Colimits in the correspondence bicategory

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Abstract. We interpret several constructions with C*-algebras as colimits in the bicategory of correspondences. This includes crossed products for actions of groups and crossed modules, Cuntz–Pimsner algebras of proper product systems, direct sums and inductive limits, and certain amalgamated free products.

1. Introduction

A basic idea of noncommutative geometry is to replace ordinary quotient spaces by noncommutative generalizations. For instance, let a group G act on a space X. The orbit space X/G is often badly behaved as a topological space. In noncommutative geometry, it is replaced by the crossed product C*-algebra $C_0(X) \rtimes G$. We may view the action of G on X as a diagram of topological spaces. The quotient space is the colimit of this diagram. We will exhibit the crossed product for a group action as a colimit as well, in an appropriate bicategory of C*-algebras. As this motivating example shows, our bicategorical colimit construction leads to noncommutative C*-algebras even when we start with a diagram of locally compact spaces.

The most concrete description of bicategories involves objects, arrows, and 2-arrows, the composition of arrows and the horizontal and vertical composition of 2-arrows. We shall emphasize a more conceptual definition: in a bicategory, *sets* of arrows between objects are replaced by *categories* of arrows, and the composition becomes a bifunctor. Associativity and unitality may hold exactly (strict 2-categories or just 2-categories) or only up to natural equivalences of categories that satisfy suitable coherence conditions (weak 2-categories or bicategories, see [3, 14]). We shall mostly work in the bicategory $\mathfrak{Corr}$ of C*-algebra correspondences. This is introduced by Landsman in [12] and studied in some depth in [7].

For simplicity, we also consider the bicategory $\mathfrak{C}^*(2)$, which is introduced in [7, §2.1.1]. Its objects are C*-algebras, its arrows $A \to B$ are nondegenerate *-homomorphisms $A \to \mathcal{M}(B)$, where $\mathcal{M}(B)$ denotes the multiplier algebra,

and its 2-arrows $f_1 \Rightarrow f_2$ for nondegenerate $*$-homomorphisms $f_1, f_2 \colon A \rightrightarrows \mathcal{M}(B)$ are unitary multipliers $u \in \mathcal{U}(B)$ with $u f_1(a) u^* = f_2(a)$ for all $a \in A$. Since unitaries are invertible, the arrows $A \to B$ and 2-arrows between them in $\mathfrak{C}^*(2)$ form a groupoid, not just a category.

By the way, we may also restrict to nondegenerate $*$-homomorphisms $A \to B$; this is like restricting to proper correspondences. Since a nonunital C*-algebra contains no unitary elements, our bicategory depends on using unitary *multipliers*. We need nondegeneracy for our arrows so that they act on unitary multipliers.

What are diagrams in categories and their colimits? Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A *diagram* in $\mathcal{D}$ of shape $\mathcal{C}$ is a functor $\mathcal{C} \to \mathcal{D}$. Such diagrams are again the objects of a category $\mathcal{D}^{\mathcal{C}}$, with natural transformations between functors as arrows. Any object x of $\mathcal{D}$ gives rise to a "constant" diagram $\mathrm{const}_x \colon \mathcal{C} \to \mathcal{D}$ of shape $\mathcal{C}$. The colimit $\mathrm{colim}\, F$ of a diagram $F \colon \mathcal{C} \to \mathcal{D}$ is an object of $\mathcal{D}$ with the following universal property: there is a natural bijection between arrows $\mathrm{colim}\, F \to x$ in $\mathcal{D}$ and natural transformations $F \Rightarrow \mathrm{const}_x$ for all objects x of $\mathcal{D}$. In brief,

$$\text{(1)} \qquad\qquad \mathcal{D}(\mathrm{colim}\, F, x) \cong \mathcal{D}^{\mathcal{C}}(F, \mathrm{const}_x).$$

Now let $\mathcal{C}$ and $\mathcal{D}$ be bicategories. As before, a *diagram* in $\mathcal{D}$ of shape $\mathcal{C}$ is a functor (or morphism) $\mathcal{C} \to \mathcal{D}$, as defined in, say, [3, 14]. The functors $\mathcal{C} \to \mathcal{D}$ are the objects of a bicategory $\mathcal{D}^{\mathcal{C}}$; its arrows and 2-arrows are the *transformations* between functors and the *modifications* between transformations, see [3, 14]. These definitions are repeated in our main reference [7] in Definition 4.1 and Sections 4.2 and 4.3.

Thus $\mathcal{D}^{\mathcal{C}}(F_1, F_2)$ for two diagrams F_1 and F_2 is now a *category*, not just a set, with transformations $F_1 \Rightarrow F_2$ as objects and modifications between them as arrows. Similarly, for two objects x_1 and x_2 of $\mathcal{D}$, there is a category $\mathcal{D}(x_1, x_2)$ of arrows $x_1 \to x_2$ and 2-arrows between them. Once again, there is a constant diagram const_x of shape $\mathcal{C}$ for any object x of $\mathcal{D}$. The bicategorical colimit is defined by the same condition (1), now interpreting $\cong$ as a natural equivalence of categories. An object $\mathrm{colim}\, F$ of $\mathcal{D}$ with this property is unique up to equivalence if it exists.

What do these definitions mean if $\mathcal{C} = G$ is a group and $\mathcal{D}$ is the bicategory $\mathfrak{C}^*(2)$ described above? First, diagrams in $\mathfrak{C}^*(2)$ are the twisted group actions in the sense of Busby and Smith; this is observed in [7]. Transformations between such diagrams are also described there. In particular, a transformation $F \Rightarrow \mathrm{const}_D$ is a *covariant representation* of the twisted G-action corresponding to F in the multiplier algebra of D. A modification is a unitary intertwiner between two covariant representations. Hence the colimit and the crossed product for the twisted action are characterized by the same universal property, forcing them to be isomorphic. As a result, if we replace the category of spaces and maps by the bicategory $\mathfrak{C}^*(2)$, we are led to enlarge the class of group actions to twisted actions, and the crossed product construction appears as the natural analog of a "quotient" in our bicategory.

Here we interpret many interesting constructions with C*-algebras as colimits. Thus our new point of view unifies several known constructions with C*-algebras. Most proofs are as trivial as above: we merely make the universal property that defines the bicategorical colimit explicit in a particular case and recognize the result as the definition of a familiar C*-algebraic construction.

Instead of $\mathfrak{C}^*(2)$, we mainly work in the correspondence bicategory $\mathfrak{Corr}$, which is defined in [7, §2.2]. Let A and B be C*-algebras. A *correspondence* from A to B is a Hilbert B-module $\mathcal{E}$ with a nondegenerate *-homomorphism from A to the C*-algebra of adjointable operators on $\mathcal{E}$. An isomorphism between two such correspondences is a unitary operator intertwining the left A-actions. We let $\mathfrak{Corr}(A, B)$ be the groupoid of correspondences from A to B and their isomorphisms. The composition is given by the bifunctors

$$(2) \qquad \mathfrak{Corr}(B, C) \times \mathfrak{Corr}(A, B) \to \mathfrak{Corr}(A, C), \quad (\mathcal{E}, \mathcal{F}) \mapsto \mathcal{F} \otimes_B \mathcal{E}.$$

This is associative and monoidal up to canonical isomorphisms, which are part of the bicategory structure (see [7]). A correspondence $\mathcal{E}$ from A to B is *proper* if the left A-module structure is through a map $A \to \mathbb{K}(\mathcal{E})$ to the C*-algebra of compact operators. Thus proper correspondences with isomorphisms between them form a subbicategory $\mathfrak{Corr}_{\mathrm{prop}}$ of $\mathfrak{Corr}$. Our main results will only hold for diagrams of proper correspondences, that is, functors to $\mathfrak{Corr}_{\mathrm{prop}}$.

Groups are categories with only one object. At the other extreme are discrete categories. These are categories where all arrows are identities, that is, sets viewed as categories. Colimits in this case are also called coproducts. Whereas coproducts need not exist in $\mathfrak{C}^*(2)$, they are given by the C_0-direct sum in the correspondence bicategory $\mathfrak{Corr}$; this statement is a standard additivity result about representations of C_0-direct sums on Hilbert modules. The nonexistence of coproducts in $\mathfrak{C}^*(2)$ is one reason to prefer the correspondence bicategory $\mathfrak{Corr}$. Moreover, since $\mathfrak{C}^*(2)$ is a subbicategory of $\mathfrak{Corr}$, we get more diagrams in $\mathfrak{Corr}$ than in $\mathfrak{C}^*(2)$.

A functor $G \to \mathfrak{Corr}$ for a group G is equivalent to a saturated Fell bundle over G (see [7]). The colimit for such a functor is the full C*-algebra of sections of the corresponding Fell bundle.

Crossed modules are a 2-categorical generalization of groups. Their actions on C*-algebras by automorphisms or correspondences have been introduced in [5, 7]. Once again, the universal property of the colimit is the same as that for the appropriate analog of the crossed product in this context.

What happens for non-reversible dynamical systems? Let P be a monoid, that is, a category with a single object. A functor $P \to \mathfrak{Corr}$ is the same as an essential product system over the opposite monoid P^{op}. The change of direction comes from (2), where we tensor in reverse order to conform to the usual conventions of composing maps. Colimits for product systems are remarkable because the universal property we get is not always but often equivalent to a standard one. More precisely, if the product system is *proper*, that is, all left actions in the product system are through compact operators, then the colimit of the corresponding diagram exists and is isomorphic to the Cuntz–Pimsner

algebra of the product system. We get the "absolute" Cuntz–Pimsner algebra, not the popular modification by Katsura, and we get there directly and never see the Cuntz–Toeplitz algebra along the way. This result on Cuntz–Pimsner algebras is the main idea of [1]. We had originally planned [1] as an applications section inside this article. We were, however, convinced by C*-algebra colleagues to write down those results separately, to make them accessible without category theory background.

Readers familiar with free products of C*-algebras may have been surprised that the bicategory $\mathfrak{C}^*(2)$ is not closed under coproducts: already in the usual category of C*-algebras with *-homomorphisms, there is a coproduct, namely the *free* product. This does not cooperate with unitary multipliers, however, and fails to satisfy the universal property for a coproduct in $\mathfrak{C}^*(2)$ or $\mathfrak{Corr}$. This situation clears up when we consider pushouts. Given two *nondegenerate* *-homomorphisms $B_1 \leftarrow A \rightarrow B_2$, their colimit in $\mathfrak{Corr}$ or $\mathfrak{C}^*(2)$ is the amalgamated free product $B_1 \star_A B_2$. Free products without amalgamation occur in the highly degenerate case $A = 0$.

Even more fundamental than pushouts are coequalizers. These are colimits of diagrams of the shape $\mathcal{E}_1, \mathcal{E}_2 \colon A \rightrightarrows B$. For instance, if $A = B = \mathbb{C}$ and $\mathcal{E}_i = \mathbb{C}^{n_i}$ for $i = 1, 2$, then the coequalizer is the universal C*-algebra generated by elements u_{jk} for $1 \leq j \leq n_1$, $1 \leq k \leq n_2$, subject to the relations

$$\sum_j u_{ij} u_{kj}^* = \delta_{i,k}, \quad \sum_i u_{ij}^* u_{ik} = \delta_{j,k}$$

for all $1 \leq i, k \leq n_1$ or all $1 \leq j, k \leq n_2$, respectively. If $n_1 = n_2$, then this is the C*-algebra U_n^{nc} introduced by Brown and studied further by McClanahan [4, 15, 16]. This example shows that coequalizers, even of very small diagrams, need not be particularly well-behaved C*-algebras.

Another situation we treat are inductive limits: the inductive limit of a chain of *-homomorphisms is also a colimit in $\mathfrak{Corr}$, even if some of these *-homomorphisms are degenerate.

We also prove one general result here: any diagram of proper correspondences, indexed by any bicategory, has a colimit. We describe this colimit by generators and relations, with the known construction of Cuntz–Pimsner algebras of product systems as a model case. This model case also shows that something may go wrong for diagrams involving non-proper correspondences.

2. Colimits in bicategories

Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories. An object of $\mathcal{D}^{\mathcal{C}}$ is a functor (or morphism) $\mathcal{C} \rightarrow \mathcal{D}$; it consists of several objects, arrows and 2-arrows in $\mathcal{D}$. In the *constant diagram*, $\mathrm{const}_x \colon \mathcal{C} \rightarrow \mathcal{D}$, all these objects are the same object x of $\mathcal{D}$, all the arrows are the identity on x, and all 2-arrows are the identity 2-arrow on id_x.

For instance, functors $G \rightarrow \mathfrak{C}^*(2)$ for a group G are identified with Busby–Smith twisted actions of G on C*-algebras in [7, §3.1.1]. The constant diagram const_A for a C*-algebra A is the trivial G-action on A, with trivial twists. Functors $G \rightarrow \mathfrak{Corr}$ are identified with saturated Fell bundles in [7, §3.1.1]. A

constant diagram const_A in $\mathfrak{Corr}$ corresponds to the constant Fell bundle with all fibers equal to A and the constant multiplication and involution.

Definition 2.1. Let $\mathcal{C}$ and $\mathcal{D}$ be bicategories and let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor. A *cone* over F is an object x of $\mathcal{D}$ with a transformation $\vartheta_x\colon F \to \mathrm{const}_x$; a *colimit* of F is a universal cone over F, that is, an object x of $\mathcal{D}$ with a transformation $\vartheta_x\colon F \to \mathrm{const}_x$, such that composition with ϑ_x induces equivalences of categories

$$\mathcal{D}(x,y) \xrightarrow{\cong} \mathcal{D}^{\mathcal{C}}(F, \mathrm{const}_y) \quad \text{for all objects } y \text{ of } \mathcal{D}.$$

If we are given natural equivalences $\mathcal{D}(x,y) \cong \mathcal{D}^{\mathcal{C}}(F, \mathrm{const}_y)$, then the identity map in $\mathcal{D}(x,x)$ gives a transformation $\vartheta_x\colon F \to \mathrm{const}_x$, which is determined uniquely up to isomorphism; naturality forces the equivalences $\mathcal{D}(x,y) \to \mathcal{D}^{\mathcal{C}}(F, \mathrm{const}_y)$ to be composition with ϑ_x. Hence a colimit may also be defined as an object x of $\mathcal{D}$ with natural equivalences of categories $\mathcal{D}(x,y) \cong \mathcal{D}^{\mathcal{C}}(F, \mathrm{const}_y)$.

Proposition 2.2. *The colimit is functorial: a transformation $\Phi\colon F_1 \to F_2$ induces an arrow $\mathrm{colim}\,\Phi\colon \mathrm{colim}\,F_1 \to \mathrm{colim}\,F_2$, and a modification $\Phi_1 \to \Phi_2$ induces a 2-arrow $\mathrm{colim}\,\Phi_1 \to \mathrm{colim}\,\Phi_2$, and these constructions are compatible with the composition bifunctor for transformations.*

Proof. Let (x_1, ϑ_1) and (x_2, ϑ_2) be colimits of F_1 and F_2, respectively. Transformations may be composed, so $\vartheta_2 \circ \Phi$ is an object of $\mathcal{D}^{\mathcal{C}}(F, \mathrm{const}_{x_2})$. By the definition of the colimit, there is an arrow $\mathrm{colim}\,\Phi\colon x_1 \to x_2$ with

$$\vartheta_2 \circ \Phi \cong (\mathrm{colim}\,\Phi) \circ \vartheta_1,$$

and this arrow is unique up to equivalence. Similarly, a modification $\Phi_1 \to \Phi_2$ induces a modification $\vartheta_2 \circ \Phi_1 \to \vartheta_2 \circ \Phi_2$, which gives a 2-arrow $\mathrm{colim}\,\Phi_1 \to \mathrm{colim}\,\Phi_2$. Thus we get a functor $\mathcal{D}^{\mathcal{C}}(F_1, F_2) \to \mathcal{D}(\mathrm{colim}\,F_1, \mathrm{colim}\,F_2)$. It is routine to check that this functor, up to equivalence, does not depend on choices and that the construction is compatible with the composition bifunctors in $\mathcal{D}^{\mathcal{C}}$ and $\mathcal{D}$. $\square$

Corollary 2.3. *Any two colimits of the same diagram are canonically equivalent.* $\square$

Equivalences in $\mathfrak{C}^*(2)$ are $*$-isomorphisms, those in $\mathfrak{Corr}$ are imprimitivity bimodules. Hence colimits in $\mathfrak{C}^*(2)$ are unique up to isomorphism if they exist, whereas colimits in $\mathfrak{Corr}$ are only unique up to Morita–Rieffel equivalence.

3. Coproducts and products

Coproducts are colimits of diagrams indexed by a category with only identity morphisms. Such a diagram is simply a map from some index set I to the objects of the category. The following proposition shows that the usual C_0-direct sum of C^*-algebras is both a coproduct and a product of the set of objects $(A_i)_{i \in I}$ in $\mathfrak{Corr}$. (We do not consider limits in this article because it

seems rare that they exist in $\mathfrak{Corr}$. We only mention the result on products because its proof and statement are so similar to the description of coproducts.)

Proposition 3.1. *Let* A_i *for* $i \in I$ *and* B *be* C*-algebras. Then*

$$\mathfrak{Corr}\bigg(\bigoplus_{i \in I} A_i, B\bigg) \cong \prod_{i \in I} \mathfrak{Corr}(A_i, B),$$

$$\mathfrak{Corr}\bigg(B, \bigoplus_{i \in I} A_i\bigg) \cong \prod_{i \in I} \mathfrak{Corr}(B, A_i).$$

Proof. Given correspondences $\mathcal{E}_i \colon A_i \to B$, we may form the Hilbert B-module $\bigoplus_{i \in I} \mathcal{E}_i$ and equip it with a nondegenerate left action of $\bigoplus_{i \in I} A_i$ to get a correspondence from $\bigoplus_{i \in I} A_i$ to B. Isomorphisms of correspondences $\mathcal{E}_i \to \mathcal{E}_i'$ may be put together to an isomorphism of correspondences $\bigoplus_{i \in I} \mathcal{E}_i \to \bigoplus_{i \in I} \mathcal{E}_i'$. Thus we get a functor

$$(3) \qquad \prod_{i \in I} \mathfrak{Corr}(A_i, B) \to \mathfrak{Corr}\bigg(\bigoplus_{i \in I} A_i, B\bigg).$$

To show that (3) is an equivalence, consider a correspondence $\mathcal{E}$ from $\bigoplus_{i \in I} A_i$ to B. Since the left action is nondegenerate, it extends to an action of the multiplier algebra of $\bigoplus_{i \in I} A_i$. The latter is $\prod_{i \in I} \mathcal{M}(A_i)$. (The product is taken in the category of C*-algebras, so it contains only bounded families.) In particular, $\mathcal{M}(\bigoplus_{i \in I} A_i)$ contains an orthogonal projection p_i onto the ith summand for each $i \in I$. We have strict convergence $\sum_{i \in I} p_i = 1$. The projections p_i act by orthogonal projections on $\mathcal{E}$. Let $\mathcal{E}_i := p_i \mathcal{E}$ be their images; these are Hilbert submodules on which A_i acts nondegenerately, respectively. Thus $\mathcal{E}_i$ is a correspondence from A_i to B. Since $\sum_{i \in I} p_i = 1$, we have $\bigoplus_{i \in I} \mathcal{E}_i = \mathcal{E}$. Thus $\mathcal{E}$ belongs to the essential range of the functor (3). Furthermore, since any intertwining operator between two correspondences commutes with the left action of the multiplier algebra and hence with the projections p_i, it comes from a family of intertwining operators on the summands $\mathcal{E}_i$; this shows that the functor (3) is fully faithful. Hence (3) is an equivalence of groupoids. This yields the first isomorphism, showing that $\bigoplus_{i \in I} A_i$ is a coproduct of $(A_i)_{i \in I}$ in $\mathfrak{Corr}$.

Now consider a family of correspondences $\mathcal{E}_i$ from B to A_i. Let $\bigoplus_{i \in I} \mathcal{E}_i$ be the set of all families $(\xi_i)_{i \in I}$ with $\xi_i \in \mathcal{E}_i$ and $(i \mapsto \|\xi_i\|) \in C_0(I)$. This is a Hilbert module over $\bigoplus_{i \in I} A_i$ by the pointwise operations. The left actions of B on the Hilbert modules $\mathcal{E}_i$ give a nondegenerate left action of B on $\bigoplus_{i \in I} \mathcal{E}_i$. Thus we get a correspondence from B to $\bigoplus_{i \in I} A_i$. This construction is natural with respect to isomorphisms of correspondences and hence gives a functor

$$(4) \qquad \prod \mathfrak{Corr}(B, A_i) \to \mathfrak{Corr}\bigg(B, \bigoplus A_i\bigg).$$

Take a correspondence $\mathcal{E}$ from B to $\bigoplus_{i \in I} A_i$. For each $i \in I$, $\mathcal{E}_i := \mathcal{E} \cdot A_i \subseteq \mathcal{E}$ is a correspondence from B to the ideal A_i in $\bigoplus_{j \in I} A_j$. Since these ideals are orthogonal, we have $\mathcal{E} \cong \bigoplus_{i \in I} \mathcal{E}_i$. Thus $\mathcal{E}$ belongs to the essential range

of (4). Since the decomposition $\mathcal{E} \cong \bigoplus_{i \in I} \mathcal{E}_i$ is natural, the functor (4) is fully faithful. □

Proposition 3.1 works because we may take direct sums of correspondences to make things orthogonal. In the category of C*-algebras with *-homomorphisms as morphisms, coproducts are free products, which are highly noncommutative. Since the coproduct in $\mathfrak{Corr}$ is unique up to isomorphism in $\mathfrak{Corr}$, that is, Morita–Rieffel equivalence, the free product is *not* a coproduct in $\mathfrak{Corr}$ any more. The reason is that it is not compatible with isomorphisms of correspondences: for a coproduct, we allow different unitaries $\mathcal{E}_i \cong \mathcal{E}_i'$ for all $i \in I$. Orthogonality of the $\mathcal{E}_i$ allows us to put two unrelated unitaries together. In the 2-category $\mathfrak{C}^*(2)$, coproducts do not exist in general for this reason: there are no orthogonal direct sums in $\mathfrak{C}^*(2)$, and free products do not behave well with respect to 2-arrows.

Example 3.2. We prove formally that the coproduct of two copies of $\mathbb{C}$ in $\mathfrak{C}^*(2)$ does not exist. Let B be a C*-algebra. There is a unique arrow $\mathbb{C} \to B$, namely the unit map of $\mathcal{M}(B)$. Thus there is a unique transformation from our coproduct diagram to const_B, given by the unit map on both copies of $\mathbb{C}$. A modification on this unique transformation is given by two unitaries $u_1, u_2 \in \mathcal{M}(B)$, one for each copy of $\mathbb{C}$, subject to no conditions. If we also take $B = \mathbb{C}$, then our groupoid of transformations is the two-torus group $\mathbb{T}^2$.

Now assume that the C*-algebra A were a coproduct of $\mathbb{C}$ and $\mathbb{C}$ in $\mathfrak{C}^*(2)$. Then the groupoid of arrows $A \to \mathbb{C}$ would be equivalent to $\mathbb{T}^2$. Its objects are nonzero characters $A \to \mathbb{C}$ and its arrows are unitaries in $\mathbb{C}$ acting on characters by conjugation, that is, trivially. So we get a disjoint union of some copies of the group $\mathbb{T}$, one for each character of A. But this is never equivalent to $\mathbb{T}^2$ because the groups $\mathbb{T}$ and $\mathbb{T}^2$ are not isomorphic. To see the latter, observe that $\mathbb{T}$ has exactly one element of order 2, namely -1, while $\mathbb{T}^2$ has exactly three of them, namely $(-1, +1)$, $(-1, -1)$, $(+1, -1)$.

The category $\mathfrak{Corr}$ has more diagrams than $\mathfrak{C}^*(2)$. Proposition 3.1 and Example 3.2 show that some very simple diagrams have a colimit in $\mathfrak{Corr}$, but not in $\mathfrak{C}^*(2)$. In the following, we therefore mostly study colimits in $\mathfrak{Corr}$.

Next we clarify the role of free products in our theory. We show that amalgamated free products are pushouts in $\mathfrak{Corr}$ under a nondegeneracy assumption; this rules out, in particular, free products without any amalgamation. Indeed, in the most degenerate case where we amalgamate over 0, Proposition 3.1 shows that the coproduct is the C_0-direct sum and not the free product.

3.3. **Pushouts.** A *pushout* in $\mathfrak{Corr}_{\text{prop}}$ is a colimit of a diagram of the form

$$
\begin{array}{ccc}
A & \xrightarrow{\mathcal{E}_1} & B_1 \\
{\scriptstyle \mathcal{E}_2} \downarrow & & \\
B_2, & &
\end{array}
$$

where A, B_1 and B_2 are C*-algebras and $\mathcal{E}_1$ and $\mathcal{E}_2$ are proper correspondences, without further data or conditions.

One extreme case is $A = 0$, where the pushout degenerates to a coproduct; this gives the direct sum $B_1 \oplus B_2$ by Proposition 3.1. Here we consider the opposite extreme case, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are associated to *nondegenerate* *-homomorphisms $A \to B_1$, $A \to B_2$; that is, $\mathcal{E}_i = B_i$ with A acting by $a \cdot b := \varphi_i(a) \cdot b$ for $i = 1, 2$.

Proposition 3.4. *Let A, B_1 and B_2 be C*-algebras and let $\varphi_1 \colon A \to B_1$ and $\varphi_2 \colon A \to B_2$ be* nondegenerate *-homomorphisms. The amalgamated free product $B_1 \star_A B_2$ is also a pushout in $\mathfrak{Corr}$.*

Proof. When we turn the *-homomorphism φ_i for $i = 1, 2$ into a correspondence $\mathcal{E}_i$, we take the right ideal $\varphi_i(A) \cdot B_i$, viewed as a Hilbert B_i-module, and equipped with the left action of A through φ_i. Our nondegeneracy assumption means that $\mathcal{E}_i = B_i$ as a right Hilbert B_i-module. Furthermore, we remark that $\varphi_i(A) \subseteq \mathbb{K}(\mathcal{E}_i) = B_i$ by assumption, so the $\mathcal{E}_i$ are proper correspondences. We will see later that properness is crucial to get colimits.

Let D be a C*-algebra. A transformation in $\mathfrak{Corr}$ from our pushout diagram to the constant diagram on D is given by correspondences $\mathcal{F}_1 \colon B_1 \to D$, $\mathcal{F}_2 \colon B_2 \to D$ and an isomorphism

$$U \colon \mathcal{F}_1 \cong B_1 \otimes_{B_1} \mathcal{F}_1 \to B_2 \otimes_{B_2} \mathcal{F}_2 \cong \mathcal{F}_2$$

of correspondences from A to D. That is, U is a unitary operator $\mathcal{F}_1 \to \mathcal{F}_2$ that intertwines the left actions of A given by composing the actions of B_i with the *-homomorphisms φ_i. Here we have used the nondegeneracy of φ_i to identify $\mathcal{E}_i = B_i$ as Hilbert B_i-modules.

A modification from $(\mathcal{F}_i, U)$ to $(\mathcal{F}'_i, U')$ is given by isomorphisms of correspondences $V_i \colon \mathcal{F}_i \to \mathcal{F}'_i$ for $i = 1, 2$ that intertwine U and U'.

Every such transformation is isomorphic to one where $\mathcal{F}_1 = \mathcal{F}_2$ as right Hilbert D-modules and U is the identity operator: the identity on $\mathcal{F}_1$ and $U \colon \mathcal{F}_1 \to \mathcal{F}_2$ is an invertible modification. Hence restricting to transformations with $\mathcal{F}_1 = \mathcal{F}_2$ and $U = \mathrm{id}$ gives an equivalent groupoid. So it does not change the colimit. The intertwining condition for modifications now simply says that the unitaries $\mathcal{F}_i \to \mathcal{F}'_i$ for $i = 1, 2$ are the *same* unitary, so we only have a single unitary that intertwines the actions of B_1 and B_2, and hence the actions of A.

If $\mathcal{F}_1 = \mathcal{F}_2$ and $U = \mathrm{id}$, then B_1 and B_2 act on the same Hilbert module, and the actions composed with φ_i coincide on A; this gives an action of the amalgamated free product $B_1 \star_A B_2$ on $\mathcal{F}_i$. Since B_1 and B_2 act nondegenerately, so does $B_1 \star_A B_2$. Hence we get a correspondence $B_1 \star_A B_2 \to D$.

Conversely, a correspondence $B_1 \star_A B_2 \to D$ gives a Hilbert module $\mathcal{F}$ with a nondegenerate left action of $B_1 \star_A B_2$. Since $A \cdot B_i = B_i$, the embedding $A \to B_1 \star_A B_2$ is nondegenerate, so the action of A on $\mathcal{F}$ is nondegenerate, and then so are the actions of B_i. Thus we get a transformation from the pushout diagram to the constant diagram on D with $\mathcal{F} = \mathcal{F}_1 = \mathcal{F}_2$ and U the

identity. Thus we have found an equivalence between the groupoid of natural transformations and modifications and the groupoid of correspondences from $B_1 \star_A B_2$ to D. This proves that $B_1 \star_A B_2$ is a colimit. $\square$

Corollary 3.5. *Let $\mathcal{E}_i$ be proper, full correspondences from A to B_i for $i = 1, 2$. The pushout in $\mathfrak{Corr}$ of $\mathcal{E}_1$ and $\mathcal{E}_2$ is the amalgamated free product $\mathbb{K}(\mathcal{E}_1) \star_A \mathbb{K}(\mathcal{E}_2)$.*

Proof. Since $\mathcal{E}_i$ is full, it gives a Morita–Rieffel equivalence between $\mathbb{K}(\mathcal{E}_i)$ and B_i. Hence the diagrams in $\mathfrak{Corr}$ given by $\mathcal{E}_1$ and $\mathcal{E}_2$ and by the *-homomorphisms $A \to \mathbb{K}(\mathcal{E}_i)$ for $i = 1, 2$ from the left A-module structures on $\mathcal{E}_i$ are isomorphic. The latter diagram has $\mathbb{K}(\mathcal{E}_1) \star_A \mathbb{K}(\mathcal{E}_2)$ as a colimit by Proposition 3.4. Since the construction of colimits is functorial by Proposition 2.2, this is also a colimit of the original diagram. $\square$

3.6. **An example of a coequalizer.** A *coequalizer* is a colimit of a diagram consisting of two parallel arrows $\alpha_1, \alpha_2 \colon A_1 \rightrightarrows A_2$. These particular colimits quickly become very complicated, as the following example shows.

Example 3.7. Consider the coequalizer of the following diagram:

$$
(5) \qquad\qquad \mathbb{C} \underset{\mathbb{C}^n}{\overset{\mathbb{C}^m}{\rightrightarrows}} \mathbb{C}.
$$

The groupoid of transformations from the above diagram to the constant diagram on a C*-algebra D is equivalent to the groupoid that has pairs $(\mathcal{F}, U)$ for a Hilbert D-module $\mathcal{F}$ and a unitary operator

$$
U \colon \mathcal{F}^n \cong \mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{F} \xrightarrow{\cong} \mathbb{C}^m \otimes_{\mathbb{C}} \mathcal{F} \cong \mathcal{F}^m
$$

as objects. We may write U as a matrix $U = (u_{i,j})$ with $u_{i,j} \in \mathbb{B}(\mathcal{F})$ for $1 \le i \le n$, $1 \le j \le m$. The operator U is unitary if and only if

$$
(6) \qquad \sum_{k=1}^{m} u_{i_1 k} u_{i_2 k}^* = \delta_{i_1, i_2}, \quad \sum_{k=1}^{n} u_{k j_1}^* u_{k j_2} = \delta_{j_1, j_2}
$$

for all $1 \le i_1, i_2 \le n$, $1 \le j_1, j_2 \le m$. Hence the universal C*-algebra $U_{m \times n}^{\mathrm{nc}}$ generated by the elements u_{ij} for $i = 1, \ldots, n$, $j = 1, \ldots, m$ that satisfy (6) is a coequalizer of (5). For $m = n$, this C*-algebra is introduced by Lawrence Brown [4] and studied further by Kevin McClanahan, who showed that it has no projections (see [15, Cor. 2.7]) and is KK-equivalent to $\mathrm{C}^*(\mathbb{Z}) \cong \mathrm{C}(\mathbb{T})$ (see [16, Prop. 5.5]). The C*-algebras $U_{m \times n}^{\mathrm{nc}}$ are C*-algebra analogs of the algebras introduced by Leavitt [13], and they are prototypical examples of separated graph C*-algebras (see [2]).

4. Colimits for group and crossed module actions

We now consider colimits where $\mathcal{C}$ is a group G or a crossed module. We consider both target bicategories $\mathfrak{C}^*(2)$ and $\mathfrak{Corr}$. In all these cases, the identification of the colimit with an appropriate "crossed product" is a mere reformulation of results in [5, 7]. Hence we will be rather brief. These results are trivial, but they are important motivation to look at colimits in bicategories.

To make the results below look more surprising, we briefly consider the colimit for a group action in the usual category of C*-algebras and *-homomorphisms, without any 2-arrows. A group action by automorphisms is, indeed, the same as a functor from G to the category of C*-algebras, given by a C*-algebra A and $\alpha_g \in \mathrm{Aut}(A)$ satisfying $\alpha_g \alpha_h = \alpha_{gh}$. A cone over this diagram is a C*-algebra B with a *-homomorphism $f \colon A \to B$ such that $f \circ \alpha_g = f$ for all $g \in G$. Thus f vanishes on the ideal I_α generated by $\alpha_g(a) - a$ for all $g \in G$, $a \in A$. Indeed, the quotient map $A \to A/I_\alpha$ is *the universal cone*. Hence the colimit is A/I_α. This is very often zero, and certainly not an object worth studying.

When working in a bicategory, we replace the *condition* $f \circ \alpha_g = f$ by *extra data*, say, by a unitary u_g with $u_g f(a) u_g^* = f(\alpha_g(a))$ for all $a \in A$. Thus the bicategorical colimit is larger than A, very much unlike A/I_α above.

The objects of $\mathfrak{C}^*(2)^G$ are described concretely in [7, §3.1.1] as Busby–Smith twisted actions of G; those of $\mathfrak{Corr}^G$ are equivalent to saturated Fell bundles over G. The transformations in $\mathfrak{C}^*(2)^G$ and $\mathfrak{Corr}^G$ are described concretely in [7, §3.2]; modifications in $\mathfrak{C}^*(2)^G$ and $\mathfrak{Corr}^G$ are described concretely in [7, §3.3]. Results in [7] immediately give the following proposition.

Proposition 4.1. *Let G be a group. Let $\alpha \colon G \to \mathrm{Aut}(A)$ and $\omega \colon G \times G \to \mathcal{U}(A)$ be a Busby–Smith twisted action of G on a C*-algebra A. The crossed product $A \rtimes_{\alpha,\omega} G$ is a colimit of the functor $F \colon G \to \mathfrak{C}^*(2)$ associated to (A, α, ω).*

Proof. Let D be a C*-algebra. The functor $\mathrm{const}_D \colon G \to \mathfrak{C}^*(2)$ corresponds to the trivial action of G on D. A transformation from F to const_D is equivalent to a covariant representation of (A, G, α, ω) in $\mathcal{M}(D)$, that is, a nondegenerate representation $\varrho \colon A \to \mathcal{M}(D)$ and a map $\pi \colon G \to \mathcal{U}(D)$ satisfying $\pi_g \varrho(a) \pi_g^* = \varrho(\alpha_g(a))$ for all $g \in G$, $a \in A$ and $\pi_{g_1} \pi_{g_2} = \varrho(\omega(g_1, g_2)) \pi_{g_1 g_2}$ for all $g_1, g_2 \in G$ (see [7, Ex. 3.8]). Modifications between such transformations are the same as unitary equivalences of covariant representations by [7, Ex. 3.13].

The crossed product is defined to be universal for covariant representations. That is, there is a bijection between transformations from F to const_D and morphisms from $A \rtimes_{\alpha,\omega} G$ to D; the modifications between the transformations corresponding to covariant representations (ϱ, π) and (ϱ', π') are unitary multipliers u of D with $u\varrho(a)u^* = \varrho'(a)$ for all $a \in A$ and $u\pi(g)u^* = \pi'(g)$ for all $g \in G$. These are exactly the unitaries that intertwine the induced representations of $A \rtimes_{\alpha,\omega} G$. Thus the groupoids $\mathfrak{C}^*(2)^G(F, \mathrm{const}_D)$ and $\mathfrak{C}^*(2)(A \rtimes_{\alpha,\omega} G, D)$ are naturally isomorphic. $\square$

For group actions by correspondences, that is, saturated Fell bundles, the section C*-algebra plays the role of the crossed product:

Proposition 4.2. *Let G be a group and let $(A_g)_{g \in G}$ be a saturated Fell bundle over G, viewed as a functor $F \colon G \to \mathfrak{Corr}$. The section C*-algebra of $(A_g)_{g \in G}$ is a colimit of F.*

Proof. Let D be a C*-algebra. Then const_D corresponds to the constant Fell bundle with fibers D, which describes the trivial action of G on D. Transformations to const_D in $\mathfrak{Corr}^G$ are in bijection with pairs $(\mathcal{E}, \pi)$, where $\mathcal{E}$ is a Hilbert D-module and $\pi \colon \bigsqcup_{g \in G} A_g \to \mathbb{B}(\mathcal{E})$ is a nondegenerate Fell bundle representation (see the discussion before [7, Def. 3.12]). Modifications between such transformations are equivalent to unitary intertwiners between Fell bundle representations.

The section C*-algebra $C := \mathrm{C}^*(A_g)_{g \in G}$ is defined as the C*-completion of the convolution algebra of sections of the Fell bundle. By definition, representations of a Fell bundle integrate to *-representations of this section C*-algebra, and all representations of C come from Fell bundle representations. Furthermore, a Fell bundle representation is nondegenerate if and only if the resulting representation of C is nondegenerate. A nondegenerate representation of C on a Hilbert D-module is the same as a correspondence from C to D. Furthermore, an operator intertwines the Fell bundle representations if and only if it intertwines the resulting representations of C, that is, is an isomorphism of correspondences. Hence the groupoids $\mathfrak{Corr}(F, \mathrm{const}_D)$ and $\mathfrak{Corr}(C, D)$ are naturally isomorphic. $\qquad\square$

Summing up, we merely have to inspect the description of transformations and modifications between functors $G \to \mathfrak{C}^*(2)$ or $G \to \mathfrak{Corr}$ in [7] to see that the colimit in either case is the crossed product or Fell bundle section algebra, respectively.

Now let $\mathcal{CM}$ be a *crossed module*; that is, $\mathcal{CM}$ consists of two groups G and H with homomorphisms $\partial \colon H \to G$ and $c \colon G \to \mathrm{Aut}(H)$, such that $\partial(c_g(h)) = g\partial(h)g^{-1}$ and $c_{\partial h}(k) = hkh^{-1}$ for all $g \in G$, $h, k \in H$.

Strict actions of crossed modules on C*-algebras and crossed products for such actions are defined in [6]. These are more special than functors $\mathcal{CM} \to \mathfrak{C}^*(2)$, which are discussed in [7, §4.1.1]. Functors $\mathcal{CM} \to \mathfrak{Corr}$ are described in [5, Thm. 2.11], generalizing the notion of a saturated Fell bundle from groups to crossed modules. The crossed product for a functor $F \colon \mathcal{CM} \to \mathfrak{Corr}$ is defined in [5, Def. 2.8] by a universal property and identified more concretely in [5, Prop. 2.17].

Proposition 4.3. *The crossed product for a crossed module action by correspondences is a colimit in $\mathfrak{Corr}$.*

Proof. Let $F \colon \mathcal{CM} \to \mathfrak{Corr}$ be a functor. As in the group case, the proof is by making explicit what transformations $F \to \mathrm{const}_D$ and modifications between them are and observing that the resulting universal property for the colimit is

the same one as the defining universal property of the crossed product. Since this is routine checking, we omit further details. □

5. A single endomorphism

Before we study colimits of arbitrary shape, we look at an important special case: let $\mathcal{C}$ be the monoid $(\mathbb{N}, +)$, viewed as a category with a single object.

A functor $\mathcal{C} \to \mathfrak{Corr}$ is given by a C*-algebra A, correspondences $\mathcal{E}_n \colon A \to A$ for $n \in \mathbb{N}$ and isomorphisms of correspondences $\mu_{n,m} \colon \mathcal{E}_n \otimes_A \mathcal{E}_m \cong \mathcal{E}_{n+m}$ for all $n, m \in \mathbb{N}$, such that $\mathcal{E}_0$ is the identity correspondence, $\mu_{0,m}$ and $\mu_{n,0}$ are the canonical transformations, and the multiplication maps $\mu_{n,m}$ are associative in a suitable sense. This is a special case of Proposition 6.2 below.

A transformation between such diagrams $(A, \mathcal{E}_n, \mu_{n,m})$ and $(B, \mathcal{F}_n, v_{n,m})$ is given by a correspondence $\mathcal{G} \colon A \to B$ and isomorphisms

$$(7) \qquad \mathcal{E}_n \otimes_A \mathcal{G} \xrightarrow[\cong]{w_n} \mathcal{G} \otimes_B \mathcal{F}_n$$

for all $n \in \mathbb{N}$, subject to compatibility conditions with the $\mu_{n,m}$ and $v_{n,m}$ for $n, m \in \mathbb{N}$ and the condition that w_0 should be the canonical isomorphism (see Proposition 6.3). A modification between two such transformations, $(\mathcal{G}, w_n)$ and $(\mathcal{G}', w'_n)$, is given by an isomorphism of correspondences $\mathcal{G} \to \mathcal{G}'$ intertwining the w_n and w'_n in the obvious sense (see also Proposition 6.4).

This data can be simplified because the monoid $(\mathbb{N}, +)$ is freely generated by $1 \in \mathbb{N}$. For a functor $\mathbb{N} \to \mathfrak{Corr}$, it is enough to give A and a single correspondence $\mathcal{E} = \mathcal{E}_1$, with no further data or conditions. We may extend this to a functor in the above sense by letting $\mathcal{E}_n := \mathcal{E}^{\otimes_A n}$ for $n \in \mathbb{N}$ (understood to be the identity correspondence if $n = 0$), and letting $\mu_{n,m}$ be the canonical map (this is the identity map up to the associators, which we have dropped from our notation). The conditions on the $\mu_{n,m}$ ensure that any functor is isomorphic to one of this form.

Next, a transformation is specified by a correspondence $\mathcal{G}$ and an isomorphism

$$w = w_1 \colon \mathcal{E} \otimes_A \mathcal{G} \cong \mathcal{G} \otimes_B \mathcal{F},$$

with no condition on w: iteration of w_1 provides the isomorphisms w_n for $n \in \mathbb{N}$ as in (7), and the compatibility conditions for the w_n say that any transformation is generated from w_1 in this way. Finally, for a modification, it is enough to require the intertwining condition for w_1, then the condition follows for w_n for all $n \in \mathbb{N}$. In brief, the bicategory of functors $\mathbb{N} \to \mathfrak{Corr}$ is equivalent to the following simpler bicategory:

(i) objects are given by a C*-algebra A and a correspondence $\mathcal{E} \to \mathcal{E}$;
(ii) arrows $(A, \mathcal{E}) \to (B, \mathcal{F})$ are given by a correspondence $\mathcal{G} \colon A \to B$ and an isomorphism of correspondences $w \colon \mathcal{G} \otimes_B \mathcal{F} \cong \mathcal{E} \otimes_A \mathcal{G}$;
(iii) 2-arrows $(\mathcal{G}, w) \to (\mathcal{G}', w')$ are isomorphisms $x \colon \mathcal{G} \to \mathcal{G}'$ such that

$$(\mathrm{id}_{\mathcal{E}} \otimes_A x) \circ w = w \circ (x \otimes_B \mathrm{id}_{\mathcal{F}}).$$

We may use the simplified data to describe colimits as well, which only require equivalences of categories.

We now analyze transformations from $(A, \mathcal{E})$ to a constant diagram const_D. First, $\text{const}_D = (D, D)$, where the second D means the identity correspondence on D. Hence the isomorphism w in a transformation may also be viewed as an isomorphism $\mathcal{G} \cong \mathcal{E} \otimes_A \mathcal{G}$; here we use the canonical isomorphism $\mathcal{G} \otimes_D D \cong \mathcal{G}$.

Roughly speaking, we want to turn an isomorphism $w \colon \mathcal{G} \xrightarrow{\sim} \mathcal{E} \otimes_A \mathcal{G}$ into a representation of a C*-algebra on $\mathcal{G}$. The necessary work is carried out in [1]. First, the isomorphism $w \colon \mathcal{G} \xrightarrow{\sim} \mathcal{E} \otimes_A \mathcal{G}$ is turned into a "representation" $\mathcal{E} \to \mathbb{B}(\mathcal{G})$ by sending $\xi \in \mathcal{E}$ to the operator

$$\mathcal{G} \ni \eta \mapsto w^*(\xi \otimes \eta) \in \mathcal{G}.$$

This is a representation of the Hilbert module $\mathcal{E}$ in the standard sense, satisfying an extra nondegeneracy condition corresponding to the surjectivity of w^*. This extra nondegeneracy condition is equivalent to the Cuntz–Pimsner covariance condition *provided $\mathcal{E}$ is a proper correspondence* by [1, Prop. 2.5]. This leads to the following theorem.

Theorem 5.1. *Let $\mathcal{E} \colon A \to A$ be a proper correspondence. The Cuntz–Pimsner algebra of $\mathcal{E}$ is a colimit of the corresponding diagrams $(\mathbb{N}, +) \to \mathfrak{Corr}_{\text{prop}}$ and $(\mathbb{N}, +) \to \mathfrak{Corr}$.*

Proof. The Cuntz–Pimsner algebra $\mathcal{O}_{\mathcal{E}}$ is characterized by the universal property that *-homomorphisms $\mathcal{O}_{\mathcal{E}} \to D$ for a C*-algebra D are in bijection with pairs (φ, ϑ), where $\varphi \colon A \to D$ is a *-homomorphism and $\vartheta \colon \mathcal{E} \to D$ is a Cuntz–Pimsner covariant representation of $\mathcal{E}$ (see [18, Thm. 3.12]). In particular, $A \subseteq \mathcal{O}_{\mathcal{E}}$, and inspection shows that this embedding is nondegenerate, that is, $A \cdot \mathcal{O}_{\mathcal{E}}$ is dense in $\mathcal{O}_{\mathcal{E}}$. It follows that the *-homomorphism $\mathcal{O}_{\mathcal{E}} \to \mathbb{B}(\mathcal{F})$ associated to $\varphi \colon A \to \mathbb{B}(\mathcal{F})$ and $\vartheta \colon \mathcal{E} \to \mathbb{B}(\mathcal{F})$ is nondegenerate if and only if φ is nondegenerate. Thus a correspondence from $\mathcal{O}_{\mathcal{E}}$ to D is the same as a correspondence $(\mathcal{F}, \varphi)$ from A to D with a map $\vartheta \colon \mathcal{E} \to \mathbb{B}(\mathcal{F})$ which, together with φ, is a Cuntz–Pimsner covariant representation.

Since $\mathcal{E}$ is proper, the Cuntz–Pimsner covariance condition for ϑ holds if and only if ϑ is "nondegenerate" in the sense that the closed linear span of $\vartheta(\mathcal{E}) \cdot (\mathcal{F})$ is $\mathcal{F}$ (see [1, Prop. 2.5]). Such nondegenerate correspondences are in bijection with isomorphisms of correspondences $\mathcal{E} \otimes \mathcal{F} \cong \mathcal{F}$ by [1, Prop. 2.3]. So a correspondence from $\mathcal{O}_{\mathcal{E}}$ to D is the same as a correspondence $\mathcal{F}$ from A to D with an isomorphism of correspondences $\mathcal{E} \otimes_A \mathcal{F} \cong \mathcal{F}$. These are exactly the simplified transformations of functors $(\mathbb{N}, +) \to \mathfrak{Corr}$, by the discussion above the theorem.

Isomorphisms of correspondences $\mathcal{O}_{\mathcal{E}} \to D$ are the same as unitaries $\mathcal{F} \to \mathcal{F}'$ that intertwine the left actions of A and $\mathcal{E}$. Intertwining the left actions of A means that they are isomorphisms of correspondences from A to D, and intertwining the representations of $\mathcal{E}$ means that they are modifications between the corresponding transformations of functors $(\mathbb{N}, +) \to \mathfrak{Corr}$. Hence the groupoid of correspondences $\mathcal{O}_{\mathcal{E}} \to D$ and their isomorphisms is naturally isomorphic

to the groupoid of simplified transformations $(A, \mathcal{E}) \to \mathrm{const}_D$ and their modifications. This says that $\mathcal{O}_{\mathcal{E}}$ has the universal property of a colimit in $\mathfrak{Corr}$.

A correspondence $\mathcal{F} \colon \mathcal{O}_{\mathcal{E}} \to D$ is proper if and only if the corresponding representation φ of A has image in the compact operators, $\varphi(A) \subseteq \mathbb{K}(\mathcal{F})$; this is because $A \cdot \mathcal{O}_{\mathcal{E}} = \mathcal{O}_{\mathcal{E}}$. Hence $\mathcal{O}_{\mathcal{E}}$ has the universal property of a colimit in $\mathfrak{Corr}_{\mathrm{prop}}$ as well. $\square$

Note that the colimit is the Cuntz–Pimsner algebra right away, the Cuntz–Toeplitz algebra plays no role; this is because of the built-in nondegeneracy properties of $\mathfrak{Corr}$.

Following Muhly and Solel [17] and Katsura [11], many authors have modified the definition of the Cuntz–Pimsner algebra by requiring the Cuntz–Pimsner covariance condition only on a suitable ideal in $\varphi_{\mathcal{E}}^{-1}(\mathbb{K}(\mathcal{E}))$. Such modifications are particularly popular if the left action of A on $\mathcal{E}$ is not faithful because in that case, the unmodified Cuntz–Pimsner algebra may well be zero. The colimit construction, however, singles out the unmodified Cuntz–Pimsner algebra.

Unlike the Cuntz–Pimsner condition, "nondegeneracy" is not a relation that we may impose on a bunch of generators. This is why there need not be a universal C*-algebra for nondegenerate representations, but there is always one for Cuntz–Pimsner covariant representations. The two properties are only equivalent if $\mathcal{E}$ is proper. This is the reason why we only understand colimits for diagrams of proper correspondences.

It seems likely that the colimit of the diagram $(\mathbb{N}, +) \to \mathfrak{Corr}$ given by the endomorphism $\ell^2(\mathbb{N})$ of $\mathbb{C}$ does not exist (see [1, Ex. 2.7]). In the following, we therefore restrict attention to colimits of diagrams of *proper* correspondences.

6. Category-shaped diagrams and product systems

We have examined enough examples that it makes sense to spell out what functors, transformations, and modifications $\mathcal{C} \to \mathfrak{Corr}$ mean for an arbitrary category $\mathcal{C}$. We are particularly interested in transformations to a constant functor, which lead to the description of the colimit of a diagram.

6.1. **Functors, transformations and modifications.** The objects of $\mathfrak{Corr}^{\mathcal{C}}$ are functors (or morphisms) $\mathcal{C} \to \mathfrak{Corr}$; arrows are transformations between such functors, and 2-arrows are modifications. We describe these things more concretely and then explain briefly how to compose transformations. These definitions are summarized succinctly in [14]. They are worked out for $\mathfrak{C}^*(2)^{\mathcal{C}}$ in [7, §4], even for an arbitrary bicategory $\mathcal{C}$. The definitions simplify if $\mathcal{C}$ is a category because part of the data does not occur any more. The following propositions already contain these simplifications. We omit the (rather trivial) proofs. Readers that do not care much about bicategory theory could take the following propositions as definitions.

Proposition 6.2. *A functor $\mathcal{C} \to \mathfrak{Corr}$ consists of*

- *C^*-algebras A_x for all objects x of $\mathcal{C}$;*
- *correspondences $\mathcal{E}_g \colon A_x \to A_y$ for all arrows $g \colon x \to y$ in $\mathcal{C}$;*
- *isomorphisms of correspondences $\mu_{g,h} \colon \mathcal{E}_h \otimes_{A_y} \mathcal{E}_g \to \mathcal{E}_{gh}$ for all pairs of composable arrows $g \colon y \to z$, $h \colon x \to y$ in $\mathcal{C}$;*

such that

(i) *$\mathcal{E}_{1_x}$ is the identity correspondence on A_x for all objects x of $\mathcal{C}$;*

(ii) *$\mu_{1_y,g} \colon \mathcal{E}_g \otimes_{A_y} A_y \to \mathcal{E}_g$ and $\mu_{g,1_x} \colon A_x \otimes_{A_x} \mathcal{E}_g \to \mathcal{E}_g$ are the canonical isomorphisms for all arrows $g \colon x \to y$ in $\mathcal{C}$;*

(iii) *for all composable arrows $g_{01} \colon x_0 \to x_1$, $g_{12} \colon x_1 \to x_2$, $g_{23} \colon x_2 \to x_3$, the following diagram commutes:*

$$
\begin{array}{ccc}
(\mathcal{E}_{g_{01}} \otimes_{A_{x_1}} \mathcal{E}_{g_{12}}) \otimes_{A_{x_2}} \mathcal{E}_{g_{23}} & \xrightarrow{\mu_{g_{12},g_{01}} \otimes_{A_{x_2}} \mathrm{id}_{\mathcal{E}_{g_{23}}}} & \mathcal{E}_{g_{02}} \otimes_{A_{x_2}} \mathcal{E}_{g_{23}} \\
\Big\downarrow{\scriptstyle \text{can.}} & & \Big\downarrow{\scriptstyle \mu_{g_{23},g_{02}}} \\
& & \mathcal{E}_{g_{03}} \\
& & \Big\uparrow{\scriptstyle \mu_{g_{13},g_{01}}} \\
\mathcal{E}_{g_{01}} \otimes_{A_{x_1}} (\mathcal{E}_{g_{12}} \otimes_{A_{x_2}} \mathcal{E}_{g_{23}}) & \xrightarrow{\mathrm{id}_{\mathcal{E}_{g_{01}}} \otimes_{A_{x_1}} \mu_{g_{23},g_{12}}} & \mathcal{E}_{g_{01}} \otimes_{A_{x_1}} \mathcal{E}_{g_{13}}.
\end{array}
$$

Here $g_{02} := g_{12} \circ g_{01}$, $g_{13} := g_{23} \circ g_{12}$, and $g_{03} := g_{23} \circ g_{12} \circ g_{01}$.

The diagram in (iii) *commutes automatically if one of the arrows g_{01}, g_{12} or g_{23} is an identity arrow.*

Proposition 6.3. *Let $(A^0_x, \mathcal{E}^0_g, \mu^0_{g,h})$ and $(A^1_x, \mathcal{E}^1_g, \mu^1_{g,h})$ be two functors from $\mathcal{C}$ to $\mathfrak{Corr}$. A transformation between them consists of*

- *correspondences γ_x from A^0_x to A^1_x for all objects x of $\mathcal{C}$;*
- *isomorphisms of correspondences $V_g \colon \gamma_x \otimes_{A^1_x} \mathcal{E}^1_g \to \mathcal{E}^0_g \otimes_{A^0_y} \gamma_y$ for all arrows $g \colon x \to y$ in $\mathcal{C}$;*

such that

(i) *$V_{1_x} \colon \gamma_x \otimes_{A^1_x} A^1_x \to A^0_x \otimes_{A^0_x} \gamma_x$ is the canonical isomorphism through γ_x for each object x in $\mathcal{C}$;*

(ii) *for each pair of composable arrows $g \colon y \to z$, $h \colon x \to y$ in $\mathcal{C}$, the following diagram commutes:*

$$
\begin{array}{ccc}
\gamma_x \otimes_{A^1_x} \mathcal{E}^1_h \otimes_{A^1_y} \mathcal{E}^1_g & \xrightarrow{V_h \otimes_{A^1_y} \mathrm{id}_{\mathcal{E}^1_g}} & \mathcal{E}^0_h \otimes_{A^0_y} \gamma_y \otimes_{A^1_y} \mathcal{E}^1_g \\
\Big\downarrow{\scriptstyle \mathrm{id}_{\gamma_x} \otimes_{A^1_x} u^1_{g,h}} & & \Big\downarrow{\scriptstyle \mathrm{id}_{\mathcal{E}^0_h} \otimes_{A^0_y} V_g} \\
& & \mathcal{E}^0_h \otimes_{A^0_y} \mathcal{E}^0_g \otimes_{A^0_z} \gamma_z \\
& & \Big\downarrow{\scriptstyle u^0_{g,h} \otimes_{A^0_z} \mathrm{id}_{\gamma_z}} \\
\gamma_x \otimes_{A^1_x} \mathcal{E}^1_{gh} & \xrightarrow{V_{gh}} & \mathcal{E}^0_{gh} \otimes_{A^0_z} \gamma_z.
\end{array}
$$

The diagram in (ii) *commutes automatically if g or h is an identity arrow.*

Proposition 6.4. *Let* $(A_x^0, \mathcal{E}_g^0, \mu_{g,h}^0)$ *and* $(A_x^1, \mathcal{E}_g^1, \mu_{g,h}^1)$ *be functors from* $\mathcal{C}$ *to* $\mathfrak{Corr}$ *and let* (γ_x^1, V_g^1) *and* (γ_x^2, V_g^2) *be transformations between them. A modification from* (γ_x^1, V_g^1) *to* (γ_x^2, V_g^2) *consists of isomorphisms of correspondences* $W_x \colon \gamma_x^1 \to \gamma_x^2$ *for all objects* x *in* $\mathcal{C}$ *such that the diagrams*

$$
\begin{array}{ccc}
\gamma_x^1 \otimes_{A_x^1} \mathcal{E}_g^1 & \xrightarrow{\ W_x \otimes_{A_x^1} \mathrm{id}_{\mathcal{E}_g^1}\ } & \gamma_x^2 \otimes_{A_x^1} \mathcal{E}_g^1 \\
{\scriptstyle V_g^1} \big\Downarrow & & \big\Downarrow {\scriptstyle V_g^2} \\
\mathcal{E}_g^0 \otimes_{A_y^0} \gamma_y^1 & \xrightarrow{\ \mathrm{id}_{\mathcal{E}_g^0} \otimes_{A_y^0} W_y\ } & \mathcal{E}_g^0 \otimes_{A_y^0} \gamma_y^2
\end{array}
$$

commute for all arrows $g \colon x \to y$ *in* $\mathcal{C}$. *This diagram commutes automatically if* g *is an identity arrow.*

The composition of transformations is defined as follows. Describe functors $\mathcal{C} \to \mathfrak{Corr}$ by $(A_x^0, \mathcal{E}_g^0, \mu_{g,h}^0)$, $(A_x^1, \mathcal{E}_g^1, \mu_{g,h}^1)$ and $(A_x^2, \mathcal{E}_g^2, \mu_{g,h}^2)$, and transformations between them by $(\gamma_x^{01}, V_g^{01})$ and $(\gamma_x^{12}, V_g^{12})$ as above. The product is given by the correspondences $\gamma_x^{02} := \gamma_x^{01} \otimes_{A_x^1} \gamma_x^{12}$ from A_x^0 to A_x^2 for objects x of $\mathcal{C}$ and by the isomorphisms of correspondences

$$
V_g^{02} \colon \gamma_x^{02} \otimes_{A_x^2} \mathcal{E}_g^2 = \gamma_x^{01} \otimes_{A_x^1} \gamma_x^{12} \otimes_{A_x^2} \mathcal{E}_g^2 \xrightarrow{\ \mathrm{id}_{\gamma_x^{01}} \otimes_{A_x^1} V_g^{12}\ } \gamma_x^{01} \otimes_{A_x^1} \mathcal{E}_g^1 \otimes_{A_y^2} \gamma_y^{12}
$$

$$
\xrightarrow{\ V_g^{01} \otimes_{A_y^2} \mathrm{id}_{\gamma_y^{12}}\ } \mathcal{E}_g^0 \otimes_{A_y^0} \gamma_y^{01} \otimes_{A_y^1} \gamma_y^{12} = \mathcal{E}_g^0 \otimes_{A_y^0} \gamma_y^{02}
$$

for arrows $g \colon x \to y$ in $\mathcal{C}$. These $(\gamma_x^{02}, V_g^{02})$ indeed form a transformation. General bicategory theory predicts that this composition turns $\mathfrak{Corr}^{\mathcal{C}}$ into a bicategory again, and this is routine to check by hand.

To understand the above definitions, consider the special case where $\mathcal{C}$ has only one object, that is, $\mathcal{C}$ is a monoid. Then we may drop all indices x above: a functor provides a single C*-algebra A, a transformation a single correspondence γ, and a modification a single isomorphism W. Furthermore, all arrows in $\mathcal{C}$ are composable, and there is only one identity morphism. Simplifying the data in Proposition 6.2 accordingly, the result is very close to a *product system* in the notation of Fowler [10].

There are only two differences. First, we require all left actions on Hilbert modules to be nondegenerate (or "essential"), whereas Fowler is careful to avoid this assumption. Secondly, we multiply in the opposite order, $\mathcal{E}_h \otimes_A \mathcal{E}_g \to \mathcal{E}_{gh}$, which corresponds to the composition of *-homomorphisms. As a result, functors $M \to \mathfrak{Corr}$ for a monoid M are the same as essential product systems over the opposite monoid M^{op}.

When we pass from monoids to categories, the only change is that we get more than one C*-algebra: one for each object of the category.

Nondegeneracy of the left actions on correspondences is necessary for unit arrows in $\mathfrak{Corr}$ to work as expected: otherwise we would not get a bicategory. The order reversal comes in because when we pass from *-homomorphisms to

correspondences, the composition of *-homomorphisms becomes the reverse-order tensor product. With our convention, monoid actions by *-endomorphisms become actions by correspondences of the same monoid. The same order-reversal also appears when translating between actions of a group by correspondences and saturated Fell bundles over the group. It is the reason why g^{-1} appears in the correspondence between functors $G \to \mathfrak{Corr}$ and saturated Fell bundles over G in the proof of [7, Thm. 3.3].

6.5. **Colimits.** Let $\mathcal{C}$ be a category, let $(A_x, \mathcal{E}_g, \mu_{g,h})$ describe a functor $F \colon \mathcal{C} \to \mathfrak{Corr}$ as in Proposition 6.2, and let D be a C*-algebra. We first describe the constant functor $\mathrm{const}_D \colon \mathcal{C} \to \mathfrak{Corr}$. Then we specialize the description of transformations and modifications to the case of a constant target. We use this to describe the colimit of a proper product system by generators and relations.

Definition 6.6. Let D be a C*-algebra. The *constant functor* $\mathrm{const}_D \colon \mathcal{C} \to \mathfrak{Corr}$ maps all objects x of $\mathcal{C}$ to D, all arrows g in $\mathcal{C}$ to the identity correspondence on D, and all pairs g, h to the canonical isomorphism $D \otimes_D D \to D$.

A transformation from the functor given by $(A_x, \mathcal{E}_g, \mu_{g,h})$ to const_D is given by correspondences γ_x from A_x to D for all objects x of $\mathcal{C}$ and isomorphisms of correspondences

$$V_g \colon \gamma_x \to \mathcal{E}_g \otimes_{A_y} \gamma_y \quad \text{for all arrows } g \colon x \to y \text{ in } \mathcal{C},$$

such that V_{1_x} for an object x is the canonical isomorphism and the diagrams

$$
\begin{array}{ccc}
\gamma_x & \xRightarrow{\ V_h\ } & \mathcal{E}_h \otimes_{A_y} \gamma_y \\
\Vert \mathrm{id}_{\gamma_x} & & \Vert \mathrm{id}_{\mathcal{E}_h} \otimes_{A_y} V_g \\
& \mathcal{E}_h \otimes_{A_y} \mathcal{E}_g \otimes_{A_z} \gamma_z & \\
& & \Vert \mu_{g,h} \otimes_{A_z} \mathrm{id}_{\gamma_z} \\
\gamma_x & \xRightarrow{\ V_{gh}\ } & \mathcal{E}_{gh} \otimes_{A_z} \gamma_z
\end{array}
$$

for composable arrows $g \colon y \to z$, $h \colon x \to y$ in $\mathcal{C}$ commute. Here we simplified the data in Proposition 6.3 using the canonical isomorphisms $\gamma_x \otimes_D D \cong \gamma_x$ for all x; we may, of course, drop the identity arrow on γ_x and redraw this diagram as a commuting square:

$$
(8) \qquad
\begin{array}{ccc}
\gamma_x & \xRightarrow{\ V_h\ } & \mathcal{E}_h \otimes_{A_y} \gamma_y \\
\Vert V_{gh} & & \Vert \mathrm{id}_{\mathcal{E}_h} \otimes_{A_y} V_g \\
\mathcal{E}_{gh} \otimes_{A_z} \gamma_z & \xLeftarrow{\ \mu_{g,h} \otimes_{A_z} \mathrm{id}_{\gamma_z}\ } & \mathcal{E}_h \otimes_{A_y} \mathcal{E}_g \otimes_{A_z} \gamma_z.
\end{array}
$$

This diagram commutes automatically if g or h is an identity arrow.

If (γ_x^1, V_g^1) and (γ_x^2, V_g^2) are two such transformations, then a modification between them is given by isomorphisms of correspondences

$$W_x \colon \gamma_x^1 \to \gamma_x^2 \quad \text{for all objects } x \text{ of } \mathcal{C},$$

such that the diagrams

(9)

$$
\begin{array}{ccc}
\gamma_x^1 & \xRightarrow{\quad W_x \quad} & \gamma_x^2 \\
\Big\Downarrow{\scriptstyle V_g^1} & & \Big\Downarrow{\scriptstyle V_g^2} \\
\mathcal{E}_g \otimes_{A_y} \gamma_y^1 & \xRightarrow{\ \mathrm{id}_{\mathcal{E}_g} \otimes_{A_y} W_y\ } & \mathcal{E}_g \otimes_{A_y} \gamma_y^2
\end{array}
$$

commute for all arrows $g \colon x \to y$ in $\mathcal{C}$. This diagram commutes automatically if g is an identity arrow.

The colimit for a functor $F \colon \mathcal{C} \to \mathfrak{Corr}$ is, by definition, a C*-algebra B such that, for each C*-algebra D, the groupoid of correspondences $B \to D$ and isomorphisms of correspondences between them is naturally equivalent to the groupoid of transformations $F \to \mathrm{const}_D$ and modifications between them.

Proposition 6.7. *There is a bijection between transformations $F \to \mathrm{const}_D$ and the following set of data:*

- *Hilbert D-modules γ_x for objects x of $\mathcal{C}$;*
- *nondegenerate *-homomorphisms $\varphi_x \colon A_x \to \mathbb{B}(\gamma_x)$ for objects x of $\mathcal{C}$;*
- *linear maps $S_g \colon \mathcal{E}_g \to \mathbb{B}(\gamma_y, \gamma_x)$ for arrows $g \colon x \to y$ in $\mathcal{C}$;*

such that

(i) *for each arrow $g \colon x \to y$, S_g is A_x-A_y-linear, compatible with inner products, and nondegenerate:*
    (a) *$S_g(a_1 \xi a_2) = \varphi_x(a_1) S_g(\xi) \varphi_y(a_2)$ for $a_1 \in A_x$, $a_2 \in A_y$;*
    (b) *$S_g(\xi_1)^* S_g(\xi_2) = \varphi_y(\langle \xi_1, \xi_2 \rangle_{A_y})$ for all $\xi_1, \xi_2 \in \mathcal{E}_g$;*
    (c) *the closed linear span of $S_g(\mathcal{E}_g) \cdot \gamma_y$ is γ_x;*
(ii) *$S_{1_x} = \varphi_x \colon A_x \to \mathbb{B}(\gamma_x)$ for all objects x;*
(iii) *for each pair of composable arrows $g \colon y \to z$, $h \colon x \to y$ in $\mathcal{C}$, $\xi \in \mathcal{E}_g$, $\eta \in \mathcal{E}_h$, we have $S_h(\eta) S_g(\xi) = S_{gh}(\mu_{g,h}(\eta \otimes \xi))$.*

Let $(\gamma_x^1, \varphi_x^1, S_g^1)$ and $(\gamma_x^2, \varphi_x^2, S_g^2)$ be two such collections. Modifications between the corresponding transformations are in natural bijection with families of unitaries $W_x \colon \gamma_x^1 \to \gamma_x^2$ such that $W_x \varphi_x^1(a) = \varphi_x^2(a) W_x$ for all objects x and all $a \in A_x$ and $W_x S_g^1(\xi) = S_g^2(\xi) W_y$ for all arrows $g \colon x \to y$ in $\mathcal{C}$ and all $\xi \in \mathcal{E}_g$.

Proof. Let (γ_x, V_g) as in Proposition 6.3 describe a transformation from F to const_D. The left A_x-module structure on γ_x is through a nondegenerate *-homomorphism $\varphi_x \colon A_x \to \mathbb{B}(\gamma_x)$, and when we record this as extra data, we may forget the left module structure on γ_x and view it simply as a Hilbert D-module. We also replace the unitary $V_g^* \colon \mathcal{E}_g \otimes_{A_y} \gamma_y \to \gamma_x$ by the linear map $S_g \colon \mathcal{E}_g \to \mathbb{B}(\gamma_y, \gamma_x)$ defined by $S_g(\xi)(\eta) := V_g^*(\xi \otimes \eta)$. The map S_g satisfies (a)–(c) in (i) and, conversely, maps S_g with these three properties are in bijection with isomorphisms of correspondences V_g^*; this is proved in [1, Prop. 2.3].

To give a transformation, the unitaries V_g for arrows g in $\mathcal{C}$ must also satisfy the two conditions in Proposition 6.3. The first one describes V_{1_x}, and it gives our condition (ii) when we translate it into S_{1_x}. The second condition in Proposition 6.3 is the commuting diagram (8) that relates V_g and V_h to V_{hg}. This is equivalent to

$$V_h^*(\eta \otimes V_g^*(\xi \otimes \zeta)) = V_{gh}^*(\mu_{g,h}(\eta \otimes \xi) \otimes \zeta)$$

for all $\xi \in \mathcal{E}_g$, $\eta \in \mathcal{E}_h$, $\zeta \in \gamma_z$. This is, in turn, equivalent to

$$S_h(\eta)S_g(\xi)(\zeta) = S_{gh}(\mu_{g,h}(\eta \otimes \xi))(\zeta),$$

which is condition (iii). All these steps may be reversed. So a family $(\gamma_x, \varphi_x, S_g)$ with the properties (i)–(iii) always comes from a unique transformation.

The last statement holds because (9) commutes for given (W_x) if and only if $W_x S_g^1(\xi)(\zeta) = S_g^2(\xi) W_y(\zeta)$ for all $\zeta \in \gamma_y^1$. $\square$

The nondegeneracy condition (i)(c) in Proposition 6.7 is the only one with an unusual form, which we cannot impose as a relation on generators of a universal C*-algebra. *If each $\mathcal{E}_g$ is proper,* then this condition is equivalent to a Cuntz–Pimsner covariance condition for each $\mathcal{E}_g$; this is slightly more general than Theorem 5.1 because we are dealing with a correspondence between two different C*-algebras. All proofs carry over to this case, however, and we can now write down a candidate for the colimit using generators and relations:

Definition 6.8. Let $\mathcal{O}(A_x, \mathcal{E}_g, \mu_{g,h})$ be the universal C*-algebra generated by the C*-algebra $\bigoplus_x A_x$ and symbols $S_g(\xi)$ for arrows $g\colon x \to y$ in $\mathcal{C}$ and $\xi \in \mathcal{E}_g$, subject to the following relations:

(i) the relations in the C*-algebra $\bigoplus_x A_x$ hold, $\mathcal{E}_g \ni \xi \mapsto S_g(\xi)$ is linear for each arrow g, and $S_{1_x}(a) = a$ for all $a \in A_x$ and all x;
(ii) if $g\colon x \to y$, $\xi \in \mathcal{E}_g$, $a \in A_z$, then

$$S_g(\xi)a = \begin{cases} S_g(\xi a) & z = y, \\ 0 & z \neq y, \end{cases} \qquad aS_g(\xi) = \begin{cases} S_g(a\xi) & z = x, \\ 0 & z \neq x; \end{cases}$$

(iii) if $g\colon x \to y$, $\xi_1, \xi_2 \in \mathcal{E}_g$, then $S_g(\xi_1)^* S_g(\xi_2) = \langle \xi_1, \xi_2 \rangle_{A_y} \in A_y$;
(iv) for $g\colon x \to y$ and $a \in A_x$ with $\varphi_{\mathcal{E}_g}(a) \in \mathbb{K}(\mathcal{E}_g)$ and for $\xi_j, \eta_j \in \mathcal{E}_g$, the norm of $a - \sum S_g(\xi_j)S_g(\eta_j)^*$ is at most the norm of $\varphi_{\mathcal{E}_g}(a) - \sum |\xi_j\rangle\langle\eta_j|$ in $\mathbb{K}(\mathcal{E}_g)$; here $\varphi_{\mathcal{E}_g}\colon A_x \to \mathbb{B}(\mathcal{E}_g)$ denotes the left action;
(v) $S_h(\eta)S_g(\xi) = S_{gh}(\mu_{g,h}(\eta \otimes \xi))$ for all $\xi \in \mathcal{E}_g$, $\eta \in \mathcal{E}_h$.

It is clear that there is a universal C*-algebra satisfying these relations. First, take the universal *-algebra U_1 on the set of generators. Secondly, let U_2 be the quotient of U_1 by the ideal generated by the conditions (i)–(iii) and (v). Thirdly, take the supremum of all C*-semi-norms on U_2 that satisfy (iv). This is the maximal C*-semi-norm on U_1 that satisfies (iv). The maximum exists because there is a maximal C*-semi-norm on the C*-subalgebra $\bigoplus A_x \subseteq U_2$ and $\|S_g(\xi)\| = \|\xi\|$ for any C*-semi-norm on U_2 by condition (iii). Finally, $\mathcal{O}(A_x, \mathcal{E}_g, \mu_{g,h})$ is the (Hausdorff) completion of U_2 in this C*-semi-norm.

Theorem 6.9. *Let $\mathcal{C}$ be a category. Let $(A_x, \mathcal{E}_g, \mu_{g,h})$ give a functor $F\colon \mathcal{C} \to \mathfrak{Corr}$. Assume that $\mathcal{E}_g$ is a proper correspondence for each arrow $g\colon x \to y$. Then the C*-algebra $\mathcal{O}(A_x, \mathcal{E}_g, \mu_{g,h})$ is a colimit of F in $\mathfrak{Corr}$ and in $\mathfrak{Corr}_{\mathrm{prop}}$.*

Proof. We abbreviate $\mathcal{O} := \mathcal{O}(A_x, \mathcal{E}_g, \mu_{g,h})$. Condition (i) in Definition 6.8 gives a *-homomorphism $f\colon \bigoplus A_x \to \mathcal{O}$ and linear maps $S_g\colon \mathcal{E}_g \to \mathcal{O}$. Condition (ii) implies $A_x S_g(\mathcal{E}_g) A_y = S_g(\mathcal{E}_g)$ and hence $A_y S_g(\mathcal{E}_g)^* A_x = S_g(\mathcal{E}_g)^*$. Since all elements in $\mathcal{O}$ may be approximated by noncommutative polynomials in elements of $S_g(\mathcal{E}_g)$, $S_g(\mathcal{E}_g)^*$ for arrows g and A_x for objects x, this implies that the *-homomorphism $f\colon \bigoplus A_x \to \mathcal{O}$ is nondegenerate.

Let $p_x \in \mathcal{M}(\bigoplus A_y)$ be the projection onto A_x and let $\gamma_x^{\mathcal{O}} := f(p_x)\mathcal{O}$; we view this right ideal as a Hilbert module over $\mathcal{O}$. Let A_x act on $\gamma_x^{\mathcal{O}}$ on the left via multiplication through f. This is nondegenerate, so $\gamma_x^{\mathcal{O}}$ becomes a correspondence from A_x to $\mathcal{O}$. We may identify $f(p_x) \cdot \mathcal{O} \cdot f(p_y)$ with $\mathbb{K}(\gamma_y^{\mathcal{O}}, \gamma_x^{\mathcal{O}}) \subseteq \mathbb{B}(\gamma_y^{\mathcal{O}}, \gamma_x^{\mathcal{O}})$.

Condition (ii) in Definition 6.8 implies $S_g(\mathcal{E}_g) \subseteq f(p_x) \cdot \mathcal{O} \cdot f(p_y)$ for $g\colon x \to y$. Conditions (ii) and (iii) say that $S_g\colon \mathcal{E}_g \to \mathbb{K}(\gamma_y^{\mathcal{O}}, \gamma_x^{\mathcal{O}})$ is a representation of the correspondence $\mathcal{E}_g$. They provide an isometric embedding of correspondences $V_g^{\mathcal{O}}\colon \mathcal{E}_g \otimes_{A_y} \gamma_y^{\mathcal{O}} \to \gamma_x^{\mathcal{O}}$ by the proof of [1, Prop. 2.3].

Our next goal is to show that this isometry is unitary or, equivalently, $S_g(\mathcal{E}_g) \cdot \gamma_Y^{\mathcal{O}}$ spans a dense subspace of $\gamma_y^{\mathcal{O}}$. This argument is essentially the same as for one direction in [1, Prop. 2.5]. It is the place where we need the correspondences $\mathcal{E}_g$ to be proper, that is, $\varphi_{\mathcal{E}_g}(A_x) \subseteq \mathbb{K}(\mathcal{E}_g)$. Let $(u_i)_{i \in I}$ be an approximate unit in A_x. For each $i \in I$ and $\epsilon > 0$ there is a finite-rank operator $T = \sum_{n=1}^k |\xi_n\rangle\langle\eta_n|$ on $\mathcal{E}_g$ with $\|\varphi_{\mathcal{E}_g}(u_i) - T\| < \epsilon$. Condition (iv) ensures that

$$\left\|\sum_{n=1}^k S_g(\xi_n)S_g(\eta_n)^* - u_i\right\| < \epsilon.$$

Thus we may approximate $u_i x$ by elements of $S_g(\mathcal{E}_g)S_g(\mathcal{E}_g)^* x \subseteq S_g(\mathcal{E}_g)\gamma_y^{\mathcal{O}}$ for any $x \in \mathcal{O}$. Since the left action of A_x on $\gamma_x^{\mathcal{O}}$ is nondegenerate, this shows that $S_g(\mathcal{E}_g)\gamma_y^{\mathcal{O}}$ spans a dense subspace of $\gamma_x^{\mathcal{O}}$, as desired.

We have verified the critical condition (i) (c) in Proposition 6.7 for the correspondences $\gamma_x^{\mathcal{O}}$ for $x \in \mathcal{C}^0$ and the maps $S_g\colon \mathcal{E}_g \to \mathbb{B}(\gamma_y^{\mathcal{O}}, \gamma_x^{\mathcal{O}})$. The remaining conditions are built into our relations very directly. So this data comes from a transformation $(\gamma_x^{\mathcal{O}}, V_g)$ from our diagram F to $\mathrm{const}_{\mathcal{O}}$.

Now let $\mathcal{F}$ be a correspondence from $\mathcal{O}$ to a C*-algebra D. Then the correspondences $\mathcal{F}_x := \gamma_x^{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{F}$ from A_x to D and the isomorphisms of correspondences $V_g \otimes_{\mathcal{O}} \mathrm{id}_{\mathcal{F}}\colon \mathcal{E}_g \otimes_{A_y} \mathcal{F}_y \to \mathcal{F}_x$ form a transformation $F \to \mathrm{const}_D$. We claim that this construction gives an equivalence between the groupoid of correspondences $\mathcal{O} \to D$ and the groupoid of transformations $F \to \mathrm{const}_D$.

Let (γ_x, S_g) be the data of a transformation to const_D for some C*-algebra D. Let $\gamma := \bigoplus_x \gamma_x$ with the canonical representation of $\bigoplus A_x$, as in Proposition 3.1. Also map $S_g(\xi) \in \mathbb{B}(\gamma_y, \gamma_x)$ to an operator on γ that vanishes on γ_z for $z \neq y$. We claim that this defines a *-homomorphism $\alpha\colon \mathcal{O} \to \mathbb{B}(\gamma)$, which

is nondegenerate because already its restriction to $\bigoplus A_x$ is nondegenerate. We want to use the universal property of $\mathcal{O}$, of course. All conditions except the fourth one are evident. To check that one, we copy the other half of the proof of [1, Prop. 2.5].

Let $g\colon x \to y$ be an arrow, let $a \in A_x$, $\xi_i, \eta_i \in \mathcal{E}_g$, and let $C > 0$ be strictly bigger than the norm of $\varphi_{\mathcal{E}_g}(a) - \sum |\xi_i\rangle\langle\eta_i|$. It is convenient to use that the map $|\xi\rangle\langle\eta| \mapsto S_g(\xi)S_g(\eta)^*$ induces a *-homomorphism $\vartheta_g\colon \mathbb{K}(\mathcal{E}_g) \to \mathbb{B}(\gamma_x)$. This is nondegenerate because Proposition 6.7 gives

$$\mathbb{K}(\mathcal{E}_g)\gamma_x = S_g(\mathcal{E}_g)S_g(\mathcal{E}_g)^*\gamma_x \supseteq S_g(\mathcal{E}_g)\gamma_y \supseteq \gamma_x.$$

Since $aS_g(\zeta) = S_g(\varphi_{\mathcal{E}_g}(a)\zeta)$ for all $a \in A_x$, we get $a\zeta = \vartheta_g(\varphi_{\mathcal{E}_g}(a))\zeta$ for all $a \in A_x$ and $\zeta \in \mathbb{K}(\mathcal{E}_g)D = D$. Thus the direct action of A_x is equal to $\vartheta_g \circ \varphi_{\mathcal{E}_g}(a)$. This easily implies the norm estimate (iv) in Definition 6.8. Hence we get the desired nondegenerate *-homomorphism $\mathcal{O} \to \mathbb{B}(\gamma)$, so γ becomes a correspondence from $\mathcal{O}$ to D. By construction, the transformation $(\gamma_x^{\mathcal{O}} \otimes_{\mathcal{O}} \gamma, V_g^{\mathcal{O}} \otimes_{\mathcal{O}} \gamma)$ associated to this correspondence γ is the transformation given by the original data (γ_x, S_g).

Let (γ_x^1, S_g^1) and (γ_x^2, S_g^2) be transformations $F \to \mathrm{const}_D$. Form the associated correspondences γ^1 and γ^2 from $\mathcal{O}$ to D. A family of isomorphisms of correspondences $W_x\colon \gamma_x^1 \to \gamma_x^2$ gives a unitary operator $\bigoplus W_x\colon \gamma^1 \to \gamma^2$ that intertwines the left actions of $\bigoplus A_x \subseteq \mathcal{O}$. Conversely, any such operator $\gamma^1 \to \gamma^2$ commutes with the projections $f(p_x)$ and therefore decomposes as $\bigoplus W_x$ for isomorphisms of correspondences $W_x\colon \gamma_x^1 \to \gamma_x^2$. The operators W_x form a modification if and only if they also intertwine the actions of $S_g^1(\xi)$ and $S_g^2(\xi)$ for all $\xi \in \mathcal{E}_g$ and all arrows g in $\mathcal{C}$. Since these elements together with $\bigoplus A_x$ generate $\mathcal{O}$, this is equivalent to intertwining the representations of $\mathcal{O}$. Thus modifications between functors $F \to \mathrm{const}_D$ are in bijection with isomorphisms of the associated correspondences $\mathcal{O} \to D$. Hence we have an equivalence of groupoids $\mathfrak{Corr}^{\mathcal{C}}(F, \mathrm{const}_D) \cong \mathfrak{Corr}(\mathcal{O}, D)$.

If the correspondences γ_x are proper, then $\bigoplus A_x \to \mathbb{K}(\gamma)$ and hence $\mathcal{O} \to \mathbb{K}(\gamma)$ because $\bigoplus A_x \to \mathcal{O}$ is nondegenerate. Thus we get a proper correspondence from $\mathcal{O}$ to D. The converse also holds because the correspondences $\gamma_x^{\mathcal{O}}$ are proper. Hence $\mathfrak{Corr}_{\mathrm{prop}}^{\mathcal{C}}(F, \mathrm{const}_D) \cong \mathfrak{Corr}_{\mathrm{prop}}(\mathcal{O}, D)$ as well, that is, $\mathcal{O}$ is also a colimit in the subcategory $\mathfrak{Corr}_{\mathrm{prop}}$. $\qquad\square$

Let us return to the notationally easier case where $\mathcal{C}$ has only one object, that is, $\mathcal{C}$ is a monoid P. By Proposition 6.2, a functor $P \to \mathfrak{Corr}$ is the same as an essential product system over the opposite monoid P^{op}.

Theorem 6.10. *Let P be a monoid. View a proper, essential product system over P^{op} as a functor $P \to \mathfrak{Corr}_{\mathrm{prop}}$. The Cuntz–Pimsner algebra of the product system is the colimit of this functor $P \to \mathfrak{Corr}_{\mathrm{prop}}$ both in $\mathfrak{Corr}_{\mathrm{prop}}$ and in $\mathfrak{Corr}$.*

Proof. The colimit is given by Theorem 6.9 and Definition 6.8. By construction, it is also universal for Cuntz–Pimsner covariant representations of the product system. $\qquad\square$

6.11. **Colimits over bicategories.** If $\mathcal{C}$ is a category, then diagrams $\mathcal{C} \to$ $\mathfrak{Corr}_{\mathrm{prop}}$ have a colimit by Theorem 6.9. We are going to extend this to the case where $\mathcal{C}$ is only a bicategory. The bicategory $\mathfrak{Corr}^{\mathcal{C}}$ for a general bicategory $\mathcal{C}$ is described, among others, in [3, 7, 14]. For the target bicategory $\mathfrak{Corr}$, there are no serious simplifications compared to the case of an arbitrary target bicategory; we will, however, often disregard associators in the following arguments because they are fairly trivial in $\mathfrak{Corr}$. For simplicity, we first assume that $\mathcal{C}$ is a strict 2-category. Any bicategory is equivalent to a strict one (see [14]), so this is no serious restriction.

If $\mathcal{C}$ is a strict 2-category, its arrows and objects form a category $\mathcal{C}_1$, and a functor $F\colon \mathcal{C} \to \mathfrak{Corr}$ contains a functor $\mathcal{C}_1 \to \mathfrak{Corr}$; the latter is given by C*-algebras A_x for objects x of $\mathcal{C}$, correspondences $\mathcal{E}_g$ from A_x to A_y for arrows $g\colon x \to y$ in $\mathcal{C}$, isomorphisms of correspondences $\mu_{g,h}\colon \mathcal{E}_h \otimes_{A_y} \mathcal{E}_g \to \mathcal{E}_{gh}$ for composable arrows $g\colon y \to z$ and $h\colon x \to y$, subject to the conditions in Proposition 6.2. In addition, a functor $F\colon \mathcal{C} \to \mathfrak{Corr}$ also provides isomorphisms of correspondences $v_a\colon \mathcal{E}_g \to \mathcal{E}_h$ for 2-arrows $a\colon g \Rightarrow h$, which are compatible with horizontal and vertical composition. We refer to [7, §4.1] for the details, which play no role in the following.

Describe two functors $F_i\colon \mathcal{C} \to \mathfrak{Corr}$ for $i = 0, 1$ by the data $(A_x^i, \mathcal{E}_g^i, \mu_{g,h}^i, v_a)$ as above. A transformation $\Phi\colon F_0 \to F_1$ between them restricts to a transformation between their restrictions to $\mathcal{C}_1$ and thus provides correspondences $\gamma_x\colon A_x^0 \to A_x^1$ and isomorphisms of correspondences $V_g\colon \gamma_x \otimes_{A_x^1} \mathcal{E}_g^1 \to \mathcal{E}_g^0 \otimes_{A_y^0} \gamma_y$ for arrows $g\colon x \to y$ in $\mathcal{C}$, subject to the conditions in Proposition 6.3. To be a transformation on the level of $\mathcal{C}$, we need no extra data, but extra conditions: the diagrams

$$(10) \qquad \begin{array}{ccc} \gamma_x \otimes_{A_x^1} \mathcal{E}_g^1 & \xrightarrow{\mathrm{id}_{\gamma_x} \otimes_{A_x^1} v_a^1} & \gamma_x \otimes_{A_x^1} \mathcal{E}_h^1 \\ {\scriptstyle V_g} \Big\Downarrow & & \Big\Downarrow {\scriptstyle V_h} \\ \mathcal{E}_g^0 \otimes_{A_y^0} \gamma_y & \xrightarrow{v_a^0 \otimes_{A_y^0} \mathrm{id}_{\gamma_y}} & \mathcal{E}_h^0 \otimes_{A_y^0} \gamma_y \end{array}$$

must commute for all 2-arrows $a\colon g \Rightarrow h$ in $\mathcal{C}$, for parallel arrows $g, h\colon x \rightrightarrows y$. This diagram commutes automatically if a is an identity 2-arrow.

A modification between two transformations $\Phi_1, \Phi_2\colon F_0 \to F_1$ is defined *exactly* as in Proposition 6.4; there is no extra data and no extra condition to be a modification on the level of $\mathcal{C}$.

Definition 6.12. Let $(A_x, \mathcal{E}_g, \mu_{g,h}, v_a)$ describe a functor from the 2-category $\mathcal{C}$ to $\mathfrak{Corr}$. The *Cuntz–Pimsner algebra* $\mathcal{O}(A_x, \mathcal{E}_g, \mu_{g,h}, v_a)$ is defined as the quotient of $\mathcal{O}(A_x, \mathcal{E}_g, \mu_{g,h})$ (see Definition 6.8) by the relations $S_h(v_a(\xi)) = S_g(\xi)$ for all 2-arrows $a\colon g \Rightarrow h$ and all $\xi \in \mathcal{E}_g$.

Theorem 6.13. *Let $\mathcal{C}$ be a (strict) 2-category and let $(A_x, \mathcal{E}_g, \mu_{g,h}, v_a)$ give a functor $F\colon \mathcal{C} \to \mathfrak{Corr}_{\mathrm{prop}}$. The C*-algebra $\mathcal{O}(A_x, \mathcal{E}_g, \mu_{g,h}, v_a)$ is a colimit of F both in $\mathfrak{Corr}$ and in $\mathfrak{Corr}_{\mathrm{prop}}$.*

Proof. Let $F_1 \colon \mathcal{C}_1 \to \mathfrak{Corr}_{\mathrm{prop}}$ denote the restriction of a diagram to the arrows and objects in $\mathcal{C}$. A transformation $F \to \mathrm{const}_D$ is also a transformation $F_1 \to \mathrm{const}_D$, and the modifications are the same in both cases. Hence the universal C*-algebra for transformations $F \to \mathrm{const}_D$ is a quotient of the one for transformations $F_1 \to \mathrm{const}_D$. The extra relations that we need to divide out are exactly the relations $S_h(v_a(\xi)) = S_g(\xi)$ for all 2-arrows $a \colon g \Rightarrow h$ and all $\xi \in \mathcal{E}_g$: this is exactly what is needed to make the diagrams (10) commute. $\square$

If $\mathcal{C}$ is only a bicategory, then functors $\mathcal{C} \to \mathfrak{Corr}$ look the same as above, except that now the "category" $\mathcal{C}_1$ is only associative and unital up to certain 2-arrows, which form part of the data. The definitions of transformations and modifications, however, do not contain the associators and unit transformations. So the proof of Theorem 6.9 extends to non-associative "categories," and Theorem 6.13 extends literally to bicategories.

7. INDUCTIVE LIMITS

Let $\mathcal{C}$ be the partially ordered set $(\mathbb{N}, \leq)$ viewed as a category, that is, with a unique arrow $m \to n$ if $m \leq n$ and no arrow otherwise. Diagrams indexed by $\mathcal{C}$ are called *inductive systems*, and their colimits *inductive limits*. Such a diagram in $\mathfrak{Corr}$ is given by C*-algebras A_n, correspondences $\mathcal{E}_m^n \colon A_m \to A_n$ for $m \leq n$, and isomorphisms of correspondences $\mu_{m,n,k} \colon \mathcal{E}_m^n \otimes_{A_n} \mathcal{E}_n^k \cong \mathcal{E}_m^k$ for all $m \leq n \leq k$, subject to the following conditions. First, $\mathcal{E}_n^n \cong A_n$ and $\mu_{m,n,k}$ has to be the canonical isomorphism if $m = n$ or $n = k$. Secondly, the maps $\mu_{m,n,k}$ are "associative" (view them as multiplication maps).

We may simplify this data, up to isomorphism of diagrams: It is enough to specify C*-algebras A_n and correspondences $\mathcal{E}_n^{n+1}$ for $n \in \mathbb{N}$, with no constraints on the $\mathcal{E}_n^{n+1}$. We may extend this to a diagram as above by taking

$$\mathcal{E}_m^n \cong \mathcal{E}_m^{m+1} \otimes_{A_{m+1}} \mathcal{E}_{m+1}^{m+2} \otimes_{A_{m+2}} \cdots \otimes_{A_{n-1}} \mathcal{E}_{n-1}^n$$

for $m \leq n$ (the empty tensor product is interpreted as A_n for $m = n$) and letting $\mu_{m,n,k}$ be the canonical isomorphisms. Conversely, any diagram is isomorphic to one of this form.

Let $(A_n, \mathcal{E}_n^m, \mu_{n,m,k})$ and $(B_n, \mathcal{F}_n^m, v_{n,m,k})$ be such diagrams. We may also simplify transformations between them. By definition, a transformation is given by correspondences $\mathcal{G}_n \colon A_n \to B_n$ and isomorphisms of correspondences

$$w_{m,n} \colon \mathcal{E}_m^n \otimes_{A_n} \mathcal{G}_n \cong \mathcal{G}_m \otimes_{B_m} \mathcal{F}_m^n \quad \text{for all } m \leq n,$$

subject to compatibility conditions with $\mu_{m,n,k}$ and $v_{m,n,k}$ for all $m \leq n \leq k$, and the condition that $w_{n,n}$ be the canonical isomorphism. It suffices to specify the isomorphisms $w_{n,n+1}$ for $n \in \mathbb{N}$, without any conditions.

Finally, a modification between two such transformations, $(\mathcal{G}_n, w_{n,n+1})$ and $(\mathcal{G}_n', w_{n,n+1}')$, is given by isomorphisms of correspondences $x_n \colon \mathcal{G}_n \to \mathcal{G}_n'$ such that $w_{m,n} \circ (\mathrm{id}_{\mathcal{E}_m^n} \otimes_{A_n} x_n) = (x_n \otimes_{B_m} \mathrm{id}_{\mathcal{F}_m^n}) \circ w_{m,n}$ for all $m \leq n$; but these conditions hold for all $m \leq n$ once they hold for all $m \in \mathbb{N}$ and $n = m + 1$.

The simplifications above say that the bicategory of functors $\mathcal{C} \to \mathfrak{Corr}$ is equivalent to the bicategory of simplified functors with simplified transformations and modifications. In particular, for colimits it makes no difference whether we work with full or simplified diagrams.

Our general existence theorem shows that any inductive system of proper correspondences has a colimit in $\mathfrak{Corr}$. We claim that for an inductive system of *-homomorphisms in the usual sense, this colimit is the same as the usual inductive limit in the category of C*-algebras. Thus we consider a diagram

$$(11) \qquad A_0 \xrightarrow{\varphi_0} A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} A_n \xrightarrow{\varphi_n} \cdots,$$

where the A_n are C*-algebras and the φ_n are *-homomorphisms. Let A_∞ be the inductive limit C*-algebra of this diagram in the usual sense, and let $\varphi_n^\infty \colon A_n \to A_\infty$ be the canonical *-homomorphisms.

Proposition 7.1. *The* C*-*algebra A_∞ with the maps φ_n^∞ is also a colimit of* (11) *in* $\mathfrak{Corr}_{\mathrm{prop}}$ *and* $\mathfrak{Corr}$.

Proof. Let D be a C*-algebra and let $\mathcal{F}_\infty \colon A_\infty \to D$ be a correspondence. For $n \in \mathbb{N}$, we define a correspondence $\mathcal{F}_n := A_n \otimes_{\varphi_n^\infty} \mathcal{F}_\infty \colon A_n \to D$. These correspondences together with the canonical isomorphisms

$$A_n \otimes_{\varphi_n} \mathcal{F}_{n+1} \cong A_n \otimes_{\varphi_n} A_{n+1} \otimes_{\varphi_{n+1}^\infty} \mathcal{F}_\infty \cong A_n \otimes_{\varphi_{n+1}^\infty \circ \varphi_n} \mathcal{F}_\infty \cong \mathcal{F}_n$$

give a transformation from (11) to const_D. An isomorphism of correspondences $\mathcal{F}_\infty \to \mathcal{F}'_\infty$ induces a modification between these associated transformations, so we get a functor from the groupoid of correspondences $A_\infty \to D$ to the groupoid of transformations in $\mathfrak{Corr}$ from the diagram (11) to the constant diagram on D. We claim that this functor is an equivalence of groupoids.

Let the correspondences $\mathcal{F}_n \colon A_n \to D$ and the isomorphisms of correspondences $\mu_n \colon A_n \otimes_{\varphi_n} \mathcal{F}_{n+1} \to \mathcal{F}_n$ form a transformation from (11) to the constant diagram on D. We are going to construct a correspondence $\mathcal{F}_\infty \colon A_\infty \to D$.

If $a \in \ker \varphi_n \subseteq A_n$, then $a \otimes_{\varphi_n} \xi = 0$ for all $\xi \in \mathcal{F}_{n+1}$ and hence $ab \otimes_{\varphi_n} \xi = a \otimes_{\varphi_n} b\xi = 0$ for all $b \in A_n$, $\xi \in \mathcal{F}_{n+1}$. Since $A_n \otimes_{\varphi_n} \mathcal{F}_{n+1} \cong \mathcal{F}_n$, $\ker \varphi_n$ acts trivially on $\mathcal{F}_n$. Similarly, the kernel of $\varphi_n^{n+m} \colon A_n \to A_{n+m}$ acts trivially on $\mathcal{F}_n$ because $\mathcal{F}_n \cong A_n \otimes_{\varphi_n} A_{n+1} \otimes_{\varphi_{n+1}} \cdots \otimes_{\varphi_{n+m-1}} \mathcal{F}_{n+m}$. The union of these kernels is dense in the kernel of φ_n^∞, which therefore also acts trivially on $\mathcal{F}_n$. Thus we may turn $\mathcal{F}_n$ into a correspondence $\mathcal{F}'_n$ from $A'_n := A_n / \ker \varphi_n^\infty$ to D. The maps φ_n become embeddings $A'_n \to A'_{n+1} \to \cdots \to A_\infty$, and the isomorphisms $\mu_n \colon A_n \otimes_{\varphi_n} \mathcal{F}_{n+1} \to \mathcal{F}_n$ induce isomorphisms $\mathcal{F}'_n \cong A'_n \otimes_{\varphi_n} \mathcal{F}'_{n+1}$. We use these isomorphisms and the embeddings $A'_n \hookrightarrow A'_{n+1}$ to view $\mathcal{F}'_n$ as a subspace of $\mathcal{F}'_{n+1}$ for each n.

Let $\mathcal{F}_\infty := \varinjlim \mathcal{F}'_n$. Then $\mathcal{F}_\infty$ is a Hilbert D-module and the C*-algebras A'_n act on $\mathcal{F}_\infty$ because $A'_n \cdot \mathcal{F}_\infty = \mathcal{F}'_n \subseteq \mathcal{F}_\infty$. The left action of A_∞ is nondegenerate because $A_\infty \cdot \mathcal{F}_\infty$ contains $A'_n \cdot \mathcal{F}_\infty = \mathcal{F}'_n$ for each $n \in \mathbb{N}$, and these subspaces are dense in $\mathcal{F}_\infty$. Thus $\mathcal{F}_\infty$ is a correspondence from A_∞ to D.

This construction is inverse to the one above because $\mathcal{F}_n \cong A_n \otimes_{\varphi_n^\infty} \mathcal{F}_\infty$. Hence A_∞ has the universal property of the colimit. $\square$

Example 7.2. Let $B \subseteq A$ be a nondegenerate C*-subalgebra and $E \colon A \to B$ a conditional expectation. Then $\langle a_1, a_2 \rangle = E(a_1^* a_2)$ and the obvious right multiplication action of B turn A into a pre-Hilbert B-module. The action of A on itself by left multiplication extends to the completion, giving a C*-correspondence A_E from A to B. If $C \subseteq B$ and $F \colon B \to C$ is a conditional expectation as well, then $F \circ E \colon A \to C$ is a conditional expectation and the map $a \otimes b \mapsto E(a) \cdot b$ extends to an isomorphism of C*-correspondences $A_E \otimes_B B_F \cong A_{F \circ E}$. Thus a decreasing chain of nondegenerate C*-subalgebras $\mathcal{R}_n \subseteq A$ with conditional expectations $\mathcal{R}_n \to \mathcal{R}_{n+1}$ defines a functor $(\mathbb{N}, \leq) \to \mathfrak{Corr}$. This situation is studied in [8, 9]. To apply our theory, we need proper C*-correspondences. Equivalently, the conditional expectations are of finite-index type as in [19]. In the proper case, the above diagram has a colimit by Theorem 6.9; in fact, this is isomorphic to the C*-algebra constructed in [8]. It is an appropriate analog of the inductive limit of a chain of *-homomorphisms by Proposition 7.1.

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