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On the reduction of the Siegel moduli space of abelian varieties of dimension 3 with Iwahori level structure

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Abstract. We study the moduli space of abelian threefolds with Iwahori level structure in positive characteristic. We explicitly determine the fibers of the canonical projection to the moduli space of principally polarized abelian varieties and draw conclusions about the relationship between the Ekedahl-Oort, the Kottwitz-Rapoport and the Newton stratification on these spaces.

1. Introduction

Fix a prime p, an integer $g \geq 1$ and an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$. Denote by $\mathcal{A}_g$ the moduli space of principally polarized abelian varieties of dimension g over $\mathbb{F}$ and by $\mathcal{A}_I$ the moduli space of abelian varieties of dimension g over $\mathbb{F}$ with Iwahori level structure (see Section 2 for details).

In this paper we determine an explicit description of the fibers of the canonical projection $\pi : \mathcal{A}_I \to \mathcal{A}_g$ in the case $g = 3$ and use this description to study the relationship between the natural stratifications on $\mathcal{A}_I$ and $\mathcal{A}_g$.

On $\mathcal{A}_g$ we have the p-rank stratification which has the property that two abelian varieties lie in the same stratum if and only if their p-ranks coincide. We have the Ekedahl-Oort stratification, originally defined in [14], which is given by the isomorphism type of the kernel of multiplication by p on the abelian variety. There is an explicit bijection from the set of EO strata to the set of final sequences of length g, that is, to the set of maps $\psi : \{0, \dots, 2g\} \to \mathbb{N}$ with $\psi(0) = 0,\ \psi(2g) = g$, such that

$$\psi(i) \leq \psi(i+1) \leq \psi(i) + 1$$

and

$$\psi(i) < \psi(i+1) \Leftrightarrow \psi(2g-i) = \psi(2g-i-1)$$

for $0 \leq i < 2g$. If ψ is a final sequence, we denote the corresponding EO stratum by EO_ψ.

Trivially the EO stratification is a refinement of the p-rank stratification.

Furthermore we have the Newton stratification, given by the isogeny type of the Barsotti-Tate group of the abelian variety. We are primarily concerned with one special Newton stratum, namely the supersingular locus $\mathcal{S}_g$. In general neither of the Newton or the EO stratification is a refinement of the other. In fact the supersingular locus is not a union of EO strata for $g \geq 3$. In [11] Harashita determines those EO strata that are entirely contained in the supersingular locus.

For $g = 3$ there are four EO strata of p-rank 0, totally ordered by their dimensions. By Harashita's result, the 0- and the 1-dimensional stratum are contained in the supersingular locus. Using a normal form for the Dieudonné module of the Barsotti-Tate group of a supersingular abelian variety, due to Harashita, we prove that the 2-dimensional EO stratum is contained in the complement of $\mathcal{S}_3$. It then follows for dimension reasons that the 3-dimensional EO stratum intersects the supersingular locus in an open dense subset of $\mathcal{S}_3$.

On $\mathcal{A}_I$ we have the Kottwitz-Rapoport stratification, given by the relative position of the chain of de Rham cohomology groups and the chain of Hodge filtrations associated with an element of $\mathcal{A}_I$. There is an explicit bijection from the set of KR strata to the set of admissible elements $\mathrm{Adm}(\mu)$, where the latter is a subset of the extended affine Weyl group of the group of symplectic similitudes GSp_{2g}. There is a unique element τ of length 0 such that $\mathrm{Adm}(\mu) \subset W_a\tau$, where W_a is the affine Weyl group of G, a Coxeter group generated by simple reflection $s_0, \ldots, s_g$ (described explicitly in Section 2.4).

We denote by $\mathcal{S}_I$ the preimage of $\mathcal{S}_g$ under π, which we also call the supersingular locus. In [8] and [7] Görtz and Yu study the dimension of $\mathcal{S}_I$ and they determine those KR strata that are entirely contained in the supersingular locus. But again it is not true in general that $\mathcal{S}_I$ is a union of KR strata and it is natural to ask which other KR strata have a nonempty intersection with the supersingular locus and what the dimension of this intersection is. Another question concerning the KR stratification deals with its relationship to the EO stratification. It is known that the image of a KR stratum under π is always a union of EO strata but it is not known which EO strata occur in the image of a given arbitrary KR stratum. This question has been studied by Ekedahl and van der Geer in [3] (cp. [7, Sec. 9]) and also by Görtz and Hoeve in [6].

To answer these questions for $g = 3$ we need to investigate the fibers of π. Classical Dieudonné theory provides us with an injective map from a fiber of π into a suitable flag variety over $\mathbb{F}$ and it can be shown that this map is actually a universally injective, finite morphism of algebraic varieties over $\mathbb{F}$. In particular it induces a universal homeomorphism onto its image and in order to study topological properties of the fibers it is therefore sufficient to study their images under these respective maps. Up to isomorphism these images only depend on the EO stratum of the basepoint and hence there are only finitely many cases that have to be considered. If ψ is a final sequence we denote this image of the fiber over a point of $EO_\psi(\mathbb{F})$ by $\mathrm{Flag}_\psi^{\perp,F,V} = \mathrm{Flag}_{\psi,2g}^{\perp,F,V}$. While the conditions that determine $\mathrm{Flag}_{\psi}^{\perp,F,V}$ as a closed subvariety of a full flag variety over $\mathbb{F}$ are easy to describe, the geometry of the resulting variety is rather complicated.

To give the reader an impression of what has to be expected let us sketch the geometry of the variety $\mathrm{Flag}_{\psi_0}^{\perp,F,V}$, where EO_{ψ_0} is the 0-dimensional EO stratum, see Section 9.1.

Theorem 1.1. *Let $g = 3$ and let $A \in \mathcal{A}_3(\mathbb{F})$ be a* superspecial *abelian variety. Then there is a universal homeomorphism from the fiber $\pi^{-1}(A)$ onto $\mathrm{Flag}_{\psi_0}^{\perp,F,V}$. The variety $\mathrm{Flag}_{\psi_0}^{\perp,F,V}$ is decomposed into irreducible components*

$$Y \cup Z \cup \coprod_{\zeta \in \widetilde{I}} T_\zeta,$$

where $\widetilde{I}$ can be chosen as $\{(x:y:z) \in \mathbb{P}^2(\mathbb{F}_{p^2}) \mid x^p z + y^{p+1} + x z^p = 0\}$ and such that

- *Y is isomorphic to the variety of full flags in $\mathbb{F}^3$,*
- *Z can be considered as a $\mathbb{P}^1_{\mathbb{F}}$-bundle over a variety Z_0, where Z_0 is itself a $\mathbb{P}^1_{\mathbb{F}}$-bundle over the irreducible curve $V_+(X_1^p X_3 + X_2^{p+1} + X_1 X_3^p) \subset \mathbb{P}^2_{\mathbb{F}}$ (for homogeneous coordinates X_1, X_2 and X_3 on $\mathbb{P}^2_{\mathbb{F}}$),*
- *each T_ζ is isomorphic to the blowing-up of $\mathbb{P}^2_{\mathbb{F}}$ in a closed point.*

Sticking to the notation of the Theorem, we see that $\dim Y = \dim Z = 3$ and $\dim T_\zeta = 2$. Furthermore the T_ζ are pairwise disjoint. The intersection $Y \cap Z$ is isomorphic to the variety Z_0. Z intersects each T_ζ in its exceptional curve, while $Y \cap T_\zeta$ is a different subvariety of T_ζ isomorphic to $\mathbb{P}^1_{\mathbb{F}}$. Finally the triple intersection $Y \cap Z \cap T_\zeta$ only consists of one point.

Concerning the fiber over abelian varieties of positive p-rank we prove the following general result (modelled on the "shuffle construction" explained in [16, 5.2]) that provides a method for reducing the case of positive p-rank to the case of p-rank 0 in lower dimensions, see Section 11.

Proposition 1.2. *Let $g \geq 1$, $k \geq 0$ and let $A \in \mathcal{A}_g(\mathbb{F})$ be of p-rank k. Let ψ be the final sequence with $A \in EO_\psi$.*

(1) *Let A be ordinary. Then the fiber over A is discrete and*

$$\#\left(\pi^{-1}(A)\right) = ON_g := 2^g \#\mathrm{Flag}_g(\mathbb{F}_p) = 2^g \frac{\prod_{l=1}^{g}(p^l - 1)}{(p-1)^g}.$$

Here we denote by $\mathrm{Flag}_g(\mathbb{F}_p)$ the set of flags $(\mathcal{F}_j)_{j=0}^g$ in $(\mathbb{F}_p)^g$ with $\dim \mathcal{F}_j = j$ for all $0 \leq j \leq g$.

(2) *Let $1 \leq k \leq g-1$. Then $\mathrm{Flag}_{\psi}^{\perp,F,V}$ is isomorphic to $\binom{g}{k} ON_k$ disjoint copies of $\mathrm{Flag}_{\widetilde{\psi},2(g-k)}^{\perp,F,V}$, where $\widetilde{\psi}$ is the final sequence of length $g-k$ determined by $\widetilde{\psi}(i) = \psi(k+i) - k$ for $0 \leq i \leq g-k$.*

This result will allow us determine the number of connected components of the fibers of π:

Proposition 1.3. *Let $g \geq 1$ and $k \geq 0$. If $A \in \mathcal{A}_g(\mathbb{F})$ is of p-rank k, the fiber $\pi^{-1}(A)$ consists of $\binom{g}{k} ON_k$ connected components. In particular it is connected if and only if $k = 0$.*

From the calculations of the varieties $\mathrm{Flag}_{\psi}^{\perp,F,V}$ for $g = 3$ it is rather easy to determine which KR strata intersect the fiber of π over a given element of $\mathcal{A}_3(\mathbb{F})$ and what the dimension of this intersection is. From this we can determine the EO strata which occur in the image of a given KR stratum:

Theorem 1.4 (Section 17). *For an element $x \in \mathrm{Adm}(\mu)$ of p-rank 0 denote by $\mathbf{ES}(x)$ the set of final sequences such that $\pi(\mathcal{A}_{I,x}) = \coprod_{\psi \in \mathbf{ES}(x)} EO_\psi$. Then Table 1.1 contains a complete list of the sets $\mathbf{ES}(x)$ in the case $g = 3$. Here ψ_i denotes the final sequence corresponding to the i-dimensional EO stratum of p-rank 0 for $0 \leq i \leq 3$.*

x	$\mathbf{ES}(x)$			
$\tau, s_1\tau, s_2\tau, s_{21}\tau, s_{12}\tau, s_{121}\tau$	ψ_0			
$s_3\tau, s_0\tau$		ψ_1		
$s_{30}\tau$	ψ_0	ψ_1		
$s_{10}\tau, s_{23}\tau, s_{20}\tau, s_{31}\tau, s_{01}\tau, s_{32}\tau$			ψ_2	
$s_{310}\tau, s_{320}\tau$	ψ_0		ψ_2	
$s_{3120}\tau$	ψ_0	ψ_1	ψ_2	
$s_{120}\tau, s_{312}\tau, s_{201}\tau, s_{231}\tau$		ψ_1	ψ_2	
$s_{010}\tau, s_{323}\tau, s_{301}\tau, s_{230}\tau$				ψ_3
$s_{2301}\tau$	ψ_0	ψ_1		ψ_3
$s_{3010}\tau, s_{3230}\tau$			ψ_2	ψ_3

TABLE 1.1. The sets $\mathbf{ES}(x)$ for $g = 3$.

The upper block of Table 1.1 contains the supersingular elements.

With Table 1.1 we can show that the inclusion

$$\coprod_{\substack{x \in \mathrm{Adm}(\mu)^{(0)} \\ \mathcal{A}_{I,x} \subset \mathcal{S}_I}} \mathcal{A}_{I,x} \subseteq \pi^{-1}\left(\coprod_{\substack{w \in W_{\mathrm{final}} \\ EO_w \subset \mathcal{S}_g}} EO_w \right), \tag{1.1}$$

which is valid for every $g \geq 1$, is a proper inclusion for $g = 3$, negatively answering a question posed in a preliminary version of [8].

Finally we show that for $g = 3$ we have $\dim(\mathcal{A}_{I,x} \cap \mathcal{S}_I) = \dim \mathcal{A}_{I,x} - 1$ for every KR stratum $\mathcal{A}_{I,x}$ with $\varnothing \subsetneq \mathcal{A}_{I,x} \cap \mathcal{S}_I \subsetneq \mathcal{A}_{I,x}$.

2. NOTATION

2.1. **Basic notation and moduli spaces.** We fix a prime p, an integer $g \geq 1$, an integer $N \geq 3$ coprime to p, an algebraic closure $\mathbb{F}$ of $\mathbb{F}_p$ and a primitive N-th root of unity ζ_N in $\mathbb{F}$. Let $\sigma : \mathbb{F} \to \mathbb{F},\ x \mapsto x^p$ denote the Frobenius morphism. We consider the moduli space $\mathcal{A}_g = \mathcal{A}_{g,N}$ of principally polarized abelian varieties of dimension g over $\mathbb{F}$ with a symplectic level-N-structure with respect to ζ_N. It is a quasi-projective scheme over $\mathbb{F}$, irreducible of dimension $g(g+1)/2$. We will usually omit the principal polarization and the level structure from our notation. We denote by $\mathcal{S}_g$ the supersingular locus inside $\mathcal{A}_g$. It is a closed subset, equidimensional of dimension $\left[\frac{g}{4}\right]$ by [12].

On the other hand, we consider the moduli space $\mathcal{A}_I$ of tuples

$$(A_0 \overset{\alpha}{\to} A_1 \overset{\alpha}{\to} \cdots \overset{\alpha}{\to} A_g, \lambda_0, \lambda_g, \eta),$$

where

- each A_i is a g-dimensional abelian variety over $\mathbb{F}$,
- α is an isogeny of degree p,
- λ_0 and λ_g are principal polarizations on A_0 and A_g, respectively, such that $(\alpha^g)^*\lambda_g = p\lambda_0$,
- η is a symplectic level-N-structure on A_0 with respect to ζ_N.

$\mathcal{A}_I$ has pure dimension $g(g+1)/2$. We will often omit η and even λ_0, λ_g from the notation.

We denote by $\pi : \mathcal{A}_I \to \mathcal{A}_g$ the morphism sending a point

$$(A_0 \overset{\alpha}{\to} A_1 \overset{\alpha}{\to} \cdots \overset{\alpha}{\to} A_g, \lambda_0, \lambda_g, \eta)$$

to the point (A_0, λ_0, η). It is proper and surjective.

Inside $\mathcal{A}_I$ we have the supersingular locus $\mathcal{S}_I$, given by $\pi^{-1}(\mathcal{S}_g)$ as a closed subset. It is shown in [7] that for g even we have $\dim \mathcal{S}_I = g^2/2$ and that $(g^2-g)/2 \le \dim \mathcal{S}_I \le (g^2-1)/2$ if g is odd. However the supersingular locus $\mathcal{S}_I$ is not equidimensional as soon as $g \ge 2$.

2.2. **The p-rank stratification.** Let X be a topological space. We call a set-theoretical decomposition $X = \coprod_{i\in I} X_i$ of X a *stratification on X* if for all $i \in I$ the set X_i is nonempty, locally closed and satisfies $\overline{X_i} = \bigcup_{j\in J_i} X_j$ for some subset $J_i \subset I$.

Let A be an abelian variety of dimension g over $\mathbb{F}$. For $n \in \mathbb{N}$ we denote by $A[n]$ the kernel of multiplication by n on A. It is a finite group scheme of rank n^{2g} over $\mathbb{F}$. There is an integer $0 \le i \le g$ with $A[p](\mathbb{F}) \simeq (\mathbb{Z}/p\mathbb{Z})^i$, called the *$p$-rank* of A. We denote by $\mathcal{A}_g^{(i)}$ the subset of $\mathcal{A}_g$ where the p-rank of the underlying abelian variety is i. Then $\mathcal{A}_g = \bigcup_{i\in\mathbb{N}} \mathcal{A}_g^{(i)}$ is a stratification on $\mathcal{A}_g$ with $\overline{\mathcal{A}_g^{(i)}} = \bigcup_{j\le i} \mathcal{A}_g^{(j)}$. Similarly we write $\mathcal{A}_I^{(i)} = \pi^{-1}(\mathcal{A}_g^{(i)})$, but these sets do not give rise to a stratification on $\mathcal{A}_I$.

2.3. **The a-number.** Let α_p be the $\mathbb{F}$-group scheme representing the functor $S \mapsto \{s \in \mathcal{O}_S(S) \mid s^p = 0\}$ on the category of $\mathbb{F}$-schemes. For an abelian variety A over $\mathbb{F}$ we write $a(A) = \dim_{\mathbb{F}} \operatorname{Hom}(\alpha_p, A)$. This integer is called the *a-number* of A.

2.4. **Group theoretic notation.** We denote by $\mathrm{G} = \mathrm{GSp}_{2g}$ the group of symplectic similitudes. We consider it as a subgroup of GL_{2g} with respect to the embedding induced by the alternating form given on the standard basis vectors $e_1 \ldots, e_{2g}$ by $(e_i, e_j) \mapsto 0$, $(e_{2g+1-i}, e_{2g+1-j}) \mapsto 0$ and $(e_i, e_{2g+1-j}) \mapsto \delta_{ij}$ for $1 \leq i, j \leq g$. We use the Borel subgroup of upper triangular matrices and the maximal torus T of diagonal matrices. We denote by W the finite Weyl group of G which we consider as a subgroup of the finite Weyl group of GL_{2g}. If we identify the latter with S_{2g} in the usual way, an element w of S_{2g} lies in W if and only if $w(i) + w(2g+1-i) = 2g+1$ for all $1 \leq i \leq 2g$. Similarly we identify $X_*(T)$ with the group $\{(a_1, \ldots, a_{2g}) \in \mathbb{Z}^{2g} \mid a_1 + a_{2g} = a_2 + a_{2g-1} = \cdots = a_g + a_{g+1}\}$. For an element $x = (x_1, \ldots, x_{2g})$ of $X_*(T)$ we also write $x(i)$ instead of x_i. W is generated by the elements $s_1, \ldots, s_g$ given by $s_g = (g, g+1)$ and $s_i = (i, i+1)(2g+1-i, 2g-i)$ for $1 \leq i \leq g-1$. Inside W we have the subset $W_{\mathrm{final},g}$ of elements w with $w(1) < w(2) < \cdots < w(g)$.

We denote by $\widetilde{W} = W \ltimes X_*(T)$ the extended affine Weyl group of G. For an element $\lambda \in X_*(T)$ we denote by t^λ the corresponding element of $\widetilde{W}$. We denote by s_0 and τ the elements of $\widetilde{W}$ given by $s_0 = (1, 2g)t^{(1,0,\ldots,0,-1)}$ and

$$\tau = \begin{pmatrix} 1 & \cdots & g & g+1 & \cdots & 2g \\ g+1 & \cdots & 2g & 1 & \cdots & g \end{pmatrix} t^{(1,\ldots,1,0,\ldots,0)}.$$

The affine Weyl group W_a of G is the subgroup of $\widetilde{W}$ generated by $s_0, \ldots, s_g$. It is an infinite Coxeter group. Our choice of generators $s_0, \ldots, s_g$ gives rise to a length function ℓ and the Bruhat order $\leq$ on W_a. We write $s_{i_1 \ldots i_n}$ instead of $s_{i_1} \cdots s_{i_n}$.

2.5. **Convention.** Let K be an algebraically closed field. A *variety (over K)* is a reduced scheme of finite type over $\operatorname{Spec} K$. A *subvariety* of a variety is a reduced subscheme. If we identify a variety X with its set $X(K)$ of K-valued points we refer to the latter object as a *classical* variety.

3. Dieudonné modules

This section introduces our notation for the Dieudonné modules associated with the p-torsion of a principally polarized abelian variety. The principal polarization induces an isomorphism from the Dieudonné module onto its dual and hence an isomorphism between co- and contravariant Dieudonné theory. For most of our statements it will therefore not matter which theory we use. For the few statements where it is of importance, we will use the *contravariant* theory. We refer to [2] and [13] for proofs of the statements below.

Given a ring R, an endomorphism $\alpha : R \to R$ and an R-module M, an additive map $\phi : M \to M$ is called *α-linear* if $\phi(r \cdot m) = \alpha(r) \cdot \phi(m)$ for all $r \in R$, $m \in M$.

Let $g \geq 1$.

3.1. **The Dieudonné module of $A[p]$.** Let $A \in \mathcal{A}_g(\mathbb{F})$ and denote by $\mathbb{D} = \mathbb{D}(A[p])$ the Dieudonné module of $A[p]$. It is a $2g$-dimensional vector space over $\mathbb{F}$, equipped with linear maps $F : \mathbb{D}^{(p)} \to \mathbb{D}$ and $V : \mathbb{D} \to \mathbb{D}^{(p)}$, called Frobenius and Verschiebung respectively, where $\mathbb{D}^{(p)}$ denotes the base change $\mathbb{D} \otimes_{\mathbb{F},\sigma} \mathbb{F}$. As σ is an isomorphism we can identify $\mathbb{D}^{(p)}$ with $\mathbb{D}$ and we will henceforth consider F as a σ-linear and V as a σ^{-1}-linear map $\mathbb{D} \to \mathbb{D}$. The principal polarization $A \to A^\vee$ induces a nondegenerate, alternating pairing $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_A$ on $\mathbb{D}$. F, V and this pairing have the following properties:

Proposition 3.2.

(1) $\operatorname{im} V = \ker F$ *and* $\operatorname{im} F = \ker V$.
(2) $\langle Fx, y\rangle = \langle x, Vy\rangle^p$ *for all* $x, y \in \mathbb{D}$.

□

For future reference we include the following

Corollary 3.3.

(1) *For any subspace* $W \subset \mathbb{D}$ *we have* $V(W^\perp) = F^{-1}(W)^\perp$.
(2) $(\operatorname{im} V)^\perp = \operatorname{im} V$.

Proof. (1) Using Proposition 3.2(2) we have

$$x \in F^{-1}(W) \Leftrightarrow \forall y \in W^\perp \ \ \langle Fx, y\rangle = 0$$
$$\Leftrightarrow \forall y \in W^\perp \ \ \langle x, Vy\rangle = 0 \Leftrightarrow x \in V(W^\perp)^\perp,$$

and the statement follows.

(2) By (1) and Proposition 3.2(1) we have

$$(\operatorname{im} V)^\perp = V(\mathbb{D})^\perp = V(0^\perp)^\perp = F^{-1}(0) = \ker F = \operatorname{im} V.$$

□

3.4. **The Dieudonné module of $A[p^\infty]$.** Let $A \in \mathcal{A}_g(\mathbb{F})$. We denote by $A[p^\infty] = \bigcup_n A[p^n]$ the Barsotti-Tate group of A. It has height $2g$. Associated to $A[p^\infty]$ is the Dieudonné module $\mathbb{D}_\infty = \mathbb{D}(A[p^\infty])$. It is a free module of rank $2g$ over the Witt ring $W(\mathbb{F})$ of $\mathbb{F}$, equipped with linear maps $F_\infty : \mathbb{D}_\infty^{(p)} \to \mathbb{D}_\infty$ and $V_\infty : \mathbb{D}_\infty \to \mathbb{D}_\infty^{(p)}$, called Frobenius and Verschiebung respectively, where $\mathbb{D}_\infty^{(p)}$ denotes the base change $\mathbb{D}_\infty \otimes_{W(\mathbb{F}),\sigma_W} W(\mathbb{F})$. Here we denote by σ_W the Frobenius morphism on $W(\mathbb{F})$. As σ_W is an isomorphism we can identify $\mathbb{D}_\infty^{(p)}$ with $\mathbb{D}_\infty$ and we will henceforth consider F_∞ as a σ_W-linear and V_∞ as a σ_W^{-1}-linear map $\mathbb{D}_\infty \to \mathbb{D}_\infty$. The principal polarization $A \to A^\vee$ induces a perfect, alternating pairing $\langle\cdot,\cdot\rangle_\infty = \langle\cdot,\cdot\rangle_{\infty,A}$ on $\mathbb{D}_\infty$. F_∞, V_∞ and this pairing have the following properties:

Proposition 3.5.

- $F_\infty V_\infty = V_\infty F_\infty = p \cdot \mathrm{id}$.
- $\langle F_\infty x, y\rangle = \langle x, V_\infty y\rangle^{\sigma_W}$ *for all* $x, y \in \mathbb{D}$.

The reduction of $(\mathbb{D}_\infty, F_\infty, V_\infty, \langle\cdot,\cdot\rangle_\infty)$ modulo p is isomorphic to $(\mathbb{D}, F, V, \langle\cdot,\cdot\rangle)$.

3.6. **Supersingular Dieudonné modules.** We recall a result by Harashita, see [11, Sec. 3]. Let $A \in \mathcal{A}_g(\mathbb{F})$ be supersingular. Then there exists a basis $(X_1, \dots, X_g, Y_1, \dots, Y_g)$ of $\mathbb{D}_\infty$ over $W(\mathbb{F})$ such that

- $\langle X_i, Y_j \rangle_\infty = \delta_{ij}, \quad \langle X_i, X_j \rangle_\infty = 0, \quad \langle Y_i, Y_j \rangle_\infty = 0 \quad \text{for } 1 \le i, j \le g.$
- Let $w = (\delta_{i,g+1-j})_{i,j} \in M^{g\times g}(W(\mathbb{F}))$. There is an $\varepsilon \in W(\mathbb{F}_{p^2})^\times$ with $\varepsilon = -\varepsilon^{\sigma w}$ and a strictly lower triangular matrix $T \in M^{g\times g}(W(\mathbb{F}))$ satisfying $Tw = (Tw)^t$, such that F_∞ and V_∞ admit the following descriptions with respect to this basis:

$$F_\infty = \begin{pmatrix} T & -p\varepsilon^{-1}w \\ \varepsilon w & 0 \end{pmatrix}, \quad V_\infty = \begin{pmatrix} 0 & -p\varepsilon^{-1}w \\ \varepsilon w & wT^{\sigma_W^{-1}}w \end{pmatrix}.$$

Reducing modulo p, we get a basis $(\overline{X}_1, \dots, \overline{X}_g, \overline{Y}_1, \dots, \overline{Y}_g)$ of $\mathbb{D}(A[p])$ over $\mathbb{F}$ such that

- $\langle \overline{X}_i, \overline{Y}_j \rangle = \delta_{ij}, \quad \langle \overline{X}_i, \overline{X}_j \rangle = 0, \quad \langle \overline{Y}_i, \overline{Y}_j \rangle = 0 \quad \text{for } 1 \le i, j \le g.$
- Let $\overline{w} = (\delta_{i,g+1-j})_{i,j} \in M^{g\times g}(\mathbb{F})$. There is an $\overline{\varepsilon} \in \mathbb{F}_{p^2}^\times$ with $\overline{\varepsilon} = -\overline{\varepsilon}^\sigma$ and a strictly lower triangular matrix $\overline{T} \in M^{g\times g}(\mathbb{F})$ satisfying $\overline{T}w = (\overline{T}w)^t$, such that F and V admit the following descriptions with respect to this basis:

$$F = \begin{pmatrix} \overline{T} & 0 \\ \overline{\varepsilon w} & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0 \\ \overline{\varepsilon w} & \overline{w}\overline{T}^{\sigma^{-1}}\overline{w} \end{pmatrix}.$$

We have $a(A) = g - \operatorname{rank}(\overline{T})$.

4. The EO stratification

This section contains the results about the EO stratification on $\mathcal{A}_g$ that we are going to use.

4.1. **Final sequences.** We recall a notion defined in [14]. Let $g \in \mathbb{N}$. A final sequence (of length g) is a map $\psi : \{0, \dots, 2g\} \to \mathbb{N}$ with $\psi(0) = 0$, $\psi(2g) = g$, such that

$$\psi(i) \le \psi(i+1) \le \psi(i) + 1$$

and

$$\psi(i) < \psi(i+1) \Leftrightarrow \psi(2g-i) = \psi(2g-i-1)$$

for $0 \le i < 2g$. Let $\mathbf{ES} = \mathbf{ES}_g$ be the set of final sequences of length g. We will identify $\mathbf{ES}$ with $W_{\text{final}} = W_{\text{final},g}$ via the bijection $W_{\text{final}} \to \mathbf{ES}$ given by $w \mapsto \psi_w$ with

$$\psi_w(i) = i - \#\{a \in \{1, \dots, g\} \mid w(a) \le i\}$$

and $\psi_w(2g-i) = \psi_w(i) + g - i$ for $0 \le i \le g$. Instead of ψ we will also write $(\psi(1), \psi(2), \dots, \psi(g))$ to denote a final sequence.

4.2. **The canonical filtration.** Let $A \in \mathcal{A}_g(\mathbb{F})$. Consider the set e of all finite words in the symbols F and $\perp$. In [14, Sec. 5], Oort shows that $\{W(\mathbb{D}) \mid W \in e\}$ is a filtration by linear subspaces

$$0 = W_0 \subset \cdots \subset W_i \subset \cdots \subset W_r \subset \cdots \subset W_{2r} = \mathbb{D}$$

such that

(1) For every $0 \leq j \leq 2r$ we have $\perp (W_j) = W_{2r-j}$.
(2) There is a surjective function $v : \{0, \dots, 2r\} \to \{0, \dots, r\}$ such that $F(W_j) = W_{v(j)}$ for every $0 \leq j \leq 2r$.

It is called the *canonical filtration of A*. Let $\rho : \{0, \dots, 2r\} \to \mathbb{N}$ be given by $\mathrm{rank}(W_i) = \rho(i)$. We associate with A a final sequence $\psi = \psi(A)$ using these data. Suppose $\{\psi(0), \psi(1), \dots, \psi(\rho(i))\}$ has been defined for some $0 \leq i$. Define $\{\psi(0), \dots, \psi(\rho(i+1))\}$ by $\psi(\rho(i)) = \psi(\rho(i)+1) = \cdots = \psi(\rho(i+1))$ if $v(i+1) = v(i)$ and by $\psi(\rho(i)) < \psi(\rho(i)+1) < \cdots < \psi(\rho(i+1))$ if $v(i+1) > v(i)$. We denote by w_A the element of W_{final} corresponding to $\psi(A)$.

The main result in this context is

Theorem 4.3. [14, Sec. 9] *Let $A_1, A_2 \in \mathcal{A}_g(\mathbb{F})$. Then $\psi(A_1) = \psi(A_2)$ if and only if $A_1[p] \simeq A_2[p]$ as finite group schemes over $\mathbb{F}$.*

We will need the following.

Lemma 4.4. *For $A \in \mathcal{A}_g(\mathbb{F})$ we have $\dim \mathrm{im}(V^2) = \psi(g)$.*

Proof. It follows from [14, Rem., p. 18] that $\dim \mathrm{im}(F^2) = \psi(g)$. Using Proposition 3.2(1) and Corollary 3.3 we see that

$$\begin{aligned} \mathrm{im}\, V^2 &= V(\mathrm{im}\, V) = V((\mathrm{im}\, V)^\perp) = (F^{-1}(\mathrm{im}\, V))^\perp \\ &= (F^{-1}(\ker F))^\perp = (\ker F^2)^\perp. \end{aligned}$$

Hence $\dim \mathrm{im}(V^2) = 2g - (2g - \dim \mathrm{im}(F^2)) = \dim \mathrm{im}(F^2)$. □

4.5. **The EO stratification.** On $\mathcal{A}_g$ we have the Ekedahl-Oort stratification (a stratification in the sense of Section 2.2)

$$\mathcal{A}_g = \coprod_{w \in W_{\mathrm{final}}} EO_w,$$

given by $A \in EO_w(\mathbb{F})$ if and only if $w = w_A$. Using the bijection from Section 4.1 we will also index the strata by elements of **ES**.

We list some properties of the EO stratification.

Proposition 4.6. *Let $w \in W_{\mathrm{final}}$.*

(1) *The stratum EO_w is contained in $\mathcal{S}_g$ if and only if $w(i) = i$ for $1 \leq i \leq g - \left[\frac{g}{2}\right]$.*
(2) *If EO_w is not contained in $\mathcal{S}_g$, then EO_w is irreducible.*
(3) *The p-rank on EO_w is given by*

$$\#\{i \in \{1, \dots, g\} \mid w(i) = g + i\}.$$

(4) *The stratum EO_ψ is equidimensional of dimension*

$$\dim EO_w = \ell(w) = \sum_{i=1}^{g} \psi_w(i).$$

Let $\psi, \widetilde{\psi} \in \mathbf{ES}$.

(5) *The a-number on EO_ψ is given by $g - \psi(g)$.*

(6) *If $\psi(i) \leq \widetilde{\psi}(i)$ for $1 \leq i \leq g$ then $EO_\psi \subset \overline{EO_{\widetilde{\psi}}}$.*

Proof. (5) can be found in [11, p. 5], (6) is shown in [14, 14.3]. See [7, Prop. 2.3–2.5] for the other points. □

In view of property (3) we denote by $W_{\text{final}}^{(i)}$ the set of final elements of p-rank i and by $\mathbf{ES}^{(i)}$ the set of final sequences of p-rank i, $0 \leq i \leq g$.

4.7. **EO strata for $g = 2$.** Table 4.1 contains all final sequences $\psi \in \mathbf{ES}$ and the corresponding elements of W_{final} for $g = 2$. We also make explicit some of the information on EO_ψ contained in Proposition 4.6.

ES	W_{final}	dim	p-rank	a-number	$\subset \mathcal{S}_2$?
$(0,0)$	id	0	0	2	$\surd$
$(0,1)$	$s_2 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$	1	0	1	$\surd$
$(1,1)$	$s_{12} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$	2	1	1	–
$(1,2)$	$s_{212} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$	3	2	0	–

TABLE 4.1. EO strata for $g = 2$.

In particular we see that for $g = 2$ the relationship between the EO stratification and the supersingular locus is very easy to describe: We have $\mathcal{S}_2 = EO_{\text{id}} \cup EO_{s_2}$ and $\mathcal{S}_2 \cap EO_{s_{12}} = \mathcal{S}_2 \cap EO_{s_{212}} = \varnothing$.

4.8. **EO strata for $g = 3$.** Table 4.2 contains all final sequences $\psi \in \mathbf{ES}$ and the corresponding elements of W_{final} for $g = 3$. We also make explicit some of the information on EO_ψ contained in Proposition 4.6.

ES	W_{final}	dim	p-rank	a-number	$\subset \mathcal{S}_3$?
$(0,0,0)$	id	0	0	3	$\surd$
$(0,0,1)$	$s_3 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 4 & 3 & 5 & 6 \end{pmatrix}$	1	0	2	$\surd$
$(0,1,1)$	$s_{23} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 5 & 2 & 4 & 6 \end{pmatrix}$	2	0	2	–
$(0,1,2)$	$s_{323} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 4 & 5 & 2 & 3 & 6 \end{pmatrix}$	3	0	1	–
$(1,1,1)$	$s_{123} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 6 & 1 & 4 & 5 \end{pmatrix}$	3	1	2	–

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ES	W_{final}	dim	p-rank	a-number	$\subset \mathcal{S}_3$?
$(1,1,2)$	$s_{3123} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 4 & 6 & 1 & 3 & 5 \end{pmatrix}$	4	1	1	–
$(1,2,2)$	$s_{23123} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 6 & 1 & 2 & 4 \end{pmatrix}$	5	2	1	–
$(1,2,3)$	$s_{323123} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{pmatrix}$	6	3	0	–

TABLE 4.2. EO strata for $g = 3$.

4.9. **The isomorphisms $\boldsymbol{\Psi_A}$.** For $n \in \mathbb{N}$ we endow $\mathbb{F}^{2n}$ with the nondegenerate alternating pairing $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{\text{def}}$ defined on the standard basis $(e_i)_{i=1}^{2n}$ by $\langle e_i, e_j\rangle = 0 = \langle e_{n+i}, e_{n+j}\rangle$ and $\langle e_i, e_{n+j}\rangle = \delta_{i,j}$ for $i, j \in \{1,\dots,n\}$ (note that this pairing is different from the one used in Section 2.4).

Let $A \in \mathcal{A}_g(\mathbb{F})$ and $w = w_A$. In [14, Sec. 9], Oort constructs an isomorphism $\Psi_A : \mathbb{F}^{2g} \to \mathbb{D}(A)$ such that the endomorphisms $F_w = \Psi_A^* F$ and $V_w = \Psi_A^* V$ of $\mathbb{F}^{2g}$ map standard basis vectors to standard basis vectors up to sign and such that $\Psi_A^* \langle\cdot,\cdot\rangle_A = \langle\cdot,\cdot\rangle_{\text{def}}$. As the notation indicates these pullbacks only depend on w. They are given as follows.

Let $w \in W_{\text{final},g}$ with corresponding final sequence $\psi \in \mathbf{ES}$. Denote by $1 \le m_1 < m_2 < \cdots < m_g \le 2g$ the set of all $m \in \{1,\dots,2g\}$ with $\psi(m-1) < \psi(m)$. Denote by $1 \le n_g < n_{g-1} < \cdots < n_1 \le 2g$ the complementary set. In particular $m_i + n_i = 2g+1$ for all $1 \le i \le g$. Now for $1 \le i, j \le g$ we have $F_w(e_{2g+1-i}) = 0$ and

$$F_w(e_i) = \begin{cases} e_j & \text{if } i = m_j, \\ e_{g+j} & \text{if } i = n_j. \end{cases}$$

Furthermore

$$V_w(e_i) = \begin{cases} -e_{g+n_i} & \text{if } n_i \le g, \\ 0 & \text{if } m_i \le g, \end{cases}$$

and

$$V_w(e_{g+i}) = \begin{cases} e_{g+m_i} & \text{if } m_i \le g, \\ 0 & \text{if } n_i \le g. \end{cases}$$

As we are going to make extensive use of these pullbacks in the cases $g = 2$ and $g = 3$ we make this description explicit in the next subsections.

4.10. $\boldsymbol{g = 2}$. Table 4.3 contains the description of the pullbacks F_w and V_w depending on $w \in W_{\text{final}}^{(0)}$ for $g = 2$ with respect to the standard basis $(e_1,\dots,e_4)$.

W_{final}	F_w	V_w
id	$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 1&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&-1&0&0\\ -1&0&0&0 \end{pmatrix}$
s_2	$\begin{pmatrix} 0&1&0&0\\ 0&0&0&0\\ 0&0&0&0\\ 1&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&-1&0&0\\ 0&0&1&0 \end{pmatrix}$

TABLE 4.3. F and V for $g = 2$.

4.11. $\boldsymbol{g = 3.}$ Table 4.4 contains the description of the pullbacks F_w and V_w depending on $w \in W^{(0)}_{\text{final}}$ for $g = 3$ with respect to the standard basis $(e_1, \dots, e_6)$.

W_{final}	F_w	V_w
id	$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&-1&0&0&0&0\\ -1&0&0&0&0&0 \end{pmatrix}$
s_3	$\begin{pmatrix} 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&1&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&-1&0&0&0&0\\ 0&0&0&1&0&0 \end{pmatrix}$
s_{23}	$\begin{pmatrix} 0&1&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&1&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&1&0&0\\ 0&-1&0&0&0&0 \end{pmatrix}$
s_{323}	$\begin{pmatrix} 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 1&0&0&0&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&0&0&0&0\\ 0&0&-1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0 \end{pmatrix}$

TABLE 4.4. F and V for $g = 3$.

5. THE EO STRATIFICATION AND $\mathcal{S}_g$ FOR $g = 3$

We determine the relationship between the EO stratification and the supersingular locus $\mathcal{S}_g$ for $g = 3$. As $\mathcal{S}_g \subset \mathcal{A}^{(0)}$ we only have to look at the EO strata of p-rank 0.

Theorem 5.1. *Let $g = 3$.*

(1) $EO_{\text{id}}, EO_{s_3} \subset \mathcal{S}_3$.
(2) $EO_{s_{23}} \cap \mathcal{S}_3 = \varnothing$.
(3) $EO_{s_{323}} \cap \mathcal{S}_3$ *is a dense open subset of $\mathcal{S}_3$ of pure dimension 2.*

Proof. (1) See Proposition 4.6 or Section 4.8.

(2) By Section 4.8 the a-number on $EO_{s_{23}}$ is equal to 2. Let $A \in \mathcal{S}_3(\mathbb{F})$ be a supersingular abelian variety with $a(A) = 2$ and choose a basis $(\overline{X}_1, \overline{X}_2, \overline{X}_3,$

$\overline{Y}_1$, $\overline{Y}_2, \overline{Y}_3$) of $\mathbb{D}(A[p])$, a matrix $\overline{T}$ and an element $\overline{\varepsilon}$ with the properties of Section 3.6. We have $\mathrm{rank}(\overline{T}) = g - a(A) = 1$ and the symmetry condition for $\overline{T}$ then implies that there is a $t \in \mathbb{F}^\times$ with $\overline{T} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ t & 0 & 0 \end{pmatrix}$. We deduce that the canonical filtration of A is given by

$$0 \subset \langle \overline{Y}_1 \rangle \subset \langle \overline{Y}_1, \overline{Y}_2 \rangle \subset \langle t\overline{X}_3 + \overline{\varepsilon}\overline{Y}_3, \overline{Y}_1, \overline{Y}_2 \rangle \subset \langle \overline{Y}_1, \overline{Y}_2, \overline{Y}_3, \overline{X}_3 \rangle$$
$$\subset \langle \overline{Y}_1, \overline{Y}_2, \overline{Y}_3, \overline{X}_2, \overline{X}_3 \rangle \subset \mathbb{D}(A)$$

with $r = g = 3$, $\rho(i) = i$ for $0 \le i \le 6$ and $v(0) = v(1) = v(2) = 0$, $v(3) = v(4) = 1$, $v(5) = 2$, $v(6) = 3$. Hence A lies in EO_{s_3} and our claim is shown.

(3) Set $U = EO_{s_{323}} \cap \mathcal{S}_3$. By Proposition 4.6 we know that $\overline{EO_{s_{323}}} = \mathcal{A}_3^{(0)}$. As $EO_{s_{323}}$ is locally closed, this implies that $EO_{s_{323}}$ is open in $\mathcal{A}_3^{(0)}$, hence U is open in $\mathcal{S}_3$. Now $\mathcal{S}_3$ is equidimensional of dimension 2, hence the same is true for every nonempty open subset of $\mathcal{S}_3$. But $\mathcal{S}_3 - U = EO_{\mathrm{id}} \cup EO_{s_3} = \overline{EO_{s_3}}$ has dimension 1 and this implies that U intersects every irreducible component of $\mathcal{S}_3$, hence it is even dense in $\mathcal{S}_3$. □

Remark 5.2. According to [11, Rem. 1, p. 8] it is true for any $g \ge 3$ that $EO_{(0,1,\dots,g-1)} \cap \mathcal{S}_g$ is open and dense in $\mathcal{S}_g$.

6. Flag varieties and corresponding notation

We want to study the fibers of $\pi : \mathcal{A}_I \to \mathcal{A}_g$. Instead of investigating them directly we will look at their image under an injective, finite morphism with values in a suitable flag variety. This is sufficient if we are only interested in their topological properties. Before introducing this morphism in the next section we have to fix some notation concerning flag varieties.

6.1. **Flag varieties.** Let $n \in \mathbb{N}$. For $0 \le i \le n$ we denote by $\mathrm{Flag}_{i,n}$ the variety of partial flags $(W_j)_{j=0}^i$ in $\mathbb{F}^n$ satisfying $\dim W_j = j$ for all $0 \le j \le i$. We write $\mathrm{Flag}_n = \mathrm{Flag}_{n,n}$. We denote by $\mathrm{Flag}_{2n}^\perp$ the variety of full symplectic flags in $\mathbb{F}^{2n}$ with respect to the pairing $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_{\mathrm{def}}$ defined in Section 4.9. If we have an element $(W_i)_{i=0}^n$ of $\mathrm{Flag}_{n,2n}(\mathbb{F})$ with W_n totally isotropic we will occasionally consider it as an element of $\mathrm{Flag}_{2n}^\perp(\mathbb{F})$ by implicitly extending it to the flag $(W_i)_{i=0}^{2n}$ with $W_{2n-i} = W_i^\perp$ for $0 \le i \le n$.

Let $g \ge 1$ and $w \in W_{\mathrm{final}}$. We denote by $\mathrm{Flag}_w^{F,V} = \mathrm{Flag}_{w,2g}^{F,V}$ the closed subvariety of Flag_{2g} whose $\mathbb{F}$-valued points are given by those flags $(W_i)_{i=0}^{2g}$ satisfying

$$(*) \qquad F_w(W_i), V_w(W_i) \subset W_i$$

for all $0 \le i \le 2g$. We write $\mathrm{Flag}_w^{\perp,F,V} = \mathrm{Flag}_{w,2g}^{\perp,F,V} = \mathrm{Flag}_{w,2g}^{F,V} \cap \mathrm{Flag}_{w,2g}^\perp$. It follows from Proposition 3.2(2) that an element $(W_i)_{i=0}^{2g} \in \mathrm{Flag}_{2g}^\perp(\mathbb{F})$ lies in $\mathrm{Flag}_w^{\perp,F,V}(\mathbb{F})$ if and only if it satisfies condition $(*)$ for $0 \le i \le g$.

6.2. **Standard charts.** For a field K and $k, l \in \mathbb{N}$ we denote by $\mathrm{FM}^{k\times l}(K)$ the set of $k \times l$-matrices with entries in K of full rank. Let n, i be as above. There is a canonical surjection $\overline{\Phi} : \mathrm{FM}^{n\times i}(\mathbb{F}) \to \mathrm{Flag}_{i,n}(\mathbb{F})$ sending a matrix B to the flag $(W_j)_{j=0}^{i}$ with W_j spanned by the first j columns of B.

Given pairwise distinct $j_1, j_2, \ldots, j_i \in \{1, 2, \ldots, n\}$ we denote by $U_{j_1,\ldots,j_i}$ the open subset of $\mathrm{Flag}_{i,n}$ whose $\mathbb{F}$-valued points are given by the image under $\overline{\Phi}$ of the set of matrices $C = (c_{kl}) \in \mathrm{FM}^{n\times i}(\mathbb{F})$ satisfying

(1) $c_{j_l l} = 1$ for all $l \in \{1, \ldots, i\}$,
(2) $c_{j_l l'} = 0$ for all $l \in \{1, \ldots, i\}$ and every $l' \in \{l + 1, \ldots, i\}$.

An open subset of this form will be called a *standard chart* for $\mathrm{Flag}_{i,n}$. Considered as open subschemes of $\mathrm{Flag}_{i,n}$ we identify them with an affine space of an appropriate dimension in the usual way. Obviously we have

$$\mathrm{Flag}_{i,n} = \bigcup_{\substack{j_1,j_2,\ldots,j_i\in\{1,2,\ldots,n\}\\ \text{pairwise distinct}}} U_{j_1,\ldots,j_i}.$$

Hence in order to prove that some subset of $\mathrm{Flag}_{i,n}$ is closed we will show that its intersection with all standard charts is closed. Furthermore morphisms into and out of $\mathrm{Flag}_{i,n}$ will be obtained by glueing morphisms into and out of standard charts respectively.

6.3. **Subvarieties of $\mathrm{Flag}^{\perp}$.** Consider the set

$$\mathrm{FM}^{2n,\perp}(\mathbb{F}) = \left\{ B \in \mathrm{FM}^{2n\times n}(\mathbb{F}) \;\middle|\; B^t \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} B = 0 \right\}.$$

Then $\overline{\Phi}$ restricts to a surjection $\Phi : \mathrm{FM}^{2n,\perp}(\mathbb{F}) \to \mathrm{Flag}_{2n}^{\perp}(\mathbb{F})$.

We need an economic notation for defining subvarieties of $\mathrm{Flag}_{2n}^{\perp}$. We think that such a notation is most easily explained via an example. Consider a table such as Table 6.1.

Z_1	Z_2	Z_3
$\begin{pmatrix} 0 & x \\ 1 & 0 \\ 0 & y \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & x \\ 1 & 0 \\ 0 & y \end{pmatrix}$	$\begin{pmatrix} 0 \\ \mathrm{GL}_2(\mathbb{F}) \end{pmatrix} \vee \begin{pmatrix} 0 & a \\ 1 & 0 \\ 0 & b \\ 0 & 0 \end{pmatrix}$
$(x, y)^t \in (\mathbb{F}_p)^2 - \{0\}$		$(a, b)^t \in \mathbb{F}^2 - \{0\}$
$\coprod_{p+1} \mathrm{Spec}\,\mathbb{F}$		$\mathbb{P}^1_{\mathbb{F}} \coprod \mathbb{P}^1_{\mathbb{F}}$
0		1

TABLE 6.1

The upper block of Table 6.1 defines subvarieties Z_1, Z_2 and Z_3 of $\mathrm{Flag}_4^{\perp}$ whose $\mathbb{F}$-valued points are given by

$$Z_1(\mathbb{F}) = \Phi\left(\left\{ B \in \mathrm{FM}^{4\times 2,\perp}(\mathbb{F}) \;\middle|\; \exists (x, y)^t \in (\mathbb{F}_p)^2 - \{0\} \text{ s.t. } B = \begin{pmatrix} 0 & x \\ 1 & 0 \\ 0 & y \\ 0 & 0 \end{pmatrix} \right\}\right),$$

$$Z_2(\mathbb{F}) = \Phi\left(\left\{ B \in \mathrm{FM}^{4\times 2,\perp}(\mathbb{F}) \;\middle|\; \exists (x,y)^t \in (\mathbb{F}_p)^2 - \{0\} \text{ s.t. } B = \begin{pmatrix} 0 & 0 \\ 0 & x \\ 1 & 0 \\ 0 & y \end{pmatrix} \right\}\right),$$

and

$$Z_3(\mathbb{F}) = \Phi\left(\left\{ B \in \mathrm{FM}^{4\times 2,\perp}(\mathbb{F}) \;\middle|\; \begin{matrix} \exists A \in \mathrm{GL}_2(\mathbb{F}) \text{ s.t. } B = \begin{pmatrix} 0 \\ A \end{pmatrix} \vee \\ \exists (a,b)^t \in \mathbb{F}^2 - \{0\} \text{ s.t. } B = \begin{pmatrix} 0 & a \\ 1 & 0 \\ 0 & b \\ 0 & 0 \end{pmatrix} \end{matrix} \right\}\right).$$

Furthermore the lower block of the Table 6.1 claims that there are isomorphisms $Z_1 \simeq \coprod_{p+1} \operatorname{Spec} \mathbb{F}$, $Z_2 \simeq \coprod_{p+1} \operatorname{Spec} \mathbb{F}$ and $Z_3 \simeq \mathbb{P}^1_{\mathbb{F}} \coprod \mathbb{P}^1_{\mathbb{F}}$ and that $\dim Z_1 = 0$, $\dim Z_2 = 0$ and $\dim Z_3 = 1$. Note our convention for rows spanning multiple columns where the contained information is to be applied to every column separately. The notation is not meant to imply any connection between the individual subvarieties.

Note that this notation is highly ambiguous. For instance we could have written Z_1 ineptly as in Table 6.2.

Z_1
$\begin{pmatrix} 0 & a \\ 1 & \alpha \\ 0 & b \\ 0 & 0 \end{pmatrix}$
$(a,b)^t \in \mathbb{F}^2 - \{0\},\ \alpha \in \mathbb{F}$ $a^p b - ab^p = 0$

TABLE 6.2

7. THE MAPS ι_A

7.1. **de Rham cohomology.** Let $g \geq 1$. Let $f : A \to S$ be an abelian scheme of relative dimension g. We denote by $\Omega^\bullet_{A/S}$ the de Rham complex of $\mathcal{O}_A$-modules. The *first de Rham cohomology sheaf* $H^1_{DR}(A/S)$ is defined by

$$H^1_{DR}(A/S) = R^1 f_*(\Omega^\bullet).$$

It is a locally free $\mathcal{O}_S$-module of rank $2g$, functorial in A, and its formation commutes with base-change by [1, Prop. 2.5.2].

Inside $H^1_{DR}(A/S)$ we have the *Hodge filtration* ω_A, a locally free $\mathcal{O}_S$-submodule of rank g, given by the image of the injection

$$R^0 f_*(\Omega^1_{A/S}) \to H^1_{DR}(A/S)$$

coming from the Hodge-de Rham spectral sequence, cp. [1, Prop. 2.5.3].

Let $A/\mathbb{F}$ be an abelian variety. In [13], Oda constructs a natural isomorphism $\mathbb{D}(A) \xrightarrow{\sim} H^1_{DR}(A/\mathbb{F})$ taking $\operatorname{im} V$ to ω_A. See in particular [13, Cor. 5.11].

7.2. **The map ι.** We recall a construction from [7, Sec. 4]. Let $f : A^{\mathrm{univ}} \to \mathcal{A}_g$ be the universal abelian scheme and consider its de Rham cohomology $\mathbb{H} = H^1_{DR}(A^{\mathrm{univ}}/\mathcal{A}_g)$. Denote by $\operatorname{Flag}(\mathbb{H}) \to \mathcal{A}_g$ the variety of full flags in $\mathbb{H}$. We define a morphism $\mathcal{A}_I \xrightarrow{\iota} \mathrm{Flag}(\mathbb{H})$ over $\mathcal{A}_g$ on S-valued points as follows: Let $(A_i)_{i=0}^{g} \in \mathcal{A}_I(S)$. We extend it to a chain $(A_i)_{i=0}^{2g}$ by setting $A_{2g-i} = A_i^\vee$ for $0 \leq i < g$. The map $A_{2g-i} \to A_{2g-i+1}$ is given by the dual isogeny of $A_{g-i-1} \to A_{g-i}$ for $0 \leq i < g$, while the map $A_g \to A_{g+1}$ is given by the composition $A_g \xrightarrow{\lambda_g} A_g^\vee \xrightarrow{\alpha^\vee} A_{g-1}^\vee = A_{g+1}$.

Then the image of $(A_i)_i$ in $\mathrm{Flag}(\mathbb{H})(S)$ is given by

$$0 = \alpha(H^1_{DR}(A_{2g})) \subset \alpha(H^1_{DR}(A_{2g-1})) \subset \cdots \subset \alpha(H^1_{DR}(A_1)) \subset H^1_{DR}(A_0),$$

where for each i, α denotes the map induced by $A_0 \to A_i$. The morphism ι is universally injective and finite, see [7, Lemma 4.3].

Definition 7.3. Let $A \in \mathcal{A}_g(\mathbb{F})$ and consider the final element $w_A \in W_{\mathrm{final}}$. We denote by ι_A the composition

$$\pi^{-1}(A) \to \mathrm{Flag}(H^1_{DR}(A/\mathbb{F})) \xrightarrow{\sim} \mathrm{Flag}(\mathbb{D}(A)) \xrightarrow{\sim} \mathrm{Flag}_{2g}$$

where the first map is obtained from ι by base-change, the second map is induced by Oda's isomorphism mentioned in the previous subsection and the third map is induced by the isomorphism Ψ_A from Section 4.9.

Let $A \in \mathcal{A}_g(\mathbb{F})$ and $w = w_A$. It follows from classical Dieudonné theory that the image of ι_A is given by $\mathrm{Flag}^{\perp,F,V}_{w,2g}$. Hence ι_A induces a universal homeomorphism $\pi^{-1}(A) \to \mathrm{Flag}^{\perp,F,V}_{w,2g}$. If we are only interested in topological properties of the fiber $\pi^{-1}(A)$, it is therefore sufficient to study the spaces $\mathrm{Flag}^{\perp,F,V}_{w,2g}$. The following sections contain a list of the varieties $\mathrm{Flag}^{\perp,F,V}_{w,2g}$ for $g = 2$ and $g = 3$.

8. The varieties $\mathrm{Flag}^{\perp,F,V}_{w,4}$ over the p-rank 0 locus

Let $g = 2$. Depending on $w \in W^{(0)}_{\mathrm{final}}$ we determine the variety $\mathrm{Flag}^{\perp,F,V}_{w} \subset \mathrm{Flag}^{\perp}_4$ by writing down its irreducible components. We use the notation explained in Section 6.3.

8.1. $w = \mathrm{id}$. Let $J = \{x \in \mathbb{F} \mid x^p = -x\}$. The irreducible components of $\mathrm{Flag}^{\perp,F,V}_{\mathrm{id}}$ are given by Y, $(\mathrm{Z}_x)_{x\in J}$ and Z_∞ as defined in Table 8.1.

Z	Z_x	Z_∞
$\begin{pmatrix} 0 \\ \mathrm{GL}_2(\mathbb{F}) \end{pmatrix}$	$\begin{pmatrix} 0 & a \\ 0 & -xa \\ x & b \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 \\ 0 & a \\ 1 & 0 \\ 0 & b \end{pmatrix}$
	$(a,b)^t \in \mathbb{F}^2 - \{0\}$	
$\mathbb{P}^1_{\mathbb{F}}$		
1		

TABLE 8.1. The irreducible components of $\mathrm{Flag}^{\perp,F,V}_{\mathrm{id},4}$.

The Z_ζ are pairwise disjoint and each Z_ζ intersects Z in precisely one point, as ζ runs through $J \cup \{\infty\}$. Hence there are $p+2$ irreducible components and $\mathrm{Flag}_{\mathrm{id}}^{\perp,F,V}$ is connected.

8.2. $\boldsymbol{w = s_2}$. $\mathrm{Flag}_{s_2}^{\perp,F,V}$ is given by the variety defined in Table 8.2.

$\begin{pmatrix} 0 & a \\ 0 & 0 \\ 0 & b \\ 1 & 0 \end{pmatrix}$
$(a,b)^t \in \mathbb{F}^2 - \{0\}$
$\mathbb{P}^1_{\mathbb{F}}$
1

TABLE 8.2. The variety $\mathrm{Flag}_{s_3,4}^{\perp,F,V}$.

9. THE VARIETIES $\mathrm{Flag}_{w,6}^{\perp,F,V}$ OVER THE p-RANK 0 LOCUS

Let $g = 3$. Depending on $w \in W_{\mathrm{final}}^{(0)}$ we determine the varieties $\mathrm{Flag}_w^{\perp,F,V} \subset \mathrm{Flag}_6^{\perp}$. We use the notation introduced in Section 6.3. For a matrix $B \in M^{3\times 2}(\mathbb{F})$ we denote by B_i the matrix obtained from B by deleting the i-th row, $i = 1,2,3$. Furthermore we denote by $\mathrm{B}_*(\mathbb{P}^2_{\mathbb{F}})$ the blowing-up of $\mathbb{P}^2_{\mathbb{F}}$ in a closed point.

9.1. $\boldsymbol{w} = \mathbf{id}$. Let $I = \{(x,y)^t \in (\mathbb{F}_{p^2})^2 \mid x^p + x + y^{p+1} = 0\}$. The irreducible components of $\mathrm{Flag}_{\mathrm{id}}^{\perp,F,V}$ are given by Y, Z, T_∞ and $(T_{x,y})_{(x,y)^t \in I}$ as defined in Table 9.1.

Y	Z
$\begin{pmatrix} 0 \\ \mathrm{GL}_3(\mathbb{F}) \end{pmatrix}$	$\begin{pmatrix} 0 & \det B_1 \\ 0 & -\det B_2 \\ 0 & \det B_3 \\ B & \mathbb{F}^3 \end{pmatrix} \vee \begin{pmatrix} & 0 & \\ B & & v \end{pmatrix}$
	$B \in \mathrm{FM}^{3\times 2}(\mathbb{F}),\ v \in \mathbb{F}^3$ $\begin{pmatrix} B & v \end{pmatrix} \in \mathrm{GL}_3(\mathbb{F})$ $\det B_1^p \det B_3 + \det B_2^{p+1} + \det B_1 \det B_3^p = 0$
$\mathrm{Flag}_3(\mathbb{F})$	
3	

continued on next page

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T_∞	$T_{x,y}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a & 0 \\ 1 & 0 & 0 \\ 0 & b & 1 \\ 0 & c & 0 \end{pmatrix} \vee \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta \end{pmatrix}$	$\begin{pmatrix} 0 & a & 0 \\ 0 & ya & 0 \\ 0 & xa & 0 \\ x^p & b & -y \\ y^p & c & 1 \\ 1 & 0 & 0 \end{pmatrix} \vee \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & y\alpha \\ 0 & 0 & x\alpha \\ x^p & -y & \beta \\ y^p & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$(a,b,c)^t \in \mathbb{F}^3 - \mathbb{F}\cdot(0,1,0)^t$ $(\alpha,\beta)^t \in \mathbb{F}^2 - \{0\}$	$(a,b,c)^t \in \mathbb{F}^3 - \mathbb{F}\cdot(0,-y,1)^t$ $(\alpha,\beta)^t \in \mathbb{F}^2 - \{0\}$
$\mathrm{B}_*(\mathbb{P}^2_{\mathbb{F}})$	$\mathrm{B}_*(\mathbb{P}^2_{\mathbb{F}})$
2	2

TABLE 9.1. The irreducible components of $\mathrm{Flag}^{\perp,F,V}_{\mathrm{id},6}$.

In order to get a better understanding of Z we look at the closed subvariety Z_0 of $\mathrm{Flag}_{2,3}$ whose $\mathbb{F}$-valued points are the image under $\overline{\Phi}$ of the set $\{B \in \mathrm{FM}^{3\times 2}(\mathbb{F}) \mid \det B_1^p \det B_3 + \det B_2^{p+1} + \det B_1 \det B_3^p = 0\}$. There is an obvious surjective morphism $\gamma : Z \to Z_0$ and Z becomes a $\mathbb{P}^1_{\mathbb{F}}$-bundle over Z_0 via γ. It is trivial over the intersection of Z_0 with any of the standard charts of $\mathrm{Flag}_{2,3}$.

Now $\mathrm{Flag}_{2,3}$ is itself a $\mathbb{P}^1_{\mathbb{F}}$-bundle over $\mathrm{Grass}_{2,3}$, the variety of 2-dimensional subspaces of $\mathbb{F}^3$, and if we identify $\mathrm{Grass}_{2,3}$ with $\mathbb{P}^2_{\mathbb{F}}$ in the usual way, the map $\delta : \mathrm{Flag}_{2,3}(\mathbb{F}) \to \mathbb{P}^2_{\mathbb{F}}(\mathbb{F})$ of this bundle is given by $\delta(\overline{\Phi}(B)) = (\det B_1 : \det B_2 : \det B_3)$, where $B \in \mathrm{FM}^{3\times 2}$. Choosing homogenous coordinates X_1, X_2 and X_3 on $\mathbb{P}^2_{\mathbb{F}}$, δ restricts to a map $\varepsilon : Z_0 \to V_+(X_1^p X_3 + X_2^{p+1} + X_1 X_3^p)$ making Z_0 a $\mathbb{P}^1_{\mathbb{F}}$-bundle over the curve $V_+(X_1^p X_3 + X_2^{p+1} + X_1 X_3^p) \subset \mathbb{P}^2_{\mathbb{F}}$.

$$Z \xrightarrow{\mathbb{P}^1_{\mathbb{F}}\text{-bundle}} Z_0 \xrightarrow{\mathbb{P}^1_{\mathbb{F}}\text{-bundle}} V_+(X_1^p X_3 + X_2^{p+1} + X_1 X_3^p)$$

As it is not immediately obvious from Table 9.1 what the intersection between the individual irreducible components are, we list them separately in Table 9.2. The T_ζ are pairwise disjoint, as ζ runs through $I \cup \{\infty\}$.

$Y \cap Z$	$Y \cap T_\infty$	$Y \cap T_{x,y}$
$\begin{pmatrix} B & \begin{matrix} 0 \\ v \end{matrix} \end{pmatrix}$	$\begin{pmatrix} & 0 \\ 1 & 0 \\ 0 & \mathrm{GL}_2(\mathbb{F}) \end{pmatrix}$	$\begin{pmatrix} & 0 \\ x^p & \\ y^p & \mathrm{GL}_2(\mathbb{F}) \\ 1 & 0 \end{pmatrix}$
$B \in \mathrm{FM}^{3\times 2}(\mathbb{F}),\ v \in \mathbb{F}^3$ $\begin{pmatrix} B & v \end{pmatrix} \in \mathrm{GL}_3(\mathbb{F})$ $\det B_1^p \det B_3 + \det B_2^{p+1} +$ $\det B_1 \det B_3^p = 0$		
Z_0	$\mathbb{P}^1_{\mathbb{F}}$	

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$Z \cap T_\infty$	$Z \cap T_{x,y}$	$Y \cap Z \cap T_\infty$	$Y \cap Z \cap T_{x,y}$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \beta \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & y\alpha \\ 0 & 0 & x\alpha \\ x^p & -y & \beta \\ y^p & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} & 0 & \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} & 0 & \\ x^p & -y & 1 \\ y^p & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$(\alpha, \beta)^t \in \mathbb{F}^2 - \{0\}$			
$\mathbb{P}^1_{\mathbb{F}}$		$\operatorname{Spec} \mathbb{F}$	
1		0	

Table 9.2. The intersections of the irreducible components of $\mathrm{Flag}^{\perp,F,V}_{\mathrm{id},6}$.

Remark 9.2. In [15, Sec. 2] Richartz studies a similar situation in order to investigate the geometry of the supersingular locus $\mathcal{S}_3$. Let us briefly explain how her situation is related to ours. Denote by $(\mathbb{F}^6, F, V, \langle\cdot,\cdot\rangle) = (\mathbb{F}^6, F_{\mathrm{id}}, V_{\mathrm{id}}, \langle\cdot,\cdot\rangle_{\mathrm{def}})$ the superspecial Dieudonné module in dimension 3. We look at the variety X of flags $(W_2 \subset W_3 \subset W_4)$ in $\mathbb{F}^6$ with $W_3^\perp = W_3$ and $\dim W_i = i$, $F(W_i), V(W_i) \subset W_i$ for all $i \in \{2,3,4\}$. Inside X we have the subvariety $\tilde{X}$ given by those $(W_2 \subset W_3 \subset W_4) \in X(\mathbb{F})$ satisfying $F(W_4), V(W_4) \subset W_2$ and $\operatorname{im} F \subset M_4$, compare [15, 2.8].

Richartz shows that the image of $\tilde{X}$ under the canonical projection $X \to \mathrm{Grass}_{2,6}$ is isomorphic to the curve $C = V_+(X_1^{p+1} + X_2^{p+1} + X_3^{p+1}) \subset \mathbb{P}^2_{\mathbb{F}}$ and that the restriction $\tilde{X} \to C$ has fibers isomorphic to $\mathbb{P}^1_{\mathbb{F}}$. Furthermore she shows that the restriction of the canonical projection $X \to \mathrm{Grass}_{3,6}$ to $\tilde{X}$ is birational onto its image. This image is denoted by $\mathcal{L}_0(N)$ in *loc.cit.* It is the key tool used in [15] to describe the structure of $\mathcal{S}_3$.

To relate these objects to our situation, look at the morphism $\zeta : \mathrm{Flag}^{\perp,F,V}_{\mathrm{id},6} \to X$ given by $(W_i)_{i=0}^6 \to (W_i)_{i=2}^4$ for $(W_i)_{i=0}^6 \in \mathrm{Flag}^{\perp,F,V}_{\mathrm{id},6}(\mathbb{F})$. Consider a point $(W_i)_{i=2}^4 \in X(\mathbb{F})$ and the corresponding endomorphisms $F_{|W_2}$ and $V_{|W_2}$ of W_2 induced by F and V, respectively and write $\mathcal{U} = \ker F_{|W_2} \cap \ker V_{|W_2}$. Then we see as in the proof of Proposition 12.1 below that the fiber of ζ over $(W_i)_{i=2}^4$ is nonempty and isomorphic to $\mathbb{P}(\mathcal{U})$. It is isomorphic to $\mathbb{P}^1_{\mathbb{F}}$ if and only if $\mathcal{U} = W_2$ and consists of one point else. The first case occurs if and only if $W_2 \subset \ker F \cap \ker V$, which is equivalent to $\operatorname{im} F \subset W_4$.

It is easily checked that $\zeta^{-1}(\tilde{X}) = Z$ and that the fibers of the restriction $\tilde{\zeta} : Z \to \tilde{X}$ of ζ are isomorphic to $\mathbb{P}^1_{\mathbb{F}}$. Hence we get the following picture:

$$\begin{array}{ccccc}
\mathrm{Flag}^{\perp,F,V}_{\mathrm{id},6} & \supseteq & Z & & \\
\downarrow & & \downarrow{\scriptstyle \mathbb{P}^1_{\mathbb{F}}} & & \\
X & \supseteq & \tilde{X} & \xrightarrow{\text{bir.}} & \mathcal{L}_0(N) \\
 & & \downarrow{\scriptstyle \mathbb{P}^1_{\mathbb{F}}} & & \\
 & & C & &
\end{array}$$

9.3. $\boldsymbol{w = s_3}$. The irreducible components of $\mathrm{Flag}_{s_3}^{\perp,F,V}$ are given by the varieties defined in Table 9.3.

X	Y
$\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \\ a & b & 0 \\ c & d & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & b & 0 \\ 0 & c & 1 \\ 1 & 0 & 0 \end{pmatrix} \vee \begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F})$ $(\alpha,\beta)^t \in \mathbb{F}^2 - \{0\}$	$(a,b,c)^t \in \mathbb{F}^3 - \mathbb{F}\cdot(0,0,1)^t$ $(\alpha,\beta)^t \in \mathbb{F}^2 - \{0\}$
$\mathbb{P}^1_{\mathbb{F}} \times \mathbb{P}^1_{\mathbb{F}}$	$\mathrm{B}_*(\mathbb{P}^2_{\mathbb{F}})$
2	2

TABLE 9.3. The irreducible components of $\mathrm{Flag}_{s_3,6}^{\perp,F,V}$.

Hence $\mathrm{Flag}_{s_3}^{\perp,F,V}$ consists of two planes intersecting in the exceptional curve of Y.

9.4. $\boldsymbol{w = s_{23}}$. The irreducible components of $\mathrm{Flag}_{s_{23}}^{\perp,F,V}$ are given by the varieties defined in Table 9.4.

X	Y_1	Y_2	Z_1	Z_2
$\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \\ a & b & 0 \\ c & d & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & a & b \\ 1 & 0 & 0 \\ 0 & c & d \end{pmatrix}$	$\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c & d \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \beta \end{pmatrix}$
$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F})$ $(\alpha,\beta)^t \in \mathbb{F}^2 - \{0\}$	$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F})$		$(\alpha,\beta)^t \in \mathbb{F}^2 - \{0\}$	
$\mathbb{P}^1_{\mathbb{F}} \times \mathbb{P}^1_{\mathbb{F}}$	$\mathbb{P}^1_{\mathbb{F}}$			
2	1			

TABLE 9.4. The irreducible components of $\mathrm{Flag}_{s_{23},6}^{\perp,F,V}$.

We have $Y_1 \cap Y_2 = Z_1 \cap Z_2 = X \cap Z_1 = X \cap Z_2 = \varnothing$. The curves Y_1 and Y_2 each intersect the plane X in precisely one point. The intersections $Y_1 \cap Z_1$ and $Y_2 \cap Z_2$ also consist of precisely one point each.

9.5. $\boldsymbol{w = s_{323}}$. The irreducible components of $\mathrm{Flag}_{s_{323}}^{\perp,F,V}$ are given by the varieties defined in Table 9.5.

X	Y	Z
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & c & d \\ 1 & 0 & 0 \end{pmatrix}$
$(\alpha, \beta)^t \in \mathbb{F}^2 - \{0\}$		$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{F})$
$\mathbb{P}^1_{\mathbb{F}}$		
1		

TABLE 9.5. The irreducible components of $\mathrm{Flag}^{\perp,F,V}_{s_{323},6}$.

We have $X \cap Y = \varnothing$ while the intersections $X \cap Z$ and $Y \cap Z$ consist of precisely one point each.

10. PROOF OF THE RESULTS OF SECTIONS 8 AND 9

The case $w = \mathrm{id}$ for $g = 3$ is obviously the most complicated one and we use it to illustrate the method. Assume that

$$C = (c_1 c_2 c_3) = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \\ c_{31} & c_{32} & c_{33} \\ c_{41} & c_{42} & c_{43} \\ c_{51} & c_{52} & c_{53} \\ c_{61} & c_{62} & c_{63} \end{pmatrix} \in \mathrm{FM}^{6\times 3,\perp}(\mathbb{F})$$

is such that $(W_i)_i = \Phi(C) \in \mathrm{Flag}^{\perp,F,V}_{\mathrm{id}}(\mathbb{F})$. We evaluate the condition that every step of the flag $\Phi(C)$ is stable under F_{id} and V_{id}, where we use the explicit description of F_{id} and V_{id} given in Section 4.9. As we are only interested in the image of C under Φ we may without loss of generality multiply columns of C by elements of $\mathbb{F}^*$ and add $\mathbb{F}^*$-multiples of c_i to c_j for $1 \leq i < j \leq 3$. We will do so below without further mentioning.

W_1 is stable under F and V if and only if

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ c_{31}^p \\ c_{21}^p \\ c_{11}^p \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ -c_{31}^{p^{-1}} \\ -c_{21}^{p^{-1}} \\ -c_{11}^{p^{-1}} \end{pmatrix} \in \mathbb{F} \cdot \begin{pmatrix} c_{11} \\ c_{21} \\ c_{31} \\ c_{41} \\ c_{51} \\ c_{61} \end{pmatrix}.$$

This is satisfied if and only if $c_{11} = c_{21} = c_{31} = 0$.

For W_2 the conditions are trivially satisfied if $(c_{12}, c_{22}, c_{32}) = 0$. If this vector is not zero, we consider several cases.

(1) $c_{61} \neq 0$: We may assume that $(c_1 c_2)$ is of the form

$$\begin{pmatrix} 0 & c_{12} \\ 0 & c_{22} \\ 0 & c_{32} \\ c_{41} & c_{42} \\ c_{51} & c_{52} \\ 1 & 0 \end{pmatrix}.$$

W_2 is stable under F and V if and only if

$$\begin{pmatrix} c_{32}^p \\ c_{22}^p \\ c_{12}^p \end{pmatrix}, \begin{pmatrix} -c_{32}^{p^{-1}} \\ -c_{22}^{p^{-1}} \\ -c_{12}^{p^{-1}} \end{pmatrix} \in \mathbb{F} \cdot \begin{pmatrix} c_{41} \\ c_{51} \\ 1 \end{pmatrix}.$$

This is the case if and only if $c_{41}, c_{51} \in \mathbb{F}_{p^2}$ and $c_{32} = c_{12}c_{41}^p$, $c_{22} = c_{12}c_{51}^p$. Hence we see that we may assume that $(c_1 c_2)$ is of the form

$$\begin{pmatrix} 0 & 1 \\ 0 & y \\ 0 & x \\ x^p & b \\ y^p & c \\ 1 & 0 \end{pmatrix}$$

for some $x, y \in \mathbb{F}_{p^2}$ and some $b, c \in \mathbb{F}$. The fact that C is supposed to be an element of $\mathrm{FM}^{\perp}(\mathbb{F})$ implies that we have $x^p + y^{p+1} + x = 0$.

(2) $c_{61} = 0$, $c_{51} \neq 0$: We may assume that $(c_1 c_2)$ is of the form

$$\begin{pmatrix} 0 & c_{12} \\ 0 & c_{22} \\ 0 & c_{32} \\ c_{41} & c_{42} \\ 1 & 0 \\ 0 & c_{62} \end{pmatrix}.$$

The stability of W_2 under F implies that

$$\begin{pmatrix} c_{32}^p \\ c_{22}^p \\ c_{12}^p \end{pmatrix} \in \mathbb{F} \cdot \begin{pmatrix} c_{41} \\ 1 \\ 0 \end{pmatrix}.$$

From this we get that $c_{12} = 0$ and $c_{22} \neq 0$, which is impossible as $C \in \mathrm{FM}^{\perp}(\mathbb{F})$ implies that $c_{22} + c_{12}c_{41} = 0$.

(3) $c_{61} = c_{51} = 0$: We may assume that $(c_1 c_2)$ is of the form

$$\begin{pmatrix} 0 & c_{12} \\ 0 & c_{22} \\ 0 & c_{32} \\ 1 & 0 \\ 0 & c_{52} \\ 0 & c_{62} \end{pmatrix}.$$

$C \in \mathrm{FM}^{\perp}(\mathbb{F})$ implies $c_{12} = 0$. W_2 is stable under F and V if and only if

$$\begin{pmatrix} c_{32}^p \\ c_{22}^p \\ 0 \end{pmatrix}, \begin{pmatrix} -c_{32}^{p^{-1}} \\ -c_{22}^{p^{-1}} \\ 0 \end{pmatrix} \in \mathbb{F} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This implies that $c_{22} = 0$ and we see that $(c_1 c_2)$ can be chosen of the form

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & b \\ 0 & c \end{pmatrix}$$

for some $b, c \in \mathbb{F}$.

For W_3 we first assume that $(c_{12}, c_{22}, c_{32}) = 0$. We write

$$B = (b_1 b_2) = \begin{pmatrix} c_{41} & c_{42} \\ c_{51} & c_{52} \\ c_{61} & c_{62} \end{pmatrix}.$$

If $(c_{13}, c_{23}, c_{33}) = 0$ the conditions are trivially satisfied. The flags of this form are contained in the set Y. If this vector is not zero, $C \in \mathrm{FM}^{\perp}(\mathbb{F})$ implies $(c_{13}, c_{23}, c_{33})^t \in (\mathbb{F} \cdot b_1 \oplus \mathbb{F} \cdot b_2)^{\perp_{can}}$, where $\perp_{can}$ refers to the canonical pairing $(x, y) \mapsto x^t y$ on $\mathbb{F}^3$. But $(\mathbb{F} \cdot b_1 \oplus \mathbb{F} \cdot b_2)^{\perp_{can}}$ is spanned by the vector $(\det B_1, -\det B_2, \det B_3)^t$ and hence we may assume that

$$(c_{13}, c_{23}, c_{33}) = (\det B_1, -\det B_2, \det B_3).$$

The stability of W_3 under F and V is equivalent to

$$\begin{pmatrix} \det B_1^p \\ -\det B_2^p \\ \det B_3^p \end{pmatrix}, \begin{pmatrix} \det B_1^{p^{-1}} \\ -\det B_2^{p^{-1}} \\ \det B_3^{p^{-1}} \end{pmatrix} \in \mathbb{F} \cdot b_1 + \mathbb{F} \cdot b_2.$$

This can also be expressed as the vanishing of the determinants of the matrices

$$M_1 = \begin{pmatrix} B & \begin{matrix} \det B_1^p \\ -\det B_2^p \\ \det B_3^p \end{matrix} \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} B & \begin{matrix} \det B_1^{p^{-1}} \\ -\det B_2^{p^{-1}} \\ \det B_3^{p^{-1}} \end{matrix} \end{pmatrix}.$$

But we have $\det M_1 = \det M_2^p$ and hence we are left with the equation $\det M_1 = 0$, which is equal to the equation $\det B_1^p \det B_3 + \det B_2^{p+1} + \det B_1 \det B_3^p = 0$. Hence we see that the flags of this form are contained in the set Z.

Finally we have to consider the case where $(c_{12}, c_{22}, c_{32}) \neq 0$. We do this accordingly to the cases introduced in the discussion of W_2 above.

(1) We may assume that C is of the form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & y & c_{23} \\ 0 & x & c_{33} \\ x^p & b & c_{43} \\ y^p & c & c_{53} \\ 1 & 0 & 0 \end{pmatrix}$$

for some $x, y \in \mathbb{F}_{p^2}$ with $x^p + y^{p+1} + x = 0$ and some $b, c \in \mathbb{F}$. We see that the stability of W_3 under F and V implies that $(c_{23}, c_{33}) = 0$. $C \in \mathrm{FM}^{\perp}(\mathbb{F})$ then implies that $c_{43} = -yc_{53}$ and we see that $c_{53} \neq 0$. Hence we may assume that C is of the form

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & y & 0 \\ 0 & x & 0 \\ x^p & b & -y \\ y^p & c & 1 \\ 1 & 0 & 0 \end{pmatrix},$$

and we see that flags of this form are contained in the set $T_{x,y}$.

(3) We may assume that C is of the form

$$\begin{pmatrix} 0 & 0 & c_{13} \\ 0 & 0 & c_{23} \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & b & c_{53} \\ 0 & c & c_{63} \end{pmatrix}$$

for some $b, c \in \mathbb{F}$. The stability of W_3 under F and V implies $c_{13} = c_{23} = 0$. $c_3 \perp c_2$ implies $c_{63} = 0$. Hence C is of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & b & 1 \\ 0 & c & 0 \end{pmatrix}$$

and flags of this type are contained in the set T_∞.

Conversely it is easily checked that the sets Y, Z, T_∞ and $(T_{x,y})_{(x,y)^t \in I}$ defined in Table 9.1 are indeed subsets of $\mathrm{Flag}^{\perp,F,V}(\mathbb{F})$. In order to show that they are closed and to construct the isomorphisms claimed in Table 9.1 one has to calculate their intersection with the standard charts.

For instance the intersections $Z \cap U_{453}$ and $Z \cap U_{456}$ are described in Table 10.1.

$Z \cap U_{453}$	$Z \cap U_{456}$
$\begin{pmatrix} 0 & 0 & b_{21}b_{32} - b_{31} \\ 0 & 0 & -b_{32} \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ b_{21} & 1 & 0 \\ b_{31} & b_{32} & \alpha \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \alpha(b_{21}b_{32} - b_{31}) \\ 0 & 0 & -\alpha b_{32} \\ 0 & 0 & \alpha \\ 1 & 0 & 0 \\ b_{21} & 1 & 0 \\ b_{31} & b_{32} & 1 \end{pmatrix}$
$b_{21}, b_{31}, b_{32}, \alpha \in \mathbb{F}$ $(b_{21}b_{32} - b_{31})^p + b_{32}^{p+1} + (b_{21}b_{32} - b_{31}) = 0$	

TABLE 10.1

They are closed subsets of the respective affine spaces.

To see that $\gamma : Z \to Z_0$ is a $\mathbb{P}^1_{\mathbb{F}}$-bundle, trivial over the intersections of Z_0 with the standard charts of $\mathrm{Flag}_{2,3}$, we note exemplarily that the preimage of $V_{45} = Z_0 \cap U_{45}$ under $\gamma : Z \to Z_0$ is given by $(Z \cap U_{453}) \cup (Z \cap U_{456})$. It is now easy to define an isomorphism $\gamma^{-1}(V_{45}) \to V_{45} \times \mathbb{P}^1_{\mathbb{F}}$ fitting into the commutative diagram

$$\begin{array}{ccc} \gamma^{-1}(V_{45}) & \overset{\simeq}{\dashrightarrow} & V_{45} \times \mathbb{P}^1_{\mathbb{F}} \, . \\ & {\scriptstyle \gamma|_{\gamma^{-1}(V_{45})}} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}} & \\ & V_{45} & \end{array}$$

Here V_{45} is the hypersurface in $\mathbb{A}^3_{\mathbb{F}}$ given by the image under $\overline{\Phi}$ of the set of matrices $B \in \mathrm{FM}^{3\times 2}(\mathbb{F})$ of the form $B = \begin{pmatrix} 1 & 0 \\ b_{21} & 1 \\ b_{31} & b_{32} \end{pmatrix}$ for $b_{21}, b_{31}, b_{32} \in \mathbb{F}$ with $(b_{21}b_{32} - b_{31})^p + b_{32}^{p+1} + (b_{21}b_{32} - b_{31}) = 0$.

To prove that Z is irreducible of dimension 3 it suffices to show that $V_+(X_1^p X_3 + X_2^{p+1} + X_1 X_3^p)$ is irreducible (see Lemma 18.4 below). If we consider $X_1^p X_3 + X_2^{p+1} + X_1 X_3^p$ as an element of $K[X_1, X_3][X_2]$, we can apply Eisenstein's criterion (using the prime element X_1 of $K[X_1, X_3]$) to see that $X_1^p X_3 + X_2^{p+1} + X_1 X_3^p$ is irreducible.

Concerning the intersections only the statement about $Z \cap T_{x,y}$ in Table 9.2 is not quite obvious, namely that it is contained in the exceptional curve of $T_{x,y}$. Let $(a, b, c)^t \in \mathbb{F}^3 - \mathbb{F} \cdot (0, -y, 1)^t$ and assume that

$$\Phi \begin{pmatrix} 0 & a & 0 \\ 0 & ya & 0 \\ 0 & xa & 0 \\ x^p & b & -y \\ y^p & c & 1 \\ 1 & 0 & 0 \end{pmatrix} \in Z.$$

First this implies $a = 0$. If $c = 0$ the condition on the determinants for elements in Z would imply $b = 0$, but $(a, b, c) \neq 0$ by assumption. Hence $c \neq 0$ and we may assume $c = 1$. Then the determinant condition becomes

$$-(x^p - by^p) + (-b)^{p+1} - (x^p - by^p)^p = 0.$$

Using $x^p+x+y^{p+1}=0$ we get $y^{p(p+1)}+by^p+b^{p+1}+b^py^{p^2}=0$. Using $y\in\mathbb{F}_{p^2}$ this becomes $y^{p+1}+by^p+b^{p+1}+b^py=0$ which is equivalent to

$$(y+b)^{p+1}=0.$$

As $b\neq -y$ by assumption this does not have a solution.

11. The case of positive p-rank

In order to investigate the fiber of π over abelian varieties of positive p-rank we need to recall some additional material concerning finite commutative group schemes over $\mathbb{F}$ and their Dieudonné theory. Our main reference is again [2].

Let $g\geq 1$ and $A\in\mathcal{A}_g(\mathbb{F})$. Then $A[p]$ is in a unique way a product of three subgroups $A[p]=G^{e,u}\times G^{i,m}\times G^{i,u}$ with $G^{e,u}$ étale unipotent, $G^{i,m}$ infinitesimal multiplicative and $G^{i,u}$ infinitesimal unipotent. One has isomorphisms $G^{e,u}\simeq(\mathbb{Z}/p\mathbb{Z})^k$ and $G^{i,m}\simeq\mu_p^k$, where k is equal to the p-rank of A. Here μ_p denotes the $\mathbb{F}$-group scheme representing the functor $S\mapsto\{s\in\mathcal{O}_S(S)\mid s^p=1\}$ on the category of $\mathbb{F}$-schemes. In terms of Dieudonné modules this corresponds to a decomposition $\mathbb{D}=W^{e,u}\oplus W^{i,m}\oplus W^{i,u}$ into subspaces stable under F and V and such that

- $F_{|W^{e,u}}$ is an isomorphism and $V_{|W^{e,u}}$ is nilpotent,
- $F_{|W^{i,m}}$ is nilpotent and $V_{|W^{i,m}}$ is an isomorphism,
- $F_{|W^{i,u}}$ and $V_{|W^{i,u}}$ are nilpotent.

Here we write $F_{|W^{e,u}}$ for the morphism $W^{e,u}\to W^{e,u}$ induced by F etc. We have $\dim_{\mathbb{F}}W^{e,u}=\dim_{\mathbb{F}}W^{i,m}=k$ and $\dim_{\mathbb{F}}W^{i,u}=2(g-k)$.

This decomposition is natural: If $\widetilde{A}\in\mathcal{A}_g(\mathbb{F})$ with decomposition $\mathbb{D}(\widetilde{A}[p])=\widetilde{W}^{e,u}\oplus\widetilde{W}^{i,m}\oplus\widetilde{W}^{i,u}$ and if $\alpha: A[p]\to\widetilde{A}[p]$ is a group homomorphism, the induced morphism $\mathbb{D}(\alpha):\mathbb{D}(\widetilde{A}[p])\to\mathbb{D}(A[p])$ splits into the direct sum of three morphisms $\widetilde{W}^{e,u}\to W^{e,u}$, $\widetilde{W}^{i,m}\to W^{i,m}$ and $\widetilde{W}^{i,u}\to W^{i,u}$. In particular

$$\begin{aligned}\operatorname{im}\mathbb{D}(\alpha)&=\operatorname{im}\mathbb{D}(\alpha)\cap(W^{e,u}\oplus W^{i,m})\oplus\operatorname{im}\mathbb{D}(\alpha)\cap W^{i,u}\\&=\operatorname{im}\mathbb{D}(\alpha)\cap W^{e,u}\oplus\operatorname{im}\mathbb{D}(\alpha)\cap W^{i,m}\oplus\operatorname{im}\mathbb{D}(\alpha)\cap W^{i,u}.\end{aligned}\tag{11.1}$$

Lemma 11.1. *Let $k\geq 0$ and $A\in\mathcal{A}_g^{(k)}(\mathbb{F})$ with decomposition $\mathbb{D}=W^{e,u}\oplus W^{i,m}\oplus W^{i,u}$ as above. Let $\mathbb{F}^{2g}=U^{e,u}\oplus U^{i,m}\oplus U^{i,u}$ be the decomposition induced via Ψ_A. Let $w=w_A$ and consider the associated data ψ and $(m_i)_{i=1}^g$ introduced in Section 4.9. Write $I=\{i\in\{1,\dots,g\}\mid i=m_i\}$ and $I^c=\{1,\dots,g\}-I$.*

(1) *We have*

$$U^{e,u} = \bigoplus_{i \in I} \mathbb{F} \cdot e_i$$
$$U^{i,m} = \bigoplus_{i \in I} \mathbb{F} \cdot e_{g+i}$$
$$U^{i,u} = \bigoplus_{i \in I^c} (\mathbb{F} \cdot e_i \oplus \mathbb{F} \cdot e_{g+i}).$$

(2) *Let* $\widetilde{w} \in W_{\mathrm{final},g-k}$ *be the final element corresponding to the final sequence* $[\psi(k+1)-k, \psi(k+2)-k, \dots, \psi(g)-k]$. *Denote by* $(\widetilde{e}_i)_{i=1}^{2(g-k)}$ *the standard basis and by* $\widetilde{\langle \cdot,\cdot \rangle} = \widetilde{\langle \cdot,\cdot \rangle_{\mathrm{def}}}$ *the pairing introduced in Section 4.9 on* $\mathbb{F}^{2(g-k)}$. *Consider the morphism* $\widetilde{\beta} : \mathbb{F}^{2(g-k)} \to U^{i,u}$ *given by* $\widetilde{e}_i \mapsto e_{k+i}$ *and* $\widetilde{e}_{g-k+i} \mapsto e_{g+k+i}$ *for* $1 \leq i \leq g-k$. *Then* $\widetilde{\beta}$ *induces an isomorphism of quadruples*

$$(\mathbb{F}^{2(g-k)}, F_{\widetilde{w}}, V_{\widetilde{w}}, \widetilde{\langle \cdot,\cdot \rangle}) \to (U^{i,u}, F_{w|U^{i,u}}, U_{w|U^{i,u}}, \langle \cdot,\cdot \rangle_{|U^{i,u}}).$$

(3) *Let* $U = U^{e,u} \oplus U^{i,m}$. *Let* $\widehat{w} \in W_{\mathrm{final},g-k}$ *be the final element corresponding to the final sequence* $[1,2,\dots,k]$. *Denote by* $(\widehat{e}_i)_{i=1}^{2k}$ *the standard basis and by* $\widehat{\langle \cdot,\cdot \rangle} = \widehat{\langle \cdot,\cdot \rangle_{\mathrm{def}}}$ *the pairing introduced in Section 4.9 on* $\mathbb{F}^{2k}$. *Consider the morphism* $\widehat{\beta} : \mathbb{F}^{2k} \to U$ *given by* $\widehat{e}_i \mapsto e_i$ *and* $\widehat{e}_{k+i} \mapsto e_{g+i}$ *for* $1 \leq i \leq k$. *Then* $\widehat{\beta}$ *induces an isomorphism of quadruples*

$$(\mathbb{F}^{2k}, F_{\widehat{w}}, V_{\widehat{w}}, \widehat{\langle \cdot,\cdot \rangle}) \to (U, F_{w|U}, U_{w|U}, \langle \cdot,\cdot \rangle_{|U}).$$

Proof. This is an easy consequence of the explicit descriptions of F_w and V_w given in Section 4.9. ☐

Corollary 11.2. *Let* $g \geq 1$ *and* $w \in W_{\mathrm{final},g}$. *Then the* p*-rank on* EO_w *is given by* $w(1)-1$.

Proof. As above, let ψ_w and $(m_i)_{i=1}^{g}$ be the data associated with w. We see from Lemma 11.1 that the p-rank on EO_w is given by $\#\{i \in \{1,\dots,g\} \mid i = m_i\} = \#\{i \in \{1,\dots,g\} \mid \psi_w(i) = i\}$. It follows from Section 4.1 that for all $i \in \{1,\dots,g\}$ we have $\psi_w(i) = i \Leftrightarrow (\forall a \in \{1,\dots,g\}\ w(a) > i)$. As w is final, this is equivalent to $w(1) > i$ and there are $w(1)-1$ elements of $\{1,\dots,g\}$ satisfying this inequality. ☐

Remark 11.3. In this way we have obtained a formula for the p-rank on an EO stratum which is considerably simpler than the one cited in Proposition 4.6(3). Given both formulas it is of course easy to show by combinatorial means that they are equivalent.

Proposition 11.4. *Let* K *be an algebraically closed field and* $\mathcal{V}$ *a vector space of finite dimension* g *over* K. *Let* $\mathcal{V}_1$ *and* $\mathcal{V}_2$ *be subspaces of* $\mathcal{V}$ *with* $\mathcal{V} = \mathcal{V}_1 \oplus \mathcal{V}_2$ *and* $\dim_K \mathcal{V}_1 = k$, $\dim_K \mathcal{V}_2 = g-k$. *Fix integers* $0 \leq i \leq g$,

$\max(0, i+k-g) \leq l \leq \min(i,k)$ and a subset $J \subset \{1,\dots,i\} =: I$ of cardinality l.

(1) *Consider the set*

$$Z_{g,i,k,J} = \left\{ (\mathcal{F}_j)_{j=0}^i \in \mathrm{Flag}_{i,g}(\mathcal{V}) \;\middle|\; \forall j \in I \begin{pmatrix} \mathcal{F}_j = \mathcal{F}_j \cap \mathcal{V}_1 \oplus \mathcal{F}_j \cap \mathcal{V}_2 \wedge \\ (\mathcal{F}_j \cap \mathcal{V}_1 \neq \mathcal{F}_{j-1} \cap \mathcal{V}_1 \Leftrightarrow j \in J) \end{pmatrix} \right\},$$

where $\mathrm{Flag}_{i,g}(\mathcal{V})$ denotes the (classical) variety of flags $(\mathcal{F}_j)_{j=0}^i$ in $\mathcal{V}$ with $\dim \mathcal{F}_j = j$ for $0 \leq j \leq i$. Then $Z_{g,i,k,J}$ is a closed subvariety of $\mathrm{Flag}_{i,g}(\mathcal{V})$.
Consider the maps $\varphi : \{0,1,\dots,i\} \to \{0,1,\dots,l\}$ and $\varphi' : \{0,1,\dots,i\} \to \{0,1,\dots,g-l\}$ with $\varphi(0) = 0 = \varphi'(0)$,

$$\varphi(j) = \begin{cases} \varphi(j-1), & j \notin J \\ \varphi(j-1)+1, & j \in J \end{cases}$$

and

$$\varphi'(j) = \begin{cases} \varphi'(j-1), & j \in J \\ \varphi'(j-1)+1, & j \notin J \end{cases}.$$

Then the map $\alpha_J : \mathrm{Flag}_{l,k}(\mathcal{V}_1) \times \mathrm{Flag}_{i-l,g-k}(\mathcal{V}_2) \to Z_{g,i,k,J}$ given by

$$((\mathcal{F}_j)_{j=0}^l, (\mathcal{G}_j)_{j=0}^{i-l}) \mapsto (\mathcal{F}_{\varphi(j)} \oplus \mathcal{G}_{\varphi'(j)})_{j=0}^i$$

is an isomorphism of (classical) varieties.

(2) *Consider the set*

$$Z_{g,i,k,l} = \left\{ (\mathcal{F}_j)_{j=0}^i \in \mathrm{Flag}_{i,g}(\mathcal{V}) \;\middle|\; \begin{matrix} \dim_K(\mathcal{F}_i \cap \mathcal{V}_1) = l \quad \wedge \\ (\forall j \in I\ \mathcal{F}_j = \mathcal{F}_j \cap \mathcal{V}_1 \oplus \mathcal{F}_j \cap \mathcal{V}_2) \end{matrix} \right\}.$$

Then $Z_{g,i,k,l}$ is a closed subvariety of $\mathrm{Flag}_{i,g}(\mathcal{V})$ and

$$Z_{g,i,k,l} = \coprod_{\substack{J \subset I \\ \#J = l}} Z_{g,i,k,J} \simeq \coprod_{\binom{i}{l}} \mathrm{Flag}_{l,k}(\mathcal{V}_1) \times \mathrm{Flag}_{i-l,g-k}(\mathcal{V}_2).$$

(3) *Consider the set*

$$Z_{g,i,k} = \left\{ (\mathcal{F}_j)_{j=0}^i \in \mathrm{Flag}_{i,g}(\mathcal{V}) \mid \forall j \in I\ \mathcal{F}_j = \mathcal{F}_j \cap \mathcal{V}_1 \oplus \mathcal{F}_j \cap \mathcal{V}_2 \right\}.$$

Then $Z_{g,i,k}$ is a closed subvariety of $\mathrm{Flag}_{i,g}(\mathcal{V})$ and

$$\begin{aligned} Z_{g,i,k} &= \coprod_{l=\max(0,i+k-g)}^{\min(i,k)} Z_{g,i,k,l} \\ &\simeq \coprod_{l=\max(0,i+k-g)}^{\min(i,k)} \coprod_{\binom{i}{l}} \mathrm{Flag}_{l,k}(\mathcal{V}_1) \times \mathrm{Flag}_{i-l,g-k}(\mathcal{V}_2). \end{aligned}$$

Consider integers $0 \leq i \leq n$. The Frobenius $\sigma : \mathbb{F} \to \mathbb{F}$ induces an automorphism $\Sigma : \mathrm{Flag}_{i,n}(\mathbb{F}) \to \mathrm{Flag}_{i,n}(\mathbb{F})$. It is given as follows: Denote by $\rho : \mathbb{F}^n \to \mathbb{F}^n$ the componentwise application of σ. Then for $(\mathcal{F}_j)_{j=0}^i \in \mathrm{Flag}_{i,n}(\mathbb{F})$ we have $\Sigma((\mathcal{F}_j)_{j=0}^i) = (\rho(\mathcal{F}_j))_{j=0}^i$. We denote by $\mathrm{Flag}_{i,n}(\mathbb{F}_p)$ the fixed point set of Σ. It is a finite subset of $\mathrm{Flag}_{i,n}(\mathbb{F})$ and it can be identified canonically with the set of flags $(\mathcal{F}_j)_{j=0}^{i}$ in $(\mathbb{F}_p)^n$ with $\dim \mathcal{F}_j = j$ for all $0 \leq j \leq i$.

Proposition 11.5. *Let $g \geq 1$, $k \geq 0$ and $A \in \mathcal{A}_g^{(k)}(\mathbb{F})$ with $w = w_A$.*

(1) *Assume that $k = g$ (i.e. A is ordinary). Then the fiber of π over A is discrete and*

$$\#\left(\pi^{-1}(A)\right) = ON_g := 2^g \#\mathrm{Flag}_g(\mathbb{F}_p) = 2^g \prod_{l=0}^{g-1} \sum_{i=0}^{l} p^i = 2^g \frac{\prod_{l=1}^{g}(p^l - 1)}{(p-1)^g}.$$

(2) *Assume that $1 \leq k \leq g-1$. Then $\mathrm{Flag}_w^{\perp,F,V}$ is isomorphic to $\binom{g}{k} ON_k$ disjoint copies of $\mathrm{Flag}_{\widetilde{w},2(g-k)}^{\perp,F,V}$, where $\widetilde{w}$ is as in point (2) of Lemma 11.1. Note that the p-rank on $EO_{\widetilde{w}}$ is equal to 0.*

Proof. We use the notation of Lemma 11.1.

(1) If $k = g$ we have $U^{i,u} = 0$, $U^{e,u} = \oplus_{i=1}^{g} \mathbb{F} \cdot e_i$ and $U^{i,m} = \oplus_{i=g+1}^{2g} \mathbb{F} \cdot e_i$. Let $w = w_A$. We use the notation of Proposition 11.4 for $\mathcal{V} = \mathbb{F}^{2g}$, $\mathcal{V}_1 = U^{e,u}$ and $\mathcal{V}_2 = U^{i,m}$. By equation (11.1) we see that

$$\mathrm{Flag}_{w,2g}^{\perp,F,V} \subset Z_{2g,g,g} = \coprod_{l=0}^{g} \coprod_{\substack{J \subset I \\ \#J = l}} \alpha_J \left(\mathrm{Flag}_{l,g}(\mathcal{V}_1) \times \mathrm{Flag}_{g-l,g}(\mathcal{V}_2)\right).$$

Let $0 \leq l \leq g$ and $J \subset I$ of cardinality l.

First note that $F_{|\mathcal{V}_1}$ is equal to the componentwise application of σ (with respect to the basis $(e_1, \ldots, e_g)$) and that $V_{|\mathcal{V}_1} = 0$. On the other hand $V_{|\mathcal{V}_2}$ is equal to the componentwise application of σ^{-1} (with respect to the basis $(e_{g+1}, \ldots, e_{2g})$) and $F_{|\mathcal{V}_2} = 0$. From this it follows immediately that $\mathrm{Flag}_w^{F,V} \cap \alpha_J \left(\mathrm{Flag}_{l,g}(\mathcal{V}_1) \times \mathrm{Flag}_{g-l,g}(\mathcal{V}_2)\right) \simeq \mathrm{Flag}_{l,g}(\mathbb{F}_p) \times \mathrm{Flag}_{g-l,g}(\mathbb{F}_p)$.

Let $\left((\mathcal{F}_j)_{j=0}^{l}, (\mathcal{G}_j)_{j=0}^{g-l}\right) \in \mathrm{Flag}_{l,g}(\mathcal{V}_1) \times \mathrm{Flag}_{g-l,g}(\mathcal{V}_2)$. Then the image $\alpha_J\left((\mathcal{F}_j)_{j=0}^{l}, (\mathcal{G}_j)_{j=0}^{g-l}\right)$ is symplectic if and only if $(\mathcal{G})_{j=0}^{g-l}$ is actually a flag in the $g-l$ dimensional space $\mathcal{V}_2 \cap \mathcal{F}_l^{\perp}$, where we consider $\mathcal{F}_l$ as a subspace of $\mathcal{V}$.

Combining these two statements it follows that

$$\mathrm{Flag}_w^{\perp,F,V} \cap \alpha_J \left(\mathrm{Flag}_{l,g}(\mathcal{V}_1) \times \mathrm{Flag}_{g-l,g}(\mathcal{V}_2)\right) \simeq \mathrm{Flag}_{l,g}(\mathbb{F}_p) \times \mathrm{Flag}_{g-l}(\mathbb{F}_p).$$

The claim now follows from a short calculation.

(2) We use the notation of Proposition 11.4 with $\mathcal{V} = \mathbb{F}^{2g}$, $\mathcal{V}_1 = U^{e,u} \oplus U^{i,m}$ and $\mathcal{V}_2 = U^{i,u}$. It follows from $\mathcal{V}_1 = \mathcal{V}_2^{\perp}$ and equation (11.1) that

$$\mathrm{Flag}_w^{\perp,F,V} \subset Z_{2g,g,2k,k} = \coprod_{\substack{J \subset I \\ \#J = k}} \alpha_J \left(\mathrm{Flag}_{k,2k}(\mathcal{V}_1) \times \mathrm{Flag}_{g-k,2(g-k)}(\mathcal{V}_2)\right).$$

Let $J \subset I$ of cardinality k. Using the notation and the isomorphisms $\widetilde{\beta}$ and $\widehat{\beta}$ of Lemma 11.1 we see that

$$\begin{aligned}\mathrm{Flag}_w^{\perp,F,V} \cap \alpha_J \left(\mathrm{Flag}_{k,2k}(\mathcal{V}_1) \times \mathrm{Flag}_{g-k,2(g-k)}(\mathcal{V}_2)\right) \\ \simeq \mathrm{Flag}_{\widehat{w},2k}^{\perp,F,V} \times \mathrm{Flag}_{\widetilde{w},2(g-k)}^{\perp,F,V}.\end{aligned}$$

By the first point $\mathrm{Flag}_{\widehat{w},2k}^{\perp,F,V}$ is discrete of cardinality ON_k and the claim follows. □

12. The number of connected components of the fibers of π

Proposition 12.1. *Let $g \geq 1$. For all $A \in \mathcal{A}_g^{(0)}(\mathbb{F})$, the fiber $\pi^{-1}(A)$ is connected.*

Remark 12.2. In [17, Prop. 5.2] Yu proves a more general statement. We rephrase his proof in our language.

Proof of Proposition 12.1. Let $w = w_A$. We have to show that $\mathrm{Flag}_w^{\perp,F,V}$ is connected. Let $0 \leq i \leq g$, $I = \{0, \dots, i\}$ and denote by $\mathrm{Flag}_{w,i}^{\perp,F,V}$ the variety whose $\mathbb{F}$-valued points are given by

$$\mathrm{Flag}_{w,i}^{\perp,F,V}(\mathbb{F}) = \left\{ (W_j)_{j=0}^i \in \mathrm{Flag}_{i,2g}(\mathbb{F}) \;\middle|\; \begin{array}{c} \forall j \in I \; V_w(W_j), F_w(W_j) \subset W_j \\ \text{and } W_i \text{ is isotropic} \end{array} \right\}.$$

Then $\mathrm{Flag}_{w,g}^{\perp,F,V} = \mathrm{Flag}_w^{\perp,F,V}$ and we will show by induction on i that $\mathrm{Flag}_{w,i}^{\perp,F,V}$ is connected for all $0 \leq i \leq g$. For each $1 \leq i \leq g$ consider the morphism $\zeta_i : \mathrm{Flag}_{w,i}^{\perp,F,V} \to \mathrm{Flag}_{w,i-1}^{\perp,F,V}$ given by $\zeta_i\left((W_j)_{j=0}^i\right) = (W_j)_{j=0}^{i-1}$ for $(W_j)_{j=0}^i \in \mathrm{Flag}_{w,i}^{\perp,F,V}(\mathbb{F})$. This is, in particular, a closed map of topological spaces and it will be sufficient to show that it is surjective with connected fibers. Fix a point $(W_j)_{j=0}^{i-1} \in \mathrm{Flag}_{w,i-1}^{\perp,F,V}(\mathbb{F})$ and write $\mathcal{W} = W_{i-1}^{\perp}/W_{i-1}$ with canonical projection $pr : W_{i-1}^{\perp} \to \mathcal{W}$. F_w and V_w induce endomorphisms $\overline{F}$ and $\overline{V}$ of $\mathcal{W}$. Our assumption on the p-rank of A implies that $\overline{F}$ and $\overline{V}$ are nilpotent. This means that a 1-dimensional subspace of $\mathcal{W}$ is stable under $\overline{F}$ or $\overline{V}$ if and only if it is contained in $\ker \overline{F}$ or $\ker \overline{V}$, respectively. Therefore consider the subspace $\mathcal{U} = \ker \overline{F} \cap \ker \overline{V}$ and denote by $\mathbb{P}(\mathcal{U})$ the (classical) projective space over $\mathcal{U}$. Consider the map $\mathbb{P}(\mathcal{U}) \to \mathrm{Flag}_{i,2g}(\mathbb{F})$, sending a subspace $U \subset \mathcal{U}$ to the flag

$$W_0 \subset W_1 \subset \cdots \subset W_{i-1} \subset pr^{-1}(U).$$

With the considerations above this map is easily seen to induce an isomorphism of (classical) varieties $\mathbb{P}(\mathcal{U}) \to \zeta_i^{-1}\left((W_j)_{j=0}^{i-1}\right)(\mathbb{F})$. Hence the fibers of ζ_i are connected. To see that they are nonempty we have to check that $\dim \mathcal{U} \geq 1$. This is automatic if $\overline{F}$ and $\overline{V}$ are the zero morphism. By Proposition 3.2(1) we know that $\operatorname{im} \overline{V} \subset \ker \overline{F}$ and $\operatorname{im} \overline{F} \subset \ker \overline{V}$. Now the nilpotency of $\overline{F}$ implies that $\operatorname{im} \overline{F} \cap \ker \overline{F} \neq 0$ if $\overline{F} \neq 0$ and the nilpotency of $\overline{V}$ implies that $\operatorname{im} \overline{V} \cap \ker \overline{V} \neq 0$ if $\overline{V} \neq 0$, whence the claim. □

Proposition 12.3. *Let $g \geq 1$ and $k \geq 0$. If $A \in \mathcal{A}_g^{(k)}(\mathbb{F})$, the fiber $\pi^{-1}(A)$ consists of $\binom{g}{k} ON_k$ connected components. In particular it is connected if and only if $k = 0$.*

Proof. Combine Proposition 11.5 and Proposition 12.1. □

13. Dimension of the fibers of π

Let $g = 2$ or $g = 3$ and let $A \in \mathcal{A}_g(\mathbb{F})$. Depending on $w_A \in W_{\mathrm{final}}$ we list the dimension of $\pi^{-1}(A) \subset \mathcal{A}_I$ in Table 13.2. It can be read off the calculations in Sections 8 and 9 and the results of Section 11.

$g = 2$	
W_{final}	dim
id	1
s_2	1
s_{12}	0
s_{212}	0

$g = 3$			
W_{final}	dim	W_{final}	dim
id	3	s_{123}	1
s_3	2	s_{3123}	1
s_{23}	2	s_{23123}	0
s_{323}	1	s_{323123}	0

TABLE 13.2. The dimension of the fibers of π depending on the EO stratum.

14. The KR stratification

This section contains the results about the KR stratification on $\mathcal{A}_I$ that we are going to use. We will use an ad hoc definition on $\mathbb{F}$-valued points and we refer to [8, Sec. 2.4] for a more comprehensive treatment of the subject.

Let $g \geq 1$.

14.1. **Relative positions.**

Proposition 14.2. [7, Sec. 3] *Let $w \in W_{\mathrm{final}}$ and $(W_i)_{i=0}^{2g} \in \mathrm{Flag}_{w,2g}^{\perp,F,V}(\mathbb{F})$. There is a unique element $x = t^\lambda \omega \in W_a \tau$ ($\omega \in W$, $\lambda \in X_*(T)$) such that there is a basis $(\varepsilon_i)_{i=0}^{2g}$ of $\mathbb{F}^{2g}$ with the following properties:*

(1) *$\lambda(i) \in \{0, 1\}$ for all i.*
(2) *For every i, W_i is spanned by $\varepsilon_1, \dots, \varepsilon_i$.*
(3) *If $V_w(W_{i-1}) \subsetneq V_w(W_i)$ for any $i \geq 1$, we have $V_w(W_i) = V_w(W_{i-1}) \oplus \mathbb{F} \cdot \varepsilon_{\omega(i)}$.*

(4)
$$\operatorname{im} V_w = \bigoplus_{\substack{i=1,\dots,2g \\ \lambda(i)=0}} \mathbb{F}\cdot\varepsilon_i.$$

We call any such basis a *KR basis* for $(W_i)_{i=0}^{2g}$ and x is called the *KR type* of $(W_i)_{i=0}^{2g}$.

The set of possible KR types (as w runs through $W_{\text{final},g}$) is denoted by $\operatorname{Adm}(\mu)$. It is a subset of $W_a\tau$. Given $w \in W_{\text{final}}$ and $x \in \operatorname{Adm}(\mu)$ we denote by $\mathcal{L}(x,w)$ the set of flags in $\operatorname{Flag}_w^{\perp,F,V}(\mathbb{F})$ with KR type equal to x.

14.3. **The KR stratification.** On $\mathcal{A}_I$ we have the Kottwitz-Rapoport stratification (a stratification in the sense of Section 2.2)
$$\mathcal{A}_I = \coprod_{x\in\operatorname{Adm}(\mu)} \mathcal{A}_{I,x},$$
given by $(A_i)_i \in \mathcal{A}_{I,x}(\mathbb{F})$ if and only if $\iota_{A_0}((A_i)_i) \in \mathcal{L}(x, w_{A_0})$.

The following Proposition lists some properties of the KR stratification.

Proposition 14.4. [7, Sec. 2.5] *Let $x, y \in \operatorname{Adm}(\mu)$ and $\omega \in W$, $\lambda \in X_*(T)$ such that $x = t^\lambda\omega$.*

1. *$\mathcal{A}_{I,x}$ is equidimensional of dimension $\ell(x)$.*
2. *The p-rank is constant on $\mathcal{A}_{I,x}$ with value $\#\operatorname{Fix}(\omega)/2$ (where $\operatorname{Fix}(\omega) = \{i \in \{1,\dots,2g\} \mid \omega(i) = i\}$).*
3. *We have $\mathcal{A}_{I,x} \subset \overline{\mathcal{A}_{I,y}}$ if and only if $x \le y$.*
4. *If $\mathcal{A}_{I,x}$ is not contained in the supersingular locus $\mathcal{S}_I$, then $\mathcal{A}_{I,x}$ is irreducible.*

In view of property (2) we denote by $\operatorname{Adm}(\mu)^{(i)}$ the set of admissible elements of p-rank i, $0 \le i \le g$.

Lemma 14.5. [7, Lemma 8.1] *The projection $\widetilde{W} \to W$ induces a bijection $\xi : \operatorname{Adm}(\mu)^{(0)} \to \{\omega \in W \mid \operatorname{Fix}(\omega) = \varnothing\}$. Its inverse is given by $\omega \mapsto t^{\lambda(\omega)}\omega$ with*
$$\lambda(\omega)(i) = \begin{cases} 0, & \omega^{-1}(i) > i \\ 1, & \omega^{-1}(i) < i \end{cases}, \quad i = 1,\dots,2g.$$

14.6. **The set $\operatorname{Adm}(\boldsymbol{\mu})^{(0)}$.** In [16] Yu gives a list of all the 29 elements of $\operatorname{Adm}(\mu)^{(0)}$ for $g = 3$. We reproduce this list in Table 14.1 as we will use it extensively.

KR	$(\lambda, w) \in X_*(T) \rtimes W$	KR	$(\lambda, w) \in X_*(T) \rtimes W$
τ	$(0,0,0,1,1,1)$, $(14)(25)(36)$	$s_{310}\tau$	$(0,0,1,0,1,1)$, (132645)
$s_0\tau$	$(0,0,0,1,1,1)$, $(1463)(25)$	$s_{120}\tau$	$(0,0,0,1,1,1)$, $(16)(2453)$
$s_1\tau$	$(0,0,0,1,1,1)$, (142635)	$s_{320}\tau$	$(0,0,1,0,1,1)$, (154623)
$s_2\tau$	$(0,0,0,1,1,1)$, (153624)	$s_{230}\tau$	$(0,1,0,1,0,1)$, (124653)
$s_3\tau$	$(0,0,1,0,1,1)$, $(1364)(25)$	$s_{201}\tau$	$(0,0,0,1,1,1)$, $(1562)(34)$

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$s_{10}\tau$	$(0,0,0,1,1,1)$, $(145)(263)$	$s_{301}\tau$	$(0,0,1,0,1,1)$, (135642)
$s_{20}\tau$	$(0,0,0,1,1,1)$, $(153)(246)$	$s_{121}\tau$	$(0,0,0,1,1,1)$, $(16)(25)(34)$
$s_{30}\tau$	$(0,0,1,0,1,1)$, $(13)(25)(46)$	$s_{231}\tau$	$(0,1,0,1,0,1)$, $(1265)(34)$
$s_{01}\tau$	$(0,0,0,1,1,1)$, $(142)(356)$	$s_{312}\tau$	$(0,0,1,0,1,1)$, $(16)(2354)$
$s_{21}\tau$	$(0,0,0,1,1,1)$, $(15)(26)(34)$	$s_{323}\tau$	$(0,1,1,0,0,1)$, (123654)
$s_{31}\tau$	$(0,0,1,0,1,1)$, $(135)(264)$	$s_{3010}\tau$	$(0,0,1,0,1,1)$, $(132)(456)$
$s_{12}\tau$	$(0,0,0,1,1,1)$, $(16)(24)(35)$	$s_{3120}\tau$	$(0,0,1,0,1,1)$, $(16)(23)(45)$
$s_{32}\tau$	$(0,0,1,0,1,1)$, $(154)(236)$	$s_{3230}\tau$	$(0,1,1,0,0,1)$, $(123)(465)$
$s_{23}\tau$	$(0,1,0,1,0,1)$, $(124)(365)$	$s_{2301}\tau$	$(0,1,0,1,0,1)$, $(12)(34)(56)$
$s_{010}\tau$	$(0,0,0,1,1,1)$, (145632)		

TABLE 14.1. The set $\mathrm{Adm}(\mu)^{(0)}$ for $g=3$.

15. KR strata and the fibers of π

Let $g \geq 1$ and $x \in \mathrm{Adm}(\mu)$. We write

$$\mathbf{ES}(x) = \{w \in W_{\mathrm{final}} \mid \pi^{-1}(EO_w) \cap A_{I,x} \neq \varnothing\}.$$

Then [7, Cor. 3.3] states that

$$\pi(\mathcal{A}_{I,x}) = \coprod_{w \in \mathbf{ES}(x)} EO_w. \tag{15.1}$$

Hence in order to understand the relationship between the EO and the KR stratification we need to understand the sets $\mathbf{ES}(x)$.

Now for all $w \in W_{\mathrm{final}}$ we have $w \in \mathbf{ES}(x) \Leftrightarrow \mathcal{L}(x,w) \neq \varnothing$ and it is therefore sufficient to study the sets $\mathcal{L}(x,w)$. We will do this for $g=3$, using our calculations of the sets $\mathrm{Flag}_{w,6}^{\perp,F,V}$. The sets $\mathcal{L}(x,\mathrm{id})$ are rather complicated and we content ourselves with determining whether they are nonempty. For the other final elements w of p-rank 0 we are able to determine the sets $\mathcal{L}(x,w)$ completely.

First we have the following general result:

Lemma 15.1. *Let $g \geq 1$.*

(1) *For $\omega \in S_g = \langle s_1, \ldots, s_{g-1}\rangle \subset W$ we have $\omega\tau \in \mathrm{Adm}(\mu)^{(0)}$ and* $\mathbf{ES}(\omega\tau) = \mathrm{id}$.
(2) *For $x = t^\lambda\omega \in \mathrm{Adm}(\mu)^{(0)}$ ($\lambda \in X_*(T)$, $\omega \in W$) we write $N_x = \{i \in \{1,\ldots,2g\} \mid \omega^2(i) < \omega(i) < i\}$. Then for $(A_i)_i \in \mathcal{A}_{I,x}(\mathbb{F})$ we have $g - a(A_0) \geq \#N_x$.*

Proof. (1) Let $\omega \in S_g$, then $\omega\tau$ is admissible by Lemma 14.5 above. Consider $(A_i)_i \in \mathcal{A}_{I,\omega\tau}(\mathbb{F})$ with image $(W_i)_i$ under ι_{A_0}. $\omega\tau$ satisfies $\xi(\omega\tau)(\{g+1,\ldots,2g\}) = \{1,\ldots,g\}$, which means that $\mathrm{im}\, V_{w_{A_0}} = \ker V_{w_{A_0}} = W_g$. By Proposition 3.21 this implies that $\mathrm{im}\, F_{w_{A_0}} = \ker F_{w_{A_0}}$, hence the canonical filtration on $\mathbb{D}(A_0)$ is given by $0 \subset F(\mathbb{D}) \subset \mathbb{D}$ which has associated final element id.

(2) By Lemma 4.4 and Proposition 4.6 our claim is equivalent to the following statement: Let $w \in W_{\mathrm{final}}^{(0)}$ and $(W_i)_{i=0}^{2g} \in \mathrm{Flag}_{w,2g}^{\perp,F,V}(\mathbb{F})$ of KR type x. Then $\dim \operatorname{im} V_w^2 \geq \# N_x$.

But if $(\varepsilon)_{i=0}^{2g}$ is a KR basis for $(W_i)_i$, the set $\{V(\varepsilon_{\omega(i)}) \mid i \in N_x\}$ is a linearly independent subset of $\operatorname{im} V_w^2$ of cardinality $\# N_x$. □

For the rest of this section g is equal to 3.

15.2. $\boldsymbol{w = \mathrm{id}}$. Let $x \in \mathrm{Adm}(\mu)^{(0)}$. By Lemma 15.1(2) we know that $\mathcal{L}(x, \mathrm{id}) \neq \varnothing$ implies that $N_x = \varnothing$. Inspecting Table 14.1 we see that this condition is only satisfied for $x \in \{\tau, s_1\tau, s_2\tau, s_{21}\tau, s_{12}\tau, s_{121}\tau, s_{30}\tau, s_{310}\tau, s_{320}\tau, s_{3120}\tau, s_{2301}\tau\}$. We claim that $\mathrm{id} \in \mathbf{ES}(x)$ for all x in this set. For $x \in S_3\tau = \{\tau, s_1\tau, s_2\tau, s_{21}\tau, s_{12}\tau, s_{121}\tau\}$ this is true by Lemma 15.1(1). For the remaining elements we write down an explicit nonempty subset $\mathcal{K}(x) \subset \mathcal{L}(x, \mathrm{id})$ in Table 15.1.

$\mathcal{K}(s_{30}\tau)$	$\mathcal{K}(s_{3120}\tau)$	$\mathcal{K}(s_{2301}\tau)$
$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$

$\mathcal{K}(s_{310}\tau)$	$\mathcal{K}(s_{320}\tau)$	
$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & b_2 \\ 0 & 0 & b_2^{p+1} + b_1 \\ b_1 & b_2 & \alpha \\ -b_2^p & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & b_2 \\ 0 & 0 & b_2^{p^{-1}} b_2 + b_1 \\ b_1 & b_2 & \alpha \\ -b_2^{p^{-1}} & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	
$b_2 \in \mathbb{F} - \mathbb{F}_{p^2},\ \alpha \in \mathbb{F}$ $b_1 \in \mathbb{F}$ a root of $T^p + T + b_2^{p(p+1)} \in \mathbb{F}[T]$	$b_2 \in \mathbb{F} - \mathbb{F}_{p^2},\ \alpha \in \mathbb{F}$ $b_1 \in \mathbb{F}$ a root of $T^{p^2} + T^p + b_2^{p+1} \in \mathbb{F}[T]$	
2		

TABLE 15.1. Nonempty subsets of $\mathcal{L}(x, \mathrm{id})$.

15.3. $\boldsymbol{w = s_3}$. In Table 15.2 list those $\mathcal{L}(x, s_3)$ which are nonempty.

$\mathcal{L}(s_{120}\tau, s_3)$	$\mathcal{L}(s_{3120}\tau, s_3)$	$\mathcal{L}(s_{312}\tau, s_3)$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ c & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ c & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ c & 1 & 0 \end{pmatrix}$
$c \in \mathbb{F}$	$c \in \mathbb{F},\ \alpha \in \mathbb{F}^\times$	$c \in \mathbb{F}$
1	2	1

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$\mathcal{L}(s_{201}\tau, s_3)$	$\mathcal{L}(s_{2301}\tau, s_3)$	$\mathcal{L}(s_{231}\tau, s_3)$
$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\0&1&0\\0&c&1\\1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&a&0\\0&0&0\\0&0&0\\0&1&0\\0&c&1\\1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&1&0\\0&0&0\\0&0&0\\0&0&0\\0&c&1\\1&0&0 \end{pmatrix}$
$c \in \mathbb{F}$	$c \in \mathbb{F},\ a \in \mathbb{F}^\times$	$c \in \mathbb{F}$
1	2	1

$\mathcal{L}(s_{30}\tau, s_3)$	$\mathcal{L}(s_{0}\tau, s_3)$	$\mathcal{L}(s_{3}\tau, s_3)$
$\begin{pmatrix} 0&0&\alpha\\0&0&0\\0&0&0\\0&0&1\\0&1&0\\1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\0&0&1\\0&1&0\\1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\0&0&0\\0&0&0\\0&0&0\\0&1&0\\1&0&0 \end{pmatrix}$
$\alpha \in \mathbb{F}^\times$		
1	0	0

TABLE 15.2. The sets $\mathcal{L}(x, s_3)$.

15.4. $\boldsymbol{w = s_{23}}$**.** In Table 15.3 list those $\mathcal{L}(x, s_{23})$ which are nonempty.

$\mathcal{L}(s_{20}\tau, s_{23})$	$\mathcal{L}(s_{320}\tau, s_{23})$	$\mathcal{L}(s_{120}\tau, s_{23})$	$\mathcal{L}(s_{3120}\tau, s_{23})$	$\mathcal{L}(s_{312}\tau, s_{23})$
$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&\alpha\\0&0&0\\0&0&0\\0&0&1\\1&0&0\\0&1&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\0&0&1\\1&0&0\\c&1&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&\alpha\\0&0&0\\0&0&0\\0&0&1\\1&0&0\\c&1&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\0&0&0\\0&0&0\\0&0&0\\1&0&0\\c&1&0 \end{pmatrix}$
	$\alpha \in \mathbb{F}^\times$	$c \in \mathbb{F}^\times$	$c, \alpha \in \mathbb{F}^\times$	$c \in \mathbb{F}^\times$
0	1	1	2	1

$\mathcal{L}(s_{32}\tau, s_{23})$	$\mathcal{L}(s_{310}\tau, s_{23})$	$\mathcal{L}(s_{10}\tau, s_{23})$	$\mathcal{L}(s_{31}\tau, s_{23})$	$\mathcal{L}(s_{01}\tau, s_{23})$
$\begin{pmatrix} 0&0&1\\0&0&0\\0&0&0\\0&0&0\\1&0&0\\0&1&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&\alpha\\0&0&0\\0&0&0\\0&0&1\\0&1&0\\1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\0&0&1\\0&1&0\\1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&1\\0&0&0\\0&0&0\\0&0&0\\0&1&0\\1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\0&1&0\\1&0&0\\0&0&1 \end{pmatrix}$
	$\alpha \in \mathbb{F}^\times$			
0	1	0	0	0

$\mathcal{L}(s_{201}\tau, s_{23})$	$\mathcal{L}(s_{23}\tau, s_{23})$	$\mathcal{L}(s_{231}\tau, s_{23})$	$\mathcal{L}(s_{3230}\tau, s_{23})$	$\mathcal{L}(s_{3010}\tau, s_{23})$
$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&0\\0&1&0\\1&0&0\\0&c&1 \end{pmatrix}$	$\begin{pmatrix} 0&1&0\\0&0&0\\0&0&0\\0&0&0\\0&0&1\\1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&1&0\\0&0&0\\0&0&0\\0&0&0\\0&c&1\\1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&1&0\\0&0&1\\0&0&0\\0&0&0\\0&0&\beta\\1&0&0 \end{pmatrix}$	$\begin{pmatrix} 0&0&0\\0&0&0\\0&0&1\\0&1&0\\1&0&0\\0&0&\beta \end{pmatrix}$
$c \in \mathbb{F}^\times$		$c \in \mathbb{F}^\times$	$\beta \in \mathbb{F}$	$\beta \in \mathbb{F}$
1	0	1	1	1

TABLE 15.3. The sets $\mathcal{L}(x, s_{23})$.

15.5. $\boldsymbol{w = s_{323}}$. In Table 15.4 list those $\mathcal{L}(x, s_{323})$ which are nonempty.

$\mathcal{L}(s_{3230}\tau, s_{323})$	$\mathcal{L}(s_{323}\tau, s_{323})$	$\mathcal{L}(s_{230}\tau, s_{323})$	$\mathcal{L}(s_{3010}\tau, s_{323})$
$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \beta \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$\beta \in \mathbb{F}^\times$			$\alpha \in \mathbb{F}^\times$
1	0	0	1

$\mathcal{L}(s_{010}\tau, s_{323})$	$\mathcal{L}(s_{301}\tau, s_{323})$	$\mathcal{L}(s_{2301}\tau, s_{323})$	
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & a & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$	
		$a \in \mathbb{F}^\times$	
0	0	1	

TABLE 15.4. The sets $\mathcal{L}(x, s_{323})$.

16. Proof of the results of Section 15

In order to illustrate the method we show that $\mathcal{K}(s_{310}\tau) \subset \mathcal{L}(s_{310}\tau, \mathrm{id})$. Let $V = V_{\mathrm{id}}$. For an element $(W_i)_{i=0}^6$ of $\mathcal{K}(s_{310}\tau)(\mathbb{F})$ choose elements $b_2 \in \mathbb{F} - \mathbb{F}_{p^2}$, $\alpha \in \mathbb{F}$ and $b_1 \in \mathbb{F}$ with $b_1^p + b_1 + b_2^{p(p+1)} = 0$ such that

$$(*) \qquad (W_i)_{i=0}^6 = \Phi \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & b_2 \\ 0 & 0 & b_2^{p+1} + b_1 \\ b_1 & b_2 & \alpha \\ -b_2^p & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

First we write down a matrix $C = (c_1 c_2 c_3 c_4 c_5 c_6) \in \mathrm{GL}_6(\mathbb{F})$ such that $W_i = \oplus_{j=1}^i \mathbb{F} \cdot c_j$ for all $0 \leq i \leq 6$. For the first three columns of C we can use the columns of the matrix of equation $(*)$ above. We find the other columns using the condition that $(W_i)_{i=0}^6$ is a symplectic flag, meaning that $c_4 \perp c_1, c_2$ and $c_5 \perp c_1$. Hence

$$C = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & b_2 & 0 & 1 & 0 \\ 0 & 0 & b_2^{p+1} + b_1 & 0 & b_2^p & 1 \\ b_1 & b_2 & \alpha & 1 & 0 & 0 \\ -b_2^p & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

satisfies our requirements.

From this matrix we can read off the images $(V(W_i))_{i=0}^6$ using the explicit description of Section 4.9 and we need to find a basis $(\varepsilon_i)_{i=1}^6$ of $\mathbb{F}^6$ such that $W_i = \oplus_{j=1}^i \mathbb{F} \cdot \varepsilon_j$ and such that $V(W_i)$ is spanned by a subset of $\{\varepsilon_1, \ldots, \varepsilon_i\}$

for each $0 \le i \le 6$. First we have $V(W_2) = 0$. Now $b_2 \notin \mathbb{F}_{p^2}$ implies that $V(W_3) \not\subseteq W_1$ and hence we can take $\varepsilon_2 = V(c_3)$. The equation for b_1 implies that $V(W_5) \subset W_2$ which means that we can use $\varepsilon_1 = c_1$ and $\varepsilon_4 = c_4$.

This means that a KR basis $(\varepsilon_i)_{i=1}^6$ of (W_i) is given by the columns of the following matrix

$$\varepsilon = \begin{pmatrix} 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & b_2 & 0 & 1 & 0 \\ 0 & 0 & b_2^{p+1} + b_1 & 0 & b_2^p & 1 \\ b_1 & b_1^{p^{-2}} & \alpha & 1 & 0 & 0 \\ -b_2^p & -b_2^{p^{-1}} & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here the equation for b_1 is needed to see that $V(\varepsilon_3) = \varepsilon_2$. We have observed that $V(W_1) = V(W_2) = 0$, $V(W_3) = V(W_4) = \langle \varepsilon_2 \rangle$, $V(W_5) = \langle \varepsilon_1, \varepsilon_2 \rangle$ and $V(W_6) = \langle \varepsilon_1, \varepsilon_2, \varepsilon_4 \rangle$. Hence if $\lambda \in X_*(T)$ and $\omega \in W$ are such that $(W_i)_i \in \mathcal{L}(t^\lambda \omega, \mathrm{id})$, we see that $\lambda = (0,0,1,0,1,1)$ and that $\omega(3) = 2$, $\omega(5) = 1$ and $\omega(6) = 4$. The $\omega \in W$ satisfying these conditions is given by

$$\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 2 & 5 & 1 & 4 \end{pmatrix}$$

and from Table 14.1 we see that $t^\lambda \omega = s_{310}\tau$.

The proof of $\mathcal{K}(s_{320}\tau) \subset \mathcal{L}(s_{320}\tau, \mathrm{id})$ is similar and in all the other cases it is very easy to write down a suitable KR basis.

17. The sets $\mathbf{ES}(x)$ for $x \in \mathrm{Adm}(\mu)^{(0)}$ in dimensions 2 and 3

Let $g = 2$. In this case the sets $\mathbf{ES}(x)$ for $x \in \mathrm{Adm}(\mu)^{(0)}$ have already been known, see for instance [7, Ex. 3.4]. We list them in Table 17.1.

x	$\mathbf{ES}(x)$
$\tau, s_1\tau$	id
$s_2\tau,\ s_0\tau$	s_2
$s_{20}\tau$	id s_2

Table 17.1. The sets $\mathbf{ES}(x)$ for $g = 2$.

Let $g = 3$. Table 17.2 contains the sets $\mathbf{ES}(x)$ for $x \in \mathrm{Adm}(\mu)^{(0)}$. They can be read off the calculations in Section 15. The upper block contains the supersingular elements.

x	$\mathbf{ES}(x)$
$\tau, s_1\tau, s_2\tau, s_{21}\tau, s_{12}\tau, s_{121}\tau$	id
$s_3\tau, s_0\tau$	s_3
$s_{30}\tau$	id s_3

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$s_{10}\mathcal{T}, s_{23}\mathcal{T}, s_{20}\mathcal{T}, s_{31}\mathcal{T}, s_{01}\mathcal{T}, s_{32}\mathcal{T}$			s_{23}	
$s_{310}\mathcal{T}, s_{320}\mathcal{T}$	id		s_{23}	
$s_{3120}\mathcal{T}$	id	s_3	s_{23}	
$s_{120}\mathcal{T}, s_{312}\mathcal{T}, s_{201}\mathcal{T}, s_{231}\mathcal{T}$		s_3	s_{23}	
$s_{010}\mathcal{T}, s_{323}\mathcal{T}, s_{301}\mathcal{T}, s_{230}\mathcal{T}$				s_{323}
$s_{2301}\mathcal{T}$	id	s_3		s_{323}
$s_{3010}\mathcal{T}, s_{3230}\mathcal{T}$			s_{23}	s_{323}

TABLE 17.2. The sets $\mathbf{ES}(x)$ for $g = 3$.

Remark 17.1. We can use Table 17.2 to answer a question posed in a preliminary version of [8]: For every $g \geq 1$ one has the following inclusion:

$$\coprod_{\substack{x \in \mathrm{Adm}(\mu)^{(0)} \\ \mathcal{A}_{I,x} \subset \mathcal{S}_I}} \mathcal{A}_{I,x} \subseteq \pi^{-1}\left(\coprod_{\substack{w \in W_{\mathrm{final}} \\ EO_w \subset \mathcal{S}_g}} EO_w \right). \tag{17.1}$$

In *loc.cit.* it was asked whether this inclusion is an equality. The answer is negative in the case $g = 3$: Let $A \in EO_{\mathrm{id}}(\mathbb{F})$. By Table 17.2 there is preimage $(A_i)_i \in (\pi^{-1}(A) \cap \mathcal{A}_{I,s_{310}\mathcal{T}})(\mathbb{F})$ of A, so that $(A_i)_i$ is contained in the right hand side of the above inclusion, but as $\mathcal{A}_{I,s_{310}\mathcal{T}} \nsubseteq \mathcal{S}_I$ it is not contained in the left hand side.

17.2. **Some informal observations.** Let $1 \leq g \leq 3$. It is interesting to note that $\mathbf{ES}\left(\xi^{-1}(\xi(x)^{-1})\right) = \mathbf{ES}(x)$ for every $x \in \mathrm{Adm}(\mu)^{(0)}$. We do not know if this is true for arbitrary g.

Compare the line

x	$\mathbf{ES}(x)$
$s_2\mathcal{T},\ s_0\mathcal{T}$	s_2

of Table 17.1 with the lines

x	$\mathbf{ES}(x)$
$s_3\mathcal{T}, s_0\mathcal{T}$	s_3
$s_{10}\mathcal{T}, s_{23}\mathcal{T}, s_{20}\mathcal{T}, s_{31}\mathcal{T}, s_{01}\mathcal{T}, s_{32}\mathcal{T}$	s_{23}

of Table 17.2. We can prove the following result, generalizing these lines: Let $2 \leq g$, $i \in \{0, 1\}$ and consider the sets $S_j = \{s_j, s_{g-j}\} \subset W$ for $0 \leq j \leq i$. Then for every element $x \in \mathrm{Adm}(\mu)^{(0)}$ of the form $\left(t_{\upsilon(0)} \cdot t_{\upsilon(1)} \cdots t_{\upsilon(i)}\right)\tau$, where $\upsilon \in S(\{0, \dots, i\})$ and $t_j \in S_j$ for all $0 \leq j \leq i$, we have $\mathbf{ES}(x) = \{s_{g-i} \cdot s_{g-i+1} \cdots s_g\}$.

18. The KR stratification and the supersingular locus

Let $g = 3$. For $x \in \mathrm{Adm}(\mu)^{(0)}$ we want to understand the intersection $\mathcal{A}_{I,x} \cap S_I$.

Proposition 18.1. *Let $g = 3$ and $x \in \mathrm{Adm}(\mu)^{(0)}$. Then $\mathcal{A}_{I,x} \cap \mathcal{S}_I = \varnothing \Leftrightarrow$ $\mathbf{ES}(x) = \{s_{23}\}$.*

Proof. This follows from the results of Section 5, using equation (15.1). □

Remark 18.2. The relationship between the KR stratification and the supersingular locus is closely related to the theory of affine Deligne-Lusztig varieties. In [10, Prop. 12.6], Haines shows that for $x \in \mathrm{Adm}(\mu)^{(0)}$ the nonemptiness of the intersection $\mathcal{A}_{I,x} \cap \mathcal{S}_I$ is equivalent to the nonemptiness of a certain affine Deligne-Lusztig variety. In *loc.cit.* this result is stated using p-adic Deligne-Lusztig varieties, but by [4, Cor. 11.3.5] the nonemptiness of an affine Deligne-Lusztig variety is equivalent in the function field and the p-adic case.

We want to get a more precise statement about the intersection $\mathcal{A}_{I,x} \cap \mathcal{S}_I$ in those cases where it is nonempty and not equal to $\mathcal{A}_{I,x}$. For this we need the following

Proposition 18.3. *Let $f : X \to Y$ be a proper morphism of algebraic varieties over an algebraically closed field K. Let $B \subset Y$ be a locally closed subset equidimensional of dimension $d \in \mathbb{N}$. Let $A \subset X$ be a locally closed subset with the property that there is a natural number $e \in \mathbb{N}$ such that $f^{-1}(b) \cap A$ is irreducible of dimension e and dense in $f^{-1}(b) \cap \overline{A}$ for every $b \in B(K)$, where we denote by $\overline{A}$ the closure of A in X. Then $f^{-1}(B) \cap A$ is equidimensional of dimension $d + e$. Furthermore the number of irreducible components of $f^{-1}(B) \cap A$ is equal to the number of irreducible components of B.*

Proof. We immediately reduce to the case $B = Y$ and $A = X$. We may also assume Y to be irreducible. Hence we are reduced to the statement of the following Lemma whose proof we include for lack of reference. □

Lemma 18.4. *Let $f : X \to Y$ be proper morphism of algebraic varieties over an algebraically closed field K. Assume that Y is irreducible of dimension $d \in \mathbb{N}$ and that the fiber $f^{-1}(y)$ is irreducible of dimension $e \in \mathbb{N}$ for every $y \in Y(K)$. Then X is irreducible of dimension $d + e$.*

Proof. Let $X = C_1 \cup C_2$ with closed subsets $C_1, C_2 \subset X$. Let f_1 and f_2 denote the restrictions of f to C_1 and C_2 respectively. By a Corollary to Chevalley's Theorem, see [9, 13.1.5], the sets $F_i = \{y \in Y \mid \dim f_i^{-1}(y) \geq e\}$ are closed subsets of Y, $i = 1, 2$. For $y \in Y(K)$ the e-dimensional fiber $f^{-1}(y)$ is the union of the closed subsets $f_1^{-1}(y)$ and $f_2^{-1}(y)$ and hence $y \in F_1 \cup F_2$. As the set $Y(K)$ is dense in Y this implies that $Y = F_1 \cup F_2$ and by the irreducibility of Y we may assume that $F_1 = Y$. Let $x \in X(K)$ be a closed point with image $y = f(x) \in Y(K)$. Then $\dim f_1^{-1}(y) = \dim f^{-1}(y) \cap C_1 = e$ and as $f^{-1}(y) \cap C_1$ is a closed subset of $f^{-1}(y)$ and the latter is irreducible of dimension e, we get $f^{-1}(y) \cap C_1 = f^{-1}(y)$ and hence $C_1 \supset f^{-1}(y) \ni x$. This implies that $C_1 = X$ and we see that X is irreducible. Furthermore the closed subset $\{y \in Y \mid \dim f^{-1}(y) \geq e+1\}$ of Y does not contain a point of $Y(K)$, hence it is empty. This means that $\dim f^{-1}(y) = e$ for all $y \in Y$. We can now apply [9, 10.6.1(iii)] to get the result. □

We are now ready to determine the dimension of the intersection $\mathcal{A}_{I,x} \cap \mathcal{S}_I$ for those $x \in \mathrm{Adm}(\mu)^{(0)}$ with $\mathcal{A}_{I,x} \not\subseteq \mathcal{S}_I$. It is clear a priori that for any such x we have $\dim \mathcal{A}_{I,x} \cap \mathcal{S}_I \leq \dim \mathcal{A}_{I,x} - 1 = \ell(x) - 1$ as this intersection is a proper closed subset of the irreducible space $\mathcal{A}_{I,x}$, see Proposition 14.4(4). Table 18.1 shows that this inequality is in fact an equality for $g = 3$.

x	$\dim \mathcal{A}_{I,x} \cap \mathcal{S}_I$	equidimensional?
$s_{310}\tau,\ s_{320}\tau$	2	?
$s_{3120}\tau$	3	?
$s_{2301}\tau$	3	?
$s_{120}\tau, s_{312}\tau, s_{201}\tau, s_{231}\tau$	2	$\surd$
$s_{010}\tau, s_{323}\tau, s_{301}\tau, s_{230}\tau$	2	$\surd$
$s_{3010}\tau, s_{3230}\tau$	3	$\surd$

TABLE 18.1. The intersections of KR strata with the supersingular locus for $g = 3$.

18.5. **Proof.** For $x \in \{s_{120}\tau, s_{312}\tau, s_{201}\tau, s_{231}\tau\}$ we apply Proposition 18.3 for π with $B = EO_{s_3}$ and $A = \mathcal{A}_{I,x}$. It is clear from the results of Section 15.3 that the conditions on the fibers (appearing in Proposition 18.3) are indeed satisfied. For example we have

$$\overline{\mathcal{A}_{I,s_{120}\tau}} = \mathcal{A}_{I,s_{120}\tau} \cup \mathcal{A}_{I,s_{12}\tau} \cup \mathcal{A}_{I,s_{20}\tau} \cup \mathcal{A}_{I,s_{10}\tau} \cup \mathcal{A}_{I,s_{1}\tau} \cup \mathcal{A}_{I,s_{2}\tau} \cup \mathcal{A}_{I,s_{0}\tau} \cup \mathcal{A}_{I,\tau},$$

hence we see from Section 15.3 that for $b \in EO_{s_3}(\mathbb{F})$ we have

$$\overline{\mathcal{A}_{I,s_{120}\tau}} \cap \pi^{-1}(b) = \mathcal{A}_{I,s_{120}\tau} \cap \pi^{-1}(b) \cup \mathcal{A}_{I,s_0\tau} \cap \pi^{-1}(b).$$

From the results of Section 15.3 we can deduce the content of Table 18.2.

$\iota_b(\pi^{-1}(b) \cap \mathcal{A}_{I,s_{120}\tau})$	$\iota_b(\pi^{-1}(b) \cap \overline{\mathcal{A}_{I,s_{120}\tau}})$
$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ c & 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ c & 1 & 0 \end{pmatrix} \vee \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$
$c \in \mathbb{F}$	

TABLE 18.2

Hence $\pi^{-1}(b) \cap \mathcal{A}_{I,s_{120}\tau}$ is irreducible of dimension 1 and dense in $\pi^{-1}(b) \cap \overline{\mathcal{A}_{I,s_{120}\tau}}$. As $\pi^{-1}(EO_{s_3}) \cap \mathcal{A}_{I,x} = \mathcal{S}_I \cap \mathcal{A}_{I,x}$ by the results of Section 5 and Table 17.2, our claim follows.

For $x \in \{s_{010}\tau, s_{323}\tau, s_{301}\tau, s_{230}\tau, s_{3010}\tau, s_{3230}\tau\}$ we apply Proposition 18.3 for π with $B = \mathcal{S}_3 \cap EO_{s_{323}}$ and $A = \mathcal{A}_{I,x}$. It follows from Section 15.5 that the conditions on the fibers are indeed satisfied and we have $\pi^{-1}(EO_{s_{323}} \cap \mathcal{S}_3) \cap \mathcal{A}_{I,x} = \mathcal{S}_I \cap \mathcal{A}_{I,x}$ by the results of Section 5 and Table 17.2.

Furthermore we have $\pi^{-1}(EO_{s_3}) \cap \mathcal{A}_{I,s_{3120}\tau} \subset \mathcal{S}_I \cap \mathcal{A}_{I,s_{3120}\tau}$ and $\pi^{-1}(EO_{s_{323}} \cap \mathcal{S}_3) \cap \mathcal{A}_{I,s_{2301}\tau} \subset \mathcal{S}_I \cap \mathcal{A}_{I,s_{2301}\tau}$ and we use Proposition 18.3 and the results of Section 15.3 and 15.5, respectively, to see that these subsets have dimension 3.

Finally let $A \in EO_{\mathrm{id}}(\mathbb{F})$. Then $\pi^{-1}(A) \cap \mathcal{A}_{I,s_{310}\tau} \subset \mathcal{S}_I \cap \mathcal{A}_{I,s_{310}\tau}$ and $\pi^{-1}(A) \cap \mathcal{A}_{I,s_{320}\tau} \subset \mathcal{S}_I \cap \mathcal{A}_{I,s_{320}\tau}$. But these subsets have dimension at least 2 because $\dim \mathcal{K}(s_{310}\tau) = \dim \mathcal{K}(s_{320}\tau) = 2$ (see Section 15.2).

Remark 18.6. If g is even it is shown in [7, Prop. 8.12] that every top-dimensional irreducible component of $\mathcal{S}_I$ is an irreducible component of the left hand side of equation (17.1). Looking at Table 18.1 we see that the corresponding statement is not true for $g = 3$, as $\dim S_I = 3$ in this case.

Remark 18.7. It is strongly expected that the relationship mentioned in Remark 18.2 extends to other properties of the intersection $\mathcal{A}_{I,x} \cap \mathcal{S}_I$, $x \in \mathrm{Adm}(\mu)^{(0)}$. In particular strong evidence suggests that $\mathcal{A}_{I,x} \cap \mathcal{S}_I$ is equidimensional of dimension n if and only if the corresponding affine Deligne-Lusztig variety (in the function field case) is equidimensional of dimension n, $n \in \mathbb{N}$. In [5], Görtz and He explain a reduction method for affine Deligne-Lusztig varieties over function fields which is completely analogous to the classical reduction method by Deligne and Lusztig. Using this reduction method one sees that the affine Deligne-Lusztig varieties corresponding to the intersections $\mathcal{A}_{I,x} \cap \mathcal{S}_I$ for $g = 3$ and $x \in \{s_{310}\tau, s_{320}\tau, s_{3120}\tau, s_{2301}\tau\}$ are equidimensional. Hence we expect that the same is true for the intersections themselves.

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