Westfälische Wilhelms-Universität Münster
Fachbereich Mathematik und Informatik

Federico B. Calderon I.

COUNTING SINGULAR POINTS OF ALGEBRAIC VARIETIES OVER FINITE FIELDS

2020

Mathematik

Inaugural-Dissertation
zur Erlangung des Doktorgrades
der Naturwissenschaften im Fachbereich
Mathematik und Informatik
der Mathematisch-Naturwissenschaftlichen Fakultät
der Westfälischen Wilhelms-Universität Münster

vorgelegt von
Federico Baldomero Calderón Idárraga
aus Bogotá, Kolumbien
- 2020 -

Dekan:	Prof. Dr. Xiaoyi Jiang
Erster Gutachter:	Prof. Dr. Christopher Deninger
Zweiter Gutachter:	PD. Dr. Jakob Scholbach
Tag der mündlichen Prüfung:	04.05.2020
Tag der Promotion:	

Abstract

For a proper singular algebraic variety X_0 over a finite field of characteristic p, we study the intersection Zeta function $IZ(X_0, t)$. We prove the rationality of $IZ(X_0, t)$. For each singular closed point $x \in X_0$, we define and study the multiplicities with which the point x has to be counted to get the intersection Zeta function $IZ(X_0, t)$ from its generating series: the *r-multiplicities* of x in X_0. We prove that these numbers are integers and independent of $l \neq p$. We also calculate them explicitly in low-dimensional cases and relate them to well-known geometric objects such as the nearby and vanishing cycles functors.

Zusammenfassung

Wir untersuchen die Schnitt-Zetafunktion $IZ(X_0, t)$ einer eigentlichen algebraischen Varietät X_0 über einem endlichen Körper der Charakteristik p. Wir beweisen, dass $IZ(X_0, t)$ rational ist. Für jeden abgeschlossenen singulären Punkt $x \in X_0$ definieren und untersuchen wir die Multiplizität, mit der der Punkt x gezählt werden muss, um die Schnitt-Zetafunktion $IZ(X_0, t)$ aus ihrer Erzeugendenfolge zu erhalten: die *r-Multiplizität* von x in X_0. Es wird bewiesen, dass diese Multiplizitäten ganze Zahlen und unabhängig von $l \neq p$ sind. Wir berechnen diese explizit in den Fällen, in denen die Dimension von X_0 klein ist, und stellen eine Beziehung zu bekannten geometrischen Objekten wie den benachbarten und verschwindenden Zykeln her.

To Ana Maria and Maria del Mar Espinosa

Acknowledgments

First and foremost I would like to thank my supervisor Professor Dr. Christopher Deninger for giving me the opportunity to work with him, for choosing such an interesting and well thought topic for my PhD thesis, but above all thank him for his patience, kindness and time by performing an outstanding supervision. I would also like to thank PD Dr. Jakob Scholbach for co-supervising this thesis, helping me out through fruitful discussions and being an example of continuous and precise work. I thank Prof. Dr. Jörg Schürmann for his guidance in topological and non-topological topics, as well as for always having a smile to share and something nice to say. I would also like to thank my office partners Maria Beckemeyer and PD Dr. Christian Serpé for helping me out every day mathematically or non-mathematically, for their patience, friendship and making every day joyful and enjoyable. I thank Gabi Dierkes (as well as Kamo and Ringo) for taking care of me since my arrival in Münster, for her friendship, great conversations, understanding and multiple advises. I thank also all my colleges at the institute of mathematics Münster for their support and suggestions. Specially, Maria Beckemeyer, Maike Frantzen and Dr. Claudius Heyer for helping me correct some mistakes at the later stages of this work.

I want to thank Laura Arias for her unconditional support, making me smile everyday no matter what and filling my life with excitement and love. I would like to thank my family for being always there for me, for they love and support, but most of all for believing in me. I thank also my friends for showing up when I need them the most, for their support, friendship and not letting me forget who I am or why I do things.

This work would not have been possible without the support of the University of Münster and the education I received at the University of Heidelberg and Universidad Nacional de Colombia, as without all the people that have

been there for me since the very beginning.

Contents

Introduction

In 1949 André Weil proposed the highly influential Weil conjectures. They led to the development of modern algebraic geometry and number theory. The conjectures consist of four different statements about properties of the Zeta function of a smooth and projective variety Y_0 over a finite field $\mathbb{F}_q$.

Let d be the dimension of Y_0. We denote by Y the base change of Y_0 to the algebraic closure of $\mathbb{F}_q$ and by $Y_0(\mathbb{F}_q)$ the set of points with coordinates in $\mathbb{F}_q$. For a set A, we write $|A|$ for the number of elements in A. The integers $|Y_0(\mathbb{F}_{q^n})|$ are numbers of solutions of equations over finite fields and are very interesting and important from an arithmetic point of view.

Definition 0.0.1. The *Zeta function of* Y_0 is the following formal power series:

$$Z(Y_0, t) := \exp\left(\sum_{n=1}^{\infty} \frac{|Y_0(\mathbb{F}_{q^n})|}{n} t^n\right) \in \mathbb{Q}[[t]].$$

The relation

$$t\frac{d}{dt}\log Z(Y_0, t) = \sum_{n\geq 1} |Y_0(\mathbb{F}_{q^n})| t^n,$$

shows that $t\frac{d}{dt}logZ(Y_0, t)$ is the generating function of the sequence $\{|Y_0(\mathbb{F}_{q^n})|\}_{n\geq 1}$.

The Weil conjectures are the following statements ([39]):

(I) (Rationality): The Zeta function $Z(Y_0, t)$ is a rational function of t. Moreover, we have

$$Z(Y_0, t) = \frac{P_1(Y_0, t)P_3(Y_0, t)\dots P_{2d-1}(Y_0, t)}{P_0(Y_0, t)P_2(Y_0, t)\dots P_{2d}(Y_0, t)},$$

where $P_0(Y_0, t) = 1 - t$, $P_{2d}(Y_0, t) = 1 - q^d t$ and each $P_i(Y_0, t)$ is a polynomial with coefficients in $\mathbb{Z}$.

(II) (Functional equation): The Zeta function $Z(Y_0, t)$ satisfies the following *functional equation*:

$$Z(Y_0, q^{-d}t^{-1}) = \pm q^{d\chi/2} t^{\chi} Z(Y_0, t),$$

where $\chi = \sum_i (-1)^i \beta_i$ for $\beta_i = \deg P_i(Y_0, t)$.

(III) (Betti numbers) If Y lifts to a variety Y_1 in characteristic 0, then the numbers β_i are the *Betti numbers* of the complex manifold $Y_1(\mathbb{C})$.

(IV) (Riemann hypothesis) For $0 \leq i \leq 2d$ we have that,

$$P_i(Y_0, t) = \prod_{j=1}^{\beta_i} (1 - \alpha_{i,j} t),$$

where $\alpha_{i,j}$ are algebraic integers of absolute value $q^{i/2}$ with respect to every embedding of $\mathbb{Q}$ into $\mathbb{C}$.

The development of the Weil conjectures dates back to the work of Carl Friedrich Gauss *Disquititiones Arithmeticae*. His study in section VII on roots of unity and Gaussian periods allowed an interpretation of the coefficients of certain product of periods as the number of points on elliptic curves defined over a finite field. In 1924 Emil Artin proposed in his PhD thesis ([1]) a definition of the Zeta function for an algebraic curve over a finite field and proposed a Riemann hypothesis for this kind of Zeta functions. These propositions amount to the Weil conjectures in the special case of algebraic curves. Weil then proved these propositions, finishing the project started by Hasse's theorem on elliptic curves over finite fields ([39]). His results gave the insight for the correct approach to generalizations. Weil proposed to look for a cohomology theory for algebraic varieties Y over a finite field with an analogue of the Lefschetz fixed-point formula for Frobenius. In particular, statement (I) suggests the Lefschetz fixed-point formula and statement (II) Poincaré duality. Statement (III) suggests that the numbers β_i are Betti numbers (dimensions of finite-dimensional cohomology groups of Y in this new theory).

Statement (I) was first proved by Dwork in 1960 ([9]) using p-adic methods. Later, Artin, Verdier and Grothendieck developed a well-suited cohomology theory for algebraic varieties over finite fields: l-adic cohomology. This cohomology theory has an analogue of Poincaré duality and of the Lefschetz fixed-point formula. In particular, we have that

$$|Y_0(\mathbb{F}_{q^n})| = \sum_{i=0}^{2d} (-1)^i \mathrm{Tr}(Fr^n | H^i(Y, \overline{\mathbb{Q}_l})), \tag{1}$$

where Fr is the *Frobenius Endomorphism* of Y_0 and $H^i(Y, \overline{\mathbb{Q}_l})$ are finite-dimensional $\overline{\mathbb{Q}_l}$-vector spaces with an action of Fr. The development of the étale cohomology led to proofs of statements (I), (II) and (III) of the Weil conjectures. Finally, in 1974 Deligne proved (IV) ([7]).

If the variety Y_0 is allowed to have singularities, then the étale cohomology does not satisfy Poincaré duality in general ([22, §1.1]). In 1980 Goresky and MacPherson introduced intersection homology on topological spaces ([13]) and later Beilinson, Bernstein, Deligne and Gabber developed an algebraic analogue: étale intersection cohomology ([3])

$$IH^\bullet(Y, \overline{\mathbb{Q}_l}) := \mathbb{H}^\bullet(Y, \pi^* IC_{Y_0}[-d]),$$

where the complex IC_{Y_0} is the *intersection complex* of Y_0 and $\pi : Y \to Y_0$ is the canonical projection. This cohomology theory fulfills Poincaré duality and it is in many ways well-suited for dealing with singular varieties.
Based on the definition of the Zeta function of Y_0 and equation (1) we make the following definition.

Definition 0.0.2. Let X_0 be a proper variety over a finite field $\mathbb{F}_q$ of dimension d. Let X be the base change of X_0 to the algebraic closure of $\mathbb{F}_q$ with projection $\pi : X \to X_0$. The *intersection Zeta function of* X_0 is defined as follows:

$$IZ(X_0, t) := \exp\left(\sum_{n=1}^{\infty} \frac{a_n}{n} t^n\right),$$

where $a_n := \sum_{i=0}^{2d} (-1)^i \mathrm{Tr}(Fr^n | IH^i(X, \overline{\mathbb{Q}_l}))$.

By Grothendieck's trace formula (Theorem 1.2.1), we have that

$$a_n = \sum_{x \in X_0(\mathbb{F}_{q^n})} \sum_i (-1)^{i+d} \mathrm{Tr}(F^n | \mathcal{H}^i(IC_{X_0})_{\overline{x}}),$$

where F is the *geometric Frobenius*. Note that for Y_0 a smooth projective variety over $\mathbb{F}_q$,

$$Z(Y_0, t) = IZ(Y_0, t).$$

Some of the assertions of the Weil conjectures remain true for $IZ(X_0, t)$. Statement (II) follows from Poincaré duality for intersection cohomology and statement (IV) is a consequence of the *purity* of IC_{X_0} ([3, Corollaire 5.3.4.]). As for (I) a standard argument shows that $IZ(X_0, t)$ lies in $\overline{\mathbb{Q}_l}(t) \cap \overline{\mathbb{Q}_l}[[t]]$. It is the alternating product of characteristic polynomials of Fr acting on $\mathbb{H}^i(X, \pi^* IC_{X_0}[d])$. We show that $IZ(X_0, t)$ actually lies in $\mathbb{Q}(t) \cap \mathbb{Q}[[t]]$. In particular, we prove statement (I) of the Weil conjectures for the intersection Zeta function $IZ(X_0, t)$ of an integral proper algebraic variety X_0 over $\mathbb{F}_q$.

Theorem. *(1.5.20) Let X_0 be a proper algebraic variety over $\mathbb{F}_q$ of dimension d. The intersection Zeta function $IZ(X_0, t)$ is a rational function of t. Moreover, we have*

$$IZ(X_0, t) = \frac{P_1(X_0, t) P_3(X_0, t) \dots P_{2d-1}(X_0, t)}{P_0(X_0, t) P_2(X_0, t) \dots P_{2d}(X_0, t)},$$

where $P_0(X_0, t) = 1 - t, P_{2d}(X_0, t) = 1 - q^d t$ and each $P_i(X_0, t)$ is polynomial with coefficients in $\mathbb{Z}$.

The focus of this thesis will be the study of the *r-multiplicities*

$$m^r_{X_0,x} := \sum_i (-1)^{i+d} \mathrm{Tr}(F^\eta | \mathcal{H}^i(IC_{X_0})_{\overline{x}}),$$

for singular closed points $x \in X_0$. Here $r \geq 1$ and $\eta = \deg(x) r$. By equation (1), each of these numbers may be viewed as the multiplicity with which the point x has to be counted to get the intersection Zeta function $IZ(X_0, t)$ from its generating series. Fitting with this intuition, the numbers $m^r_{X_0,x}$ will be shown to be integers.

In Chapter 1 we establish a general framework for calculating the numbers

$$\sum_i (-1)^i \mathrm{Tr}(F^r|\mathcal{H}^i(K_0)_{\overline{x}})$$

for any $K_0 \in \mathcal{D}_c^b(X_0, \overline{\mathbb{Q}_l})$ and any closed point x using the Grothendieck group of constructible $\overline{\mathbb{Q}_l}$-sheaves. In particular, we will reduce the calculation of $m^r_{X_0,x}$ to the case where X_0 is integral affine and normal (Lemma 1.4.3 and Lemma 1.4.2).
In section 1.5 we prove that the r-multiplicity of a closed singular point $m^r_{X_0,x}$ is an integer independent of l (Proposition 1.5.7 and Corollary 1.5.15) using previous results of Deligne (SGA 7, XXI) and the concept of (E, I)-compatibility due to Gabber ([12]).

We then try to understand $m^r_{X_0,x}$ either through explicit calculations or by relating it to well known geometric objects. Our approach for calculating $m^r_{X_0,x}$ can be summarized as follows: Substitute X_0 by a "better" scheme $\tilde{X}_0$ through *modifications* or *alterations* and try to establish a relation between IC_{X_0} and $IC_{\tilde{X}_0}$. The best possible modification that we could expect would be a desingularization of X_0 *in the strong sense*. Unfortunately, such desingularizations of algebraic varieties over finite fields are only known in low dimensions.

In Chapter 2 we calculate $m^r_{X_0,x}$ for $\dim(X_0) = 1, 2, 3$. For these cases, resolution of singularities of X_0 in the strong sense is known ([25] and [5]). We conclude general results in dimension 1 and 2. We give also a general result in dimension 3 assuming that X_0 has only isolated singularities.
Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension 2 or 3 and $x \in X_0$ a closed isolated singular point. Let $\pi : \tilde{X}_0 \to X_0$ be a desingularization of X_0 in the strong sense such that the exceptional divisor $D_0 := \pi^{-1}(\{x\})$ is a divisor with *strict normal crossings*. Let D be the base change of D_0 to the algebraic closure of $\mathbb{F}_q$ and $\{E_s\}_s$ be the set of irreducible components of D. Denote by F_{D_0} the q-Frobenius morphism of D_0.

Theorem. *(2.2.3) Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension 2. Let $x \in X_0$ be a closed singular point. Then for D_0 as above and for each*

$r \geq 1$,

$$m^r_{X_0,x} = |D_0(\mathbb{F}_{q^\eta})| - |\{E_s : (F^\eta_{D_0} \times \mathrm{Id})(E_s) = E_s\}|\ q^\eta,$$

where $\eta = \deg(x)r$.

In his work [40], J. Wildeshaus studied the intersection cohomology groups of a singular normal surface X over $\mathbb{C}$. He concluded that the intersection cohomology groups of X, different from the second one, agree with the intersection cohomology groups of a desingularization $\tilde{X}$ of X. On the other hand, the second intersection cohomology group of $\tilde{X}$ is a direct sum of the second intersection cohomology group of X and the second cohomology group of the exceptional divisor D over the singular locus of X ([40, Theorem 1.1]). The theorem above was thus inspired by this result of J. Wildeshaus and it is in a way a "local" version of it, since it deals with the cohomology groups of the intersection complex localized at a point instead of dealing with the whole cohomology groups of the intersection complex.

Theorem. *(2.4.2) Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension 3 and $x \in |X_0|$ an isolated singularity. Then for D_0 as above and $r \geq 1$,*

$$m^r_{X_0,x} = |D_0(\mathbb{F}_{q^\eta})| - \mathrm{Tr}(Fr^\eta | H^3(D, \overline{\mathbb{Q}_l})) + |\{E_s : (F^\eta_{D_0} \times \mathrm{Id})(E_s) = E_s\}|\ (q^\eta + q^{2\eta}),$$

where $\eta = \deg(x)r$.

In [36, §2], E. Tasso calculated the intersection cohomology of a projective singular three-dimensional variety X over $\mathbb{C}$ with an isolated singular point. She concluded that for $\pi : \tilde{X} \to X$ a desingularization of X and $\delta \neq 2, 3, 4$, the intersection cohomology groups $IH^\delta(\tilde{X}, \mathbb{Q}) = IH^\delta(X, \mathbb{Q})$ and $IH^\rho(\tilde{X}, \mathbb{Q}) = IH^\rho(X, \mathbb{Q}) \oplus H^\rho(D, \mathbb{Q})$, where $\rho = 2, 3, 4$ and D is the exceptional divisor over the singular locus of X ([36, Theorem 2.1.4]). The theorem above is again a "local" version of this result of E. Tasso and was inspired by it.

We also give some examples that illustrate the behavior of $m^r_{X_0,x}$. In particular, we show that for a closed singular point x, the r-multiplicity $m^r_{X_0,x}$ can be a negative integer (Example 2.2.4) or 1 (Example 2.5.1). We also give results on $m^r_{X_0,x}$ when X_0 has dimension 2 and x is a *pseudo-rational* singularity of X_0 (Definition 2.5.5 and Corollary 2.5.6).

In Chapter 3 we recall the definition of the (unipotent) nearby and (unipotent) vanishing cycles functors made by Beilinson ([2]) and Deligne (SGA 7, I and XIII). We express the r-multiplicity $m^r_{X_0,x}$ using the unipotent nearby and unipotent vanishing cycles functors (Equation (3.3)). Thus, we achieve a *geometric* interpretation of $m^r_{X_0,x}$. Using then the unipotent vanishing cycles functor, we relate the r-multiplicity $m^r_{X_0,x}$ to another important geometric object, namely, to the Milnor fiber of a morphism $g : X_0 \to \mathbb{A}^1_{\mathbb{F}_q}$ (Equation (3.5) and Theorem 3.1.8).
We then associate the r-multiplicity $m^r_{X_0,x}$ to the unipotent nearby and vanishing cycles functors of an alteration $\phi_1 : \tilde{X}_0 \to X_0$ (a weaker modification than a desingularization of X_0 in the strong sense). This association allows an explicit calculation of $m^r_{X_0,x}$ in terms of the unipotent vanishing cycles functor of $\tilde{X}_0$.
An alteration $\phi_1 : \tilde{X}_0 \to X_0$ comes with a center Z_0 (a closed subscheme of X_0) and with an inclusion $j_1 : \tilde{X}_0 \to \overline{X_0}$ to a projective scheme $\overline{X_0}$ such that $Y_0 := j_1(\phi_1^{-1}(Z_0)) \cup \overline{X_0} \setminus j_1(\tilde{X}_0)$ is a divisor with strict normal crossings of $\overline{X_0}$. Let $((Y_0)_s)_{s\in I}$ be the set of irreducible components of Y_0. For $E \subset I$, let

$$(Y_0)_E := \bigcap_{s\in E} (Y_0)_s$$

and

$$Y_0^{(m)} := \coprod_{|E|=m} (Y_0)_E.$$

Further let $a_m : Y_0^{(m)} \to Y_0$ be the projection and $a_0 = \mathrm{Id}$. Omit the zero subscript on the objects just defined to denote the base change to the algebraic closure of $\mathbb{F}_q$ and set $\overline{a}_m : Y^{(m)} \to Y$ to be the projection.

Theorem. *(3.2.5) Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension d. Let $x \in |X_0|$ be a singular point. Consider an alteration $\phi_1 : \tilde{X}_0 \to X_0$ of X_0 and $y \in \tilde{X}_0$ such that $\phi_1(y) = x$. Let $g : X_0 \to \mathbb{A}^1_{\mathbb{F}_q}$ be a morphism such that $(X_0)_{sing} \subseteq g^{-1}(\{0\})$. Define $h := g \circ \phi_1 : \tilde{X}_0 \to \mathbb{A}^1_{\mathbb{F}_q}$. Then for any $r \geq 1$ and $\eta = \deg(y)r$,*

$$(1-q^\eta)m^{r+\nu}_{X_0,x} = \sum_{i\geq 0}(-1)^i q^{ir}|\overline{a}_i^{-1}(y)| - (1-q^\eta)t^r_{\phi^u_h(\phi_1^* IC_{X_0})}(y),$$

where $\nu = [\kappa(y) : \kappa(x)]$.

Here $t^r_{\phi^u_h(\phi_1^* IC_{X_0})}(y)$ are *Frobenius traces* of the complex $\phi^u_h(\phi_1^* IC_{X_0})$ at the point y.
We end Chapter 3 by expressing $m^r_{X_0,x}$ through Frobenius traces of intermediate extensions of smooth sheaves coming from local systems on $\mathbb{G}_{m,\mathbb{F}_q}$ (Theorem 3.3.6).

We then finish by giving a general method for calculating the r-multiplicity $m^r_{X_0,x}$ in a normal algebraic variety X_0 through an alteration $\tilde{X}_0$ of an irreducible component containing x. Given an alteration $\phi_1 : \tilde{X}_0 \to X_0$ there exists an open subscheme $j : U_0 \hookrightarrow X_0$ such that $\phi_1|_{\phi_1^{-1}(U_0)}$ is étale, finite and flat. There exists a Galois cover of U_0 with finite Galois group W such that the push-forward of $IC_{\tilde{X}_0}$ under $\phi_1|_{\phi_1^{-1}(U_0)}$ is a W*-equivariant sheaf*. It decomposes into perverse sheaves $\mathcal{F}_\chi[d]$ given by a χ-isotypic decomposition of characters. Denote by χ_0 the trivial character.

Theorem. *(4.0.2) Let X_0 be a normal and proper algebraic variety over $\mathbb{F}_q$ of dimension d with a singular closed point x. Let $\phi : \tilde{X}_0 \to X_0$ be an alteration of an irreducible component containing x and define $D_0 := \phi^{-1}(\{x\})$. Then for $r \geq 1$,*

$$\begin{aligned} m^r_{X_0,x} =& |D(\mathbb{F}_{q^\eta})| - \sum_{j=0}^{r(\phi)} (-1)^j \mathrm{Tr}(Fr^\eta | H^{d+j}(D, \overline{\mathbb{Q}_l})) \\ & - \sum_{j=-r(\phi)}^{-1} (-1)^j \mathrm{Tr}(Fr^\eta | H^{d+j}(D, \overline{\mathbb{Q}_l}))\ q^{j\eta} - \sum_{\substack{\chi \in \hat{W} \\ \chi \neq \chi_0}} t^r_{j_{!*}\mathcal{F}_\chi[d]}(x), \end{aligned}$$

where $\eta = \deg(x) r$ *and* $r(\phi)$ *is the defect of semi-smallness of* ϕ.

Chapter 1

General Results

1.1 Notations and Conventions

1. Let p be a prime number and denote by q a power of p. Let $\mathbb{F}_q$ be the finite field of order q. Fix an algebraic closure of $\mathbb{F}_q$ and denote it by $\overline{\mathbb{F}_q}$. Further, let l be a prime number different from p. Fix an algebraic closure of $\mathbb{Q}_l$ and denote it by $\overline{\mathbb{Q}_l}$.

2. Let X_0 be a separated scheme of finite type over $\mathbb{F}_q$. For a closed point x in X_0, let $\kappa(x)$ be the residue field of x. We write $\deg(x)$ for the degree of $\kappa(x)$ over $\mathbb{F}_q$.

3. For $x \in X_0$ (not necessarily closed), we will write $\overline{x}$ for a geometric point with image x ([7, (0.3)]). If $x \in X_0(\mathbb{F}_q)$, then we choose $\overline{x}$ to be the composition $\mathrm{Spec}(\overline{\mathbb{F}}_q) \to \mathrm{Spec}(\mathbb{F}_q) \xrightarrow{x} X_0$.

4. Let X_0 be a separated scheme of finite type over a field k such that l is invertible on X_0. We denote by $\mathcal{D}_c^b(X_0, \overline{\mathbb{Q}_l})$ the derived category of bounded constructible l-adic sheaves defined by Deligne ([7, (1.1.2) and (1.1.3)]).

5. Let $Z_0 \subseteq X_0$ be a closed subset. The reduced induced scheme structure on Z_0 is denoted by $((Z_0)_{red}, \mathcal{O}_{Z_0})$. The category $\mathcal{D}_c^b(Z_0, \overline{\mathbb{Q}_l})$ only depends on $(Z_0)_{red}$, i.e, $\mathcal{D}_c^b(Z_0, \overline{\mathbb{Q}_l}) = \mathcal{D}_c^b((Z_0)_{red}, \overline{\mathbb{Q}_l})$. Thus, we assume all schemes X_0 to be reduced.

6. Let S be a scheme of dimension less than or equal to 1. For two separated schemes of finite type over $\mathbb{F}_q$, X_0 and Y_0, and an S-morphism $f : X_0 \to Y_0$,

there are two internal operations $\otimes^L$ and $\mathrm{R}Hom$ in $\mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l})$ and four functors

$$\begin{aligned} Rf_!, Rf_* :&\mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l}) \to \mathcal{D}^b_c(Y_0, \overline{\mathbb{Q}_l}) \text{ and} \\ f^*, f^! :&\mathcal{D}^b_c(Y_0, \overline{\mathbb{Q}_l}) \to \mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l}). \end{aligned}$$

We will denote the functors $Rf_!$ and Rf_* by $f_!$ and f_*, respectively. Let

$$D_{X_0/S} : \mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l})^{opp} \to \mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l})$$

be the Verdier dual functor $D(-)_{X_0/S} := \mathrm{R}Hom(-, a^!\overline{\mathbb{Q}_l})$, where $a : X_0 \to S$ is the structure morphism.

7. A t-structure in a triangulated category D consists of two strictly full subcategories $D^{\leq 0}$ and $D^{\geq 0}$, such that with the definitions $D^{\leq n} := D^{\leq 0}[-n]$ and $D^{\geq n} := D^{\geq 0}[-n]$ we have

(i) $\mathrm{Hom}(D^{\leq 0}, D^{\geq 1}) = 0$.

(ii) $D^{\leq 0} \subset D^{\leq 1}$ and $D^{\geq 1} \subset D^{\geq 0}$.

(iii) For every object E in D there exists an exact triangle (A, E, B) with $A \in D^{\leq 0}$ and $B \in D^{\geq 1}$.

D is said to be bounded with respect to the t-structure, if every object D is contained in some $D^{\geq a}$ and some $D^{\leq b}$ for certain integers a, b.

8. Given a t-structure $(D^{\leq 0}, D^{\geq 0})$ in a triangulated category D we get a functor

$$\tau_{\leq 0} : D \to D^{\leq 0},$$

which is right adjoint to the inclusion functor of $D^{\leq 0}$ into D. We also get a functor

$$\tau_{\geq 0} : D \to D^{\geq 0},$$

which is left adjoint to the inclusion functor of $D^{\geq 0}$ into D. The functors $\tau_{\leq 0}$, $\tau_{\geq 0}$ are the truncation functors for the given t-structure.

9. The core $Core(D) = D^{\leq 0} \cap D^{\geq 0}$ attached to a t-structure of a triangulated category D is an abelian category. A sequence in $Core(D)$

$$0 \to X \stackrel{u}{\to} Y \stackrel{v}{\to} Z \to 0$$

is exact if and only if there exists an exact triangle (X, Y, Z, u, v, w) in D ([3, §1.3]).

10. Let D be a triangulated category with a t-structure in it. One defines

$$H^0(X) := \tau_{\leq 0}\tau_{\geq 0}X \in Core(D).$$

More generally one defines for $n \in \mathbb{Z}$ the n-th cohomology functors

$$H^n : D \to Core(D)$$

by $H^n(X) = H^0(X[n])$; and similar $H^n(u) = \tau_{\leq 0}\tau_{\geq 0}(u[n])$ for morphisms u. Note that

$$H^n(X)[-n] = \tau_{\leq n}\tau_{\geq n}X.$$

11. Let T be an additive, translation preserving functor between triangulated categories $\mathcal{A}$ and $\mathcal{B}$ with t-structures, which transforms exact triangles in exact triangles. Such a functor is called

$$t\textit{-right exact} \text{ if and only if } T(D^{\leq 0}(\mathcal{A})) \subset D^{\leq 0}(\mathcal{B})$$

and

$$t\textit{-left exact} \text{ if and only if } T(D^{\geq 0}(\mathcal{A})) \subset D^{\geq 0}(\mathcal{B}).$$

Finally T is called t-exact, if T is t-left and t-right exact.

12. The *standard t-structure* in $\mathcal{D}_c^b(X_0, \overline{\mathbb{Q}_l})$ is given by

$$D^{\leq 0}(\mathcal{D}_c^b(X_0, \overline{\mathbb{Q}_l})) := \{K_0 \in \mathcal{D}_c^b(X_0, \overline{\mathbb{Q}_l}) : \mathcal{H}^i K_0 = 0 \ \forall i > 0\}$$
$$D^{\geq 0}(\mathcal{D}_c^b(X_0, \overline{\mathbb{Q}_l})) := \{K_0 \in \mathcal{D}_c^b(X_0, \overline{\mathbb{Q}_l}) : \mathcal{H}^i K_0 = 0 \ \forall i < 0\},$$

where $\mathcal{H}^i K_0$ denote the i-th cohomology sheaf of K_0.
The core of this t-structure is

$$Core(standard) = \{K_0 \in \mathcal{D}_c^b(X_0, \overline{\mathbb{Q}_l}) : \mathcal{H}^i K_0 = 0 \ i \neq 0\}.$$

We denote by $\tau_{\leq 0}$, $\tau_{\geq 0}$ the truncation functors for the standard t-structure.

13. The *perverse t-structure* in $\mathcal{D}_c^b(X_0,\overline{\mathbb{Q}_l})$ ([3, Théorème 1.4.10.] or [21, §III]) is given by

$$
\begin{aligned}
{}^pD^{\leq 0}(\mathcal{D}_c^b(X_0,\overline{\mathbb{Q}_l})) &:= \{K_0 \in \mathcal{D}_c^b(X_0,\overline{\mathbb{Q}_l}) : dimsupp(\mathcal{H}^{-i}K_0) \leq i \ \forall i \in \mathbb{Z}\},\\
{}^pD^{\geq 0}(\mathcal{D}_c^b(X_0,\overline{\mathbb{Q}_l})) &:= \{K_0 \in \mathcal{D}_c^b(X_0,\overline{\mathbb{Q}_l}) : dimsupp(\mathcal{H}^{-i}DK_0) \leq i \ \forall i \in \mathbb{Z}\}.
\end{aligned}
$$

Here D is the Verdier dual functor $D_{X_0/\mathrm{Spec}(\mathbb{F}_q)}$. The core of this t-structure is the abelian category of perverse sheaves on X_0:

$$Core(perverse) = Perv(X_0).$$

It is artinian and noetherian ([21, Corollary 5.7]). We denote by ${}^p\tau_{\leq 0}$, ${}^p\tau_{\geq 0}$ the truncation functors for the perverse t-structure. One defines the perverse cohomology functor as follows:

$${}^pH^0 := {}^p\tau_{\leq 0} \circ {}^p\tau_{\geq 0}.$$

14. Let X_0 be a separated scheme finitely generated over a finite or algebraically closed field. Let $j : U_0 \to X_0$ be an open embedding with closed complement $i : Y_0 \to X_0$. Then the functors

$$
\begin{gathered}
j_!, i^* \text{ are } t\text{-right exact},\\
i_*, j^* \text{ are } t\text{-exact},\\
j_*, i^! \text{ are } t\text{-left exact}
\end{gathered}
$$

for the perverse t-structures on $\mathcal{D}_c^b(X_0,\overline{\mathbb{Q}_l})$ respectively $\mathcal{D}_c^b(U_0,\overline{\mathbb{Q}_l})$ and $\mathcal{D}_c^b(Y_0,\overline{\mathbb{Q}_l})$.

15. Let X_0 be a noetherian scheme. An l-adic sheaf $\mathcal{F}$ on X is an inverse system $\{\mathcal{F}_n\}_{n\geq 1}$ where

(a) each sheaf $\mathcal{F}_n$ is a constructible $\mathbb{Z}/l^n\mathbb{Z}$-module on the étale site of X_0, and

(b) the transition maps $\mathcal{F}_{n+1} \to \mathcal{F}_n$ induce isomorphisms $\mathcal{F}_{n+1} \otimes_{\mathbb{Z}/l^{n+1}\mathbb{Z}} \mathbb{Z}/l^n\mathbb{Z} \cong \mathcal{F}_n$.

We say that $\mathcal{F}$ is *smooth* if each $\mathcal{F}_n$ is locally constant. A complex $K_0 \in \mathcal{D}_c^b(X_0,\overline{\mathbb{Q}_l})$ is *smooth* if each $\mathcal{H}^i(K_0)$ is smooth.

16. Let X_0 be a scheme over some arbitrary base field. Let X denote the scheme obtained from X_0 by base change to the algebraic closure of the base field. The scheme X_0 is called essentially smooth, if the reduced scheme X_{red} is smooth. Let X_0 be an essentially smooth separated scheme of finite type over a field k such that l is invertible on X_0. Let $K_0 \in \mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l})$ be a smooth complex on X_0. We get in this case ([21, Remark 2.2]): $K_0 \in Perv(X_0)$ if and only if

$$K_0 = \mathcal{L}[\dim X_0]$$

for a smooth sheaf $\mathcal{L}$. In particular, the cohomology sheaves of K_0 are trivial in degrees different from $-\dim X_0$.

17. Let X_0 be a separated scheme of finite type over a finite or algebraically closed field. Let $j : U_0 \hookrightarrow X_0$ be an embedding of an open subscheme. Let $i : Y_0 \hookrightarrow X_0$ denote the (reduced) closed complement of U_0. Let K_0 be a perverse sheaf on U_0. A perverse sheaf $\overline{K_0}$ on X_0 is called an extension of K_0, if

$$j^*\overline{K_0} = K_0.$$

There is (up to quasi-isomorphism) a unique extension $\overline{K_0} \in Perv(X_0)$ of a perverse sheaf $K_0 \in Perv(U_0)$, such that $\overline{K_0}$ has neither quotients nor subobjects of type i_*A_0 for $A_0 \in Perv(Y_0)$. This unique extension will be called the *intermediate extension*

$$j_{!*}K_0$$

of K_0 and defines a functor

$$j_{!*} : Perv(U_0) \to Perv(X_0).$$

The intermediate extension of a perverse sheaf $K_0 \in Perv(U_0)$ is characterized, up to isomorphism, as the unique perverse sheaf $\overline{K_0} \in Perv(X_0)$ with the following properties:

$$j^*\overline{K_0} = K_0 \text{ and } {}^pH(i^*\overline{K_0}) = {}^pH(i^!\overline{K_0}) = 0.$$

18. Let X_0 be a separated equidimensional scheme of finite type over a finite or algebraically closed field of dimension d. Let $j : U_0 \hookrightarrow X_0$ be an

embedding of an open smooth dense subscheme with (reduced) closed complement $i : Y_0 \hookrightarrow X_0$. The *intersection complex* IC_{X_0} of X_0 is defined as the intermediate extension

$$j_{!*}\overline{\mathbb{Q}_l}[d]$$

of the perverse sheaf $\overline{\mathbb{Q}_l}[d] \in Perv(U_0)$. It does not depend on the choice of U_0 and it is characterized (up to quasi-isomorphism) as the unique perverse sheaf $\overline{K_0} \in Perv(X_0)$ such that

$$j^*\overline{K_0} = \overline{\mathbb{Q}_l}[d] \text{ and } {}^pH^0(i^*\overline{K_0}) = {}^pH^0(i^!\overline{K_0}) = 0.$$

1.2 r-Multiplicity of a Closed Point

In this section we define the main object of study of this thesis; that is, the r-multiplicity of a closed singular point.

Let X_0 be a separated scheme of finite type over $\mathbb{F}_q$ of dimension d, and let F_{X_0} be its q-Frobenius; defined by the identity on X_0 and $f \mapsto f^q$ on $\mathcal{O}_{X_0}$. For any étale sheaf $\mathcal{F}_0$ on X_0, there is a canonical isomorphism

$$F_{\mathcal{F}_0} : F^*_{X_0}\mathcal{F}_0 \overset{\sim}{\to} \mathcal{F}_0$$

given by the Frobenius correspondence ([SGA 5 XIV]). Consider now the pullback to the algebraic closure of $\mathbb{F}_q$, i.e., $X := X_0 \times_{\mathbb{F}_q} \mathrm{Spec}(\overline{\mathbb{F}_q})$ and let $\mathcal{F}$ denote the pullback of $\mathcal{F}_0$ to X. Since F_{X_0} being finite is a proper morphism, there exists an induced endomorphism

$$Fr : H^i_c(X, \mathcal{F}) \overset{(F_{X_0}\times \mathrm{Id})^*}{\to} H^i_c(X, (F_{X_0} \times \mathrm{Id})^*\mathcal{F}) \overset{F_{\mathcal{F}_0}}{\to} H^i_c(X, \mathcal{F})$$

called the *Frobenius Endomorphism.* Let $K_0 \in D^b_c(X_0, \overline{\mathbb{Q}_l})$ and let K denote the pullback of K_0 to X. The Frobenius Endomorphism then extends to $\mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l})$:

$$Fr : \mathbb{H}^i_c(X, K) \to \mathbb{H}^i_c(X, K).$$

The *Frobenius substitution* $\varphi \in Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ is the automorphism $x \mapsto x^q$ on $\overline{\mathbb{F}_q}$. The *geometric Frobenius* $F \in Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ is the inverse of φ. Let $\mathcal{F}$ be a constructible $\overline{\mathbb{Q}_l}$-sheaf over X_0. For every closed point $x \in X_0$, the stalk $\mathcal{F}_{\overline{x}}$ is a finite dimensional $\overline{\mathbb{Q}_l}$-vector space ([19, §7 Théorème 1.1]) with an action of $Gal(\kappa(\overline{x})/\kappa(x))$. The field $\kappa(x)$ is a finite field with $q^{\deg(x)}$ elements, hence $F_x := F^{\deg(x)}$ can be considered as an element of $Gal(\kappa(\overline{x})/\kappa(x))$. Thus, the element $F_x \in Gal(\kappa(\overline{x})/\kappa(x))$ acts $\overline{\mathbb{Q}_l}$-linearly on the stalk $\mathcal{F}_{\overline{x}}$ of the sheaf $\mathcal{F}$ at the point $\overline{x}$

$$F_x : \mathcal{F}_{\overline{x}} \to \mathcal{F}_{\overline{x}}.$$

Up to isomorphism this action only depends on the closed point x and not on the choice of the geometric point $\overline{x}$ over x ([7, (1.1.8)]). It follows that $\mathrm{Tr}(F_x|\mathcal{F}_{\overline{x}}) \in \overline{\mathbb{Q}_l}$ is independent of the choice of $\overline{x}$.
Grothendieck's trace formula is the following result.

Theorem 1.2.1. *(Grothendieck [SGA5 XIV]) Let X_0 be a separated scheme of finite type over $\mathbb{F}_q$ and let $K_0 \in D^b_c(X_0, \overline{\mathbb{Q}_l})$. Then for all $\nu \geq 1$,*

$$\sum_n (-1)^n \mathrm{Tr}(Fr^\nu | \mathbb{H}^n_c(X, K)) = \sum_{x \in X_0(\mathbb{F}_{q^\nu})} \sum_i (-1)^i \mathrm{Tr}(F^\nu | \mathcal{H}^i(K_0)_{\overline{x}}).$$

Consequently, for X_0 equidimensional and $K_0 = IC_{X_0}$ we get the following equation:

$$\begin{aligned} a_r :&= \sum_n (-1)^n \mathrm{Tr}(Fr^r | \mathbb{H}^n_c(X, \pi^* IC_{X_0}[-d])) \\ &= \sum_{x \in X_0(\mathbb{F}_{q^r})} \sum_i (-1)^{i+d} \mathrm{Tr}(F^r | \mathcal{H}^i(IC_{X_0})_{\overline{x}}), \end{aligned}$$

where $\pi : X \to X_0$ is the projection.

Definition 1.2.2. Let X_0 be a separated equidimensional scheme of finite type over $\mathbb{F}_q$ of dimension d. Let $x \in X_0$ be a closed point. For $r \geq 1$, we define the *r-multiplicity of x in X_0* as follows:

$$m^r_{X_0,x} := \sum_i (-1)^{i+d} \mathrm{Tr}(F^r_x | \mathcal{H}^i(IC_{X_0})_{\overline{x}}).$$

Remark 1.2.3. (i) If $x \in X_0$ is a non-singular closed point of X_0, then $m^r_{X_0,x} = 1$ for all $r \geq 1$. Indeed, in this case we have $\mathcal{H}^i(IC_{X_0})_{\overline{x}} = 0$ for $i \neq -d$ and $\mathcal{H}^i(IC_{X_0})_{\overline{x}} = \overline{\mathbb{Q}_l}$ for $i = -d$ since $(IC_{X_0})_{\overline{x}} = \overline{\mathbb{Q}_l}[d]$.

(ii) By Corollary 1.5.17 below, each r-multiplicity of a closed singular point x in X is an integer. We assume this fact for the remainder of this chapter. They can be positive and negative; see Example 2.2.4.

(iii) We will prove later (Example 2.5.1) that the r-multiplicity of a singular closed point x in X_0 can also be equal to 1.

(iv) Since the normalization of X_0 is finite over X_0 and the direct image functor under finite morphisms is exact for the perverse t-structure ([3, Corollaire 2.2.6 (i)]), the intersection cohomology of X_0 is invariant under passage to the normalization. It follows that one may assume X_0 to be normal while calculating its intersection cohomology groups. Nontheless, the calculation of the r-multiplicities of a singular point is not invariant under normalization (see the next section). This is due to the fact that the r-multiplicity of a singular point is defined locally and not globally.

1.3 r-Frobenius Trace

We introduce the framework for calculating the r-multiplicity of a closed singular point x in a separated scheme X_0 of finite type over $\mathbb{F}_q$. Let k be a perfect field.

Convention 1.3.1. *Let X_0 be a scheme over a field k. We call X_0 an algebraic variety over k if X_0 is separated, reduced, equidimensional and of finite type as a scheme over k. We do not assume that X_0 is irreducible.*

By the convention above, the intersection complex IC_{X_0} is well-defined for every algebraic variety X_0.

Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension d. Let $K(X_0, \overline{\mathbb{Q}_l})$ be the Grothendieck group of constructible $\overline{\mathbb{Q}_l}$-sheaves. It is also the Grothendieck group of the abelian category of perverse $\overline{\mathbb{Q}_l}$-sheaves over X_0 ([23, (0.8)]). For a $\overline{\mathbb{Q}_l}$-sheaf $\mathcal{F}_0$ over X_0, let $[\mathcal{F}_0]$ be its class in $K(X_0, \overline{\mathbb{Q}_l})$. The map $\mathcal{F}_0 \mapsto [\mathcal{F}_0]$ extends to a surjective map

$$\begin{aligned}\mathrm{Ob}(\mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l})) &\to K(X_0, \overline{\mathbb{Q}_l})\\ K_0 &\mapsto [K_0] = \sum_i (-1)^i [\mathcal{H}^q(K_0)] = \sum_j (-1)^j [{}^pH^j(K_0)].\end{aligned}$$

For every $K_0 \in \mathrm{Ob}(\mathcal{D}_c^b(X_0,\overline{\mathbb{Q}_l}))$, let $K_{K_0}(X_0,\overline{\mathbb{Q}_l})$ be the subgroup of $K(X_0,\overline{\mathbb{Q}_l})$ generated by $[K_0]$.

Definition 1.3.2. Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension d. Let $|X_0|$ denote the set of closed points of X and let $\mathcal{C}(|X_0|,\overline{\mathbb{Q}_l})$ be the $\overline{\mathbb{Q}_l}$-algebra of maps $t : |X_0| \to \overline{\mathbb{Q}_l}$. One associates to every $K_0 \in \mathrm{Ob}(\mathcal{D}_c^b(X_0,\overline{\mathbb{Q}_l}))$ and every $r \geq 1$ its *r-Frobenius trace* $t^r_{K_0} \in \mathcal{C}(|X_0|,\overline{\mathbb{Q}_l})$ defined by:

$$t^r_{K_0}(x) := \sum_i (-1)^i \mathrm{Tr}(F^r_x | \mathcal{H}^i(K_0[-d])_{\overline{x}}), \text{ for all } x \in |X_0|.$$

We now mention some of the properties of the r-Frobenius trace ([23, §1]). For every $K'_0, K''_0 \in \mathcal{D}_c^b(X_0,\overline{\mathbb{Q}_l})$, we have

$$t^r_{K'_0 \otimes K''_0}(x) = t^r_{K'_0}(x) t^r_{K''_0}(x).$$

Let $f : X_0 \to Y_0$ be a morphism of algebraic varieties over $\mathbb{F}_q$. For any $L_0 \in \mathcal{D}_c^b(Y_0,\overline{\mathbb{Q}_l})$ and every $r \geq 1$, we have the following equation:

$$t^r_{f^* L_0}(x) = t^{r+\nu}_{L_0}(f(x)),$$

where $\nu = [\kappa(x) : \kappa(f(x))]$.
For an exact triangle

$$K'_0 \to K_0 \to K''_0 \xrightarrow{[1]}$$

in $\mathcal{D}_c^b(X_0,\overline{\mathbb{Q}_l})$, we have that

$$t^r_{K_0}(x) = t^r_{K'_0}(x) + t^r_{K''_0}(x).$$

It follows that for every $K \in \mathcal{D}_c^b(X,\overline{\mathbb{Q}_l})$ and every $r \geq 1$:

$$t^r_{K_0}(x) = \sum_i (-1)^i t^r_{\mathcal{H}^i(K_0)}(x) = \sum_j (-1)^j t^r_{{}^pH^j(K_0)}(x).$$

Thus, $m^r_{X_0,x} = t^r_{IC_{X_0}}(x)$ for any $x \in |X_0|$.

Remark 1.3.3. For a fixed $r \geq 1$, the r-Frobenius trace of a complex $K_0 \in \mathcal{D}_c^b(X_0,\overline{\mathbb{Q}_l})$ does not determine the complex K_0; the set $X_0(\mathbb{F}_{q^r})$ could be, for example, empty. On the other hand, the tuple $(t^r_{K_0})_{r\geq 1}$ in $\prod \mathcal{C}(X_0(\mathbb{F}_{q^r}),\overline{\mathbb{Q}_l}))$ of all Frobenius traces allow us to retrieve $[K_0]$ in

$K(X_0, \overline{\mathbb{Q}_l})$. To be more precise, the map $K_0 \mapsto (t^r_{K_0})_{r\geq 1}$ induces an injective ([23, Théorème 1.1.2]) group homomorphism

$$t_\bullet : K(X_0, \overline{\mathbb{Q}_l}) \to \prod_{r\geq 1} \mathcal{C}(X_0(\mathbb{F}_{q^r}), \overline{\mathbb{Q}_l}).$$

The following proposition then follows.

Proposition 1.3.4. *Let K_0', K_0'' be two perverse semi-simple $\overline{\mathbb{Q}_l}$-sheaves on X_0. If $t_\bullet[K_0'] = t_\bullet[K_0'']$, then K_0' and K_0'' are isomorphic.*

Proof. [23, Proposition 1.1.2.1] □

Thus, the intersection complex IC_{X_0} is characterized in $K(X_0, \overline{\mathbb{Q}_l})$ by the tuples $(m^r_{X_0,x})_{r\geq 1}$ for all $x \in |X_0|$.

We finish this section by giving a description of the r-multiplicity of a closed singular point x in an algebraic variety X_0 with zero-dimensional singular locus in terms of the strict henselization of X_0 at x. Let R be a local ring. We denote by R^{sh} the strict henselization of R.

Lemma 1.3.5. *Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension d. If the singular locus of X_0 is zero-dimensional, then for any $x \in |(X_0)_{sing}|$ and $r \geq 1$,*

$$m^r_{X_0,x} = \sum_{i=-d}^{-1} (-1)^{i+d} \mathrm{Tr}(Fr^\eta | H^{i+d}(\mathrm{Spec}(\mathcal{O}^{sh}_{X_0,\overline{x}_0}) \times U, \overline{\mathbb{Q}_l})),$$

where $\eta = \deg(x)r$ and U is an open dense smooth subscheme of X_0.

Proof. By [3, Proposition 2.1.11], the intersection complex of X_0 has the following form:

$$IC_{X_0} \cong \tau_{\leq -1} j_* \overline{\mathbb{Q}_l}[d],$$

where $j : U \hookrightarrow X_0$ is an open immersion of a dense smooth subscheme of X_0. Thus, it is enough to calculate $t^r_{j_*\overline{\mathbb{Q}_l}[d]}(x_0)$ in order to calculate $t^r_{IC_{X_0}}(x_0) = m^r_{X_0,x}$. Now consider the isomorphism

$$(j_*\overline{\mathbb{Q}_l}[d])_{\overline{x}} \cong R\Gamma(\mathrm{Spec}(\mathcal{O}^{sh}_{X_0,\overline{x}}) \times U, \overline{\mathbb{Q}_l}[d]),$$

which implies that

$$m^r_{X_0,x} = t^r_{\tau_{\leq -1} j_* \overline{\mathbb{Q}_l}[d]}(x) = \sum_{i=-d}^{-1} (-1)^{i+d} \mathrm{Tr}(Fr^\eta | H^{i+d}(\mathrm{Spec}(\mathcal{O}^{sh}_{X_0,\overline{x}}) \times U, \overline{\mathbb{Q}_l})),$$

where $\eta = \deg(x) r$. □

1.4 Reduction Steps

We discuss reduction steps for the calculation of the r-multiplicity of a closed singular point in an algebraic variety X_0.

Let $(X_0)_{sing}$ be the singular locus of X_0. Consider the canonical immersion $i : (X_0)_{sing} \to X_0$ and let $x \in |(X_0)_{sing}|$. Then we have the following equality:

$$m^r_{X_0,x} = t^r_{i^* IC_{X_0}}(x).$$

Consider now the normalization $\pi : \tilde{X}_0 \to X_0$ of X_0. It is the disjoint union of the normalizations of the irreducible components of X_0, i.e., $\tilde{X}_0 = \sqcup_i \tilde{Z}_i$, where $\tilde{Z}_i \to Z_i$ is the normalization of the i-th irreducible component of X_0.

In what follows, we will consider fibers of closed points of algebraic varieties under their normalization. We show that points lying over closed points are again closed.

Lemma 1.4.1. *Let X_0 be an algebraic variety over k and let $\pi : \tilde{X}_0 \to X_0$ be its normalization. For every $x \in |X_0|$, let $y \in \tilde{X}_0$ be such that $\pi(y) = x$. Then $y \in |\tilde{X}_0|$.*

Proof. Let $U := \mathrm{Spec}(A)$ be an affine neighborhood of x. Then x represents a prime ideal p of A and $\tilde{X}_0 \times_{X_0} U \cong \mathrm{Spec}(A')$. Since X_0 is equidimensional, one may assume X_0 to be integral and A' to be the integral closure of A in $\mathrm{Frac}(A)$. Consider the injective canonical morphism $\pi^\# : A \hookrightarrow A'$, which is integral. We need to show that every prime ideal q of $(A)'$ such that $(\pi^\#)^{-1}(q) = p$ is a maximal ideal of $(A)'$. This is true by [33, Lemma 10.35.20]. □

According to Grothendieck's trace formula (Theorem 1.2.1), for $K_0 \in \mathcal{D}_c^b(\tilde{X}_0, \overline{\mathbb{Q}_l})$ and $x \in |X_0|$ one gets the following equality:

$$t^r_{\pi_! K_0}(x) = \sum_{\substack{y \in \tilde{X}_0(\mathbb{F}_{q^\eta}) \\ \pi(y)=x}} \sum_i (-1)^i \mathrm{Tr}(F^\eta | \mathcal{H}^i(K_0)_{\overline{y}}), \tag{1.1}$$

where $\eta = \deg(x)r$. Since the normalization $\pi : \tilde{X}_0 \to X_0$ over $\mathbb{F}_q$ is finite, the sum above is finite.

Lemma 1.4.2. *Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension d. For any open subscheme U' of X_0 with canonical immersion $j' : U' \to X_0$,*

$$m^r_{X_0,x} = m^r_{U',x}, \text{ for all } x \in U'.$$

Proof. Let U be a dense smooth open subscheme of X_0. Consider the following cartesian diagram:

$$\begin{array}{ccccc} Y \cap U' & \xhookrightarrow{i''} & U' & \xhookleftarrow{j''} & U \cap U' \\ \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j'} & & \downarrow{\scriptstyle j'} \\ Y & \xhookrightarrow{i} & X_0 & \xhookleftarrow{j} & U \end{array} \quad \text{with } Y = X_0 \setminus U.$$

We have the following isomorphisms:

$$j''^* j'^* IC_{X_0} = j'^* j^* IC_{X_0} = \overline{\mathbb{Q}_l}[d].$$
$${}^pH^0(i'^* j'^* IC_{X_0}) = {}^pH^0(j'^* i^* IC_{X_0}) = j'^* {}^pH^0(i^* IC_{X_0}) = 0, \text{ since } {}^pH^0(i^* IC_{X_0}) = 0.$$
$${}^pH^0(i'^! j'^! IC_{X_0}) = {}^pH^0(j'^! i^! IC_{X_0}) = j'^! {}^pH^0(i^! IC_{X_0}) = 0, \text{ since } {}^pH^0(i^! IC_{X_0}) = 0.$$

Since $U \cap U' \neq \emptyset$, we have that $U \cap U'$ is a dense smooth open subscheme of U'. This means that $j'^* IC_{X_0} = IC_{U'}$ and the lemma is proved. □

Lemma 1.4.3. *Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension d and let $\pi : \tilde{X}_0 \to X_0$ be the normalization of X_0. For any $x \in |X_0|$ and $r \geq 1$, we have that*

$$m^r_{X_0,x} = \sum_{\substack{y \in \tilde{X}_0(\mathbb{F}_{q^\eta}) \\ \pi(y)=x}} m^\nu_{\tilde{X}_0,y},$$

where $\eta = \deg(x)r$ and $\nu = [\mathbb{F}_{q^\eta} : \kappa(y)]$. For any $y \in |\tilde{X}_0|$, let $\tilde{Z}_i$ be an irreducible component of $\tilde{X}_0$ containing y. Then for $r \geq 1$

$$m^r_{\tilde{X}_0,y} = m^r_{\tilde{Z}_i,y}.$$

Proof. 1. By [3, Corollaire 2.2.6 (i)], we have $\pi_* IC_{\tilde{X}_0} = IC_{X_0}$. Thus, letting $K_0 = IC_{\tilde{X}_0}$ in equation (1.1) yields the first assertion of the lemma.

2. This is a direct consequence of Lemma 1.4.2.

□

Corollary 1.4.4. *Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension d and let $\pi : \tilde{X}_0 \to X_0$ be the normalization of X_0. Let $\tilde{X}_0 = \sqcup_i \tilde{Z}_i$ be its decomposition into irreducible components. Then*

1. The canonical map:

$$K_{IC_{\tilde{X}_0}}(\tilde{X}_0, \overline{\mathbb{Q}_l}) \to K_{IC_{X_0}}(X_0, \overline{\mathbb{Q}_l})$$
$$[K_0] \mapsto [\pi_* K_0]$$

is surjective.

2. The canonical map:

$$\bigoplus_i K_{IC_{\tilde{Z}_i}}(\tilde{Z}_i, \overline{\mathbb{Q}_l}) \to K_{IC_{\tilde{X}_0}}(\tilde{X}_0, \overline{\mathbb{Q}_l})$$
$$\bigoplus_i [(K_0)_i] \mapsto \bigoplus_i [\alpha_{i*}(K_0)_i]$$

is surjective.

It follows that we may reduce the calculation of the r-multiplicity of x in X_0 to an irreducible component of $\tilde{X}_0$; any irreducible component containing $y \in |\tilde{X}_0|$, such that $\pi(y) = x$, will suffice. Thus, we may always assume that X_0 is an integral normal algebraic variety over $\mathbb{F}_q$ of dimension d while calculating $m^r_{X_0,x}$.

Let $x \in |X_0|$ be a singular point and let W be an affine open neighborhood of x. By Lemma 1.4.2, it is enough to calculate the r-multiplicity of x in W. Thus, we may assume that X_0 is in addition affine.

Lemma 1.4.5. *Let W be an affine integral algebraic variety over $\mathbb{F}_q$ of dimension d. There exists $g \in \mathcal{O}_W(W)$ such that $g \neq 0$, $Z := V(g) \supseteq W_{sing}$ and $U := W \setminus Z$ is an open dense smooth subscheme of W.*

Proof. Since W is an affine algebraic variety over k, there exists $n \in \mathbb{N}$ such that $W = V(I)$ as a closed subvariety of $\mathbb{A}^n_k$. The singular locus W_{sing} has then a canonical scheme structure given by the d-th Fitting ideal I_d of $\Omega^1_{W/k}$, which is the ideal in $\mathcal{O}_W$ generated by the $(n-d) \times (n-d)$ minors of the Jacobi matrix. Choose any element $g \neq 0$ of $\Gamma(W, I_d)$. Then $\langle g \rangle \subseteq I_d(W)$. This shows the relation

$$Z = V(g) \supseteq V(I_d(W)) = W_{sing}. \tag{1.2}$$

Since W is integral, the subscheme U is dense in W. By the relation (1.2), the scheme U does not contain any singular points. It follows that U is smooth and dense. □

1.5 Integrality and Independence of *l*

In this section we show that the r-multiplicity of a singular closed point x of an algebraic variety X_0 is an integer independent of l. We thank T. Saito for suggesting Prof. Deninger some of the literature relevant to this section.

Definition 1.5.1. Let X_0 be an algebraic variety over $\mathbb{F}_q$ and consider $K_0 \in \mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l})$. The complex K_0 is called *integral* if for each closed point $x \in |X_0|$ and each $i \in \mathbb{Z}$, the eigenvalues of the geometric Frobenius F_x acting on $\mathcal{H}^i(K_0)_{\overline{x}}$ are algebraic integers.

Remark 1.5.2. The integral property of a complex $K_0 \in \mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l})$ is stable under $\otimes^L$ and f^* ([SGA 7, XXIa, 5.2.2]).

Example 1.5.3. For any $m \in \mathbb{Z}$, consider the complex $\overline{\mathbb{Q}_l}[m] \in \mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l})$. For any $x \in |X_0|$, we have

$$\begin{aligned} \mathcal{H}^i(\overline{\mathbb{Q}_l}[m])_{\overline{x}} &= 0 \quad \text{if } i \neq -m \\ \mathcal{H}^i(\overline{\mathbb{Q}_l}[m])_{\overline{x}} &= \overline{\mathbb{Q}_l} \quad \text{if } i = -m. \end{aligned}$$

Thus, the geometric Frobenius F_x operates on $\mathcal{H}^{-m}(\overline{\mathbb{Q}_l}[m])_{\overline{x}}$ as the identity. It follows that the complex $\overline{\mathbb{Q}_l}[m] \in \mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l})$ is integral.

We prove that the intersection complex IC_{X_0} is integral using the following result by Deligne ([SGA 7, XXI, Appendice]).

Theorem 1.5.4. *Let $f : X_0 \to Y_0$ be a morphism between separated of finite type $\mathbb{F}_q$-schemes, and let $K_0 \in \mathcal{D}_c^b(X_0, \overline{\mathbb{Q}_l})$ be an integral complex. Then $f_! K_0$ and $f_* K_0$ are integral as well.*

Remark 1.5.5. The theorem can be found stated in this form in [17, Theorem 4.2].

Theorem 1.5.6. *Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension d. The intersection complex IC_{X_0} of X_0 is integral.*

Proof. The intersection complex IC_{X_0} is isomorphic in $\mathcal{D}_c^b(X_0, \overline{\mathbb{Q}_l})$ to a composition of pushforwards and truncations for a given stratification of X_0 ([3, Proposition 2.1.11]), i.e.,

$$IC_{X_0} \cong \tau_{\leq -1}(j_{-1})_* \dots \tau_{\leq -d}(j_d)_* \overline{\mathbb{Q}_l}[d].$$

For all $m \in \mathbb{Z}$ and any $K_0 \in \mathcal{D}_c^b(X_0, \overline{\mathbb{Q}_l})$ we have that

$$\mathcal{H}^i(\tau_{\leq m} K_0) = \begin{cases} \mathcal{H}^i(K_0) & \text{for } i \leq m, \\ 0 & \text{for } i > m, \end{cases}$$

for all $i \in \mathbb{Z}$. We apply now iteratively Theorem 1.5.4 and conclude that the intersection complex IC_{X_0} is integral. □

Proposition 1.5.7. *Let X_0 be an algebraic variety over $\mathbb{F}_q$ and let $x \in |X_0|$ be a singular point. For every integral $K_0 \in \mathcal{D}_c^b(X_0, \overline{\mathbb{Q}_l})$, the number*

$$\sum_i (-1)^i \mathrm{Tr}(F_x^r | \mathcal{H}^i(K_0)_{\overline{x}})$$

is an algebraic integer for all $r \geq 1$.

Proof. Since the complex K_0 is integral, the characteristic polynomial of the geometric Frobenius acting on $\mathcal{H}^i(K_0)_{\overline{x}}$ has integer coefficients for all $i \in \mathbb{Z}$. This implies that $\mathrm{Tr}(F_x | \mathcal{H}^i(K_0)_{\overline{x}})$ is an integer for all $i \in \mathbb{Z}$. It follows that $t_{K_0}(x)$ is an algebraic integer.
Set $\nu \geq 1$. Let $(X_0)_\nu$ be the base change of X_0 from $\mathrm{Spec}(\mathbb{F}_q)$ to $\mathrm{Spec}(\mathbb{F}_{q^{\deg(x)\nu}})$ and let $\pi_\nu : (X_0)_\nu \to X_0$ be the projection. For all $x \in |X_0|$ and all $i \in \mathbb{Z}$, we have that

$$\mathcal{H}^i(\pi_\nu^* K_0)_{\overline{y}} = \mathcal{H}^i(K_0)_{\overline{x}},$$

where $\pi_\nu(y) = x$. Since the complex $\pi_\nu^* K_0$ is integral, the characteristic polynomial of the geometric Frobenius $F_y = F_x^\nu$ acting on $\mathcal{H}^i(K_0)_{\overline{x}}$ has integer coefficients for all $i \in \mathbb{Z}$. It follows that $\sum_i (-1)^i \mathrm{Tr}(F^\eta | \mathcal{H}^i(K_0)_{\overline{x}})$ is an algebriac integer and the proposition is proved. □

Corollary 1.5.8. *Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension d and let $x \in |X_0|$ be a singular point. Then for all $r \geq 1$,*

$$m_{X_0,x}^r = \sum_i (-1)^{i+d} \mathrm{Tr}(F_x^r | \mathcal{H}^i(IC_{X_0})_{\overline{x}})$$

is an algebraic integer.

Proof. Apply Proposition 1.5.7 to the intersection complex IC_{X_0}. □

We now show that the r-multiplicity of a singular point x of an algebraic variety X_0 over $\mathbb{F}_q$ is independent of l. We therefore introduce the concept of (E, I)-compatibility. For this part we follow [34].

Let E be a number field, and let I be a set of pairs (l, ι); where l is a rational prime number different from the prime number $p := char(\mathbb{F}_q)$, and $\iota : E \hookrightarrow \overline{\mathbb{Q}_l}$ is an embedding of fields. Note that, to give an embedding $E \to \overline{\mathbb{Q}_l}$ is the same as giving a finite place λ of E over l.

Definition 1.5.9. Let X_0 be an $\mathbb{F}_q$-scheme. For each $(l, \iota) \in I$, consider $(K_0)_{(l,\iota)} \in \mathcal{D}_c^b(X_0, \overline{\mathbb{Q}_l})$.

(i) The system $\{(K_0)_{(l,\iota)}\}_I$ is called an *(E, I)-compatible system*, if for every integer $r \geq 1$ and for every point $x \in X_0(\mathbb{F}_{q^r})$, there exists a number $e_x \in E$ such that

$$\iota(e_x) = t_{(K_0)_{(l,\iota)}}(x),$$

for all $(l, \iota) \in I$.

(ii) Assume $(K_0)_{(l,\iota)} = P_{(l,\iota)}$ are perverse sheaves. The system $\{P_{(l,\iota)}\}_I$ is called *perverse (E, I)-compatible*, if there exist a finite number of essentially smooth irreducible locally closed subschemes $X_\alpha \hookrightarrow X_0$, and for each α a (E, I)-compatible system $\{L_\alpha^{(l,\iota)}\}_I$ of semisimple local systems on X_α, such that each irreducible factor of $L_\alpha^{(l,\iota)}$ has X_α as its maximal support (inside $\overline{X}_\alpha$), and that

$$P_{(l,\iota)}^{ss} \cong \bigoplus_\alpha IC_{\overline{X}_\alpha}(L_\alpha^{(l,\iota)})$$

for all $(l,\iota) \in I$. Here $P_{(l,\iota)}^{ss}$ denotes the semi-simplification of $P_{(l,\iota)}$ in the abelian category of perverse sheaves.

Example 1.5.10. The system $\{\overline{\mathbb{Q}}_{l(l,\iota)}\}_I$ is a $(\mathbb{Q}, I)$-compatible system. Generally, for every $b \in \mathbb{Q}^\times$ such that $\iota(b)$ is an l-adic unit for all $(l,\iota) \in I$, the system $\{\overline{\mathbb{Q}}_{l(l,\iota)}^{(\iota(b))}\}_I$ is a $(\mathbb{Q}, I)$-compatible system, since the local traces are $\iota(b)$. Here $\overline{\mathbb{Q}}_l^{(\iota(b))}$ denotes the sheaf $\overline{\mathbb{Q}}_l$ twisted by $\iota(b)$ ([7, (1.2.7)]).

Theorem 1.5.11. *Let $\{(K_0)_{(l,\iota)}\}_I$ be an (E, I)-compatible system. Then for a morphism $f : Y_0 \to X_0$ of separated schemes of finite type over $\mathbb{F}_q$, $\{f_!(K_0)_{(l,\iota)}\}_I$, $\{f_*(K_0)_{(l,\iota)}\}_I$ are also (E, I)-compatible systems. Similar results hold for $f^*, f^!, \otimes^L, Hom, D$.*

Proof. [12, Theorem 2]. □

Definition 1.5.12. Let X_0 be a separated scheme of finite type over $\mathbb{F}_q$ and let $\mathcal{F}$ be a $\overline{\mathbb{Q}}_l$-sheaf over X_0. The $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{F}$ is called *pointwise pure of weight* w ($w \in \mathbb{Z}$) if for every $n \geq 1$ and every $x \in X_0(\mathbb{F}_{q^n})$, the eigenvalues of the geometric Frobenius F_x acting on $\mathcal{F}_{\overline{x}}$ are algebraic numbers where all of the complex conjugates have absolute value $(q^n)^{w/2}$. The $\overline{\mathbb{Q}}_l$-sheaf $\mathcal{F}$ is called *mixed* if it admits a finite filtration with pointwise pure quotients. The non-zero weights of the quotients are called the *pointwise weights* of $\mathcal{F}$.

A complex $K_0 \in \mathcal{D}_c^b(X_0, \overline{\mathbb{Q}}_l)$ is called of *weight* $\leq w$ if for each i, the pointwise weights of $\mathcal{H}^i K_0$ are $\leq w + i$. Further, a complex $K_0 \in \mathcal{D}_c^b(X_0, \overline{\mathbb{Q}}_l)$ is of *weight* $\geq w$ if its Verdier dual DK_0 is of weight $\leq -w$. Finally, a complex $K_0 \in \mathcal{D}_c^b(X_0, \overline{\mathbb{Q}}_l)$ is called *pure of weight* w if it is of weight $\leq w$ and $\geq w$.

Example 1.5.13. Let X_0 be a smooth algebraic variety over $\mathbb{F}_q$. Then the $\overline{\mathbb{Q}}_l$-sheaf $\overline{\mathbb{Q}}_l$ over X_0 is pointwise pure of weight 0. This implies that $\overline{\mathbb{Q}}_l[z] \in \mathcal{D}_c^b(X_0, \overline{\mathbb{Q}}_l)$, $z \in \mathbb{Z}$, is pure of weight z.

Theorem 1.5.14. *Let $j : U_0 \to X_0$ be an open immersion of separated schemes of finite type over $\mathbb{F}_q$ and let $\{(K_0)_{(l,\iota)}\}_I$ be an (E, I)-compatible system on U_0. Assume that each $(K_0)_{(l,\iota)}$ is pure and perverse. Then the system of intermediate extensions $\{j_{!*}(K_0)_{(l,\iota)}\}_I$ is also an (E, I)-compatible system.*

Proof. [12, Theorem 3]. □

Corollary 1.5.15. *Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension d and let $x \in |X_0|$ be a singular point. Moreover, let $\iota : \mathbb{Q} \to \overline{\mathbb{Q}_l}$ be the canonical embedding. Then for all $r \geq 1$ and every l, there exists a number $e_x \in \mathbb{Q}$ such that*

$$m^r_{X_0,x} = \iota(e_x).$$

Proof. By Example 1.5.13, the complex $\overline{\mathbb{Q}_l}[d]$ is pure for every l. Theorem 1.5.14 shows that the system $\{(IC_{X_0})_{(l,\iota)}\}_I = \{j_{!*}\overline{\mathbb{Q}_l}_{(l,\iota)}[d]\}_I$, for $j : U_0 \to X_0$ an open immersion of a dense and smooth subscheme of X_0, is a $(\mathbb{Q}, I)$-compatible system. Set $\nu \geq 1$. Let $(X_0)_\nu$ denote the base change of X_0 from $\mathrm{Spec}(\mathbb{F}_q)$ to $\mathrm{Spec}(\mathbb{F}_{q^{\deg(x)\nu}})$ and let $\pi_\nu : (X_0)_\nu \to X_0$ be the projection. By Theorem 1.5.11, the system $\{\pi^*_\nu(IC_{X_0})_{(l,\iota)}\}_I$ is a $(\mathbb{Q}, I)$-compatible system for all $\nu \geq 1$. The corollary now follows. □

This proves the independence of l of the r-multiplicity of x in X_0. We actually proved that the system $\{(IC_{X_0})_{(l,\iota)}\}_I$ is a system of mixed perverse sheaves on X_0 that is $(\mathbb{Q}, I)$-compatible.
The following result implies that the system $\{(IC_{X_0})_{(l,\iota)}\}_I$ is also a perverse $(\mathbb{Q}, I)$-compatible system.

Proposition 1.5.16. *Let $\{P_{(l,\iota)}\}_I$ be a system of mixed perverse sheaves on an $\mathbb{F}_q$-scheme X_0. If they are (E, I)-compatible, then they are perverse (E, I)-compatible.*

Proof. [34, Proposition 2.7] □

Corollary 1.5.17. *Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension d and let $x \in |X_0|$ be a singular point. Then for all $r \geq 1$,*

$$m^r_{X_0,x} = \sum_i (-1)^{i+d} \mathrm{Tr}(F^r_x | \mathcal{H}^i(IC_{X_0})_{\overline{x}})$$

is an integer.

Proof. Apply Proposition 1.5.7 to the intersection complex IC_{X_0}. By Corollary 1.5.15, the algebraic integers $m^r_{X_0,x}$ are rationals. The corollary now follows. □

We recall the definition of the intersection Zeta function

Definition 1.5.18. Let X_0 be a proper variety over a finite field $\mathbb{F}_q$ of dimension d. Let X be the base change of X_0 to the algebraic closure of $\mathbb{F}_q$ with projection $\pi : X \to X_0$. The *intersection Zeta function of* X_0 is defined as follows:

$$IZ(X_0,t) := \exp\left(\sum_{n=1}^{\infty} \frac{a_n}{n} t^n\right),$$

where $a_n := \sum_{i=0}^{2d}(-1)^i \mathrm{Tr}(Fr^n | IH^i(X, \overline{\mathbb{Q}_l}))$.

For X_0 a proper integral algebraic variety over $\mathbb{F}_q$ of dimension d, the results above (Proposition 1.5.7 and Corollary 1.5.15) prove that the intersection Zeta function of X_0 is a power series with rational coefficients. As with the regular Zeta function of X_0 we then conclude that the intersection Zeta function $IZ(X_0,t)$ is a rational function. Moreover, our definition of the intersection Zeta function of X_0 shows immediately that $IZ(X_0,t)$ is a quotient of polynomials with $\overline{\mathbb{Q}_l}$ coefficients after applying the following lemma.

Lemma 1.5.19. *Let ϕ be an endomorphism of a finite-dimensional vector space V over a field K. Then we have an identity of formal power series in t, with coefficients in K,*

$$exp\left(\sum_{n=1}^{\infty} \mathrm{Tr}(\phi^n|V)\frac{t^n}{n}\right) = det(1-\phi t|V)^{-1}.$$

Proof. [15, Appendix C Lemma 4.1]. □

Theorem 1.5.20. *Let X_0 be a proper algebraic variety over $\mathbb{F}_q$ of dimension d. The intersection Zeta function $IZ(X_0,t)$ is a rational function of t. Moreover, we have*

$$IZ(X_0,t) = \frac{P_1(X_0,t)P_3(X_0,t)\dots P_{2d-1}(X_0,t)}{P_0(X_0,t)P_2(X_0,t)\dots P_{2d}(X_0,t)},$$

where

$$P_i(X_0,t) = det(1 - t\ \mathrm{Fr}|IH^i(X,\overline{\mathbb{Q}_l}))$$

such that $P_0(X_0,t) = 1-t, P_{2d}(X_0,t) = 1-q^dt$ *and each* $P_i(X_0,t)$ *is a polynomial with coefficients in* $\mathbb{Z}$*.*

Proof. The definition of each polynomial shows that $P_0(X_0,t) = 1-t$. The Frobenius morphism Fr is a finite morphism of degree q^d. Hence it acts as multiplication by q^d on a generator of $IH^{2d}(X,\overline{\mathbb{Q}_l})$. So $P_{2d}(X_0,t) = 1-q^dt$. The only thing left to prove is that each $P_i(X_0,t)$ has coefficients in $\mathbb{Z}$. The intersection complex IC_{X_0} is pure of weight d (Example 1.5.13 and [3, Corollaire 5.3.2]). This means that for each $i \in \mathbb{Z}$ the eigenvalues of the Frobenius Endomorphism Fr acting on $IH^i(X,\mathbb{Q}_l)$ are algebraic numbers where all of the complex conjugates have absolute value $q^{i/2}$. We then use the same argument given by Deligne for the Zeta function in the case where X_0 is smooth and projective ([6, preuve de (1.7) $\Rightarrow$ (1.6)]). □

Remark 1.5.21. In [12] Gabber proves that each polynomial $P_i(X_0,t)$ is independent of l. If we do not assume X_0 to be equidimesional, then the independence of l of each polynomial $P_i(X_0,t)$ is not known in general.

1.6 Estimate

We finish this chapter by giving a general estimate of the r-multiplicity of a singular closed point in an algebraic variety using the theory of weights on $\mathcal{D}_c^b(X_0,\overline{\mathbb{Q}_l})$.

Proposition 1.6.1. *Let* $f : X_0 \to Y_0$ *be a quasi-finite morphism of algebraic varieties over* $\mathbb{F}_q$*. Then the intermediate extension functor* $f_{!*}$ *transforms perverse sheaves of weight* $\leq w$ *(resp.* $\geq w$*) over* X *into perverse sheaves of weight* $\leq w$ *(resp.* $\geq w$*) over* Y_0*. In particular it transforms pure perverse sheaves over* X *into pure perverse sheaves over* Y_0*.*

Proof. This is a consequence of [3, Corollaire 5.4.3]. □

Corollary 1.6.2. *Let* X_0 *be an algebraic variety over* $\mathbb{F}_q$ *of dimension* d*. The intersection complex* IC_{X_0} *is pure of weight* d*.*

Proof. The intersection complex IC_{X_0} is per definition the intermediate extension of $\overline{\mathbb{Q}_l}[d]$ under an open immersion, which is then a quasi-finite morphism between algebraic varieties over $\mathbb{F}_q$. Since $\overline{\mathbb{Q}_l}[d]$ is pure of weight d, it follows that the intersection complex IC_{X_0} is pure of weight d. □

Theorem 1.6.3. *Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension d and let $x \in |X_0|$ be a singular point. Then for any $r \geq 1$ and $\eta = \deg(x)r$,*

$$|m^r_{X_0,x}| \leq \sum_{i=-d}^{-1} \dim(\mathcal{H}^i(IC_{X_0})_{\overline{x}})\ (q^\eta)^{\frac{i+d}{2}}.$$

Proof. The intersection complex IC_{X_0} is isomorphic in $\mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l})$ to a composition of pushforwards and truncations for a given stratification of X_0 ([3, Proposition 2.1.11]), i.e.,

$$IC_{X_0} \cong \tau_{\leq -1}(j_{-1})_* \dots \tau_{\leq -d}(j_d)_* \overline{\mathbb{Q}_l}[d].$$

This shows that $IC_{X_0} \in \mathcal{D}^{[-d,-1]}_c(X_0, \overline{\mathbb{Q}_l})$. Since IC_{X_0} is pure of weight d, the cohomology sheaves $\mathcal{H}^i(IC_{X_0})$ are of weight $i+d$ for every $-d \leq i \leq -1$. Let $\alpha_{i_h,r}$ be the h-th eigenvalue of the geometric Frobenius F_x composed r times acting on $\mathcal{H}^i(IC_{X_0})_{\overline{x}}$. Thus,

$$\begin{aligned}
|m^r_{X_0,x}| &= \Big| \sum_{i=-d}^{-1} (-1)^{i+d} t^r_{\mathcal{H}^i(IC_{X_0})}(x) \Big| \\
&\leq \sum_{i=-d}^{-1} \Big| \sum_{h=1}^{\dim(\mathcal{H}^i(IC_{X_0})_{\overline{x}})} \alpha_{i_h,r} \Big| \\
&\leq \sum_{i=-d}^{-1} \dim(\mathcal{H}^i(IC_{X_0})_{\overline{x}}))\ (q^\eta)^{\frac{i+d}{2}},
\end{aligned}$$

where $\eta = deg(x)r$. □

The theorem implies that knowing the dimension of the $\overline{\mathbb{Q}_l}$-vector spaces given by $\mathcal{H}^i(IC_{X_0})_{\overline{x}}$ for each $-d \leq i \leq -1$, leads to estimates of the r-multiplicity of x in X_0.

Chapter 2

Low-dimensional Computations

Let X_0 be an algebraic variety over $\mathbb{F}_q$. In this chapter we calculate the r-multiplicity of a closed singular point x in X_0 for the three cases where resolution of singularities is known; that is, when X_0 has dimension 1,2 or 3.

Definition 2.0.1. Let X_0 be a reduced (locally) Noetherian scheme. A proper birational morphism $\pi : \tilde{X_0} \to X_0$ with $\tilde{X_0}$ regular is called a *desingularization of* X_0 (or a *resolution of singularities of* X_0). If the morphism π is an isomorphism above every regular point of X_0, we say it is a *desingularization in the strong sense.*

A *strict normal crossings divisor on* X_0 is an effective Cartier divisor $D_0 \subset X_0$ such that for every $p \in D_0$ the local ring $\mathcal{O}_{X_0,p}$ is regular and there exists a regular system of parameters $x_1, \ldots, x_d \in \mathfrak{m}_p$ and $1 \leq r \leq d$ such that D_0 is cut out by $x_1 \ldots x_r$ in $\mathcal{O}_{X_0,p}$.

For any strict normal crossings divisor on X_0 we have that

1. for any $s \in D_0$ the local ring $\mathcal{O}_{X_0,s}$ is regular,

2. the scheme D_0 is reduced, i.e., $D_0 = \bigcup(E_0)_{i\in I}$ (scheme-theoretically), where $(E_0)_i$ are the reduced irreducible components of D_0, and

3. for any non-empty subset $J \subset I$, the closed subscheme $D_J = \bigcap_{j\in J}(E_0)_j$ is a regular scheme of codimension $|J|$ in X_0.

2.1 Dimension 1

In this section we calculate the r-multiplicity of a closed singular point x in an algebraic variety X_0 of dimension 1.

Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension 1. Let $\pi : \tilde{X}_0 \to X_0$ be the normalization of X_0, i.e., a desingularization in the strong sense of X_0. Moreover, for a separated scheme Y_0 of finite type over $\mathbb{F}_q$ let F_{Y_0} be its q-Frobenius.

Lemma 2.1.1. *Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension 1. Let $x \in |X_0|$. Then for any $r \geq 1$,*

$$m^r_{X_0,x} = |\{y \in \pi^{-1}_{\overline{\mathbb{F}_q}}(x) : (F^{\eta}_{\pi^{-1}(x)} \times \mathrm{Id})(y) = y\}|,$$

where $\pi^{-1}_{\overline{\mathbb{F}_q}}(x)$ denotes the base change of $\pi^{-1}(x)$ to the algebraic closure of $\mathbb{F}_q$ and $\eta = \deg(x)r$.

Proof. By Lemma 1.4.3 and Theorem 1.2.1, we have that

$$m^r_{X_0,x} = \mathrm{Tr}(Fr^{\eta}|\mathbb{H}^0(\pi^{-1}_{\overline{\mathbb{F}_q}}(x), \overline{\mathbb{Q}_l})),$$

where $\eta = \deg(x)r$. The cohomology group $\mathbb{H}^0(\pi^{-1}_{\overline{\mathbb{F}_q}}(x), \overline{\mathbb{Q}_l})$ is the direct sum of copies of $\overline{\mathbb{Q}_l}$ indexed by the connected components of $\pi^{-1}_{\overline{\mathbb{F}_q}}(x)$, i.e.,

$$\mathbb{H}^0(\pi^{-1}_{\overline{\mathbb{F}_q}}(x), \overline{\mathbb{Q}_l}) = \bigoplus_{(E)_i} \overline{\mathbb{Q}_l},$$

where E_i is a connected component of $\pi^{-1}_{\overline{\mathbb{F}_q}}(x)$. The q-Frobenius $F_{\pi^{-1}(x)}$ is a universal homeomorphism ([33, Lemma 32.35.3]). It follows that the morphism

$$F_{\pi^{-1}(x)} \times \mathrm{Id} : \pi^{-1}_{\overline{\mathbb{F}_q}}(x) \to \pi^{-1}_{\overline{\mathbb{F}_q}}(x),$$

which defines the geometric Frobenius correspondence ([SGA 5, XIII]), permutes the connected components of $\pi^{-1}_{\overline{\mathbb{F}_q}}(x)$. The fixed connected components of $\pi^{-1}_{\overline{\mathbb{F}_q}}(x)$ under $(F^{\eta}_{\pi^{-1}(x)} \times \mathrm{Id})$ correspond to points $y \in \pi^{-1}_{\overline{\mathbb{F}_q}}(x)(\mathbb{F}_{q^{\eta}})$. Thus,

$$\mathrm{Tr}(Fr^{\eta}|\mathbb{H}^0(\pi^{-1}_{\overline{\mathbb{F}_q}}(x), \overline{\mathbb{Q}_l})) = |\{y \in \pi^{-1}_{\overline{\mathbb{F}_q}}(x) : (F^{\eta}_{\pi^{-1}(x)} \times \mathrm{Id})(y) = y\}|.$$

□

Remark 2.1.2. The lemma shows that in dimension 1 the r-multiplicity of a closed singular point $x \in |X_0|$ is a positive integer independent of l.

In the following example, we show that for r big enough, the r-multiplicity of a singularity can be arbitrarily big.

Example 2.1.3. Consider the scheme

$$X_0 := \operatorname{Spec}\left(\mathbb{F}_{q^n}[x,y]/\prod_{\alpha\in\mathbb{F}_{q^n}}\langle x-\alpha y\rangle\right),$$

where $n \in \mathbb{N}$. By the Jacobian criterion [26, Theorem 4.2.19], the scheme X_0 has a singularity at $x_0 := V(I)$ with $I = \langle x, y\rangle$. Let $\pi : \tilde{X}_0 \to X_0$ be the normalization of X_0. Then, $\tilde{X}_0 = \sqcup \tilde{Z}_i$, where $\tilde{Z}_i \to Z_i$ is the normalization of the i-th irreducible component Z_i of X_0; these are given by the factors $(x-\alpha y)$, $\alpha \in \mathbb{F}_{q^n}$. Thus, the cardinality of the inverse image of x_0 under π is q^n, i.e., $|\pi^{-1}(x_0)| = q^n$. By Lemma 2.1.1, for any $r \geq 1$,

$$m^r_{X_0,x_0} = q^n.$$

We conclude that the dimension 1 case is rather easy. Nevertheless, it is very helpful for constructing examples in order to determine the behavior of the r-multiplicities.

2.2 Dimension 2

We calculate the r-multiplicity of a closed singular point x in a normal surface.

Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension 2. Let $x \in X_0$ be a closed singular point. Denote by X the base change of X_0 with respect to $\operatorname{Spec}(\overline{\mathbb{F}}_q)$. Choose a desingularization of X_0 in the strong sense ([24]), which will be denoted by $\tilde{X}_0$. We obtain the following cartesian diagram:

$$\begin{array}{ccccc}
\tilde{X}_0 \setminus D_0 & \xrightarrow{j'} & \tilde{X}_0 & \xleftarrow{i'} & D_0 \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
X_0 \setminus Z_0 & \xrightarrow{j} & X_0 & \xleftarrow{i} & Z_0,
\end{array}$$

where $Z_0 := \{x\}$. The omission of the zero subscript on the objects of the diagram represents, as for X_0 and X, the base change with respect to $\mathrm{Spec}(\overline{\mathbb{F}}_q)$. Let $\{E_s\}_s$ be the set of the irreducible components of D. We begin with the following Lemma where X_0 is assumed to be proper in order to have

$$\mathbb{H}^i(X_0, K_0) = \mathbb{H}^i_c(X_0, K_0),$$

for each $K_0 \in \mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l})$ and every $i \in \mathbb{Z}$.

Lemma 2.2.1. *Let X_0 be a proper and normal algebraic variety over $\mathbb{F}_q$ of dimension 2 with one isolated closed singular point $x \in X_0$. Let $\pi : \tilde{X}_0 \to X_0$ be a desingularization of X_0 in the strong sense such that $D_0 := \pi^{-1}(\{x\})$ is a strict normal crossings divisor over x and let $\{E_s\}_s$ be the set of the irreducible components of D as above. For each $r \geq 1$,*

$$m^r_{X_0,x} = |D_0(\mathbb{F}_{q^\eta})| - |\{E_s : (F^\eta_{D_0} \times \mathrm{Id})(E_s) = E_s\}|\ q^\eta,$$

where $\eta = \deg(x)r$.

Remark 2.2.2. The condition of D_0 being a strict normal crossings divisor is a mild one, since we can always achieve this after appropriate blow-ups ([40, Remark 1.4]).

Proof. Consider the intersection cohomology groups of X_0 and $\tilde{X}_0$: $\mathbb{H}^i(X_0, IC_{X_0})$ and $\mathbb{H}^i(\tilde{X}, IC_{\tilde{X}_0})$. For all $i \neq 2$, we have that

$$\mathbb{H}^i(X_0, IC_{X_0}) = \mathbb{H}^i(\tilde{X}_0, IC_{\tilde{X}_0}),$$

and for $i = 2$,

$$\mathbb{H}^i(X_0, IC_{X_0}) \oplus H^2(D_0, \overline{\mathbb{Q}_l}) \cong \mathbb{H}^i(\tilde{X}_0, IC_{\tilde{X}_0});$$

see [40, Theorem 1.1]. It follows that for $\nu \geq 1$

$$\begin{aligned} a_\nu &:= \sum_{i=0}^{4} (-1)^i \mathrm{Tr}(Fr^\nu | \mathbb{H}^i_c(\tilde{X}, \pi_1^* IC_{\tilde{X}_0})) \\ &= \sum_{i=0}^{4} (-1)^i \mathrm{Tr}(Fr^\nu | \mathbb{H}^i_c(X, \pi_2^* IC_{X_0})) + \mathrm{Tr}(Fr^\nu | H^2(D, \overline{\mathbb{Q}_l})), \end{aligned}$$

where $\pi_1 : \tilde{X} \to \tilde{X}_0$ and $\pi_2 : X \to X_0$ are the projections. By Theorem 1.2.1, we get the following equations:

$$\sum_{i=0}^{4}(-1)^i \mathrm{Tr}(Fr^\nu|\mathbb{H}_c^i(X, \pi_2^* IC_{X_0})) = \sum_{x' \in X_0(\mathbb{F}_{q^\nu})} \sum_j (-1)^{j+2}\mathrm{Tr}(F^\nu|\mathcal{H}^j(IC_{X_0})_{\overline{x'}})$$

$$\sum_{i=0}^{4}(-1)^i \mathrm{Tr}(Fr^\nu|\mathbb{H}_c^i(\tilde{X}, \pi_1^* IC_{\tilde{X}_0})) = \sum_{x' \in \tilde{X}_0(\mathbb{F}_{q^\nu})} \sum_j (-1)^{j+2}\mathrm{Tr}(F^\nu|\mathcal{H}^j(IC_{\tilde{X}_0})_{\overline{x'}}).$$

It follows that

$$\begin{aligned} a_\nu =& \sum_{i=0}^{4}(-1)^i \mathrm{Tr}(Fr^\nu|\mathbb{H}_c^i(\tilde{X}, \pi_1^* IC_{\tilde{X}_0})) \\ =& \sum_{x' \in D_0(\mathbb{F}_{q^\nu})} \sum_j (-1)^{j+2}\mathrm{Tr}(F^\nu|\mathcal{H}^j(IC_{\tilde{X}_0})_{\overline{x'}}) \\ &+ \sum_{x' \in \tilde{X}_0 \setminus D_0(\mathbb{F}_{q^\nu})} \sum_j (-1)^{j+2}\mathrm{Tr}(F^\nu|\mathcal{H}^j(IC_{\tilde{X}_0})_{\overline{x'}}) \\ =& \sum_{i=0}^{4}(-1)^i \mathrm{Tr}(Fr^\nu|\mathbb{H}_c^i(X, \pi_2^* IC_{X_0})) + \mathrm{Tr}(Fr^\nu|H^2(D, \overline{\mathbb{Q}_l})) \\ =& \sum_{x' \in (X_0)_{reg}(\mathbb{F}_{q^\nu})} \sum_j (-1)^{j+2}\mathrm{Tr}(F^\nu|\mathcal{H}^j(IC_{X_0})_{\overline{x'}}) \\ &+ \sum_{x' \in (X_0)_{sing}(\mathbb{F}_{q^\nu})} \sum_j (-1)^{j+2}\mathrm{Tr}(F^\nu|\mathcal{H}^j(IC_{X_0})_{\overline{x'}}) + \mathrm{Tr}(Fr^\nu|H^2(D, \overline{\mathbb{Q}_l})). \end{aligned}$$

Since $(X_0)_{reg} \cong \tilde{X}_0 \setminus D_0$, we get after comparing the second and fourth equality above that

$$\sum_{x' \in D_0(\mathbb{F}_{q^\nu})} 1 = \sum_{x' \in (X_0)_{sing}(\mathbb{F}_{q^\nu})} \sum_j (-1)^{j+2}\mathrm{Tr}(F^\nu|\mathcal{H}^j(IC_{X_0})_{\overline{x'}}) + \mathrm{Tr}(Fr^\nu|H^2(D, \overline{\mathbb{Q}_l})).$$

If $\deg(x) = \nu$, then we have

$$\sum_{x' \in D_0(\mathbb{F}_{q^\nu})} 1 = \sum_{x' \in (X_0)_{sing}(\mathbb{F}_{q^\nu})} m_{X_0, x'} + \mathrm{Tr}(Fr^\nu|H^2(D, \overline{\mathbb{Q}_l})).$$

Since the surface has only one isolated singularity, we have that for $r \geq 1$

$$m^r_{X_0,x} = \sum_{x' \in (X_0)_{sing}(\mathbb{F}_{q^{\nu r}})} m^r_{X_0,x'} = \sum_{x' \in D_0(\mathbb{F}_{q^{\nu r}})} 1 - \mathrm{Tr}(Fr^{\nu r}|H^2(D,\overline{\mathbb{Q}_l}))$$
$$= |D_0(\mathbb{F}_{q^{\nu r}})| - \mathrm{Tr}(Fr^{\nu r}|H^2(D,\overline{\mathbb{Q}_l})).$$

The exceptional divisor D_0 is the union of its irreducible components, i.e., $D_0 = \bigcup (E_0)_{s'}$ such that each $(E_0)_{s'}$ is smooth. We define $S := \cup_{i,j} S_{i,j}$ with $S_{i,j} := (E_0)_i \cap (E_0)_j$ and $U := D_0 \setminus S$. Consider the long exact sequence of cohomology

$$\cdots \to H^1_c(U,\overline{\mathbb{Q}_l}) \to H^1(D_0,\overline{\mathbb{Q}_l}) \to H^1(S,\overline{\mathbb{Q}_l}) \to H^2_c(U,\overline{\mathbb{Q}_l}) \to \dots$$

Let $\pi : \tilde{D_0} \to D_0$ be the normalization of D_0. Then $\tilde{D_0} = \oplus_{s'} (E_0)_{s'}$ with $\tilde{D_0} \times_{\mathbb{F}_q} U \cong U$ and $\tilde{S} := \tilde{D_0} \times_{\mathbb{F}_q} S$. Since $\dim(S) = \dim(\tilde{S}) = 0$, we have that $H^1(S,\overline{\mathbb{Q}_l}) = H^1(\tilde{S},\overline{\mathbb{Q}_l}) = H^2(S,\overline{\mathbb{Q}_l}) = H^2(\overline{S},\overline{\mathbb{Q}_l}) = 0$ and $H^2(\tilde{D_0},\overline{\mathbb{Q}_l}) \cong H^2_c(U,\overline{\mathbb{Q}_l}) \cong H^2(D_0,\mathbb{Q}_l)$. Since $H^2(\tilde{D_0},\overline{\mathbb{Q}_l}) = \oplus_{s'} H^2((E_0)_{s'},\overline{\mathbb{Q}_l})$, we conclude that

$$H^2(D,\overline{\mathbb{Q}_l}) \cong \bigoplus_s H^2(E_s,\overline{\mathbb{Q}_l}).$$

This isomorphism is compatible with the action of Fr on cohomology groups by the functoriality of the Frobenius correspondence ([19, §2.1.3]), since it holds over $\mathrm{Spec}(\mathbb{F}_q)$. The q-Frobenius F_{D_0} is a universal homeomorphism ([33, Lemma 32.35.3]). It follows that the morphism

$$F_{D_0} \times \mathrm{Id} : D \to D,$$

which defines the geometric Frobenius correspondence ([SGA 5, XIII]), permutes the irreducible components of D. We conclude that

$$\mathrm{Tr}\left(Fr^{\nu r}|\bigoplus_s H^2(E_s,\overline{\mathbb{Q}_l})\right) = |\{E_s : (F^{\nu r}_{D_0} \times \mathrm{Id})(E_s) = E_s\}| \ \mathrm{Tr}(Fr^{\nu r}|H^2(E_s,\overline{\mathbb{Q}_l})).$$

For each smooth proper curve E_s, one knows that

$$\mathrm{Tr}(Fr^{\nu r}|H^2(E_s,\overline{\mathbb{Q}_l})) = q^{\nu r}.$$

Let $\eta = \deg(x)r$. Considering all the calculations above, we have

$$m^r_{X_0,x} = |D_0(\mathbb{F}_{q^\eta})| - \mathrm{Tr}(Fr^\eta : H^2(D,\overline{\mathbb{Q}_l}))$$
$$= |D_0(\mathbb{F}_{q^\eta})| - |\{E_s : (F^\eta_{D_0} \times \mathrm{Id})(E_s) = E_s\}| \ \mathrm{Tr}(Fr^\eta : H^2(E_s,\overline{\mathbb{Q}_l}))$$
$$= |D_0(\mathbb{F}_{q^\eta})| - |\{E_s : (F^\eta_{D_0} \times \mathrm{Id})(E_s) = E_s\}| \ q^\eta.$$

□

We now present the main result of this section.

Theorem 2.2.3. *Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension 2. Let $x \in X_0$ be a closed singular point. Then for D_0 as above and for each $r \geq 1$,*

$$m^r_{X_0,x} = |D_0(\mathbb{F}_{q^\eta})| - |\{E_s : (F^\eta_{D_0} \times \mathrm{Id})(E_s) = E_s\}|\ q^\eta,$$

where $\eta = \deg(x)r$.

Proof. By Lemma 1.4.3 and Lemma 1.4.2, we may assume X_0 to be an integral affine normal algebraic variety over $\mathbb{F}_q$ of dimension 2. Since X_0 is normal, every singular point is an isolated singular point. Thus, there exists an affine open subscheme U_x of X_0 such that x is the only isolated closed singular point of U_x. By Lemma 1.4.2, we have that for each $r \geq 1$

$$m^r_{X_0,x} = m^r_{U_x,x}.$$

Thus, we may assume further that x is the only singular point of X_0. Consider the projective closure Y_0 of X_0 in $\mathbb{P}^n_{\mathbb{F}_q}$, i.e., the closed subscheme of $\mathbb{P}^n_{\mathbb{F}_q}$ defined by the homogenization of the ideal I defining X_0 as a closed subscheme of $\mathbb{A}^n_{\mathbb{F}_q}$ ([15, §I Exercise 2.9]). Since X_0 is integral, we may assume Y_0 to be integral as well. Thus, Y_0 is an integral projective scheme such that X_0 is an affine open subscheme of Y_0 containing x. As normalization commutes with smooth morphisms ([33, Tag 03GV]), we have the following cartesian diagram:

$$\begin{array}{ccc} X_0' & \hookrightarrow & Y_0' \\ \downarrow & & \downarrow \\ X_0 & \hookrightarrow & Y_0, \end{array}$$

where X_0' and Y' are the normalizations of X_0 and Y, respectively. Since X_0 is normal, we have that $X_0' = X_0$. It follows that we may assume X_0 to be an open affine subscheme of an integral (Lemma 1.4.3) normal projective variety, call it again Y_0. According to [26, Theorem 8.3.44], the resolution of singularities of a two dimensional algebraic variety is given by a finite sequence

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_1 \to X,$$

where each step is the composition of the blow-up $X_i' \to X_i$ of the singular locus of X_i endowed with the reduced scheme structure, and of the normalization $X_{i+1} \to X_i'$. The sequence above is dominated by a sequence of normalized blow-ups ([33, Tag OBBS and Tag OBBT]); that is a sequence

$$\cdots \to X_{n+1} \to X_n \to \cdots \to X_1 \to X,$$

where each step is the composition of the blow-up $X_i' \to X_i$ of a closed point of X_i, and of the normalization $X_{i+1} \to X_i'$. This implies that we can desingularize Y_0 one closed singular point at a time through blow-ups along the point and normalization of the resulting scheme. Apply then this new sequence to the closed points of $Y_0 \setminus X_0$ and we get an integral normal projective scheme $\tilde{Y_0}$ with an isomorphic copy of X_0 as an open subscheme, such that the only singularity of $\tilde{Y_0}$ is the singularity of X_0, i.e., the point x. By Lemma 1.4.2, we have that for each $r \geq 1$

$$m^r_{X_0,x} = m^r_{Y_0,x}.$$

Thus, we may assume X_0 to be a proper algebraic variety over $\mathbb{F}_q$ of dimension 2 with one isolated singularity. We now apply Lemma 2.2.1 to X_0 and the theorem follows. □

This shows that for a normal algebraic variety X_0 over $\mathbb{F}_q$ of dimension 2, the r-multiplicity of a closed singular point x in X_0 is an integer independent of l. It also shows that it would be possible to have negative numbers for the r-multiplicities. In the following example we show that this is indeed the case.

Example 2.2.4. Consider the following algebraic variety:

$$X_0 := \operatorname{Spec}(\mathbb{F}_q[x,y,z]/\langle x^3y + xy^3 - z^2xy - z^5\rangle),$$

where $char(\mathbb{F}_q) \neq 2,3,5$. By the Jacobian criterion [26, Theorem 4.2.19], the scheme X_0 has a singularity at $x_0 := V(I)$ with $I = \langle x,y,z\rangle$. Consider the blow-up of X_0 along $V(I)$, which will be denoted by $\tilde{X_0}$. The scheme $\tilde{X_0}$ is the union of the affine open subschemes $(\tilde{X_0})_1$, $(\tilde{X_0})_2$, $(\tilde{X_0})_3$ given as follows:

$$(\tilde{X_0})_1 := \operatorname{Spec}\left(\mathbb{F}_q\left[z, \frac{x}{z}, \frac{y}{z}\right] \Big/ \left\langle \left(\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1\right)\frac{x}{z}\frac{y}{z} - z \right\rangle\right)$$

$$(\tilde{X}_0)_2 := \operatorname{Spec}\left(\mathbb{F}_q\left[x, \frac{y}{x}, \frac{z}{x}\right] \Big/ \left\langle \frac{y}{x} + \left(\frac{y}{x}\right)^3 - \left(\frac{z}{x}\right)^2 \frac{y}{x} - \left(\frac{z}{x}\right)^5 x \right\rangle\right)$$

$$(\tilde{X}_0)_3 := \operatorname{Spec}\left(\mathbb{F}_q\left[y, \frac{x}{y}, \frac{z}{y}\right] \Big/ \left\langle \frac{x}{y} + \left(\frac{x}{y}\right)^3 - \left(\frac{z}{y}\right)^2 \frac{x}{y} - \left(\frac{z}{y}\right)^5 y \right\rangle\right).$$

In order to prove that $\tilde{X}_0$ is a resolution of singularities of X_0 we use the Jacobian criterion on each of the $(\tilde{X}_0)_i$, $i = 1, 2, 3$, and conclude that each $(\tilde{X}_0)_i$, $i = 1, 2, 3$ is regular. By [26, Proposition 8.1.12], the morphism $\pi : \tilde{X}_0 \to X_0$ induces an isomorphism $\pi^{-1}(X_0 \setminus V(I)) \to X_0 \setminus V(I)$. Since $\tilde{X}_0$ is regular, the scheme $\pi^{-1}(X_0 \setminus V(I))$, as an open subscheme of $\tilde{X}_0$, is also regular. It follows that $X_0 \setminus V(I)$ is regular as well. Thus, the morphism $\pi : \tilde{X}_0 \to X_0$ is a desingularization of X_0 in the strong sense and the point x_0 is the only singularity of X_0.
Consider the exceptional divisor $D_0 := \pi^{-1}(x_0)$ in $\tilde{X}_0$, which is given by the union of the following affine schemes:

$$(\tilde{X}_0)_1 \cap D_0 = V(z) = \operatorname{Spec}\left(\mathbb{F}_q\left[\frac{x}{z}, \frac{y}{z}\right] \Big/ \left\langle \left(\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1\right) \frac{x}{z} \frac{y}{z} \right\rangle\right)$$

$$(\tilde{X}_0)_2 \cap D_0 = V(x) = \operatorname{Spec}\left(\mathbb{F}_q\left[\frac{y}{x}, \frac{z}{x}\right] \Big/ \left\langle \frac{y}{x}\left(1 + \left(\frac{y}{x}\right)^2 - \left(\frac{z}{x}\right)^2\right) \right\rangle\right)$$

$$(\tilde{X}_0)_3 \cap D_0 = V(y) = \operatorname{Spec}\left(\mathbb{F}_q\left[\frac{x}{y}, \frac{z}{y}\right] \Big/ \left\langle \frac{x}{y}\left(1 + \left(\frac{x}{y}\right)^2 - \left(\frac{z}{y}\right)^2\right) \right\rangle\right).$$

The scheme $(\tilde{X}_0)_1 \cap D_0 = V(z)$ has three irreducible components that meet at the points $[0 : 0 : 1], [1 : 0 : 1], [-1 : 0 : 1], [0 : 1 : 1], [0 : -1 : 1]$. The irreducible components of $(\tilde{X}_0)_2 \cap D_0 = V(x)$ meet at $[1 : 0 : 1]$ and $[1 : 0 : -1] = [-1 : 0 : 1]$. Accordingly, the irreducible components of $(\tilde{X}_0)_3 \cap D_0 = V(y)$ meet at $[0 : 1 : 1]$ and $[0 : 1 : -1] = [0 : -1 : 1]$. It follows that the exceptional divisor D_0 has three irreducible components that intersect in at least 5 points.

In order to apply Theorem 2.2.3, we prove that X_0 is normal. For this we use Serre's criterion for normality. Consider the projective closure Y_0 of X_0 in $\mathbb{P}^3_{\mathbb{F}_q}$ with canonical morphism given as follows:

$$Y_0 := \operatorname{Proj}\left(\mathbb{F}_q[x, y, z, w] / \left\langle (x^3yw + xy^3w - z^2xyw - z^5)\right\rangle\right) \hookrightarrow \mathbb{P}^3_{\mathbb{F}_q},$$

compare [15, §I Proposition2.2]. Since Y_0 as a closed subscheme of $\mathbb{P}^3_{\mathbb{F}_p}$ is defined by one homogeneous polynomial, the scheme Y_0 is a complete intersection ([26, Ex. 5.3.3]). As Y_0 is a complete intersection over $\mathrm{Spec}(\mathbb{F}_q)$, the scheme Y_0 is Cohen-Macaulay ([26, Corollary 8.2.18]). Thus, the local ring $\mathcal{O}_{Y_0,x_0}$ is Cohen-Macaulay. It follows that $\mathcal{O}_{X_0,x_0} \cong \mathcal{O}_{Y_0,x_0}$ is also Cohen-Macaulay. Since the point x_0 is the only singularity of X_0, we have that for $x \neq x_0$, the local ring $\mathcal{O}_{X_0,x}$ is regular. By [26, Example 8.2.14], every regular noetherian local ring is Cohen-Macaulay. It follows that X_0 is Cohen-Macaulay and therefore satisfies (S_2) ([26, Example 8.2.20]).
To prove that X_0 fulfills (R_1) notice that $\mathrm{codim}(x_0) = 2$, since $\dim X = 2$ and x_0 is closed point of X_0. As seen above, for $x \neq x_0$ the local ring $\mathcal{O}_{X_0,x}$ is regular. In particular, if $\mathrm{codim}(x) = 1$, then $\mathcal{O}_{X_0,x}$ is regular. This means that $\mathcal{O}_{X_0,x}$ has dimension 1 and is normal. Hence the scheme X_0 fulfills (R_1). By Serre's criterion for normality, we have that X_0 is normal.

Let X and $\tilde{X}$ be the base change of X_0 and $\tilde{X}_0$ with respect to $\mathrm{Spec}(\overline{\mathbb{F}_q})$, respectively. Accordingly, let D be the corresponding divisor in $\tilde{X}$ associated to $D_0 := \pi^{-1}(x_0)$ in $\tilde{X}_0$ after base change with respect to $\mathrm{Spec}(\overline{\mathbb{F}_q})$. Define $X_1 := \left((\tilde{X}_0)_1 \cap D_0\right) \times_{\mathbb{F}_q} \mathrm{Spec}(\overline{\mathbb{F}_q})$. In order to study the behavior of D under the morphism $F_{D_0} \times \mathrm{Id}$ it is enough to consider its restriction to X_1:

$$F_{(\tilde{X}_0)_1 \cap D_0} \times \mathrm{Id} : X_1 \to X_1.$$

In this case the projection $\mathrm{Pr}_1 : X_1 \to (\tilde{X}_0)_1 \cap D_0$ is given by the ring homomorphism

$$\phi : \left(\mathbb{F}_q\left[\frac{x}{z}, \frac{y}{z}\right] / \left\langle \left(\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1\right) \frac{x}{z}\, \frac{y}{z} \right\rangle\right) \to \left(\overline{\mathbb{F}_q}\left[\frac{x}{z}, \frac{y}{z}\right] / \left\langle \left(\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1\right) \frac{x}{z}\, \frac{y}{z} \right\rangle\right).$$

The morphism

$$(F_{X_1} \times \mathrm{Id})^{\#} : \left(\overline{\mathbb{F}_q}\left[\frac{x}{z}, \frac{y}{z}\right] / \left\langle \left(\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1\right) \frac{x}{z}\, \frac{y}{z} \right\rangle\right) \to \left(\overline{\mathbb{F}_q}\left[\frac{x}{z}, \frac{y}{z}\right] / \left\langle \left(\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1\right) \frac{x}{z}\, \frac{y}{z} \right\rangle\right)$$

is defined by

$$\sum a_i b_j \left(\frac{x}{z}\right)^i \left(\frac{y}{z}\right)^j \mapsto \sum a_i b_j \left(\left(\frac{x}{z}\right)^i \left(\frac{y}{z}\right)^j\right)^q.$$

It follows that the irreducible components $Z_1 := V\left(\left(\frac{y}{z}\right)\right)$ and $Z_2 := V\left(\left(\frac{x}{z}\right)\right)$ of $(\tilde{X}_0)_1 \cap D_0$ are still irreducible in X_1 and are fixed under $F_{X_1} \times \mathrm{Id}$. Notice that $Z_3 := V\left(\left(\left(\frac{x}{z}\right)^2 + \left(\frac{y}{z}\right)^2 - 1\right)\right)$ is irreducible in X_1 and therefore D has three irreducible components that are fixed under $F_{(\tilde{X}_0)_1 \cap D_0} \times \mathrm{Id}$.

We have that the irreducible components Z_1, Z_2 and Z_3 in D_0 are projective lines over $\mathbb{F}_q$, i.e., $Z_1 = Z_2 = Z_3 = \mathbb{P}^1_{\mathbb{F}_q}$. By Theorem 2.2.3, we have for $r \geq 1$

$$m^r_{X_0,x_0} = |D_0(\mathbb{F}_{q^r})| - 3q^r = 3(q^r + 1) - 5 - 3q^r = -2.$$

2.3 Local Ring

Theorem 2.2.3 lets us calculate the r-multiplicity of a closed point $x \in X_0$ by knowing the $\mathbb{F}_{q^\eta}$-rational points of the exceptional divisor D_0 and the irreducible components that are fixed under $F^\eta_{D_0} \times \mathrm{Id}$ on D. In this section we establish a correspondence between these irreducible components and ideals of the local ring of X_0 at x.

Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension 2 with a closed singular point x. By Lemma 1.4.3, we may assume X_0 to be a normal integral algebraic variety over $\mathbb{F}_q$ of dimension 2 with a closed singular point x. Let R be a two-dimensional normal integral ring (the localization of X_0 at x) with maximal ideal $\mathfrak{m}$ and residue field k. Moreover, let $f : \tilde{X}_0 \to \mathrm{Spec}(R)$ be a desingularization of $\mathrm{Spec}(R)$ as in [26, Theorem 8.3.44] given by a finite sequence

$$\tilde{X}_0 = X_n \to X_{n-1} \to \cdots \to \mathrm{Spec}(R), \tag{2.1}$$

where $X_{i+1} \to X_i$ is the composition of the blow-up $X'_i \to X_i$ of the singular locus $(X_i)_{sing}$ endowed with the reduced scheme structure, and the normalization $X_{i+1} \to X'_i$.
Following the construction of a blow-up along $\mathfrak{m}$ of an affine integral scheme, we conclude that X'_1 is integral and projective with respect to $\mathrm{Spec}(R)$. It follows that the morphism $X_2 \to X'_1$ given by the normalization is a birational morphism with X_2 an integral scheme and X'_1 integral and projective. Thus, the morphism $X_1 \to X'_1$ is a blow-up along a quasi-coherent sheaf of ideals $\mathcal{I}_1$ ([26, Exercise 8.1.8]). This implies that the morphism $X_2 \to X'_1$ is projective with respect to X'_1 and the sheaf $\mathcal{I}_1\mathcal{O}_{X_1}$ is very ample relative to

this morphism ([26, Proposition 8.1.22]).
Applying this reasoning to the whole sequence 2.1 we get that $f : X \to \mathrm{Spec}(R)$ is a blow-up along a quasi-coherent sheaf of ideals $\mathcal{I}$. Thus, the morphism f is projective and the sheaf $\mathcal{I}\mathcal{O}_{\tilde{X}_0}$ is very ample relative to f. Let $(E_0)_1, \ldots, (E_0)_n$ be the irreducible components of the exceptional divisor $D_0 := f^{-1}(\{\mathfrak{m}\})$.

The exceptional divisor D_0 is defined as follows:

$$D_0 := \mathrm{Proj} \bigoplus_{d \geq 0} \mathcal{I}^d / \mathcal{I}^{d+1}.$$

By Theorem 2.2.3, the sheaf $\mathcal{I}$ has all the information needed to calculate

$$m^r_{X_0,x_0} = |D(\mathbb{F}_{q^\eta})| - |\{E_s : (F^\eta_{D_0} \times \mathrm{Id})(E_s) = E_s\}| \; q^\eta,$$

where $\eta = \deg(x)r$.
We now give a similar formula for calculating $m^r_{X_0,x}$ where the part corresponding to the irreducible components of D will be replaced by a similar expression depending on ideals of R.

Theorem 2.3.1. *Let X_0 be an integral normal algebraic variety over $\mathbb{F}_q$ of dimension 2 and $x \in |X_0|$ be a closed singular point. Let R be the localization of X_0 at x with maximal ideal $\mathfrak{m}$ and $\mathcal{I}$ be a quasi-coherent sheaf of ideals of* $\mathrm{Spec}(R)$ *such that the blow-up along $\mathcal{I}$ is a desingularization of* $\mathrm{Spec}(R)$. *Then for all $r \geq 1$ we have*

$$m^r_{X_0,x} = |\mathrm{Proj} \bigoplus_{d \geq 0} \mathcal{I}^d / \mathcal{I}^{d+1}(\mathbb{F}_{q^\eta})| - |\{V(I_i)_{\overline{\mathbb{F}_q}} : (F^\eta_{\mathrm{Spec}(R)} \times \mathrm{Id})(V(I_i)_{\overline{\mathbb{F}_q}}) = V(I_i)_{\overline{\mathbb{F}_q}}\}|,$$

where $\eta = \deg(x)r$ and each $V(I_i)_{\overline{\mathbb{F}_q}}$ is a closed subscheme of $\mathrm{Spec}(R \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q})$ *defined by a complete $\mathfrak{m}$-primary ideal I_i of R (i.e. I_i is primary and $\sqrt{I_i} = \mathfrak{m}$).*

Proof. Let $\mathbb{E}$ be the group of divisors of $\tilde{X}_0$ of the form $\sum_{i=1}^n z_i (E_0)_i$ with $z_i \in \mathbb{Z}$ and let $E^\#$ be the set of divisors $G \in \mathbb{E}$, $G \neq 0$, such that $\mathcal{O}(-G)$ is generated by global sections over $\tilde{X}_0$. There is a one-to-one correspondence between members of $E^\#$ and complete $\mathfrak{m}$-primary ideals in R which generate invertible $O_{\tilde{X}_0}$-ideals ([24, §18]). For a description of complete ideals see [24, II §5]. Since the sheaf $\mathcal{I}\mathcal{O}_{\tilde{X}_0}$ is very ample relative to f, there exists an $m \in \mathbb{N}$ such that

$$mD_0 + (E_0)_i \in E^{\#}$$

for all $i = 1, \ldots, n$. Indeed, let $m := \max\{m_i \in \mathbb{N} : i = 1, \ldots, n\}$, such that $\mathcal{O}(-(E_0)_i) \otimes (\mathcal{I}\mathcal{O}_{\tilde{X}_0})^{m_i}$ is generated by global sections over $\tilde{X}_0$. One then establishes a one-to-one correspondence between the irreducible components of the exceptional divisor D_0 and some complete $\mathfrak{m}$-primary ideals I_i in R, the ones corresponding to $mD_0 + (E_0)_i$ for every $i = 1, \ldots, n$. Let E_i be the irreducible components of D. We then have

$$|\{E_i : (F_{D_0}^{\eta} \times \mathrm{Id})(E_i) = E_i\}| = |\{V(I_i)_{\overline{\mathbb{F}_q}} : (F_{\mathrm{Spec}(R)}^{\eta} \times \mathrm{Id})(V(I_i)_{\overline{\mathbb{F}_q}}) = V(I_i)_{\overline{\mathbb{F}_q}}\}|,$$

where $\eta = \deg(x)r$ and each $V(I_i)_{\overline{\mathbb{F}_q}}$ is the closed subscheme of $\mathrm{Spec}(R \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q})$ defined by base change of $V(I_i)$ to $\overline{\mathbb{F}_q}$. □

Remark 2.3.2. Alternatively, we could also find the complete $\mathfrak{m}$-primary ideals I_i by tensoring $\mathcal{O}(-D_0)$ with $(\mathcal{I}\mathcal{O}_{\tilde{X}_0})^{m'}$ for some $m' \in \mathbb{N}$ such that $\mathcal{O}(-D_0) \otimes (\mathcal{I}\mathcal{O}_{\tilde{X}_0})^{m'}$ is generated by global sections over $\tilde{X}_0$ and then looking at the factorization of $\Gamma(\tilde{X}_0, \mathcal{O}(-D_0) \otimes (\mathcal{I}\mathcal{O}_X)^{m'})$ into complete ideals; see [24, p. 239].

2.4 Dimension 3

In this section we calculate the r-multiplicity of an isolated closed singular point in an algebraic variety of dimension 3. We also give an alternative proof of Theorem 2.2.3.

Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension 3. If X_0 is quasi-projective, there exists a projective morphism $\pi : \tilde{X}_0 \to X_0$ such that

i) $\tilde{X}_0$ is regular.

ii) π induces an isomorphism $\tilde{X}_0 \setminus \pi^{-1}((X_0)_{sing}) \cong X_0 \setminus (X_0)_{sing}$.

iii) $\pi^{-1}((X_0)_{sing}) \subset \tilde{X}_0$ is a divisor with strict normal crossings, call it D_0.

See [5, Theorem]. Thus, we have a desingularization of X_0 in the strong sense. Let D be the base change of D_0 to the algebraic closure of $\mathbb{F}_q$ and let $\{E_s\}_s$ be the set of its irreducible components.
Before we state the main theorem of this section, we recall the following important result.

Theorem 2.4.1. *(Hard Lefschetz Formula) Let $f : X_0 \to Y_0$ be a projective morphism of separated schemes of finite type over $\mathbb{F}_q$. Let K_0 be a pure (Definition 1.5.12) perverse sheaf over X_0. For $\delta \geq 0$, we have that*

$$^pH^{-\delta} f_* K_0 = {}^pH^{\delta} f_* K_0(\delta).$$

Proof. [3, Théorème 5.4.10]. □

Theorem 2.4.2. *Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension 3 and $x \in |X_0|$ an isolated singularity. Then for D_0 as above and $r \geq 1$,*

$$m^r_{X_0,x} = |D_0(\mathbb{F}_{q^\eta})| - \mathrm{Tr}(Fr^\eta | H^3(D, \overline{\mathbb{Q}_l})) + |\{E_s : (F^\eta_{D_0} \times \mathrm{Id})(E_s) = E_s\}|\ (q^\eta + q^{2\eta}),$$

where $\eta = \deg(x) r$.

Proof. By Lemma 1.4.3 and Lemma 1.4.2, we may assume X_0 to be an integral affine normal algebraic variety over $\mathbb{F}_q$. By [EGA II, 5.3.4 (i)], the scheme X_0 is quasi-projective and there exists a desingularization of X_0 in the strong sense as above. Since $x \in |X_0|$ is an isolated singular point, there exists an affine open subscheme U_x of X_0 such that x is the only isolated closed singular point of U_x. By Lemma 1.4.2, we have that for each $r \geq 1$

$$m^r_{X_0,x} = m^r_{U_x,x}.$$

Thus, we may assume further that x is the only singular point of X_0. Since $\pi : \tilde{X}_0 \to X_0$ is a projective morphism, we may apply Theorem 2.4.1 in this case. Consider the following cartesian diagram:

$$\begin{array}{ccccc}
\tilde{X}_0 \setminus D_0 & \xhookrightarrow{j'} & \tilde{X}_0 & \xhookleftarrow{i'} & D_0 \\
\downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\
U_0 := X_0 \setminus Z_0 & \xhookrightarrow{j} & X_0 & \xhookleftarrow{i} & Z_0,
\end{array}$$

where Z_0 represents the singular locus of X_0, i.e., $Z_0 := \{x\} = (X_0)_{sing}$. The omission of the zero subscript on the objects of the diagram represents as before the base change to the algebraic closure of $\mathbb{F}_q$. We have that

$$t^r_{\pi_*\overline{\mathbb{Q}_l}[3]}(x) = \sum_\delta (-1)^\delta t^r_{{}^pH^\delta(\pi_*\overline{\mathbb{Q}_l}[3])}(x).$$

Since restricting to U_0 is a t-exact functor, it commutes with perverse cohomology. It follows that

$$j^{*p}H^\delta(\pi_*\overline{\mathbb{Q}_l}[3]) = \overline{\mathbb{Q}_l}[3] \text{ for } \delta = 0 \text{ and}$$
$$j^{*p}H^\delta(\pi_*\overline{\mathbb{Q}_l}[3]) = 0 \text{ for } \delta \neq 0.$$

This implies that the support of ${}^pH^\delta(\pi_*\overline{\mathbb{Q}_l}[3])$ with $\delta \neq 0$ is only the singular point x. By [3, Remarque 5.4.9], the complex $\pi_*\overline{\mathbb{Q}_l}[3]$ is pure, which implies that every ${}^pH^\delta(\pi_*\overline{\mathbb{Q}_l}[3])$ is pure ([3, Corollaire 5.4.4]). By [3, Corollaire 5.3.11], every ${}^pH^\delta(\pi_*\overline{\mathbb{Q}_l}[3])$ admits a unique decomposition

$${}^pH^\delta(\pi_*\overline{\mathbb{Q}_l}[3]) = j_{!*}K_0 \oplus i_*\mathcal{L}_\delta,$$

where $\delta \in \mathbb{Z}$ and $K_0 \in Perv(U_0), \mathcal{L}_\delta \in Perv(\{x\})$. Thus

$${}^pH^\delta(\pi_*\overline{\mathbb{Q}_l}[3]) = i_*\mathcal{L}_\delta,$$

for $\delta \neq 0$ and each $\mathcal{L}_\delta$ is a local system on $\{x\}$. We define the defect of semi-smallness of π as follows:

$$r(\pi) := \max_{\{a \in \mathbb{N} : X_0^a \neq \emptyset\}} \{2a + \dim X_0^a - \dim X_0\},$$

where $X_0^a = \{x \in X_0 : \dim \pi^{-1}(x) = a\}$. It follows that $r(\pi) = 1$. This implies that ${}^pH^\delta(\pi_*\overline{\mathbb{Q}_l}[3]) = 0$ for $\delta \neq 0, 1, -1$ ([4, §4.1]) and ${}^pH^{-1}(\pi_*\overline{\mathbb{Q}_l}[3]) \cong {}^pH^1(\pi_*\overline{\mathbb{Q}_l}[3])(1)$ (Hard Lefschetz Formula). For $\delta = -1, 0, 1$, we have exact triangles

$${}^p\tau_{<\delta}\pi_*\overline{\mathbb{Q}_l}[3] \to {}^p\tau_{\leq\delta}\pi_*\overline{\mathbb{Q}_l}[3] \to {}^pH^\delta(\pi_*\overline{\mathbb{Q}_l}[3])[-\delta] \xrightarrow{[1]}.$$

For $\delta \geq 0$, we have that

$${}^pH^\delta(i^{*}\, {}^p\tau_{\leq\delta}\pi_*\overline{\mathbb{Q}_l}[3]) = {}^pH^\delta(i^{*p}H^\delta(\pi_*\overline{\mathbb{Q}_l}[3])[-\delta]) \text{ and}$$
$${}^pH^\delta(i^{*}\, {}^p\tau_{\leq\delta}\pi_*\overline{\mathbb{Q}_l}[3]) = {}^pH^\delta(i^{*}\, {}^p\tau_{\leq\delta+1}\pi_*\overline{\mathbb{Q}_l}[3]).$$

The first equality is a direct consequence of the exact triangles above and the t-right exactness of the functor i^*. The second equality is due to the fact that

$$\begin{aligned} {}^pH^{\delta}(\iota^* i^{*p}H^{\delta+1}(\pi_*\overline{\mathbb{Q}_l}[3])[-(\delta+1)]) &= {}^pH^{\delta}(\mathcal{L}_{\delta+1}[-(\delta+1)]) = 0 \text{ and} \\ {}^pH^{\delta-1}(\iota^* i^{*p}H^{\delta+1}(\pi_*\overline{\mathbb{Q}_l}[3])[-(\delta+1)]) &= {}^pH^{\delta-1}(\mathcal{L}_{\delta+1}[-(\delta+1)]) = 0 \end{aligned}$$

for $\delta \geq 0$. Since $\pi_*\overline{\mathbb{Q}_l}[3] = {}^p\tau_{\leq 1}\pi_*\overline{\mathbb{Q}_l}[3]$, the two equalities above imply that for $\delta = 0$ and $\delta = 1$,

$$\begin{aligned} \mathcal{L}_{\delta} &= {}^pH^{\delta}(\iota^* i^{*p}H^{\delta}(\phi_*\overline{\mathbb{Q}_l}[d])[-\delta]) \\ &= {}^pH^{\delta}(\iota^* i^{*}\, {}^p\tau_{\leq 1}\phi_*\overline{\mathbb{Q}_l}[d]) \\ &= {}^pH^{\delta}(\iota^* i^{*}\phi_*\overline{\mathbb{Q}_l}[d]). \end{aligned} \tag{2.2}$$

By the Hard Lefschetz Formula, we have for $\delta \geq 1$ the equality

$$\mathcal{L}_{-\delta} = \mathcal{L}_{\delta}(\delta). \tag{2.3}$$

It follows that

$$\begin{aligned} t^r_{\pi_*\overline{\mathbb{Q}_l}[3]}(x) &= \sum_{\delta} (-1)^{\delta} t^r_{{}^pH^{\delta}(\pi_*\overline{\mathbb{Q}_l}[3])}(x) \\ &= \left(\sum_{\delta=-1}^{1} (-1)^{\delta} t^r_{\mathcal{L}_{\delta}}(x) \right) + m^r_{X_0,x}. \end{aligned}$$

By equation (1.1), the left hand side is equal to $|D_0(\mathbb{F}_{q^{\eta}})|$ for $\eta = \deg(x)r$. By equations (2.2) and (2.3), we have that

$$\begin{aligned} t^r_{\mathcal{L}_1}(x) &= \mathrm{Tr}(Fr^{\eta}|H^4(D,\overline{\mathbb{Q}_l})) \\ t^r_{\mathcal{L}_{-1}}(x) &= \mathrm{Tr}(Fr^{\eta}|H^4(D,\overline{\mathbb{Q}_l}))\ q^{-\eta} \\ t^r_{\mathcal{L}_0}(x) &= \mathrm{Tr}(Fr^{\eta}|H^3(D,\overline{\mathbb{Q}_l})). \end{aligned} \tag{2.4}$$

Here $\eta = \deg(x)r$. Since D_0 is a divisor with strict normal crossings, it is the union of its irreducible components, i.e., $D_0 = \cup(E_0)_{s'}$ where each $(E_0)_{s'}$ is smooth. Using the same argument as in the proof of Lemma 2.2.1, we conclude that

$$H^4(D,\overline{\mathbb{Q}_l}) \cong \bigoplus_{s} H^4(E_s,\overline{\mathbb{Q}_l}).$$

This isomorphism is compatible with the action of Fr on cohomology groups as before. We conclude that

$$\mathrm{Tr}(Fr^{\eta}|H^4(D,\overline{\mathbb{Q}_l})) = |\{E_s : (F^{\eta}_{D_0} \times \mathrm{Id})(E_s) = E_s\}|\ \mathrm{Tr}(Fr^{\eta}|H^4(E_s,\overline{\mathbb{Q}_l})).$$

For each smooth proper surface E_s, one knows that

$$\mathrm{Tr}(Fr^{\eta}|H^4(E_s,\overline{\mathbb{Q}_l})) = q^{2\eta}.$$

The theorem now follows. □

Remark 2.4.3. In [36, §2], E. Tasso calculates the intersection cohomology of a projective singular three-dimensional variety X_0 over $\mathbb{C}$ with an isolated singular point. In particular, she describes ${}^pH^{\delta}(\pi_*\mathbb{Q}[3])$ for $\delta = 0, 1, -1$ where $\pi : \tilde{X}_0 \to X_0$ is a desingularization of X_0 ([36, §2 (2.2)]). The cohomology groups on which we consider our traces in equation (2.4) are the characteristic p analogues of the cohomology groups calculated in [36, §2 (2.2)].

We now refine the theorem above by giving a more precise description of $\mathrm{Tr}(Fr^{\eta}|H^3(D,\overline{\mathbb{Q}_l}))$. In the following calculations we write the subscript $\overline{\mathbb{F}_q}$ under a scheme to denote the base change to the algebraic closure of $\mathbb{F}_q$. Define $S := \cup_{i<j} S_{i,j}$ with $S_{i,j} := (E_0)_i \cap (E_0)_j$ and $U := D_0 \setminus S$. Let $\{(Z_0)_{m'}\}_{m'}$ be the set of irreducible components of S and $\{Z_m\}_m$ be the set of irreducible components of $S_{\overline{\mathbb{F}_q}}$.

Theorem 2.4.4. *Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension 3 and $x \in |X_0|$ an isolated singularity. Then for D_0 as above and $r \geq 1$,*

$$\begin{aligned}&\mathrm{Tr}(Fr^{\eta}|H^3(D,\overline{\mathbb{Q}_l}))\\ =&|\{E_s : (F^{\eta}_{D_0} \times \mathrm{Id})(E_s) = E_s\}|\ \mathrm{Tr}(Fr^{\eta}|H^3(E_s,\overline{\mathbb{Q}_l}))\\ &+ \mathrm{Tr}(Fr^{\eta}|\mathrm{Im}\,(\oplus_m H^2(Z_m,\overline{\mathbb{Q}_l}) \to H^3_c(U_{\overline{\mathbb{F}_q}},\overline{\mathbb{Q}_l})))\end{aligned} \tag{2.5}$$

Proof. Consider the following long exact sequence of cohomology:

$$\cdots \to H^2(S,\overline{\mathbb{Q}_l}) \xrightarrow{\phi} H^3_c(U,\overline{\mathbb{Q}_l}) \xrightarrow{\alpha} H^3(D_0,\overline{\mathbb{Q}_l}) \to H^3(S,\overline{\mathbb{Q}_l}) \to \ldots$$

Since $\dim(S) = 1$, we have that $H^3(S,\overline{\mathbb{Q}_l}) = 0$. It follows that

$$\begin{aligned}\mathrm{Tr}(Fr^{\eta}|H^3(D,\overline{\mathbb{Q}_l})) &= \mathrm{Tr}(Fr^{\eta}|H^3_c(U_{\overline{\mathbb{F}_q}},\overline{\mathbb{Q}_l})) - \mathrm{Tr}(Fr^{\eta}|\ker(\alpha)) \text{ and}\\ \mathrm{Tr}(Fr^{\eta}|\ker(\alpha)) &= \mathrm{Tr}(Fr^{\eta}|\mathrm{Im}\,(\phi))\\ &= \mathrm{Tr}(Fr^{\eta}|\mathrm{Im}\,(\oplus_m H^2(Z_m,\overline{\mathbb{Q}_l}) \to H^3_c(U_{\overline{\mathbb{F}_q}},\overline{\mathbb{Q}_l}))).\end{aligned}$$

Let $\pi : \tilde{D}_0 \to D_0$ be the normalization of D_0. Then $\tilde{D}_0 = \sqcup_i (E_0)_i$ with $\tilde{D}_0 \times U \cong U$ and $\tilde{S} := \tilde{D}_0 \times S = S \sqcup S$. Notice that each irreducible component $(Z_0)_{m'}$ appears twice as an irreducible component of $\tilde{S}$. We have the following long exact sequence of cohomology:

$$\cdots \to H^2(\tilde{S}, \overline{\mathbb{Q}_l}) \xrightarrow{\tilde{\phi}} H^3_c(U, \overline{\mathbb{Q}_l}) \xrightarrow{\tilde{\alpha}} H^3(\tilde{D}_0, \overline{\mathbb{Q}_l}) \to H^3(\tilde{S}, \overline{\mathbb{Q}_l}) \to \dots$$

It follows that

$$\begin{aligned}
\mathrm{Tr}(Fr^\eta | H^3_c(U_{\overline{\mathbb{F}_q}}, \overline{\mathbb{Q}_l})) &= \mathrm{Tr}(Fr^\eta | H^3(\tilde{D}, \overline{\mathbb{Q}_l})) + \mathrm{Tr}(Fr^\eta | \ker(\tilde{\alpha})) \text{ and} \\
\mathrm{Tr}(Fr^\eta | \ker(\tilde{\alpha})) &= \mathrm{Tr}(Fr^\eta | \mathrm{Im}\,(\tilde{\phi})) \\
&= 2\ \mathrm{Tr}(Fr^\eta | \mathrm{Im}\,(\oplus_m H^2(Z_m, \overline{\mathbb{Q}_l}) \to H^3_c(U_{\overline{\mathbb{F}_q}}, \overline{\mathbb{Q}_l}))).
\end{aligned}$$

Bringing everything together we get the following equation:

$$\begin{aligned}
&\mathrm{Tr}(Fr^\eta | H^3(D, \overline{\mathbb{Q}_l})) \\
=&|\{E_s : (F^\eta_{D_0} \times \mathrm{Id})(E_s) = E_s\}|\ \mathrm{Tr}(Fr^\eta | H^3(E_s, \overline{\mathbb{Q}_l})) \\
&+ \mathrm{Tr}(Fr^\eta | \mathrm{Im}\,(\oplus_m H^2(Z_m, \overline{\mathbb{Q}_l}) \to H^3_c(U_{\overline{\mathbb{F}_q}}, \overline{\mathbb{Q}_l})))
\end{aligned}$$

□

Since each E_s is a smooth integral proper surface over an algebraically closed field, we have Poincaré-Verdier duality:

$$H^3(E_s, \overline{\mathbb{Q}_l}) \cong H^1(E_s, \overline{\mathbb{Q}_l})(2)^\vee \cong H^1(E_s, \overline{\mathbb{Q}_l})^\vee \otimes_{\mathbb{Z}_l} \mathbb{Z}_l(2)^\vee.$$

Thus,

$$\mathrm{Tr}(Fr^\eta | H^3(E_s, \overline{\mathbb{Q}_l})) = q^{2\eta} \mathrm{Tr}((Fr^{-1})^\eta | H^1(E_s, \overline{\mathbb{Q}_l})).$$

We could classify the possible calculations of $\mathrm{Tr}((Fr^{-1})^\eta | H^1(E_s, \overline{\mathbb{Q}_l}))$ that could occur following the classification for proper and integral surfaces made by Bombieri and Mumford in [10]. One could then try to obtain a classification of the r-multiplicities in the setting of Theorem 2.4.2 by replacing equation (2.5) in Theorem 2.4.2. Under the classification of Bombieri and Mumford, one can look into some simple cases.

Consider the case where the Kodaira dimension of E_s equals zero. By [10, p. 25], there are three different possibilities for the first Betti number, namely

$b_1 = 0, 4, 2$.
In the first case we have that $H^3(E_s, \overline{\mathbb{Q}_l}) = 0$ and there is nothing to calculate. If $b_1 = 4$, then E_s is an abelian surface and the characteristic polynomial of the Frobenius Endomorphism action over $H^1(E_s, \overline{\mathbb{Q}_l})$ is completely determined by [27, Theorem 2.9]. The last case, where $b_1 = 2$, gives that the Albanese variety $\mathrm{Alb}(E_s)$ is an elliptic curve. Thus, the calculation of $\mathrm{Tr}(Fr^\eta | H^3(E_s, \overline{\mathbb{Q}_l}))$ depends on the genus of $\mathrm{Alb}(E_s)$.

We now present an alternative proof of Theorem 2.2.3 following the proof of Theorem 2.4.2.

Lemma 2.4.5. *Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension 2 with one isolated singularity $x \in |X_0|$. Then for each $r \geq 1$,*

$$m^r_{X_0,x} = |D_0(\mathbb{F}_{q^\eta})| - |\{E_s : (F^\eta_{D_0} \times \mathrm{Id})(E_s) = E_s\}|\ q^\eta,$$

where $\eta = \deg(x)r$.

Proof. Let $\pi : \tilde{X}_0 \to X_0$ be a resolution of singularities of X_0 in the strong sense such that $\pi^{-1}((X_0)_{sing}) \subset \tilde{X}_0$ is a divisor with strict normal crossings, call it D_0 ([24] and [40, Remark 1.4]). Let $\{(E_0)_{s'}\}_{s'}$ be the set of irreducible components of D_0. Let D be the base change of D_0 to the algebraic closure of $\mathbb{F}_q$ and $\{E_s\}_s$ be the set of its irreducible components. Consider the following cartesian diagram:

$$\begin{array}{ccccc}
\tilde{X}_0 \setminus D_0 & \xhookrightarrow{j'} & \tilde{X}_0 & \xhookleftarrow{i'} & D_0 \\
\mathrm{Id}\downarrow & & \downarrow\pi & & \downarrow\pi \\
U_0 := X_0 \setminus Z_0 & \xhookrightarrow{j} & X_0 & \xhookleftarrow{i} & Z_0,
\end{array}$$

where Z_0 represents the singular locus of X_0, i.e., $Z_0 := \{x\} = (X_0)_{sing}$. The omission of the zero subscript on the objects of the diagram represents as before the base change to the algebraic closure of $\mathbb{F}_q$. The defect of semi-smallness in this case is equal to 0, i.e., the morphism π is semi-small ([21, Definition §III 7.3]). Thus, ${}^pH^\delta(\pi_*\overline{\mathbb{Q}_l}[2]) = 0$ for $\delta \neq 0$. By [3, Corollaire 5.3.11], the complex $\pi_*\overline{\mathbb{Q}_l}[2]$ has a unique decomposition

$$\pi_*\overline{\mathbb{Q}_l}[2] = j_{!*}\overline{\mathbb{Q}_l}[2] \oplus i_*\mathcal{L}_0,$$

where $\mathcal{L}_0 \in Perv(\{x\})$. It follows that

$$\begin{aligned} t^r_{\pi_*\overline{\mathbb{Q}_l}[2]}(x) &= t^r_{\mathcal{L}_0}(x) + m^r_{X_0,x} \\ &= t^r_{{}^pH^0(i^*\pi_*\overline{\mathbb{Q}_l}[2])}(x) + m^r_{X_0,x}. \end{aligned}$$

By equation (1.1), the left hand side is equal to $|D(\mathbb{F}_{q^\eta})|$ for $\eta = \deg(x)r$. Since $t^r_{{}^pH^0(i^*\pi_*\overline{\mathbb{Q}_l}[2])}(x) = \mathrm{Tr}(Fr^\eta | H^2(D, \overline{\mathbb{Q}_l}))$ for $\eta = \deg(x)r$, we have that

$$m^r_{X_0,x} = |D_0(\mathbb{F}_{q^\eta})| - |\{E_s : (F^\eta_{D_0} \times \mathrm{Id})(E_s) = E_s\}|\ q^\eta.$$

□

This lemma relaxes the normality condition on X_0 made in Theorem 2.2.3. If we assume X_0 to be normal, we may apply the lemma to prove Theorem 2.2.3.

Theorem 2.4.6. *Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension 2. Let $x \in X_0$ be a closed singular point. Then for D_0 as above and for each $r \geq 1$,*

$$m^r_{X_0,x} = |D(\mathbb{F}_{q^\eta})| - |\{E_s : (F^\eta_{D_0} \times \mathrm{Id})(E_s) = E_s\}|\ q^\eta,$$

where $\eta = \deg(x)r$.

Proof. By Lemma 1.4.3 and Lemma 1.4.2, we may assume X_0 to be an integral affine normal algebraic variety over $\mathbb{F}_q$ of dimension 2. Since X_0 is normal, every singular point is an isolated singular point. Thus, there exists an affine open subscheme U_x of X_0 such that x is the only isolated closed singular point of U_x. By Lemma 1.4.2, we have that for each $r \geq 1$

$$m^r_{X_0,x} = m^r_{U_x,x}.$$

Thus, we may assume further that x is the only singular point of X_0. Apply Lemma 2.4.5 to X_0 and the theorem follows. □

2.5 Examples

We give an example of a singular algebraic variety X_0 over $\mathbb{F}_q$ with a singular point $x \in |X_0|$ such that for all $r \geq 1$, we have $m^r_{X_0,x} = 1$.

Example 2.5.1. Consider the following algebraic variety:

$$W := \mathrm{Spec}(\mathbb{F}_q[x, y, z]/\langle xy - z^2 \rangle),$$

with $char(\mathbb{F}_q) \neq 2$. Let x_0 denote the maximal ideal $\mathfrak{m} = \langle x, y, z \rangle$. Let $\pi : \tilde{W} \to W$ be the blow-up with center x_0. Then $\tilde{W} = \cup_{i=1}^{3} \mathrm{Spec}(A_i)$ with

$$\begin{aligned} A_1 &= \mathbb{F}_q \left[x, \frac{z}{x} \right] \\ A_2 &= \mathbb{F}_q \left[y, \frac{z}{y} \right] \\ A_3 &= \mathbb{F}_q \left[z, \frac{y}{z}, \frac{x}{z} \right] \text{ with } \left(\frac{x}{z} \right) \left(\frac{y}{z} \right) = 1. \end{aligned}$$

Since $\tilde{W}$ is a union of open subschemes that are isomorphic to open subschemes of $\mathbb{A}^2_{\mathbb{F}_q}$, we have that $\tilde{W}$ is regular over $\mathbb{F}_q$.
Now, the fiber $E_0 := \pi^{-1}(x_0)$ is given by the following affine schemes:

$$\begin{aligned} E_0 \cap A_1 &= \mathrm{Spec} \left(\mathbb{F}_q \left[\frac{z}{x} \right] \right) \\ E_0 \cap A_2 &= \mathrm{Spec} \left(\mathbb{F}_q \left[\frac{z}{y} \right] \right) \\ E_0 \cap A_3 &= \mathrm{Spec} \left(\mathbb{F}_q \left[\frac{x}{z}, \frac{y}{z} \right] / \left(\frac{x}{z} \frac{y}{z} - 1 \right) \right). \end{aligned}$$

Thus, the exceptional divisor $E_0 = \mathbb{P}^1_{\mathbb{F}_q}$. We apply Lemma 2.4.5 to W. It follows that for $r \geq 1$,

$$m^r_{W,x_0} = q^r + 1 - q^r = 1.$$

Another example of this phenomenon are *rationally smooth* varieties.

Definition 2.5.2. Let X_0 be a scheme over $\mathbb{F}_q$ of dimension d. The scheme X_0 is said to be *rationally smooth* if

$$IC_{X_0} = \overline{\mathbb{Q}_l}[d].$$

Example 2.5.3. For $G = GL_2$ or $G = PGL_2$, the affine Schubert varieties $Gr^{\leq \mu}_{G}$ are rationally smooth ([14, Theorem 1.1]).

Lemma 2.5.4. *Let X_0 be a rationally smooth variety over $\mathbb{F}_q$ of dimension d. Let $x \in |X_0|$ be a singular point. Then for any $r \geq 1$,*

$$m^r_{X_0,x} = 1.$$

Proof. Same argument as in Remark 1.2.3(i). □

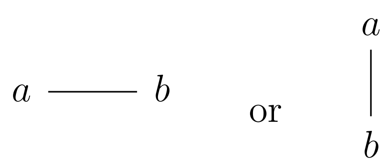
Definition 2.5.5. Let R be a two-dimensional normal local domain. The ring R is said to have a *pseudo-rational singularity* if the following condition holds:

For any projective birational map $g : W \to \operatorname{Spec}(R)$ *there exists a normal surface* Z*, proper and birational over* $\operatorname{Spec}(R)$*, such that* Z *dominates* W *and* $H^1(Z, \mathcal{O}_Z) = 0$.

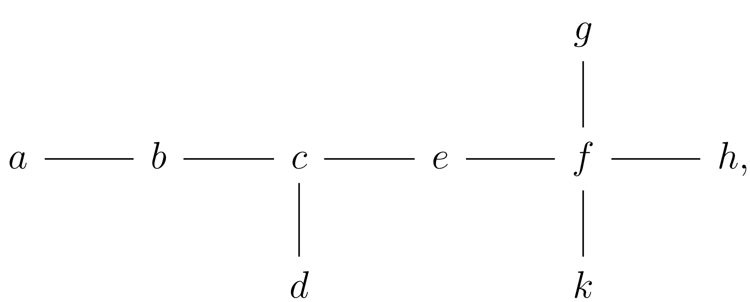
Contrary to a *rational* singularity, one does not assume $Z \to \operatorname{Spec}(R)$ to be a desingularization of $\operatorname{Spec}(R)$ ([24, Definition 1.1]).

In [24, §24] Lipman characterizes all normal local domains R of dimension 2 and multiplicity 2 having a pseudo-rational singularity by classifying the possible configuration diagrams on the minimal desingularization of $\operatorname{Spec}(R)$. These diagrams describe the types of exceptional divisors that could occur by desingularazing $\operatorname{Spec}(R)$ through successive monoidal transformations.

A symbol of either of the following types



where a, b are positive integers will stand for a pair of integral exceptional curves E, F on a desingularization of $\operatorname{Spec}(R)$, such that $h^0(E) := \operatorname{length}(H^0(E, \mathcal{O}_E)) = a, h^0(F) := \operatorname{length}(H^0(F, \mathcal{O}_F)) = b$, and $E \cap F$ is non-empty. Thus, we have as *intersection number* $(E.F) = \max\{a, b\}$ ([24, Lemma (22.2)]). We combine these symbols into diagrams such as



which stands for a configuration of nine exceptional curves $E_1, E_2 \ldots, E_9$ such that $h^0(E_1) = a, h^0(E_1) = b, \ldots, h^0(E_9) = k$, and such that the non-empty intersections of pairs of E_i's, with $1 \leq i \leq 9$, are those (and only those) indicated by the short straight lines.

For a normal local domain R of dimension 2 and multiplicity 2 having a pseudo-rational singularity the following types (and no others) of exceptional curves can occur:

A_n : 1 —— 1 —— 1 —— ⋯ —— 1 ($n \geq 2$ components).

B_n : 1 —— 2 —— 2 —— ⋯ —— 2 ($n \geq 1$ components, including the first one).

C_n : 1 —— 1 —— ⋯ —— 1 —— 2 ($n \geq 3$ components, including the last one).

D_n :
```
                        1
                        |
1 —— 1 —— 1 —— ⋯ —— 1
                        |
                        1
```
($n \geq 4$ components, including the last two).

G_2 : 1 —— 3

F_4 : 1 —— 1 —— 2 —— 2

E_6 :
```
            1
            |
1 —— 1 —— 1 —— 1 —— 1
```

E_7 :
```
                 1
                 |
1 —— 1 —— 1 —— 1 —— 1 —— 1
```

E_8 :
```
                      1
                      |
1 —— 1 —— 1 —— 1 —— 1 —— 1 —— 1
```

Accordingly, we then may classify the possible r-multiplicities in an integral normal algebraic variety X_0 over $\mathbb{F}_q$ of dimension 2 having one pseudo-rational doble-point depending on the possible configuration diagrams of the exceptional divisor of the minimal desingularization of X_0.

Corollary 2.5.6. *Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension 2 and $x \in |X_0|$ be a pseudo-rational doble-point. Let $\pi : \tilde{X}_0 \to X_0$ be the minimal desingularization of X_0 with exceptional divisor D_0. Then for r big enough we have the following:*

1. If D_0 is of a type different from C_n, then

$$m^r_{X_0,x} = 1.$$

2. If D_0 is of type C_n, then

$$m^r_{X_0,x} = 0.$$

Proof. This is a direct consequence of Theorem 2.2.3. By [24, pp. 259–268], each configuration diagram is of type A_n, D_n, E_6, E_7, E_8 or C_n after base changing D_0 to the algebraic closure of $\mathbb{F}_q$, i.e., each configuration diagram of D is of this type. The corollary now follows. □

Example 2.5.7. Consider the following algebraic variety:

$$X_0 := \mathrm{Spec}(\mathbb{F}_p[x, y, z]/\langle z^2 + y^3 + x^5\rangle),$$

where p is a prime number different from $2, 3$ and 5. Let x_0 be the maximal ideal $\mathfrak{m} = \langle x, y, z\rangle$. Let $\pi : \tilde{X}_0 \to X_0$ be the minimal desingularization of X_0 given by successive monoidal transformations with exceptional divisor D_0. Let D be the base change of D_0 to the algebraic closure of $\mathbb{F}_q$. By [24, Theorem 25.1] and [24, Remark 25.3], the exceptional divisors D_0 and D are of type E_8. It follows that for r big enough

$$m^r_{X_0,x_0} = 1.$$

We now give a more generic example than in Example 2.2.4 of a situation where the r-multiplicity of a singular closed point in an algebraic variety X_0 is negative.

Example 2.5.8. For $n \geq 1$, consider the following scheme:

$$X_0 := \operatorname{Spec}\left(\mathbb{F}_{q^n}[x,y,z]/\left\langle\left(\prod_{\alpha\in\mathbb{F}_{q^n}\setminus\{\pm\sqrt{-1},0\}}(y-\alpha x)(x^2+y^2-z^2)\right)+z^{3q^n-8}\right\rangle\right),$$

where $char(\mathbb{F}_{q^n}) \neq 2$.
By the Jacobian criterion, the scheme X_0 has only one singularity at $\{x_0\} = V(I)$ with $I = \langle x, y, z\rangle$. Let $\tilde{X}_0$ be the blow-up of X_0 along $V(I)$. The projective scheme $\tilde{X}_0$ is the union of the affine open subschemes $\tilde{X}_0^1, \tilde{X}_0^2, \tilde{X}_0^3$ given as follows:

$$\tilde{X}_0^1 := \operatorname{Spec}\left(\mathbb{F}_{q^n}\left[z,\frac{x}{z},\frac{y}{z}\right]/\left\langle\left(\prod_{\alpha\in\mathbb{F}_{q^n}\setminus\{\pm\sqrt{-1},0\}}\left(\frac{y}{z}-\alpha\frac{x}{z}\right)\left(\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2-1\right)\right)+z\right\rangle\right),$$

$$\tilde{X}_0^2 := \operatorname{Spec}\left(\mathbb{F}_{q^n}\left[x,\frac{z}{x},\frac{y}{x}\right]/\left\langle\left(\prod_{\alpha\in\mathbb{F}_{q^n}\setminus\{\pm\sqrt{-1},0\}}\left(\frac{y}{x}-\alpha\right)\left(\left(\frac{y}{x}\right)^2-\left(\frac{z}{x}\right)^2+1\right)\right)+\left(\frac{z}{x}\right)^{3q^n-8}x\right\rangle\right),$$

$$\tilde{X}_0^3 := \operatorname{Spec}\left(\mathbb{F}_{q^n}\left[y,\frac{z}{y},\frac{x}{y}\right]/\left\langle\left(\prod_{\alpha\in\mathbb{F}_{q^n}\setminus\{\pm\sqrt{-1},0\}}\left(1-\alpha\frac{x}{y}\right)\left(\left(\frac{x}{y}\right)^2-\left(\frac{z}{y}\right)^2+1\right)\right)+\left(\frac{z}{y}\right)^{3q^n-8}y\right\rangle\right).$$

The exceptional divisor $D_0 := \pi^{-1}(x_0)$ in $\tilde{X}_0$, where $\pi : \tilde{X}_0 \to X$ denotes the blow-up along x_0, is then given by the following affine schemes:

$$\tilde{X}_0^1\cap D_0 = V(z) = \operatorname{Spec}\left(\mathbb{F}_{q^n}\left[\frac{x}{z},\frac{y}{z}\right]/\left\langle\prod_{\alpha\in\mathbb{F}_{q^n}\setminus\{\pm\sqrt{-1},0\}}\left(\frac{y}{z}-\alpha\frac{x}{z}\right)\left(\left(\frac{x}{z}\right)^2+\left(\frac{y}{z}\right)^2-1\right)\right\rangle\right),$$

$$\tilde{X}_0^2\cap D_0 = V(x) = \operatorname{Spec}\left(\mathbb{F}_{q^n}\left[\frac{z}{x},\frac{y}{x}\right]/\left\langle\prod_{\alpha\in\mathbb{F}_{q^n}\setminus\{\pm\sqrt{-1},0\}}\left(\frac{y}{x}-\alpha\right)\left(\left(\frac{y}{x}\right)^2-\left(\frac{z}{x}\right)^2+1\right)\right\rangle\right),$$

$$\tilde{X}_0^3\cap D_0 = V(y) = \operatorname{Spec}\left(\mathbb{F}_{q^n}\left[\frac{z}{y},\frac{x}{y}\right]/\left\langle\prod_{\alpha\in\mathbb{F}_{q^n}\setminus\{\pm\sqrt{-1},0\}}\left(1-\alpha\frac{x}{y}\right)\left(\left(\frac{x}{y}\right)^2-\left(\frac{z}{y}\right)^2+1\right)\right\rangle\right).$$

By the Jacobian criterion again, the schemes $\tilde{X}_0^1, \tilde{X}_0^2, \tilde{X}_0^3$ are regular schemes. Thus, the scheme $\tilde{X}_0$ is a desingularization of X_0 in the strong sense with exceptional strict normal crossings divisor D_0. It is enough to consider the scheme $\tilde{X}_0^1 \cap D_0$ for analyzing the behavior of D, the base change of D_0 to the algebraic closure of $\mathbb{F}_{q^n}$.
Each irreducible component of D_0 is geometrically irreducible. The divisor D has $q^n - 2$ irreducible components with at least $2(q^n - 2) - 2$ intersection points between them (for each $\alpha \in \mathbb{F}_{q^n} \setminus \{\pm\sqrt{-1}, 0\}$ we consider the points $[\pm\beta : \pm\alpha\beta : 1]$, where $\beta^2 = (\alpha^2 + 1)^{-1}$). For any $r \geq 1$, each irreducible

component of D has at least as many $\mathbb{F}_{q^{nr}}$-valued points as a projective line. We then may consider each irreducible component of D_0 as a projective line over $\mathbb{F}_{q^n}$. By Lemma 2.4.5, for every r big enough,

$$m^r_{X_0,x_0} \leq (q^n-2)(q^{nr}+1) - (2(q^n-2)-2) - (q^n-2)\, q^{nr} = -q^n+4.$$

This shows that for every $z \in \mathbb{Z}$ there exists $n_0 \in \mathbb{N}$ such that for $r \geq n_0$,

$$m^r_{X_0,x_0} \leq z.$$

For X_0 an algebraic variety over $\mathbb{F}_q$ with closed point x, Example 2.1.3 together with Example 2.5.8 showed that the r-multiplicity $m^r_{X_0,x}$ may be any number $z \in \mathbb{Z}$ for r big enough. Given any number $n \in \mathbb{N}$, a slight variation in Example 2.1.3 shows that $m^r_{X_0,x} = n$ for $r \geq 1$.

Example 2.5.9. Let $n \in \mathbb{N}$. Consider the scheme

$$X_0 := \operatorname{Spec}\left(\mathbb{F}_q[x,y]/\prod_{\alpha_i \in \mathbb{F}_q} \langle x - \alpha_i y\rangle\right),$$

where $q \geq n$ and $1 \leq i \leq n$. The scheme X_0 has a singularity at $\{x_0\} = V(I)$ with $I = \langle x, y\rangle$. Then for $r \geq 1$,

$$m^r_{X_0,x_0} = n.$$

Given any number $z \in \mathbb{Z}$, it is still not clear if we can construct an algebraic variety X_0 over $\mathbb{F}_q$ with a singular closed point x such that for r big enough $m^r_{X_0,x} = z$.

Chapter 3

Nearby and Vanishing Cycles

In this chapter we relate the r-multiplicity of a closed singular point x in an algebraic variety X_0 to the theory of nearby and vanishing cycles functors.

Let X_0 be an algebraic variety over $\mathbb{F}_q$ of dimension d. By Lemma 1.4.3 and Lemma 1.4.2, we may assume X_0 to be an integral affine normal algebraic variety. By Lemma 1.4.5, there exists an element $g \in \mathcal{O}_{X_0}(X_0)$ such that $g \neq 0$, a closed subscheme $Z_0 := V(g) \supseteq (X_0)_{sing}$ and an open dense smooth subscheme $U_0 := X_0 \setminus Z_0$. Thus, the element g defines a morphism $g : X_0 \to \mathbb{A}^1_{\mathbb{F}_q}$. Under this morphism the *nearby cycles* functor and the *vanishing cycles* functor are defined in [SGA 7 I and XIII and SGA 4 1/2 (Th. finitude)].
Specifically, let $i : Z_0 \hookrightarrow X_0$ and $j : U_0 \hookrightarrow X_0$ be the canonical embeddings. Fix a topological generator T of the prime-to-p quotient of $\pi_1^{\text{geom}}(\mathbb{G}_{m,\mathbb{F}_q}, 1)$ (where $p = char(\mathbb{F}_q)$). Deligne constructs the nearby cycles functor

$$\psi_g : \mathcal{D}^b_c(U_0, \overline{\mathbb{Q}_l}) \to \mathcal{D}^b_c(Z, \overline{\mathbb{Q}_l}),$$

with a functorial action of $\pi_1(\mathbb{G}_{m,\mathbb{F}_q}, 1)$ on ψ_g, compatible with the action of $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ on Z, and a functorial exact triangle in $\mathcal{D}^b_c(X_0, \overline{\mathbb{Q}_l})$ given by

$$i^* \to \psi_g j^* \to \phi_g \xrightarrow{[1]},$$

where the first term is base changed to $\overline{\mathbb{F}_q}$ and the last is the vanishing cycles functor. Here Z denotes the base change of Z_0 to the algebraic closure of $\mathbb{F}_q$.

It is worth mentioning that some authors shift the nearby cycles functor defined in SGA 7 XIII by $[-1]$ and call it the same. The reason for this is that the functor $\psi_f[-1]$ is t-exact and so preserves perverse sheaves. However, we will only work with the definition of the vanishing cycles functor made in SGA 7.

3.1 Unipotent Cycles

We define the unipotent versions of the nearby and vanishing cycles functors.

Proposition 3.1.1. *There exists a functorial T-equivariant direct sum decomposition $\psi_g = \psi_g^u \oplus \psi_g^{nu}$ such that, for every $K_0 \in \mathcal{D}_c^b(U_0, \overline{\mathbb{Q}_l})$, $T-1$ acts nilpotently on $\psi_g^u(K_0)$ and invertibly on $\psi_g^{nu}(K_0)$.*

Proof. [28, Proposition 1.1] or [30, Lemma 1.1]. □

Definition 3.1.2. The functor ψ_g^u is called the *unipotent nearby cycles* functor.

One can prove that the functor $\psi_g^u[-1]$ is t-exact looking at the exact triangle

$$i^* j_* \to \psi_g^u \xrightarrow{T-1} \psi_g^u \xrightarrow{[1]};$$

see [28, Proposition 1.3]. Moreover, one can also prove that ψ_g^u is actually defined as a functor from $\mathcal{D}_c^b(U_0, \overline{\mathbb{Q}_l})$ to $\mathcal{D}_c^b(Z_0, \overline{\mathbb{Q}_l})$ using the *logarithm of the unipotent part of the monodromy*; see [28, Corollary 4.3].
It is then possible to define the *unipotent vanishing cycles* functor ϕ_g^u. This functor shifted by $[-1]$ is again t-exact and is part of a functorial exact triangle

$$i^* \to \psi_g^u j^* \xrightarrow{can} \phi_g^u \xrightarrow{[1]} . \tag{3.1}$$

We now relate the nearby and vanishing cycles functors (and its unipotent versions) to the r-multiplicity of a closed singular point in X_0.

Consider the following cartesian diagram:

$$\begin{array}{ccccc} Z_0 & \xhookrightarrow{i} & X_0 & \xhookleftarrow{j} & U_0 \\ \downarrow & & \downarrow g & & \downarrow \\ 0 & \xhookrightarrow{i'} & \mathbb{A}^1_{\mathbb{F}_q} & \xhookleftarrow{j'} & \mathbb{G}_{m,\mathbb{F}_q}. \end{array}$$

Since $j' : \mathbb{G}_{m,\mathbb{F}_q} \to \mathbb{A}^1_{\mathbb{F}_q}$ is affine, the embedding $j : U_0 \to X_0$ is an affine open morphism and the functors $j_*, j_!$ are t-exact by Artin's Theorem ([21, Corollary 6.2]). Thus, we have the following short exact sequence:

$$0 \to i_* {}^pH^{-1}(i^* IC_{X_0}) \to j_! \overline{\mathbb{Q}_l}[d] \to IC_{X_0} \to 0.$$

It follows that for any $x \in |Z_0|$,

$$m^r_{X_0,x} = -t^r_{{}^pH^{-1}(i^* IC_{X_0})}(x). \tag{3.2}$$

Proposition 3.1.3. *There are canonical isomorphisms* $\ker(can) \cong {}^pH^{-1}(i^* IC_{X_0})$ *and* $\operatorname{coker}(can) \cong {}^pH^0(i^* IC_{X_0})$*.*

Proof. [28, Proposition 6.2]. □

Thus, coker(can)$= 0$ and we have

$$m^r_{X_0,x} = t^r_{\psi^u_g(\overline{\mathbb{Q}_l}[d])}(x) - t^r_{\phi^u_g(IC_{X_0})}(x). \tag{3.3}$$

If the field of coefficients is algebraically closed, then Beilinson observed that the full nearby cycles functor ψ_g can be recovered from ψ^u_g as applied to variations of $\overline{\mathbb{Q}_l}[d]$. In [30] this is proved for $\mathbb{C}$ ([30, Lemma 4.2]). We will prove this fact for $\overline{\mathbb{Q}_l}$.

For any $K_0 \in \mathcal{D}^b_c(U_0, \overline{\mathbb{Q}_l})$ we may decompose $\psi_g(K_0)$ into its generalized eigenspaces. This decomposition generalizes to the whole functor ψ_g.

Lemma 3.1.4. *There exists a unique isomorphism of functors* $\mathcal{D}^b_c(U_0, \overline{\mathbb{Q}_l}) \to \mathcal{D}^b_c(Z, \overline{\mathbb{Q}_l})$

$$\psi_g \cong \bigoplus_{\lambda \in \overline{\mathbb{Q}_l}^\times} \psi^\lambda_g,$$

where for any complex $K_0 \in \mathcal{D}^b_c(U_0, \overline{\mathbb{Q}_l})$*,* $\lambda - T$ *is nilpotent on* $\psi^\lambda_g(K_0)$*.*

Proof. This is a consequence of [30, Lemma 4.2]. □

We now want some information about the possible λ's that can appear in the Lemma 3.1.4. For this we recall some theory.
Let R be a henselian discrete valuation ring with fraction field K and residue field k. Let p be the characteristic of k. We choose an algebraic closure $\overline{K}$ of K and $\overline{k}$ of k. We denote by I the *inertia group* given by the short exact sequence

$$1 \to I \to Gal(\overline{K}/K) \to Gal(\overline{k}/k) \to 1.$$

An *l-adic representation* of $Gal(\overline{K}/K)$ (or a profinite group G) is a homomorphism $\rho : G \to GL(V)$, where V is a finite dimensional $\overline{\mathbb{Q}_l}$-vector space such that there exists a finite extion E of $\mathbb{Q}_l$ contained in $\overline{\mathbb{Q}_l}$ and an E-structure on V_E such that ρ factorizes through a continuous homomorphism $G \to GL(V_E)$ (where $GL(V_E)$ is given its natural topology as an l-adic Lie group).
Let $G = Gal(\overline{K}/K)$. An l-adic representation ρ of G is *quasi-unipotent* if there exists an open subgroup I_1 of I such that the restriction of ρ to I_1 is unipotent (i.e. such that $\rho(g)$ is unipotent of all $g \in I_1$). A fundamental result due to Grothendieck asserts that this property is fulfilled if k is not too big:

Proposition 3.1.5. *(Grothendieck) Assume that no finite extension of the field k contains all roots of unity of order a power of l. Then every l-adic representation of G is quasi-unipotent.*

Proof. [32, Appendice]. □

Proposition 3.1.6. *The λ's appearing in Lemma 3.1.4 are all roots of unity.*

Proof. Consider the localization of $\mathbb{A}^1_{\mathbb{F}_q}$ at zero, i.e., $\mathbb{F}_q[x]_{(x)}$. The henselization of this local ring is a discrete valuation henselian ring R with fraction field K and residue field $\mathbb{F}_q$. For any $K_0 \in \mathcal{D}^b_c(U_0, \overline{\mathbb{Q}_l})$ the action of $\pi_1(\mathbb{G}_{m,\mathbb{F}_q}, 1)$ on the finite dimensional $\overline{\mathbb{Q}_l}$-vector space $\psi_g(K_0)$ is an l-adic representation of $\pi_1(\mathbb{G}_{m,\mathbb{F}_q}, 1)$. Thus, it is an l-adic representation of $G = Gal(\overline{K}/K)$ given through the canonical homomorphism $\phi : G \to Gal(\overline{K}/\mathbb{F}_q(x)) \to \pi_1(\mathbb{G}_{m,\mathbb{F}_q}, 1)$. By Proposition 3.1.5, this l-adic representation is quasi-unipotent. Let P be the *wild inertia group*, which sits in a short exact sequence

$$1 \to P \to I \to I_t \to 1.$$

Let $\hat{\mathbb{Z}}(1) := \varprojlim_n \mu_n(\overline{\mathbb{F}_q})$. Since $\pi_1^{geom}(\mathbb{G}_{m,\mathbb{F}_q}, 1) = \hat{\mathbb{Z}}(1)$ ([31, §6.5 Proposition 8], compare with [SGA 1 XI, Théorème 2.1]) and $I/P \cong \hat{\mathbb{Z}}(1)$ ([SGA 7 I, (0.3)]), we have a canonical isomorphism

$$\pi_1^{geom}(\mathbb{G}_{m,\mathbb{F}_q}, 1) \cong I/P,$$

such that the action of I on $\psi_g(K_0)$ given through this isomorphism agrees with the action of I on $\psi_g(K_0)$ given by ϕ. It follows that $T = \phi(g)$ for some $g \in I$ and its action on $\psi_g(K_0)$ is quasi-unipotent for any $K_0 \in \mathcal{D}_c^b(U_0, \overline{\mathbb{Q}_l})$. Thus, the only possible eigenvalues of the action of T on $\psi_g(K_0)$ are roots of unity, which implies that the λ's appearing in Lemma 3.1.4 are all roots of unity. □

We now describe the functors ψ_g^λ of Lemma 3.1.4.
Consider the short exact sequence

$$1 \to \pi_1^{geom}(\mathbb{G}_{m,\mathbb{F}_q}) \to \pi_1(\mathbb{G}_{m,\mathbb{F}_q}) \to Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q) \to 1.$$

Any $\mathbb{F}_q$-rational point of $\mathbb{G}_{m,\mathbb{F}_q}$ gives a section of the last map. Using the section given by the point 1, we get an isomorphism $\pi_1(\mathbb{G}_{m,\mathbb{F}_q}, 1) \cong \pi_1^{geom}(\mathbb{G}_{m,\mathbb{F}_q}, 1) \rtimes Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$, where $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ acts on the prime-to-p quotients of $\pi_1^{geom}(\mathbb{G}_{m,\mathbb{F}_q}, 1)$ by multiplication by the cyclotomic character. Thus, any l-adic character $\chi : \pi_1(\mathbb{G}_{m,\mathbb{F}_q}, 1) \to \overline{\mathbb{Q}_l}^\times$ is a product of l-adic characters $\chi_1 : \pi_1^{geom}(\mathbb{G}_{m,\mathbb{F}_q}, 1) \to \overline{\mathbb{Q}_l}^\times$ and $\chi_2 : Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q) \to \overline{\mathbb{Q}_l}^\times$. Each of these characters is uniquely determined by the image of T and F, where $\chi_1(T) = \lambda$ is a root of unity and $\chi_2(F) = b$ is any unit in $\overline{\mathbb{Q}_l}$ ([21, Theorem I.3.1] and [7, (1.1.7)]). Define $\mathcal{L}_\lambda$ as the local system corresponding to the l-adic character given by $\chi_1(T) = \lambda$ and $\chi_2(F) = 1$.

Lemma 3.1.7. *For any λ a root of unity in $\overline{\mathbb{Q}_l}$ and any $K_0 \in Perv(U_0)$ we have*

$$\psi_g^\lambda(K_0) = \psi_g^u(K_0 \otimes g^*\mathcal{L}_{\lambda^{-1}}) \otimes \overline{\mathbb{Q}_l}^{(\lambda)},$$

where $\mathcal{L}_{\lambda^{-1}}$ is a local system of rank 1 of $\pi_1(\mathbb{G}_{m,\mathbb{F}_q})$ with monodromy λ^{-1}, i.e., on the l-adic representation corresponding to $\mathcal{L}_{\lambda^{-1}}$ the element T acts as multiplication by λ^{-1}, and $\overline{\mathbb{Q}_l}^{(\lambda)}$ is the underlying associated one-dimensional $\overline{\mathbb{Q}_l}$-vector space of $\mathcal{L}_\lambda$.

Proof. Tensoring K_0 with $g^*\mathcal{L}_{\lambda^{-1}}$ just replaces the action of T by $T' := \lambda^{-1}T$. Applying then the unipotent vanishing cycles functor gives the generalized eigenspace of 1, i.e.,

$$\psi_g^u(K_0 \otimes g^*\mathcal{L}_{\lambda^{-1}}) = \ker((\mathrm{Id} - T')^n | \psi_g(K_0 \otimes g^*\mathcal{L}_{\lambda^{-1}}))$$

for $n >> 0$. Thus,

$$\begin{aligned} \psi_g^\lambda(K_0) &= \ker((\lambda - T)^n | \psi_g(K_0 \otimes g^*\mathcal{L}_{\lambda^{-1}} \otimes g^*\mathcal{L}_\lambda)) \\ &= \ker(\lambda^n(\mathrm{Id} - T')^n | \psi_g(K_0 \otimes g^*\mathcal{L}_{\lambda^{-1}})) \otimes \overline{\mathbb{Q}_l}^{(\lambda)} \\ &= \ker((\mathrm{Id} - T')^n | \psi_g(K_0 \otimes g^*\mathcal{L}_{\lambda^{-1}})) \otimes \overline{\mathbb{Q}_l}^{(\lambda)} \\ &= \psi_g^u(K_0 \otimes g^*\mathcal{L}_{\lambda^{-1}}) \otimes \overline{\mathbb{Q}_l}^{(\lambda)} \end{aligned}$$

for $n >> 0$. The second equality is due to the fact that the canonical morphism

$$\psi_g(K_0) \otimes g^*\psi_{\mathrm{Id}}(\mathcal{L}_\lambda) \to \psi_g(K_0) \otimes \psi_g(g^*\mathcal{L}_\lambda) \to \psi_g(K_0 \otimes g^*\mathcal{L}_\lambda)$$

is an isomorphism. Compare with [28, Lemma 3.3]. □

It follows that the full nearby cycles functor ψ_g may be defined as a functor from $\mathcal{D}_c^b(U_0, \overline{\mathbb{Q}_l})$ to $\mathcal{D}_c^b(Z_0, \overline{\mathbb{Q}_l})$. For any $x \in |Z_0|$, the stalk $\psi_g(\overline{\mathbb{Q}_l}[d]))_{\overline{x}}$ gets an action of $Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ through the isomorphism of Lemma 3.1.4. The exact triangle 3.1 implies that, for any $\lambda \in \overline{\mathbb{Q}_l}^\times$ a root of unity, the following triangle is also exact:

$$i^*(IC_{X_0} \otimes j_*g^*\mathcal{L}_{\lambda^{-1}}) \to \psi_g^u(\overline{\mathbb{Q}_l}[d] \otimes g^*\mathcal{L}_{\lambda^{-1}}) \xrightarrow{can} \phi_g^u(IC_{X_0} \otimes j_*g^*\mathcal{L}_{\lambda^{-1}}) \xrightarrow{[1]} .$$

It follows that

$$i^*(IC_{X_0} \otimes j_*g^*\mathcal{L}_{\lambda^{-1}}) \otimes \overline{\mathbb{Q}_l}^{(\lambda)} \to \psi_g^u(\overline{\mathbb{Q}_l}[d] \otimes g^*\mathcal{L}_{\lambda^{-1}}) \otimes \overline{\mathbb{Q}_l}^{(\lambda)} \xrightarrow{can} \phi_g^u(IC_{X_0} \otimes j_*g^*\mathcal{L}_{\lambda^{-1}}) \otimes \overline{\mathbb{Q}_l}^{(\lambda)} \xrightarrow{[1]}$$

is an exact triangle, since tensoring with $\overline{\mathbb{Q}_l}^{(\lambda)}$ is an exact functor. Taking then direct sums yields an exact triangle

$$\bigoplus_{\lambda \in \overline{\mathbb{Q}_l}^\times} i^*(IC_{X_0} \otimes j_*g^*\mathcal{L}_{\lambda^{-1}}) \otimes \overline{\mathbb{Q}_l}^{(\lambda)} \to \psi_g(\overline{\mathbb{Q}_l}[d]) \xrightarrow{\oplus can} \bigoplus_{\lambda \in \overline{\mathbb{Q}_l}^\times} \phi_g^u(IC_{X_0} \otimes j_*g^*\mathcal{L}_{\lambda^{-1}}) \otimes \overline{\mathbb{Q}_l}^{(\lambda)} \xrightarrow{[1]} .$$

Since $i^*(IC_{X_0} \otimes j_* g^* \mathcal{L}_{\lambda^{-1}}) \cong i^*(IC_{X_0}) \otimes i^*(j_* g^* \mathcal{L}_{\lambda^{-1}})$, we have the following equation:

$$\sum_{\lambda \in \overline{\mathbb{Q}_l}^\times} m^r_{X_0,x}\ t^r_{i^*(j_* g^* \mathcal{L}_{\lambda-1}) \otimes \overline{\mathbb{Q}_l}^{(\lambda)}}(x) = t^r_{\psi_g(\overline{\mathbb{Q}_l}[d])}(x) - \sum_{\lambda \in \overline{\mathbb{Q}_l}^\times} t^r_{\phi^u_g(IC_{X_0} \otimes \mathcal{L}_{\lambda-1}) \otimes \overline{\mathbb{Q}_l}^{(\lambda)}}(x). \tag{3.4}$$

Note that the vanishing cycles functor preserves constructability. This implies that the sums on both sides of the equation are finite sums.

Now the henselization of the localization of $\mathbb{A}^1_{\mathbb{F}_q}$ at zero yields a henselian trait $S = (S, s, \eta)$. In [SGA 7 XIII, Proposition 2.1.4] it is proved that for any geometric point $\overline{x}$ over $s \in S$,

$$\psi_g(\overline{\mathbb{Q}_l}[d])_{\overline{x}} \cong R\Gamma(((X_0)_{(\overline{x})})_{\overline{\eta}}, \overline{\mathbb{Q}_l}[d]), \tag{3.5}$$

where the scheme $((X_0)_{(\overline{x})})_{\overline{\eta}}$, the geometric generic fiber of the strict localization of X_0 at $\overline{x}$, plays the role of a *Milnor fiber* of g at $\overline{x}$. We then get the following result.

Theorem 3.1.8. *Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension d. Let $x \in |X_0|$ be a singular point. Then for any $r \geq 1$,*

$$\sum_{\lambda \in \overline{\mathbb{Q}_l}^\times} m^r_{X_0,x}\ t^r_{i^*(j_* g^* \mathcal{L}_{\lambda-1}) \otimes \overline{\mathbb{Q}_l}^{(\lambda)}}(x) = t^r_{R\Gamma(((X_0)_{(\overline{x})})_{\overline{\eta}}, \overline{\mathbb{Q}_l}[d])}(x) - \sum_{\lambda \in \overline{\mathbb{Q}_l}^\times} t^r_{\phi^u_g(IC_{X_0} \otimes \mathcal{L}_{\lambda-1}) \otimes \overline{\mathbb{Q}_l}^{(\lambda)}}(x).$$

Proof. By Lemma 1.4.3 and Lemma 1.4.2 we may assume X_0 to be integral and affine. The theorem now follows by equation (3.4). □

Remark 3.1.9. In this lemma, the action of F over $R\Gamma(((X_0)_{(\overline{x})})_{\overline{\eta}}, \overline{\mathbb{Q}_l}[d])$ is given through the isomorphism (3.5).

The theorem above allow us to relate the r-multiplicity of x in X_0 to a well known object in geometry, namely, the Milnor fiber of g at $\overline{x}$. Hence it facilitates a more geometric interpretation of the r-multiplicity of a closed singular point in an algebraic variety over $\mathbb{F}_q$.

We finish this section by making some general observations. Consider the exact functorial triangle

$$i^* j_* \to \psi^u_g \xrightarrow{N} \psi^u_g(-1) \xrightarrow{[1]}, \tag{3.6}$$

where N is the logarithm of the unipotent part of the monodromy; see [28, §1] for the existence of such a triangle. Since $\psi_g^u(\overline{\mathbb{Q}_l}[d])$ is defined over Z_0 as mentioned above and is perverse of degree -1, we have for any $x \in |Z_0|$,

$$-t^r_{{}^pH^{-1}(i^*j_*\overline{\mathbb{Q}_l}[d])}(x) = t^r_{\psi_g^u(\overline{\mathbb{Q}_l}[d])}(x) - t^r_{\mathrm{Im}\,(N)}(x). \tag{3.7}$$

According to [21, Lemma 5.12], ${}^pH^{-1}(i^*j_*\overline{\mathbb{Q}_l}[d]) \cong {}^pH^{-1}(i^*IC_{X_0})$. This shows that

$$m^r_{X_0,x} = -t^r_{{}^pH^{-1}(i^*IC_{X_0})}(x) = -t^r_{{}^pH^{-1}(i^*j_*\overline{\mathbb{Q}_l}[d])}(x).$$

These observations imply the following result.

Proposition 3.1.10. *Let X_0 be a normal variety over $\mathbb{F}_q$ of dimension d. Let $x \in |X_0|$ be a singular point. Then for any $r \geq 1$,*

$$t^r_{\phi_g^u(IC_{X_0})}(x) = t^r_{\mathrm{Im}\,(N)}(x),$$
$$m^r_{X_0,x} = t^r_{i^*j_*\overline{\mathbb{Q}_l}[d]}(x) - t^r_{{}^pH^0(i^*j_*\overline{\mathbb{Q}_l}[d])}.$$

Proof. By Lemma 1.4.3 and Lemma 1.4.2 we may assume X_0 to be integral and affine. The first equation follows from equation (3.3) and equation (3.7). The second follows from the following equality

$$t^r_{i^*j_*\overline{\mathbb{Q}_l}[d]}(x) = t^r_{{}^pH^0(i^*j_*\overline{\mathbb{Q}_l}[d])} - t^r_{{}^pH^{-1}(i^*j_*\overline{\mathbb{Q}_l}[d])}(x).$$

□

3.2 Alterations

We recall the definition of an alteration made in [20]:

Definition 3.2.1. [20, 2.20] Let S be a noetherian integral scheme. An *alteration* S' of S is an integral scheme S', together with a morphism $\phi : S' \to S$, which is dominant, proper and such that for some non-empty open $U \subset S$, the morphism $\phi^{-1}(U) \to U$ is finite. (This last condition is equivalent to the condition $\dim S = \dim S'$, at least if these are finite.)

We apply [20, Theorem 4.1] to X_0 and Z_0 to obtain an alteration

$$\phi_1 : \tilde{X}_0 \to X_0$$

and an open immersion $j_1 : \tilde{X}_0 \to \overline{X_0}$ such that

(i) $\overline{X_0}$ is a projective integral variety and is a regular scheme, and

(ii) the closed subset $Y_0 := j_1(\phi^{-1}(Z_0)) \cup \overline{X_0} \setminus j_1(\tilde{X}_0)$ is a strict normal crossings divisor in $\overline{X_0}$.

Consider the following cartesian diagram:

$$\begin{array}{ccccc} \tilde{Z}_0 & \xrightarrow{\tilde{i}} & \tilde{X}_0 & \xleftarrow{\tilde{j}} & \tilde{U}_0 \\ \downarrow & & \downarrow \phi_1 & & \downarrow \\ Z_0 & \xrightarrow{i} & X_0 & \xleftarrow{j} & U_0. \end{array}$$

Then for every $y \in |\tilde{Z}_0|$ such that $\phi_1(y) = x$, with $x \in |Z_0|$, we have

$$m^{r+\nu}_{X_0,x} = t^{r+\nu}_{IC_{X_0}}(x) = t^r_{\phi_1^* IC_{X_0}}(y),$$

where $\nu = [\kappa(y) : \kappa(x)]$. One can always find such a point y, since any proper and dominant morphism is surjective. The exact triangle (3.1) implies that

$$m^{r+\nu}_{X_0,x} = t^r_{\phi_1^* IC_{X_0}}(y) = t^r_{\psi^u_h(\overline{\mathbb{Q}}_l[d])}(y) - t^r_{\phi^u_h(\phi_1^* IC_{X_0})}(y), \tag{3.8}$$

where $h = g \circ \phi_1 : \tilde{X}_0 \to \mathbb{A}^1_{\mathbb{F}_q}$. Since the closed subset $j_1(\phi_1^{-1}(Z_0)) \cup \overline{X_0} \setminus j_1(\tilde{X}_0)$ is a strict normal crossings divisor in $\overline{X_0}$, we have that a local description of $\overline{X_0}$ at $y \in \tilde{Z}_0$ is given by an equation of the form $t_1 \cdots t_\gamma = 0$, where $t_1, \ldots, t_\gamma$ are part of a system of regular local parameters at y.

In what follows we calculate $t^r_{\psi^u_h(\overline{\mathbb{Q}}_l[d])}(y)$ explicitly using the fact that $\tilde{X}_0$ is an alteration of X_0. Before we do this we recall Grothendieck's purity conjecture:

Conjecture 3.2.2. *(SGA 5, I 3.1.4) Let X be a noetherian regular scheme and $i : Y \to X$ a closed immersion. Suppose that Y is regular, and that $Y \subseteq X$ has codimension d at each point. Let l be a prime number invertible on X. Then the local cohomology sheaves are given by:*

$$H^i_Y(X, \mathbb{Z}/l^n) = \begin{cases} 0, & i \neq 2d \\ i_* \mathbb{Z}/l^n(-d), & i = 2d. \end{cases}$$

The conjecture was known to be true if X was smooth over a perfect field ([SGA 4, XVI 3.9]), or if X was an excellent noetherian scheme of equicharacteristic 0 ([SGA 4, XIX 3.2]). The conjecture was later proved by Gabber ([11, Theorem 2.1.1]).

Remark 3.2.3. The conjecture is also valid for $\mathbb{Q}_l$ or $\overline{\mathbb{Q}_l}$ in place of $\mathbb{Z}/l^n$ ([37, Corollary 3.9]).

Let $Y_0 := j_1(\phi_1^{-1}(Z_0)) \cup \overline{X_0} \setminus j_1(\tilde{X}_0)$ and $((Y_0)_s)_{s\in I}$ be the set of irreducible components of Y_0. For $E \subset I$, let

$$(Y_0)_E := \bigcap_{s\in E} (Y_0)_s$$

and

$$Y_0^{(m)} := \coprod_{|E|=m} (Y_0)_E.$$

Further let $a_m : Y_0^{(m)} \to Y_0$ be the projection and $a_0 = \mathrm{Id}$. We will omit the zero subscript on the objects just defined to denote the base change to the algebraic closure of $\mathbb{F}_q$. Finally, let $\overline{a}_m : Y^{(m)} \to Y$ be the projection.

Lemma 3.2.4. *For every* $r \geq 1$ *and* $\eta = \deg(y)r$*,*

$$(1-q^\eta)t^r_{\psi_h^u(\overline{\mathbb{Q}_l}[d])}(y) = \sum_{i\geq 0}(-1)^i q^{i\eta}|\overline{a}_i^{-1}(y)|.$$

Proof. By the exact triangle (3.6), we have for every $r \geq 1$ and $\eta = \deg(y)r$ that

$$(1-q^\eta)\ t^r_{\psi_h^u(\overline{\mathbb{Q}_l}[d])}(y) = t^r_{\tilde{i}^*\tilde{j}_*\overline{\mathbb{Q}_l}[d]}(y).$$

Consider the following cartesian diagram:

$$\begin{array}{ccccc} \tilde{Z}_0 & \xrightarrow{\tilde{i}} & \tilde{X}_0 & \xleftarrow{\tilde{j}} & \tilde{U}_0 \\ \downarrow & & \downarrow{\scriptstyle j_1} & & \downarrow{\scriptstyle \mathrm{Id}} \\ Y_0 & \xrightarrow{\overline{i}} & \overline{X_0} & \xleftarrow{\overline{j}} & \tilde{U}_0. \end{array}$$

Since the diagram is commutative and $j_1 \circ \tilde{j} = \overline{j}$, we have that

$$\begin{aligned} j_1^* \overline{i}^* \overline{j}_*(\overline{\mathbb{Q}_l}[d]) &= \tilde{i}^* j_1^* \overline{j}_*(\overline{\mathbb{Q}_l}[d]) \\ &= \tilde{i}^* j_1^* (j_1)_* \tilde{j}_*(\overline{\mathbb{Q}_l}[d]) \\ &= \tilde{i}^* \tilde{j}_*(\overline{\mathbb{Q}_l}[d]). \end{aligned}$$

The point y is an element of $\tilde{Z}_0$ which is an open subscheme of Y_0 under j_1. It follows that

$$t^r_{\tilde{i}^* \tilde{j}_* \overline{\mathbb{Q}_l}[d]}(y) = t^r_{\overline{i}^* \overline{j}_* \overline{\mathbb{Q}_l}[d]}(y).$$

It follows that for $i \geq 1$,

$$\mathcal{H}^{i-d} \overline{j}_* \overline{\mathbb{Q}_l}[d] = (a_i)_* \overline{\mathbb{Q}_l}[d](-i);$$

see [29, 2.8 Satz]. Thus,

$$t^r_{\overline{i}^* \overline{j}_* \overline{\mathbb{Q}_l}[d]}(y) = \sum_{i \geq 0} (-1)^i q^{i\eta} |\overline{a}_i^{-1}(y)|,$$

where $\overline{a}_0 = \mathrm{Id}$. The lemma now follows. □

Theorem 3.2.5. *Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension d. Let $x \in |X_0|$ be a singular point. Consider an alteration $\phi_1 : \tilde{X}_0 \to X_0$ of an irreducible component of X_0 containing x and $y \in \tilde{X}_0$ such that $\phi_1(y) = x$. Let $g : X_0 \to \mathbb{A}^1_{\mathbb{F}_q}$ be a morphism such that $(X_0)_{sing} \subseteq g^{-1}(\{0\})$. Define $h := g \circ \phi_1 : \tilde{X}_0 \to \mathbb{A}^1_{\mathbb{F}_q}$. Then for any $r \geq 1$ and $\eta = \deg(y)r$,*

$$(1 - q^\eta) m^{r+\nu}_{X_0,x} = \sum_{i \geq 0} (-1)^i q^{i\eta} |\overline{a}_i^{-1}(y)| - (1 - q^\eta) t^r_{\phi^u_h(\phi_1^* IC_{X_0})}(y),$$

where $\nu = [\kappa(y) : \kappa(x)]$.

Proof. By Lemma 1.4.3 and Lemma 1.4.2 we may assume X_0 to be integral and affine. The theorem now follows by equation (3.8). □

Remark 3.2.6. The theorem also shows that for $r \geq 1$ and $\eta = \deg(y)r$, the number $(1 - q^\eta) t^r_{\phi^u_h(\phi_1^* IC_{X_0})}(y)$ is an integer independent of l.

The strategy used in chapter 2 for studying the r-multiplicity $m^r_{X_0,x}$ was to consider the behavior of the intersection complex of a desingularization of

X_0 under push-forward (the complex $\pi_*\overline{\mathbb{Q}_l}[d]$). Theorem 3.2.5 is the dual approach, namely, it explains the relation of the r-multiplicity $m^r_{X_0,x}$ to the intersection complex of an alteration of X_0 under pull-back (the complex $\phi_1^*(IC_{X_0})$).

Example 3.2.7. Let $h : \tilde{X}_0 \to \mathbb{A}^1_{\mathbb{F}_q}$ be *strictly semi-stable* ([20, 2.16]). Then the scheme $\tilde{Z}_0$ is a divisor with strict normal crossings in $\tilde{X}_0$; call it Y_0 and use the notations defined above for its irreducible components. In this case we have that

1. The topological generator $T \in \pi_1^{geom}(\mathbb{G}_{m,\mathbb{F}_q}, 1)$ acts trivially on each $\mathcal{H}^i\psi_h(\overline{\mathbb{Q}_l}[d])$, $i \in \mathbb{Z}$.

2. $\mathcal{H}^{-d}\psi_h(\overline{\mathbb{Q}_l}[d]) = \overline{\mathbb{Q}_l}[d]$.

3. If $C^\bullet$ denotes the augmented (acyclic) Čech complex in the étale site of Y with abelian presheaves defined by $\overline{a}_1 : Y^{(1)} \to Y$,

$$C^\bullet = (0 \to \overline{\mathbb{Q}_l}[d] \to (\overline{a}_1)_*\overline{\mathbb{Q}_l}[d] \to (\overline{a}_2)_*\overline{\mathbb{Q}_l}[d] \to \dots),$$

(where $\overline{\mathbb{Q}_l}[d]$ sits in degree -1), then we have (for $i \geq 1$)

$$\mathcal{H}^{i-d}\psi_h(\overline{\mathbb{Q}_l}[d])(i) = \mathrm{Coker}(C^{i-2} \to C^{i-1}) = \mathrm{Ker}(C^i \to C^{i+1}).$$

see [16, Théorème 3.2]. Since $T \in \pi_1^{geom}(\mathbb{G}_{m,\mathbb{F}_q}, 1)$ acts trivially on each $\mathcal{H}^i\psi_h(\overline{\mathbb{Q}_l}[d])$, we have that

$$t^r_{\psi_h(\overline{\mathbb{Q}_l}[d])}(y) = t^r_{\psi_h^u(\overline{\mathbb{Q}_l}[d])}(y).$$

The following equation now follows by 2. and 3.

$$t^r_{\psi_h^u(\overline{\mathbb{Q}_l}[d])}(y) = \sum_{i \geq 0}(-1)^i q^{i\eta} t^r_{\mathrm{Ker}(C^i \to C^{i+1})}(y),$$

where $\eta = \deg(y)r$. We then verify Lemma 3.2.4. Indeed, one has in this case the following equation for every $r \geq 1$ and $\eta = \deg(y)r$,

$$(1 - q^\eta)\sum_{i \geq 0}(-1)^i q^{i\eta} t^r_{\mathrm{Ker}(C^i \to C^{i+1})}(y) = \sum_{i \geq 0}(-1)^i q^{i\eta}|\{\overline{a}_i^{-1}(y)\}|.$$

Since the map $\phi_1 : \tilde{X}_0 \to X_0$ is proper, the canonical map

$$\phi_{1*}\psi_h(\overline{\mathbb{Q}_l}[d]) \to \psi_g(\phi_{1*}\overline{\mathbb{Q}_l}[d])$$

is an isomorphism ([18, (1.2.2)]). In particular, if ϕ_1 is an isomorphism over U_0 (as in a desingularization of X_0) we get the following result.

Corollary 3.2.8. *Let X_0 be an integral and affine algebraic variety over $\mathbb{F}_q$ of dimension d. Let $x \in |X_0|$ be a singular point. Assume that X_0 admits an alteration $\phi_1 : \tilde{X}_0 \to X_0$ that is an isomorphism over U_0. Consider a point $y \in \tilde{X}_0$ such that $\phi_1(y) = x$. Then for every $r \geq 1$,*

$$\begin{aligned}(1 - q^{\deg(y)\nu})m^r_{X_0,x} = &\sum_{\substack{y\in\tilde{X}_0(\mathbb{F}_{q^\eta})\\ \phi_1(y)=x}} \sum_{i\geq 0}(-1)^i q^{i\deg(y)\nu}|\overline{a}_i^{-1}(y)| \\ &- (1 - q^{\deg(y)\nu})t^r_{\phi^u_g(IC_{X_0})}(x),\end{aligned}$$

where $\eta = \deg(x)r$ and $\nu = [\mathbb{F}_{q^\eta} : \kappa(y)]$.

Proof. Since $\phi_{1*}\psi_h(\overline{\mathbb{Q}_l}[d]) \to \psi_g(\overline{\mathbb{Q}_l}[d])$ is an isomorphism, we have for $r \geq 1$ and $\eta = deg(x)r$ that

$$t^r_{\psi^u_g(\overline{\mathbb{Q}_l}[d])}(x) = \sum_{\substack{y\in\tilde{X}_0(\mathbb{F}_{q^\eta})\\ \phi_1(y)=x}} t^\nu_{\psi_h(\overline{\mathbb{Q}_l}[d])}(y),$$

where $\nu = [\mathbb{F}_{q^\eta} : \kappa(y)]$. The result now follows by Lemma 3.2.4 and equation (3.3). □

Remark 3.2.9. In the corollary above, if X_0 is of dimension $1, 2$, or 3, we may apply the results of the previous chapter and calculate $(1 - q^{\deg(y)\nu})t^r_{\phi^u_g(IC_{X_0})}(x)$. Indeed, any desingularization $\pi : \tilde{X}_0 \to X_0$ of X_0 such that $\pi^{-1}(x)$ is a strict normal crossings divisor is also an alteration of X_0 as in the corollary.

3.3 Local Systems on $\mathbb{G}_{m,\mathbb{F}_q}$

By Proposition 3.1.10, we have for every $r \geq 1$ the following equation

$$m^r_{X_0,x} = t^r_{\psi^u_g(\overline{\mathbb{Q}_l}[d])}(x) - t^r_{\mathrm{Im}(N)}(x). \tag{3.9}$$

We can not expect such a good description for $t^r_{\psi^u_g(\overline{\mathbb{Q}_l}[d])}(x)$ as in Lemma 3.2.4. Nevertheless, we will give an alternative calculation of $t^r_{\psi^u_g(\overline{\mathbb{Q}_l}[d])}(x)$.

We begin by giving a characterization of $\psi^u_g(\overline{\mathbb{Q}_l}[d])$ through some local systems defined on $\mathbb{G}_{m,\mathbb{F}_q}$. For this next part we follow [28, §2 and §3].

Fix $a \in \mathbb{N}$. Let $\overline{\mathbb{Q}}_{l,a} := \overline{\mathbb{Q}_l} \oplus \overline{\mathbb{Q}_l}(-1) \oplus \cdots \oplus \overline{\mathbb{Q}_l}(-a)$, and let $N : \overline{\mathbb{Q}}_{l,a} \to \overline{\mathbb{Q}}_{l,a}(-1)$ be the (nilpotent) morphism given by

$$N = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}.$$

We define an action of the group $\pi_1(\mathbb{G}_{m,\mathbb{F}_q}, 1) = \pi_1^{geom}(\mathbb{G}_{m,\mathbb{F}_q}, 1) \rtimes Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$ on $\overline{\mathbb{Q}}_{l,a}$ in the following way: an element $u \rtimes \sigma$ acts by the matrix

$$\exp(t(u)N) \begin{pmatrix} 1 & & & \\ & \chi(\sigma)^{-1} & & \\ & & \ddots & \\ & & & \chi(\sigma)^{-a} \end{pmatrix},$$

where $\chi : Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q) \to \hat{\mathbb{Z}}(1)$ is the cyclotomic character and t is the usual surjective map from $\pi_1^{geom}(\mathbb{G}_{m,\mathbb{F}_q}, 1)$ to $\mathbb{Z}_l(1)$. We denote by $\mathcal{G}_a$ the local system on $\mathbb{G}_{m,\mathbb{F}_q}$ associated to $\overline{\mathbb{Q}}_{l,a}$.

For $a \leq b$, one has an injection $\alpha_{a,b} : \overline{\mathbb{Q}}_{l,a} \to \overline{\mathbb{Q}}_{l,b}$ and a surjection $\beta_{b,a} : \overline{\mathbb{Q}}_{l,b} \to \overline{\mathbb{Q}}_{l,a}(a-b)$, and these maps are $\pi_1(\mathbb{G}_{m,\mathbb{F}_q}, 1)$-equivariant, hence they define morphism of local systems $\alpha_{a,b} : \mathcal{G}_a \hookrightarrow \mathcal{G}_b$ and $\beta_{b,a} : \mathcal{G}_b \twoheadrightarrow \mathcal{G}_a(a-b)$. Note that the composition $\overline{\mathbb{Q}}_{l,a} \xrightarrow{\alpha_{a,a+1}} \overline{\mathbb{Q}}_{l,a+1} \xrightarrow{\beta_{a+1,a}} \overline{\mathbb{Q}}_{l,a}(-1)$ is equal to N.

For every perverse sheaf K_0 on U_0 or $\tilde{U_0}$, the sheaves $K_0 \otimes g^*\mathcal{G}_a$ on U_0 or $K_0 \otimes h^*\mathcal{G}_a$ on $\tilde{U_0}$ are also perverse; we will denote both of these perverse sheaves by $K_0 \otimes \mathcal{G}_a$.

Proposition 3.3.1. *For any $K_0 \in Perv(U_0)$ and every $a \in \mathbb{N}$ such that $N^{a+1}(\psi_g^u(K_0)) = 0$ (in particular, for a big enough), there is a natural isomorphism*

$$\psi_g^u(K_0)[-1] \cong {}^pH^{-1}i^*j_*(K_0 \otimes \mathcal{G}_a) = i^*j_{!*}(K_0 \otimes \mathcal{G}_a)[-1],$$

where $N : \psi_g^u(K_0) \to \psi_g^u(K_0)(-1)$ is given by the logarithm of the unipotent part of the monodromy.

Proof. This is a consequence of [28, Corollary 3.2]. □

Remark 3.3.2. The Proposition also holds for any $K_0 \in Perv(\tilde{U_0})$ and any $a \in \mathbb{N}$ such that $N^{a+1}(\psi_h^u(K_0)) = 0$ replacing i, j and $\psi_g^u(K_0)$ by $\tilde{i}$, $\tilde{j}$ and $\psi_h^u(K_0)$ respectively.

Until now we have been denoting the logarithm of the unipotent part of the monodromy and the nilpotent morphism $\beta_{a+1,a} \circ \alpha_{a,a+1}$ for a fix $a \in \mathbb{N}$ by the same symbol N. The next result shows that they are indeed equal.

Proposition 3.3.3. *Let $K_0 \in \mathrm{Perv}(U_0)$. Let $a \in \mathbb{N}$ such that $N^{a+1}(\psi_g^u(\overline{\mathbb{Q}_l}[d])) = 0$. Then the following diagram is commutative:*

$$\begin{array}{ccc}
\psi_g^u(\overline{\mathbb{Q}_l}[d])[-1] & \xrightarrow{\cong} & i^*j_{!*}(\overline{\mathbb{Q}_l}[d] \otimes \mathcal{G}_a)[-1] \\
\| & & \downarrow \alpha_{a,a+1} \\
\psi_g^u(\overline{\mathbb{Q}_l}[d])[-1] & \xrightarrow{\cong} & i^*j_{!*}(\overline{\mathbb{Q}_l}[d] \otimes \mathcal{G}_{a+1})[-1] \\
N \downarrow & & \downarrow \beta_{a+1,a} \\
\psi_h^u(\overline{\mathbb{Q}_l}[d])[-1](-1) & \xrightarrow{\cong} & (i^*j_{!*}(\overline{\mathbb{Q}_l}[d] \otimes \mathcal{G}_a)[-1])(-1)
\end{array}$$

Proof. This is a consequence of [28, Proposition 3.4]. □

Remark 3.3.4. The Proposition also holds for any $K_0 \in Perv(\tilde{U_0})$ and any $a \in \mathbb{N}$ such that $N^{a+1}(\psi_h^u(K_0)) = 0$ replacing i, j and $\psi_g^u(K_0)$ by $\tilde{i}$, $\tilde{j}$ and $\psi_h^u(K_0)$ respectively.

The next result follows.

Lemma 3.3.5. *For any $a \in \mathbb{N}$ such that $N^{a+1}\psi_g^u(\overline{\mathbb{Q}_l}[d]) = 0$ and $r \geq 1$,*

$$t^r_{\psi_g^u(\overline{\mathbb{Q}_l}[d])}(x) = t^r_{i^*j_{!*}\mathcal{G}_a[d]}(x).$$

Theorem 3.3.6. *Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension d. Let $x \in |X_0|$ be a singular point. Then for any $a \in \mathbb{N}$ such that $N^{a+1}\psi_g^u(\overline{\mathbb{Q}_l}[d]) = 0$ and any $r \geq 1$,*

$$m^r_{X_0,x} = t^r_{i^*j_{!*}\mathcal{G}_a[d]}(x) - t^r_{\mathrm{Im}(\beta_{a+1,a})}(x).$$

Proof. By Lemma 1.4.3 and Lemma 1.4.2 we may assume X_0 to be integral and affine. The theorem now follows by equation (3.9) and Proposition 3.3.3. □

Let us assume that $\mathcal{G}_a$ is a semi-simple smooth $\overline{\mathbb{Q}_l}$-sheaf on U_0. This is equivalent to the assumption that T acts trivially on $\psi_g(\overline{\mathbb{Q}_l}[d])$. By Example 3.2.7, this would be the case if $h : \tilde{X}_0 \to \mathbb{A}^1_{\mathbb{F}_q}$ is strictly semi-stable. In those cases we get the following result.

Proposition 3.3.7. *Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension d. Let $x \in |X_0|$ be a singular point. Assume that $N\psi_g^u(\overline{\mathbb{Q}_l}[d]) = 0$ (or assume that T acts trivially on $\psi_g(\overline{\mathbb{Q}_l}[d])$). Then for any $r \geq 1$,*

1.

$$m^r_{X_0,x} = t^r_{\psi_g^u(\overline{\mathbb{Q}_l}[d])}(x) = t^r_{R\Gamma(((X_0)_{(\overline{x})})_{\overline{\eta}},\overline{\mathbb{Q}_l}[d])}(x).$$

2.

$$0 = t^r_{\mathrm{Im}(\beta_{a+1,a})}(x) = t^r_{\mathrm{Im}(N)}(x) = t^r_{\phi_g^u(IC_{X_0})}(x)$$

Proof. The first part of the proposition follows by equation (3.6), Proposition 3.1.10 and equation (3.5).
The second part is a consequence of Theorem 3.3.6 and Proposition 3.1.10. □

3.4 *q*-Divisibility

In [17] L. Illusie mentions that another interesting question to ask is the q-divisibility of

$$\sum(-1)^i \mathrm{Tr}(g|H^i_c(X,\overline{\mathbb{Q}_l})),$$

for $g \in Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q)$. Here X_0 is an algebraic variety of dimension d over $\mathbb{F}_q$ and X denotes the base of X_0 to the algebraic closure of $\mathbb{F}_q$.

Accordingly, we ask ourselves about the q-divisibility of the r-multiplicity of a singular closed point $x \in X_0$. The results of the last two sections yield partial answers to this question.

Corollary 3.4.1. *Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension d. Let $x \in |X_0|$ be a singular point. Then for any $a \in \mathbb{N}$ such that $N^{a+1}\psi_g^u(\overline{\mathbb{Q}_l}[d]) = 0$ and any $r \geq 1$,*

$$m^{r+\nu}_{X_0,x} \equiv 1 - t^r_{\phi^u_h(\phi_1^* IC_{X_0})}(y) \qquad \text{mod } q$$
$$m^{r}_{X_0,x} \equiv t^r_{i^* j_{!*}\mathcal{G}_a[d]}(x) - t^r_{\mathrm{Im}(\beta_{a+1,a})}(x). \qquad \text{mod } q,$$

where $\nu = [\kappa(y) : \kappa(x)]$.

Proof. This is a consequence of Theorem 3.2.5 and Theorem 3.3.6. □

Corollary 3.4.2. *Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension d. Let $x \in |X_0|$ be a singular point. Assume that $N\psi_g^u(\overline{\mathbb{Q}_l}[d]) = 0$ (or assume that T acts trivially on $\psi_g(\overline{\mathbb{Q}_l}[d])$). Then for any $r \geq 1$,*

$$m^r_{X_0,x} \equiv t^r_{\psi^u_g(\overline{\mathbb{Q}_l}[d])}(x) \equiv t^r_{R\Gamma(((X_0)_{(\overline{x})})_{\overline{\eta}}, \overline{\mathbb{Q}_l}[d])}(x) \text{ mod } q.$$

Proof. This is a consequence of Proposition 3.3.7. □

Chapter 4

General Method

The description of $m^r_{X_0,x}$ given in chapter 2 through complete $\mathfrak{m}$-primary ideals of $\mathcal{O}_{X_0,x}$ may be true for higher dimensions. But already in dimension 3 this is hard to prove. On the other hand, an internal characterization through the cohomology groups $H^\bullet(\mathrm{Spec}(\mathcal{O}^{sh}_{X_0,\overline{x}_0}) \times U, \overline{\mathbb{Q}_l}[d])$ as in Lemma 1.3.5 would be general. Unfortunately, schemes such as $\mathrm{Spec}(\mathcal{O}^{sh}_{X_0,\overline{x}_0}) \times U$ are not even algebraic varieties in general and little is known about their cohomology groups.

A general characterization through *geometric* methods seems more plausible as shown by the relations of $m^r_{X_0,x}$ to the unipotent vanishing/nearby cycles functors (equation (3.3)) and the unipotent vanishing/nearby cycles functors of an alteration of X_0 (Theorem 3.2.5). We finish this thesis by giving a general method for calculating the r-multiplicity of a closed singular point in a normal algebraic variety.

Let X_0 be a normal algebraic variety over $\mathbb{F}_q$ of dimension d with a singular closed point x. By Lemma 1.4.3 we may assume X_0 to be integral. Given an alteration $\phi : \tilde{X}_0 \to X_0$ of X_0, there exists a non-empty open subscheme $V_0 \subset X_0$ such that $\phi^{-1}(V_0) \to V_0$ is a finite, étale and flat morphism ([20, Definition 2.20]). Let $U_0 := V_0 \cap (X_0)_{reg}$ and $Z_0 := X_0 \setminus U_0$ such that $j : U_0 \hookrightarrow X_0$ and $i : Z_0 \hookrightarrow X_0$ are the canonical morphisms. Consider then the morphism $\phi|_{\phi^{-1}(U_0)} : \phi^{-1}(U_0) \to U_0$ which is finite, étale and flat. We would like to know how many points are in the geometric fiber of any closed point $u \in U_0$. For this we define the following.

Let $f : X \to Y$ be a morphism of schemes. Let

$$n_{X/Y} : Y \to \{0, 1, \ldots, \infty\}$$

be the function which associates to $y \in Y$ the number of irreducible components of $(X_y)_K$ where K is a separably closed extension of $\kappa(y)$. This is well defined ([33, Lemma 36.25.3]).

Thus, the number $n_{\phi^{-1}(U_0)/U_0}$ counts the points in the geometric fiber of any closed point $u \in U_0$. We may assume that this number is constant due to the following result.

Proposition 4.0.1. *Let $f : X \to Y$ be a morphism of schemes. Assume f is of finite type. Let $y \in Y$ be a point. Then there exists a non-empty open $V \subset \overline{\{y\}}$ such that $n_{X/Y}|_V$ is constant.*

Proof. [33, Lemma 36.25.6]. □

Consider the generic point η of U_0 in the Proposition above. Then there exists a non-empty open $V \subset U_0$ such that $n_{\phi^{-1}(U_0)/U_0}|_V$ is constant. By shrinking U_0 to the intersection of U_0 with V, we may assume that $n_{\phi^{-1}(U_0)/U_0}$ is constant.

The morphism $\phi|_{\phi^{-1}(U_0)} : \phi^{-1}(U_0) \to U_0$ is then a connected étale cover of U_0. The group $\mathrm{Aut}(\phi^{-1}(U_0)/U_0)$ is finite of order less or equal than $n_{\phi^{-1}(U_0)/U_0}$. By [35, Proposition 5.3.9], there exists a Galois cover $\psi : C_0 \to \phi^{-1}(U_0)$ such that $\pi := \phi|_{\phi^{-1}(U_0)} \circ \psi : C_0 \to U_0$ is also a Galois cover with finite Galois group W. There exists an equivalence between W-equivariant sheaves over C_0 and sheaves over U_0 ([38, Theorem 4.46]). Thus, the complex $(\phi|_{\phi^{-1}(U_0)})_*\overline{\mathbb{Q}_l}[d]$ is a W-equivariant sheaf. By [21, Theorem III.15.4], there is a representation of W on $(\phi|_{\phi^{-1}(U_0)})_*\overline{\mathbb{Q}_l}[d]$, i.e., there exists a group homomorphism

$$W \to \mathrm{Aut}((\phi|_{\phi^{-1}(U_0)})_*\overline{\mathbb{Q}_l}[d]).$$

It follows that $(\phi|_{\phi^{-1}(U_0)})_*\overline{\mathbb{Q}_l}[d]$ can be decomposed into χ-isotypic components, where χ runs over the irreducible characters of W ([8, (1.3.4)])

$$(\phi|_{\phi^{-1}(U_0)})_*\overline{\mathbb{Q}_l}[d] = \bigoplus_{\chi \in \hat{W}} \mathcal{F}_\chi[d].$$

Here $\mathcal{F}_\chi$ are smooth $\overline{\mathbb{Q}_l}$-sheaves on U_0. Since $\overline{\mathbb{Q}_l}[d]$ is contained in $(\phi|_{\phi^{-1}(U_0)})_*\overline{\mathbb{Q}_l}[d]$ and is W-equivariant, it is also a direct summand of the

χ-isotypic decomposition of $(\phi|_{\phi^{-1}(U_0)})_*\overline{\mathbb{Q}_l}[d]$ (it corresponds to the trivial character). It follows that

$$(\phi|_{\phi^{-1}(U_0)})_*\overline{\mathbb{Q}_l}[d] = \overline{\mathbb{Q}_l}[d] \oplus \bigoplus_{\substack{\chi\in\hat{W}\\ \chi\neq\chi_0}} \mathcal{F}_\chi[d], \tag{4.1}$$

where χ_0 denotes the trivial character.

Theorem 4.0.2. *Let X_0 be a normal and proper algebraic variety over $\mathbb{F}_q$ of dimension d with a singular closed point x. Let $\phi : \tilde{X}_0 \to X_0$ be an alteration of an irreducible component containing x and define $D_0 := \phi^{-1}(\{x\})$. Then for $r \geq 1$,*

$$\begin{aligned} m^r_{X_0,x} =& |D(\mathbb{F}_{q^\eta})| - \sum_{j=0}^{r(\phi)} (-1)^j \mathrm{Tr}(Fr^\eta | H^{d+j}(D, \overline{\mathbb{Q}_l})) \\ & - \sum_{j=-r(\phi)}^{-1} (-1)^j \mathrm{Tr}(Fr^\eta | H^{d+j}(D, \overline{\mathbb{Q}_l}))\ q^{j\eta} - \sum_{\substack{\chi\in\hat{W}\\ \chi\neq\chi_0}} t^r_{j_{!*}\mathcal{F}_\chi[d]}(x), \end{aligned}$$

where $\eta = \deg(x)r$ and $r(\phi)$ is the defect of semi-smallness of ϕ.

Proof. Let $V_0 \subset X_0$ be a largest non-empty open subscheme such that $\phi^{-1}(V_0) \to V_0$ is a finite, étale and flat morphism. Define $U_0 := V_0 \cap (X_0)_{reg}$ and $Z_0 := X_0 \setminus U_0$. Consider the following cartesian diagram:

$$\begin{array}{ccccccc} D_0 & \xrightarrow{\tilde{\iota}} & \tilde{Z}_0 & \xrightarrow{\tilde{i}} & \tilde{X}_0 & \xleftarrow{\tilde{j}} & \tilde{U}_0 \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\phi} & & \downarrow \\ \{x\} & \xrightarrow{\iota} & Z_0 & \xrightarrow{i} & X_0 & \xleftarrow{j} & U_0. \end{array}$$

The omission of the zero subscript on the objects of the diagram represents as before the base change to the algebraic closure of $\mathbb{F}_q$. We have that

$$t^r_{\phi_*\overline{\mathbb{Q}_l}[d]}(x) = \sum_\delta (-1)^\delta t^r_{{}^pH^\delta(\phi_*\overline{\mathbb{Q}_l}[d])}(x).$$

Since restricting to U_0 is a t-exact functor, it commutes with perverse cohomology. It follows that

$$j^{*\,p}H^\delta(\phi_*\overline{\mathbb{Q}_l}[d]) = {}^pH^\delta((\phi|_{\tilde{U}_0})_*\overline{\mathbb{Q}_l}[d]) \text{ for } \delta \in \mathbb{Z}.$$

The defect of semi-smallness of $\phi|_{\tilde{U_0}}$ is zero, i.e., $r(\phi|_{\tilde{U_0}}) = 0$. It follows that $\phi|_{\tilde{U_0}}$ is semi-small and hence $(\phi|_{\tilde{U_0}})_*\overline{\mathbb{Q}_l}[d]$ is a perverse sheaf on U_0 ([21, Lemma III.7.5]). Thus,

$$j^{*p}H^\delta(\phi_*\overline{\mathbb{Q}_l}[d]) = \overline{\mathbb{Q}_l}[d] \text{ for } \delta = 0 \text{ and}$$
$$j^{*p}H^\delta(\phi_*\overline{\mathbb{Q}_l}[d]) = 0 \text{ for } \delta \neq 0.$$

By [3, Remarque 5.4.9], the complex $\phi_*\overline{\mathbb{Q}_l}[d]$ is pure, which implies that every ${}^pH^\delta(\phi_*\overline{\mathbb{Q}_l}[d])$ is pure ([3, Corollaire 5.4.4]). By [3, Corollaire 5.3.11], every ${}^pH^\delta(\phi_*\overline{\mathbb{Q}_l}[d])$ admits a unique decomposition

$${}^pH^\delta(\phi_*\overline{\mathbb{Q}_l}[d]) = j_{!*}K_0 \oplus i_*B_\delta,$$

where $\delta \in \mathbb{Z}$ and $K_0 \in Perv(U_0), B_\delta \in Perv(Z_0)$. Thus,

$${}^pH^\delta(\phi_*\overline{\mathbb{Q}_l}[d]) = i_*\mathcal{L}_\delta,$$

for $\delta \neq 0$ and each $B_\delta \in Perv(Z_0)$. By [3, Corollaire 5.3.11] again, we have for $\delta \neq 0$ that

$$\iota^*B_\delta = \mathcal{L}_\delta,$$

where $\mathcal{L}_\delta$ are local systems on $\{x\}$. For $|\delta| \leq r(\phi)$, we have exact triangles

$${}^p\tau_{<\delta}\phi_*\overline{\mathbb{Q}_l}[d] \to {}^p\tau_{\leq\delta}\phi_*\overline{\mathbb{Q}_l}[d] \to {}^pH^\delta(\phi_*\overline{\mathbb{Q}_l}[d])[-\delta] \xrightarrow{[1]} .$$

For $\delta \geq 0$, we have that

$${}^pH^\delta(\iota^*i^*\ {}^p\tau_{\leq\delta}\phi_*\overline{\mathbb{Q}_l}[d]) = {}^pH^\delta(\iota^*i^{*p}H^\delta(\phi_*\overline{\mathbb{Q}_l}[d])[-\delta]) \text{ and}$$
$${}^pH^\delta(\iota^*i^*\ {}^p\tau_{\leq\delta}\phi_*\overline{\mathbb{Q}_l}[d]) = {}^pH^\delta(\iota^*i^*\ {}^p\tau_{\leq\delta+1}\phi_*\overline{\mathbb{Q}_l}[d]).$$

The first equality is a direct consequence of the exact triangles above and the t-right-exactness of the functors ι^* and i^*. The second equality is due to the fact that

$${}^pH^\delta(\iota^*i^{*p}H^{\delta+1}(\phi_*\overline{\mathbb{Q}_l}[d])[-(\delta+1)]) = {}^pH^\delta(\mathcal{L}_{\delta+1}[-(\delta+1)]) = 0 \text{ and}$$
$${}^pH^{\delta-1}(\iota^*i^{*p}H^{\delta+1}(\phi_*\overline{\mathbb{Q}_l}[d])[-(\delta+1)]) = {}^pH^{\delta-1}(\mathcal{L}_{\delta+1}[-(\delta+1)]) = 0$$

for $\delta \geq 0$. Since $\phi_*\overline{\mathbb{Q}_l}[d] = {}^p\tau_{\leq r(\phi)}\pi_*\overline{\mathbb{Q}_l}[d]$, the two equalities above imply that for $0 \leq \delta \leq r(\phi)$,

$$\begin{aligned}\mathcal{L}_\delta &= {}^pH^\delta(\iota^*i^{*p}H^\delta(\phi_*\overline{\mathbb{Q}_l}[d])[-\delta]) \\ &= {}^pH^\delta(\iota^*i^*\ {}^p\tau_{\leq r(\phi)}\phi_*\overline{\mathbb{Q}_l}[d]) \qquad (4.2) \\ &= {}^pH^\delta(\iota^*i^*\phi_*\overline{\mathbb{Q}_l}[d]).\end{aligned}$$

Since X_0 is proper, the variety $\tilde{X_0}$ is also proper and the open immersion $j_1 : \tilde{X_0} \to \overline{X_0}$ is an isomorphism. Thus $\tilde{X_0}$ is projective. It follows that the morphism $\phi : \tilde{X_0} \to X_0$ is projective. By the Hard Lefschetz Formula, we have for $\delta \geq 1$ the equality

$$\mathcal{L}_{-\delta} = \mathcal{L}_{\delta}(\delta). \tag{4.3}$$

It follows that

$$\begin{aligned} t^r_{\pi_* \overline{\mathbb{Q}_l}[d]}(x) &= \sum_{\delta} (-1)^{\delta} t^r_{{}^pH^{\delta}(\pi_* \overline{\mathbb{Q}_l}[d])}(x) \\ &= \left(\sum_{|\delta| \leq r(\phi)} (-1)^{\delta} t^r_{\mathcal{L}_{\delta}}(x) \right) + t^r_{j_{!*}(\phi|_{\tilde{U_0}})_* \overline{\mathbb{Q}_l}[d]}(x). \end{aligned}$$

By equation (1.1), the left hand side is equal to $|D_0(\mathbb{F}_{q^{\eta}})|$ for $\eta = \deg(x)r$. By equations (4.2) and (4.3), we have that

$$t^r_{\mathcal{L}_{\delta}}(x) = \begin{cases} \mathrm{Tr}(Fr^{\eta} | H^{d+\delta}(D, \overline{\mathbb{Q}_l})), & \text{if } \delta \geq 0 \\ \mathrm{Tr}(Fr^{\eta} | H^{d+\delta}(D, \overline{\mathbb{Q}_l}))\ q^{\delta\eta}, & \text{if } \delta < 0. \end{cases}$$

Equation (4.1) says that

$$(\phi|_{\phi^{-1}(U_0)})_* \overline{\mathbb{Q}_l}[d] = \overline{\mathbb{Q}_l}[d] \oplus \bigoplus_{\substack{\chi \in \hat{W} \\ \chi \neq \chi_0}} \mathcal{F}_{\chi}[d],$$

where χ_0 denotes the trivial character. It follows that,

$$t^r_{j_{!*}(\phi|_{\tilde{U_0}})_* \overline{\mathbb{Q}_l}[d]}(x) = m^r_{X_0,x} + \sum_{\substack{\chi \in \hat{W} \\ \chi \neq \chi_0}} t^r_{j_{!*}\mathcal{F}_{\chi}[d]}(x).$$

The theorem now follows. □

Remark 4.0.3. If the order of $\mathrm{Aut}(\phi^{-1}(U_0)/U_0)$ is equal to $n_{\phi^{-1}(U_0)/U_0}$, then the morphism $\phi|_{\phi^{-1}(U_0)} : \phi^{-1}(U_0) \to U_0$ is a Galois cover and we may replace W by $\mathrm{Aut}(\phi^{-1}(U_0)/U_0)$ in the theorem.

Corollary 4.0.4. *Let X_0 be a normal and proper algebraic variety over $\mathbb{F}_q$ of dimension d with a singular closed point x. Assume $\phi : \tilde{X}_0 \to X_0$ to be a desingularization of X_0 in the strong sense. Let $D_0 := \phi^{-1}(\{x\})$. Then for $r \geq 1$,*

$$\begin{aligned} m^r_{X_0,x} =& |D(\mathbb{F}_{q^\eta})| - \sum_{j=0}^{r(\phi)} (-1)^j \mathrm{Tr}(Fr^\eta | H^{d+j}(D, \overline{\mathbb{Q}_l})) \\ & - \sum_{j=-r(\phi)}^{-1} (-1)^j \mathrm{Tr}(Fr^\eta | H^{d+j}(D, \overline{\mathbb{Q}_l}))\ q^{j\eta}, \end{aligned}$$

where $\eta = \deg(x) r$ and $r(\phi)$ is the defect of semi-smallness of ϕ.

Proof. Since $\phi : \tilde{X}_0 \to X_0$ is a desingularization of X_0 in the strong sense, the morphism $\phi|_{\phi^{-1}(U_0)} : \phi^{-1}(U_0) \to U_0$ is an isomorphism. The corollary now follows. □

Bibliography

[1] E. Artin. “Quadratische Körper im Gebiete der höheren Kongruenzen. II. Analytischer Teil”. In: *Mathematische Zeitschrift* (1924). 19, pp. 207–246.

[2] A. A. Beilinson. “How to glue perverse sheaves”. In: *K-theory, arithmetic and geometry* (1984–1986). volume 1289 of Lecture Notes in Math, pp. 42–51.

[3] A. Beilinson, J. Bernstein and P. Deligne. “Faisceaux pervers”. In: *Astérisque* (1982). Société Mathématique de France, Paris. 100.

[4] M. A. de Cataldo. “Perverse sheaves and the topology of algebraic varieties (2015 PCMI)”. In: *arXiv:1506.03642* (2015).

[5] V. Cossart and O. Piltant. “Resolution of singularities of threefolds in positive characteristic. II”. In: *Journal of Algebra **321** (7)* (2009). doi:10.1016/j.jalgebra.2008.11.030 MR 2494751, pp. 1836–1976.

[6] P. Deligne. “La conjecture de Weil I”. In: *Inst. Hautes Études Sci., Pul. Math.* 43 (1974).

[7] P. Deligne. “La conjecture de Weil II”. In: *Inst. Hautes Études Sci., Pul. Math.* 52 (1980).

[8] P. Deligne. “Catégories tensorielles”. In: *Mosc. Math. J.* 2 (2002), pp. 227–248.

[9] B. Dwork. “On the rationality of the zeta function of an algebraic variety”. In: *Amer. J. Math. 82* (1960), pp. 631–648.

[10] D. Mumford E. Bombieri. “Enriques’ Classification of Surfaces in Char. p, II.” In: *Complex analysis and algebraic geometry* (1977). Iwanami Shoten, Tokyo, pp. 23–42.

[11] K. Fujiwara. “A Proof of the Absolute Purity Conjecture (after Gabber)”. In: *Advance Studies in Pure mathematics* (2002). 36, pp. 153–183.

[12] K. Fujiwara. “Independence of l for intersection cohomology (after Gabber)”. In: *Algebraic Geometry 2000, Azumino* (2002).

[13] M. Goresky and R. MacPherson. “Intersection Homology Theory”. In: *Topology vol.19* (1980). doi:10.1016/0040-9383(80)90003-8, pp. 135–161.

[14] T. Haines and T. Richarz. “Smoothness of Schubert varieties in twisted affine Grassmannians”. In: *https://arxiv.org/abs/1809.08464.* (2018).

[15] R. Hartshorne. *Algebraic Geometry.* Graduate Texts in Math. 52. New York: Springer-Verlag, 1977.

[16] L. Illusie. "Autour du théorème de monodromie locale". In: *Astérisque* 223 (1988), pp. 9–57.

[17] L. Illusie. "Miscellany on traces in l-adic cohomology: a survey". In: *Jpn. J. Math.* 1(1) (2006), pp. 107–136.

[18] L. Illusie. "Grothendieck and vanishing cycles". In: *AMS* 223 (). DOI: 01A65.

[19] P. Deligne, J. F. Boutot, A. Grothendieck, L. Illusie and J. L. Verdier. *Séminaire de Géométrie Algébrique du Bois Marie - Cohomologie étale -SGA4.5*. ISBN 978-3-540-08066-4. New York: Springer-Verlag, 1977.

[20] A. J. de Jong. "Smoothness, semi-stability and alterations," in: *Publications Mathematiques I.H.E.S.* Tome 83 (1996), pp. 51–93.

[21] R. Kiehl and R. Weissauer. *Weil Conjectures, Perverse Sheaves and l'adic Fourier Transform*. ISBN 3-540-41457-6. Germany: Springer-Verlag, 2001.

[22] F. Kirwan and J.Woolf. *An Introduction to Intersection Homology Theory*. Second Edition. Boca Raton, FL: Taylor and Francis Group, 2006.

[23] G. Laumon. "Transformation de Fourier, constantes d'équations fonctionelles et conjecture de Weil". In: *Publ. math. IHÉS 65* (1987), pp. 131–210.

[24] J. Lipman. "Rational singularities, with applications to algebraic surfaces and unique factorization". In: *Publications mathématiques* 36 (1969), pp. 195–279. DOI: 10.1007/BF02684604.

[25] J. Lipman. "Desingularization of two-dimensional schemes". In: *Ann. Math.* 107 (1978), pp. 151–207.

[26] Q. Liu. *Algebraic Geometry and Arithmetic Curves*. ISBN 0 19 8502842. Oxford: Oxford University Press, 2002.

[27] D. Maisner and E. Nart. "Abelian surfaces over finite fields as Jacobians". In: *Experiment. Math.* ***11*** (2002), pp. 321–337.

[28] S. Morel. *Beilinson's construction of nearby cycles and gluing*. https://web.math.princeton.edu/~smorel/gluing.pdf. 2018.

[29] M. Rapoport and T. Zink. "Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik." In: *In. Math.* (1982). 68, pp. 21–201.

[30] R. Reich. "Notes on Beilinson's "How to glue perverse sheaves"". In: *Journal of Singularities* 1 (2010), pp. 94–115.

[31] J. P. Serre. "Groupes proalgébriques". In: *Publications Mathématiques de l'IHES* (1960). 7, pp. 1–67.

[32] J. P. Serre and J. Tate. "Good reduction of abelian varieties". In: *Ann. of Math.* (1968). 88, pp. 492–517.

[33] The Stacks project authors. *The Stacks project*. https://stacks.math.columbia.edu. 2018.

[34] S. Sun. "Independence of l for the supports in the Decomposition Theorem". In: *arXiv:1512.01447* (2018).

[35] T. Szamuely. *Galois Groups and Fundamental Groups*. Online ISBN: 9780511627064. Cambridge University Press, 2009.

[36] E. Tasso. "On the Motive of the Intersection Cohomology of a Singular Three Dimensional Variety". In: (2005). Università degli Studi di Trento Dottorato di Ricerca in Matematica XV Ciclo.

[37] R.W. Thomason. "Absolute cohomological purity". In: *Bulletin de la S. M. F.* (1984). tome 112, pp. 397–406.

[38] A. Vistoli. "Notes on Grothendieck topologies, fibered categories and descent theory". In: (2007). DOI: `arXiv:math/0412512`.

[39] A. Weil. "Number of solutions of equations in finite fields". In: *Bull. AMS 55* (1949), pp. 497–508.

[40] J. Wildeshaus. "Pure motives, mixed motives and extensions of motives associated to singular surfaces". In: *arXiv:0706.4447* (2011).