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Filtering the Assembly Map in Algebraic K-Theory and Transfer Reducibility of $\mathbb{Z}^n \rtimes \mathbb{Z}$

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Filtering the Assembly Map in Algebraic K-Theory and Transfer Reducibility of $\mathbb{Z}^n \rtimes \mathbb{Z}$

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Abstract

The present thesis addresses two aspects of the Farrell-Jones Conjecture. First, we generalise certain categories of resolutions due to Waldhausen to construct a spectral sequence converging to the K-theory of a given group ring, and show that this spectral sequence is compatible with the Atiyah-Hirzebruch spectral sequence of the classifying space under the assembly map. Second, we apply a theorem due to Oliver on fixed-point free actions of finite groups to show that $\mathbb{Z}^n \rtimes \mathbb{Z}$ is transfer reducible in the sense of Bartels-Lück-Reich.

Zusammenfassung

Diese Arbeit verfolgt zwei Fragen, die im Zusammenhang mit der Farrell-Jones-Vermutung stehen. Zum Einen werden gewisse Kategorien von Auflösungen, welche auf Waldhausen zurückgehen, verallgemeinert, um eine Spektralsequenz zu konstruieren, die gegen die K-Theorie eines gegebenen Gruppenrings konvergiert. Diese Spektralsequenz ist vermöge der Assemblyabbildung kompatibel mit der Atiyah-Hirzebruch-Spektralsequenz des klassifizierenden Raums. Zum Anderen wird ein Satz von Oliver über fixpunktfreie Wirkungen endlicher Gruppen dazu verwendet, die Transferreduzibilität von $\mathbb{Z}^n \rtimes \mathbb{Z}$ im Sinne von Bartels-Lück-Reich zu zeigen.

Contents

Introduction

The computation of Whitehead groups, reduced class groups and L-groups of discrete groups belongs among the key algebraic problems arising in surgery theory. At the heart of this question lies the Farrell-Jones Conjecture, which in the K-theory case predicts that the assembly map

$$A_{VCyc}\colon H_*^G(E_{VCyc}G;\mathbb{K}_R^{-\infty}) \to K_*(R[G])$$

is an isomorphism for all discrete groups G and unital rings R. Additional interest in this conjecture was sparked by the observation that, combining the K-theory and L-theory case, several important other conjectures can be derived from the validity of the Farrell-Jones Conjecture, among these the Borel Conjecture and the Novikov Conjecture (see [LR05] for details).

Let us assume that we are willing to believe in the validity of the conjecture. Closer inspection reveals a certain disparity between the domain and target of the assembly map. For example, the classifying space $E_{VCyc}G$ is a G-CW-complex, and therefore comes with the canonical filtration given by its skeleta. This does not only turn the G-homology of $E_{VCyc}G$ into a filtered graded abelian group, but leads via standard procedures to a spectral sequence converging to $H_*^G(E_{VCyc}G;\mathbb{K}_R^{-\infty})$ [DL98, Thm. 4.7]. In stark contrast, there is no related filtration on $K_*(R[G])$, regardless of whether we consider the K-theory of $R[G]$ to be given as the homotopy groups of (a delooping of) $BGL(R[G])^+$, $\mathsf{Fr}_{R[G]}^{-1}\mathsf{Fr}_{R[G]}$, $Q\mathsf{Fr}_{R[G]}$ or $iS_\bullet\mathsf{Fr}_{R[G]}$ (where $\mathsf{Fr}_{R[G]}$ is the category of finitely generated free $R[G]$-modules).

The first part of this thesis addresses this "defect". Building on the ideas of Waldhausen [Wal78a, Wal78b, Wal] and how they were transported to the setting of Waldhausen categories by Schwänzl and Staffeldt [SS95], we define in Section 2.1 a category $\mathcal{MV}_G(X)$ of resolutions of finitely generated free $R[G]$-modules which are in a certain way parametrised by a semisimplicial G-set X.

In general, we have no control over the K-theory of these categories. However, in Section 2.2 we then go on to show that $K_*(\mathcal{MV}_G(X))$ does in fact model the K-theory of $R[G]$, provided the semisimplicial G-set X has sufficiently nice homotopical properties (Theorem 2.2.1). After this has been done, it is completely painless to define a filtration of $K_*(R[G])$, since the skeletal filtration of X induces a filtration of the category $\mathcal{MV}_G(X)$.

This filtration even gives rise to a spectral sequence, and Section 2.3 makes a first attempt at shedding some light on the E^1-term of this spectral sequence. While our results are far from complete, they suffice to recover Waldhausen's computations (Theorem 2.3.6 and Remark 2.3.8).

Up to this point, the assembly map will have been absent from our discussion. It will only enter again in Chapter 3, where we give a description which exhibits the assembly map as a map of spectra that preserves the respective filtrations on both sides (Theorem 3.12).

Unfortunately, it turns out that the two spectral sequences behave quite differently, so one cannot expect the assembly map to induce an isomorphism of spectral sequences. It remains an open question whether this can be remedied, e.g. by an appropriate modification of the definition of $\mathcal{MV}_G(X)$. Nevertheless, one may hope that this approach is viable to gain some new insights into the behaviour of the assembly map; this is exemplified by the application of Waldhausen's results in [DQR11].

The second part of this thesis, albeit also concerned with the Farrell-Jones Conjecture, pursues another problem. Over the last few years, the conjecture has been proved for hyperbolic groups [BLR08], groups acting nicely on $\mathsf{CAT}(0)$-spaces [BL12b, Weg12] and virtually polycyclic groups [BFL14, BL12a]. Each of these results made use of a different criterion implying the validity of the Farrell-Jones Conjecture, custom-tailored to suit the peculiarities of the class of groups under consideration. Recently, similar methods have been employed to prove the conjecture for even larger classes of groups (e.g. linear groups [BLRR] and soluble[1] groups [Weg]).

The prototypical examples of polycyclic groups are semidirect products of the form $\mathbb{Z}^n \rtimes \mathbb{Z}$. We will show in Theorem 5.2.2 that these groups do not require the use of the criterion for polycyclic groups, but can instead be shown to satisfy the Farell-Jones Conjecture by invoking that criterion which was manufactured to establish the conjecture for hyperbolic groups.

On the way, we will introduce in Section 4.1 a construction on simplicial complexes with group action which allows a kind of "resolution of fixed points". This construction has another interesting application in the algebraic K-theory of spaces: It enables us to prove an induction result for the "A-theoretic Swan group" (Proposition 5.1.5), which entails a corresponding induction theorem for the A-theory of classifying spaces of finite groups (Corollary 5.1.6).

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Beyond the confines of the maths department, I would like to express my gratitude to my parents and family, to the many friends I have found in Münster, and to everyone else dear to me.

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[1] solvable

Part I.

Filtering the Assembly Map in Algebraic K-Theory

1. Preliminaries

Before the start of the thesis proper, let us recall some of the fundamental concepts and results that will play a role in subsequent chapters. The theory of Waldhausen categories is central to the content of this thesis, and we outline the basics of it in Section 1.1. Waldhausen's machinery produces only a connective K-theory spectrum, but the context in which we are working forces us to consider non-connective K-theory spectra. Unfortunately, there seems to be no construction of a non-connective K-theory spectrum for an arbitrary Waldhausen category. Therefore, after a short technical digression in Section 1.2, we will discuss relevant models of the non-connective K-theory of additive categories in Section 1.3.

I claim no originality for the content of this chapter; its sole purpose lies in making this thesis as self-contained as possible, and I will try to give precise references for the presented material whenever this is possible.

1.1. Waldhausen categories and algebraic K-theory

Even though it is already 30 years old, Waldhausen's article [Wal85] is still the definite reference on the K-theory of Waldhausen categories (excepting maybe the proof of the Additivity Theorem, which has received a shorter proof by now, see [Sta89]).

Let us call a category $\mathcal{C}$ *pointed* if it comes equipped with a choice of zero object. For the purpose of this chapter, the chosen zero object will always be denoted by $*$.

1.1.1 Definition ([Wal85, p. 320]).
A *category with cofibrations* is a pointed category $\mathcal{C}$ together with a subcategory $co\mathcal{C}$, called the *subcategory of cofibrations*, which has the following properties:

(CF1) The unique map $* \to A$ is in $co\mathcal{C}$ for all $A \in \mathcal{C}$.

(CF2) Every isomorphism of $\mathcal{C}$ is in $co\mathcal{C}$.

(CF3) For every diagram of the form $B_1 \leftarrow A \to B_2$ in which the morphism $A \to B_1$ is in $co\mathcal{C}$, the pushout $B_1 \cup_A B_2$ exists and the morphism $B_2 \to B_1 \cup_A B_2$ is in $co\mathcal{C}$.

The cofibrations of $\mathcal{C}$ will usually be denoted by feathered arrows $\rightarrowtail$.

1.1.2 Definition ([Wal85, p. 326]).
A *Waldhausen category* is a category with cofibrations $(\mathcal{C}, *, co\mathcal{C})$ together with a subcategory $w\mathcal{C}$, called the *subcategory of weak equivalences*, such that the following two properties are satisfied:

(WE1) Every isomorphism in $\mathcal{C}$ is a weak equivalence.

(WE2) Suppose we have a morphism of pushout diagrams

$$\begin{array}{ccccc} B_1 & \leftarrowtail & A & \longrightarrow & B_2 \\ \downarrow & & \downarrow & & \downarrow \\ B_1' & \leftarrowtail & A' & \longrightarrow & B_2' \end{array}$$

If all three vertical arrows are weak equivalences, then so is the induced morphism $B_1 \cup_A B_2 \to B_1' \cup_{A'} B_2'$.

The weak equivalences of $\mathcal{C}$ will usually be denoted by arrows of the form $\xrightarrow{\sim}$. In some situations, one category will come equipped with several choices of weak equivalences. In that case, the symbol $\sim$ will receive a decoration (usually the name of the category of weak equivalences) to indicate what kind of weak equivalence it is.

Suppose $(\mathcal{C}, *, co\mathcal{C}, w\mathcal{C})$ is a Waldhausen category. Let us record some easy consequences of the axioms.

- Finite coproducts exist in $\mathcal{C}$: Take the pushout of $A \leftarrowtail * \rightarrowtail B$. Denote such a pushout by $A \vee B$.
- If $A \rightarrowtail B$ is a cofibration, we can form a *quotient object* B/A by taking the pushout of the diagram $B \leftarrowtail A \to *$. The resulting sequence of morphisms $A \rightarrowtail B \to B/A$ will sometimes be called a *cofibration sequence* or an *exact sequence*. The morphism $B \to B/A$ will be called a *quotient map* or *projection*, and we will often denote such maps by two-headed arrows $\twoheadrightarrow$.

1.1.3 Definition ([Wal85, p. 327]).
Let $(\mathcal{C}, *, co\mathcal{C}, w\mathcal{C})$ be a Waldhausen category.

We call $\mathcal{C}$ *saturated* if, given two composable arrows f and g in $\mathcal{C}$, whenever two out of the three morphisms f, g and gf are weak equivalences, so is the third.

We call $\mathcal{C}$ *extensional* if, for any map between exact sequences

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & B/A \\ \downarrow & & \downarrow & & \downarrow \\ A' & \rightarrowtail & B' & \twoheadrightarrow & B'/A' \end{array}$$

in which the left and right vertical arrows are weak equivalences, the middle vertical arrow is a weak equivalence as well.

1.1.4 Definition ([Wal85, pp. 321,327]).
Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between categories with cofibrations.

Call F *exact* if F maps $*_\mathcal{C}$ to $*_\mathcal{D}$, preserves cofibrations and takes the pushout diagrams appearing in (CF3) to pushout diagrams.

If $\mathcal{C}$ and $\mathcal{D}$ are Waldhausen categories, F is called *exact* if it is an exact functor of categories with cofibrations and preserves the weak equivalences. If the choice of weak equivalences w is ambiguous, we will sometimes refer to *w-exact* functors to stress which kind of weak equivalences are preserved.

The following lemma sums up some ways how one can construct new Waldhausen structures. Its validity is tacitly assumed in [Wal85], probably for the reason that the proof is completely straightforward.

1.1.5 Lemma.
Let $\mathcal{C}$ and $\mathcal{D}$ be categories with cofibrations.

- Let $v\mathcal{C}$ and $w\mathcal{C}$ be two subcategories of weak equivalences for $\mathcal{C}$. Denote by $vw\mathcal{C}$ the intersection $v\mathcal{C} \cap w\mathcal{C}$.

  Then $vw\mathcal{C}$ is also a category of weak equivalences for $\mathcal{C}$. If both $v\mathcal{C}$ and $w\mathcal{C}$ are saturated/extensional, the same holds for $vw\mathcal{C}$.

- Let $v\mathcal{C}$ and $w\mathcal{C}$ be two subcategories of weak equivalences for $\mathcal{C}$, and suppose that $v\mathcal{C} \subseteq w\mathcal{C}$. Let $\mathcal{C}^w$ denote the full subcategory of those objects A such that the unique morphism $* \to A$ is in $w\mathcal{C}$ – we call these objects *w-contractible*.

  Then $co\mathcal{C}^w := co\mathcal{C} \cap \mathcal{C}^w$ is a subcategory of cofibrations, and both $v\mathcal{C}^w := v\mathcal{C} \cap \mathcal{C}^w$ and $w\mathcal{C}^w := w\mathcal{C} \cap \mathcal{C}^w$ are categories of weak equivalences for $\mathcal{C}^w$.

- Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor of categories with cofibrations. Let $w\mathcal{D}$ be a category of weak equivalences for $\mathcal{D}$. Let $w\mathcal{C}$ be the subcategory of $\mathcal{C}$ consisting exactly of those morphisms f such that $F(f) \in w\mathcal{D}$. We call this construction *pulling back the equivalences in $w\mathcal{D}$ along F*.

  Then $w\mathcal{C}$ is a category of weak equivalences for $\mathcal{C}$. If $\mathcal{D}$ is saturated or extensional with respect to $w\mathcal{D}$, the same is true for $w\mathcal{C}$.

□

We allow ourselves to be a bit imprecise about the definition of the K-theory space of a Waldhausen category. The point is that we will never have to dig into the actual definition, but are comfortable with using only the main structural results.

Let $(\mathcal{C}, *, co\mathcal{C}, w\mathcal{C})$ be a small Waldhausen category. Then the category $S_n\mathcal{C}$ is (roughly) given by sequences of cofibrations

$$A_1 \rightarrowtail A_2 \rightarrowtail \cdots \rightarrowtail A_n,$$

with morphisms being the natural transformations of such diagrams (see [Wal85, p. 328] for the precise definition). The category $S_n\mathcal{C}$ inherits a natural Waldhausen structure from $\mathcal{C}$ by declaring a morphism to be a cofibration (resp. weak equivalence) if every constituent arrow of the morphism is a cofibration (resp. weak equivalence). Letting n vary, the categories $S_n\mathcal{C}$ assemble to a simplicial Waldhausen category $S_\bullet\mathcal{C}$ (the 0-th face map only makes sense with the proper definition of $S_n\mathcal{C}$). From this, we obtain a simplicial category $wS_\bullet\mathcal{C}$ by restricting to the subcategory of weak equivalences in each degree. This construction is functorial, i.e. an exact functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a simplicial functor $wS_\bullet F\colon wS_\bullet\mathcal{C} \to wS_\bullet\mathcal{D}$.

1.1.6 Definition ([Wal85, p. 330])**.**
Let $\mathcal{C}$ be a small Waldhausen category. The *algebraic K-theory space of* $\mathcal{C}$ is

$$K(\mathcal{C}) := \Omega\, |d(N(wS_\bullet\mathcal{C}))|\,,$$

where $d(N(wS_\bullet\mathcal{C}))$ denotes the diagonal of the bisimplicial set $N(wS_\bullet\mathcal{C})$.

If we want to emphasise the choice of weak equivalences, we also write $K(\mathcal{C}, w)$ for $K(\mathcal{C})$. An exact functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a continuous map between the K-theory spaces of $\mathcal{C}$ and $\mathcal{D}$ which we denote by $K(F)$.

1.1.7 Remark.
Taking coproducts induces an H-space structure on $K(\mathcal{C})$ (see [Wal85, 1.6.2]). We denote the H-space structure on $\mathcal{C}$ also by $\vee$.

We will now collect the main properties of the algebraic K-theory space.

1.1.8 Definition.
Let $\tau\colon F \to G$ be a natural transformation of exact functors $\mathcal{C} \to \mathcal{D}$ between Waldhausen categories. We call τ a *natural equivalence* if $\tau_A\colon F(A) \to G(A)$ is a weak equivalence in $\mathcal{D}$ for all $A \in \mathcal{C}$.

1.1.9 Proposition ([Wal85, Prop. 1.3.1])**.**
Any natural equivalence $\tau\colon F \to G$ induces a natural transformation $wS_\bullet F \to wS_\bullet G$, and therefore a homotopy between $K(F)$ and $K(G)$.

□

1.1.10 Definition ([Wal85, p. 331])**.**
Let $F, G_1, G_2\colon \mathcal{C} \to \mathcal{D}$ be exact functors, and let $\tau_1\colon G_1 \to F$ and $\tau_2\colon F \to G_2$ be natural transformations. We call $G_1 \xrightarrow{\tau_1} F \xrightarrow{\tau_2} G_2$ a *cofibration sequence (of exact functors)*, denoted $G_1 \rightarrowtail F \twoheadrightarrow G_2$, if the following holds:

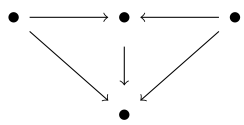
- For every $A \in \mathcal{C}$, the sequence $G_1(A) \to F(A) \to G_2(A)$ is a cofibration sequence.
- If $A \rightarrowtail B$ is a cofibration, then the canonical map $G_1(B) \cup_{G_1(A)} F(A) \to F(B)$ is also a cofibration.

1.1.11 Theorem (Additivity Theorem)**.**
Let $G_1 \rightarrowtail F \twoheadrightarrow G_2$ be a cofibration sequence of exact functors.
Then there is a homotopy

$$|wS_\bullet F| \simeq |wS_\bullet G_1| \vee |wS_\bullet G_2|\,.$$

Proof. [Wal85, Prop. 1.3.2 and Thm. 1.4.2] □

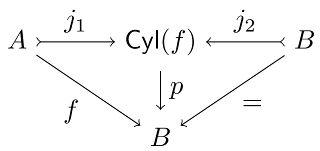
1.1.12 Definition ([Wal85, p. 348f]).
Let $\mathcal{Z}$ be the category represented by the graph



Let $\mathcal{C}$ be a Waldhausen category. A *cylinder functor* Cyl is a functor

$$\mathsf{Cyl}\colon \mathsf{Fun}(\underline{1}, \mathcal{C}) \to \mathsf{Fun}(\mathcal{Z}, \mathcal{C})$$

from the category of arrows and commutative squares in $\mathcal{C}$ to the category of diagrams of shape $\mathcal{Z}$ in $\mathcal{C}$ with the following properties:
The image of an arrow $f\colon A \to B$ is of the form



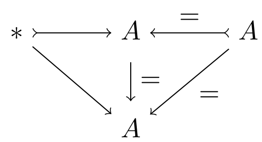
where j_1 is a cofibration called the *front inclusion*, j_2 is a cofibration called the *back inclusion*, and p is called the *projection*. Additionally, we require the following:

(CY1) The front and back inclusion combine to an exact functor

$$\mathsf{Fun}(\underline{1}, \mathcal{C}) \to F_1\mathcal{C}, \quad (f\colon A \to B) \mapsto (A \vee B \xrightarrow{j_1 \vee j_2} \mathsf{Cyl}(f)),$$

where $F_1\mathcal{C}$ is the full subcategory of $\mathsf{Fun}(\underline{1}, \mathcal{C})$ whose objects are the cofibrations in $\mathcal{C}$. It has a Waldhausen structure given by those commutative diagrams $(A \rightarrowtail B) \to (A' \rightarrowtail B')$ as cofibrations for which both $A \to A'$ and $B \cup_A A' \to B'$ are cofibrations, and those commutative diagrams as weak equivalences for which $A \to A'$ and $B \to B'$ are weak equivalences in $\mathcal{C}$.

(CY2) For every $A \in \mathcal{C}$, the unique morphism $* \to A$ is mapped to



A cylinder functor Cyl is called *good (with respect to the category of weak equivalences)* if the projection p is a weak equivalence for all morphisms $f\colon A \to B$.

1.1.13 Proposition ([Wal85, Prop. 1.6.2]).
Suppose $\mathcal{C}$ is a small Waldhausen category and has a good cylinder functor. Define the *suspension* ΣA of an object A via the cofibration sequence

$$A \rightarrowtail \mathsf{Cyl}(A \to *) \twoheadrightarrow \Sigma A.$$

The suspension construction defines an exact functor and thus gives rise to a map

$$\Sigma \colon wS_\bullet \mathcal{C} \to wS_\bullet \mathcal{C}$$

which represents a homotopy inverse to the H-space structure induced by taking sums.

□

1.1.14 Theorem (Fibration Theorem).
Let $\mathcal{C}$ be a small category with cofibrations, and let $v\mathcal{C} \subseteq w\mathcal{C}$ be two categories of weak equivalences for $\mathcal{C}$. Assume $\mathcal{C}$ is saturated and extensional with respect to $w\mathcal{C}$. Suppose that $\mathcal{C}$ admits a cylinder functor which is good with respect to $w\mathcal{C}$.

Then the obvious inclusion functors induce a homotopy pullback square

$$\begin{array}{ccc} vS_\bullet \mathcal{C}^w & \longrightarrow & wS_\bullet \mathcal{C}^w \simeq * \\ \downarrow & & \downarrow \\ vS_\bullet \mathcal{C} & \longrightarrow & wS_\bullet \mathcal{C} \end{array}$$

Proof. [Wal85, Thm. 1.6.4] □

1.1.15 Theorem (Approximation Theorem).
Let $\mathcal{C}$ and $\mathcal{D}$ be saturated small Waldhausen categories. Suppose that $\mathcal{C}$ has a good cylinder functor.

Let $F \colon \mathcal{C} \to \mathcal{D}$ be an exact functor which has the *approximation property*, i.e.:

(AP1) Weak equivalences are detected by F, meaning if $F(f)$ is a weak equivalence in $\mathcal{D}$, then f is a weak equivalence in $\mathcal{C}$.

(AP2) For any $A \in \mathcal{C}$ and morphism $g \colon F(A) \to B$ in $\mathcal{D}$, there is a morphism $f \colon A \to A'$ in $\mathcal{C}$ and a weak equivalence $e \colon F(A') \to B$ such that $g = e \circ F(f)$.

Then the induced maps

$$wF \colon w\mathcal{C} \to w\mathcal{D} \quad \text{and} \quad wS_\bullet F \colon wS_\bullet \mathcal{C} \to wS_\bullet \mathcal{D}$$

are weak equivalences.

Proof. [Wal85, Thm. 1.6.7] □

1.1.16 Remark.
Waldhausen actually has a slightly stronger condition which he calls the approximation property; namely, he requires in (AP2) the morphism f to be a cofibration in $\mathcal{C}$. As observed by Thomason [TT90, Proof of 1.9.1], this stronger approximation property follows from (AP2) by applying the cylinder functor to f since $\mathcal{C}$ is assumed to have a good cylinder functor.

1.2. Constructions on simplicial sets and spectra

This section serves mainly the purpose of fixing notation and terminology. After introducing a number of important constructions (e.g. homotopy colimits) on simplicial sets, we discuss how these translate to spectra.

We assume that the reader is familiar with the basic properties of simplicial sets.

Following common practice, let us call a covariant functor from a small category $\mathcal{C}$ to the category of (pointed) simplicial sets a *covariant (pointed) $\mathcal{C}$-space*. Analogously, a *contravariant (pointed) $\mathcal{C}$-space* is a contravariant functor from $\mathcal{C}$ to the category of pointed simplicial sets. A *map* of $\mathcal{C}$-spaces is a natural transformation of such functors.

1.2.1 Definition ([DL98, Def. 1.4])**.**
Let X be a contravariant pointed $\mathcal{C}$-space and Y a covariant pointed $\mathcal{C}$-space. Their *tensor product* $X \otimes_{\mathcal{C}} Y$ *over* $\mathcal{C}$ is defined as

$$X \otimes_{\mathcal{C}} Y := \bigvee_{C \in \mathcal{C}} X(C) \wedge Y(C)/\sim,$$

where $\sim$ is the equivalence relation generated by $(X(c)(x), y) \sim (x, Y(c)(y))$ for $c\colon C \to C'$ in $\mathcal{C}$, $x \in X(C')$ and $y \in Y(C)$.

We denote the equivalence class of a pair (x, y) in $X \otimes_{\mathcal{C}} Y$ by $x \otimes y$.

1.2.2 Remark.
The tensor product $X \otimes_{\mathcal{C}} Y$ of two pointed $\mathcal{C}$-spaces is a coequaliser in the category of pointed simplicial sets:

$$\bigvee_{c \in \operatorname{mor} \mathcal{C}} X(\operatorname{ran}(c)) \wedge Y(\operatorname{dom}(c)) \overset{L}{\underset{R}{\rightrightarrows}} \bigvee_{C \in \mathcal{C}} X(C) \wedge Y(C) \longrightarrow X \otimes_{\mathcal{C}} Y$$

Here, L denotes the map whose c-component is given by

$$X(c) \wedge \mathrm{id}\colon X(\operatorname{ran}(c)) \times Y(\operatorname{dom}(c)) \to X(\operatorname{dom}(c)) \wedge Y(\operatorname{dom}(c)),$$

and R is similarly given by evaluating Y. In particular, since geometric realisation commutes with arbitrary colimits ([GJ09, p. 7]), there is a natural homeomorphism

$$|X \otimes_{\mathcal{C}} Y| \cong |X| \otimes_{\mathcal{C}} |Y|,$$

where the tensor product of compactly generated Hausdorff spaces is defined via the analogous coequaliser diagram.

Consequently, all constructions we introduce by means of the tensor product for simplicial sets are also defined for topological spaces, and geometric realisation preserves them.

1.2.3 Remark.
Let X be a pointed topological space and S a topological space. Define the "half-smash" $X \rtimes S$ by

$$X \rtimes S := \frac{X \times S}{\{*\} \times S}.$$

The evident map $X \rtimes S \to X \wedge S_+$ sending $x \rtimes s$ to $x \wedge s$ is a natural homeomorphism. In some cases, it will be more convenient to use $X \rtimes S$ instead of $X \wedge S_+$.

1.2.4 Definition ([DL98, Def. 1.8]).
Let $F\colon \mathcal{C} \to \mathcal{D}$ be a covariant functor.

If X is a covariant pointed $\mathcal{C}$-space, then the *induction* $\operatorname{ind}_F X$ *of* X *along* F is defined to be the covariant $\mathcal{D}$-space

$$\operatorname{ind}_F X := \hom_{\mathcal{D}}(F(-), ?)_+ \otimes_{\mathcal{C}} X(-).$$

If X is contravariant, the induction is given by

$$\operatorname{ind}_F X := X(-) \otimes_{\mathcal{C}} \hom_{\mathcal{D}}(?, F(-))_+.$$

If Y is a $\mathcal{D}$-space (no matter whether covariant or contravariant), the *restriction* $\operatorname{res}_F Y$ *of* Y *along* F (which is a $\mathcal{C}$-space of the same variance as Y) is simply the composition

$$\operatorname{res}_F Y := Y \circ F.$$

1.2.5 Lemma.
Let X be a contravariant pointed $\mathcal{C}$-space. Then there is a natural isomorphism

$$\operatorname{ind}_{\mathsf{Id}} X = X(-) \otimes_{\mathcal{C}} \hom_{\mathcal{C}}(?, -)_+ \xrightarrow{\cong} X.$$

Proof. For $C, D \in \mathcal{C}$, define maps $X(D) \rtimes \hom_{\mathcal{C}}(C, D) \xrightarrow{\cong} X(C)$ by $x \rtimes c \mapsto X(c)(x)$; these are well-defined since $X(c)$ is a pointed map for every c, and induce a natural surjective map $X(-) \otimes_{\mathcal{C}} \hom_{\mathcal{C}}(C, -) \xrightarrow{\cong} X(C)$. Injectivity follows from the fact that $x \otimes c = X(c)(x) \otimes \mathrm{id}$ for any point $x \otimes c \in X(-) \otimes_{\mathcal{C}} \hom_{\mathcal{C}}(C, -)_+$.

The inverse, given by the maps $X(C) \to X(-) \otimes_{\mathcal{C}} \hom(C, -), x \mapsto x \otimes \mathrm{id}$, is clearly continuous. □

1.2.6 Proposition ([DL98, Lem. 1.9]).
Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor.

If X is a contravariant pointed $\mathcal{C}$-space and Y a covariant pointed $\mathcal{D}$-space, there is a natural adjunction isomorphism

$$\operatorname{ind}_F X \otimes_{\mathcal{D}} Y \xrightarrow{\cong} X \otimes_{\mathcal{C}} \operatorname{res}_F Y.$$

Similarly, for X a contravariant pointed $\mathcal{D}$-space and Y a covariant pointed $\mathcal{C}$-space, there is a natural adjunction isomorphism

$$X \otimes_{\mathcal{D}} \operatorname{ind}_F Y \xrightarrow{\cong} \operatorname{res}_F X \otimes_{\mathcal{C}} Y.$$

Proof. There is no difficulty in checking that the required isomorphisms are given by

$$\operatorname{ind}_F X \otimes_{\mathcal{D}} Y \to X \otimes_{\mathcal{C}} \operatorname{res}_F Y, \quad x \otimes d \otimes y \mapsto x \otimes Y(d)(y),$$
$$X \otimes_{\mathcal{D}} \operatorname{ind}_F Y \to \operatorname{res}_F X \otimes_{\mathcal{C}} Y, \quad x \otimes c \otimes y \mapsto X(c)(x) \otimes y.$$

□

1.2.7 Definition ([DL98, Def. 3.13]).
Let X be a covariant pointed $\mathcal{C}$-space. Define the *homotopy colimit* of X to be

$$\operatorname*{hocolim}_{\mathcal{C}} X := N(-/\mathcal{C})_+ \otimes_{\mathcal{C}} X(-).$$

1.2.8 Proposition.
Let $f\colon X \to Y$ be a map of covariant pointed $\mathcal{C}$-spaces. If $f(C)\colon X(C) \to Y(C)$ is a weak equivalence for all $C \in \mathcal{C}$, then the induced map

$$\operatorname*{hocolim}_{\mathcal{C}} f\colon \operatorname*{hocolim}_{\mathcal{C}} X \to \operatorname*{hocolim}_{\mathcal{C}} Y$$

is also a weak equivalence.

Proof. As we observed in Remark 1.2.2, there is a natural homeomorphism

$$\left|\operatorname*{hocolim}_{\mathcal{C}} X\right| = |N(-/\mathcal{C})_+ \otimes_{\mathcal{C}} X(-)| \cong |N(-/\mathcal{C})|_+ \otimes_{\mathcal{C}} |X(-)|.$$

Therefore, the proposition follows from [DL98, Thm. 3.11]. □

1.2.9 Remark.
We have analogous constructions for unpointed $\mathcal{C}$-spaces, in which case the one-point unions $\bigvee$ have to be replaced by disjoint unions $\coprod$, and the smash products $\wedge$ have to be replaced by direct products $\times$. Everything we have said carries over to this setting, and can also be found in the original reference [DL98, Section 1].

These constructions extend easily to spectra. By a spectrum $\mathbb{X}$, we mean a sequence of pointed (compactly generated Hausdorff) spaces $\mathbb{X} = \{X_n\}_{n\in\mathbb{N}}$ together with structure maps $\sigma_n\colon X_n \wedge S^1 \to X_{n+1}$. We call a spectrum $\mathbb{X}$ an Ω-spectrum if the adjoints of the structure maps are weak equivalences for all n. A map of spectra $f\colon \mathbb{X} \to \mathbb{Y}$ is a sequence of pointed continuous maps $\{f_n\colon X_n \to Y_n\}_n$ which are compatible with the structure maps in the sense that $f_{n+1} \circ \sigma_n = \sigma_n \circ (f_n \wedge S^1)$.

A *levelwise equivalence* of spectra is a map of spectra $f\colon \mathbb{X} \to \mathbb{Y}$ such that f_n is a weak equivalence for all n.

Building on these definitions, we obtain all the usual notions; for example, the homotopy groups $\pi_*(\mathbb{X})$ of a spectrum $\mathbb{X}$ are given by

$$\pi_n(\mathbb{X}) := \operatorname*{colim}_{k\to\infty} \pi_{n+k}(X_k).$$

We call a map of spectra a *weak equivalence* if it induces an isomorphism on π_n for all n.

1.2.10 Definition.
Let $\mathbb{X}$ be a spectrum. For every n, the structure maps of $\mathbb{X}$ give rise to a directed system

$$X_n \xrightarrow{\sigma_n} \Omega X_{n+1} \xrightarrow{\Omega\sigma_{n+1}} \Omega^2 X_{n+2} \to \dots$$

Define the *Ω-spectrification* $\mathbb{X}^\Omega$ *of* $\mathbb{X}$ to be the spectrum given by

$$\mathbb{X}^\Omega_n := \operatorname*{hocolim}_{k\to\infty} \Omega^k X_n.$$

In fact, taking Ω-spectrifications defines an endofunctor on the category of spectra. As the name suggests, $\mathbb{X}^\Omega$ is always an Ω-spectrum. The particularly nice property of this functor is that it converts weak equivalences into levelwise equivalences.

The tensor product construction on simplicial sets generalises to a tensor product between simplicial sets and spectra. By a covariant $\mathcal{C}$-spectrum, we mean a covariant functor from $\mathcal{C}$ to the category of spectra.

1.2.11 Definition ([DL98, p. 207])**.**
Let X be a contravariant pointed $\mathcal{C}$-space, and let $\mathbb{E}$ be a covariant $\mathcal{C}$-spectrum. Then define their *tensor product* to be the spectrum

$$X \otimes_{\mathcal{C}} \mathbb{E} := \Big\{ |X| \otimes_{\mathcal{C}} \mathbb{E}_n \Big\}_n,$$

whose structure maps are given by $\mathrm{id}_{|X|} \otimes_{\mathcal{C}} \sigma_n$.

In particular, the notion of *homotopy colimit* over a small category $\mathcal{C}$ extends naturally to the category of spectra since it was defined by taking the tensor product with the contravariant pointed $\mathcal{C}$-space $N(-/\mathcal{C})_+$.

Maps of $\mathcal{C}$-spectra $f\colon \mathbb{X} \to \mathbb{Y}$ induce maps between their homotopy colimits, and it follows from Proposition 1.2.8 that, if $f(C)$ is a levelwise equivalence for every $C \in \mathcal{C}$, then $\mathrm{hocolim}_{\mathcal{C}}\, f$ is a levelwise equivalence as well.

We have no desire to delve into the intricacies of actual stable homotopy theory. Instead, we resort to the rather primitive convention (which is sufficient for our purposes) that a *homotopy fibration of spectra* refers to a sequence $\mathbb{X} \to \mathbb{Y} \to \mathbb{Z}$ such that on each level n, the sequence $X_n \to Y_n \to Z_n$ is a homotopy fibration. Since taking homotopy colimits over directed systems preserves homotopy fibrations, applying the Ω-spectrification functor preserves homotopy fibrations of spectra.

1.3. Models for the K-theory of additive categories

Our original intent is to understand the algebraic K-theory of rings; that is to say, the algebraic K-theory of the category of finitely generated free or projective modules over a ring. These categories are additive and essentially small. They fit into the picture of Waldhausen categories since we may regard any small additive category $\mathcal{A}$ as a Waldhausen category by declaring those morphisms which are isomorphic to the inclusion of a direct summand to be cofibrations, and letting the weak equivalences consist exactly of the isomorphisms in $\mathcal{A}$.

However, if we wish to apply the machinery summarised in Section 1.1, it is not a good idea to consider $K(\mathcal{A})$: Both the Fibration and Approximation Theorem require cylinder functors, and we do not have these available for an arbitrary additive category. The remedy for this problem lies in passing to the category of (co)chain complexes over $\mathcal{A}$.

We let $\mathsf{Ch}(\mathcal{A})$ denote the category of finite **cochain** complexes over $\mathcal{A}$, i.e. diagrams of the form

$$\cdots \to A^{n-1} \xrightarrow{a^{n-1}} A^n \xrightarrow{a^n} A^{n+1} \to \dots$$

such that $a^{n+1}a^n = 0$ for all $n \in \mathbb{Z}$ and such that the set $\{n \in \mathbb{Z} \mid A^n \neq 0\}$ is finite. Note that we do allow non-zero components in negative degrees.

We could work equally well with chain complexes over $\mathcal{A}$ (and this is what most other people do), given that the categories of finite chain complexes and finite cochain complexes are isomorphic (by reversing the indices). But using cochain complexes is more convenient for what we are planning to do in subsequent chapters.

We may equip $\mathsf{Ch}(\mathcal{A})$ with a Waldhausen structure: Let $co\mathsf{Ch}(\mathcal{A})$ be the subcategory consisting of those chain maps[1] which are isomorphic to a chain map consisting of inclusions of direct summands in each degree (the chain map itself is *not* required to split). Defining chain homotopies by the usual formulas (being careful about the index shifts: homotopies consist now of morphisms of degree -1), we obtain the notion of chain equivalence of cochain complexes. Let the weak equivalences $h\mathsf{Ch}(\mathcal{A})$ consist of exactly the chain equivalences.

Then $\mathsf{Ch}(\mathcal{A})$ is a Waldhausen category, and the natural functor $\mathcal{A} \to \mathsf{Ch}(\mathcal{A})$ mapping an object to a cocomplex concentrated in degree 0 is an exact functor. Moreover, $\mathsf{Ch}(\mathcal{A})$ has a cylinder functor, given by the usual mapping cylinder. The following theorem tells us that we may use the K-theory of $\mathsf{Ch}(\mathcal{A})$ as a model for the K-theory of $\mathcal{A}$:

1.3.1 Theorem (Gillet-Waldhausen).
The canonical functor $\mathcal{A} \to \mathsf{Ch}(\mathcal{A})$ induces a weak equivalence

$$iS_\bullet\mathcal{A} \xrightarrow{\sim} hS_\bullet\mathsf{Ch}(\mathcal{A}).$$

Proof. [TT90, Thm. 1.11.7], [CP97, Prop. 6.1] □

[1] Even though “cochain map” would probably be slightly more appropriate, we prefer the shorter term in the interest of readibility.

While Theorem 1.3.1 makes the Fibration and Approximation Theorems available to us in the context of additive categories, it has a downside: When dealing with the algebraic K-theory of group rings, negative K-groups enter the picture quite naturally (via the Bass-Heller-Swan-Theorem); so we would like to have a non-connective K-theory spectrum at our disposal. Unfortunately, there is no general construction of a non-connective K-theory spectrum for Waldhausen categories (Waldhausen's machinery does produce a delooping of the K-theory space, but this yields a connective spectrum). Nevertheless, there is a workaround for this in the form of the Pedersen-Weibel delooping (see [PW85], cf. [CP97] or [CP95, p. 737f.]). In what follows, we are not intending to give a full exposition of the delooping process, but only wish to remind the reader of the construction. We assume some familiarity with the concepts involved, even though some of them are explained in Appendix A.

Let $\mathcal{A}$ be a small additive category. Define a category $\mathcal{CA}$, which has as objects collections $A = (A_z)_{z\in\mathbb{Z}}$ with $A_z \in \mathcal{A}$ for all z, and whose morphisms are the so-called *bounded morphisms*: These are collections of morphisms $f_{z,z'}\colon A_z \to B_{z'}$ in $\mathcal{A}$, indexed over $\mathbb{Z}^2$, such that there is some $R > 0$ for which $f_{z,z'} = 0$ as soon as $|z - z'| \geqslant R$; one may think about these as infinite matrices whose non-zero entries are only a bounded distance away from the diagonal. Composition of morphisms is then given by matrix multiplication.

The category $\mathcal{CA}$ inherits an additive structure from $\mathcal{A}$. Let $\mathcal{C}_+\mathcal{A}$ and $\mathcal{C}_-\mathcal{A}$ denote the full additive subcategories of $\mathcal{CA}$ which consist of those objects whose support (the set of integers for which $A_z \neq 0$) is bounded below/above. These categories possess an Eilenberg swindle, i.e. an endofunctor S such that $S \oplus \mathsf{Id}$ is naturally isomorphic to S. It is a consequence of the Additivity Theorem that the existence of such a functor implies the contractibility of the K-theory space.

Since the natural inclusion of $\mathcal{A}$ into $\mathcal{CA}$ factors through both $\mathcal{C}_+\mathcal{A}$ and $\mathcal{C}_-\mathcal{A}$, we obtain two canonical nullhomotopies of the map $K(\mathcal{A}) \to K(\mathcal{CA})$, which together define a map

$$K(\mathcal{A}) \wedge S^1 \to K(\mathcal{CA}).$$

Iterating this construction, we obtain a spectrum $\mathbb{K}(\mathcal{A}) := \{K(\mathcal{C}^{n-1}\mathcal{A})\}_n$, where we interpret $K(\mathcal{C}^{-1}\mathcal{A})$ as a point. The non-connective K-theory spectrum of $\mathcal{A}$ can now be obtained by taking the Ω-spectrification of this spectrum.

We do a slight variation on this: The functoriality of taking cochain complexes over a small additive category yields Eilenberg swindles on $\mathsf{Ch}(\mathcal{C}_\pm\mathcal{A})$, which again induce canonical contractions of the associated K-theory spaces (by the Additivity Theorem). Therefore, the map $K(\mathsf{Ch}(\mathcal{A})) \to K(\mathsf{Ch}(\mathcal{C}_\pm\mathcal{A}))$ is also nullhomotopic in two canonical ways, and we construct a spectrum $\mathbb{K}(\mathsf{Ch}(\mathcal{A})) := \{K(\mathsf{Ch}(\mathcal{C}^{n-1}\mathcal{A}))\}_n$ as before. Again by functoriality, the inclusion maps $\mathcal{C}^n\mathcal{A} \to \mathsf{Ch}(\mathcal{C}^n\mathcal{A})$ induce a map of spectra

$$\mathbb{K}(\mathcal{A}) \to \mathbb{K}(\mathsf{Ch}(\mathcal{A}))$$

which is a levelwise equivalence by the Gillet-Waldhausen theorem.

1.3.2 Definition.

The *non-connective algebraic K-theory spectrum of $\mathcal{A}$*, denoted $\mathbb{K}^{-\infty}(\mathcal{A})$, is by definition the Ω-spectrification of $\mathbb{K}(\mathsf{Ch}(\mathcal{A}))$.

The previous discussion exemplifies quite nicely how we will deal with the problem of generalising results from the connective to the non-connective case; whenever we would like to do so, we will be able to use the categories $\mathcal{C}^n\mathcal{A}$ to transfer our argument to negative K-theory.

We are particularly interested in the K-theory of "group ring" categories, by which we mean the following:

1.3.3 Definition ([BR07, Def. 2.1]).
Let G be a discrete group. Let $\mathcal{A}$ be a small additive category on which G acts from the right. Let T be a (left) G-set. We let

$$\mathcal{A} *_G T$$

denote the small additive category whose objects are collections $A = \{A_t\}_{t \in T}$ for which the set $\{t \in T \mid A_t \neq 0\}$ is finite, and in which a morphism $\varphi \colon A \to B$ consists of a collection $\{\varphi_{g,t} \colon A_{g^{-1}t} \to B_t g\}_{g \in G, t \in T}$ of morphisms in $\mathcal{A}$ such that the set $\{(g,t) \in G \times T \mid \varphi_{g,t} \neq 0\}$ is finite. Composition of morphisms is given by the formula

$$(\psi \circ \varphi)_{g,t} = \sum_{kh=g} \psi_{k,t} h \circ \varphi_{h,k^{-1}t}.$$

If $T = G/G$, we will usually just write $\mathcal{A}[G]$ instead of $\mathcal{A} *_G G/G$.

1.3.4 Remark.
Our definition of $\mathcal{A} *_G T$ does not match the definition in [BR07] exactly: Morphisms are indexed in a slightly different way. Of course, this does not really change the category. There is an obvious isomorphism between $\mathcal{A} *_G T$ and the category bearing the same name in *loc. cit.*

1.3.5 Remark.
The construction $\mathcal{A} *_G T$ is related to group rings in the following way: Suppose $\mathcal{A}$ is a small model (e.g. a skeletal subcategory) for the category of finitely generated free R-modules, where R is some unital ring. Letting G act trivially on $\mathcal{A}$, the category $\mathcal{A} *_G G/H$ can be easily seen to be additively equivalent to the category of finitely generated free $R[H]$-modules. Since a decomposition $T = \coprod_r T_r$ of T into transitive G-sets induces an isomorphism $\bigoplus_r \mathcal{A} *_G T_r \cong \mathcal{A} *_G T$, the category $\mathcal{A} *_G T$ is only built from categories of modules over a group ring. See [BR07, Ex. 2.4–2.6, Section 6] for a more detailed discussion.

If we consider $\mathcal{A} *_G T$ as a rule associating an additive category to every pair $(\mathcal{A}, T)$, this construction is functorial in $\mathcal{A}$; it is nearly functorial in T, modulo some problems regarding choices of direct sums and the braiding $A \oplus B \cong B \oplus A$. The ambiguity in choosing a direct sum can be eliminated by passing to an equivalent additive category which possesses a functorial direct sum. A possible remedy for the remaining problems is given in [BR07, Rem. 2.3]; but since this would burden us with additional (essentially superfluous) notation, we follow the authors in *loc. cit.* and "prefer to ignore this problem".

We have already discussed how to obtain a non-connective K-theory spectrum $\mathbb{K}^{-\infty}(\mathcal{A} *_G T)$ for any additive category with right G-action $\mathcal{A}$ and G-set T. It will be important to know that we can construct an equivalent spectrum in a slightly different fashion:

If $\mathcal{A}$ carries a right G-action, so does $\mathcal{C}\mathcal{A}$. Consequently, we can form the category $(\mathcal{C}\mathcal{A}) *_G T$, and we have the associated subcategories $(\mathcal{C}_\pm\mathcal{A}) *_G T$. By functoriality, we obtain Eilenberg swindles on $(\mathcal{C}_\pm\mathcal{A}) *_G T$, and therefore a map

$$K(\mathcal{A} *_G T) \wedge S^1 \to K((\mathcal{C}\mathcal{A}) *_G T).$$

By the same reasoning, there are maps $K((\mathcal{C}^n\mathcal{A}) *_G T) \wedge S^1 \to K((\mathcal{C}^{n+1}\mathcal{A}) *_G T)$ for every n; these serve as structure maps in a spectrum

$$\mathbb{K}(\mathcal{A}, T) = \left\{K((\mathcal{C}^{n-1}\mathcal{A}) *_G T)\right\}_n.$$

1.3.6 Proposition.
There is a natural levelwise equivalence

$$\mathbb{K}(\mathcal{A}, T) \xrightarrow{\sim} \mathbb{K}(\mathcal{A} *_G T).$$

Proof. The proof relies on the fact that some inclusions of additive subcategories induce particularly nice homotopy fibration sequences of K-theory spaces, and is more of an aside to our present discussion. See Appendix A for details. □

As before, we can use cochain complexes over $(\mathcal{C}^n\mathcal{A}) *_G T$ instead of the category itself to define another spectrum $\mathbb{K}'(\mathcal{A}, T)$. Since we know that the other three maps in the commutative square

$$\begin{array}{ccc} \mathbb{K}(\mathcal{A}, T) & \xrightarrow{\sim} & \mathbb{K}(\mathcal{A} *_G T) \\ \sim\downarrow & & \downarrow\sim \\ \mathbb{K}'(\mathcal{A}, T) & \longrightarrow & \mathbb{K}(\mathsf{Ch}(\mathcal{A} *_G T)) \end{array}$$

are levelwise equivalences, it follows that the map $\mathbb{K}'(\mathcal{A}, T) \to \mathbb{K}(\mathsf{Ch}(\mathcal{A} *_G T))$ is also a levelwise equivalence.

1.3.7 Definition.
Define $\mathbb{K}^{-\infty}(\mathcal{A}, T)$ to be the Ω-spectrification of $\mathbb{K}'(\mathcal{A}, T)$.

1.3.8 Corollary.
There is a natural levelwise equivalence

$$\mathbb{K}^{-\infty}(\mathcal{A}, T) \xrightarrow{\sim} \mathbb{K}^{-\infty}(\mathcal{A} *_G T).$$

□

2. Filtering the algebraic K-theory of group rings

As outlined in the introduction, this chapter is concerned with establishing a potentially interesting filtration on the K-theory of group rings of discrete groups. After defining the appropriate category $\mathcal{MV}_G(X)$, we give a criterion for when the K-theory of this category turns out to be the K-theory of a group ring. The chapter closes with a discussion of a spectral sequence associated to the K-theory of $\mathcal{MV}_G(X)$.

2.1. The category $\mathcal{W}_G(X;\mathcal{K})$

Our main goal for this section is the definition of the category $\mathcal{MV}_G(X)$. This category is a generalisation of the category of "Mayer-Vietoris presentations" considered by Waldhausen in [Wal78a, Wal78b]. Our definition takes some inspiration from [SS95], where Waldhausen's exact categories were promoted to Waldhausen categories with cylinder functor, thus making it possible to study their K-theory via Waldhausen's machinery. Still, our definition does not coincide exactly with that of Schwänzl and Staffeldt; instead of simplicial modules, we choose to work with cochain complexes. Interestingly enough, this comes again closer to Waldhausen's original take on the subject as contained in his preprint [Wal], in which "Mayer-Vietoris presentations" still consisted of chain complexes. However, the observant reader will notice that the contractibility condition in 2.1.15 is somewhat different.

To give the reader some intuition which kinds of objects we intend to study, let us first give an informal description of a Mayer-Vietoris resolution. So let G be a discrete group, R a ring, and X a simplicial complex with G-action. Suppose we are given a finitely generated free $R[G]$-module M. When we speak of a resolution $M \to M^*$ of such a module, we think of a contractible, finite, free augmented $R[G]$-cochain complex. What needs further clarification is what we mean by saying that this is a "resolution over X". First of all, we think of the module M^n as based on the n-simplices of X; i.e., M^n comes equipped with an $R[G]$-basis, and each basis element is assigned an n-simplex of X. Thus, each basis element generates a free $R[G_x]$-module, x being the simplex assigned to the basis element, which lives entirely on a single simplex. The actions of other group elements translate this module to other simplices in the same orbit. So we can regard each M^n as a direct sum of induced modules, where the subgroups from which the modules are induced are controlled by the stabilisers of the n-simplices of X. As soon as we wish to define what a morphism between such resolutions is, we will of course also want the morphism in degree n to respect this direct sum decomposition, and to be induced from the appropriate stabiliser groups.

Moreover, the differentials in the resolution $M \to M^*$ are not allowed to be arbitrary. While we have used the simplices of X themselves as well as their stabilisers in the characterisation of the modules M^n, the simplicial structure of X has not yet entered the picture. The differentials are supposed to arise as alternating sums of coface maps. By an i-th coface map, we mean an $R[G]$-linear map $M^n \to M^{n+1}$ such that the image of a given basis element of M^n, located on a simplex x, is only allowed to be based on those $(n+1)$-simplices whose i-th face is x. Additionally, we will again want to take the stabiliser of x into consideration. Naturally, interpreting the differentials as alternating sums of coface maps will force us to require the coface maps to satisfy the usual cosimplicial identities (otherwise we would not be defining a cochain complex).

In practice, we want to resolve cochain complexes instead of just modules: Remember that the Fibration and Approximation Theorem are only available to us when the categories under consideration possess a cylinder functor.

2.1.1 Definition.
A *G-coefficient system* is a covariant functor $\mathcal{K}\colon G\text{-}\mathsf{Sets} \to \mathsf{Add}$ from the category of G-sets to the category of small additive categories which is monoidal in the following sense: For every G-set T, the canonical functor $\bigoplus_r \mathcal{K}(T_r) \to \mathcal{K}(T)$ (induced by the decomposition $T = \coprod_r T_r$ of T into transitive G-sets) is an isomorphism of categories, and these isomorphisms combine to a natural isomorphism $\mathcal{K} \to \widetilde{\mathcal{K}}$, where $\widetilde{\mathcal{K}}$ is the functor $G\text{-}\mathsf{Sets} \to \mathsf{Add}$ given by $\widetilde{\mathcal{K}}(T) := \bigoplus_r \mathcal{K}(T_r)$.

We assume that $\mathcal{K}(\emptyset)$ is always the trivial additive category containing only a zero object.

2.1.2 Remark.
Let $\mathcal{K}$ be a G-coefficient system. If $T = \coprod_r T_r$ is any decomposition of a G-set T into G-invariant subsets, it follows that the canonical functor $\bigoplus_r \mathcal{K}(T_r) \to \mathcal{K}(T)$ is an isomorphism.

We also obtain induced isomorphisms $\bigoplus_r \mathsf{Ch}(\mathcal{K}(T_r)) \xrightarrow{\cong} \mathsf{Ch}(\mathcal{K}(T))$. Consequently, we can decompose cocomplexes and their maps into smaller pieces. When we speak about the T_r*-component* of a cocomplex or map, we mean the summand living in $\mathsf{Ch}(\mathcal{K}(T_r))$.

2.1.3 Example.
Let $\mathcal{A}$ be a small additive category with right G-action. Then $\mathcal{A}$ induces a G-coefficient system $\mathcal{K}_\mathcal{A}$ given by

$$\mathcal{K}_\mathcal{A}(T) := \mathcal{A} *_G T.$$

Fix the following data: Let G be a discrete group, $\mathcal{K}$ a G-coefficient system, and let X be a semisimplicial G-set. As a permanent convention, we agree to think of X as a trivially augmented semisimplicial G-set. That is, there is a set of (-1)-simplices $X_{-1} = G/G$, and there is a unique boundary map $d\colon X_0 \to X_{-1}$ given by the obvious projection map. Moreover, when we speak about ordinal number maps $\underline{m} \hookrightarrow \underline{n}$ in Δ_{inj}, we usually also allow the unique map $\emptyset = \underline{-1} \to \underline{n}$ which induces the augmentation $X_n \to X_{-1}$.

2.1.4 Definition.
We define a category $\mathcal{W}_G(X;\mathcal{K})$ as follows: An object (A, a) consists of a collection $A = \{A^n\}_{n \geqslant -1}$ of objects $A^n \in \mathsf{Ch}(\mathcal{K}(X_n))$, together with a collection $a = \{a_n^i\}_{n \geqslant -1, 0 \leqslant i \leqslant n+1}$ of morphisms $a_n^i \colon A^n \to \mathcal{K}(d_i)(A^{n+1})$ in $\mathsf{Ch}(\mathcal{K}(X_n))$ which satisfy the cosimplicial identities in the sense that

$$\mathcal{K}(d_i)(a_{n+1}^j) \circ a_n^i = \mathcal{K}(d_{j-1})(a_{n+1}^i) \circ a_n^{j-1} \tag{2.1}$$

whenever $i < j$. Moreover, we require that $A^n = 0$ for all but finitely many n.

A morphism $\varphi\colon (A, a) \to (B, b)$ is a sequence $\varphi = \{\varphi^n \colon A^n \to B^n\}_{n \geqslant -1}$, with each φ^n a morphism in $\mathsf{Ch}(\mathcal{K}(X_n))$, such that $\mathcal{K}(d_i)(\varphi^{n+1}) \circ a_n^i = b_n^i \circ \varphi^n$ for all n and i.

The composition of two morphisms $\varphi\colon A \to B$ and $\psi\colon B \to C$ is defined to be $\psi \circ \varphi = \{\psi^n \circ \varphi^n\}_n$.

2.1.5 Remark.
Let (A, a) be an object in $\mathcal{W}_G(X;\mathcal{K})$. Given a morphism $\mu\colon \underline{m} \hookrightarrow \underline{n}$ in Δ_{inj}, define

$$a(\mu) := \mathcal{K}(d_{i_m} \circ \cdots \circ d_{i_{n-2}})(a^{i_{n-1}}) \circ \cdots \circ \mathcal{K}(d_{i_m})(a^{i_{m+1}}) \circ a^{i_m},$$

where $\mu = d^{i_{n-1}} \circ \cdots \circ d^{i_m}$ is a factorisation of μ into coface maps; because of equality 2.1, this is a well-defined morphism

$$a(\mu)\colon A^m \to \mathcal{K}(\mu^*)(A^n).$$

If $\varphi\colon (A, a) \to (B, b)$ is a morphism in $\mathcal{W}_G(X;\mathcal{K})$, it follows that for any $\mu\colon \underline{m} \hookrightarrow \underline{n}$, we have

$$\mathcal{K}(\mu)(\varphi^n) \circ a(\mu) = b(\mu) \circ \varphi^m.$$

Note that this applies in particular to the morphism $\emptyset \to \underline{n}$.

We would like to equip $\mathcal{W}_G(X;\mathcal{K})$ with a Waldhausen structure. As far as cofibrations are concerned, there is a fairly canonical choice: Let $co\mathcal{W}_G(X;\mathcal{K})$ be the subcategory of those morphisms φ with the property that φ^n is a cofibration of cocomplexes for every n.

2.1.6 Lemma.
The triple $(\mathcal{W}_G(X;\mathcal{K}), 0, co\mathcal{W}_G(X;\mathcal{K}))$ is a category with cofibrations.

Proof. Axiom `(CF1)` is clearly satisfied. If φ is an isomorphism, then each φ^n is an isomorphism, so `(CF2)` holds. Suppose a diagram of the form $(B, b) \leftarrowtail (A, a) \to (C, c)$

is given. Then we obtain for each n a pushout diagram in $\mathsf{Ch}(\mathcal{K}(X_n))$

$$\begin{array}{ccc} A^n & \rightarrowtail & B^n \\ \downarrow & & \downarrow \\ C^n & \rightarrowtail & P^n \end{array}$$

The morphisms $p_n^i\colon P^n \to \mathcal{K}(d_i)(P^{n+1})$ are induced by the universal property of the pushout construction, and the cosimplicial identities also hold by virtue of the universal property. Each morphism $C^n \rightarrowtail P^n$ is a cofibration. This proves `(CF3)`. □

There is a seemingly canonical choice of weak equivalences on $\mathcal{W}_G(X;\mathcal{K})$ where we require every component of a morphism to be a chain equivalence (in fact, we will study these weak equivalences in the course of Section 2.3, and it will turn out that the result is somewhat boring). However, our freedom in the choice of weak equivalences on $\mathcal{W}_G(X;\mathcal{K})$ is what makes this category really interesting. A typical notion of weak equivalence is of the following type:

2.1.7 Definition.
Let $T \subseteq \coprod_{n\geqslant -1} X_n$ be a non-empty G-subset. Define $T^n := T \cap X_n$, and write $T^n = \coprod_r T_r^n$ for the decomposition of T^n into transitive G-sets.

Call a morphism φ in $\mathcal{W}_G(X;\mathcal{K})$ a T-equivalence if, for every $n \geqslant -1$, all T_r^n-components of φ^n are chain equivalences (equivalently, the T^n-component of φ^n is a chain equivalence for all n).

The composition of two T-equivalences is a T-equivalence, so the collection of T-equivalences forms a subcategory which we denote by $h_T\mathcal{W}_G(X;\mathcal{K})$. In the special case where $T = \coprod_{n\geqslant -1} X_n$ is the set of all simplices, we denote the corresponding subcategory of weak equivalences by $h\mathcal{W}_G(X;\mathcal{K})$.

2.1.8 Proposition.
The quadruple $(\mathcal{W}_G(X;\mathcal{K}), 0, co\mathcal{W}_G(X;\mathcal{K}), h_T\mathcal{W}_G(X;\mathcal{K}))$ is a saturated and extensional Waldhausen category for every non-empty G-subset $T \subseteq \coprod_{n\geqslant -1} X_n$.

The mapping cylinder construction on the category of cocomplexes over an additive category induces a good cylinder functor on $\mathcal{W}_G(X;\mathcal{K})$ for every choice of T.

Proof. If φ is an isomorphism, every φ^n is an isomorphism, so `(WE1)` holds regardless of the choice of T. If we have a transformation of pushout diagrams which consists of chain equivalences in all degrees specified by T, the induced morphism on the pushouts is also a chain equivalence in all degrees given by T (since `(WE2)` holds in any category of cochain complexes). This proves `(WE2)`.

The category $\mathcal{W}_G(X;\mathcal{K})$ is saturated and extensional because the category of cochain complexes over an additive category has these properties. Given a morphism $\varphi\colon (A,a) \to (B,b)$, we can apply the cylinder functor to each φ^n separately; i.e. we define

$$\mathsf{Cyl}(\varphi)^n := \mathsf{Cyl}(\varphi^n).$$

Since additive functors preserve the cylinder functor, there is a natural isomorphism $\mathcal{K}(d_i)(\mathsf{Cyl}(\varphi^{n+1})) \cong \mathsf{Cyl}(\mathcal{K}(d_i)(\varphi^{n+1}))$. This yields the coboundary maps $c_n^i\colon \mathsf{Cyl}(\varphi^n) \to \mathcal{K}(d_i)(\mathsf{Cyl}(\varphi^{n+1}))$ by functoriality, and also by functoriality we get an object $(\mathsf{Cyl}(\varphi), c) \in \mathcal{W}_G(X;\mathcal{K})$. The cylinder diagrams

$$\begin{array}{ccccc} (A,a) & \rightarrowtail & \mathsf{Cyl}(\varphi) & \leftarrowtail & (B,b) \\ & \overset{\varphi}{\searrow} & \downarrow & \overset{=}{\swarrow} & \\ & & (B,b) & & \end{array}$$

are of the required shape, and (CY2) is clearly satisfied. The induced functor

$$\mathsf{Fun}(\underline{1}, \mathcal{C}) \to F_1\mathcal{C}, \quad (\varphi\colon (A,a) \to (B,b)) \mapsto ((A,a) \oplus (B,b) \rightarrowtail \mathsf{Cyl}(\varphi))$$

preserves the zero object. Suppose $(\psi_A, \psi_B)\colon (A \xrightarrow{\varphi} B) \to (A' \xrightarrow{\varphi'} B')$ is a commutative square in $\mathcal{W}_G(X;\mathcal{K})$, and both ψ_A and ψ_B are cofibrations. Then $\psi_A \oplus \psi_B$ is a cofibration, and the morphism $\mathsf{Cyl}(\varphi) \cup_{A\oplus B} (A' \oplus B') \to \mathsf{Cyl}(\varphi')$ is a cofibration since Cyl is a cylinder functor on cochain complexes; this shows that the above functor preserves cofibrations. For the same reason, the above functor preserves pushout diagrams. Clearly, it also preserves weak equivalences, so it is exact (verifying (CY1)).

The projection map is a chain equivalence in all degrees, so it is a T-equivalence for any choice of T. This proves goodness. □

2.1.1. Functoriality

The construction of $\mathcal{W}_G(X;\mathcal{K})$ depends on the choice of a discrete group G, a semisimplicial G-set X, and a G-coefficient system $\mathcal{K}$. Let us quickly discuss in which sense the category $\mathcal{W}_G(X;\mathcal{K})$ is functorial with respect to $\mathcal{K}$, X and G; the proofs of the statements in this section are all not difficult, and boil down to unravelling the definition of $\mathcal{W}_G(X;\mathcal{K})$ until the claim is obvious. Therefore, we skip most of the proofs.

2.1.9 Definition.
Let $\mathcal{K}$, $\mathcal{L}$ be G-coefficient systems. A *map of coefficient systems* $\kappa\colon \mathcal{K} \to \mathcal{L}$ is a natural transformation $\mathcal{K} \to \mathcal{L}$ via additive functors such that for every G-set T, the diagram

$$\begin{array}{ccc} \mathcal{K}(T) & \xrightarrow{\kappa_T} & \mathcal{L}(T) \\ \cong\downarrow & & \downarrow\cong \\ \bigoplus_r \mathcal{K}(T_r) & \xrightarrow[\oplus_r \kappa_{T_r}]{} & \bigoplus_r \mathcal{L}(T_r) \end{array}$$

commutes, where $T = \coprod_r T_r$ is the decomposition of T into transitive G-sets.

2.1.10 Example.
Suppose $F\colon \mathcal{A} \to \mathcal{B}$ is a G-equivariant, additive functor of small additive categories with right G-action. Then F induces a map of coefficient systems

$$\kappa_F\colon \mathcal{K}_{\mathcal{A}} \to \mathcal{K}_{\mathcal{B}}.$$

2.1.11 Proposition.
Let $T \subseteq \coprod_n X_n$ be non-empty. A map of G-coefficient systems $\kappa\colon \mathcal{K} \to \mathcal{L}$ induces an h_T-exact functor

$$\mathcal{W}_G(\kappa)\colon \mathcal{W}_G(X;\mathcal{K}) \to \mathcal{W}_G(X;\mathcal{L})$$

which maps (A, a) to the object given by

$$\mathcal{W}_G(\kappa)(A)^n := \kappa_{X_n}(A^n), \quad \mathcal{W}_G(\kappa)(a)^i_n := \kappa_{X_n}(a^i_n).$$

□

2.1.12 Proposition.
Let $f\colon X \to Y$ be a map of semisimplicial G-sets. Let $S \subseteq \coprod_n X_n$ and $T \subseteq \coprod_n Y_n$ be non-empty such that $f\big(\coprod_n X_n \setminus S\big) \cap T = \emptyset$. Then f induces an exact functor

$$\mathcal{W}_G(f)\colon (\mathcal{W}_G(X;\mathcal{K}), h_S) \to (\mathcal{W}_G(Y;\mathcal{K}), h_T)$$

which maps (A, a) to the object given by

$$\mathcal{W}_G(f)(A)^n := \mathcal{K}(f)(A^n), \quad \mathcal{W}_G(f)(a)^i_n := \mathcal{K}(f)(a^i_n).$$

□

2.1.13 Proposition.
Let $T \subseteq \coprod_n X_n$ be non-empty, and let $\kappa\colon \mathcal{K} \to \mathcal{L}$ be a map of G-coefficient systems.
If κ_S is an equivalence for all G-sets S, then the functor $\mathcal{W}_G(\kappa)$ is an h_T-exact equivalence.

Proof. The equivalences κ_S induce exact equivalences $\mathsf{Ch}(\mathcal{K}(S)) \to \mathsf{Ch}(\mathcal{L}(S))$. This suffices to check that $\mathcal{W}_G(\kappa)$ is fully faithful and essentially surjective. □

As far as functoriality in G is concerned, we have analogues of the usual induction maps. For the sake of simplicity, we discuss only induction along monomorphisms: Let $H \leqslant G$ be a subgroup. Recall the "induction functor"

$$\operatorname{ind}_H^G\colon H\text{-}\mathsf{Sets} \to G\text{-}\mathsf{Sets}, \quad T \mapsto G \times_H T.$$

This functor extends (regarding G-sets as discrete semisimplicial G-sets) naturally to a functor

$$\operatorname{ind}_H^G\colon \mathsf{ss}H\text{-}\mathsf{Sets} \to \mathsf{ss}G\text{-}\mathsf{Sets}$$

by applying ind_H^G degreewise. For a semisimplicial H-set X, we have in fact an equality

$$\mathcal{W}_H(X;\mathcal{K} \circ \operatorname{ind}_H^G) = \mathcal{W}_G(\operatorname{ind}_H^G X;\mathcal{K}).$$

If $\mathcal{K} = \mathcal{K}_\mathcal{A}$ for some small additive category $\mathcal{A}$ with right G-action, this gives rise to an h-exact equivalence

$$\mathrm{Ad}_H^G\colon \mathcal{W}_H(X; \mathcal{K}_{\mathrm{res}_H^G \mathcal{A}}) \to \mathcal{W}_G(\mathrm{ind}_H^G X; \mathcal{K}_\mathcal{A}).$$

This follows from Proposition 2.1.13 in view of the natural additive equivalences

$$(\mathrm{res}_H^G \mathcal{A}) *_H T \xrightarrow{\sim} \mathcal{A} *_G (\mathrm{ind}_H^G T)$$

given in [BR07, Prop. 2.8].

2.1.2. The category of Mayer-Vietoris resolutions $\mathcal{MV}_G(X; \mathcal{K})$

Let $\bullet$ denote the terminal object in the category of semisimplicial G-sets, i.e. $\bullet$ is given by $\bullet_n = G/G$ for every n. By functoriality, we have for every X, prior to any choice of subcategory of weak equivalences, an exact functor of categories with cofibrations $\mathcal{W}_G(X; \mathcal{K}) \to \mathcal{W}_G(\bullet; \mathcal{K})$. Observe that $\mathcal{W}_G(\bullet; \mathcal{K})$ is merely another name for the category of cosemisimplicial objects in $\mathsf{Ch}(\mathcal{K}(G/G))$ which are non-zero in only finitely many degrees. To each such cosemisimplicial object, we can functorially associate a finite cochain complex over $\mathcal{K}(G/G)$ by retaining the underlying graded object and taking alternating sums of coface maps; this is sometimes called the associated *Moore complex*. Taking Moore complexes produces double cochain complexes, i.e. defines a functor $\mathcal{W}_G(\bullet; \mathcal{K}) \to \mathsf{Ch}(\mathsf{Ch}(\mathcal{K}(G/G)))$. Denote the composition of these two functors by

$$\sigma_X\colon \mathcal{W}_G(X; \mathcal{K}) \to \mathsf{Ch}(\mathsf{Ch}(\mathcal{K}(G/G))).$$

For an object $(A, a) \in \mathcal{W}_G(X; \mathcal{K})$, note that the two gradings on $\sigma_X(A, a)$ arise in two different ways: One grading comes from the dimension of the simplices on which the object was based; we will sometimes refer to this as the *geometric degree*. The other grading will go by the name of *algebraic degree*.

Composing with the totalisation functor

$$\mathsf{Tot}\colon \mathsf{Ch}(\mathsf{Ch}(\mathcal{K}(G/G))) \to \mathsf{Ch}(\mathcal{K}(G/G))$$

produces an exact functor of categories with cofibrations

$$\mathsf{Tot} \circ \sigma_X\colon \mathcal{W}_G(X; \mathcal{K}) \to \mathsf{Ch}(\mathcal{K}(G/G)).$$

2.1.14 Definition.
Define the subcategory $w_{\mathsf{tot}}\mathcal{W}_G(X; \mathcal{K})$ by pulling back the weak equivalences in $\mathsf{Ch}(\mathcal{K}(G/G))$ along $\mathsf{Tot} \circ \sigma_X$ (see 1.1.5); i.e., a morphism φ is in $w_{\mathsf{tot}}\mathcal{W}_G(X; \mathcal{K})$ if and only if $\mathsf{Tot}(\sigma_X(\varphi))$ is a chain equivalence.

Note that $(\mathcal{W}_G(X; \mathcal{K}), 0, co\mathcal{W}_G(X; \mathcal{K}), w_{\mathsf{tot}}\mathcal{W}_G(X; \mathcal{K}))$ is a saturated and extensional Waldhausen category. Since σ_X and Tot preserve the canonical cylinder functors, $\mathcal{W}_G(X; \mathcal{K})$ possesses a good cylinder functor with respect to w_{tot} as well.

2.1.15 Definition.
An object in $\mathcal{W}_G(X;\mathcal{K})$ is called a *Mayer-Vietoris resolution* if it is w_{tot}-contractible. We denote the category of Mayer-Vietoris resolutions by

$$\mathcal{MV}_G(X;\mathcal{K}) := \mathcal{W}_G(X;\mathcal{K})^{w_{\mathsf{tot}}}.$$

If the choice of G-coefficient system $\mathcal{K}$ is clear, we sometimes shorten notation and write $\mathcal{MV}_G(X)$ for $\mathcal{MV}_G(X;\mathcal{K})$.

Observe that the notion of being a T-equivalence restricts to the category of Mayer-Vietoris resolutions. Formally, we may consider the subcategory of weak equivalences $h_T w_{\mathsf{tot}}\mathcal{W}_G(X;\mathcal{K}) \subseteq w_{\mathsf{tot}}\mathcal{W}_G(X;\mathcal{K})$. Applying 1.1.5, we see that the category $h_T\mathcal{MV}_G(X;\mathcal{K}) := h_T w_{\mathsf{tot}}\mathcal{W}_G(X;\mathcal{K})^{w_{\mathsf{tot}}}$ is also a subcategory of weak equivalences for $\mathcal{MV}_G(X;\mathcal{K})$. Note that we can safely suppress w_{tot} in the notation: Since $w_{\mathsf{tot}}\mathcal{W}_G(X;\mathcal{K})$ is saturated, all morphisms in $\mathcal{MV}_G(X;\mathcal{K})$ are automatically w_{tot}-equivalences.

For an arbitrary small Waldhausen category, we have only defined a K-theory space. We want to compare the K-theory of $\mathcal{MV}_G(X;\mathcal{K})$ with that of $\mathcal{K}(G/G)$ (which is just an additive category), so we require a non-connective K-theory spectrum associated to $\mathcal{MV}_G(X;\mathcal{K})$. Fortunately, the functoriality properties of $\mathcal{W}_G(X;\mathcal{K})$ translate in the obvious fashion to $\mathcal{MV}_G(X;\mathcal{K})$:

2.1.16 Proposition.

1. Let $T \subseteq \coprod_n X_n$, and let $\kappa\colon \mathcal{K} \to \mathcal{L}$ be a map of G-coefficient systems. Then $\mathcal{W}_G(\kappa)$ restricts to an h_T-exact functor $\mathcal{MV}_G(\kappa)$.

   If κ_S is an equivalence of additive categories for all G-sets S, then $\mathcal{MV}_G(\kappa)$ is an h_T-exact equivalence.

2. Let $f\colon X \to Y$ be a map of semisimplicial G-sets. Suppose that $S \subseteq \coprod_n X_n$ and $T \subseteq \coprod_n Y_n$ are non-empty such that $f\big(\coprod_n X_n \setminus S\big) \cap T = \emptyset$. Then $\mathcal{W}_G(f)$ restricts to an exact functor $\mathcal{MV}_G(f)\colon (\mathcal{MV}_G(X;\mathcal{K}), h_S) \to (\mathcal{MV}_G(X;\mathcal{K}), h_T)$.

3. Let $H \leqslant G$ be a subgroup, and let X be a semisimplicial H-set. Then $\mathcal{MV}_H(X;\mathcal{K} \circ \operatorname{ind}_H^G) = \mathcal{MV}_G(\operatorname{ind}_H^G X;\mathcal{K})$.

   If $\mathcal{K} = \mathcal{K}_{\mathcal{A}}$ for some small additive category $\mathcal{A}$ with right G-action, the h-exact equivalence $\operatorname{Ad}_H^G\colon \mathcal{W}_H(X;\mathcal{K}_{\operatorname{res}_H^G \mathcal{A}}) \to \mathcal{W}_G(\operatorname{ind}_H^G X;\mathcal{K})$ restricts to an h-exact equivalence $\operatorname{Ad}_H^G\colon \mathcal{MV}_H(X;\mathcal{K}_{\operatorname{res}_H^G \mathcal{A}}) \xrightarrow{\sim} \mathcal{MV}_G(\operatorname{ind}_H^G X;\mathcal{K})$.

□

In particular, we can exploit the first part of this proposition to produce a delooping of $K(\mathcal{MV}_G(X;\mathcal{K}), h_T)$ as we did for additive categories. We focus on the case where $\mathcal{K} = \mathcal{K}_{\mathcal{A}}$ for some small additive category $\mathcal{A}$ with right G-action. For every n, we have a map of G-coefficient systems $\mathcal{K}_{\mathcal{C}^n\mathcal{A}} \to \mathcal{K}_{\mathcal{C}^{n+1}\mathcal{A}}$, which factors through both $\mathcal{K}_{\mathcal{C}_+\mathcal{C}^n\mathcal{A}}$ and $\mathcal{K}_{\mathcal{C}_-\mathcal{C}^n\mathcal{A}}$. The Eilenberg swindles on $\mathcal{C}_\pm\mathcal{C}^n\mathcal{A}$ induce h-exact swindle functors on $\mathcal{MV}_G(X;\mathcal{K}_{\mathcal{C}_\pm\mathcal{C}^n\mathcal{A}})$, so we obtain two canonical nullhomotopies of the map $K(\mathcal{MV}_G(X;\mathcal{K}_{\mathcal{C}^n\mathcal{A}}),h_T) \to K(\mathcal{MV}_G(X;\mathcal{K}_{\mathcal{C}^{n+1}\mathcal{A}}),h_T)$ (regardless of the choice of T); these combine to give a map

$$K(\mathcal{MV}_G(X;\mathcal{K}_{\mathcal{C}^n\mathcal{A}}),h_T) \wedge S^1 \to K(\mathcal{MV}_G(X;\mathcal{K}_{\mathcal{C}^{n+1}\mathcal{A}}),h_T),$$

which is one of the structure maps of the spectrum

$$\mathbb{K}\left(\mathcal{MV}_G(X;\mathcal{A}),h_T\right) = \left\{K\left(\mathcal{MV}_G(X;\mathcal{K}_{\mathcal{C}^{n-1}\mathcal{A}}),h_T\right)\right\}_n.$$

We let $\mathbb{K}^{-\infty}\left(\mathcal{MV}_G(X;\mathcal{A}),h_T\right)$ denote its Ω-spectrification; this is the *non-connective K-theory spectrum of $\mathcal{MV}_G(X;\mathcal{K}_\mathcal{A})$ with respect to the T-equivalences.*

2.1.17 Remark.
The Pedersen-Weibel delooping does not only generalise to coefficient systems of the form $\mathcal{K}_\mathcal{A}$. It is equally well possible to define coefficient systems $\mathcal{C}^n\mathcal{K} = \mathcal{C}^n \circ \mathcal{K}$ in order to obtain a spectrum

$$\left\{K(\mathcal{MV}_G(X;\mathcal{C}^{n-1}\mathcal{K}),h_T)\right\}_n$$

which is another candidate for a non-connective K-theory spectrum. In the case where $\mathcal{K} = \mathcal{K}_\mathcal{A}$, there are functors $\mathcal{MV}_G(X;\mathcal{K}_{\mathcal{C}^n\mathcal{A}}) \to \mathcal{MV}_G(X;\mathcal{C}^n\mathcal{K}_\mathcal{A})$ as we saw in Section 1.3. I conjecture that these maps are equivalences.

It is somewhat harder to picture an object in $\mathcal{MV}_G(X;\mathcal{K})$ than it is to think of one in $\mathcal{W}_G(X;\mathcal{K})$. For this reason, we close this section by describing a method which produces some "generic" Mayer-Vietoris resolutions.

Let T be a G-set. We define a functor

$$\rho_T^p\colon \mathsf{Ch}(\mathcal{K}(T)) \to \mathcal{W}_G(T \times \Delta^p;\mathcal{K})$$

as follows: Recall that the set of n-simplices of Δ^p is given by $\hom_{\Delta_{\mathsf{inj}}}(\underline{n},\underline{p})$. Observe that

$$\mathcal{K}\big(T \times \hom_{\Delta_{\mathsf{inj}}}(\underline{n},\underline{p})\big) \cong \bigoplus_{\mu\colon \underline{n}\hookrightarrow\underline{p}} \mathcal{K}(T).$$

Given a cocomplex K over $\mathcal{K}(T)$, set

$$A^n := (K)_\mu \in \bigoplus_{\mu\colon \underline{n}\hookrightarrow\underline{p}} \mathsf{Ch}(\mathcal{K}(T))$$

for $n > -1$, and let A^{-1} be the image of K in $\mathsf{Ch}(\mathcal{K}(G/G))$.

Let $0 \leqslant i \leqslant n+1$. Similar to the objects, the morphism a_n^i decomposes into several components, one for each injective map $\mu\colon \underline{n} \to \underline{p}$. We let the μ-component of a_n^i be given by the diagonal morphism

$$\Delta_K\colon K \to \bigoplus_{\substack{\nu\colon \underline{n+1}\hookrightarrow\underline{p},\\ \nu d^i=\mu}} K.$$

We need to check that this defines an object in $\mathcal{W}_G(T \times \Delta^p; \mathcal{K})$. If we regard a_n^i as a $\hom_{\Delta_{\mathsf{inj}}}(\underline{n+1}, \underline{p}) \times \hom_{\Delta_{\mathsf{inj}}}(\underline{n}, \underline{p})$-indexed matrix[1], it is given by

$$a_n^i = \left(\delta_{\nu d^i, \mu}\right)_{\nu,\mu},$$

where $\delta_{\Box,\Diamond}$ is the Kronecker delta. The cosimplicial identities are now immediate. Define $\rho_T^p(K) := (A, a)$. Sending a chain map φ to the sequence

$$\rho_T^p(\varphi)^n := \bigoplus_\mu \varphi, \quad n > -1,$$

together with the image of φ in $\mathsf{Ch}(\mathcal{K}(G/G))$ as $\rho_T^p(\varphi)^{-1}$, turns ρ_T^p into a functor $\mathsf{Ch}(\mathcal{K}(T)) \to \mathcal{W}_G(T \times \Delta^p; \mathcal{K})$. We want to show that ρ_T^p actually maps to the category of Mayer-Vietoris resolutions $\mathcal{MV}_G(T \times \Delta^p; \mathcal{K})$.

Recall [BL12b, p. 660] that the category of finitely generated free $\mathbb{Z}$-modules acts on any additive category: We let $\mathsf{Fr}_\mathbb{Z}$ denote the following small model for the category of finitely generated free $\mathbb{Z}$-modules: The set of objects is $\{\mathbb{Z}^n \mid n \in \mathbb{N}\}$, and for $m, n \in \mathbb{N}$ we set

$$\mathrm{mor}(\mathbb{Z}^n, \mathbb{Z}^m) := M_{m,n}(\mathbb{Z}),$$

with composition of morphisms given by matrix multiplication. This is clearly an additive category. For any additive category $\mathcal{A}$, we may define a functor

$$\otimes \colon \mathcal{A} \times \mathsf{Fr}_\mathbb{Z} \to \mathcal{A},$$

additive in both variables, by setting $A \otimes \mathbb{Z}^n := \bigoplus_{i=1}^n A$ and $f \otimes M := (m_{i,j} f)_{i,j}$. If the morphism f is an identity morphism id_A, we abbreviate the notation and write $A \otimes M$ for $\mathrm{id}_A \otimes M$.

This functor extends to a pairing

$$\otimes \colon \mathcal{A} \times \mathsf{Ch}(\mathsf{Fr}_\mathbb{Z}) \to \mathsf{Ch}(\mathcal{A})$$

by defining $(A \otimes C)^n := A \otimes C^n$.

2.1.18 Lemma.
Let $A \in \mathcal{A}$, let $C, D \in \mathsf{Ch}(\mathsf{Fr}_\mathbb{Z})$, and suppose that $s \colon c \simeq c'$ is a chain homotopy of chain maps $c, c' \colon C \to D$.

Then s induces a chain homotopy $A \otimes s \colon A \otimes c \simeq A \otimes c'$. In particular, if C is a contractible cochain complex, then $A \otimes C$ is contractible in $\mathsf{Ch}(\mathcal{A})$.

Proof. This is a straightforward calculation. We define a chain homotopy $A \otimes s$ by $(A \otimes s)^n := A \otimes s^n \colon A \otimes C^n \to A \otimes C^{n-1}$. Then

$$\begin{aligned}(A \otimes d^{n-1})(A \otimes s^n) + (A \otimes s^{n+1})(A \otimes d^n) &= A \otimes (d^{n-1}s^n + s^{n+1}d^n) \\ &= A \otimes (c^n - (c')^n) \\ &= (A \otimes c^n) - (A \otimes (c')^n).\end{aligned}$$

[1] A word of warning: Not every such matrix defines a morphism in $\bigoplus_{\mu \in \hom_{\Delta_{\mathsf{inj}}}(\underline{n},\underline{p})} \mathcal{K}(T)$ (only certain block matrices). Whenever two matrices do correspond to morphisms, matrix multiplication is the same as composition of the morphisms.

The "in particular"-part of the claim is the special case where $D = C$, $c = \mathrm{id}_C$, $c' = 0$, and s is a contraction of C. □

Going back to the definition of $\rho_T^p(K)$, it is straightforward to check that in each algebraic degree k, $\sigma_{T\times\Delta^p}(\rho_T^p(K))$ is given by the tensor product of the image of K^k in $\mathcal{K}(G/G)$ with the augmented simplicial cochain complex $C_\varepsilon^*(\Delta^p)$. So by 2.1.18, $\sigma_{T\times\Delta^p}(\rho_T^p)$ is contractible in each algebraic degree, and it follows that $\mathsf{Tot}(\sigma_{T\times\Delta^p}(\rho_T^p))$ is contractible; therefore, ρ_T^p is a functor

$$\rho_T^p\colon \mathsf{Ch}(\mathcal{K}(T)) \to \mathcal{MV}_G(T\times\Delta^p;\mathcal{K}).$$

This enables us to construct Mayer-Vietoris resolutions over arbitrary spaces: If $\chi\colon G/H\times\Delta^p \to X$ is the characteristic map of an equivariant p-simplex, the composition $\mathcal{MV}_G(\chi)\circ\rho_{G/H}^p$ converts an arbitrary cocomplex over $\mathcal{K}(G/H)$ into a Mayer-Vietoris resolution over X.

The functor ρ_T^p enjoys the following properties:

2.1.19 Lemma.

1. The functor ρ_T^p is h-exact; it is natural in T and $\mathcal{K}$.

2. Let $F_p\colon \mathcal{MV}_G(T\times\Delta^p;\mathcal{K}) \to \mathsf{Ch}(\mathcal{K}(T))$ be the forgetful functor sending an object (A,a) to A^p. This functor is h_S-exact, where S is the set of p-simplices of $T\times\Delta^p$. Additionally, $F_p\circ\rho_T^p = \mathsf{Id}$, and there is a natural transformation $\eta\colon \mathsf{Id} \to \rho_T^p\circ F_p$ whose component in degree p is the identity morphism.

   If $p = 0$, the natural transformation η is a natural h-equivalence.

3. Suppose that $\mathcal{K} = \mathcal{K}_\mathcal{A}$ for some additive category $\mathcal{A}$ with G-action. Let $H \leqslant G$ be a subgroup, and T an H-set. Then the following diagram commutes:

$$\begin{array}{ccc}
\mathsf{Ch}((\mathrm{res}_H^G\mathcal{A}) *_G T) & \xrightarrow{\ \rho_T^p\ } & \mathcal{MV}_H(T\times\Delta^p;\mathcal{K}_{\mathrm{res}_H^G\mathcal{A}}) \\
\Big\downarrow \mathrm{Ad}_H^G & & \Big\downarrow \mathrm{Ad}_H^G \\
\mathsf{Ch}(\mathcal{A} *_G (\mathrm{ind}_H^G T)) & & \mathcal{MV}_G(\mathrm{ind}_H^G(T\times\Delta^p);\mathcal{K}_\mathcal{A}) \\
& \searrow^{\rho^p_{\mathrm{ind}_H^G T}} \qquad \swarrow^{\cong} & \\
& \mathcal{MV}_G((\mathrm{ind}_H^G T)\times\Delta^p;\mathcal{K}_\mathcal{A}) &
\end{array}$$

Proof.

1. This is straightforward.

2. The equality $F_p\circ\rho_T^p = \mathsf{Id}$ is immediate from the definitions.

Let $(A, a) \in \mathcal{MV}_G(T \times \Delta^p; \mathcal{K})$. Set $K := A^p$. We have to specify a chain map $\eta^n \colon A^n \to \rho_T^p(K)^n$ for each $n \leqslant p$. Set

$$\eta^n := (a(\mu))_\mu \colon A^n \to \bigoplus_{\mu \in \hom_{\Delta_{\mathsf{inj}}}(\underline{n}, \underline{p})} A^p.$$

Let $0 \leqslant i \leqslant n+1$, and denote the coface maps of $\rho_T^p(K)$ by r_n^i. Then

$$\begin{aligned} r_n^i \circ \eta^n &= r_n^i \circ (a(\mu))_{\mu \colon \underline{n} \to \underline{p}} \\ &= \big(\Delta_K \circ a(\mu)\big)_{\mu \colon \underline{n} \to \underline{p}} \\ &= \big((a(\mu))_{\nu \colon \nu d^i = \mu}\big)_{\mu \colon \underline{n} \to \underline{p}} \\ &= \big(\big(\mathcal{K}(d_i)(a(\nu))\big)_{\nu \colon \nu d^i = \mu}\big)_{\mu \colon \underline{n} \to \underline{p}} \circ a_n^i \\ &= \mathcal{K}(d_i)(\eta^{n+1}) \circ a_n^i. \end{aligned}$$

This shows that η is a morphism in $\mathcal{MV}_G(X; \mathcal{K})$. It is now easy to see that we have defined a natural transformation $\eta \colon \mathsf{Id} \to \rho_T^p \circ F$; each such morphism η satisfies $\eta^p = \mathrm{id}$.

If $p = 0$, the natural transformation $\eta \colon \mathsf{Id} \to \rho_T^p \circ F$ is in fact an h-equivalence: Given $(A, a) \in \mathcal{MV}_G(T; \mathcal{K})$, the morphism a_{-1} is a chain equivalence since $\mathsf{Tot}(\sigma_T(A, a)) = \mathsf{Cone}(a_{-1})$. Consequently, a_{-1} is an equivalence. Since $\eta_{(A,a)}$ is given by $\eta^0_{(A,a)} = \mathrm{id}$ and $\eta^{-1}_{(A,a)} = a^{-1}$, this morphism is an h-equivalence.

3. The additive equivalence

$$\mathrm{Ad}_H^G \colon (\mathrm{res}_H^G \mathcal{A}) *_G (T \times \Delta^p) \to \mathcal{A} *_G \mathrm{ind}_H^G (T \times \Delta^p) \cong \mathcal{A} *_G ((\mathrm{ind}_H^G T) \times \Delta^p)$$

is natural with respect to H-maps $T \to T'$. Therefore, we have

$$\begin{aligned} \mathrm{Ad}_H^G(\rho_T^p(K))^n &= \mathrm{Ad}_H^G(\rho_T^p(K)^n) \\ &= \big(\mathrm{Ad}_H^G(K)\big)_\mu \\ &= \rho^p_{\mathrm{ind}_H^G}(\mathrm{Ad}_H^G(K)), \end{aligned}$$

and the claim follows from that.

□

2.2. The forgetful functor $\mathcal{MV}_G(X) \to \mathsf{Ch}(\mathcal{A}[G])$

Let us concentrate on a more specific situation: Fix an additive category $\mathcal{A}$ with right G-action, and let $\mathcal{MV}_G(X)$ stand for $\mathcal{MV}_G(X; \mathcal{K}_{\mathcal{A}})$ for the rest of this section. Moreover, our focus lies on the very coarse notion of weak equivalence

$$w\mathcal{MV}_G(X) := h_{X_{-1}}\mathcal{MV}_G(X).$$

Under certain circumstances, the K-theory of $\mathcal{MV}_G(X)$ with respect to this notion of weak equivalence will turn out to be the K-theory of $\mathcal{A}[G]$.

2.2.1 Theorem.
Let X be a semisimplicial G-set in which every finite sub-semisimplicial set is contained in a finite contractible sub-semisimplicial set.

Then the forgetful functor $F\colon \mathcal{MV}_G(X) \to \mathsf{Ch}(\mathcal{A}[G])$ which maps a Mayer-Vietoris resolution (A, a) to A^{-1} induces a weak equivalence

$$wS_\bullet F\colon wS_\bullet\mathcal{MV}_G(X) \xrightarrow{\sim} hS_\bullet\mathsf{Ch}(\mathcal{A}[G]).$$

2.2.2 Remark.
Examples of such semisimplicial G-sets are more frequent than one might be inclined to think at first. To name a few, the theorem applies to:

- Finite contractible semisimplicial G-sets, in particular any standard simplex Δ^p with trivial G-action.
- Oriented trees with G-action, i.e. 1-dimensional, contractible, locally ordered simplicial complexes on which G acts without inversions. In this case, the theorem was already proved in [SS95, Thm. 2], and the proof of Theorem 2.2.1 is adapted from there.
- Specific models for classifying spaces: Let EG be given by the Milnor construction, i.e.
$$EG = \operatorname*{colim}_{n\to\infty} *_{i=0}^n G.$$
Then EG is a contractible simplicial complex, and there is a natural local ordering which allows us to consider EG as a semisimplicial G-set. If $X \subseteq EG$ is a finite subcomplex, X is necessarily contained in a finite join $*_{i=0}^n G$. Adding an addtional copy of G to the join, each element of G allows us to find a copy of the cone on X in $*_{i=0}^{n+1} G \subseteq EG$ (since the join of X with a single vertex is exactly the cone on X).
- Specific models for the classifying space of proper G-actions: Take a free G-set S (e.g. G itself) and form the full simplex on S; this is the simplicial complex whose vertex set is S, and in which every finite collection of vertices spans a simplex. The first barycentric subdivision of this complex is a semisimplicial G-set in a natural way. Any finite subcomplex Y is contained in a finite contractible subcomplex since Y is contained in the simplex spanned by its vertices.

- Other classifying spaces: We can repeat the previous construction with an arbitrary G-set T. The resulting semisimplicial G-set is a classifying space for the family of subgroups which are virtually stabilisers of T, see [Bar, Ex. 1.2].

The proof of Theorem 2.2.1 consists of an application of the Approximation Theorem to F. To show that F satisfies `(AP2)`, we will have to lift objects in $\mathsf{Ch}(\mathcal{A}[G])$ to Mayer-Vietoris resolutions. This will be done via a "resolution" construction we introduce next.

For this purpose, it will be convenient to speak about the category of "decompositions" over X: An object in $\mathsf{Dec}_G(X;\mathcal{A})$ is defined exactly like an object in $\mathcal{W}_G(X;\mathcal{K}_{\mathcal{A}})$, the only difference being that each A^n is an object in $\mathcal{A} *_G X_n$ instead of a cochain complex. Observe that objects in $\mathcal{W}_G(X;\mathcal{K}_{\mathcal{A}})$ are precisely the cochain complexes over $\mathsf{Dec}_G(X;\mathcal{A})$. In particular, we can regard every object in $\mathsf{Dec}_G(X;\mathcal{A})$ as an object in $\mathcal{W}_G(X;\mathcal{K}_{\mathcal{A}})$ (as a cocomplex concentrated in degree 0).

Recall from Definition 1.3.3 that an object $A^n \in \mathcal{A} *_G X_n$ is in fact a collection $A^n = \{A^n_x\}_{x\in X_n}$, and a morphism $\varphi^n \colon A^n \to B^n$ in $\mathcal{A} *_G X_n$ is given by a collection $\varphi^n = \{\varphi^n_{g,x}\}_{(g,x)\in G\times X_n}$. Since the coface maps a^i_n of an object $(A,a) \in \mathsf{Dec}_G(X;\mathcal{A})$ are morphisms $A^n \to (d_i)_*(A^{n+1})$, we can decompose each component $(a^i_n)_{g,x}$ further into a column vector

$$\big((a^i_n)^y_{g,x}\big)_y \colon A^n_{g^{-1}x} \to (d_i)_*(A^{n+1})_x g = \bigoplus_{d_i y = x} A^{n+1}_y g.$$

2.2.3 Definition.
Let $Y \subseteq X$ be a finite sub-semisimplicial set, and let $A \in \mathcal{A}[G]$ (this means, in fact, that A is simply an object in $\mathcal{A}$). Define the *canonical decomposition of A given by Y*

$$(\Delta_X(A;Y), \delta_X(A;Y)) \in \mathsf{Dec}_G(X;\mathcal{A})$$

to be

$$\Delta_X(A;Y)^n_x := \begin{cases} A & x \in Y_n \text{ or } n = -1, \\ 0 & \text{otherwise,} \end{cases}$$
$$\big(\delta_X(A;Y)^i_n\big)^{x'}_{g,x} := \begin{cases} \mathrm{id}_A & g = e; x' \in Y, d_i(x') = x, \\ 0 & \text{otherwise.} \end{cases}$$

2.2.4 Lemma.
Suppose Y is contractible. Then $\sigma_X(\Delta_X(A;Y)) \simeq 0$.

Proof. We observe that $\sigma_X(\Delta_X(A;Y)) = A \otimes C^*_\varepsilon(Y)$; the lemma follows from Lemma 2.1.18. □

Before we start with the proof of Theorem 2.2.1, we establish a sufficient criterion to extend morphisms in $\mathcal{A}[G]$ to morphisms in $\mathsf{Dec}_G(X;\mathcal{A})$ (cf. [Wal78a, Prop. 1.1] and [SS95, Prop. 3.2]). This will be the crucial part in proving `(AP2)` for F.

2.2.5 Lemma.
Let $A \in \mathcal{A}[G]$, $(B, b) \in \mathsf{Dec}_G(X; \mathcal{A})$, and let $\varphi\colon A \to B^{-1}$ be a morphism in $\mathcal{A}[G]$. Let μ_n be the unique function $\emptyset \to \underline{n}$. Define

$$\operatorname{supp}_n(\varphi) := \left\{x \in X_n \mid \exists (k,h) \in G \times G\colon \varphi_h \neq 0, b(\mu_n)_k^{khx} \neq 0\right\}.$$

If $\operatorname{supp}_n(\varphi) \subseteq Y_n$ for all $n \in \mathbb{N}$, the formula

$$\Delta_X(\varphi)^n_{g,x} := \begin{cases} (b(\mu_n) \circ \varphi)^x_g & g^{-1}x \in \operatorname{supp}_n(\varphi), \\ 0 & \text{otherwise}, \end{cases}$$

defines a morphism $\Delta_X(\varphi)\colon \Delta_X(A; Y) \to (B, b)$ in $\mathsf{Dec}_G(X; \mathcal{A})$.

Proof. Suppose that $g^{-1}x \in \operatorname{supp}_n(\varphi) \subseteq Y_n$. Then $A^n_{g^{-1}x} = A$, so $\Delta_X(\varphi)^n_{g,x}$ is indeed a morphism $A^n_{g^{-1}x} \to B^n_x g$. Moreover, since

$$\Delta_X(\varphi)^n_{g,x} = (b(\mu_n) \circ \varphi)^x_g = \sum_{kh=g} b(\mu_n)^x_k h \circ \varphi_h,$$

and both φ and b have finite supports, this shows that $\Delta_X(\varphi)^n$ has finite support, too.

We need to check that the morphisms $\Delta_X(\varphi)^n$ are compatible with the coface maps; i.e., we have to show that for all $0 \leqslant i \leqslant n+1$

$$(d_i)_* \Delta_X(\varphi)^{n+1} \circ \delta_X(A; Y)^i_n = b^i_n \circ \Delta_X(\varphi)^n$$

holds. For the right hand side, we get

$$\begin{aligned}
(b^i_n \circ \Delta_X(\varphi)^n)_{g,x} &= \sum_{kh=g} (b^i_n)_{k,x} h \circ \Delta_X(\varphi)^n_{h,k^{-1}x} \\
&= \sum_{kh=g} \left((b^i_n)^y_{k,x} h\right)_{d_i y = x} \circ \Delta_X(\varphi)^n_{h,k^{-1}x} \\
&= \sum_{kh=g} \left((b^i_n)^y_{k,x} h \circ \Delta_X(\varphi)^n_{h,k^{-1}x}\right)_{d_i y = x} \\
&= \Big(\sum_{kh=g} (b^i_n)^y_{k,x} h \circ \big(\sum_{ml=h} b(\mu_n)^{k^{-1}x}_m l \circ \varphi_l \big) \Big)_{d_i y = x} \\
&= \Big(\sum_{kml=g} (b^i_n)^y_{k,x} ml \circ b(\mu_n)^{k^{-1}x}_m l \circ \varphi_l \Big)_{d_i y = x} \\
&= \Big(\sum_{hl=g} \big(\sum_{km=h} (b^i_n)^y_{k,x} m \circ b(\mu_n)^{k^{-1}x}_m \big) l \circ \varphi_l \Big)_{d_i y = x} \\
&= \Big(\sum_{hl=g} b(\mu_{n+1})^y_h l \circ \varphi_l \Big)_{d_i y = x} \\
&= \big(\Delta_X(\varphi)^{n+1}_{g,y}\big)_{d_i y = x}
\end{aligned}$$

Recalling the definition of $\delta_X(A;Y)$, we see that the latter is indeed what we want to get:

$$\begin{aligned}\big((d_i)_*\Delta_X(\varphi)^{n+1}\circ\delta_X(A;Y)^i_n\big)_{g,x} &= \sum_{kh=g}\Big(\bigoplus_{d_iy=x}\Delta_X(\varphi)^{n+1}_{k,y}\Big)\circ(\delta_X(A;Y)^i_n)_{h,k^{-1}x}\\ &= \Big(\bigoplus_{d_iy=x}\Delta_X(\varphi)^{n+1}_{g,y}\Big)\circ(\delta_X(A;Y)^i_n)_{e,g^{-1}x}\\ &= \big(\Delta_X(\varphi)^{n+1}_{g,y}\big)_{d_iy=x}\end{aligned}$$

This proves that $\Delta_X(\varphi)$ is a morphism in $\mathsf{Dec}_G(X;\mathcal{K}_\mathcal{A})$. □

2.2.6 Remark.
In the claim of Lemma 2.2.5, we may equivalently assume that the (finite) sub-semisimplicial set of X spanned by $\bigcup_n \operatorname{supp}_n(\varphi)$ is contained in Y.

Proof of Theorem 2.2.1. As stated before, it suffices to prove that F satisfies both `(AP1)` and `(AP2)`. Property `(AP1)` holds by definition.

So let (A,a) be in $\mathcal{MV}_G(X)$, let $B\in\mathsf{Ch}(\mathcal{A}[G])$, and suppose we are given a chain map $\varphi\colon A^{-1}\to B$ over $\mathcal{A}[G]$. Take the mapping cylinder of φ to factor this morphism into a cofibration and a weak equivalence. We will extend the natural inclusion $A^{-1}\rightarrowtail\mathsf{Cyl}(\varphi)$ to a cofibration of Mayer-Vietoris resolutions. For the purpose of this proof, write $\mathsf{Dec}_G(X)$ instead of $\mathsf{Dec}_G(X;\mathcal{A})$. To be more precise, we claim the following:

There are sequences $(C^k)_{k\in\mathbb{Z}}$, $(\gamma^k)_{k\in\mathbb{Z}}$ and $(i^k)_{k\in\mathbb{Z}}$ such that

- C^k is an object in $\mathsf{Dec}_G(X)$, and $C^{k,-1}=\mathsf{Cyl}(\varphi)^k$.
- $\gamma^k\colon C^k\to C^{k+1}$ is a morphism in $\mathsf{Dec}_G(X)$ with $\gamma^{k+1}\circ\gamma^k=0$, and $\gamma^{k,-1}$ is the k-th differential of the mapping cylinder $\mathsf{Cyl}(\varphi)$.
- $i^k\colon A^k\rightarrowtail C^k$ is the inclusion of a direct summand (in particular, a cofibration), and $\gamma^k\circ i^k=i^{k+1}\circ\alpha^k$.
- the quotient of $\sigma_X(C^k)$ by $\sigma_X(A^k)$ is contractible.

The proof of the claim requires only a finite downward induction: In sufficiently high degree k, we can set $C^k=0$, $\gamma^k=0$ and $i^k=0$. The same is true in sufficiently small degrees. In effect, we need only consider $k\in[-K,K]$ for some $K\in\mathbb{N}$.

Let us assume that C^l, γ^l and i^l have been constructed with the properties listed above for $l>k$. Define

$$C^{k,-1}:=\mathsf{Cyl}(\varphi)^k=A^{-1,k}\oplus A^{-1,k+1}\oplus B^k,$$

and let $\gamma^{k,-1}$ be the k-th differential of the mapping cylinder $\mathsf{Cyl}(\varphi)$. Recall that

$$\gamma^{k,-1}=\begin{pmatrix}\alpha^k & -\operatorname{id} & 0\\ 0 & -\alpha^{k+1} & 0\\ 0 & \varphi^{k+1} & \beta^k\end{pmatrix},$$

where $\alpha^k\colon A^{-1,k} \to A^{-1,k+1}$ and $\beta^k\colon B^k \to B^{k+1}$ denote the differentials in A^{-1} and B. Let $\zeta^k\colon A^{-1,k+1} \oplus B^k \to A^{-1,k+1} \oplus A^{-1,k+2} \oplus B^{k+1}$ be the morphism given by

$$\zeta^k := \begin{pmatrix} -\operatorname{id} & 0 \\ -\alpha^{k+1} & 0 \\ \varphi^{k+1} & \beta^k \end{pmatrix}.$$

By assumption, there is a finite contractible sub-semisimplicial set $Y \subseteq X$ such that the sub-semisimplicial set spanned by $\bigcup_n \operatorname{supp}_n(\zeta^k)$ is contained in Y. Set

$$C^k := A^k \oplus \Delta_X(A^{-1,k+1} \oplus B^k; Y).$$

Applying Lemma 2.2.5, we obtain an extension of ζ^k to a morphism

$$\Delta_X(\zeta^k)\colon \Delta_X(A^{-1,k+1} \oplus B^k; Y) \to C^{k+1},$$

and we let γ^k be the morphism in $\mathsf{Dec}_G(X)$ given by

$$\gamma^k := \begin{pmatrix} i^{k+1}\alpha^k & \Delta_X(\zeta^k) \end{pmatrix}.$$

By definition, C^k contains A^k as a direct summand; let i^k denote this inclusion. Obviously, $\gamma^k i^k = i^{k+1}\alpha^k$. The quotient of $\sigma_X(C^k)$ by $\sigma_X(A^k)$ is given precisely by $\sigma_X(\Delta_X(A^{-1,k+1} \oplus B^k; Y))$, which is contractible by Lemma 2.2.4.

What is left to show is that $\gamma^{k+1}\gamma^k = 0$. We compute

$$\begin{aligned} \gamma^{k+1}\gamma^k &= \gamma^{k+1} \begin{pmatrix} i^{k+1}\alpha^k & \Delta_X(\zeta^k) \end{pmatrix} \\ &= \begin{pmatrix} i^{k+1}\alpha^{k+1}\alpha^k & \gamma^{k+1}\Delta_X(\zeta^k) \end{pmatrix}. \end{aligned}$$

The first component vanishes since α is a differential. For the second component, we have for $g \in G$ and $x \in X_n$

$$\begin{aligned} (\gamma^{k+1,n}\Delta_X(\zeta^k)^n)_{g,x} &= \sum_{g_1 g_2 = g} \gamma^{k+1,n}_{g_1,x} g_2 \circ \Delta_X(\zeta^k)^n_{g_2, g_1^{-1}x} \\ &= \sum_{g_1 g_2 = g} \gamma^{k+1,n}_{g_1,x} g_2 \circ \Big(\sum_{g_3 g_4 = g_2} c^{k+1}(\mu_n)^{g_1^{-1}x}_{g_3} g_4 \circ \zeta^k_{g_4} \Big) \\ &= \sum_{g_1 g_3 g_4 = g} (\gamma^{k+1,n}_{g_1,x} g_3 \circ c^{k+1}(\mu_n)^{g_1^{-1}x}_{g_3}) g_4 \circ \zeta^k_{g_4} \\ &= \sum_{g_2 g_4 = g} \Big(\sum_{g_1 g_3 = g_2} \gamma^{k+1,n}_{g_1,x} g_3 \circ c^{k+1}(\mu_n)^{g_1^{-1}x}_{g_3} \Big) g_4 \circ \zeta^k_{g_4} \\ &= \sum_{g_2 g_4 = g} ((\mu_n)^*(\gamma^{k+1,n}) \circ c^{k+1}(\mu_n))^x_{g_2} g_4 \circ \zeta^k_{g_4} \\ &\overset{2.1.5}{=} \sum_{g_2 g_4 = g} (c^{k+2}(\mu_n) \circ \gamma^{k+1,-1})^x_{g_2} g_4 \circ \zeta^k_{g_4} \\ &= (c^{k+2}(\mu_n) \circ \gamma^{k+1,-1} \circ \zeta^k)^x_g. \end{aligned}$$

Finally, we use the fact that α^k and β^k are differentials to get

$$\begin{aligned}\gamma^{k+1,-1} \circ \zeta^k &= \begin{pmatrix} \alpha^{k+1} & -\operatorname{id} & 0 \\ 0 & -\alpha^{k+2} & 0 \\ 0 & \varphi^{k+2} & \beta^{k+1} \end{pmatrix} \begin{pmatrix} -\operatorname{id} & 0 \\ -\alpha^{k+1} & 0 \\ \varphi^k & \beta^k \end{pmatrix} \\ &= \begin{pmatrix} -\alpha^{k+1} + \alpha^{k+1} & 0 \\ \alpha^{k+2}\alpha^{k+1} & 0 \\ -\varphi^{k+2}\alpha^{k+1} + \beta^{k+1}\varphi^{k+1} & \beta^{k+1}\beta^k \end{pmatrix} \\ &= 0.\end{aligned}$$

This finishes the induction step. By now, we have constructed a cofibration $i\colon A \rightarrowtail C$ in $\mathcal{W}_G(X)$ such that $F(C) = \mathsf{Cyl}(\varphi)$, and i extends the canonical inclusion of A^{-1} into the mapping cylinder. Therefore, verification of (AP2) has been reduced to showing that C is actually a Mayer-Vietoris resolution. Indeed, if we apply σ_X to the quotient of the cofibration $i\colon A \rightarrowtail C$, the resulting double complex is contractible in each algebraic degree; so its total complex is contractible, too. Since A was a Mayer-Vietoris resolution in the first place, we also have $\mathsf{Tot}(\sigma_X(A)) \simeq 0$. The contractibility of $\mathsf{Tot}(\sigma_X(C))$ follows. This proves that (AP2) holds, so we are finished. □

2.2.7 Corollary.
In the situation of Theorem 2.2.1, the forgetful functor F induces a levelwise equivalence

$$\mathbb{K}^{-\infty}(\mathcal{MV}_G(X;\mathcal{K}_\mathcal{A}), w) \xrightarrow{\sim} \mathbb{K}^{-\infty}(\mathcal{A}[G]).$$

Proof. By Theorem 2.2.1, the map of spectra

$$\mathbb{K}(\mathcal{MV}_G(X;\mathcal{A}), w) \to \mathbb{K}'(\mathcal{A}, G/G)$$

is a levelwise equivalence. Composing with the map $\mathbb{K}'(\mathcal{A}, G/G) \xrightarrow{\sim} \mathbb{K}(\mathsf{Ch}(\mathcal{A}[G]))$, which is a levelwise equivalence, and taking Ω-spectrifications produces the desired equivalence. □

2.2.8 Corollary.
Let X be a finite contractible semisimplicial G-set. Then the forgetful functor F induces a levelwise equivalence

$$\mathbb{K}^{-\infty}(F)\colon \mathbb{K}^{-\infty}(\mathcal{MV}_G(X;\mathcal{K}_\mathcal{A}), w) \xrightarrow{\sim} \mathbb{K}^{-\infty}(\mathcal{A}[G]).$$

Proof. The assumptions of Theorem 2.2.1 are satisfied for trivial reasons. □

2.2.9 Corollary.
The functor $\rho^p_T\colon \mathsf{Ch}(\mathcal{A} *_G T) \to \mathcal{MV}_G(T \times \Delta^p;\mathcal{K}_\mathcal{A})$ induces a levelwise equivalence on K-theory.

Proof. It suffices to show the claim for transitive G-sets. By Lemma 2.1.19, we have a commutative diagram

$$\begin{array}{ccc} \mathsf{Ch}((\operatorname{res}_H^G \mathcal{A}) *_G H/H) & \xrightarrow{\rho^p_{H/H}} & \mathcal{MV}_H(\Delta^p; \mathcal{K}_{\operatorname{res}_H^G \mathcal{A}}) \\ \operatorname{Ad}_H^G \downarrow & & \downarrow \operatorname{Ad}_H^G \\ \mathsf{Ch}(\mathcal{A} *_G G/H) & \xrightarrow[\rho^p_{G/H}]{} & \mathcal{MV}_G(G/H \times \Delta^p; \mathcal{K}_{\mathcal{A}}) \end{array}$$

The functors Ad_H^G are exact equivalences. The functor $\rho^p_{H/H}$ is clearly a section of the forgetful functor F, and therefore induces a levelwise equivalence on K-theory by Corollary 2.2.8. This implies that $\rho^p_{G/H}$ induces a levelwise equivalence on K-theory. □

2.3. A glimpse of a spectral sequence

The category $\mathcal{MV}_G(X;\mathcal{K})$ always comes with a filtration induced by the skeletal filtration of X. Namely, every category $\mathcal{MV}_G(X^{(p)};\mathcal{K})$ is a full subcategory of $\mathcal{MV}_G(X;\mathcal{K})$, and, taking the colimit over p, we have

$$\mathcal{MV}_G(X;\mathcal{K}) = \bigcup_p \mathcal{MV}_G(X^{(p)};\mathcal{K}).$$

Since K-theory commutes with directed colimits, we get an induced filtration of the K-theory of $\mathcal{MV}_G(X;\mathcal{K})$; this allows us to construct a spectral sequence abutting to $K_*(\mathcal{MV}_G(X;\mathcal{K}))$. In view of Theorem 2.2.1, this becomes of some interest; the resulting spectral sequence gives us some means of calculating the K-theory of $\mathcal{A}[G]$ if $\mathcal{K} = \mathcal{K}_\mathcal{A}$ and if X is sufficiently nice.

Our goal for this section is to provide some information on the E^1-term of this spectral sequence. Our treatment will be nowhere near complete; in particular, we will make no attempt at a characterisation of the differentials.

Suppose that $\mathcal{K}$ is of the form $\mathcal{K} = \mathcal{K}_\mathcal{A}$ for some small addditive category $\mathcal{A}$ with right G-action. Some arguments in this section work for arbitrary coefficient systems, but we are only interested in the case $\mathcal{K} = \mathcal{K}_\mathcal{A}$ anyway.

For the sake of completeness, let us indicate the construction of the spectral sequence. Starting from the filtration

$$\mathcal{MV}_G(X^{(0)};\mathcal{K}) \subseteq \mathcal{MV}_G(X^{(1)};\mathcal{K}) \subseteq \mathcal{MV}_G(X^{(2)};\mathcal{K}) \subseteq \ldots \subseteq \mathcal{MV}_G(X;\mathcal{K}),$$

we obtain an exact couple as follows: Let $I_p\colon \mathcal{MV}_G(X^{(p-1)};\mathcal{K}) \hookrightarrow \mathcal{MV}_G(X^{(p)};\mathcal{K})$ denote the inclusion functor. Set

$$\mathcal{H}_p := \operatorname{hofib}(\mathbb{K}^{-\infty}(I_p)),$$

which is the same as the Ω-spectrification of the spectrum obtained from taking the levelwise homotopy fibres of $\mathbb{K}(I_p)\colon \mathbb{K}(\mathcal{MV}_G(X^{(p-1)};\mathcal{K}), w) \to \mathbb{K}(\mathcal{MV}_G(X^{(p)};\mathcal{K}), w)$. Then define

$$\begin{aligned} D^1_{p,q} &:= \pi_{p+q}(\mathbb{K}^{-\infty}(\mathcal{MV}_G(X^{(p)};\mathcal{A}))), \\ E^1_{p,q} &:= \pi_{p+q-1}(\mathcal{H}_p) \end{aligned}$$

to get the exact couple

$$\begin{array}{ccc} \bigoplus_{p,q} D^1_{p,q} =: D^1 & \xrightarrow{\quad i \quad} & D^1 := \bigoplus_{p,q} D^1_{p,q} \\ {\scriptstyle\partial}\nwarrow & & \swarrow{\scriptstyle j} \\ & \bigoplus_{p,q} E^1_{p,q} =: E^1 & \end{array}$$

in which the homomorphisms induced by $\mathbb{K}^{-\infty}(I_p)$ form a morphism i of bidegree $(1,-1)$, the homomorphisms induced by $\mathcal{H}_p \to \mathbb{K}^{-\infty}(\mathcal{MV}_G(X^{(p-1)};\mathcal{A}))$ form a morphism ∂ of bidegree $(-1,0)$, and the boundary morphisms in the long exact sequences of homotopy groups form a morphism j of bidegree $(0,0)$. Hopefully, the reader is not too confused about the shifted gradings; this comes from the fact that the "relative" homotopy groups are given by the shifted homotopy groups of the homotopy fibre.

The standard machinery applies from this point on (as explained for example in [Wei94, Sec. 5.9]) to produce a spectral sequence. We call this the *X-resolution spectral sequence*.

Note that this exact couple is bounded below (because the skeletal filtration of X is bounded below) and that the filtration is exhaustive (because K-theory commutes with directed colimits). By the usual spectral sequence yoga (e.g. [Wei94, Classical Convergence Theorem 5.9.7]) we get:

2.3.1 Proposition.

The X-resolution spectral sequence converges strongly to $K_*(\mathcal{MV}_G(X;\mathcal{A}))$.

□

The main goal of this section is to obtain a slightly better description of the bigraded group underlying the E^1-term. In the end, the description we give will at least suffice to recover Waldhausen's results in [Wal78a] and [Wal78b] from the X-resolution spectral sequence.

Before we can set the Waldhausen machinery to work, we have to introduce a new category of weak equivalences. For $p \in \mathbb{N}$, set

$$h_p\mathcal{MV}_G(X;\mathcal{K}) := h_{\coprod_{n>p} X_n}\mathcal{MV}_G(X;\mathcal{K}).$$

For each p, we get the following diagram of inclusions of categories (recall Lemma 1.1.5 for the notation wh_p):

$$\begin{array}{ccccc}
h\mathcal{MV}_G(X;\mathcal{K}) & \subseteq & wh_p\mathcal{MV}_G(X;\mathcal{K}) & \subseteq & h_p\mathcal{MV}_G(X;\mathcal{K}) \\
 & & \cap\vert & & \cap\vert \\
 & & wh_{p+1}\mathcal{MV}_G(X;\mathcal{K}) & \subseteq & h_{p+1}\mathcal{MV}_G(X;\mathcal{K}) \\
 & & \cap\vert & & \\
 & & w\mathcal{MV}_G(X;\mathcal{K}) & &
\end{array}$$

By virtue of Lemma 1.1.5, we may apply the Fibration Theorem to any of these inclusions.

First of all, we will identify $\mathcal{H}_p$ with another homotopy fibre that we obtain by applying the Fibration Theorem. To do this, we need to show that it is irrelevant whether we consider resolutions over the p-skeleton of X or arbitrary resolutions which are contractible above the p-skeleton.

2.3.2 Definition.
Let $(A, a) \in \mathcal{W}_G(X;\mathcal{K})$. Let $p \in \mathbb{N}$. Define the *upper p-truncation* $\mathrm{tr}_p^+(A,a)$ *of* (A,a) to be the object in $\mathcal{W}_G(X;\mathcal{K})$ given by $(A\big|_{X^{(p)}}, a\big|_{X^{(p)}})$, where

$$A\big|^n_{X^{(p)}} := \begin{cases} A^n & n \leqslant p \\ 0 & n > p, \end{cases}$$

$$\big(a\big|_{X^{(p)}}\big)^i_n := \begin{cases} a^i_n & n < p \\ 0 & n \geqslant p. \end{cases}$$

Similarly, we let the *lower p-truncation* $\mathrm{tr}_p^-(A,a)$ *of* (A,a) be given by the object $(A\big|_{X\setminus X^{(p)}}, a\big|_{X\setminus X^{(p)}})$, where

$$A\big|^n_{X\setminus X^{(p)}} := \begin{cases} A^n & n > p \\ 0 & n \leqslant p, \end{cases}$$

$$\Big(a\big|_{X\setminus X^{(p)}}\Big)^i_n := \begin{cases} a^i_n & n > p \\ 0 & n \leqslant p. \end{cases}$$

2.3.3 Lemma.
Let $T \subseteq \coprod_n X_n$ be non-empty. Then the inclusion functor induces a weak equivalence

$$h_T S_\bullet \mathcal{MV}_G(X^{(p)};\mathcal{K}) \xrightarrow{\sim} h_T S_\bullet \mathcal{MV}_G(X;\mathcal{K})^{h_p}.$$

Taking upper p-truncations induces a homotopy inverse to this map.

Proof. As indicated in the claim, observe first that taking upper p-truncations defines an exact functor

$$\mathrm{tr}_p^+ \colon \mathcal{MV}_G(X;\mathcal{K})^{h_p} \to \mathcal{MV}_G(X^{(p)};\mathcal{K}).$$

The objects in the image of tr_p^+ are indeed Mayer-Vietoris resolutions since there is a cofibration sequence

$$\mathrm{tr}_p^-(A,a) \rightarrowtail (A,a) \twoheadrightarrow \mathrm{tr}_p^+(A,a)$$

and $\mathsf{Tot}(\sigma_X(\mathrm{tr}_p^-(A,a)))$ is contractible (the double complex is contractible in each geometric degree).

If I is the inclusion functor, the composition $\mathrm{tr}_p^+ \circ I$ equals the identity functor. Additionally, the projection $(A,a) \twoheadrightarrow \mathrm{tr}_p^+(A,a)$ is an h-equivalence since A^n is contractible for $n > p$; so $\big|h_T S_\bullet(I \circ \mathrm{tr}_p^+)\big|$ is homotopic to the identity. □

2.3.4 Corollary.
Let $T \subseteq \coprod_n X_n$ be non-empty. Then the inclusion functor induces a levelwise equivalence

$$\mathbb{K}^{-\infty}(\mathcal{MV}_G(X^{(p)};\mathcal{A}), h_T) \xrightarrow{\sim} \mathbb{K}^{-\infty}(\mathcal{MV}_G(X;\mathcal{A})^{h_p}, h_T).$$

Proof. Applying Lemma 2.3.3 for each coefficient system $\mathcal{K}_{\mathcal{C}^n\mathcal{A}}$, we get a levelwise equivalence $\mathbb{K}(\mathcal{MV}_G(X^{(p)};\mathcal{A}),h_T) \xrightarrow{\sim} \mathbb{K}(\mathcal{MV}_G(X;\mathcal{A})^{h_p},h_T)$, and consequently a levelwise equivalence between the Ω-spectrifications. $\square$

Setting

$$\widetilde{\mathcal{H}_p} := \operatorname{hofib}\left(\mathbb{K}^{-\infty}(\mathcal{MV}_G(X;\mathcal{A})^{h_{p-1}},w) \to \mathbb{K}^{-\infty}(\mathcal{MV}_G(X;\mathcal{A})^{h_p},w)\right),$$

we obtain a natural levelwise equivalence $\mathcal{H}_p \xrightarrow{\sim} \widetilde{\mathcal{H}_p}$. By virtue of the Fibration Theorem, the rows and columns containing three entries in the commutative diagram

$$\begin{array}{ccccc}
 & & h_{p-1}S_\bullet\mathcal{MV}_G(X;\mathcal{K})^{wh_p} & \longrightarrow & h_{p-1}S_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_p} \\
 & & \downarrow & & \downarrow \\
wS_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_{p-1}} & \longrightarrow & wh_{p-1}S_\bullet\mathcal{MV}_G(X;\mathcal{K}) & \longrightarrow & h_{p-1}S_\bullet\mathcal{MV}_G(X;\mathcal{K}) \\
\downarrow & & \downarrow & & \downarrow \\
wS_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_p} & \longrightarrow & wh_pS_\bullet\mathcal{MV}_G(X;\mathcal{K}) & \longrightarrow & h_pS_\bullet\mathcal{MV}_G(X;\mathcal{K})
\end{array}$$

are homotopy fibrations. So we may equally well compute $\widetilde{\mathcal{H}_p}$ by investigating the homotopy fibre of the map $h_{p-1}S_\bullet\mathcal{MV}_G(X;\mathcal{K})^{wh_p} \to h_{p-1}S_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_p}$. Let us focus our attention on $h_{p-1}S_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_p}$. The following square is a homotopy pullback:

$$\begin{array}{ccc}
\Omega\left|h_{p-1}S_\bullet\mathcal{MV}_G(X;\mathcal{K})\right| & \longrightarrow & \left|hS_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_{p-1}}\right| \\
\downarrow & & \downarrow \\
\Omega\left|h_pS_\bullet\mathcal{MV}_G(X;\mathcal{K})\right| & \longrightarrow & \left|hS_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_p}\right|
\end{array}$$

This is easy to check as the horizontal homotopy fibres are both $\Omega\left|hS_\bullet\mathcal{MV}_G(X;\mathcal{K})\right|$ by the Fibration Theorem. It follows that there is a canonical homotopy equivalence between the vertical homotopy fibres (identifying the left one by another application of the Fibration Theorem)

$$\Omega\left|h_{p-1}S_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_p}\right| \xrightarrow{\sim} \operatorname{hofib}\left(\left|hS_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_{p-1}}\right| \to \left|hS_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_p}\right|\right).$$

Using Lemma 2.3.3 a second time, we see that there is a natural homotopy equivalence

$$\begin{aligned}
&\operatorname{hofib}\left(\left|hS_\bullet\mathcal{MV}_G(X^{(p-1)};\mathcal{K})\right| \to \left|hS_\bullet\mathcal{MV}_G(X^{(p)};\mathcal{K})\right|\right) \\
&\quad\xrightarrow{\sim} \operatorname{hofib}\left(\left|hS_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_{p-1}}\right| \to \left|hS_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_p}\right|\right).
\end{aligned}$$

The homotopy fibre of $\left|hS_\bullet\mathcal{MV}_G(X^{(p-1)};\mathcal{K})\right| \to \left|hS_\bullet\mathcal{MV}_G(X^{(p)};\mathcal{K})\right|$ is readily computable:

2.3.5 Theorem.
There is a weak equivalence

$$hS_\bullet\mathcal{MV}_G(X^{(p+1)};\mathcal{K}) \xrightarrow{\sim} hS_\bullet\mathcal{MV}_G(X^{(p)};\mathcal{K}) \times hS_\bullet\mathsf{Ch}(\mathcal{A} *_G X_{p+1}),$$

natural in X and $\mathcal{A}$, which gives rise to a weak equivalence

$$hS_\bullet\mathcal{MV}_G(X;\mathcal{K}) \xrightarrow{\sim} hS_\bullet \bigoplus_{n\geqslant 0} \mathsf{Ch}(\mathcal{A} *_G X_n).$$

The map $hS_\bullet\mathcal{MV}_G(X^{(p+1)};\mathcal{K}) \to hS_\bullet\mathcal{MV}_G(X^{(p)};\mathcal{K})$ is split by the inclusion functor.

Proof. Write $\mathcal{MV}_G(X)$ for $\mathcal{MV}_G(X;\mathcal{K}) = \mathcal{MV}_G(X;\mathcal{K}_\mathcal{A})$. We will define a functor

$$\begin{pmatrix} T \\ F_{p+1} \end{pmatrix} : \mathcal{MV}_G(X^{(p+1)}) \to \mathcal{MV}_G(X^{(p)}) \times \mathsf{Ch}(\mathcal{A} *_G X_{p+1})$$

which is exact with respect to the h-equivalences and induces a weak equivalence in K-theory. The component F_{p+1} is given by the forgetful functor sending (A, a) to A^{p+1}; the functor T is yet to be defined. In fact, we will first define the prospective inverse

$$\begin{pmatrix} I & S \end{pmatrix} : \mathcal{MV}_G(X^{(p)}) \times \mathsf{Ch}(\mathcal{A} *_G X_{p+1}) \to \mathcal{MV}_G(X^{(p+1)}).$$

Here, I is the inclusion functor $\mathcal{MV}_G(X^{(p)}) \hookrightarrow \mathcal{MV}_G(X^{(p+1)})$ (i.e. we drop the index $p+1$ for the purpose of this proof). We define the functor S next.

The characteristic maps $\chi_x \colon \Delta^{p+1} \to X^{(p+1)}$ of the $(p+1)$-simplices $x \in X_{p+1}$ combine to a map of semisimplicial G-sets

$$\chi_{p+1} \colon X_{p+1} \times \Delta^{p+1} \to X^{(p+1)}.$$

Using naturality, we define S as the composition

$$S \colon \mathsf{Ch}(\mathcal{A} *_G X_{p+1}) \xrightarrow{\rho^{p+1}_{X_{p+1}}} \mathcal{MV}_G(X_{p+1} \times \Delta^{p+1}) \xrightarrow{\mathcal{MV}_G(\chi_{p+1})} \mathcal{MV}_G(X^{(p+1)}).$$

We claim the existence of a natural transformation $\tau \colon \mathsf{Id} \to SF_{p+1}$: Assume (A, a) is a fixed object in $\mathcal{MV}_G(X^{(p+1)})$. Plugging in the definitions, we have that

$$SF_{p+1}(A)^n = \bigoplus_{\mu \colon \underline{n} \hookrightarrow \underline{p+1}} \mathcal{K}(\mu^*)(A^{p+1}).$$

Define

$$\tau^n := (a(\mu))_{\mu \colon \underline{n} \hookrightarrow \underline{p+1}} : A^n \to \bigoplus_{\mu \colon \underline{n} \hookrightarrow \underline{p+1}} \mathcal{K}(\mu^*)(A^{p+1}).$$

If $\mu = \mathrm{id}$, we interpret $a(\mathrm{id})$ as the identity morphism. We have to check that $(\tau^n)_n$ defines a morphism of Mayer-Vietoris resolutions: Let $0 \leqslant i \leqslant n+1$. We can regard the i-th coboundary map in degree n of $SF_{p+1}(A)$ as a certain $\hom_{\Delta_{\mathsf{inj}}}(\underline{n+1}, \underline{p+1}) \times$

$\hom_{\Delta_{\mathsf{inj}}}(\underline{n}, \underline{p+1})$-indexed matrix; to be explicit, the i-th coboundary map is given by the matrix

$$\left(\delta_{\nu d^i,\mu}\right)_{\nu\colon \underline{n+1}\hookrightarrow \underline{p+1}, \mu\colon \underline{n}\hookrightarrow\underline{p+1}},$$

where $\delta_{\Box,\Diamond}$ denotes the Kronecker delta.

Therefore, the following calculation proves that τ is a morphism:

$$\begin{aligned}\left(\delta_{\nu d^i,\mu}\right)_{\nu,\mu} \circ \tau^n &= \left(\delta_{\nu d^i,\mu}\right)_{\nu,\mu} \circ (a(\mu))_\mu \\ &= \left(a(\nu d^i)\right)_\nu \\ &= \left((d_i)_* a(\nu) \circ a^i_n\right)_\nu \\ &= (d_i)_*(\tau^{n+1}) \circ a^i_n.\end{aligned}$$

Now let $\varphi\colon (A,a) \to (B,b)$ be a morphism in $\mathcal{MV}_G(X^{(p+1)})$. Then

$$\begin{aligned}\tau^n_B \circ \varphi^n &= (b(\mu)\circ\varphi^n)_\mu \\ &= \left(\mathcal{K}(\mu^*)(\varphi^{p+1}) \circ a(\mu)\right)_\mu \\ &= SF_{p+1}(\varphi)\circ\tau^n_A,\end{aligned}$$

so τ is indeed a natural transformation.

Having the natural transformation τ at our disposal, we are finally ready to define T. Set

$$\overline{T} := \Sigma^{-1}\mathsf{Cone}(\tau).$$

The object $\overline{T}(A,a)$ is h_p-contractible for every $(A,a) \in \mathcal{MV}_G(X^{(p+1)})$ as $\tau^{p+1} = \mathrm{id}$. The functor T is now given by the composition

$$T\colon \mathcal{MV}_G(X^{(p+1)}) \xrightarrow{\overline{T}} \mathcal{MV}_G(X^{(p+1)})^{h_p} \xrightarrow{\mathrm{tr}^+_p} \mathcal{MV}_G(X^{(p)}).$$

Since

$$\begin{pmatrix} T \\ F_{p+1}\end{pmatrix}\begin{pmatrix} I & S\end{pmatrix} = \begin{pmatrix} TI & TS \\ F_{p+1}I & F_{p+1}S\end{pmatrix} = \begin{pmatrix}\mathsf{Id} & TS \\ 0 & \mathsf{Id}\end{pmatrix},$$

the induced map on K-theory is a self-homotopy equivalence.

So what is left to show is that if we compose these functors the other way around, the resulting map on K-theory is homotopic to the identity. Letting J denote the inclusion $\mathcal{MV}_G(X^{(p+1)})^{h_p} \hookrightarrow \mathcal{MV}_G(X^{(p+1)})$, we have a cofibration sequence of exact functors

$$\mathsf{Id} \rightarrowtail \mathsf{Cyl}(\tau) \twoheadrightarrow \Sigma J\overline{T},$$

which induces a homotopy $|hS_\bullet\mathsf{Id}| \vee \left|hS_\bullet\Sigma J\overline{T}\right| \simeq |hS_\bullet\mathsf{Cyl}(\tau)|$ by the Additivity Theorem. Since the cylinder functor is good, the projection maps of the cylinder functor induce a homotopy $|hS_\bullet\mathsf{Cyl}(\tau)| \simeq |hS_\bullet SF_{p+1}|$. Moreover, the (de-)suspension functor implements the inversion operation with respect to the H-space structure (Proposition 1.1.13), so in combination we obtain a homotopy

$$|hS_\bullet\mathsf{Id}| \simeq |hS_\bullet SF_{p+1}| \vee \left|hS_\bullet J\overline{T}\right|.$$

Finally, the natural equivalence $I\,\mathrm{tr}_p^+ \to J$ from the proof of Lemma 2.3.3 yields a homotopy

$$\begin{aligned}
|hS_\bullet SF_{p+1}| \vee |hS_\bullet IT| &\simeq |hS_\bullet SF_{p+1}| \vee \left|hS_\bullet I\,\mathrm{tr}_p^+\,\overline{T}\right| \\
&\simeq |hS_\bullet SF_{p+1}| \vee \left|hS_\bullet J\overline{T}\right| \\
&\simeq |hS_\bullet \mathsf{Id}|\,,
\end{aligned}$$

so we have proved the first part of the theorem.

For finite-dimensional X, the second part of the theorem follows from the first by induction, where the start of the induction is provided by Lemma 2.1.19. To generalise the result to arbitrary X, observe that the diagram

$$\begin{array}{ccc}
\mathcal{MV}_G(X^{(p)}) & \xrightarrow{\quad I \quad} & \mathcal{MV}_G(X^{(p+1)}) \\
& \searrow & \Big\downarrow \begin{pmatrix} T \\ F_{p+1} \end{pmatrix} \\
& & \mathcal{MV}_G(X^{(p)}) \times \mathsf{Ch}(\mathcal{A} *_G X_{p+1})
\end{array}$$

is strictly commutative. So the weak equivalences on each finite skeleton are preserved under the structure maps of the colimit system, inducing an equivalence

$$hS_\bullet \mathcal{MV}_G(X) \xrightarrow{\sim} hS_\bullet \bigoplus_{n \geqslant 0} \mathsf{Ch}(\mathcal{A} *_G X_n).$$

□

Applying Theorem 2.3.5, we can identify

$$\mathrm{hofib}\left(\left|hS_\bullet \mathcal{MV}_G(X^{(p-1)};\mathcal{K})\right| \to \left|hS_\bullet \mathcal{MV}_G(X^{(p)};\mathcal{K})\right| \right) \simeq \Omega\,|hS_\bullet \mathsf{Ch}(\mathcal{A} *_G X_p)|\,.$$

Consequently, we get a zig-zag of equivalences

$$\Omega\left|h_{p-1}S_\bullet \mathcal{MV}_G(X;\mathcal{K})^{h_p}\right| \simeq \Omega\,|hS_\bullet \mathsf{Ch}(\mathcal{A} *_G X_p)|\,.$$

2.3.6 Theorem.
There is a homotopy fibration

$$\mathcal{H}_p \to \mathbb{K}^{-\infty}(\mathcal{MV}_G(X^{(p)};\mathcal{A})^w, h_{p-1}) \xrightarrow{\mathbb{K}^{-\infty}(F_p)} \mathbb{K}^{-\infty}(\mathcal{A} *_G X_p).$$

Proof. Since the previous discussion applies to all small additive categories with right G-action, it suffices to prove the theorem in the connective case.

So far, we have established the existence of a homotopy fibration sequence

$$\begin{aligned}
&\mathrm{hofib}\left(K(\mathcal{MV}_G(X^{(p-1)}), w) \to K(\mathcal{MV}_G(X^{(p)}), w)\right) \\
&\qquad \to K(\mathcal{MV}_G(X)^{wh_p}, h_{p-1}) \to K(\mathsf{Ch}(\mathcal{A} *_G X_p)).
\end{aligned}$$

Repeating the argument from the proof of Lemma 2.3.3 allows us to identify

$$K(\mathcal{MV}_G(X)^{wh_p}, h_{p-1}) = \Omega\left|h_{p-1}S_\bullet\mathcal{MV}_G(X)^{wh_p}\right| \simeq \Omega\left|h_{p-1}S_\bullet\mathcal{MV}_G(X^{(p)})^w\right|.$$

The only thing left to show is that the map

$$\Omega\left|h_{p-1}S_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_p}\right| \to \Omega\left|hS_\bullet\mathsf{Ch}(\mathcal{A} *_G X_p)\right|$$

is homotopic to the map induced by the forgetful functor F_p. Set

$$\begin{aligned} H_1 &:= \operatorname{hofib}\left(\left|hS_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_{p-1}}\right| \to \left|hS_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_p}\right|\right), \\ H_2 &:= \operatorname{hofib}\left(\left|hS_\bullet\mathcal{MV}_G(X^{(p-1)})\right| \to \left|hS_\bullet\mathcal{MV}_G(X^{(p)})\right|\right). \end{aligned}$$

Then we can wrap up the situation in the following commutative diagram in which the two rows are homotopy fibrations:

$$\begin{array}{ccccc}
 & & & & \Omega\left|h_{p-1}S_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_p}\right| \\
 & & & \nearrow^{j} & e_1\downarrow\wr \\
\Omega\left|hS_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_{p-1}}\right| & \longrightarrow & \Omega\left|hS_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_p}\right| & \xrightarrow{\quad q_1 \quad} & H_1 \\
\mathrm{tr}^+_{p-1}\downarrow\uparrow i_{p-1} & & \mathrm{tr}^+_p\downarrow\uparrow i_p & & \widetilde{\mathrm{tr}}\downarrow\uparrow\widetilde{i} \\
\Omega\left|hS_\bullet\mathcal{MV}_G(X^{(p-1)})\right| & \longrightarrow & \Omega\left|hS_\bullet\mathcal{MV}_G(X^{(p)})\right| & \xrightarrow{\quad q_2 \quad} & H_2 \\
 & & & \underset{s}{\overset{f_p}{\rightleftarrows}} & e_2\downarrow\wr \\
 & & & & \Omega\left|hS_\bullet\mathsf{Ch}(\mathcal{A} *_G X_p))\right|
\end{array}$$

The maps in this diagram arise as follows: The maps i_{p-1} and i_p are induced by the inclusion functors I_{p-1} and I_p. By abuse of notation, tr^+_p denotes the map induced by the functor with that name. The maps $\widetilde{i}$ and $\widetilde{\mathrm{tr}}$ are induced by i_{p-1} and i_p, respectively tr^+_{p-1} and tr^+_p. The maps q_1, q_2, e_1 and e_2 all arise from certain homotopy fibration sequences. The obvious inclusion functor induces the map j, and f_p is induced by the forgetful functor F_p. Theorem 2.3.5 asserts the existence of a section (up to homotopy) s of f_p.

Define now $s'\colon \Omega\left|h_{p-1}S_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_p}\right| \to \Omega\left|hS_\bullet\mathcal{MV}_G(X;\mathcal{K})^{h_p}\right|$ by

$$s' := i_p \circ s \circ e_2 \circ \widetilde{\mathrm{tr}} \circ e_1.$$

Then s' is a section of j up to homotopy:

$$\begin{aligned} j \circ s' &\simeq e_1^{-1} \circ q_1 \circ i_p \circ s \circ e_2 \circ \widetilde{\mathrm{tr}} \circ e_1 \\ &\simeq e_1^{-1} \circ \widetilde{i} \circ q_2 \circ s \circ e_2 \circ \widetilde{\mathrm{tr}} \circ e_1 \\ &\simeq e_1^{-1} \circ \widetilde{i} \circ e_2^{-1} \circ e_2 \circ q_2 \circ s \circ e_2 \circ \widetilde{\mathrm{tr}} \circ e_1 \\ &\simeq e_1^{-1} \circ \widetilde{i} \circ e_2^{-1} \circ f_p \circ s \circ e_2 \circ \widetilde{\mathrm{tr}} \circ e_1 \\ &\simeq \mathrm{id}\,. \end{aligned}$$

There is another forgetful functor $\mathcal{MV}_G(X;\mathcal{K})^{h_p} \to \mathsf{Ch}(\mathcal{A} *_G X_p)$ mapping (A,a) to A^p, which we also denote by F_p. Note that the map $f_p \circ \mathrm{tr}_p^+$ is induced by a forgetful functor as well, and that $f_p \circ \mathrm{tr}_p^+ = F_p \circ j$. Therefore, we get that

$$\begin{aligned} e_2 \circ \widetilde{\mathrm{tr}} \circ e_1 &\simeq e_2 \circ \widetilde{\mathrm{tr}} \circ e_1 \circ j \circ s' \simeq e_2 \circ \widetilde{\mathrm{tr}} \circ q_1 \circ s' \\ &\simeq e_2 \circ q_2 \circ \mathrm{tr}_p^+ \circ s' \simeq f_p \circ \mathrm{tr}_p^+ \circ s' = F_p \circ j \circ s' \\ &\simeq F_p. \end{aligned}$$

This proves that the map which implements the identification

$$\Omega \left| h_{p-1} S_\bullet \mathcal{MV}_G(X;\mathcal{K})^{h_p} \right| \simeq \Omega \left| hS_\bullet \mathsf{Ch}(\mathcal{K}(X_p)) \right|$$

is homotopic to the map induced by the forgetful functor F_p. □

2.3.7 Remark.
In Theorem 2.3.6, we can also replace the category of w-contractible Mayer-Vietoris resolutions by the full subcategory consisting only of resolutions of the zero complex. Argue as in Lemma 2.3.3 with the lower (-1)-truncation in place of the upper p-truncation.

2.3.8 Remark.
Theorem 2.3.6 allows us to relate the X-resolution spectral sequence to Waldhausen's results on the K-theory of group rings over amalgamated products and HNN extensions (in fact, we will get the non-connective generalisation of Waldhausen's result as discussed in [BL06, Sec. 10]). Continue to write $\mathcal{MV}_G(X)$ instead of $\mathcal{MV}_G(X;\mathcal{K})$.

Take the example of an amalgamated product $G = G_1 *_H G_2$, and let X be the Bass-Serre tree of G, which can be described as the semisimplicial G-set obtained by the pushout

$$\begin{array}{ccc} G/H \times \partial\Delta^1 & \longrightarrow & G/G_1 \coprod G/G_2 \\ \downarrow & & \downarrow \\ G/H \times \Delta^1 & \longrightarrow & X \end{array}$$

The attaching map $G/H \times \partial\Delta^1 = G/H \coprod G/H \to G/G_1 \coprod G/G_2$ is given by the coproduct of the obvious projection maps $G/H \to G/G_i$.

The X-resolution spectral sequence is concentrated in the columns $p=0$ and $p=1$, with entries given by the homotopy groups of

$$\begin{aligned} \mathcal{H}_0 &= \Omega\mathbb{K}^{-\infty}(\mathcal{MV}_G(X^{(0)}), w) \simeq \Omega\mathbb{K}^{-\infty}(\mathcal{A} *_G X_0) \\ &\simeq \Omega\mathbb{K}^{-\infty}(\mathcal{A}[G_1]) \vee \Omega\mathbb{K}^{-\infty}(\mathcal{A}[G_2]), \\ \mathcal{H}_1 &= \mathrm{hofib}\left(\mathbb{K}^{-\infty}(\mathcal{MV}_G(X^{(0)}), w) \to \mathbb{K}^{-\infty}(\mathcal{MV}_G(X^{(1)}), w)\right). \end{aligned}$$

By Theorem 2.3.6, we have a homotopy fibration

$$\mathcal{H}_1 \to \mathbb{K}^{-\infty}(\mathcal{MV}_G(X)^w, h_0) \to \mathbb{K}^{-\infty}(\mathcal{A} *_G X_1).$$

As usual, we can identify $\mathbb{K}^{-\infty}(\mathcal{A}*_G X_1)$ with $\mathbb{K}^{-\infty}(\mathcal{A}[H])$. Arguing as in the proof of Lemma 2.3.3, w-contractibility can be strengthened to the condition that we are only considering objects (A, a) for which $A^{-1} = 0$; see Remark 2.3.7. Denote the category of these Mayer-Vietoris resolutions by $\mathcal{S}_G(X)$ (the category of "split modules" - we can deloop the K-theory of this category as we did with the K-theory of $\mathcal{MV}_G(X)$). Since we are only working with Mayer-Vietoris resolutions, all h_0-equivalences are then automatically h-equivalences, so that we end up with a homotopy fibration

$$\mathcal{H}_1 \to \mathbb{K}^{-\infty}(\mathcal{S}_G(X), h) \to \mathbb{K}^{-\infty}(\mathcal{A}[H]).$$

In effect, this proves that the square

$$\begin{array}{ccc} \mathbb{K}^{-\infty}(\mathcal{S}_G(X), h) & \longrightarrow & \mathbb{K}^{-\infty}(\mathcal{A}[G_1]) \vee \mathbb{K}^{-\infty}(\mathcal{A}[G_2]) \\ \big\downarrow & & \big\downarrow \\ \mathbb{K}^{-\infty}(\mathcal{A}[H]) & \longrightarrow & \mathbb{K}^{-\infty}(\mathcal{A}[G]) \end{array}$$

is a homotopy pullback, which is one way of stating Waldhausen's main result in [Wal78a, Wal78b], cf. [SS95, Thm. 4] and [BL06, Thm. 10.2].

Observe that Waldhausen's result is stronger than what we have shown: He provides an identification of $\mathcal{S}_G(X)$ with a certain category of nilpotent morphisms, which is for example necessary if one wants to prove vanishing results like [Wal78b, Thm. 11.2]. Moreover, Waldhausen's theorems are valid for certain amalgamations of rings which are not necessarily group rings of amalgamated products.

In order to get a first idea of the X-resolution spectral sequence, we conclude by considering an elementary example: Assume that G is the trivial group (which we are going to omit from notation), that $\mathcal{K} = \mathcal{K}_{\mathcal{A}}$ for some small additive category $\mathcal{A}$, and that X is a standard simplex Δ^s. By Corollary 2.2.8, we know that the Δ^s-resolution spectral sequence converges to $\mathbb{K}^{-\infty}(\mathcal{A})$.

In case $s = 0$, the filtration on $\mathcal{MV}(\Delta^0)$ is trivial, and nothing interesting happens. For $s = 1$, we have to consider the two homotopy fibres

$$\begin{aligned} H_0 &= \operatorname{hofib}\left(* \to \left|wS_\bullet\mathcal{MV}(\partial\Delta^1)\right|\right) \simeq \Omega\left|hS_\bullet\mathsf{Ch}(\mathcal{A})\right| \times \Omega\left|hS_\bullet\mathsf{Ch}(\mathcal{A})\right|, \\ H_1 &= \operatorname{hofib}\left(\left|wS_\bullet\mathcal{MV}(\partial\Delta^1)\right| \to \left|wS_\bullet\mathcal{MV}(\Delta^1)\right|\right). \end{aligned}$$

For a given object $(A, a) \in \mathcal{MV}(\partial\Delta^1)$, the morphism a^{-1} is a chain equivalence $A^{-1} \to A^0$ by definition. Therefore, the non-commutative diagram

$$\begin{array}{ccc} \mathcal{MV}(\partial\Delta^1) & \xrightarrow{I_1} & \mathcal{MV}(\Delta^1) \\ {\scriptstyle F_0}\big\downarrow & & \big\downarrow{\scriptstyle F} \\ \mathsf{Ch}(\mathcal{A}) \times \mathsf{Ch}(\mathcal{A}) & \xrightarrow{\oplus} & \mathsf{Ch}(\mathcal{A}) \end{array}$$

becomes homotopy commutative upon application of the $wS_\bullet$-construction. Since the forgetful functors F_0 and F induce equivalences, we can identify the homotopy fibre H_1 with $hS_\bullet \mathsf{Ch}(\mathcal{A})$, and the map

$$H_1 \to \left|wS_\bullet \mathcal{MV}(\partial\Delta^1)\right| \simeq |hS_\bullet \mathsf{Ch}(\mathcal{A})| \times |hS_\bullet \mathsf{Ch}(\mathcal{A})|$$

induces the skew-diagonal on homotopy groups (the direct sum operation $\oplus$ induces the H-space addition on $hS_\bullet \mathsf{Ch}(\mathcal{A})$).

In total, letting $\nabla = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ denote the skew-diagonal, the E^1-term of the Δ^1-resolution spectral sequence takes the following shape:

$$\begin{array}{cccc}
2 & 2K_2(\mathcal{A}) & \xleftarrow{\nabla} & K_2(\mathcal{A}) \\
 & 2K_1(\mathcal{A}) & \xleftarrow{\nabla} & K_1(\mathcal{A}) \\
0 & 2K_0(\mathcal{A}) & \xleftarrow{\nabla} & K_0(\mathcal{A}) \\
 & 2K_{-1}(\mathcal{A}) & \xleftarrow{\nabla} & K_{-1}(\mathcal{A}) \\
 & 0 & & 1
\end{array}$$

Notably, the Δ^1-spectral sequence coincides with the Atiyah-Hirzebruch spectral sequence for $H_*(\Delta^1; \mathbb{K}^{-\infty}(\mathcal{A}))$ in this case.

Things become more interesting for $s = 2$, and also more complicated. The E^1-term of the $X := \Delta^2$-resolution spectral sequence arises from the homotopy fibres

$$\begin{aligned}
H_0 &\simeq \Omega \left|wS_\bullet \mathcal{MV}(X^{(0)})\right| \simeq \Omega\,|hS_\bullet \mathsf{Ch}(\mathcal{A})| \times \Omega\,|hS_\bullet \mathsf{Ch}(\mathcal{A})| \times \Omega\,|hS_\bullet \mathsf{Ch}(\mathcal{A})|\,, \\
H_1 &= \operatorname{hofib}\big(\left|wS_\bullet \mathcal{MV}(X^{(0)})\right| \to \left|wS_\bullet \mathcal{MV}(X^{(1)})\right|\big), \\
H_2 &= \operatorname{hofib}\big(\left|wS_\bullet \mathcal{MV}(X^{(1)})\right| \to |wS_\bullet \mathcal{MV}(X)|\big).
\end{aligned}$$

Let us agree that (0), (1) and (2) are the 0-simplices of Δ^2, that (01), (02) and (12) are the 1-simplices, and that (012) is the unique 2-simplex. The face maps are obvious. Let F be the forgetful functor that maps a resolution (A, a) to the complex A^{-1}, let F_p be the forgetful functors projecting to $\mathcal{A} * X_p$, and denote by ρ_i the functor given by the composition

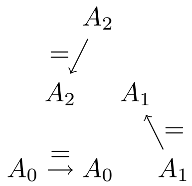
$$\rho_i \colon \mathsf{Ch}(\mathcal{A}) \xrightarrow{\rho^0} \mathcal{MV}(\Delta^0) \xrightarrow{\chi_{(i)}} \mathcal{MV}(\Delta^2),$$

where $\chi_{(i)}$ is the characteristic map of the simplex (i).

The identification of H_0 is made via the forgetful functor F_0, whose inverse is given by the functor $\rho_0 \times \rho_1 \times \rho_2$. The homotopy fibre H_1 fits into the following diagram, in which the rows and columns with three entries are homotopy fibrations (combine the diagram right below Corollary 2.3.4 with Theorem 2.3.5):

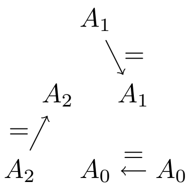
$$
\begin{array}{ccccc}
H_1 & \xrightarrow{\quad \eta \quad} & hS_\bullet\mathcal{MV}(X^{(1)})^w & \xrightarrow{hS_\bullet F_1} & hS_\bullet\mathsf{Ch}(\mathcal{A})^3 \\
{\scriptstyle \iota}\big\downarrow & & {\scriptstyle j_2}\big\downarrow & & \\
wS_\bullet\mathcal{MV}(X^{(0)}) & \xrightarrow{\quad j_1 \quad} & hS_\bullet\mathcal{MV}(X^{(1)}) & & \\
{\scriptstyle wS_\bullet I_1}\big\downarrow & & \big\downarrow & & \\
wS_\bullet\mathcal{MV}(X^{(1)}) & \xrightarrow{\quad = \quad} & wS_\bullet\mathcal{MV}(X^{(1)}) & &
\end{array}
$$

Define an h-exact functor $R\colon \mathsf{Ch}(\mathcal{A})^3 \to \mathcal{MV}(X^{(1)})^w$ as follows. Given a triple $(A_0, A_1, A_2) \in \mathsf{Ch}(\mathcal{A})^3$, let $R(A_1, A_2, A_3)$ be the resolution



The arrangement of the picture is hopefully suggestive enough to indicate how the complexes are to be based on $X^{(1)} = \partial\Delta^2$: one copy of the complex A_i is based on the 0-simplex (i), another copy on one additional 1-simplex, namely (01) for A_0, (12) for A_1 and (02) for A_2. All coboundary maps except those depicted above are zero. This is obviously a resolution of 0, and this mapping is easily extended to form a functor.

We define another functor $R'\colon \mathsf{Ch}(\mathcal{A})^3 \to \mathcal{MV}(X^{(1)})^w$ in analogous fashion by sending a triple (A_0, A_1, A_2) to the resolution



Now define

$$r := |hS_\bullet R| - |hS_\bullet R'| : \ \left|hS_\bullet \mathsf{Ch}(\mathcal{A})^3\right| \to \left|hS_\bullet \mathcal{MV}(X^{(1)})^w\right|.$$

Depending on the reader's taste, this map is either only defined up to homotopy or taking inverses is implemented using the suspension functor (cf. Proposition 1.1.13).

Since $F_1 \circ R = F_1 \circ R'$, the composition $|hS_\bullet F_1| \circ r$ is canonically nullhomotopic, and we obtain a lift $\widetilde{r}\colon \left|hS_\bullet\mathsf{Ch}(\mathcal{A})^3\right| \to H_1$, well-defined up to homotopy. In the next step, we are interested in the map that we obtain from composing $\widetilde{r}$ with the canonical map $\iota\colon H_1 \to wS_\bullet\mathcal{MV}(X^{(0)})$. Note that $wS_\bullet\mathcal{MV}(X^{(0)}) = hS_\bullet\mathcal{MV}(X^{(0)})$. The natural inclusion j_1 splits the retraction $T\colon hS_\bullet\mathcal{MV}(X^{(1)}) \to hS_\bullet\mathcal{MV}(X^{(0)})$ from the proof of Theorem 2.3.5. In particular, we get that

$$\iota \circ \widetilde{r} = |hS_\bullet T| \circ j_1 \circ \iota \circ \widetilde{r} = |hS_\bullet T| \circ j_2 \circ \eta \circ \widetilde{r} = |hS_\bullet T| \circ j_2 \circ r.$$

Observe that the latter map is the difference of two maps induced by functors $\mathsf{Ch}(\mathcal{A})^3 \to \mathcal{MV}(X^{(0)})$. Namely, if $J\colon \mathcal{MV}(X^{(1)})^w \to \mathcal{MV}(X^{(1)})$ is the inclusion functor, we have

$$|hS_\bullet T| \circ j_2 \circ r = |hS_\bullet(T \circ J \circ R)| - |hS_\bullet(T \circ J \circ R')|\,.$$

It is an easy computation to see that

$$\begin{aligned}(F_0 \circ T \circ J \circ R)(A_0, A_1, A_2) &\simeq (A_2, A_0, A_1),\\ (F_0 \circ T \circ J \circ R')(A_0, A_1, A_2) &\simeq (A_0, A_1, A_2).\end{aligned}$$

Consequently, if we identify $wS_\bullet\mathcal{MV}(X^{(0)})$ with $hS_\bullet\mathsf{Ch}(\mathcal{A})^3$ via F_0, the map $\iota \circ \widetilde{r}$ induces the homomorphism $3K_n(\mathcal{A}) \to 3K_n(\mathcal{A})$ represented by the matrix

$$M := \begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

This matrix has rank 2, and its kernel is given by the diagonal in $3K_n(\mathcal{A})$; in particular, the kernel can be implemented by the diagonal functor $\Delta\colon \mathsf{Ch}(\mathcal{A}) \to \mathsf{Ch}(\mathcal{A})^3$. In total, the map $\iota \circ \widetilde{r} \circ |hS_\bullet\Delta|$ is nullhomotopic, and this yields a lift

$$r_\sharp\colon\ |hS_\bullet\mathsf{Ch}(\mathcal{A})| \to \Omega\left|wS_\bullet\mathcal{MV}(X^{(1)})\right|.$$

Suppose for a moment that we know that $\widetilde{r} \circ |hS_\bullet\Delta|$ (and thus also $r_\sharp$) is not the zero map on homotopy groups. If this is the case, it follows that there is a non-trivial element in some homotopy group of H_1 which lies in the kernel of the differential d^1, i.e. we can find a cycle on the column $p = 1$ of the E^1-page. However, it follows from the fact that $wS_\bullet\mathcal{MV}(\Delta^2) \simeq hS_\bullet\mathsf{Ch}(\mathcal{A})$ that the only permanent cycles of the spectral sequence lie on the column $p = 0$. So any cycle on the column $p = 1$ cannot survive to the E^2-page. Therefore, we can conclude that there must be non-trivial entries on the column $p = 2$.

We can even prove a slightly stronger statement. Consider the composition

$$\begin{aligned}c\colon\ &|hS_\bullet\mathsf{Ch}(\mathcal{A})| \xrightarrow{r_\sharp} \Omega\left|wS_\bullet\mathcal{MV}(X^{(1)})\right| \to \Omega\left|wS_\bullet\mathcal{MV}(X)\right|\\ &\xrightarrow{F} \Omega\left|hS_\bullet\mathsf{Ch}(\mathcal{A})\right| \xrightarrow{\rho_0} \Omega\left|wS_\bullet\mathcal{MV}(X^{(0)})\right| \to \Omega\left|wS_\bullet\mathcal{MV}(X^{(1)})\right|.\end{aligned}$$

Replace $r_\sharp$ by $r_\sharp - c$. Then the redefined $r_\sharp$ is still a lift of $\widetilde{r} \circ |hS_\bullet\Delta|$ (since the map c factors via the 0-skeleton). Moreover, the new $r_\sharp$ becomes nullhomotopic upon composition with the map $\left|wS_\bullet\mathcal{MV}(X^{(1)})\right| \to |wS_\bullet\mathcal{MV}(X)|$: This can be seen by further composing with the forgetful functor F and using that ρ_0 is a section to F. Therefore, we can lift $r_\sharp$ to a map $|hS_\bullet\mathsf{Ch}(\mathcal{A})| \to \Omega H_2$.

This entire discussion can be summed up by saying that we have provided a non-trivial morphism of differential bigraded abelian groups

$$\begin{array}{ccccc} 2 & 3K_2(\mathcal{A}) \xleftarrow{M} 3K_2(\mathcal{A}) \xleftarrow{\Delta} K_2(\mathcal{A}) & & & \\ & 3K_1(\mathcal{A}) \xleftarrow{M} 3K_1(\mathcal{A}) \xleftarrow{\Delta} K_1(\mathcal{A}) & & & \\ & & \Longrightarrow & & E^1_{*,*} \\ 0 & 3K_0(\mathcal{A}) \xleftarrow{M} 3K_0(\mathcal{A}) \xleftarrow{\Delta} K_0(\mathcal{A}) & & & \\ & 3K_{-1}(\mathcal{A}) \xleftarrow{M} 3K_{-1}(\mathcal{A}) \xleftarrow{\Delta} K_{-1}(\mathcal{A}) & & & \\ & 0 \qquad\qquad\qquad 1 & & & \end{array}$$

If we compare this to the case $s = 1$, it is noteworthy that the left hand side is isomorphic to the E^1-page of the Atiyah-Hirzebruch spectral sequence for $\mathbb{H}(\Delta^2; \mathbb{K}_{\mathcal{A}}^{-\infty})$. The results of the next chapter will make it clear that it should come as no surprise that such a morphism exists.

However, we still need to provide a reason why $\widetilde{r} \circ |hS_\bullet\Delta|$ is not always weakly nullhomotopic. To this end, we will provide a procedure to detect classes in the image of $\pi_1(r)$; since r factors through $r_\sharp$, this will be sufficient to prove the claim.

Consider the subcategory $\mathcal{MV}(\partial\Delta^2)^0 \subseteq \mathcal{MV}(\partial\Delta^2)^w$ of those objects (A, a) for which $A^{-1} = 0$; as we pointed out before, the K-theory of these categories coincides. Let $(A, a) \in \mathcal{MV}(\partial\Delta^2)^0$. We can decompose $\sigma_{\partial\Delta^2}(A, a)^0$ as $A^0_0 \oplus A^0_1 \oplus A^0_2$, where A^0_i is the complex based on the 0-simplex (i). Similarly, $\sigma_{\partial\Delta^2}(A, a)^1$ can be decomposed into $A^1_{12} \oplus A^1_{02} \oplus A^1_{01}$. Then $\sigma_{\partial\Delta^2}(A, a)$ is a chain equivalence of the form

$$\alpha = \begin{pmatrix} 0 & a_{01} & a_{02} \\ a_{10} & 0 & a_{12} \\ a_{20} & a_{21} & 0 \end{pmatrix} : A^0_0 \oplus A^0_1 \oplus A^0_2 \xrightarrow{\sim} A^1_{12} \oplus A^1_{02} \oplus A^1_{01}.$$

The off-diagonal entries of this matrix are induced by the coface maps of (A, a). Let

$$C^0 := \mathsf{Cone}\Big(A^0_1 \oplus A^0_2 \xrightarrow{\begin{pmatrix} a_{01} & a_{02} \end{pmatrix}} A^1_{12}\Big),$$

$$C^1 := \Sigma\mathsf{Cone}\Big(A^0_0 \xrightarrow{\begin{pmatrix} a_{10} \\ a_{20} \end{pmatrix}} A^1_{02} \oplus A^1_{01}\Big).$$

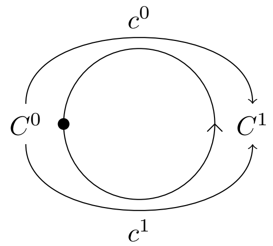
Now define the map $c\colon C^0 \to C^1$ as the composition

$$c\colon C^0 \xrightarrow{\pi} \Sigma(A^0_1 \oplus A^0_2) \xrightarrow{\begin{pmatrix} 0 & -a_{12} \\ -a_{21} & 0 \end{pmatrix}} \Sigma(A^1_{02} \oplus A^1_{01}) \xrightarrow{\iota} C^1,$$

where π is the obvious projection and ι is the obvious inclusion map. By inspection, the mapping cone of c is (up to permutation of summands) $\Sigma\mathsf{Cone}(\alpha)$; since $\mathsf{Cone}(\alpha)$ is contractible by assumption, the map c is a chain equivalence. Moreover, there is an easy way to write c as the difference of two chain maps, namely

$$c = c^0 - c^1 := \iota \circ \begin{pmatrix} 0 & 0 \\ -a_{21} & 0 \end{pmatrix} \circ \pi - \iota \circ \begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix} \circ \pi.$$

We can now think of the data (C^0, C^1, c^0, c^1) as a resolution of the trivial complex based on a semisimplicial circle, i.e. as an object in the category $\mathcal{MV}(S^1)^0$:



It is a consequence of the naturality of the cone construction that the assignment $(A, a) \mapsto (C^0, C^1, c^0, c^1)$ extends to a functor $S\colon \mathcal{MV}(\partial\Delta^2)^0 \to \mathcal{MV}(S^1)^0$. The zero object is preserved by S. Exactness as a functor of categories with cofibrations follows from the fact that $\mathsf{Id} \rightarrowtail \mathsf{Cyl} \twoheadrightarrow \mathsf{Cone}$ is a cofibration sequence of exact functors. Since $\mathsf{Ch}(\mathcal{A})$ is extensional, h-equivalences are also preserved. Therefore, we have defined an h-exact functor

$$S\colon \mathcal{MV}(\partial\Delta^2)^0 \to \mathcal{MV}(S^1)^0.$$

Consider two more categories: Let $\mathsf{End}(\mathcal{A})$ be the category whose objects are endomorphisms $A \xrightarrow{\alpha} A$ of objects in $\mathcal{A}$, and whose morphisms are given by those morphisms $f\colon A \to B$ in $\mathcal{A}$ such that the appropriate square commutes. This category becomes an exact category by pulling back the split exact structure on $\mathcal{A}$ along the forgetful functor $\mathsf{End}(\mathcal{A}) \to \mathcal{A}$. The weak equivalences in this category are the isomorphisms, which we denote $i\mathsf{End}(\mathcal{A})$. Similarly, define $\mathsf{End}(\mathsf{Ch}(\mathcal{A}))$ to be the category of endomorphisms of cochain complexes over $\mathcal{A}$. This category can be equipped with a Waldhausen structure by pulling back the corresponding subcategories along the forgetful functor $\mathsf{End}(\mathsf{Ch}(\mathcal{A})) \to \mathsf{Ch}(\mathcal{A})$; we denote its category of weak equivalences by $h\mathsf{End}(\mathsf{Ch}(\mathcal{A}))$.

There is an exact inclusion functor $J\colon \mathsf{End}(\mathcal{A}) \hookrightarrow \mathsf{End}(\mathsf{Ch}(\mathcal{A}))$ induced by the canonical inclusion $\mathcal{A} \hookrightarrow \mathsf{Ch}(\mathcal{A})$. Moreover, mapping an object $A \xrightarrow{\alpha} A$ in $\mathsf{End}(\mathsf{Ch}(\mathcal{A}))$ to the Mayer-Vietoris resolution

$$0 \longrightarrow A \underset{\alpha}{\overset{\alpha+1}{\rightrightarrows}} A$$

defines an h-exact functor $E\colon \mathsf{End}(\mathsf{Ch}(\mathcal{A})) \to \mathcal{MV}(S^1)^0$. Finally, we also define two exact functors $I_1, I_2\colon \mathcal{A} \to \mathsf{End}(\mathcal{A})$; the functor I_1 sends an object A to the endomorphism $A \xrightarrow{-1} A$, whereas the functor I_2 maps A to $A \xrightarrow{0} A$.

All this data can be arranged into a single diagram:

$$
\begin{array}{ccccc}
hS_\bullet \mathsf{Ch}(\mathcal{A}) & \underset{R' \circ \Delta}{\overset{R \circ \Delta}{\rightrightarrows}} & hS_\bullet \mathcal{MV}(\partial\Delta^2)^0 & \xrightarrow{S} & hS_\bullet \mathcal{MV}(S^1)^0 \\
\uparrow & & & & \uparrow E \\
 & & & & hS_\bullet \mathsf{End}(\mathsf{Ch}(\mathcal{A})) \\
 & & & & \uparrow \Sigma \circ J \\
iS_\bullet \mathcal{A} & & \underset{I_2}{\overset{I_1}{\rightrightarrows}} & & iS_\bullet \mathsf{End}(\mathcal{A})
\end{array}
$$

Suppose that we pick the dotted arrows in both the top and bottom row. Let $A \in \mathcal{A}$. Considering A as a complex concentrated in degree 0, the composition $S \circ R \circ \Delta$ maps A to the Mayer-Vietoris resolution

$$
0 \longrightarrow \Sigma A \oplus \mathsf{Cone}(A) \underset{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}{\overset{0}{\rightrightarrows}} \Sigma A \oplus \mathsf{Cone}(A)
$$

The resolution $(E \circ \Sigma \circ J \circ I_1)(A)$ admits an obvious cofibration into $(S \circ R)(A)$:

$$
\begin{array}{ccccc}
0 & \longrightarrow & \Sigma A \oplus \mathsf{Cone}(A) & \underset{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}}{\overset{0}{\rightrightarrows}} & \Sigma A \oplus \mathsf{Cone}(A) \\
\uparrow & & \begin{pmatrix} 1 \\ 0 \end{pmatrix} \uparrow & & \uparrow \begin{pmatrix} -1 \\ 0 \end{pmatrix} \\
0 & \longrightarrow & \Sigma A & \underset{-1}{\overset{0}{\rightrightarrows}} & \Sigma A
\end{array}
$$

This morphism is natural in A, and also happens to be an h-equivalence ($\mathsf{Cone}(A)$ is contractible). Hence, the diagram above commutes up to homotopy. An analogous argument works in case we pick the dashed arrows in place of the dotted ones.

We are now going to show that all vertical arrows in the diagram induce isomorphisms on π_1. This will reduce our problem to studying the difference between I_1 and I_2.

By virtue of the Gillet-Waldhausen theorem, the map $iS_\bullet \mathcal{A} \to hS_\bullet \mathsf{Ch}(\mathcal{A})$ is a weak equivalence, and for similar reasons, the map $iS_\bullet \mathsf{End}(\mathcal{A}) \to hS_\bullet \mathsf{End}(\mathsf{Ch}(\mathcal{A}))$ is a weak equivalence as well (see the discussion in Section 4 and the proof of Lemma 8.24 in [LS]). We know that the suspension functor Σ induces a self-equivalence in K-theory.

2.3.9 Lemma.
The functor E induces an isomorphism on π_1.

Proof. Before we start with the actual proof, we prove an easy auxiliary result that will be used repeatedly. Suppose that we are given two cochain maps $c^0, c^1 \colon C \to C$, another two cochain maps $d^0, d^1 \colon D \to D$ as well as a cochain map $\varphi \colon C \to D$ such that $d^0\varphi$ is homotopic to φc^0, and $d^1\varphi$ is homotopic to φc^1.

Then we claim that there are two endomorphisms $z^0, z^1 \colon \mathsf{Cyl}(\varphi) \to \mathsf{Cyl}(\varphi)$ such that the diagram

$$
\begin{array}{ccccc}
C & \rightarrowtail & \mathsf{Cyl}(\varphi) & \leftarrowtail & D \\
{\scriptstyle c^i}\downarrow & & {\scriptstyle z^i}\downarrow & & {\scriptstyle d^i}\downarrow \\
C & \rightarrowtail & \mathsf{Cyl}(\varphi) & \leftarrowtail & D
\end{array}
$$

commutes for $i = 0, 1$, where the indicated cofibrations are the obvious ones.

To prove this, pick homotopies $s^0 \colon d^0\varphi \simeq \varphi c^0$ and $s^1 \colon d^1\varphi \simeq \varphi c^1$. Then

$$z^i := \begin{pmatrix} d^i & 0 & s^i \\ 0 & c^i & 0 \\ 0 & 0 & c^i \end{pmatrix}$$

defines an endomorphism $z^i \colon \mathsf{Cyl}(\varphi) \to \mathsf{Cyl}(\varphi)$ with the desired properties.

Note that in this situation, the map $D \rightarrowtail \mathsf{Cyl}(\varphi)$ is always a chain equivalence, and the cofibration $C \rightarrowtail \mathsf{Cyl}(\varphi)$ is a chain equivalence whenever φ is one. We are now ready to turn to the proof of the lemma.

The proof relies on an explicit description of π_1 (see e.g. [Wei13, Prop. IV.8.4]). Namely, if $\mathcal{C}$ is an arbitrary small Waldhausen category, $\pi_1(|wS_\bullet\mathcal{C}|)$ (a.k.a. $K_0(\mathcal{C})$) is given as the quotient of the free abelian group generated by the objects of $\mathcal{C}$ by the following two relations:

- $[A] = [B]$ whenever there is a weak equivalence $A \xrightarrow{\sim} B$.
- $[B] = [A] + [C]$ whenever there is a cofibration sequence $A \rightarrowtail B \twoheadrightarrow C$.

We claim now that

$$\overline{E} \colon K_0(\mathcal{MV}(S^1)^0, h) \to K_0(\mathsf{End}(\mathsf{Ch}(\mathcal{A})), h)$$
$$[C^0, C^1, c^0, c^1] \mapsto [C^0 \xrightarrow{\gamma c^1} C^0],$$

where γ is an arbitrary inverse to $c^0 - c^1$, is a well-defined homomorphism. Suppose for a moment that this is the case. Then $\overline{E} \circ K_0(E) = \mathrm{id}$ since we can choose id_A as an inverse to $\alpha + \mathrm{id}_A - \alpha = \mathrm{id}_A$. On the other hand, the image of an element $[C^0, c^1, c^0, c^1] \in K_0(\mathcal{MV}(S^1)^0, h)$ under $K_0(E) \circ \overline{E}$ is of the form $[C^0, C^0, \gamma c^1 + 1, \gamma c^1]$, where γ is some inverse to $c^0 - c^1$. First of all, note that

$$\begin{array}{ccc} C^0 & \underset{\gamma c^1}{\overset{\gamma c^0}{\rightrightarrows}} & C^0 \\ {\scriptstyle 1}\uparrow & & \uparrow{\scriptstyle \gamma} \\ C^0 & \underset{c^1}{\overset{c^0}{\rightrightarrows}} & C^1 \end{array}$$

is an h-equivalence, so $[C^0, C^1, c^0, c^1] = [C^0, C^0, \gamma c^0, \gamma c^1]$. From $1 \simeq \gamma(c^0 - c^1)$, it follows that $\gamma c^1 + 1 \simeq \gamma c^0$. Hence, we can apply our auxiliary claim for $\varphi = \mathrm{id}_{C^0}$ to obtain a zig-zag of h-equivalences between $(C^0, C^0, \gamma c^0, \gamma c^1)$ and $(C^0, C^0, \gamma c^1{+}1, \gamma c^1)$; this shows that $K_0(E) \circ \overline{E} = \mathrm{id}$, too.

So we only need to show that $\overline{E}$ is well-defined. Let (C^0, C^1, c^0, c^1) be given, and suppose that γ and γ' are both inverses to $c^0 - c^1$. Then γ is homotopic to γ', and we can apply the auxiliary claim (with $\varphi = \mathrm{id}_{C^0}$, and the second endomorphism also being the identity) to obtain a zig-zag of equivalences between $C^0 \xrightarrow{\gamma c^1} C^0$ and $C^0 \xrightarrow{\gamma' c^1} C^0$ in $\mathsf{End}(\mathsf{Ch}(\mathcal{A}))$. Consequently, $[C^0 \xrightarrow{\gamma c^1} C^0] = [C^0 \xrightarrow{\gamma' c^1} C^0]$.

Let $\varphi\colon (C, c) \xrightarrow{\sim} (D, d)$ be an h-equivalence in $\mathcal{MV}(S^1)^0$. Choose inverses γ and δ to $c^0 - c^1$ and $d^0 - d^1$. Then

$$\delta d^1 \varphi^0 = \delta \varphi^1 c^1 \simeq \delta \varphi^1 (c^0 - c^1) \gamma c^1 = \delta(d^0 - d^1)\varphi^0 \gamma c^1 \simeq \varphi^0 \gamma c^1.$$

Another application of the auxiliary claim shows that $[C^0 \xrightarrow{\gamma c^1} C^0] = [D^0 \xrightarrow{\delta d^1} D^0]$.

Now suppose that $(C, c) \overset{\iota}{\rightarrowtail} (D, d) \overset{\pi}{\twoheadrightarrow} (E, e)$ is a cofibration sequence in $\mathcal{MV}(S^1)^0$. Let γ and ε be inverses to $c^0 - c^1$ and $e^0 - e^1$, respectively. By assumption, we know that we can write

$$d^i = \begin{pmatrix} c^i & \overline{d}^i \\ 0 & e^i \end{pmatrix}.$$

Therefore, the morphism

$$\delta := \begin{pmatrix} \gamma & -\gamma(\overline{d}^0 - \overline{d}^1)\varepsilon \\ 0 & \varepsilon \end{pmatrix}$$

is an inverse to $d^0 - d^1$. With these choices,

$$(C^0 \xrightarrow{\gamma c^1} C^0) \overset{\iota}{\rightarrowtail} (D^0 \xrightarrow{\delta d^1} D^0) \overset{\pi}{\twoheadrightarrow} (E^0 \xrightarrow{\varepsilon e^1} E^0)$$

is a cofibration sequence in $\mathsf{End}(\mathsf{Ch}(\mathcal{A}))$, showing that

$$[C^0 \xrightarrow{\gamma c^1} C^0] + [E^0 \xrightarrow{\varepsilon e^1} E^0] = [D^0 \xrightarrow{\delta d^1} D^0] \in K_0(\mathsf{End}(\mathsf{Ch}(\mathcal{A}))).$$

Thus, $\overline{E}$ is a well-defined homomorphism, and we are finished. □

From the information we have gathered, we can deduce that non-trivial elements in the image of $\pi_1(r)$ may be exhibited by showing that $\pi_1(|iS_\bullet I_1| - |iS_\bullet I_2|)$ is non-trivial. Assume that $\mathcal{A}$ is (a skeletal subcategory of) the category of finitely generated projective R-modules for some commutative ring R. Identifying π_1 of the K-theory space with K_0 of the respective category, the homomorphism of interest is

$$\begin{aligned} D\colon K_0(\mathcal{A}) &\to K_0(\mathsf{End}(\mathcal{A})) \\ [A] &\mapsto [A \xrightarrow{-1} A] - [A \xrightarrow{0} A] \end{aligned}$$

Using Almkvist's computation of $K_0(\mathsf{End}(\mathcal{A}))$, it is quite easy to show that this homomorphism is non-trivial. Let A be a finitely generated projective R-module, and let A^* be its dual. Define the *trace homomorphism* Tr of A to be the image of id_{A^*} under the homomorphism

$$\begin{aligned} \hom_R(A^*, A^*) = \hom_R(A^*, \hom_R(A, R)) &\xrightarrow{\cong} \hom_R(A^* \otimes_R A, R) \\ &\xrightarrow{\cong} \hom_R(\hom_R(A, A), R). \end{aligned}$$

The *characteristic polynomial* $\chi(\alpha)$ of an endomorphism $\alpha\colon A \to A$ is then given by

$$\chi(\alpha) := \sum_{i \geqslant 0} \mathrm{Tr}(\Lambda^i \alpha) X^i,$$

where $\Lambda^i \alpha$ denotes the i-th exterior power of α. Consider the set

$$\widetilde{R}_0 := \left\{ \frac{1 + r_1 X + \cdots + r_m X^m}{1 + s_1 X + \cdots + s_n X^n} \mid r_i, s_j \in R \right\}.$$

This becomes an abelian group via multiplication. Almkvist's theorem now states the following:

2.3.10 Theorem ([Alm74])**.**
Let R be a commutative ring. Let $\mathcal{A}$ be (a skeletal subcategory of) the category of finitely generated projective R-modules. Then

$$\begin{aligned} K_0(\mathsf{End}(\mathcal{A})) &\xrightarrow{\cong} K_0(\mathcal{A}) \oplus \widetilde{R}_0 \\ [A \xrightarrow{\alpha} A] &\mapsto ([A], \chi(\alpha)) \end{aligned}$$

is an isomorphism of abelian groups.

□

By virtue of Almkvist's theorem, we need only check that the characteristic polynomials of the zero morphism and the identity morphism do not coincide. For the zero morphism $0\colon A \to A$, we have

$$\chi(0) = \sum_{i \geqslant 0} \mathrm{Tr}(\Lambda^i(0)) X^i = 1.$$

On the other hand, it is known that

$$\chi(\mathrm{id}_A) = \sum_{i=1}^{m} e_i (1+X)^i$$

for some pairwise orthogonal idempotents $e_1, \ldots, e_m \in R$ such that $e_1 + \cdots + e_m = 1_R$; see [Alm73, Thm. 2.2 on p. 271] for the statement and proof. For example, in the easiest case that $A = R^m$, we have $\chi(\mathrm{id}_A) = (1+X)^m$ (see [Alm73, Thm. 2.2 on p. 270]). In any case, we can conclude that $D = \pi_1(r)$ is non-trivial.

3. Filtering the assembly map

Up to this point we have considered the K-theory of $\mathcal{MV}_G(X;\mathcal{K})$ isolated for itself. Given that it can serve as a model for the K-theory of group rings (Theorem 2.2.1), we are now going to tie up our discussion with the main computational conjecture about the K-theory of group rings, the *Farrell-Jones Conjecture.*

The standard treatment of assembly in the equivariant case is [DL98], where the authors also show how their characterisations of the assembly map relate to the Weiss-Williams picture of assembly [WW95]. Since both our models for spaces and for the K-theory of group rings differ, we repeat the key points about equivariant homology and the assembly map. The knowledgeable reader will notice some subtle differences which are necessitated by the fact that semisimplicial sets allow far fewer maps than simplicial sets. Also, since our homotopy colimit description of the assembly map uses semisimplicial G-sets while the Davis-Lück description has to rely on simplicial sets prevents us from a direct application of the universal property of the assembly map. Our arguments in this section are mostly borrowed from [DL98], even though in a certain sense we are closer to [WW95] in spirit.

We recapitulate the construction of G-homology spectra from [DL98]: Recall that every simplicial G-set X gives rise to a contravariant $\mathsf{Or}(G)$-space $\operatorname{map}_G(-,X)$:

$$G/H \mapsto \operatorname{map}_G(G/H, X) \cong X^H.$$

3.1 Definition ([DL98, Def. 4.1]).
Let $\mathbb{E}$ be an $\mathsf{Or}(G)$-spectrum, and let X be a simplicial G-set. The *G-equivariant homology spectrum of X with coefficients in* $\mathbb{E}$ is defined to be

$$\mathbb{H}^G(X;\mathbb{E}) := \operatorname{map}_G(-,X)_+ \otimes_{\mathsf{Or}(G)} \mathbb{E}.$$

3.2 Remark.
Definition 3.1 can be extended to pairs of simplicial G-sets (X,A) using the $\mathsf{Or}(G)$-space $\operatorname{map}_G(-, X \cup_A \mathsf{Cone}(A))$ instead of $\operatorname{map}_G(-,X)$. It follows from classical theorems that $(X,A) \mapsto \pi_*\mathbb{H}^G(X,A;\mathbb{E})$ defines a G-homology theory ([DL98, Lem. 4.2]).

In particular, there is for every G-simplicial set X an Atiyah-Hirzebruch spectral sequence which arises from the skeletal filtration of X and converges to the G-homology of X with coefficients in $\mathbb{E}$; see [DL98, Thm. 4.7], or [Ros04, Sec. 8] for a more detailed treatment.

We are particularly interested in the G-homology theory arising from the $\mathsf{Or}(G)$-spectrum $\mathbb{K}_{\mathcal{A}}^{-\infty}$, which is defined by the rule

$$G/H \mapsto \mathbb{K}^{-\infty}(\mathcal{A} *_G G/H).$$

3.3 Definition ([DL98, Section 5]).
Let X be a simplicial G-set. The *assembly map* for X is the map

$$A_X : \mathbb{H}^G(X; \mathbb{K}_{\mathcal{A}}^{-\infty}) \to \mathbb{H}^G(G/G; \mathbb{K}_{\mathcal{A}}^{-\infty}) \simeq \mathbb{K}^{-\infty}(\mathcal{A}[G])$$

induced by the projection map $X \to G/G$.

3.4 Conjecture (Farrell-Jones Conjecture).
Let G be a discrete group, and $\mathcal{A}$ a small additive category with right G-action. Then

$$A_{VCyc} \colon \mathbb{H}^G(E_{VCyc}G; \mathbb{K}_{\mathcal{A}}^{-\infty}) \to \mathbb{K}^{-\infty}(\mathcal{A}[G])$$

is a weak equivalence, where $E_{VCyc}G$ denotes the classifying space for the family of virtually cyclic subgroups.

See [LR05] for a comprehensive survey on the Farrell-Jones Conjecture; be aware that the section on the status of the conjecture is outdated. We will now give an alternative description of the assembly map as in [WW95], and identify this map with the assembly map as defined in 3.3 using arguments from [DL98, Section 5].

3.5 Definition ([DL98, p. 246]).
Let X be a semisimplicial G-set. The *equivariant simplex category* $\mathsf{simp}_G(X)$ *of* X has as objects maps of semisimplicial G-sets $\chi \colon G/H \times \Delta^n \to X$, where $n \in \mathbb{N}$ and H ranges over all subgroups of G. The morphisms in $\mathsf{simp}_G(X)$ are given by maps over X, i.e. a morphism $f \colon \chi \to \chi'$ is a map $f \colon G/H \times \Delta^n \to G/H' \times \Delta^{n'}$ of semisimplicial G-sets such that the following diagram commutes:

$$\begin{array}{ccc} G/H \times \Delta^n & \xrightarrow{\quad f \quad} & G/H' \times \Delta^{n'} \\ & {\scriptstyle\chi} \searrow \quad \swarrow {\scriptstyle\chi'} & \\ & X & \end{array}$$

The composition law in $\mathsf{simp}_G(X)$ is the obvious one.

Let X be a semisimplicial G-set. Define a functor

$$\mathbb{K}_X^{-\infty} \colon \mathsf{simp}_G(X) \to \mathsf{Spectra}$$

by $\mathbb{K}_X^{-\infty}(\chi \colon G/H \times \Delta^n \to X) := \mathbb{K}^{-\infty}(\mathcal{MV}_G(G/H \times \Delta^n; \mathcal{K}), w)$. There is a natural transformation $\widetilde{\alpha}$ between $\mathbb{K}_X^{-\infty}$ and the constant functor which always takes the value $\mathbb{K}^{-\infty}(\mathcal{MV}_G(X; \mathcal{K}), w)$:

$$\widetilde{\alpha}_\chi \colon \mathbb{K}^{-\infty}(\mathcal{MV}_G(G/H \times \Delta^n; \mathcal{K}), w) \xrightarrow{\mathbb{K}^{-\infty}(\mathcal{MV}_G(\chi))} \mathbb{K}^{-\infty}(\mathcal{MV}_G(X; \mathcal{K}), w).$$

This gives rise to a map

$$\alpha_X \colon \operatorname*{hocolim}_{\mathsf{simp}_G(X)} \mathbb{K}_X^{-\infty} \to \mathbb{K}^{-\infty}(\mathcal{MV}_G(X; \mathcal{K}), w) \xrightarrow{\mathbb{K}^{-\infty}(F)} \mathbb{K}^{-\infty}(\mathcal{K}(G/G)),$$

where F is as usual the forgetful functor mapping a Mayer-Vietoris resolution (A, a) to A^{-1}.

Suppose for the rest of this section that $\mathcal{K} = \mathcal{K}_{\mathcal{A}}$ for some small additive category $\mathcal{A}$ with right G-action. We wish to identify α_X with the assembly map as defined previously in 3.3. To do this, we will first show that there is a certain zig-zag of equivalences

$$\operatorname*{hocolim}_{\mathsf{simp}_G(X)} \mathbb{K}_X^{-\infty} \xleftarrow{\sim} \dots \xrightarrow{\sim} \mathbb{H}^G(\widehat{\mathsf{sd}(X)}; \mathbb{K}_{\mathcal{A}}^{-\infty}),$$

where the latter is the equivariant homology of the first barycentric subdivision of X turned into a simplicial set; we will explain a bit later what this precisely means. Once this is done, we can proceed to identify the two competing definitions of assembly.

Let $\widetilde{\mathbb{K}_{\mathcal{A}}^{-\infty}} \colon \mathsf{simp}_G(X) \to \mathsf{Spectra}$ be the functor mapping $G/H \times \Delta^p \xrightarrow{\chi} X$ to $\mathbb{K}^{-\infty}(\mathcal{A} *_G G/H)$. The main hurdle we have to overcome is replacing $\mathbb{K}_X^{-\infty}$ by the simpler functor $\widetilde{\mathbb{K}_{\mathcal{A}}^{-\infty}}$.

3.6 Proposition.
There is a zig-zag of natural levelwise equivalences

$$\operatorname*{hocolim}_{\mathsf{simp}_G(X)} \mathbb{K}_X^{-\infty} \xleftarrow{\sim} \dots \xrightarrow{\sim} \operatorname*{hocolim}_{\mathsf{simp}_G(X)} \widetilde{\mathbb{K}_{\mathcal{A}}^{-\infty}}.$$

Proof. Since homotopy colimits commute among each other, it suffices to prove the claim in the connective case. Consider the two diagrams of (small) Waldhausen categories

$$D_X \colon \mathsf{simp}_G(X) \to \mathsf{WaldhCat}, \quad (G/H \times \Delta^p \to X) \mapsto (\mathcal{MV}_G(G/H \times \Delta^p; \mathcal{K}_{\mathcal{A}}), w)$$
$$D_{\mathcal{A}} \colon \mathsf{simp}_G(X) \to \mathsf{WaldhCat}, \quad (G/H \times \Delta^p \to X) \mapsto \mathsf{Ch}(\mathcal{A} *_G G/H)$$

which give rise to $\mathbb{K}_X^{-\infty}$ and $\widetilde{\mathbb{K}_{\mathcal{A}}^{-\infty}}$ upon applying $\mathbb{K}^{-\infty}$. We do not have a natural transformation of these diagrams at hand, but something nearly as good. For every map $G/H \times \Delta^p \xrightarrow{\chi} X$, let

$$\tau_\chi := \rho^p_{G/H} \colon \mathsf{Ch}(\mathcal{A} *_G G/H) \to \mathcal{MV}_G(G/H \times \Delta^p; \mathcal{K}_{\mathcal{A}}),$$

where $\rho^p_{G/H}$ is the functor defined at the end of Section 2.1. These are exact functors. Given a morphism $f = (f_G, f_\Delta) \colon \chi \to \chi'$ in $\mathsf{simp}_G(X)$, the diagram

$$\begin{array}{ccc} \mathsf{Ch}(\mathcal{A} *_G G/H) & \xrightarrow{\rho^p_{G/H}} & \mathcal{MV}_G(G/H \times \Delta^p; \mathcal{K}_{\mathcal{A}}) \\ {\scriptstyle (f_G)_*}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{MV}_G(f)} \\ \mathsf{Ch}(\mathcal{A} *_G G/H') & \xrightarrow[\rho^{p'}_{G/H'}]{} & \mathcal{MV}_G(G/H' \times \Delta^{p'}; \mathcal{K}_{\mathcal{A}}) \end{array}$$

does not commute. However, the inclusion f_Δ induces a natural transformation

$$\vartheta_f \colon \rho^{p'}_{G/H'} \circ (f_G)_* \to \mathcal{MV}_G(f) \circ \rho^p_{G/H}$$

which corresponds to the projection $A \otimes C^*_\varepsilon(\Delta^{p'}) \twoheadrightarrow A \otimes C^*_\varepsilon(\Delta^p)$. Observe that ϑ_f consists entirely of w-equivalences. Additionally, for every two composable morphisms f and f' in $\mathsf{simp}_G(X)$, we have $\vartheta_{f'f} = \vartheta_{f'} \circ \vartheta_f$.

Consider $\operatorname{hocolim}_{\mathsf{simp}_G(X)} d(NwS_\bullet D_X)$ (recall the definition of the K-theory space 1.1.6 if you are uncertain why this diagram of simplicial sets concerns us); by [BK72, XII.4.3 and XII.3.3], we have natural equivalences

$$\begin{aligned}\operatorname*{hocolim}_{\mathsf{simp}_G(X)} d(NwS_\bullet D_X) &\xleftarrow{\simeq} \operatorname*{hocolim}_{\mathsf{simp}_G(X)} \operatorname*{hocolim}_{\Delta} NwS_\bullet D_X \\ &\cong \operatorname*{hocolim}_{\Delta} \operatorname*{hocolim}_{\mathsf{simp}_G(X)} NwS_\bullet D_X,\end{aligned}$$

where $\operatorname{hocolim}_\Delta$ is taken with respect to the $S_\bullet$-direction.

Recall the *Grothendieck construction* of a diagram of small categories: Given a diagram of small categories $D\colon \mathcal{I} \to \mathsf{Cat}$, the category $\int_{\mathcal{I}} D$ (the Grothendieck construction of D) has as objects pairs (I, X) with $I \in \mathcal{I}$ and $X \in D(I)$, while a morphism $(i, x)\colon (I, X) \to (I', X')$ consists of a morphism $i\colon I \to I'$ in $\mathcal{I}$ and a morphism $x\colon D(i)(X) \to X'$ in $D(I')$.

By Thomason's theorem on homotopy colimits [Tho79, Thm. 1.2], there is a natural equivalence

$$\operatorname*{hocolim}_{\mathsf{simp}_G(X)} NwS_n D_X \xrightarrow{\simeq} N \int_{\mathsf{simp}_G(X)} wS_n D_X.$$

The functors τ_χ induce functors $wS_n\tau_\chi\colon wS_n D_{\mathcal{A}}(\chi) \to wS_n D_X(\chi)$ by functoriality of the $wS_\bullet$-construction, and we also have natural transformations $wS_n\vartheta_f$ satisfying similar relations as the transformations ϑ_f.

By [Wal85, Lem. 1.6.6], every functor wS_nF is a weak equivalence by virtue of the Approximation Theorem because we had shown that F has the approximation property. Arguing as in the proof of Corollary 2.2.9, we see that $wS_n\tau_\chi$ induces an equivalence of nerves for each n individually. It follows that the data $\{wS_n\tau_\chi, wS_n\vartheta_f\}$ induces a weak equivalence

$$\left| \int_{\mathsf{simp}_G(X)} wS_n D_{\mathcal{A}} \right| \xrightarrow{\simeq} \left| \int_{\mathsf{simp}_G(X)} wS_n D_X \right|,$$

see Appendix B for details. Again using the naturality of the $S_\bullet$-construction, we obtain a levelwise equivalence of simplicial spaces, inducing an equivalence

$$\operatorname*{hocolim}_{\Delta} \left| \int_{\mathsf{simp}_G(X)} wS_\bullet D_{\mathcal{A}} \right| \xrightarrow{\simeq} \operatorname*{hocolim}_{\Delta} \left| \int_{\mathsf{simp}_G(X)} wS_\bullet D_X \right|.$$

Invoking Thomason's theorem a second time, using that hocolim commutes with geometric realisation, and repeating our initial argument, we arrive at a a zig-zag of equivalences

$$\left| \operatorname*{hocolim}_{\mathsf{simp}_G(X)} d(NwS_\bullet D_X) \right| \simeq \left| \operatorname*{hocolim}_{\mathsf{simp}_G(X)} d(NwS_\bullet D_X) \right|.$$

□

The rest of the argument is formal, and follows very closely the discussion in [DL98]. From this point until stating Theorem 3.12, we will be solely concerned with making successive identifications connecting $\operatorname{hocolim}_{\mathsf{simp}_G(X)} \widetilde{\mathbb{K}_{\mathcal{A}}^{-\infty}}$ to $\mathbb{H}^G(X;\mathbb{K}_{\mathcal{A}}^{-\infty})$.

We have to introduce some more notation: Set

$$\mathsf{OD} := \mathsf{Or}(G) \times \Delta_{\mathsf{inj}},$$

and let V denote the forgetful functor

$$V\colon \mathsf{simp}_G(X) \to \mathsf{OD}, \quad (G/H \times \Delta^p \to X) \mapsto (G/H, \underline{p}).$$

Define also an inclusion functor

$$J\colon \mathsf{Or}(G) \to \mathsf{OD}, \quad G/H \mapsto (G/H, \underline{0}),$$

as well as two projection functors

$$P_1\colon \mathsf{OD} \to \mathsf{Or}(G), \quad (G/H, \underline{p}) \mapsto G/H,$$
$$P_2\colon \mathsf{OD} \to \Delta_{\mathsf{inj}}, \quad (G/H, \underline{p}) \mapsto \underline{p}.$$

Moreover, consider the functor

$$\overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}\colon \mathsf{OD} \to \mathsf{Spectra}, \quad (G/H, \underline{p}) \mapsto \mathbb{K}^{-\infty}(\mathcal{A} *_G G/H).$$

Observe that $\widetilde{\mathbb{K}_{\mathcal{A}}^{-\infty}} = \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}} \circ V$ and $\mathbb{K}_{\mathcal{A}}^{-\infty} = \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}} \circ J$. Applying Proposition 3.6, it suffices to consider

$$\operatorname*{hocolim}_{\mathsf{simp}_G(X)} \widetilde{\mathbb{K}_{\mathcal{A}}^{-\infty}} = N\big(?/\mathsf{simp}_G(X)\big)_+ \otimes_{\mathsf{simp}_G(X)} \widetilde{\mathbb{K}_{\mathcal{A}}^{-\infty}}(?).$$

The adjunction 1.2.6 yields a levelwise equivalence

$$\begin{aligned} N\big(?/\mathsf{simp}_G(X)\big)_+ \otimes_{\mathsf{simp}_G(X)} \widetilde{\mathbb{K}_{\mathcal{A}}^{-\infty}}(?) &= N\big(?/\mathsf{simp}_G(X)\big)_+ \otimes_{\mathsf{simp}_G(X)} \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}} \circ V(?) \\ &\simeq \big(\operatorname{ind}_V N\big(?/\mathsf{simp}_G(X)\big)\big)_+ \otimes_{\mathsf{OD}} \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}(-). \end{aligned}$$

3.7 Lemma ([DL98, p. 246f.]).
There is an isomorphism of OD-simplicial sets

$$N\big(?/\mathsf{simp}_G(X)\big) \otimes_{\mathsf{simp}_G(X)} \hom(-, V(?)) \xrightarrow{\cong} \hom(?, X) \otimes_{\mathsf{OD}} N(-/\mathsf{OD}/?).$$

Proof. For each $(G/K, \underline{q})$, we have to specify a simplicial map

$$\begin{aligned} N\big(?/\mathsf{simp}_G(X)\big) \otimes_{\mathsf{simp}_G(X)} &\hom\big((G/K,\underline{q}), V(?)\big) \\ &\to \hom(?, X) \otimes_{\mathsf{OD}} N\big((G/K,\underline{q})/\mathsf{OD}/?\big). \end{aligned}$$

We claim that the rule

$$\begin{aligned} (G/H \times \Delta^p \to G/H_0 \times \Delta^{p_0} \to \cdots \to G/H_n \times \Delta^{p_n} \to X) \otimes \big((G/K,\underline{q}) \to (G/H,\underline{p})\big) \\ \mapsto (G/H_n \times \Delta^{p_n} \to X) \otimes \big((G/K,\underline{q}) \to (G/H,\underline{p}) \to \cdots \to (G/H_n,\underline{p_n})\big) \end{aligned}$$

defines such a map.

It is obvious that the given rule is compatible with the degneracy maps, and also with all face maps except the n-th face map. In that case, taking the n-th face and then applying the rule yields

$$\begin{aligned}(G/H_{n-1} \times \Delta^{p_{n-1}} \to X) &\otimes \big((G/K,\underline{q}) \to (G/H,\underline{p}) \to \dots \to (G/H_{n-1},\underline{p_{n-1}})\big) \\ = (G/H_n \times \Delta^{p_n} \to X) &\otimes \big((G/K,\underline{q}) \to (G/H,\underline{p}) \to \dots \\ &\qquad \dots \to (G/H_{n-2},\underline{p_{n-2}}) \to (G/H_n,\underline{p_n})\big),\end{aligned}$$

and the latter is the image after first applying the rule and then taking the n-th face. So we have in fact defined a simplicial map. To see that it is an isomorphism it suffices to observe (we have already used this in our notation) that all the maps $G/H_i \times \Delta^{p_i} \to X$ are determined via their map to $G/H_n \times \Delta^{p_n}$ by the reference map $G/H_n \times \Delta^{p_n} \to X$. □

Applying the lemma, we get levelwise equivalences

$$\begin{aligned}&\big(\mathrm{ind}_V N\big(?/\mathsf{simp}_G(X)\big)\big)_+ \otimes_{\mathsf{OD}} \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}(-) \\ &\quad = N\big(?/\mathsf{simp}_G(X)\big) \otimes_{\mathsf{simp}_G(X)} \hom(-,V(?)) \otimes_{\mathsf{OD}} \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}(-) \\ &\quad \xrightarrow{\simeq} \big(\hom(?,X) \otimes_{\mathsf{OD}} N(-/\mathsf{OD}/?)\big)_+ \otimes_{\mathsf{OD}} \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}(-) \\ &\quad \simeq \hom(?,X)_+ \otimes_{\mathsf{OD}} N(-/\mathsf{OD}/?)_+ \otimes_{\mathsf{OD}} \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}(-).\end{aligned}$$

3.8 Lemma ([DL98, Proof of Thm. 6.3])**.**
There is a natural levelwise equivalence

$$\begin{aligned}&N(-/\mathsf{OD}/?)_+ \otimes_{\mathsf{OD}} \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}(-) \\ &\qquad \xrightarrow{\simeq} \Big(N\big(-/\mathsf{Or}(G)/P_1(?)\big)_+ \wedge N\big(\Delta_{\mathsf{inj}}/P_2(?)\big)_+\Big) \otimes_{\mathsf{Or}(G)} \mathbb{K}_{\mathcal{A}}^{-\infty}(-).\end{aligned}$$

Proof. Since the nerve construction commutes with products, we have a levelwise equivalence

$$\begin{aligned}&N\big(-/\mathsf{OD}/?\big)_+ \otimes_{\mathsf{OD}} \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}(-) \\ &\quad \simeq \Big(N\big(P_1(-)/\mathsf{Or}(G)/P_1(?)\big) \times N\big(P_2(-)/\Delta_{\mathsf{inj}}/P_2(?)\big)\Big)_+ \otimes_{\mathsf{OD}} \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}(-) \\ &\quad \simeq \Big(N\big(P_1(-)/\mathsf{Or}(G)/P_1(?)\big)_+ \wedge N\big(P_2(-)/\Delta_{\mathsf{inj}}/P_2(?)\big)_+\Big) \otimes_{\mathsf{OD}} \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}(-) \\ &\quad \simeq N\big(-/\mathsf{Or}(G)/P_1(?)\big)_+ \otimes_{\mathsf{Or}(G)} \Big(N\big(\Box/\Delta_{\mathsf{inj}}/P_2(?)\big)_+ \otimes_{\Delta_{\mathsf{inj}}} \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}(-,\Box)\Big).\end{aligned}$$

Now it suffices to give a natural isomorphism

$$\big(N(\Box/\Delta_{\mathsf{inj}}/?)_+ \otimes_{\Delta_{\mathsf{inj}}} \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}(G/H,\Box)\big)_n \to |N(\Delta_{\mathsf{inj}}/?)|_+ \wedge (\mathbb{K}_{\mathcal{A}}^{-\infty}(G/H))_n.$$

Both sides are realisations of simplicial sets (recall 1.2.2), so we can even define a simplicial isomorphism by

$$(\underline{p} \to \underline{p_0} \to \cdots \to \underline{p_n} \to \underline{q} \otimes x) \mapsto (\underline{p_0} \to \cdots \to \underline{p_n} \to \underline{q} \otimes x),$$

where we have used that $\overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}(G/H, \underline{p_0}) = \mathbb{K}_{\mathcal{A}}^{-\infty}(G/H)$. This map commutes with face and degeneracy maps and is clearly surjective. For injectivity, observe that $\overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}(\mathrm{id}, \mu)(x) = x$ for any map μ in Δ_{inj}. Therefore,

$$\begin{aligned}
&\underline{p} \to \underline{p_0} \to \cdots \to \underline{p_n} \to \underline{q} \otimes x \\
&\quad = \underline{p_0} \to \underline{p_0} \to \cdots \to \underline{p_n} \to \underline{q} \otimes \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}}(\mathrm{id}, \underline{p} \to \underline{p_0})(x) \\
&\quad = \underline{p_0} \to \underline{p_0} \to \cdots \to \underline{p_n} \to \underline{q} \otimes x.
\end{aligned}$$

□

Thus, we have levelwise equivalences (suppressing the coherence isomorphisms)

$$\begin{aligned}
&\hom(?, X)_+ \otimes_{\mathsf{OD}} N(-/\mathsf{OD}/?)_+ \otimes_{\mathsf{OD}} \overline{\mathbb{K}_{\mathcal{A}}^{-\infty}} \\
&\quad \xrightarrow{\sim} \hom(?, X)_+ \otimes_{\mathsf{OD}} \Big(N\big(-/\mathsf{Or}(G)/P_1(?)\big)_+ \wedge N\big(\Delta_{\mathsf{inj}}/P_2(?)\big)_+\Big) \otimes_{\mathsf{Or}(G)} \mathbb{K}_{\mathcal{A}}^{-\infty} \\
&\quad \xrightarrow{\sim} \hom(? \times ??, X)_+ \otimes_{\Delta_{\mathsf{inj}}} N(\Delta_{\mathsf{inj}}/??)_+ \otimes_{\mathsf{Or}(G)} N\big(-/\mathsf{Or}(G)/?\big)_+ \otimes_{\mathsf{Or}(G)} \mathbb{K}_{\mathcal{A}}^{-\infty} \\
&\quad \xrightarrow{\sim} \big(\hom(? \times ??, X) \otimes_{\Delta_{\mathsf{inj}}} N(\Delta_{\mathsf{inj}}/??)\big)_+ \otimes_{\mathsf{Or}(G)} N\big(-/\mathsf{Or}(G)/?\big)_+ \otimes_{\mathsf{Or}(G)} \mathbb{K}_{\mathcal{A}}^{-\infty}.
\end{aligned}$$

3.9 Lemma.
There is an isomorphism of $\mathsf{Or}(G)$-simplicial sets

$$\hom(? \times ??, X) \otimes_{\Delta_{\mathsf{inj}}} N(\Delta_{\mathsf{inj}}/??) \xrightarrow{\cong} N\big(\mathsf{simp}(\mathrm{map}_G(?, X))\big).$$

Proof. Fix $G/H \in \mathsf{Or}(G)$. Recall that there is a natural bijection between maps of semisimplicial G-sets $G/H \times \Delta^p \to X$ and p-simplices in $X^H = \mathrm{map}_G(G/H, X)$. Also, every morphism $\underline{m} \hookrightarrow \underline{n}$ corresponds to a map $\Delta^m \to \Delta^n$. We claim that the rule

$$\begin{aligned}
&(G/H \times \Delta^p \to X) \otimes (\underline{p_0} \to \cdots \to \underline{p_n} \to \underline{p}) \\
&\qquad \mapsto \big(\Delta^{p_0} \to \cdots \to \Delta^{p_n} \to \mathrm{map}_G(G/H, X)\big)
\end{aligned}$$

defines a natural isomorphism, where the map $\Delta^{p_n} \to \mathrm{map}_G(G/H, X)$ is obtained as the composition of the obvious map $\Delta^{p_n} \to \Delta^p$ with the adjoint of the map $G/H \times \Delta^p \to X$. This map is simplicial. Bijectivity follows from the preliminary comment and the fact that the Yoneda embedding $\Delta_{\mathsf{inj}} \to \mathsf{ssSets}$ is fully faithful. □

3.10 Lemma.
Let $\mathcal{C}$ be a small category. Each homomorphism set $\hom_{\mathcal{C}}(-,?)$ can be regarded as a discrete category, whose nerve is precisely $\hom_{\mathcal{C}}(-,?)$ again, now regarded as a discrete simplicial set.

The "projection" map

$$N(-/\mathcal{C}/?) \to \hom_{\mathcal{C}}(-,?)$$

induced by the functor sending $C \xrightarrow{f} C_0 \xrightarrow{g} D$ to the composition $C \xrightarrow{gf} D$ defines a natural weak equivalence of $\mathcal{C}^{op} \times \mathcal{C}$-simplicial sets.

Proof. Let C and D be objects in $\mathcal{C}$. By Quillen's Theorem A ([Qui73, p. 85]), it suffices to show that $(C/\mathcal{C}/D)/f$ has a contractible nerve for every morphism $f\colon C \to D$. Indeed, the category $(C/\mathcal{C}/D)/f$ has both an initial and a terminal object, given by $C \xrightarrow{\mathrm{id}} C \xrightarrow{f} D$ and $C \xrightarrow{f} D \xrightarrow{\mathrm{id}} D$, respectively. □

The previous two lemmas combine to a levelwise equivalence

$$\begin{aligned}\big(\hom(?\times??,X) \otimes_{\Delta_{\mathsf{inj}}} N(\Delta_{\mathsf{inj}}/??)\big)_+ \otimes_{\mathsf{Or}(G)} N\big(-/\mathsf{Or}(G)/?\big)_+ \otimes_{\mathsf{Or}(G)} \mathbb{K}_{\mathcal{A}}^{-\infty} \\ \xrightarrow{\sim} N\big(\mathsf{simp}(\mathrm{map}_G(?,X))\big)_+ \otimes_{\mathsf{Or}(G)} \hom_{\mathsf{Or}(G)}(-,?)_+ \otimes_{\mathsf{Or}(G)} \mathbb{K}_{\mathcal{A}}^{-\infty},\end{aligned}$$

and the latter term is equivalent to $N\big(\mathsf{simp}(\mathrm{map}_G(?,X))\big)_+ \otimes_{\mathsf{Or}(G)} \mathbb{K}_{\mathcal{A}}^{-\infty}$ by Lemma 1.2.5.

3.11 Lemma (see [WW95, p. 333])**.**
There is a natural isomorphism of $\mathsf{Or}(G)$-simplicial sets

$$N\big(\mathsf{simp}(\mathrm{map}_G(?,X))\big) \xrightarrow{\cong} \mathrm{map}_G\big(?,\widehat{\mathsf{sd}(X)}\big).$$

Proof. Recall the relevant definitions: Let Y be a semisimplicial set. An n-simplex in the barycentric subdivison $\mathsf{sd}(Y)$ is a sequence of proper inclusions of semisimplicial sets $\Delta^{p_0} \to \cdots \to \Delta^{p_n} \to Y$; the i-th face map is given by dropping Δ^{p_i}.

The functor $\widehat{\ }\colon \mathsf{ssSets} \to \mathsf{sSets}$ adjoins degeneracies to a semisimplicial set Y as follows: The set of n-simplices of $\widehat{Y}$ is given by

$$\widehat{Y}_n := \big\{(y,\mu) \mid y \in Y_p, \mu\colon \underline{n} \twoheadrightarrow \underline{p}\big\}.$$

The degeneracy maps in $\widehat{X}$ are simply given by $s_i(y,\mu) := (y, s^i \circ \mu)$. For the i-th face map, one uses the fact that the morphism $\mu \circ d^i$ has a unique mono-epi-factorisation $\mu \circ d^i = \mathsf{Sur}(\mu d^i) \circ \mathsf{Inj}(\mu d^i)$, and sets $d_i(y,\mu) := \big(\mathsf{Inj}(\mu d^i)^*(y), \mathsf{Sur}(\mu d^i)\big)$.

Since

$$\begin{aligned}\mathrm{map}_G(G/H,\widehat{\mathsf{sd}(X)})_n &= \hom(\Delta^n, \mathrm{map}_G(G/H,\widehat{\mathsf{sd}(X)})) \\ &\cong \hom(\Delta^n, \widehat{\mathsf{sd}(X)}^H) \\ &\cong \big(\widehat{\mathsf{sd}(X)}^H\big)_n,\end{aligned}$$

it suffices to show that for every semisimplicial set Y, there is a natural isomorphism of simplicial sets

$$N\big(\mathsf{simp}(Y)\big) \xrightarrow{\cong} \widehat{\mathsf{sd}(Y)}.$$

Given an n-simplex $\sigma = \Delta^{p_0} \to \cdots \to \Delta^{p_n} \to Y$ in $N\big(\mathsf{simp}(Y)\big)$, every arrow $\Delta^{p_i} \to \Delta^{p_{i+1}}$ is either a face inclusion or the identity. We let $nd(\sigma)$ denote the face of σ in which all the identity maps have been dropped from σ; if $n-p$ is the number of identity maps in σ denote by $\mu(\sigma)$ the monotone surjection $\underline{n} \to \underline{p}$ given by

$$i \mapsto \text{number of proper face inclusions in the sequence } \Delta^{p_0} \to \cdots \to \Delta^{p_i}.$$

We claim that the rule

$$\sigma \mapsto (nd(\sigma), \mu(\sigma))$$

defines such a natural isomorphism of simplicial sets. It is easy to check that we have defined bijections $N_n(\mathsf{simp}(Y)) \cong \widehat{\mathsf{sd}(Y)}_n$, and that these are compatible with the degeneracy maps. Compatibility with face maps is a bit more interesting: We have that

$$nd(d_i(\sigma)) = \begin{cases} nd(\sigma) & \Delta^{p_{i-1}} \to \Delta^{p_i} = \mathrm{id} \text{ or } \Delta^{p_i} \to \Delta^{p_{i+1}} = \mathrm{id} \\ d_i(nd(\sigma)) & \text{otherwise.} \end{cases}$$

$$\mu(d_i(\sigma)) = \begin{cases} \mu(\sigma)d^i & \Delta^{p_{i-1}} \to \Delta^{p_i} = \mathrm{id} \text{ or } \Delta^{p_i} \to \Delta^{p_{i+1}} = \mathrm{id} \\ s^i\mu(\sigma) & \text{otherwise.} \end{cases}$$

This equals exactly the definition of $d_i(nd(\sigma), \mu(\sigma))$ we have given in the description of $\widehat{\mathsf{sd}(Y)}$. This proves the lemma. □

We have finally arrived at a levelwise equivalence

$$\begin{aligned} N\big(\mathsf{simp}(\mathrm{map}_G(?, X))\big)_+ \otimes_{\mathsf{Or}(G)} \mathbb{K}_{\mathcal{A}}^{-\infty} &\xrightarrow{\sim} \mathrm{map}_G\big(?, \widehat{\mathsf{sd}(X)}\big)_+ \otimes_{\mathsf{Or}(G)} \mathbb{K}_{\mathcal{A}}^{-\infty} \\ &= \mathbb{H}^G(\widehat{\mathsf{sd}(X)}; \mathbb{K}_{\mathcal{A}}^{-\infty}) \end{aligned}$$

This finishes the identification of $\mathrm{hocolim}_{\mathsf{simp}_G(X)} \mathbb{K}_X^{-\infty}$ with the equivariant homology theory appearing in the definition of the Davis-Lück assembly map.

3.12 Theorem.
The map α_X is the assembly map for $\widehat{\mathsf{sd}(X)}$, i.e. the diagram

$$\begin{array}{ccc} \mathrm{hocolim}_{\mathsf{simp}_G(X)} \mathbb{K}_X^{-\infty} \xleftarrow{\sim} & \cdots & \xrightarrow{\sim} \mathbb{H}^G(\widehat{\mathsf{sd}(X)}; \mathbb{K}_{\mathcal{A}}^{-\infty}) \\ {\scriptstyle \alpha_X} \searrow & & \swarrow {\scriptstyle A_{\widehat{\mathsf{sd}(X)}}} \\ & \mathbb{K}^{-\infty}(\mathcal{A}[G]) & \end{array}$$

commutes.

Proof. We have to trace through the sequence of levelwise equivalences which constitute the zig-zag at the top of the diagram. There is a natural transformation from $\widetilde{\mathbb{K}_{\mathcal{A}}^{-\infty}}$ to the functor taking the constant value $\mathsf{Ch}(\mathcal{A}[G])$, which is given by the obvious map $\mathsf{Ch}(\mathcal{A} *_G G/H) \to \mathsf{Ch}(\mathcal{A} *_G G/G) = \mathsf{Ch}(\mathcal{A}[G])$. This transformation induces a map

$$\widetilde{\alpha}_X \colon \operatorname*{hocolim}_{\mathsf{simp}_G(X)} \widetilde{\mathbb{K}_{\mathcal{A}}^{-\infty}} \to \mathbb{K}^{-\infty}(\mathcal{A}[G]).$$

Via the zig-zag established in Proposition 3.6, we can identify α_X with $\widetilde{\alpha}_X$, essentially because the diagram

$$\begin{array}{ccc} \mathsf{Ch}(\mathcal{A} *_G G/H) & \xrightarrow{\rho^p_{G/H}} & \mathcal{MV}_G(G/H \times \Delta^p; \mathcal{K}_{\mathcal{A}}) \\ & \searrow \quad \swarrow F & \\ & \mathsf{Ch}(\mathcal{A}[G]) & \end{array}$$

commutes. The remaining equivalences let us identify $\widetilde{\alpha}_X$ with the map

$$\widetilde{A}_{\widehat{\mathsf{sd}(X)}} \colon \operatorname{map}_G(?, \widehat{\mathsf{sd}(X)})_+ \otimes_{\mathsf{Or}(G)} \mathbb{K}_{\mathcal{A}}^{-\infty} \to \mathbb{K}^{-\infty}(\mathcal{A}[G])$$

which is induced by the obvious natural transformation from $\mathbb{K}_{\mathcal{A}}^{-\infty}$ to the functor taking the constant value $\mathbb{K}^{-\infty}(\mathcal{A}[G])$. To be a bit more precise, this natural transformation induces a map $\operatorname{map}_G(?, \widehat{\mathsf{sd}(X)})_+ \otimes_{\mathsf{Or}(G)} \mathbb{K}_{\mathcal{A}}^{-\infty} \to \operatorname{map}_G(?, \widehat{\mathsf{sd}(X)})_+ \otimes_{\mathsf{Or}(G)} \mathbb{K}^{-\infty}(\mathcal{A}[G])$, and $\widetilde{A}_{\widehat{\mathsf{sd}(X)}}$ is obtained form this map by further composition with the projection map $\operatorname{map}_G(?, \widehat{\mathsf{sd}(X)})_+ \to S^0$. It is therefore sufficient to observe that the triangle

$$\begin{array}{ccc} \operatorname{map}_G(?, \widehat{\mathsf{sd}(X)})_+ \otimes_{\mathsf{Or}(G)} \mathbb{K}_{\mathcal{A}}^{-\infty} & & \\ \downarrow & \searrow^{\alpha_X} & \\ \operatorname{map}_G(?, \widehat{\mathsf{sd}(X)})_+ \otimes_{\mathsf{Or}(G)} \mathbb{K}^{-\infty}(\mathcal{A}[G]) & \to & S^0 \otimes_{\mathsf{Or}(G)} \mathbb{K}^{-\infty}(\mathcal{A}[G]) \cong \mathbb{K}^{-\infty}(\mathcal{A}[G]) \end{array}$$

commutes; indeed, this is true as $\bullet \otimes x = \pi^*(\bullet) \otimes x = \bullet \otimes \pi_* x$, where $\pi \colon G/H \to G/G$ is the projection map and $S^0 = \bullet_+$. $\square$

Recall from earlier on that the map α_X can be factored into two maps:

$$\alpha_X \colon \operatorname*{hocolim}_{\mathsf{simp}_G(X)} \mathbb{K}_X^{-\infty} \xrightarrow{\overline{\alpha_X}} \mathbb{K}^{-\infty}(\mathcal{MV}_G(X; \mathcal{K}), w) \xrightarrow{\mathbb{K}^{-\infty}(F)} \mathbb{K}^{-\infty}(\mathcal{A}[G]).$$

Suppose now that X is a semisimplicial G-set in which every finite sub-semisimplicial set is contained in a contractible finite sub-semisimplicial set. Then Theorem 2.2.1

applies to show that $\mathbb{K}^{-\infty}(F)$ is an equivalence. Therefore, we can identify the assembly map with the first map $\operatorname{hocolim}_{\mathsf{simp}_G(X)} \mathbb{K}_X^{-\infty} \to \mathbb{K}^{-\infty}(\mathcal{MV}_G(X;\mathcal{K}), w)$.

The evident filtration of the left-hand side by the skeleta of X can now be seen to be closely related to the filtration of $\mathbb{K}^{-\infty}(\mathcal{A}[G])$ we obtained via Theorem 2.2.1: For every p, the square

$$\begin{array}{ccc} \operatorname*{hocolim}_{\mathsf{simp}_G(X^{(p)})} \mathbb{K}_X^{-\infty} & \longrightarrow & \mathbb{K}^{-\infty}(\mathcal{MV}_G(X^{(p)}), w) \\ \downarrow & & \downarrow \\ \operatorname*{hocolim}_{\mathsf{simp}_G(X)} \mathbb{K}_X^{-\infty} & \longrightarrow & \mathbb{K}^{-\infty}(\mathcal{MV}_G(X), w) \end{array}$$

commutes, exhibiting $\overline{\alpha_X}$ as a filtered map. Consequently, we get:

3.13 Corollary.
The map $\overline{\alpha_X}$ induces a morphism from the Atiyah-Hirzebruch spectral sequence converging to $\pi_* \operatorname{hocolim}_{\mathsf{simp}_G(X)} \mathbb{K}_X^{-\infty} \cong \pi_* \mathbb{H}^G(X; \mathbb{K}_{\mathcal{A}}^{-\infty})$ (associated to the skeletal filtration of X) to the X-resolution spectral sequence.

Proof. Our observations so far provide a morphism of spectral sequences from the Atiyah-Hirzebruch spectral sequence of $\mathbb{H}^G(\widehat{\mathsf{sd}(X)}; \mathbb{K}_{\mathcal{A}}^{-\infty})$ to the X-resolution spectral sequence. The corollary follows since there is a cellular homeomorphism between the realisation of $\widehat{\mathsf{sd}(X)}$ and the realisation of X. □

At first glance, there is significant temptation to use Corollary 3.13 as justification to formulate a "filtered isomorphism conjecture": This conjecture would predict that the morphism of spectral sequences whose existence has been asserted is an isomorphism on E^2 (the E^1-page is out of the question, see our discussion of the Δ^2-resolution spectral sequence in Section 2.3). This would be a much stronger assertion than the Farrell-Jones Conjecture, but also looks more tractable than the original conjecture (even though we do not understand the E^2-term yet).

However, such delusions of grandeur are quickly dispersed by the following observation due to Arthur Bartels: Let G be an arbitrary discrete group, $\mathcal{A}$ a small additive category on which G acts trivially, and let X be a semisimplicial G-set satisfying the assumptions of Theorem 2.2.1. We claim that the forgetful functor $F\colon \mathcal{MV}_G(X^{(1)}; \mathcal{K}_{\mathcal{A}}) \to \mathsf{Ch}(\mathcal{A}[G])$ always induces a surjection on K_1 (with respect to the w-equivalences on the left hand side).

By work of Muro and Tonks [MT08] any element in K_1 of a small Waldhausen category $\mathcal{C}$ can be represented by a diagram of the form

$$\begin{array}{ccccc} & & C_1 & & \\ & \overset{b_1}{\nearrow} & & \nwarrow{\scriptstyle\sim} & \\ A \underset{a_2}{\overset{a_1}{\rightrightarrows}} B & & & & C \\ & \underset{b_2}{\searrow} & & \swarrow{\scriptstyle\sim} & \\ & & C_2 & & \end{array}$$

where $A \overset{a_i}{\rightarrowtail} B \overset{b_i}{\twoheadrightarrow} C_i$ are cofibration sequences for $i = 1, 2$. In case of $K_1(\mathsf{Ch}(\mathcal{A}[G]))$, it even suffices to consider diagrams of the form

$$\begin{array}{ccccc} & & A & & \\ & \overset{=}{\nearrow} & & \overset{\alpha}{\nwarrow} & \\ 0 \rightrightarrows A & & & & A \\ & \underset{=}{\searrow} & & \underset{=}{\swarrow} & \\ & & A & & \end{array} \tag{3.1}$$

where A is concentrated in degree 0 (i.e., it is an object in $\mathcal{A}[G]$) and α is an automorphism; to show surjectivity, it is enough to lift these diagrams along the functor F (see Appendix C for a proof of both of these claims).

Pick a 0-simplex $x \in X_0$, and form the canonical resolution $\Delta_X(A; \{x\})$ (see Definition 2.2.3). The question is whether the two maps $\alpha, \mathrm{id}\colon A \to A$ extend to morphisms in $\mathcal{MV}_G(X; \mathcal{K}_\mathcal{A})$ with target $\Delta_X(A; \{x\})$. Observe that $\mathrm{supp}_0(\alpha) \cup \mathrm{supp}_0(\mathrm{id})$ (see Lemma 2.2.5) is a finite subset of X_0, and $\mathrm{supp}_n(\alpha) \cup \mathrm{supp}_n(\mathrm{id}) = \emptyset$ for $n > 0$. Since X is necessarily connected, we can choose a finite subtree T of the 1-skeleton which contains $\mathrm{supp}_0(\alpha) \cup \mathrm{supp}_0(\mathrm{id})$. Then we may apply Lemma 2.2.5 to obtain morphisms

$$\Delta_X(\alpha), \Delta_X(\mathrm{id})\colon \Delta_X(A; T) \to \Delta_X(A; \{x\}).$$

Thus, we can lift diagram 3.1 to the following diagram in $\mathcal{MV}_G(X^{(1)}; \mathcal{K}_\mathcal{A})$:

$$\begin{array}{ccccc} & & \Delta_X(A;\{x\}) & & \\ & \overset{\mathrm{id}}{\nearrow} & & \overset{\Delta_X(\alpha)}{\nwarrow} & \\ 0 \rightrightarrows \Delta_X(A;\{x\}) & & & & \Delta_X(A;T) \\ & \underset{\mathrm{id}}{\searrow} & & \underset{\Delta_X(\mathrm{id})}{\swarrow} & \\ & & \Delta_X(A;\{x\}) & & \end{array}$$

Note that $0 \to \Delta_X(A; \{x\}) \xrightarrow{\mathrm{id}} \Delta_X(A; \{x\})$ is clearly a cofibration sequence, and that $\Delta_X(\alpha)$ and $\Delta_X(\mathrm{id})$ are w-equivalences. So this proves that

$$K_1(F)\colon K_1(\mathcal{MV}_G(X^{(1)}; \mathcal{K}_\mathcal{A}), w) \twoheadrightarrow K_1(\mathsf{Ch}(\mathcal{A}[G]))$$

is a surjection.

This provides immediate counterexamples to a prospective "filtered" conjecture: Say we specialise to the case $G = \mathbb{Z}^2$, $\mathcal{A}$ is (a small model for) the category of finitely generated free R-modules for some ring R, and $X = \mathbb{R}^2$ (considered as an ordered simplicial complex). In the X-resolution spectral sequence, all permanent cycles contributing to K_1 are concentrated on the columns $p = 0$ and $p = 1$, whereas the Atiyah-Hirzebruch spectral sequence has permanent cycles on the column $p = 2$ as soon as R has non-trivial K_{-1}.

That negative K-theory appears to be a problem in general is further exemplified if we investigate the delooping construction: The first structure map of the non-connective K-theory spectrum (before taking Ω-spectrifications) fits into a commutative diagram

$$\begin{array}{ccc} K(\mathcal{M}\mathcal{V}_G(X^{(1)};\mathcal{K}_{\mathcal{A}}),w) & \longrightarrow & \Omega K(\mathcal{M}\mathcal{V}_G(X^{(1)};\mathcal{K}_{\mathcal{C}\mathcal{A}}),w) \\ \downarrow & & \downarrow \\ K(\mathsf{Ch}(\mathcal{A}[G])) & \longrightarrow & \Omega K(\mathsf{Ch}((\mathcal{C}\mathcal{A})[G])) \end{array}$$

The bottom arrow is an isomorphism on π_1, so that the right vertical map is also surjective on π_1. Indeed, this asserts that the forgetful functor induces a surjection

$$K_2(\mathcal{M}\mathcal{V}_G(X^{(1)};\mathcal{K}_{\mathcal{C}\mathcal{A}}),w) \twoheadrightarrow K_2(\mathsf{Ch}((\mathcal{C}\mathcal{A})[G])).$$

Similarly, considering $\mathcal{C}^n\mathcal{A}$ as coefficients provides examples where the forgetful functor induces surjections $K_{n+1}(\mathcal{M}\mathcal{V}_G(X^{(1)};\mathcal{K}_{\mathcal{C}^n\mathcal{A}}),w) \twoheadrightarrow K_{n+1}(\mathsf{Ch}((\mathcal{C}^n\mathcal{A})[G]))$ for any $n \geqslant 1$. This leaves the following question:

Are there reasonable assumptions under which it is sensible to formulate a "filtered isomorphism conjecture"?

A possible candidate for such a set of assumptions would be to require G to be torsionfree and to assume that $\mathcal{A}$ has no negative K-theory (e.g. $\mathcal{A} = \mathsf{Fr}_R$ for some regular ring R).

A. The homotopy fibration of a filtering subcategory

We omitted the proof of Proposition 1.3.6 in the main text. The reason is that we require certain fibration sequences which have been formulated in different ways by various authors. The notion of "Karoubi filtration" is usually employed to produce homotopy fibration sequences for the K-theory of additive categories (see for example [CP97]), but Schlichting has shown that this notion can be relaxed quite a bit, and we would basically like to quote his theorem [Sch04, Thm. 2.1].

However, Schlichting's theorem contains assumptions about idempotent completeness, whereas we would rather prefer to have a version of his theorem that makes no additional assumptions. Fortunately, it takes little effort do derive our preferred variation of the theorem (which has the conclusion drawn in [CP97]) from Schlichting's theorem. After this has been done, we give the proof of Proposition 1.3.6.

A.1 Definition.
Let $\mathcal{A}$ be a small additive category, and let $\mathcal{U} \subseteq \mathcal{A}$ be a full additive subcategory. For any two objects A, $B \in \mathcal{A}$, let

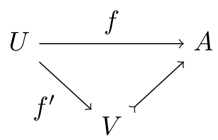
$$\hom_{\mathcal{A}}(A, \mathcal{U}, B) := \{f \in \hom_{\mathcal{A}}(A, B) \mid f \text{ factors through an object in } \mathcal{U}\}.$$

This is a subgroup of $\hom_{\mathcal{A}}(A, B)$. Define the *quotient category* $\mathcal{A}/\mathcal{U}$ to be the category whose objects are those of $\mathcal{A}$, and whose morphism sets are given by

$$\hom_{\mathcal{A}/\mathcal{U}}(A, B) := \hom_{\mathcal{A}}(A, B)/\hom_{\mathcal{A}}(A, \mathcal{U}, B).$$

The quotient category is again an additive category (direct sums in $\mathcal{A}$ are also direct sums in $\mathcal{A}/\mathcal{U}$).

A.2 Definition ([Sch04, Def. 1.3]).
Let $\mathcal{A}$ be a small additive category, and let $\mathcal{U} \subseteq \mathcal{A}$ be a full additive subcategory. We say that $\mathcal{U} \subseteq \mathcal{A}$ is a *weakly filtering subcategory* if, whenever $f \colon U \to A$ is a morphism from an object $U \in \mathcal{U}$ to an object $A \in \mathcal{A}$, there is a direct sum decomposition $A \cong A' \oplus V$ into $A' \in \mathcal{A}$ and $V \in \mathcal{U}$ and a morphism $f' \colon U \to V$ such that the following diagram is commutative:



We call $\mathcal{U} \subseteq \mathcal{A}$ a *strongly filtering subcategory* (in Schlichting's terms *left s-filtering*) if it is a weakly filtering subcategory and in addition the following holds: If $U \in \mathcal{U}$ and V is a direct summand of U in $\mathcal{A}$, then V is isomorphic to an object in $\mathcal{U}$.

A.3 Definition.
Let $\mathcal{A}$ be an additive category. The *idempotent completion* $\mathcal{A}^\wedge$ of $\mathcal{A}$ is the following category: Objects are pairs (A,p) with $A \in \mathcal{A}$ and $p = p^2 \colon A \to A$ an idempotent endomorphism. A morphism $f \colon (A,p) \to (B,q)$ is given by a morphism $f \colon A \to B$ in $\mathcal{A}$ with the property that $qfp = f$.

The idempotent completion is again an additive category.

We call an additive category *idempotent complete* if it is equivalent to its own idempotent completion. The idempotent completion of an additive category is idempotent complete. Our intention is to apply the following instance of Schlichting's theorem:

A.4 Theorem ([Sch04, Thm. 2.1])**.**
Let $\mathcal{A}$ be a small additive category, and $\mathcal{U} \subseteq \mathcal{A}$ a strongly filtering, idempotent subcategory. Then there is a homotopy fibration

$$iS_\bullet\mathcal{U} \to iS_\bullet\mathcal{A} \to iS_\bullet\mathcal{A}/\mathcal{U}.$$

□

For the purpose of this section, we let $K_0(\mathcal{A})$ denote the set of path components of the algebraic K-theory space $K(\mathcal{A}, i)$. This set is, in fact, an abelian group; in general, it does not coincide with π_0 of the non-connective algebraic K-theory spectrum. Recall that there is an explicit presentation of $K_0(\mathcal{A})$ as the abelian group generated by isomorphism classes of objects in $\mathcal{A}$, subject to the relation $[A] = [A_1] + [A_2]$ whenever $A \cong A_1 \oplus A_2$. The canonical homomorphism $K_0(\mathcal{A}) \to K_0(\mathcal{A}^\wedge)$ is always injective, so we may regard $K_0(\mathcal{A})$ as a subgroup of $K_0(\mathcal{A}^\wedge)$.

Let $\mathcal{U} \subseteq \mathcal{A}$ be a full additive subcategory of $\mathcal{A}$, and suppose that $K \subseteq K_0(\mathcal{U}^\wedge)$ is a subgroup. Then we define $\mathcal{U}^{\wedge K} \subseteq \mathcal{U}$ to be the full subcategory whose objects are those objects $(U,p) \in \mathcal{U}^\wedge$ for which $[(U,p)] \in K$. Moreover, we define $\mathcal{A}^{\wedge K}$ to be the full subcategory of $\mathcal{A}$ which consists of those objects which are a direct sum $A \oplus (U,p)$ with $A \in \mathcal{A}$ and $(U,p) \in \mathcal{U}^{\wedge K}$. The partial idempotent completion $\mathcal{U}^{\wedge K}$ is a full additive subcategory of $\mathcal{A}^{\wedge K}$.

As stated in the introduction, we want to get rid of the idempotent completeness assumption on $\mathcal{U}$. The correct statement then reads as follows:

A.5 Proposition ([CP97, Thm. 7.1])**.**
Let $\mathcal{A}$ be a small additive category and $\mathcal{U} \subseteq \mathcal{A}$ a weakly filtering subcategory. Let $K \subseteq K_0(\mathcal{U}^\wedge)$ be the preimage of $K_0(\mathcal{A})$ under the canonical homomorphism $K_0(\mathcal{U}^\wedge) \to K_0(\mathcal{A}^\wedge)$. Then there is a homotopy fibration

$$K(\mathcal{U}^{\wedge K}) \to K(\mathcal{A}) \to K(\mathcal{A}/\mathcal{U}).$$

Proof. Let $\mathcal{A}^{\wedge\mathcal{U}}$ denote the full subcategory of $\mathcal{A}^\wedge$ which consists of those objects (A,p) for which there is some $(U,q) \in \mathcal{U}^\wedge$ such that $(A,p) \oplus (U,q)$ is isomorphic to an object in $\mathcal{A}$. Note that $\mathcal{U}^\wedge$ is a full additive subcategory of $\mathcal{A}^{\wedge\mathcal{U}}$.

We claim that the commutative square

$$\begin{array}{ccc} iS_\bullet\mathcal{U}^{\wedge K} & \longrightarrow & iS_\bullet\mathcal{U}^{\wedge} \\ \downarrow & & \downarrow \\ iS_\bullet\mathcal{A}^{\wedge K} & \longrightarrow & iS_\bullet\mathcal{A}^{\wedge\mathcal{U}} \end{array} \tag{A.1}$$

is a homotopy pullback. To prove this, recall the notion of a *cofinal subcategory*: Given an exact inclusion of Waldhausen categories $\mathcal{C} \subseteq \mathcal{D}$, we call $\mathcal{C}$ cofinal in $\mathcal{D}$ if for every $D \in \mathcal{D}$ there is some $D' \in \mathcal{D}$ such that $D \vee D'$ is isomorphic to an object in $\mathcal{C}$. Cofinal inclusions induce monomorphisms on K_0. Clearly, if $\mathcal{C} \subseteq \mathcal{C}' \subseteq \mathcal{D}$ are exact inclusions and $\mathcal{C} \subseteq \mathcal{D}$ is cofinal, so is $\mathcal{C}' \subseteq \mathcal{D}$. Since any additive category is cofinal in its idempotent completion (because $(A, p) \oplus (A, 1 - p) \cong A$ for every A), this implies that the horizontal maps in A.1 are induced by cofinal inclusions. Set

$$H := K_0(\mathcal{U}^{\wedge})/K_0(\mathcal{U}^{\wedge K}), \qquad G := K_0(\mathcal{A}^{\wedge\mathcal{U}})/K_0(\mathcal{A}^{\wedge K}).$$

By Staffeldt's version of the cofinality theorem [Sta89, Thm. 2.1], we can extend diagram A.1 to a map of homotopy fibrations:

$$\begin{array}{ccccc} iS_\bullet\mathcal{U}^{\wedge K} & \longrightarrow & iS_\bullet\mathcal{U}^{\wedge} & \longrightarrow & BH \\ \downarrow & & \downarrow & & \downarrow \\ iS_\bullet\mathcal{A}^{\wedge K} & \longrightarrow & iS_\bullet\mathcal{A}^{\wedge\mathcal{U}} & \longrightarrow & BG \end{array}$$

So as soon as we know that the induced homomorphism $H \to G$ is an isomorphism, we have shown that A.1 is a homotopy pullback.

Let $(A, p) \in \mathcal{A}^{\wedge\mathcal{U}}$ be arbitrary. By definition, there is some $(U, q) \in \mathcal{U}^{\wedge}$ such that $(A, p) \oplus (U, q) \cong B$ for some $B \in \mathcal{A}$. This implies that $[A, p] = -[U, q]$ in G, so the homomorphism $H \to G$ is surjective.

To show injectivity, observe the following: If $(U, q) \in \mathcal{U}^{\wedge K}$ there are $A_1, A_2 \in \mathcal{A}$ such that $[U, q] = [A_1] - [A_2]$ in $K_0(\mathcal{A}^{\wedge})$. That is, there is some $(B, p) \in \mathcal{A}^{\wedge}$ such that $(U, q) \oplus A_2 \oplus (B, p) \cong A_1 \oplus (B, p)$. Adding $(B, 1 - p)$ on both sides, we see that $(U, q) \oplus A_2 \oplus B \cong A_1 \oplus B \in \mathcal{A}$. Consequently, if $A \oplus (U, q) \in \mathcal{A}^{\wedge K}$, there exists some $B \in \mathcal{A}$ such that $A \oplus (U, q) \oplus B$ is isomorphic to an object in $\mathcal{A}$. This implies that $K_0(\mathcal{A}) = K_0(\mathcal{A}^{\wedge K})$. Now suppose that $[U_1, q_1] - [U_2, q_2]$ lies in the kernel of the homomorphism $H \to G$. Then there are A_1, A_2 in $\mathcal{A}$ such that

$$[U_1, q_1] - [U_2, q_2] = [A_1] - [A_2]$$

in $K_0(\mathcal{A}^{\wedge\mathcal{U}})$, which means that there is some $(B, p) \in \mathcal{A}^{\wedge\mathcal{U}}$ such that

$$(U_1, q_1) \oplus A_2 \oplus (B, p) \cong (U_2, q_2) \oplus A_1 \oplus (B, p).$$

Since $(B,p) \in \mathcal{A}^{\wedge\mathcal{U}}$, we can find $(U',q') \in \mathcal{U}^{\wedge}$ such that $(B,p)\oplus(U',q')$ is isomorphic to an object in $\mathcal{A}$. That is, there is some $A' \in \mathcal{A}$ such that

$$(U_1,q_1) \oplus A_2 \oplus A' \cong (U_2,q_2) \oplus A_1 \oplus A'.$$

Set $A'_i := A_i \oplus A'$. Adding $(U_1, 1-q_1)$ on both sides, we have

$$U_1 \oplus A'_2 \cong (U_1, 1-q_1) \oplus (U_2,q_2) \oplus A'_1,$$

so $[(U_1,1-q_1) \oplus (U_2,q_2)] = [U_1,1-q_1] + [U_2,q_2] = [U_1 \oplus A'_2] - [A'_1] \in K_0(\mathcal{A})$. In particular, $(U_1,1-q_1) \oplus (U_2,q_2) \in \mathcal{U}^{\wedge K}$. Since

$$[U_1,q_1] - [U_2,q_2] = [U_1] - [(U_1,1-q_1) \oplus (U_2,q_2)],$$

this shows that $[U_1,q_1] - [U_2,q_2] \in K_0(\mathcal{U}^{\wedge K})$, and therefore represents 0 in H. This finishes the proof of injectivity, and thus shows that A.1 is a homotopy pullback.

Next, we claim that $\mathcal{U}^{\wedge} \subseteq \mathcal{A}^{\wedge\mathcal{U}}$ is strongly filtering (we adopt the proof from [HS04, Lem. 2.4]): Let $f\colon (U,q) \to (A,p)$ be a morphism in $\mathcal{A}^{\wedge\mathcal{U}}$ with $(U,q) \in \mathcal{U}^{\wedge}$ and $(A,p) \in \mathcal{A}^{\wedge\mathcal{U}}$. There is some $(U',q') \in \mathcal{U}^{\wedge}$ such that $(A,p)\oplus(U',q')$ is isomorphic to an object in $\mathcal{A}$. Consider the morphism

$$\widetilde{f} := \begin{pmatrix} f & 0 & 0 & 0 \\ 0 & q' & 0 & 0 \end{pmatrix} : \widetilde{U} := (U,q)\oplus(U',q')\oplus(U,1-q)\oplus(U',1-q') \to (A,p)\oplus(U',q').$$

The domain of this morphism lies in $\mathcal{U}$, the codomain is isomorphic to an object in $\mathcal{A}$. Therefore, we find an object $V \in \mathcal{U}$, a cofibration

$$\iota = \begin{pmatrix} \iota_A \\ \iota_U \end{pmatrix} : V \rightarrowtail (A,p) \oplus (U',q')$$

and a morphism $g\colon \widetilde{U} \to V$ such that $\iota g = \widetilde{f}$. Let

$$\pi = \begin{pmatrix} \pi_A & \pi_U \end{pmatrix} : (A,p) \oplus (U',q') \to V$$

be a retraction. Then $\pi\widetilde{f} = \pi\iota g = g$, so

$$g = \begin{pmatrix} \pi_A f & \pi_U q' & 0 & 0 \end{pmatrix}.$$

Since $\iota g = \widetilde{f}$, it follows that $\iota_A\pi_A f = f$, $\iota_U\pi_U q' = q'$ and $\iota_A\pi_U q' = 0$. Note that $q' = \mathrm{id}_{(U',q')}$, which implies that

$$\iota\pi = \begin{pmatrix} \iota_A\pi_A & \iota_A\pi_U \\ \iota_U\pi_A & \iota_U\pi_U \end{pmatrix} = \begin{pmatrix} \iota_A\pi_A & 0 \\ \iota_U\pi_A & \iota_U\pi_U \end{pmatrix}.$$

Since $\iota\pi$ is idempotent, it follows that $\iota_A\pi_A$ is idempotent, and $v := (\pi_A\iota_A)^2\colon V \to V$ is an idempotent endomorphism. Then

$$\iota_A\pi_A\iota_A\colon (V,v) \to (A,p) \quad \text{and} \quad \pi_A\iota_A\pi_A\colon (A,p) \to (V,v)$$

define morphisms in $\mathcal{A}^{\wedge}$, and we have

$$(\pi_A \iota_A \pi_A)(\iota_A \pi_A \iota_A) = \pi_A(\iota_A \pi_A)^2 \iota_A = v = \mathrm{id}_{(V,v)}\,.$$

So $\iota_A\pi_A\iota_A$ is a cofibration in $\mathcal{A}^{\wedge}$; let (A', p') denote its complement. By definition, $(A', p') \oplus (V, v) \cong (A, p)$, and it follows that (A', p') lies in $\mathcal{A}^{\wedge\mathcal{U}}$. This shows that $\iota_A\pi_A\iota_A$ is also a cofibration in $\mathcal{A}^{\wedge\mathcal{U}}$. Now observe that $\pi_A\iota_A\pi_A f\colon (U, q) \to (V, v)$ defines a morphism in $\mathcal{U}^{\wedge}$, and since

$$(\iota_A \pi_A \iota_A)(\pi_A \iota_A \pi_A f) = \iota_A \pi_A f = f,$$

we have found the desired factorisation

$$\begin{array}{ccc} (U,q) & \xrightarrow{\quad f \quad} & (A,p) \\ & {\scriptstyle \pi_A\iota_A\pi_A f}\searrow \qquad \nearrow{\scriptstyle \iota_A\pi_A\iota_A} & \\ & (V,v) & \end{array}$$

This proves that $\mathcal{U}^{\wedge} \subseteq \mathcal{A}^{\wedge\mathcal{U}}$ is weakly filtering.

Now let $(U, q) \in \mathcal{U}^{\wedge}$ and suppose that $(A, p) \in \mathcal{A}^{\wedge\mathcal{U}}$ is a direct summand of (U, p). Then (A, p) is isomorphic to an object of the form (U, q') for some idempotent $q'\colon U \to U$, and the latter clearly lies in $\mathcal{U}^{\wedge}$. So $\mathcal{U}^{\wedge} \subseteq \mathcal{A}^{\wedge\mathcal{U}}$ is even strongly filtering.

We can now apply Theorem A.4 to the inclusion $\mathcal{U}^{\wedge} \subseteq \mathcal{A}^{\wedge\mathcal{U}}$ to identify the vertical homotopy fibres in A.1 as $K(\mathcal{A}^{\wedge\mathcal{U}}/\mathcal{U}^{\wedge})$.

It is straightforward to check that the functor $\mathcal{A}/\mathcal{U} \to \mathcal{A}^{\wedge\mathcal{U}}/\mathcal{U}^{\wedge}$ is an additive equivalence. Also, we have shown earlier that $K_0(\mathcal{A}) = K_0(\mathcal{A}^{\wedge K})$, so $\mathcal{A} \hookrightarrow \mathcal{A}^{\wedge K}$ induces a weak equivalence $iS_{\bullet}\mathcal{A} \to iS_{\bullet}\mathcal{A}^{\wedge K}$ by the cofinality theorem. This finally yields the desired homotopy fibration

$$K(\mathcal{U}^{\wedge K}) \to K(\mathcal{A}) \to K(\mathcal{A}/\mathcal{U}).$$

□

With Proposition A.5 at our disposal, we can now turn to the proof of Proposition 1.3.6. As a preliminary note, suppose that $\mathcal{A}$ is an additive category with right G-action, and let $\mathcal{U}$ be a G-invariant weakly filtering subcategory. We claim that $\mathcal{U} *_G T \subseteq \mathcal{A} *_G T$ is also weakly filtering: Let $\varphi\colon U \to A$ be a morphism from $U = \{U_t\}_t \in \mathcal{U} *_G T$ to $A = \{A_t\}_t \in \mathcal{A} *_G T$. Recall that $\varphi = \{\varphi_{g,t}\colon U_{g^{-1}t} \to A_t g\}_{g,t}$ is a collection of morphisms in $\mathcal{A}$. Fix $t \in T$. Let $S_t := \{g \in G \mid \varphi_{g,t} \neq 0\}$. This is a finite set. Since $\mathcal{U} \subseteq \mathcal{A}$ is filtering, there is a factorisation

$$\begin{array}{ccc} \bigoplus_{g \in S_t} U_{g^{-1}t}g^{-1} & \xrightarrow{\ \left(\varphi_{g,t}g^{-1}\right)_g\ } & A_t \\ & {\scriptstyle \psi_t}\searrow \qquad \nearrow{\scriptstyle \iota_t} & \\ & V_t & \end{array}$$

via an object $V_t \in \mathcal{U}$. Letting t vary, we define $V := \{V_t\}_t \in \mathcal{U} *_G T$. There is a canonical cofibration $\iota\colon V \to A$ given by $\iota_{e,t} := \iota_t$ and $\iota_{g,t} := 0$ for $g \neq e$. Moreover, we let $\psi_{g,t} := \psi_t g$ and set $\psi := \{\psi_{g,t}\}_{g,t}$. This defines a morphism $U \to V$ in $\mathcal{U} *_G T$. We can compute

$$(\iota\psi)_{g,t} = \sum_{kh=g} \iota_{k,t} h \circ \psi_{h,k^{-1}t} = \iota_{e,t} g \circ \psi_{g,t} = \iota_t g \circ \psi_t g = (\iota_t \psi_t) g = \varphi_{g,t},$$

so we have found the desired factorisation $\varphi = \iota\psi$.

Proof of Proposition 1.3.6. Observe that there is a canonical additive functor

$$F\colon (\mathcal{C}\mathcal{A}) *_G T \to \mathcal{C}(\mathcal{A} *_G T), \quad \big\{\{A_t^z\}_z\big\}_t \mapsto \big\{\{A_t^z\}_t\big\}_z.$$

We want to prove that F induces a weak equivalence $K((\mathcal{C}\mathcal{A}) *_G T) \to K(\mathcal{C}(\mathcal{A} *_G T))$. Let $\mathcal{C}_{fin}\mathcal{A} \subseteq \mathcal{C}\mathcal{A}$ denote the full subcategory of objects with finite support. The inclusions $\mathcal{C}_{fin}\mathcal{A} \subseteq \mathcal{C}_+\mathcal{A}$ and $\mathcal{C}_-\mathcal{A} \subseteq \mathcal{C}\mathcal{A}$ are weakly filtering. Therefore, the functor F induces a map between homotopy pullback squares (the horizontal homotopy fibres are given by the loops on the K-theory of equivalent categories):

$$\begin{array}{ccc} K\big(((\mathcal{C}_{fin}\mathcal{A}) *_G T)^\wedge\big) & \rightarrow & K\big((\mathcal{C}_+\mathcal{A}) *_G T\big) \\ \downarrow & & \downarrow \\ K\big(((\mathcal{C}_-\mathcal{A}) *_G T)^\wedge\big) & \longrightarrow & K\big((\mathcal{C}\mathcal{A}) *_G T\big) \end{array} \Longrightarrow \begin{array}{ccc} K\big((\mathcal{C}_{fin}(\mathcal{A} *_G T))^\wedge\big) & \rightarrow & K\big(\mathcal{C}_+(\mathcal{A} *_G T)\big) \\ \downarrow & & \downarrow \\ K\big((\mathcal{C}_-(\mathcal{A} *_G T))^\wedge\big) & \longrightarrow & K\big(\mathcal{C}(\mathcal{A} *_G T)\big) \end{array}$$

In both squares, the top right and bottom left corners are contractible since the categories posssess an Eilenberg swindle. Since $\mathcal{C}_{fin}\mathcal{A}$ is additively equivalent to $\mathcal{A}$, it follows that the map on the top left corner is a homotopy equivalence.

Therefore, we conclude that the map

$$K(F)\colon K((\mathcal{C}\mathcal{A}) *_G T) \to K(\mathcal{C}(\mathcal{A} *_G T))$$

is an equivalence. If we iterate the functor F, we get functors

$$(\mathcal{C}^n\mathcal{A}) *_G T \to \mathcal{C}((\mathcal{C}^{n-1}\mathcal{A}) *_G T) \to \cdots \to \mathcal{C}^n(\mathcal{A} *_G T)$$

which induce equivalences on K-theory spaces. This proves the claim of the proposition. ∎

B. Maps of Grothendieck constructions

Recall again the *Grothendieck construction* of a diagram of small categories: If $D\colon \mathcal{I} \to \mathsf{Cat}$ is a diagram of small categories, the category $\int_{\mathcal{I}} D$ (the Grothendieck construction of D) has as objects pairs (I, X) with $I \in \mathcal{I}$ and $X \in D(I)$, while a morphism $(i, x)\colon (I, X) \to (I', X')$ consists of a morphism $i\colon I \to I'$ in $\mathcal{I}$ and a morphism $x\colon D(i)(X) \to X'$ in $D(I')$. Composition of morphisms is given by $(i', x') \circ (i, x) = (i'i, x' \circ D(i')(x))$.

We wish to justify the following result which we used in the proof of Proposition 3.6; I thank Thomas Nikolaus for a helpful discussion on this topic.

B.1 Proposition.
Let $F, G\colon \mathcal{I} \to \mathsf{Cat}$ be two diagrams in the category of small categories. Suppose that we are given

- for each $I \in I$, a functor $\tau_I\colon F(I) \to G(I)$ inducing a weak equivalence
$$N\tau_I\colon NF(I) \xrightarrow{\sim} NG(I).$$
- for each morphism $i\colon I \to I'$ a natural transformation $\vartheta_i\colon \tau_{I'} \circ F(i) \to G(i) \circ \tau_I$ such that $\vartheta_{\mathrm{id}} = \mathrm{id}$ and $\vartheta_{i'i} = \vartheta_{i'} \circ \vartheta_i$.

Then there is a natural weak equivalence
$$\left| N \int_{\mathcal{I}} F \right| \xrightarrow{\sim} \left| N \int_{\mathcal{I}} G \right|.$$

Let $\check{F}\colon \mathcal{I} \to \mathsf{Cat}$ denote the diagram given by $\check{F}(I) := F(I)^{op}$. In view of the natural homeomorphisms $B(F(I)) \cong B(F(I)^{op})$, it suffices to prove that there is a weak equivalence $\int_{\mathcal{I}} \check{F} \to \int_{\mathcal{I}} \check{G}$.

As observed by Thomason [Tho79, 1.3.1], the given data defines a functor
$$\begin{aligned} T\colon \int_{\mathcal{I}} \check{F} &\to \int_{\mathcal{I}} \check{G} \\ (I, X) &\mapsto (I, \tau_I(X)) \\ (i, x)\colon (I, X) \to (I', X') &\mapsto (i, \vartheta_i \circ \tau_{I'}(x)). \end{aligned}$$

Our goal is to show that the induced map on the nerves is an equivalence. Let us introduce some more language:

B.2 Definition.
Let $P\colon \mathcal{E} \to \mathcal{B}$ be a functor of small categories.

A morphism $e\colon E' \to E$ is *Cartesian* if for every morphism $f\colon F \to E$ in $\mathcal{E}$ and every morphism $b\colon P(F) \to P(E')$ with $P(e) \circ b = P(f)$, there exists a unique lift $\widetilde{b}\colon F \to E'$ such that $P(\widetilde{b}) = b$ and $e \circ \widetilde{b} = f$.

The functor $P\colon \mathcal{E} \to \mathcal{B}$ is called a *fibred category* if, for all $E \in \mathcal{E}$ and morphisms $b\colon B \to P(E)$, there is a Cartesian morphism e such that $P(e) = b$.

B.3 Lemma.
Let $D\colon \mathcal{I} \to \mathsf{Cat}$ be a diagram of small categories. Let $P\colon \int_{\mathcal{I}} D \to \mathcal{I}$ be the projection functor mapping $(I, X) \mapsto I$.

Then $P\colon (\int_{\mathcal{I}} D)^{op} \to \mathcal{I}^{op}$ is a fibred category.

Proof. The proof is very elementary, but it is easy to get confused about the correct directions of arrows, which is why we include a proof.

Let $(I, X) \in (\int_{\mathcal{I}} D)^{op}$ and $I' \xrightarrow{i} I = P(I, X)$ be an arrow in $\mathcal{I}^{op}$, i.e. i is a morphism $I \to I'$ in $\mathcal{I}$. The pair $(i, \mathrm{id}_{D(i)(X)})$ defines a morphism $(I', D(i)(X)) \to (I, X)$ in $(\int_{\mathcal{I}} D)^{op}$, and clearly $P(i, \mathrm{id}_{D(i)(X)}) = i$. We claim that $(i, \mathrm{id}_{D(i)(X))})$ is Cartesian: Let $(j, y)\colon (J, Y) \to (I, X)$ be another morphism in $(\int_{\mathcal{I}} D)^{op}$, and suppose $j'\colon J \to I'$ is a morphism in $\mathcal{I}^{op}$ such that $i \circ j' = j$. This translates to the equation $j' \circ i = j$ in $\mathcal{I}$; moreover, y is a morphism $D(j)(X) = D(j')D(i)(X) \to Y$. Therefore, the pair (j', y) defines a morphism $(J, Y) \to (I', D(i)(X))$ in $(\int_{\mathcal{I}} D)^{op}$; clearly, the equation $(i, \mathrm{id}_{D(i)(X)}) \circ (j', y) = (j, y)$ holds.

Any other lift of j' must necessarily have j' as its first component. In addition, if such a lift (j', y') satisfies $(i, \mathrm{id}_{D(i)(X)}) \circ (j', y') = (j, y)$, we have $y' \circ \mathrm{id}_{D(i)(X)} = y$, so $y' = y$. This shows that the given lift is unique. $\square$

Our proposition now follows immediately by applying the following result to the functor $T\colon (\int_{\mathcal{I}} \check{F})^{op} \to (\int_{\mathcal{I}} \check{G})^{op}$:

B.4 Theorem ([dH09, dH12, Prop. 4.4.1 resp. Prop. 4.2.1])**.**
Let $P\colon \mathcal{E} \to \mathcal{B}$ and $P'\colon \mathcal{E}' \to \mathcal{B}$ be fibred categories. Suppose that $T\colon P \to P'$ is a map of fibred categories (i.e. $P'T = P$) and that the induced functor $P^{-1}(B) \to (P')^{-1}(B)$ induces an equivalence on nerves for every $B \in \mathcal{B}$.

Then NT is a weak equivalence.

$\square$

C. Representing elements in the K_1-group of an additive category

Our goal is to justify two claims made at the end of Section 3: First, we claimed that elements in $K_1(\mathsf{Ch}(\mathcal{A}[G]))$ have particularly simple representatives; second, it was asserted that we only need to lift these representatives along a given functor to show that this functor induces a surjection on K_1.

Suppose for the beginning that $\mathcal{C}$ is an arbitrary small Waldhausen category. Theorem 2.2 in [MT08] asserts that any element in K_1 of a small Waldhausen category $\mathcal{C}$ can be represented by a diagram of the form

$$\begin{array}{ccccc} & & C_1 & & \\ & \overset{b_1}{\nearrow} & & \nwarrow^{\sim} & \\ A \overset{a_1}{\underset{a_2}{\rightrightarrows}} B & & & & C \\ & \underset{b_2}{\searrow} & & \swarrow_{\sim} & \\ & & C_2 & & \end{array}$$

where $A \overset{a_i}{\rightarrowtail} B \overset{b_i}{\twoheadrightarrow} C_i$ are cofibration sequences for $i = 1, 2$. Such diagrams are called *pairs of weak cofibre sequences.*

If F is an exact functor, the induced map on K_1 is given by applying F to these diagrams. This makes it clear that, if we have a particular set of representatives for elements in K_1, it suffices to lift these representatives along F to show that $K_1(F)$ is surjective. Thus, we need only show the following:

C.1 Proposition.
Let $\mathcal{A}$ be a small additive category. Then any element in $K_1(\mathsf{Ch}(\mathcal{A}))$ can be represented by a diagram of the form

$$\begin{array}{ccccc} & & A & & \\ & \overset{\mathrm{id}}{\nearrow} & & \nwarrow^{\alpha} & \\ 0 \rightrightarrows A & & & & A \\ & \underset{\mathrm{id}}{\searrow} & & \swarrow_{\mathrm{id}} & \\ & & A & & \end{array}$$

where A is an object in $\mathcal{A}$ and α is an automorphism.

Proof. Theorem 3.1 in [MT08] provides the following relations among pairs of weak cofibre sequences: Suppose we are given six pairs of weak cofibre sequences

$$\begin{array}{ccccc} & & A_1'' & & \\ & \overset{r_1^A}{\nearrow} & & \overset{w_1^A}{\nwarrow} & \\ A' \overset{j_1^A}{\underset{j_2^A}{\rightrightarrows}} A & & & & A'' \\ & \underset{r_2^A}{\searrow} & & \underset{w_2^A}{\swarrow} & \\ & & A_2'' & & \end{array} \quad \begin{array}{ccccc} & & B_1'' & & \\ & \overset{r_1^B}{\nearrow} & & \overset{w_1^B}{\nwarrow} & \\ B' \overset{j_1^B}{\underset{j_2^B}{\rightrightarrows}} B & & & & B'' \\ & \underset{r_2^B}{\searrow} & & \underset{w_2^B}{\swarrow} & \\ & & B_2'' & & \end{array} \quad \begin{array}{ccccc} & & \check{C}_1'' & & \\ & \overset{r_1^C}{\nearrow} & & \overset{w_1^C}{\nwarrow} & \\ C' \overset{j_1^C}{\underset{j_2^C}{\rightrightarrows}} C & & & & C'' \\ & \underset{r_2^C}{\searrow} & & \underset{w_2^C}{\swarrow} & \\ & & \check{C}_2'' & & \end{array}$$

$$\begin{array}{ccccc} & & C_1' & & \\ & \overset{r_1'}{\nearrow} & & \overset{w_1'}{\nwarrow} & \\ A' \overset{j_1'}{\underset{j_2'}{\rightrightarrows}} B' & & & & C' \\ & \underset{r_2'}{\searrow} & & \underset{w_2'}{\swarrow} & \\ & & C_2' & & \end{array} \quad \begin{array}{ccccc} & & C_1 & & \\ & \overset{r_1}{\nearrow} & & \overset{w_1}{\nwarrow} & \\ A \overset{j_1}{\underset{j_2}{\rightrightarrows}} B & & & & C \\ & \underset{r_2}{\searrow} & & \underset{w_2}{\swarrow} & \\ & & C_2 & & \end{array} \quad \begin{array}{ccccc} & & \widehat{C}_1'' & & \\ & \overset{r_1''}{\nearrow} & & \overset{w_1''}{\nwarrow} & \\ A'' \overset{j_1''}{\underset{j_2''}{\rightrightarrows}} B'' & & & & C'' \\ & \underset{r_2''}{\searrow} & & \underset{w_2''}{\swarrow} & \\ & & \widehat{C}_2'' & & \end{array}$$

and we denote by λ_A, λ_B, λ_C, λ', λ and λ'' the corresponding elements they represent in $K_1(\mathcal{C})$. If for $i = 1, 2$ there is a commutative diagram

$$\begin{array}{ccccccc} A' & \overset{j_i^A}{\rightarrowtail} & A & \overset{r_i^A}{\twoheadrightarrow} & A_i'' & \overset{w_i^A}{\longleftarrow} & A'' \\ {\scriptstyle j_i'}\downarrow & & {\scriptstyle j_i}\downarrow & & {\scriptstyle \check{j}_i}\downarrow & & {\scriptstyle j_i''}\downarrow \\ B' & \overset{j_i^B}{\rightarrowtail} & B & \overset{r_i^B}{\twoheadrightarrow} & B_i'' & \overset{w_i^B}{\longleftarrow} & B'' \\ {\scriptstyle r_i'}\downarrow & & {\scriptstyle r_i}\downarrow & & {\scriptstyle \check{r}_i}\downarrow & & {\scriptstyle r_i''}\downarrow \\ C_i' & \overset{\widehat{j}_i}{\rightarrowtail} & C_i & \overset{\widehat{r}_i}{\twoheadrightarrow} & \widehat{C}_i'' & \overset{\widehat{w}_i}{\longleftarrow} & \widehat{C}_i'' \\ {\scriptstyle w_i'}\uparrow & & {\scriptstyle w_i}\uparrow & & {\scriptstyle \check{w}_i}\uparrow & & {\scriptstyle w_i''}\uparrow \\ C' & \overset{j_i^C}{\rightarrowtail} & C & \overset{r_i^C}{\twoheadrightarrow} & \check{C}_i'' & \overset{w_i^C}{\longleftarrow} & C'' \end{array}$$

in which all arrows whose name is a variation of the letter w are weak equivalences and the other morphisms are of the type indicated in the diagram, then the following equality holds in $K_1(\mathcal{C})$:

$$\lambda_A - \lambda_B + \lambda_C = \lambda' - \lambda + \lambda''.$$

Moreover, according to Proposition 3.2 in [MT08], a pair of weak cofibre sequences is zero in K_1 if the top and bottom half of the pair are identical.

By virtue of the Gillet-Waldhausen Theorem 1.3.1, the functor $\mathcal{A} \to \mathsf{Ch}(\mathcal{A})$ which embeds $\mathcal{A}$ as cochain complexes concentrated in degree 0 induces an isomorphism on K_1. Consequently, it suffices to consider pairs of weak cofibre sequences in the additive category $\mathcal{A}$. So let

$$\begin{array}{ccccccc}
 & & & & C_1 & & \\
 & & & {\scriptstyle b_1}\nearrow & & \nwarrow{\scriptstyle c_1} & \\
A & \overset{a_1}{\underset{a_2}{\rightrightarrows}} & B & & & & C \\
 & & & {\scriptstyle b_2}\searrow & & \swarrow{\scriptstyle c_2} & \\
 & & & & C_2 & &
\end{array}$$

be an arbitrary pair of weak cofibre sequences in $\mathcal{A}$. Note that c_1 and c_2 are isomorphisms, so we may define

$$b_1' := c_1^{-1}b_1, b_2' := c_2^{-1}b_2 \colon B \twoheadrightarrow C.$$

For $i = 1, 2$ the diagram

$$\begin{array}{ccccccc}
A & \overset{\mathrm{id}}{\rightarrowtail} & A & \twoheadrightarrow & 0 & \longleftarrow & 0 \\
{\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle a_i}\downarrow & & \downarrow & & \downarrow \\
A & \overset{a_i}{\rightarrowtail} & B & \overset{b_i'}{\twoheadrightarrow} & C & \overset{\mathrm{id}}{\longleftarrow} & C \\
\downarrow & & {\scriptstyle b_i}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow & & {\scriptstyle \mathrm{id}}\downarrow \\
0 & \rightarrowtail & C_i & \overset{c_i^{-1}}{\twoheadrightarrow} & C & \overset{\mathrm{id}}{\longleftarrow} & C \\
\uparrow & & {\scriptstyle c_i}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow & & {\scriptstyle \mathrm{id}}\uparrow \\
0 & \rightarrowtail & C & \overset{\mathrm{id}}{\twoheadrightarrow} & C & \overset{\mathrm{id}}{\longleftarrow} & C
\end{array}$$

commutes; this gives us the relation

$$\left[\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & \nearrow & & \nwarrow & \\
A & \overset{\mathrm{id}}{\underset{\mathrm{id}}{\rightrightarrows}} & A & & & & 0 \\
 & & & \searrow & & \swarrow & \\
 & & & & 0 & &
\end{array}\right]
-
\left[\begin{array}{ccccccc}
 & & & & C_1 & & \\
 & & & {\scriptstyle b_1}\nearrow & & \nwarrow{\scriptstyle c_1} & \\
A & \overset{a_1}{\underset{a_2}{\rightrightarrows}} & B & & & & C \\
 & & & {\scriptstyle b_2}\searrow & & \swarrow{\scriptstyle c_2} & \\
 & & & & C_2 & &
\end{array}\right]
+
\left[\begin{array}{ccccccc}
 & & & & C & & \\
 & & & {\scriptstyle \mathrm{id}}\nearrow & & \nwarrow{\scriptstyle \mathrm{id}} & \\
0 & \rightrightarrows & C & & & & C \\
 & & & {\scriptstyle \mathrm{id}}\searrow & & \swarrow{\scriptstyle \mathrm{id}} & \\
 & & & & C & &
\end{array}\right]$$

$$= \left[\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & \nearrow & & \nwarrow & \\
A & \overset{\mathrm{id}}{\underset{\mathrm{id}}{\rightrightarrows}} & A & & & & 0 \\
 & & & \searrow & & \swarrow & \\
 & & & & 0 & &
\end{array}\right]
-
\left[\begin{array}{ccccccc}
 & & & & C & & \\
 & & & {\scriptstyle b_1'}\nearrow & & \nwarrow{\scriptstyle \mathrm{id}} & \\
A & \overset{a_1}{\underset{a_2}{\rightrightarrows}} & B & & & & C \\
 & & & {\scriptstyle b_2'}\searrow & & \swarrow{\scriptstyle \mathrm{id}} & \\
 & & & & C & &
\end{array}\right]
+
\left[\begin{array}{ccccccc}
 & & & & C & & \\
 & & & {\scriptstyle \mathrm{id}}\nearrow & & \nwarrow{\scriptstyle \mathrm{id}} & \\
0 & \rightrightarrows & C & & & & C \\
 & & & {\scriptstyle \mathrm{id}}\searrow & & \swarrow{\scriptstyle \mathrm{id}} & \\
 & & & & C & &
\end{array}\right],$$

and consequently we have

$$\left[\begin{array}{ccccc} & & C_1 & & \\ & \overset{b_1}{\nearrow} & & \overset{c_1}{\nwarrow} & \\ A \underset{a_2}{\overset{a_1}{\rightrightarrows}} B & & & & C \\ & \underset{b_2}{\searrow} & & \underset{c_2}{\swarrow} & \\ & & C_2 & & \end{array}\right] = \left[\begin{array}{ccccc} & & C & & \\ & \overset{b'_1}{\nearrow} & & \overset{\mathrm{id}}{\nwarrow} & \\ A \underset{a_2}{\overset{a_1}{\rightrightarrows}} B & & & & C \\ & \underset{b'_2}{\searrow} & & \underset{\mathrm{id}}{\swarrow} & \\ & & C & & \end{array}\right].$$

The two cofibration sequences in the latter diagram give rise to two direct sum systems

$$A \underset{p_1}{\overset{a_1}{\rightleftarrows}} B \underset{s_1}{\overset{b'_1}{\rightleftarrows}} C$$
$$A \underset{p_2}{\overset{a_2}{\rightleftarrows}} B \underset{s_2}{\overset{b'_2}{\rightleftarrows}} C$$

Define

$$\beta_1 := a_2 p_1 + s_2 b'_1 \colon B \to B.$$

This is an automorphism of B since

$$\begin{aligned} \beta_1(a_1 p_2 + s_1 b'_2) &= (a_2 p_1 + s_2 b'_1)(a_1 p_2 + s_1 b'_2) \\ &= a_2 p_1 a_1 p_2 + a_2 p_1 s_1 b'_2 + s_2 b'_1 a_1 p_2 + s_2 b'_1 s_1 b'_2 \\ &= a_2 p_2 + s_2 b'_2 = \mathrm{id}\,. \end{aligned}$$

We also have $(a_1 p_2 + s_1 b'_2)\beta_1 = \mathrm{id}$ for reasons of symmetry. Moreover,

$$\begin{aligned} \beta_1 a_1 &= a_2 p_1 a_1 + s_2 b'_1 a_1 = a_2, \\ b'_2 \beta_1 &= b'_2 a_2 p_1 + b'_2 s_2 b'_1 = b'_1. \end{aligned}$$

Set $\beta_2 := \mathrm{id}$. The computations we have just done show that for $i = 1, 2$ there is a commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrowtail & A & \xrightarrow{\mathrm{id}} & A & \xleftarrow{\mathrm{id}} & A \\ \downarrow & & \downarrow{\scriptstyle a_i} & & \downarrow{\scriptstyle a_2} & & \downarrow{\scriptstyle a_2} \\ 0 & \rightarrowtail & B & \xrightarrow{\beta_i} & B & \xleftarrow{\mathrm{id}} & B \\ \downarrow & & \downarrow{\scriptstyle b'_i} & & \downarrow{\scriptstyle b'_2} & & \downarrow{\scriptstyle b'_2} \\ 0 & \rightarrowtail & C & \xrightarrow{\mathrm{id}} & C & \xleftarrow{\mathrm{id}} & C \\ \uparrow & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} & & \uparrow{\scriptstyle \mathrm{id}} \\ 0 & \rightarrowtail & C & \xrightarrow{\mathrm{id}} & C & \xleftarrow{\mathrm{id}} & C \end{array}$$

This yields the relation

$$
\left[\begin{array}{cccc} & \nearrow & 0 & \nwarrow & \\ 0 \rightrightarrows 0 & & & & 0 \\ & \searrow & 0 & \swarrow & \end{array}\right]
-
\left[\begin{array}{cccc} & b_1' \nearrow & C & \nwarrow \text{id} & \\ A \overset{a_1}{\underset{a_2}{\rightrightarrows}} B & & & & C \\ & b_2' \searrow & C & \swarrow \text{id} & \end{array}\right]
+
\left[\begin{array}{cccc} & b_2' \nearrow & C & \nwarrow \text{id} & \\ A \overset{a_2}{\underset{a_2}{\rightrightarrows}} B & & & & C \\ & b_2' \searrow & C & \swarrow \text{id} & \end{array}\right]
$$

$$
= \left[\begin{array}{cccc} & \text{id} \nearrow & A & \nwarrow \text{id} & \\ 0 \rightrightarrows A & & & & A \\ & \text{id} \searrow & A & \swarrow \text{id} & \end{array}\right]
-
\left[\begin{array}{cccc} & \beta_1 \nearrow & B & \nwarrow \text{id} & \\ 0 \rightrightarrows B & & & & B \\ & \text{id} \searrow & B & \swarrow \text{id} & \end{array}\right]
+
\left[\begin{array}{cccc} & \text{id} \nearrow & C & \nwarrow \text{id} & \\ 0 \rightrightarrows C & & & & C \\ & \text{id} \searrow & C & \swarrow \text{id} & \end{array}\right]
$$

in K_1; i.e., ignoring those diagrams that represent 0, we have

$$
\left[\begin{array}{cccc} & b_1' \nearrow & C & \nwarrow \text{id} & \\ A \overset{a_1}{\underset{a_2}{\rightrightarrows}} B & & & & C \\ & b_2' \searrow & C & \swarrow \text{id} & \end{array}\right]
=
\left[\begin{array}{cccc} & \beta_1 \nearrow & B & \nwarrow \text{id} & \\ 0 \rightrightarrows B & & & & B \\ & \text{id} \searrow & B & \swarrow \text{id} & \end{array}\right]
$$

Finally, the diagrams

$$
\begin{array}{ccccccc}
0 & \rightarrowtail & 0 & \twoheadrightarrow & 0 & \longleftarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrowtail & B & \xrightarrow{\text{id}} & B & \xleftarrow{\beta_i} & B \\
\downarrow & & \text{id}\downarrow & & \text{id}\downarrow & & \beta_i\downarrow \\
0 & \rightarrowtail & B & \xrightarrow{\text{id}} & B & \xleftarrow{\text{id}} & B \\
\uparrow & & \text{id}\uparrow & & \text{id}\uparrow & & \text{id}\uparrow \\
0 & \rightarrowtail & B & \xrightarrow{\text{id}} & B & \xleftarrow{\text{id}} & B
\end{array}
$$

witness that

$$
\left[\begin{array}{ccccc} & & 0 & & \\ & \nearrow & & \nwarrow & \\ 0 \rightrightarrows 0 & & & & 0 \\ & \searrow & & \swarrow & \\ & & 0 & & \end{array}\right]
-
\left[\begin{array}{ccccc} & & B & & \\ & \mathrm{id}\nearrow & & \nwarrow\mathrm{id} & \\ 0 \rightrightarrows B & & & & B \\ & \mathrm{id}\searrow & & \swarrow\mathrm{id} & \\ & & B & & \end{array}\right]
+
\left[\begin{array}{ccccc} & & B & & \\ & \beta_1\nearrow & & \nwarrow\mathrm{id} & \\ 0 \rightrightarrows B & & & & B \\ & \mathrm{id}\searrow & & \swarrow\mathrm{id} & \\ & & B & & \end{array}\right]
$$

$$
=
\left[\begin{array}{ccccc} & & 0 & & \\ & \nearrow & & \nwarrow & \\ 0 \rightrightarrows 0 & & & & 0 \\ & \searrow & & \swarrow & \\ & & 0 & & \end{array}\right]
-
\left[\begin{array}{ccccc} & & B & & \\ & \mathrm{id}\nearrow & & \nwarrow\beta_1 & \\ 0 \rightrightarrows B & & & & B \\ & \mathrm{id}\searrow & & \swarrow\mathrm{id} & \\ & & B & & \end{array}\right]
+
\left[\begin{array}{ccccc} & & B & & \\ & \mathrm{id}\nearrow & & \nwarrow\mathrm{id} & \\ 0 \rightrightarrows B & & & & B \\ & \mathrm{id}\searrow & & \swarrow\mathrm{id} & \\ & & B & & \end{array}\right],
$$

so that we arrive at the equality

$$
\left[\begin{array}{ccccc} & & B & & \\ & \beta_1\nearrow & & \nwarrow\mathrm{id} & \\ 0 \rightrightarrows B & & & & B \\ & \mathrm{id}\searrow & & \swarrow\mathrm{id} & \\ & & B & & \end{array}\right]
= -
\left[\begin{array}{ccccc} & & B & & \\ & \mathrm{id}\nearrow & & \nwarrow\beta_1 & \\ 0 \rightrightarrows B & & & & B \\ & \mathrm{id}\searrow & & \swarrow\mathrm{id} & \\ & & B & & \end{array}\right]
$$

If we now start with a given element $x \in K_1(\mathsf{Ch}(\mathcal{A}[G]))$, we represent $-x$ by an appropriate diagram, and our discussion shows that x can be represented by a diagram of the required form. $\square$

Part II.

Transfer Reducibility of $\mathbb{Z}^n \rtimes \mathbb{Z}$

4. Finite group actions with small stabilisers

As outlined in the introduction, the second part of this thesis has the purpose to indicate how some proofs of the Farrell-Jones Conjecture may be unified to a certain extent. An important component of the argument is the answer to the following question:

What are the minimal stabilisers of a finite group action on a finite, contractible simplicial complex?

Admittedly, this question looks quite unrelated on first sight. If the reader wishes to learn how the answer to this question relates to algebraic K-theory, it is advisable to read Section 4.1, to accept Theorem 4.2.3 *bona fide*, to have a look at its Corollary 4.2.4, and then to continue with the next chapter.

Throughout this part of the thesis, our attention will be focussed on actions of discrete groups G on simplicial complexes X. If not mentioned otherwise, we always require such an action to satisfy the following property: If $x = \{x_0, \dots, x_n\}$ is a simplex in X and $g \in G$ is an element with $gx = x$, then g fixes the simplex x pointwise, i.e. $gx_i = x_i$ for all $0 \leqslant i \leqslant n$. A simplicial complex with G-action which satisfies this property will be called a *G-simplicial complex*.

We let $\mathcal{F}_X$ denote the family of stabilisers of a G-simplicial complex. Note that $\mathcal{F}_X = \mathcal{F}_{X_0}$ for every G-simplicial complex.

4.1. Resolving fixed points of simplicial complexes

Instead of constructing a finite contractible complex with minimal stabilisers in one go, we will proceed inductively. Namely, suppose we are given a finite contractible complex with an action of a group G; if we know that the stabilisers of this action again allow actions on finite contractible complexes with even smaller stabilisers, we may try to patch these together to reduce the size of the stabilisers of the original action. Our goal in this section is to provide a construction which accomplishes precisely this.

Recall that the *transport groupoid* $\mathsf{Tr}_G(T)$ of a G-set T is the category whose set of objects is T, and in which a morphism $t \to t'$ is an element $g \in G$ such that $gt = t'$.

4.1.1 Definition.

Let X be a G-simplicial complex, and let R be a functor $\mathsf{Tr}_G(X_0) \to \mathsf{SCplx}$ to the category of simplicial complexes. We call R a *set of resolution data for* X if, for every endomorphism $g\colon x \to x$ in $\mathsf{Tr}_G(X_0)$, the simplicial map $R(g)\colon R(x) \to R(x)$ has the property that, if $R(g)(y) = y$ for some n-simplex y, the map $R(g)$ fixes y pointwise.

4.1.2 Definition.

Given a G-simplicial complex X and a set of resolution data R for X, we define the *resolution of* X *by* R to be the simplicial complex $X[R]$ whose vertex set is given by

$$X[R]_0 := \coprod_{x \in X_0} R(x)_0,$$

and in which $y = \{y_0, \ldots, y_n\}$ is an n-simplex if and only if the following holds:

- For all $x \in X_0$, the set $S_x(y) := \{y_i \mid 0 \leqslant i \leqslant n,\ y_i \in R(x)_0\}$ is either empty or a simplex in $R(x)$.
- The set $S(y) := \{x \in X_0 \mid S_x(y) \neq \emptyset\}$ is a simplex in X.

For $g \in G$ and $y = \{y_0, \ldots, y_n\}$ a simplex in $X[R]$, set

$$g \cdot y = \{R(g)(y_0), \ldots, R(g)(y_n)\}.$$

4.1.3 Remark.

While the definition of $X[R]$ is quite workable for proofs, it does not really elucidate the construction. The intuition behind Definitions 4.1.1 and 4.1.2 is the following:

We would like to resolve the fixed points of X by other simplicial complexes with smaller stabilisers. If R is a set of resolution data for X, this gives us for every vertex x in X a simplicial complex $R(x)$. Since R is a functor whose source is the transport groupoid of X_0, we have for every element in the stabiliser of x a simplicial automorphism $R(x) \to R(x)$; our requirements on R assure that these maps assemble to give a G_x-simplicial complex $R(x)$. Since the transport groupoid encodes all translation actions (not just stabilisers), we also have simplicial isomorphisms $R(x) \cong R(gx)$, exhibiting $R(gx)$ essentially as $R(x)$ equipped with the conjugate action.

To be a bit more precise: Let $\overline{x}$ be the orbit of a single vertex, and let $x_0 \in \overline{x}$ be a fixed vertex. Observe that the entire group G acts on $\coprod_{x \in \overline{x}} R(x)$ by $g \cdot y := R(g)(y)$. Then we obtain G-equivariant simplicial isomorphisms $\coprod_{x \in \overline{x}} R(x) \cong G \times_{G_{x_0}} R(x_0)$ by sending a simplex $y \in R(x) \subseteq \coprod_{x \in \overline{x}} R(x)$ to the simplex $(g, R(g^{-1})y)$, where $gx_0 = x$. The inverse map sends $(g, y) \in G \times_{G_{x_0}} R(x_0)$ to $R(g)(y) \in R(gx_0) \subseteq \coprod_{x \in \overline{x}} R(x)$.

The "resolution" construction $X[R]$ now replaces every vertex x of X by the G_x-simplicial complex $R(x)$, and then takes successive joins of these individual complexes according to the simplicial structure of X (i.e., for a simplex $\{x_0, \ldots, x_n\}$ in X, we find the join $R(x_0) * \cdots * R(x_n)$ as a subcomplex of $X[R]$).

4.1.4 Proposition.
Let X be a G-simplicial complex, and let R be a set of resolution data for X.

Then $X[R]$ is a G-simplicial complex, and for every isotropy group H of $X[R]$, there is some $x \in X_0$ with $H \in \mathcal{F}_{R(x)}$. Moreover, if X is N-dimensional and the dimension of each $R(x)$ is bounded by K, then $X[R]$ is at most $(NK + N + K)$-dimensional.

Proof. We begin by showing that $X[R]$ is a simplicial complex. Let $y = \{y_0, \ldots, y_n\}$ be a simplex in $X[R]$, and suppose that $y' = \{y_{i_0}, \ldots, y_{i_m}\} \subseteq y$ is a non-empty subset. Then for every $x \in X_0$, the set $S_x(y')$ is a subset of $S_x(y)$. Since $S_x(y)$ is a simplex, $S_x(y')$ is also a simplex if it is non-empty. Similarly, the set $S(y')$ is a subset of $S(y)$, and therefore a simplex in X.

We have defined an action of G on $X[R]$: Let again $g \in G$ and let y be a simplex in $X[R]$. Then for every $x \in X_0$,

$$\begin{aligned} S_x(g \cdot y) &= \{g \cdot y_i \mid g \cdot y_i \in R(x)\} \\ &= g \cdot \{y_i \mid g \cdot y_i \in R(x)\} \\ &= g \cdot \{y_i \mid y_i \in R(g^{-1}x)\} \\ &= g \cdot S_{g^{-1}x}(y). \end{aligned}$$

This implies that $S(g \cdot y) = g^{-1} \cdot S(y)$; so $g \cdot y$ is also a simplex in $X[R]$.

To show that $X[R]$ is actually a G-simplicial complex, suppose that $g \cdot y = y$ for some $g \in G$ and a simplex $y \in X[R]$. As we have seen, $S(y) = S(g \cdot y) = g^{-1} \cdot S(y)$. If $x \in X_0$ with $S_x(y) \neq \emptyset$, then $g^{-1} \cdot x = x$ because X was assumed to be a G-simplicial complex. Therefore, $S_x(y) = S_x(g \cdot y) = g \cdot S_{g^{-1}x}(y) = g \cdot S_x(y)$; i.e., g acts on each set $S_x(y)$ individually. But since $S_x(y)$ is a simplex in $R(x)$, this means that g fixes y pointwise.

To determine the stabilisers of $X[R]$, it suffices to consider the stabilisers of individual vertices. Indeed, since $X[R] = \coprod_{x \in X_0} R(x)_0$, if H is the stabiliser of a vertex in $X[R]$, then there is some $x \in X_0$ such that $H \in \mathcal{F}_{R(x)}$.

For the dimension estimate, observe that we can write every simplex $y \in X[R]$ as a disjoint union

$$y = \coprod_{x \in S(y)} S_x(y).$$

If X is N-dimensional, the set $S(y) \in X$ can contain at most $N + 1$ elements. Similarly, the size of each $S_x(y)$ is bounded by $K + 1$. Consequently, a simplex y can contain at most $(N + 1)(K + 1)$ elements, so $X[R]$ has at most dimension $(N + 1)(K + 1) - 1 = NK + N + K$. □

Recall that the geometric realisation $|Y|$ of a simplicial complex Y consists of formal sums $\eta = \sum_{y \in Y_0} \lambda_y y$ for which $\lambda_y \geqslant 0$, the set $\operatorname{supp}(\eta) := \{y \mid \lambda_y \neq 0\}$ is a simplex in Y (in particular, it is finite), and $\sum_y \lambda_y = 1$. The geometric realisation is topologised as a subspace of $\mathbb{R}^{Y_0}$. Then we can describe the geometric realisation of a resolution as follows.

4.1.5 Lemma.
Let X be a G-simplicial complex, and let R be a set of resolution data for X. Define the topological space $|X,R|$ as the set of formal sums

$$|X,R| := \left\{ \sum_{x \in X_0} \lambda_x \cdot \eta_x \mid \lambda_x \geqslant 0, \{x \mid \lambda_x \neq 0\} \in X, \sum_{x \in X_0} \lambda_x = 1, \eta_x \in |R(x)| \right\},$$

topologised as a subspace of $\prod_{x \in X_0}(\mathbb{R} \times |R(x)|) \subseteq \prod_{x \in X_0}(\mathbb{R} \times \mathbb{R}^{R(x)_0})$.

Then the map

$$F\colon |X[R]| \to |X,R|, \qquad \sum_{y \in X[R]_0} \lambda_y \cdot y \mapsto \sum_{x \in X_0} \lambda_x \cdot \Big(\sum_{y \in R(x)_0} \frac{\lambda_y}{\lambda_x} \cdot y \Big),$$

where $\lambda_x := \sum_{y \in R(x)_0} \lambda_y$[1], is a homeomorphism.

Proof. F is well-defined: Write $\eta := \sum_{y \in X[R]_0} \lambda_y \cdot y$ and $\eta_x := \sum_{y \in R(x)_0} \frac{\lambda_y}{\lambda_x} \cdot y$. Every λ_x is non-negative and $\sum_{x \in X_0} \lambda_x = \sum_{y \in X[R]_0} \lambda_y = 1$. The set $\{x \mid \lambda_x \neq 0\}$ is exactly the set $S(\mathrm{supp}(\eta))$, and therefore a simplex in X. Moreover, we can identify $\mathrm{supp}(\eta_x) = S_x(\mathrm{supp}(\eta_x)) \in R(x)$, and $\sum_{y \in R(x)_0} \frac{\lambda_y}{\lambda_x} = \frac{1}{\lambda_x} \cdot \sum_{y \in R(x)_0} \lambda_y = 1$, so η_x is a point in $|R(x)|$. F is continuous.

We define an inverse map. If $y \in R(x)_0 \subseteq \coprod_{x \in X_0} R(x)_0$, write $x(y)$ for x. For $x \in X_0$ and $\eta_x \in |R(x)|$, write $\eta_x = \sum_{y \in R(x)_0} \eta_{x,y} \cdot y$. Then set

$$F'\colon |X,R| \to |X[R]|, \qquad \eta = \sum_{x \in X_0} \lambda_x \cdot \eta_x \mapsto \sum_{y \in X[R]_0} \lambda_{x(y)} \eta_{x(y),y} \cdot y.$$

The image of a point under F' lies in $|X[R]|$: The coefficient $\lambda_{x(y)}\eta_{x(y),y}$ is always non-negative, and

$$\sum_{y \in X[R]_0} \lambda_{x(y)}\eta_{x(y),y} = \sum_{x \in X_0} \sum_{y \in R(x)_0} \lambda_{x(y)}\eta_{x(y),y} = \sum_{x \in X_0} \lambda_x \sum_{y \in R(x)_0} \eta_{x,y} = \sum_{x \in X_0} \lambda_x = 1.$$

The set $\mathrm{supp}(F'(\eta))$ consists of all those $y \in X[R]$ such that both $\lambda_{x(y)}$ and $\eta_{x(y),y}$ are non-zero. For a fixed x, we have $S_x(\mathrm{supp}(F'(\eta)) = \mathrm{supp}(\eta_x) \in R(x)$, and similarly $S(\mathrm{supp}(F'(\eta))) = \{x \mid \lambda_x \neq 0\} \in X$. This shows that $F'(\eta)$ is a point in $|X[R]|$. This map is also continuous.

It is straightforward to check that $F \circ F' = \mathrm{id}_{|X,R|}$ and $F' \circ F = \mathrm{id}_{|X[R]|}$. ☐

If $\tau\colon R \to R'$ is a natural transformation of sets of resolution data for X, we get an induced map $X[\tau]\colon X[R] \to X[R']$ by sending $y \in R(x)_0$ to $\tau_x(y) \in R'(x)_0$. This is a G-equivariant simplicial map.

[1] We are quite liberal with our notation: Since we effectively ignore all summands whose coefficient λ_x is zero and think of the formal sums in $|X,R|$ as finite formal sums, we do not worry about the fact that the "point" we specify for λ_x is not defined if $\lambda_x = 0$.

4.1.6 Proposition.
Let R, R' be sets of resolution data for the G-simplicial complex X. Suppose that there is a natural transformation $\tau\colon R \to R'$ such that every $\tau_x\colon R(x) \to R'(x)$ is a homotopy equivalence.

Then the induced map $X[\tau]\colon X[R] \to X[R']$ is a homotopy equivalence.

Proof. Since we are only interested in the non-equivariant homotopy type, we ignore all group actions. Let $f_x\colon |R'(x)| \to |R(x)|$ be a homotopy inverse to $|\tau_x|$ for every $x \in X_0$, and pick homotopies $H_x\colon |R(x)| \times I \to |R'(x)|$ and $K_x\colon |R'(x)| \times I \to |R(x)|$ witnessing $f_x \circ |\tau_x| \simeq \mathrm{id}_{R(x)}$ and $|\tau_x| \circ f_x \simeq \mathrm{id}_{R'(x)}$.

Using Lemma 4.1.5, the map $|X[\tau]|$ is given by $|X[\tau]| \left(\sum_x \lambda_x \eta_x\right) = \sum_x \lambda_x |\tau_x| (\eta_x)$. Similarly, the maps f_x induce a map $f\colon |X[R']| \to |X[R]|, \sum_x \lambda_x \eta_x \mapsto \sum_x \lambda_x f_x(\eta_x)$. We can now define a homotopy $H\colon |X[R]| \times [0,1] \to |X[R]|$ by

$$H\Big(\sum_x \lambda_x \eta_x, t\Big) := \sum_x \lambda_x H_x(\eta_x, t).$$

This homotopy has the property that

$$H\Big(\sum_x \lambda_x \eta_x, 0\Big) = \sum_x \lambda_x H_x(\eta_x, 0) = \sum_x \lambda_x (f_x \circ |\tau_x|)(\eta_x) = (f \circ |X[\tau]|)\Big(\sum_x \lambda_x \eta_x\Big),$$

and $H(\sum_x \lambda_x \eta_x, 1) = \sum_x \lambda_x H_x(\eta_x, 1) = \sum_x \lambda_x \eta_x$; so H is a witnessing homotopy for $f \circ |X[\tau]| \simeq \mathrm{id}_{|X[R]|}$. The same argument applied to the collection of homotopies $\{K_x\}_x$ proves that $|X[\tau]|$ is a homotopy equivalence. □

4.1.7 Remark.
It is evident from the proof that we can relax the assumptions in Proposition 4.1.6 a bit: Instead of a natural transformation $R \to R'$, it suffices to have a natural transformation of functors $\mathsf{Tr}_1(X_0) \to \mathsf{SCplx}$ which is a homotopy equivalence in each component. Then there is still an induced (non-equivariant) map, and the proof goes through to show that this map is a homotopy equivalence. This should come as no surprise, as we are only interested in the non-equivariant homotopy type.

Letting $\bullet$ denote the set of resolution data which assigns Δ^0 to every vertex, we see that we can recover X as $X[\bullet]$. If R is any set of resolution data, the unique natural transformation $R \to \bullet$ induces a map $\varepsilon\colon X[R] \to X[\bullet] \cong X$.

4.1.8 Corollary.
Let R be a set of resolution data for the G-simplicial complex X. If $R(x)$ is contractible for all $x \in X_0$, then the natural map $X[R] \to X[\bullet]$ is a homotopy equivalence.

□

4.2. Oliver's theorem

Since we can iterate the resolution of fixed points on a G-simplicial complex, the question about the minimal stabilisers of an action on a finite contractible complex reduces to the question which groups admit actions on such complexes without a global fixed point. It is precisely this question that Oliver fully answered in [Oli75].

4.2.1 Definition.
We consider the following families of finite groups:

- Let $\mathcal{C}yc$ denote the class of finite cyclic groups.
- Let p be a prime. Denote by

  $$\begin{aligned}\mathcal{C}yc_p := \{H \mid H &\text{ is finite and there is an extension } 1 \to P \to H \to C \to 1\\ &\text{such that } P \text{ is a } p\text{-group and } C \in \mathcal{C}yc.\}\end{aligned}$$

  the class of finite groups which are *cyclic mod p*.
- We call

  $$\begin{aligned}\mathcal{D} := \{G \mid &\text{ There are primes } p \text{ and } q \text{ such that there is an extension}\\ &1 \to H \to G \to Q \to 1 \text{ with } H \in \mathcal{C}yc_p \text{ and } Q \text{ a } q\text{-group.}\}\end{aligned}$$

  the *Dress family* (instead of some rather awkward name like "groups which are q-hyper-(cyclic mod p) for some primes p and q").

4.2.2 Definition.
Let G be a finite group. Define the *depth of* G to be

$$\begin{aligned}d(G) := \sup\{n \mid &\text{ There is a properly descending chain of subgroups}\\ &G_1 \supsetneq G_2 \supsetneq \cdots \supsetneq G_n \text{ in } G.\}\end{aligned}$$

4.2.3 Theorem ([Oli75])**.**
There is a monotone function $\mathrm{bd}\colon \mathbb{N}_+ \to \mathbb{N}_+$ such that the following holds:

- $\mathrm{bd} \in \mathcal{O}(n)$[2].
- For every finite group G which does not lie in $\mathcal{D}$, there is a finite, contractible G-simplicial complex X such that the dimension of X is bounded by $\mathrm{bd}(d(G))$ and $X^G = \emptyset$.

This theorem is stated in [Oli75] without any mention of a bound on the dimension of X. However, our second application of Oliver's theorem in Section 5.2 does rely on the existence of such a bound; it can be read off Oliver's proof, but has not been recorded explicitly.

Therefore, we will first apply Oliver's theorem to answer our original question, and then revisit the proof of the theorem.

[2] Recall the Landau $\mathcal{O}$-notation: Let $f, g\colon \mathbb{N} \to \mathbb{N}$ be two functions. We say that $f \in \mathcal{O}(g)$ if there is some constant $C > 0$ such that $f(n) \leqslant C \cdot g(n)$ for almost all n.

4.2.4 Corollary.
For every finite group G, there is a finite, contractible G-simplicial complex X of dimension at most

$$\delta(d(G)) := \sum_{\emptyset \neq M \subseteq \{1,\dots,d(G)\}} \prod_{m \in M} \mathrm{bd}(m),$$

all whose stabiliser subgroups lie in $\mathcal{D}$. In particular, the dimension can be bounded by $2^{d(G)} \cdot \mathrm{bd}(d(G))^{d(G)}$.

Proof. We proceed by induction on $d(G)$. If $d(G) = 1$, the group G is trivial and we can pick $X = \Delta^0$.

Suppose that $d(G) = d+1$. If $G \in \mathcal{D}$, we can again pick X to be a point. Otherwise, we apply Theorem 4.2.3 to obtain a finite, contractible G-simplicial complex X' such that the dimension of X' is bounded by $\mathrm{bd}(d(G))$ and X' does not have a global fixed point. Let $\mathcal{X}$ be a set of representatives of the G-orbits in X_0. For every $x \in \mathcal{X}$, the stabiliser G_x is a proper subgroup of G, and therefore $d(G_x) \leqslant d$. By the induction hypothesis, there is for every $x \in \mathcal{X}$ a finite, contractible G_x-simplicial complex R_x whose dimension is bounded by $\delta(d)$ and whose stabilisers all lie in $\mathcal{D}$. According to Remark 4.1.3, the collection $\{G \times_{G_x} R_x\}_{x \in \mathcal{X}}$ defines a set of resolution data R. Set $X := X'[R]$. Corollary 4.1.8 implies that X is contractible. Then X is a finite G-simplicial complex whose stabilisers all lie in $\mathcal{D}$, and whose dimension is at most

$$\begin{aligned}
&\mathrm{bd}(d+1) \cdot \delta(d) + \mathrm{bd}(d+1) + \delta(d) \\
&= \left(\mathrm{bd}(d+1) \cdot \sum_{\emptyset \neq M \subseteq \{1,\dots,d\}} \prod_{m \in M} \mathrm{bd}(m) \right) + \mathrm{bd}(d+1) + \delta(d) \\
&= \sum_{\emptyset \neq M \subseteq \{1,\dots,d\}} \left(\mathrm{bd}(d+1) \cdot \prod_{m \in M} \mathrm{bd}(m) \right) + \mathrm{bd}(d+1) + \delta(d) \\
&= \sum_{\substack{\emptyset \neq M, \\ d+1 \in M}} \prod_{m \in M} \mathrm{bd}(m) + \sum_{\substack{\emptyset \neq M, \\ d+1 \notin M}} \prod_{m \in M} \mathrm{bd}(m) \\
&= \sum_{\emptyset \neq M \subseteq \{1,\dots,d+1\}} \prod_{m \in M} \mathrm{bd}(m) \\
&= \delta(d+1).
\end{aligned}$$

To obtain the simpler (and less precise) dimension bound, observe that for every non-empty set $M \subseteq \{1, \dots, d(G)\}$, we have

$$\prod_{m \in M} \mathrm{bd}(m) \leqslant \prod_{m \in M} \mathrm{bd}(d(G)) = \mathrm{bd}(d(G))^{|M|} \leqslant \mathrm{bd}(d(G))^{d(G)},$$

so that

$$\begin{aligned}
\delta(d(G)) &\leqslant \sum_{\emptyset \neq M \subseteq \{1,\dots,d(G)\}} \mathrm{bd}(d(G))^{d(G)} = (2^{d(G)} - 1) \cdot \mathrm{bd}(d(G))^{d(G)} \\
&\leqslant 2^{d(G)} \cdot \mathrm{bd}(d(G))^{d(G)}.
\end{aligned}$$

□

4.2.5 Remark.
The G-simplicial complex whose existence is asserted in Corollary 4.2.4 will typically not be a model of $E_{\mathcal{D}}G$ for the following reason:

Suppose that E is a finite classifying space of a finite group G with respect to the family $\mathcal{F}$. Let $H \in \mathcal{F}$, and suppose that $H \trianglelefteq K \leqslant G$ such that K/H is a cyclic group of order p for some prime p. The normaliser $N_G(H)$ acts on the fixed point set $E^H \simeq *$. Consequently, we get an action of the quotient K/H on E^H. If this action does not have a fixed point, we have found a finite model for $E(K/H)$, which is nonsense. So the fixed point set must be non-empty. It follows that the family $\mathcal{F}$ must be closed under extensions by p-groups.

This observation is due to Oliver, see [Oli76, bottom of p. 93].

The rest of this section is devoted to the proof of Theorem 4.2.3. The material covered in this section is taken from [Oli75], with some additional ideas from [Oli78]. I claim no originality for the content; the sole purpose of the proof is to make the dimension bound explicit. In particular, we do not reprove Oliver's results in full generality, but only as much as we need to obtain Theorem 4.2.3.

Let G be a finite group. Define an equivalence relation on finite G-CW-complexes by setting $X \sim_\chi Y$ if and only if $\chi(X^H) = \chi(Y^H)$ for all subgroups $H \leqslant G$. The *Burnside group* $\Omega(G)$ of G is the set of $\sim_\chi$-equivalence classes of finite G-CW-complexes with group operation induced by taking disjoint unions. The Burnside group becomes a commutative ring when equipped with the product induced by taking products of complexes.

Let $\Delta(G) \subseteq \Omega(G)$ be the subset given by

$$\Delta(G) := \{x \in \Omega(G) \mid \text{There is a finite contractible } G\text{-CW-complex } X \text{ with } x = [X] - 1\}.$$

4.2.6 Lemma ([Oli76, p. 90])**.**
$\Delta(G)$ is an ideal in $\Omega(G)$.

Proof. Suppose for the beginning that X is an arbitrary finite G-CW-complex. The suspension ΣX is again a finite G-CW-complex. If $H \leqslant G$ is a subgroup such that $X^H \neq \emptyset$, then $(\Sigma X)^H = \Sigma X^H$; otherwise, the fixed point set is $(\Sigma X)^H = S^0$, given by the two suspension points. Consequently, the double suspension $\Sigma^2 X$ represents the same element in $\Omega(G)$ since either $\chi((\Sigma^2 X)^H) = \chi(\Sigma^2 X^H) = \chi(X^H)$ if $X^H \neq \emptyset$ or $\chi((\Sigma^2 X)^H) = \chi(\Sigma S^0) = \chi(S^1) = 0 = \chi(\emptyset)$ if $X^H = \emptyset$. Note also that the suspension of a complex always has a global fixed point.

We are now ready to prove the claim. The 0-simplex Δ^0 is a finite contractible G-CW-complex which represents the unit; so $0 = [\Delta^0] - 1$ is a member of $\Delta(G)$. If $x, x' \in \Delta(G)$, pick X and X' such that $x = [X] - 1$ and $x' = [X'] - 1$. By the preliminary remark, $[\Sigma^2 X] - 1 = x$ and $[\Sigma^2 X'] - 1 = x'$. Pick a global fixed point in $\Sigma^2 X$ and $\Sigma^2 X'$, and form their wedge sum $\Sigma^2 X \vee \Sigma^2 X'$. The wedge sum of two contractible complexes is still contractible, and we can compute

$$[\Sigma^2 X \vee \Sigma^2 X'] - 1 = [\Sigma^2 X] + [\Sigma^2 X'] - 1 - 1 = x + x'.$$

To prove the existence of inverses in $\Delta(G)$, observe that for a finite contractible G-CW-complex X, we always have $\chi((\Sigma X)^H) = 2 - \chi(X^H)$. Given $x = [X] - 1 \in \Delta(G)$, set $y := [\Sigma X] - 1$. Then

$$x + y = [X] - 1 + [\Sigma X] - 1 = [X \sqcup \Sigma X] - 2 = 2 - 2 = 0,$$

since $\chi((X \sqcup \Sigma X)^H) = \chi(X^H) + 2 - \chi(X^H) = 2 = \chi((S^0)^H)$. So $\Delta(G) \subseteq \Omega(G)$ is an abelian subgroup.

To show that $\Delta(G)$ is an ideal, let $x = [X] - 1$ be an element in $\Delta(G)$, and let Y be an arbitrary finite G-CW-complex. As we have seen before, we can assume without loss of generality that both X and Y have a global fixed point, and that Y is connected. Pick one fixed point for each complex. Then their smash product $X \wedge Y$ is again a finite contractible G-CW-complex. By the Künneth theorem,

$$\begin{aligned}
\chi((X \wedge Y)^H) &= \sum_i (-1)^i \dim_{\mathbb{R}} H^i((X \wedge Y)^H; \mathbb{R}) \\
&= \sum_i (-1)^i \dim_{\mathbb{R}} \widetilde{H}^i((X \wedge Y)^H; \mathbb{R}) + 1 \\
&= \Big(\sum_i \sum_{m+n=i} (-1)^m \dim_{\mathbb{R}} \widetilde{H}^m(X^H; \mathbb{R}) \cdot (-1)^n \widetilde{H}^n(Y^H; \mathbb{R}) \Big) + 1 \\
&= \Big(\sum_i (-1)^i \dim_{\mathbb{R}} \widetilde{H}^i(X; \mathbb{R}) \Big) \cdot \Big(\sum_i (-1)^i \dim_{\mathbb{R}} \widetilde{H}^i(Y; \mathbb{R}) \Big) + 1 \\
&= (\chi(X^H) - 1) \cdot (\chi(Y^H) - 1) + 1 \\
&= \chi(X^H)\chi(Y^H) - \chi(X^H) - \chi(Y^H) + 2 \\
&= \chi((X \times Y)^H) - \chi(X^H) - \chi(Y^H) + 2.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
[(X \wedge Y) \vee X] - 1 &= [X \wedge Y] + [X] - 2 = [X][Y] - [X] - [Y] + 2 + [X] - 2 \\
&= [X][Y] - [Y] = ([X] - 1)[Y].
\end{aligned}$$

Since $(X \wedge Y) \vee X$ is contractible, this shows that $([X] - 1)[Y] \in \Delta(G)$. □

For a subgroup $H \leqslant G$, define a map $\mathsf{gh}_H \colon \Omega(G) \to \mathbb{Z}$ by

$$\mathsf{gh}_H([X]) := \chi(X^H).$$

This is a well-defined ring homomorphism. Clearly, if H and K are conjugate subgroups, then $\mathsf{gh}_H = \mathsf{gh}_K$. Letting H vary, these homomorphisms combine to the *ghost map*

$$\mathsf{gh} \colon \Omega(G) \to \prod_{(H)} \mathbb{Z},$$

where (H) ranges over all conjugation classes of subgroups of G. The map gh is easily seen to be injective. The image $\mathsf{gh}_G(\Delta(G))$ of the ideal $\Delta(G)$ under the homomorphism gh_G is an ideal in $\mathbb{Z}$. Define $n_G \in \mathbb{N}$ to be the unique natural number such that

$$\mathsf{gh}_G(\Delta(G)) = n_G\mathbb{Z}.$$

The number n_G is an obstruction to the existence of a finite contractible G-CW-complex without a global fixed point: Suppose that X is such a complex. Then $[X] - 1 \in \Delta(G)$, and therefore $\mathsf{gh}_G([X] - 1) = \chi(X^G) - 1 = \chi(\emptyset) - 1 = -1$. Hence, if such a complex exists, n_G must equal 1. The proof of Theorem 4.2.3 amounts to showing that the converse is also true, and then calculating $n_G = 1$ whenever $G \notin \mathcal{D}$.

Suppose that $n_G = 1$. Then there is some $x \in \Delta(G)$ such that $\mathsf{gh}_G(x) = -1$, i.e. we can find a finite contractible G-CW-complex X such that $\chi(X^G) = 0$. The crucial observation is that the Euler characteristics of the fixed point sets of X provide us with sufficiently much information to construct a complex without a global fixed point.

4.2.7 Definition ([Oli75, p. 159])**.**
Let G be a finite group, and let $\mathcal{S}(G)$ denote the poset of subgroups of G. A function $\varphi\colon \mathcal{S}(G) \to \mathbb{Z}$ is called a *resolving function* if the following holds:

- φ is constant on conjugacy classes of subgroups.
- For all $H \leqslant G$, the order of the Weyl group $[N_G(H)\colon H]$ divides $\varphi(H)$.
- If $H \in \mathcal{C}yc_p$ for some prime p, then $\sum_{K \supseteq H} \varphi(K) = 0$.

4.2.8 Lemma.
Let X be a finite G-CW-complex. Then there is a unique function $\varphi_X\colon \mathcal{S}(G) \to \mathbb{Z}$ such that for all $H \leqslant G$,

$$\chi(X^H) = 1 + \sum_{K \supseteq H} \varphi_X(K). \tag{4.1}$$

The function φ is constant on conjugacy classes of subgroups and for all $H \leqslant G$, the order of the Weyl group $[N_G(H)\colon H]$ divides $\varphi_X(H)$.

Furthermore, if $S \subseteq \mathcal{S}(G)$ is a non-empty subset closed under taking supergroups and ψ is a function which satisfies 4.1 for all $H \in S$, then

$$\chi(\bigcup_{H \in S} X^H) = 1 + \sum_{H \in S} \psi_X(H).$$

Proof. [Oli75, Lem. 2]. □

4.2.9 Corollary.
If X is a finite, contractible G-CW-complex, the function φ_X from Lemma 4.2.8 is a resolving function.

Proof. If X is contractible, we have by [Oli75, Prop. 2], whenever $H \in \mathcal{C}yc_p$ for some prime p,

$$1 = \chi(X^H) = 1 + \sum_{K \supseteq H} \varphi_X(K).$$

□

The sum of two resolving functions is again a resolving function, and if φ is a resolving function, so is $-\varphi$; so the set

$$\{\varphi(G) \mid \varphi \text{ is a resolving function for } G\} \subseteq \mathbb{Z}$$

forms a subgroup, and we can define r_G to be the unique natural number generating this subgroup. By Corollary 4.2.9, if $[X] - 1$ is a preimage of $n_G = \chi(X^G) - 1$, then the associated resolving function φ_X has the property that $\chi(X^G) = 1 + \varphi(G)$, i.e. $n_G = \varphi(G)$. It follows that $n_G\mathbb{Z} \subseteq r_G\mathbb{Z}$, or in other words $r_G \mid n_G$.

4.2.10 Theorem.
If $r_G = 1$ and G is not a p-group for any prime p, then there is a finite, contractible G-CW-complex X with $X^G = \emptyset$ whose dimension is bounded by $4 \cdot d(G) + 2$.

Proof. As a first step, we will construct a G-CW-complex X_0 such that $X_0^G = \emptyset$, the dimension $n > 1$ of X_0 is bounded by $2 \cdot d(G)$, the space X_0 is $(n-1)$-connected and the top-dimensional homology $H_n(X_0; \mathbb{Z})$ is finitely generated projective over $\mathbb{Z}[G]$ (Oliver calls such spaces *G-resolutions of* $\emptyset$). This will be a reproduction of [Oli75, Thm. 2]. Once we have the space X_0 at our disposal, we will deviate slightly from the treatment in [Oli75] to construct the space X in such a way that we have better control on the dimension bound.

To get started, pick a resolving function $\varphi \colon \mathcal{S}(G) \to \mathbb{Z}$ with $\varphi(G) = -1$. The existence of such a function is guaranteed by the assumption that $r_G = 1$.

We need some additional notation which will allow us to relate our dimension bounds to the order type of $\mathcal{S}(G)$ (which we consider as a poset with respect to $\supseteq$). Define the *rank* of a subgroup $H \in \mathcal{S}(G)$ to be

$$\operatorname{rk}(H) := \max\{\operatorname{rk}(K) \mid K \in \mathcal{S}(G), K \supsetneq H\} + 1,$$

where $\max \emptyset := 0$. Observe that the depth $d(G)$ of G as we defined it in 4.2.2 is simply the rank of the trivial subgroup, which in turn is the order type of $(\mathcal{S}(G), \supseteq)$. Conjugate subgroups have equal rank. We can now define a linear order $\prec$ on $\mathcal{S}(G)$ as follows: Declare two subgroups H and K of G to be equivalent if they have equal rank. This is clearly an equivalence relation. For each r, pick an arbitrary linear order $\preceq_r$ on the equivalence class of subgroups of rank r, Then we obtain the linear order $\preceq$ by setting

$$H \preceq K :\Longleftrightarrow \operatorname{rk}(H) < \operatorname{rk}(K) \text{ or } (\operatorname{rk}(H) = \operatorname{rk}(K) \text{ and } H \preceq_{\operatorname{rk}(H)} K).$$

Write $H \prec K$ if $H \preceq K$ and $H \neq K$. Note that the $\prec$-minimal element is G, while the $\prec$-maximal element is given by the trivial subgroup. In particular, the

$\prec$-minimal element is not a p-group for any prime p. We proceed by induction along $(\mathcal{S}(G) \setminus \{1\}, \prec)$. To be more precise, we prove the following:

Claim: For every $H \in \mathcal{S}(G)$, there is a finite G-complex $X(H)$ such that

- the complex $X(H)$ contains only cells of type G/K, where $G \neq K \preceq H$; by this we mean that all equivariant cells in $X(H)$ are of the form $G/K \times D^d$ for some natural number d. Note that the type of a cell is only well-defined up to conjugation. In particular, $X(H)$ has no global fixed point.
- the dimension of $X(H)$ is bounded by $2 \cdot \mathrm{rk}(H)$.
- the dimension of the K-fixed point set $X(H)^K$ is bounded by $2 \cdot \mathrm{rk}(K)$ for $K \preceq H$.
- $\chi(X(H)^K) = 1 + \sum_{K' \supseteq K} \varphi(K')$ for all $K \preceq H$.
- $X(H)^K$ is $\mathbb{Z}/p\mathbb{Z}$-acyclic for all non-trivial p-groups $K \preceq H$ (p any prime number).

For the start of the induction, we can set $X(G) := \emptyset$. Now suppose that $H \in \mathcal{S}(G)$ is arbitrary. Let H^- be the immediate $\prec$-predecessor of H, and assume that the complex $X(H^-)$ has been constructed.

If there is an element $g \in G$ such that $gHg^{-1} \prec H$, set $X(H) := X(H^-)$. This complex contains only cells of the allowed type. Since $gHg^{-1} \preceq H^- \prec H$, we have $\mathrm{rk}(gHg^{-1}) \leqslant \mathrm{rk}(H^-) \leqslant \mathrm{rk}(H) = \mathrm{rk}(gHg^{-1})$, so $\mathrm{rk}(H^-) = \mathrm{rk}(H)$, which shows that the dimension bound on $X(H)$ holds. The dimension bound on the fixed point sets is automatic. If $K \preceq H$, then either $K \preceq H^-$ or $K = H$. In the first case, the required condition on $\chi(X(H)^K)$ is obvious; in the case that $K = H$, we use the fact that the set of subgroups H' with $H' \supseteq H$ consists exactly of the g^{-1}-conjugates of the subgroups containing gHg^{-1} and the invariance of φ under conjugation to get

$$\begin{aligned}\chi(X(H)^H) = \chi(X(H)^{gHg^{-1}}) &= 1 + \sum_{H' \supseteq gHg^{-1}} \varphi(H') \\ &= 1 + \sum_{H' \supseteq gHg^{-1}} \varphi(g^{-1}H'g) = 1 + \sum_{H' \supseteq H} \varphi(H').\end{aligned}$$

The condition on fixed point sets of p-groups is equally easy.

Suppose now that there is no conjugate of H which is $\prec$-smaller than H. The following congruences hold (the second holds since φ is a resolving function):

$$\begin{aligned}\chi(X(H^-)^H) \equiv \chi\Big(\bigcup_{K \supsetneq H} X(H^-)^K\Big) &\overset{4.2.8}{=} 1 + \sum_{K \supsetneq H} \varphi(K) \\ &\equiv 1 + \sum_{K \supseteq H} \varphi(K) \mod [N_G(H) \colon H].\end{aligned}$$

Write $\chi(X(H^-)^H) + n \cdot [N_G(H) \colon H] = 1 + \sum_{K \supseteq H} \varphi(K)$. We have to distinguish between two cases:

Assume first that H is not a p-group for any prime p. If $n = 0$, set $X(H) := X(H^-)$ and note that all conditions are satisfied. So let $n \neq 0$.

Define $X(H)$ by taking the disjoint union of $X(H^-)$ and $\coprod_{i=1}^{|n|} G/H \times C$, where $C = *$ if n is positive and $C = S^1 \vee S^1$ if n is negative. Then $X(H)$ has no global fixed point and the dimension bounds are satisfied. For $1 \neq K \prec H$, the fixed point set $(G/H \times C)^K$ is empty, so the condition on $\chi(X(H)^K)$ holds, and the fourth condition has also been preserved. For H itself,

$$\begin{aligned}\chi(X(H)^H) &= \chi(X(H^-)^H) + |n| \cdot \chi(N_G(H)/H \times C) \\ &= \chi(X(H^-)^H) + n \cdot [N_G(H)\colon H] = 1 + \sum_{K \supseteq H} \varphi(K).\end{aligned}$$

We are left with the case that H is a p-group for some prime p. Consider the H-fixed point set $Y_0 := X(H^-)^H$. If Y_0 has several path components, pick a 0-cell in every path component, and attach 1-cells of type G/H to $X(H^-)$ along these 0-cells to obtain a finite G-CW-complex Z_1 whose H-fixed point set Y_1 is path-connected. Proceed by induction along the skeleta (making sure that $(d+1)$-cells are only attached to the d-skeleton) to produce a finite G-CW-complex Z whose fixed point set Y has dimension bigger than $\dim(X(H^-)^K)$ for all $K \supsetneq H$ and which is $(\dim(Y) - 1)$-connected. If we avoid adding superfluous cells, we can arrange that

$$\begin{aligned}\dim(Y) &= 1 + \max\{\dim(X(H^-)^K) \mid K \supsetneq H\} \leqslant 1 + \max\{2 \cdot \mathrm{rk}(K) \mid K \supsetneq H\} \\ &\leqslant 2 \cdot \mathrm{rk}(H) - 1.\end{aligned}$$

Repeating the argument on p. 162 of [Oli75], it follows that $H_n(Y; \mathbb{Z}/p\mathbb{Z})$ is free. Therefore, we can glue on a set of $(\dim(Y) + 1)$-cells of type G/H to get a finite G-CW-complex $X(H)$ whose H-fixed point set is $\mathbb{Z}/p\mathbb{Z}$-acyclic. This complex contains only cells of the type we are allowed to use. Its dimension can be bounded by $\max(\dim(X(H^-)), 2 \cdot \mathrm{rk}(H)) \leqslant 2 \cdot \mathrm{rk}(H)$. As before, the conditions regarding subgroups $K \prec H$ need not be checked since we only added cells of type G/H. By construction, $X(H)^H$ is $\mathbb{Z}/p\mathbb{Z}$-acyclic; this, together with the fact that φ is a resolving function and p-groups are cyclic mod p, implies that

$$\chi(X(H)^H) = 1 = 1 + \sum_{K \supseteq H} \varphi(K).$$

At the end of the induction along $\prec$, we obtain a finite G-CW-complex X' which has no global fixed point, whose dimenson n' is bounded by $2(\mathrm{rk}(1) - 1) = 2(d(G) - 1)$, whose fixed point sets satisfy $\chi((X')^H) = 1 + \sum_{K \supseteq H} \varphi(K)$ for all $H \neq 1$, and whose fixed point sets under non-trivial p-groups are $\mathbb{Z}/p\mathbb{Z}$-acyclic.

By another induction along the skeleta, glue free G-cells onto X' to build an $(n'+1)$-dimensional and n'-connected finite G-CW-complex X_0. Set $n := n' + 1 \leqslant 2 \cdot d(G)$. Since we only added free cells, X_0 still does not have a global fixed point. As in the proof of [Oli75, Thm. 2], it follows that $H_n(X_0; \mathbb{Z})$ is finitely generated and projective over $\mathbb{Z}[G]$.

Recall the join of two spaces Y and Z is the quotient of $Y \times [0,1] \times Z$ obtained by identifying $(y,0,z)$ with $(y,0,z')$ for $y \in Y$, $z, z' \in Z$ and $(y,1,z)$ with $(y',1,z)$ for $y, y' \in Y$, $z \in Z$. The map

$$Y * Z \to \big(C(Y) \times Z\big) \cup_{Y \times Z} \big(Y \times C(Z)\big)$$
$$(y,t,z) \mapsto \begin{cases} (y,(1-2t,z)) \in Y \times C(Y) & \text{if } t \leqslant \frac{1}{2} \\ ((y,2t-1),z) \in C(Y) \times Z & \text{if } t \geqslant \frac{1}{2} \end{cases}$$

is a homeomorphism. The latter description allows us to see that $Y * Z$ is a G-CW-complex if Y and Z are G-CW-complexes.

Consider now the join $X_1 := X_0 * X_0$ of two copies of X_0. This is a G-CW-complex of dimension $2n+1 \leqslant 4 \cdot d(G) + 1$, and referring to the alternative description of the join makes it obvious that this complex does not have a global fixed point.

X_1 is $2n$-connected: Either use the Seifert-van Kampen theorem to show that $\pi_1(X_1)$ is trivial and work your way up using the Hurewicz theorem and the Mayer-Vietoris sequence for homology, or use the fact that $X_0 * X_0$ is non-equivariantly homotopy equivalent to the suspension of $X_0 \wedge X_0$ (after an arbitrary choice of basepoint) and apply the Freudenthal suspension theorem.

Since X_0 is $(n-1)$-connected, the Mayer-Vietoris sequence for $X_0 * X_0$ yields an isomorphism $H_{2n+1}(X_1;\mathbb{Z}) \cong H_{2n}(X_0 \times X_0;\mathbb{Z})$ as $\mathbb{Z}$-modules. Indeed, checking the construction of the boundary map, which is based on the fact that we have a short exact sequence of $\mathbb{Z}$-chain complexes

$$0 \longrightarrow C_*(X_0 \times X_0) \longrightarrow C_*(C(X_0) \times X_0) \oplus C_*(X_0 \times C(X_0)) \longrightarrow C_*(X_1) \longrightarrow 0$$

which is even an exact sequence of $\mathbb{Z}[G]$-chain complexes, it is easy to see that the isomorphism $H_{2n+1}(X_1;\mathbb{Z}) \cong H_{2n}(X_0 \times X_0;\mathbb{Z})$ is an isomorphism of $\mathbb{Z}[G]$-modules. By the Künneth theorem, we have a short exact sequence of $\mathbb{Z}$-modules

$$0 \to \bigoplus_{p+q=2n} H_p(X_0;\mathbb{Z}) \otimes_{\mathbb{Z}} H_q(X_0;\mathbb{Z}) \to H_{2n}(X_0 \times X_0;\mathbb{Z})$$
$$\to \bigoplus_{p+q=2n-1} \mathsf{Tor}_{\mathbb{Z}}(H_p(X_0;\mathbb{Z}), H_q(X_0;\mathbb{Z})) \to 0.$$

The third term vanishes automatically since $H_p(X_0;\mathbb{Z}) = 0$ or $H_q(X_0;\mathbb{Z}) = 0$ for all p and q, and $H_n(X_0;\mathbb{Z}) \otimes_{\mathbb{Z}} H_n(X_0;\mathbb{Z})$ is the only non-trivial summand in the first term. Taking singular chains on the two copies of X_0 give rise to a bisimplicial object in the category of $\mathbb{Z}[G]$-modules; it follows that the Eilenberg-Zilber isomorphism

$$H_*(X_0 \times X_0;\mathbb{Z}) \cong H_*(C_*(X_0) \otimes_{\mathbb{Z}} C_*(X_0))$$

is an isomorphism of $\mathbb{Z}[G]$-modules (see [Wei94, Thm. 8.5.1]). The algebraic cross product is seen to be $\mathbb{Z}[G]$-linear by inspection of the defining formula. Therefore, the isomorphism $H_n(X_0;\mathbb{Z}) \otimes_{\mathbb{Z}} H_n(X_0;\mathbb{Z}) \xrightarrow{\cong} H_{2n}(X_0 \times X_0;\mathbb{Z})$ given by the cross product map is also $\mathbb{Z}[G]$-linear. To sum up the discussion, we have proved that the top-dimensional homology group $H_{2n+1}(X_1; \mathbb{Z})$ is isomorphic as a $\mathbb{Z}[G]$-module to the tensor product $H_n(X_0; \mathbb{Z}) \otimes_{\mathbb{Z}} H_n(X_0; \mathbb{Z})$ of two finitely generated projective $\mathbb{Z}[G]$-modules.

Applying [OS02, Prop. C.3], we get that $H_{2n+1}(X_1; \mathbb{Z})$ is a stably free $\mathbb{Z}[G]$-module. Therefore, we can glue free $(2n+1)$- and $(2n+2)$-cells to X_1 to obtain a finite, $\mathbb{Z}$-acyclic G-CW-complex X with $X^G = \emptyset$. Since X is simply connected, it is contractible. Note that $\dim(X) = 2n + 2 \leqslant 4 \cdot d(G) + 2$; so we are done. □

4.2.11 Corollary.
The following are equivalent:

1. $r_G = 1$.
2. There is a finite, contractible G-CW-complex X with $X^G = \emptyset$ whose dimension is bounded by $4 \cdot d(G) + 2$.
3. $n_G = 1$

Proof. Theorem 4.2.10 tells us that $r_G = 1$ implies the existence of a finite, contractible G-CW-complex X with $X^G = \emptyset$ whose dimension is bounded by $4 \cdot d(G) + 2$. If such a complex exists, this entails $n_G = 1$ as we discussed earlier. If $n_G = 1$, then $r_G = 1$ since $r_G \mid n_G$. □

Oliver has proved that $r_G = 1$ if and only if $G \notin \mathcal{D}$ [Oli75, Thm. 5]. Thus, we have nearly established the refined version of Oliver's theorem. The only missing step consists in replacing the finite G-CW-complex by a G-simplicial complex. That this can be done is a standard fact in the topology of cell complexes, but seems difficult to track down in the literature: The non-equivariant case was already considered by Whitehead [Whi49, Thm. 13], see for example also [LW69, Prop. IV.7.1] or [Hat02, Thm. 2C.5]. The equivariant case is recorded, with a very short indication of proof, in [OS02, Prop. A.4]. We consider this an encouragement to include a proof.

4.2.12 Proposition.
Let X be a finite G-CW-complex. Then X is G-homotopy equivalent to a finite G-simplicial complex K of equal dimension.

The main technical component of the proof is to show the existence of suitable mapping cylinders. We shall use a model for the mapping cylinder which is due to Conner and Floyd [CF59, Def. 5.1].

4.2.13 Definition.
Suppose that $f\colon K \to L$ is an equivariant simplicial map between (abstract) finite G-simplicial complexes. Assume further that K and L are G-locally ordered, i.e. both K and L are locally ordered in such a way that the bijections of finite sets given by translation with a group element are order-preserving maps, and that f is order-preserving. We think of simplices in K and L as ordered sequences. If $(x_0, \dots, x_n)$ is a simplex in K, we write $f(x_0, \dots, x_n)$ for its image (and still think of this as an ordered sequence).

Define the *mapping cylinder* $\mathsf{Cyl}(f)$ of f to be the simplicial complex whose vertex set is the disjoint union $K_0 \sqcup L_0$, and whose vertices are those sequences which define a simplex in K or L, or are a subsequence of a sequence $(x_0, \ldots, x_k, f(x_k, ..., x_n))$, where $(x_0, \ldots, x_n)$ is a simplex in K.

The simplicial mapping cylinder $\mathsf{Cyl}(f)$ is a locally ordered simplicial complex of dimension $\max(\dim(X) + 1, \dim(Y))$. The given action of G on $\mathsf{Cyl}(f)_0$ induces a simplicial action on $\mathsf{Cyl}(f)$ (this uses the fact that the complexes are G-locally ordered). We wish to show that $\mathsf{Cyl}(f)$ has the property we expect from a mapping cylinder.

4.2.14 Lemma.
Let f be a map as in 4.2.13. Let $i_K \colon |K| \hookrightarrow |\mathsf{Cyl}(f)|$ and $i_L \colon |L| \hookrightarrow |\mathsf{Cyl}(f)|$ be the inclusion maps.

Then there is an equivariant deformation retraction H of $|\mathsf{Cyl}(f)|$ onto the subspace $|L|$ such that $(H \circ i_K)\big|_{t=1} = i_L \circ |f|$.

Proof. The pair $(K_0, \leqslant)$ is a finite poset. Define for $x \in K_0$ its *rank* as

$$\mathrm{rk}(x) := \max\{\mathrm{rk}(x') \mid x < x'\} + 1,$$

where we let $\max \emptyset = 0$. Let r be the largest rank attained by some vertex in K_0. Set

$$K(s) := \{x \in K_0 \mid \mathrm{rk}(x) \geqslant s\},$$

and define $C(s) \subseteq \mathsf{Cyl}(f)$ to be the subcomplex spanned by $K(s) \sqcup L_0$. Since we have $\mathrm{rk}(gx) = \mathrm{rk}(x)$ for all $x \in K_0$ and $g \in G$, the subcomplex $C(s)$ is G-invariant. Note that $C(1) = \mathsf{Cyl}(f)$, $C(s+1)$ is a subcomplex of $C(s)$ and $L \subseteq C(s)$ for all s, and $L = C(r+1)$.

We are going to define a sequence of homotopies $\{H_s\}_{1 \leqslant s \leqslant r}$ such that H_s is an equivariant deformation retraction of $C(s)$ onto $C(s+1)$. Fix $1 \leqslant s \leqslant r$. Let H_s be the constant homotopy on $|L|$. If $\sum_i \lambda_i x_i$ is a simplex in $|K| \cap |C(s)|$, set

$$H_s\Big(\sum_{i=0}^{n} \lambda_i x_i, t\Big) := \begin{cases} \sum_{i=0}^{n-1} \lambda_i x_i + (1-t)\lambda_n x_n + t\lambda_n f(x_n) & \text{if } \mathrm{rk}(x_n) = s, \\ \sum_{i=0}^{n} \lambda_i x_i & \text{if } \mathrm{rk}(x_n) > s. \end{cases}$$

Suppose $(x_0, \ldots, x_k, y_{k+1}, \ldots, y_{k+l})$ is a subsequence of $(x'_0, \ldots, x'_{k'}, f(x'_k, \ldots, x'_n))$ for some vertex $(x'_0, \ldots, x'_n) \in C(s)$. In this case, define

$$H_s\Big(\sum_{i=0}^{k} \lambda_i x_i + \sum_{j=k+1}^{k+l} \mu_j y_j, t\Big)$$
$$:= \begin{cases} \sum_{i=0}^{k-1} \lambda_i x_i + (1-t)\lambda_k x_k + t\lambda_k f(x_k) + \sum_{j=k}^{k+l} \mu_j y_j & \text{if } \mathrm{rk}(x_k) = s, \\ \sum_{i=0}^{k} \lambda_i x_i + \sum_{j=k+1}^{k+l} \mu_j y_j & \text{if } \mathrm{rk}(x_k) > s. \end{cases}$$

This defines H_s on the entirety of $|C(s)| \times [0,1]$, and it is straightforward to check that all image points lie in $|C(s)|$. Moreover, if $t = 1$, no point in the image of H_s is supported on a vertex with rank s, so $H_s\big|_{t=1}$ maps to $C(s+1)$. The function H_s leaves nearly all coordinates untouched. The only coordinates that vary are those of vertices whose rank is s, and their images. In these coordinates, H_s clearly varies continuously, so H_s is in fact a homotopy.

Concatenating these homotopies (i.e. deformation retracting $|\mathsf{Cyl}(f)| = |C(1)|$ onto $|C(2)|$, then deformation retracting $|C(2)|$ onto $|C(3)|$ etc.), we obtain an equivariant deformation retraction H of $|\mathsf{Cyl}(f)| = |C(1)|$ onto $|C(r+1)| = |L|$.

Finally, let $\xi = \sum_{i=0}^{n} \lambda_i x_i \in |K|$, and let $r_i := \mathrm{rk}(x_i)$. Then $H_s(\xi, 1) = \xi$ for all $s < r_n$. In case $s = r_n$, we have

$$\xi_n := H_{r_n}(\xi, 1) = \sum_{i=0}^{n-1} \lambda_i x_i + \lambda_n f(x_n).$$

For $r_n < s < r_{n-1}$, the homotopies H_s leave ξ_n unchanged. Then

$$\xi_{n-1} := H_{r_{n-1}}(\xi_n, 1) = \sum_{i=0}^{n-2} \lambda_i x_i + \lambda_{n-1} f(x_{n-1}) + \lambda_n f(x_n).$$

Things go on like this to produce a sequence $(\xi_s)_{n \geqslant s \geqslant 0}$, where ultimately

$$H(\xi, 1) = \xi_n = \sum_{i=0}^{n} \lambda_i f(x_i) = (i_L \circ |f|)\Big(\sum_{i=0}^{n} \lambda_i x_i\Big).$$

This verifies that H has all the desired properties. □

Proof of Proposition 4.2.12. We proceed by induction along the skeleta to produce finite G-simplicial complexes K_n with $\dim(K_n) = n$ and G-homotopy equivalences $f_n \colon X^{(n)} \to K_n$. Then K is given by the complex $K_{\dim(X)}$.

For the start of the induction, set $K_0 := X^{(0)}$ and let f_0 be the identity map.

Suppose K_n and f_n have been constructed. The $(n+1)$-skeleton of X is obtained from the n-skeleton by a pushout

$$\begin{array}{ccc} \coprod_{i \in I_{n+1}} G/H_i \times S^n & \xrightarrow{\coprod_i \alpha_i} & X^{(n)} \\ \downarrow & & \downarrow \\ \coprod_{i \in I_{n+1}} G/H_i \times D^{n+1} & \longrightarrow & X^{(n+1)} \end{array}$$

where I_{n+1} is some indexing set for the equivariant $(n+1)$-cells of X. By the adjunction isomorphism

$$\mathrm{map}_G(G/H_i \times S^n, X^{(n)}) \cong \mathrm{map}(S^n, \mathrm{map}_G(G/H_i, X^{(n)}) \cong \mathrm{map}(S^n, (X^{(n)})^{H_i}),$$

each map α_i is determined by a non-equivariant map $\overline{\alpha}_i\colon S^n \to (X^{(n)})^{H_i}$. Let $\overline{\alpha}\colon \coprod_i S^n \to X^{(n)}$ be the non-equivariant map defined by the collection $\{\overline{\alpha}_i\}_i$. Define $\widehat{\alpha}$ as the composition $\widehat{\alpha}\colon \coprod_i S^n \xrightarrow{\overline{\alpha}} X^{(n)} \xrightarrow{f_n} K_n$. Equip S^n with the structure of a (finite) simplicial complex, so $\coprod_i S^n$ also has the structure of a finite simplicial complex. After an appropriate subdivision, there is a simplicial map $\widetilde{\alpha}\colon \coprod_i \Sigma_i \to K_n$ such that the realisation of each Σ_i is an n-sphere, and the realisation of the restriction of $\widetilde{\alpha}$ to Σ_i is homotopic to the restriction of $\widehat{\alpha}$ to the i-th component. The map $\widetilde{\alpha}$ extends to an equivariant simplicial map $\alpha_\sharp\colon \coprod_i G/H_i \times \Sigma_i \to K_n$ in the obvious fashion. Apply the barycentric subdivision functor to get a map $\alpha'_\sharp$. This is now an equivariant simplicial map between G-equivariantly locally ordered simplicial complexes. Then set

$$K_{n+1} := \left|\mathsf{Cyl}(\alpha'_\sharp) \cup_{\coprod_i \Sigma'_i} C(\Sigma'_i)\right|,$$

where $\mathsf{Cyl}(\alpha'_\sharp)$ denotes the simplicial mapping cylinder, and $C(\Sigma'_i)$ is the cone on Σ'_i. Note that $\dim(K_{n+1}) = n + 1$. The map f_{n+1} is now defined as follows: Let $f_{n+1}\big|_{X^{(n)}}$ be given by f_n composed with a G-homeomorphism h between $|K_n|$ and $|K'_n|$. Divide the $(n+1)$-disk D^{n+1} into a smaller $(n+1)$-disk d^{n+1} around 0, say of radius $\frac{1}{2}$, and the closure of the complement c^{n+1}. We can use the homotopy between $\widehat{\alpha}$ and its simplicial approximation to define f_{n+1} on $X^{(n)} \cup_{\coprod_i \alpha_i} G/H_i \times c^{n+1}$ (the time coordinate of the homotopy corresponds to moving along the line given by radial projection). Then define f_{n+1} on the interiors of the smaller disks d^{n+1} by the obvious extension to the cones on Σ'_i.

What is left to check is that the map f_{n+1} is a G-homotopy equivalence, so we can forget about the simplicial structures. Let L_{n+1} be the space obtained from K'_n by attaching the given $(n+1)$-cells along the map $h \circ f_n \circ \left(\coprod_i \alpha_i\right)$. Then we get an induced G-homotopy equivalence $g\colon X^{(n+1)} \to L_{n+1}$. Moreover, the deformation retraction from Lemma 4.2.14 induces, upon restriction to $t = 1$, a G-homotopy equivalence $e\colon K_{n+1} \to L_{n+1}$ such that $e \circ f_{n+1}$ and g are homotopic. In particular, f_{n+1} is a G-homotopy equivalence. □

This gives us finally the proof of Theorem 4.2.3: Set $\mathrm{bd}(n) := 4n + 2$. Then $\mathrm{bd} \in \mathcal{O}(n)$ is a monotone function. If $G \notin \mathcal{D}$, it follows that $r_G = 1$, so there is a finite, contractible G-CW-complex X' without fixed point whose dimension is bounded by $4 \cdot d(G) + 2 = \mathrm{bd}(d(G))$. By Proposition 4.2.12, we find a finite G-simplicial complex X of equal dimension which is G-homotopy equivalent to X'. In particular, X cannot have a global fixed point, so 4.2.3 has been proven.

5. Applications

5.1. An application in the algebraic K-theory of spaces

We would like to sketch an application of Corollary 4.2.4 in A-theory. This application does not yet use the bound on the dimension, so we will try to be brief, and omit some details. Recall the definition of $A(X)$ from [Wal85, Sec. 2.1]: For a topological space X, the category $\mathcal{R}^f(X)$ of *retractive spaces* over X consists of finite CW-complexes Y relative to X together with maps $X \xrightarrow{i} Y \xrightarrow{r} X$ such that $ri = \mathrm{id}_X$; morphisms are given by cellular maps $Y \to Y'$ which are compatible with the reference maps. The category $\mathcal{R}^f(X)$ becomes a Waldhausen category by picking $X \xrightarrow{=} X \xrightarrow{=} X$ as a zero object, declaring those morphisms which are isomorphic to cellular inclusions to be cofibrations, and picking the (weak) homotopy equivalences as weak equivalences. The A-theory of X is then defined to be $K(\mathcal{R}^f(X))$. This space can be delooped to a connective spectrum $A(X)$ as in [Wal85, p. 329f.].

Let G be a discrete group. Consider G as a one-object groupoid, then the space $BG = |NG|$ is a functorial model for the classifying space of G. We will use the following alternative description of $A(BG)$: Let $\mathcal{R}^f(W, G)$ be the category of cocompact free G-CW-complexes relative to the G-space W, and cellular G-equivariant maps relative to W. We declare morphisms isomorphic to cellular inclusions to be cofibrations, and let the (weak) homotopy equivalences be the category of weak equivalences (note that G-homotopy equivalences and homotopy equivalences coincide in this case since the action is free). This is again a Waldhausen category.

Propositions 2.1.1 and 2.1.4 in [Wal85] combine to show that there is a natural homotopy equivalence

$$hS_\bullet\mathcal{R}^f(BG) \xrightarrow{\sim} hS_\bullet\mathcal{R}^f(*, G).$$

We will therefore use the K-theory of $\mathcal{R}^f(*, G)$ as a model for the A-theory of BG.

Let $\mathcal{R}ep(G)$ be the category of *finite* pointed G-CW-complexes and pointed cellular equivariant maps between them. Let us agree that G acts from the right on these spaces. Define $co\mathcal{R}ep(G)$ to consist of all maps isomorphic to a cellular inclusion, and let $h\mathcal{R}ep(G)$ be the subcategory of those morphisms which are (ordinary) homotopy equivalences.

5.1.1 Lemma.
The quadruple $(\mathcal{R}ep(G), *, co\mathcal{R}ep(G), h\mathcal{R}ep(G))$ is a Waldhausen category.

□

5.1.2 Definition.
Call $\mathsf{Sw}^A(G) := K_0(\mathcal{R}ep(G))$ the *A-theoretic Swan group.*

Suppose that $H \leqslant G$ is a subgroup. Then there are functors

$$\begin{aligned} \operatorname{ind}_H^G \colon \mathcal{R}^f(*, H) \to \mathcal{R}^f(*, G), &\quad X \mapsto G_+ \wedge_H X, \\ \operatorname{res}_H^G \colon \mathcal{R}ep(H) \to \mathcal{R}ep(G), &\quad D \mapsto \operatorname{res}_H^G D. \end{aligned}$$

If $[G\colon H] < \infty$, we can also define

$$\begin{aligned} \operatorname{res}_H^G \colon \mathcal{R}^f(*, G) \to \mathcal{R}^f(*, H), &\quad X \mapsto \operatorname{res}_H^G X, \\ \operatorname{ind}_H^G \colon \mathcal{R}ep(H) \to \mathcal{R}ep(G), &\quad D \mapsto D \wedge_H G_+. \end{aligned}$$

Moreover, there are pairings

$$\begin{aligned} \wedge \colon \mathcal{R}ep(G) \times \mathcal{R}ep(G) \to \mathcal{R}ep(G), &\quad (D, D') \mapsto D \wedge D', \\ \wedge \colon \mathcal{R}ep(G) \times \mathcal{R}^f(*, G) \to \mathcal{R}^f(*, G), &\quad (D, X) \mapsto D \wedge X, \end{aligned}$$

where we equip the resulting space with the diagonal action in both cases (in the second case, $g \in G$ acts on D from the left via $g \cdot d := d \cdot g^{-1}$). The following lemma is straightforward (compare the discussion in [Wal82]).

5.1.3 Lemma.
Let G be a group, and $H \leqslant G$ a subgroup.

1. All restriction and induction functors we have just defined are exact (as long as they exist).
2. The functor $\wedge \colon \mathcal{R}ep(G) \times \mathcal{R}ep(G) \to \mathcal{R}ep(G)$ is biexact.
3. The functor $\wedge \colon \mathcal{R}ep(G) \times \mathcal{R}^f(*, G) \to \mathcal{R}^f(*, G)$ is biexact.

□

It follows from Lemma 5.1.3, using [Wal85, p. 342] that we have induced pairings

$$\begin{aligned} &\mathsf{Sw}^A(G) \times \mathsf{Sw}^A(G) \to \mathsf{Sw}^A(G), \\ &\mathsf{Sw}^A(G) \times A_n(BG) \to A_n(BG), \quad n \in \mathbb{N}, \end{aligned}$$

both of which we denote by a multiplication symbol.

5.1.4 Lemma (Frobenius reciprocity)**.**
Let G be a finite group and $H \leqslant G$ a subgroup. Then for all $s \in \mathsf{Sw}^A(H)$, $t \in \mathsf{Sw}^A(G)$ and $a \in A_n(BG)$ the following holds:

$$\begin{aligned} \operatorname{ind}_H^G(s) \cdot t &= \operatorname{ind}_H^G(s \cdot \operatorname{res}_H^G t), \\ \operatorname{ind}_H^G(s) \cdot a &= \operatorname{ind}_H^G(s \cdot \operatorname{res}_H^G a). \end{aligned}$$

Proof. Suppose that $D \in \mathcal{R}ep(H)$, $E \in \mathcal{R}ep(G)$ and $X \in \mathcal{R}^f(*, G)$. It suffices to observe that there are natural G-equivariant isomorphisms

$$\begin{aligned}(D \wedge_H G_+) \wedge E &\xrightarrow{\cong} (D \wedge \operatorname{res}^G_H E) \wedge_H G_+,\\ ((d,g),e) &\mapsto ((d, eg^{-1}), g).\end{aligned}$$

and

$$\begin{aligned}(D \wedge_H G_+) \wedge X &\xrightarrow{\cong} G_+ \wedge_H (D \wedge \operatorname{res}^G_H X)\\ ((d,g),x) &\mapsto (g^{-1}, (d, gx)).\end{aligned}$$

□

5.1.5 Proposition.
The canonical homomorphism

$$\sum_H \operatorname{ind}^G_H \colon \bigoplus_{\substack{H \leqslant G,\\ H \in \mathcal{D}}} \mathsf{Sw}^A(H) \to \mathsf{Sw}^A(G)$$

is surjective.

Proof. Suppose that the unit element 1_G lies in the image of the map. If this is the case, write $1_G = \sum_H \operatorname{ind}(s_H)$. Then we have by Lemma 5.1.4 for every $s \in \mathsf{Sw}^A(G)$

$$s = 1_G \cdot s = \left(\sum_H \operatorname{ind}(s_H)\right) \cdot s = \sum_H \big(\operatorname{ind}^G_H(s_H) \cdot s\big) = \sum_H \operatorname{ind}^G_H(s_H \cdot \operatorname{res}^G_H(s)),$$

and the latter element is clearly a member of the image of the induction map. Therefore, it suffices to show that $1_G = [S^0]$ lies in the image.

For the beginning, let X be an arbitrary object in $\mathcal{R}ep(G)$. We filter X by its skeleta $X^{(0)} \subseteq X^{(1)} \subseteq \ldots \subseteq X^{(n)} = X$. Two consecutive filtration steps give rise to a cofibration sequence $X^{(k)} \rightarrowtail X^{(k+1)} \twoheadrightarrow X^{(k+1)}/X^{(k)}$. Let I_{k+1} be an indexing set for the equivariant $(k+1)$-cells of X. Then

$$X^{(k+1)}/X^{(k)} \cong \bigvee_{i \in I_{k+1}} (K_i \backslash G)_+ \wedge S^{k+1} \cong \Sigma^{k+1}\Big(\bigvee_{i \in I_{k+1}} (K_i \backslash G)_+\Big),$$

which gives us the relation $[X^{(k+1)}] = [X^{(k)}] + \big[\Sigma^{k+1}\big(\bigvee_{i \in I_{k+1}} (K_i \backslash G)_+\big)\big]$.

Since $Y \rightarrowtail CY \twoheadrightarrow \Sigma Y$ is a cofibration sequence for every object $Y \in \mathcal{R}ep(G)$, and $CY \simeq *$, we have $[\Sigma Y] = [*] - [Y] = -[Y]$ for every Y. It follows by induction that $[\Sigma^k Y] = (-1)^k [Y]$.

Again by induction, we see that the class $[X] \in \mathsf{Sw}^A(G)$ is given by

$$[X] = [X^{(0)} + \sum_{k=1}^{n} (-1)^k [\Big(\bigvee_{i \in I_k} (K_i \backslash G)_+\Big)] = [X^{(0)}] + \sum_{k=1}^{n} (-1)^k \sum_{i \in I_k} [(K_i \backslash G)_+].$$

The 0-skeleton $X^{(0)}$ is itself a pointed (right) G-set, so we can write it as a wedge sum of pointed G-sets $X^{(0)} = \bigvee_{i \in I_0} (K_i \backslash G)_+$. This shows that the class of X in $\mathsf{Sw}^A(G)$ equals its Euler characteristic:

$$[X] = \sum_{k=0}^{n} (-1)^k \sum_{i \in I_k} [(K_i \backslash G)_+].$$

Now let X be a finite, contractible G-simplicial complex whose isotropy groups lie in $\mathcal{D}$; such a complex exists by Corollary 4.2.4. Then we have

$$1_G = [S^0] = [X_+].$$

Each equivariant cell in X is of type $H \backslash G$ for some $H \in \mathcal{D}$. Moreover, if $H \in \mathcal{D}$, the pointed G-set $(H \backslash G)_+$ equals $\operatorname{ind}_H^G((H \backslash H)_+)$. Since the class of $[X_+]$ is given by its Euler characteristic, this proves that $1_G = [X_+]$ lies in the image of the induction homomorphism. □

5.1.6 Corollary.
Let G be a finite group. Then the homomorphism

$$\sum_H \operatorname{ind}_H^G \colon \bigoplus_{\substack{H \leqslant G, \\ H \in \mathcal{D}}} A_n(BH) \to A_n(BG)$$

is surjective for all $n \in \mathbb{N}$.

Proof. Write $1_G = \sum_H \operatorname{ind}_H^G(s_H) \in \mathsf{Sw}^A(G)$. Then for every $a \in A_n(BG)$ we have by Lemma 5.1.4

$$a = 1_G \cdot a = \Big(\sum_H \operatorname{ind}_H^G(s_H) \Big) \cdot a = \sum_H \big(\operatorname{ind}_H^G(s_H) \cdot a \big) = \sum_H \operatorname{ind}_H^G(s_H \cdot \operatorname{res}_H^G(a)).$$

□

5.1.7 Remark.
Arguing as in the case of linear algebraic K-theory, Corollary 5.1.6 can be strengthened to give a statement about isomorphism of groups if we replace the domain by an appropriate homology theory. Define an $\mathsf{Or}(G)$-spectrum $\mathbb{A}$ by

$$\mathbb{A}(G/H) := A(|N(\mathsf{Tr}_G(G/H))|),$$

where $\mathsf{Tr}_G(\ \ )$ denotes the transport groupoid. Note that $\mathsf{Tr}_G(G/H)$ is equivalent to the one-object groupoid H, so $|N(\mathsf{Tr}_G(G/H))| \simeq BH$; consequently, both spaces have the same A-theory, see [Wal85, Prop. 2.1.7]. Then the assembly map

$$H_*^G(E_{\mathcal{D}}G; \mathbb{A}) \to A_*(BG)$$

is an isomorphism for all finite groups G: This can be seen by invoking Theorem 2.9 from [BL07].

Proposition 5.1.5 also has a consequence in the linear setting. To state this, recall the linear Swan group: We can consider any discrete group G as a groupoid $\underline{G}$. The category of integral G-representations $\mathcal{R}ep^{\mathbb{Z}}(G)$ can then be defined to be

$$\mathcal{R}ep^{\mathbb{Z}}(G) := \mathsf{Fun}(\underline{G}, \mathsf{Fr}_{\mathbb{Z}}),$$

where $\mathsf{Fr}_{\mathbb{Z}}$ is the category of finitely generated free $\mathbb{Z}$-modules. Equipping $\mathsf{Fr}_{\mathbb{Z}}$ with the split exact structure, $\mathcal{R}ep^{\mathbb{Z}}(G)$ inherits an exact structure from $\mathsf{Fr}_{\mathbb{Z}}$. The *linear Swan group of* G is then

$$\mathsf{Sw}(G) := K_0(\mathcal{R}ep^{\mathbb{Z}}(G)).$$

A classical result of Swan [Swa60, Cor. 4.2] states that the induction map

$$\sum_H \mathrm{ind}_H^G \colon \bigoplus_{\substack{H \leqslant G, \\ H \in \mathcal{H}}} \mathsf{Sw}(G) \to \mathsf{Sw}(G)$$

is a surjection, where $\mathcal{H}$ is the family of hyperelementary subgroups. We can prove the following variation of this result:

5.1.8 Corollary.
Let G be a finite group. Under the canonical homomorphism

$$\sum_H \mathrm{ind}_H^G \colon \bigoplus_{\substack{H \leqslant G, \\ H \in \mathcal{D}}} \mathsf{Sw}(H) \to \mathsf{Sw}(G),$$

the unit element $1_G \in \mathsf{Sw}(G)$ has a preimage which can be represented by permutation modules. In particular, this homomorphism is surjective.

Proof. Surjectivity follows again from the fact that 1_G lies in the image of the map, but is also obvious from Swan's result.

Consider the linearisation map defined by

$$\begin{aligned} \lambda \colon \mathsf{Sw}^A(G) &\to \mathsf{Sw}(G) \\ [X] &\mapsto \sum_{k=0}^{\infty} \big[\widetilde{C}_k(X)\big], \end{aligned}$$

where $\widetilde{C}_*(X)$ denotes the reduced cellular chain complex. This map is well-defined: We have seen in the proof of Proposition 5.1.5 that the class $[X] \in \mathsf{Sw}^A(G)$ is given by the Euler characteristic. It is easy to check that λ is additive, preserves the products, and that the unit $[S^0] \in \mathsf{Sw}^A(G)$ is sent to the unit $[\mathbb{Z}] \in \mathsf{Sw}(G)$.

Moreover, λ is compatible with induction, so for every subgroup $H \leqslant G$, we get a commutative diagram

$$\begin{array}{ccc} \mathsf{Sw}^A(H) & \xrightarrow{\mathrm{ind}_H^G} & \mathsf{Sw}^A(G) \\ \lambda_H \big\downarrow & & \big\downarrow \lambda_G \\ \mathsf{Sw}(H) & \xrightarrow[\mathrm{ind}_H^G]{} & \mathsf{Sw}(G) \end{array}$$

Now the only thing left to observe is that the image of λ consists entirely of elements which are represented by permutation modules. $\square$

5.1.9 Remark.
Corollary 5.1.8 can be given the following slightly more conceptual proof:

Define a "chain complex version" of the Swan group (cf. [BR05, Sec. 8.3]): Let

$$\mathcal{R}ep^{lin}(G) := \mathsf{Ch}(\mathsf{Fun}(\underline{G}, \mathsf{Fr}_{\mathbb{Z}})).$$

This category can be endowed with a Waldhausen structure; cofibrations are the degreewise admissible monomorphisms, and the weak equivalences $h\mathcal{R}ep^{lin}(G)$ are defined by pulling back the weak equivalences in $\mathsf{Ch}(\mathsf{Fr}_{\mathbb{Z}})$ along the exact restriction functor

$$\mathrm{res}_1^G \colon \mathcal{R}ep^{lin}(G) \to \mathcal{R}ep^{lin}(1) = \mathsf{Ch}(\mathsf{Fun}(\underline{1}, \mathsf{Fr}_{\mathbb{Z}})) \cong \mathsf{Ch}(\mathsf{Fr}_{\mathbb{Z}}).$$

Taking tensor products of complexes defines a biexact functor

$$\mathcal{R}ep^{lin}(G) \times \mathcal{R}ep^{lin}(G) \to \mathcal{R}ep^{lin}(G),$$

which turns $\mathsf{Sw}^{ch}(G) := K_0(\mathcal{R}ep^{lin}(G))$ into a ring.

The inclusion functor $\mathcal{R}ep^{\mathbb{Z}}(G) \hookrightarrow \mathcal{R}ep^{lin}(G)$ induces a ring homomorphism

$$j \colon \mathsf{Sw}(G) \to \mathsf{Sw}^{ch}(G).$$

Then there is a linearisation *functor* $L \colon \mathcal{R}ep(G) \to \mathcal{R}ep^{lin}(G)$ which sends an object X to its reduced cellular chain complex $\widetilde{C}_*(X)$. From here on, the argument proceeds as before. However, this argument needs the additional input that j is an isomorphism.

5.2. An application in linear algebraic K-theory

We are now going to turn to an application of Corollary 4.2.4 in the context of the K-theoretic Farrell-Jones Conjecture. This time, we do require the bound on the dimension.

Using a criterion called the *Farrell-Hsiang condition*, Bartels, Farrell and Lück show in [BFL14] that virtually polycyclic groups satisfy the Farrell-Jones Conjecture. The special case of semidirect products of the form $\mathbb{Z}^n \rtimes \mathbb{Z}$ is further elaborated on in [Bar], where the Farrell-Hsiang condition bears the name "Theorem C".

For this particular case, we will now proceed to show that it is possible to bypass the Farrell-Hsiang condition, and instead prove the conjecture for $\mathbb{Z}^n \rtimes \mathbb{Z}$ by means of [Bar, Theorem A]. This criterion originally served the purpose of proving the Farrell-Jones Conjecture for hyperbolic groups, and went under the name of "transfer reducibility".

In order to give the statement of "Theorem A", we have to recall the definition of the ℓ^1-metric on a simplicial complex. Given a simplicial complex X, its realisation $|X|$ carries a metric d^1 given by

$$d^1\big(\sum_x \lambda_x \cdot x, \sum_x \mu_x \cdot x\big) := \sum_x |\lambda_x - \mu_x|\,.$$

If X is a G-simplicial complex, this is a G-invariant metric (i.e. G acts by isometries).

5.2.1 Theorem ([Bar, Thm. A]).
Let G be a group generated by the finite set S. Let $\mathcal{F}$ be a family of subgroups of G. Let $\mathcal{A}$ be a small additive category with right G-action. Assume that there is $N \in \mathbb{N}$ such that for every $\varepsilon > 0$ there are

- an *N-transfer space* X equipped with a G-action, i.e. a compact contractible metric space such that for every $\delta > 0$ there is a simplicial complex K of dimension at most N together with maps $i\colon X \to K$ and $p\colon K \to X$, and a homotopy $H\colon p \circ i \simeq \mathrm{id}_X$ such that the diameter of $\{H(x,t) \mid t \in [0,1]\}$ is at most δ for all $x \in X$,
- a G-simplicial complex E of dimension at most N whose isotropy groups lie in $\mathcal{F}$,
- a map $f\colon X \to E$ which is G-equivariant up to ε in the sense that for all $s \in S$ and $x \in X$,
$$d^1(f(sx), sf(x)) \leqslant \varepsilon.$$

Then the assembly map $\alpha_{\mathcal{F}}\colon H_*^G(E_{\mathcal{F}}G; \mathbb{K}_{\mathcal{A}}^{-\infty}) \to K_*(\mathcal{A}[G])$ is an isomorphism.

□

Our goal is to show that we can produce the required data for every group of the form $\mathbb{Z}^n \rtimes \mathbb{Z}$, i.e. we want to prove:

5.2.2 Theorem.
Every group of the form $\mathbb{Z}^n \rtimes_M \mathbb{Z}$ satisfies the assumptions of Theorem 5.2.1 with respect to the family of abelian subgroups.

The proof of this theorem occupies the rest of this section. The basic idea goes as follows: We obtain transfer spaces by taking a finite quotient F of $\mathbb{Z}^n \rtimes_M \mathbb{Z}$, picking an F-simplicial complex X as in Corollary 4.2.4, and restricting the group action along the quotient map $\mathbb{Z}^n \rtimes_M \mathbb{Z} \twoheadrightarrow F$. If we are sufficiently careful about the choice of F (see Proposition 5.2.5), we can arrange things so that we can construct almost equivariant maps on the 0-skeleton of the transfer space - this is asserted essentially by Propositions 5.2.7 and 5.2.8. After that, we have to extend these to higher skeleta (see 5.2.9).

We start by proving the existence of certain "normal forms" of extensions for groups in $\mathcal{C}yc_p$ and $\mathcal{D}$.

5.2.3 Lemma.
Let $H \in \mathcal{C}yc_p$. Then there is a group extension $1 \to P \to H \to C \to 1$ with P a p-group, $C \in \mathcal{C}yc$ and $p \nmid |C|$.

Proof. Let $1 \to P' \to H \xrightarrow{\pi} C' \to 1$ be an extension with P a p-group and C' a finite cyclic group. Let S_p be the Sylow p-subgroup of C. Then $C' \cong C \times S_p$. Let $P := \pi^{-1}(S_p) \trianglelefteq H$. Then we have a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & P' & \longrightarrow & P & \longrightarrow & S_p & \longrightarrow & 1 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & P' & \longrightarrow & H & \xrightarrow{\pi} & C' & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H/P & \xrightarrow{\cong} & C & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

P is a p-group because it is an extension of the p-group P' by S_p. Since $H/P \cong C$, this group is cyclic and its order is not divisible by p. □

5.2.4 Lemma.
Let $G \in \mathcal{D}$. Suppose that there is a group extension $1 \to H' \to G \to Q' \to 1$ with $H \in \mathcal{C}yc_p$, Q a q-group and $p \neq q$.

Then there is a group extension $1 \to H \to G \to Q \to 1$ with $H \in \mathcal{C}yc_p$, Q a q-group and $q \nmid |H|$.

Proof. Let $1 \to P \to H' \xrightarrow{\pi} C' \to 1$ be an extension with P a p-group and C' a finite cyclic group. Let S_q be the Sylow q-group of C, so that $C' \cong C \times S_q$. Let $H := \pi^{-1}(C)$. Then we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & & & 1 & & 1 & & \\
 & & & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & P & \longrightarrow & H & \longrightarrow & C & \longrightarrow & 1 \\
 & & \downarrow{=} & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & P & \longrightarrow & H' & \xrightarrow{\pi} & C' & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & H'/H & \xrightarrow{\cong} & S_q & \longrightarrow & 1 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 1 & & 1 & &
\end{array}$$

It follows that H is cyclic mod p, and the order of H is not divisible by q. We claim that H is normal in G: Let $g \in G$ and $h \in H$. Suppose $ghg^{-1} \notin H$. We know that ghg^{-1} must lie in H' because H' is normal in G. Since $q \nmid |H|$, the order of h (and therefore also of ghg^{-1}) is not divisible by q. Then the cyclic group generated by ghg^{-1} surjects onto a non-trivial cyclic group in $H'/H \cong S_q$. This implies that $q \mid \left|ghg^{-1}\right|$, which is a contradiction.

Thus, we have an extension $1 \to H \to G \to G/H =: Q \to 1$. We can compute the order of Q:

$$|Q| = \frac{|G|}{|H|} = \frac{|H'| \cdot |Q'|}{|P| \cdot |C|} = \frac{|H'| \cdot |Q'| \cdot |S_q|}{|P| \cdot |C'|} = \frac{|H'| \cdot |Q'| \cdot |S_q|}{|H'|} = |Q'| \cdot |S_q|\,,$$

and this is clearly a power of q. □

The following proposition is a variant of the results in Section 3.4 of [BFL14]; our proof is also a direct adaptation of the arguments given there.

5.2.5 Proposition.
Let $n \in \mathbb{N}$. Then there is a natural number B such that for all $M \in GL_n(\mathbb{Z})$ and $\nu \in \mathbb{N}$ there are natural numbers r and s (which depend only on n and ν) with the following properties:

- The order of $GL_n(\mathbb{Z}/s)$ divides r; so the semidirect product $(\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r$ is defined, $\overline{M}$ denoting the reduction of M modulo s. Let $\pi\colon (\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r \twoheadrightarrow \mathbb{Z}/r$ be the projection.
- The order of $(\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r$ contains at most B prime factors, counted with their multiplicities.

- For all subgroups $G \leqslant (\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r$ which lie in the Dress family $\mathcal{D}$, there is some $\nu' \geqslant \nu$ such that $G \cap (\mathbb{Z}/s)^n \subseteq \nu'(\mathbb{Z}/s)^n$ or $\pi(G) \subseteq \nu'(\mathbb{Z}/r)$.

Proof. We start with some preparatory observations. Suppose s is a product of pairwise distinct prime numbers $s = p_1 \dots p_k$. Then $\mathbb{Z}/s \cong \mathbb{Z}/p_1 \times \cdots \times \mathbb{Z}/p_k$. The diagram

$$\begin{array}{ccc} M_n(\mathbb{Z}/s) & \xrightarrow{\cong} & M_n(\mathbb{Z}/p_1) \times \cdots \times M_n(\mathbb{Z}/p_k) \\ \det_s \downarrow & & \downarrow \det_{p_1} \times \cdots \times \det_{p_k} \\ \mathbb{Z}/s & \xrightarrow{\cong} & \mathbb{Z}/p_1 \times \cdots \times \mathbb{Z}/p_k \end{array}$$

commutes. Since $(\mathbb{Z}/p_1 \times \dots \mathbb{Z}/p_k)^* = (\mathbb{Z}/p_1)^* \times \cdots \times (\mathbb{Z}/p_k)^*$, the top homomorphism (co)restricts to an isomorphism

$$GL_n(\mathbb{Z}/s) \xrightarrow{\cong} GL_n(\mathbb{Z}/p_1) \times \cdots \times GL_n(\mathbb{Z}/p_k).$$

For a single prime p, the order of $GL_n(\mathbb{Z}/p)$ is $(p^n - 1)(p^n - p) \dots (p^n - p^{n-1})$. Let $o_n(X)$ denote the polynomial

$$o_n(X) = (X^n - 1)(X^n - X) \dots (X^n - X^{n-1}) \in \mathbb{Z}[X].$$

The order of $GL_n(\mathbb{Z}/s)$ is then given by

$$|GL_n(\mathbb{Z}/s)| = o_n(p_1) \dots o_n(p_k).$$

We would like to bound the number of prime factors (counted with their multiplicities) appearing in $|GL_n(\mathbb{Z}/s)|$. This can be accomplished by appealing to the following number-theoretic result.

5.2.6 Theorem ([Mie65]).
Let $f(X) \in \mathbb{Z}[X]$ be a polynomial. Then there is a constant $C > 0$ such that there are infinitely many primes p for which the number of prime factors of $f(p)$, counted with their multiplicities, is bounded by C. □

The result of [Mie65] is even stronger: We can additionally fix integers r and m with $(r, m) = 1$, and still find infinitely many primes of this sort such that $f(p)$ is also congruent to r modulo m. We will not have to use this.

We apply the theorem to $o_n(X)$, so there is a constant C such that the set $\mathcal{P}$ of prime numbers p for which $|GL_n(\mathbb{Z}/p)| = o_n(p)$ has at most C prime factors is infinite. If we pick the primes $p_1, \dots, p_k$ from $\mathcal{P}$, it follows that the order of $GL_n(\mathbb{Z}/s)$ contains at most $k \cdot C$ prime factors (counted with their multiplicities).

We are now ready for the proof of the proposition. Pick three pairwise distinct primes p_1, p_2 and p_3 from $\mathcal{P}$, each one of them greater than ν. Set $s := p_1p_2p_3$, and let $r := s \cdot |GL_n(\mathbb{Z}/s)|$. Since $\left|\overline{M}\right| \mid |GL_n(\mathbb{Z}/s)|$, the semidirect product $(\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r$ is defined. Moreover, we have

$$|(\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r| = |(\mathbb{Z}/s)^n| \cdot |\mathbb{Z}/r| = (p_1p_2p_3)^{n+1} \cdot |GL_n(\mathbb{Z}/s)|,$$

so the order of the semidirect product contains at most $3(n+1)+3\cdot C = 3\cdot(C+n+1)$ prime factors (this is the B whose existence was claimed). We are left with checking the index estimates for subgroups which lie in the Dress family $\mathcal{D}$.

Let us first consider the case of a subgroup $H \leqslant (\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r$ which is cyclic mod p for some prime p. By Lemma 5.2.3, there is an extension $1 \to P \to H \to C \to 1$ such that P is a p-group, C is a finite cyclic group and $p \nmid |C|$. Let $c \in C$ be a generator, and pick a preimage $vt^j \in H$ of c; i.e. $v \in (\mathbb{Z}/s)^n$ and $\mathbb{Z}/r = \langle t \rangle$. Since p does not divide $|C|$, the element $(vt^j)^{|P|}$ gets mapped to another generator $d = c^{|P|}$ of C. Write $(vt^j)^{|P|}$ as wt^l. Put $x := (wt^l)^{[\pi(H) \colon \pi(P)]} \in H$.

Suppose that $H \cap (\mathbb{Z}/s)^n \neq P \cap (\mathbb{Z}/s)^n$. Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & P \cap (\mathbb{Z}/s)^n & \longrightarrow & H \cap (\mathbb{Z}/s)^n & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & P & \longrightarrow & H & \longrightarrow & C & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi(P) & \longrightarrow & \pi(H) & \longrightarrow & \pi(H)/\pi(P) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}$$

All rows and columns are exact. The element x lies in the kernel of π because

$$x = ((vt^j)^{|P\cap(\mathbb{Z}/s)^n|\cdot|\pi(P)|})^{[\pi(H) \colon \pi(P)]} = (vt^j)^{|\pi(H)|\cdot|P\cap(\mathbb{Z}/s)^n|}.$$

So $x \in H \cap (\mathbb{Z}/s)^n$, and its image in Q is $d^{[\pi(H) \colon \pi(P)]} \neq 0$. It follows that x does not lie in $P \cap (\mathbb{Z}/s)^n$.

Let $s' := |x|$. We calculate

$$x^{|Q|} = (vt^j)^{|\pi(H)|\cdot|P\cap(\mathbb{Z}/s)^n|\cdot|Q|} = (vt^j)^{|\pi(H)|\cdot|H\cap(\mathbb{Z}/s)^n|} = (vt^j)^{|H|} = e,$$

so s' has to divide the order of Q. It follows that $p \nmid s'$ (because $Q \leqslant C$ and $p \nmid |C|$).

Note that $s' \mid s$; we write $s = \sigma \cdot s'$. Since every prime factor of s has multiplicity 1, the numbers σ and s' are coprime. Let k be the product of the powers of all primes appearing in $r' := |GL_n(\mathbb{Z}/s)|$ but not in s'. Then we still have $(k\sigma, s') = 1$. It follows that $(wt^l)^{k\sigma[\pi(H) \colon \pi(P)]} = x^{k\sigma} \neq 0 \in H \cap (\mathbb{Z}/s)^n$. On the other hand, we compute (see [BFL14, Lem. 3.19]):

$$
\begin{aligned}
(wt^l)^{sr'} &= \Big(\sum_{i=0}^{sr'-1} \overline{M}^{il}(w)\Big)t^{lsr'} \overset{sr'=r}{=} \Big(\sum_{i=0}^{s-1}\sum_{j=0}^{r'-1} \overline{M}^{(j+ir')l}(w)\Big) = \Big(\sum_{i=0}^{s-1}\sum_{j=0}^{r'-1} \overline{M}^{jl}(w)\Big) \\
&= \Big(s \cdot \sum_{j=0}^{r'-1} \overline{M}^{jl}(w)\Big) = \Big(\sum_{j=0}^{r'-1} \overline{M}^{jl}(sw)\Big) = \Big(\sum_{j=0}^{r'-1} \overline{M}^{jl}(0)\Big) = e.
\end{aligned}
$$

So $sr' \nmid k\sigma[\pi(H)\colon \pi(P)]$. Dividing by $k\sigma$ on both sides, we get $s'\overline{r} \nmid [\pi(H)\colon \pi(P)]$, where $\overline{r}$ is some natural number containing only prime factors which are also prime factors of s'. Therefore, there is some $i \in \{1,2,3\}$ and a natural number $N \geqslant 1$ such that $p_i \mid s'$, $p_i^N \mid s'r$, and $p_i^N \nmid [\pi(H)\colon \pi(P)]$.

As $p \nmid s'$, we must have $p \neq p_i$. Clearly, $s' \mid |H|$, so p_i is a divisor of $|H|$. Since we set $r = sr'$, the number r must be divisible by p_i^N. Then $[\mathbb{Z}/r\colon \pi(P)]$ is still divisible by p_i^N because $p \neq p_i$. From the equality

$$[\mathbb{Z}/r\colon \pi(H)] \cdot [\pi(H)\colon \pi(P)] = [\mathbb{Z}/r\colon \pi(P)],$$

it follows that p_i must appear at least once as a prime factor of $[\mathbb{Z}/r\colon \pi(H)]$. Consequently, $[\mathbb{Z}/r\colon \pi(H)] \geqslant p_i \geqslant \nu$.

So far, we have shown that for every group H which is cyclic mod p for some prime p, we can find an extension $1 \to P \to H \to C \to 1$ with $C \in \mathcal{C}yc$, P a p-group such that $p \nmid |C|$, and one of the following statements is true:

- $H \cap (\mathbb{Z}/s)^n = P \cap (\mathbb{Z}/s)^n$.
- There is an $i \in \{1,2,3\}$ such that $p \neq p_i$, $p_i \mid |H|$ and $p_i \mid [\mathbb{Z}/r\colon \pi(H)]$.

Let now $G \leqslant (\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r$ be a subgroup which lies in $\mathcal{D}$. Using Lemma 5.2.4, write G as an extension $1 \to H \to G \to Q \to 1$ such that $H \in \mathcal{C}yc_p$ for some prime p, the quotient Q is a q-group for some prime q, and we may assume $q \nmid |H|$ if $p \neq q$. As before, we obtain a commutative diagram

$$
\begin{array}{ccccccccc}
 & & 1 & & 1 & & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H \cap (\mathbb{Z}/s)^n & \longrightarrow & G \cap (\mathbb{Z}/s)^n & \longrightarrow & Q' & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & Q & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & \pi(H) & \longrightarrow & \pi(G) & \longrightarrow & \pi(G)/\pi(H) & \longrightarrow & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array}
$$

with exact rows and columns. It follows that $[G \cap (\mathbb{Z}/s)^n : H \cap (\mathbb{Z}/s)^n]$ is a power of q, and that $[\pi(G) : \pi(H)]$ is also a q-power. Using what we have done so far, we write H as an extension $1 \to P \to H \to C \to 1$ with the properties we had just listed.

Assume first that $H \cap (\mathbb{Z}/s)^n = P \cap (\mathbb{Z}/s)^n$. Then $|G \cap (\mathbb{Z}/s)^n| = p^k q^l$ for some natural numbers k and l. Pick $i \in \{1, 2, 3\}$ such that $p \neq p_i \neq q$. Choose an arbitrary element $(g_1, \dots, g_n) \in G \cap (\mathbb{Z}/s)^n$, and write $\mathbb{Z}/s = \langle \gamma \rangle$. Then $g_j = \gamma^{a_j}$ for some a_j, and $\gamma^{a_j p^k q^l} = e$. Since $s = p_1 p_2 p_3$, the exponent a_j must be divisible by p_i. That is, the subgroup $G \cap (\mathbb{Z}/s)^n$ must be contained in the subgroup $p_i(\mathbb{Z}/s)^n \leqslant (\mathbb{Z}/s)^n$.

Consider now the case that there is an $i \in \{1, 2, 3\}$ such that $p \neq p_i$, $p_i \mid |H|$ and $p_i \mid [\mathbb{Z}/r : \pi(H)]$. If $q = p$, the inequality $q \neq p_i$ is automatic. If $q \neq p$, we can still deduce that $q \neq p_i$ because q does not divide the order of H, but p_i does. Looking at the equality

$$[\mathbb{Z}/r : \pi(G)] \cdot [\pi(G) : \pi(H)] = [\mathbb{Z}/r : \pi(H)],$$

it follows that p_i must divide $[\mathbb{Z}/r : \pi(G)]$, so that $\pi(G) \subseteq p_i(\mathbb{Z}/r)$. This finishes the proof. □

We quote the following results, which will allow us to produce almost equivariant maps on the 0-skeleton of our transfer spaces.

5.2.7 Proposition ([Bar, Prop. 6.7]).
Let $S \subseteq \mathbb{Z}^n \rtimes_M \mathbb{Z}$ be finite. There is a natural number N (depending only on n) with the property that for every $\varepsilon > 0$ there is a natural number ν_0 such that for all $\nu \geqslant \nu_0$ the following holds:

If $\overline{G} \leqslant \mathbb{Z}^n \rtimes_M \mathbb{Z}$ is a subgroup which is contained in $(\nu\mathbb{Z})^n \rtimes_M \mathbb{Z}$, then there is an N-dimensional $(\mathbb{Z}^n \rtimes_M \mathbb{Z})$-simplicial complex E with cyclic isotropy groups and a $\overline{G}$-equivariant map $f\colon \mathbb{Z}^n \rtimes_M \mathbb{Z} \to E$ such that $d^1(f(g), f(h)) \leqslant \varepsilon$ whenever $g^{-1}h \in S$. □

5.2.8 Proposition ([Bar, Prop. 6.6]).
Let $S \subseteq \mathbb{Z}^n \rtimes_M \mathbb{Z}$ be finite. For every $\varepsilon > 0$ there is a natural number ν_0 such that for all $\nu \geqslant \nu_0$ the following holds:

If $\overline{G} \leqslant \mathbb{Z}^n \rtimes_M \mathbb{Z}$ is a subgroup which is contained in $\mathbb{Z}^n \rtimes_M (\nu\mathbb{Z})$, then there is a 1-dimensional $(\mathbb{Z}^n \rtimes_M \mathbb{Z})$-simplicial complex E with abelian isotropy groups and a $\overline{G}$-equivariant map $f\colon \mathbb{Z}^n \rtimes_M \mathbb{Z} \to E$ such that $d^1(f(g), f(h)) \leqslant \varepsilon$ whenever $g^{-1}h \in S$. □

We claim that these almost equivariant maps can be patched together in such a way that we obtain almost equivariant maps on the entire transfer space.

5.2.9 Lemma.
Let G be a group, $S \subseteq G$ a finite subset, and let X be a locally finite G-simplicial complex. Let R be a set of resolution data for X such that every $R(x)$ is locally finite. Let $\varepsilon > 0$. Suppose that $f_0\colon X^{(0)} = X_0 \to |X[R]|$ is a map which is G-equivariant up to ε.

Then there is a map $f\colon |X| \to |X[R]|$ extending f_0 which is G-equivariant up to ε.

Proof. We use Lemma 4.1.5 another time to define f as a map $|X| \to |X, R|$ by

$$f\big(\sum_{x \in X_0} \lambda_x \cdot x\big) := \sum_{x \in X_0} \lambda_x \cdot f_0(x).$$

In order to show that this map is G-equivariant up to ε, we have to determine how the ℓ^1-metric is given on $|X, R|$. This will be accomplished by mapping two arbitrary points in $|X, R|$ via the map F' from the proof of Lemma 4.1.5 to $|X[R]|$ and calculating the distance of the image points (i.e., we are forcing F and F' from 4.1.5 to be isometries; we can do this since the ℓ^1-metric induces the usual topology on the realisation of a locally finite simplicial complex, see [LW69, p. 130]).

Let $\eta = \sum_x \lambda_x \cdot \eta_x$ and $\vartheta = \sum_x \mu_x \cdot \vartheta_x$ be two points in $|X, R|$. Then

$$\begin{aligned} d^1\big(F'\big(\sum_{x \in X_0} \lambda_x \cdot \eta_x\big), F'\big(\sum_{x \in X_0} \mu_x \cdot \vartheta_x\big)\big) \\ &= d^1\big(\sum_{y \in X[R]_0} \lambda_{x(y)} \eta_{x(y),y} \cdot y, \sum_{y \in X[R]_0} \mu_{x(y)} \vartheta_{x(y),y} \cdot y\big) \\ &= \sum_{y \in X[R]_0} \left|\lambda_{x(y)} \eta_{x(y),y} - \mu_{x(y)} \vartheta_{x(y),y}\right|. \end{aligned}$$

This is the only precise statement we can make. However, this value can always be bounded:

$$\begin{aligned} \sum_{y \in X[R]_0} & \left|\lambda_{x(y)} \eta_{x(y),y} - \mu_{x(y)} \vartheta_{x(y),y}\right| \\ &= \sum_{y \in X[R]_0} \left|\lambda_{x(y)} \eta_{x(y),y} - \lambda_{x(y)} \vartheta_{x(y),y} + \lambda_{x(y)} \vartheta_{x(y),y} - \mu_{x(y)} \vartheta_{x(y),y}\right| \\ &\leqslant \sum_{y \in X[R]_0} \left|\lambda_{x(y)} \eta_{x(y),y} - \lambda_{x(y)} \vartheta_{x(y),y}\right| + \sum_{y \in X[R]_0} \left|\lambda_{x(y)} \vartheta_{x(y),y} - \mu_{x(y)} \vartheta_{x(y),y}\right| \\ &= \sum_{x \in X_0} \Big(\lambda_x \cdot \sum_{y \in R(x)_0} |\eta_{x,y} - \vartheta_{x,y}|\Big) + \sum_{x \in X_0} |\lambda_x - \mu_x| \Big(\sum_{y \in R(x)_0} \vartheta_{x,y}\Big) \\ &= \sum_{x \in X_0} \lambda_x \cdot d^1(\eta_x, \vartheta_x) + \sum_{x \in X_0} |\lambda_x - \mu_x| \end{aligned}$$

Let $s \in S$. Then we deduce

$$\begin{aligned} d^1\big(f\big(s\big(\sum_{x \in X_0} \lambda_x \cdot x\big)\big), sf\big(\sum_{x \in X_0} \lambda_x \cdot x\big)\big) \\ &= d^1\big(\sum_{x \in X_0} \lambda_x \cdot f_0(sx), \sum_{x \in X_0} \lambda_x \cdot sf_0(x)\big) \\ &\leqslant \sum_{x \in X_0} \lambda_x \cdot d^1(f_0(sx), sf_0(x)) + \sum_{x \in X_0} |\lambda_x - \lambda_x| \\ &\leqslant \sum_{x \in X_0} \lambda_x \varepsilon \\ &= \varepsilon. \end{aligned}$$

□

Everything is now in place to prove Theorem 5.2.2.

Proof of Theorem 5.2.2. Fix some finite generating set S, and let $\varepsilon > 0$. Let $\nu \in \mathbb{N}$ be larger than the maximum of the two numbers dubbed ν_0 in Propositions 5.2.7 and 5.2.8. By Proposition 5.2.5, there is a natural number B (depending on n) and natural numbers r and s (depending on n and ν) with the following properties:

- The order of $GL_n(\mathbb{Z}/s)$ divides r, so $(\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r$ is defined.
- The order of $(\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r$ contains at most B prime factors (counted with multiplicities).
- For all subgroups $G \leqslant (\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r$ which lie in $\mathcal{D}$, there is some $\nu' \geqslant \nu$ such that $G \cap (\mathbb{Z}/s)^n \subseteq \nu'(\mathbb{Z}/s)^n$ or $\pi(G) \subseteq \nu'(\mathbb{Z}/r)$.

The second property implies that $d((\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r)$ is at most $B+1$. We apply the corollary to Oliver's theorem to obtain a finite, (non-equivariantly) contractible $((\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r)$-simplicial complex X whose dimension is bounded by $\beta := 2^{B+1}\,\mathrm{bd}(B+1)^{B+1}$, and whose isotropy groups lie in $\mathcal{D}$. This complex qualifies as an N-transfer space for every $N \geqslant \beta$ (we can take $X \xrightarrow{=} X \xrightarrow{=} X$ as a finite δ-controlled domination for every $\delta > 0$). Define an action of $\mathbb{Z}^n \rtimes_M \mathbb{Z}$ on X by restricting along the projection map $\mathbb{Z}^n \rtimes_M \mathbb{Z} \twoheadrightarrow (\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r$.

Let $X_0 = \coprod_{\overline{x} \in G\backslash X_0} \overline{x}$ be the decomposition of X_0 into transitive G-sets. Fix $\overline{x} \in G\backslash X_0$. Upon choice of a base point $x_0 \in \overline{x}$, we can identify $\overline{x} \cong (\mathbb{Z}^n \rtimes_M \mathbb{Z})/\overline{G}$, where $\overline{G} = (\mathbb{Z}^n \rtimes \mathbb{Z})_{x_0}$ is the preimage of some group $G \leqslant (\mathbb{Z}/s)^n \rtimes_{\overline{M}} \mathbb{Z}/r$ which lies in $\mathcal{D}$.

The following argument is taken from the proof of Proposition 6.1 in [Bar]:

Case 1: $G \cap (\mathbb{Z}/s)^n \subseteq \nu'(\mathbb{Z}/s)^n$ for some $\nu' \geqslant \nu$. Apply Proposition 5.2.7 to obtain a $\overline{G}$-equivariant map $f'_{\overline{x}} \colon \mathbb{Z}^n \rtimes_M \mathbb{Z} \to E'$ such that $d^1(f(g), f(h)) \leqslant \varepsilon$ whenever $g^{-1}h \in S$, where E' is a $(\mathbb{Z}^n \rtimes_M \mathbb{Z})$-simplicial complex of dimension at most N' (the number N' depending only on n) with cyclic isotropy. Set $E_{\overline{x}} := (\mathbb{Z}^n \rtimes_M \mathbb{Z}) \times_{\overline{G}} E'$, and define a map

$$f_{\overline{x}} \colon (\mathbb{Z}^n \rtimes_M \mathbb{Z})/\overline{G} \to E_{\overline{x}}, \quad g\overline{G} \mapsto (g, f'_{\overline{x}}(g^{-1})).$$

If $s \in S$, we get

$$\begin{aligned} d^1(f_{\overline{x}}(sg\overline{G}), sf_{\overline{x}}(g\overline{G})) &= d^1((sg, f'_{\overline{x}}(g^{-1}s^{-1})), (sg, f'_{\overline{x}}(g^{-1}))) \\ &= d^1(f'_{\overline{x}}(g^{-1}s^{-1}), f'_{\overline{x}}(g^{-1})) \\ &\leqslant \varepsilon \end{aligned}$$

since $(g^{-1}s^{-1})^{-1}g^{-1} = sgg^{-1} = s \in S$.

Case 2: $\pi(G) \subseteq \nu'(\mathbb{Z}/r)$ for some $\nu' \geqslant \nu$. We apply Proposition 5.2.8 to obtain a $\overline{G}$-equivariant map $f'_{\overline{x}} \colon \mathbb{Z}^n \rtimes_M \mathbb{Z} \to E'$ with the properties listed there. Again, define

$E_{\overline{x}} := (\mathbb{Z}^n \rtimes_M \mathbb{Z})_{\overline{G}} E'$ and $f_{\overline{x}} \colon \mathbb{Z}^n \rtimes_M \mathbb{Z}/\overline{G} \to E_{\overline{x}}$ by $f_{\overline{x}}(g\overline{G}) := (g, f'_{\overline{x}}(g^{-1})$. The same calculation as in Case 1 shows that the map $f_{\overline{x}}$ is G-equivariant up to ε.

We are nearly done: The collection of $(\mathbb{Z}^n \rtimes_M \mathbb{Z})$-simplicial complexes $\{E_{\overline{x}}\}_{\overline{x} \in G \backslash X_0}$ defines a set of resolution data E as we observed in Remark 4.1.3; to be more precise, if we choose abstract G-simplicial complexes $\mathcal{E}_{\overline{x}}$ whose realisation is $E_{\overline{x}}$, these define a set of resolution data $\mathcal{E}$. The coproduct of the maps $f_{\overline{x}}$ is a map

$$f_0 \colon X_0 \to \coprod_{\overline{x} \in G \backslash X_0} E_{\overline{x}} \subseteq |X[\mathcal{E}]|$$

which is G-equivariant up to ε. By Lemma 5.2.9, the map f_0 extends to a map $f \colon X \to |X[E]|$ which is G-equivariant up to ε. Moreover, Proposition 4.1.4 tells us that the isotropy of the simplicial complex $X[\mathcal{E}]$ is abelian, and that its dimension is bounded by $N := \beta N' + \beta + N'$.

Since both β and N' depend only on n, this is all we have to show. □

5.2.10 Remark.
The Farrell-Jones Conjecture for $\mathbb{Z}^n \rtimes_M \mathbb{Z}$ follows from Theorem 5.2.2 by the transitivity principle [BFL14, Thm. 1.11] since abelian groups satisfy the Farrell-Jones Conjecture by [BL12b, Weg12].

We have implicitly proven a criterion when a Farrell-Hsiang group (see [BL12a, Def. 1.1], [BFL14, Def. 1.15] or [Bar, Thm. C]) satisfies the assumptions of Theorem A. To round up our discussion, we make this more explicit.

5.2.11 Definition.
Let G be a discrete group together with a finite generating set S. Let $\mathcal{F}$ be a family of subgroups of G, and let $\mathcal{A}$ be a small additive category with right G-action.

Say that (G, S) is a *Farrell-Hsiang group of bounded depth with respect to* $\mathcal{F}$ if there are $\nu \in \mathbb{N}$ and $B \in \mathbb{N}$ such that for every $\varepsilon > 0$ there are

- an epimorphism $\pi \colon G \twoheadrightarrow F$ to a finite group with depth $d(F) \leqslant B$ and
- for every subgroup $H \leqslant F$ which is in $\mathcal{D}$, a G-simplicial complex E_H of dimension at most ν whose isotropy groups lie in $\mathcal{F}$, such that there is an $\overline{H} := \pi^{-1}(H)$-equivariant map $f_H \colon G \to E_H$ such that $d^1(f_H(g), f_H(g')) \leqslant \varepsilon$ whenever $g^{-1}g' \in S$.

5.2.12 Proposition.
Let G be a discrete group together with a finite generating set S. Let $\mathcal{F}$ be a family of subgroups of G, and let $\mathcal{A}$ be a small additive category with right G-action.

If (G, S) is a Farrell-Hsiang group of bounded depth with respect to $\mathcal{F}$, then G satisfies the assumptions of Theorem A with respect to $\mathcal{F}$.

Proof. Let $\varepsilon > 0$ be given. Pick an epimorphism $\pi_\varepsilon \colon G \twoheadrightarrow F_\varepsilon$ to a finite group F_ε with $d(F_\varepsilon) \leqslant B$. Then there is by Corollary 4.2.4 a finite, contractible F_ε-simplicial complex X whose dimension is bounded by $\beta := 2^B \cdot \mathrm{bd}(B)^B$ and whose isotropy groups lie in $\mathcal{D}$. We obtain a G-action on X by restricting the F_ε-action along π_ε. Then the complex X is an N-transfer space for every $N \geqslant \beta$.

Consider the set of vertices X_0 of X. This set decomposes into transitive G-sets

$$X_0 = \coprod_i G/\overline{H}_i,$$

where $\overline{H}_i$ is the preimage of some subgroup $H_i \in \mathcal{D}$ of F_ε under π_ε. For each i, pick a G-simplicial complex E'_i of dimension at most ν whose isotropy groups lie in $\mathcal{F}$ as well as an $\overline{H}_i$-equivariant map $f'_i \colon G \to E'_i$ such that $d^1(f'(g), f'(g')) \leqslant \varepsilon$ whenever $g^{-1}g' \in S$.

Define $E_i := G \times_{\overline{H}_i} E'_i$, and let

$$f_i \colon G/\overline{H}_i \to E_i, \quad g\overline{H}_i \mapsto (g, f'_i(g^{-1})).$$

This map is well-defined, and for $s \in S$ we have

$$\begin{aligned} d^1(f_i(sg\overline{H}_i), sf_i(g\overline{H}_i)) &= d^1((sg, f'_i(g^{-1}s^{-1}), (sg, f'_i(g^{-1})) \\ &= d^1(f'_i(g^{-1}s^{-1}), f'_i(g^{-1})) \\ &\leqslant \varepsilon \end{aligned}$$

since $(g^{-1}s^{-1})^{-1}g^{-1} = sgg^{-1} = s \in S$. The collection $\{E_i\}_i$ determines a set of resolution data $\mathcal{E}$ for X, and the map

$$f \colon X_0 = \coprod_i G/\overline{H}_i \xrightarrow{\coprod_i f_i} \coprod_i E_i = |X[\mathcal{E}]|^{(0)} \subseteq |X[\mathcal{E}]|$$

is G-equivariant up to ε (as we have just verified). Lemma 5.2.9 applies to give a map $f_\varepsilon \colon |X| \to |X[\mathcal{E}]|$ which is G-equivariant up to ε. The simplicial complex $X[\mathcal{E}]$ has stabilisers in $\mathcal{F}$, and its dimension is bounded by $N := \beta\nu + \beta + \nu$. Observe that β depends only on B, which is given globally, and ν is also a global constant, so N is independent of the choice of ε. This proves that all assumptions of Theorem A are satisfied. $\square$

Index of notation

Categories

Add	category of small additive categories and additive functors
Δ	category of finite ordinal numbers and monotone maps
Δ_{inj}	category of finite ordinal numbers and monotone injections
G-Sets	category of G-sets
$\mathsf{s}G$-Sets	category of simplicial G-sets
$\mathsf{ss}G$-Sets	category of semisimplicial G-sets
$\mathsf{Ch}(\mathcal{A})$	category of finite cochain complexes over $\mathcal{A}$
$\mathcal{A} *_G \mathcal{C}$	Bartels-Reich construction
$\mathcal{C}\mathcal{A}$	category of $\mathcal{A}$-objects based over $\mathbb{Z}$ and bounded morphisms
$\mathcal{W}_G(X;\mathcal{K})$	category of "decompositions" over X
$\mathcal{MV}_G(X;\mathcal{K})$	category of Mayer-Vietoris resolutions over X
$-/\mathcal{C}$	category of objects under $-$
$\mathcal{C}/-$	category of objects over $-$
$-/\mathcal{C}/?$	category of objects under $-$ and over ?
$\mathsf{Or}(G)$	orbit category of the discrete group G
$\int_{\mathcal{I}} F$	Grothendieck construction of F over $\mathcal{I}$
$\mathsf{Tr}_G(T)$	transport category of the G-set T
$\mathcal{R}^f(X)$	category of finite retractive spaces over X
$\mathcal{R}^f(*,G)$	category of cocompact free G-CW-complexes relative $*$
$\mathcal{R}ep(G)$	category of finite pointed G-CW-complexes
$\mathcal{R}ep^{\mathbb{Z}}(G)$	category of integral G-representations
SCplx	category of simplicial complexes

Functors

$N-$	nerve of a category
$wS_\bullet$	Waldhausen's $wS_\bullet$-construction
$K(-)$	algebraic K-theory space
$\mathbb{K}^{-\infty}(-)$	(non-connective) algebraic K-theory spectrum
Σ	suspension
$\otimes_{\mathcal{C}}$	tensor product/balanced product over $\mathcal{C}$

ind	induction
res	restriction
hocolim	homotopy colimit
$\lvert - \rvert$	geometric realisation
σ_X	"forget basedness over X"-functor
Tot	totalisation functor from double complexes to cochain complexes
ρ_T^p	resolution functor associated to $T \times \Delta^p$
$\Delta_X(A;Y)$	canonical decomposition of A over X given by Y
$\mathrm{tr}^{\pm}\, p$	upper/lower p-truncation functor
$C_\varepsilon^*(-)$	augmented simplicial or cellular cochain complex
$\mathbb{H}^G(X;\mathbb{E})$	G-homology of X with coefficients in the $\mathsf{Or}(G)$-spectrum $\mathbb{E}$
$\Omega(G)$	Burnside ring of G
$A(X)$	algebraic K-theory ("A-theory") of X
$\mathsf{Sw}^A(G)$	A-theoretic Swan group of G
$\mathsf{Sw}(G)$	linear Swan group of G

Families of groups

$\mathcal{C}yc$	class of finite cyclic groups
$\mathcal{C}yc_p$	class of finite groups which are cyclic mod p
$\mathcal{H}$	family of hyperelementary groups
$\mathcal{D}$	Dress family

Miscellaneous

$\underline{n}$	(ordered) set $\{0,\ldots,n\}$
Δ^p	standard p-simplex, usually considered as a semisimplicial set
$a(\mu)$	"coaugmentation" map associated to the ordinal number map μ
$\mathrm{supp}_n(\varphi)$	support of the morphism φ over the n-simplices
A_X	Davis-Lück assembly map of X
α_X	Weiss-Williams assembly map of X
$X[R]$	resolution of X by the set of resolution data R
$\lvert X, R \rvert$	alternative realisation of $X[R]$
d^1	ℓ^1-metric of a simplicial complex
$N_G(H)$	normaliser of the subgroup H in G
$d(G)$	depth of the group G
$\Delta(G)$	Oliver ideal in $\Omega(G)$
gh	ghost map of the Burnside ring

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