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Representations attached to vector bundles on curves over finite and p-adic fields, a comparison

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Abstract. For a vector bundle E on a model of a smooth projective curve over a p-adic number field a p-adic representation of the geometric fundamental group of X has been defined in work with Annette Werner if the reduction of E is strongly semistable of degree zero. In the present note we calculate the reduction of this representation using the theory of Nori's fundamental group scheme.

1. The comparison

In [5] and [7] a partial analogue of the classical Narasimhan-Seshadri correspondence between vector bundles and representations of the fundamental group was developed. See also [8] for a p-adic theory of Higgs bundles. Let $\mathfrak{o}$ be the ring of integers in $\mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$ and let $k = \mathfrak{o}/\mathfrak{m} = \overline{\mathbb{F}}_p$ be the common residue field of $\overline{\mathbb{Z}}_p$ and $\mathfrak{o}$. Consider a smooth projective (connected) curve X over $\overline{\mathbb{Q}}_p$ and let E be a vector bundle of degree zero on $X_{\mathbb{C}_p} = X \otimes \mathbb{C}_p$. If E has potentially strongly semistable reduction in the sense of [7, Def. 2], then for any $x \in X(\mathbb{C}_p)$ according to [7, Thm. 10] there is a continuous representation

$$\rho_{E,x} : \pi_1(X, x) \longrightarrow \mathrm{GL}(E_x). \tag{1}$$

We now describe a special case of the theory where one can define the reduction of $\rho_{E,x} \bmod \mathfrak{m}$. Assume that we are given the following data:

i) A model $\mathfrak{X}$ of X i.e. a finitely presented proper flat $\overline{\mathbb{Z}}_p$-scheme $\mathfrak{X}$ with $X = \mathfrak{X} \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p$,

ii) A vector bundle $\mathcal{E}$ over $\mathfrak{X}_{\mathfrak{o}} = \mathfrak{X} \otimes_{\overline{\mathbb{Z}}_p} \mathfrak{o}$ extending E.

Such models $\mathfrak{X}$ and $\mathcal{E}$ always exist. Consider the special fiber $\mathfrak{X}_k = \mathfrak{X} \otimes_{\overline{\mathbb{Z}}_p} k = \mathfrak{X}_{\mathfrak{o}} \otimes_{\mathfrak{o}} k$ and set $\mathcal{E}_k = \mathcal{E} \otimes_{\mathfrak{o}} k$, a vector bundle on $\mathfrak{X}_k$. We assume that $\mathcal{E}_k$ restricted to $\mathfrak{X}_k^{\mathrm{red}}$ is strongly semistable of degree zero in the sense of Section 2 below.

In this case we say that $\mathcal{E}$ has strongly semistable reduction of degree zero on $\mathfrak{X}_{\mathfrak{o}}$. Then [5] provides a continuous representation

$$\rho_{\mathcal{E},x_{\mathfrak{o}}} : \pi_1(X,x) \longrightarrow \mathrm{GL}(\mathcal{E}_{x_{\mathfrak{o}}}), \tag{2}$$

which induces (1). Here $x_{\mathfrak{o}} \in \mathfrak{X}(\mathfrak{o}) = X(\mathbb{C}_p)$ is the section of $\mathfrak{X}$ corresponding to x and $\mathcal{E}_{x_{\mathfrak{o}}} = \Gamma(\operatorname{spec}\mathfrak{o}, x_{\mathfrak{o}}^*\mathcal{E})$ is an $\mathfrak{o}$-lattice in $\mathcal{E}_x$.

Denoting by $x_k \in \mathfrak{X}_k(k) = \mathfrak{X}_k^{\mathrm{red}}(k)$ the reduction of $x_{\mathfrak{o}}$, we have $\mathcal{E}_{x_{\mathfrak{o}}} \otimes_{\mathfrak{o}} k = \mathcal{E}_{x_k}$ the fiber over x_k of the vector bundle $\mathcal{E}_k$.

The aim of this note is to describe the reduction $\operatorname{mod}\mathfrak{m}$ of $\rho_{\mathcal{E},x_{\mathfrak{o}}}$ i.e. the representation

$$\rho_{\mathcal{E},x_{\mathfrak{o}}} \otimes k : \pi_1(X,x) \longrightarrow \mathrm{GL}(\mathcal{E}_{x_k}) \tag{3}$$

using Nori's fundamental group scheme [13].

Let us recall some of the relevant definitions. Consider a perfect field k and a reduced complete and connected k-scheme Z with a point $z \in Z(k)$. A vector bundle H on Z is *essentially finite* if there is a torsor $\lambda : P \to Z$ under a finite group scheme over k such that $\lambda^* H$ is a trivial bundle. Nori has defined a profinite algebraic group scheme $\pi(Z,z)$ over k classifying the essentially finite bundles H on Z. Every such bundle corresponds to an algebraic representation

$$\boldsymbol{\lambda}_{H,z} : \pi(Z,z) \longrightarrow \mathbf{GL}_{H_z}. \tag{4}$$

The group scheme $\pi(Z,z)$ also classifies the pointed torsors under finite group schemes on Z. If k is algebraically closed, it follows that the group of k-valued points of $\pi(Z,z)$ can be identified with Grothendieck's fundamental group $\pi_1(Z,z)$. On k-valued points the representation $\boldsymbol{\lambda}_{H,z}$ therefore becomes a continuous homomorphism

$$\lambda_{H,z} = \boldsymbol{\lambda}_{H,z}(k) : \pi_1(Z,z) \longrightarrow \mathrm{GL}(H_z). \tag{5}$$

We will show the following result:

Theorem 1. *With notations as above, consider a vector bundle $\mathcal{E}$ on $\mathfrak{X}_{\mathfrak{o}}$ with strongly semistable reduction of degree zero. Then $\mathcal{E}_k^{\mathrm{red}}$, the bundle $\mathcal{E}_k$ restricted to $\mathfrak{X}_k^{\mathrm{red}}$ is essentially finite. For the corresponding representation:*

$$\lambda = \lambda_{\mathcal{E}_k^{\mathrm{red}},x_k} : \pi_1(\mathfrak{X}_k^{\mathrm{red}}, x_k) \longrightarrow \mathrm{GL}(\mathcal{E}_{x_k}),$$

the following diagram is commutative:

$$\begin{array}{ccc}
\pi_1(X,x) & \xrightarrow{\rho_{\mathcal{E},x_{\mathfrak{o}}}\otimes k} & \mathrm{GL}(\mathcal{E}_{x_k}) \\
\downarrow & & \Vert \\
\pi_1(\mathfrak{X},x) & & \Vert \\
\Vert & & \Vert \\
\pi_1(\mathfrak{X}_k^{\mathrm{red}}, x_k) & \xrightarrow{\lambda} & \mathrm{GL}(\mathcal{E}_{x_k}).
\end{array}$$

In particular, the reduction mod$\mathfrak{m}$ of $\rho_{\mathcal{E},x_o}$ factors over the specialization map $\pi_1(X,x) \to \pi_1(\mathfrak{X}_k^{\mathrm{red}}, x_k)$. In general this is not true for $\rho_{\mathcal{E},x_o}$ itself according to Example 5.

This note originated from a question of Vikram Mehta. I am very thankful to him and also to Hélène Esnault who once drew my attention to Nori's fundamental group.

2. *sss*-bundles on curves over finite fields

In this section we collect a number of definitions and results related to Nori's fundamental group [13]. The case of curves over finite fields presents some special features.

Consider a reduced complete and connected scheme Z over a perfect field k with a rational point $z \in Z(k)$. According to [13] the $\otimes$-category of essentially finite vector bundles H on Z with the fiber functor $H \mapsto H_z$ is a neutral Tannakian category over k. By Tannakian duality it is equivalent to the category of algebraic representations of an affine group scheme $\pi(Z,z)$ over k which turns out to be a projective limit of finite group schemes.

Let $f : Z \to Z'$ be a morphism of reduced complete and connected k-schemes. The pullback of vector bundles induces a tensor functor between the categories of essentially finite bundles on Z' and Z which is compatible with the fiber functors in $f(z)$ and z. By Tannakian functoriality we obtain a morphism $f_* : \pi(Z,z) \to \pi(Z', f(z))$ of group schemes over k. If k is algebraically closed the induced map on k-valued points

$$\pi_1(Z,z) = \pi(Z,z)(k) \to \pi(Z', f(z))(k) = \pi_1(Z', f(z))$$

is the usual map f_* between the Grothendieck fundamental groups.

We will next describe the homomorphism

$$\lambda_{H,z} = \boldsymbol{\lambda}_{H,z}(k) : \pi_1(Z,z) = \pi(Z,z)(k) \to \mathrm{GL}(H_z)$$

in case H is trivialized by a finite étale covering. Consider a scheme S with a geometric point $s \in S(\Omega)$. We view $\pi_1(S,s)$ as the automorphism group of the fiber functor F_s which maps any finite étale covering $\pi : S' \to S$ to the set of points $s' \in S'(\Omega)$ with $\pi(s') = s$.

Proposition 2. *Let Z be a reduced complete and connected scheme over the algebraically closed field k with a point $z \in Z(k)$. Consider a vector bundle H on Z for which there exists a connected finite étale covering $\pi : Y \to Z$ such that $\pi^* H$ is a trivial bundle. Then H is essentially finite and the map $\lambda_{H,z} : \pi_1(Z,z) \to \mathrm{GL}(H_z)$ in* (5) *has the following description. Choose a point $y \in Y(k)$ with $\pi(y) = z$. Then for every $\gamma \in \pi_1(Z,z)$ there is a commutative* *diagram:*

$$
(6)\qquad
\begin{array}{ccccc}
(\pi^* H)_y & \xleftarrow[\sim]{y^*} & \Gamma(Y, \pi^* H) & \xrightarrow[\sim]{(\gamma y)^*} & (\pi^* H)_{\gamma y} \\
\Vert & & & & \Vert \\
H_z & & \xrightarrow{\qquad\lambda_{H,z}(\gamma)\qquad} & & H_z
\end{array}
$$

Proof. The covering $\pi : Y \to Z$ can be dominated by a finite étale Galois covering $\pi' : Y' \to Z$. Let $y' \in Y'(k)$ be a point above y. If the diagram (6) with π, Y, y replaced by π', Y', y' is commutative, then (6) itself commutes. Hence we may assume that $\pi : Y \to Z$ is Galois with group G. In particular H is essentially finite. Consider the surjective homomorphism $\pi_1(Z, z) \to G$ mapping γ to the unique $\sigma \in G$ with $\gamma y = y^\sigma$. The right action of G on Y induces a left action on $\Gamma(Y, \pi^* H)$ by pullback and it follows from the definitions that $\lambda_{H,z}$ is the composition

$$\lambda_{H,z} : \pi_1(Z, z) \to G \to \mathrm{GL}(\Gamma(Y, \pi^* H)) \xrightarrow[\sim]{\text{via } y^*} \mathrm{GL}(H_x).$$

Now the equations

$$
\begin{aligned}
(\gamma y)^* \circ (y^*)^{-1} = (y^\sigma)^* \circ (y^*)^{-1} &= (\sigma \circ y)^* \circ (y^*)^{-1} \\
&= y^* \circ \sigma^* \circ (y^*)^{-1} = \lambda_H(k)(\gamma)
\end{aligned}
$$

imply the assertion. Here σ^* is the automorphism of $\Gamma(Y, \pi^* H)$ induced by σ. □

The following class of vector bundles contains the essentially finite ones. A vector bundle H on a reduced connected and complete k-scheme Z is called strongly semistable of degree zero (*sss*) if for all k-morphisms $f : C \to Z$ from a smooth connected projective curve C over k the pullback bundle $f^*(H)$ is semistable of degree zero, cp. [3, (2.34)]. It follows from [13, Lem. (3.6)] that the *sss*-bundles form an abelian category. Moreover a result of Gieseker shows that it is a tensor category, cp. [9]. If Z is purely one-dimensional, a bundle H is *sss* if and only if the pullback of H to the normalization $\tilde{C}_i$ of each irreducible component C_i of Z is strongly semistable of degree zero in the usual sense on the smooth projective curve $\tilde{C}_i$ over k, see e.g. [6, Prop. 4].

Generalizing results of Lange-Stuhler and Subramanian slightly we have the following fact, where $\mathbb{F}_q$ denotes the field with $q = p^r$ elements.

Theorem 3. *Let Z be a reduced complete and connected purely one-dimensional scheme over $\mathbb{F}_q$. Then the following three conditions are equivalent for a vector bundle H on Z.*
1) *H is strongly semistable of degree zero.*
2) *There is a finite surjective morphism $\varphi : Y \to Z$ with Y a complete and purely one-dimensional scheme over $\mathbb{F}_q$ such that $\varphi^* H$ is a trivial bundle.*
3) *There are a finite étale covering $\pi : Y \to Z$ and some $s \geq 0$ such that for the composition $\varphi : Y \xrightarrow{F^s} Y \xrightarrow{\pi} Z$ the pullback $\varphi^* H$ is a trivial bundle. Here*

$F = \mathrm{Fr}_q = \mathrm{Fr}_p^r$ *is the q-linear Frobenius morphism on Y.*
If Z has an $\mathbb{F}_q$-rational point, these conditions are equivalent to
4) *H is essentially finite.*

Remark If $Z(\mathbb{F}_q) \neq \varnothing$, then according to **4)** the trivializing morphism $\varphi : Y \to Z$ in **2)** can be chosen to be a G-torsor under a finite group scheme $G/\mathbb{F}_q$.

Proof. The equivalence of **1)** to **3)** is shown in [5, Thm. 18] by slightly generalizing a result of Lange and Stuhler. It is clear that **4)** implies **2)**. Over a smooth projective curve $Z/\mathbb{F}_q$ the equivalence of **1)** and **4)** was shown by Subramanian in [14, Thm. (3.2)], with ideas from [12] and [2]. His proof works also over our more general bases Z and shows that **1)** implies **4)**. Roughly the argument goes as follows: Using the fiber functor in a point $z \in Z(\mathbb{F}_q)$ the abelian tensor category $\mathcal{T}_Z$ of sss-bundles on Z becomes a neutral Tannakian category over $\mathbb{F}_q$. Note by the way that the characterization **2)** of sss-bundles shows without appealing to [9] that $\mathcal{T}_Z$ is stable under the tensor product. Consider the Tannakian subcategory generated by H. Its Tannakian dual is called the monodromy group scheme M_H in [2]. Let n be the rank of H. The GL_n-torsor associated to H allows a reduction of structure group to M_H. Hence we obtain an M_H-torsor $\alpha : P \to Z$ such that $\alpha^* H$ is a trivial bundle. We have $\mathrm{Fr}_q^{s*} H = \mathrm{Fr}_q^{t*} H$ for some $s > t \geq 0$ because there are only finitely many isomorphism classes of semistable vector bundles of degree zero on a smooth projective curve over a finite field. See [5, Proof of Thm. 18] for more details. A short argument as in [14] now implies that M_H is a finite group scheme and we are done. □

Later on we will need the following fact:

Proposition 4. *Let S_0 be a scheme over $\mathbb{F}_q$ and let $F = \mathrm{Fr}_q$ be the q-linear Frobenius morphism on S_0. Set $k = \overline{\mathbb{F}}_q$ and let $\overline{F} = F \otimes_{\mathbb{F}_q} k$ be the base extension of F to a morphism of $S = S_0 \otimes_{\mathbb{F}_q} k$. Then for any geometric point $s \in S(\Omega)$ the induced map $\overline{F}_* : \pi_1(S, s) \to \pi_1(S, \overline{F}(s))$ is an isomorphism.*

Proof. Let F_k be the automorphism of k with $F_k(x) = x^q$ for all $x \in k$. Then $\psi = \mathrm{id}_{S_0} \otimes F_k$ is an automorphism of the scheme S and hence it induces isomorphisms on fundamental groups. It suffices therefore to show that

$$(\psi \circ \overline{F})_* : \pi_1(S, s) \to \pi_1(S, \psi(\overline{F}(s)))$$

is an isomorphism. The morphism $\psi \circ \overline{F}$ is the q-linear Frobenius morphism Fr_q on S. For any finite étale covering $\pi : T \to S$ the relative Frobenius morphism is known to be an isomorphism and hence the commutative diagram

$$\begin{array}{ccc} T & \xrightarrow{\mathrm{Fr}_q} & T \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ S & \xrightarrow{\mathrm{Fr}_q} & S \end{array}$$

is *cartesian*. It follows that $\mathrm{Fr}_{q*} = (\psi \circ \overline{F})_*$ is an isomorphism on fundamental groups. □

3. Proof of Theorem 1

For the proof of Theorem 1 we first give a description of the representation $\rho_{\mathcal{E},x_o} \otimes k$ which follows immediately from the construction of $\rho_{\mathcal{E},x_o}$ in [5, Sec. 3].

We assume that we are in the situation of Theorem 1. By assumption $\mathcal{E}_k^{\mathrm{red}}$ is strongly semistable of degree zero on $\mathfrak{X}_k^{\mathrm{red}}$. According to [5, Thm. 17] there is a proper morphism $\pi : \mathcal{Z} \to \mathfrak{X}$ with the following properties:

a) The generic fiber $Z = \mathcal{Z} \otimes_{\overline{\mathbb{Z}}_p} \overline{\mathbb{Q}}_p$ is a smooth projective connected $\overline{\mathbb{Q}}_p$-curve.

b) The induced morphism $\pi : Z \to X$ is finite and for an open dense subscheme $U \subset X$ the restriction $\pi : \pi^{-1}(U) = W \to U$ is étale. Moreover we have $x \in U(\mathbb{C}_p)$ for the chosen base point $x \in X(\mathbb{C}_p)$.

c) The scheme $\mathcal{Z}$ is a model of Z over $\overline{\mathbb{Z}}_p$ whose special fiber $\mathcal{Z}_k$ is reduced. In particular $\mathcal{Z}/\overline{\mathbb{Z}}_p$ is cohomologically flat in degree zero.

d) The pullback $\pi_k^*\mathcal{E}_k$ is a trivial vector bundle on $\mathcal{Z}_k$.

The following construction gives a representation of $\pi_1(U,x)$ on $\mathcal{E}_{x_k}$. For $\gamma \in \pi_1(U,x) = \mathrm{Aut}(F_x)$ choose a point $z \in W(\mathbb{C}_p)$ with $\pi(z) = x$. Then $\gamma z \in W(\mathbb{C}_p)$ is another point over x. From z and γz in $W(\mathbb{C}_p) \subset Z(\mathbb{C}_p)$ we obtain points z_k and $(\gamma z)_k$ in $\mathcal{Z}_k(k)$ as in the introduction. Consider the diagram

$$\mathcal{E}_{x_k} = (\pi_k^*\mathcal{E}_k)_{z_k} \xleftarrow[\sim]{z_k^*} \Gamma(\mathcal{Z}_k, \pi_k^*\mathcal{E}_k) \xrightarrow[\sim]{(\gamma z)_k^*} (\pi_k^*\mathcal{E}_k)_{(\gamma z)_k} = \mathcal{E}_{x_k}. \tag{7}$$

Here the pullback morphisms along $z_k : \mathrm{spec} k \to \mathcal{Z}_k$ and $(\gamma z)_k : \mathrm{spec} k \to \mathcal{Z}_k$ are isomorphisms because $\pi_k^*\mathcal{E}_k$ is a trivial bundle and $\mathcal{Z}/\overline{\mathbb{Z}}_p$ is cohomologically flat in degree zero.

It turns out that the map

$$\rho : \pi_1(U,x) \to \mathrm{GL}(\mathcal{E}_{x_k}) \text{ defined by } \rho(\gamma) = (\gamma z)_k^* \circ (z_k^*)^{-1} \tag{8}$$

is a homomorphism of groups which (by construction) factors over a finite quotient of $\pi_1(U,x)$. Thus ρ is continuous if $\mathrm{GL}(\mathcal{E}_{x_k})$ is given the discrete topology. Moreover ρ does not depend on either the choice of the point z above x nor on the choice of morphism $\pi : \mathcal{Z} \to \mathfrak{X}$ satisfying **a)-d)**. It follows from [5, Thm. 17 and Prop. 35] that ρ factors over $\pi_1(X,x)$. The resulting representation $\rho : \pi_1(X,x) \to \mathrm{GL}(\mathcal{E}_{x_k})$ agrees with $\rho_{\mathcal{E},x_o} \otimes k$.

In order to prove Theorem 1 we will now construct given $\mathcal{E}_k$ a suitable morphism $\mathcal{Z} \to \mathfrak{X}$. We use a modification of the method from the proof of Theorem 17 in [5]. In that proof the singularities were resolved at the level of $\mathcal{Y}$ which is too late for our present purposes because it creates an extension of $\mathcal{Y}_k$ which is hard to control discussing the Nori fundamental group. Instead, we will resolve the singularities of a model of X. Then $\mathcal{Y}$ does not have to be changed later. We proceed with the details:

Choose a finite extension $K/\mathbb{Q}_p$ with ring of integers $\mathfrak{o}_K$ and residue field κ such that $(\mathfrak{X}, \mathcal{E}_k, x_k)$ descends to $(\mathfrak{X}_{\mathfrak{o}_K}, \mathcal{E}_0, x_0)$. Here $\mathfrak{X}_{\mathfrak{o}_K}$ is a proper and flat

$\mathfrak{o}_K$-scheme with $\mathfrak{X}_{\mathfrak{o}_K} \otimes_{\mathfrak{o}_K} \overline{\mathbb{Z}}_p = \mathfrak{X}$ and $\mathcal{E}_0$ a vector bundle on $\mathfrak{X}_0 = \mathfrak{X}_{\mathfrak{o}_K} \otimes \kappa$ with $\mathcal{E}_0 \otimes_\kappa k = \mathcal{E}_k$. Since $\mathcal{E}_k^{\mathrm{red}}$ is an sss-bundle on $\mathfrak{X}_k^{\mathrm{red}}$ the restriction $\mathcal{E}_0^{\mathrm{red}}$ of $\mathcal{E}_0$ to $\mathfrak{X}_0^{\mathrm{red}}$ is an sss-bundle as well. Finally $x_0 \in \mathfrak{X}_0(\kappa)$ is a point which induces x_k after base change to k. Theorem 3 implies that $\mathcal{E}_0^{\mathrm{red}}$ is essentially finite and hence $\mathcal{E}_k^{\mathrm{red}}$ is essentially finite as well.

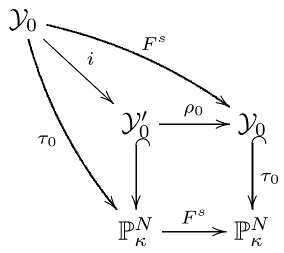
After replacing K by a finite extension and performing a base extension to the new K we can find a semistable model $\mathfrak{X}'_{\mathfrak{o}_K}$ of the smooth projective curve $X_K = \mathfrak{X}_{\mathfrak{o}_K} \otimes K$ together with a morphism $\alpha_{\mathfrak{o}_K} : \mathfrak{X}'_{\mathfrak{o}_K} \to \mathfrak{X}_{\mathfrak{o}_K}$ extending the identity on the generic fiber X_K. This is possible by the semistable reduction theorem, cp. [1] for a comprehensive account. By Lipman's desingularization theorem we may assume that $\mathfrak{X}'_{\mathfrak{o}_K}$ besides being semistable is also regular, cp. [11, 10.3.25 and 10.3.26]. The irreducible regular surface $\mathfrak{X}'_{\mathfrak{o}_K}$ is proper and flat over $\mathfrak{o}_K$.

Let $\mathcal{E}'_0$ be the pullback of $\mathcal{E}_0$ along the morphism $\alpha_0 : \mathfrak{X}'_0 = \mathfrak{X}' \otimes \kappa \to \mathfrak{X}_0$. Since $\mathfrak{X}'_0$ is reduced the map factors as $\alpha_0 : \mathfrak{X}'_0 \to \mathfrak{X}_0^{\mathrm{red}} \subset \mathfrak{X}_0$ and $\mathcal{E}'_0$ is also the pullback of the sss-bundle $\mathcal{E}_0^{\mathrm{red}}$. Hence $\mathcal{E}'_0$ is an sss-bundle as well.

Using Theorem 3 we find a finite étale covering $\pi_0 : \mathcal{Y}_0 \to \mathfrak{X}'_0$ by a complete and one-dimensional κ-scheme $\mathcal{Y}_0$ and an integer $s \geq 0$ such that under the composed map $\varphi : \mathcal{Y}_0 \xrightarrow{F^s} \mathcal{Y}_0 \xrightarrow{\pi_0} \mathfrak{X}'_0$ the pullback $\varphi^* \mathcal{E}'_0$ is a trivial bundle. Here $F = \mathrm{Fr}_q$ is the $q = |\kappa|$-linear Frobenius morphism on $\mathcal{Y}_0$. Let $\tilde{\kappa}$ be a finite extension of κ such that all connected components of $\mathcal{Y}_0 \otimes_\kappa \tilde{\kappa}$ are geometrically connected. Let $\tilde{K}/K$ be the unramified extension with residue field $\tilde{\kappa}$. We replace $\mathfrak{X}_{\mathfrak{o}_K}, \mathfrak{X}'_{\mathfrak{o}_K}$ and $\mathcal{E}_0, \mathcal{E}'_0$ by their base extensions with $\mathfrak{o}_{\tilde{K}}$ resp. $\tilde{\kappa}$ and F by the $|\tilde{\kappa}|$-linear Frobenius morphism. We also replace $\mathcal{Y}_0$ be a connected component of $\mathcal{Y}_0 \otimes_\kappa \tilde{\kappa}$ and π_0 by the induced morphism. Then the new $\mathfrak{X}_{\mathfrak{o}_K}, \mathfrak{X}'_{\mathfrak{o}_K}, \varphi, \ldots$ keep the previous properties and $\mathcal{Y}_0$ is now geometrically connected. Using [15, IX Thm. 1.10] we may lift $\pi_0 : \mathcal{Y}_0 \to \mathfrak{X}'_0$ to a finite étale morphism $\pi_{\mathfrak{o}_K} : \mathcal{Y}_{\mathfrak{o}_K} \to \mathfrak{X}'_{\mathfrak{o}_K}$. The proper flat $\mathfrak{o}_K$-scheme $\mathcal{Y}_{\mathfrak{o}_K}$ is regular with geometrically reduced fibers over $\mathfrak{o}_K$ because $\mathfrak{X}'_{\mathfrak{o}_K}$ has these properties. In particular, the morphism $\mathcal{Y}_{\mathfrak{o}_K} \to \operatorname{spec} \mathfrak{o}_K$ is cohomologically flat in degree zero. Since the special fiber $\mathcal{Y}_0$ is geometrically connected and reduced it follows that the generic fiber Y_K of $\mathcal{Y}_{\mathfrak{o}_K}$ is geometrically connected and hence a smooth projective geometrically irreducible curve over K. In particular $\mathcal{Y}_{\mathfrak{o}_K}$ is irreducible in addition to being regular and proper flat over $\mathfrak{o}_K$. By a theorem of Lichtenbaum [10] there is thus a closed immersion $\tau_K : \mathcal{Y}_{\mathfrak{o}_K} \hookrightarrow \mathbb{P}^N_{\mathfrak{o}_K}$ for some N. Composing with a suitable automorphism of $\mathbb{P}^N_{\mathfrak{o}_K}$ we may assume that $\tau_K^{-1}(\mathbb{G}^N_{m,K}) \subset Y_K$ contains all points in $Y_K(\mathbb{C}_p)$ over $x \in X_K(\mathbb{C}_p) = X(\mathbb{C}_p)$. In particular, $\tau_K^{-1}(\mathbb{G}^N_{m,K})$ is open and dense in Y_K with a finite complement. Thus there is an open subscheme $U_K \subset X_K$ with $x \in U_K(\mathbb{C}_p)$ and such that $V_K = \pi_K^{-1}(U_K)$ is contained in $\tau_K^{-1}(\mathbb{G}^N_{m,K})$.

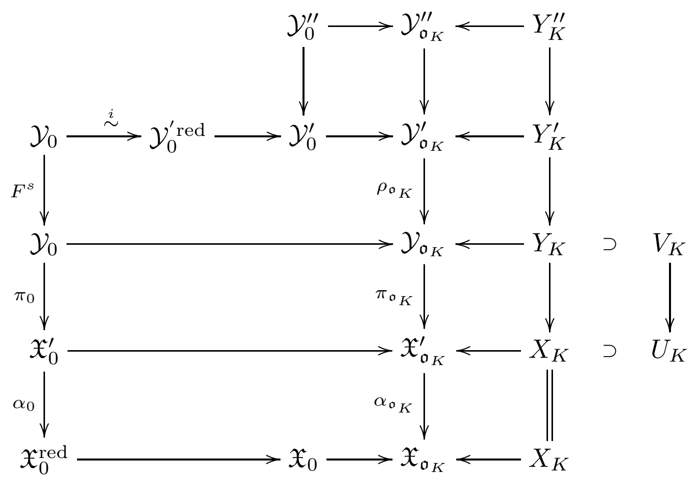
Consider the finite morphism $F_{\mathfrak{o}_K} : \mathbb{P}^N_{\mathfrak{o}_K} \to \mathbb{P}^N_{\mathfrak{o}_K}$ given on A-valued points where A is any $\mathfrak{o}_K$-algebra, by sending $[x_0 : \ldots : x_N]$ to $[x_0^q : \ldots : x_N^q]$. The reduction of $F_{\mathfrak{o}_K}$ is the q-linear Frobenius morphism on $\mathbb{P}^N_\kappa$.

Let $\rho_{\mathfrak{o}_K} : \mathcal{Y}'_{\mathfrak{o}_K} \to \mathcal{Y}_{\mathfrak{o}_K}$ be the base change of $F^s_{\mathfrak{o}_K}$ via τ_K. It is finite and its generic fiber $\rho_K : Y'_K \to Y_K$ is étale over V_K. Now we look at the reductions and we define a morphism $i : \mathcal{Y}_0 \to \mathcal{Y}'_0$ over κ by the commutative diagram



In [5, Lem. 19] it is shown that i induces an isomorphism $i : \mathcal{Y}_0 \xrightarrow{\sim} \mathcal{Y}_0'^{\mathrm{red}}$. Here the index 0 always refers to the special fiber over specκ.

Taking the normalization of $\mathcal{Y}'_{\mathfrak{o}_K}$ in the function field of an irreducible component of Y'_K we get a proper, flat $\mathfrak{o}_K$-scheme $\mathcal{Y}''_{\mathfrak{o}_K}$ which is finite over $\mathcal{Y}'_{\mathfrak{o}_K}$. Its generic fiber Y''_K is a smooth projective connected curve over K (maybe not geometrically connected). The following diagram summarizes the situation



For a suitable finite extension $\tilde{K}/K$ all connected components of $Y''_K \otimes_K \tilde{K}$ will be geometrically connected. Let $Y'''_{\tilde{K}}$ be one of them and let $\mathcal{Y}'''_{\mathfrak{o}_{\tilde{K}}}$ be its closure with the reduced scheme structure in $\mathcal{Y}''_{\mathfrak{o}_K} \otimes \mathfrak{o}_{\tilde{K}}$. By the semistable reduction theorem there are a finite extension $L/\tilde{K}$ and a semistable model $\mathcal{Z}_{\mathfrak{o}_L}$ of $Y'''_{\tilde{K}} \otimes_{\tilde{K}} L$ over $\mathcal{Y}'''_{\mathfrak{o}_{\tilde{K}}}$. Base extending $\mathfrak{X}_{\mathfrak{o}_K}, \ldots, \mathcal{Y}''_{\mathfrak{o}_K}$ over $\mathfrak{o}_K$ to $\overline{\mathbb{Z}}_p$ and $\mathcal{Y}'''_{\mathfrak{o}_{\tilde{K}}}$ over $\mathfrak{o}_{\tilde{K}}$ and $\mathcal{Z}_{\mathfrak{o}_L}$ over $\mathfrak{o}_L$ we get a commutative diagram, where δ is the composition $\delta : \mathcal{Z} \to \mathcal{Y}''' \to \mathcal{Y}'' \to \mathcal{Y}' \xrightarrow{\rho} \mathcal{Y}$,

(9)
$$
\begin{array}{ccccccccc}
\mathcal{Z}_k & & \longrightarrow & & \mathcal{Z} & \longleftarrow & Z & & \\
{\scriptstyle\beta_k}\downarrow & & & & & & & & \\
\mathcal{Y}_k & & & & \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta_{\overline{\mathbb{Q}}_p}} & & \\
{\scriptstyle F^s\otimes_\kappa k}\downarrow & & & & & & & & \\
\mathcal{Y}_k & & \longrightarrow & & \mathcal{Y} & \longleftarrow & Y & \supset & V \\
{\scriptstyle\pi_k}\downarrow & & & & {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi_{\overline{\mathbb{Q}}_p}} & & \downarrow \\
\mathfrak{X}'_k & & \longrightarrow & & \mathfrak{X}' & \longleftarrow & X & \supset & U \\
{\scriptstyle\alpha_k}\downarrow & & & & {\scriptstyle\alpha}\downarrow & & \Vert & & \\
\mathfrak{X}_k^{\mathrm{red}} & \hookrightarrow & \mathfrak{X}_k & \longrightarrow & \mathfrak{X} & \longleftarrow & X & &
\end{array}
$$

Here the morphism $\beta_k : \mathcal{Z}_k \to \mathcal{Y}_k$ comes about as follows: Since $\mathcal{Z}_k$ is reduced, the composition $\mathcal{Z}_k \to \mathcal{Y}_k''' \to \mathcal{Y}_k'' \to \mathcal{Y}_k'$ factors over $\mathcal{Y}_k^{\prime\mathrm{red}} \xleftarrow{\sim} \mathcal{Y}_k$ and this defines β_k. By construction, the map $\pi_{\overline{\mathbb{Q}}_p} \circ \delta_{\overline{\mathbb{Q}}_p} : Z \to X$ is finite and such that its restriction to a map $W = (\pi_{\overline{\mathbb{Q}}_p} \circ \delta_{\overline{\mathbb{Q}}_p})^{-1}(U) \to U$ is finite and étale. By construction the bundle $\mathcal{E}_k' = \alpha_k^* \mathcal{E}_k = \mathcal{E}_0' \otimes_\kappa k$ is trivialized by pullback along $\pi_k \circ (F^s \otimes_\kappa k)$ and hence also along $(\pi \circ \delta)_k = \pi_k \circ (F^s \otimes_\kappa k) \circ \beta_k$. For later purposes note that we have a commutative diagram

(10)
$$
\begin{array}{ccc}
\mathcal{Y}_k & \xrightarrow{F^s \otimes_\kappa k} & \mathcal{Y}_k \\
{\scriptstyle\pi_k}\downarrow & & \downarrow{\scriptstyle\pi_k} \\
\mathfrak{X}'_k & \xrightarrow{F^s \otimes_\kappa k} & \mathfrak{X}'_k
\end{array}
$$

obtained by base changing the corresponding diagram over κ:

$$
\begin{array}{ccc}
\mathcal{Y}_0 & \xrightarrow{F^s} & \mathcal{Y}_0 \\
{\scriptstyle\pi_0}\downarrow & & \downarrow{\scriptstyle\pi_0} \\
\mathfrak{X}'_0 & \xrightarrow{F^s} & \mathfrak{X}'_0 .
\end{array}
$$

The inclusion $\mathfrak{X}_k \to \mathfrak{X}$ induces a natural isomorphism $\pi_1(\mathfrak{X}_k, x_k) \xrightarrow{\sim} \pi_1(\mathfrak{X}, x_k)$. This follows from [15, Exp. X, Thm. 2.1] together with an argument to reduce the finitely presented case to a Noetherian one as in the proof of [15, Exp. IX, Thm. 6.1, p. 254] above.

Next we note that there is a canonical isomorphism

$$\pi_1(\mathfrak{X}, x_k) = \mathrm{Aut}(F_{x_k}) = \mathrm{Aut} F_x = \pi_1(\mathfrak{X}, x).$$

Namely, for a finite étale covering $\mathcal{Y} \to \mathfrak{X}$, by the infinitesimal lifting property, any point $y_k \in \mathcal{Y}_k(k)$ over x_k determines a unique section $y_{\mathfrak{o}} \in \mathcal{Y}(\mathfrak{o})$ over

$x_{\mathfrak{o}} \in \mathfrak{X}(\mathfrak{o})$ and hence a point $y \in Y(\mathbb{C}_p)$ over $x \in X(\mathbb{C}_p)$. In this way one obtains a bijection between the points y_k over x_k and the points y over x. Thus the fiber functors F_{x_k} and F_x are canonically isomorphic.

Finally, by [15, Exp. IX, Prop. 1.7], the inclusion $\mathfrak{X}_k^{\mathrm{red}} \hookrightarrow \mathfrak{X}_k$ induces an isomorphism $\pi_1(\mathfrak{X}_k^{\mathrm{red}}, x_k) \xrightarrow{\sim} \pi_1(\mathfrak{X}_k, x_k)$. Thus we get an isomorphism

$$\pi_1(\mathfrak{X}_k^{\mathrm{red}}, x_k) \xrightarrow{\sim} \pi_1(\mathfrak{X}_k, x_k) = \pi_1(\mathfrak{X}, x_k) = \pi_1(\mathfrak{X}, x)$$

and hence a commutative diagram

$$\begin{array}{ccccc} & \nearrow & \pi_1(\mathfrak{X}', x) & = & \pi_1(\mathfrak{X}'_k, x'_k) \\ \pi_1(X, x) & & \downarrow{\scriptstyle \alpha_*} & & \downarrow{\scriptstyle \alpha_{k*}} \\ & \searrow & \pi_1(\mathfrak{X}, x) & = & \pi_1(\mathfrak{X}_k^{\mathrm{red}}, x_k). \end{array} \tag{11}$$

For $\overline{\gamma} \in \pi_1(X, x)$ choose an element $\gamma \in \pi_1(U, x)$ which maps to $\overline{\gamma}$ and let $\overline{\gamma}_k$ be the image of $\overline{\gamma}$ in $\pi_1(\mathfrak{X}'_k, x'_k)$. Fix a point $z \in W(\mathbb{C}_p)$ which maps to $x \in U(\mathbb{C}_p)$ in diagram (9). As explained at the beginning of this section the automorphism $\rho_{\mathcal{E}, x_{\mathfrak{o}}}(\overline{\gamma}) \otimes k$ of $\mathcal{E}_{x_k}$ is given by the formula

$$\rho_{\mathcal{E}, x_{\mathfrak{o}}}(\overline{\gamma}) \otimes k = (\gamma z)_k^* \circ (z_k^*)^{-1}. \tag{12}$$

Here the isomorphisms z_k^* and $(\gamma z)_k^*$ are the ones in the upper row of the following commutative diagram, where we have set $\mathcal{F}_k = (\pi_k \circ (F^s \otimes_\kappa k))^* \mathcal{E}'_k$, so that $(\alpha \circ \pi \circ \delta)_k^* \mathcal{E}_k = \beta_k^* \mathcal{F}_k$. Moreover $\overline{y}_1 := \beta_k(z_k)$ and $\overline{y}_2 := \beta_k((\gamma z)_k)$ in $\mathcal{Y}_k(k)$,

$$\begin{array}{ccccccccc} \mathcal{E}_{x_k} & = & (\beta_k^* \mathcal{F}_k)_{z_k} & \xleftarrow[\sim]{z_k^*} & \Gamma(\mathcal{Z}_k, \beta_k^* \mathcal{F}_k) & \xrightarrow[\sim]{(\gamma z)_k^*} & (\beta_k^* \mathcal{F}_k)_{(\gamma z)_k} & = & \mathcal{E}_{x_k} \\ & & \| & & \uparrow{\scriptstyle \beta_k^*}\wr & & \| & & \\ \mathcal{E}_{x_k} & = & (\mathcal{F}_k)_{\overline{y}_1} & \xleftarrow[\sim]{\overline{y}_1^*} & \Gamma(\mathcal{Y}_k, \mathcal{F}_k) & \xrightarrow[\sim]{\overline{y}_2^*} & (\mathcal{F}_k)_{\overline{y}_2} & = & \mathcal{E}_{x_k}. \end{array} \tag{13}$$

Note here that $\mathcal{F}_k$ is already a trivial bundle and that $\mathcal{Y}_k$ and $\mathcal{Z}_k$ are both reduced and connected. It follows that all maps in this diagram are isomorphisms. Using (12) we therefore get the formula:

$$\rho_{\mathcal{E}, x_{\mathfrak{o}}}(\overline{\gamma}) \otimes k = \overline{y}_2^* \circ (\overline{y}_1^*)^{-1}. \tag{14}$$

The point $y = \delta_{\overline{\mathbb{Q}}_p}(z)$ in $V(\mathbb{C}_p) \subset Y(\mathbb{C}_p)$ lies above x and we have $\gamma y = \delta_{\overline{\mathbb{Q}}_p}(\gamma z)$. Moreover the relations

$$(F^s \otimes_\kappa k)(\overline{y}_1) = y_k \quad \text{and} \quad (F^s \otimes_\kappa k)(\overline{y}_2) = (\gamma y)_k = \overline{\gamma}_k(y_k) \tag{15}$$

hold because $\gamma y = \overline{\gamma} y$ implies that $(\gamma y)_k = (\overline{\gamma} y)_k = \overline{\gamma}_k(y_k)$. Setting $\mathcal{G}_k = (F^s \otimes_\kappa k)^* \mathcal{E}'_k$, a bundle on $\mathfrak{X}'_k$, we have $\mathcal{F}_k = \pi_k^* \mathcal{G}_k$.

Next we look at representations of Nori's fundamental group. For the point $\overline{x}_1 = \pi_k(\overline{y}_1)$ in $\mathfrak{X}'_k(k)$ we have $(F^s \otimes k)(\overline{x}_1) = x'_k$.

Consider the commutative diagram:

$$(16)\qquad \begin{array}{ccc} \pi_1(\mathfrak{X}'_k, \overline{x}_1) & \xrightarrow{\lambda_{\mathcal{G}_k, \overline{x}_1}} & \mathrm{GL}((\mathcal{G}_k)_{\overline{x}_1}) \\ {\scriptstyle (F^s \otimes k)_*} \downarrow \wr & & \| \\ \pi_1(\mathfrak{X}'_k, x'_k) & \xrightarrow{\lambda_{\mathcal{E}'_k, x'_k}} & \mathrm{GL}((\mathcal{E}'_k)_{x'_k}) \\ {\scriptstyle \alpha_{k*}} \downarrow & & \| \\ \pi_1(\mathfrak{X}^{\mathrm{red}}_k, x_k) & \xrightarrow{\lambda_{\mathcal{E}^{\mathrm{red}}_k, x_k}} & \mathrm{GL}(\mathcal{E}_{x_k}). \end{array}$$

It is obtained by passing to the groups of k-valued points in the corresponding diagram for representations of Nori's fundamental group schemes. Recall that as observed above $\mathcal{E}^{\mathrm{red}}_k$ is an essentially finite bundle on $\mathfrak{X}^{\mathrm{red}}_k$. The fact that $(F^s \otimes k)_*$ is an isomorphism on fundamental groups was shown in Proposition 4. Let $\tilde{\gamma}_k \in \pi_1(\mathfrak{X}'_k, \overline{x}_1)$ be the element with $(F^s \otimes k)_*(\tilde{\gamma}_k) = \overline{\gamma}_k$. Using the diagrams (11) and (16), Theorem 1 will follow once we have shown the equation

$$(17)\qquad \rho_{\mathcal{E}, x_o}(\overline{\gamma}) \otimes k = \lambda_{\mathcal{G}_k, \overline{x}_1}(\tilde{\gamma}_k) \quad \text{in} \quad \mathrm{GL}(\mathcal{E}_{x_k}).$$

We now use the description of $\rho_{\mathcal{E}, x_o} \otimes k$ in formula (14) and the one of $\lambda_{\mathcal{G}_k, \overline{x}_1}$ in Proposition 2 applied to the finite étale covering $\pi_k : \mathcal{Y}_k \to \mathfrak{X}'_k$ which trivializes $\mathcal{G}_k$. It follows that (17) is equivalent to the following diagram being commutative where we recall that $\mathcal{F}_k = \pi_k^* \mathcal{G}_k$:

$$(18)\qquad \begin{array}{ccccccccc} \mathcal{E}_{x_k} & = & (\mathcal{F}_k)_{\overline{y}_1} & \xleftarrow[\sim]{\overline{y}_1^*} & \Gamma(\mathcal{Y}_k, \mathcal{F}_k) & \xrightarrow[\sim]{\overline{y}_2^*} & (\mathcal{F}_k)_{\overline{y}_2} & = & \mathcal{E}_{x_k} \\ \| & & & & & & & & \| \\ \mathcal{E}_{x_k} & = & (\mathcal{F}_k)_{\overline{y}_1} & \xleftarrow[\sim]{\overline{y}_1^*} & \Gamma(\mathcal{Y}_k, \mathcal{F}_k) & \xrightarrow[\sim]{\tilde{\gamma}_k(\overline{y}_1)^*} & (\mathcal{F}_k)_{\tilde{\gamma}_k(\overline{y}_1)} & = & \mathcal{E}_{x_k}. \end{array}$$

But this is trivial since we have $\overline{y}_2 = \tilde{\gamma}_k(\overline{y}_1)$. Namely (15) implies the equations:

$$(F^s \otimes k)(\overline{y}_2) = \overline{\gamma}_k(y_k) = \overline{\gamma}_k((F^s \otimes k)(\overline{y}_1)) = (F^s \otimes k)(\tilde{\gamma}_k(\overline{y}_1))$$

and $F^s \otimes k$ is injective on k-valued points because F is universally injective.

Example 5. The following example shows that in general the representation

$$\rho_{\mathcal{E}, x_o} : \pi_1(X, x) \to \mathrm{GL}(\mathcal{E}_{x_o})$$

in Theorem 1 does not factor over the specialization map $\pi_1(X, x) \to \pi_1(\mathfrak{X}^{\mathrm{red}}_k, x_k)$. Let $\mathfrak{X}$ be an elliptic curve over $\overline{\mathbb{Z}}_p$ whose reduction $\mathfrak{X}_k$ is supersingular. Then we have $\mathfrak{X}^{\mathrm{red}}_k = \mathfrak{X}_k$ and $\pi_1(\mathfrak{X}_k, 0)(p) = 0$. The exact functor $E \mapsto \rho_{E,0}$ of [5] or [7] induces a homomorphism

$$\rho_* : \mathrm{Ext}^1_{X_{\mathbb{C}_p}}(\mathcal{O}, \mathcal{O}) \longrightarrow \mathrm{Ext}^1_{\pi_1(X,0)}(\mathbb{C}_p, \mathbb{C}_p) = \mathrm{Hom}(\pi_1(X, 0), \mathbb{C}_p).$$

Here the second Ext-group refers to the category of finite dimensional $\mathbb{C}_p$-vector spaces with a continuous $\pi_1(X,0)$-operation. Moreover, Hom refers to continuous homomorphisms. In [4, Cor. 1], by comparing with Hodge-Tate theory it is shown that ρ_* is injective. For an extension of vector bundles $0 \to \mathcal{O} \to E \to \mathcal{O} \to 0$ on $X_{\mathbb{C}_p}$ the corresponding representation $\rho_{E,0}$ of $\pi_1(X,0)$ on $\mathrm{GL}(E_0)$ is unipotent of rank 2 and described by the additive character

$$\rho_*([E]) \in \mathrm{Hom}(\pi_1(X,0), \mathbb{C}_p) = \mathrm{Hom}(\pi_1(X,0)(p), \mathbb{C}_p).$$

In particular $\rho_{E,0}$ factors over $\pi_1(X,0)(p)$ and $\rho_{E,0}$ is trivial if and only if $[E] = 0$. Thus any extension $[\mathcal{E}]$ in $H^1(\mathfrak{X}, \mathcal{O})$ whose restriction to $H^1(X, \mathcal{O})$ is non-trivial has a non-trivial associated representation

$$\rho_{\mathcal{E},0} : \pi_1(X,0) \longrightarrow \mathrm{GL}(\mathcal{E}_0).$$

Since $\rho_{\mathcal{E},0}$ factors over $\pi_1(X,0)(p)$ it cannot factor over $\pi_1(\mathfrak{X}_k, 0)$ because then it would factor over $\pi_1(\mathfrak{X}_k, 0)(p) = 0$.

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