}}$ such that the pair $(G, \mathfrak{g})$, together with this group $\mathfrak{w}$ and this character $\epsilon \in \widehat{\mathfrak{w}}$, satisfies all the properties of Definition 2.2.5. In particular, there exists a natural number k, such that the pair $(G, \mathfrak{g})$ defines a nilpotent k-Lie pair.

Since $\mathfrak{g}$ is a Lie algebra over $\mathbb{Q}_p$, it defines a Lie algebra over the ring $\mathbb{Z}[\frac{1}{k!}]$ for every $k \geq 1$ and property (i) of Definition 2.2.5 holds for every $k \geq 1$. Moreover, the map $\log : G \to \mathfrak{g}$ is a homeomorphism of groups with inverse map $\exp : \mathfrak{g} \to G$, satisfying the Campbell-Hausdorff formula (see [25], §4).

Define $\mathfrak{w} := \mathbb{Q}_p$. Then $\mathfrak{w}$ is a locally compact abelian group and clearly a Λ_k-module. Recall that each p-adic number x can be written uniquely as

$$x = \sum_{j=m}^{\infty} c_j\, p^j \quad \text{for some } m \in \mathbb{Z}, \text{ where } c_j \in \{0, 1, \ldots, p-1\},$$

and we have $x \in \mathbb{Z}_p$ (the p-adic integers) if and only if $c_j = 0$ for all $j < 0$. The map

$$\epsilon : \mathbb{Q}_p \to \mathbb{T},\ \sum_{j=m}^{\infty} c_j\, p^j \mapsto \exp(2\pi i \sum_{j=-\infty}^{-1} c_j)$$

defines a character of the additive group $\mathbb{Q}_p$.

We claim that the Λ_k-module $\mathfrak{w}$ and the character $\epsilon \in \widehat{\mathfrak{w}}$ satisfy properties $(b)-(d)$ of Definition 2.2.5 for every $k \geq l$.

For this, let $k \geq l$. Since the abelian group $\mathfrak{g}$ is a vector space over $\mathbb{Q}_p$, we have $\mathrm{Hom}(\mathfrak{g}, \mathbb{Q}_p) = \mathbb{Q}_p^*$ (the space of continuous linear functionals of $\mathfrak{g}$) and it follows from Theorem 2.11.1 that the map

$$\Phi_\epsilon : \mathfrak{g}^* \to \widehat{\mathfrak{g}},\ f \mapsto \epsilon \circ f$$

is an isomorphism of groups.

Moreover, if $\mathfrak{h}$ is any closed subalgebra of $\mathfrak{g}$, then $\mathfrak{h}$ is a vector subspace of $\mathfrak{g}$ and if $f \in \mathrm{Hom}(\mathfrak{h}, \mathfrak{w}) = \mathfrak{h}^*$ is a continuous linear functional of $\mathfrak{h}$, then there exists by the Hahn-Banach Theorem a functional $\tilde{f} \in \mathfrak{g}^*$, such that $\tilde{f}|_{\mathfrak{h}} = f$.

So it remains to prove that there does not exist a non-trivial Λ_k-submodule of $\mathbb{Q}_p$ inside the kernel of the character ϵ. But we have $\ker(\epsilon) = \mathbb{Z}_p$, and the group of p-adic integers is an integral domain, in which an element $x = \sum_{j=0}^{\infty} c_j\, p^j$ is a unit if and only if $c_0 \neq 0$. In particular, the prime number p is not invertible in $\mathbb{Z}_p$ and since $k \geq p$, there can not exist a nontrivial Λ_k-module inside $\mathbb{Z}_p = \ker(\epsilon)$.

Furthermore, every unipotent linear algebraic group over the local field $K = \mathbb{Q}_p$ satisfies the property that the product of any normal, exponentiable subgroup and any exponentiable subgroup is closed. In fact, every such group satisfies the more general property that the product of any two closed subgroups is closed. To see this, replace the norm $\|.\|$ by the p-norm $\|.\|_p$ in Example 2.11.4. (Recall that if $x = \sum_{j=m}^{\infty} c_j\, p^j \in \mathbb{Q}_p$ then $\|x\|_p := p^{-m}$.)

Therefore, we obtain for every unipotent linear algebraic group G over $\mathbb{Q}_p$ with Lie algebra $\mathfrak{g}$ a homeomorphism

$$\tilde{\kappa} : \mathfrak{g}^* /_\sim \longrightarrow \operatorname{Prim}(C^*(G)),\ \mathcal{O} \mapsto \ker(\operatorname{ind}_R^G \varphi_f),$$

where $f \in \mathfrak{g}^*$ is any chosen representative of the coadjoint quasi-orbit $\mathcal{O}$.

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Lebenslauf

Helma Charlotte Klüver

Geburtsdatum und -ort: 05.10.1978, Neumünster

Familienstand: ledig

Eltern: Wulf Klüver und Monika Klüver-Weitenhagen

Schulbildung	08/1984 – 08/1988	Grundschule Wattenbek, Wattenbek
	08/1988 – 06/1997	Alexander-von-Humboldt Gymnasium, Neumünster
Hochschulreife	07.06.1997	Alexander-von-Humboldt Gymnasium, Neumünster
Studium	10/1997 – 08/2000	Diplom-Mathematik an der Christian-Albrechts-Universität zu Kiel
	08/2000 – 08/2001	Mathematik, Austauschstipendiat der *International Friendship Foundation* an der Arizona State University, Tempe, AZ, U.S.A.
	08/2001 – 08/2002	Master-Programm in Mathematik, Arizona State University, Tempe, AZ, U.S.A.
Prüfungen	10/2000	Vordiplom in Mathematik, Christian-Albrechts-Universität zu Kiel
	08/2002	*Master of Arts in Mathematics*, Arizona-State-University, Tempe, AZ, U.S.A.
Tätigkeiten	11/1999 – 07/2000	Studentische Hilfskraft, Christian-Albrechts-Universität zu Kiel
	08/2001 – 08/2002	Teaching Assistant, Arizona State University, Tempe, AZ, U.S.A.
	seit 2002	Wissenschaftliche Mitarbeiterin am Sonderforschungsbereich 478 “Geometrische Strukturen in der Mathematik”, Westfälische Wilhelms-Universität Münster

Beginn der Dissertation	10/2002	Westfälische Wilhelms-Universität Münster, Mathematisches Institut, Betreuer: Prof. Dr. Siegried Echterhoff