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# Axiomatic homology and duality revisited

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**Abstract.** We associate to each homology theory in an elementary and canonical manner a tautological cohomology theory on Cartesian spaces such that the classical Alexander duality holds. The duality isomorphisms obtained from cap-products yield an isomorphism of cohomology theories. Guided by our methods we also introduce the new category of dualizible maps.

## 1. Introduction

Alexander duality asserts an isomorphism of the type

$$h^k(X, A) \cong h_{n-k}(\mathbb{R}^n \setminus A, \mathbb{R}^n \setminus X)$$

between homology and cohomology groups for (suitable) pairs of spaces $A \subset X \subset \mathbb{R}^n$. A statement of this type is only sensible if the cohomology $h^*$ and the homology $h_*$ are related in a specific way. If the spaces $X$ and $A$ are closed subsets of the Euclidean space, then their complements are relatively harmless, being open subsets of a Euclidean space. But closed subsets can be quite complicated and this requires certain continuity properties for the cohomology theory, in order that Alexander duality holds in that generality.

Starting with a homology theory we construct a cohomology theory such that Alexander duality holds by definition (see Theorem 4.1). Our approach is elementary, natural and canonical. No advanced techniques like stable homotopy or the notion of spectra are needed. Our method also adds insight to the classical theory: The duality isomorphisms obtained from cap-products turn out to be an isomorphism of cohomology theories (see Theorem 5.1).

Dually, we could start with a cohomology theory and obtain a dual version of Alexander duality $h_k(X, A) \cong h^{n-k}(\mathbb{R}^n \setminus A, \mathbb{R}^n \setminus X)$.

The presentation of our theory has text book style and can be used in a topology course. Standard results are not included in this paper, and I refer the reader to the text books [3] and [2]. The book [3] contains a definite treatment of duality for manifolds based on singular homology and cohomology.

## 2. Tautological cohomology

The cohomology theories we are going to construct are defined on a somewhat nonstandard category: The category of embedded Cartesian spaces and proper maps with an associated notion of proper homotopy. We begin by explaining these terms.

The category **EC** of embedded Cartesian spaces has as objects the pairs $(\mathbb{R}^n, X)$ with $X$ a closed subset of $\mathbb{R}^n$. A morphism $f$ from $(\mathbb{R}^n, X)$ to $(\mathbb{R}^m, Y)$ is a proper map $f : X \to Y$. Composition is the composition of continuous maps. In a similar manner we have the category $\mathbf{EC(2)}$ of triples $(\mathbb{R}^n, X, A)$ with closed sets $A \subset X \subset \mathbb{R}^n$ and proper maps of pairs $(X, A) \to (Y, B)$. Note that if $X \to Y$ is proper, then so is the restriction $A \to B$. In both categories we use the notion of proper homotopy as explained below. In some cases we use the subcategory with objects $(\mathbb{R}^n, X)$ for compact $X$.

A proper homotopy is a proper map $h : X \times I \to Y$ (as usual $I = [0,1]$). The constant homotopy of a proper map is a proper homotopy. The inverse of a proper homotopy is a proper homotopy. The usual concatenation of proper homotopies is a proper homotopy. If $f_1, f_2 : X \to Y$ and $g_1, g_2 : Y \to Z$ are properly homotopic, then $g_1 f_1$ and $g_2 f_2$ are properly homotopic. The embedding $j_t : X \to X \times I$, $x \mapsto (x,t)$ is proper; hence, if $h$ is a proper homotopy, then the partial maps $h_t = h \circ j_t$ are proper. Being properly homotopic is compatible with products. All this holds for maps between arbitrary spaces if one uses the notion of [1] for proper maps. In the case of **EC** we view $X \times I \subset \mathbb{R}^n \times \mathbb{R}$ if $X \subset \mathbb{R}^n$.

The homotopy $h : \mathbb{R} \times I \to \mathbb{R}$, $(x,t) \mapsto tx^2 + x$ is not proper, since the preimage of 0 is not compact. The partial maps $h_t$ are proper because a nonconstant polynomial is a proper map.

Let $X, Y$ be locally compact Hausdorff spaces, and denote by $X^+ = X \cup \{\infty\}$ the one-point compactification of $X$. The canonical map $q : X^+ \times I \to (X \times I)^+$ is a quotient map. A proper homotopy $h : X \times I \to Y$ induces a continuous pointed map $h^+ : (X \times I)^+ \to Y^+$ and therefore a pointed homotopy $h^+ \circ q : X^+ \times I \to Y^+$.

Let $h : \mathbb{R}^n \times I \to \mathbb{R}^n$ be a homotopy such that each partial map $h_t$ is a homeomorphism. Then $h$ is a proper homotopy. For the proof consider $(x,t) \mapsto (h(x,t), t)$ and apply the open mapping theorem [2, 10.3.7].

**Definition 2.1** (Cohomology)**.** A cohomology theory on **EC** consists, as usual, of the data:

(1) A family of contravariant functors $h^k : \mathbf{EC(2)} \to R$-MOD, $k \in \mathbb{Z}$.
(2) A family of natural transformations (coboundary operators) $\delta : h^{k-1} \circ \iota \to h^k$ with $\iota$ the usual transformation, $\iota(X, A) = (A, \varnothing)$ on objects.

Here $R$-MOD is the category of modules over a commutative ring $R$. These data are assumed to satisfy the following version of the axioms of Eilenberg and Steenrod. The value of $h^k$ on $(\mathbb{R}^n, X, A)$ will be denoted $h^k(X, A; n)$, and $h^k(A; n) = h^k(A, \varnothing; n)$ as usual.

**Exactness.** For each pair of closed subsets $(X,A)$ in $\mathbb{R}^n$ the cohomology sequence $\ldots \to h^{k-1}(A;n) \xrightarrow{\delta} h^k(X,A;n) \to h^k(X;n) \to \ldots$ is exact.
**Excision.** Let $A$ and $B$ be closed subsets of $\mathbb{R}^n$. Then the inclusion induces an isomorphism $h^k(A \cup B, A; n) \cong h^k(B, A \cap B; n)$.
**Homotopy invariance.** Let $\varphi : (X,A) \times I \to (Y,B)$ be a proper homotopy. Then the induced morphisms $h^k(\varphi_t)$ do not depend on $t$. ♢

The use of embedded spaces is a convenient technical device for our constructions. A cohomology theory on **EC** induces in a canonical formal way a cohomology theory on the category **C** of Cartesian spaces and proper maps. We explain this in Section 6.

Note that the excision axiom does not use a condition "excisive". We work with proper maps and proper homotopies. In other settings these cohomology groups are said to have compact support, see [3, VIII.6.22], [6], [7].

In the next definition we start with the same data $h^k$ and $\delta$ as in Definition 2.1, but we require different properties for them. We will show in 2.3 that the axioms of Eilenberg and Steenrod are implied by the axioms (1)–(3) in 2.2. This then justifies the term "cohomology". The term "tautological" refers to the fact that Alexander duality holds by definition.

**Definition 2.2** (Tautological cohomology)**.** Let $h_*$ be a homology theory. A tautological cohomology theory associated to $h_*$ consists of a family of contravariant functors $h^k : \mathbf{EC(2)} \to R\text{-}\mathrm{MOD}$, $k \in \mathbb{Z}$ and a family of natural transformations $\delta : h^{k-1} \circ \iota \to h^k$ as in 2.1 with the properties:

(1) $h^k$ assigns to $(\mathbb{R}^n, X, A)$ the module

$$h^k(X,A;n) = h_{n-k}(\mathbb{R}^n \setminus A, \mathbb{R}^n \setminus X).$$

(2) $h^k$ assigns to an inclusion $(X,A) \xrightarrow{\subset} (Y,B)$ of closed subspaces of $\mathbb{R}^n$ the morphism $h_{n-k}(\mathbb{R}^n \setminus B, \mathbb{R}^n \setminus Y) \to h_{n-k}(\mathbb{R}^n \setminus A, \mathbb{R}^n \setminus X)$ induced by the dual inclusion.
(3) The coboundary operator $\delta : h^{k-1}(A, \varnothing; n) \to h^k(X,A;n)$ is defined as $(-1)^n \partial : h_{n-k+1}(\mathbb{R}^n, \mathbb{R}^n \setminus A) \to h_{n-k}(\mathbb{R}^n \setminus A, \mathbb{R}^n \setminus X)$ in terms of the given boundary operator $\partial$ of the theory $h_*$. ♢

Note that the functors $h^k$ and the coboundary transformations $\delta$ in 2.2 are already given on objects. Moreover, the values of the functors are determined on the subcategory of inclusions. By naturality of $\partial$, the axioms (2) and (3) are compatible on this subcategory. For the sign in the definition of $\delta$ see the proof of 4.1. In the construction of a tautological theory we thus have to extend the functors $h^k$ to $\mathbf{EC(2)}$ and to verify the naturality of $\delta$ on this larger category.

**Proposition 2.3.** *A contravariant functor $h^k$ on $\mathbf{EC(2)}$ with the properties (1) and (2) of Definition 2.2 is homotopy invariant and satisfies excision. A tautological cohomology theory is a cohomology theory in the sense of Definition 2.1.*

*Proof. Excision.* The corresponding map in homology

$$h_{n-k}(\mathbb{R}^n \setminus A, \mathbb{R}^n \setminus (A \cup B)) \to h_{n-k}(\mathbb{R}^n \setminus (A \cap B), \mathbb{R}^n \setminus B)$$

is an excision isomorphism, since we are working with open subsets of Euclidean spaces (compare [2, 10.7.5,10.7.6]).

*Homotopy invariance.* The map $j_t$ is the composition of the homeomorphism $a : X \to X \times \{t\}$ and the inclusion $b : X \times \{t\} \subset X \times I$. Therefore $h^k(a)$ is an isomorphism (functoriality) and $h^k(b)$ is induced by the inclusion of the complements. For each space $X$ the inclusion $\mathbb{R}^{n+1} \setminus X \times I \to \mathbb{R}^{n+1} \setminus X \times \{t\}$ induces an isomorphism in homology (a proof can be based on 3.9). Therefore $h^k(b)$ is an isomorphism. Hence $h^k(j_t)$ is an isomorphism. The projection $\mathrm{pr} : X \times I \to X$ is proper and $h^k(\mathrm{pr})$ is an inverse of $h^k(j_t)$ which is independent of $t$. The general case follows by functoriality. Similarly for pairs of spaces.

*Exactness.* The exactness axiom for the cohomology theory is a direct consequence of the exact homology sequences of triples. □

The first main topic of this paper is the construction of a tautological cohomology theory on **EC** from a given homology theory (Theorem 4.1). The next section is devoted to the main technical results used in this construction.

## 3. Dualizible maps

Guided by our methods we generalize the category **EC** to the category **ED** of dualizible maps and construct (tautological) cohomology functors on **ED**. The first main result of this paper is Theorem 3.2.

We use the notations: $(X, A) \times (Y, B) = (X \times Y, X \times B \cup A \times Y)$ and $(Z, Z \setminus C) = Z|C$. The latter makes diagrams smaller. With this notation we also have the convenient rule $(Z_1|C_1) \times (Z_2|C_2) = (Z_1 \times Z_2)|(C_1 \times C_2)$. A map $f : X \to Y$ induces a map of pairs $f : X|A \to Y|B$ if and only if $f(x) \in B$ implies $x \in A$. The reader should use the last remark in subsequent proofs.

Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ be arbitrary subsets and $f : X \to Y$ a continuous map. A Tietze extension of $f$ is a continuous map $f_\bullet : \mathbb{R}^n \to \mathbb{R}^m$ which extends $f$. A Tietze extension exists if and only if $f$ has an extension $\overline{f} : \overline{X} \to \overline{Y}$ to the respective closures.

A Tietze extension yields a homeomorphism

$$\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m, \quad (x, y) \mapsto (x, y - f_\bullet(x))$$

with inverse $(x, y) \mapsto (x, y + f_\bullet(x))$. It induces a homeomorphism of the graph $G(f) = \{(x, f(x) \mid x \in X\}$ with $X \times 0$. If $\alpha^0$ and $\alpha^1$ are Tietze extensions, then $(x, t) \mapsto (1 - t)\alpha^0(x) + t\alpha^1(x)$ is a one-parameter family of Tietze extensions, and this family induces a homotopy between the homeomorphism within the set of homeomorphisms. We call $f$ extendible, if $f$ has a Tietze extension. Composition and product of two extendible maps are again extendible. We denote the homeomorphism of pairs of spaces constructed above by

$$f^2_\flat : (\mathbb{R}^n \times \mathbb{R}^m)|G(f) \to (\mathbb{R}^n \times \mathbb{R}^m)|(X \times 0)$$

although it depends on the choice of $f_\bullet$.

A scaling function for $f$ is a map $\varphi : \mathbb{R}^m \to \,]0,\infty[$ such that $\|x\| \leq \varphi(f(x))$ holds for each $x \in X$. A scaling function induces a homeomorphism

$$\mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^m, \quad (x,y) \mapsto (\varphi(y)\cdot x, y)$$

with inverse $(x,y) \mapsto (\varphi(y)^{-1}\cdot x, y)$. It induces a homeomorphism of $D^n \times Y$, $D^n = \{x \in \mathbb{R}^n \mid 1 \geq \|x\|\}$ with $N_\varphi = \{(x,y) \in \mathbb{R}^n \times Y \mid \|x\| \leq \varphi(y)\}$. The graph $G(f)$ is contained in $N_\varphi$. Therefore we obtain a map between pairs of spaces

$$f_\flat^1 : (\mathbb{R}^n \times \mathbb{R}^m)|(D^n \times Y) \to (\mathbb{R}^n \times \mathbb{R}^m)|G(f).$$

Its homotopy class does not depend on the choice of $\varphi$, since scaling functions $\varphi^0, \varphi^1$ induce a homotopy $(x,t) \mapsto (1-t)\varphi^0 + t\varphi^1$ of scaling functions.

We call $f$ bounded, if the counter-image $f^{-1}(C)$ of each bounded set $C \subset Y$ is bounded in $X$. Composition and product of two bounded maps are again bounded.

**Lemma 3.1.** *$f$ has a scaling function if and only if $f$ is bounded.*

*Proof.* Let $\varphi$ be a scaling function. Then the image of $D(t) = \{y \in \mathbb{R}^m \mid \|y\| \leq t\}$ is bounded, say $\varphi(y) \leq s$ for $\|y\| \leq t$. Let $x \in f^{-1}(Y \cap D(t))$. Then $\|x\| \leq \varphi(f(x)) \leq s$. Hence this counter-image is bounded.

Conversely, assume that $f$ is bounded. Let $\tilde{\psi}(t)$ be the sup-norm of

$$f^{-1}(Y \cap D(t)) = \{x \in X \mid \|f(x)\| \leq t\}.$$

Then

$$\tilde{\psi}(\|f(x)\|) = \sup\{\|a\| \mid a \in X, \|f(a)\| \leq \|f(x)\|\} \geq \|x\|.$$

The function $\tilde{\psi}$ is increasing. There exists a continuous increasing function $\psi : [0,\infty[\, \to \,]0,\infty[$ such that $\psi \geq \tilde{\psi}$. The map $y \mapsto \psi(\|y\|)$ is a scaling function for $f$. □

We call $f$ dualizible, if $f$ is extendible and has a scaling function. A map $f$ is dualizible if and only if it has an extension $\overline{f} : \overline{X} \to \overline{Y}$ which is proper. A dualizible map yields a geometric duality map

$$f_\flat = f_\flat^2 \circ f_\flat^1 : (\mathbb{R}^n \times \mathbb{R}^m)|(D^n \times Y) \to (\mathbb{R}^n \times \mathbb{R}^m)|(X \times 0)$$

which is unique up to homotopy. The category **ED** of dualizible maps has as objects the pairs $(\mathbb{R}^n, X)$, $X \subset \mathbb{R}^n$ and as morphisms $f : (\mathbb{R}^n, X) \to (\mathbb{R}^m, Y)$ the dualizible maps $f : X \to Y$. An inclusion $(\mathbb{R}^n, A) \to (\mathbb{R}^n, B)$, $A \subset B$ is dualizible. A dualizible homotopy is a dualizible map $(\mathbb{R}^n \times \mathbb{R}, X \times I) \to (\mathbb{R}^m, Y)$. It is the restriction of a proper homotopy $\overline{X} \times I \to \overline{Y}$. The category **EC** is contained in **ED**.

The definitions

$$h^k(X, A; n) = h_{n-k}(\mathbb{R}^n \setminus A, \mathbb{R}^n \setminus X), \quad h^k(X, \varnothing; n) = h^k(X; n)$$

can be used for arbitrary triples $A \subset X \subset \mathbb{R}^n$. From a geometric duality map $f_\flat$ we obtain an induced morphism $h^k(f)$ via the commutative diagram

$$\begin{array}{ccc} h^k(Y;m) = h_{m-k}(\mathbb{R}^m|Y) & \xrightarrow{\Sigma^n} & h_{m-k+n}(\mathbb{R}^m|Y \times \mathbb{R}^n|D^n) \\ \downarrow{\scriptstyle h^k(f)} & & \downarrow{\scriptstyle (-1)^{nm} h_*(f_\flat \circ \tau_{mn})} \\ h^k(X;n) = h_{n-k}(\mathbb{R}^n|X) & \xrightarrow{\Sigma^m} & h_{n-k+m}(\mathbb{R}^n|X \times \mathbb{R}^m|0) \end{array}$$

in which $\Sigma^m$ and $\Sigma^n$ are iterated suspension isomorphisms and $t_{mn}$ interchanges the factors of $\mathbb{R}^m \times \mathbb{R}^n$. For some technical details about suspension isomorphisms see 3.9 and 3.10. In the following proofs we need the relations $\Sigma^a \Sigma^b = \Sigma^{a+b}$ and the naturality of the suspension. Since the homotopy class of $f_\flat$ is unique, $h^k(f)$ is well-defined.

The main result of this section is 3.2. We divide its proof into several propositions.

**Theorem 3.2.** *Let $h_*$ be a homology theory. The previously defined data $h^k$ are a contravariant functor* $\mathbf{ED} \to R$-MOD *(see 3.3 and 3.4). This functor is compatible with inclusions 3.3, homeomorphisms 3.5 and suspensions 3.6.*

**Proposition 3.3.** *Let $i : A \subset X \subset \mathbb{R}^n$ be an inclusion. Then $h^k(i)$ is the morphism $h_{n-k}(\mathbb{R}^n|X) \to h_{n-k}(\mathbb{R}^n|A)$ induced by the dual inclusion. In particular* $h^k(\mathrm{id}) = \mathrm{id}$.

*Proof.* We take the scaling function $y \mapsto \|y\| + 1$ and extend $i$ by the identity. Then $i_\flat$ is the map $(x,y) \mapsto ((\|y\|+1) \cdot x, y - (\|y\|+1) \cdot x)$. The map $i^1_\flat$ is $(x,y) \mapsto ((\|y\|+1) \cdot x, y)$ and $((1-t)(\|y\|+1) \cdot x + t(x+y), y)$ is a homotopy to $(x,y) \mapsto (x+y,y)$. Hence $i_\flat$ is homotopic to $(x,y) \mapsto (x+y,-x)$ and the homotopy $((1-t)x+y,-x)$ shows it to be homotopic to $(x,y) \mapsto (y,-x)$. Now interchange the factors and observe that $x \mapsto -x$ has degree $(-1)^n = (-1)^{n \cdot n}$. ☐

**Proposition 3.4.** *Let $Z \subset \mathbb{R}^p$ and $g : Y \to Z$. Then $h^k(f) \circ h^k(g) = h^k(gf)$.*

*Proof.* We verify that the following diagram is homotopy commutative.

$$\begin{array}{ccc} \mathbb{R}^n|D \times \mathbb{R}^m|D \times \mathbb{R}^p|Z & \xrightarrow{\tau_{nm} \times 1} & \mathbb{R}^m|D \times \mathbb{R}^n|D \times \mathbb{R}^p|Z \\ \downarrow{\scriptstyle 1 \times g_\flat} & & \downarrow{\scriptstyle 1 \times (gf)_\flat} \\ \mathbb{R}^n|D \times \mathbb{R}^m|Y \times \mathbb{R}^p|0 & & \mathbb{R}^m|D \times \mathbb{R}^n|X \times \mathbb{R}^p|0 \\ \downarrow{\scriptstyle f_\flat \times 1} & & \downarrow{\scriptstyle \cap} \\ \mathbb{R}^n|X \times \mathbb{R}^m|0 \times \mathbb{R}^p|0 & \xrightarrow{\tau_{nm} \times 1} & \mathbb{R}^m|0 \times \mathbb{R}^n|X \times \mathbb{R}^p|0 \end{array}$$

The morphisms $\tau_{nm}$ are the appropriate interchange maps. The proof is based on the next diagram. We write $V = \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$, $W = \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^p$ and use the intermediate map $h : X \to G(g)$, $x \mapsto (f(x), gf(x))$. We have the factorization $h^2_\flat = h^{22}_\flat h^{21}_\flat$ with $h^{21}_\flat(x,y,z) = (x, y - f_\bullet(x), z)$, $h^{22}_\flat(x,y,z) =$

$(x, y, z - g_\bullet f_\bullet(x))$, where $h_\bullet = (f_\bullet, g_\bullet f_\bullet)$. We write $G'(gf) = \{(x, 0, gf(x)) \mid x \in X\}$; and $D$ denotes any of the appropriate disks. The index of a scaling function indicates the map it belongs to.

$$\begin{array}{ccccccc}
V|D \times D \times Z & & \xrightarrow{\qquad\qquad\qquad \tau_{nm}\times 1 \qquad\qquad\qquad} & & & & W|D \times D \times Z \\
\downarrow{\scriptstyle 1\times g^1_\flat} & & & & & & \downarrow{\scriptstyle 1\times (gf)^1_\flat} \\
V|D \times G(g) & \xrightarrow{h^1_\flat} & V|G(h) & \xrightarrow{h^{21}_\flat} & V|G'(gf) & \xrightarrow{\tau_{nm}\times 1} & W|0 \times G(gf) \\
\downarrow{\scriptstyle 1\times g^2_\flat} & & & & \downarrow{\scriptstyle h^{22}_\flat} & & \downarrow{\scriptstyle 1\times (gf)^2_\flat} \\
V|D \times Y \times 0 & & \xrightarrow[f_\flat\times 1]{} & & V|X \times 0 \times 0 & \xrightarrow[\tau_{nm}\times 1]{} & W|0 \times X \times 0
\end{array}$$

The lower right rectangle commutes. For the upper rectangle we use the homotopy $(\varphi_{gf}(z) \cdot x, s(\varphi_g(z) \cdot y - f_\bullet(\varphi_{gf}(z) \cdot x) + (1-s)y, z)$; here we use the scaling function $\varphi_h : (y, z) \mapsto \varphi_{gf}(z)$. For the lower left rectangle we use the homotopy $z - g_\bullet((1-t)y + tf_\bullet(\varphi_f(y) \cdot x))$ in the third component; here we use the scaling function $\varphi_h : (y, z) \mapsto \varphi_f(y)$. One has to verify that the subspaces are respected.

Finally one has to rewrite the commutativity of the diagram above into the equality to be proved. This is done as follows. Let us write $D(f) = f_\flat \circ \tau_{mn}$ and similarly for $D(g)$ and $D(gf)$. Then one translates the diagram in the beginning of the proof into the homotopy commutative diagram

$$\begin{array}{ccccc}
\bullet & \xrightarrow{1\times\tau_{mn}} & \bullet & \xrightarrow{D(gf)\times 1} & \bullet \\
{\scriptstyle D(g)\times 1}\downarrow & & & & \downarrow{\scriptstyle 1\times\tau_{pm}} \\
\bullet & \xrightarrow[1\times\tau_{np}]{} & \bullet & \xrightarrow[D(f)\times 1]{} & \bullet
\end{array}$$

in which the $\tau_{mn}$ on top also uses the inclusion $\mathbb{R}^m|D \subset \mathbb{R}^m|0$. One applies homology to this diagram, uses the naturality of the suspension and verifies that the signs involved in the definition of the $h^k(f)$ etc. and produced by the interchange maps altogether cancel. □

**Proposition 3.5.** *Let $\alpha$ be a homeomorphism of $\mathbb{R}^n$. Then the morphism*

$$h^k(\alpha) : h^k(\alpha X; n) \to h^k(X; n)$$

*coincides with the morphism*

$$h_*(\alpha^{-1}) : h_{n-k}(\mathbb{R}^n|\alpha X) \to h_{n-k}(\mathbb{R}^n|X)$$

*times the degree of $\alpha$.*

*Proof.* We first show that $\alpha^1_\flat$ is homotopic to $(x, y) \mapsto (x + \alpha^{-1}(y), y)$. We use the linear homotopy $((1-t)(x + \alpha^{-1}(y)) + t\varphi(\|y\|) \cdot x, y)$ where $\varphi(r) = 1 + \psi(r)$ and $\psi : [0, \infty[ \to ]0, \infty[$ is a function such that $\psi(\|\alpha(x)\|) \geq \|x\|$ for $x \in X$. Hence $\alpha_\flat$ is homotopic to $(x, y) \mapsto (x + \alpha^{-1}(y), y - \alpha(x + \alpha^{-1}(y)))$. The latter is the restriction of a map $\mathbb{R}^n|0 \times \mathbb{R}^n|\alpha X \to \mathbb{R}^n|X \times \mathbb{R}^n|0$. To this map we first apply the homotopy

$$(x, y, t) \mapsto ((1-t)x + \alpha^{-1}(y), y - \alpha(x + \alpha^{-1}(y)))$$

and then the homotopy

$$(x, y, t) \mapsto (\alpha^{-1}(y), (1-t)y - \alpha(x + \alpha^{-1}((1-t)y))).$$

Then the final map is $(x, y) \mapsto (\alpha^{-1}(y), -\alpha(x + \alpha(0)))$. Now use the definition of $h^k(\alpha)$ and the fact that a homeomorphism $\beta$ of $\mathbb{R}^n|0$ induces the multiplication by the degree of $\beta$ on $h_*((P, Q) \times \mathbb{R}^n|0)$. □

**Proposition 3.6.** *Let $f : Z \times 0 \to Z, (z, 0) \mapsto z$ be the standard morphism for $Z \subset \mathbb{R}^k$ and $0 \subset \mathbb{R}^l$. Then the induced morphism $h^a(f)$ coincides with the suspension $\Sigma^l : h_{k-a}(\mathbb{R}^k|Z) \to h_{k-a+l}(\mathbb{R}^k|Z \times \mathbb{R}^l|0)$.*

*Proof.* We use the scaling function $\varphi : \mathbb{R}^k \to ]0, \infty[, z \mapsto \|z\| + 1$ and the Tietze extension $f_\bullet(z, u) = z$. Then $f_\flat^1$ is homotopic, via a linear homotopy, to $(z', u, z) \mapsto (z' + z, u, z)$ and $f_\flat$ therefore homotopic to $(z', u, z) \mapsto (z' + z, u, -z')$, and this is homotopic to $(z, u, -z')$. Now use the definition of $h^a(f)$. □

**Proposition 3.7.** *A contravariant functor* **ED** $\to R$-MOD *is determined by its effect on four special types of morphisms:*

*(1) Inclusions $j : (\mathbb{R}^n, X) \to (\mathbb{R}^n, Y)$.*
*(2) Homeomorphisms $j : (\mathbb{R}^n, X) \to (\mathbb{R}^n \times \mathbb{R}^m, X \times 0),\ x \mapsto (x, 0)$.*
*(3) Homeomorphisms $\beta : (\mathbb{R}^l, X) \to (\mathbb{R}^l, Y)$ obtained by restriction from a homeomorphism $\beta : \mathbb{R}^l \to \mathbb{R}^l$.*
*(4) Projections $q : (\mathbb{R}^n \times \mathbb{R}^m, D \times Y) \to (\mathbb{R}^m, Y),\ (x, y) \mapsto y$ where $D \subset \mathbb{R}^n$ is a closed disk.*

(*If the functor is homotopy invariant one can dispense with morphisms of type (4), because then the projection induces an isomorphism which is inverse to a morphism induced by an embedding.*)

*Proof.* We write an arbitrary morphism $f : (\mathbb{R}^n, X) \to (\mathbb{R}^m, Y)$ of **ED** as a composition of the four special types of morphisms:

$$(\mathbb{R}^n, X) \xrightarrow{\text{(i)}} (\mathbb{R}^n \times \mathbb{R}^m, X \times 0) \xrightarrow{\text{(ii)}} (\mathbb{R}^n \times \mathbb{R}^m, G(f))$$
$$\xrightarrow{\text{(iii)}} (\mathbb{R}^n \times \mathbb{R}^m, N_\varphi) \xrightarrow{\text{(iv)}} (\mathbb{R}^n \times \mathbb{R}^m, D^n \times Y) \xrightarrow{\text{(v)}} (\mathbb{R}^m, Y).$$

(i) is $x \mapsto (x, 0)$; (ii) is $(x, 0) \mapsto (x, f(x))$; (iii) is an inclusion; (iv) is $(x, y) \mapsto (\varphi(y)^{-1} \cdot x, y)$; (v) is the projection $(x, y) \mapsto y$. For (i) we use (2), for (ii) we use (3), for (iii) we use (1), for (iv) we use (3), for (v) we use (4). □

Proposition 3.7 also holds for functors **EC** $\to R$-MOD.

From the proof of 2.3 we see that $h^k$ is homotopy invariant. We then use 3.7 and obtain:

**Corollary 3.8.** *The functor* $h^k : \mathbf{ED} \to R\text{-MOD}$ *is uniquely determined by the properties 3.3, 3.5 and 3.6.* □

As a final ingredient we have to explain the special form of the suspension isomorphisms which are needed in the construction.

**Definition 3.9** (Suspension)**.** Let $V = ]a,b[$ , $D = [c,d]$, with $-\infty \le a < c \le d < b \le \infty$, $W = V \setminus D = W_+ \cup W_-$, $W_+ = ]d,b[$ , $W_- = ]a,c[$ . For each pair $(X,A)$ of topological spaces we have a suspension isomorphism $\Sigma$ defined by the commutative diagram

$$\begin{array}{ccc} h_*(X,A) & \xrightarrow[(1)]{\cong} & h_*(X \times W_+, A \times W_+) \\ \downarrow \Sigma & & (2) \downarrow \cong \\ h_*(X \times V, X \times W \cup A \times V) & \xrightarrow[\partial]{\cong} & h_*(X \times W \cup A \times V, X \times W_- \cup A \times V) \end{array}$$

with (1) induced by the embedding $x \mapsto (x,d)$ and (2) induced by the inclusion. The morphism $\partial$ is the boundary operator of the triple

$$(X \times V, X \times W \cup A \times V, X \times W_+ \cup A \times V).$$

$\Sigma$ is natural in the variable $(X,A)$ with respect to maps between pairs and natural with respect to inclusions obtained by increasing $V$ and decreasing $D$.

We now apply this suspension to the pair $(U \setminus A, U \setminus X)$ where $A \subset X \subset \mathbb{R}^n$ are closed and $U$ is an open neighborhood of $X$ in $\mathbb{R}^n$. Then we compose this suspension with the excision

$$(U \setminus A, U \setminus X) \times (V,W) \to (U \times V \setminus A \times D, U \times V \setminus X \times D)$$

(for which we now use that $X$ and $A$ are closed in order to have an excision isomorphism in homology). The resulting map

$$\Sigma : h_s(U \setminus A, U \setminus X) \to h_{s+1}(U \times V \setminus A \times D, U \times V \setminus X \times D)$$

is also a natural suspension isomorphism. ◊

**Proposition 3.10.** *Both types of isomorphisms* $\Sigma$ *anticommute with* $\partial$.

*The diagram*

$$\begin{array}{ccc} h_*(X,A) & \xrightarrow{\partial} & h_*(A) \\ \downarrow \Sigma & & \downarrow -\Sigma \\ h_*((X,A) \times (V,W)) & \xrightarrow{\partial} & h_*(A \times (V,W)) \end{array}$$

*is commutative. This is analogous to* [2, 10.9.2].

*Let* $C \subset B \subset A \subset \mathbb{R}^n$ *be closed subsets and* $U$ *an open neighborhood of* $A$. *Let* $V$ *and* $D$ *be as before. Then the diagram*

$$\begin{array}{ccc} h_{s+1}(U \setminus C, U \setminus B) & \xrightarrow{\Sigma} & h_{s+2}(U \times V \setminus C \times D, U \times V \setminus B \times D) \\ \downarrow \partial & & \downarrow -\partial \\ h_s(U \setminus B, U \setminus A) & \xrightarrow{\Sigma} & h_{s+1}(U \times V \setminus B \times D, U \times V \setminus A \times D) \end{array}$$

*is commutative.* □

**Remark 3.11.** Suppose $f : (\mathbb{R}^n, X) \to (\mathbb{R}^m, Y)$ is an extendible map with bounded $X$ contained in some disk $D$. Then one can construct $h^k(f)$ with the morphism

$$\mathbb{R}^n|D \times \mathbb{R}^m|Y \subset \mathbb{R}^{n+m}|G(f) \to \mathbb{R}^n|X \times \mathbb{R}^m|0, \quad (x,y) \mapsto (x, y - f_\bullet(x)).$$

The size of the disk plays no role and it is not necessary to reparametrize by a scaling function. Some of the previous proofs simplify by using these geometric duality maps. ♢

Homeomorphic subsets of $\mathbb{R}^n$ may not be isomorphic in **ED**. If a homeomorphism extends to the closures, then the extension may not be a homeomorphism even if it is bijective. One can extend some classical results to the category **ED**, for instance the component theorem [2, 10.3.3]. See also [2, 7.3.1].

**Theorem 3.12.** *If $A$ and $B$ are subsets of $\mathbb{R}^n$ which are isomorphic in* **ED**, *then $\mathbb{R}^n \setminus A$ and $\mathbb{R}^n \setminus B$ have the same number of path components.* □

## 4. Construction of cohomology theories

This section is devoted to the construction of a tautological cohomology theory associated to a given homology theory (see Definition 2.2 and Proposition 2.3).

The basic technical work has already been done in the previous section. The contravariant functors $h^k$ (restricted to **EC**) will be part of our cohomology theory. What remains is to extend these functors to the category $\mathbf{EC}(\mathbf{2})$ and to show that the coboundary maps are a natural transformation (see 2.3). Since we now work with closed subspaces of Euclidean spaces, proper maps are dualizible.

Let $f : (\mathbb{R}^n, X, A) \to (\mathbb{R}^m, Y, B)$ be a morphism in $\mathbf{EC}(\mathbf{2})$. A scaling function for $f : X \to Y$ is also a scaling function for the restriction $f|A : A \to B$, and similarly for Tietze extensions. We therefore have geometric duality maps with $P = \mathbb{R}^n$, $Q = \mathbb{R}^m$, $R = \mathbb{R}^n \times \mathbb{R}^m$

$$\begin{array}{c} (R, R \setminus D^n \times B, R \setminus D^n \times Y) \\ \big\downarrow {\scriptstyle f^1_\flat} \\ (R, R \setminus G(f|A), R \setminus G(f)) \\ \big\downarrow {\scriptstyle f^2_\flat} \\ (R, R \setminus A \times 0, R \setminus X \times 0). \end{array}$$

The induced cohomology morphism $h^k(f)$ is defined by the diagram

$$\begin{array}{ccc} h^k(Y,B;m) = h_{m-k}(Q \setminus B, Q \setminus Y) & \xrightarrow{\Sigma^n} & h_{m-k+n}(R \setminus B \times D^n, R \setminus Y \times D^n) \\ \downarrow{\scriptstyle h^k(f)} & & \downarrow{\scriptstyle (-1)^{nm} h_*(f_\flat \circ \tau)} \\ h^k(X,A;n) = h_{n-k}(P \setminus A, P \setminus X) & \xrightarrow{\Sigma^m} & h_{n-k+m}(R \setminus A \times 0, R \setminus X \times 0). \end{array}$$

In this diagram we use the second form of the suspension isomorphism 3.9 in the case that $U$ is a Euclidean space, $V = \mathbb{R}$ $D = D_z = \,]-z,z[$. When we iterate this $l$ times we have to use the disk $D^l = D^l_z$ in order to obtain the rule $\Sigma^k \circ \Sigma^l = \Sigma^{k+l}$; but other disks also do the job up to some natural isomorphism.

The induced morphisms still satisfy 3.3 and 3.4; the homotopies involved in the proofs respect the subspaces, since they are defined by linear connection. Also 3.5 and 3.6 hold for pairs of subspaces.

**Theorem 4.1.** *The functors* $h^k : \mathbf{EC}(\mathbf{2}) \to R\text{-MOD}$ *just defined and the coboundary operators* $\delta : h^{k-1}(A;n) \to h^k(X,A;n)$*, defined as*

$$(-1)^n\partial : h_{n-1+k}(\mathbb{R}^n, \mathbb{R}^n \setminus A) \to h_{n-k}(\mathbb{R}^n \setminus A, \mathbb{R}^n \setminus X),$$

*where* $\partial$ *is the boundary operator of the given homology theory applied to the appropriate triple, form a tautological cohomology theory.*

*Proof.* It only remains to verify that the coboundary maps constitute a natural transformation. One unravels the definition and sees that the naturality of $\delta$ is a direct consequence of the anticommutation rules 3.10 and the naturality of $\partial$. □

From the definitions it is clear that a natural transformation of homology theories induces a natural transformation of the associated tautological cohomology theories.

For further properties of tautological cohomology see Sections 7 and 8.

## 5. Duality and cap-product

A classical proof of duality uses cap-products. For a proof in an axiomatic context see [2, Chap. 18]. We follow this exposition and compare it with with our present approach. For this purpose we have to recall some definitions and notations. We assume given a homology theory $h_*$, a cohomology theory $h^*$ and a duality pairing (cap-product) in the sense of [2, 18.2]. From the theory $h^*$ one constructs Čech cohomology groups $\check{h}^k(X,A)$ via colimits over neighborhoods, e.g. by the method of [3, VIII.6]. If $A \subset X$ is a compact pair in $\mathbb{R}^n$, then $\check{h}^k(X,A)$ depends, by the definition via colimits, on the embedding into $\mathbb{R}^n$, and in accordance with our approach we should, for the time being, denote this group $\check{h}^k(X,A;n)$. The independence of the embedding will be shown in Section 6.

The cap-product is used to define duality morphisms $D_{X,A}$ for compact $(X,A)$ in $\mathbb{R}^n$. The tautological groups will now be denoted as $\hat{h}^*$. The duality morphisms are then isomorphisms $D_{X,A} : \check{h}^k(X, A; n) \to \hat{h}^k(X, A; n)$, see [2, 18.3.3] (the manifold $M$ in that reference is now $\mathbb{R}^n$ with its canonical orientation). We will investigate naturality and stability properties of duality morphisms and show:

**Theorem 5.1.** *The homomorphisms*

$$d_{X,A} = (-1)^{kn} D_{X,A} : \check{h}^k(X, A; n) \to \hat{h}^k(X, A; n)$$

*constitute a morphism of cohomology theories, i.e., they commute with induced morphisms and with coboundary operators. (The sign in the definition of $d_{X,A}$ takes care of the sign in 5.2).*

*Proof.* In the proof we use the two special properties 5.2 and 5.3 of the duality morphism $D_{X,A}$ which have not been proved in [2]. In order to cut down the size of a diagram we write out the proof of the naturality for morphisms $f : (\mathbb{R}^n, X) \to (\mathbb{R}^m, Y)$ for compact $X$ and $Y$. We also use the simpler description 3.11 of the geometric duality morphisms to simplify the next diagram.

Consider the following diagram with $P = \mathbb{R}^n$, $Q = \mathbb{R}^m$, $R = \mathbb{R}^{n+m}$. We use $\check{h}$ for cohomology and $h$ for homology. The indices are determined by the context and will not be displayed.

$$\begin{array}{ccc}
\check{h}(Y) & \longrightarrow & h(Q|Y) \\
{\scriptstyle \check{h}(\mathrm{pr})}\downarrow & & \downarrow{\scriptstyle \Sigma} \\
\check{h}(Y \times 0) & \longrightarrow & h(R|Y \times 0) \\
{\scriptstyle (i)}\uparrow{\scriptstyle \cong} & & {\scriptstyle (ii)}\uparrow{\scriptstyle \cong} \\
\check{h}(Y \times D) & \longrightarrow & h(R|Y \times D) \\
{\scriptstyle \check{h}(\tau)}\downarrow & & \downarrow{\scriptstyle (-1)^{nm} h(\tau)} \\
\check{h}(D \times Y) & \longrightarrow & h(R|D \times Y) \\
\downarrow & & \downarrow \\
\check{h}(G(f)) & \longrightarrow & h(R|G(f)) \\
\downarrow & & \downarrow{\scriptstyle h(f_\flat^2)} \\
\check{h}(X \times 0) & \longrightarrow & h(R|X \times 0) \\
\uparrow & & \uparrow{\scriptstyle \Sigma} \\
\check{h}(X) & \longrightarrow & h(P|X)
\end{array}$$

The horizontal maps are morphisms of the type $d_K$ for the appropriate $K$. The isomorphisms (i) and (ii) are induced by the inclusion $Y \times 0 \subset Y \times D$, and (i) is inverse to the morphism induced by the projection. The disk $D$ contains $X$, and $\tau$ interchanges the factors $\mathbb{R}^n$ and $\mathbb{R}^m$. The composition of $\Sigma$ with (ii) is again a $\Sigma$, by naturality of $\Sigma$.

The rectangles (1)–(6), from top to bottom, commute for the following reasons:

(1) 5.2;
(2) $d$ is compatible with inclusions [2, 18.5.1];
(3) 5.3, since $\tau$ has degree $(-1)^{nm}$;
(4) $d$ is compatible with inclusions;
(5) 5.3, since $f_\flat^2$ has degree one;
(6) 5.2.

The left downward composition is $\check{h}(f)$, the right downward composition is $h(f)$. This finishes the proof of the naturality of $d$.

The compatibility with the coboundary operators is a consequence of [2, 18.5.5]. □

In the proof of 5.1 we do not use the fact that the $D_{X,A}$ are isomorphisms. But as the reader certainly knows: A natural transformation which yields an isomorphism for points induces formally an isomorphism for finite CW-complexes (these are compact Cartesian spaces).

**Proposition 5.2.** *For each compact pair $(K, L)$ in $\mathbb{R}^n$ the diagram*

$$\begin{array}{ccc} \check{h}^k(K,L;n) & \xrightarrow{D_{K,L}} & h_{n-k}(\mathbb{R}^n \setminus L, \mathbb{R} \setminus K) \\ \downarrow{\scriptstyle \check{h}^k(\mathrm{pr})} & & \downarrow{\scriptstyle \Sigma} \\ \check{h}^k(K\times 0, L\times 0; n+1) & \xrightarrow{D_{K\times 0, L\times 0}} & h_{n-k+1}(\mathbb{R}^{n+1} \setminus L\times 0, \mathbb{R}^{n+1} \setminus K \times 0) \end{array}$$

*is $(-1)^k$-commutative. By 3.6 the right hand morphism $\Sigma$ is $\hat{h}^k(\mathrm{pr})$.*

*Proof.* The duality morphism $D_{K,L}$ is obtained via a colimit process over neighborhoods $(U, V)$ of $(K, L)$ from morphisms

$$D^{UV}_{KL} : h^k(U, V) \longrightarrow h^k(U \setminus L, V \setminus L) \xrightarrow{\frown z^{UV}_{KL}} h_{n-k}(U \setminus L, U \setminus K)\ ,$$

see [2, p. 445]. One now verifies the relation $\Sigma(z^{UV}_{KL}) = z^{U\times W, V\times W}_{K\times 0, L\times 0}$ for the orientation elements. Then one uses the next diagram which is analogous to the first one in [2, 18.2.1]; for its statement we take the freedom to use notations like $U \setminus L = U_L$ and $U \times V = UV$ in order to cut down its size. The diagram

$$\begin{array}{ccc} h^k(U_L, V_L) \otimes h_*(U_L, U_K \cup V_L) & \xrightarrow{\frown} & h_*(U_L, U_K) \\ \downarrow{\scriptstyle h^k(\mathrm{pr})\otimes\Sigma} & & \downarrow{\scriptstyle \Sigma} \\ h^k(UW_{L0}, VW_{L0}) \otimes h_*(UW_{L0}, UW_{K0} \cup VW_{L0}) & \xrightarrow{\frown} & h_*(UW_{L0}, UW_{K0}) \end{array}$$

is $(-1)^k$-commutative. From the resulting relation

$$\Sigma \circ D^{UV}_{KL} = (-1)^k D^{U\times W, V\times W}_{K\times 0, L\times 0} \circ h^k(\mathrm{pr}),$$

where $U, V, W$ are appropriate neighborhoods of $K, L, 0$, one obtains the claim of the proposition by passage to colimits. The somewhat lengthy verification of these statements from the axioms will not be carried out in this paper. □

**Proposition 5.3.** *Let $\alpha$ be a homeomorphism of $\mathbb{R}^n$ with degree $d$ and $K \subset \mathbb{R}^n$. Then the diagram*

$$\begin{array}{ccc} \check{h}^k(\alpha(K);n) & \xrightarrow{D_{\alpha(K)}} & h_{n-k}(\mathbb{R}^n|\alpha(K)) \\ \downarrow{\scriptstyle \check{h}^k(\alpha)} & & \downarrow{\scriptstyle d\cdot h_{n-k}(\alpha^{-1})} \\ \check{h}^k(K;n) & \xrightarrow{D_K} & h_{n-k}(\mathbb{R}^n|K) \end{array}$$

*commutes.*

*Proof.* The canonical orientation of $\mathbb{R}^n$ is a family of elements $o_K \in h_n(\mathbb{R}^n|K)$ for compact $K$ as explained in Problem 4 of [2, p. 446]. One verifies that $\alpha_*(o_K) = d \cdot \alpha_{\alpha(K)}$. Then one uses the naturality of the cap-product [2, p. 441]. □

Note that 5.3 is analogous to 3.5.

**Remark 5.4.** The duality morphisms for locally compact subsets are defined via a colimit process from compact pairs, see [3, VIII.7.12]. The naturality is a consequence of 5.1. ◊

**Remark 5.5.** 5.2 can be generalized to the product of an oriented manifold $M$ with $\mathbb{R}$ and a suitable product orientation. ◊

## 6. Cartesian spaces

We call $X$ a Cartesian space if it admits an embedding $i : X \to \mathbb{R}^n$ into some Euclidean space as a closed subset. Cartesian spaces have an abstract characterization: Locally compact Hausdorff spaces with countable basis and finite covering dimension. We do not use this characterization.

We use 3.3 and 3.4 to define an invariant object $h^k(X)$ by a universal property which does not depend on the choice of an embedding $i : X \to \mathbb{R}^n$. For let $j : X \to \mathbb{R}^m$ be another embedding. The homeomorphism $ji^{-1} : iX \to jX$ induces an isomorphism $h^k(ji^{-1}) : h^k(jX;m) \to h^k(iX;n)$. We define $h^k(X)$ to be a group together with a family of isomorphisms $\alpha(i) : h^k(X) \to h^k(iX;n)$ such that the diagrams

$$\begin{array}{ccccc} & & h^k(X) & & \\ & {\scriptstyle \alpha(j)}\swarrow & & \searrow{\scriptstyle \alpha(i)} & \\ h^k(jX;m) & & \xrightarrow{\quad h^k(ji^{-1})\quad} & & h^k(iX;n) \end{array}$$

are commutative. It suffices of course to fix a single isomorphism $\alpha(i)$, the remaining ones are then uniquely determined, and 3.3 and 3.4 show that this definition makes sense, after all. (Formally the family is a categorical limit. Thus $h^k(X)$ is determined up to a unique isomorphism by the universal property.) Given now $f : X \to Y$ between Cartesian spaces choose embeddings

$i : X \to \mathbb{R}^n$ and $j : Y \to \mathbb{R}^m$. Then there exists a unique homomorphism $f^* : h^k(Y) \to h^k(X)$ such that all diagrams

$$\begin{array}{ccc} h^k(Y) & \xrightarrow{\quad f^* \quad} & h^k(X) \\ \downarrow{\scriptstyle \alpha(j)} & & \downarrow{\scriptstyle \alpha(i)} \\ h^k(jY; m) & \xrightarrow{h^k(jfi^{-1})} & h^k(iX; n) \end{array}$$

are commutative.[1] In this manner $h^k$ becomes a contravariant functor on the category of Cartesian spaces and proper maps.

A similar argument can be applied to the Čech-cohomology groups and the duality morphisms.

## 7. Continuity

Traditionally the proof of the Alexander duality for arbitrary compact Cartesian spaces uses continuity properties of the cohomology theory. They hold either for a Čech type cohomology or by construction via colimits (for the latter see e.g. [3, VIII.6] [2, 18]).

We assume in addition to the axioms of Eilenberg and Steenrod that the homology theory is additive (compatible with disjoint union), see [2, p. 245], and verify that tautological cohomology has appropriate continuity properties.

Let $A \subset X$ be a compact pair in $\mathbb{R}^n$. Suppose a system of compact sets $(V_1, W_1) \supset (V_2, W_2) \supset \cdots \supset (X, A)$ is given such that $\bigcap_j V_j = X$, $\bigcap_j W_j = A$. A continuity property of tautological cohomology is:

**Proposition 7.1.** *The inclusions* $h^k(V_j, W_j) \to h^k(X, A)$ *induce an isomorphism* $\operatorname{colim}_j h^k(V_j, W_j) \cong h^k(X, A)$.

*Proof.* According to the definitions the statement amounts to

$$\operatorname{colim} h_{n-k}(\mathbb{R}^n \setminus W_j, \mathbb{R}^n \setminus V_j) \cong h_{n-k}(\mathbb{R}^n \setminus A, \mathbb{R}^n \setminus X).$$

By assumption, $\bigcup_j (\mathbb{R}^n \setminus W_j) = \mathbb{R}^n \setminus A$ and similarly for $X$. We can now apply the Milnor telescope construction to obtain the desired result [2, 10.8.1,10.8.3]. □

## 8. Further properties

We collect a few further properties of tautological cohomology.

**Proposition 8.1.** *Let $V \subset \mathbb{R}^n$ be open and $B \subset Z$ be closed in $V$. Then there exists an isomorphism* $h^k(Z, B) \cong h_{n-k}(V \setminus B, V \setminus Z)$.

*Proof.* In view of the duality theory for manifolds in general, there should exist a duality isomorphism for the manifold $V$ as stated. In our setting we first have to consider $V$ as a Cartesian space. This is done by a closed embedding

$$j : V \to \mathbb{R}^n \times \mathbb{R}, \quad v \mapsto (v, d(v, \mathbb{R}^n \setminus V)^{-1})$$

[1] The reader should note that the situation is exactly analogous to the definition of a tangent space and a differential. See [2, p. 361].

where $d(x,C)$ denotes the Euclidean distance from $x$ to $C$. The image $j(V)$ is contained in $V \times \mathbb{R}$. This yields the homological excision isomorphism

$$h^k(Z,B) = h_{n-k+1}(\mathbb{R}^{n+1} \setminus jB, \mathbb{R}^{n+1} \setminus jZ) \cong h_{n-k+1}(V \times \mathbb{R} \setminus jB, V \times \mathbb{R} \setminus jZ).$$

We now apply the homeomorphism

$$\varphi : V \times \mathbb{R} \to V \times \mathbb{R}, \quad (v,\lambda) \mapsto (v, \lambda - d(v, \mathbb{R}^n \setminus V)^{-1})$$

which sends $j(V)$ onto $V \times 0$. Altogether we obtain ($s = n - k + 1$)

$$\begin{aligned} h_s(V \times \mathbb{R} \setminus jB, V \times \mathbb{R} \setminus jZ) &\xrightarrow{\varphi_*} h_s(V \times \mathbb{R} \setminus B \times 0, V \times \mathbb{R} \setminus Z \times 0) \\ &\cong h_{s-1}(V \setminus B, V \setminus Z), \end{aligned}$$

the latter isomorphism by suspension. □

**Proposition 8.2.** *Let $Y \subset X \subset \mathbb{R}^n$ be closed subsets. Then there exists an isomorphism $h^k(X,Y) \cong h^k(X \setminus Y)$.*

*Proof.* Since $Y \subset \mathbb{R}^n$ is closed, there exists a continuous function $f : \mathbb{R}^n \to [0,\infty[$ with $f^{-1}(0) = Y$, e.g. $f(x) = d(x,Y)$ with the Euclidean distance $d$. Let $V = f^{-1}\,]0,\infty[$. Then $X \cap V = X \setminus Y$ and $X \setminus Y$ is closed in $V$, since $X$ is closed. Note that $V = \mathbb{R}^n \setminus Y$ and $\mathbb{R}^n \setminus X = V \setminus (X \setminus Y)$. We therefore have the (homological) equality

$$h^k(X,Y) = h_{n-k}(\mathbb{R}^n \setminus Y, \mathbb{R}^n \setminus X) = h_{n-k}(V, V \setminus (X \setminus Y).$$

We abbreviate $Z = X \setminus Y$. Then $V$ is open in $\mathbb{R}^n$ and $Z$ closed in $V$. Now we apply 8.1. □

**Proposition 8.3.** *Let $X$ be a closed subset of $\mathbb{R}^n$ and $A$ a compact subset of $X$. Then the quotient map $q : X \to X/A$ induces an isomorphism $q^* : h^k(X/A, \{A\}) \to h^k(X,A)$.*

*Proof.* One can imitate the proof of [3, VIII.6.20]. □

**Remark 8.4.** Let $A \subset X \subset \mathbb{R}^n$ be closed and assume that $X \setminus A$ is contained in a compact set $C$. Then the excision $h^k(X,A) \cong h^k(X \cap C, A \cap C)$ holds. Working through the definitions one verifies that this is an application of ordinary homology excision. ◊

## 9. Product structures

The construction of the tautological cohomology theory did not use product structures. We now start with a multiplicative homology theory and describe a consequence of this additional structure for our setting. A product structure of a homology theory consists of morphisms

$$h_k(X,A) \otimes h_l(Y,B) \to h_{k+l}((X,A) \times (Y,B)), \quad x \otimes y \mapsto x \times y$$

which satisfy some standard axioms (naturality, compatibility with the boundary operator) which we do not recall at this point. From this product structure one can construct slant-, cap-, and cup-products involving tautological cohomology.

A slant-product is defined to be the following composition. In the diagram we use the difference map $d : (x,y) \mapsto x - y$ and $\Sigma_l$ denotes a left-handed suspension. The indices are denoted by an unspecified symbol $\bullet$.

$$\begin{array}{c} h^\bullet(X,A) \otimes h_\bullet((X,A)\times(Y,B)) \\ \Big\downarrow {\scriptstyle =} \\ h_\bullet(\mathbb{R}^n \setminus A, \mathbb{R}^n \setminus X) \otimes h_\bullet((X,A)\times(Y,B)) \\ \Big\downarrow {\scriptstyle \text{product}} \\ h_\bullet((\mathbb{R}^n \setminus A, \mathbb{R}^n \setminus X)\times(X,A)\times(Y,B)) \\ \Big\downarrow {\scriptstyle (d\times 1)_*} \\ h_\bullet((\mathbb{R}^n, \mathbb{R}^n \setminus 0)\times(Y,B)) \\ \Big\downarrow {\scriptstyle \Sigma_l^{-n}} \\ h_\bullet(Y,B) \end{array}$$

From the slant-product we obtain the cap-product by composition with a diagonal $\Delta : (X, A\cup B) \to (X,A)\times(X,B)$

$$\begin{array}{ccc} h^\bullet(X,A)\otimes h_\bullet(X,A\cup B) & \ni & u\otimes v \\ \Big\downarrow {\scriptstyle 1\otimes\Delta_*} & & \Big\downarrow \\ h^\bullet(X,A)\otimes h_\bullet((X,A)\times(X,B)) & & \\ \Big\downarrow {\scriptstyle \text{slant}} & & \\ h_\bullet(X,B) & \ni & u\frown v. \end{array}$$

With this cap-product we arrive at a tautological situation: The natural transformation 5.1 now consists of morphisms

$$h_s(\mathbb{R}^n \setminus A, \mathbb{R}^n \setminus X) \to h_s(\mathbb{R}^n \setminus A, \mathbb{R}^n \setminus X).$$

If everything is arranged correctly it turns out to be the identity.

The definition of exterior cohomology products, associated cup-products and dual slant-products follows the same pattern and is left to the reader. The verification of all details for the product structures in an axiomatic context is a long story.

## 10. Concluding remarks

Our approach can be generalized in many ways: Equivariant setting for compact Lie groups; parametrized spaces. One only needs a situation in which a Tietze extension theorem holds.

If one starts with a cohomology theory, one obtains a dual homology theory with continuity properties known from Steenrod homology (Milnor $\lim - \lim^1$ sequence). One can apply this to (equivariant) $K$-theory in order to obtain $K$-homology. Another input theory could be equivariant Borel cohomology. Since the input uses only the values of the theory on open subsets of Euclidean spaces, one could try de Rham cohomology. Another interesting theory is geometric cobordism [4].

In the context of oriented manifolds one can reduce Poincaré–Lefschetz duality via tubular neighborhoods and a Thom isomorphism for the normal bundle to Alexander duality.

Since the basic setting of dualizible maps is independent of the homology theory one could start with a setting in (semi-)stable homotopy theory. The result is then a fundamental structure in homotopy theory and related to [5].

Final question: Can the setting be extended to infinite dimensional spaces (in analogy to degree theory and fixed point theory, say)?

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