Mathematik

On the Classifying Space for the Family of Virtually Cyclic Subgroups

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Preface

Let G be a discrete group. A model for the classifying space $E_{\mathcal{F}}(G)$ with respect to a family $\mathcal{F}$ of subgroups of G is just a terminal object in the G-homotopy category of G-CW-complexes whose isotropy groups belong to $\mathcal{F}$. It can be shown that such a model always exists, and it is obvious from this definition that any two models are G-homotopy equivalent. However, it is often desirable to find explicit models which are "small" in some sense.

For instance, if $\mathcal{T}r$ is the family which consists only of the trivial subgroup, then a model for $E_{\mathcal{T}r}(G) = EG$ can be characterized up to G-homotopy equivalence as being a free G-CW-complex which is non-equivariantly contractible. These spaces, as well as their quotients $G\backslash EG = BG$, have been studied for a long time. A well-known theorem of Eilenberg and Ganea states that the minimal dimension of a model for EG equals the cohomological dimension $\mathrm{cd}_{\mathbb{Z}}(G)$ of G except possibly if $\mathrm{cd}_{\mathbb{Z}}(G) = 2$ when the minimal dimension might be three.

Similarly, for the family $\mathcal{F}in$ of all finite subgroups of G, questions on the type of models for $E_{\mathcal{F}in}(G) = \underline{E}G$ have been closely investigated by many authors (see [Lüc05] for a survey), and in numerous situations models for $E_{\mathcal{F}in}(G)$ arise in a natural geometrical way. In this thesis, we focus on the problem of constructing explicit models for $E_{\mathcal{V}\mathcal{C}yc}(G)$, where $\mathcal{V}\mathcal{C}yc$ is the family of all virtually cyclic subgroups: this case does not seem to be very well understood. One reason why it is interesting to study these classifying spaces is that they appear in the formulation of the Baum-Connes isomorphism conjecture about the topological K-theory of reduced group C^*-algebras and in the Farrell-Jones isomorphism conjecture about the algebraic K- and L-theory of group rings, respectively. These conjectures predict that one may compute these K- and L-groups by evaluating certain equivariant homology theories at the aforementioned classifying spaces.

In the first chapter, this will be explained in more detail among other things we will need later on. Then, in the next chapter, we will review some of the constructions of models for $E_{\mathcal{F}in}(G)$ before dealing with models for $E_{\mathcal{V}\mathcal{C}yc}(G)$. In particular, we will construct such a model if G is locally virtually cyclic. The third chapter is based on the observation that, for some classes of groups, it is possible to produce a model for $E_{\mathcal{V}\mathcal{C}yc}(G)$ from a given model for $E_{\mathcal{F}in}(G)$. This not only leads to a computation of the relative homology groups which are direct summands of the source of the Farrell-Jones assembly map, but also yields bounds on the dimension that models for $E_{\mathcal{V}\mathcal{C}yc}(G)$ can have. The last chapter is devoted to an explanation of the relation of amenable group actions and the Baum-Connes and Farrell-Jones isomorphism conjectures. We will see that the classifying spaces $E_{\mathcal{F}}(G)$ are amenable

G-spaces if and only if $\mathcal{F}$ consists of amenable groups.

Conventions

We will always work in the category of compactly generated spaces introduced in [Ste67]. In this category, the adjunction $\mathrm{map}(X \times Y, Z) \to \mathrm{map}\big(X, \mathrm{map}(Y, Z)\big)$ is always a homeomorphism, and the product of two CW-complexes is again a CW-complex.

Furthermore, groups will always be assumed to be discrete, and all group actions on spaces are actions from the left unless otherwise stated.

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Münster, October 2006   Michael Weiermann

Contents

1 Classifying Spaces

In this chapter, we will introduce some basic notions that will turn up throughout the work at hand. First of all, this includes the classifying space $E_{\mathcal{F}}(G)$ with respect to a family $\mathcal{F}$ of subgroups of a discrete group G.

Next, we want to explain what the isomorphism conjectures of Baum-Connes and Farrell-Jones have to do with these spaces. These conjectures state that so-called assembly maps should be isomorphisms, which are maps on certain equivariant homology theories induced by the projection $E_{\mathcal{F}}(G) \to \mathrm{pt}$, the family $\mathcal{F}$ consisting of all finite (in the Baum-Connes case) or all virtually cyclic (in the Farrell-Jones case) subgroups of G.

Finally, we will define the homotopy colimit of a space over a category and reprove some of its well-known properties, using the notion of a classifying space of a category.

1.1 Classifying Spaces for Families of Subgroups

The purpose of the following is to define the classifying space $E_{\mathcal{F}}(G)$. To be more precise, we will define it to be a G-CW-complex. Moreover, the proofs of its existence and universal property will briefly be reviewed.

Definition 1.1 (Family of subgroups). A *family $\mathcal{F}$ of subgroups* of a group G is a collection of subgroups of G which is closed under conjugation and taking subgroups, i.e. if $H \in \mathcal{F}$, then also $g^{-1}Hg \in \mathcal{F}$ for every $g \in G$, and $K \in \mathcal{F}$ for every subgroup $K \subset H$.

Examples of such families $\mathcal{F}$ are

$$\mathcal{T}r, \quad \mathcal{F}in, \quad \mathcal{C}yc, \quad \mathcal{V}Cyc, \quad \mathcal{A}ll,$$

denoting the families consisting only of the trivial subgroup, all finite subgroups, all cyclic subgroups, all virtually cyclic subgroups and all subgroups of G, respectively. Bear in mind that a group is virtually cyclic if it contains a cyclic subgroup of finite index.

The restriction of $\mathcal{F}$ to a subgroup $H \subset G$ is $\mathcal{F} \cap H := \{K \cap H \mid K \in \mathcal{F}\}$, and we set $\mathcal{S}ub(H) := \mathcal{A}ll \cap H$.

Definition 1.2 (G-CW-complex). A *G-CW-complex* is a G-space X together with a G-invariant filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset \bigcup_{n \geq 0} X_n = X$ such that

$X = \operatorname{colim}_{n \in \mathbb{N}} X_n$ and X_n is obtained from X_{n-1} by attaching equivariant G-cells, i.e. there is a pushout

$$
\begin{array}{ccc}
\coprod_{i \in I_n} G/H_i \times S^{n-1} & \longrightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{i \in I_n} G/H_i \times D^n & \longrightarrow & X_n
\end{array}
$$

A G-CW-complex is the same as a CW-complex with a G-action by cellular maps such that for each open cell e and each $g \in G$ with $ge \cap e \neq \emptyset$ one has $gx = x$ for all $x \in e$.

A G-CW-complex X is said to be *finite* if $G \backslash X$ is compact, or, equivalently, if it has only finitely many equivariant cells $G/H_i \times D^n$. It is called *of finite type* if every n-skeleton X_n is finite and *n-dimensional* if $X = X_n$ but $X \neq X_{n-1}$.

Definition 1.3 (Classifying space for a family of subgroups). Let $\mathcal{F}$ be a family of subgroups of G. A *model for the classifying space $E_{\mathcal{F}}(G)$* is a G-CW-complex X such that the fixed-point set X^H is empty if $H \notin \mathcal{F}$ and is contractible if $H \in \mathcal{F}$.

Lemma 1.4 (Universal property of $E_{\boldsymbol{\mathcal{F}}}(G)$). *The G-CW-complex X is a model for $E_{\mathcal{F}}(G)$ if and only if the following holds:*

- *The isotropy groups of X belong to $\mathcal{F}$, and*

- *if Y is any G-CW-complex with isotropy groups belonging to $\mathcal{F}$, then there is precisely one G-map $Y \to X$ up to G-homotopy.*

Proof. If X is as in the assumptions, then it remains to show that X^H is contractible for $H \in \mathcal{F}$. Since X^H is a CW-complex, it suffices to show that all its homotopy groups vanish. However, by assumption, there is a G-map $G/H \times S^n \to X$, which is, furthermore, unique up to G-homotopy. Its adjoint is a map $S^n \to \operatorname{map}(G/H, X) = X^H$ which is unique up to homotopy. Hence $\pi_n(X^H)$ is trivial.

If X is a model for $E_{\mathcal{F}}(G)$, then the projection $X^H \to \mathrm{pt}$ is a homotopy equivalence for all $H \in \mathcal{F}$. This implies by the Whitehead theorem for families (cf. e.g. [Lüc89, Prop. 2.3]) that the induced map $[Y, X]_G \to [Y, \mathrm{pt}]_G$ between G-homotopy classes of G-maps of G-CW-complexes is bijective. $\qquad\square$

Thus, a model for $E_{\mathcal{F}}(G)$ is just a terminal object in the G-homotopy category of G-CW-complexes whose isotropy groups are in $\mathcal{F}$. This implies immediately that any two models for $E_{\mathcal{F}}(G)$ must be G-homotopy equivalent. The existence of models for $E_{\mathcal{F}}(G)$ is also not difficult to show:

Proposition 1.5. *There exists a model for $E_{\mathcal{F}}(G)$ for any group G and family of subgroups $\mathcal{F}$.*

Proof. Let $X_0 := \coprod_{H \in \mathcal{F}} G/H$ and assume by induction that X_n is a G-CW-complex with isotropy groups belonging to $\mathcal{F}$ such that $\pi_k(X_n^H)$ is trivial for $H \in \mathcal{F}$ and $0 \leq k \leq n - 1$. For $H \in \mathcal{F}$, we choose a collection $\{f_{H,i} \colon S^n \to X_n^H \mid i \in I\}$ of cellular maps which constitutes a complete system of representatives of the elements in $\pi_n(X_n^H)$. Then, the G-CW-complex X_{n+1} is defined by the G-pushout

$$
\begin{array}{ccc}
\coprod_{H \in \mathcal{F}, i \in I} G/H \times S^n & \xrightarrow{\;\amalg_{H,i} \widetilde{f_{H,i}}\;} & X_n \\
\downarrow & & \downarrow \\
\coprod_{H \in \mathcal{F}, i \in I} G/H \times D^{n+1} & \xrightarrow{\hspace{3cm}} & X_{n+1}
\end{array}
$$

in which the maps $\widetilde{f_{H,i}}$ denote the adjoints of the maps $f_{H,i}$. This completes the induction step since for $H \in \mathcal{F}$ any map $S^n \to X_{n+1}^H$ is homotopic to a map into X_n^H by the cellular approximation theorem, and any such map can be extended to D^{n+1} by construction. Finally, by taking the colimit of the X_n, we obtain a model for $E_{\mathcal{F}}(G)$. $\qquad\square$

There is also a functorial construction of models for $E_{\mathcal{F}}(G)$, see Example 1.26.

Example 1.6. The following is a list of immediate examples of classifying spaces:

- The one point space G/G is a model for $E_{\mathcal{F}}(G)$ if and only if $\mathcal{F} = \mathcal{A}\ell\ell$.

- $EG := E_{\mathcal{T}r}(G)$ is just a free G-CW-complex which is contractible after forgetting the G-action. It also occurs as the total space of the universal principal G-bundle $G \to EG \to BG$.

- $\underline{E}G := E_{\mathcal{F}in}(G)$ is sometimes called the universal G-CW-complex for proper G-actions. If G is torsion-free, then $\underline{E}G = EG$.

We will end this section by stating a lemma we will frequently use.

Lemma 1.7. *Let $H \subset G$ be an inclusion of groups and $s \colon G/H \to G$ a (set-theoretic) section of the projection.*

(1) *Let X be an H-space. Then the induced G-space $G \times_H X$ is naturally G-homeomorphic to the G-space $G/H \times X$ which is defined by $g \cdot (\alpha, x) := \big(g\alpha, s(g\alpha)^{-1} g s(\alpha) x\big)$.*

In particular, if $K \subset G$ is another subgroup, then

$$
(G \times_H X)^K \cong \coprod_{\substack{\alpha \in G/H, \\ s(\alpha)^{-1} K s(\alpha) \subset H}} X^{s(\alpha)^{-1} K s(\alpha)}
$$

are naturally homeomorphic spaces.

(2) *Let Y be a G-space. Then the G-space $G \times_H \operatorname{res}_G^H Y$ is naturally G-homeo-morphic to the diagonal G-space $G/H \times Y$.*

Proof. In the situation of (1), the following G-maps are inverse to each other:

$$G \times_H X \rightleftarrows G/H \times X$$
$$[g, x] \mapsto \big(gH, s(gH)^{-1}gx\big)$$
$$[s(\alpha), x] \leftarrow (\alpha, x)$$

The homeomorphism of the fixed-point sets then follows from the observation that for $\alpha \in G/H$ one has $k\alpha = \alpha$ for every $k \in K$ if and only if $s(\alpha)^{-1}Ks(\alpha) \subset H$.

As for (2), the following G-maps are inverse to each other:

$$G \times_H \operatorname{res}_G^H Y \rightleftarrows G/H \times Y$$
$$[g, y] \mapsto (gH, gy)$$
$$[s(\alpha), s(\alpha)^{-1}y] \leftarrow (\alpha, y) \qquad \square$$

1.2 Equivariant Homology Theories

The goal of this section is to formulate the Baum-Connes and Farrell-Jones conjectures. In order to do so, we will explain a general way of constructing homology theories on pairs of spaces over a category.

1.2.1 Spaces over a Category

Definition 1.8 (Space over a category). Let $\mathcal{C}$ be a small category. A *covariant* (or *contravariant*) *$\mathcal{C}$-space* is a covariant (or contravariant) functor $X \colon \mathcal{C} \to$ *Spaces* from $\mathcal{C}$ to the category of compactly generated spaces. A *map $X \to Y$ of $\mathcal{C}$-spaces* is a natural transformation of functors. The space $\hom_{\mathcal{C}}(X, Y)$ of such maps is equipped with the subspace topology of the obvious inclusion into $\prod_{c \in \mathrm{ob}(\mathcal{C})} \operatorname{map}\big(X(c), Y(c)\big)$.

One can take coproducts, colimits, etc. in the category of $\mathcal{C}$-spaces by applying the usual constructions for spaces objectwise. Furthermore, it becomes clear what a homotopy of maps of $\mathcal{C}$-spaces should mean once we have said that from a $\mathcal{C}$-space X we obtain the $\mathcal{C}$-space $X \times [0, 1]$ by sending $c \in \mathrm{ob}(\mathcal{C})$ to $X(c) \times [0, 1]$.

Example 1.9 (Orbit category). Let G be a group and $\mathcal{F}$ a family of subgroups. The *orbit category* $\operatorname{Or}(G, \mathcal{F})$ *with respect to* $\mathcal{F}$ is the category with homogeneous G-spaces G/H for $H \in \mathcal{F}$ as objects and G-maps as morphisms. Note that a map $G/H \to G/K$ is a G-map if and only if it is of the form $r_{g_0} \colon gH \mapsto gg_0K$, where $g_0 \in G$ is such that $g_0^{-1}Hg_0 \subset K$. We abbreviate $\operatorname{Or}(G) := \operatorname{Or}(G, \mathcal{A}ll)$.

Every left G-space X yields a contravariant $\operatorname{Or}(G, \mathcal{F})$-space $\operatorname{map}_G(-, X) = X^-$ by assigning the space $\operatorname{map}_G(G/H, X) = X^H$ of fixed-points to an object G/H of $\operatorname{Or}(G, \mathcal{F})$.

Definition 1.10 (Balanced product of $\mathcal{C}$-spaces). Let X be a contravariant and Y a covariant $\mathcal{C}$-space. The *balanced product of X and Y over $\mathcal{C}$* is defined to be the space

$$X \times_\mathcal{C} Y := \coprod_{c \in \mathrm{ob}(\mathcal{C})} X(c) \times Y(c) / \sim,$$

where $\sim$ denotes the equivalence relation generated by $\big(X(f)(x), y\big) \sim \big(x, Y(f)(y)\big)$ for all $x \in X(d)$, $y \in Y(c)$ and morphisms $f \colon c \to d$ in $\mathcal{C}$.

In the following note that $\mathrm{mor}_\mathcal{C}(-, -)$ can be considered as a covariant $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$-space by equipping $\mathrm{mor}_\mathcal{C}(c, c')$ with the discrete topology. Then, for a $\mathcal{D}$-space X and a covariant functor $F \colon \mathcal{C} \to \mathcal{D}$, the *restriction of X by F* is the $\mathcal{C}$-space $\mathrm{res}_F X = F^* X$ given by $c \mapsto X\big(F(c)\big)$. If X is a co- or contravariant $\mathcal{C}$-space, the *induction of X by F* is the co- or contravariant $\mathcal{D}$-space $\mathrm{ind}_F X$ given by

$$d \mapsto \mathrm{mor}_\mathcal{D}(F, d) \times_\mathcal{C} X \qquad \text{or} \qquad d \mapsto X \times_\mathcal{C} \mathrm{mor}_\mathcal{D}(d, F)$$

respectively.

Lemma 1.11 (Adjointness of induction and restriction). *Suppose that Y is a covariant $\mathcal{D}$-space and X a $\mathcal{C}$-space of the required variance to make the following statements meaningful, and that $F \colon \mathcal{C} \to \mathcal{D}$ is a covariant functor. Then:*

(1) There are homeomorphisms

$$\mathrm{hom}_\mathcal{D}\big(\mathrm{mor}_\mathcal{D}(F, -), Y\big) \xleftarrow{\cong} \mathrm{res}_F Y \xrightarrow{\cong} \mathrm{mor}_\mathcal{D}(-, F) \times_\mathcal{D} Y$$

of covariant $\mathcal{C}$-spaces.

(2) There are natural homeomorphisms

$$\mathrm{ind}_F X \times_\mathcal{D} Y \xrightarrow{\cong} X \times_\mathcal{C} \mathrm{res}_F Y,$$
$$\mathrm{hom}_\mathcal{D}(\mathrm{ind}_F X, Y) \xrightarrow{\cong} \mathrm{hom}_\mathcal{C}(X, \mathrm{res}_F Y).$$

Analogous results hold for a contravariant $\mathcal{D}$-space Y.

Proof. The second homeomorphism of (1) comes from the mutually inverse maps which are given by

$$Y\big(F(c)\big) \ni y \mapsto [\mathrm{id}_{F(c)}, y]$$
$$Y(f)(y) \leftarrow\!\shortmid (f, y) \in \mathrm{mor}_\mathcal{D}\big(d, F(c)\big) \times Y(d)$$

and the first is the Yoneda lemma.

Now (2) follows from (1) using the associativity of the balanced product of spaces over a category (see [Mac98, section IX.8]) and the adjointness of the balanced product of $\mathcal{C}$-spaces and $\mathrm{hom}_\mathcal{D}$ (cf. [DL98, Lemma 1.5]). $\qquad\square$

Lemma 1.12. *Let $\alpha\colon H \hookrightarrow G$ be an injective group homomorphism and X an H-space. Then α defines a functor $\alpha\colon \mathrm{Or}(H) \to \mathrm{Or}(G)$ in the obvious way, and*

$$\mathrm{ind}_\alpha(X^-) \xrightarrow{\cong} (G \times_\alpha X)^-$$

are homeomorphic $\mathrm{Or}(G)$-spaces.

Proof. A homeomorphism $\nu\colon X^{--} \times_{\mathrm{Or}(H)} \mathrm{map}_G\big(-, G/\alpha(--)\big) \to (G \times_\alpha X)^-$ is defined by setting

$$\nu(G/K)[x, r_g] := [g, x].$$

To put it differently, ν corresponds under the natural homeomorphism

$$\hom_{\mathrm{Or}(G)}\big(\mathrm{ind}_\alpha X, (G \times_\alpha X)^-\big) \cong \hom_{\mathrm{Or}(H)}\big(X^-, \mathrm{res}_\alpha(G \times_\alpha X)^-\big)$$

of Lemma 1.11 (2) to $\widetilde{\nu}\colon X^- \to \mathrm{res}_\alpha(G \times_\alpha X)^-$ given by $\widetilde{\nu}(H/K)(x) = [1, x]$. $\qquad\square$

Definition 1.13 ($\mathcal{C}$-CW-complex). A *(contravariant) $\mathcal{C}$-CW-complex* is a contravariant $\mathcal{C}$-space X together with a filtration $\emptyset = X_{-1} \subset X_0 \subset X_1 \subset \ldots \subset \bigcup_{n \geq 0} X_n = X$ by contravariant $\mathcal{C}$-spaces such that $X = \mathrm{colim}_{n\to\infty} X_n$ and X_n is obtained from X_{n-1} by attaching free $\mathcal{C}$-n-cells for any $n \geq 0$, i.e. there is a pushout of $\mathcal{C}$-spaces

$$
\begin{array}{ccc}
\coprod_{i \in I_n} \mathrm{mor}_{\mathcal{C}}(-, c_i) \times S^{n-1} & \longrightarrow & X_{n-1} \\
\downarrow & & \downarrow \\
\coprod_{i \in I_n} \mathrm{mor}_{\mathcal{C}}(-, c_i) \times D^n & \longrightarrow & X_n
\end{array}
$$

the c_i being objects in $\mathcal{C}$ for every element i of an index set I_n, and the vertical maps being inclusions of $\mathcal{C}$-spaces.

Example 1.14. If X is a G-CW-complex, then the $\mathrm{Or}(G)$-space X^- is an $\mathrm{Or}(G)$-CW-complex. This is because a pushout telling how X_n is obtained from X_{n-1} by attaching equivariant n-cells remains a pushout after taking fixed-points. Thus, we get a pushout of $\mathcal{C}$-spaces telling how X_n^- is obtained from X_{n-1}^-, an $\mathrm{Or}(G)$-cell of the form $\mathrm{map}_G(-, G/H) \times D^n$ corresponding to a G-cell $G/H \times D^n$ of X.

The next lemma indicates that standard results for CW-complexes have straightforward analogues for $\mathcal{C}$-CW-complexes. We remark that a *$\mathcal{C}$-CW-approximation* of a $\mathcal{C}$-space X consists of a $\mathcal{C}$-CW-complex Y and a map $f\colon Y \to X$ of $\mathcal{C}$-spaces which is a *weak homotopy equivalence*, meaning that $f(c)$ is a weak homotopy equivalence of spaces for all $c \in \mathrm{ob}(\mathcal{C})$.

Lemma 1.15.

(1) *Any $\mathcal{C}$-space X possesses a $\mathcal{C}$-CW-approximation (Y, f). Moreover, if (Y', f') is another $\mathcal{C}$-CW-approximation of X, then there is a homotopy equivalence $g\colon Y \to Y'$ which is uniquely determined up to homotopy by the property that $f' \circ g$ is homotopic to f.*

(2) *A weak homotopy equivalence of $\mathcal{C}$-CW-complexes is already a homotopy equivalence.*

(3) *Let Z be a $\mathcal{C}$-CW-complex and $f \colon X \to Y$ a weak homotopy equivalence of covariant $\mathcal{C}$-spaces. Then $\mathrm{id}_Z \times_\mathcal{C} f \colon Z \times_\mathcal{C} X \to Z \times_\mathcal{C} Y$ is a weak homotopy equivalence.*

Proof. This is taken from [DL98, Thm. 3.7, Cor. 3.5 and Thm. 3.11]. □

1.2.2 Construction of Equivariant Homology Theories

Analogously to the notion of a space over a category, one can speak of pointed spaces, spectra, etc. over a category. For instance, the balanced product $X \wedge_\mathcal{C} Y$ of two pointed $\mathcal{C}$-spaces is given as in Definition 1.10, merely replacing the disjoint union by a one-point union and the cross product by a smash product. Of course, results like Lemma 1.11 carry over.

We fix notation and emphasize that a spectrum $\mathbf{E}$ is a collection of pointed spaces $\{E_n\}_{n\in\mathbb{Z}}$ together with pointed maps $\sigma_n \colon E_n \wedge S^1 \to E_{n+1}$, the structure maps, while a map of spectra $\mathbf{f} \colon \mathbf{E} \to \mathbf{F}$ is given by pointed maps $f_n \colon E_n \to F_n$ that are compatible with these structure maps, i.e. $\sigma_n^F \circ (f_n \wedge \mathrm{id}_{S^1}) = f_{n+1} \circ \sigma_n^E$. The homotopy groups of a spectrum are given by

$$\pi_k(\mathbf{E}) := \operatorname*{colim}_{n\to\infty} \pi_{k+n}(E_n)$$

for $k \in \mathbb{Z}$. Here the required maps $\pi_{k+n}(E_n) \to \pi_{k+n+1}(E_{n+1})$ come from the composition of the suspension homomorphism and the homomorphism induced by the structure map.

A covariant $\mathcal{C}$-spectrum $\mathbf{E}$ can also be viewed as a collection $\{\mathbf{E}(-)_n\}_{n\in\mathbb{Z}}$ of pointed $\mathcal{C}$-spaces, the structure maps being maps of pointed $\mathcal{C}$-spaces. Thus it is clear that for any pointed $\mathcal{C}$-space X we get a spectrum $X \wedge_\mathcal{C} \mathbf{E}$. Now, setting

$$\mathcal{H}_n^\mathcal{C}(X, A; \mathbf{E}) := \pi_n\big(Y_+ \cup_{B_+} \mathrm{cone}(B_+) \wedge_\mathcal{C} \mathbf{E}\big)$$

defines an unreduced homology theory on pairs of $\mathcal{C}$-spaces satisfying the disjoint union axiom such that weak homotopy equivalences of such pairs induce isomorphisms on homology, see [DL98, Lemma 4.4]. Here (Y, B) is a $\mathcal{C}$-CW-approximation of (X, A), and $Y_+ := Y \amalg \mathrm{pt}$ is the pointed $\mathcal{C}$-space obtained from Y by adjoining the trivial $\mathcal{C}$-space as a base point.

Example 1.16 (Borel homology). The set of morphisms of $\mathrm{Or}(G, \mathcal{Tr})$ can be identified with G by sending $r_g \colon G/1 \to G/1$ to $g^{-1} \in G$. Then, a contravariant $\mathrm{Or}(G, \mathcal{Tr})$-space is the same as a right G-space (analogously for $\mathrm{Or}(G, \mathcal{Tr})$-spectra), whereas an $\mathrm{Or}(G, \mathcal{Tr})$-CW-complex is the same as a free G-space, and

$$\mathcal{H}_n^{\mathrm{Or}(G,\mathcal{Tr})}(X; \mathbf{E}) = \pi_n\big((X \times EG)_+ \wedge_G \mathbf{E}\big) =: H_n^G(X; \mathbf{E})$$

can be identified with Borel homology.

Let S be a G-set. The corresponding *transport groupoid* $\mathcal{G}^G(S)$ is the groupoid having as its set of objects the set S, while the morphisms from s_0 to s_1 are the elements $g \in G$ that satisfy $gs_0 = s_1$. This yields a functor $\mathcal{G}^G \colon \mathrm{Or}(G) \to \textit{Groupoids}$ in the obvious way.

In the case of a G-CW-pair (X, A) and a covariant functor $\mathbf{E} \colon \textit{Groupoids} \to \textit{Spectra}$ which sends equivalences of groupoids to maps of spectra inducing an isomorphism on homotopy groups, we define

$$\mathcal{H}_n^G(X, A; \mathbf{E}) := \mathcal{H}_n^{\mathrm{Or}(G)}\big(X^-, A^-; \mathbf{E} \circ \mathcal{G}^G\big). \tag{1.17}$$

It is shown in [Sau02]:

Proposition 1.18. *The various functors $\mathcal{H}_n^G(-; \mathbf{E})$ for all groups G, which are defined in* (1.17)*, match up to form an equivariant homology theory.*

This means in particular that there is an induction structure, i.e. if $\alpha \colon H \to G$ is a group homomorphism and (X, A) an H-CW-pair such that $\ker(\alpha)$ acts freely on X, there are natural isomorphisms

$$\mathrm{ind}_\alpha \colon \mathcal{H}_n^H(X, A; \mathbf{E}) \xrightarrow{\cong} \mathcal{H}_n^G\big(G \times_\alpha (X, A); \mathbf{E}\big)$$

for $n \in \mathbb{Z}$ which have certain properties like being functorial in α and being compatible with the boundary homomorphisms.

Notation 1.19. For a cellular map $f \colon X \to Y$ of G-CW-complexes, we set

$$\mathcal{H}_n^G(f \colon X \to Y; \mathbf{E}) := \mathcal{H}_n^G\big(\mathrm{cyl}(f), X; \mathbf{E}\big),$$

where X is considered as a G-subcomplex of the mapping cylinder of f.

1.2.3 Formulation of the Isomorphism Conjectures

We will adopt the point of view of [DL98] to formulate the isomorphism conjectures [BCH94, Conj. 3.15] of Baum-Connes and [FJ93, Conj. 1.6] of Farrell-Jones.

Let R be an associative ring with unit and involution. One can construct covariant *Groupoids*-spectra $\mathbf{K}^{\mathrm{top}}$, $\mathbf{K}_R$ and $\mathbf{L}_R^{\langle -\infty \rangle}$ which send equivalences of groupoids to maps of spectra inducing an isomorphism on homotopy groups, such that

$$\begin{aligned}
\pi_n\big(\mathbf{K}^{\mathrm{top}}(\mathcal{G}^G(G/H))\big) &= K_n(C_r^* H), \\
\pi_n\big(\mathbf{K}_R(\mathcal{G}^G(G/H))\big) &= K_n(RH), \\
\pi_n\big(\mathbf{L}_R^{\langle -\infty \rangle}(\mathcal{G}^G(G/H))\big) &= L_n^{\langle -\infty \rangle}(RH),
\end{aligned}$$

see [DL98, section 2] and [Joa03]. These groups denote the topological K-theory of the reduced group C^*-algebra of H and the algebraic K- and L-theory of the group ring RH, respectively.

Conjecture 1.20. *The* Baum-Connes isomorphism conjecture *for a group G states that the assembly map*

$$\mathcal{H}_n^G\big(E_{\mathcal{F}in}(G); \mathbf{K}^{\mathrm{top}}\big) \to \mathcal{H}_n^G(\mathrm{pt}; \mathbf{K}^{\mathrm{top}}) = K_n(C_r^*G), \tag{1.21}$$

which is the map induced by the projection $E_{\mathcal{F}in}(G) \to \mathrm{pt}$, is an isomorphism for $n \in \mathbb{Z}$.

The Farrell-Jones isomorphism conjecture *for the group ring RG states that the assembly maps*

$$\mathcal{H}_n^G\big(E_{\mathcal{VC}yc}(G); \mathbf{K}_R\big) \to \mathcal{H}_n^G(\mathrm{pt}; \mathbf{K}_R) = K_n(RG), \tag{1.22}$$

$$\mathcal{H}_n^G\big(E_{\mathcal{VC}yc}(G); \mathbf{L}_R^{\langle-\infty\rangle}\big) \to \mathcal{H}_n^G(\mathrm{pt}; \mathbf{L}_R^{\langle-\infty\rangle}) = L_n^{\langle-\infty\rangle}(RG) \tag{1.23}$$

coming from the projection $E_{\mathcal{VC}yc}(G) \to \mathrm{pt}$ are isomorphisms for $n \in \mathbb{Z}$.

While the Baum-Connes conjecture is known to be true for quite a large class of groups, not that much is known in the Farrell-Jones case, reflecting the fact that there the family $\mathcal{VC}yc$ of virtually cyclic subgroups of G has to be taken into account, which usually is harder to handle than the family $\mathcal{F}in$. The point behind these conjectures is to compute the target of the assembly map (the group which is of interest) by looking at the source which might be more accessible to calculations. For a survey on this matter, we encourage the reader to consult [LR05].

Note that, because of the universal property of $E_{\mathcal{VC}yc}(G)$, there is a G-map $E_{\mathcal{F}in}(G) \to E_{\mathcal{VC}yc}(G)$ which is unique up to G-homotopy. We mention the following result taken from [Bar03]:

Proposition 1.24 (The relative homology groups split off). *The canonical G-map $E_{\mathcal{F}in}(G) \to E_{\mathcal{VC}yc}(G)$ induces a split injection on $\mathcal{H}_n^G(-; \mathbf{K}_R)$. Hence there is an isomorphism*

$$\mathcal{H}_n^G\big(E_{\mathcal{F}in}(G); \mathbf{K}_R\big) \oplus \mathcal{H}_n^G\big(E_{\mathcal{F}in}(G) \to E_{\mathcal{VC}yc}(G); \mathbf{K}_R\big) \xrightarrow{\cong} \mathcal{H}_n^G\big(E_{\mathcal{VC}yc}(G); \mathbf{K}_R\big).$$

The same holds for $\mathcal{H}_n^G(-; \mathbf{L}_R^{\langle-\infty\rangle})$, provided that for any virtually cyclic subgroup $V \subset G$ one has $K_{-i}(RV) = 0$ for sufficiently large i.

1.3 Homotopy Colimits

In section 2.3, we will establish a concrete model for $E_{\mathcal{F}}(G)$ in case that G is a colimit of subgroups $\{G_i\}_{i \in I}$ of which models for $E_{\mathcal{F} \cap G_i}(G_i)$ are given. Below, we will compile the necessary facts on homotopy colimits that will be needed there.

Definition 1.25 (Classifying space of a category). A model for the *classifying space $E\mathcal{C}$ of the category $\mathcal{C}$* is a $\mathcal{C}$-CW-approximation of the trivial contravariant $\mathcal{C}$-space pt. $E\mathcal{C}$ is uniquely determined up to homotopy equivalence of $\mathcal{C}$-spaces (see Lemma 1.15 (1)).

There is a functorial construction for $E\mathcal{C}$ which will be explained now. Recall that the geometric realization $B^{\mathrm{bar}}\mathcal{C}$ of $\mathcal{C}$ is the geometric realization $|N.\mathcal{C}|$ of its nerve. In particular, $B^{\mathrm{bar}}\mathcal{C}$ has a canonical CW-structure, the n-cells corresponding to sequences $c_0 \to c_1 \to \ldots \to c_n$ of morphisms in $\mathcal{C}$ none of which is the identity. Any functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a cellular map $B^{\mathrm{bar}}F\colon B^{\mathrm{bar}}\mathcal{C} \to B^{\mathrm{bar}}\mathcal{D}$, and a natural transformation of two functors F and G induces a homotopy between $B^{\mathrm{bar}}F$ and $B^{\mathrm{bar}}G$. One says that $\mathcal{C}$ is contractible if $B^{\mathrm{bar}}\mathcal{C}$ is.

For fixed $c_1, c_2 \in \mathrm{ob}(\mathcal{C})$, the objects of the category $c_1 \downarrow \mathcal{C} \downarrow c_2$ are all diagrams in $\mathcal{C}$ of the form $c_0 \xrightarrow{f} c \xrightarrow{g} c_2$. A morphism from $c_0 \xrightarrow{f} c \xrightarrow{g} c_2$ to $c_0 \xrightarrow{f'} c' \xrightarrow{g'} c_2$ is given by a morphism $h\colon c \to c'$ in $\mathcal{C}$ such that $h \circ f = f'$ and $g' \circ h = g$.

If $F\colon \mathcal{C} \to \mathcal{D}$ is a functor and $d \in \mathrm{ob}(\mathcal{D})$, the objects of the *category $d \downarrow F$ of objects F-under d* are all morphisms in $\mathcal{D}$ of the form $f\colon d \to F(c)$, and the morphisms from $f\colon d \to F(c)$ to $f'\colon d \to F(c')$ are all morphisms $h\colon c \to c'$ in $\mathcal{C}$ such that $F(h) \circ f = f'$. Similarly, one can define the *category $F \downarrow d$ of objects F-over d*. If F is the identity functor on $\mathcal{D}$, these categories are denoted by $d \downarrow \mathcal{D}$ and $\mathcal{D} \downarrow d$, respectively.

Moreover, in the obvious way we obtain covariant $\mathcal{C}^{\mathrm{op}} \times \mathcal{C}$-, $\mathcal{D}^{\mathrm{op}}$- and $\mathcal{D}$- categories $- \downarrow \mathcal{C} \downarrow -$, $- \downarrow F$ and $F \downarrow -$, respectively. Now we set

$$E^{\mathrm{bar}}\mathcal{C} := \mathrm{pt} \times_{\mathcal{C}} B^{\mathrm{bar}}(- \downarrow \mathcal{C} \downarrow -) = B^{\mathrm{bar}}(- \downarrow \mathcal{C}).$$

This gives a model for $E\mathcal{C}$ by [DL98, Lemma 3.19(3)].

Example 1.26 (Functorial construction for $E_{\mathcal{F}}(G)$). A model for $E_{\mathcal{F}}(G)$ yields a model $E_{\mathcal{F}}(G)^-$ for $E\,\mathrm{Or}(G, \mathcal{F})$. On the other hand, consider the covariant $\mathrm{Or}(G, \mathcal{F})$-space I which is given by sending G/H to itself. Then a model for $E\,\mathrm{Or}(G, \mathcal{F})$ defines the G-CW-complex $E\,\mathrm{Or}(G, \mathcal{F}) \times_{\mathrm{Or}(G,\mathcal{F})} I$, which is a model for $E_{\mathcal{F}}(G)$, see [DL98, Lemma 7.6].

Definition 1.27 (Colimit and homotopy colimit). Let X be a covariant $\mathcal{C}$-space. Its *colimit* and *homotopy colimit* are defined by

$$\operatorname*{colim}_{\mathcal{C}} X := \mathrm{pt} \times_{\mathcal{C}} X \qquad \text{and} \qquad \operatorname*{hocolim}_{\mathcal{C}} X := E\mathcal{C} \times_{\mathcal{C}} X$$

respectively, where pt is the trivial $\mathcal{C}$-space. Note that $\mathrm{hocolim}_{\mathcal{C}} X$ is only defined up to homotopy equivalence.

The following theorem lists the main properties the notion of homotopy colimit should have. They are well-known, cf. [HV92], but we will give a proof in our context below. As for assertion (2) of the theorem, recall that a non-empty category $\mathcal{C}$ is *filtered* if for any two objects c_0 and c_1 in $\mathcal{C}$ there is an object c in $\mathcal{C}$ together with morphisms $c_0 \to c$ and $c_1 \to c$, and if for any two morphisms $f, g\colon c_0 \to c_1$ in $\mathcal{C}$ there is a morphism $h\colon c_1 \to c_2$ in $\mathcal{C}$ such that $h \circ f = h \circ g$.

Theorem 1.28 (Properties of hocolim). *Let X and Y be covariant $\mathcal{D}$-spaces and $F\colon \mathcal{C} \to \mathcal{D}$ a covariant functor. Then:*

(1) *(Homotopy invariance)*
Every weak homotopy equivalence $X \to Y$ of $\mathcal{D}$-spaces induces a weak homotopy equivalence
$$\operatorname*{hocolim}_{\mathcal{D}} X \to \operatorname*{hocolim}_{\mathcal{D}} Y.$$

(2) *(Cofinality)*
If for all $d \in \operatorname{ob}(\mathcal{D})$ the category $d \downarrow F$ is contractible, e.g. if $\mathcal{C}$ is filtered and F cofinal, then there is a weak homotopy equivalence
$$\operatorname*{hocolim}_{\mathcal{C}} F^* X \to \operatorname*{hocolim}_{\mathcal{D}} X.$$

(3) *(Reduction)*
There is a homotopy equivalence
$$\operatorname*{hocolim}_{\mathcal{C}} F^* X \xrightarrow{\cong} B^{\mathrm{bar}}(- \downarrow F) \times_{\mathcal{D}} X.$$

Proof. The proof of (1) can be found in [DL98, Thm. 3.11].

As for (3), there are homeomorphisms

$$E^{\mathrm{bar}} \mathcal{C} \times_{\mathcal{C}} F^* X \xrightarrow{\cong} B^{\mathrm{bar}}(- \downarrow \mathcal{C}) \times_{\mathcal{C}} \operatorname{mor}_{\mathcal{D}}(-, F) \times_{\mathcal{D}} X \xrightarrow{\cong} B^{\mathrm{bar}}(- \downarrow F) \times_{\mathcal{D}} X, \quad (1.29)$$

the first due to Lemma 1.11 (1) and the second coming from the homeomorphism

$$B^{\mathrm{bar}}(- \downarrow \mathcal{C}) \times_{\mathcal{C}} \operatorname{mor}_{\mathcal{D}}(-, F) \xrightarrow{\cong} B^{\mathrm{bar}}(- \downarrow F) \qquad (1.30)$$

of contravariant $\mathcal{D}$-spaces defined as follows. Let us denote by $\overline{\operatorname{mor}_{\mathcal{D}}(-, F)}$ the $\mathcal{D}^{\mathrm{op}} \times \mathcal{C}$-category given by sending an object (d, c) of $\mathcal{D}^{\mathrm{op}} \times \mathcal{C}$ to the category with set of objects equal to $\operatorname{mor}_{\mathcal{D}}(d, F(c))$ and set of morphisms equal to the identities. Then $B^{\mathrm{bar}}\overline{\operatorname{mor}_{\mathcal{D}}(-, F)} = \operatorname{mor}_{\mathcal{D}}(-, F)$. We have that $B^{\mathrm{bar}} = |-| \circ N.$, where $|-|$ has a right adjoint and therefore (cf. [Mac98, section V.5]) preserves arbitrary colimits such as the balanced product of two $\mathcal{C}$-spaces (cf. [Mac98, Prop. IX.5.1]). This means that we only need to construct a natural equivalence $N.(- \downarrow \mathcal{C}) \times_{\mathcal{C}} N.\big(\overline{\operatorname{mor}_{\mathcal{D}}(-, F)}\big) \to N.(- \downarrow F)$ of simplicial $\mathcal{D}^{\mathrm{op}}$-spaces, and one such is induced by the mutually inverse maps

$$N_n(- \downarrow \mathcal{C}) \times_{\mathcal{C}} N_n\left(\overline{\operatorname{mor}_{\mathcal{D}}\big(d, F(-)\big)}\right) \rightleftarrows N_n(d \downarrow F)$$

$$\begin{bmatrix} \begin{array}{ccccc} c & = & c & = \cdots = & c \\ \downarrow & & \downarrow & & \downarrow \\ c_0 & \to & c_1 & \to \cdots \to & c_n \end{array} , \begin{array}{c} d \\ \downarrow \\ F(c) \end{array} \end{bmatrix} \mapsto \begin{array}{ccccc} d & = & d & = \cdots = & d \\ \downarrow & & \downarrow & & \downarrow \\ F(c_0) & \to & F(c_1) & \to \cdots \to & F(c_n) \end{array}$$

$$\begin{bmatrix} \begin{array}{ccccc} c_0 & = & c_0 & = \cdots = & c_0 \\ \| & & \downarrow & & \downarrow \\ c_0 & \to & c_1 & \to \cdots \to & c_n \end{array} , \begin{array}{c} d \\ \downarrow \\ F(c_0) \end{array} \end{bmatrix} \hookleftarrow \begin{array}{ccccc} d & = & d & = \cdots = & d \\ \downarrow & & \downarrow & & \downarrow \\ F(c_0) & \to & F(c_1) & \to \cdots \to & F(c_n) \end{array}$$

Concerning (2), we first remark that if $\mathcal{C}$ is filtered and F cofinal, then the category $d \downarrow F$ is filtered for every $d \in \mathrm{ob}(\mathcal{D})$, and any filtered category is contractible, see e.g. [Qui73, Cor. 2 in § 1 on p. 93]. Now let us prove the assertion. It is not difficult to check that, for any $c \in \mathrm{ob}(\mathcal{C})$, the natural transformation $\overline{\mathrm{mor}_{\mathcal{D}}(-, F(c))} \to - \downarrow \mathcal{D} \downarrow F(c)$ of contravariant $\mathcal{D}$-categories which is given on objects by sending $f \colon d \to F(c)$ to $f \circ \mathrm{id}_d$ induces a weak homotopy equivalence $\mathrm{mor}_{\mathcal{D}}(-, F(c)) \to B^{\mathrm{bar}}(- \downarrow \mathcal{D} \downarrow F(c))$ of $\mathcal{D}$-CW-complexes. By Lemma 1.15 (2), this is even a homotopy equivalence. Using Lemma 1.11 (1), it follows that there is weak homotopy equivalence $F^* X \to B^{\mathrm{bar}}(- \downarrow \mathcal{D} \downarrow F) \times_{\mathcal{D}} X$ of covariant $\mathcal{C}$-spaces and thus, by Lemma 1.15 (3), a weak homotopy equivalence

$$E^{\mathrm{bar}}\mathcal{C} \times_{\mathcal{C}} F^* X \to E^{\mathrm{bar}}\mathcal{C} \times_{\mathcal{C}} B^{\mathrm{bar}}(- \downarrow \mathcal{D} \downarrow F) \times_{\mathcal{D}} X.$$

Furthermore, because of Lemma 1.11 (1) and $E^{\mathrm{bar}}\mathcal{C} = B^{\mathrm{bar}}(- \downarrow \mathcal{C})$ together with (1.30), there is a homeomorphism

$$E^{\mathrm{bar}}\mathcal{C} \times_{\mathcal{C}} B^{\mathrm{bar}}(- \downarrow \mathcal{D} \downarrow F) \times_{\mathcal{D}} X \xrightarrow{\cong} B^{\mathrm{bar}}(- \downarrow F) \times_{\mathcal{D}} B^{\mathrm{bar}}(- \downarrow \mathcal{D} \downarrow -) \times_{\mathcal{D}} X.$$

The contravariant $\mathcal{D}$-space $B^{\mathrm{bar}}(- \downarrow F)$ is, by assumption, weakly homotopy equivalent to the trivial contravariant $\mathcal{D}$-space pt. As $B^{\mathrm{bar}}(d \downarrow \mathcal{D} \downarrow -)$ is a $\mathcal{D}^{\mathrm{op}}$-$CW$-complex for $d \in \mathrm{ob}(\mathcal{D})$, we obtain, by the analogue of Lemma 1.15 (3), a weak homotopy equivalence $\eta \colon B^{\mathrm{bar}}(- \downarrow F) \times_{\mathcal{D}} B^{\mathrm{bar}}(- \downarrow \mathcal{D} \downarrow -) \to \mathrm{pt} \times_{\mathcal{D}} B^{\mathrm{bar}}(- \downarrow \mathcal{D} \downarrow -)$ of contravariant $\mathcal{D}$-spaces. One can actually show that the source and target of η carry the structure of $\mathcal{D}$-CW-complexes. Hence Lemma 1.15 (2) implies that η is even a homotopy equivalence. Thus we finally obtain a homotopy equivalence

$$B^{\mathrm{bar}}(- \downarrow F) \times_{\mathcal{D}} B^{\mathrm{bar}}(- \downarrow \mathcal{D} \downarrow -) \times_{\mathcal{D}} X \xrightarrow{\cong} \mathrm{pt} \times_{\mathcal{D}} B^{\mathrm{bar}}(- \downarrow \mathcal{D} \downarrow -) \times_{\mathcal{D}} X,$$

the latter being equal to $E^{\mathrm{bar}}\mathcal{D} \times_{\mathcal{D}} X$ by definition. $\qquad\square$

Corollary 1.31. *Let $\mathcal{C}$ be a filtered category and X a covariant $\mathcal{C}$-space. Then there is a homotopy equivalence*

$$\operatorname*{hocolim}_{\mathcal{C}} X \xrightarrow{\cong} \operatorname*{colim}_{c \in \mathcal{C}} E^{\mathrm{bar}}(\mathcal{C} \downarrow c) \times_{\mathcal{C} \downarrow c} I_c^* X,$$

where $I_c \colon \mathcal{C} \downarrow c \to \mathcal{C}$ denotes the functor given on objects by sending $c' \to c$ to c'.

Proof. The statement follows from the following homeomorphisms, which will be explained below:

$$E^{\mathrm{bar}}\mathcal{C} \times_{\mathcal{C}} X = B^{\mathrm{bar}}(- \downarrow \mathcal{C}) \times_{\mathcal{C}} X$$

$$\cong B^{\mathrm{bar}}\big(\operatorname*{colim}_{c \in \mathcal{C}} - \downarrow \mathcal{C} \downarrow c\big) \times_{\mathcal{C}} X \tag{1.32}$$

$$\cong \operatorname*{colim}_{c \in \mathcal{C}}\big(B^{\mathrm{bar}}(- \downarrow \mathcal{C} \downarrow c) \times_{\mathcal{C}} X\big) \tag{1.33}$$

$$\cong \operatorname*{colim}_{c \in \mathcal{C}} E^{\mathrm{bar}}(\mathcal{C} \downarrow c) \times_{\mathcal{C} \downarrow c} I_c^* X. \tag{1.34}$$

The natural equivalence $\eta_1\colon \operatorname{colim}_{c\in\mathcal{C}} - \downarrow \mathcal{C} \downarrow c \to - \downarrow \mathcal{C}$ of $\mathcal{C}$-categories which is given by $\eta_1(c')[c' \to c'' \to c] := (c' \to c'')$ yields the homeomorphism of (1.32). The one of (1.34) comes from (1.29) and the natural equivalence $\eta_2\colon - \downarrow I_c \to - \downarrow \mathcal{C} \downarrow c$ of contravariant $\mathcal{C}$-categories given by $\eta_2(c')\big(c' \to I_c(c'' \to c)\big) := (c' \to c'' \to c)$.

Finally, the homeomorphism of (1.33) holds because $- \times_{\mathcal{C}} X$ has a right adjoint (cf. [DL98, Lemma 1.5]) and hence preserves arbitrary colimits (cf. [Mac98, section V.5]), and because B^{bar} commutes with $\operatorname{colim}_{\mathcal{C}}$. The latter is true since $B^{\mathrm{bar}} = |-| \circ N.$, where $|-|$ has a right adjoint, and $N.$ preserves filtered colimits, too, by the following argument. Let $s, t\colon \operatorname{mor}(\mathcal{D}) \to \operatorname{ob}(\mathcal{D})$ be the maps that assign to a morphism in $\mathcal{D}$ its source and target respectively. Then $N_n(\mathcal{D})$ is the n-fold pullback of the diagram $\operatorname{mor}(\mathcal{D}) \xrightarrow{s} \operatorname{ob}(\mathcal{D}) \xleftarrow{t} \operatorname{mor}(\mathcal{D})$. The claim follows since $\operatorname{ob}(-)$ and $\operatorname{mor}(-)$ obviously preserve $\operatorname{colim}_{\mathcal{C}}$ and filtered colimits preserve finite limits, see [Mac98, Thm. IX.2.1]. $\qquad\square$

2 Models for Classifying Spaces

One reason why finding concrete models for the classifying spaces $\underline{E}G$ and $E_{\mathcal{VCyc}}(G)$ is of interest is that they appear in the formulation of the isomorphism conjectures 1.20 of Baum-Connes and Farrell-Jones. In this chapter, after reviewing well-known constructions for $\underline{E}G$ in case G acts on a tree or is word-hyperbolic, we address the general question whether there are finite-dimensional models for $E_{\mathcal{VCyc}}(G)$ or models of finite type.

Moreover, we will present a model for $E_{\mathcal{F}}(G)$ in the last section which leads to a model for $E_{\mathcal{VCyc}}(G)$ if G is locally virtually cyclic.

2.1 The Case of the Family of Finite Subgroups

In a number of situations it is possible to construct nice models for $\underline{E}G$. We will explain three of them to be able to refer to them later. A survey on models for $\underline{E}G$ for various groups G is given in [Lüc05], where also type questions are discussed.

2.1.1 Groups acting on Trees

In this section, we first want to explain the notion of a graph of groups and its associated Bass-Serre tree.

Given two sets V and E, the vertices and edges, and a map $r \colon E \times \{-1, 1\} \to V$, assigning to an edge its initial and terminal vertices, let X be the one-dimensional CW-complex given by the pushout

$$
\begin{array}{ccc}
E \times \{-1, 1\} & \overset{r}{\longrightarrow} & V \\
\downarrow & & \downarrow \\
E \times [-1, 1] & \longrightarrow & X
\end{array}
\tag{2.1}
$$

A *graph of groups* $\mathcal{G}$ *on* X consists of collections of groups $\{G_v\}_{v \in V}$ and $\{G_e\}_{e \in E}$, together with injective group homomorphisms $f_{e,\varepsilon} \colon G_e \to G_{r(e,\varepsilon)}$ for $e \in E$ and $\varepsilon \in \{\pm 1\}$. The *fundamental group* $\pi = \pi_1(\mathcal{G}, X, X_0)$ *of* $\mathcal{G}$ *with respect to a maximal subtree* $X_0 \subset X$ (i.e. X_0 is a contractible subcomplex, and if $X_0 \subset Y \subset X$ such that Y is contractible, then $X_0 = Y$) is the following group. Let $\{t_e\}_{e \in E}$ be a set of abstract symbols indexed by E. Then π is generated by the set $\bigcup_{v \in V} G_v \cup \{t_e\}_{e \in E}$, and the relations in π are the relations in G_v for all $v \in V$, the relation $t_e = 1$ whenever $e \in E$ is an edge of X_0, and the relation $t_e^{-1} f_{e,-1}(g) t_e = f_{e,1}(g)$ for every $e \in E$ and $g \in G_e$.

It is shown in [Ser80, Cor. I.5.2.1] that the obvious maps $G_v \to \pi$ are injective. Thus we can identify G_v with its image in π. Now we define the one-dimensional π-CW-complex T by the π-pushout

$$
\begin{array}{ccc}
\coprod_{e \in E} \pi/f_{e,-1}(G_e) \times \{-1,1\} & \xrightarrow{q} & \coprod_{v \in V} \pi/G_v \\
\downarrow & & \downarrow \\
\coprod_{e \in E} \pi/f_{e,-1}(G_e) \times [-1,1] & \longrightarrow & T
\end{array}
$$

where the restriction of q to $\pi/f_{e,-1}(G_e) \times \{-1\}$ is the projection $\pi/f_{e,-1}(G_e) \to \pi/G_{r(e,-1)}$, and the restriction to $\pi/f_{e,-1}(G_e) \times \{1\}$ is the π-map $\pi/f_{e,-1}(G_e) \to \pi/G_{r(e,1)}$ sending $gf_{e,-1}(G_e)$ to $gt_e G_{r(e,1)}$. Then T is contractible after forgetting the π-action (see [Ser80, Thm. I.5.12]) and is called the *Bass-Serre tree of $\mathcal{G}$ with respect to X_0*.

Conversely, suppose T is a one-dimensional G-CW-complex that is non-equivariantly contractible. We choose a G-pushout

$$
\begin{array}{ccc}
\coprod_{e \in E} G/G_e \times \{-1,1\} & \xrightarrow{q} & \coprod_{v \in V} G/G_v \\
\downarrow & & \downarrow \\
\coprod_{e \in E} G/G_e \times [-1,1] & \longrightarrow & T
\end{array}
\tag{2.2}
$$

Then $X := G \backslash T$ is given by a pushout as in (2.1) if we define $r \colon E \times \{-1,1\} \to V$ to be the map which sends (e, ε) to the unique element $v \in V$ for which $q\big(G/G_e \times \{\varepsilon\}\big)$ is equal to G/G_v. Keep in mind that there is a $g_{e,\varepsilon} \in G$ satisfying $g_{e,\varepsilon}^{-1} G_e g_{e,\varepsilon} \subset G_{r(e,\varepsilon)}$ such that q is determined on $G/G_e \times \{\varepsilon\}$ by $q(1G_e, \varepsilon) = g_{e,\varepsilon} G_{r(e,\varepsilon)}$. Now consider the graph of groups $\mathcal{G}$ on X defined by the collections of groups $\{G_v\}_{v \in V}$ and $\{G_e\}_{e \in E}$, together with the injective homomorphisms $f_{e,\varepsilon} \colon G_e \to G_{r(e,\varepsilon)}$ given by conjugation with $g_{e,\varepsilon}$. It follows from [Ser80, Thm. I.5.13] that, after a choice of a maximal subtree $X_0 \subset X$, one obtains an isomorphism $G \cong \pi_1(\mathcal{G}, X, X_0)$. Moreover, up to isomorphism, this construction is inverse to the one above.

Remark 2.3 (Model for $\underline{E}G$). By "inflating the equivariant cells" of T in (2.2), one gets a model for $\underline{E}G$. More precisely, consider the G-pushout

$$
\begin{array}{ccc}
\coprod_{e \in E} G \times_{G_e} \underline{E}G_e \times \{-1,1\} & \xrightarrow{Q} & \coprod_{v \in V} G \times_{G_v} \underline{E}G_v \\
\downarrow & & \downarrow \\
\coprod_{e \in E} G \times_{G_e} \underline{E}G_e \times [-1,1] & \longrightarrow & X
\end{array}
\tag{2.4}
$$

in which the restriction of Q to $G \times_{G_e} \underline{E}G_e \times \{\varepsilon\}$ is the G-map sending $[g, x, \varepsilon]$ to $[g, \underline{E}f_{e,\varepsilon}(x)] \in G \times_{G_{r(e,\varepsilon)}} \underline{E}G_{r(e,\varepsilon)}$, where $\underline{E}f_{e,\varepsilon} \colon \underline{E}G_e \to \underline{E}G_{r(e,\varepsilon)}$ denotes an $f_{e,\varepsilon}$-equivariant map. Then, by Lemma 1.7 (2), X has the same G-homotopy type as

the diagonal G-space $T \times \underline{E}G$. Hence X is a model for $\underline{E}G$. This is due to the fact that, in the situation of a finite group acting on a tree, the fixed-point set is contractible, see e.g. [Lüc05, Thm. 4.7].

Suppose, for instance, $G = G_1 *_H G_2$ is an amalgamated product of groups G_1 and G_2 over a common subgroup H. Endow $[0, 1]$ with the obvious CW-structure consisting of two 0-cells v_1, v_2 and one 1-cell e. Then G is the fundamental group $\pi_1\big(\mathcal{G}, [0, 1], [0, 1]\big)$ of the graph of groups $\mathcal{G}$ given by $G_{v_i} := G_i$ for $i = 1, 2$ and $G_e := H$, together with the inclusions of H into G_1 and G_2 respectively. Inflating the equivariant cells of the associated Bass-Serre tree yields a model for $\underline{E}G$ which is built of models for $\underline{E}H$, $\underline{E}G_1$ and $\underline{E}G_2$, and the pushout (2.4) can be written

$$
\begin{array}{ccc}
G \times_H \underline{E}H & \longrightarrow & G \times_{G_1} \underline{E}G_1 \\
\downarrow & & \downarrow \\
G \times_{G_2} \underline{E}G_2 & \longrightarrow & \underline{E}G
\end{array}
$$

Similarly, suppose $G = H \rtimes_\alpha \mathbb{Z}$ is a semidirect product with respect to an automorphism $\alpha \colon H \to H$. Consider S^1 with the CW-structure consisting of a single 0-cell v and a single 1-cell e. Obviously, G is the fundamental group $\pi_1\big(\mathcal{G}, S^1, \{v\}\big)$ of the graph of groups $\mathcal{G}$ given by $G_v := H$ and $G_e := H$, together with the homomorphisms id_H and α. The model for $\underline{E}G$ of (2.4) is then just a mapping telescope of the α-equivariant map $\underline{E}\alpha \colon \underline{E}H \to \underline{E}H$ that is infinite to both sides.

2.1.2 Word-hyperbolic Groups

Let G be a finitely generated group. We choose a finite symmetric subset $S \subset G$ that generates G, where symmetric means $S = S^{-1}$. We can impose a metric d_S on G by setting

$$
d_S(g_1, g_2) := \min \big\{ n \in \mathbb{N} \mid g_1^{-1} g_2 = s_1 \cdots s_n \text{ for } s_i \in S \big\},
$$

which is obviously invariant under left translation by elements of G and called word metric.

To a pair (G, S) as above one can then associate its Rips complex:

Definition 2.5 (Rips complex). For $r \in \mathbb{N}$, the *Rips complex* $P_r(G, S)$ is the geometric realization of the simplicial complex whose n-simplices are given by $(n + 1)$-tuples $(g_0, \dots, g_n)$ of pairwise distinct elements of G such that $d_S(g_i, g_j) \le r$ for $1 \le i, j \le n$.

Remark 2.6. One can make the following easy observations concerning the Rips complex:

- Because of the left invariance of the word metric, we have a simplicial G-action on $P_r(G, S)$ by setting $g \cdot (g_0, \dots, g_n) := (gg_0, \dots, gg_n)$.

- The 0-skeleton of $P_r(G, S)$ coincides with G.

- $P_1(G, S) =: \Gamma(G, S)$ is the *Cayley graph of G with respect to S*. It is the graph with vertex set G in which there is exactly one edge from g_1 to g_2 if and only if $d_S(g_1, g_2) = 1$. By requiring that any edge be isometric to the unit interval $[0, 1]$, the Cayley graph becomes a metric space in which any two points can be joined by a geodesic.

  In general, the 1-skeleton of $P_r(G, S)$ is just the Cayley graph $\Gamma(G, S')$ for $S' = \{g \in G \mid 0 < d_S(g, 1) \leq r\}$.

The class of word-hyperbolic groups consists of those groups whose Cayley graphs "resemble a tree". More precisely:

Definition 2.7 (Word-hyperbolic group). Let G be a finitely generated group and $S \subset G$ a finite symmetric set generating G. Then (G, S) is said to be *δ-hyperbolic* if there is a real number $\delta \geq 0$ with the property that any triangle in $\Gamma(G, S)$ whose sides are geodesics is *δ-slim*, i.e. the δ-neighbourhood of the union of any two of the sides contains the third.

A finitely generated group G is *word-hyperbolic* if there is a finite symmetric set $S \subset G$ generating G and a $\delta \geq 0$ such that (G, S) is δ-hyperbolic.

If S and S' are two finite symmetric sets generating G, then it is not very difficult to show that $\Gamma(G, S)$ is quasi-isometric to $\Gamma(G, S')$. This implies that $\Gamma(G, S)$ satisfies the slim triangle condition if and only if $\Gamma(G, S')$ does, see [BH99, Thm. 1.9] (of course, the required δ varies in general).

Example 2.8. There are two classes of groups for which it is immediate that they are word-hyperbolic.

- One can see directly from Definition 2.7 that finite groups are word-hyperbolic. This is because all geodesic triangles in $\Gamma(G, S)$ are δ-slim for δ the diameter of $\Gamma(G, S)$.

- Free groups are word-hyperbolic since, for the canonical choice of S, the graph $\Gamma(G, S)$ is a tree and thus 0-hyperbolic.

The following result is proved in [MS02]:

Theorem 2.9. *Let G be a group and $S \subset G$ a finite symmetric set generating G such that (G, S) is δ-hyperbolic for some $\delta \geq 0$. Then the second barycentric subdivision of the Rips complex $P_r(G, S)$ is a finite model for $\underline{E}G$, provided that $r \geq 16\delta + 8$.*

2.1.3 Crystallographic Groups

The symmetry group of $\mathbb{R}^n$ is among the most studied groups, at least for $n \leq 3$. We will now consider certain subgroups:

Definition 2.10 (Crystallographic group). An *n-dimensional crystallographic group* is a discrete cocompact subgroup of the Lie group of all isometries of $\mathbb{R}^n$.

A concise treatment on this matter is given in [Far81]. It follows from [Abe78, Cor. 4.14] (see also [Lüc05, Thm. 4.4]):

Theorem 2.11. *Let G be an n-dimensional crystallographic group. Then $\mathbb{R}^n$ can be endowed with the structure of a G-CW-complex in such a way that one gets a finite model for $\underline{E}G$.*

2.2 The Case of the Family of Virtually Cyclic Subgroups

We will now turn to the investigation of questions about the type of the classifying spaces $E_{\mathcal{VCyc}}(G)$. Firstly, it is shown in general how statements about the type of $E_{\mathcal{G}}(G)$ lead to statements about $E_{\mathcal{F}}(G)$ if $\mathcal{F} \subset \mathcal{G}$ are two families of subgroups of G.

Proposition 2.12. *Let $\mathcal{F} \subset \mathcal{G}$ be families of subgroups of G. Suppose that for any $H \in \mathcal{G}$ there is an n-dimensional model for $E_{\mathcal{F} \cap H}(H)$. Then the existence of an m-dimensional model for $E_{\mathcal{G}}(G)$ implies the existence of an $(n+m)$-dimensional model for $E_{\mathcal{F}}(G)$.*

The same is true if one replaces "k-dimensional" everywhere by "finite" or "finite type".

Proof. Let Z be an m-dimensional G-CW-complex with isotropy groups in $\mathcal{G}$. We will show that then $Z \times E_{\mathcal{F}}(G)$ is G-homotopy equivalent to an $(n+m)$-dimensional G-CW-complex, which implies the claim of the proposition as $E_{\mathcal{G}}(G) \times E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F}}(G)$.

We utilize induction over the dimension d of Z. If $Z = \emptyset$, then there is nothing to show, so let $d \geq 0$. Crossing the G-pushout telling how Z_d arises from Z_{d-1} with $E_{\mathcal{F}}(G)$ yields a G-pushout

$$
\begin{array}{ccc}
\coprod_{i \in I_d} G/H_i \times E_{\mathcal{F}}(G) \times S^{d-1} & \overset{q}{\longrightarrow} & Z_{d-1} \times E_{\mathcal{F}}(G) \\
\downarrow & & \downarrow \\
\coprod_{i \in I_d} G/H_i \times E_{\mathcal{F}}(G) \times D^d & \longrightarrow & Z_d \times E_{\mathcal{F}}(G)
\end{array}
\tag{2.13}
$$

Due to Lemma 1.7 (2) and the fact that $\mathrm{res}_G^{H_i} E_{\mathcal{F}}(G)$ is a model for $E_{\mathcal{F} \cap H_i}(H_i)$, there is a G-homotopy equivalence $f_i \colon G \times_{H_i} E_{\mathcal{F} \cap H_i}(H_i) \to G/H_i \times E_{\mathcal{F}}(G)$. We set $f_0 := \coprod_i f_i \times \mathrm{id}_{S^{d-1}}$. Furthermore, by induction hypothesis, there is a G-homotopy equivalence $f_1 \colon Z' \to Z_{d-1} \times E_{\mathcal{F}}(G)$, where Z' is an $(n+d-1)$-dimensional G-CW-complex. We denote the G-homotopy inverse of f_1 by g_1.

For $i \colon \coprod_i G \times_{H_i} E_{\mathcal{F} \cap H_i}(H_i) \times S^{d-1} \to \mathrm{cyl}(g_1 \circ q \circ f_0)$ and $p \colon \mathrm{cyl}(g_1 \circ q \circ f_0) \to Z'$ the obvious inclusion and projection, $f_1 \circ p \circ i \simeq_G q \circ f_0$ holds. Since i is a G-cofibration, $f_1 \circ p$ can be altered within its G-homotopy class to yield a G-map

$f_1' \colon \operatorname{cyl}(g_1 \circ q \circ f_0) \to Z_{d-1} \times E_{\mathcal{F}}(G)$ such that $f_1' \circ i = q \circ f_0$. Now consider the $(n+d)$-dimensional G-CW-complex Z'' which is defined by the G-pushout

$$
\begin{array}{ccc}
\coprod\limits_{i \in I_d} G \times_{H_i} E_{\mathcal{F} \cap H_i}(H_i) \times S^{d-1} & \xrightarrow{\ i\ } & \operatorname{cyl}(g_1 \circ q \circ f_0) \\
\downarrow & & \downarrow \\
\coprod\limits_{i \in I_d} G \times_{H_i} E_{\mathcal{F} \cap H_i}(H_i) \times D^d & \xrightarrow{\hspace{2cm}} & Z''
\end{array}
\tag{2.14}
$$

The G-homotopy equivalences f_0 and f_1' induce a map of G-pushouts from (2.14) to (2.13), and, as the left vertical arrows in these diagrams are G-cofibrations, $Z_d \times E_{\mathcal{F}}(G)$ is G-homotopy equivalent to Z'' by [Lüc89, Lemma 2.13]. $\qquad\square$

As an application, recall that every virtually cyclic group V has a finite one-dimensional model for $\underline{E}V$. In fact, if V is infinite virtually cyclic, then V admits a surjection with finite kernel either to $\mathbb{Z}$ or to $\mathbb{Z}/2 * \mathbb{Z}/2$, see [FJ95, Lemma 2.5]. Hence a model for $\underline{E}\mathbb{Z}$, or for $\underline{E}(\mathbb{Z}/2 * \mathbb{Z}/2)$ respectively, yields a model for $\underline{E}V$ by restriction.

However, a model for $\underline{E}\mathbb{Z}$ is the real line on which $\mathbb{Z}$ acts by translation, the $\mathbb{Z}$-CW-structure consisting of one free equivariant 0- and 1-cell. On the other hand, note that $\mathbb{Z}/2 * \mathbb{Z}/2$ is isomorphic to $\mathbb{Z} \rtimes \mathbb{Z}/2 = \langle a, b \mid bab = a^{-1}, b^2 = 1 \rangle$. The subgroups generated by b and ab represent the two conjugacy classes of finite subgroups. One can define a model for $\underline{E}(\mathbb{Z} \rtimes \mathbb{Z}/2)$ with two equivariant 0-cells and one free equivariant 1-cell by the equivariant pushout

$$
\begin{array}{ccc}
\mathbb{Z} \rtimes \mathbb{Z}/2 \times \{-1, 1\} & \xrightarrow{\ \mathrm{pr}_{-1} \amalg \mathrm{pr}_1\ } & (\mathbb{Z} \rtimes \mathbb{Z}/2)/\langle b \rangle \amalg (\mathbb{Z} \rtimes \mathbb{Z}/2)/\langle ab \rangle \\
\downarrow & & \downarrow \\
\mathbb{Z} \rtimes \mathbb{Z}/2 \times [-1, 1] & \xrightarrow{\hspace{3cm}} & \underline{E}(\mathbb{Z} \rtimes \mathbb{Z}/2)
\end{array}
$$

where $\mathrm{pr}_{-1} \colon \mathbb{Z} \rtimes \mathbb{Z}/2 \times \{-1\} \to (\mathbb{Z} \rtimes \mathbb{Z}/2)/\langle b \rangle$ and $\mathrm{pr}_1 \colon \mathbb{Z} \rtimes \mathbb{Z}/2 \times \{1\} \to (\mathbb{Z} \rtimes \mathbb{Z}/2)/\langle ab \rangle$ are the canonical projections and the left vertical arrow is the inclusion. Explicitly, this model for $\underline{E}(\mathbb{Z} \rtimes \mathbb{Z}/2)$ is the real line with the action $a^n b^m \cdot x = n + (-1)^m x$.

It follows with the help of Proposition 2.12:

Corollary 2.15. *Let G be a group and $n \geq 2$. If no model for $\underline{E}G$ is of dimension less than n (or finite, or of finite type), then no model for $E_{\mathcal{V}Cyc}(G)$ is of dimension less than $n - 1$ (or finite, or of finite-type).*

This expresses in a way that finding models for $E_{\mathcal{V}Cyc}(G)$ must be more demanding than for $\underline{E}G$. Furthermore, the question arises whether one can also derive an upper bound for the minimal dimension of models for $E_{\mathcal{V}Cyc}(G)$ from the minimal dimension of models for $\underline{E}G$. An answer to this problem will be given in section 3.1 if G satisfies a certain condition.

The proof of the next result is along the lines of the proof of [Lüc00, Thm. 3.1 and Thm. 3.2].

Theorem 2.16 (Behaviour under group extensions). *Suppose we are given an exact sequence $1 \to K \to G \xrightarrow{p} Q \to 1$ of groups.*

(1) *Assume that Q is a torsion group that possesses an upper bound b on the orders of its finite subgroups, and that there is a k-dimensional model for $E_{\mathcal{VCyc}}(K)$ and a q-dimensional model for $\underline{E}Q$. Then there is a $(kb + q)$-dimensional model for $E_{\mathcal{VCyc}}(G)$.*

(2) *Assume that for any virtually cyclic subgroup $V \subset Q$ there is a finite model for $E_{\mathcal{VCyc}}\big(p^{-1}(V)\big)$, and that there is a finite model for $E_{\mathcal{VCyc}}(Q)$. Then there is also a finite model for $E_{\mathcal{VCyc}}(G)$.*

The same is true if one replaces "finite" everywhere by "finite type".

We remark that, concerning the condition on Q in Theorem 2.16 (1), there are indeed infinite torsion groups which have an upper bound on the orders of their finite subgroups. For instance, the so-called Tarski monster groups constructed in [Ol'82] have this property.

As for Theorem 2.16 (2), it is not clear which groups (if any but the virtually cyclic ones) possess a model for $E_{\mathcal{VCyc}}(G)$ of finite type, let alone a finite model. This is illustrated by the following example, which shows in particular that models for $E_{\mathcal{VCyc}}(G)$ behave badly with respect to direct products of groups even in the most basic situations (but also cf. Corollary 3.11).

Example 2.17. Let G be an extension $1 \to \mathbb{Z} \to G \to \mathbb{Z} \to 1$. This sequence splits, so $G = \mathbb{Z} \rtimes \mathbb{Z}$ is a semidirect product, and either $G = \langle a, b \mid ba = ab \rangle = \mathbb{Z} \oplus \mathbb{Z}$ or $G = \langle a, b \mid ba = a^{-1}b \rangle$.

Assume first that $G = \mathbb{Z} \oplus \mathbb{Z}$. We get an explicit model for $E_{\mathcal{VCyc}}(G)$ as follows. Note that the maximal cyclic subgroups of G are precisely those which are generated by $a^n b^m$, where n, $m \in \mathbb{Z}$ are coprime. Let $\{C_i\}_{i \in \mathbb{N}}$ be the collection of all the maximal cyclic subgroups and $p_i \colon G \to G/C_i$ the projections. Since the quotients G/C_i are infinite cyclic, we can choose models X_i for $E(G/C_i)$ whose underlying space is the real line. Every X_i carries a G-action coming from p_i. As $C_i \cap C_j = \{1\}$ if $i \neq j$, the map $p_i \times p_j \colon G \to G/C_i \times G/C_j$ is injective in this case. Via this map, $X_i \times X_j$ is a model for EG. Now the G-CW-complex X is defined to be the G-pushout

$$
\begin{array}{ccc}
\coprod_{i \in \mathbb{N}} X_i \times X_{i+1} \times \{0,1\} & \xrightarrow{\amalg_i \mathrm{pr}_i} & \coprod_{j \in \mathbb{N}} X_j \\
\downarrow & & \downarrow \\
\coprod_{i \in \mathbb{N}} X_i \times X_{i+1} \times [0,1] & \longrightarrow & X
\end{array}
\tag{2.18}
$$

where $\mathrm{pr}_i \colon X_i \times X_{i+1} \times \{0,1\} \to \coprod_{j \in \mathbb{N}} X_j$ is the map which projects $X_i \times X_{i+1} \times \{0\}$ to X_i and $X_i \times X_{i+1} \times \{1\}$ to X_{i+1}, while the left vertical arrow is the inclusion.

From (2.18) it is easy to see that X is a three-dimensional model for $E_{\mathcal{VCyc}}(G)$. Since $G\backslash(X_i \times X_{i+1})$ as well as $G\backslash X_i \times G\backslash X_{i+1}$ are models for BG, it is also clear

that dividing out the G-action in (2.18) yields a pushout

$$
\begin{array}{ccc}
\coprod\limits_{i \in \mathbb{N}} G \backslash X_i \times G \backslash X_{i+1} \times \{0,1\} & \xrightarrow{\ \amalg_i \mathrm{pr}_i\ } & \coprod\limits_{j \in \mathbb{N}} G \backslash X_j \\[2mm]
\downarrow & & \downarrow \\[2mm]
\coprod\limits_{i \in \mathbb{N}} G \backslash X_i \times G \backslash X_{i+1} \times [0,1] & \xrightarrow{\hspace{3cm}} & Y
\end{array}
$$

such that Y is homotopy equivalent to $G\backslash X$. Applying the Mayer-Vietoris sequence shows that $H_3(Y)$ is isomorphic to a free abelian group of infinite rank. In fact, one can deduce from the above pushout that Y is built of a countable number of copies of S^1, the join construction $S^1 * S^1 = S^3$ being applied to two consecutive copies. In particular, any model for $E_{\mathcal{V}Cyc}(G)$ must at least be of dimension three (more generally, it will be shown in Example 3.10 that if G is finitely generated abelian of rank n, the minimal dimension of models for $E_{\mathcal{V}Cyc}(G)$ is $n+1$).

If $G = \langle a, b \mid ba = a^{-1}b \rangle$, then note that a and b^2 generate a free abelian normal subgroup of index two in G. Hence at any rate, by Theorem 2.16 there is a six-dimensional model for $E_{\mathcal{V}Cyc}(G)$ in this case. In anticipation of section 3.1, however, we briefly want to indicate why the minimal dimension of a model for $E_{\mathcal{V}Cyc}(G)$ is actually three. Of course, it cannot be smaller since $\mathbb{Z} \oplus \mathbb{Z} \subset G$, which means that from every model for $E_{\mathcal{V}Cyc}(G)$ one obtains a model for $E_{\mathcal{V}Cyc}(\mathbb{Z} \oplus \mathbb{Z})$ by restricting the G-action. On the other hand, G certainly satisfies the assumptions of Theorem 3.7. Using the fact that there is a two-dimensional model for $\underline{E}G$ due to [Lüc05, Ex. 5.26], this yields a three-dimensional model for $E_{\mathcal{V}Cyc}(G)$.

We remark that in neither of the above cases there can be a model for $E_{\mathcal{V}Cyc}(G)$ of finite type, cf. the next lemma.

Lemma 2.19. *Let G be a group and $\mathcal{F}$ a family of subgroups. Suppose that there is a model for $E_{\mathcal{F}}(G)$ with a finite 0-skeleton. Then there is a finite subset $\mathcal{F}' \subset \mathcal{F}$ such that any $H \in \mathcal{F}$ is subconjugated to an element of $\mathcal{F}'$.*

In particular, if there is a model for $\underline{E}G$ with finite 0-skeleton, then G contains only finitely many conjugacy classes of finite subgroups.

Proof. Let $G/H_1, \ldots, G/H_n$ be the finitely many equivariant 0-cells of $E_{\mathcal{F}}(G)$. For any $H \in \mathcal{F}$, the set of G-maps from G/H to $E_{\mathcal{F}}(G)$ must be non-empty. However, every G-map $G/H \to E_{\mathcal{F}}(G)$ is G-homotopic to a G-map $G/H \to G/H_i$ for some i by the equivariant version of the cellular approximation theorem. Hence H must be subconjugated to one of the H_i. $\qquad\square$

We want to conclude this section with an explanation of how L^2-Betti numbers are related to the minimal dimension of models for $E_{\mathcal{F}}(G)$ in case $\mathcal{F}$ is a family of amenable subgroups of G. First of all, let us recall some definitions (a good reference for the matter in this section is [Lüc02, ch. 6]).

We denote by $l^2(G)$ the Hilbert space of square-summable formal sums over G with complex coefficients, where the scalar product is given by

$$\Big\langle \sum_{g \in G} \lambda_g \cdot g, \sum_{g \in G} \mu_g \cdot g \Big\rangle := \sum_{g \in G} \lambda_g \overline{\mu_g}.$$

Left multiplication with elements in G induces an isometric G-action on $l^2(G)$. The *group von Neumann algebra* $\mathcal{N}(G)$ is, by definition, the algebra $\mathcal{B}(l^2(G))^G$ of G-equivariant bounded operators $l^2(G) \to l^2(G)$.

To any module M over the ring $\mathcal{N}(G)$ one can assign its *von Neumann dimension* $\dim_{\mathcal{N}(G)}(M) \in [0, \infty]$. If, for instance, M is finitely generated projective, then $\dim_{\mathcal{N}(G)}(M) = \sum_{i=1}^n \langle a_{i,i}(1), 1 \rangle$ for any $(n \times n)$-matrix $A = (a_{i,j})$ with entries in $\mathcal{N}(G)$ such that $A^2 = A$ and the image of the $\mathcal{N}(G)$-homomorphism $\mathcal{N}(G) \to \mathcal{N}(G)$ given by right multiplication with A is $\mathcal{N}(G)$-isomorphic to M.

The homology groups $H_n^G\big(X; \mathcal{N}(G)\big)$ of the G-CW-complex X with coefficients in $\mathcal{N}(G)$ are the homology groups of the $\mathcal{N}(G)$-chain complex $\mathcal{N}(G) \otimes_{\mathbb{Z}G} C_*(X)$, where $C_*(X)$ denotes the cellular $\mathbb{Z}G$-chain complex of X. The *n-th L^2-Betti number of* X then is

$$b_n^{(2)}\big(X; \mathcal{N}(G)\big) := \dim_{\mathcal{N}(G)}\big(H_n^G(X; \mathcal{N}(G))\big).$$

For a group G, one sets $b_n^{(2)}(G) := b_n^{(2)}\big(EG; \mathcal{N}(G)\big)$.

Theorem 2.20 (L^2-Betti numbers and minimal dimension of $E_{\mathcal{F}}(G)$). *Let G be a group and $n \in \mathbb{N}$ such that $b_n^{(2)}(G) \neq 0$. Let $\mathcal{F}$ be a family of subgroups of G such that every $H \in \mathcal{F}$ is amenable. Then any model for $E_{\mathcal{F}}(G)$ must be at least n-dimensional.*

Proof. Since $EG \times E_{\mathcal{F}}(G)$ equipped with the diagonal G-action is a model for EG, it suffices to show that $b_n^{(2)}\big(X; \mathcal{N}(G)\big) = b_n^{(2)}\big(EG \times X; \mathcal{N}(G)\big)$ for all G-CW-complexes X with amenable isotropy groups and all $n \in \mathbb{N}$ (see also [Lüc02, Th. 6.54(2)]). Moreover, since the dimension function satisfies $\dim_{\mathcal{N}(G)}(N) = \dim_{\mathcal{N}(G)}(M) + \dim_{\mathcal{N}(G)}(Q)$ whenever there is a short exact sequence $0 \to M \to N \to Q \to 0$ of $\mathcal{N}(G)$-modules, one only needs to show that the map $H_n^G\big(EG \times X; \mathcal{N}(G)\big) \to H_n^G\big(X; \mathcal{N}(G)\big)$ which comes from the projection has a kernel and a cokernel of trivial dimension. We can furthermore assume that X, being the directed colimit of its finite G-CW-subcomplexes, is itself finite, as directed colimits are compatible with $H_n^G\big(-; \mathcal{N}(G)\big)$ and exact sequences and behave nicely with respect to $\dim_{\mathcal{N}(G)}$. Finally, by induction over the number of equivariant cells, this reduces to $X = G/H$ for an amenable subgroup H.

Using Lemma 1.7 (2) together with the fact that both $H_n^G\big(-; \mathcal{N}(G)\big)$ and $\dim_{\mathcal{N}(G)}$ are compatible with induction, it remains to show that for an amenable group H the projection induces a map $H_n^H\big(EH; \mathcal{N}(H)\big) \to H_n^H\big(\mathrm{pt}; \mathcal{N}(H)\big)$ whose kernel and cokernel have trivial dimension. Suppose first that H is finite. In this case $\mathcal{N}(H) = \mathbb{C}H$ and the claim is obvious. If H is infinite amenable, then it is known that all its L^2-Betti numbers vanish. On the other hand, it is not hard to show that $\dim_{\mathcal{N}(H)}\big(\mathcal{N}(H) \otimes_{\mathbb{Z}H} \mathbb{Z}\big) = 0$, which finishes the proof. $\qquad \square$

Example 2.21. Let G be such that $b_1^{(2)}(G) \neq 0$ but $b_i^{(2)}(G) = 0$ for all $i \neq 1$, e.g. $G = \mathbb{Z} * \mathbb{Z}$. Then it follows from the Künneth formula for L^2-Betti numbers that

$$b_n^{(2)}\left(\prod_{i=1}^n G\right) = b_1^{(2)}(G)^n \neq 0.$$

2.3 A Model for Colimits of Groups

In the following keep in mind that, for an inclusion $H \subset K \subset G$ of groups and a family $\mathcal{F}$ of subgroups of G, there is a canonical cellular G-map $f \colon G \times_H E_{\mathcal{F} \cap H}(H) \to G \times_K E_{\mathcal{F} \cap K}(K)$. Explicitly, one can simply put $f := \mathrm{id}_G \times_K f'$ for a cellular K-map $f' \colon K \times_H E_{\mathcal{F} \cap H}(H) \to E_{\mathcal{F} \cap K}(K)$, which one obtains from the universal property of $E_{\mathcal{F} \cap K}(K)$ and the equivariant version of the cellular approximation theorem.

We will also consider directed sets as filtered categories in the obvious way.

Theorem 2.22. *Assume that the group $G = \bigcup_{i \in I} G_i$ is a directed union of subgroups. Let $\mathcal{F}$ be a family of subgroups of G with the property that any $H \in \mathcal{F}$ is contained in some G_i, e.g. $\mathcal{F}$ consists of finitely generated subgroups of G. If models X_i for $E_{\mathcal{F} \cap G_i}(G_i)$ are chosen, then $\mathrm{hocolim}_{i \in I} G \times_{G_i} X_i$ will be a model for $E_{\mathcal{F}}(G)$.*

Proof. We define the I-space F by $F(i) := G \times_{G_i} X_i$ for $i \in I$. Whenever Y is an I-CW-complex, the map $\lambda_g \colon F \to F$ of I-spaces which is given by left multiplication with $g \in G$ induces a G-action on $Y \times_I F$ by requiring that g acts as $\mathrm{id}_Y \times_I \lambda_g$. It will be shown below that for all subgroups $H \subset G$ the natural inclusion $F^H \to F$ induces a homotopy equivalence

$$\left(\underset{I}{\mathrm{hocolim}}\, F\right)^H \xleftarrow{\simeq} \underset{I}{\mathrm{hocolim}}\, F^H. \tag{2.23}$$

From this it follows that the isotropy groups of $\mathrm{hocolim}_I F$ are contained in $\mathcal{F}$ because $\mathcal{F}$ is closed with respect to taking subgroups.

Furthermore, if $H \in \mathcal{F}$, it suffices to prove the contractibility of $\mathrm{hocolim}_I F^H$. Note that $F(i)^H \simeq (G/G_i)^H$ for $i \in I$ by Lemma 1.7 (1), in fact naturally in i. Thus, according to Theorem 1.28 (1), we only need to show that $\mathrm{hocolim}_{i \in I}(G/G_i)^H \simeq \mathrm{pt}$. This is, however, equivalent to showing that

$$C := \underset{i \in I}{\mathrm{colim}}\, E^{\mathrm{bar}}(I \downarrow i) \times_{I \downarrow i} (G/G_{J_i(-)})^H \simeq \mathrm{pt}, \tag{2.24}$$

where for $i \in I$ the functor $J_i \colon I \downarrow i \to I$ is given on objects by sending $j \to i$ to j, see Corollary 1.31. Certainly C is connected due to the directedness of I. Next we explain why for $n \geq 1$ any map $f \colon S^n \to C$ must be null-homotopic. Recall (cf. page 10) that for $i' \leq i$ an n-cell of $E^{\mathrm{bar}}(I \downarrow i)(i' \to i)$ corresponds to a sequence $i' \leq i_0 < i_1 < \ldots < i_n \leq i$ in I. Hence, for each pair $i \leq j \in I$, the structure map

$$\coprod_{i' \leq i} E^{\mathrm{bar}}(I \downarrow i)(i' \to i) \times (G/G_{i'})^H / \sim \; \to \coprod_{i' \leq j} E^{\mathrm{bar}}(I \downarrow j)(i' \to j) \times (G/G_{i'})^H / \sim$$

in the above colimit is an inclusion of CW-complexes because it simply comes from the collection of maps of the form

$$(i' \le i_0 < i_1 < \ldots < i_n \le i,\, gG_{i'}) \mapsto (i' \le i_0 < i_1 < \ldots < i_n \le j,\, gG_{i'}).$$

Since $f(S^n)$ is contained in a finite subcomplex of C, it is therefore already contained in $E^{\mathrm{bar}}(I \downarrow i) \times_{I \downarrow i} (G/G_{J_i(-)})^H$ for an appropriate $i \in I$. However, since $I \downarrow i$ has a terminal object, the trivial contravariant $I \downarrow i$-space pt is an $I \downarrow i$-CW-complex and hence homotopy equivalent to $E^{\mathrm{bar}}(I \downarrow i)$ by Lemma 1.15 (2). This implies that the projection induces a homotopy equivalence $E^{\mathrm{bar}}(I \downarrow i) \times_{I \downarrow i} (G/G_{J_i(-)})^H \to$ $\mathrm{pt} \times_{I \downarrow i}(G/G_{J_i(-)})^H = (G/G_i)^H$, the latter being a discrete space. This settles (2.24).

It remains to show the rather obvious claim of (2.23). If we choose a classifying space EI for I, then, by definition, $\mathrm{hocolim}_I F = EI \times_I F$ and $\mathrm{hocolim}_I F^H = EI \times_I F^H$, both being well-defined up to homotopy equivalence. Suppose first that EI is finite-dimensional, then the assertion will be proved using induction over the skeleta of EI. Since $EI_{-1} = \emptyset$, the induction start is trivial. Assuming $n \ge 0$, there is a pushout

$$
\begin{array}{ccc}
\coprod\limits_{j \in J_n} \mathrm{mor}_I(-, i_j) \times S^{n-1} & \longrightarrow & EI_{n-1} \\
\downarrow & & \downarrow \\
\coprod\limits_{j \in J_n} \mathrm{mor}_I(-, i_j) \times D^n & \longrightarrow & EI_n
\end{array}
$$

to which we can apply $- \times_I F^H$, or we can apply $- \times_I F$ and then take H-fixed points. In either case, the resulting squares

$$
\begin{array}{ccc}
\coprod\limits_{j \in J_n} \mathrm{mor}_I(-, i_j) \times_I F^H \times S^{n-1} & \longrightarrow & EI_{n-1} \times_I F^H \\
\downarrow & & \downarrow \\
\coprod\limits_{j \in J_n} \mathrm{mor}_I(-, i_j) \times_I F^H \times D^n & \longrightarrow & EI_n \times_I F^H
\end{array}
\tag{2.25}
$$

and

$$
\begin{array}{ccc}
\coprod\limits_{j \in J_n} \big(\mathrm{mor}_I(-, i_j) \times_I F\big)^H \times S^{n-1} & \longrightarrow & (EI_{n-1} \times_I F)^H \\
\downarrow & & \downarrow \\
\coprod\limits_{j \in J_n} \big(\mathrm{mor}_I(-, i_j) \times_I F\big)^H \times D^n & \longrightarrow & (EI_n \times_I F)^H
\end{array}
\tag{2.26}
$$

are again pushouts as the balanced product over a category by a fixed functor possesses a right adjoint and thus preserves arbitrary colimits, cf. [DL98, Lemma 1.5] and [Mac98, section V.5]. Now the natural inclusion $F^H \to F$ induces a map of

pushouts from (2.25) to (2.26), where

$$\mathrm{mor}_I(-, i_j) \times_I F^H \times S^{n-1} \to \left(\mathrm{mor}_I(-, i_j) \times_I F\right)^H \times S^{n-1},$$

$$\mathrm{mor}_I(-, i_j) \times_I F^H \times D^n \to \left(\mathrm{mor}_I(-, i_j) \times_I F\right)^H \times D^n$$

are homeomorphisms by Lemma 1.11 (1), and $EI_{n-1} \times_I F^H \to (EI_{n-1} \times_I F)^H$ is a homeomorphism by the induction hypothesis. Hence $EI_n \times_i F^H \to (EI_n \times_I F)^H$ is also a homeomorphism.

In the general case we can write $EI = \mathrm{colim}_{n \in \mathbb{N}} EI_n$, so that there are homeomorphisms $EI \times_I F^H \cong \mathrm{colim}_{n \in \mathbb{N}}(EI_n \times_I F^H) \cong \mathrm{colim}_{n \in \mathbb{N}}(EI_n \times_I F)^H$, the second by what we have just shown. Now (2.23) follows from the fact that $\mathrm{colim}_{n \in \mathbb{N}}$ and $(-)^H$ commute. $\qquad\square$

Corollary 2.27. *Assume, in the situation of Theorem 2.22, that there is an $n \in \mathbb{N}$ such that all the $E_{\mathcal{F} \cap G_i}(G_i)$ have models of dimension not exceeding n. If there is a d-dimensional model for EI, then there is an $(n+d)$-dimensional model for $E_{\mathcal{F}}(G)$.*

Lemma 2.28. *Let I be a directed set and $d \in \mathbb{N} \cup \{\infty\}$ the minimal dimension of a model for EI. Then:*

(1) I has got a maximal element if and only if $d = 0$.

(2) If there is a countable cofinal subset of I, then $d \le 1$.

Proof. If i_0 is the maximal element of I, then $\mathrm{mor}_I(-, i_0)$ is a zero-dimensional model for EI. Conversely, the existence of a zero-dimensional model for EI means that there is a family $\{i_k\}_{k \in K}$ in I such that for every $i \in I$ one has $i \le i_k$ for a unique $k \in K$. However, since I is directed, this implies $K = \{k_0\}$, and i_{k_0} is the maximal element of I. This settles (1).

As for (2), if I does not have a maximal element, let $J = \{i_k \mid k \in \mathbb{N}\} \subset I$ be cofinal. For $n \in \mathbb{N}$ let $J_n := \{i_k \mid k \le n\}$, then J_n is finite and $J = \bigcup_{n \in \mathbb{N}} J_n$. We set $j_0 := i_0$ and choose $j_n \in I$ for $n \ge 1$ such that j_n is strictly greater than any element in $J_n \cup \{j_{n-1}\}$. Obviously, $\{j_n \mid n \in \mathbb{N}\} \subset I$ is cofinal and isomorphic to $\mathbb{N}$ as a directed set. Now a one-dimensional model for EI is given by the pushout

$$
\begin{array}{ccc}
\coprod_{n \in \mathbb{N}} \mathrm{mor}_I(-, j_n) \times \{-1, 1\} & \xrightarrow{\coprod_n q_n} & \coprod_{m \in \mathbb{N}} \mathrm{mor}_I(-, j_m) \\
\downarrow & & \downarrow \\
\coprod_{n \in \mathbb{N}} \mathrm{mor}_I(-, j_n) \times [-1, 1] & \longrightarrow & EI
\end{array}
$$

where $q_n \colon \mathrm{mor}_I(-, j_n) \times \{-1, 1\} \to \coprod_{m \in \mathbb{N}} \mathrm{mor}_I(-, j_m)$ is the natural transformation which is given on $\mathrm{mor}_I(-, j_n) \times \{-1\}$ by composition with id_{j_n} and on $\mathrm{mor}_I(-, j_n) \times \{1\}$ by composition with $j_n \to j_{n+1}$. $\qquad\square$

Of course, one may ask oneself whether a generalized version of Lemma 2.28 holds, saying that there is an $(n + 1)$-dimensional model for EI whenever I has a cofinal subset of cardinality $\aleph_n$. This could neither be proved nor disproved. We refer to Theorem 2.32 instead.

Example 2.29. Let G be a countable group and $\mathcal{F}$ the family of all finitely generated subgroups. It follows from Corollary 2.27 and Lemma 2.28 that there is a one-dimensional model for $E_{\mathcal{F}}(G)$.

Furthermore, in this case we can write $G = \bigcup_{n \in \mathbb{N}} G_n$ for an ascending chain $G_0 \subset G_1 \subset \ldots$ of finitely generated subgroups of G. The model $\operatorname{hocolim}_{n \in \mathbb{N}} G/G_n$ for $E_{\mathcal{F}}(G)$ of Theorem 2.22 is then the Bass-Serre tree of the following graph of groups (see section 2.1.1). Its underlying CW-complex is just the ray of non-negative real numbers with the natural CW-structure. The vertex groups are the groups $\{G_n\}_{n \in \mathbb{N}}$, at the edge from n to $n + 1$ is the group G_n, and the corresponding edge homomorphisms are given by id_{G_n} and the inclusion of G_n into G_{n+1}, respectively.

Note that the assumption in Example 2.29 that G be countable is necessary. In fact, suppose that G is a locally finite group of cardinality $\aleph_n$. Then [KT97, Thm. A] implies $\operatorname{cd}_{\mathbb{Q}}(G) = n + 1$. However, it is known that for any group K there is a one-dimensional model for $\underline{E}K$ if and only if $\operatorname{cd}_{\mathbb{Q}}(K) \leq 1$, see [Dun79, Thm 1.1].

Proposition 2.30 (Torsion-free locally virtually cyclic groups). *The family of countable locally virtually cyclic groups that are torsion-free consists precisely of all subgroups of the additive rationals $\mathbb{Q}$.*

Proof. If G is a subgroup of $\mathbb{Q}$, then any finitely generated subgroup of G is infinite cyclic. Conversely, let G be a countable locally virtually cyclic group that is torsion-free. Then every finitely generated subgroup of G is trivial or infinite cyclic, from which it follows that G is abelian. Moreover, by [Bie76, Thm. 4.7(b)], we have $\operatorname{cd}_{\mathbb{Z}}(G) \leq \operatorname{cd}_{\mathbb{Z}}(\mathbb{Z}) + 1 = 2$. Now the claim follows from the classification of solvable groups of cohomological dimension two in [Gil79, Thm. 5]. $\qquad\square$

Example 2.31. For a prime number $p \in \mathbb{N}$, consider the colimit of the sequence $\mathbb{Z} \xrightarrow{p} \mathbb{Z} \xrightarrow{p} \mathbb{Z} \rightarrow \ldots$, all maps being multiplication by p. It can be identified with the subgroup $\mathbb{Z}[1/p] \subset \mathbb{Q}$ consisting of all fractions whose denominators are powers of p. By the results of this section, there is a one-dimensional model for $E_{\mathcal{VC}yc}\big(\mathbb{Z}[1/p]\big)$.

There is something more we want to show:

Theorem 2.32 (Groups that are locally $\mathcal{F}$). *Let G be a group of cardinality $\aleph_n$ and $\mathcal{F}$ the family of all finitely generated subgroups of G. Then $E_{\mathcal{F}}(G)$ has a model of dimension $n + 1$.*

Proof. We use induction on $n \in \mathbb{N}$. The case $n = 0$ has already been settled in Example 2.29, so let $n > 0$. We can write $G = \bigcup_{\alpha < \omega_n} G_\alpha$ for subgroups G_α of cardinality $\aleph_{n-1}$ such that $G_\alpha \subset G_\beta$ if $\alpha \leq \beta$. By induction hypothesis, for $\alpha < \omega_n$ there are models X_α for $E_{\mathcal{F} \cap G_\alpha}(G_\alpha)$ of dimension n. The induction step now must

provide us with an $(n+1)$-dimensional model for $E_{\mathcal{F}}(G)$. If we set $G_{\omega_n} := G$, this will be accomplished by using transfinite induction for $\alpha \leq \omega_n$ to construct $(n+1)$-dimensional models Y_α for $E_{\mathcal{F} \cap G_\alpha}(G_\alpha)$ such that $G_\gamma \times_{G_\alpha} Y_\alpha \subset G_\gamma \times_{G_\beta} Y_\beta$ is a G_γ-subcomplex if $\alpha \leq \beta \leq \gamma$.

Let $Y_0 := X_0$. Now suppose that α has got a predecessor. Then the universal property of $E_{\mathcal{F} \cap G_\alpha}(G_\alpha)$ yields a G_α-map $f_\alpha \colon G_\alpha \times_{G_{\alpha-1}} X_{\alpha-1} \to X_\alpha$, which we can assume to be cellular by the equivariant version of the cellular approximation theorem. We define Y_α by the G_α-pushout

$$
\begin{array}{ccc}
G_\alpha \times_{G_{\alpha-1}} X_{\alpha-1} & \xrightarrow{\ i_\alpha\ } & \mathrm{cyl}(f_\alpha) \\
{\scriptstyle \mathrm{id} \times g_{\alpha-1}} \big\downarrow & & \big\downarrow \\
G_\alpha \times_{G_{\alpha-1}} Y_{\alpha-1} & \longrightarrow & Y_\alpha
\end{array}
$$

in which i_α is the obvious inclusion into the mapping cylinder of f_α and $g_{\alpha-1}$ the up to $G_{\alpha-1}$-homotopy unique homotopy equivalence which comes from the universal property of $E_{\mathcal{F} \cap G_{\alpha-1}}(G_{\alpha-1})$. It follows that Y_α is G_α-homotopy equivalent to X_α and hence a model for $E_{\mathcal{F} \cap G_\alpha}(G_\alpha)$. Moreover, Y_α is clearly $(n+1)$-dimensional. Finally, if α is a limit ordinal, we define Y_α to be the union of the $G_\alpha \times_{G_\beta} Y_\beta$ for $\beta < \alpha$. $\qquad\square$

3 Constructing Models for $E_{\mathcal{VCyc}}(G)$ from $E_{\mathcal{Fin}}(G)$

This chapter deals with the observation that for certain classes of groups G it is possible to obtain a model for $E_{\mathcal{VCyc}}(G)$ from a model for $E_{\mathcal{Fin}}(G)$ by attaching equivariant cells.

This not only leads to a computation of the relative homology groups which are direct summands of the source of the Farrell-Jones assembly map (1.22) (and sometimes (1.23), cf. Proposition 1.24), but also yields bounds on the dimension of models for $E_{\mathcal{VCyc}}(G)$.

3.1 Constructing Models out of Given Ones

We start by fixing notation. Recall that, given an inclusion $H \subset G$ of groups, the normalizer of H in G is the subgroup $N_G H := \left\{ g \in G \mid g^{-1} H g = H \right\}$ of G. Moreover, we point out that in our context the Weyl group is defined to be the quotient $W_G H := N_G H / H$.

Notation 3.1. Let $\mathcal{F} \subset \mathcal{G}$ be families of subgroups of a group G. We shall say that G *satisfies* $(M\mathcal{G}, \mathcal{F})$ if every subgroup $H \in \mathcal{G} \setminus \mathcal{F}$ is contained in a unique $M \in \mathcal{G} \setminus \mathcal{F}$ that is maximal with this property, i.e. $M \subset M'$ for an $M' \in \mathcal{G} \setminus \mathcal{F}$ implies $M = M'$.

We now state and prove an important result of this section.

Theorem 3.2. *Let G be a group which satisfies $(M\mathcal{G}, \mathcal{F})$. We denote by $\mathcal{M}$ a complete system of representatives of the conjugacy classes of maximal subgroups $M \in \mathcal{G} \setminus \mathcal{F}$. Assume that $\mathcal{F} \cap N_G M \subset \mathcal{Sub}(M)$ for every such M. Then there is a cellular G-pushout*

$$
\begin{array}{ccc}
\coprod_{M \in \mathcal{M}} G \times_{N_G M} E_{\mathcal{F} \cap N_G M}(N_G M) & \xrightarrow{\;\;i\;\;} & E_{\mathcal{F}}(G) \\
\downarrow{\scriptstyle \coprod_M \mathrm{id} \times f_M} & & \downarrow \\
\coprod_{M \in \mathcal{M}} G \times_{N_G M} EW_G M & \longrightarrow & E_{\mathcal{G}}(G)
\end{array}
$$

in which the $W_G M$-spaces $EW_G M$ are considered as $N_G M$-spaces via the canonical projections $N_G M \twoheadrightarrow W_G M$, the $N_G M$-maps $f_M \colon E_{\mathcal{F} \cap N_G M}(N_G M) \to EW_G M$ are unique up to $N_G M$-homotopy, and i is an inclusion of G-CW-complexes.

Proof. Let $M \in \mathcal{M}$. Since the $N_G M$-space $EW_G M$ is a model for $E_{\mathcal{S}ub(M)}(N_G M)$, the assumption yields an $N_G M$-map $f_M \colon E_{\mathcal{F} \cap N_G M}(N_G M) \to EW_G M$ which is unique up to $N_G M$-homotopy. Similarly, since all the isotropy groups of the G-CW-complex $\coprod_M G \times_{N_G M} E_{\mathcal{F} \cap N_G M}(N_G M)$ belong to $\mathcal{F}$, there is precisely one G-map $i \colon \coprod_M G \times_{N_G M} E_{\mathcal{F} \cap N_G M}(N_G M) \to E_{\mathcal{F}}(G)$ up to G-homotopy. By the equivariant version of the cellular approximation theorem, we can assume f_M and i to be cellular. Moreover, by replacing i with the inclusion into its mapping cylinder, it can be arranged for i to be an inclusion of G-CW-complexes.

Now let us define the G-CW-complex X to be the G-pushout

$$
\begin{array}{ccc}
\coprod\limits_{M \in \mathcal{M}} G \times_{N_G M} E_{\mathcal{F} \cap N_G M}(N_G M) & \overset{i}{\longrightarrow} & E_{\mathcal{F}}(G) \\
{\scriptstyle \coprod_M \mathrm{id} \times f_M} \downarrow & & \downarrow \\
\coprod\limits_{M \in \mathcal{M}} G \times_{N_G V} EW_G M & \longrightarrow & X
\end{array}
\tag{3.3}
$$

We claim that X is a model for $E_{\mathcal{G}}(G)$. In order to prove this, let $s_M \colon G/N_G M \to G$ be sections of the projections, and let a subgroup $H \subset G$ be given. Then, taking H-fixed points of (3.3) and applying Lemma 1.7 (1) yields a pushout

$$
\begin{array}{ccc}
\coprod\limits_{M \in \mathcal{M}} \coprod\limits_{\substack{\alpha \in G/N_G M, \\ s_M(\alpha)^{-1} H s_M(\alpha) \subset N_G M}} E_{\mathcal{F} \cap N_G M}(N_G M)^{s_M(\alpha)^{-1} H s_M(\alpha)} & \overset{i}{\longrightarrow} & E_{\mathcal{F}}(G)^H \\
{\scriptstyle \coprod_M \coprod_\alpha f_{M,\alpha}} \downarrow & & \downarrow \\
\coprod\limits_{M \in \mathcal{M}} \coprod\limits_{\substack{\alpha \in G/N_G M, \\ s_M(\alpha)^{-1} H s_M(\alpha) \subset N_G M}} EW_G M^{s_M(\alpha)^{-1} H s_M(\alpha)} & \longrightarrow & X^H
\end{array}
\tag{3.4}
$$

in which i is an inclusion of CW-complexes.

Assume that $H \notin \mathcal{G}$. Then the entries in the upper row of (3.4) are clearly empty. However, so is the lower left entry because $EW_G M^{s_M(\alpha)^{-1} H s_M(\alpha)} = \emptyset$ unless $s_M(\alpha)^{-1} H s_M(\alpha) \subset M$. Hence in this case $X^H = \emptyset$.

If $H \in \mathcal{G} \setminus \mathcal{F}$, the entries in the upper row of (3.4) are again empty. Thus it suffices to show that the lower left entry is contractible. By assumption, there is a unique $M \in \mathcal{M}$ to which H is subconjugated, say $g^{-1} H g \subset M$ for an appropriate $g \in G$, whose projection to $G/N_G M$ we denote by γ. Then $g^{-1} s_M(\gamma) \in N_G M$ so that $s_M(\gamma)^{-1} H s_M(\gamma) \subset M$. Moreover, if $s_M(\alpha)^{-1} H s_M(\alpha) \subset M$, we have

$$
M = \left((s_M(\alpha)^{-1} H s_M(\alpha) \right)_{\max} = s_M(\alpha)^{-1} H_{\max} s_M(\alpha) = s_M(\alpha)^{-1} g M g^{-1} s_M(\alpha),
$$

where we write $K_{\max}$ for the unique maximal element in $\mathcal{G} \setminus \mathcal{F}$ containing a given element K in $\mathcal{G} \setminus \mathcal{F}$. This means that $g^{-1} s_M(\alpha) \in N_G M$, hence $\alpha = \gamma$. It follows that the lower left entry of (3.4) is equal to $EW_G M$, and this is non-equivariantly contractible.

Finally, if $H \in \mathcal{F}$, the upper right entry of (3.4) is contractible. This implies that the same will hold for X^H if we can show that all the maps $f_{M,\alpha}$ are homotopy equivalences. But this is clear since the source and target spaces are contractible, the latter because $\mathcal{F} \cap N_G M \subset \mathcal{Sub}(M)$. $\qquad\square$

In order to derive the conclusion we are interested in from Theorem 3.2, we need the following simple observation:

Lemma 3.5. *Let G be a group and $V \subset G$ a maximal virtually cyclic subgroup. Then $W_G V$ is torsion-free. In particular, every finite subgroup of $N_G V$ is already contained in V.*

Proof. Let $p \colon N_G V \twoheadrightarrow W_G V$ be the projection. If $H \subset W_G V$ is a finite subgroup, then $p^{-1}(H)$ contains V as a subgroup of finite index. Thus, $p^{-1}(H)$ is virtually cyclic and therefore equal to V because V is maximal virtually cyclic. This implies that H is trivial. $\qquad\square$

Corollary 3.6. *Let G be a group which satisfies $(M\mathcal{Fin}, \mathcal{Tr})$ or $(M\mathcal{VCyc}, \mathcal{Fin})$. We denote by $\mathcal{M}$ a complete system of representatives of the conjugacy classes of maximal finite subgroups $F \subset G$ or of infinite maximal virtually cyclic subgroups $V \subset G$, respectively. Then there are cellular G-pushouts*

$$
\begin{array}{ccc}
\coprod\limits_{F \in \mathcal{M}} G \times_{N_G F} EN_G M & \overset{i}{\longrightarrow} & EG \\
\Big\downarrow{\scriptstyle \coprod_F \mathrm{id} \times f_F} & & \Big\downarrow \\
\coprod\limits_{F \in \mathcal{M}} G \times_{N_G F} EW_G F & \longrightarrow & \underline{E}G
\end{array}
\quad or \quad
\begin{array}{ccc}
\coprod\limits_{V \in \mathcal{M}} G \times_{N_G V} \underline{E}N_G V & \overset{i}{\longrightarrow} & \underline{E}G \\
\Big\downarrow{\scriptstyle \coprod_V \mathrm{id} \times f_V} & & \Big\downarrow \\
\coprod\limits_{V \in \mathcal{M}} G \times_{N_G V} EW_G V & \longrightarrow & E_{\mathcal{VCyc}}(G)
\end{array}
$$

respectively, the maps being as in Theorem 3.2.

Proof. In order to apply Theorem 3.2, we have to show that $\mathcal{Tr} \cap N_G F \subset \mathcal{Sub}(F)$ for any maximal finite subgroup $F \subset G$, and that $\mathcal{Fin} \cap N_G V \subset \mathcal{Sub}(V)$ for any infinite maximal virtually cyclic subgroup $V \subset G$. The former is trivial, and the latter follows from Lemma 3.5. $\qquad\square$

For examples of groups satisfying one of the conditions of Corollary 3.6, we refer to Remark 3.14 and Examples 3.10 and 3.22. The next goal is to show that one can actually prove a version of the above result for groups that only virtually satisfy $(M\mathcal{VCyc}, \mathcal{Fin})$.

Theorem 3.7. *Suppose that $1 \to K \to G \to Q \to 1$ is an exact sequence of groups such that K satisfies $(M\mathcal{VCyc}, \mathcal{Fin})$ and Q is finite. We identify K with its image in G and denote by $\mathcal{M}$ a complete system of representatives of the conjugacy classes in G of infinite subgroups $V \subset K$ that are maximal virtually cyclic in K. Then there*

is a cellular G-pushout

$$
\begin{array}{ccc}
\coprod\limits_{V \in \mathscr{M}} G \times_{N_G V} \underline{E}N_G V & \xrightarrow{\ i\ } & \underline{E}G \\[2mm]
\Big\downarrow{\scriptstyle \coprod_V \mathrm{id} \times \underline{E}\pi_V} & & \Big\downarrow \\[2mm]
\coprod\limits_{V \in \mathscr{M}} G \times_{N_G V} \underline{E}W_G V & \longrightarrow & E_{\mathcal{V}\mathit{Cyc}}(G)
\end{array}
$$

in which the $W_G V$-spaces $\underline{E}W_G V$ are regarded as $N_G V$-spaces via the canonical projections $\pi_V \colon N_G V \twoheadrightarrow W_G V$, the maps $\underline{E}\pi_V \colon \underline{E}N_G V \to \underline{E}W_G V$ are $N_G V$-equivariant, and i is an inclusion of G-CW-complexes.

Proof. With the help of a functorial construction of classifying spaces, the projections π_V induce the desired $N_G V$-maps $\underline{E}\pi_V \colon \underline{E}N_G V \to \underline{E}W_G V$. Furthermore, since all the isotropy groups of the G-CW-complex $\coprod_V G \times_{N_G V} \underline{E}N_G V$ are obviously finite, there is precisely one G-map $i \colon \coprod_V G \times_{N_G V} \underline{E}N_G V \to \underline{E}G$ up to G-homotopy. By the equivariant version of the cellular approximation theorem, we can assume that all the $\underline{E}\pi_V$ and i are cellular. After replacing i with the inclusion into its mapping cylinder, we can moreover assume that i is an inclusion of G-CW-complexes.

Now, we define the G-CW-complex X to be the G-pushout

$$
\begin{array}{ccc}
\coprod\limits_{V \in \mathscr{M}} G \times_{N_G V} \underline{E}N_G V & \xrightarrow{\ i\ } & \underline{E}G \\[2mm]
\Big\downarrow{\scriptstyle \coprod_V \mathrm{id} \times \underline{E}\pi_V} & & \Big\downarrow \\[2mm]
\coprod\limits_{V \in \mathscr{M}} G \times_{N_G V} \underline{E}W_G V & \longrightarrow & X
\end{array}
$$

and want to show that X is a model for $E_{\mathcal{V}\mathit{Cyc}}(G)$. To do so, we choose sections $s_V \colon G/N_G V \to W_G V$ of the projections and a subgroup $H \subset G$. Taking H-fixed points of the above G-pushout then yields, in connection with Lemma 1.7 (1), a pushout

$$
\begin{array}{ccc}
\coprod\limits_{V \in \mathscr{M}} \ \coprod\limits_{\substack{\alpha \in G/N_G V, \\ s_V(\alpha)^{-1} H s_V(\alpha) \subset N_G V}} \underline{E}N_G V^{s_V(\alpha)^{-1} H s_V(\alpha)} & \xrightarrow{\ i\ } & \underline{E}G^H \\[3mm]
\Big\downarrow{\scriptstyle \coprod_V \coprod_\alpha \underline{E}\pi_V} & & \Big\downarrow \\[3mm]
\coprod\limits_{V \in \mathscr{M}} \ \coprod\limits_{\substack{\alpha \in G/N_G V, \\ s_V(\alpha)^{-1} H s_V(\alpha) \subset N_G V}} \underline{E}W_G V^{s_V(\alpha)^{-1} H s_V(\alpha)} & \longrightarrow & X^H
\end{array}
\tag{3.8}
$$

in which i is an inclusion of CW-complexes.

Let us first assume that H is not virtually cyclic. If $V \in \mathscr{M}$ and $\alpha \in G/N_G V$ are such that $H_\alpha := s_V(\alpha)^{-1} H s_V(\alpha) \subset N_G V$, then $\pi_V(H_\alpha) \cong H_\alpha / H_\alpha \cap V$ must be an infinite subgroup of $W_G V$ as $H_\alpha \cap V$ is virtually cyclic. This means that the lower

left entry of (3.8) is empty. Clearly, the entries in the upper row are also empty, hence $X^H = \emptyset$ in this case.

If H is infinite virtually cyclic, we claim that there is precisely one infinite maximal virtually cyclic subgroup $V \subset K$ such that $H \subset N_G V$ and $|\pi_V(H)| < \infty$. As for its existence, note that $H \cap K$ is infinite virtually cyclic since $G/K = Q$ is finite by assumption. Hence, $H \cap K$ is contained in an infinite maximal virtually cyclic subgroup $V \subset K$ because K satisfies $(M\mathcal{VCyc}, \mathcal{Fin})$. For every $h \in H$, however, $H \cap K = h^{-1}H \cap Kh$ is contained in $h^{-1}Vh$ as well. Thus, again because K satisfies $(M\mathcal{VCyc}, \mathcal{Fin})$, we have that $H \subset N_G V$. Moreover, $|\pi_V(H)| = |H/H \cap V| < \infty$ since $H \cap V$ is infinite virtually cyclic. To establish the uniqueness of such a V, let $V' \subset K$ be another infinite maximal virtually cyclic subgroup with the according properties. In particular, $H \cap V'$ is also of finite index in H, which implies that the same holds for $H \cap V \cap V'$. Thus, $V \cap V'$ is infinite virtually cyclic, and it follows that $V = V'$ as K satisfies $(M\mathcal{VCyc}, \mathcal{Fin})$. From what we have just shown, it can readily be deduced that there is precisely one possible choice of $V \in \mathcal{M}$ and $\alpha \in W_G V$ such that $s_V(\alpha)^{-1}Hs_V(\alpha) \subset N_G V$ and $\left|\pi_V\big(s_V(\alpha)^{-1}Hs_V(\alpha)\big)\right| < \infty$. So the lower left entry of (3.8) is contractible, whereas the entries in the upper row are empty, and it follows that X^H is contractible.

Finally, if H is finite, then the left vertical arrow in (3.8) clearly is a homotopy equivalence. Thus X^H is homotopy equivalent to $\underline{E}G^H$, which is contractible. $\qquad\square$

Remark 3.9 (Dimension of the constructed models). In the situation of Theorem 3.2, suppose one takes a k-dimensional model for $E_{\mathcal{F}}(G)$ and, for $M \in \mathcal{M}$, models for $E_{\mathcal{F} \cap N_G M}(N_G M)$ of dimension $l(M)$ and models for $EW_G M$ of dimension $m(M)$. Then the model that is constructed for $E_{\mathcal{G}}(G)$ is of dimension n, where

$$n = \sup\{k, \, l(M) + 1, \, m(M) \mid M \in \mathcal{M}\}.$$

The analogous statement holds in the situation of Theorem 3.7.

We are now prepared to dwell on the problem of finding models for $E_{\mathcal{VCyc}}(G)$ in the case of a finitely generated abelian group G (see also Example 2.17).

Example 3.10 (Finitely generated abelian groups). Let G be a finitely generated abelian group of rank $n \geq 2$, i.e. $G = \mathbb{Z}^n \oplus F$, where F is finite abelian. Pick elements $a_1, \ldots, a_n \in G$ that generate $\mathbb{Z}^n$. Then the maximal virtually cyclic subgroups of G are precisely those $V(r_1, \ldots, r_n) := \langle a_1^{r_1} \cdots a_n^{r_n}\rangle \oplus F$ for which there is some choice of distinct $i, j \in \{1, \ldots, n\}$ such that r_i and r_j are coprime integers. Furthermore, it is immediate that either $V(r_1, \ldots, r_n) = V(s_1, \ldots, s_n)$ or $V(r_1, \ldots, r_n) \cap V(s_1, \ldots, s_n) = F$ for all such maximal virtually cyclic subgroups. It follows that no model for $E_{\mathcal{VCyc}}(G)$ can be of finite type, cf. Lemma 2.19. Moreover, G satisfies $(M\mathcal{VCyc}, \mathcal{Fin})$, and we deduce from Remark 3.9 that there exists an $(n+1)$-dimensional model for $E_{\mathcal{VCyc}}(G)$ since there is an n-dimensional model for $\underline{E}G$.

However, there cannot exist a model of lesser dimension. This is due to the fact that dividing out the G-action in the pushout of Corollary 3.6 and keeping in mind that the normalizer of any subgroup of G is G itself yields the pushout

$$
\begin{array}{ccc}
\coprod\limits_{V \in \mathcal{M}} G\backslash \underline{E}G & \xrightarrow{\ \amalg_V \operatorname{id}\ } & G\backslash \underline{E}G \\
{\scriptstyle \amalg_V \, G\backslash f_V} \downarrow & & \downarrow \\
\coprod\limits_{V \in \mathcal{M}} B(G/V) & \longrightarrow & G\backslash E_{\mathcal{VC}yc}(G)
\end{array}
$$

While $G\backslash \underline{E}G$ is homotopy equivalent to the n-dimensional CW-complex $T^n = \prod_{i=1}^n S^1$, the space $B(G/V)$ is homotopy equivalent to the $(n-1)$-dimensional CW-complex T^{n-1}. Thus the Mayer-Vietoris sequence belonging to this diagram reads

$$
\ldots \to 0 \to H_{n+1}\big(G\backslash E_{\mathcal{VC}yc}(G)\big) \to \bigoplus_{V \in \mathcal{M}} H_n(G\backslash \underline{E}G) \xrightarrow{\ \oplus_V \operatorname{id}\ } H_n(G\backslash \underline{E}G) \to \ldots,
$$

which implies that $H_{n+1}\big(G\backslash E_{\mathcal{VC}yc}(G)\big)$ is free abelian of infinite rank, proving the claim.

Provided that $\mathcal{F} = \mathcal{T}r$ or $\mathcal{F}in$, it is obvious that whenever one has models for $E_{\mathcal{F}}(G_1)$ and $E_{\mathcal{F}}(G_2)$, their product will be a model for $E_{\mathcal{F}}(G_1 \times G_2)$. As we have already pointed out in section 2.2, the analogous statement fails drastically in the case of $\mathcal{F} = \mathcal{VC}yc$. However, one can say a bit more under the assumption that $G_1 \times G_2$ virtually satisfies $(M\mathcal{VC}yc, \mathcal{F}in)$ (note that the property $(M\mathcal{VC}yc, \mathcal{F}in)$ is not stable under forming direct products).

Corollary 3.11. *Let $G = G_1 \times G_2$ be a group which fits into an exact sequence $1 \to K \to G \to Q \to 1$ such that K satisfies $(M\mathcal{VC}yc, \mathcal{F}in)$ and Q is finite. We identify K with its image in G and denote by $\mathcal{M}$ a complete system of representatives of the conjugacy classes in G of infinite subgroups $V \subset K$ that are maximal virtually cyclic in K. Let*

$$
\begin{aligned}
l &:= \sup_{V \in \mathcal{M}} \{\text{minimal dimension of } \underline{E}N_G V\}, \\
m &:= \sup_{V \in \mathcal{M}} \{\text{minimal dimension of } \underline{E}W_G V\}.
\end{aligned}
$$

Suppose there are models for $E_{\mathcal{VC}yc}(G_1)$ and $E_{\mathcal{VC}yc}(G_2)$ of dimension d_1 and d_2 respectively. Then $l \leq d_1 + d_2 + 2$, and there is an n-dimensional model for $E_{\mathcal{VC}yc}(G)$, where

$$
n = \max\{d_1 + d_2 + 2,\, l + 1,\, m\}.
$$

Moreover, this estimate for n is best possible.

Proof. Proposition 2.12 shows that for $i = 1, 2$ there are $(d_i + 1)$-dimensional models for $\underline{E}G_i$. Their product then constitutes a model for $\underline{E}G$ of dimension $d_1 + d_2 + 2$. Thus, by Remark 3.9, there is a model for $E_{\mathcal{VCyc}}(G)$ of the dimension claimed. Furthermore, since a model for $\underline{E}G$ yields by restriction a model for $\underline{E}N_GV$, the inequality $l \leq d_1 + d_2 + 2$ holds.

Finally, the given estimate on the dimension of $E_{\mathcal{VCyc}}(G)$ is sharp as can be seen in the case when Q is the trivial group and $G_1 = G_2 = \mathbb{Z}$, in which $d_1 = d_2 = 0$, $l = 2$ and $m = 1$, while there is no model for $E_{\mathcal{VCyc}}(G)$ of dimension less than three, see Example 2.17 or Example 3.10. □

We end this section by explaining how the results in [CFH06] on the dimension of models for $E_{\mathcal{VCyc}}(G)$ for crystallographic groups G fit into our framework.

Example 3.12 (Crystallographic groups). For given $n \geq 2$, let G be a crystallographic subgroup of the isometry group of $\mathbb{R}^n$. Let us denote by $T \subset G$ the group consisting of all the translations in G. Then T is normal in G. Furthermore, the Bieberbach theorem (see e.g. [Far81, Thm. 14]) states that T is finitely generated free abelian of rank n and that G/T is finite. In particular, Example 3.10 shows that there cannot be a model for $E_{\mathcal{VCyc}}(G)$ of dimension less than $n + 1$.

On the other hand, Theorem 3.7 applied to the exact sequence $1 \to T \to G \to G/T \to 1$ can be utilized as follows to produce an $(n + 1)$-dimensional model for $E_{\mathcal{VCyc}}(G)$. First note that in Theorem 2.11 we have already seen that $\mathbb{R}^n$, with an appropriate G-CW-structure, is a model for $\underline{E}G$. For every subgroup $C \subset T$ which is maximal cyclic in T, the restriction of $\mathbb{R}^n$ to the N_GC-operation can then be used as a model for $\underline{E}N_GC$. As C is generated by a single non-trivial translation of $\mathbb{R}^n$, there is a unique line in $\mathbb{R}^n$ which contains $0 \in \mathbb{R}^n$ and is invariant under the action of C. Let $\mathbb{R}^{n-1}_C \subset \mathbb{R}^n$ be the hyperplane which is orthogonal to this line, and let $p_C \colon \mathbb{R}^n \to \mathbb{R}^{n-1}_C$ be the orthogonal projection. It is immediate from the construction that one can impose a W_GC-action on $\mathbb{R}^{n-1}_C$ by setting $\pi_C(n) \cdot p_C(x) \coloneqq p_C(nx)$, where $\pi_C \colon N_GC \twoheadrightarrow W_GC$ is the canonical projection.

Now, in [CFH06] it is shown that one can choose a CW-structure for $\mathbb{R}^{n-1}_C$ which turns this space into a model for $\underline{E}W_GC$. Hence, Theorem 2.16 yields indeed a model for $E_{\mathcal{VCyc}}(G)$ of dimension $n + 1$.

3.2 Computation of the Relative Homology Groups

As a further application of Corollary 3.6, we will see in this section that the relative homology groups which split off from the source of the Farrell-Jones assembly map simplify considerably if the involved group satisfies $(M\mathcal{VCyc}, \mathcal{Fin})$.

First of all, suppose $\mathbf{E} \colon \mathcal{Groupoids} \to \mathcal{Spectra}$ is a functor that sends equivalences of groupoids to maps of spectra inducing an isomorphism on homotopy groups (the main example to keep in mind here is when $\mathbf{E}$ is one of $\mathbf{K}^{\mathrm{top}}$, $\mathbf{K}_R$ or $\mathbf{L}_R^{\langle -\infty \rangle}$, cf. section 1.2). If $H \subset G$ is an inclusion of groups and $\mathcal{F}$ a family of subgroups of

$N_G H$, let $p[H, \mathcal{F}]$ be the map

$$\left(\operatorname{res}^H_{N_G H} E_{\mathcal{F}}(N_G H)\right)^-_+ \wedge_{\operatorname{Or}(H)} \operatorname{res}^{\operatorname{Or}(H)}_{\operatorname{Or}(N_G H)} \mathbf{E}\left(\mathcal{G}^{N_G H}(N_G H/-)\right)$$
$$\downarrow$$
$$\operatorname{pt}^-_+ \wedge_{\operatorname{Or}(H)} \operatorname{res}^{\operatorname{Or}(H)}_{\operatorname{Or}(N_G H)} \mathbf{E}\left(\mathcal{G}^{N_G H}(N_G H/-)\right)$$
$$\|$$
$$\mathbf{E}\left(\mathcal{G}^{N_G H}(W_G H)\right)$$

of spectra which is induced by the projection on the first and the identity on the second factor. We will now implement an action of $W_G H$ on the source and target of $p[H, \mathcal{F}]$ with respect to which $p[H, \mathcal{F}]$ is equivariant. This follows ideas of Lück.

For a fixed element $n_0 \in N_G H$, we denote by $c_{n_0}(n) := n_0 n n_0^{-1}$ conjugation in $N_G H$ by n_0. Then c_{n_0} induces a functor $c_{n_0} \colon \operatorname{Or}(N_G H) \to \operatorname{Or}(N_G H)$ by sending an object $N_G H/K$ to $N_G H/c_{n_0}(K)$ and a morphism r_n to $r_{c_{n_0}(n)}$. Analogously, we get a functor $c_{n_0} \colon \operatorname{Or}(H) \to \operatorname{Or}(H)$. Furthermore, there is a natural equivalence $\rho_{n_0^{-1}} \colon \mathcal{G}^{N_G H} \to c^*_{n_0} \mathcal{G}^{N_G H}$ of $\operatorname{Or}(N_G H)$-groupoids which for an object $N_G H/K$ of $\operatorname{Or}(N_G H)$ is given by the functor

$$
\begin{array}{ccc}
nK & & nn_0^{-1}c_{n_0}(K) \\
{\scriptstyle n'}\Big\downarrow & \mapsto & {\scriptstyle n'}\Big\downarrow \\
n'nK & & n'nn_0^{-1}c_{n_0}(K)
\end{array}
$$

which is an isomorphism of categories. Lastly, we have an H-homeomorphism $\phi_{n_0} \colon H \times_{c_{n_0}} \operatorname{res}^H_{N_G H} E_{\mathcal{F}}(N_G H) \to \operatorname{res}^H_{N_G H} E_{\mathcal{F}}(N_G H)$ defined by $[h, z] \mapsto hn_0 z$.

Now, let n_0 act on the source of $p[H, \mathcal{F}]$ as the composition

$$\left(\operatorname{res}^H_{N_G H} E_{\mathcal{F}}(N_G H)\right)^-_+ \wedge_{\operatorname{Or}(H)} \operatorname{res}^{\operatorname{Or}(H)}_{\operatorname{Or}(N_G H)} \mathbf{E}\left(\mathcal{G}^{N_G H}(N_G H/-)\right)$$
$${\scriptstyle \operatorname{id} \wedge \operatorname{res} \mathbf{E}(\rho_{n_0^{-1}})}\Big\downarrow$$
$$\left(\operatorname{res}^H_{N_G H} E_{\mathcal{F}}(N_G H)\right)^-_+ \wedge_{\operatorname{Or}(H)} \operatorname{res}^{\operatorname{Or}(H)}_{\operatorname{Or}(N_G H)} \mathbf{E}\left(c^*_{n_0} \mathcal{G}^{N_G H}(N_G H/-)\right)$$
$$\|$$
$$\left(\operatorname{res}^H_{N_G H} E_{\mathcal{F}}(N_G H)\right)^-_+ \wedge_{\operatorname{Or}(H)} c^*_{n_0} \operatorname{res}^{\operatorname{Or}(H)}_{\operatorname{Or}(N_G H)} \mathbf{E}\left(\mathcal{G}^{N_G H}(N_G H/-)\right)$$
$${\scriptstyle \operatorname{adj}}\Big\downarrow {\scriptstyle \text{(cf. Lemma 1.11 (2))}}$$
$$\operatorname{ind}_{c_{n_0}}\left(\operatorname{res}^H_{N_G H} E_{\mathcal{F}}(N_G H)\right)^-_+ \wedge_{\operatorname{Or}(H)} \operatorname{res}^{\operatorname{Or}(H)}_{\operatorname{Or}(N_G H)} \mathbf{E}\left(\mathcal{G}^{N_G H}(N_G H/-)\right)$$
$${\scriptstyle \nu \wedge \operatorname{id}}\Big\downarrow {\scriptstyle \text{(cf. Lemma 1.12)}}$$
$$\left(H \times_{c_{n_0}} \operatorname{res}^H_{N_G H} E_{\mathcal{F}}(N_G H)\right)^-_+ \wedge_{\operatorname{Or}(H)} \operatorname{res}^{\operatorname{Or}(H)}_{\operatorname{Or}(N_G H)} \mathbf{E}\left(\mathcal{G}^{N_G H}(N_G H/-)\right)$$
$${\scriptstyle \phi_{n_0} \wedge \operatorname{id}}\Big\downarrow$$
$$\left(\operatorname{res}^H_{N_G H} E_{\mathcal{F}}(N_G H)\right)^-_+ \wedge_{\operatorname{Or}(H)} \operatorname{res}^{\operatorname{Or}(H)}_{\operatorname{Or}(N_G H)} \mathbf{E}\left(\mathcal{G}^{N_G H}(N_G H/-)\right),$$

and, similarly, on the target of $p[H, \mathcal{F}]$, merely replacing $E_{\mathcal{F}}(N_G H)$ by pt. Explicitly, we have $n_0 \cdot z \wedge e = n_0 z \wedge \text{res } \mathbf{E}(\rho_{n_0^{-1}})(e)$, and it is straightforward to show that this action is trivial on $H \subset N_G H$. Hence it descends to an action of $W_G H$ which is surely compatible with $p[H, \mathcal{F}]$.

Notation 3.13. We write $\mathbf{E}[H, \mathcal{F}]$ for the homotopy cofibre of the $W_G H$-equivariant map $p[H, \mathcal{F}]$. In particular, $\mathbf{E}[H, \mathcal{F}]$ is a $W_G H$-spectrum.

Remark 3.14 (The p-chain spectral sequence). Let G be a group and assume that either

(1) G satisfies $(M\mathcal{F}in, \mathcal{T}r)$, and for every maximal finite subgroup $F \subset G$ we have $N_G F = F$, or

(2) G is torsion-free, satisfies $(M\mathcal{C}yc, \mathcal{T}r)$, and for every maximal cyclic subgroup $C \subset G$ we have $N_G C = C$.

Examples of (1) are

- groups G that are an extension $1 \to \mathbb{Z}^n \to G \to H \to 1$ where H is finite and the conjugation action of H on Z^n is free outside $0 \in \mathbb{Z}^n$ (see [LS00, Lemma 6.1 and Lemma 6.3]),

- Fuchsian groups (see [LS00, Lemma 4.5]), and

- one-relator groups (cf. [LS77, Prop. 5.17, Prop. 5.18 and Prop. 5.19]),

whereas (2) holds, for instance, for torsion-free word-hyperbolic groups, see Example 3.22.

We denote by $\mathcal{M}$ in the situation of (1) a complete system of representatives of the conjugacy classes of maximal finite subgroups, or of maximal cyclic subgroups in the situation of (2). Then, Theorem 3.16 will yield an exact sequence

$$\ldots \to \bigoplus_{H \in \mathcal{M}} \pi_n \big(EH_+^- \wedge_{\text{Or}(H)} \mathbf{E}\big(\mathcal{G}^H(H/-) \big) \big) \to \bigoplus_{H \in \mathcal{M}} \pi_n \big(\mathbf{E}\big(\mathcal{G}^H(H/H) \big)$$

$$\to \mathcal{H}_n^G \big(EG \to E_{\mathcal{F}}(G); \mathbf{E} \big) \to \bigoplus_{H \in \mathcal{M}} \pi_{n-1} \big(EH_+^- \wedge_{\text{Or}(H)} \mathbf{E}\big(\mathcal{G}^H(H/-) \big) \big) \to \ldots,$$

where $\mathcal{F} = \mathcal{F}in$ in the situation of (1), and $\mathcal{F} = \mathcal{C}yc$ in the situation of (2). This is precisely what one gets from the p-chain spectral sequence, cf. [DL03, Cor. 3.13].

Lemma 3.15. *Let $H \subset G$ be an inclusion of groups and $\pi \colon N_G H \twoheadrightarrow W_G H$ the projection. For a free $W_G H$-CW-complex X and a family $\mathcal{F}$ of subgroups of $N_G H$ consider the projection $q \colon \pi^* X \times E_{\mathcal{F}}(N_G H) \to \pi^* X$ to the first factor. Then for every $n \in \mathbb{Z}$ there is an isomorphism*

$$\mathcal{H}_n^{N_G H}(q; \mathbf{E}) \xrightarrow{\cong} H_n^{W_G H} \big(X; \mathbf{E}[H, \mathcal{F}] \big)$$

which is natural in X.

Proof. We will show below that if Z is an $N_G H$-CW-complex, then there are isomorphisms

$$\left(\pi^* X \times Z\right)_+^- \wedge_{\mathrm{Or}(N_G H)} \mathbf{E}\!\left(\mathcal{G}^{N_G H}(N_G H/-)\right)$$

$$\xrightarrow{f_1} X_+ \wedge_{W_G H} \left(\left(\pi^*(W_G H) \times Z\right)_+^- \wedge_{\mathrm{Or}(N_G H)} \mathbf{E}\!\left(\mathcal{G}^{N_G H}(N_G H/-)\right)\right)$$

$$\xrightarrow{f_2} X_+ \wedge_{W_G H} \left(\left(N_G H \times_H \mathrm{res}_{N_G H}^H Z\right)_+^- \wedge_{\mathrm{Or}(N_G H)} \mathbf{E}\!\left(\mathcal{G}^{N_G H}(N_G H/-)\right)\right)$$

$$\xrightarrow{f_3} X_+ \wedge_{W_G H} \left(\left(\mathrm{res}_{N_G H}^H Z\right)_+^- \wedge_{\mathrm{Or}(H)} \mathrm{res}_{\mathrm{Or}(N_G H)}^{\mathrm{Or}(H)} \mathbf{E}\!\left(\mathcal{G}^{N_G H}(N_G H/-)\right)\right)$$

of spectra which are natural in Z and X. Thus, taking Z to be $E_{\mathcal{F}}(N_G H)$ and pt yields a map from the long exact homology sequence

$$\ldots \to \mathcal{H}_n^{N_G H}\!\left(\pi^* X \times E_{\mathcal{F}}(N_G H); \mathbf{E}\right) \to \mathcal{H}_n^{N_G H}(\pi^* X; \mathbf{E}) \to \mathcal{H}_n^{N_G H}(q; \mathbf{E})$$
$$\to \mathcal{H}_{n-1}^{N_G H}\!\left(\pi^* X \times E_{\mathcal{F}}(N_G H); \mathbf{E}\right) \to \ldots$$

to the sequence

$$\vdots$$
$$\downarrow$$

$$H_n^{W_G H}\!\left(X; \left(\mathrm{res}_{N_G H}^H E_{\mathcal{F}}(N_G H)\right)_+^- \wedge_{\mathrm{Or}(H)} \mathrm{res}_{\mathrm{Or}(N_G H)}^{\mathrm{Or}(H)} \mathbf{E}\!\left(\mathcal{G}^{N_G H}(N_G H/-)\right)\right)$$
$$\downarrow$$
$$H_n^{W_G H}\!\left(X; \mathrm{pt}_+^- \wedge_{\mathrm{Or}(H)} \mathrm{res}_{\mathrm{Or}(N_G H)}^{\mathrm{Or}(H)} \mathbf{E}\!\left(\mathcal{G}^{N_G H}(N_G H/-)\right)\right)$$
$$\downarrow$$
$$H_n^{W_G H}\!\left(X; \mathbf{E}[H, \mathcal{F}]\right)$$
$$\downarrow$$
$$H_{n-1}^{W_G H}\!\left(X; \left(\mathrm{res}_{N_G H}^H E_{\mathcal{F}}(N_G H)\right)_+^- \wedge_{\mathrm{Or}(H)} \mathrm{res}_{\mathrm{Or}(N_G H)}^{\mathrm{Or}(H)} \mathbf{E}\!\left(\mathcal{G}^{N_G H}(N_G H/-)\right)\right)$$
$$\downarrow$$
$$\vdots$$

which is exact by construction of $\mathbf{E}[H, \mathcal{F}]$, and, according to the five lemma, the maps $\mathcal{H}_n^{N_G H}(q; \mathbf{E}) \to H_n^{W_G H}\!\left(X; \mathbf{E}[H, \mathcal{F}]\right)$ must be isomorphisms.

It remains to show that there are maps f_1, f_2 and f_3 as above which are isomorphisms of spectra. Starting with f_1, we first have to establish suitable $W_G H$-actions on X and $\left(\pi^*(W_G H) \times Z\right)_+^- \wedge_{\mathrm{Or}(N_G H)} \mathbf{E}\!\left(\mathcal{G}^{N_G H}(N_G H/-)\right)$. The former becomes a right $W_G H$-space by setting $x \cdot w := w^{-1} x$, and the latter a left $W_G H$-spectrum as follows. For $w \in W_G H$, consider the homeomorphism $\eta_w \colon \left(\pi^*(W_G H) \times Z\right)_+^- \to \left(\pi^*(W_G H) \times Z\right)_+^-$ of $\mathrm{Or}(N_G H)$-spaces which comes from the $N_G H$-homeomorphism $(w', z) \mapsto (w' w^{-1}, z)$. Then w acts as the automorphism $\eta_w \wedge \mathrm{id}$ of spectra. Now we define f_1 as the map that is given by

$$(x, z) \wedge e \mapsto x \wedge (1, z) \wedge e.$$

Then f_1 is indeed an isomorphism, its inverse being given by $x \wedge (w, z) \wedge e \mapsto (wx, z) \wedge e$.

As for the definition of f_2, let $\phi \colon \pi^*(W_G H) \times Z \to N_G H \times_H \mathrm{res}^H_{N_G H} Z$ be the $N_G H$-homeomorphism

$$\big(\pi(n), z\big) \mapsto [n, n^{-1} z]$$

of Lemma 1.7 (2). The induced map $\phi \wedge \mathrm{id} \colon \big(\pi^*(W_G H) \times Z\big)^-_+ \wedge_{\mathrm{Or}(N_G H)} \mathbf{E}(\dots) \to \big(N_G H \times_H \mathrm{res}^H_{N_G H} Z\big)^-_+ \wedge_{\mathrm{Or}(N_G H)} \mathbf{E}(\dots)$ is an isomorphism of spectra. Next, we define a left $W_G H$-action on the target of $\phi \wedge \mathrm{id}$ by requiring that $\pi(n_0) \in W_G H$ acts as the automorphism which makes the following diagram commute:

$$
\begin{array}{ccc}
\big(\pi^*(W_G H) \times Z\big)^-_+ \wedge_{\mathrm{Or}(N_G H)} \mathbf{E}(\dots) & \xrightarrow[\cong]{\phi \wedge \mathrm{id}} & \big(N_G H \times_H \mathrm{res}^H_{N_G H} Z\big)^-_+ \wedge_{\mathrm{Or}(N_G H)} \mathbf{E}(\dots) \\
{\scriptstyle \eta_{\pi(n_0)} \wedge \mathrm{id}} \downarrow {\scriptstyle \cong} & & \Big\downarrow \\
\big(\pi^*(W_G H) \times Z\big)^-_+ \wedge_{\mathrm{Or}(N_G H)} \mathbf{E}(\dots) & \xrightarrow[\cong]{\phi \wedge \mathrm{id}} & \big(N_G H \times_H \mathrm{res}^H_{N_G H} Z\big)^-_+ \wedge_{\mathrm{Or}(N_G H)} \mathbf{E}(\dots)
\end{array}
$$

Explicitly, $\pi(n_0) \cdot [n, z] \wedge e := [nn_0^{-1}, n_0 z] \wedge e$. We eventually obtain a well-defined map $f_2 := \mathrm{id} \wedge \phi \wedge \mathrm{id}$, which is an isomorphism with inverse given by $x \wedge [n, z] \wedge e \mapsto x \wedge \big(\pi(n), nz\big) \wedge e$.

Finally, in order to get f_3, let $\Psi \colon \big(N_G H \times_H \mathrm{res}^H_{N_G H} Z\big)^-_+ \wedge_{\mathrm{Or}(N_G H)} \mathbf{E}(\dots) \to \big(\mathrm{res}^H_{N_G H} Z\big)^-_+ \wedge_{\mathrm{Or}(H)} \mathrm{res}^{\mathrm{Or}(H)}_{\mathrm{Or}(N_G H)} \mathbf{E}(\dots)$ be the isomorphism of spectra which comes from the homeomorphism $\big(N_G H \times_H \mathrm{res}^H_{N_G H} Z\big)^-_+ \cong \mathrm{ind}^{\mathrm{Or}(N_G H)}_{\mathrm{Or}(H)} \big(\mathrm{res}^{\mathrm{Or}(H)}_{\mathrm{Or}(N_G H)} Z\big)^-_+$ of $\mathrm{Or}(N_G H)$-spaces (see Lemma 1.12) and Lemma 1.11 (2). This means that Ψ is given by

$$[n, z] \wedge e \mapsto z \wedge \mathbf{E}\big(\mathcal{G}^{N_G H}(r_n)\big)(e).$$

We introduce a left $W_G H$-action on the target of Ψ in the same way we did when defining $\mathbf{E}[H, \mathcal{F}]$, see page 35. That is, $\pi(n_0) \cdot z \wedge e := n_0 z \wedge \mathrm{res}\, \mathbf{E}(\rho_{n_0^{-1}})(e)$. It is then immediate that Ψ is compatible with the $W_G H$-actions on its source and target, so $f_3 := \mathrm{id} \wedge \Psi$ is a well-defined isomorphism of spectra. The inverse of f_3 is given by $x \wedge z \wedge e \mapsto x \wedge [1, z] \wedge e$. $\qquad \square$

Theorem 3.16. *Let G be a group which satisfies $(M\mathcal{F}in, \mathcal{T}r)$ or $(M\mathcal{VC}yc, \mathcal{F}in)$. We denote by $\mathcal{M}$ a complete system of representatives of the conjugacy classes of maximal finite subgroups $F \subset G$ or of infinite maximal virtually cyclic subgroups $V \subset G$, respectively. Then for every $n \in \mathbb{Z}$ there are natural isomorphisms*

$$\mathcal{H}^G_n\big(EG \to E_{\mathcal{F}in}(G); \mathbf{E}\big) \xrightarrow{\cong} \bigoplus_{F \in \mathcal{M}} H^{W_G F}_n\big(EW_G F; \mathbf{E}[F, \mathcal{T}r]\big) \qquad \textit{or}$$

$$\mathcal{H}^G_n\big(E_{\mathcal{F}in}(G) \to E_{\mathcal{VC}yc}(G); \mathbf{E}\big) \xrightarrow{\cong} \bigoplus_{V \in \mathcal{M}} H^{W_G V}_n\big(EW_G V; \mathbf{E}[V, \mathcal{F}in]\big) \quad \textit{respectively.}$$

Proof. We carry out the proof only when G satisfies ($M\mathcal{VC}yc$, $\mathcal{F}in$) because the other case works analogously. Using the model for $E_{\mathcal{VC}yc}(G)$ of Corollary 3.6, it follows from the properties of an equivariant homology theory that

$$\mathcal{H}_n^G\big(E_{\mathcal{F}in}(G) \to E_{\mathcal{VC}yc}(G); \mathbf{E}\big) \xrightarrow{\cong} \bigoplus_{V \in \mathcal{M}} \mathcal{H}_n^{N_G V}(f_V : \underline{E}N_G V \to EW_G V; \mathbf{E})$$

are naturally isomorphic. Since, by Lemma 3.5, the diagonal $N_G V$-space $EW_G V \times \underline{E}N_G V$ is a model for $\underline{E}N_G V$, the projection $p : EW_G V \times \underline{E}N_G V \to \underline{E}N_G V$ is an $N_G V$-homotopy equivalence. Hence we get a natural isomorphism

$$\mathcal{H}_n^{N_G V}(f_V \circ p; \mathbf{E}) \xrightarrow{\cong} \mathcal{H}_n^{N_G V}(f_V; \mathbf{E}),$$

cf. [Lüc89, Lemma 4.17]. Now $f_V \circ p$, by the universal property of $EW_G V$, must be $N_G V$-homotopic to the projection $q : EW_G V \times \underline{E}N_G V \to EW_G V$, so that an application of Lemma 3.15 finishes the proof. $\square$

For the following, which has already been stated in [LJP05], let R be an associative ring with unit and recall that the *n-th Whitehead group of RG* is $\mathrm{Wh}_n^R(G) := \mathcal{H}_n^G(EG \to \mathrm{pt}; \mathbf{K}_R)$. If $R = \mathbb{Z}$, then $\mathrm{Wh}_1^{\mathbb{Z}}(G) = \mathrm{Wh}(G)$ is the classical Whitehead group.

Corollary 3.17. *Let G be a torsion-free group satisfying ($M\mathcal{C}yc$, $\mathcal{T}r$) such that $N_G C = C$ holds for every maximal cyclic subgroup $C \subset G$. If $\mathcal{M}$ is a complete system of representatives of the conjugacy classes of maximal cyclic subgroups of G, then for all $n \in \mathbb{N}$ we have*

$$\mathcal{H}_n^G\big(EG \to E_{\mathcal{VC}yc}(G); \mathbf{K}_R\big) \xrightarrow{\cong} \bigoplus_{C \in \mathcal{M}} \mathrm{Wh}_n^R(C).$$

In particular, if G is a torsion-free word-hyperbolic group, then

$$\mathrm{Wh}_n^R(G) \cong \bigoplus_{C \in \mathcal{M}} \mathrm{Wh}_n^R(C).$$

Proof. The first statement follows from Theorem 3.16 since $H_n^{\{1\}}\big(\mathrm{pt}; \mathbf{K}_R[C; \mathcal{T}r]\big) = \pi_n\big(\mathbf{K}_R[C; \mathcal{T}r]\big)$, while the latter is equal to $\mathrm{Wh}_n^R(C)$ by definition, due to the assumptions on G.

The addendum is true since, by recent work of Bartels-Lück-Reich ([BLR]), word-hyperbolic groups are known to satisfy the Farrell-Jones conjecture for algebraic K-theory. It has already been proved in [BR05, Theorem 1.4] that this conjecture holds for fundamental groups of Riemannian manifolds with strictly negative sectional curvature. $\square$

3.3 A Class of Groups

Let G be a countable group all of whose non-virtually cyclic subgroups contain a copy of the free group $\mathbb{Z} * \mathbb{Z}$ on two generators. In [LJP05, Prop. 6] it is shown that G then satisfies $(M\mathcal{VCyc}, \mathcal{Fin})$, while every infinite maximal virtually cyclic subgroup of G is self-normalizing.

We want to enlarge the class of groups considered in [Lüc05, Thm. 8.11] to incorporate groups with these properties.

Theorem 3.18. *Suppose that the countable group G satisfies the following two conditions:*

- *Every infinite cyclic subgroup $C \subset G$ has finite index $\big[C_G(C) : C\big]$ in its centralizer.*

- *Every ascending chain $H_1 \subset H_2 \subset \ldots$ of finite subgroups of G becomes stationary, i.e. there is an $n_0 \in \mathbb{N}$ such that $H_n = H_{n_0}$ for all $n \geq n_0$.*

Then every infinite virtually cyclic subgroup $V \subset G$ is contained in a unique maximal virtually cyclic subgroup $V_{\max} \subset G$. Moreover, $V_{\max}$ is equal to its normalizer $N_G(V_{\max})$, and

$$V_{\max} = \bigcup_{C \subset V} N_G(C),$$

where the union is over all infinite cyclic normal subgroups C of V.

Proof. We fix an infinite virtually cyclic subgroup $V \subset G$ and denote by $\{C_n\}_{n \in \mathbb{N}}$ the collection of its infinite cyclic normal subgroups. Note that since every index $[V : C_n]$ is finite, $[V : C_1 \cap \ldots \cap C_n]$ must also be finite for $n \in \mathbb{N}$. Thus, if we set $Z_n := C_1 \cap \ldots \cap C_n$, then $Z_n \subset C_n$, and $Z_1 \supset Z_2 \supset \ldots$ is a descending chain of infinite cyclic normal subgroups of V.

If $C' \subset C$ are two infinite cyclic subgroups of G, then $N_G(C) \subset N_G(C')$ because if $C = \langle c \rangle$, then $C' = \langle c^k \rangle$ for some $k \in \mathbb{N}$, and for $g \in N_G(C)$ we have $gc^k g^{-1} = (gcg^{-1})^k = c^{\pm k}$, hence $g \in N_G(C')$. It follows in our situation that $N_G(C_n) \subset N_G(Z_n)$, which implies

$$\bigcup_{n=1}^{\infty} N_G(C_n) = \bigcup_{n=1}^{\infty} N_G(Z_n). \tag{3.19}$$

Furthermore, $N_G(Z_1) \subset N_G(Z_2) \subset \ldots$ is an ascending chain, which becomes stationary by the following argument. Namely, we can estimate

$$\big[N_G(Z_n) : N_G(Z_1)\big] \leq \big[N_G(Z_n) : C_G(Z_1)\big]$$
$$= \big[N_G(Z_n) : C_G(Z_n)\big] \cdot \big[C_G(Z_n) : C_G(Z_1)\big],$$

and the first factor on the right is not greater than 2 since there is an injection $N_G(Z_n)/C_G(Z_n) \hookrightarrow \mathrm{aut}(Z_n)$, while the second does not exceed an appropriate constant as we will show below. Summarizing, we see that for a sufficiently large $n_0 \in \mathbb{N}$ the right hand side of (3.19) is equal to $N_G(Z_{n_0}) =: V_{\max}$.

With this definition, it is obvious that $V_{\max}$ is virtually cyclic since the finite index subgroup $C_G(Z_{n_0})$ has got the same property due to the assumption imposed on G. In addition, $V \subset V_{\max}$ since $Z_{n_0} \subset V$ is normal. Now suppose $W \subset G$ is an infinite virtually cyclic subgroup such that $V \subset W$. We claim that $V_{\max} = W_{\max}$. In order to prove this, let $Z_W \subset W$ be infinite cyclic normal such that $W_{\max} = N_G(Z_W)$. Then $Z_W \cap V$ is a finite index subgroup of W and so must again be infinite cyclic normal. Thus, there is a $Z_V \subset Z_W \cap V$ such that $V_{\max} = N_G(Z_V)$, compare the above construction. Since $W_{\max} = N_G(Z_V)$ as well, the claim follows. From this we can deduce immediately that $V_{\max}$ is indeed maximal among virtually cyclic subgroups of G containing V and that it is uniquely determined by this property.

Finally, we will show that $N_G(V_{\max})$ is virtually cyclic, so that it is equal to $V_{\max}$. Let $C \subset V_{\max}$ be infinite cyclic and note that $V_{\max}$ contains only finitely many subgroups of index $[V_{\max} : C]$. This implies that the group D which we define as the intersection of all conjugates of C in $N_G(V_{\max})$ has finite index in $V_{\max}$ and is therefore infinite cyclic as well. Obviously, D is normal in $N_G(V_{\max})$, so that $N_G(V_{\max}) \subset N_G(D)$ holds, the latter being virtually cyclic since $C_G(D)$ is so by assumption.

It remains to prove that $\big\{ \big[C_G(Z_n) : C_G(Z_1)\big] \,\big|\, n \in \mathbb{N}\big\}$ possesses an upper bound in $\mathbb{N}$. Let $Q_n := C_G(Z_n)/Z_n$. Then the Hochschild-Serre spectral sequence which belongs to the group extension $1 \to Z_n \to C_G(Z_n) \to Q_n \to 1$ yields an exact sequence

$$H_2\big(Q_n; H_0(Z_n)\big) \to H_0\big(Q_n; H_1(Z_n)\big) \to H_1\big(C_G(Z_n)\big) \to H_1\big(Q_n; H_0(Z_n)\big) \to 1$$

for every $n \in \mathbb{N}$, cf. [Bro82, Cor. VII.6.4]. Here, the action of Q_n on $H_*(Z_n)$ is induced from conjugation, hence it is trivial as Z_n is central in $C_G(Z_n)$. For this reason, $H_1\big(Q_n; H_0(Z_n)\big)$ is just the abelianization of Q_n and thus finite. Furthermore, $H_0\big(Q_n; H_1(Z_n)\big) \cong H_1(Z_n) \cong Z_n$ while $H_2\big(Q_n; H_0(Z_n)\big)$ is a torsion group, whence it follows that we get an injection $Z_n \hookrightarrow H_1\big(C_G(Z_n)\big)$ with finite cokernel. In particular, if we denote by T_n the torsion subgroup of $H_1\big(C_G(Z_n)\big)$, then $H_1\big(C_G(Z_n)\big)/T_n$ is infinite cyclic. Let $p_n \colon C_G(Z_n) \to C_G(Z_n)_{\mathrm{ab}} \cong H_1\big(C_G(Z_n)\big) \to H_1\big(C_G(Z_n)\big)/T_n$ be the canonical projection, and let $c_n \in C_G(Z_n)$ be such that $p(c_n)$ is a generator.

Consider the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \ker(p_1) & \longrightarrow & C_G(Z_1) & \xrightarrow{\ p_1\ } & H_1\big(C_G(Z_1)\big)/T_1 & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \ker(p_2) & \longrightarrow & C_G(Z_2) & \xrightarrow{\ p_2\ } & H_1\big(C_G(Z_2)\big)/T_2 & \longrightarrow & 1 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \vdots & & \vdots & & \vdots & &
\end{array}
$$

which has exact rows and in which all the vertical arrows are inclusions. Since all the $\ker(p_n)$ are certainly finite, $\bigcup_{n=1}^\infty \ker(p_n)$ is again finite, say of order $a \in \mathbb{N}$, by

the assumption on G. Then in particular $\big[\ker(p_n) : \ker(p_1)\big] \leq a$ for all $n \in \mathbb{N}$, so $\big\{\big[C_G(Z_n) : C_G(Z_1)\big] \,\big|\, n \in \mathbb{N}\big\}$ will be bounded if we can show that the index of $H_1\big(C_G(Z_1)\big)/T_1$ in $H_1\big(C_G(Z_n)\big)/T_n$ is less than a constant which does not depend on $n \in \mathbb{N}$.

In order to construct such a constant, let $r_n \in \mathbb{Z}$ be such that $p_1(c_1)$ is mapped to $p_n(c_n)^{r_n}$ under the inclusions in the above diagram. Then, by exactness, there is a $k_n \in \ker(p_n)$ such that $c_1 = k_n c_n^{r_n}$. Since the order of $\ker(p_n)$ divides a, the group $\mathrm{aut}\big(\ker(p_n)\big)$ contains at most $a!$ elements, so that any $\phi_n \in \mathrm{aut}\big(\ker(p_n)\big)$ satisfies $\phi_n^{a!} = \mathrm{id}$. Setting $d := a \cdot a!$, we thus get for such a ϕ_n the identity

$$\prod_{i=0}^{d-1} \phi_n^i(k) = \prod_{j=1}^{a} \prod_{i=(j-1)\cdot a!}^{j\cdot a!-1} \phi_n^i(k) = \left(\prod_{i=0}^{a!-1} \phi_n^i(k)\right)^a = 1$$

for all $k \in \ker(p_n)$. Specializing to $\phi_n(k) := c_n^{r_n} k c_n^{-r_n}$ yields

$$c_1^d = (k_n c_n^{r_n})^d = \left(\prod_{i=0}^{d-1} \phi_n^i(k_n)\right) \cdot c_n^{r_n d} = c_n^{r_n d}.$$

If z_1 is a generator of Z_1, then there exists an $s \in \mathbb{Z}$ such that $p_1(z_1) = p_1(c_1)^s$. This means $z_1^{-1} c_1^s \in \ker(p_1)$, hence $z_1^d = c_1^{sd}$. Altogether this implies that if dZ_1 denotes the cyclic group generated by z_1^d, we have $dZ_1 = \langle c_n^{r_n ds}\rangle$ and thus $c_n \in C_G(dZ_1)$.

We can finally define $b := \big[C_G(dZ_1) : dZ_1\big]$, which is finite by assumption and constitutes the required constant. This is due to the fact that $c_n^b \in dZ_1$, so there is a $t_n \in \mathbb{Z}$ such that $c_n^b = c_n^{r_n ds t_n}$. Hence r_n divides b, and the index of $H_1\big(C_G(Z_1)\big)/T_1$ in $H_1\big(C_G(Z_n)\big)/T_n$ equals $|r_n|$ by construction. $\qquad\square$

Lemma 3.20. *Let G be a group with the property that every non-virtually cyclic subgroup of G contains a copy of $\mathbb{Z} * \mathbb{Z}$. Then G satisfies the conditions of Theorem 3.18.*

Proof. It is obvious that any ascending chain $H_1 \subset H_2 \subset \ldots$ of finite subgroups of G must become stationary since, otherwise, $\bigcup_n H_n$ would be an infinite torsion subgroup of G, contradicting the assumptions on G. To prove the first condition of Theorem 3.18, let $C \subset G$ be infinite cyclic. Its centralizer $C_G(C)$ is virtually cyclic, provided it does not contain $\mathbb{Z} * \mathbb{Z}$. Assuming it does, then $\mathbb{Z} * \mathbb{Z} \cap C = \{1\}$ as $\mathbb{Z} * \mathbb{Z}$ does not commute with any of its infinite cyclic subgroups. Hence one of the generators of $\mathbb{Z} * \mathbb{Z}$ together with a generator of C generate a copy of $\mathbb{Z} \oplus \mathbb{Z}$ inside G, which contradicts the assumptions imposed on G. $\qquad\square$

Remark 3.21. It follows from the Kurosh subgroup theorem (see e.g. [Ser80, Thm. I.5.14]) that the class of groups satisfying the conditions of Theorem 3.18 is closed under arbitrary free products, whereas this is not the case for the class of groups considered in [Lüc05, Thm. 8.11].

Example 3.22 (Word-hyperbolic groups). Any word-hyperbolic group G satisfies the conditions of Theorem 3.18. The first is satisfied by [BH99, Cor. 3.10(2)]. Furthermore, G contains only finitely many conjugacy classes of finite subgroups. This follows from Lemma 2.19 in conjunction with the fact that the second barycentric subdivision of the Rips complex $P_r(G)$ for a sufficiently large r is a finite model for $\underline{E}G$, see Theorem 2.9.

In this situation, Corollary 3.6 implies that a model for $E_{\mathcal{V}Cyc}(G)$ can be obtained from a model for $\underline{E}G$ via the pushout

$$
\begin{array}{ccc}
\coprod_{V \in \mathcal{M}} G \times_V \underline{E}V & \xrightarrow{\; i \;} & \underline{E}G \\
{\scriptstyle \amalg_V \mathrm{pr}_V} \downarrow & & \downarrow \\
\coprod_{V \in \mathcal{M}} G/V & \longrightarrow & E_{\mathcal{V}Cyc}(G)
\end{array}
\tag{3.23}
$$

where pr_V is induced from the projection $\underline{E}V \to \mathrm{pt}$. Recall there are one-dimensional models for $\underline{E}V$ (see page 19). Hence the existence of an n-dimensional model for $\underline{E}G$ implies the existence of a model for $E_{\mathcal{V}Cyc}(G)$ of dimension $\max\{n, 2\}$.

However, if G is not virtually cyclic, then it contains infinitely many conjugacy classes of maximal infinite virtually cyclic subgroups, see [Gro87, Cor. 5.1.B], and thus no model for $E_{\mathcal{V}Cyc}(G)$ can be of finite type due to Lemma 2.19.

Example 3.24 (Free groups). As a special case of Example 3.22, consider a free group G. We remind the reader that the Cayley graph $\Gamma(G, S)$ (see section 2.1.2) for the canonical choice of a generating set S of G is a tree on which G acts freely, so it is a one-dimensional model for $EG = \underline{E}G$. We have already seen in Example 3.22 that then a two-dimensional model for $E_{\mathcal{V}Cyc}(G)$ exists.

This is, at the same time, a model of minimal dimension. To show this, we may assume that G be finitely generated because for any subgroup $H \subset G$ a model for $E_{\mathcal{V}Cyc}(G)$ yields a model for $E_{\mathcal{V}Cyc}(H)$ by restriction. Now we divide out the G-action in (3.23) and consider the Mayer-Vietoris sequence

$$
\ldots \to 0 \to H_2\big(G \backslash E_{\mathcal{V}Cyc}(G)\big) \to \bigoplus_{C \in \mathcal{M}} H_1(C \backslash \underline{E}C) \to H_1(G \backslash \underline{E}G) \to \ldots
$$

which belongs to the resulting pushout. Note that all the spaces $C \backslash \underline{E}C$ are homotopy equivalent to S^1, while $G \backslash \underline{E}G$ is homotopy equivalent to $\bigvee_{i=1}^{r} S^1$, where $r < \infty$ denotes the rank of G. Thus, $H_1(G \backslash \underline{E}G) \cong \bigoplus_{i=1}^{r} H_1(S^1)$, and since the set $\mathcal{M}$ of representatives of the conjugacy classes of infinite cyclic subgroups $C \subset G$ is infinite, we conclude that $H_2\big(G \backslash E_{\mathcal{V}Cyc}(G)\big)$ is free abelian of infinite rank. The claim follows.

4 Amenable Actions

The notion of an amenable group (which is a group carrying an invariant mean) can be generalized to the notion of an amenable action of a group. There are two different versions in the literature, one introduced by Greenleaf (see [Gre69]), the other by Zimmer (see e.g. [Zim77]). For a comparison of the two as well as for a thorough survey on the matter, we refer to [ADR00].

In this chapter, we concentrate on the latter definition and explain in how far there is a relation to the Baum-Connes and Farrell-Jones isomorphism conjectures. Moreover, we prove some stability properties of amenable actions and show that any G-CW-complex with amenable isotropy groups is amenable as a G-space.

We stress that, in this chapter, groups are supposed to be countable and discrete, and spaces are Hausdorff.

4.1 Definition of Amenable Actions

Let us recall some notions from measure theory. Let X be a space equipped with its Borel σ-algebra $\mathfrak{B}$. A *Radon measure* on X is a measure $\mu\colon \mathfrak{B} \to [0,\infty]$ that is locally finite, i.e. every point of X has an open neighbourhood of finite measure, and inner regular, i.e. for every $B \in \mathfrak{B}$ we have

$$\mu(B) = \sup\{\mu(K) \mid K \subset B,\ K \subset X \text{ compact}\}.$$

Notation 4.1. We denote by $\operatorname{prob}(X)$ the set of all Radon probability measures on X. If, moreover, X carries an action of a group G, we can define a G-action on $\operatorname{prob}(X)$ by setting

$$(g \cdot \mu)(A) := \mu(g^{-1}A)$$

for $A \in \mathfrak{B}$.

A *signed measure* on X is a σ-additive map $\mu\colon \mathfrak{B} \to [-\infty,\infty]$ such that $\mu(\emptyset) = 0$ and not both ∞ and $-\infty$ are contained in the image of μ. One gets obvious examples of signed measures by taking the difference $\mu - \lambda$ of two ordinary measures μ and λ, one of which has to be finite for this to make sense. As it turns out, these are at the same time the only examples, which is a consequence of the Hahn decomposition theorem (cf. [Coh80, Thm. 4.14 and Cor. 4.15]): any signed measure μ on X can be expressed as $\mu = \mu^+ - \mu^-$, where μ^+ and μ^- are ordinary measures on X, at least one of them being finite. Moreover, μ^+ and μ^- are uniquely determined by μ. The *variation* of μ is defined to be the measure $|\mu| := \mu^+ + \mu^-$.

Now consider the space $M(X)$ of all finite signed measures on X. We can impose a norm on it by setting $\|\mu\|_1 := |\mu|(X)$, which turns $M(X)$ into a Banach space. If X is locally compact, the Riesz representation theorem states that the Banach subspace $M_R(X)$ of all $\mu \in M(X)$ such that μ^+ and μ^- are Radon measures is the dual of $C_0(X)$, cf. [Coh80, Thm. 7.2.8]. To be more precise, there is an isometric isomorphism

$$M_R(X) \xrightarrow{\cong} C_0^*(X), \quad \mu \mapsto \left(f \mapsto \int_X f \, d\mu \right).$$

If X is even countable and discrete, any $\mu \in \mathrm{prob}(X) \subset M_R(X)$ can and will be regarded as a map $X \to [0,1]$ such that $\sum_{x \in X} \mu(x) = 1$. In this case it is not hard to show that the topologies of pointwise, weak-* and norm convergence on $\mathrm{prob}(X)$ all coincide.

Definition 4.2 (Amenable action). Let X be a G-space. A sequence $(\mu_n)_{n \in \mathbb{N}}$ of continuous maps $X \to \mathrm{prob}(G)$ is called an *approximate invariant continuous mean* for the given action (*a.i.c.m.* in short) if

$$\forall g \in G \quad \forall K \subset X \text{ compact:} \quad \sup_{x \in X} \left\| g \cdot \mu_n^x - \mu_n^{gx} \right\|_1 \xrightarrow{n \to \infty} 0,$$

where we write μ_n^x for the probability measure $\mu_n(x)$ on G.

An action $G \curvearrowright X$ is called *amenable* if it admits an approximate invariant continuous mean.

This is the definition of [ADR00, Ex. 2.2.14(2)]. However, there only amenable actions on locally compact, second countable spaces (e.g. manifolds) are considered, and not all results carry over directly to amenable actions on arbitrary spaces. Consider instead the following property a space X may or may not have:

$(*)$ There exist compact subsets $K_n \subset X$ for $n \in \mathbb{N}$ such that $X = \bigcup_{n \in \mathbb{N}} K_n$ and every compact $K \subset X$ is contained in some K_n.

For instance, any locally compact, second countable space has this property, as follows from [Sch69, Satz I.7.8.2]. Another example are CW-complexes with countably many cells.

Lemma 4.3 (Characterization of amenability for spaces satisfying $(*)$). *Consider the following two statements about a G-space X:*

(1) $G \curvearrowright X$ is amenable.

(2) For every finite subset $F \subset G$, compact subset $K \subset X$ and $\varepsilon > 0$, there is a map $\mu \colon X \to \mathrm{prob}(G)$ such that

$$\sup_{x \in K} \left\| g \cdot \mu^x - \mu^{gx} \right\|_1 \leq \varepsilon$$

holds for all $g \in F$.

Then (1) implies (2). If X satisfies (∗), the converse is also true.

In particular, the action $G \curvearrowright \mathrm{pt}$ is amenable if and only if G is an amenable group.

Proof. This is straightforward. For the addendum, note that if $X = \mathrm{pt}$, then (2) is nothing but Reiter's condition (P_1), which characterizes the amenability of a group, see [Pat88, Thm. 4.4]. □

4.2 Relations to Assembly Maps

4.2.1 Baum-Connes Assembly Map

In this section, we want to explain how one can prove that for a word-hyperbolic group G the Baum-Connes assembly map (1.21) is injective, namely by constructing a compact amenable G-space.

First of all, recall the definition of word-hyperbolicity of groups in Definition 2.7. For doing concrete calculations, it is, however, often more convenient to consider an equivalent notion. So let G be a finitely generated group and d the word metric on G with respect to some finite symmetric subset that generates G. Then we denote by

$$(x \cdot y) := \frac{1}{2}\big(d(x,1) + d(y,1) - d(x,y)\big)$$

the *Gromov product of* $x, y \in G$. If, for instance, the Cayley graph $\Gamma(G, S)$ is a tree, then $(x \cdot y)$ equals the distance from $1 \in G \subset \Gamma(G, S)$ to the geodesic joining x and y in $\Gamma(G, S)$. It follows from [BH99, Prop. 1.22]:

Lemma 4.4. *A finitely generated group G is word-hyperbolic if and only if there is a $\delta \geq 0$ such that*

$$(x \cdot y) \geq \min\big\{(x \cdot z), (y \cdot z)\big\} - \delta$$

holds for all $x, y, z \in G$.

In the following we fix a word-hyperbolic group G and a $\delta \geq 0$ as in Lemma 4.4. We say that a sequence (x_i) in G *converges at infinity* if $(x_i \cdot x_j) \to \infty$ as $i, j \to \infty$, and that two such sequences (x_i) and (y_i) are *equivalent* if $(x_i \cdot y_j) \to \infty$ as $i, j \to \infty$. The *Gromov boundary* ∂G of G is then defined to be the set of equivalence classes of sequences in G that converge at infinity. If an $a \in \partial G$ is represented by (x_i), then we write $a = \lim x_i$.

One can extend the Gromov product to $a, b \in G \cup \partial G$ by

$$(a \cdot b) := \sup\Big\{\liminf_{i,j\to\infty}(x_i \cdot y_j) \mid \lim x_i = a, \lim y_i = b\Big\},$$

where it is understood that an element of G is represented by the constant sequence at this element.

Now, keeping in mind the invariance of the word metric d under left translation, it is easy to see that for $x, y, g \in G$ we have the inequality $\left|(x \cdot y) - (gx \cdot gy)\right| \leq d(g, 1)$. In particular, it makes sense to define a G-action on ∂G by setting

$$g \cdot a := \lim gx_i$$

for any sequence (x_i) in G such that $\lim x_i = a$.

Let $P_r(G)$ be the Rips complex of G. Consider the space $\overline{P_r(G)} := P_r(G) \cup \partial G$ equipped with the topology in which the neighbourhoods of an $a \in \partial G$ are the sets $U_R(a)$ for $R > 0$, where $U_R(a)$ consists of all elements $x \in G \cup \partial G \subset P_r(G) \cup \partial G$ such that $(a \cdot x) \geq R$, together with the simplices of $P_r(G)$ that they span.

Proposition 4.5. *With the topology defined above, $\overline{P_r(G)}$ is a compact metrizable space that contains $P_r(G)$ as a dense open subset.*

Proof. This is intrinsic in [Gro87] (see also [BH99, III.H.3.18(4)]). $\qquad\square$

Let $a \in \partial G$, $k \in \mathbb{N}$ and denote by $I(a, k)$ the set of all geodesics $\gamma \colon [0, \infty[\to P_r(G)$ in the 1-skeleton of $P_r(G)$ such that $d\big(\gamma(0), 1\big) < k$ and $\lim \gamma(i) = a$. Moreover, for $l > 0$ let $\chi(a, k, l)$ be the characteristic function on $\bigcup_{\gamma \in I(a,k)} \gamma\big([l, 2l]\big)$. Finally, we define for $n \in \mathbb{N}$ Borel maps $\mu_n \colon \partial G \to \mathrm{prob}(G)$ by

$$\mu_n^a(g) := \frac{1}{\sqrt{n}} \sum_{k < \sqrt{n}} \chi(a, k, n)(g).$$

Then it is carried out in [ADR00, App. B] that the collection of these μ_n satisfies $\left\|\mu_n^a\right\|_1 > 0$ for $a \in \partial G$ and

$$\forall g \in G : \quad \sup_{a \in \partial G} \frac{\left\|g \cdot \mu_n^a - \mu_n^{ga}\right\|_1}{\left\|\mu_n^a\right\|_1} \xrightarrow{n \to \infty} 0,$$

which implies by [ADR00, Cor. 3.3.8] that $G \curvearrowright \partial G$ is an amenable action. This is relevant because of the following result taken from [Hig00, Thm. 1.1]:

Theorem 4.6. *If the group G acts amenably on some compact space, then the Baum-Connes assembly map* (1.21) *for G is split injective.*

More recently, it has been shown that the Baum-Connes assembly map is actually an isomorphism for word-hyperbolic groups, see [MY02].

4.2.2 Assembly Maps in algebraic K- and L-Theory

Given a group G, the Baum-Connes assembly map can also be shown to be injective provided that $E_{\mathcal{F}in}(G)$ is a finite G-CW-complex and has a suitable metrizable compactification $\overline{E_{\mathcal{F}in}(G)}$ to which the G-action extends, cf. [Hig00, Thm. 1.2]. Here "compactification" means that $\overline{E_{\mathcal{F}in}(G)}$ is compact and contains $E_{\mathcal{F}in}(G)$ as a dense open subset.

A similar statement can be made in the context of algebraic K- and L-theory:

Theorem 4.7. *Let G be a group having a finite model for $E_{\mathcal{F}in}(G)$ which admits a metrizable compactification $\overline{E_{\mathcal{F}in}(G)}$ to which the G-action extends. Assume that*

- $\overline{E_{\mathcal{F}in}(G)}^H$ *is contractible and $E_{\mathcal{F}in}(G)^H \subset \overline{E_{\mathcal{F}in}(G)}^H$ is dense for any finite subgroup $H \subset G$, and*

- *every compact $K \subset E_{\mathcal{F}in}(G)$ becomes small at infinity, i.e. for every neighbourhood $U \subset \overline{E_{\mathcal{F}in}(G)}$ of $a \in \overline{E_{\mathcal{F}in}(G)} \setminus E_{\mathcal{F}in}(G)$ there exists a neighbourhood $V \subset \overline{E_{\mathcal{F}in}(G)}$ of a such that $gK \cap V \neq \emptyset$ implies $gK \subset U$ for $g \in G$.*

Then, for any associative ring R with unit, the projection $E_{\mathcal{F}in}(G) \to \mathrm{pt}$ induces an assembly map

$$\mathcal{H}_n^G\big(E_{\mathcal{F}in}(G); \mathbf{K}_R\big) \to \mathcal{H}_n^G(\mathrm{pt}; \mathbf{K}_R) = K_n(RG)$$

in algebraic K-theory which is split injective. Moreover, the according assembly map in algebraic L-theory is split injective provided that for every finite subgroup $H \subset G$ one has $K_{-i}(RH) = 0$ for sufficiently large i.

Proof. This is proved in [Ros04] and [Ros06], generalizing a theorem of Carlsson-Pederson in [CP95]. $\square$

The purpose of [RS05] is to show that if G is word-hyperbolic and one takes the Rips complex $P_r(G)$ as a model for $E_{\mathcal{F}in}(G)$ (see Theorem 2.9), together with its compactification considered in Proposition 4.5, then the assumptions of Theorem 4.7 will be satisfied. We mention also that, lately, the Farrell-Jones assembly map 1.22 for algebraic K-theory has been shown to be an isomorphism for word-hyperbolic groups, see [BLR].

We finally refer to [BR06], where the authors generalize results of [Ros04] and are in that way able to prove the claim of Theorem 4.7 for discrete subgroups of virtually connected Lie groups, too.

4.3 Properties of Amenable Actions

Some of the stability properties of amenable actions are collected in this section. The more or less straightforward proofs of these results are included for the sake of completeness.

Proposition 4.8. *For G-spaces X and Y, the following holds:*

(1) *If $G \curvearrowright X$ is amenable and $A \subset X$ is G-invariant, then $G \curvearrowright A$ is also amenable.*

(2) *If $f \colon X \to Y$ is a G-equivariant map and $G \curvearrowright Y$ is amenable, then $G \curvearrowright X$ is also amenable.*

   In particular, if G is an amenable group, then $G \curvearrowright X$ is always amenable.

(3) *If $G \curvearrowright X$ is amenable, then so is $G \curvearrowright X \times Y$, where G acts diagonally.*

Proof. For the addendum in (2), one takes $Y = \text{pt}$ and uses Lemma 4.3. Everything else is completely obvious. $\qquad\square$

Theorem 4.9 (Amenability under induction and restriction). *Let $H \subset G$ be an inclusion of groups. Then the following holds:*

(1) *(Induction)*
  The H-space X is amenable if and only if the G-space $G \times_H X$ is amenable.

(2) *(Restriction)*
  If the G-space Y is amenable, then so is the H-space $\text{res}_G^H Y$.

Proof. In order to prove (1), we first choose a section $s\colon H\backslash G \to G$ of the projection. Assume now that $G \curvearrowright G \times_H X$ is amenable, and that an a.i.c.m. is given by $\big(\mu_n\colon G \times_H X \to \text{prob}(G)\big)_{n\in\mathbb{N}}$. With the help of the H-equivariant embedding $i\colon X \to G \times_H X$, $x \mapsto [1, x]$, we define for $n \in \mathbb{N}$ maps

$$\nu_n\colon X \to \text{prob}(H), \quad x \mapsto \Big(h \mapsto \sum_{\alpha \in H\backslash G} \mu_n^{i(x)}\big(hs(\alpha)\big)\Big)$$

which are indeed continuous due to the estimate $\big\|\nu_n^x - \nu_n^y\big\|_1 \le \big\|\mu_n^{i(x)} - \mu_n^{i(y)}\big\|_1$. Since for $h \in H$ and a compact $K \subset X$ we have

$$\sup_{x\in K}\big\|h \cdot \nu_n^x - \nu_n^{hx}\big\|_1 = \sup_{y\in i(K)} \sum_{h'\in H}\bigg|\sum_{\alpha\in H\backslash G}\Big(\mu_n^y\big(h^{-1}h's(\alpha)\big) - \mu_n^{hy}\big(h's(\alpha)\big)\Big)\bigg|$$

$$\le \sup_{y\in i(K)}\big\|h \cdot \mu_n^y - \mu_n^{hy}\big\|_1$$

and the latter tends to 0 as $n \to \infty$, the H-space X is amenable.

To show the converse, let $\big(\mu_n\colon X \to \text{prob}(H)\big)_{n\in\mathbb{N}}$ be an a.i.c.m. for $H \curvearrowright X$. Consider the space $G/H \times X$ with the G-action $g \cdot (\alpha, x) := \big(g\alpha, s(g\alpha)^{-1}gs(\alpha)x\big)$. We define for each $n \in \mathbb{N}$ maps $\nu_n\colon G/H \times X \to \text{prob}(G)$ by

$$(\alpha, x) \mapsto \nu_n^{(\alpha,x)} := \begin{cases} s(\alpha) \cdot \mu_n^x & \text{on } s(\alpha)H \subset G, \\ 0 & \text{on } G \setminus s(\alpha)H. \end{cases}$$

Now let $g \in G$ and $L \subset G/H \times X$ be compact. We can choose a finite $F \subset G/H$ and a compact $K \subset X$ such that $L \subset F \times X$. For any $\varepsilon > 0$, there is an $N \in \mathbb{N}$ with the property

$$\forall n \ge N \quad \forall \alpha \in F\colon \quad \sup_{x\in K}\big\|s(g\alpha)^{-1}gs(\alpha)\mu_n^x - \mu_n^{s(g\alpha)^{-1}gs(\alpha)x}\big\|_1 \le \varepsilon,$$

49

which implies that for $n \geq N$ the following holds:

$$\sup_{y \in L} \left\| g \cdot \nu_n^y - \nu_n^{gy} \right\|_1$$

$$\leq \sup_{\alpha \in F} \sup_{x \in K} \left\| g \cdot \nu_n^{(\alpha, x)} - \nu_n^{g(\alpha, x)} \right\|_1$$

$$= \sup_{\alpha \in F} \sup_{x \in K} \sum_{\substack{g' \in gs(\alpha)H \\ = s(g\alpha)H}} \left| s(\alpha) \cdot \mu_n^x(g^{-1}g') - s(g\alpha) \cdot \mu_n^{s(g\alpha)^{-1}gs(\alpha)x}(g') \right|$$

$$= \sup_{\alpha \in F} \sup_{x \in K} \left\| s(g\alpha)^{-1}gs(\alpha) \cdot \mu_n^x - \mu_n^{s(g\alpha)^{-1}gs(\alpha)x} \right\|_1$$

$$\leq \varepsilon.$$

The assertion follows as by Lemma 1.7 (1) the G-space $G/H \times X$ defined above is G-homeomorphic to $G \times_H X$.

It remains to show (2). In this case, let $G/H \times Y$ be the G-space with the diagonal action, which according to Lemma 1.7 (2) is G-homoeomorphic to $G \times_H \mathrm{res}_G^H Y$. It is amenable by Proposition 4.8 (3) since the same holds for Y. Hence $\mathrm{res}_G^H Y$ is an amenable H-space as follows from (1). □

Setting $X = \mathrm{pt}$ in Theorem 4.9 (1) yields:

Corollary 4.10 (Amenable G-sets). *Let $H \subset G$ be an inclusion of groups. Then $G \curvearrowright G/H$ is amenable if and only if H is an amenable group.*

Recall that the class of amenable groups is closed with respect to forming directed unions and group extensions. The following two propositions show that similar statements hold when dealing with amenable actions.

Proposition 4.11 (Amenability and colimits). *Let X be a G-space satisfying $(*)$. Assume that $G = \bigcup_{i \in I} G_i$ is a directed union of subgroups such that all $G_i \curvearrowright X$ are amenable. Then $G \curvearrowright X$ is also amenable.*

Proof. We prove the proposition by verifying (2) of Lemma 4.3. So let a finite subset $F \subset G$, a compact $K \subset X$ and an $\varepsilon > 0$ be given. We can choose an $i \in I$ such that $F \subset G_i$, and a map $\nu \colon X \to \mathrm{prob}(G_i)$ such that

$$\forall g \in F \colon \quad \sup_{x \in K} \left\| g \cdot \nu^x - \nu^{gx} \right\|_1 \leq \varepsilon.$$

Then $\mu \colon X \to \mathrm{prob}(G)$ defined by

$$x \mapsto \mu^x := \begin{cases} \nu^x & \text{on } G_i, \\ 0 & \text{on } G \setminus G_i \end{cases}$$

will certainly have the desired property. □

Proposition 4.12 (Amenability and group extensions). *Let X be a G-space. We denote by $A := \{g \in G \mid gx = x \text{ for all } x \in X\}$ the kernel of the action and set $Q := G/A$. Consider the following statements:*

(1) $G \curvearrowright X$ is amenable.

(2) A is an amenable group and $Q \curvearrowright X$ is amenable.

Then (1) implies (2). If X satisfies $(*)$, the converse is also true.

Proof. Let $p \colon G \twoheadrightarrow Q$ be the projection and $s \colon Q \to G$ a set-theoretic map such that $p \circ s = \mathrm{id}_Q$. We will first show $(1) \Rightarrow (2)$. If $G \curvearrowright X$ is amenable, then so is $A \curvearrowright \{x\}$ for any $x \in X$ according to Theorem 4.9 (2) and Proposition 4.8 (1). Thus A is amenable by Lemma 4.3.

As for the second assertion, we choose an a.i.c.m. $\big(\mu_n \colon X \to \mathrm{prob}(G)\big)_{n \in \mathbb{N}}$ for $G \curvearrowright X$ and define for $n \in \mathbb{N}$ maps

$$\nu_n \colon X \to \mathrm{prob}(Q), \quad x \mapsto \Big(\beta \mapsto \sum_{g \in p^{-1}(\beta)} \mu_n^x(g)\Big).$$

Then, if $K \subset X$ is compact and $\beta \in Q$, we choose a $g_0 \in G$ such that $p(g_0) = \beta$, and it follows that

$$\sup_{x \in K} \big\|\beta \cdot \nu_n^x - \nu_n^{\beta x}\big\|_1 = \sup_{x \in K} \sum_{\beta' \in Q} \Big| \sum_{g \in p^{-1}(\beta')} \mu_n^x(g_0^{-1}g) - \mu_n^{g_0 x}(g)\Big|$$

$$\leq \sup_{x \in K} \big\|g_0 \cdot \mu_n^x - \mu_n^{g_0 x}\big\|_1,$$

the latter tending to 0 as $n \to \infty$. Hence $Q \curvearrowright X$ is amenable.

In order to verify $(2) \Rightarrow (1)$ when X satisfies $(*)$, we will check (2) of Lemma 4.3. So let a finite subset $F \subset G$, a compact $K \subset X$ and an $\varepsilon > 0$ be given. Since $Q \curvearrowright X$ is amenable, there is a $\mu_Q \colon X \to \mathrm{prob}(Q)$ such that

$$\forall g \in F \colon \quad \sup_{x \in K} \big\|g \cdot \mu_Q^x - \mu_Q^{gx}\big\|_1 \leq \frac{\varepsilon}{2}. \tag{4.13}$$

We now want to construct a finite subset $L \subset Q$ with the property that

$$\forall g \in F \quad \forall x \in K \colon \quad \sum_{\beta \in L} \mu_Q^x(g^{-1}\beta) > 1 - \frac{\varepsilon}{8}. \tag{4.14}$$

To do so, we first fix $x \in K$. Then there is a finite $L_x \subset Q$ such that $\sum_{\beta \in L_x} \mu_Q^x(\beta) > 1 - \varepsilon/8$. Because of the continuity of μ_Q, the set

$$U_x := \Big\{y \in X \,\Big|\, \big\|\mu_Q^x - \mu_Q^y\big\|_1 < \frac{\sum_{\beta \in L_x} \mu_Q^x(\beta) - (1 - \frac{\varepsilon}{8})}{|L_x|}\Big\}$$

is an open neighbourhood of x, and

$$\sum_{\beta \in L_x} \mu_Q^y(\beta) > \sum_{\beta \in L_x} \left(\mu_Q^x(\beta) - \frac{\sum_{\beta' \in L_x} \mu_Q^x(\beta') - (1 - \frac{\varepsilon}{8})}{|L_x|} \right) = 1 - \frac{\varepsilon}{8}$$

holds for all $y \in U_x$. As K is compact, that way we can choose $x_1, \ldots, x_n \in K$ such that $K \subset \bigcup_{i=1}^n U_{x_i}$. Then $L' \coloneqq \bigcup_{i=1}^n L_{x_i} \subset Q$ is finite and $\sum_{\beta \in L'} \mu_Q^x(\beta) > 1 - \varepsilon/8$ for $x \in K$. Now obviously $L \coloneqq \bigcup_{g \in F} gL'$ satisfies (4.14).

Furthermore, we set $E \coloneqq \left\{ s(\beta)^{-1} g s(g^{-1}\beta) \mid \beta \in L, \, g \in F \right\}$, which is a finite subset of A, and pick $\mu_A \in \mathrm{prob}(A)$ such that

$$\forall a \in E: \quad \| a \cdot \mu_A - \mu_A \|_1 \le \frac{\varepsilon}{4}. \tag{4.15}$$

We can at last define $\mu_G \colon X \to \mathrm{prob}(G)$ by

$$\mu_G^x(g) \coloneqq \mu_A\big(s(gA)^{-1}g\big)\, \mu_Q^x(gA)$$

and compute for $g \in F$ and $x \in K$ that

$$\begin{aligned}
\big\| g \cdot \mu_G^x - \mu_G^{gx} \big\|_1 &\le \sum_{g' \in G} \big| \mu_A\big(s(g^{-1}g'A)^{-1}g^{-1}g'\big) - \mu_A\big(s(g'A)^{-1}g'\big) \big| \cdot \mu_Q^x(g^{-1}g'A) \\
&\quad + \sum_{g' \in G} \mu_A\big(s(g'A)^{-1}g'\big) \cdot \big| \mu_Q^x(g^{-1}g'A) - \mu_Q^{gx}(g'A) \big| \\
&= \sum_{\beta \in Q} \sum_{a \in A} \big| \mu_A\big(s(g^{-1}\beta)^{-1}g^{-1}s(\beta)a\big) - \mu_A(a) \big| \cdot \mu_Q^x(g^{-1}\beta) \\
&\quad + \sum_{\beta \in Q} \sum_{a \in A} \mu_A(a) \cdot \big| \mu_Q^x(g^{-1}\beta) - \mu_Q^{gx}(\beta) \big| \\
&= \sum_{\beta \in L} \big\| s(\beta)^{-1}g s(g^{-1}\beta) \cdot \mu_A - \mu_A \big\|_1 \cdot \mu_Q^x(g^{-1}\beta) \\
&\quad + \sum_{\beta \in Q \setminus L} \big\| s(\beta)^{-1}g s(g^{-1}\beta) \cdot \mu_A - \mu_A \big\|_1 \cdot \mu_Q^x(g^{-1}\beta) \\
&\quad + \big\| g \cdot \mu_Q^x - \mu_Q^{gx} \big\|_1.
\end{aligned}$$

Combining (4.13), (4.14) and (4.15) now yields the desired inequality

$$\big\| g \cdot \mu_G^x - \mu_G^{gx} \big\|_1 \le \frac{\varepsilon}{4} \cdot \sum_{\beta \in L} \mu_Q^x(g^{-1}\beta) + 2 \cdot \frac{\varepsilon}{8} + \frac{\varepsilon}{2} \le \varepsilon$$

for $g \in F$ and $x \in K$. $\hfill\square$

The assumptions of the following theorem are, for instance, satisfied when X is locally compact and second countable, see [Bre93, Prop. III.7.2 and Thm. I.12.12]. Under this hypothesis, an analogous result is proved in [ADR00, Cor. 2.1.17]. Another situation in which the assumptions are satisfied is when X is a free G-CW-complex. The result is then a special case of Theorem 4.21.

Theorem 4.16 (Amenability of free proper actions). *Let X be a G-space such that $G\backslash X$ is paracompact. If the action on X is free and proper, then it is also amenable.*

Proof. With G acting freely and properly on the Hausdorff space X, it is easy to see that any $x \in X$ possesses an open neighbourhood U such that $gU \cap U = \emptyset$ whenever $g \neq 1$. So by restricting the projection $p \colon X \to G\backslash X$, we get homeomorphisms $U \to p(U)$ and in this way an open covering $\{V_i\}_{i \in I}$ of $G\backslash X$ together with continuous sections $s_i \colon V_i \to X$ of $p|_{p^{-1}(V_i)}$. By refining $\{V_i\}_{i \in I}$ if necessary, we may assume that this covering is locally finite, so that we can choose a subordinate partition of unity $\{h_i \colon G\backslash X \to [0,1]\}_{i \in I}$.

Now consider, for $x \in X$ and $g \in G$, the subset

$$I(x,g) := \big\{ i \in I \mid p(x) \in V_i, \ \ gs_i\big(p(x)\big) = x \big\}$$

of I. We have $I(gx, g') = I(x, g^{-1}g')$, and the finite set $I(x) := \{i \in I \mid p(x) \in V_i\}$ is the disjoint union

$$I(x) = \coprod_{g \in G} I(x,g).$$

Therefore, setting

$$\mu^x(g) := \sum_{i \in I(x,g)} h_i\big(p(x)\big)$$

for $x \in X$ and $g \in G$ gives rise to a map $\mu \colon X \to \mathrm{prob}(G)$ such that $\mu^{gx} = g \cdot \mu^x$. If the continuity of μ can be shown, then the assertion of the theorem will follow.

In order to carry this out, let $x \in X$ and an $\varepsilon > 0$ be given. We can choose an open neighbourhood V of $p(x)$ which meets only finitely many V_i non-trivially. Let $J := \big\{i \in I \mid V \cap V_i \neq \emptyset\big\}$. As for all $i \in J \setminus I(x)$ we have $h_i\big(p(x)\big) = 0$, there exists an open neighbourhood W_1 of x with the property

$$\forall y \in W_1 \colon \qquad p(y) \in V \quad \text{and} \quad \sum_{i \in J \setminus I(x)} h_i\big(p(y)\big) \leq \frac{\varepsilon}{2}.$$

Furthermore, there is an open neighbourhood W_2 of x such that

$$\forall y \in W_2 \quad \forall i \in I(x) \colon \quad \big| h_i\big(p(x)\big) - h_i\big(p(y)\big) \big| \leq \frac{\varepsilon}{2 \cdot |I(x)|}.$$

Finally, it follows from the construction of the $s_i \colon V_i \to X$ that for every $i \in I(x)$ there is exactly one $g_i \in G$ such that $g_i s_i\big(p(x)\big) = x$. Then $W_3 := \bigcap_{i \in I(x)} g_i s_i(V_i)$ is an open neighbourhood of x and

$$\forall y \in W_3 \quad \forall g \in G \colon \quad I(x,g) \subset I(y,g).$$

All of these estimates can eventually be assembled so that for all $y \in \bigcap_{k=1}^{3} W_k$ we get

$$
\begin{aligned}
\left\| \mu^x - \mu^y \right\|_1 &= \sum_{g \in G} \Big| \sum_{i \in I(x,g)} h_i\big(p(x)\big) - \sum_{i \in I(y,g)} h_i\big(p(y)\big) \Big| \\
&\leq \sum_{g \in G} \sum_{i \in I(x,g)} \big| h_i\big(p(x)\big) - h_i\big(p(y)\big) \big| + \sum_{g \in G} \sum_{\substack{i \in I(y,g) \setminus \\ I(x,g)}} h_i\big(p(y)\big) \\
&\leq \varepsilon.
\end{aligned}
$$

This finishes the proof. $\qquad\square$

4.4 Isotropy Groups of Amenable Actions

Although all isotropy groups of an amenable action must be amenable groups, it is, in general, not true that every action with amenable isotropy groups is already amenable itself. Both will be explained in this section as well as the fact that the latter statement does hold in the world of *G-CW*-complexes.

Theorem 4.17 (Isotropy groups of amenable actions). *If the G-space X is amenable, then the isotropy groups G_x are amenable for every $x \in X$.*

Proof. By Theorem 4.9 (2), the action of $G_x \subset G$ on X is amenable, hence so is $G_x \curvearrowright \{x\}$ by Proposition 4.8 (1). Now the claim follows from Lemma 4.3. $\qquad\square$

If G is an amenable group, any action $G \curvearrowright X$ is amenable (see Lemma 4.3 (2)), and it is easy to obtain a G-invariant measure in $\mathrm{prob}(X)$ from an invariant mean on G. As we will see now, the reverse implication is also true. We remark that results corresponding to the following are well-known, cf. e.g. [AD79, Cor. 4.3] for the case of a free ergodic G-action on X.

Theorem 4.18. *Let X be an amenable G-space and assume that there is a sequence $(\mu_n)_{n \in N}$ in $\mathrm{prob}(X)$ such that*

$$
\forall g \in G: \quad \left\| g \cdot \mu_n - \mu_n \right\|_1 \xrightarrow{n \to \infty} 0.
$$

Then the group G must be amenable.

Proof. The proof is accomplished by showing that G satisfies Reiter's condition (P_1). So let a finite $F \subset G$ and an $\varepsilon > 0$ be given. By assumption, we can choose an $m \in N$ such that $\| g \cdot \mu_m - \mu_m \|_1 \leq \varepsilon/4$ for all $g \in F$. Since μ_m is a regular measure, there is a compact $K \subset X$ such that $\mu_m(X \setminus K) \leq \varepsilon/4$.

Furthermore, the amenablity of $G \curvearrowright X$ gives rise to maps $\nu_n \colon X \to \mathrm{prob}(G)$ for $n \in \mathbb{N}$ with the property

$$
\forall g \in G \quad \forall L \subset X \text{ compact:} \quad \sup_{x \in L} \left\| g \cdot \nu_n^x - \nu_n^{gx} \right\|_1 \xrightarrow{n \to \infty} 0.
$$

Let us choose an $l \in \mathbb{N}$ such that $\left\| g \cdot \nu_l^x - \nu_l^{gx} \right\|_1 \leq \varepsilon/4$ on K for all $g \in F$.

Now we define $\lambda \in \mathrm{prob}(G)$ by setting

$$\lambda(g) := \int_X \nu_l^x(g) \, d\mu_m(x).$$

Then we get for all $g \in F$ the inequality

$$\|g \cdot \lambda - \lambda\|_1 \leq \sum_{g' \in G} \left| \int_X g \cdot \nu_l^x(g') \, d\mu_m(x) - \int_X \nu_l^{gx}(g') \, d\mu_m(x) \right|$$

$$+ \sum_{g' \in G} \left| \int_X \nu_l^{gx}(g') \, d\mu_m(x) - \int_X \nu_l^{gx}(g') \, d(g^{-1} \cdot \mu_m)(x) \right|.$$

The first sum on the right can be estimated by

$$\ldots \leq \sum_{g' \in G} \int_X \left| g \cdot \nu_l^x(g') - \nu_l^{gx}(g') \right| d\mu_m(x)$$

$$= \int_X \left\| g \cdot \nu_l^x - \nu_l^{gx} \right\|_1 d\mu_m(x)$$

$$\leq 2 \cdot \mu_m(X \setminus K) + \frac{\varepsilon}{4} \cdot \mu_m(K)$$

$$\leq \frac{3}{4} \varepsilon$$

and the second by

$$\ldots = \sum_{g' \in G} \left| \int_X \nu_l^x(g') \, d(g \cdot \mu_m)(x) - \int_X \nu_l^x(g') \, d\mu_m(x) \right|$$

$$\leq \sum_{g' \in G} \int_X \nu_l^x(g') \, d|g \cdot \mu_m - \mu_m|(x)$$

$$= \|g \cdot \mu_m - \mu_m\|_1$$

$$\leq \frac{\varepsilon}{4}.$$

This means that $\|g \cdot \lambda - \lambda\|_1 \leq \varepsilon$ for all $g \in F$, hence Reiter's condition (P_1) is indeed satisfied. $\qquad \square$

In the following example, Theorem 4.18 is applied to show that, in general, the converse of Theorem 4.17 does not hold: an action of a group on a space with amenable isotropy groups need not be amenable.

Example 4.19 (A non-amenable free action). Suppose that G is a countably infinite group. It acts on the space $\{0,1\}^G = \prod_{g \in G} \{0,1\}$ via shifting, i.e.

$$g_0 \cdot (x_g)_{g \in G} := (x_{g_0 g})_{g \in G}.$$

Note that $\{0,1\}^G$ is a Polish space, as is any countable product of Polish spaces, cf. [Coh80, Prop. 8.1.3]. Consider the Borel subspace

$$
\begin{aligned}
X &:= \left\{ x \in \{0,1\}^G \mid gx \neq x \text{ for all } g \neq 1 \right\} \\
&= \bigcap_{g \neq 1} \left(X^{\langle g \rangle} \right)^c,
\end{aligned}
\tag{4.20}
$$

where the $X^{\langle g \rangle} = \left\{ x \in \{0,1\}^G \mid gx = x \right\}$ are closed in $\{0,1\}^G$. The induced action of G on X is free by definition of X, but we will show now that it is not amenable except in the trivial case where G is an amenable group.

First of all, it follows from [Coh80, Prop. 8.1.10] that the measure μ on $\{0,1\}^G$ which comes from the equiprobability on $\{0,1\}$ is a Radon measure since $\{0,1\}^G$ is Polish. Then this is also true for its restriction to X. Moreover, μ is obviously G-invariant. Finally, we will show that X has positive measure with respect to μ, in fact $\mu(X) = 1$.

Let us prove that the complement of X is a null set. Regarding (4.20), it is enough to show that $\mu\big(X^{\langle g \rangle}\big) = 0$ for any $g \neq 1$. For a subset $S \subset G$ we denote by $X(S)$ the Borel set of all elements $x \in X$ for which either $x_g = 0$ or $x_g = 1$ holds for all $g \in S$. Then, if $\{g_i\}_{i \in I}$ is a system of representatives of $\langle g \rangle \backslash G$, we have

$$
X^{\langle g \rangle} = \bigcap_{i \in I} X\big(\langle g \rangle \cdot g_i \big),
$$

and

$$
\mu\Big(X\big(\langle g \rangle \cdot g_i \big) \Big) = \begin{cases} 2^{1 - |\langle g \rangle|} & \text{if } |\langle g \rangle| < \infty, \\ 0 & \text{if } |\langle g \rangle| = \infty \end{cases}
$$

for every $i \in I$. Thus, in the case of an infinite $\langle g \rangle$, it is immediate that $\mu\big(X^{\langle g \rangle}\big) = 0$. Otherwise, we can assume $I = \mathbb{N}$ and calculate

$$
\mu\big(X^{\langle g \rangle}\big) = \lim_{n \to \infty} \mu\left(\bigcap_{i=1}^{n} X\big(\langle g \rangle \cdot g_i \big) \right) = \lim_{n \to \infty} \big(2^{1 - |\langle g \rangle|} \big)^n = 0.
$$

Having shown that μ is a G-invariant element of $\mathrm{prob}(X)$, Theorem 4.18 implies that $G \curvearrowright X$ is an example of a non-amenable free action whenever G is not amenable.

The above example should be compared to [Zim77, Thm. 2.4], where free ergodic group actions are characterized as being amenable if and only if the associated Murray-von Neumann construction yields a factor that is hyperfinite.

In contrast to this, if the space being acted upon by a group G has the structure of a G-CW-complex, then it suffices to look at the isotropy groups in order to determine whether the action is amenable or not:

Theorem 4.21. *Let X be a G-CW-complex with amenable isotropy groups. Then $G \curvearrowright X$ is amenable.*

Proof. In case X is finite-dimensional, we can use induction over the skeleta of X to prove the statement. The induction start is trivial since X_{-1} is empty. For $n \geq 0$ there is a pushout

$$
\begin{array}{ccc}
\coprod_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\;\amalg_i q_i\;} & X_{n-1} \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle J} \\
\coprod_{i \in I_n} G/H_i \times D^n & \xrightarrow{\;\amalg_i Q_i\;} & X_n
\end{array}
$$

and, by induction hypothesis, there are maps $\mu_k \colon X_{n-1} \to \mathrm{prob}(G)$ for $k \in \mathbb{N}$ such that

$$
\forall g \in G \quad \forall K \subset X_{n-1} \text{ compact:} \quad \sup_{x \in K} \left\| g \cdot \mu_k^x - \mu_k^{gx} \right\|_1 \xrightarrow{k \to \infty} 0.
$$

Note that the maps $\mu_k \circ q_i \colon G/H_i \times S^{n-1} \to \mathrm{prob}(G)$ for $k \in \mathbb{N}$ have the corresponding property.

Since all the groups H_i are amenable by assumption, it follows from Corollary 4.10 that the G-spaces G/H_i are amenable. Hence we can choose for all $k \in \mathbb{N}$ maps $\nu_{i,k} \colon G/H_i \to \mathrm{prob}(G)$ such that

$$
\forall g \in G \quad \forall F \subset G/H_i \text{ finite:} \quad \sup_{\alpha \in F} \left\| g \cdot \nu_{i,k}^\alpha - \nu_{i,k}^{g\alpha} \right\|_1 \xrightarrow{k \to \infty} 0.
$$

From now on, we will identify the cone on S^{n-1}, which is by definition $CS^{n-1} :=$ $S^{n-1} \times [0,1]/S^{n-1} \times \{1\}$, with the disk D^n using the homeomorphism

$$
CS^{n-1} \xrightarrow{\cong} D^n, \quad [z,t] \mapsto (1-t)z.
$$

Under this identification, $S^{n-1} \times \{0\} \subset CS^{n-1}$ corresponds to $S^{n-1} \subset D^n$. We fix an $i \in I_n$ and consider for each $k \in \mathbb{N}$ the map

$$
\widetilde{\nu_{i,k}} \colon G/H_i \times D^n \to \mathrm{prob}(G)
$$
$$
(\alpha, [z,t]) \mapsto (1-t) \cdot (\mu_k \circ q_i)(\alpha, z) + t \cdot \nu_{i,k}(\alpha)
$$

It is well-defined since $\mathrm{prob}(G)$ is a convex space, and it agrees with $\mu_k \circ q_i$ on $G/H_i \times S^{n-1} \subset G/H_i \times D^n$. Furthermore, the following holds for every $g \in G$ and every finite subset $F \subset G/H_i$:

$$
\sup_{\substack{(\alpha,[z,t]) \\ \in F \times D^n}} \left\| g \cdot \widetilde{\nu_{i,k}}^{(\alpha,[z,t])} - \widetilde{\nu_{i,k}}^{(g\alpha,[z,t])} \right\|_1
$$
$$
= \sup_{\substack{(\alpha,[z,t]) \\ \in F \times D^n}} \left\| (1-t) \cdot g \cdot (\mu_k \circ q_i)^{(\alpha,z)} + t \cdot g \cdot \nu_{i,k}^\alpha - (1-t) \cdot (\mu_k \circ q_i)^{(g\alpha,z)} - t \cdot \nu_{i,k}^{g\alpha} \right\|_1
$$
$$
\leq \sup_{\substack{(\alpha,z) \in \\ F \times S^{n-1}}} \left\| g \cdot (\mu_k \circ q_i)^{(\alpha,z)} - (\mu_k \circ q_i)^{(g\alpha,z)} \right\|_1 + \sup_{\alpha \in F} \left\| g \cdot \nu_{i,k}^\alpha - \nu_{i,k}^{g\alpha} \right\|_1
$$
$$
\xrightarrow{k \to \infty} 0.
$$

Let $\widetilde{\nu}_k \colon \coprod_{i \in I_n} G/H_i \times D^n \to \mathrm{prob}(G)$ be defined as the disjoint union of the maps $\widetilde{\nu_{i,k}}$. Then the universal property of pushouts provides us with maps $\widetilde{\mu_k} \colon X_n \to \mathrm{prob}(G)$ which make the following diagram commute:

$$
\begin{array}{ccc}
\coprod\limits_{i \in I_n} G/H_i \times S^{n-1} & \xrightarrow{\coprod_i q_i} & X_{n-1} \\
\downarrow{\scriptstyle j} & & \downarrow{\scriptstyle J} \\
\coprod\limits_{i \in I_n} G/H_i \times D^n & \xrightarrow{\coprod_i Q_i} & X_n \\
\end{array}
$$

Now we can show that X_n is an amenable G-space. Let $g \in G$ and a compact $K \subset X_n$ be given, and let us choose compact subsets $K_1 \subset \coprod_i G/H_i \times D^n$ and $K_2 \subset X_{n-1}$ such that $K \subset (\coprod_i Q_i)(K_1) \cup J(K_2)$. Then

$$
\sup_{y \in K} \left\| g \cdot \widetilde{\mu_k}^y - \widetilde{\mu_k}^{gy} \right\|_1 \leq \sup_{x \in K_1} \left\| g \cdot \left(\widetilde{\mu_k} \circ \coprod_i Q_i \right)^x - \left(\widetilde{\mu_k} \circ \coprod_i Q_i \right)^{gx} \right\|_1
$$
$$
+ \sup_{x \in K_2} \left\| g \cdot (\widetilde{\mu_k} \circ J)^x - (\widetilde{\mu_k} \circ J)^{gx} \right\|_1
$$
$$
= \sup_{x \in K_1} \left\| g \cdot \widetilde{\nu}_k^{\,x} - \widetilde{\nu}_k^{\,gx} \right\|_1 + \sup_{x \in K_2} \left\| g \cdot \mu_k^x - \mu_k^{gx} \right\|_1,
$$

while both of the latter terms tend to 0 as $k \to \infty$.

Finally, we will treat the general case, in which $X = \bigcup_{n \in \mathbb{N}} X_n$ is the colimit of its skeleta. By what we have just shown, we obtain for all $n \in \mathbb{N}$ maps $\mu_k^n \colon X_n \to \mathrm{prob}(G)$ indexed by $k \in \mathbb{N}$ such that $\mu_k^n|_{X_{n-1}} = \mu_k^{n-1}$ and $(\mu_k^n)_{k \in \mathbb{N}}$ is an a.i.c.m. for $G \curvearrowright X_n$. If we define $\mu_k := \mathrm{colim}_n \mu_k^n$ for each $k \in \mathbb{N}$, then these maps will form an a.i.c.m. for $G \curvearrowright X$. This is due to the fact that any compact $K \subset X$ is already contained in X_n for any n that is sufficiently large. $\qquad\square$

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