Münster J. of Math. **10** (2017), 485–521
DOI 10.17879/70299517944
urn:nbn:de:hbz:6-70299518227

Münster Journal of Mathematics

The real C^*-algebra of a graph with involution

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(Communicated by Joachim Cuntz)

Abstract. We introduce a real C^*-algebra $C^*_{\mathbb{R}}(E;\gamma)$ associated to a graph E with an involution γ. In the case that γ is the identity, this algebra is isomorphic to the real graph algebra $C^*_{\mathbb{R}}(E)$ studied in [5]. By allowing nontrivial involutions we obtain much larger class of real C^*-algebras. We develop some basic theory, and we prove a Pimsner–Voiculescu type long exact sequence for the real K-theory of $C^*_{\mathbb{R}}(E;\gamma)$. We present several examples, some that we analyze directly and others that we analyze by combining K-theory computations with classification theorems.

1. Introduction

In this paper, we introduce and study the real C^*-algebra $C^*_{\mathbb{R}}(E;\gamma)$ associated to a graph E with an involution γ. Our motivation for this project arises first from the desire for more examples of real C^*-algebras to draw upon, but it is also clear that this is a very natural and compelling class of real C^* that warrant study.

The construction generalizes the construction of the real C^*-algebra $C^*_{\mathbb{R}}(E)$ associated to a graph E studied in [5]. The class of C^*-algebras obtained from graphs with involution contains the class of real graph C^*-algebras; but we will see that it is significantly larger, providing examples of multiple real structures on the same complex graph C^*-algebra. Furthermore, we prove the existence of a Pimsner–Voiculescu type long exact sequence on K-theory (generalizing [5, Thm. 13]) that allows us to compute the K-theory of $C^*_{\mathbb{R}}(E;\gamma)$. In the simple purely infinite case this enables us to identify the isomorphism class of $C^*_{\mathbb{R}}(E;\gamma)$. We will present several examples of K-theory computations of all eight groups of $KO_*(C^*_{\mathbb{R}}(E;\gamma))$ to demonstrate the facility of the long exact sequence and to show how tractable the K-theory computations can be.

Recall that there is an equivalence between the category $\mathcal{R}$ of real C^*-algebras and the category $\mathcal{C}^\tau$ of $C^{*,\tau}$-algebras. A $C^{*,\tau}$-algebra is a pair (A,τ) where A is a complex C^*-algebra and τ is a involutive anti-automorphism of A. A morphism in $\mathcal{C}^\tau$ is a C^*-algebra homomorphism that commutes with the involutions. Given a real C^*-algebra A, the corresponding $C^{*,\tau}$-algebra is the complexification $A_{\mathbb{C}}$ with anti-automorphism $a + ib \mapsto a^* + ib^*$. Conversely, given a $C^{*,\tau}$-algebra (A, τ) the corresponding real C^*-algebra is

$$A^\tau = \{a \in A \mid \tau(a) = a^*\}.$$

See [10, §2] for a detailed discussion of this categorical equivalence.

Two involutions τ_1 and τ_2 on a C^*-algebra A are equivalent if there is an automorphism α of A such that $\alpha \circ \tau_1 = \tau_2 \circ \alpha$. Then the question of classifying (up to isomorphism) all the real C^*-algebras whose complexification is isomorphic to a given C^*-algebra A translates under the categorical equivalence described in the previous paragraph to the question of classifying involutive anti-automorphisms of A up to equivalence.

The complexification functor from the category $\mathcal{R}$ to the category $\mathcal{C}$ of all complex C^*-algebras is neither surjective nor injective in general. On one hand it is not surjective since a complex C^*-algebra is isomorphic to the complexification of a real C^*-algebra only if it is isomorphic to its own opposite algebra. Thus any C^*-algebra that is not isomorphic to its own opposite algebra, such as those described in [12] and [13], is not the complexification of any real C^*-algebra.

On the other hand, there are many C^*-algebras B known to have more than one real structure. Examples of such C^*-algebras include finite-dimensional C^*-algebras (such as $M_n(\mathbb{C})$ for n even), commutative C^*-algebras (such as $C_0(X)$ when X is locally compact and has a nontrivial involution), stably finite C^*-algebras (such as $C(S^1, M_n(\mathbb{C}))$), and purely infinite simple C^*-algebras (such as $\mathcal{O}_n$ when n is odd).

Focusing our discussion on this last class of examples, let B be a simple, purely infinite, nuclear, C^*-algebra satisfying the UCT. Then we know that B is isomorphic to the complexification of at least one real C^*-algebra by [4, Thm. 17]. Furthermore, we know that the collection of such real C^*-algebras is classified up to isomorphism by united K-theory by [7, Cor. 10.3]. The set of real structures of a given C^*-algebra B is parametrized by the set of abstract CRT-modules that are compatible with the K-theory of B.

However, though we can classify the real structures of B in terms of the united K-theory, we may not have familiar representations of the corresponding real C^*-algebra; and the realization theorem in [4] is not especially constructive. For example, we know that the Cuntz algebra $\mathcal{O}_n$ for n even or $n = \infty$ has only one real structure, namely that corresponding to the real Cuntz algebra $\mathcal{O}_n^{\mathbb{R}}$. For n odd, $\mathcal{O}_n$ has two real structures up to isomorphism. One corresponds to the real Cuntz algebra $\mathcal{O}_n^{\mathbb{R}}$ and the other corresponds to an exotic real C^*-algebra known as $\mathcal{E}_n$. The algebras $\mathcal{E}_n$ are known to exist by their united K-theory. They are known to be simple and purely infinite, but we do not have a familiar representation for them. For $k > 1$, the C^*-algebra $M_k(\mathcal{O}_n)$ is even richer with potential real structures.

In [5] we studied the real C^*-algebra $C^*_{\mathbb{R}}(E)$ associated to a graph. Corollary 16 there implies that for a simple purely infinite nuclear UCT C^*-algebra B there is one and only one real C^*-algebra A arising as a graph algebra such that $A_{\mathbb{C}} \cong B$. So this construction does not turn out to be a very rich source of examples, and certainly not a source of examples of multiple real structures on a given C^*-algebra. In contrast, we will find that the class of real C^*-algebras $C^*(E;\gamma)$ associated to graphs with involution is much richer and does result in multiple real structures on the same complex graph C^*-algebra.

Although we find that $\mathcal{E}_n$ in particular cannot be realized as the real C^*-algebra arising from a graph with involution, we do find a family of real C^*-algebras that are KK-equivalent to $\mathcal{E}_n$ up to suspension (for odd integers $n \geq 3$). We also find examples to show that there exist real structures on $M_2(\mathbb{C}) \otimes \mathcal{O}_n$, isomorphic to neither $M_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathcal{O}_n^{\mathbb{R}}$ nor $M_2(\mathbb{R}) \otimes_{\mathbb{R}} \mathcal{E}_n$, for each $n \equiv 1 \mod 4$.

2. Definition and basic structure theorems

Definition 2.1. A *graph* E is a quadruple $E = (E^0, E^1, r, s)$ where E^0 is a countable vertex set, E^1 is a countable edge set, and $r, s\colon E^1 \to E^0$ are the range and source functions.

Definition 2.2. An *involution* γ of a graph $E = (E^0, E^1, r, s)$ is a pair $\gamma = (\gamma_0, \gamma_1)$ of functions satisfying

- $\gamma_i\colon E^i \to E^i$ $(i = 0, 1)$,
- $(\gamma_i)^2 = \mathrm{id}_{E^i}$ $(i = 0, 1)$,
- $r \circ \gamma_1 = \gamma_0 \circ r$,
- $s \circ \gamma_1 = \gamma_0 \circ s$.

In the C^*-algebra literature, there are two differing definitions of a graph C^*-algebra $C^*(E)$, for a given directed graph E. While the two definitions may yield nonisomorphic C^*-algebras for a given directed graph E, they are categorically equivalent since each construction is isomorphic to the other one applied to the opposite graph E^{op}, which is the graph obtained by reversing the orientation of each edge. Thus, any theory developed in the language of one definition can be translated into language using the other definition by "reversing the arrows." The definition that we take below is chosen in an attempt to conform to what we perceive to be the recently prevailing convention, although this differs from the author's previous work [5] and from the commonly used reference [14].

We take the C^*-algebra $C^*(E)$ associated to a directed graph E to be the universal C^*-algebra generated by mutually orthogonal projections p_v and partial isometries s_e (for all $v \in E^0$ and all $e \in E^1$) such that

- $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,
- $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$,
- $\sum_{s(e)=v} s_e s_e^* = p_v$ for all $v \in E^0$ such that $0 < |s^{-1}(v)| < \infty$,
- $(s_e s_e^*)(s_f s_f^*) = 0$ for $e, f \in E^1$ and $e \neq f$.

Given an involution γ on a graph E, there is a unique anti-multiplicative involution γ on $C^*(E)$ that satisfies

- $\gamma(p_v) = p_{\gamma(v)}$ for all $v \in E^0$,
- $\gamma(s_e) = s^*_{\gamma(e)}$ for all $e \in E^1$.

Indeed, the homomorphism γ exists by the universal property of $C^*(E)$ since the elements $p_{\gamma(v)}$ and $s^*_{\gamma(e)}$ viewed in $(C^*(E))^{\rm op}$ form a Cuntz–Krieger system for the graph E. Thus the pair $(C^*(E), \gamma)$ is a $C^{*,\tau}$-algebra, and we let

$$C^*_{\mathbb{R}}(E;\gamma) = \{a \in C^*(E) \mid \gamma(a) = a^*\}$$

denote the corresponding real C^*-algebra. Note that the projection p_v is in $C^*_{\mathbb{R}}(E;\gamma)$ if and only if $\gamma(v) = v$ and similarly the partial isometry s_e is in $C^*_{\mathbb{R}}(E;\gamma)$ if and only if $\gamma(e) = e$. More generally, $C^*_{\mathbb{R}}(E;\gamma)$ contains elements of the form $\lambda p_v + \overline{\lambda} p_{\gamma(v)}$ and $\lambda s_e + \overline{\lambda} s_{\gamma(e)}$ for $\lambda \in \mathbb{C}$. Note that in the special case $\gamma = \mathrm{id}$, then $C^*_{\mathbb{R}}(E;\mathrm{id}) \cong C_{\mathbb{R}}(E)$.

As expected, the pair $(C^*(E), \gamma)$ is characterized by the following universal property in the category of $C^{*,\tau}$-algebras.

Theorem 2.3. *Let (E, γ) be a directed graph with involution. Then for any $C^{*,\tau}$-algebra (A, τ) with mutually orthogonal projections q_v (for $v \in E^0$) and partial isometries t_e (for $e \in E^1$) that satisfy*

- $t_e^* t_e = q_{r(e)}$ *for all* $e \in E^1$,
- $t_e t_e^* \leq q_{s(e)}$ *for all* $e \in E^1$,
- $\sum_{s(e)=v} t_e t_e^* = q_v$ *for all* $v \in E^0$ *such that* $0 < |s^{-1}(v)| < \infty$,
- $(t_e t_e^*)(t_f t_f^*) = 0$ *for* $e, f \in E^1$ *and* $e \neq f$,
- $\tau(q_v) = q_{\gamma(v)}$ *for all* $v \in E^0$,
- $\tau(t_e) = t^*_{\gamma(e)}$ *for all* $e \in E^1$.

there is a unique $C^{*,\tau}$-algebra homomorphism

$$\alpha\colon (C^*(E), \gamma) \to (A, \tau)$$

satisfying

- $\alpha(p_v) = q_v$ *for all* $v \in E^0$,
- $\alpha(s_e) = t_e$ *for all* $e \in E^1$.

Proof. The first four conditions and the universal property of $C^*(E)$ guarantee the existence of a unique C^*-algebra homomorphism α that satisfies $\alpha(p_v) = q_v$ and $\alpha(e_e) = t_e$ (by [14, Prop. 1.21]). It remains to show that $\alpha(\gamma(x)) = \tau(\alpha(x))$ for all $x \in C^*(E)$. It suffices to prove this formula for $x = p_v$ and $x = s_e$ for all $v \in E^0$ and $e \in E^1$ since such elements generate $C^*(E)$. Using $\tau(q_v) = q_{\gamma(v)}$ and $\tau(t_e) = t^*_{\gamma(e)}$, we have

$$\alpha(\gamma(p_v)) = \alpha(p_{\gamma(v)}) = q_{\gamma(v)} = \tau(q_v) = \tau(\alpha(p_v))$$

and

$$\alpha(\gamma(s_e)) = \alpha(s^*_{\gamma(e)}) = t^*_{\gamma(e)} = \tau(t_e) = \tau(\alpha(s_e)).$$ □

The next theorem implies that the isomorphism class of $C^*_{\mathbb{R}}(E;\gamma)$ depends only on the action of the involution γ_0 on the set of vertices of E and not the involution γ_1 on the set of edges. Therefore, in what follows it will be sufficient to specify the action of an involution γ on the vertex set, provided the given action is consistent with a full graph involution.

For a graph (E^0, E^1) with vertices $v, w \in E^0$, let $E_{v,w}$ be the set of edges from vertex v to vertex w. Let $(E^0)^\gamma = \{v \in E^0 \mid \gamma(v) = v\}$.

Theorem 2.4. *Let E be a graph.*

(i) *If γ is an involution on E such that $\gamma(v) = v$ for all $v \in E^0$ then*
$$C^*_{\mathbb{R}}(E;\gamma) \cong C^*_{\mathbb{R}}(E).$$

(ii) *Suppose γ and δ are involutions on E such that $\gamma(v) = \delta(v)$ for all $v \in E^0$ and that $\gamma(e) = \delta(e) = \mathrm{id}(e)$ for all edges e in*
$$\bigcup_{v,w \in (E^0)^\gamma} E_{v,w}.$$
Then $(E,\gamma) \cong (E,\delta)$, as graphs with involution.

(iii) *If γ and δ are involutions on E such that $\gamma(v) = \delta(v)$ for all $v \in E^0$ then*
$$C^*_{\mathbb{R}}(E;\gamma) \cong C^*_{\mathbb{R}}(E;\delta).$$

Proof. First consider the situation in which there are two distinct edges e_1 and e_2 both from a vertex v to a vertex w, and that γ is an involution on E such that $\gamma(v) = v$, $\gamma(w) = w$, and $\gamma(e_1) = e_2$ where $e_1 \neq e_2$. Write $s_1 = s_{e_1}$ and $s_2 = s_{e_2}$. Then we have
$$\begin{aligned}
&s_1^* s_1 = s_2^* s_2 = p_w,\\
&s_1 s_1^* + s_2 s_2^* \leq p_v,\\
&\gamma(p_v) = p_v, \quad \gamma(p_w) = p_v,\\
&\gamma(s_1) = s_2^*, \quad \gamma(s_2) = s_1^*.
\end{aligned}$$
Let $t_1 = 1/\sqrt{2}(s_1 + s_2)$ and $t_2 = i/\sqrt{2}(s_1 - s_2)$. Then the reader can easily compute that
$$\begin{aligned}
&t_1^* t_1 = t_2^* t_2 = p_w,\\
&t_1 t_1^* + t_2 t_2^* = s_1 s_1^* + s_2 s_2^* \leq p_v,\\
&\gamma(t_1) = t_1^*, \quad \gamma(t_2) = t_2^*.
\end{aligned}$$
Let $\widetilde{\gamma}$ be the modified involution on E that satisfies $\widetilde{\gamma}(e_1) = e_1$ and $\widetilde{\gamma}(e_2) = e_2$, and otherwise is the same as γ. It follows that if
$$\chi = \{p_v, s_e \mid v \in E^0, e \in E^1\}$$
is a Cuntz–Krieger system for the graph E with involution γ, then
$$\widetilde{\chi} = (\chi - \{s_1, s_2\}) \cup \{t_1, t_2\}$$
is a Cuntz–Krieger system for E with involution $\widetilde{\gamma}$. Since both χ and $\widetilde{\chi}$ generate the same C^*-algebra, it follows that $C^*_{\mathbb{R}}(E;\gamma) \cong C^*_{\mathbb{R}}(E;\widetilde{\gamma})$.

Now, suppose that γ is an involution on E that fixes all vertices as in part (i) of the theorem. Then we can apply the reasoning of the previous paragraph simultaneously to any and all pairs of edges that are interchanged by γ (there will be countably many) to show that

$$C^*_{\mathbb{R}}(E;\gamma) \cong C^*_{\mathbb{R}}(E;\mathrm{id}) \cong C^*_{\mathbb{R}}(E).$$

This completes the proof of part (i).

Under the assumptions of part (ii), we show there is an isomorphism α in the category of graphs with involution from (E,γ) to (E,δ). Start by defining α to be the identity on the vertex set and on the edges in $E_{v,w}$ where v and w are both in $(E^0)^\gamma$. Now select two vertices v and w not both in $(E^0)^\gamma$. We extend the definition of α to both $E_{v,w}$ and $E_{\gamma(v),\gamma(w)}$ by the formulas

$$\alpha(e) = \begin{cases} \delta\gamma(e) & \text{for } e \in E_{v,w}, \\ e & \text{for } e \in E_{\gamma(v),\gamma(w)}. \end{cases}$$

We check that this satisfies $\alpha\gamma = \delta\alpha$ on the set $E_{v,w} \cup E_{\gamma(v),\gamma(w)}$. Indeed, we have

$$\begin{aligned} \alpha\gamma(e) &= \gamma(e) = \delta\delta\gamma(e) = \delta\alpha(e) \quad \text{for } e \in E_{v,w}, \\ \alpha\gamma(e) &= \delta\gamma\gamma(e) = \delta(e) = \delta\alpha(e) \quad \text{for } e \in E_{\gamma(v),\gamma(w)}. \end{aligned}$$

Note that the definition of α on $E_{v,w} \cup E_{\gamma(v),\gamma(w)}$ is based on the choice of vertices v and w. In this way we extend α to the entire edge set E^1 by making a choice of a preferred vertex pair (v,w) over $(\gamma(v),\gamma(w))$ for every orbit of order 2 in $E^0 \times E^0$. The result is an isomorphism that satisfies $\alpha\gamma = \delta\alpha$. Therefore $(E,\gamma) \cong (E,\delta)$.

For part (iii), suppose that γ and δ are two involutions of E that are equal on the set of vertices, but may act differently on the set of edges. First use the result of the first paragraph to replace γ and δ with involutions that fix all edges from v to w where $v,w \in (E^0)^\gamma = (E^0)^\delta$. Then the result follows by part (ii). $\square$

The next theorem describes the structure of $C^*_{\mathbb{R}}(E;\gamma)$ when E is finite with no cycles. Recall from [14, Prop. 1.18] that if E has no cycles then

$$C^*(E) \cong \bigoplus_{v \in E^0_{\text{sink}}} M_{n(v)}(\mathbb{C})$$

where E^0_{sink} is the set of sinks in E and $n(v)$ is the cardinality of Λ_v, the set of directed paths α such that $r(\alpha) = v$. Our result below generalizes this result to graphs with involution; but we also extend the result using a broader finiteness condition on E.

Since γ restricts to an involution on E^0_{sink}, we write the set of sinks as a disjoint union

$$E^0_{\text{sink}} = F_1 \sqcup F_2 \sqcup F_3$$

where F_1 is the set of sinks that are fixed by γ, and the remaining sinks are partitioned into F_2 and F_3 in such a way that $\gamma(F_2) = F_3$ and $\gamma(F_3) = F_2$ (we can take F_2 to be any maximal set of vertices of E^0_{sink} such that $F_2 \cap \gamma(F_2) = \varnothing$).

Theorem 2.5. *Let E be a directed graph with no cycles and an involution γ. Suppose that for each vertex v, there is a finite number of directed paths originating at v. Then*

$$C^*_{\mathbb{R}}(E;\gamma) \cong \Big(\bigoplus_{v\in F_1} \mathcal{K}^{\mathbb{R}}(\ell^2_{\mathbb{R}}(\Lambda_v))\Big) \oplus \Big(\bigoplus_{v\in F_2} \mathcal{K}(\ell^2(\Lambda_v))\Big).$$

Note that the finiteness condition implies that every maximal path is finite and terminates at a sink. It also implies that $|s^{-1}(v)| < \infty$ for all $v \in E^0$. If E is a finite directed graph with no cycles then the condition holds automatically and the statement of Theorem 2.5 reduces to

$$C^*_{\mathbb{R}}(E;\gamma) \cong \Big(\bigoplus_{v\in F_1} M_{n(v)}(\mathbb{R})\Big) \oplus \Big(\bigoplus_{v\in F_2} M_{n(v)}(\mathbb{C})\Big)$$

which is the graph-with-involution generalization of [14, Thm. 1.18].

Proof. Let $A = C^*(E)$. For each $v \in E^0_{\text{sink}}$, let A_v be the closed *-ideal generated by the projection p_v. If E is finite, then [14, Thm. 1.18] gives

$$A = \bigoplus_{v\in E^0_{\text{sink}}} A_v \quad \text{and} \quad A_v \cong M_{n(v)}(\mathbb{C})$$

where a system of matrix units in A_v is $s_\alpha s_\beta^*$ for $\alpha, \beta \in \Lambda_v$.

Now we claim that under the more general finiteness condition stated in the theorem, we have

$$A = \bigoplus_{v\in E^0_{\text{sink}}} A_v \quad \text{and} \quad A_v \cong \mathcal{K}(\ell^2(\Lambda_v))$$

with the same statement about matrix units. Consider any element of the form $s_\alpha s_\beta^*$ where α, β satisfy $r(\alpha) = r(\beta) = w$. The finite paths condition at w implies that $s_\alpha s_\beta^*$ can be written as a finite sum of elements of the form $s_{\alpha_i} s_{\beta_i}^*$ where $r(\alpha_i) = r(\beta_i)$ is a sink for each i. It follows that every element in A can be written as a sum of elements, each in A_v where v is a sink. With this modification, the proof of [14, Thm. 1.18] carries through the same to establish the claim.

Now let γ be an involution on E and let v be a sink. We first consider the case that $\gamma(v) = v$ (that is, $v \in F_1$). Then γ restricts to an anti-automorphism on A_v satisfying $\gamma(s_\alpha) = s^*_{\gamma(\alpha)}$. We analyze the structure of the real C^*-algebra associated to (A_v, γ). Define a new set of partial isometries t_α for $\alpha \in \Lambda_v$ as follows. If $\gamma(\alpha) = \alpha$, let $t_\alpha = s_\alpha$. For each pair $\alpha, \beta \in \Lambda_v$ such that $\gamma(\alpha) = \beta$, let

$$t_\alpha = \frac{1}{\sqrt{2}}(s_\alpha + s_\beta), \quad t_\beta = \frac{i}{\sqrt{2}}(s_\alpha - s_\beta).$$

(Note that this requires a fixed choice of α and β for each orbit of order two in Λ_v.) For each such pair α, β we have

$$\begin{aligned} t_\alpha t_\alpha^* + t_\beta t_\beta^* &= s_\alpha s_\alpha^* + s_\beta s_\beta^*, \\ t_\alpha^* t_\alpha = t_\beta^* t_\beta &= p_v, \\ t_\alpha^* t_\beta = t_\beta^* t_\alpha &= 0. \end{aligned}$$

The collection of partial isometries t_α for $\alpha \in \Lambda_v$ generates the same C^*-algebra as s_α, namely A_v. But since $\gamma(s_\alpha) = s_{\gamma(\alpha)}^*$, it follows that $\gamma(t_\alpha) = t_\alpha^*$ for all $\alpha \in \Lambda_v$. Therefore each t_α is in the real C^*-algebra associated to (A_v, γ). Furthermore, the calculations above show that the elements $t_\alpha t_\beta^*$ for $\alpha, \beta \in \Lambda_v$ form a collection of matrix units for A_v, showing that the real C^*-algebra corresponding to (A_v, γ) is isomorphic to $\mathcal{K}^{\mathbb{R}}(\ell^2_{\mathbb{R}}(\Lambda_v))$.

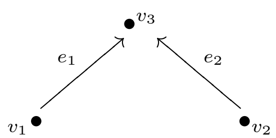
Finally, consider the case that $\gamma(v) \neq v$. Then $\gamma(A_v) = A_{\gamma(v)}$, $\gamma(\Lambda_v) = \Lambda_{\gamma(v)}$, and $n(v) = n(\gamma(v))$. We will analyze the structure of the real C^*-algebra associated to the restriction of γ to $\mathcal{K}(\ell^2(\Lambda_v)) \oplus \mathcal{K}(\ell^2(\Lambda_{\gamma(v)}))$. Again, a collection of matrix units for A_v is $s_\alpha s_\beta^*$ for $\alpha, \beta \in \Lambda_v$. Then a collection of matrix units for $A_{\gamma(v)}$ is $s_{\gamma(\alpha)} s_{\gamma(\beta)}^*$. Since

$$\gamma(s_\alpha s_\beta^*) = s_{\gamma(\beta)} s_{\gamma(\alpha)}^*,$$

we see that the action of γ on $A_v \oplus A_{\gamma(v)}$ corresponds to the action on $\mathcal{K}(\ell^2(\Lambda_v)) \oplus \mathcal{K}(\ell^2(\Lambda_{\gamma(v)}))$ given by $(x, y) \mapsto (y^{\mathrm{Tr}}, x^{\mathrm{Tr}})$ where Tr is the generalized trace operator. It follows that the real C^*-algebra corresponding to this action is isomorphic to $\mathcal{K}(\ell^2(\Lambda_v))$. □

3. EXAMPLES

Example 3.1. Let E be the graph shown below and let γ be the involution that interchanges v_1 and v_2.



Then there are isomorphisms

$$C^*(E) \cong M_3(\mathbb{C}), \quad C^*_{\mathbb{R}}(E; \mathrm{id}) \cong M_3(\mathbb{R}), \quad C^*_{\mathbb{R}}(E; \gamma) \cong M_3(\mathbb{R}).$$

Proof. The first statement follows from [14, Prop. 1.18] and the last two statements follow from Theorem 2.5 since the single sink is fixed in both cases. □

We remark that both $C^*(E)$ and $C^*_{\mathbb{R}}(E; \mathrm{id})$ above contain projections p_{v_1} and p_{v_2} which are each Murray–von Neumann equivalent to p_{v_3} via partial isometries s_{e_1} and s_{e_2}.

On the other hand, $C^*_{\mathbb{R}}(E;\gamma)$ contains orthogonal projections

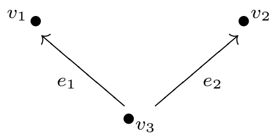
$$\begin{aligned} q_1 &= \frac{1}{2}(p_{v_1} + p_{v_2} + s_{e_1}s_{e_2}^* + s_{e_2}s_{e_1}^*), \\ q_2 &= \frac{1}{2}(p_{v_1} + p_{v_2} - s_{e_1}s_{e_2}^* - s_{e_2}s_{e_1}^*), \\ q_3 &= p_3. \end{aligned}$$

From the proof of Theorem 2.5 we see that q_1 and q_2 are Murray–von Neumann equivalent to q_3 via the partial isometries

$$\frac{1}{\sqrt{2}}(s_{e_1} + s_{e_2}) \quad \text{and} \quad \frac{i}{\sqrt{2}}(s_{e_1} - s_{e_2}).$$

It is also possible to deduce that $C^*_{\mathbb{R}}(E;\mathrm{id}) \cong C^*_{\mathbb{R}}(E;\gamma) \cong M_3(\mathbb{R})$ simply due to the fact that there is only one simple real C*-algebra of dimension 9. However in the next example we have $C^*_{\mathbb{R}}(E;\mathrm{id}) \ncong C^*_{\mathbb{R}}(E;\gamma)$.

Example 3.2. Let E be the graph shown below and let γ be the involution that interchanges v_1 and v_2.



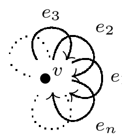
Then there are isomorphisms

$$\begin{aligned} C^*(E) &\cong M_2(\mathbb{C}) \oplus M_2(\mathbb{C}), \\ C^*_{\mathbb{R}}(E;\mathrm{id}) &\cong M_2(\mathbb{R}) \oplus M_2(\mathbb{R}), \\ C^*_{\mathbb{R}}(E;\gamma) &\cong M_2(\mathbb{C}). \end{aligned}$$

Proof. All three statements follow from [14, Prop. 1.18] and Theorem 2.5. □

The orthogonal projections $p_{v_1} + p_{v_2}$ and p_{v_3} in $C^*_{\mathbb{R}}(E;\gamma)$ above are Murray–von Neumann equivalent to each other via the partial isometry $s_{e_1} + s_{e_2}$.

Example 3.3. Let E be the graph shown below, with one vertex and n edges, and let γ be an involution corresponding to any order two permutation of the edges.



Then there are isomorphisms

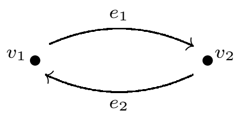
$$C^*(E) \cong \mathcal{O}_{n-1}, \quad C^*_{\mathbb{R}}(E;\mathrm{id}) \cong \mathcal{O}^{\mathbb{R}}_{n-1}, \quad C^*_{\mathbb{R}}(E;\gamma) \cong \mathcal{O}^{\mathbb{R}}_{n-1}.$$

Proof. It is well known that the graph algebra $C^*(E)$ is isomorphic to the Cuntz algebra $\mathcal{O}_{n-1}$, generated by partial isometries $s_1, s_2, \ldots, s_n$. In the case that $\gamma = \mathrm{id}$, these partial isometries satisfy $\gamma(s_i) = s_i^*$ so they are all elements of the real C^*-algebra $C^*_{\mathbb{R}}(E;\mathrm{id})$. But as in [16, p. 4], the real C^*-algebra generated by the elements s_i (inside $\mathcal{O}_{n-1}$) is exactly the real Cuntz algebra $\mathcal{O}^{\mathbb{R}}_{n-1}$.

Finally, Theorem 2.4 implies that, since γ is the identity on the vertex set, regardless of the action of γ on the edge set we have

$$C^*_{\mathbb{R}}(E;\gamma) \cong C^*_{\mathbb{R}}(E;\mathrm{id}) \cong \mathcal{O}^{\mathbb{R}}_{n-1}. \qquad \square$$

Example 3.4. Let E be the graph shown below and let γ be the involution that interchanges v_1 and v_2.



Then there are isomorphisms

$$\begin{aligned} C^*(E) &\cong C(S^1, M_2(\mathbb{C})), \\ C^*_{\mathbb{R}}(E;\mathrm{id}) &\cong \{f \in C(S^1, M_2(\mathbb{C})) \mid f(\overline{z}) = \overline{f(z)}\} \\ &\cong \{f \in C([0,1], M_2(\mathbb{C})) \mid f(0), f(1) \in M_2(\mathbb{R})\}, \\ C^*_{\mathbb{R}}(E;\gamma) &\cong \{f \in C([0,1], M_2(\mathbb{C})) \mid f(0) \in M_2(\mathbb{R}), f(1) \in \mathbb{H}\}. \end{aligned}$$

Proof. The first statement follows from [14, Ex. 2.14]. Specifically, the elements

$$s_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad s_2 = \begin{pmatrix} 0 & 0 \\ z & 0 \end{pmatrix}$$

generate $C(S^1, M_2(\mathbb{C}))$ and form a Cuntz–Krieger system associated with the graph E (and this algebra is universal).

For the second statement, it is enough to note that $C^*_{\mathbb{R}}(E;\mathrm{id}) = C^*_{\mathbb{R}}(E)$ is a real C^*-algebra generated by s_1 and s_2 and that the elements s_1 and s_2 both satisfy $f(\overline{z}) = \overline{f(z)}$. The second isomorphism associated with $C^*_{\mathbb{R}}(E;\mathrm{id})$ is given by restriction to the top half of the circle which is then identified with the interval $[0,1]$. We further remark that we therefore have an isomorphism $C^*_{\mathbb{R}}(E;\mathrm{id}) \cong M_2((S^{-1}\mathbb{R})^{\sim})$ where the desuspension algebra $S^{-1}\mathbb{R}$ is defined as in [2, §1] by

$$S^{-1}\mathbb{R} = \{f \in C_0((0,1),\mathbb{C}) \mid f(t) = \overline{f(1-t)}\}.$$

For the third statement, we carefully analyze the induced involution γ on $C^*(E) \cong C(S^1, M_2(\mathbb{C}))$, which satisfies $\gamma(s_1) = s_2^*$ and $\gamma(s_2) = s_1^*$. As in the proof of [17, Thm. 2.3], it suffices to identify the action of γ on the maximal ideal space of $C(S^1, M_2(\mathbb{C}))$ and on the fibers. For any $\lambda \in S^1$, let

$$t_\lambda = s_1 - \lambda s_2^* = \begin{pmatrix} 0 & 1 - \lambda\overline{z} \\ 0 & 0 \end{pmatrix},$$

which generates the ideal

$$I_\lambda = \{f \in C(S^1, M_2(\mathbb{C})) \mid f(\lambda) = 0\} \subset C(S^1, M_2(\mathbb{C})).$$

Since $\gamma(t_\lambda) = s_2^* - \lambda s_1 = -\lambda(s_1 - \overline{\lambda} s_2^*) = -\lambda t_{\overline{\lambda}}$, it follows that $\gamma(I_\lambda) = I_{\overline{\lambda}}$.

Therefore γ induces an involution $\lambda \mapsto \overline{\lambda}$ on S^1, the maximal ideal space. The involution γ on $C(S^1, M_2(\mathbb{C}))$ has the form $\gamma(f)(z) = \gamma_z(f(\overline{z}))$ where γ_z is an involution on $M_2(\mathbb{C})$ depending on z. As in the analysis of [17, §2], we can restrict functions to the upper half circle and it suffices to analyze the involution γ_λ at the end points $\lambda = \pm 1$. At each of the fixed points $\lambda = \pm 1$, the involution γ_λ acts on the quotient space $C(S^1, M_2(\mathbb{C}))/I_\lambda = M_2(\mathbb{C})$. Let $\mathrm{ev}_\lambda \colon C(S^1, M_2(\mathbb{C})) \to M_2(\mathbb{C})$ be the map defined by evaluation at λ.

At $\lambda = 1$, we have $\gamma_\lambda([s_1]) = [s_2^*]$. Since

$$\mathrm{ev}_1(s_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{ev}_1(s_2^*) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},$$

it follows that

$$\gamma_1 \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Using the fact that γ_1 is antimultiplicative and that $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ generates $M_2(\mathbb{C})$, we deduce that

$$\gamma_1 \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

This is the involution $\widetilde{\mathrm{Tr}}$ on $M_2(\mathbb{C})$, which is isomorphic (as an antimultiplicative involution on a C^*-algebra) to Tr, as described for example in [6, §2]. It corresponds to the real C^*-algebra $M_2(\mathbb{R})$.

At $\lambda = -1$, we have $\gamma_\lambda([s_1]) = [s_2^*]$ as before, but

$$\mathrm{ev}_{-1}(s_1) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{ev}_{-1}(s_2^*) = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix},$$

from which it follows that

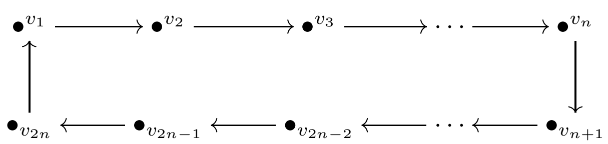
$$\gamma_{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

which is the involution (denoted $\sharp$ in [6]) of $M_2(\mathbb{C})$ corresponding to the real C^*-algebra of quaternions $\mathbb{H}$.

Therefore, the real C^*-algebra associated to γ on $C(S^1, M_2(\mathbb{C}))$ is isomorphic (by restriction of a function to the top half of the circle and by identifying the top half of the circle with the unit interval) to

$$\big\{ f \in C([0,1], M_2(\mathbb{C})) \mid f(0) \in M_2(\mathbb{R}), f(1) \in \mathbb{H} \big\}. \qquad \square$$

Example 3.5. Generalizing Example 3.4, let E be the graph shown below, a simple cycle with $2n$ vertices. Let γ be the involution on E that satisfies $\gamma(v_i) = v_{i+n}$ (using addition is modulo $2n$).



Then there are isomorphisms

$$\begin{aligned} C^*(E) &\cong C(S^1, M_{2n}(\mathbb{C})), \\ C^*_{\mathbb{R}}(E; \mathrm{id}) &\cong \big\{ f \in C(S^1, M_{2n}(\mathbb{C})) \mid f(\overline{z}) = \overline{f(z)} \big\}, \\ C^*_{\mathbb{R}}(E; \gamma) &\cong \big\{ f \in C([0,1], M_{2n}(\mathbb{C})) \mid f(0) \in M_{2n}(\mathbb{R}), f(1) \in M_n(\mathbb{H}) \big\}. \end{aligned}$$

Proof. Define partial isometries $s_i \in C(S^1, M_{2n}(\mathbb{C}))$ by

$$s_i = \begin{cases} e_{(i,i+1)} & \text{for } 1 \le i \le 2n-1, \\ z e_{(2n,1)} & \text{for } i = 2n \end{cases}$$

where $e_{(i,j)}$ is the matrix unit, having a 1 in position (i,j) and 0 elsewhere. These partial isometries form a Cuntz–Krieger family for the graph E; and according to [14, Ex. 2.14] they are universal showing that

$$C^*(E) \cong C(S^1, M_{2n}(\mathbb{C})).$$

As above, since $\gamma(s_i) = s_i^*$ for all i and since each s_i satisfies $f(\overline{z}) = \overline{f(z)}$, it follows that

$$C^*_{\mathbb{R}}(E; \mathrm{id}) \cong \big\{ f \in C(S^1, M_{2n}(\mathbb{C})) \mid f(\overline{z}) = \overline{f(z)} \big\}.$$

Now the involution γ on $C(S^1, M_{2n}(\mathbb{C}))$ satisfies $\gamma(s_i) = s_{i+n}^*$ for all i. Let

$$t_\lambda = s_1 s_2 \dots s_{2n-1} - \lambda s_{2n}^* = (1 - \lambda \overline{z}) e_{(1,2n)}$$

for $\lambda \in S^1$, so that

$$\begin{aligned} \gamma(t_\lambda) &= \gamma(s_1 s_2 \dots s_{2n-2} s_{2n-1} - \lambda s_{2n}^*) \\ &= \gamma(s_{2n-1}) \gamma(s_{2n-2}) \dots \gamma(s_2) \gamma(s_1) - \lambda \gamma(s_{2n}^*) \\ &= s_{n-1}^* s_{n-2}^* \dots s_1^* s_{2n}^* \dots s_{n+2}^* s_{n+1}^* - \lambda s_n \\ &= \overline{z} e_{(n,n-1)} e_{(n-1,n-2)} \dots e_{(2,1)} e_{(1,2n)} \dots e_{(n+3,n+2)} e_{(n+2,n+1)} - \lambda e_{(n,n+1)} \\ &= (\overline{z} - \lambda) e_{(n,n+1)}. \end{aligned}$$

This shows that $\gamma(I_\lambda) = I_{\overline{\lambda}}$ as in Example 3.4. Again it suffices to analyze the induced involution γ_z on $M_{2n}(\mathbb{C})$ for $z = \pm 1$.

First let $\lambda = 1$. We claim that the involution γ_1 on $M_{2n}(\mathbb{C})$ is given by

$$\gamma_1 \colon \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} D^{\mathrm{Tr}} & B^{\mathrm{Tr}} \\ C^{\mathrm{Tr}} & A^{\mathrm{Tr}} \end{pmatrix}.$$

Indeed, for $i \neq n, i \neq 2n$ we have $\gamma(s_i) = s_{i+n}^*$ which implies that

$$\gamma_\lambda(e_{(i,i+1)}) = e_{(i+n,i+n+1)}^{\mathrm{Tr}}.$$

For $i = n$, we have $\gamma(s_n) = s_{2n}^*$, which after evaluating at λ gives

$$\gamma_\lambda(e_{(n+1,n)}) = e_{(2n,1)}.$$

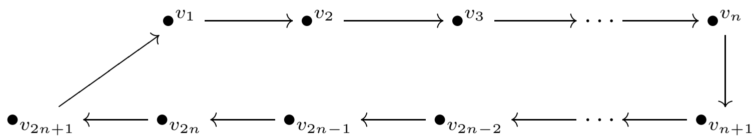
Thus we have verified that γ_1 agrees with the given formula on a set that generates $M_{2n}(\mathbb{C})$, proving the claim. Therefore γ_1 is equivalent to the involution $\mathrm{Tr}_n \otimes \widetilde{\mathrm{Tr}}$ on $M_n(\mathbb{C}) \otimes M_2(\mathbb{C})$, which is in turn isomorphic to Tr_{2n}. Therefore, the real C^*-algebra associated with $\lambda = 1$ is $M_{2n}(\mathbb{R})$.

For $\lambda = -1$, we claim that the involution γ_{-1} on $M_{2n}(\mathbb{C})$ is given by

$$\gamma_{-1}\colon \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} D^{\mathrm{Tr}} & -B^{\mathrm{Tr}} \\ -C^{\mathrm{Tr}} & A^{\mathrm{Tr}} \end{pmatrix}.$$

The only difference is that the formula $\gamma(s_n) = s_{2n}^*$ after evaluating at $\lambda = -1$ gives $\gamma_\lambda(e_{(n+1,n)}) = -e_{(2n,1)}$ which leads to γ_{-1} as given. This is the involution $\mathrm{Tr}_n \otimes \sharp$ on $M_n(\mathbb{C}) \otimes M_2(\mathbb{C})$ and the associated real C^*-algebra is $M_n(\mathbb{R}) \otimes \mathbb{H}$. □

Example 3.6. Let E be the cyclic graph below with $2n+1$ vertices. There are no nontrivial involutions on E. So we only have one real structure on $C^*(E)$ arising from this graph as shown.



Then there are isomorphisms

$$C^*(E) \cong C(S^1, M_{2n+1}(\mathbb{C})),$$
$$C^*_{\mathbb{R}}(E; \mathrm{id}) \cong \big\{ f \in C(S^1, M_{2n+1}(\mathbb{C})) \mid f(\overline{z}) = \overline{f(z)} \big\}.$$

Proof. The proof is the same as the first paragraph of the proof of Example 3.5. □

4. K-THEORY

In this section, we develop a long exact sequence that generalizes [14, Thm. 7.16] or [15, Thm. 3.2] for complex graph algebras and at the same time generalizes [5, Thm. 13] for real graph algebras. Our sequence will allow us to compute the K-theory of $C^*_{\mathbb{R}}(E; \gamma)$, in the case of row-finite graphs with no sinks. We postpone for further work the analogous results that we believe can be obtained for graphs with involution in which there may be sinks or vertices that emit infinitely many edges along the lines of [8, Thm. 3.1].

Before stating the main theorem, we must develop a few preliminary constructions and definitions, regarding K-theory for real C^*-algebras. For any C^*-algebra A, we let $K^{CRT}(A)$ be the united K-theory – the algebraic object consisting of the groups of real, complex, and self-conjugate K-theory, as well as the natural transformations between them, as defined in [2]. From the universal coefficient theorem in [3] and the subsequent classification result in [7], we know that united K-theory classifies a large class of real C^*-algebras up to KK-equivalence and that it classifies up to isomorphism in the case of real Kirchberg algebras satisfying the UCT.

We will also transfer the definition of united K-theory to the category of $C^{*,\tau}$-algebras using the categorical equivalence between real C^*-algebras and $C^{*,\tau}$-algebras. Thus $K^{CRT}(A, \tau)$ is by definition the united K-theory of the real C^*-algebra A^τ corresponding to (A, τ).

Our main result below, Theorem 4.1, is stated for the full united K-theory $K^{CRT}(C^*_{\mathbb{R}}(E;\gamma))$, but in our calculations we will focus on a somewhat smaller and more manageable invariant $K^{CR}(C^*_{\mathbb{R}}(E;\gamma))$. For a real C^*-algebra A, let $K^{CR}(A)$ be the algebraic object consisting of the groups of real and complex K-theory, as well as the natural transformations between them. In other words,

$$K^{CR}(A) = \{KO_*(A), KU_*(A), r, c, \psi_U\}.$$

Here $KO_*(A)$ is the period 8 real K-theory (with its $KO_*(\mathbb{R})$-module structure) and $KU_*(A) = K_*(A_{\mathbb{C}})$ is the period 2 complex K-theory of the complexification. The natural transformations are

$$\begin{aligned} r_i &\colon KU_i(A) \to KO_i(A) && \text{induced by } r\colon \mathbb{C} \to M_2(\mathbb{R}), \\ c_i &\colon KO_i(A) \to KU_i(A) && \text{induced by } c\colon \mathbb{R} \to \mathbb{C}, \\ \psi_i &\colon KU_i(A) \to KU_i(A) && \text{induced by conjugation}\colon \mathbb{C} \to \mathbb{C}. \end{aligned}$$

In addition, the $KO_*(\mathbb{R})$-module structure of $KO_*(A)$ involves a degree 1 endomorphism η on $KO_*(A)$, a degree 4 endomorphism ξ, and a degree 8 Bott automorphism β_O. We let β_U represent the degree 2 Bott automorphism on $KU_*(A)$. These natural transformations satisfy the relations

$$\begin{aligned} rc &= 2, & cr &= 1 + \psi, & 2\eta &= 0, \\ r\psi &= r, & \psi^2 &= \mathrm{id}, & \eta^3 &= 0, \\ \psi c &= c, & \psi\beta_U &= -\beta_U\psi, & \xi &= r\beta_U^2 c, \end{aligned}$$

and form a long exact sequence

$$\cdots \to KO_i(C^*_{\mathbb{R}}(E;\gamma)) \xrightarrow{\eta} KO_{i+1}(C^*_{\mathbb{R}}(E;\gamma)) \xrightarrow{c} KU_{i+1}(C^*_{\mathbb{R}}(E;\gamma)) \xrightarrow{r\beta_U^{-1}} KO_{i-1}(C^*_{\mathbb{R}}(E;\gamma)) \to \cdots .$$

As proven in [9], the invariant $K^{CR}(A)$ actually carries the same information as the bulkier invariant $K^{CRT}(A)$. Therefore two real Kirchberg algebras A and B satisfying the UCT are stably isomorphic if and only if $K^{CR}(A) \cong K^{CR}(B)$. The disadvantage of this invariant is that not every abstract acyclic $\mathcal{CR}$-module can be realized as isomorphic to $K^{CR}(A)$ for some real C^*-algebra A, whereas it is known that every abstract acyclic $\mathcal{CRT}$-module can be so realized by the main result of [4]. See [9] for a full account of the relationship between the category of abstract acyclic $\mathcal{CR}$-modules and that of abstract acyclic $\mathcal{CRT}$-modules.

Our approach to computing the K-theory of $C^*_{\mathbb{R}}(E;\gamma)$ follows the approach of [14, Chap. 7]. This includes the use of the constructions $E \times_1 \mathbb{Z}$ and $E \times_1 \mathbb{N}$, which can be extended to the context of graphs with involutions as follows.

Let (E,γ) be a graph with involution. Define a graph $E \times_1 \mathbb{Z}$ with involution γ by

$$(E \times_1 \mathbb{Z})^0 = E^0 \times \mathbb{Z}, \quad (E \times_1 \mathbb{Z})^1 = E^1 \times \mathbb{Z},$$

and

$$
\begin{aligned}
s(e,n) &= (s(e), n+1) && \text{for all } (e,n) \in (E \times_1 \mathbb{Z})^1, \\
r(e,n) &= (r(e), n) && \text{for all } (e,n) \in (E \times_1 \mathbb{Z})^1, \\
\gamma(v,n) &= (\gamma(v), n) && \text{for all } (v,n) \in (E \times_1 \mathbb{Z})^0, \\
\gamma(e,n) &= (\gamma(e), n) && \text{for all } (e,n) \in (E \times_1 \mathbb{Z})^1.
\end{aligned}
$$

Similarly, we define a graph $E \times_1 \mathbb{N}$ with involution γ by

$$(E \times_1 \mathbb{N})^0 = E^0 \times \mathbb{N}, \quad (E \times_1 \mathbb{N})^1 = E^1 \times \mathbb{N},$$

and the same formulas for s, r and γ as above.

There are maps

$$
\begin{aligned}
\beta &\colon C^*_{\mathbb{R}}(E \times_1 \mathbb{Z}; \gamma) \to C^*_{\mathbb{R}}(E \times_1 \mathbb{Z}; \gamma), \\
\beta &\colon C^*_{\mathbb{R}}(E \times_1 \mathbb{N}; \gamma) \to C^*_{\mathbb{R}}(E \times_1 \mathbb{N}; \gamma)
\end{aligned}
$$

defined by the formulas $\beta(p_{(v,n)}) = p_{(v,n+1)}$ and $\beta(s_{(v,n)}) = s_{(v,n+1)}$ using the universal property of graph algebras with involution, Theorem 2.3.

For this section, we also introduce the notational shorthand

$$M(E;\gamma) := K^{CRT}(C^*_{\mathbb{R}}(E \times_1 \mathbb{N}; \gamma)).$$

Write E^0 as a disjoint union

$$E^0 = F_1 \sqcup F_2 \sqcup F_3$$

where F_1 is the set of vertices of E fixed by γ, and F_2 and F_3 are sets such that $\gamma(F_2) = F_3$ and $\gamma(F_3) = F_2$. Assume that E is row finite and has no sinks. Then the sinks of $E \times_1 \mathbb{N}$ are the vertices of the form $(v, 1)$ where $v \in E^0$. Furthermore, the number of paths originating at any vertex of $E \times_1 \mathbb{N}$ is finite. Thus by Theorem 2.5 we have

$$M(E;\gamma) \cong K^{CRT}(\mathbb{R})^{|F_1|} \oplus K^{CRT}(\mathbb{C})^{|F_2|}.$$

Furthermore,

$$C^*_{\mathbb{R}}(E \times_1 \mathbb{Z}; \gamma) \cong \lim_{k \to \infty} C^*_{\mathbb{R}}(E \times_1 \mathbb{N}; \gamma)$$

where the connecting homomorphisms of the inductive limit are given by β. The map from the direct limit is induced by the family of homomorphisms

$$\alpha_k \colon C^*_{\mathbb{R}}(E \times_1 \mathbb{N}) \to C^*_{\mathbb{R}}(E \times_1 \mathbb{Z})$$

given by $\alpha_k(s_{(v,n)}) = s_{(v,n-k)}$ and $\alpha_k(p_{(v,n)}) = p_{(v,n-k)}$ for $k \geq 0$. Hence

$$K^{CRT}(C^*(E \times_1 \mathbb{Z}; \gamma)) \cong \lim_{k \to \infty} (M(E;\gamma), \beta_*).$$

We are ready now to state and prove our main result.

Theorem 4.1. *Let (E, γ) be a row-finite graph with involution and with no sinks. Then there is a long exact sequence*

$$\cdots \to K^{CRT}(C^*_{\mathbb{R}}(E;\gamma)) \xrightarrow{\iota} M(E;\gamma) \xrightarrow{\beta_* - 1} M(E;\gamma) \xrightarrow{\pi} K^{CRT}(C^*_{\mathbb{R}}(E;\gamma)) \to \cdots$$

where ι has degree -1 and π has degree 0.

Proof. Let λ be the gauge action of the circle $\mathbb{T}$ on $C^*(E)$. Explicitly, we have $\lambda_z(p_v) = p_v$ and $\lambda_z(s_e) = zs_e$ for all $z \in \mathbb{T}$, $v \in E^0$, and $e \in E^1$. Then we have

$$\gamma(\lambda_z(s_e)) = \gamma(zs_e) = zs^*_{\gamma(e)} = (\overline{z}s_{\gamma(e)})^* = (\lambda_{\overline{z}}s_{\gamma(e)})^* = \lambda_{\overline{z}}s^*_{\gamma(e)} = \lambda_{\overline{z}}(\gamma(s_e))$$

for all $z \in \mathbb{T}$ and $e \in E^1$, showing that $\gamma \circ \lambda_z = \lambda_{\overline{z}} \circ \gamma$ (as automorphisms of $C^*(E)$). Therefore, with the involution on $\mathbb{T}$ given by $z \mapsto \overline{z}$, the quintuple $(C^*(E), \gamma, \mathbb{T}, \bar{\cdot}, \lambda_z)$ is a real C^* dynamical system in the sense of [5, §2], and gives rise to an induced real structure $\widetilde{\gamma}$ on the crossed product $C^*(E) \rtimes_\lambda \mathbb{T}$, which is given by $\widetilde{\gamma}(f)(z) = \gamma(f(\overline{z}))$ for $f \in C(\mathbb{T}, C^*(E)) \subset C^*(E) \rtimes_\lambda \mathbb{T}$. Then [5, Thm. 11] gives a long exact sequence

$$\begin{aligned}(1) \qquad \cdots \to K^{CRT}(C^*(E), \gamma) &\xrightarrow{\partial} K^{CRT}(C^*(E;\gamma) \rtimes_\lambda \mathbb{T}, \widetilde{\gamma}) \\ &\xrightarrow{\widehat{\lambda}_* - 1} K^{CRT}(C^*(E;\gamma) \rtimes_\lambda \mathbb{T}, \widetilde{\gamma}) \\ &\xrightarrow{\pi} K^{CRT}(C^*(E), \gamma) \to \cdots\end{aligned}$$

where $\widehat{\lambda}$ is the dual action of $\mathbb{Z}$ on $C^*_{\mathbb{R}}(E) \rtimes_\lambda \mathbb{T}$.

Now, define a homomorphism $\phi\colon C^*(E \times_1 \mathbb{Z}) \to C^*(E) \rtimes_\lambda \mathbb{T}$ by the formula $\phi(s_{(e,n)}) = t_{(e,n)}$ where $t_{(e,n)} \in C(\mathbb{T}, C^*(E)) \subset C^*(E) \rtimes_\lambda \mathbb{T}$ is defined by $t_{(e,n)}(z) = z^n s_e$. We know from [14, Lem. 7.10] that this is an isomorphism and we know also from the proof of [5, Thm. 13] that this isomorphism intertwines the homomorphism β_* on $C^*(E \times_1 \mathbb{Z})$ with $\widehat{\lambda}_*$ on $C^*(E) \rtimes_\lambda \mathbb{T}$. For our present purposes, we also need to know that ϕ commutes with the real structures on these C^*-algebras induced by γ. That is, we need to show that $\phi \circ \gamma = \widetilde{\gamma} \circ \phi$. In this formula γ represents the involution on $C^*(E \times_1 \mathbb{Z})$ given by $s_{(e,n)} \mapsto s^*_{(\gamma(e),n)}$ and $\widetilde{\gamma}$ is as described in the previous paragraph.

It is enough to check

$$(\phi \circ \gamma)(s_{(e,n)})(z) = (\widetilde{\gamma} \circ \phi)(s_{(e,n)})(z)$$

for $e \in E^1$, $n \in \mathbb{Z}$, and $z \in \mathbb{T}$. Indeed, we have

$$\begin{aligned}\phi(\gamma(s_{(e,n)}))(z) &= \phi(s^*_{(\gamma(e),n)})(z) \\ &= t^*_{(\gamma(e),n)}(z) \\ &= (t_{(\gamma(e),n)}(z))^* \\ &= (z^n s_{\gamma(e)})^* \\ &= \overline{z}^n s^*_{\gamma(e)}\end{aligned} \qquad \text{and} \qquad \begin{aligned}\widetilde{\gamma}(\phi(s_{(e,n)}))(z) &= \widetilde{\gamma}(t_{(e,n)})(z) \\ &= \gamma(t_{(e,n)}(\overline{z})) \\ &= \gamma(\overline{z}^n s_e) \\ &= \overline{z}^n \gamma(s_e) \\ &= \overline{z}^n s^*_{\gamma(e)}.\end{aligned}$$

Therefore, ϕ is an isomorphism of $C^{*,\tau}$-algebras between $(C^*(E \times_1 \mathbb{Z}), \gamma)$ and $(C^*(E) \rtimes_\lambda \mathbb{T}, \widetilde{\gamma})$, and equivalently ϕ is an isomorphism on the corresponding real C^*-algebras. Thus sequence (1) above becomes

$$\begin{aligned}(2) \qquad \cdots \to K^{CRT}(C^*_{\mathbb{R}}(E;\gamma)) &\xrightarrow{\partial} K^{CRT}(C^*_{\mathbb{R}}(E \times_1 \mathbb{Z};\gamma)) \\ &\xrightarrow{\beta_* - 1} K^{CRT}(C^*_{\mathbb{R}}(E \times_1 \mathbb{Z};\gamma)) \\ &\xrightarrow{\pi} K^{CRT}(C^*_{\mathbb{R}}(E;\gamma)) \to \cdots .\end{aligned}$$

The inductive limit $K^{CRT}(C^*_{\mathbb{R}}(E \times_1 \mathbb{Z}; \gamma)) \cong \lim_{n\to\infty}(M(E;\gamma), \beta_*)$ leads us to consider the following diagram.

$$\begin{array}{ccccccccc} M(E;\gamma) & \xrightarrow{\beta_*} & M(E;\gamma) & \xrightarrow{\beta_*} & M(E;\gamma) & \xrightarrow{\beta_*} & \cdots & \longrightarrow & K^{CRT}(C^*_{\mathbb{R}}(E \times_1 \mathbb{Z};\gamma)) \\ \downarrow{\scriptstyle \beta_*-1} & & \downarrow{\scriptstyle \beta_*-1} & & \downarrow{\scriptstyle \beta_*-1} & & & & \downarrow{\scriptstyle \beta_*-1} \\ M(E;\gamma) & \xrightarrow{\beta_*} & M(E;\gamma) & \xrightarrow{\beta_*} & M(E;\gamma) & \xrightarrow{\beta_*} & \cdots & \longrightarrow & K^{CRT}(C^*_{\mathbb{R}}(E \times_1 \mathbb{Z};\gamma)) \end{array}$$

The same argument as in [14, Lem. 7.15] shows that the homomorphism

$$\alpha_0 \colon M(E;\gamma) \to K^{CRT}(C^*_{\mathbb{R}}(E \times_1 \mathbb{Z};\gamma))$$

induces an isomorphism from the kernel and cokernel of the vertical homomorphism

$$M(E;\gamma) \xrightarrow{\beta_*-1} M(E;\gamma)$$

to the kernel and cokernel, respectively, of

$$K^{CRT}(C^*_{\mathbb{R}}(E \times_1 \mathbb{Z};\gamma)) \xrightarrow{\beta_*-1} K^{CRT}(C^*_{\mathbb{R}}(E \times_1 \mathbb{Z};\gamma)),$$

completing the proof. □

Unsplicing the long exact sequence of Theorem 4.1 yields the following.

Corollary 4.2. *Let (E,γ) be a row-finite graph with involution and with no sinks. Then there is a short exact sequence*

$$0 \to \operatorname{coker}(\beta_* - 1) \xrightarrow{\widetilde{\pi}} K^{CRT}(C^*_{\mathbb{R}}(E;\gamma)) \xrightarrow{\iota} \ker(\beta_* - 1) \to 0$$

where ι has degree -1 and π has degree 0.

Focusing our attention on the real K-theory groups $KO_i(C^*_{\mathbb{R}}(E;\gamma))$, we obtain the following result.

Corollary 4.3. *Let (E,γ) be a row-finite graph with involution and with no sinks. Then*

$$\begin{aligned} KO_i(C^*_{\mathbb{R}}(E;\gamma)) &\cong \operatorname{coker}(\beta_* - 1)_i && \textit{for } i = 0,4,6, \\ KO_i(C^*_{\mathbb{R}}(E;\gamma)) &\cong \ker(\beta_* - 1)_{i-1} && \textit{for } i = 3,5,7, \\ KO_i(C^*_{\mathbb{R}}(E;\gamma)) &\textit{ is free} && \textit{for } i = 5,7. \end{aligned}$$

Proof. This follows from Corollary 4.2 and the fact that the real part of $M(E;\gamma)_i$ vanishes for $i = 3,5,7$ and is free for $i = 0,4,6$ (and the fact that any subgroup of a free group is free). □

To compute $KO_*(C^*_{\mathbb{R}}(E;\gamma))$, it helps to write the homomorphism

$$\beta_* \colon M(E;\gamma) \to M(E;\gamma)$$

in each of the degrees, in both real and complex parts, in terms of the adjacency matrix of the graph. Theorem 4.4 below shows how to accomplish this. For a row-finite graph, let A_E be the adjacency matrix defined by

$$(A_E)_{v,w} = \text{number of edges from vertex } v \text{ to vertex } w.$$

n	0	1	2	3	4	5	6	7
$KO_n(\mathbb{R})$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	0	0	0
$KU_n(\mathbb{R})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
c_n	1	0	0	0	2	0	0	0
r_n	2	0	1	0	1	0	0	0
ψ_n	1	0	-1	0	1	0	-1	0
η_n	1	1	0	0	0	0	0	0
ξ_n	1	0	0	0	4	0	0	0
n	0	1	2	3	4	5	6	7
$KO_n(\mathbb{C})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
$KU_n(\mathbb{C})$	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	0
c_n	$\begin{pmatrix}1\\1\end{pmatrix}$	0	$\begin{pmatrix}1\\-1\end{pmatrix}$	0	$\begin{pmatrix}1\\1\end{pmatrix}$	0	$\begin{pmatrix}1\\-1\end{pmatrix}$	0
r_n	$\begin{pmatrix}1 & 1\end{pmatrix}$	0	$\begin{pmatrix}1 & -1\end{pmatrix}$	0	$\begin{pmatrix}1 & 1\end{pmatrix}$	0	$\begin{pmatrix}1 & -1\end{pmatrix}$	0
ψ_n	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	0	$\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$	0	$\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$	0	$\begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$	0
η_n	0	0	0	0	0	0	0	0
ξ_n	2	0	0	0	2	0	0	0

TABLE 1. $K^{CR}(\mathbb{R})$ (top) and $K^{CR}(\mathbb{C})$ (bottom).

n	0	1	2	3	4	5	6	7
$M(E;\gamma)^O_n$	$\mathbb{Z}^{\|F_1\|}\oplus\mathbb{Z}^{\|F_2\|}$	$\mathbb{Z}_2^{\|F_1\|}$	$\mathbb{Z}_2^{\|F_1\|}\oplus\mathbb{Z}^{\|F_2\|}$	0	$\mathbb{Z}^{\|F_1\|}\oplus\mathbb{Z}^{\|F_2\|}$	0	$\mathbb{Z}^{\|F_2\|}$	0
$M(E;\gamma)^U_n$	$\mathbb{Z}^{\|E_0\|}$	0	$\mathbb{Z}^{\|E_0\|}$	0	$\mathbb{Z}^{\|E_0\|}$	0	$\mathbb{Z}^{\|E_0\|}$	0

TABLE 2. $M(E;\gamma)$.

Write the set of vertices of E as

$$\begin{aligned}E^0 &= F_1 \sqcup F_2 \sqcup F_3\\ &= \{v_{1,1}, v_{1,2}, \dots, v_{1,m}, v_{2,1}, v_{2,2}, \dots, v_{2,n}, v_{3,1}, v_{3,2}, \dots, v_{3,n}\}\end{aligned}$$

where $v_{i,j} \in F_i$ and we have $\gamma(v_{1,j}) = v_{1,j}$ and $\gamma(v_{2,j}) = v_{3,j}$. We use this order on the vertex set to order the rows and columns of the adjacency matrix A_E. The fact that the number of edges from vertex v to vertex w is same as the number of edges from $\gamma(v)$ to $\gamma(w)$ implies a certain symmetry to the adjacency matrix (and its transpose). Specifically, we can write the matrix in block form as

$$A_E^{\mathrm{Tr}} = B = \begin{pmatrix} B_{11} & B_{12} & B_{13}\\ B_{21} & B_{22} & B_{23}\\ B_{31} & B_{32} & B_{33}\end{pmatrix} = \begin{pmatrix} B_{11} & B_{12} & B_{12}\\ B_{21} & B_{22} & B_{23}\\ B_{21} & B_{23} & B_{22}\end{pmatrix}$$

where submatrix B_{ij} represents edges from vertices in F_j to vertices in F_i.

Since

$$M(E;\gamma) = K^{CRT}(\mathbb{R})^{|F_1|} \oplus K^{CRT}(\mathbb{C})^{|F_2|},$$

the real and complex parts of $M(E;\gamma)$ are determined by the $\mathcal{CR}$-modules $K^{CR}(\mathbb{R})$ and $K^{CR}(\mathbb{C})$ in Table 1. We will be using these structures in the proof of Theorem 4.4. Thus the abelian groups making up the real and complex parts of $M(E;\gamma)$ are as in Table 2.

The automorphism β_* acts on each of the groups given in this table. Then the theorem below shows how to write down a matrix representation of β_* in each degree. From that, one can identify $\beta_* - \mathrm{id}$ and use the Smith normal form to identify the kernel and cokernel in each degree, which can be used to compute $KO_i(C^*_{\mathbb{R}}(E;\gamma))$ and $KU_i(C^*_{\mathbb{R}}(E;\gamma))$.

Theorem 4.4. *Let (E,γ) be a finite graph with involution and let A_E be the incidence matrix. Then the homomorphism $\beta_*\colon M(E;\gamma) \to M(E;\gamma)$ in each degree (real and complex part) is given in Table 3.*

degree	*matrix representation of* β_*	
0	$\begin{pmatrix} B_{11} & 2B_{12} \\ B_{21} & B_{22}+B_{32} \end{pmatrix}$	$\mathbb{Z}^{\lvert F_1\rvert} \oplus \mathbb{Z}^{\lvert F_2\rvert} \to \mathbb{Z}^{\lvert F_1\rvert} \oplus \mathbb{Z}^{\lvert F_2\rvert}$
1	B_{11}	$\mathbb{Z}_2^{\lvert F_1\rvert} \to \mathbb{Z}_2^{\lvert F_1\rvert}$
2	$\begin{pmatrix} B_{11} & B_{12} \\ 0 & B_{22}-B_{32} \end{pmatrix}$	$\mathbb{Z}_2^{\lvert F_1\rvert} \oplus \mathbb{Z}^{\lvert F_2\rvert} \to \mathbb{Z}_2^{\lvert F_1\rvert} \oplus \mathbb{Z}^{\lvert F_2\rvert}$
3	*0*	0
4	$\begin{pmatrix} B_{11} & B_{12} \\ 2B_{21} & B_{22}+B_{32} \end{pmatrix}$	$\mathbb{Z}^{\lvert F_1\rvert} \oplus \mathbb{Z}^{\lvert F_2\rvert} \to \mathbb{Z}^{\lvert F_1\rvert} \oplus \mathbb{Z}^{\lvert F_2\rvert}$
5	*0*	0
6	$B_{22}-B_{32}$	$\mathbb{Z}^{\lvert F_2\rvert} \to \mathbb{Z}^{\lvert F_2\rvert}$
7	*0*	0
0	$\begin{pmatrix} B_{11} & B_{12} & B_{12} \\ B_{21} & B_{22} & B_{23} \\ B_{21} & B_{23} & B_{22} \end{pmatrix}$	$\mathbb{Z}^{\lvert F_1\rvert} \oplus \mathbb{Z}^{\lvert F_2\rvert} \oplus \mathbb{Z}^{\lvert F_3\rvert} \to \mathbb{Z}^{\lvert F_1\rvert} \oplus \mathbb{Z}^{\lvert F_2\rvert} \oplus \mathbb{Z}^{\lvert F_3\rvert}$
1	*0*	0

Table 3. Real part (top) and complex part (bottom).

Proof. First of all, in the complex part in degree 0, we claim that the map

$$\beta_*\colon KU_0(C^*_{\mathbb{R}}(E\times_1\mathbb{N})) \xrightarrow{\beta_*} KU_0(C^*_{\mathbb{R}}(E\times_1\mathbb{N}))$$

coincides with the map

$$\mathbb{Z}^{\lvert E^0\rvert} \xrightarrow{B} \mathbb{Z}^{\lvert E^0\rvert}$$

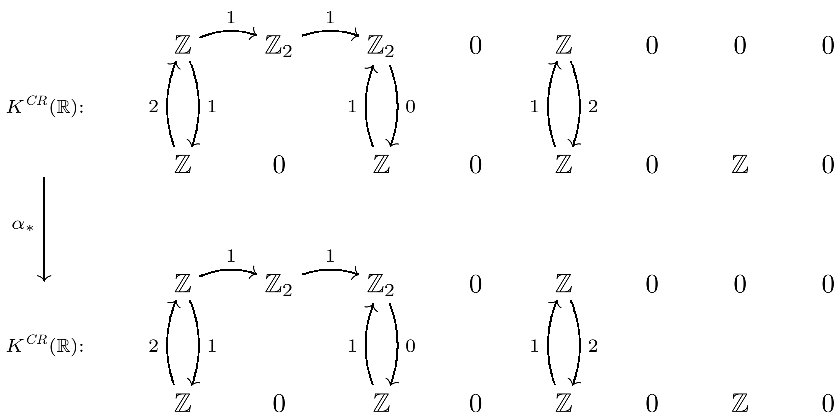
given by matrix $A_E^T = B$ (compare to the results leading up to [14, Thm. 7.16]). Indeed, the free generators of $KU_0(C^*_{\mathbb{R}}(E\times_1\mathbb{N})) \cong \mathbb{Z}^{\lvert E^0\rvert}$ are $[p_{(v,1)}]$ for $v\in E^0$. Then for each vertex v we have

$$\begin{aligned}\beta_*([p_{(v,1)}]) &= [p_{(v,2)}] \\ &= \sum_{e\in s^{-1}(v)} [s_{(e,1)}s^*_{(e,1)}] = \sum_{e\in s^{-1}(v)} [s^*_{(e,1)}s_{(e,1)}] = \sum_{e\in s^{-1}(v)} [p_{(r(e),1)}] \\ &= B([p_{(v,1)}]).\end{aligned}$$

In principle, the behavior of β_* in the complex part in degree 0 of $M(E;\gamma)$ entirely determines algebraically the behavior of β_* in the real part in all gradings. This is because $M(E;\gamma)$ is a free $\mathcal{CRT}$-module; and, though the generators may be in both the real and complex part of $M(E;\gamma)$, the complexification map from the real part to the complex part in degree 0 is injective. This will enable us to work out the behavior of β_* in the real part, obtaining the result. The rest of the proof consists of the details of working out the maps in the real part from the complex part.

Since the homomorphism β_* is linear over $\mathbb{Z}$, it suffices to consider the behavior of β_* restricted to each summand, say M_1, of $M(E;\gamma)$ and projected onto another summand, say M_2, of $M(E;\gamma)$. We will consider all four cases separately, depending on whether each of these summands has the form $K^{CR}(\mathbb{R})$ or $K^{CR}(\mathbb{C})$.

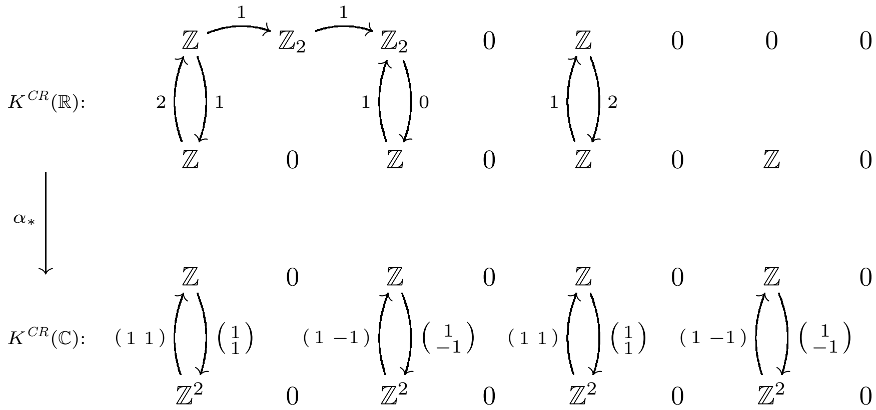
The first case is a homomorphism α of the form $K^{CR}(\mathbb{R}) \to K^{CR}(\mathbb{R})$. Let α_i^O denote the real part of α_* in degree i and let α_i^U denote the complex part in degree i. The diagram below helps us visualize α_* as a $\mathcal{CR}$-module homomorphism. The first two lines represent the real and complex part of $K^{CR}(\mathbb{R})$ where the vertical maps are c_i and r_i and the horizonal maps are η_i. The second two lines represent the second target copy of $K^{CR}(\mathbb{R})$. Then α is a graded homomorphism that must commute with all of these internal natural transformations.



Assume that $\alpha_0^U : \mathbb{Z} \to \mathbb{Z}$ is multiplication by some integer b (corresponding to an entry of the B_{11} portion of the matrix B), then by Bott periodicity α_i^U is multiplication by b for all even i. Then the diagram (specifically the fact that $\alpha_0^U c_0 = c_0 \alpha_0^O$) implies that $\alpha_0^O : \mathbb{Z} \to \mathbb{Z}$ is also multiplication by b. Similarly using the η maps and the ξ map, we see that α_i^O is multiplication by b in all nontrivial gradings (that is $i = 0, 1, 2, 4$).

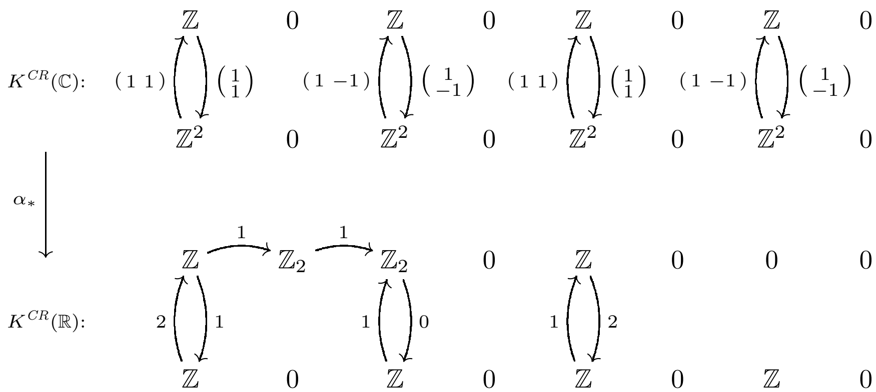
From this we conclude that B_{11} is exactly the matrix that describes the homomorphism β_* on the real part of the $K^{CR}(\mathbb{R})$ summands of $M(E;\gamma)$. Therefore, the matrix B_{11} appears in the real part in degrees 0, 1, 2, 4 as in Table 3.

Next, consider a homomorphism $\alpha\colon K^{CR}(\mathbb{R}) \to K^{CR}(\mathbb{C})$. The diagram below shows the $\mathcal{CR}$-modules.



If we assume that $\alpha_0^U : \mathbb{Z} \to \mathbb{Z} \oplus \mathbb{Z}$ is multiplication by a 2×1 matrix $\binom{b}{b}$ (where b corresponds to an entry of the B_{21} portion of the matrix B), then using the homomorphisms c_0, r_2, and r_4, we find that $\alpha_0^O = b$, $\alpha_1^O = \alpha_2^O = 0$, and $\alpha_4^O = 2b$. This shows that the matrix B_{21} from the incidence matrix appears as B_{21} and $2B_{21}$ in degrees 0 and 4 as in Table 3.

Now, consider a homomorphism $\alpha_*\colon K^{CR}(\mathbb{C}) \to K^{CR}(\mathbb{R})$ that is described in degree 0 as multiplication by a matrix $(\,b\ b\,)$ from $\mathbb{Z} \oplus \mathbb{Z}$ to $\mathbb{Z}$. Use the diagram below.



Using Bott periodicity and the natural transformations r_i for i even, we find that $\alpha_0^O = 2b$, $\alpha_2^O = \alpha_4^O = b$, and $\alpha_6^O = 0$. This accounts for the matrix $2B_{12}$ in the formula for β_* in degree 0 and B_{12} in degrees 2 and 4.

Finally, consider a homomorphism $\alpha\colon K^{CR}(\mathbb{C}) \to K^{CR}(\mathbb{C})$ and assume that $\alpha_0^U : \mathbb{Z}^2 \to \mathbb{Z}^2$ is given by a 2×2 matrix of the form $\left(\begin{smallmatrix} b & c \\ c & b \end{smallmatrix}\right)$. Observe the following diagram:

$$
\begin{array}{lcccccccc}
 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 \\
K^{CR}(\mathbb{C}): & (1\ 1)\Big\uparrow\Big\downarrow\begin{pmatrix}1\\1\end{pmatrix} & & (1\ {-1})\Big\uparrow\Big\downarrow\begin{pmatrix}1\\-1\end{pmatrix} & & (1\ 1)\Big\uparrow\Big\downarrow\begin{pmatrix}1\\1\end{pmatrix} & & (1\ {-1})\Big\uparrow\Big\downarrow\begin{pmatrix}1\\-1\end{pmatrix} & \\
 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0 \\
\Big\downarrow \alpha_* & & & & & & & & \\
 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 & \mathbb{Z} & 0 \\
K^{CR}(\mathbb{C}): & (1\ 1)\Big\uparrow\Big\downarrow\begin{pmatrix}1\\1\end{pmatrix} & & (1\ {-1})\Big\uparrow\Big\downarrow\begin{pmatrix}1\\-1\end{pmatrix} & & (1\ 1)\Big\uparrow\Big\downarrow\begin{pmatrix}1\\1\end{pmatrix} & & (1\ {-1})\Big\uparrow\Big\downarrow\begin{pmatrix}1\\-1\end{pmatrix} & \\
 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0 & \mathbb{Z}^2 & 0
\end{array}
$$

Using this diagram and arguments similar to the previous cases, we can deduce that $\alpha_0^O = \alpha_4^O = b + c$ and $\alpha_2^O = \alpha_6^O = b - c$. Thus the matrix $B_{22} + B_{32}$ is correct in degrees 0 and 4, and the matrix $B_{22} - B_{32}$ is correct in degrees 2 and 6. □

Recall that if a vertex v is fixed by γ, then p_v is a projection in $C^*_{\mathbb{R}}(E;\gamma)$; and if v is not fixed by γ, then the projection $p_v + p_{\gamma(v)}$ is in $C^*_{\mathbb{R}}(E;\gamma)$. We finish this section with a result that explicitly identifies the class of each of these elements $KO_0(C^*_{\mathbb{R}}(E;\gamma))$.

From Theorem 4.4 and Corollary 4.3, there is an isomorphism

$$\rho\colon \operatorname{coker}(B_0 - 1) \to KO_0(C^*_{\mathbb{R}}(E;\gamma))$$

where

$$B_0 = \begin{pmatrix} B_{11} & 2B_{12} \\ B_{21} & B_{22} + B_{32} \end{pmatrix} : \mathbb{Z}^{|F_1|} \oplus \mathbb{Z}^{|F_2|} \to \mathbb{Z}^{|F_1|} \oplus \mathbb{Z}^{|F_2|}$$

is the matrix describing the map β_* in degree 0. For any vertex $v \in F_1 \sqcup F_2$, let δ_v be the vector in $\mathbb{Z}^{|F_1|} \oplus \mathbb{Z}^{|F_2|}$ with a 1 in the entry corresponding to vertex v and a 0 elsewhere, and let $[\delta_v]$ denote the equivalence class of δ_v in $\operatorname{coker}(1 - B_0)$.

Theorem 4.5. *Let (E, γ) be a row-finite graph with involution and with no sinks. Then*

(i) $\rho[\delta_v] = [p_v] \in KO_0(C^*_{\mathbb{R}}(E;\gamma))$ *for any vertex $v \in F_1$;*
(ii) $\rho[\delta_v] = [p_v + p_w] \in KO_0(C^*_{\mathbb{R}}(E;\gamma))$ *for any vertex $v \in F_2$ and $w = \gamma(v) \in F_3$; and*
(iii) $\rho[\delta_v] = [p_v] \in KU_0(C^*_{\mathbb{R}}(E;\gamma))$ *for any vertex $v \in E^1$.*

Corollary 4.6. *If E is finite, then* $[1] = \rho\big[\sum_{v \in F_1 \sqcup F_2} \delta_v\big]$.

Corollary 4.7. *$KO_0(C^*_{\mathbb{R}}(E;\gamma))$ is generated by the classes*

$$\{[p_v] \mid v \in F_1\} \cup \{[p_v + p_{\gamma(v)}] \mid v \in F_2\}.$$

Part (iii) of Theorem 4.5 is a well-known result in the complex case (see [15, Thm. 3.2]), but our proof will take a different approach than that in [15]. In the real case, as in the complex case, the key is to identify the homomorphism

$$KO_0(C^*_{\mathbb{R}}(E \times_1 \mathbb{Z})) \to KO_0(C^*_{\mathbb{R}}(E))$$

which is a composition of several homomorphisms including one of the homomorphisms of the original Pimsner–Voiculescu exact sequence, the isomorphism $C^*(E \times_1 \mathbb{Z}) \cong C^*(E) \rtimes_\gamma \mathbb{T}$, and the isomorphism of Takai duality. Rather than chase through these maps explicitly, in our proof below we obtain a description by using naturality and a couple of key universal examples.

Proof of Theorem 4.5. First we make the following observation from the proof of Theorem 4.1. If E is a graph with no sinks, then the homomorphism

$$(\alpha_0)_*\colon \mathbb{Z}^{|F_1 \sqcup F_2|} \cong KO_0(C^*_{\mathbb{R}}(E \times_1 \mathbb{N}; \gamma)) \to KO_0(C^*_{\mathbb{R}}(E \times_1 \mathbb{Z}; \gamma))$$

induces an isomorphism $\widetilde{\alpha}$ from $\operatorname{coker}(1 - B_0)$ to $\operatorname{coker}(1 - \beta_*)$ and this isomorphism satisfies

$$\widetilde{\alpha}([\delta_v]) = \begin{cases} [p_{(v,0)}] & \text{for } v \in F_1, \\ [p_{(v,0)} + p_{(\gamma(v),0)}] & \text{for } v \in F_2. \end{cases} \tag{3}$$

Now, for the moment, we drop the assumption that E has no sinks. Even so, there is a homomorphism

$$\pi\colon KO_0(C^*_{\mathbb{R}}(E \times_1 \mathbb{Z}; \gamma)) \to KO_0(C^*_{\mathbb{R}}(E; \gamma))$$

as in sequence (2) in the proof of Theorem 4.1. Note that π is natural with respect to homomorphisms of graph algebras $C^*_{\mathbb{R}}(E;\gamma) \to C^*_{\mathbb{R}}(F;\gamma)$ that arise from suitable graph morphisms.

We claim that for any vertex $(v, n) \in E \times_1 \mathbb{Z}$ with $\gamma(v) = v$ we have

$$\pi[p_{(v,n)}] = [p_v]. \tag{4}$$

Consider the graph EO consisting of only a single vertex v_0 and no edges (where the involution γ is obviously trivial). Then

$$C^*_{\mathbb{R}}(EO; \gamma) \cong \mathbb{R} \cdot p_{v_0} \quad \text{and} \quad KO_0(C^*_{\mathbb{R}}(EO;\gamma)) \cong \mathbb{Z} \cdot [p_{v_0}] \cong \mathbb{Z}.$$

Furthermore

$$C^*_{\mathbb{R}}(EO \times_1 \mathbb{Z}; \gamma) \cong \bigoplus_{n \in \mathbb{Z}} \mathbb{R} \cdot p_{(v_0, n)}$$

and thus

$$KO_0(C^*_{\mathbb{R}}(EO \times_1 \mathbb{Z}; \gamma)) \cong \bigoplus_{n \in \mathbb{Z}} \mathbb{Z} \cdot [p_{(v_0,n)}] \cong \mathbb{Z}^{|\mathbb{Z}|}.$$

Also $KO_{-1}(C^*_{\mathbb{R}}(EO \times_1 \mathbb{Z}; \gamma)) = 0$. Therefore sequence (2) becomes

$$\begin{aligned} KO_0(C^*_{\mathbb{R}}(EO \times_1 \mathbb{Z}; \gamma)) &\xrightarrow{\beta_* - 1} KO_0(C^*_{\mathbb{R}}(EO \times_1 \mathbb{Z}; \gamma)) \\ &\xrightarrow{\pi} KO_0(C^*_{\mathbb{R}}(EO; \gamma)) \to 0 \end{aligned}$$

or

$$\bigoplus_{n\in\mathbb{Z}} \mathbb{Z}\cdot[p_{(v_0,n)}] \xrightarrow{\beta_*-1} \bigoplus_{n\in\mathbb{Z}} \mathbb{Z}\cdot[p_{(v_0,n)}] \xrightarrow{\pi} \mathbb{Z}\cdot[p_{v_0}] \to 0.$$

Since we have

$$(\beta_*-1)[p_{(v_0,n)}] = [p_{(v_0,n+1)}] - [p_{(v_0,n)}],$$

exactness of the sequence forces $\pi([p_{(v_0,n)}]) = [p_{v_0}]$ for all n (at least up to a universal change in sign).

Now, let (E,γ) be an arbitrary graph with a specified vertex v fixed by γ. There are a homomorphism $g : C^*_{\mathbb{R}}(EO;\gamma) \to C^*_{\mathbb{R}}(E;\gamma)$ given by $g(p_{v_0}) = p_v$ and a commutative diagram

$$\begin{array}{ccc} KO_0(C^*_{\mathbb{R}}(EO\times_1\mathbb{Z};\gamma)) & \xrightarrow{\pi_{EO}} & KO_0(C^*_{\mathbb{R}}(EO;\gamma)) \\ \downarrow{\scriptstyle (g\times\mathrm{id})_*} & & \downarrow{\scriptstyle g_*} \\ KO_0(C^*_{\mathbb{R}}(E\times_1\mathbb{Z};\gamma)) & \xrightarrow{\pi_E} & KO_0(C^*_{\mathbb{R}}(E;\gamma)) \end{array}$$

which can be written as

$$\begin{array}{ccc} \bigoplus_{n\in\mathbb{Z}}\mathbb{Z}\cdot[p_{(v_0,n)}] & \xrightarrow{\pi_{EO}} & \mathbb{Z}\cdot[p_{v_0}] \\ \downarrow{\scriptstyle (g\times\mathrm{id})_*} & & \downarrow{\scriptstyle g_*} \\ KO_0(C^*_{\mathbb{R}}(E\times_1\mathbb{Z};\gamma)) & \xrightarrow{\pi_E} & KO_0(C^*_{\mathbb{R}}(E;\gamma)). \end{array}$$

We have $(g\times\mathrm{id})_*([p_{(v_0,n)}]) = [p_{(v,n)}]$ and $g_*([p_{v_0}]) = [p_v]$; and from the previous paragraph we have $\pi_{EO}([p_{(v_0,n)}]) = [p_{v_0}]$; so it follows that

$$\pi_E([p_{(v,n)}]) = [p_v].$$

This proves claim (4).

Now, returning to the case that E is a graph that is row finite with no sinks and with a vertex v fixed by γ, we use the composition $\rho = \pi\circ\widetilde{\alpha}$ and equation (3) to show that

$$\rho([\delta_v]) = \pi\widetilde{\alpha}([\delta_v]) = \pi([p_{(v,0)}]) = [p_v]$$

proving part (i) of the theorem.

Turning to part (ii), we first prove that for any vertex $(v,n)\in E\times_1\mathbb{Z}$ with $\gamma(v)=w$ and $v\neq w$ we have

$$\pi[p_{(v,n)} + p_{(w,n)}] = [p_v + p_w] \tag{5}$$

This time let EO be the graph with no edges and two vertices, v_0 and w_0, with involution γ satisfying $\gamma(v_0)=w_0$. Then

$$C^*_{\mathbb{R}}(EO;\gamma) \cong \{\lambda p_{v_0} + \overline{\lambda} p_{w_0} \mid \lambda\in\mathbb{C}\} \cong \mathbb{C}$$

and

$$KO_0(C^*_{\mathbb{R}}(EO;\gamma)) \cong \mathbb{Z}\cdot[p_{v_0}+p_{w_0}] \cong \mathbb{Z}.$$

Furthermore

$$C^*_{\mathbb{R}}(EO \times_1 \mathbb{Z}; \gamma) \cong \bigoplus_{n \in \mathbb{Z}} \{\lambda p_{(v_0,n)} + \overline{\lambda} p_{(w_0,n)} \mid \lambda \in \mathbb{C}\} \cong \mathbb{C}^{|\mathbb{Z}|}$$

and thus

$$KO_0(C^*_{\mathbb{R}}(EO \times_1 \mathbb{Z}; \gamma)) \cong \bigoplus_{n \in \mathbb{Z}} \mathbb{Z} \cdot [p_{(v_0,n)} + p_{(w_0,n)}] \cong \mathbb{Z}^{|\mathbb{Z}|}.$$

Also $KO_{-1}(C^*_{\mathbb{R}}(EO \times_1 \mathbb{Z}; \gamma)) = 0$ so again sequence (2) becomes

$$KO_0(C^*_{\mathbb{R}}(EO \times_1 \mathbb{Z}; \gamma)) \xrightarrow{\beta_* - 1} KO_0(C^*_{\mathbb{R}}(EO \times_1 \mathbb{Z}; \gamma)) \xrightarrow{\pi} KO_0(C^*_{\mathbb{R}}(EO; \gamma)) \to 0$$

or

$$\bigoplus_{n \in \mathbb{Z}} \mathbb{Z} \cdot [p_{(v_0,n)} + p_{(w_0,n)}] \xrightarrow{\beta_* - 1} \bigoplus_{n \in \mathbb{Z}} \mathbb{Z} \cdot [p_{(v_0,n)} + p_{(w_0,n)}] \xrightarrow{\pi} \mathbb{Z} \cdot [p_{v_0} + p_{w_0}] \to 0.$$

As before, exactness of the sequence forces the homomorphism π to satisfy

$$\pi([p_{(v,n)} + p_{(w,n)}]) = [p_v + p_w]$$

for all n.

Now, let (E, γ) be an arbitrary graph with a specified vertex v not fixed by γ. Let $w = \gamma(v)$. Let $g : C^*_{\mathbb{R}}(EO; \gamma) \to C^*_{\mathbb{R}}(E; \gamma)$ be given by $g(p_{v_0}) = p_v$ and $g(p_{w_0}) = p_w$. Then using a similar naturality argument to the one proving claim (4), we obtain

$$\pi_E([p_{(v_0,n)} + p_{(w_0,n)}]) = [p_{v_0} + p_{w_0}]$$

for all n, proving claim (5).

For any row-finite graph E with no sinks and with a vertex v not fixed by γ, let $w = \gamma(v)$. So we have

$$\rho([\delta_v]) = \pi\widetilde{\alpha}([\delta_v]) = \pi([p_{(v,0)} + p_{(w,0)}]) = [p_v + p_w],$$

proving part (ii).

For part (iii), we mention that the same proof works as for part (i) but placed in the context of complex graph algebras. Alternatively, one can prove part (iii) from part (i) as follows. First replace the involution γ with the identity knowing that $KU_0(C^*(E; \gamma)) \cong KU_0(C^*_{\mathbb{R}}(E; \mathrm{id}))$. In this case, it follows from [5, Thm. 13] that the complexification map induces an isomorphism

$$c_0 \colon KO_0(C^*_{\mathbb{R}}(E; \gamma)) \to KU_0(C^*_{\mathbb{R}}(E; \gamma)).$$

Since c_0 carries $[p_v]$ to $[p_v]$ for any vertex v and since it commutes with ρ, part (iii) of the theorem follows. □

Proof of Corollaries 4.6 and 4.7. Corollary 4.6 follows immediately from Theorem 4.5 since

$$[1] = \sum_{v \in F_1} [p_v] + \sum_{v \in F_2} [p_v + p_{\gamma(v)}];$$

and Corollary 4.7 follows since the elements $[\delta_v]$ for $v \in F_1 \sqcup F_2$ generate $\operatorname{coker}(B_0 - I)$. □

5. Examples: Real structures on circle algebras

In this section we return to the circle algebras examined in Examples 3.5 and 3.6. Although the K-theory of these algebras can be computed using the isomorphisms identified in Section 3, we will here show that we can compute the K-theory directly from the graph structure using Theorems 4.1 and 4.4.

For context, we note that in [17, §2], Stacey classifies the real structures of complex circle C^*-algebras. If n is odd, there are four real structures on $C(S^1, M_n(\mathbb{C}))$ up to isomorphism. The algebra $C^*_{\mathbb{R}}(E; \mathrm{id})$ in Example 5.1 below is one of these, listed as A_5 in [17, p. 1518]. In the even case, there are seven real structures on $C(S^1, M_n(\mathbb{C}))$ up to isomorphism. In addition to $C^*_{\mathbb{R}}(E; \mathrm{id}) \cong A_5$, the algebra $C^*_{\mathbb{R}}(E; \gamma)$ in Example 5.2 below is listed as A_7.

The algebras A_3, A_6, and A_7 cannot be realized as graph algebras or as arising from a graph with involution by Corollary 4.3 since they each have torsion in either KO_5 or KO_7 (see [17, §2]). In fact, with a little more detailed analysis we can also eliminate A_1 and A_2 as possibly arising from a finite graph with involution. For any such algebra A, Corollary 4.3 and Theorem 4.4 imply that the rank of the free part of $KO_6(A)$ and $KO_7(A)$ must be the same since one is the cokernel and the other is the kernel of a homomorphism from $\mathbb{Z}^{|F_2|}$ to $\mathbb{Z}^{|F_2|}$. But from [17, §2], we have $KO_6(A_1) = KO_6(A_2) = 0$ and $KO_7(A_1) = KO_7(A_2) = \mathbb{Z}$.

Example 5.1. Let E be the cyclic graph with n vertices as in Examples 3.5 and 3.6 and let $\gamma = \mathrm{id}$. Then $K^{CR}(C^*(E; \gamma))$ is given in Table 4.

k	0	1	2	3	4	5	6	7
$KO_k(C^*(E;\gamma))$	$\mathbb{Z}$	$\mathbb{Z} \oplus \mathbb{Z}_2$	$\mathbb{Z}_2 \oplus \mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	$\mathbb{Z}$	0	0
$KU_k(C^*(E;\gamma))$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

TABLE 4. $K^{CR}(C^*(E;\gamma))$.

Proof. We have

$$A_E^{\mathrm{Tr}} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Since all the vertices are fixed by γ, this is the matrix $B = B_{11}$ used to compute $\beta_* - \mathrm{id}$ in degrees $k = 0, 1, 2, 4$ as in Theorem 4.4. The Smith normal form of

$$B_{11} - I_n = \begin{pmatrix} -1 & 0 & \dots & 0 & 1 \\ 1 & -1 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -1 \end{pmatrix}$$

is easily found to be the diagonal matrix $\mathrm{diag}(1, 1, \dots, 1, 0)$. Hence

$$\begin{aligned} \ker(A_E^{\mathrm{Tr}} - I_n) &\cong \mathrm{coker}(A_E^{\mathrm{Tr}} - I_n) \cong \mathbb{Z} \quad \text{in degree } k = 0, 4, \\ \ker(A_E^{\mathrm{Tr}} - I_n) &\cong \mathrm{coker}(A_E^{\mathrm{Tr}} - I_n) \cong \mathbb{Z}_2 \quad \text{in degree } k = 1, 2. \end{aligned}$$

In other degrees ($i = 3, 5, 6, 7$) we have

$$\ker(A_E^{\mathrm{Tr}} - I_n) = \mathrm{coker}(A_E^{\mathrm{Tr}} - I_n) = 0.$$

Then the long exact sequence from Theorem 4.1 immediately gives all of the KO-groups, except $KO_2(C^*(E;\gamma))$ which is an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_2$. The $\mathcal{CR}$-module structure can be used to show that $KO_2(C^*(E;\gamma)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$. □

This calculation in Example 5.1 confirms that

$$K^{CR}(C^*(E;\gamma)) \cong K^{CR}(\mathbb{R}) \oplus \Sigma^{-1} K^{CR}(\mathbb{R}),$$

which follows also from the analysis in Examples 3.5 and 3.6 giving the isomorphism $C^*(E;\gamma) \cong (S^{-1}\mathbb{R})^{\sim} \otimes M_n(\mathbb{R})$.

Example 5.2. Let E be the cyclic graph with $2n$ vertices and let γ be the nontrivial involution on E as in Example 3.5. Then $K^{CR}(C^*_{\mathbb{R}}(E;\gamma))$ is given in Table 5.

k	0	1	2	3	4	5	6	7
$KO_k(C^*(E;\gamma))$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}_2$	0
$KU_k(C^*(E;\gamma))$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}$

TABLE 5. $K^{CR}(C^*_{\mathbb{R}}(E;\gamma))$.

Proof. Since there are no vertices fixed by γ, we arrange the vertices into two disjoint sets F_2 and F_3 so that we have

$$A_E^{\mathrm{Tr}} = B = \begin{pmatrix} B_{22} & B_{23} \\ B_{32} & B_{22} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Then the matrix description of $\beta_* - \mathrm{id}$ in degree 0 and degree 4 is

$$D_0 = D_4 = B_{22} + B_{23} - I_n = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 & 1 \\ 1 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix}$$

which has Smith normal form equal to $\mathrm{diag}(1, 1, \dots, 1, 0)$ (as in Example 5.1). So

$$\operatorname{coker} D_0 \cong \ker D_0 \cong \operatorname{coker} D_4 \cong \ker D_4 \cong \mathbb{Z}.$$

In degrees 2 and 6 the matrix description is

$$D_2 = D_6 = B_{22} - B_{23} - I_n = \begin{pmatrix} -1 & 0 & 0 & \dots & 0 & -1 \\ 1 & -1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \end{pmatrix}$$

which has Smith normal form equal to $\mathrm{diag}(1, 1, \dots, 1, 2)$. So

$$\ker D_2 \cong \ker D_6 \cong 0 \quad \text{and} \quad \operatorname{coker} D_2 \cong \operatorname{coker} D_6 \cong \mathbb{Z}_2.$$

For all k odd, $D_k \colon 0 \to 0$. From this all the KO_* groups are determined as shown. □

6. Examples: Real structures on Cuntz algebras

We now turn to some examples of graphs whose C^*-algebras are simple and purely infinite, and therefore classified by K-theory [7, 11]. While Theorem 6.1 below shows that we cannot realize all real structures of $M_k(\mathcal{O}_n)$ using these constructions, we can find a rich variety of different real structures for Cuntz algebras and matrix algebras over Cuntz algebras. Below we present just three families of examples of graphs with involutions. Each gives rise to new real structure not previously known. In Example 6.2, we find a graph with involution producing a real structure on $M_n(\mathbb{C}) \otimes \mathcal{O}_{2n+1}$ where the corresponding real C^*-algebras are KK-equivalent to a six-fold suspension of $\mathcal{E}_{2n+1}$. In Examples 6.3 and 6.4 (taken together), we find graphs with involution producing real structures on $M_2(\mathbb{C}) \otimes \mathcal{O}_{4n+1}$ (for all positive n) that differ from the previously known examples even up to suspension.

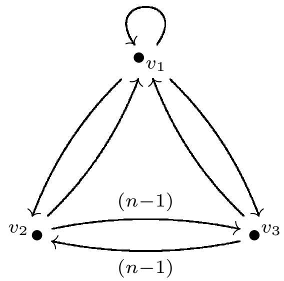
Theorem 6.1. *There does not exist a graph E with involution γ such that $C^*_{\mathbb{R}}(E;\gamma)$ is stably isomorphic to $\mathcal{E}_n$. More generally, if $C^*_{\mathbb{R}}(E;\gamma)$ is KK-equivalent to $\Sigma^k K^{CR}(\mathcal{E}_n)$ for some integers k and n, then $k \equiv 6 \pmod 8$.*

Proof. The more general statement follows from the fact that $KO_i(\mathcal{E}_n)$ has torsion for all i except $i = 3, 5$ from [7, Tab. 2]. Meanwhile we know that $KO_i(C^*_{\mathbb{R}}(E;\gamma))$ must be free for $i = 5, 7$ by Corollary 4.3. Thus

$$KO_i(C^*_{\mathbb{R}}(E;\gamma)) \cong \Sigma^k KO_i(\mathcal{E}_n) \cong KO_i(S^k \mathcal{E}_n)$$

is only possible for $k \equiv 6 \pmod 8$. □

Example 6.2. For $n \geq 2$, let E be the graph shown below with the associated incidence matrix. Let γ be the involution that fixes v_1 and interchanges v_2 and v_3.



$$A_E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & n-1 \\ 1 & n-1 & 0 \end{pmatrix}.$$

Then there are isomorphisms

$$\begin{aligned} C^*(E) &\cong M_n(\mathbb{C}) \otimes \mathcal{O}_{2n+1}, \\ C^*_{\mathbb{R}}(E; \mathrm{id}) &\cong M_n(\mathbb{R}) \otimes \mathcal{O}^{\mathbb{R}}_{2n+1}, \\ C^*_{\mathbb{R}}(E; \gamma) &\cong \text{the unique real form of } M_n(\mathbb{C}) \otimes \mathcal{O}_{2n+1} \text{ that is} \\ &\qquad KK\text{-equivalent to } S^6\mathcal{E}_{2n+1}. \end{aligned}$$

Proof. First note that E satisfies the conditions that ensure that $C^*(E)$ is simple and purely infinite (see for example [1, Prop. 5.3] and [14, Prop. 4.2]). This also ensures that the real C^*-algebra associated to any real structure on $C^*(E)$ is simple and purely infinite (by [7, Thm. 3.9]). Therefore both $C^*_{\mathbb{R}}(E; \mathrm{id})$ and $C^*_{\mathbb{R}}(E; \gamma)$ are classified by united K-theory.

First we compute the K-theory of $C^*(E)$ by identifying the kernel and cokernel of the map on $\mathbb{Z}^3$ induced by the matrix $D = A_E^{\mathrm{Tr}} - I_3$, using the Smith normal form. If

$$U = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ n & 1 & -1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & 0 & -n \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix},$$

then we have

$$UDV = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2n \end{pmatrix}.$$

Since U and V are invertible (in $GL_3(\mathbb{Z})$), the matrix UDV is the Smith normal form, so it follows that $K_0(C^*(E)) \cong \mathbb{Z}_{2n}$ and $K_1(C^*(E)) \cong 0$. It also follows that $C^*(E)$ is stably isomorphic to $\mathcal{O}_{2n+1}$. To identify the class of the unit in $K_0(C^*(E)) \cong \mathbb{Z}_{2n}$, we must analyze the class represented by $(\,1\;1\;1\,)^{\mathrm{Tr}}$ in the cokernel of $D\colon \mathbb{Z}^3 \to \mathbb{Z}^3$. The isomorphism from the cokernel of D to the cokernel of UDV is given by left multiplication by U. Thus we consider the element represented by

$$\left[U \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right] = \left[\begin{pmatrix} 2 \\ 1 \\ n \end{pmatrix} \right] = \left[\begin{pmatrix} 0 \\ 0 \\ n \end{pmatrix} \right]$$

in the cokernel of $UDV\colon \mathbb{Z}^3 \to \mathbb{Z}^3$. This calculation implies that the class of the unit corresponds to n times a generator of $K_0(C^*(E)) \cong \mathbb{Z}_{2n}$. Therefore $C^*(E) \cong M_n(\mathbb{C}) \otimes \mathcal{O}_{2n+1}$.

The K-theory of the real C^*-algebra $C^*_{\mathbb{R}}(E; \mathrm{id})$ can be computed as in [5, §4]. In degree 0 the calculation is the same as in the previous paragraph, so we have $KO_0(C^*_{\mathbb{R}}(E; \mathrm{id})) \cong \mathbb{Z}_{2n}$, the unit corresponds to n times a generator, and the complexification map c is an isomorphism in degree 0. This information, with the $\mathcal{CR}$-structure, is enough to compute the rest of the KO-groups. (Alternatively, one can continue to compute $KO_i(C^*_{\mathbb{R}}(E; \mathrm{id}))$ for each i using Theorem 4.4.)

Now we compute $KO_*(C^*_{\mathbb{R}}(E; \gamma))$. Applying Theorem 4.4, we obtain the matrix descriptions given in Table 6 for the real part of $(\beta_* - \mathrm{id})$ in degrees 0 through 7.

degree	matrix representation of $\beta_* - \mathrm{id}$	
0	$D_0 = \begin{pmatrix} 0 & 2 \\ 1 & n-2 \end{pmatrix}$	$\mathbb{Z}^2 \to \mathbb{Z}^2$
1	$D_1 = 0$	$\mathbb{Z}_2 \to \mathbb{Z}_2$
2	$D_2 = \begin{pmatrix} 0 & 1 \\ 0 & -n \end{pmatrix}$	$\mathbb{Z}_2 \oplus \mathbb{Z} \to \mathbb{Z}_2 \oplus \mathbb{Z}$
3	$D_3 = 0$	$0 \to 0$
4	$D_4 = \begin{pmatrix} 0 & 1 \\ 2 & n-2 \end{pmatrix}$	$\mathbb{Z}^2 \to \mathbb{Z}^2$
5	$D_5 = 0$	$0 \to 0$
6	$D_6 = -n$	$\mathbb{Z} \to \mathbb{Z}$
7	$D_7 = 0$	$0 \to 0$

TABLE 6. $(\beta_* - \mathrm{id})\colon M^O(E; \gamma) \to M^O(E; \gamma)$.

First we find $\ker D_0$ and $\operatorname{coker} D_0$. If n is odd, let

$$U_0 = \begin{pmatrix} \frac{-1}{2}(n-2) & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad V_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

otherwise, let

$$U_0 = \begin{pmatrix} \frac{-1}{2}(n-3) & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad V_0 = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

These are invertible matrices in $GL_2(\mathbb{Z})$ that satisfy

$$U_0 D_0 V_0 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

giving the Smith normal form of D_0 in all cases. It follows that

$$KO_0(C^*_{\mathbb{R}}(E; \gamma)) \cong \operatorname{coker} D_0 \cong \mathbb{Z}_2,$$

and since $\ker D_0 = 0$, there is no contribution to $KO_1(C^*_{\mathbb{R}}(E;\gamma))$, thus

$$KO_1(C^*_{\mathbb{R}}(E;\gamma)) \cong \operatorname{coker} D_1 = \mathbb{Z}_2.$$

Furthermore,

$$U_0 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} * \\ 1 \end{pmatrix}$$

from which it follows that the class of the unit is the nontrivial element of $KO_0(C^*_{\mathbb{R}}(E;\gamma))$. Therefore the map $c\colon KO_0(C^*_{\mathbb{R}}(E;\gamma)) \to KU_0(C^*_{\mathbb{R}}(E;\gamma))$ (which must carry the class of the unit to the class of the unit) is the nontrivial homomorphism $\mathbb{Z}_2 \to \mathbb{Z}_{2n}$.

In degree 2 the matrix D_2 gives a homomorphism from $\mathbb{Z}_2 \oplus \mathbb{Z} \to \mathbb{Z}_2 \oplus \mathbb{Z}$. The reader can verify that $\ker D_2 \cong \mathbb{Z}_2$ and $\operatorname{coker} D_2 \cong \mathbb{Z}_{2n}$. Thus

$$KO_3(C^*_{\mathbb{R}}(E;\gamma)) \cong \mathbb{Z}_2$$

and $KO_2(C^*_{\mathbb{R}}(E;\gamma))$ has a subgroup isomorphic to $\mathbb{Z}_{2n}$ (with corresponding quotient isomorphic to $\ker D_1 \cong \mathbb{Z}_2$).

In degree 4 we have

$$U_4 D_4 V_4 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

where

$$U_4 = \begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix} \quad \text{and} \quad V_4 = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Therefore $KO_4(C^*_{\mathbb{R}}(E;\gamma)) \cong \mathbb{Z}_2$ and $KO_5(C^*_{\mathbb{R}}(E;\gamma)) = 0$.

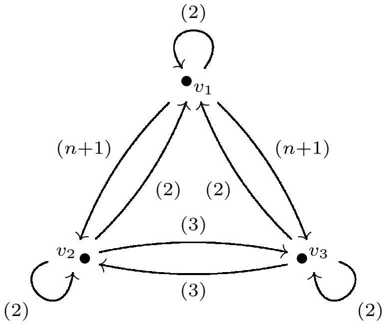
The kernel and cokernel of D_i are directly calculated in all other degrees, and this yields the KO_*-groups as given in Table 7. There is a slight complication in degree 2 in which we have an extension of $\mathbb{Z}_2$ by $\mathbb{Z}_{2n+2}$. The internal $\mathcal{CR}$ structure forces the particular solution to the extension problem up to isomorphism. The internal $\mathcal{CR}$ structure can also be used to compute the natural transformations. The full description of the $\mathcal{CR}$-module is given in Table 7. Comparing this to the K-theory of $\mathcal{E}_{2n+1}$ from [7, §11] we have

$$K^{CR}(C^*_{\mathbb{R}}(E;\gamma)) \cong \Sigma^6 K^{CR}(\mathcal{E}_{2n+1}). \qquad \square$$

k	0	1	2	3	4	5	6	7
$KO_k(\mathbb{R})$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_{4n}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	0	$\mathbb{Z}_n$	0
$KU_k(\mathbb{R})$	$\mathbb{Z}_{2n}$	0	$\mathbb{Z}_{2n}$	0	$\mathbb{Z}_{2n}$	0	$\mathbb{Z}_{2n}$	0
c	n	0	1	0	0	0	2	0
r	0	0	2	0	1	0	1	0
ψ	-1	0	1	0	-1	0	1	0
η	1	$2n$	1	1	0	0	0	0
ξ	0	0	1	0	0	0	4	0

TABLE 7. $K^{CR}(C^*_{\mathbb{R}}(E;\gamma))$.

Example 6.3. For any positive integer n, let E be the graph shown below with the associated incidence matrix. Let γ be the involution that fixes v_1 and interchanges v_2 and v_3.



$$A_E = \begin{pmatrix} 2 & n+1 & n+1 \\ 2 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}.$$

Then there are isomorphisms

$$\begin{aligned} C^*(E) &\cong M_2(\mathbb{C}) \otimes \mathcal{O}_{8n+1}, \\ C^*_{\mathbb{R}}(E; \mathrm{id}) &\cong M_2(\mathbb{R}) \otimes \mathcal{O}^{\mathbb{R}}_{8n+1}, \\ C^*_{\mathbb{R}}(E;\gamma) &\cong \text{the unique real form of } M_2(\mathbb{C}) \otimes \mathcal{O}_{8n+1} \text{ with} \\ &\quad\ \text{the } K\text{-theory given by Table 8.} \end{aligned}$$

k	0	1	2	3	4	5	6	7
$KO_k(\mathbb{R})$	$\mathbb{Z}_{4n}$	0	$\mathbb{Z}_2$	0	$\mathbb{Z}_{4n}$	0	$\mathbb{Z}_2$	0
$KU_k(\mathbb{R})$	$\mathbb{Z}_{8n}$	0	$\mathbb{Z}_{8n}$	0	$\mathbb{Z}_{8n}$	0	$\mathbb{Z}_{8n}$	0
c_k	2	0	$2n$	0	2	0	$2n$	0
r_k	$2n+1$	0	1	0	$2n+1$	0	1	0
ψ_k	$4n+1$	0	$4n-1$	0	$4n+1$	0	$4n-1$	0

TABLE 8. $K^{CR}(C^*(E;\gamma))$.

Proof. Let

$$D = A_E^{\mathrm{Tr}} - I_3 = \begin{pmatrix} 1 & n+1 & n+1 \\ 2 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}.$$

If n is odd, then set $n = 2k-1$ and

$$U = \begin{pmatrix} 1 & k & -k \\ 2 & 2k-2 & -2k+1 \\ 4 & 4k-3 & -4k+1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & 0 & -4k \\ 0 & 1 & -8k+5 \\ 0 & 0 & 1 \end{pmatrix}.$$

If n is even, then set $n = 2k$ and

$$U = \begin{pmatrix} 3 & 3k & -3k-1 \\ 2 & 2k-1 & -2k \\ 4 & 4k-1 & -4k-1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & 0 & -12k-2 \\ 0 & 1 & -8k+1 \\ 0 & 0 & 1 \end{pmatrix}.$$

In either case, we find that

$$UDV = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 8n \end{pmatrix} \quad \text{and} \quad U \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} * \\ * \\ 2 \end{pmatrix}.$$

Thus $K_0(C^*(E)) \cong \mathbb{Z}_{8n}$, $K_1(C^*(E)) \cong 0$, and the class of the unit is given by $2 \in \mathbb{Z}_{8n}$. This proves that $C^*(E) \cong M_2(\mathbb{C}) \otimes \mathcal{O}_{8n+1}$.

We skip the calculation of $KO_*(C^*_{\mathbb{R}}(E; \mathrm{id}))$ which is similar (at least in degree 0) to the complex case.

Turning to the calculation of $KO_*(C^*_{\mathbb{R}}(E; \gamma))$, the matrix descriptions for the real part of $(\beta_* - \mathrm{id})$ in degrees 0 through 7 are as in Table 9.

degree	matrix representation of $\beta_* - \mathrm{id}$	
0	$D_0 = \begin{pmatrix} 1 & 2n+2 \\ 2 & 4 \end{pmatrix}$	$\mathbb{Z}^2 \to \mathbb{Z}^2$
1	$D_1 = 1$	$\mathbb{Z}_2 \to \mathbb{Z}_2$
2	$D_2 = \begin{pmatrix} 1 & n+1 \\ 0 & -2 \end{pmatrix}$	$\mathbb{Z}_2 \oplus \mathbb{Z} \to \mathbb{Z}_2 \oplus \mathbb{Z}$
3	$D_3 = 0$	$0 \to 0$
4	$D_4 = \begin{pmatrix} 1 & n+1 \\ 4 & 4 \end{pmatrix}$	$\mathbb{Z}^2 \to \mathbb{Z}^2$
5	$D_5 = 0$	$0 \to 0$
6	$D_6 = -2$	$\mathbb{Z} \to \mathbb{Z}$
7	$D_7 = 0$	$0 \to 0$

TABLE 9. $(\beta_* - \mathrm{id})\colon M^O(E;\gamma) \to M^O(E;\gamma)$.

In degree 0 let

$$U_0 = \begin{pmatrix} -1 & 1 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad V_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

If $n = 1$ and otherwise, let

$$U_0 = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \quad \text{and} \quad V_0 = \begin{pmatrix} 1 & -2n-2 \\ 0 & 1 \end{pmatrix}.$$

Then

$$U_0 D_0 V_0 = \begin{pmatrix} 1 & 0 \\ 0 & 4n \end{pmatrix} \quad \text{and} \quad U_0 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} * \\ 1 \end{pmatrix}$$

which shows that $KO_0(C^*_{\mathbb{R}}(E;\gamma)) \cong \mathbb{Z}_{4n}$ and that the class of the identity is a generator. Furthermore, it follows immediately that

$$c_0\colon KO_0(C^*_{\mathbb{R}}(E;\gamma)) \to KU_0(C^*_{\mathbb{R}}(E;\gamma))$$

is isomorphic to multiplication by 2 on $\mathbb{Z}_{4n} \to \mathbb{Z}_{8n}$.

In the other degrees we use these same techniques to compute that D_i is injective for all i, is surjective for all odd i, and that

$$\operatorname{coker} D_2 \cong \operatorname{coker} D_6 \cong \mathbb{Z}_2 \quad \text{and} \quad \operatorname{coker} D_4 \cong \mathbb{Z}_{4n}.$$

This completes the calculation of the groups of $KO_*(C^*_{\mathbb{R}}(E;\gamma))$.

We now show how to compute the relations r_0 and ψ_0, using the map

$$\pi\colon M(E;\gamma) \to K^{CR}(C^*_{\mathbb{R}}(E;\gamma))$$

which is surjective on both KO_0 and KU_0. Since this map respects the homomorphism r_0, we have the diagram

$$\begin{array}{ccc} M_0^U(E;\gamma) & \xrightarrow{\pi} & KU_0(C^*_{\mathbb{R}}(E;\gamma)) \\ \downarrow{\scriptstyle r_0} & & \downarrow{\scriptstyle r_0} \\ M_0^O(E;\gamma) & \xrightarrow{\pi} & KO_0(C^*_{\mathbb{R}}(E;\gamma)) \end{array} \quad \text{isomorphic to} \quad \begin{array}{ccc} \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}_{8n} \\ \downarrow & & \downarrow \\ \mathbb{Z}^2 & \longrightarrow & \mathbb{Z}_{4n} \end{array}$$

It is the vertical map on the right that we wish to identify. Since

$$M(E;\gamma) \cong K^{CR}(\mathbb{R}) \oplus K^{CR}(\mathbb{C}),$$

we use what we know about the map r_0 for $K^{CR}(\mathbb{R})$ and $K^{CR}(\mathbb{C})$ (see the tables in Section 4) to write the vertical map on the left as given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}.$$

The horizontal maps π are given by multiplication by U or U_0 followed by the modular reduction on the final coordinate. In the odd case $n = 2k-1$, the vector

$$\begin{pmatrix} k \\ 0 \\ 1 \end{pmatrix} \in \mathbb{Z}^3$$

maps to the generator $[1] \in \mathbb{Z}_{8n}$. Following the diagram the other way around gives $[2n+1] \in \mathbb{Z}_{4n}$. This shows that c_0 is multiplication by $2n+1$. In the even case $n = 2k$, we get the same result using the same argument starting with the vector

$$\begin{pmatrix} -k \\ 0 \\ -1 \end{pmatrix} \in \mathbb{Z}^3.$$

For the homomorphism ψ_0 we have the diagram

$$\begin{array}{ccc} M_0^U(E;\gamma) & \xrightarrow{\pi} & KU_0(C^*_{\mathbb{R}}(E;\gamma)) \\ \downarrow{\scriptstyle \psi_0} & & \downarrow{\scriptstyle \psi_0} \\ M_0^U(E;\gamma) & \xrightarrow{\pi} & KU_0(C^*_{\mathbb{R}}(E;\gamma)) \end{array} \quad \text{isomorphic to} \quad \begin{array}{ccc} \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}_{8n} \\ \downarrow & & \downarrow \\ \mathbb{Z}^3 & \longrightarrow & \mathbb{Z}_{8n} \end{array}$$

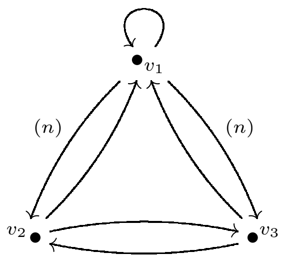
This time the vertical map on the left is given by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

We start with the same vectors in $\mathbb{Z}^3$ as in the previous paragraph and find that ψ_0 is multiplication by $4n+1$.

The rest of the natural transformations c, r, ψ can then be calculated by similar techniques in the other even degrees or from just the $\mathcal{CR}$-structure of $K^{CR}(C^*_{\mathbb{R}}(E;\gamma))$. □

Example 6.4. Let n be a positive odd integer and let E be the graph shown below with the associated incidence matrix. Let γ be the involution that fixes v_1 and interchanges v_2 and v_3.



$$A_E = \begin{pmatrix} 1 & n & n \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then there are isomorphisms

$$\begin{aligned} C^*(E) &\cong M_2(\mathbb{C}) \otimes \mathcal{O}_{4n+1}, \\ C^*_{\mathbb{R}}(E;\mathrm{id}) &\cong M_2(\mathbb{R}) \otimes \mathcal{O}^{\mathbb{R}}_{4n+1}, \\ C^*_{\mathbb{R}}(E;\gamma) &\cong \text{the unique real form of } M_2(\mathbb{C}) \otimes \mathcal{O}_{4n+1} \text{ with} \\ &\qquad \text{the } K\text{-theory given by Table 10.} \end{aligned}$$

In Table 10, the sign $r_2 = \pm 2$ must be taken to satisfy $r_2 \circ c_2 \equiv 2 \pmod 8$.

Proof. Let

$$D = A_E^{\mathrm{Tr}} - I_3 = \begin{pmatrix} 0 & n & n \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}.$$

k	0	1	2	3	4	5	6	7
$KO_k(\mathbb{R})$	$\mathbb{Z}_{2n}$	$\mathbb{Z}_2$	$\mathbb{Z}_8$	$\mathbb{Z}_2$	$\mathbb{Z}_{2n}$	0	$\mathbb{Z}_2$	0
$KU_k(\mathbb{R})$	$\mathbb{Z}_{4n}$	0	$\mathbb{Z}_{4n}$	0	$\mathbb{Z}_{4n}$	0	$\mathbb{Z}_{4n}$	0
c_k	2	0	n	0	4	0	$2n$	0
r_k	$n+1$	0	± 2	0	$-\frac{1}{2}(n-1)$	0	1	0
ψ_k	$2n+1$	0	$2n-1$	0	$2n+1$	0	$2n-1$	0
η_k	1	4	1	n	0	0	0	0
ξ_k	$n+1$	0	1	0	4	0	4	0

TABLE 10. $K^{CR}(C^*_{\mathbb{R}}(E;\gamma))$.

Set $n = 2k+1$ for $k \geq 0$ and

$$U = \begin{pmatrix} 1 & k+1 & -k \\ 1 & k & -k \\ 2 & 2k+1 & -2k-1 \end{pmatrix} \quad \text{and} \quad V = \begin{pmatrix} 1 & 0 & -2n \\ 0 & 1 & -2n+1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then

$$UDV = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4n \end{pmatrix} \quad \text{and} \quad U \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}.$$

Thus $KU_0(C^*_{\mathbb{R}}(E)) \cong \mathbb{Z}_{4n}$ and the class of the unit is 2 times a generator, proving the claim about $C^*(E)$.

For $KO_*(C^*_{\mathbb{R}}(E;\gamma))$, the matrices for the real part of $(\beta_* - \mathrm{id})$ in degrees 0 through 7 are given in Table 11.

degree	matrix representation of $\beta_* - \mathrm{id}$	
0	$D_0 = \begin{pmatrix} 0 & 2n \\ 1 & 0 \end{pmatrix}$	$\mathbb{Z}^2 \to \mathbb{Z}^2$
1	$D_1 = 0$	$\mathbb{Z}_2 \to \mathbb{Z}_2$
2	$D_2 = \begin{pmatrix} 0 & n \\ 0 & -2 \end{pmatrix}$	$\mathbb{Z}_2 \oplus \mathbb{Z} \to \mathbb{Z}_2 \oplus \mathbb{Z}$
3	$D_3 = 0$	$0 \to 0$
4	$D_4 = \begin{pmatrix} 0 & n \\ 2 & 0 \end{pmatrix}$	$\mathbb{Z}^2 \to \mathbb{Z}^2$
5	$D_5 = 0$	$0 \to 0$
6	$D_6 = -2$	$\mathbb{Z} \to \mathbb{Z}$
7	$D_7 = 0$	$0 \to 0$

TABLE 11. $(\beta_* - \mathrm{id})\colon M^O(E;\gamma) \to M^O(E;\gamma)$.

Focusing on degree 0, using similar methods as in the previous examples, it is easily determined that D_0 is injective and the cokernel is isomorphic to $\mathbb{Z}_{2n}$. Thus $KO_0(C^*_{\mathbb{R}}(E;\gamma)) \cong \mathbb{Z}_{2n}$ and, furthermore, the class of the unit is a generator. Since the unit is 2 times a generator of $KU_0(C^*_{\mathbb{R}}(E;\gamma)) \cong \mathbb{Z}_{4n}$, it follows that c_0 is multiplication by 2. Now using the same method as in Example 6.3, we find that $r_0 = n+1$ and $\psi_0 = 2n+1$.

From the matrices D_i, the kernel and cokernel of $(\beta_* - \mathrm{id})$ for the rest oft the groups $KO_*(C^*_{\mathbb{R}}(E;\gamma))$ are determined up to isomorphism except for $KO_2(C^*_{\mathbb{R}}(E;\gamma))$ which is isomorphic either to $\mathbb{Z}_8$ or to $\mathbb{Z}_2 \oplus \mathbb{Z}_4$. Using the $\mathcal{CR}$-structure, we find $KO_2(C^*_{\mathbb{R}}(E;\gamma)) \cong \mathbb{Z}_8$. □

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Received March 24, 2017; accepted September 7, 2017

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