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Diagonal classes and the Bloch–Kato conjecture

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Dedicated to Christopher Deninger on the occasion of his 60th birthday

Abstract. The aim of this note is twofold. Firstly, we prove an explicit reciprocity law for certain diagonal classes in the étale cohomology of the triple product of a modular curve, stated in [8] and used there as a crucial ingredient in the proof of the main results. Secondly, we apply the aforementioned reciprocity law to address the rank-zero case of the equivariant Bloch–Kato conjecture for the self-dual motive of an elliptic newform of weight $k \geqslant 2$. In the special case $k = 2$, our result gives a self-contained and simpler proof of the main result of [15].

1. Introduction

Let $p \geqslant 5$ be a rational prime and let $N \geqslant 1$ be an integer. Fix algebraic closures $\bar{\mathbf{Q}}$ and $\bar{\mathbf{Q}}_p$ of $\mathbf{Q}$ and $\mathbf{Q}_p$, respectively, embeddings $i_\infty\colon \bar{\mathbf{Q}} \hookrightarrow \mathbf{C}$ and $i_p\colon \bar{\mathbf{Q}} \hookrightarrow \bar{\mathbf{Q}}_p$ and a finite extension L of $\mathbf{Q}_p(\mu_N)$. For each positive integers n and u, denote by $M_u(n,\chi)_L$ the space of complex modular forms of weight u, level $\Gamma_1(n)$, character $\chi\colon (\mathbf{Z}/n\mathbf{Z})^* \to L^*$ and Fourier coefficients in $\bar{\mathbf{Q}} \cap L$, and by $S_u(n,\chi)_L$ the subspace of cuspidal modular forms.

In the rest of the introduction, assume that $p \nmid N$ and consider three (nonzero) cusp forms

$$f \in S_k(N,\chi_f)_L, \quad g \in S_l(N,\chi_g)_L \quad \text{and} \quad h \in S_m(N,\chi_h)_L$$

of weights $k \geqslant 2$, $l \geqslant 1$ and $m \geqslant 1$, respectively, which are eigenvectors for the Hecke operator T_ℓ for each prime ℓ which does not divide N, and satisfy the *self-duality condition*

$$\chi_f \cdot \chi_g \cdot \chi_h = 1. \tag{1}$$

Denote by $\mathtt{D}(f)$ the Deligne p-adic representation of (the primitive form associated with) f, and by $V(f)$ the tensor product of $\mathtt{D}(f)$ with the f-isotypic component of $S_k(N,\chi_f)_L$. If $G_{\mathbf{Q}} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$, the $L[G_{\mathbf{Q}}]$-module $V(f)$ is then (non-canonically) isomorphic to the direct sum of a finite number of copies of $\mathtt{D}(f)$. If ξ denotes either g or h, define similarly $V(\xi)$, after replacing $\mathtt{D}(\xi)$ with the Deligne–Serre representation $\mathtt{DS}(\xi)$ if the weight of ξ is equal to one. Equation (1) implies that $k+l+m$ is even and that the $G_{\mathbf{Q}}$-representation

$$V(f,g,h) = V(f) \otimes_L V(g) \otimes_L V(h) \otimes_{\mathbf{Z}_p} \mathbf{Z}_p\big((k+l+m-2)/2\big)$$

is Kummer self-dual, viz. it is isomorphic to its L-linear dual representation twisted by $\mathbf{Z}_p(1)$.

1.1. **The geometric and balanced case.** Assume in this section that the triple (k,l,m) is *geometric* and *balanced*, that is, $l \geqslant 2, m \geqslant 2$ and k,l and m are the lengths of the sides of a triangle. In this setting [8] associates to (f,g,h) a *diagonal class* $\kappa(f,g,h)$ in the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ of the $G_{\mathbf{Q}}$-representation $V(f,g,h)$. (Its construction is recalled in Section 2.) The first aim of this note is to prove Theorem A below, a generalization of the *explicit reciprocity law* for $\kappa(f,g,h)$ stated as Proposition 3.5 in [8] and used as a crucial ingredient in the proof of the main results of [8] and [9].

We first introduce the relevant notations. Assume that p does not divide N, and denote by ξ one of f,g and h. Let α_ξ and β_ξ be the roots of the Hecke polynomial $h_{p,\xi}(X) = X^2 - \lambda_p(\xi)\cdot X + \chi_\xi(p)p^{u-1}$, where $T_p\xi = \lambda_p(\xi)\cdot\xi$ and u is the weight of ξ. Enlarging L if necessary, assume it contains $\mathbf{Q}_p(\alpha_\xi, \beta_\xi, \mu_N)$. Assume in the rest of the paper that

$$\alpha_\xi \neq \beta_\xi.$$

Assume moreover that $\mathrm{ord}_p(\alpha_\xi) < k-1$. Denote by $V_{\mathrm{dR}}(f,g,h)$ the filtered L-module $D_{\mathrm{dR}}(V(f,g,h))$ associated by Fontaine to $V(f,g,h)$. The Faltings comparison isomorphism and (a suitably twisted) Poincaré duality identify the Bloch–Kato p-adic logarithm of (the restriction at p of) $\kappa(f,g,h)$ with a linear functional

$$\log_p(\kappa(f,g,h))\colon \mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h) \to L$$

(cp. Section 3.1.2). The L-module $\mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h)$ has dimension four, and contains a distinguished class

$$\eta_f^\alpha \otimes \omega_g \otimes \omega_h \in \mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h).$$

Here ω_ξ is de Rham class in $V_{\mathrm{dR}}(\xi) = D_{\mathrm{cris}}(V(\xi))$ corresponding to ξ under the Faltings comparison isomorphism and η_f^α is a natural element in $V_{\mathrm{dR}}(f)^{\varphi=\alpha_f}$ associated with f, where φ is the crystalline Frobenius. (We refer to Section 3.1.3 for precise definitions.) The explicit reciprocity law relates the value of $\log_p(\kappa(f,g,h))$ at $\eta_f^\alpha \otimes \omega_g \otimes \omega_h$ to a *p-adic period* $I_p(f,g,h)$ which we now define.

Let $f^w = w_N f$ in $M_k(N, \bar\chi_f)_L$ be the image of f under the Atkin–Lehner operator $w = w_N$. One has $T_p f^w = \bar\chi(p)\lambda_p(f)\cdot f^w$, so that $\bar\chi(p)\cdot\alpha_f$ and $\bar\chi(p)\cdot\beta_f$ are the roots of the p-th Hecke polynomial $h_{p,f^w}(X)$. Define

$$f_\alpha^w \in S_k(Np, \bar\chi_f)_L \tag{2}$$

to be the p-stabilizations of f^w satisfying $U_p f^w_\alpha = \bar{\chi}_f(p)\alpha_f \cdot f^w_\alpha$. Regard g and h as p-adic modular forms and let

$$\Xi_k(g,h) = d^{(k-l-m)/2} g^{[p]} \times h,$$

where $g^{[p]}$ and $d^{(k-l-m)/2} g^{[p]}$ are defined as follows. If g has q-expansion $\sum_{n\geqslant 0} a_n(g)\cdot q^n$, then its *$p$-depletion* $g^{[p]}$ is the weight-l p-adic modular form with q-expansion $\sum_{n\nmid p} a_n(g)\cdot q^n$ (cp. Equation (15)). Let $d = q\frac{d}{dq}$ be Serre's derivative operator on $L[\![q]\!]$, which sends (the q-expansion of) a p-adic modular form of weight u to a p-adic modular form of weight $u+2$. For each integer n (not necessarily positive), the sequence of p-adic modular forms $d^{n+(p-1)p^m} g^{[p]}$, then converges, for $m \to \infty$, to a p-adic modular form $d^n g^{[p]}$ of weight $l+2n$. It follows that $\Xi_k(g,h)$ defines a p-adic modular form of weight k. As proved in Section 4.7 (see in particular Equation (46)) the form $\Xi_k(g,h)$ belongs to the space $M_k^{\text{n-o}}(N,L)$ of nearly-overconvergent forms of weight k defined over L (cp. Section 3.3 or [41, 14]). Under the additional assumption $\mathrm{ord}_p(\alpha_f) < k-1$, the work of Coleman defines a natural *f^w_α-isotypic projection*

$$e_{f^w_\alpha} : M_k^{\text{n-o}}(N,L) \to S_k(Np,L)_{f^w_\alpha},$$

where $S_k(N,L)_{f^w_\alpha}$ is the f^w_α-isotypic component of $S_k(Np,\chi_f)_L$ (cp. Section 3.3). In this case define

$$I_p(f,g,h) = \frac{(f^w_\alpha, e_{f^w_\alpha}\cdot \Xi_k(g,h))_{Np}}{(f^w_\alpha, f^w_\alpha)_{Np}},$$

where $(\zeta,\xi)_M = \int_{Y_1(M)} \zeta(z)\bar{\xi}(z) y^{u-2}\,dx\,dy$ is the Petersson scalar product on $S_u(M,\mathbf{C})$.[1] It is easily seen that the p-adic period $I_p(f,g,h)$ is algebraic and belongs to L.

Theorem A. *Assume that $p\nmid N$ and that $\mathrm{ord}_p(\alpha_f) < k-1$. Then*

$$\log_p(\kappa(f,g,h))(\eta_f^\alpha \otimes \omega_g \otimes \omega_h)$$

is equal to

$$\frac{(-1)^k N^{c-2}(c-k)!(1-\frac{\beta_f}{\alpha_f})(1-\frac{\beta_f}{p\alpha_f})}{(1-\frac{\beta_f\alpha_g\alpha_h}{p^c})(1-\frac{\beta_f\alpha_g\beta_h}{p^c})(1-\frac{\beta_f\beta_g\alpha_h}{p^c})(1-\frac{\beta_f\beta_g\beta_h}{p^c})}\cdot I_p(f,g,h),$$

where $c = c(k,l,m)$ denotes the positive integer $(k+l+m-2)/2$.

The proof of Theorem A is given in Section 4. It uses the work of Bannai, Bannai–Kings, Besser, Nekovář, Nizioł [1, 2, 10, 11, 31, 35, 36, 33] in an essential way. See also [5, 4, 14, 6, 7, 26] for related results.

[1]If f^w_α is ordinary (i.e., $\mathrm{ord}_p(\alpha_f)=0$), there is no need to prove that $\Xi_k(g,h)$ is nearly-overconvergent in order to define $e_{f^w_\alpha}\cdot\Xi_k(g,h)$ and $I_p(f,g,h)$. In this case Hida [22] defines an ordinary projector e_{ord} from the space $\mathbf{M}_k(N,L)$ of weight-k p-adic modular forms over L to the space $M_k^{\text{ord}}(Np,L)$ of classical p-ordinary modular forms. The composition of e_{ord} with the natural projection $M_k^{\text{ord}}(Np,L) \to S_k(Np,L)_{f^w_\alpha}$ onto the f^w_α-isotypic component is an extension of the Coleman morphism $e_{f^w_\alpha}$ to $\mathbf{M}_k(N,L)$.

1.2. **Applications to the Bloch–Kato conjecture.** Throughout this section, (f,g,h) is a triple of *newforms* of weights $(k,l,m)=(k,1,1)$ and conductors (N_f,N_g,N_h). The following assumption is in force.

Assumption 1.3.

1. The product of χ_f,χ_g and χ_h is the trivial character.
2. p does not divide $N_f\cdot N_g\cdot N_h$ and $(N_f,N_g,N_h)=1$.
3. For $\xi=g,h$ the p-th Hecke polynomial $X^2-a_p(\xi)\cdot X+\chi_\xi(p)$ is separable.
4. f is p-ordinary (that is its p-th Fourier coefficient is a p-adic unit).

Let

$$\mathrm{Sel}(\mathbf{Q},V(f,g,h))\hookrightarrow H^1(\mathbf{Q},V(f,g,h))$$

be the Bloch–Kato Selmer group of the $G_{\mathbf{Q}}$-representation $V(f,g,h)$ and let

$$H^1_{\mathrm{str}}(\mathbf{Q},V(f,g,h))=\ker\big(\mathrm{res}_p\colon \mathrm{Sel}(\mathbf{Q},V(f,g,h))\to H^1(\mathbf{Q}_p,V(f,g,h))\big)$$

be its *strict* Selmer subgroup. Write $L(f\otimes g\otimes h,s)$ for the complex L-series of the tensor product of the motives of f,g and h. Under Assumptions 1.3.1 and 1.3.2, it admits an analytic continuation and satisfies a functional equation with sign $+1$ at the central critical point $s=k/2$. The following theorem (proved in Section 5) is the main result of this note.

Theorem B. *If $L(f\otimes g\otimes h,s)$ does not vanish at $s=k/2$, then the Selmer group $\mathrm{Sel}(\mathbf{Q},V(f,g,h))$ is equal to the strict Selmer group $H^1_{\mathrm{str}}(\mathbf{Q},V(f,g,h))$.*

The Bloch–Kato conjecture predicts that the Selmer group $\mathrm{Sel}(\mathbf{Q},V(f,g,h))$ is trivial if (and only if) the L-series $L(f\otimes g\otimes h,s)$ does not vanish at the central critical point $s=k/2$. As explained below, the methods of this paper fall short of proving this conjecture. Nonetheless, the previous result provides strong evidence in support of it.

When $k=2$, Theorem B gives a significantly simpler proof of the main result proved by Darmon and Rotger in [15] (cp. Section 1.3.1 below) and has important applications to the equivariant Birch and Swinnerton-Dyer conjecture. Let A be an elliptic curve defined over the rationals and let $L=L_\varrho$ be the splitting field of the tensor product $\varrho=\varrho_1\otimes\varrho_2$ of two irreducible, odd Artin representations satisfying $\det(\varrho_1)=\det(\varrho_2)^{-1}$. Then Theorem B and the Serre modularity conjecture prove that the non-vanishing of the L-series $L(A,\varrho,s)$ at $s=1$ implies the triviality of the ϱ-isotypic component $A(L)^\varrho=(A(L)\otimes_{\mathbf{Z}}V_\varrho)^{\mathrm{Gal}(L/\mathbf{Q})}$ of the Mordell–Weil group of A over L. Indeed, $L(A,\varrho,s)=L(f\otimes g\otimes h,s)$, where f,g and h are the cusp forms associated with A, ϱ_1 and ϱ_2 by modularity, and a non-torsion element of $A(L)^\varrho$ gives rise, via the p-adic Kummer map, to a class in $\mathrm{Sel}(\mathbf{Q},V(f,g,h))$ with nontrivial restriction at p, id est not in $H^1_{\mathrm{str}}(\mathbf{Q},V(f,g,h))$. One can then apply Theorem B with any (carefully choosen) prime p for which Assumption 1.3 is satisfied.

More generally, let f be a newform of weight $k\geqslant 2$ and let $\varrho=\varrho_1\otimes\varrho_2$ be as above. The representation $V(f)$ can be realized in the middle cohomology $\mathscr{V}_k=H^{k-1}_{\text{ét}}(\mathcal{E}^{k-2}\otimes_{\mathbf{Q}}\bar{\mathbf{Q}},\mathbf{Q}_p)$ of the i-fold fibre product $\mathcal{E}^i=\mathcal{E}_1(N)^i$ of the universal elliptic curve $\mathcal{E}_1(N)\to Y_1(N)$ over the open modular curve of level

$\Gamma_1(N)$ over $\mathbf{Q}$. The p-adic Abel–Jacobi map and the f-isotypic projection $\mathscr{V}_k \to V(f)$ gives a morphism

$$r_p\colon \mathrm{CH}^{k/2}(\mathcal{E}_L^{k-2})_0 \to \mathrm{Sel}(L, V_f),$$

where $\mathcal{E}_L^i = \mathcal{E}^i \otimes_{\mathbf{Q}} L$, the $\mathrm{Gal}(L/\mathbf{Q})$-module $\mathrm{CH}^i(\,\cdot\,)_0$ is the Chow group of homologically trivial codimension i cycles on $\cdot$ modulo rational equivalence and V_f denotes the $k/2$-th Tate twist of $V(f)$. If (Assumption 1.3 is satisfied and) $L(f, \varrho, s) = L(f \otimes g \otimes h, s)$ does not vanish at $s = k/2$, Theorem B proves that r_p maps the ϱ-component $\mathrm{CH}^{k/2}(\mathcal{E}_L^{k-2})_0^{\varrho} = H^0(\mathrm{Gal}(L/\mathbf{Q}), \mathrm{CH}^{k/2}(\mathcal{E}_L^{k-2})_0 \otimes_{\mathbf{Z}} V_{\varrho})$ to the restricted Selmer group $H^1_{\mathrm{str}}(\mathbf{Q}, V(f, g, h))$. In contrast with the weight two case, when $k > 2$, this is far from proving the (conjectural) vanishing of the f-isotypic component of $\mathrm{CH}^{k/2}(\mathcal{E}_L^{k-2})_0^{\varrho}$, as the injectivity of the Abel–Jacobi maps is arguably the deepest aspect of the Beilinson–Bloch–Kato conjectures. Despite this, Theorem B still provides strong evidence in support of the Bloch–Kato conjecture for the ϱ-twist of the self-dual motive associated with f.

1.3.1. *Outline of the proof and comparison with* [15]. The general strategy underlying the proof of Theorem B dates back to Kato's work on the cyclotomic main conjecture, as revisited and extended in a series of recent works, including [14, 4, 28, 7, 42, 15, 27]. It can be summarized as follows. (We refer the reader to Section 5 for the actual proof of Theorem B.)

For $\xi = g, h$, fix a root α_ξ of the Hecke polynomial $X^2 - a_p(\xi) \cdot X + \chi_\xi(p)$ and write $\xi_\alpha(q) = \xi(q) - (\chi_\xi(p)/\alpha) \cdot \xi(q^p)$ for the corresponding p-stabilization of ξ. According to a result of Wiles, there exist Hida families $\boldsymbol{g} = \boldsymbol{g}_\alpha$ and $\boldsymbol{h} = \boldsymbol{h}_\alpha$ specializing, respectively, to g_α and h_α in weight one. For each integer u in a dense subset of a small p-adic disc U centered at one, the constructions outlined in the previous section associate to f and the weight-u specializations $\boldsymbol{g}_u$ and $\boldsymbol{h}_u$ an algebraic number $I_p(f, \boldsymbol{g}_u, \boldsymbol{h}_u)$. A method due to Hida (cp. [23]) shows that these algebraic numbers are p-adically interpolated by an analytic function $\mathscr{L}_p(f, \boldsymbol{gh})$ on U. Thanks to the proof by Harris–Kudla of a conjecture of Jacquet, the value of $\mathscr{L}_p(f, \boldsymbol{gh})$ at $u = 1$ is related to the complex special value $L(f \otimes g \otimes h, k/2)$. The key technical step in the proof of Theorem B consists in showing that there exists a class $\kappa(f, \boldsymbol{gh})$, in a suitable big Selmer group with coefficients in the Tate algebra of analytic functions on U, such that

$$\mathscr{L}_p(f, \boldsymbol{gh}) = \mathcal{L}\big(\mathrm{res}_p(\kappa(f, \boldsymbol{gh}))\big), \tag{3}$$

where $\mathcal{L}$ is a branch of the appropriate Perrin-Riou big logarithm map. (We refer to Theorem 5.3 for a precise statement of this result.) Once this is proved, the previous discussion relates $L(f \otimes g \otimes h, k/2)$ to the value at $u = 1$ of the right-hand side of Equation (3), which in turn is related by results of Colmez–Perrin-Riou to the Bloch–Kato dual exponential of the specialization $\kappa(f, g_\alpha, h_\alpha)$ of $\kappa(f, \boldsymbol{gh})$ at $u = 1$. Assuming that $L(f \otimes g \otimes h, s)$ does not vanish at $s = k/2$, this produces a *ramified class* $\kappa(f, g_\alpha, h_\alpha)$ in the relaxed-at-p Selmer group of $V(f, g, h)$ over $\mathbf{Q}$. Under Assumption 1.3.3, one actually produces *four* ramified classes $\kappa(f, g_i, h_j)$, one for each choice of the roots i and j of the p-th Hecke polynomials of g and h. The p-adic residues of these classes are easily seen to be linearly independent, hence Theorem B follows from an application of Poitou–Tate duality.

Theorem 5.3 (or better its proof) shows that Equation (3) can be deduced directly from Theorem A and a simple density argument. More precisely, take a sequence u_i of integers congruent to 1 modulo $p-1$, which converges to infinity in the ordinary topology and to 1 in the p-adic topology (e.g., take $u_i = 1 + (p-1)p^i$). We prove that the *existence* of a class $\kappa(f, \boldsymbol{gh})$ satisfying Equation (3) is a direct consequence of the explicit reciprocity law at each crystalline weight-u_i specialization $(f, \boldsymbol{g}_{u_i}, \boldsymbol{h}_{u_i})$ of the triple $(f, \boldsymbol{g}, \boldsymbol{h})$. For this strategy to work, it is crucial to use the good integrality properties enjoyed by the diagonal classes introduced in [8] (cp. Section 2 and the proof of Theorem 5.3). This simple method applies to the study of the analytic rank-zero case of the equivariant Bloch–Kato conjecture in many other interesting settings (e.g., the one considered in [7]).

In the significant special case $k = 2$, Theorem B recasts the main result of [15]. The proof of the latter follows a different pattern. More precisely, *loc. cit.* constructs an explicit class $\kappa(f, \boldsymbol{gh})$ satisfying the identity (3) by using delicate geometric arguments. For each positive integer s, a *twisted diagonal cycle* is defined in the Chow group of codimension two cycles in the triple product of the modular curve $X_1(Np^s)$ of level $\Gamma_1(Np^s)$ over $\mathbf{Q}$. The p-adic Abel–Jacobi images of these cycles satisfy certain compatibilities under the natural maps from $X_1(Np^{s+1})$ to $X_1(Np^s)$, from which $\kappa(f, \boldsymbol{gh})$ arises as the inverse limit of classes in the ordinary parts of the middle étale cohomology with constant coefficients of the cubes of the curves $X_1(Np^s)$. Once $\kappa(f, \boldsymbol{gh})$ is constructed, reciprocity laws for its specializations at triples of the form $(f, \boldsymbol{g}_{2,\chi}, \boldsymbol{h}_{2,\chi^{-1}})$ are proved, where $\boldsymbol{g}_{2,\chi}$ denotes the *non-crystalline* specialization of $\boldsymbol{g}$ at an arithmetic point of weight 2 and character χ of conductor divisible by p. This entails working on varieties with *bad* reduction at p, which makes it harder to obtain the reciprocity laws directly. In this special setting, Equation (3) follows from these reciprocity laws and the properties of the Perrin-Riou logarithm.

2. DIAGONAL CLASSES

This section recalls the definition of the diagonal classes introduced in [8], to which we refer for more details.

Let $N \geqslant 3$ be a positive integer and let $Y_1(N)$ be the affine modular curve of level $\Gamma_1(N)$ over $\mathbf{Z}[1/N]$, classifying isomorphism classes of pairs (E, P), where E is an elliptic curve over a $\mathbf{Z}[1/N]$-scheme S and P is a section in $E(S)$ of exact order N. Let R be a $\mathbf{Z}[1/N]$-algebra, let $Y = Y_1(N)_R$ be the base change of $Y_1(N)$ to R and let $v\colon E \to Y$ be the universal elliptic curve over Y. There is a natural functor $\cdot_{\text{ét}}$ from the category of p-adic representations of $\mathrm{GL}_2(\mathbf{Z}_p)$ to the category of p-adic étale sheaves on Y. If St denotes the standard representation of $\mathrm{GL}_2(\mathbf{Z}_p)$, then $\mathscr{S} = \mathrm{St}_{\text{ét}}$ is equal to the relative étale cohomology $R^1 v_* \mathbf{Z}_p$ of E over Y. In particular, one has $\det_{\text{ét}} = \mathbf{Z}_p(-1)$ for the determinant det of St (see [8, Section 3] and the references therein, in particular, [19, Prop. A I.8] for more details). For each nonnegative integer u, denote by $S_u = \mathrm{Symm}^u_{\mathbf{Z}_p}(\mathrm{St})$ the symmetric quotient of the u-fold tensor power of St and by $\mathscr{S}_u = \mathrm{Symm}^u_{\mathbf{Z}_p}\mathscr{S}$ the étale sheaf corresponding to S_u under $\cdot_{\text{ét}}$. Write $H^{\cdot}_{\text{ét}}(Y, \mathscr{S}_u)$ for the continuous étale cohomology groups (in the sense of Janssen [24]) of Y with coefficients in $\mathscr{S}_u$.

Notation. In this rest of this section $Y = Y_1(N)_{\mathbf{Q}}$ denotes the modular curve over $\mathbf{Q}$. We also fix a rational prime $p > 3$.

Let (k, l, m) be a *balanced* triple in $(\mathbf{Z}_{\geqslant 2})^3$ such that $k + l + m$ is *even*. (Balanced means that k, l and m are the lengths of the sides of a triangle.) The Clebsch–Gordan decomposition of classical invariant theory gives a canonical generator $\mathtt{Det}_{\boldsymbol{r}}$ of $H^0(\mathrm{GL}_2(\mathbf{Z}_p), S_{\boldsymbol{r}} \otimes \det^{-r})$, where $\boldsymbol{r} = (r_1, r_2, r_3)$ is equal to $(k-2, l-2, m-2)$, r is equal to $(r_1 + r_2 + r_3)/2$ and $S_{\boldsymbol{r}}$ is a shorthand for $S_{r_1} \otimes_{\mathbf{Z}_p} S_{r_2} \otimes_{\mathbf{Z}_p} S_{r_3}$. After setting $\mathscr{S}_{\boldsymbol{r}} = \mathscr{S}_{r_1} \otimes_{\mathbf{Z}_p} \mathscr{S}_{r_2} \otimes_{\mathbf{Z}_p} \mathscr{S}_{r_3}$, the invariant $\mathtt{Det}_{\boldsymbol{r}}$ corresponds (under $\cdot_{\text{ét}}$) to a global section

$$\mathtt{Det}^{\text{ét}}_{\boldsymbol{r}} = \mathtt{Det}^{\text{ét}}_{N,\boldsymbol{r}} \in H^0_{\text{ét}}(Y, \mathscr{S}_{\boldsymbol{r}}(r)).$$

Let $d \colon Y \hookrightarrow Y^3$ be the diagonal embedding and let

$$\mathscr{S}_{[\boldsymbol{r}]} = \mathscr{S}_{r_1} \boxtimes \mathscr{S}_{r_2} \boxtimes \mathscr{S}_{r_3},$$

so that $d^* \mathscr{S}_{[\boldsymbol{r}]} = \mathscr{S}_{\boldsymbol{r}}$. The push-forward of $\mathtt{Det}^{\text{ét}}_{\boldsymbol{r}}$ along d gives a class in $H^4_{\text{ét}}(Y^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2))$, and the Hochschild–Serre spectral sequence yields a natural map $\mathtt{HS}_{\text{ét}}$ from $H^4_{\text{ét}}(Y^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2))$ to the global Galois cohomology group $H^1(\mathbf{Q}, \mathtt{W}_{N,\boldsymbol{r}})$ of the lattice

$$\mathtt{W}_{N,\boldsymbol{r}} = H^3_{\text{ét}}(Y^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{[\boldsymbol{r}]})(r+2)$$

in the p-adic representation $W_{N,\boldsymbol{r}} = \mathtt{W}_{N,\boldsymbol{r}} \otimes_{\mathbf{Z}} \mathbf{Q}$. The class

$$\kappa_{N,\boldsymbol{r}} = \mathtt{HS}_{\text{ét}} \circ d_*(\mathtt{Det}^{\text{ét}}_{\boldsymbol{r}}) \in H^1(\mathbf{Q}, \mathtt{W}_{N,\boldsymbol{r}}) \tag{4}$$

is called the *diagonal class* of level N and weights (k, l, m). The results of [33] imply that (after inverting p) $\kappa_{N,\boldsymbol{r}}$ belongs to the Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, W_{N,\boldsymbol{r}})$ of $W_{N,\boldsymbol{r}}$ over $\mathbf{Q}$ (cp. [8] and Section 4.1 below).

Let L be a finite extension of $\mathbf{Q}_p$ and consider a triple of modular forms

$$f \in S_k(N, \chi_f)_L, \quad g \in S_l(N, \chi_g)_L \quad \text{and} \quad h \in S_m(N, \chi_h)_L,$$

where (k, l, m) is a balanced triple with $k, l, m \geqslant 2$ and $k + l + m$ even. Assume that f, g and h are (nonzero) eigenforms for the Hecke operator T_ℓ with eigenvalues $\lambda_\ell(f), \lambda_\ell(g)$ and $\lambda_\ell(h)$, for each prime ℓ not dividing N. As in the introduction, assume in addition that they satisfy the self-duality condition Equation (1), namely, that the product of the characters of f, g and h is the trivial character modulo N. Let

$$\mathrm{pr}_{fgh} \colon W_{N,\boldsymbol{r}} \otimes_{\mathbf{Q}_p} L \to V(f, g, h)$$

be the maximal L-quotient of $W_{N,\boldsymbol{r}} \otimes_{\mathbf{Q}_p} L$ on which the Hecke operator $T_\ell \otimes \mathrm{id} \otimes \mathrm{id}$ (resp., $\mathrm{id} \otimes T_\ell \otimes \mathrm{id}$, $\mathrm{id} \otimes \mathrm{id} \otimes T_\ell$) acts as multiplication by $\lambda_\ell(f)$ (resp., $\lambda_\ell(g)$, $\lambda_\ell(h)$) for each prime ℓ not dividing $N/p^{\mathrm{ord}_p(N)}$, and $\langle d_1\rangle \otimes \langle d_2\rangle \otimes \langle d_3\rangle$ acts as multiplication by $\chi_f(d_1)\cdot\chi_g(d_2)\cdot\chi_h(d_3)$ for each d_i in $(\mathbf{Z}/N\mathbf{Z})^*$. The $L[G_{\mathbf{Q}}]$-module $V(f,g,h)$ is a direct summand of $W_{N,\boldsymbol{r}} \otimes_{\mathbf{Q}_p} L$, isomorphic to the direct sum of a finite number of copies of the $(r+2)$-th Tate twist of the tensor product of the L-adic Deligne representations of f, g and h. Define

$$\kappa(f,g,h) = \mathrm{pr}_{fgh*}(\kappa_{N,\boldsymbol{r}}) \in \mathrm{Sel}(\mathbf{Q}, V(f,g,h))$$

to be the image of $\kappa_{N,\boldsymbol{r}}$ under the map induced in cohomology by pr_{fgh}.

3. Cohomology and modular forms

This section briefly recalls the needed facts on the de Rham and rigid cohomology of modular curves over $\mathbf{Z}_p$. We refer to [25, 39, 13, 2, 5] for the details.

Notation. In this section $Y = Y_1(N)_{\mathbf{Q}_p}$ and $X = X_1(N)_{\mathbf{Q}_p}$ denote the open and compact modular curves of level $\Gamma_1(N)$ over $\mathbf{Q}_p$. Let $C = X - Y$ and let $u\colon E \to Y$ be the universal elliptic curve. Let L be a finite extension of $\mathbf{Q}_p(\zeta_N)$, where $\zeta_N = e^{2\pi i/N}$.

3.1. **De Rham cohomology.** Let $\boldsymbol{\omega} = u_*\Omega^1_{E/Y}$ and $\mathscr{S}_{\mathrm{dR}} = \mathbf{R}^1 u_*\Omega^\bullet_{E/Y}$ denote, respectively, the line bundle of relative differentials and the first relative de Rham cohomology of E/Y, extended to vector bundles on X as in [39, Section 2.3]. For $i \geqslant 0$, set $\mathscr{S}_{\mathrm{dR},i} = \mathrm{Symm}^i_{\mathscr{O}_X}\mathscr{S}_{\mathrm{dR}}$ and $\boldsymbol{\omega}^i = \boldsymbol{\omega}^{\otimes i}$; one has a natural isomorphism between $\boldsymbol{\omega}^2$ and $\Omega^1_X(\log C)$, called the Kodaira–Spencer isomorphism. For $0 \leqslant q \leqslant i$, denote by $\mathrm{Fil}^q\mathscr{S}_{\mathrm{dR},i} = \boldsymbol{\omega}^q \otimes_{\mathscr{O}_X} \mathscr{S}_{\mathrm{dR},i-q}$ the q-th step in the Hodge filtration and by $\mathscr{S}^{\cdot}_{\mathrm{dR},i} = \mathscr{S}^{\cdot}_{\mathrm{dR},i}(X)$ the logarithmic de Rham complex of X:

$$\mathscr{S}^{\cdot}_{\mathrm{dR},i} = \big[\nabla\colon \mathscr{S}_{\mathrm{dR},i} \to \mathscr{S}_{\mathrm{dR},i} \otimes_{\mathscr{O}_X} \Omega^1_X(\log C)\big]$$

(concentrated in degrees zero and one), where ∇ is the Gauß–Manin connection. For each open subscheme U of X, write $\mathscr{S}^{\cdot}_{\mathrm{dR},i}(U)$ for the restriction of $\mathscr{S}^{\cdot}_{\mathrm{dR},i}$ to U. Write

$$H_{\mathrm{dR}}(Y,\mathscr{S}_i) = H\Gamma(Y,\mathscr{S}^{\cdot}_{\mathrm{dR},i}(Y)) \tag{5}$$

for the de Rham cohomology of Y with values in $(\mathscr{S}_{\mathrm{dR},i}(Y), \mathrm{Fil}^\bullet, \nabla)$. According to [16, Cor. II.3.15], this is naturally isomorphic to the de Rham cohomology $H_{\mathrm{dR}}(X,\mathscr{S}_i) = H_{\mathrm{dR}}(X,\mathscr{S}^{\cdot}_{\mathrm{dR},i})$, viz. to the cohomology groups of the derived complex $\mathbf{R}\Gamma(X,\mathscr{S}^{\cdot}_{\mathrm{dR},i})$. The Hodge filtration and the Kodaira–Spencer isomorphism then give a natural isomorphism

$$M_{i+2}(N,L) = \mathrm{Fil}^1 H^1_{\mathrm{dR}}(Y,\mathscr{S}_i)_L,$$

where $M_i(N,L) = \Gamma(X,\boldsymbol{\omega}^i)_L$ is the space of weight-i modular forms of level $\Gamma_1(N)$ defined over L.

3.1.1. *Comparison with étale cohomology.* Let $k \geqslant 2$ and let f in $S_k(N,\chi_f)_L$ be an eigenvector for the Hecke operator T_ℓ, with eigenvalue $\lambda_\ell(f)$, for each prime ℓ not dividing $N_o = N/p^{\mathrm{ord}_p(N)}$. Denote by

$$V_{\mathrm{dR}}(f) = H^0(\mathbf{Q}_p, B_{\mathrm{dR}} \otimes_{\mathbf{Q}_p} V(f))$$

the de Rham module of the restriction to $G_{\mathbf{Q}_p}$ of the $G_{\mathbf{Q}}$-representation $V(f)$ defined in the introduction. The comparison isomorphism between étale and de Rham cohomology proved by Faltings–Tsuji [18, 40] yields a natural isomorphism of filtered modules

$$V_{\mathrm{dR}}(f) \cong H^1_{\mathrm{dR}}(Y, \mathscr{S}_{k-2})_f, \tag{6}$$

where the right-hand side is the direct summand of $H^1_{\mathrm{dR}}(Y,\mathscr{S}_{k-2})_L$ on which the Hecke operator T_ℓ (resp., diamond operator $\langle d\rangle$) acts as multiplication by $\lambda_\ell(f)$ (resp., $\chi_f(d)$) for each prime ℓ not dividing N_o (resp., each unit d in $\mathbf{Z}/N\mathbf{Z}$). We identify $V_{\mathrm{dR}}(f)$ with a direct summand of $H^1_{\mathrm{dR}}(Y,\mathscr{S}_{k-2})_L$ under the previous isomorphism, so that the f-isotypic component $S_k(N,L)_f$ of $M_k(N,L)$ becomes identified with $\mathrm{Fil}^1 V_{\mathrm{dR}}(f)$. Define

$$\omega_f \in \mathrm{Fil}^1 V_{\mathrm{dR}}(f)$$

to be the element corresponding to the modular form f in $M_k(N,L)_f$ under these identifications.

If (f,g,h) is a triple of modular forms as in Section 2, the isomorphism (6) and the Künneth decomposition for de Rham cohomology induce a natural isomorphism of filtered modules (considered as an equality)

$$V_{\mathrm{dR}}(f,g,h) \cong H^3_{\mathrm{dR}}(Y^3, \mathscr{S}_{[\boldsymbol{r}]})_{fgh} \otimes_{\mathbf{Q}_p} \mathbf{Q}_p[r+2]. \tag{7}$$

Here $V_{\mathrm{dR}}(f,g,h) = H^0(\mathbf{Q}_p, V(f,g,h)\otimes_{\mathbf{Q}_p} B_{\mathrm{dR}})$ and $\mathbf{Q}_p[n] = D_{\mathrm{dR}}(\mathbf{Q}_p(n))$ for each n in $\mathbf{Z}$. The filtered vector bundle with connection $\mathscr{S}_{[\boldsymbol{r}],\mathrm{dR}}$ on Y^3 is defined by $\mathscr{S}_{\mathrm{dR},k-2} \boxtimes \mathscr{S}_{\mathrm{dR},l-2} \boxtimes \mathscr{S}_{\mathrm{dR},m-2}$. Finally, the fgh-isotypic component $H^3_{\mathrm{dR}}(Y^3,\mathscr{S}_{\boldsymbol{r}})_{fgh}$ of $H^3_{\mathrm{dR}}(Y^3,\mathscr{S}_{[\boldsymbol{r}]})_L = H^3_{\mathrm{dR}}(Y^3,\mathscr{S}_{\mathrm{dR},[\boldsymbol{r}]})_L$ is defined as in Section 2.

3.1.2. *Duality.* Let

$$(\,\cdot\,,\cdot\,)\colon \mathscr{S}_{\mathrm{dR}} \otimes_{\mathscr{O}_Y} \mathscr{S}_{\mathrm{dR}} \to \mathscr{O}_Y(-1)$$

be the perfect relative Poincaré duality pairing, arising from the dualities $(\,\cdot\,,\cdot\,)_x\colon H^1_{\mathrm{dR}}(E_x/k)\otimes_{\mathbf{Q}_p} H^1_{\mathrm{dR}}(E_x/k) \to k$ on the fibres at $x\colon \mathrm{Spec}(k)\to Y$ (with k a field extension of $\mathbf{Q}_p$). Here $\mathscr{O}_Y(n)$ (for n in $\mathbf{Z}$) denotes the sheaf $\mathscr{O}_Y$, equipped with the trivial connection and with the filtration $\mathrm{Fil}^\bullet\mathscr{O}_Y(n)$, given by $\mathrm{Fil}^q\mathbf{Q}_p(n) = \mathscr{O}_Y$ for $q \leqslant -n$ and $\mathrm{Fil}^q\mathscr{O}_Y(n) = 0$ for $q \geqslant 1-n$. For each $i \geqslant 0$, the pairing $(\,\cdot\,,\cdot\,)$ induces a duality

$$(\,\cdot\,,\cdot\,)_i\colon \mathscr{S}_{\mathrm{dR},i} \otimes_{\mathscr{O}_Y} \mathscr{S}_{\mathrm{dR},i} \to \mathscr{O}_Y(-i), \tag{8}$$

whose restriction to the fibre at $x\colon \mathrm{Spec}(k)\to Y$ is given by

$$(\boldsymbol{\alpha},\boldsymbol{\beta})_{i,x} = \frac{1}{i!}\sum_{\sigma\in S_i} (\alpha_1,\beta_{\sigma(1)})_x \cdots (\alpha_i,\beta_{\sigma(i)})_x \tag{9}$$

for each $\boldsymbol{\alpha} = \alpha_1 \cdots \alpha_i$ and $\boldsymbol{\beta} = \beta_1 \cdots \beta_i$ in $\mathrm{Symm}^i_k H^1_{\mathrm{dR}}(E_x/k)$. This in turn induces a perfect duality

$$(10)\quad (\,\cdot\,,\cdot\,)_i \colon H^1_{\mathrm{dR}}(Y,\mathscr{S}_i)\otimes_{\mathbf{Q}_p} H^1_{\mathrm{dR},c}(Y,\mathscr{S}_i) \to H^2_{\mathrm{dR},c}(Y,\mathscr{O}_Y(-i)) \cong \mathbf{Q}_p[-i-1].$$

Let (f,g,h) be as in Section 2 and (as in the introduction) set $\xi^w = w_N\xi$, for ξ equal to f, g and h. As ξ^w is cuspidal, the morphism $H^1_{\mathrm{dR},c} \to H^1_{\mathrm{dR}}$ maps the ξ^w-isotypic component of $H^1_{\mathrm{dR},c}(Y,\mathscr{S}_i)_L$ isomorphically onto $V_{\mathrm{dR}}(\xi^w)$ (cp. Equation (6)), and $(\,\cdot\,,\cdot\,)_{u+2}$ induces a perfect pairing

$$(11)\qquad (\,\cdot\,,\cdot\,)_\xi \colon V_{\mathrm{dR}}(\xi)\otimes_L V_{\mathrm{dR}}(\xi^w) \to L[1-u],$$

where u is the weight of ξ. With a slight abuse of notation, write again

$$w_N \colon H^1_{\mathrm{dR},\cdot}(Y,\mathscr{S}_i) \to H^1_{\mathrm{dR},\cdot}(Y,\mathscr{S}_i)$$

for the geometric Atkin–Lehner isomorphism (cp. [8, Section 2.3.1]), which induces an isomorphism $w_N \colon V_{\mathrm{dR}}(\xi) \to V_{\mathrm{dR}}(\xi^w)$. The composition of $(\,\cdot\,,\cdot\,)_\xi$ and $\mathrm{id}\otimes w_N$ then yields a perfect duality

$$\langle\,\cdot\,,\cdot\,\rangle_\xi \colon V_{\mathrm{dR}}(\xi)\otimes_L V_{\mathrm{dR}}(\xi) \to L[1-u],$$

under which $S_u(N,L)_\xi = \mathrm{Fil}^1 V_{\mathrm{dR}}(\xi)$ is the orthogonal complement of itself.

Define the perfect duality

$$(12)\qquad \langle\,\cdot\,,\cdot\,\rangle_{fgh} \colon V_{\mathrm{dR}}(f,g,h)\otimes_L V_{\mathrm{dR}}(f,g,h) \to L[1]$$

to be the tensor product of the pairings $\langle\,\cdot\,,\cdot\,\rangle_\xi$ for $\xi = f,g,h$. As easily checked, the Bloch–Kato exponential gives an isomorphism $\exp_p$ between the tangent space $\mathrm{tg}_{\mathrm{dR}}(f,g,h)$ of $V_{\mathrm{dR}}(f,g,h)$ and the finite part $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V^*(f,g,h))$ of the local cohomology group $H^1(\mathbf{Q}_p, V(f,g,h))$. After identifying $\mathrm{tg}_{\mathrm{dR}}(f,g,h)$ with the L-linear dual of $\mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h)$ via the perfect duality $\langle\,\cdot\,,\cdot\,\rangle_{fgh}$, the inverse of $\exp_p$ then gives rise to an L-linear isomorphism

$$\log_p \colon H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(f,g,h)) \cong \mathrm{Hom}_L\big(\mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h), L\big).$$

In particular, the image under $\log_p$ of (the restriction at p of) the Selmer class $\kappa(f,g,h)$ yields a functional

$$(13)\qquad \log_p(\kappa(f,g,h)) \colon \mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h) \to L.$$

3.1.3. *The class η^α_f.* Assume in this section $\mathrm{ord}_p(N) \leqslant 1$ and let f be as in Section 3.1.1. Assume in addition that p does not divide the conductor of the character of f. Then $V(f)$ is a semi-stable representation of $G_{\mathbf{Q}_p}$. As a consequence, $V_{\mathrm{dR}}(f) = H^0(\mathbf{Q}_p, B_{\mathrm{st}}\otimes_{\mathbf{Q}_p} V(f))$ is equipped with a semi-stable Frobenius endomorphism φ. As in the introduction, let α_f and β_f be the roots of the p-th Hecke polynomial $h_{f,p}(X) = X^2 - \lambda_p(f)\cdot X + \chi_f(p)p^{k-1}$ and assume that L contains $\mathbf{Q}_p(\alpha_f,\beta_f)$. Under the assumptions of Section 1.1 the characteristic polynomial of φ is a power of $h_{f,p}(X)$ and $V_{\mathrm{dR}}(f)$ is the direct sum of $\mathrm{Fil}^1 V_{\mathrm{dR}}(f) = S_k(N,L)_f$ and the φ-eigenspace $V_{\mathrm{dR}}(f)^{\varphi=\alpha_f}$ (cp. [38]). It follows from this and Section 3.1.2 that there exists a unique de Rham class

$$\eta^\alpha_f \in V_{\mathrm{dR}}(f)^{\varphi=\alpha_f}$$

such that, for each ξ in $S_k(N,L)_f$, one has (cp. the introduction)

$$\langle \eta_f^\alpha, \omega_\xi\rangle_f = \frac{(\xi^w, f^w)_N}{(f^w, f^w)_N}.$$

If (f,g,h) is a triple of modular forms as in Section 2, the Künneth product of η_f^α, ω_g and ω_h defines a class

$$\eta_f^\alpha \otimes \omega_g \otimes \omega_h \in \mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h). \tag{14}$$

(To show that the class $\eta_f^\alpha \otimes \omega_g \otimes \omega_h$ indeed belongs to the zeroth step of the Hodge filtration of $V_{\mathrm{dR}}(f,g,h)$, note that $\mathrm{Fil}^1 V_{\mathrm{dR}}(\xi) = \mathrm{Fil}^{u-1} V_{\mathrm{dR}}(\xi)$ for a modular form ξ of weight u and recall that the triple (k,l,m) is *balanced*.)

3.2. ***p*-adic modular forms.** Let X^{rig} and Y^{rig} be the rigid analytic varieties over $\mathbf{Q}_p$ associated with X and Y, respectively, and let X^{ord} and Y^{ord} be their ordinary loci. Let L be a finite extension of $\mathbf{Q}_p(\boldsymbol{\mu}_N)$ and fix a generator ζ_N of $\boldsymbol{\mu}_N(L)$. For each integer s, denote by

$$\mathbf{M}_s(N,L) = \Gamma(X^{\mathrm{ord}}, \boldsymbol{\omega}^s)_L$$

the space of Katz p-adic modular forms of weight s and level $\Gamma_1(N)$ defined over L. Let $R_N = \mathcal{O}_L[\![q]\!] \otimes_{\mathbf{Z}} \mathbf{Q}$ and let $\mathrm{Tate}(q) = (\mathbf{G}_m/q^{\mathbf{Z}}, \zeta_N)$ be the Tate generalized elliptic curve with $\Gamma_1(N)$-level structure over R_N. As $\mathrm{Tate}(q)$ is defined by a global affine equation $y^2 + xy = x^3 + b(q)\cdot x + c(q)$ over $\mathbf{Z}[\![q]\!]$, the invertible sheaf $\boldsymbol{\omega}|_{\mathrm{Tate}(q)} = \imath^*\boldsymbol{\omega}$ has a canonical generator $\omega_{\mathrm{can}} = dx/(2y+x)$ (cp. [25, Section A.1.2]). Given a section ω of $\boldsymbol{\omega}^s$ over a neighborhood of $\mathrm{Tate}(q)$, its restriction $\omega|_{\mathrm{Tate}(q)}$ to $\mathrm{Tate}(q)$ is then of the form $f_\omega \cdot \omega_{\mathrm{can}}^s$ for a unique element f_ω in R_N, called the q-expansion of ω. The q-expansion map indeed gives an *injective* morphism

$$\mathbf{M}_s(N,L) \hookrightarrow R_N,$$

which we consider as an inclusion. If f in R_N is the q-expansion of a p-adic modular form of weight s, we write ω_f for the corresponding section of $\boldsymbol{\omega}^{s-2}$ over the ordinary locus (so that $\omega = \omega_{f_\omega}$).

The module $\mathbf{M}_s(N,L)$ is equipped with the action of the Hecke operator $U = U_p$ and of the Verschiebung V, defined on q-expansions by

$$U\Big(\sum_{n\geqslant 0} a_n \cdot q^n\Big) = \sum_{n\geqslant 0} a_{np}\cdot q^n \quad \text{and} \quad V\Big(\sum_{n\geqslant 0} a_n \cdot q^n\Big) = \sum_{n\geqslant 0} a_n \cdot q^{np},$$

respectively. In particular, for each p-adic modular form $f = \sum_{n\geqslant 0} a_n(f)\cdot q^n$ in $\mathbf{M}_s(N,L)$, its *p-depletion*

$$f^{[p]} = (1 - VU)f = \sum_{p\nmid n} a_n(f)\cdot q^n \tag{15}$$

is again a p-adic modular form of weight s. The derivation $d = q\frac{d}{dq}$ on R_N restricts to *Serre's operator*

$$d\colon \mathbf{M}_s(N,L) \to \mathbf{M}_{s+2}(N,L).$$

In addition, $\mathbf{M}_s(N,L)$ is equipped with the Hecke operators T_ℓ and $\langle d\rangle$ for primes ℓ not dividing Np and units d in $\mathbf{Z}/N\mathbf{Z}$, which restrict to the usual Hecke operators on the space $M_s(N,L)$ of *classical* modular forms if $s \geqslant 0$.

3.3. **Rigid cohomology.** In this section p does not divide N, so that $Y_1(N)_{\mathbf{Z}_p}$ and $X_1(N)_{\mathbf{Z}_p}$ are smooth models of Y and X, respectively, over $\mathbf{Z}_p$.

Denote by $\imath\colon Y^{\mathrm{rig}} \hookrightarrow X^{\mathrm{rig}}$ and by $\jmath\colon X^{\mathrm{ord}} \hookrightarrow X^{\mathrm{rig}}$ the natural inclusions and by $\imath^\dagger$ and $\jmath^\dagger$ the corresponding Berthelot functors from the category of abelian sheaves on X^{rig} to itself [3]. If $\mathscr{F}$ is a coherent sheaf on X and $\kappa = \imath, \jmath$, we write $\kappa^\dagger\mathscr{F}$ for the image of the analytic sheaf $\mathscr{F}|_{X^{\mathrm{rig}}}$ under $\kappa^\dagger$. Set

$$\mathscr{S}^{\cdot}_{\mathrm{rig},i} = \imath^\dagger \mathscr{S}^{\cdot}_{\mathrm{dR},i}$$

and denote again by $\mathrm{Fil}^{\cdot}$ and ∇ the filtration and connection on

$$\mathscr{S}_{\mathrm{rig},i} = \mathscr{S}^{0}_{\mathrm{rig},i}$$

induced by the corresponding structures on $\mathscr{S}_{\mathrm{dR},i}$. The abelian sheaf $\mathscr{S}_{\mathrm{rig},i}$ is also equipped with a Frobenius endomorphism φ, such that $(\mathscr{S}_{\mathrm{rig},i}, \mathrm{Fil}^{\cdot}, \nabla, \varphi)$ is an overconvergent filtered φ-isocrystal on the special fibre $Y_{\mathbf{F}_p}$ of $Y_1(N)_{\mathbf{Z}_p}$ (cp. [2, Appendix A]). According to a result of Dwork [25, Thm. A2.3.6], the restriction of $\mathscr{S}_{\mathrm{rig}} = \mathscr{S}_{\mathrm{rig},1}$ to the ordinary locus admits a unique φ-equivariant splitting $\mathrm{spl}^{ur}\colon \mathscr{S}_{\mathrm{rig}}|_{Y^{\mathrm{ord}}} \to \mathrm{Fil}^1\mathscr{S}_{\mathrm{rig}}|_{Y^{\mathrm{ord}}} = \boldsymbol{\omega}|_{Y^{\mathrm{ord}}}$ of the Hodge filtration such that the Frobenius φ acts invertibly on its kernel. Write again

$$\mathrm{spl}^{ur}\colon \mathscr{S}_{\mathrm{rig},i}|_{Y^{\mathrm{ord}}} \to \boldsymbol{\omega}^i|_{Y^{\mathrm{ord}}}$$

for the map induced on the i-th symmetric powers, called the *unit root splitting.*

The cohomology of $\mathbf{R}\Gamma(X^{\mathrm{rig}}, \imath^\dagger\mathscr{S}^{\cdot}_{\mathrm{dR},i})$ and $\mathbf{R}\Gamma(X^{\mathrm{rig}}, \jmath^\dagger\mathscr{S}^{\cdot}_{\mathrm{dR},i})$ compute the rigid cohomology groups

$$H_{\mathrm{rig}}(Y_{\mathbf{F}_p}, \mathscr{S}_i) = H_{\mathrm{rig}}(Y_{\mathbf{F}_p}/\mathbf{Q}_p, \imath^\dagger\mathscr{S}_{\mathrm{dR},i})$$

and

$$H_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_i) = H_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}/\mathbf{Q}_p, \jmath^\dagger\mathscr{S}_{\mathrm{dR},i}),$$

respectively, where $Y_{\mathbf{F}_p} = Y_1(N)_{\mathbf{F}_p}$ and $Y^{\mathrm{ord}}_{\mathbf{F}_p}$ is the complement in $Y_{\mathbf{F}_p}$ of the finitely many $\mathbf{F}_{p^2}$-rational supersingular points. Theorem 5.4 of [13] proves that the Hodge filtration induces an isomorphism

$$[\,\cdot\,]_{i+2}\colon \frac{M^\dagger_{i+2}(N,L)}{d^{i+1}M^\dagger_{-i}(N,L)} \cong H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_i)_L. \tag{16}$$

Here $M^\dagger_s(N,L) = \Gamma(X^{\mathrm{rig}}, \jmath^\dagger\boldsymbol{\omega}^s)_L$ is the space of overconvergent modular forms of level weight $s \in \mathbf{Z}$ and level $\Gamma_1(N)$ defined over L, and d^{i+2} is the $(i+2)$-th iterate of the Serre derivative operator d (denote by θ in *loc. cit.*). The L-submodule $M^\dagger_s(N,L)$ of $\mathbf{M}_s(N,L)$ is invariant under the action of the Hecke operators U, T_ℓ for primes ℓ not dividing Np, $\langle d\rangle$ for units d in $(\mathbf{Z}/N\mathbf{Z})^*$, and under the action of the Verschiebung V. Loc. cit. proves that the isomorphism

$[\,\cdot\,]_{i+2}$ intertwines the action of the rigid Frobenius φ on $H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_i)$ with that of $p^{i+1}\langle p\rangle V$ on overconvergent modular forms, that is,

(17) $$\varphi \circ [\,\cdot\,]_{i+2} = [\,\cdot\,]_{i+2} \circ p^{i+1}\langle p\rangle V.$$

(Note that our model $Y_1(N)$ of the modular curve of level $\Gamma_1(N)$, in which Tate(q) is *not* defined over $\mathbf{Q}$, differs from the one used in [13]. This explains the appearance of the diamond operator $\langle p\rangle$ in the previous equation.)

The restriction of the unit-root splitting to the global sections of $\mathscr{S}_{\mathrm{rig},i}$ and the Kodaira–Spencer isomorphism induce an injective map

$$\mathrm{spl}^{ur} \colon \Gamma\big(X^{\mathrm{rig}}, \mathscr{S}^1_{\mathrm{rig},i}\big)_L \hookrightarrow \mathbf{M}_{i+2}(N,L).$$

Its image $M^{\text{n-o}}_{i+2}(N,L)$ is called the space of nearly-overconvergent modular forms. The composition of the inverse of $[\,\cdot\,]_{i+2}$ with the natural map

$$\Gamma\big(X^{\mathrm{rig}}, \mathscr{S}^1_{\mathrm{rig},i}\big) \to H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_i)$$

then yields a morphism

(18) $$e^\dagger \colon M^{\text{n-o}}_{i+2}(N,L) \to M^\dagger_{i+2}(N,L)/d^{i+1}M^\dagger_{-i}(N,L).$$

Let f in $S_k(N,\chi_f)_L$ be a cusp form of weight $k \geqslant 2$, level $\Gamma_1(N)$, character $\chi_f \colon (\mathbf{Z}/N\mathbf{Z})^* \to L^*$ and Fourier coefficients in L. Assume that f is an eigenvector of the Hecke operator T_ℓ, with eigenvalue $a_\ell(f)$, for each prime ℓ not dividing N. Let α_f, β_f and $f^w_\alpha \in S_k(Np,\bar\chi_f)_L$ be as in Section 1.1 (see in particular Equation (2)). Define

(19) $$H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{k-2})_L \twoheadrightarrow H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{k-2})_{f^w_\alpha}$$

to be the maximal quotient on which

$$\varphi = \bar\chi_f(p)\cdot\beta_f, \quad T_\ell = \bar\chi_f(p)\cdot a_\ell(f) \quad \text{and} \quad \langle d\rangle = \bar\chi_f(d)$$

for each prime ℓ not dividing Np and each unit d in $\mathbf{Z}/N\mathbf{Z}$. According to Equation (17), the inclusion $S_k(Np,L) \hookrightarrow M^\dagger_k(N,L)$ and the Coleman isomorphism $[\,\cdot\,]_k$ defined in Equation (16) induce a morphism

$$[\,\cdot\,]^\alpha_f \colon S_k(Np,L)_{f^w_\alpha} \to H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{k-2})_{f^w_\alpha},$$

where $S_k(Np,L)_{f^w_\alpha}$ is the f^w_α-isotypic quotient of $S_k(Np,L)$.

If one further assumes that f^w_α has *small slope*, viz. $\mathrm{ord}_p(\alpha_f) < k-1$, then $[\,\cdot\,]^\alpha_f$ is an *isomorphism*:

(20) $$[\,\cdot\,]^\alpha_f \colon S_k(Np,L)_{f^w_\alpha} \cong H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{k-2})_{f^w_\alpha}.$$

Indeed, [13, Thm. 6.1 and Lem. 6.3] proves that the natural map

$$S_k(Np,L) \to M^\dagger_k(N,L)/d^{k-1}M^\dagger_{2-k}(M,L)$$

induces an isomorphism on the f^w_α-isotypic quotients, provided that f^w_α has small slope. In this case, define

(21) $$e_{f^w_\alpha} \colon M^{\text{n-o}}_k(N,L) \to S_k(Np,L)_{f^w_\alpha}$$

to be the composition of the morphism $e^{\dagger}$ defined in Equation (18) with the projection to the f_α^w-isotypic quotient. The morphism $e_{f_\alpha^w}$ is the (*Coleman*) *f_α^w-isotypic projector* mentioned in Section 1.1.

3.4. **Explicit formulas (cp.** [2, Section 4]**).** Let $\tilde{\mathcal{Y}} \to Y^{\mathrm{ord}}$ be the affine formal scheme over $\mathbf{Z}_p$ which classifies trivialized elliptic curves with $\Gamma_1(N)$-level structure defined over p-rings. (We recall that a *trivialization* on an elliptic $E \to S$ is an S-isomorphism between the formal multiplicative group $\hat{\mathbf{G}}_m$ over S and the formal completion $\hat{E}$ of E along the zero section.) Let $\tilde{\mathbf{M}}(N, \mathbf{Z}_p)$ be the coordinate ring of $\tilde{\mathcal{Y}}$, the space of *Katz generalized p-adic modular forms* of level $\Gamma_1(N)$. Write $\tilde{R}_N$ for the p-adic completion of $\mathbf{Z}_p[\zeta_N]((q))$. Evaluation at the Tate curve $\mathrm{Tate}(q)$ over $\tilde{R}_N$ gives a q-expansion map

$$\tilde{\mathbf{M}}(N, \mathbf{Z}_p) \hookrightarrow \tilde{R}_N,$$

which we consider as an inclusion. Then $\tilde{\mathbf{M}}(N, \mathbf{Z}_p)$ is invariant under the action on $\tilde{R}_N$ of the Hecke operator U, of the Verschiebung V and of Serre's derivative operator $d = q\frac{d}{dq}$.

Denote by $\tilde{\boldsymbol{\omega}}$ and $\tilde{\mathscr{S}}_{\mathrm{rig},i}$ the restrictions of $\boldsymbol{\omega}$ and $\mathscr{S}_{\mathrm{rig},i}$, respectively, to $\tilde{\mathcal{Y}}$. These are *free* $\tilde{\mathbf{M}}(N, \mathbf{Z}_p)$-modules. More precisely, let $\mathcal{E} \to \tilde{\mathcal{Y}}$ be the universal elliptic curve with trivialization $\psi\colon \hat{\mathbf{G}}_m \cong \hat{\mathcal{E}}$. The line bundle $\tilde{\boldsymbol{\omega}}$ is then generated by the global section $\tilde{\omega}_{\mathrm{can}}$ satisfying $\psi^*\tilde{\omega}_{\mathrm{can}} = dT/(1+T)$ (with $\mathbf{G}_m = \mathrm{Spec}(\mathbf{Z}[T, T^{-1}])$), which specializes to ω_{can} on $\mathrm{Tate}(q)$. Let $\tilde{\Omega}$ be the module of Kähler differentials of the $\mathbf{Z}_p$-algebra $\tilde{\mathbf{M}}(N, \mathbf{Z}_p)$ and denote by $\tilde{\delta}_{\mathrm{can}}$ the differential in $\tilde{\Omega}$ corresponding to $\tilde{\omega}_{\mathrm{can}}^2$ under the Kodaira–Spencer isomorphism. The derivation of $\tilde{M}(N, \mathbf{Z}_p)$ corresponding to $\tilde{\delta}_{\mathrm{can}}$ is Serre's operator d. After setting $\tilde{\eta}_{\mathrm{can}} = \nabla_d(\tilde{\omega}_{\mathrm{can}})$, one has

$$\tilde{\mathscr{S}}_{\mathrm{rig}} = \tilde{\mathbf{M}}(N, \mathbf{Z}_p) \cdot \tilde{\omega}_{\mathrm{can}} \oplus \tilde{\mathbf{M}}(N, \mathbf{Z}_p) \cdot \tilde{\eta}_{\mathrm{can}},$$

and the action of the Gauß–Manin connection ∇ is described by the formula

$$\nabla(f \cdot \tilde{\omega}_{\mathrm{can}} + g \cdot \tilde{\eta}_{\mathrm{can}}) = \big(df \cdot \tilde{\omega}_{\mathrm{can}} + (f + dg) \cdot \tilde{\eta}_{\mathrm{can}}\big) \otimes \tilde{\delta}_{\mathrm{can}}. \tag{22}$$

The action of the Frobenius φ can also be described explicitly (paying some attention to the fact that $\mathrm{Tate}(q)$ is not defined over $\mathbf{Q}$). In particular,

$$\varphi \begin{pmatrix} \tilde{\omega}_{\mathrm{can}} \\ \tilde{\eta}_{\mathrm{can}} \end{pmatrix} = \begin{pmatrix} p & \\ & 1 \end{pmatrix} \begin{pmatrix} \tilde{\omega}_{\mathrm{can}} \\ \tilde{\eta}_{\mathrm{can}} \end{pmatrix}. \tag{23}$$

Let i be an integer, let L be a finite extension of $\mathbf{Q}_p[\zeta_N]$ and write $\tilde{\mathbf{M}}(N, L)$ for the base change of $\tilde{\mathbf{M}}(N, \mathbf{Z}_p)$ to L. Identify $\Gamma(\tilde{\mathcal{Y}}, \tilde{\boldsymbol{\omega}}^i)$ with $\tilde{\mathbf{M}}(N, \mathbf{Z}_p)$ via $\tilde{\omega}_{\mathrm{can}}$, and $\boldsymbol{\omega}^2$ with $\Omega_X^1(\log C)$ under the Kodaira–Spencer isomorphism. Then restriction to $\tilde{\mathcal{Y}}$ gives an injective map $\mathbf{M}_i(N, L) \hookrightarrow \tilde{\mathbf{M}}(N, L)$ compatible with the q-expansion maps, which we consider as an inclusion. As the pullback of $\tilde{\delta}_{\mathrm{can}}$ to the Tate curve is equal to dq/q, one deduces that the restriction to $\tilde{\mathcal{Y}}$ of a classical modular form f in $M_{i+2}(N, L)$ is given by $f(q) \cdot \tilde{\omega}_{\mathrm{can}}^i \otimes \tilde{\delta}_{\mathrm{can}}$.

4. PROOF OF THEOREM A

This section proves Theorem A stated in Section 1.1.

Notation. Let the notations and assumptions be as in *loc. cit.* In particular, $N \geqslant 1$ is a positive integer not divisible by p and (k, l, m) is a geometric balanced triple in $(\mathbf{Z}_{\geqslant 2})^3$. Throughout this section one writes $Y = Y_1(N)_{\mathbf{Z}_p}$ and $X = X_1(N)_{\mathbf{Z}_p}$ for the open and closed modular curves over $\mathbf{Z}_p$, respectively. Moreover, (as in Section 2), $\boldsymbol{r} = (r_1, r_2, r_3)$ equals $(k-2, l-2, m-2)$ and r denotes the nonnegative integer $(r_1 + r_2 + r_3)/2$. To ease notation, in this section only we write $\mathscr{S}_{\cdot} = \mathscr{S}_{\text{ét},\cdot}$ for the $\mathbf{Q}_p$-linear extensions of the p-adic étale sheaves denoted by the same symbol in Section 2. (For example, the étale cohomology groups $H^{\cdot}_{\text{ét}}(Y, \mathscr{S}_i) = H^{\cdot}_{\text{ét}}(Y, \mathscr{S}_{\text{ét},i})$ are $\mathbf{Q}_p$-vector spaces).

4.1. **Syntomic and finite polynomial cohomology.** This section recalls the needed facts on rigid syntomic and finite polynomial cohomology. We use [12] and [2, Appendix A] as main references.

For each smooth pair $\mathscr{U} = (U, \bar{U})$ over $\mathbf{Z}_p$, write $S(\mathscr{U})$ for the category of *admissible* filtered overconvergent φ-isocrystals on $\mathscr{U}$ defined in [2, Def. A.2]. We also call an element of $S(\mathscr{U})$ a *syntomic sheaf* on $\mathscr{U}$. For each syntomic sheaf $\mathscr{F}$ on $\mathscr{U}$ and each polynomial $P(t)$ in $1 + t \cdot L[t]$, denote by $H^{\cdot}_P(\mathscr{U}, \mathscr{F})$ the Besser rigid finite-polynomial cohomology groups of $\mathscr{U}$ with values in $\mathscr{F}$. In the special case $P(t) = 1 - t$, these are the *syntomic* cohomology groups defined in *loc. cit.* and denoted by $H^{\cdot}_{\text{syn}}(\mathscr{U}, \mathscr{F})$. The definition given there readily generalizes to the more general setting considered here (cp. [10, 12]). Moreover, one can define finite polynomial cohomology groups with compact support $H^{\cdot}_{P,c}(\mathscr{U}, \mathscr{F})$ as in [12].

4.1.1. *Syntomic sheaves I: the case* $\mathscr{U} = \mathbf{Z}_p$. Write $\mathbf{Z}_p$ for the smooth pair $(\mathrm{Spec}(\mathbf{Z}_p), \mathrm{Spec}(\mathbf{Z}_p))$ and let $P(t) = \prod_i (1 - \alpha_i t)$ and $Q(t) = \prod_j (1 - \beta_j t)$ be polynomials in $1 + t \cdot L[t]$ (with α_i, β_j in $\bar{\mathbf{Q}}_p$).

The category $S(\mathbf{Z}_p)$ of syntomic sheaves on $\mathbf{Z}_p$ is simply the one of filtered φ-modules over $\mathbf{Q}_p$. For F in $S(\mathbf{Z}_p)$ consider on $F_L = F \otimes_{\mathbf{Q}_p} L$ the (induced filtration and the) L-linear endomorphism $\varphi = \varphi \otimes_{\mathbf{Q}_p} L$. Then the finite polynomial cohomology group $H^i_P(\mathbf{Z}_p, F)$ vanishes when $i \neq 0, 1$ and one has

$$H^0_P(\mathbf{Z}_p, F) = F_L^{P(\varphi)=0} \cap \mathrm{Fil}^0 F_L \quad \text{and} \quad H^1_P(\mathbf{Z}_p, F) = F_L / P(\varphi) \cdot \mathrm{Fil}^0 F_L$$

(where $F_L^{P(\varphi)=0}$ denotes the kernel of $P(\varphi)$.) Let $P \star Q(t) = \prod_{i,j}(1 - \alpha_i \beta_j t)$ and let $a(x, y)$ and $b(x, y)$ be any pair of two-variable polynomials satisfying

$$P \star Q(xy) = a(x, y) \cdot P(x) + b(x, y) \cdot Q(y).$$

Let F, G and H be filtered φ-modules, let $\gamma \colon F \otimes_{\mathbf{Q}_p} G \to H$ be a morphism of filtered φ-modules and let i, j be nonnegative integers which sum to one. Define the cup-product pairing

$$\cup_{\text{fp}} \colon H^i_P(\mathbf{Z}_p, F) \otimes_L H^j_Q(\mathbf{Z}_p, G) \to H^1_{P \star Q}(\mathbf{Z}_p, H)$$

by $cl(f) \cup_{\mathrm{fp}} g = cl(\gamma(a(x,y) \cdot f \otimes g))$ when $i = 1$, respectively, $f \cup_{\mathrm{fp}} cl(g) = cl(\gamma(b(x,y) \cdot f \otimes g))$ when $j = 1$, for each f in F and g in G, where the variables x and y act on $F \otimes_{\mathbf{Q}_p} G$ as $\varphi \otimes \mathrm{id}$ and $\mathrm{id} \otimes \varphi$, respectively.

4.1.2. *Syntomic sheaves II: the general case.* Let $\mathscr{U}$ be a smooth pair over $\mathbf{Z}_p$. A syntomic sheaf $\mathscr{F}$ in $S(\mathscr{U})$ admits (and is characterized by) de Rham and rigid realisations $\mathscr{F}_{\mathrm{dR}}$ and $\mathscr{F}_{\mathrm{rig}}$. The de Rham realization $\mathscr{F}_{\mathrm{dR}}$ is a filtered coherent $\mathscr{O}_{\bar{U}_{\mathbf{Q}_p}}$-module equipped with an integrable connection with logarithmic singularities along $\bar{U} - U$. Write $H^{\cdot}_{\mathrm{dR}}(U_{\mathbf{Q}_p}, \mathscr{F})$ for the de Rham cohomology groups $H^{\cdot}_{\mathrm{dR}}(U_{\mathbf{Q}_p}, \mathscr{F}_{\mathrm{dR}}) \cong H^{\cdot}_{\mathrm{dR}}(\bar{U}_{\mathbf{Q}_p}, \mathscr{F}_{\mathrm{dR}})$ (cp. [2, Def. A.2] and the discussion surrounding Equation (5)). The rigid realization $\mathscr{F}_{\mathrm{rig}}$ is an overconvergent filtered φ-isocrystal (in the sense of Berthelot) on the special fibre $U_{\mathbf{F}_p}$ of U. (If $\jmath\colon \mathcal{U}_{\mathbf{Q}_p} \hookrightarrow \bar{\mathcal{U}}_{\mathbf{Q}_p}$ is the natural inclusion of the Raynaud generic fibre of the p-adic completion of U into that of $\bar{U}$, then $\mathscr{F}_{\mathrm{rig}} = \jmath^{\dagger}(\mathscr{F}_{\mathrm{dR}}|U^{\mathrm{rig}})$ as a coherent $\jmath^{\dagger}\mathscr{O}_{U^{\mathrm{rig}}}$-module with connection, where U^{rig} is the rigid space over $\mathbf{Q}_p$ associated with $U_{\mathbf{Q}_p}$. See *loc. cit.* for more detials.) Denote by $H^{\cdot}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})$ the Berthelot rigid cohomology groups $H^{\cdot}_{\mathrm{rig}}(U_{\mathbf{F}_p}/\mathbf{Q}_p, \mathscr{F}_{\mathrm{rig}})$. By the admissibility of $\mathscr{F}$, the natural map from de Rham to rigid cohomology gives an isomorphism

$$H^{\cdot}_{\mathrm{dR}}(U_{\mathbf{Q}_p}, \mathscr{F}) \cong H^{\cdot}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F}),$$

which allows us to view $H^{\cdot}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})$ as a filtered φ-module, i.e., an element of $S(\mathbf{Z}_p)$. Indeed, $H^{i}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})$ is the i-th direct image $R^i\pi_*\mathscr{F}$ of $\mathscr{F}$ under the structural morphism $\pi\colon \mathscr{U} \to \mathbf{Z}_p$, and the Leray spectral sequence

$${}^{\mathrm{syn}}E_2^{p,q} = H^p_{\mathrm{syn}}(\mathbf{Z}_p, H^q_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})) \Longrightarrow H^i_{\mathrm{syn}}(\mathscr{U}, \mathscr{F})$$

degenerates into the short exact sequences

$$\begin{aligned}(24)\qquad 0 \to H^1_{\mathrm{syn}}(\mathbf{Z}_p, H^{i-1}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})) &\xrightarrow{\mathbf{i}_{\mathrm{syn}}} H^i_{\mathrm{syn}}(\mathscr{U}, \mathscr{F}) \\ &\xrightarrow{\mathbf{p}_{\mathrm{syn}}} H^0_{\mathrm{syn}}(\mathbf{Z}_p, H^i_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})) \to 0.\end{aligned}$$

More generally, for any polynomial $P(t)$ in $1 + t \cdot L[t]$ one has short exact sequences

$$\begin{aligned}(25)\qquad 0 \to H^1_P(\mathbf{Z}_p, H^{i-1}_{\mathrm{rig},\cdot}(U_{\mathbf{F}_p}, \mathscr{F})_L) &\xrightarrow{\mathbf{i}_P} H^i_{P,\cdot}(\mathscr{U}, \mathscr{F}) \\ &\xrightarrow{\mathbf{p}_P} H^0_P(\mathbf{Z}_p, H^i_{\mathrm{rig},\cdot}(U_{\mathbf{F}_p}, \mathscr{F})_L) \to 0,\end{aligned}$$

(where $\blacktriangle_L = \blacktriangle \otimes_{\mathbf{Q}_p} L$ and "$,\cdot$" $= \varnothing$, "$,c$"). If P is clear from the context, we simply write $\mathbf{i} = \mathbf{i}_P$ and $\mathbf{p} = \mathbf{p}_P$.

Let P and Q be polynomials in $1 + t \cdot L[t]$ and let $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$ be syntomic sheaves on $\mathscr{U}$. To a morphism $\mathscr{F} \otimes \mathscr{G} \to \mathscr{H}$ in $S(\mathscr{U})$, one associates as in [12, Section 2] finite polynomial cup product pairings

$$\cup_{\mathrm{fp}}\colon H^i_P(\mathscr{U}, \mathscr{F}) \otimes_L H^j_{Q,c}(\mathscr{U}, \mathscr{G}) \to H^{i+j}_{P\star Q,c}(\mathscr{U}, \mathscr{F} \otimes \mathscr{G}).$$

These are compatible with the Leray spectral sequence, viz. the diagram

$$
\begin{array}{ccccc}
H^1_P(\mathbf{Z}_p, H^{i-1}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})) & \otimes_L & H^0_Q(\mathbf{Z}_p, H^j_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathscr{G})) & \xrightarrow{\cup_{\mathrm{fp}}} & H^1_{P\star Q}(\mathbf{Z}_p, H^{i+j-1}_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathscr{H})) \\
\downarrow{\scriptstyle \mathbf{i}_P} & & \uparrow{\scriptstyle \mathbf{p}_Q} & & \downarrow{\scriptstyle \mathbf{i}_{P\star Q}} \\
H^i_P(\mathscr{U}, \mathscr{F}) & \otimes_L & H^j_{Q,c}(\mathscr{U}, \mathscr{G}) & \xrightarrow{\cup_{\mathrm{fp}}} & H^{i+j}_{P\star Q,c}(\mathscr{U}, \mathscr{H}) \\
\downarrow{\scriptstyle \mathbf{p}_P} & & \uparrow{\scriptstyle \mathbf{i}_Q} & & \uparrow{\scriptstyle \mathbf{i}_{P\star Q}} \\
H^0_P(\mathbf{Z}_p, H^{i}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F})) & \otimes_L & H^1_Q(\mathbf{Z}_p, H^{j-1}_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathscr{G})) & \xrightarrow{\cup_{\mathrm{fp}}} & H^1_{P\star Q}(\mathbf{Z}_p, H^{i+j-1}_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathscr{H}))
\end{array}
\tag{26}
$$

commutes, where the top and bottom cup-products $\cup_{\mathrm{fp}}$ are the ones associated in Section 4.1.1 with

$$\cup_{\mathrm{rig}} \colon H^{i-1}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathscr{F}) \otimes_{\mathbf{Q}_p} H^j_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathscr{G}) \to H^{i+j-1}_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathscr{H}).$$

For each integer n, denote by $\mathbf{Q}_p(n)$ the n-th Tate object in $S(\mathscr{U})$. The de Rham realization of $\mathbf{Q}_p(n)$ is the free rank-one $\mathscr{O}_{\bar{U}}$-module $\mathscr{O}_{\bar{U}} \cdot t_n$, with trivial connection and decreasing filtration given by $\mathrm{Fil}^{1-n}\mathbf{Q}_p(n) = 0$ and $\mathrm{Fil}^{-n}\mathbf{Q}_p(n) = \mathbf{Q}_p(n)$, and the Frobenius on $\mathbf{Q}_p(n)_{\mathrm{rig}}$ is defined by $\varphi(t_n) = p^{-n} \cdot t_n$. (When $\mathscr{U} = \mathbf{Z}_p$ the filtered φ-module $\mathbf{Q}_p(1)$ is then equal to $D_{\mathrm{dR}}(\mathbf{Q}_p(1))$.) If U is geometrically connected of relative dimension d over $\mathbf{Z}_p$, the trace $\mathrm{tr}_{\mathrm{rig}}$ in rigid cohomology gives an isomorphism between $H^{2d}_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathbf{Q}_p(d+1))$ and $\mathbf{Q}_p(1)$ and $\mathbf{i}_P$ is an isomorphism between $H^{2d}_{\mathrm{rig},c}(U_{\mathbf{F}_p}, \mathbf{Q}_p(d+1))_L$ and $H^{2d+1}_{P,c}(\mathscr{U}, \mathbf{Q}_p(d+1))$. Assuming that $P(t)$ does not vanish at $t = p^{-1}$, define the (normalized) trace isomorphism

$$\mathrm{tr}_P = P(p^{-1})^{-1} \cdot \mathrm{tr}_{\mathrm{rig}} \circ \mathbf{i}_P^{-1} \colon H^{2d+1}_{P,c}(\mathscr{U}, \mathbf{Q}_p(d+1)) \cong L(1).$$

Given a morphism $\mathscr{F} \otimes_{\mathbf{Q}_p} \mathscr{G} \to \mathbf{Q}_p(d+1)$ in $S(\mathscr{U})$ and polynomials P and Q in $1 + t \cdot L[t]$ such that $P \star Q(t)$ does not vanish at $t = p^{-1}$, the composition of $\cup_{\mathrm{fp}}$ and $\mathrm{tr}_{P\star Q}$ then yields cup-product pairings

$$\langle \cdot , \cdot \rangle_{\mathscr{U}} \colon H^i_P(\mathscr{U}, \mathscr{F}) \otimes_L H^{2d+1-i}_{Q,c}(\mathscr{U}, \mathscr{G}) \to L(1).$$

4.1.3. *Syntomic sheaves III: modular curves.* We are mainly interested in the smooth pairs

$$\mathscr{Y} = (Y, X) \quad \text{and} \quad \mathscr{Y}^{\mathrm{ord}} = (Y^{\mathrm{ord}}, X),$$

where $Y^{\mathrm{ord}} = Y_1(N)^{\mathrm{ord}}_{\mathbf{Z}_p}$ is the open subscheme of Y on which the Hasse invariant E_{p-1} is invertible. For $i \geqslant 0$, the sheaves $\mathscr{S}_{\mathrm{dR},i}$ and $\mathscr{S}_{\mathrm{rig},i}$ arise as the de Rham and rigid realisations of a syntomic sheaf $\mathscr{S}_{\mathrm{syn},i}$ on $\mathscr{Y}$ (cp. [2]). More precisely, let $\mathscr{E}^i$ denote the smooth pair $(\mathcal{E}^i, \bar{\mathcal{E}}^i)$ over $\mathbf{Z}_p$, where $\mathcal{E}^i$ is the i-fold fibre product of the universal elliptic curve $\mathcal{E} \to Y$ and $\bar{\mathcal{E}}^i$ is the corresponding Kuga–Sato variety (viz. Deligne's canonical desingularization of the i-fold fibre product of the universal generalized elliptic curve $\bar{\mathcal{E}} \to X$). Then

$$\mathscr{S}_{\mathrm{syn},i} = R^1(\mathscr{E}^i \to \mathscr{Y})_* \mathbf{Q}_p$$

is the first higher direct image of the trivial syntomic sheaf on $\mathscr{E}^i$ under the smooth proper morphism $\mathscr{E}^i \to \mathscr{Y}$ attached to the structural map $\bar{\mathcal{E}}^i \to X$. We denote by the same symbol $\mathscr{S}_{\mathrm{syn},i}$ its restriction to $\mathscr{Y}^{\mathrm{ord}}$.

Define the syntomic sheaves $\mathscr{S}_{\mathrm{syn},\boldsymbol{r}}$ and $\mathscr{S}_{\mathrm{syn},[\boldsymbol{r}]}$ on $\mathscr{Y}$ and $\mathscr{Y}^3 = (Y^3, X^3)$, respectively, as in Section 2. Set $\mathscr{E}^{\boldsymbol{r}} = \mathscr{E}^{r_1} \times_{\mathbf{Z}_p} \mathscr{E}^{r_2} \times_{\mathbf{Z}_p} \mathscr{E}^{r_3}$. The Leray spectral sequences associated with $\mathscr{E}^{2r} \to \mathscr{Y}$ and $\mathscr{E}^{\boldsymbol{r}} \to \mathscr{Y}^3$ induce, respectively, natural isomorphisms ("Lieberman's trick", cp. the proof of [17, Lem. 5.3])

$$H^i_{\mathrm{syn}}(\mathscr{Y}, \mathscr{S}_{\boldsymbol{r}}(j)) = H^{i+2r}_{\mathrm{syn}}(\mathscr{E}^{2r}, \mathbf{Q}_p(j))(\varepsilon_{\boldsymbol{r}}) \tag{27}$$

and

$$H^i_{\mathrm{syn}}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(j)) = H^{i+2r}_{\mathrm{syn}}(\mathscr{E}^{\boldsymbol{r}}, \mathbf{Q}_p(j))(\varepsilon_{\boldsymbol{r}}),$$

where $\cdot(\varepsilon_{\boldsymbol{r}})$ are defined as follows. Let S_i denote the symmetric group on i letters. The semi-direct product $\mathfrak{S}_i = S_i \rtimes \mu_2^i$ acts naturally as a group of automorphisms of $\mathscr{E}^i$ (the nontrivial element of the i-th factor of μ_2 acting as multiplication by -1 on the i-th factor $\mathscr{E}$ of $\mathscr{E}^i$). As a consequence, the subgroup $\mathfrak{S}_{\boldsymbol{r}} = \mathfrak{S}_{r_1} \times \mathfrak{S}_{r_2} \times \mathfrak{S}_{r_3}$ of $\mathfrak{S}_{2r}$ acts by automorphisms on both $\mathscr{E}^{2r}$ and $\mathscr{E}^{\boldsymbol{r}}$. For any $\mathbf{Q}[\mathfrak{S}_{\boldsymbol{r}}]$-module $\cdot$, one defines $\cdot(\varepsilon_{\boldsymbol{r}})$ to be the submodule of elements of $\cdot$ on which $\mathfrak{S}_{\boldsymbol{r}}$ acts via the character $\varepsilon_{\boldsymbol{r}} = \varepsilon_{r_1} \times \varepsilon_{r_2} \times \varepsilon_{r_3}$, where $\varepsilon_i \colon \mathfrak{S}_i \to \mu_2$ maps $\sigma \rtimes (s_1, \dots, s_i)$ to $\mathrm{sign}(\sigma) \cdot s_1 \cdots s_i$. Similarly, in p-adic étale cohomology there are natural isomorphisms

$$H^i_{\text{ét}}(Y_{\mathbf{Q}_p}, \mathscr{S}_{\boldsymbol{r}}(j)) = H^{i+2r}_{\text{ét}}(\mathcal{E}^{2r}_{\mathbf{Q}_p}, \mathbf{Q}_p(j))(\varepsilon_{\boldsymbol{r}}) \tag{28}$$

and

$$H^i_{\text{ét}}(Y_{\mathbf{Q}_p}, \mathscr{S}_{[\boldsymbol{r}]}(j)) = H^{i+2r}_{\text{ét}}(\mathcal{E}^{\boldsymbol{r}}_{\mathbf{Q}_p}, \mathbf{Q}_p(j))(\varepsilon_{\boldsymbol{r}}).$$

One has analogues of the isomorphisms (27) and (28) after replacing $(\mathscr{Y}, \mathscr{E}^{\cdot})$ with $(\mathscr{X}, \bar{\mathscr{E}}^{\cdot})$, where $\mathscr{X}$ and $\bar{\mathscr{E}}$ denote the proper smooth pairs (X, X) and $(\bar{\mathcal{E}}, \bar{\mathcal{E}})$ over $\mathbf{Z}_p$, respectively.

The Hecke correspondences on X and $\bar{\mathcal{E}}$ equip the syntomic and finite polynomial cohomology groups which appear in this section with the action of Hecke operators away from Np, which make the exact sequences (24)–(25) and the isomorphisms (27) Hecke equivariant.

4.1.4. *Comparison with étale cohomology.* Let $\mathscr{U} = (U, \bar{U})$ be a smooth pair over $\mathbf{Z}_p$. The work of Nekovář and Nizioł [35, 36, 31, 33] gives *comparison morphisms*

$$\varrho_{\mathrm{syn}} \colon H^i_{\mathrm{syn}}(\mathscr{U}, \mathbf{Q}_p(n)) \to H^i_{\text{ét}}(U_{\mathbf{Q}_p}, \mathbf{Q}_p(n)),$$

satisfying the following properties. (See [11, Section 9] and the references quoted there for more details):

- The maps ϱ_{syn} are compatible with pullbacks and proper pushforwards.
- If U is proper over $\mathbf{Z}_p$, then the following diagram commutes.

$$\begin{array}{ccccc} & & H^1_{\mathrm{syn}}(\mathbf{Z}_p, H^{i-1}_{\mathrm{rig}}(U_{\mathbf{F}_p}, \mathbf{Q}_p(n))) & \xrightarrow{\mathrm{i}_{\mathrm{syn}}} & F^1 H^i_{\mathrm{syn}}(\mathscr{U}, \mathbf{Q}_p(n)) \\ & \overset{1-\varphi}{\nearrow} & & & \downarrow \varrho_{\mathrm{syn}} \\ \mathrm{tg}(H^{i-1}_{\mathrm{dR}}(U_{\mathbf{Q}_p}, \mathbf{Q}_p(n))) & & & & \\ & \underset{\exp_p}{\searrow} & & & \\ & & H^1(\mathbf{Q}_p, H^{i-1}_{\text{ét}}(U_{\bar{\mathbf{Q}}_p}, \mathbf{Q}_p(n))) & \xleftarrow{\mathrm{HS}_{\text{ét}}} & F^1 H^i_{\text{ét}}(U_{\mathbf{Q}_p}, \mathbf{Q}_p(n)) \end{array} \tag{29}$$

Here $F^1H^i_{\text{ét}}(U_{\mathbf{Q}_p},\cdot\,)$ is the kernel of $H^i_{\text{ét}}(U_{\mathbf{Q}_p},\cdot\,) \to H^i_{\text{ét}}(U_{\bar{\mathbf{Q}}_p},\cdot\,)$ and $F^1H^i_{\text{syn}}(\mathscr{U},\cdot\,)$ is the kernel of $\mathbf{p}_{\text{syn}}$ (that is the image of $\mathbf{i}_{\text{syn}}$, cp. Equation (24)). Moreover, $\exp_p$ denotes the composition

$$\operatorname{tg}\big(H^{i-1}_{\text{dR}}(U_{\mathbf{Q}_p},\cdot\,)\big) \to D_{\text{dR}}(H^{i-1}_{\text{ét}}(U_{\bar{\mathbf{Q}}_p},\cdot\,))/\text{Fil}^0 \to H^1(\mathbf{Q}_p, H^{i-1}_{\text{ét}}(U_{\bar{\mathbf{Q}}_p},\cdot\,))$$

of Faltings' comparison isomorphism and the Bloch–Kato exponential.

In light of Equations (27)–(28) and the first property above, the maps ϱ_{syn} for $\mathscr{U} = \mathscr{E}^r$ and $\mathscr{U} = \mathscr{E}^{\boldsymbol{r}}$ induce, respectively, Hecke equivariant comparison morphisms (denoted again by the same symbol)

$$\varrho_{\text{syn}}\colon H^i_{\text{syn}}(\mathscr{Y},\mathscr{S}_r) \to H^i_{\text{ét}}(Y_{\mathbf{Q}_p},\mathscr{S}_r) \tag{30}$$

and

$$\varrho_{\text{syn}}\colon H^i_{\text{syn}}(\mathscr{Y}^3,\mathscr{S}_{[\boldsymbol{r}]}) \to H^i_{\text{ét}}(Y^3_{\mathbf{Q}_p},\mathscr{S}_{[\boldsymbol{r}]}),$$

which are compatible with the pullback d^* and pushforward d_* along the diagonal $d\colon \mathscr{Y} \to \mathscr{Y}^3$. (There are similar comparison morphisms for $\mathscr{X}$ and $\mathscr{X}^3$ in place of $\mathscr{Y}$ and $\mathscr{Y}^3$, induced, respectively, by the maps ϱ_{syn} for $\mathscr{U} = \bar{\mathscr{E}}^r$ and $\mathscr{U} = \bar{\mathscr{E}}^{\boldsymbol{r}}$, cp. Section 4.1.3.) In particular,

$$\varrho_{\text{syn}}\colon H^0_{\text{syn}}(\mathscr{Y},\mathscr{S}_{\boldsymbol{r}}(r)) \to H^0_{\text{ét}}(Y_{\mathbf{Q}_p},\mathscr{S}_{\boldsymbol{r}}(r))$$

is an isomorphism, given by the composition of the canonical isomorphisms

$$\begin{aligned} H^0_{\text{syn}}(\mathscr{Y},\mathscr{S}_{\boldsymbol{r}}(r)) &= \text{Fil}^0 H^0_{\text{rig}}(Y_{\mathbf{F}_p},\mathscr{S}_{\boldsymbol{r}}(r))^{\varphi=1} \\ &= \text{Fil}^0 D_{\text{cris}}(H^0_{\text{ét}}(Y_{\bar{\mathbf{Q}}_p},\mathscr{S}_{\boldsymbol{r}}(r)))^{\varphi=1} \\ &= H^0(\mathbf{Q}_p, H^0_{\text{ét}}(Y_{\bar{\mathbf{Q}}_p},\mathscr{S}_{\boldsymbol{r}}(r))) \\ &= H^0_{\text{ét}}(Y_{\mathbf{Q}_p},\mathscr{S}_{\boldsymbol{r}}(r)), \end{aligned}$$

where the first equality arises from $\mathbf{p}_{\text{syn}}$, the second is the comparison isomorphism, the third follows from the well-known equality $\text{Fil}^0 B_{\text{cris}} \cap B_{\text{cris}}^{\varphi=1} = \mathbf{Q}_p$ and the forth is defined by the inverse of the base change along the morphism $\text{Spec}(\bar{\mathbf{Q}}_p) \to \text{Spec}(\mathbf{Q}_p)$ (i.e., by the Hochschild–Serre spectral sequence). Let

$$\mathtt{Det}^{\text{syn}}_{\boldsymbol{r}} \in H^0_{\text{syn}}(\mathscr{Y},\mathscr{S}_{\boldsymbol{r}}(r)) \quad \text{and} \quad \mathtt{Det}^{\text{rig}}_{\boldsymbol{r}} \in \text{Fil}^0 H^0_{\text{rig}}(\mathscr{Y},\mathscr{S}_{\boldsymbol{r}}(r))^{\varphi=1} \tag{31}$$

be defined by the identities $\varrho_{\text{syn}}(\mathtt{Det}^{\text{syn}}_{\boldsymbol{r}}) = \mathtt{Det}^{\text{ét}}_{\boldsymbol{r}}$ and $\mathbf{p}_{\text{syn}}(\mathtt{Det}^{\text{syn}}_{\boldsymbol{r}}) = \mathtt{Det}^{\text{rig}}_{\boldsymbol{r}}$, respectively. (Here we write again $\mathtt{Det}^{\text{ét}}_{\boldsymbol{r}}$ in $H^0_{\text{ét}}(Y_{\mathbf{Q}_p},\mathscr{S}_{\boldsymbol{r}}(r))$ for the $\mathbf{Q}_p$-base change of the Clebsch–Gordan invariant $\mathtt{Det}^{\text{ét}}_{\boldsymbol{r}}$ in $H^0_{\text{ét}}(Y_1(N)_{\mathbf{Q}},\mathscr{S}_{\boldsymbol{r}}(r))$.)

4.2. **The syntomic Abel–Jacobi map.** Because $Y^3_{\mathbf{Q}_p}$ is a smooth affine threefold, the de Rham cohomology group $H^4_{\text{dR}}(Y^3_{\mathbf{Q}_p},\mathscr{S}_{[\boldsymbol{r}]}(r+2))$ vanishes. As a consequence the inverse of $\mathbf{i}_{\text{syn}}$ gives an isomorphism

$$\mathtt{HS}_{\text{syn}}\colon H^4_{\text{syn}}(\mathscr{Y}^3,\mathscr{S}_{[\boldsymbol{r}]}(r+2)) \cong H^1_{\text{syn}}(\mathbf{Z}_p, H^3_{\text{rig}}(Y^3_{\mathbf{F}_p},\mathscr{S}_{[\boldsymbol{r}]}(r+2))_L).$$

After setting $V^*_{\text{dR}}(f,g,h) = V_{\text{dR}}(f^w,g^w,h^w)$, composing $\mathtt{HS}_{\text{syn}}$ with the map induced by the natural projection

$$\text{pr}_{f^w g^w h^w}\colon H^3_{\text{rig}}(Y^3_{\mathbf{F}_p},\mathscr{S}_{[\boldsymbol{r}]}(r+2))_L \twoheadrightarrow V^*_{\text{dR}}(f,g,h)$$

(arising from the comparison isomorphism between rigid and de Rham cohomology) gives a surjective map

$$H^4_{\mathrm{syn}}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) \twoheadrightarrow H^1_{\mathrm{syn}}(\mathbf{Z}_p, V^*_{\mathrm{dR}}(f,g,h)) = \frac{V^*_{\mathrm{dR}}(f,g,h)}{(1-\varphi)\cdot \mathrm{Fil}^0 V^*_{\mathrm{dR}}(f,g,h)},$$

which we denote by $\mathtt{HS}^{fgh}_{\mathrm{syn}}$. As $p \nmid N$, the Ramanujan–Petersson conjecture implies that $1-\varphi$ is an automorphism of $V^*_{\mathrm{dR}}(f,g,h)$. Denote by $\mathrm{tg}^*_{\mathrm{dR}}(f,g,h)$ the tangent space of $V^*_{\mathrm{dR}}(f,g,h)$ and define the *syntomic Abel–Jacobi map*

$$\mathrm{AJ}^{fgh}_{\mathrm{syn}} \colon H^4_{\mathrm{syn}}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) \twoheadrightarrow \mathrm{tg}^*_{\mathrm{dR}}(f,g,h)$$

to be the composition of $\mathtt{HS}^{fgh}_{\mathrm{syn}}$ with the inverse of $1-\varphi$. Then the following diagram commutes:

$$\begin{array}{ccc} H^4_{\mathrm{syn}}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) & \xrightarrow{\mathrm{AJ}^{fgh}_{\mathrm{syn}}} & \mathrm{tg}^*_{\mathrm{dR}}(f,g,h) \\ \downarrow{\scriptstyle \varrho_{\mathrm{syn}}} & & \downarrow{\scriptstyle \exp_p} \\ H^4_{\mathrm{\acute{e}t}}(Y^3_{\mathbf{Q}_p}, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) & \xrightarrow{\mathrm{AJ}^{fgh}_{\mathrm{\acute{e}t}}} & H^1(\mathbf{Q}_p, V^*(f,g,h)), \end{array} \tag{32}$$

where $\mathrm{AJ}^{fgh}_{\mathrm{\acute{e}t}} = \mathrm{pr}_{f^w g^w h^w *} \circ \mathtt{HS}_{\mathrm{\acute{e}t}}$ (cp. Section 2), $V^*(f,g,h) = V(f^w, g^w, h^w)$ and $\exp_p$ is the composition of the Faltings comparison isomorphism

$$\mathrm{tg}^*_{\mathrm{dR}}(f,g,h) \cong D_{\mathrm{dR}}(V^*(f,g,h))/\mathrm{Fil}^0$$

with the Bloch–Kato exponential. This is a consequence of Equation (29) for $i = 4$ and $\mathscr{U} = \bar{\mathscr{E}}^{\boldsymbol{r}}$ (so that $U = \bar{\mathcal{E}}^{\boldsymbol{r}}$ is smooth and proper over $\mathbf{Z}_p$). Indeed, by construction, the map $\mathrm{AJ}^{fgh}_{\mathrm{syn}}$ (resp., $\mathrm{AJ}^{fgh}_{\mathrm{\acute{e}t}}$) factors through the (f^w, g^w, h^w)-isotypic component of $H^4_{\mathrm{syn}}(\mathscr{Y}^3, \cdot\,)$ (resp., $H^4_{\mathrm{\acute{e}t}}(Y^3_{\mathbf{Q}_p}, \cdot\,)$), which is naturally isomorphic to that of $H^4_{\mathrm{syn}}(\mathscr{X}^3, \cdot\,)$ (resp., $H^4_{\mathrm{\acute{e}t}}(X^3_{\mathbf{Q}_p}, \cdot\,)$), since f, g and h are cuspidal forms. Similarly, $V^*(f,g,h)$ and $V^*_{\mathrm{dR}}(f,g,h)$ can be realized, respectively, in the étale and de Rham cohomology of the Kuga–Sato variety $\bar{\mathcal{E}}^{\boldsymbol{r}}$ (via Equation (28) and its analog for the de Rham cohomology). By the definition of the maps ϱ_{syn} (cp. Equation (30)), the previous diagram can then be rewritten in terms of cohomology groups of $\bar{\mathscr{E}}^{\boldsymbol{r}}$, and once this is done its commutativity is a direct consequence of Equation (29) and the definitions.

The commutative diagram (32) and the compatibility of ϱ_{syn} with d_* (cp. Equation (30)) yield the equality

$$\log_p(\kappa(f,g,h)) = N^r \cdot \mathrm{AJ}^{fgh}_{\mathrm{syn}}\big(d_*(\mathtt{Det}^{\mathrm{syn}}_{\boldsymbol{r}})\big) \tag{33}$$

of L-valued linear forms on $\mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h)$, cp. Equations (13) and (31). More precisely, we remind that the left-hand side of the previous equation is identified with an L-linear form on $\mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h)$ via the *twisted* Poincaré duality $\langle\,\cdot\,,\cdot\,\rangle_{fgh}$ introduced in Equation (12). On the other hand, we identify the right-hand side of the previous equation with a linear functional on $\mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h)$ via the perfect duality

$$(\,\cdot\,,\cdot\,)_{fgh} \colon V^*_{\mathrm{dR}}(f,g,h) \otimes_L V_{\mathrm{dR}}(f,g,h) \to L(1)$$

induced by the pairings $(\,\cdot\,,\cdot\,)_i$ defined in Equation (10). Equation (33) then follows from Equations (32), because (as easily checked)

$$N^r \cdot \kappa(f^w, g^w, h^w) = \mathrm{AJ}^{fgh}_{\text{ét}}(d_*(\mathtt{Det}^{\text{ét}}_{\boldsymbol{r}})) \in H^1(\mathbf{Q}, V^*(f,g,h))$$

is the image of the diagonal class

$$\kappa(f,g,h) \in H^1(\mathbf{Q}, V(f,g,h))$$

under the map induced in cohomology by the $G_{\mathbf{Q}}$-equivariant isomorphism

$$w_N^{\otimes 3} \colon V(f,g,h) \cong V^*(f,g,h).$$

Here $w_N^{\otimes 3}$ arises from the Künneth decomposition and the product of the geometric Atkin–Lehner automorphisms w_N of $H^1_{\text{ét}}(Y_1(N)_{\bar{\mathbf{Q}}}, \mathscr{S}_i)$, for $i+2$ equal to k, l and m. (Recall that $\chi_f \cdot \chi_g \cdot \chi_h$ is equal to the trivial character.)

Because $H^2_{\mathrm{rig},c}(Y^3_{\mathbf{F}_p}, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) = 0$, each class

$$\omega \in \mathrm{Fil}^0 V_{\mathrm{dR}}(f,g,h) \subset \mathrm{Fil}^0 H^3_{\mathrm{dR},c}(Y^3_{\mathbf{Q}_p}, \mathscr{S}_{[\boldsymbol{r}]}(r+2)),$$

which is killed by a polynomial $P_\omega(T) \in 1 + T \cdot L[T]$ has a unique lift $\tilde{\omega}$ in the (f,g,h)-isotypic component of the finite-polynomial cohomology group $H^3_{P_\omega,c}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2))$. Assuming that $P_\omega(p^{-1})$ is nonzero (so that the trace on $H^7_{P_\omega,c}(\mathscr{Y}^3, \mathbf{Q}_p(4))$ is defined), the compatibility of the finite polynomial cup-product with the Leray spectral sequence, viz. Equation (26), gives the following identity of functionals on $H^4_{\mathrm{syn}}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2))$:

$$\mathrm{AJ}^{fgh}_{\mathrm{syn}}(\,\cdot\,)(\omega) = \langle\,\cdot\,, \tilde{\omega}\rangle_{\mathscr{Y}^3}. \tag{34}$$

Here the finite polynomial cup product pairing

$$\begin{aligned}\langle\,\cdot\,,\cdot\,\rangle_{\mathscr{Y}^3} \colon H^4_{\mathrm{syn}}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) &\otimes_L H^3_{P_\omega,c}(\mathscr{Y}^3, \mathscr{S}_{[\boldsymbol{r}]}(r+2)) \\ &\to H^7_{P_\omega,c}(\mathscr{Y}^3, \mathbf{Q}_p(4)) \cong L(1)\end{aligned}$$

is the one arising from the perfect relative Poincaré dualities of syntomic sheaves (cp. Equations (8))

$$(\,\cdot\,,\cdot\,)_i \colon \mathscr{S}_{\mathrm{syn},i} \otimes_{\mathbf{Q}_p} \mathscr{S}_{\mathrm{syn},i} \to \mathbf{Q}_p(-i).$$

(Unless otherwise stated, all the cup-product pairings which appear below arise from the dualities $(\,\cdot\,,\cdot\,)_i$.) Since the pullback $d^* = d^*_{\mathrm{syn}}$ and push-forward $d_* = d_{\mathrm{syn},*}$, associated with the diagonal embedding d in finite polynomial cohomology, satisfy the projection formula, Equations (33) and (34) yield

$$\log_p(\kappa(f,g,h))(\omega) = N^r \cdot \langle \mathtt{Det}^{\mathrm{syn}}_{\boldsymbol{r}}, d^*(\tilde{\omega})\rangle_{\mathscr{Y}}. \tag{35}$$

Take ω equal to the class $\eta^\alpha_f \otimes \omega_g \otimes \omega_h$ defined in Equation (14) and P_ω equal to

$$P_{fgh}(T) = \Big(1 - \frac{p^{r+2}T}{\alpha_f\alpha_g\alpha_h}\Big)\Big(1 - \frac{p^{r+2}T}{\alpha_f\alpha_g\beta_h}\Big)\Big(1 - \frac{p^{r+2}T}{\alpha_f\beta_g\alpha_h}\Big)\Big(1 - \frac{p^{r+2}T}{\alpha_f\beta_g\beta_h}\Big).$$

As by assumption $\chi_f\chi_g\chi_h$ is the trivial character, a direct computation shows that $P_{fgh}(p^{-1})$ equals

$$\mathcal{E}(f,g,h)=\Big(1-\frac{\beta_f\alpha_g\alpha_h}{p^{r+2}}\Big)\Big(1-\frac{\beta_f\alpha_g\beta_h}{p^{r+2}}\Big)\Big(1-\frac{\beta_f\beta_g\alpha_h}{p^{r+2}}\Big)\Big(1-\frac{\beta_f\beta_g\beta_h}{p^{r+2}}\Big),$$

which is nonzero by the Ramanujan–Petersson conjecture under the current hypothesis $p\nmid N$.

Let ξ denote either g or h and set

$$P_f(T)=1-\frac{p^{r_1-r}T}{\alpha_f}\quad\text{and}\quad P_\xi(T)=\Big(1-\frac{p^{u+1}T}{\alpha_\xi}\Big)\Big(1-\frac{p^{u+1}T}{\beta_\xi}\Big),$$

so that $P_{fgh}=P_f\star P_g\star P_h$. Let

$$\tilde{\eta}_f^{\alpha}\in H^1_{P_f,c}(\mathscr{Y},\mathscr{S}_{r_1}(r_1-r)),\quad \text{resp. } \tilde{\omega}_\xi\in H^1_{P_\xi}(\mathscr{Y},\mathscr{S}_u(u+1)) \tag{36}$$

(with $u+2$ the weight of ξ), denote the unique lift of

$$\eta_f^{\alpha}\in \mathrm{Fil}^0H^1_{\mathrm{dR},c}(Y_{\mathbf{Q}_p},\mathscr{S}_{r_1}(r_1-r))^{P_f(\varphi)=0},$$

resp. a lift of

$$\omega_\xi\in \mathrm{Fil}^0H^1_{\mathrm{dR}}(Y_{\mathbf{Q}_p},\mathscr{S}_u(u+1))^{P_\xi(\varphi)=0},$$

under $\mathbf{p}$. Equation (35) can then be rewritten as

$$\log_p(\kappa(f,g,h))\big(\eta_f^{\alpha}\otimes\omega_g\otimes\omega_h\big)=N^r\cdot\langle\tilde{\eta}_f^{\alpha},\mathtt{Det}_{\boldsymbol{r}}^{\mathrm{syn}}\cup\tilde{\omega}_g\cup\tilde{\omega}_h\rangle_{\mathscr{Y}}. \tag{37}$$

Write $\mathscr{S}_{gh}=\mathscr{S}_{\mathrm{syn},r_2}\otimes\mathscr{S}_{\mathrm{syn},r_3}(r_2+r_3+2)$ and $P_{gh}=P_g\star P_h$. After noting that $H^2_{\mathrm{dR}}(Y_{\mathbf{Q}_p},\mathscr{S}_{gh})$ vanishes, let

$$\Phi\in H^1_{P_{gh}}\big(\mathbf{Z}_p,H^1_{\mathrm{rig}}(Y_{\mathbf{F}_p},\mathscr{S}_{gh})\big)$$

be the class defined by the identity

$$\mathbf{i}(\Phi)=\tilde{\omega}_g\cup\tilde{\omega}_h.$$

Equation (37) and a direct computation using Equation (26) then prove the following (cp. Equation (31)).

Proposition 4.3. *One has*

$$\log_p(\kappa(f,g,h))\big(\eta_f^{\alpha}\otimes\omega_g\otimes\omega_h\big)=\frac{N^r}{\mathcal{E}(f,g,h)}\cdot\big\langle\eta_f^{\alpha}(r_1-r),\mathtt{Det}_{\boldsymbol{r}}^{\mathrm{rig}}\cup\Phi\big\rangle_{Y,\mathrm{rig}}.$$

4.4. **Restriction to the ordinary locus.** Given a $\mathbf{Q}_p$-vector space V, a $\mathbf{Q}_p$-linear endomorphism e of V and a nonzero element a of L, denote by $V_{e=a}$ (resp., $V^{e=a}$) the maximal L-quotient (resp., L-submodule) of $V\otimes_{\mathbf{Q}_p}L$ on which e acts as multiplication by a. As explained in the proof of Proposition III.1.4 of [34], the restriction map $\cdot_{\mathrm{ord}}\colon H^1_{\mathrm{rig}}(Y_{\mathbf{F}_p},\mathscr{S}_{r_1})\to H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p},\mathscr{S}_{r_1})$ induces an isomorphism

$$H^1_{\mathrm{rig}}(Y_{\mathbf{F}_p},\mathscr{S}_{r_1})_{\varphi=\bar{\chi}_f(p)\cdot\beta_f}\cong H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p},\mathscr{S}_{r_1})_{\varphi=\bar{\chi}_f(p)\cdot\beta_f},$$

which commutes with the action of the Hecke operators T_ℓ and $\langle d\rangle$ for $\ell\nmid Np$ and $d\in(\mathbf{Z}/N\mathbf{Z})^*$. (This follows from weight considerations, recalling that the square of β_f has complex absolute value $k-1$ under the running assumption

$p \nmid N$.) Taking the duals and using Poincaré duality, this induces an isomorphism

$$\cdot^{\mathrm{ord}}\colon H^1_{\mathrm{rig},c}(Y_{\mathbf{F}_p}, \mathscr{S}_{r_1})^{\varphi=\alpha_f} \cong H^1_{\mathrm{rig},c}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})^{\varphi=\alpha_f}.$$

After setting

$$\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}} = (\mathtt{Det}^{\mathrm{rig}}_{\boldsymbol{r}})_{\mathrm{ord}} \otimes t_{-2-r} \in H^0_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{\boldsymbol{r}}(-2)) \tag{38}$$

(so that $\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}} \cup \Phi_{\mathrm{ord}}$ belongs to $H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})$), Proposition 4.3 then gives the following.

Proposition 4.5. *One has*

$$\log_p(\kappa(f,g,h))(\eta^\alpha_f \otimes \omega_g \otimes \omega_h) = \frac{N^r}{\mathcal{E}(f,g,h)} \cdot \big\langle \eta^{\alpha,\mathrm{ord}}_f(r_1+2), \mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}} \cup \Phi_{\mathrm{ord}} \big\rangle_{Y^{\mathrm{ord}},\mathrm{rig}}.$$

The linear form

$$\langle \eta^{\alpha,\mathrm{ord}}_f(r_1+2), \cdot \rangle_{Y^{\mathrm{ord}},\mathrm{rig}}\colon H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})_L \to L$$

factors through the quotient

$$H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})_L \twoheadrightarrow H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})_{f^w_\alpha}$$

defined in Equation (19). As by assumption $f^w_\alpha = (f^w)_\alpha$ has small slope (i.e., $\mathrm{ord}_p(\alpha_f) < k-1$), Equation (20) shows that the latter is isomorphic to $S_k(Np,L)_{f^w_\alpha}$ under the Coleman map $[\cdot]^\alpha_f$. Let

$$\Xi \in S_k(Np,L)_{f^w_\alpha}$$

be the cusp form satisfying

$$[\Xi]^\alpha_f = \big[\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}} \cup \Phi_{\mathrm{ord}}\big]_{f^w_\alpha}, \tag{39}$$

where $[\cdot]_{f^w_\alpha}$ denotes the projection of $H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})$ onto $H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})_{f^w_\alpha}$.

Proposition 4.6. *After setting $\mathcal{E}^*(f) = 1 - \frac{\beta_f}{\alpha_f}$, one has*

$$\log_p(\kappa(f,g,h))(\eta^\alpha_f \otimes \omega_g \otimes \omega_h) = \frac{N^r \mathcal{E}^*(f)}{\mathcal{E}(f,g,h)} \frac{(f^w_\alpha, \Xi)_{Np}}{(f^w_\alpha, f^w_\alpha)_{Np}}.$$

Proof. One has $\Xi = (1 - \bar{\chi}_f(p)\beta_f \cdot V) \cdot \xi$ for a cusp form $\xi \in S_k(N,L)$. Let

$$\omega_\xi \in H^1_{\mathrm{rig}}(Y_{\mathbf{F}_p}, \mathscr{S}_{r_1})$$

be the class associated with ξ and let $\omega_{\xi,\mathrm{ord}} \in H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p}, \mathscr{S}_{r_1})$ be the restriction of ω_ξ to the ordinary locus. Then

$$[\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}} \cup \Phi_{\mathrm{ord}}]_{f^w_\alpha} = [\Xi]^\alpha_f = \Big[\Big(1 - \frac{\beta_f \cdot \varphi}{p^{k-1}}\Big) \cdot \omega_{\xi,\mathrm{ord}}\Big]_{f^w_\alpha} = \mathcal{E}^*(f) \cdot [\omega_{\xi,\mathrm{ord}}]_{f^w_\alpha},$$

hence

$$\begin{aligned}\big\langle \eta^{\alpha,\mathrm{ord}}_f(r_1+2), \mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}} \cup \Phi_{\mathrm{ord}} \big\rangle_{Y^{\mathrm{ord}},\mathrm{rig}} &= \mathcal{E}^*(f) \cdot \big\langle \eta^{\alpha,\mathrm{ord}}_f(r_1+2), \omega_{\xi,\mathrm{ord}} \big\rangle_{Y^{\mathrm{ord}},\mathrm{rig}} \\ &= \mathcal{E}^*(f) \cdot \big\langle \eta^{\alpha}_f(r_1+2), \omega_{\xi} \big\rangle_{Y,\mathrm{rig}} \\ &= \mathcal{E}^*(f) \cdot (\xi, f^w)_N / (f^w, f^w)_N,\end{aligned}$$

by the definitions of η_f^α and $\eta_f^{\alpha,\mathrm{ord}}$ As easily checked $(\xi, f^w)_N/(f^w, f^w)_N$ is equal to $(\Xi, f_\alpha^w)_{Np}/(f_\alpha^w, f_\alpha^w)_{Np}$. The statement then follows from the previous equation and Proposition 4.5. □

4.7. **Conclusion of the proof.** This section concludes the proof of Theorem A.

Let ξ in $M_{u+2}(N, L)$ denote either g or h and let

$$\omega_\xi \in \mathrm{Fil}^0 H^1_{\mathrm{dR}}(Y, \mathscr{S}_u(u+1))_L$$

be the corresponding de Rham class. With a slight abuse of notation, denote by ω_ξ in $\Gamma(X, \boldsymbol{\omega}^u(u+1) \otimes \Omega^1(C))_L$ also the section representing ω_ξ, so that

$$\omega_\xi|_{\tilde{\mathcal{Y}}} = \xi \cdot \tilde{\omega}^u_{\mathrm{can}} \otimes \tilde{\delta}_{\mathrm{can}} \otimes t_{u+1}$$

in $\Gamma(\tilde{\mathcal{Y}}, \tilde{\boldsymbol{\omega}}^u(u+1) \otimes \tilde{\Omega}^1)_L$ (cp. Section 3.4).

Let $\tilde{\omega}_{\xi,\mathrm{ord}}$ in $H^1_{P_\xi}(\mathscr{Y}^{\mathrm{ord}}, \mathscr{S}_u(u+1))$ be the restriction to the ordinary locus of $\tilde{\omega}_\xi$ (cp. Equation (36)). By construction $\tilde{\omega}_{\xi,\mathrm{ord}}$ is a lift under $\mathbf{p}$ of the restriction of ω_ξ to the ordinary locus. (If $u \geqslant 1$ such a lift is unique, cp. [2, Lem. 4.2]). According to [2, Prop. A.16], the class $\tilde{\omega}_{\xi,\mathrm{ord}}$ is uniquely represented by (the restriction to the ordinary locus $\mathscr{Y}^{\mathrm{ord}}$ of) a pair (F_ξ, ω_ξ), where the overconvergent section

$$F_\xi \in \Gamma\big(X^{\mathrm{rig}}_{\mathbf{Q}_p}, \mathscr{S}_{\mathrm{rig},u}(u+1)\big)_L \quad \text{satisfies } \nabla F_\xi = P_\xi(\varphi) \cdot \omega_\xi.$$

As explained in [5, Sections 3.6–3.8] (see in particular Proposition 3.24), one can, and will, choose $\tilde{\omega}_\xi$ in such a way that $\tilde{\omega}_{\xi,\mathrm{ord}}$ is represented by the pair (F_ξ, ω_ξ) with

$$F_\xi|_{\tilde{\mathcal{Y}}} = \sum_{j=0}^{u} (-1)^j j! \binom{u}{j} d^{-1-j} \xi^{[p]}(q) \cdot \tilde{\omega}^{u-j}_{\mathrm{can}} \tilde{\eta}^j_{\mathrm{can}} \otimes t_{u+1} \tag{40}$$

in $\Gamma(\tilde{\mathcal{Y}}, \tilde{\mathscr{S}}_{u,\mathrm{rig}}(u+1))_L$. (The equality $\nabla F = P_\xi(\varphi) \cdot \omega_\xi$ over $\tilde{\mathcal{Y}}$ can be easily checked using Equations (22) and (23). Note that the lift $\tilde{\omega}_\xi$ of ω_ξ, and then F_ξ, is unique if the weight of ξ is strictly greater than two, cp. [2, Lem. 4.2].)

The finite polynomial cup product $\tilde{\omega}_{g,\mathrm{ord}} \cup \tilde{\omega}_{h,\mathrm{ord}} = (\tilde{\omega}_g \cup \tilde{\omega}_h)_{\mathrm{ord}}$ is represented by any 2-cocycle of the form

$$\bigcup\big(a(x,y) \cdot F_g \otimes \omega_h - b(x,y) \cdot \omega_g \otimes F_h, \omega_g \otimes \omega_h\big), \tag{41}$$

where $a(x,y)$ and $b(x,y)$ are polynomials in $L[x,y]$ satisfying

$$P_{gh}(xy) = a(x,y) \cdot P_g(x) + b(x,y) \cdot P_h(y)$$

and x and y act via $\varphi \otimes \mathrm{id}$ and $\mathrm{id} \otimes \varphi$, respectively (cp. [10, Rem. 4.3]). Proposition 5.2.5 of [29] shows that one can take $a(x,y)$ and $b(x,y)$ of the form

$$a(x,y) = 1 - \chi_f(p) p^{r_2+r_3+2} \cdot x^2 y^2 + y \cdot a_o(x,y) \quad \text{and} \quad b(x,y) = x \cdot b_o(x,y), \tag{42}$$

with $a_o(x,y)$ and $b_o(x,y)$ in $L[x,y]$. (Recall that $\chi_g \chi_h$ equals χ_f^{-1}.)

Let $F_{g,j} \in \tilde{\mathbf{M}}(N,L)$ be the $\tilde{\omega}_{\mathrm{can}}^{j}\tilde{\eta}_{\mathrm{can}}^{r_2-j}$-coefficient of $F_g|_{\tilde{\mathcal{Y}}}$. The section $F_{g,j}$ is p-depleted, viz. the n-th Fourier coefficient of its q-expansion is zero if p divides n (cp. Equation (40)). On the other hand, the n-th Fourier coefficient of the q-expansion $\sum_{n\geqslant 0} a_n(h)\cdot q^{pn}$ of $V(h)$ is zero if p does not divide n. It follows that $F_{g,j}\cdot V(h)$ is p-depleted, hence so is each coefficient of $F_g\otimes\varphi(\omega_h)$ (as the restriction of $\varphi(\omega_h)$ to $\tilde{\mathcal{Y}}$ is a multiple of $V(h)\cdot\tilde{\omega}_{\mathrm{can}}^{r_3}\otimes\tilde{\delta}_{\mathrm{can}}\otimes t_{r_3+1}$). This implies that U_p kills the class in $H^1_{\mathrm{rig}}(Y^{\mathrm{ord}}_{\mathbf{F}_p},\mathscr{S}_{gh})_L$ represented by $F_g\cup\varphi(\omega_h)$. Because U_p is an isomorphism, one deduces that the section $F_g\cup\varphi(\omega_h)$ is exact. Similarly, one proves that $\varphi(\omega_g)\cup F_h$ is exact. Together with Equations (41) and (42) this proves that $(\tilde{\omega}_g\cup\tilde{\omega}_h)_{\mathrm{ord}}$ is represented by

$$\big((1-\chi_f(p)p^{r_2+r_3+2}\cdot\varphi^2)\cdot F_g\cup\omega_h,0\big).$$

As Φ_{ord} is characterized by the equality $\mathbf{i}(\Phi_{\mathrm{ord}})=(\tilde{\omega}_g\cup\tilde{\omega}_h)_{\mathrm{ord}}$, the previous equation then yields

$$\Phi_{\mathrm{ord}} = \text{class of } (1-\chi_f(p)p^{r_2+r_3+2}\cdot\varphi^2)\cdot F_g\cup\omega_h. \tag{43}$$

Identify the $\tilde{\mathbf{M}}(N,\mathbf{Z}_p)$-module of global sections of $\tilde{\mathscr{S}}_{\mathrm{rig},r_i}$ with the set of two-variable homogeneous polynomials of degree r_i in $\tilde{\mathbf{M}}(N,\mathbf{Z}_p)[x_i,y_i]$ via $x_i^j y_i^{r_i-j}=\tilde{\omega}_{\mathrm{can}}^{j}\tilde{\eta}_{\mathrm{can}}^{r_i-j}$. Then $\tilde{\mathscr{S}}_{\mathrm{rig},\boldsymbol{r}}=\tilde{\mathscr{S}}_{\mathrm{rig},r_1}\otimes\tilde{\mathscr{S}}_{\mathrm{rig},r_2}\otimes\tilde{\mathscr{S}}_{\mathrm{rig},r_3}$ becomes identified with a submodule of $\tilde{\mathbf{M}}(N,\mathbf{Z}_p)[x_i,y_i\colon 1\leqslant i\leqslant 3]$ and (cp. Equation (38))

$$\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}}|_{\tilde{\mathcal{Y}}}=(x_1y_2-y_1x_2)^{r-r_3}\cdot(x_1y_3-y_1x_3)^{r-r_2}\cdot(x_2y_3-y_2x_3)^{r-r_1}\otimes t_{-2} \tag{44}$$

in $\Gamma(\tilde{\mathcal{Y}},\tilde{\mathscr{S}}_{\mathrm{rig},\boldsymbol{r}}(-2))_L$. Note that the rigid Frobenius acts on $\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}}$ as multiplication by p^{2+r}, hence (cp. Equation (39))

$$\begin{aligned}[\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}}\cup\Phi_{\mathrm{ord}}]_{f^w_\alpha} &= \Big[\Big(1-\frac{\chi_f(p)\cdot\varphi^2}{p^{r_1+2}}\Big)\cdot\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}}\cup F_g\cup\omega_h\Big]_{f^w_\alpha}\\ &= \Big(1-\frac{\beta_f}{p\alpha_f}\Big)\cdot[\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}}\cup F_g\cup\omega_h]_{f^w_\alpha},\end{aligned} \tag{45}$$

by Equation (43). According to Equations (40) and (44) the restriction of $\mathtt{Det}^{\mathrm{ord}}_{\boldsymbol{r}}\cup F_g\cup\omega_h$ to $\tilde{\mathcal{Y}}$ is equal to

$$\begin{aligned}\sum_{i_1,i_2,i_3,j}&(-1)^{r-i_1-i_2-i_3+j}j!\binom{r-r_3}{i_1}\binom{r-r_2}{i_2}\binom{r-r_1}{i_3}\binom{r_2}{j}\\ &\cdot d^{-1-j}g^{[p]}\cdot h\cdot x_1^{i_1+i_2}y_1^{r_1-i_1-i_2}\otimes\tilde{\delta}_{\mathrm{can}}\\ &\otimes x_2^{r-r_3-i_1+i_3}y_2^{r-r_1-i_3+i_1}x_3^{r_3-i_2-i_3}y_3^{i_2+i_3}\cup x_2^{r_2-j}y_2^{j}x_3^{r_3}\otimes t_{r_2+r_3},\end{aligned}$$

where the sum runs over the tuples (i_1,i_2,i_3,j), with $0\leqslant j\leqslant r_2$ and $0\leqslant i_s\leqslant r_s$ for $s=1,2,3$. The only contribution to the $x_1^{r_1}\otimes\tilde{\delta}_{\mathrm{can}}$-component comes from $(i_1,i_2,i_3,j)=(r-r_3,r-r_2,r-r_1,r-r_1)$ and is equal to (cp.

Equation (9))

$$(-1)^{r-r_1}(r-r_1\binom{r_2}{r-r_1}\cdot d^{-1-r+r_1}g^{[p]}\cdot h\cdot x_1^{r_1}\otimes\tilde{\delta}_{\mathrm{can}}$$
$$\otimes x_2^{r-r_1}y_2^{r-r_3}y_3^{r_3}\cup x_2^{r-r_3}y_2^{r-r_1}x_3^{r_3}\otimes t_{r_2+r_3}.$$

As

$$x_2^{r-r_1}y_2^{r-r_3}y_3^{r_3}\cup x_2^{r-r_3}y_2^{r-r_1}x_3^{r_3}=(-1)^r\binom{r_2}{r-r_1}^{-1}\cdot t_{-r_2-r_3},$$

one deduces

$$\mathrm{spl}^{ur}(\mathtt{Det}_{\boldsymbol{r}}^{\mathrm{ord}}\cup F_g\cup\omega_h)|_{\tilde{\mathcal{Y}}}=(-1)^{r_1}(r-r_1)!\cdot d^{-1-r+r_1}g^{[p]}(q)\cdot h(q)\cdot x_1^{r_1}\otimes\tilde{\delta}_{\mathrm{can}}.$$

This proves that (as claimed in the discussion preceding the statement of Theorem A) the p-adic modular form

$$d^{-1-r+r_1}g^{[p]}\cdot h=\mathrm{spl}^{ur}\big((-1)^{r_1}(r-r_1)!^{-1}\cdot\mathtt{Det}_{\boldsymbol{r}}^{\mathrm{ord}}\cup F_g\cup\omega_h\big)\tag{46}$$

is nearly-overconvergent, and (after unwinding the definitions, cp. Equations (21), (39) and (45)) yields the identity

$$(-1)^{r_1}(r-r_1)!\Big(1-\frac{\beta_f}{p\alpha_f}\Big)\cdot e_{f_\alpha^w}(d^{-1-r+r_1}g^{[p]}\cdot h)=\Xi.$$

Theorem A follows from Proposition 4.6 and the previous equation.

5. Proof of Theorem B

This section proves Theorem B stated in Section 1.2. Let the notations and assumptions be as in *loc. cit.*

5.1. **Hida theory.** Let L be a finite extension of $\mathbf{Q}_p$ and let U be an L-rational affinoid disc in the weight space $\mathcal{W}$ over $\mathbf{Q}_p$, centered at an integer $u_o\geqslant 1$. Let $\mathcal{O}(U)$ denote the ring of analytic functions on U. It can be identified with a subring of $L[\![\boldsymbol{u}-u_o]\!]$, where $\boldsymbol{u}-u_o$ is a uniformiser at u_o. Write U^{cl} for the set of positive integers in U which are congruent to u_o modulo $2(p-1)$, and let χ be an L-valued Dirichlet character modulo N. Denote by $S_U^{\mathrm{ord}}(N,\chi)$ the set of formal q-expansions $\boldsymbol{\xi}=\sum_{n\geqslant 0}r_n\cdot q^n$ in $\mathcal{O}(U)[\![q]\!]$ satisfying the following property: For each *classical point* u in $U^{\mathrm{cl}}\cap\mathbf{Z}_{\geqslant 2}$, the *weight-$u$ specialization* $\boldsymbol{\xi}_u=\sum_{n\geqslant 0}r_n(u)\cdot q^n$ is the q-expansion of a cusp form in $S_u(Np,\chi)_L$, which is an eigenvector for the Hecke operator T_ℓ, for each prime ℓ not dividing Np, and for the Hecke operator U_p with eigenvalue a *p-adic unit* in L. For each classical point $u>2$, the form $\boldsymbol{\xi}_u$ is indeed the ordinary p-stabilization of a p-ordinary eigenform ξ_u in $S_k(N,\chi)_L$. If $u=2$, then either p divides the level of $\boldsymbol{\xi}_u$, in which case one sets $\xi_u=\boldsymbol{\xi}_u$, or $\boldsymbol{\xi}_u$ is the p-stabilization of a p-ordinary eigenform ξ_u of level $\Gamma_1(N)$.

An element of $S_U^{\mathrm{ord}}(N,\chi)$ (for some U as above) is called a (cuspidal) Hida family of tame level N, character χ and center u_o. One says that $\boldsymbol{\xi}$ is *primitive* if ξ_u is a primitive form of conductor N for all classical points $u>2$. Let $\boldsymbol{\xi}^\sharp$ be a primitive Hida family in $S_U^{\mathrm{ord}}(N_o,\chi)$ and let N be a multiple of N_o.

A level-N test vector for $\boldsymbol{\xi}^\sharp$ is a Hida family $\boldsymbol{\xi}$ in $S_U^{\mathrm{ord}}(N,\chi)$ such that, for all $u \geqslant 2$ in U^{cl}, the specializations $\boldsymbol{\xi}_u^\sharp$ and $\boldsymbol{\xi}_u$ have the same eigenvalues under the action of the Hecke operators U_p and T_ℓ, for all primes ℓ not dividing Np.

Let N denote the least common multiple of N_f, N_g and N_h. For $\xi = f, g, h$, write α_ξ and β_ξ for the roots of the p-th Hecke polynomial

$$X^2 - a_p(\xi) \cdot X + \chi_\xi(p)p^{u-1}$$

of ξ, where u is the weight of ξ. Assume that L contains α_ξ and β_ξ and order α_f and β_f in such a way that α_f is a p-adic unit. This is possible by Assumption 1.3.4. According to a theorem of Wiles [43], there exist primitive Hida families

$$\boldsymbol{g}^\sharp = \sum_{n\geqslant 0} b_n(\boldsymbol{u}) \cdot q^n \in S_U^{\mathrm{ord}}(N_g, \chi_g) \quad \text{and} \quad \boldsymbol{h}^\sharp = \sum_{n\geqslant 0} c_n(\boldsymbol{u}) \cdot q^n \in S_U^{\mathrm{ord}}(N_h, \chi_h)$$

of levels N_g and N_h, common center $u_o = 1$ and tame characters χ_g and χ_h, specializing, respectively, to the p-stabilized cusp forms g_α and h_α at weight one, namely, satisfying

$$\boldsymbol{g}_1^\sharp = g_\alpha \quad \text{and} \quad \boldsymbol{h}_1^\sharp = h_\alpha.$$

(Recall that $\xi_\alpha(q) = \xi(q) - \beta_\xi \cdot \xi(q^p)$ is an eigenvector for U_p with eigenvalue α_ξ.) Note that $\boldsymbol{g}^\sharp = \boldsymbol{g}_\alpha^\sharp$ and $\boldsymbol{h}^\sharp = \boldsymbol{h}_\alpha^\sharp$ depend on the choice of the roots α_g and α_h of the p-th Hecke polynomials of g and h, respectively.

Let $\boldsymbol{g}$ and $\boldsymbol{h}$ be level-N test vectors for $\boldsymbol{g}^\sharp$ and $\boldsymbol{h}^\sharp$, respectively. Moreover, let $\boldsymbol{f}_k$ be the *ordinary* p-stabilization of a cusp form f_k in $S_k(N, \chi_f)_L$, which is an eigenvector of the Hecke operator T_ℓ, with the same eigenvalue $a_\ell(f)$ as f, for each prime ℓ not dividing N. (We call $\boldsymbol{f}_k$ and f_k *level-N test vectors* for f.) For each $u \geqslant 2$ in U^{cl}, set

$$\mathtt{W}_{Np}(u) = H^3_{\mathrm{\acute{e}t}}(Y_1(Np)^3_{\bar{\mathbf{Q}}}, \mathscr{S}_{k-2} \boxtimes \mathscr{S}_{u-2} \boxtimes \mathscr{S}_{u-2}) \otimes_{\mathbf{Z}_p} \mathscr{O}_L(k/2 + u - 1).$$

Denote by

$$\mathrm{pr}_{\boldsymbol{f}_k \boldsymbol{g}_u \boldsymbol{h}_u} : \mathtt{W}_{Np}(u) \to \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u) \tag{47}$$

the maximal $\mathscr{O}_L$-quotient on which the Hecke operators $U_p \otimes 1 \otimes 1$, $1 \otimes U_p \otimes 1$ and $1 \otimes 1 \otimes U_p$ (resp., $T_\ell \otimes 1 \otimes 1, 1 \otimes T_\ell \otimes 1, 1 \otimes 1 \otimes T_\ell$ and $\langle d_1 \rangle \otimes \langle d_2 \rangle \otimes \langle d_3 \rangle$) act as multiplication by $\alpha_f, b_p(u)$ and $c_p(u)$ (resp., $a_\ell(f), b_\ell(u), c_\ell(u)$ and $\chi_f(d_1) \cdot \chi_g(d_2) \cdot \chi_h(d_3)$ for any prime $\ell \nmid Np$ and units $d_i \in (\mathbf{Z}/N\mathbf{Z})^*$), and set

$$V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u) = \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u) \otimes_{\mathbf{Z}} \mathbf{Q}.$$

Note that $V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)$ depends only on the level N and on the primitive forms $f, g_u^\sharp$ and $h_u^\sharp$.

Let $\|\cdot\|_U$ be the supremum norm on $\mathcal{O}(U)$ and let $\Lambda = \Lambda_U$ be the corresponding unit ball. The work of Hida, Perrin-Riou et al. yields a free Λ-module $\mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}\boldsymbol{h})$, equipped with a continuous Λ-linear action of $G_{\mathbf{Q}}$, satisfying the following properties (cp. [8, Sections 4 and 6]).

- For each $u \geqslant 2$ in U^{cl}, evaluation at u on Λ induces a natural isomorphism of $\mathscr{O}_L[G_{\mathbf{Q}}]$-modules

$$\rho_u \colon \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh}) \otimes_u \mathscr{O}_L \cong \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u). \tag{48}$$

  The representation $V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)$ is isomorphic to

$$\bigoplus_{i=1}^{a} V(f) \otimes_L V(g_u^\sharp) \otimes_L V(h_u^\sharp)((k/2+u-1)),$$

  where $V(\,\cdot\,) = \mathtt{D}(\,\cdot\,)$ is the L-adic Deligne representation $\cdot$ and the positive integer $a = a_N$ is independent of u. If $u = 1$, the previous formula holds with $\mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ isomorphic to a lattice in $V(f,g,h)^a$.
- Let U^{bal} be the set of $u \geqslant 2$ in U^{cl} with $k < 2u$. There exists a $\Lambda[G_{\mathbf{Q}_p}]$-submodule

$$i_{\mathrm{bal}} \colon \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh})_{\mathrm{bal}} \to \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh}),$$

  free of rank $\frac{1}{2}\mathrm{rank}_\Lambda \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh})$ over Λ, such that for all u in U^{bal}, the Bloch–Kato finite subspace

$$H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$$

  of $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$ is equal to the image of the map

$$H^1(\mathbf{Q}_p, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh})_{\mathrm{bal}} \otimes_u L) \to H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)) \tag{49}$$

  induced by ρ_u.

  The morphism induced in cohomology by i_{bal} is injective, and its image $H^1_{\mathrm{bal}}(\mathbf{Q}_p, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh}))$ is called the *balanced* subspace. Similarly, for u in U^{cl}, one defines the *balanced* subspace $H^1_{\mathrm{bal}}(\mathbf{Q}_p, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$ of $H^1(\mathbf{Q}_p, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$ as the image of $H^1(\mathbf{Q}_p, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh})_{\mathrm{bal}} \otimes_u \mathscr{O}_L)$ under the morphism induced by ρ_u. The *balanced Selmer group*

$$H^1_{\mathrm{bal}}(\mathbf{Q}, \mathtt{V}(\,\cdot\,)) \hookrightarrow H^1(\mathbf{Q}, \mathtt{V}(\,\cdot\,))$$

  is the module of global classes which are balanced at p and unramified at any prime $\ell \neq p$. Set $H^1_{\mathrm{bal}}(\mathbf{Q}, \mathtt{V}(\,\cdot\,) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p) = H^1_{\mathrm{bal}}(\mathbf{Q}, \mathtt{V}(\,\cdot\,)) \otimes_{\mathbf{Z}_p} \mathbf{Q}_p$.
- There exists a (unique) morphism of $\mathcal{O}(U)$-modules

$$\mathcal{L} = \mathcal{L}_{\boldsymbol{f}_k, \boldsymbol{gh}} \colon H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{gh})) \to \mathcal{O}(U)$$

  such that, for each $u \geqslant 1$ in U^{cl} and $\mathfrak{z}$ in $H^1_{\mathrm{bal}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{gh}))$, one has

$$\mathcal{L}(\mathfrak{z}, u) = \frac{(1 - \frac{\beta_{\boldsymbol{f}_k}\alpha_{\boldsymbol{g}_u}\alpha_{\boldsymbol{h}_u}}{p^{k/2+u-1}})}{(1 - \frac{\alpha_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_u}\beta_{\boldsymbol{h}_u}}{p^{k/2+u-1}})} \cdot \begin{cases} \frac{(-1)^{u-k/2-1}}{(u-k/2-1)!} \log_p(\mathfrak{z}_u)_f & \text{if } k < 2u, \\ (k/2-u)! \exp_p^*(\mathfrak{z}_u)_f & \text{if } k \geqslant 2u, \end{cases} \tag{50}$$

  where the notations are as follows. One writes $\alpha_{\boldsymbol{f}_k}$ for the unit root of the p-th Hecke polynomial of f and $\beta_{\boldsymbol{f}_k} = p^{k-1}/\alpha_{\boldsymbol{f}_k}$. Similarly, $\alpha_{\boldsymbol{g}_u} = b_p(u)$, $\alpha_{\boldsymbol{h}_u} = c_p(u)$, $\beta_{\boldsymbol{g}_u} = \frac{\chi_g(p)\cdot p^{u-1}}{\alpha_{\boldsymbol{g}_u}}$ and $\beta_{\boldsymbol{h}_u} = \frac{\chi_h(p)\cdot p^{u-1}}{\alpha_{\boldsymbol{h}_u}}$. The class $\mathfrak{z}_u$ is the image of $\mathfrak{z}$ in $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$ under the morphism induced by ρ_u, so that $\mathfrak{z}_u$ belongs to $H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$ if u is in U^{bal} (cp. Equation (49)). One writes

$$\log_p\colon H^1_{\mathrm{fin}}(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)) \to \mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)^\vee$$

(where $\vee$ denotes the L-linear dual) and

$$\exp_p^*\colon H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)) \to V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)^\vee$$

for the Bloch–Kato logarithm and dual exponential, respectively, and $\log_p(\,\cdot\,)_f$ and $\exp_p^*(\,\cdot\,)_f$ for their evaluations on the class

$$\mho_u = \eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_u} \otimes \omega_{\boldsymbol{h}_u}.$$

When $u \geqslant 2$, this is the class defined in Section 3.1.3, which belongs to $\mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)$ if u is balanced, i.e., $k < 2u$ (cp. Equation (14)). Moreover, in the definition of $\log_p$ and $\exp_p^*$, we identify $V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)$ with its L-linear dual under the product of the w_N-twisted Poincaré dualities $(\,\cdot\,, w_N(\,\cdot\,))_\xi$ for ξ equal to $\boldsymbol{f}_k$, $\boldsymbol{g}_l$ and $\boldsymbol{h}_m$ (cp. Equation (11), noting that here N is the tame level of the relevant modular curves).

When $u = 1$, the differential $\mho_1$ in $V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ is defined as above, using a suitable canonical generator $\omega_{\boldsymbol{\xi}_1}$ of $D_{\mathrm{cris}}(V(\boldsymbol{\xi}_1))^{\varphi=\beta_\xi}$, for $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$. The latter is the weight-1 specialization of a *big* differential $\omega_{\boldsymbol{\xi}}$ interpolating $\omega_{\boldsymbol{\xi}_u}$ at weight $u \geqslant 2$. Similarly, in the definition of $\log_p$ and $\exp_p^*$, we identify $V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ with its dual under a suitable perfect canonical pairing $\langle\,\cdot\,,\cdot\,\rangle_{\boldsymbol{f}_k\boldsymbol{g}_1\boldsymbol{h}_1}$, arising as the weight-1 specialization of a twisted Poincaré duality on $V(\boldsymbol{\xi})$. We refer to [8, Section 6.3] and its references for the details.

5.2. ***p*-adic *L*-functions and reciprocity laws.** The notations and assumptions are as in the previous section. Hida's method (cp. [23]) shows that the p-adic periods (cp. Section 1.1)

$$I_p(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u) = I_p(f_k, g_u, h_u),$$

for u in U^{cl}, are interpolated by an analytic function $\mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh})$ in $\mathcal{O}(U)$.

Theorem 5.3. *Shrinking U if necessary, there exists a global balanced class $\kappa(\boldsymbol{f}_k, \boldsymbol{gh})$ in $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{gh}))$ such that*

$$\mathcal{L}_{\boldsymbol{f}_k,\boldsymbol{gh}}\big(\mathrm{res}_p(\kappa(\boldsymbol{f}_k, \boldsymbol{gh}))\big) = \mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh}).$$

Proof. *Step 1.* There exist an integer $A \geqslant 0$ and, for each *balanced* point u in U^{bal}, a global cohomology class $\kappa(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)$ in $H^1(\mathbf{Q}, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$, such that $p^A \cdot \kappa(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)$ belongs to $H^1_{\mathrm{bal}}(\mathbf{Q}, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))$ and

$$\log_p\big(\mathrm{res}_p(\kappa(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u))\big)(\eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_u} \otimes \omega_{\boldsymbol{h}_u})$$

is equal to

$$(-1)^{u-k/2-1}(u-k/2-1)!\frac{(1 - \frac{\alpha_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_u}\beta_{\boldsymbol{h}_u}}{p^{k/2+u-1}})}{(1 - \frac{\beta_{\boldsymbol{f}_k}\alpha_{\boldsymbol{g}_u}\alpha_{\boldsymbol{h}_u}}{p^{k/2+u-1}})} \cdot I_p(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u). \tag{51}$$

Proof of Step 1. Denote by $\kappa_{Np}(u)$ the diagonal class of level Np and weights (k,u,u) (cp. Equation (4)). Let

$$\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)\in H^1(\mathbf{Q}(\mu_p),\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))$$

be the image of $\frac{\bar{\chi}_f(p)}{N^r}\cdot\kappa_{Np}(u)$ under the composition (cp. Equation (47))

$$\mathtt{W}_{Np}(u)\xrightarrow{w_p'\otimes\mathrm{id}\otimes\mathrm{id}}\mathtt{W}_{Np}(u)\xrightarrow{\mathrm{pr}_{\boldsymbol{f}_k\boldsymbol{g}_u\boldsymbol{h}_u}}\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u),$$

where w_p' is the dual p-th Atkin–Lehner endomorphism of $H^1_{\text{ét}}(Y_1(Np)_{\bar{\mathbf{Q}}},\mathscr{S}_{k-2})$ as defined in [8, Section 2.3.1].

The image $\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)\otimes 1$ of $\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)$ in $H^1(\mathbf{Q}(\mu_p),V(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))$ is a *Selmer* class (cp. Section 2), invariant under the action of $\mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q})$ (as $\boldsymbol{f}_k$ is p-old), hence can be identified with a class in the balanced Selmer group $H^1_{\mathrm{bal}}(\mathbf{Q},V(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))$ by Equation (49).

The explicit computations carried out in Proposition 7.3 and Lemma 7.4 of [8] prove that

$$\log_p\bigl(\mathrm{res}_p(\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))\bigr)(\eta^{\alpha}_{\boldsymbol{f}_k}\otimes\omega_{\boldsymbol{g}_u}\otimes\omega_{\boldsymbol{h}_u})$$

is equal to the product of

$$\frac{(1-\frac{\beta_{\boldsymbol{f}_k}\alpha_{\boldsymbol{g}_u}\beta_{\boldsymbol{h}_u}}{p^{k/2+u-1}})(1-\frac{\beta_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_u}\alpha_{\boldsymbol{h}_u}}{p^{k/2+u-1}})(1-\frac{\beta_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_u}\beta_{\boldsymbol{h}_u}}{p^{k/2+u-1}})}{N^r(1-\frac{\beta_{\boldsymbol{f}_k}}{\alpha_{\boldsymbol{f}_k}})(1-\frac{\beta_{\boldsymbol{f}_k}}{p\alpha_{\boldsymbol{f}_k}})}$$

and

$$\log_p\bigl(\mathrm{res}_p(\kappa(f_k,g_u,h_u))\bigr)(\eta^{\alpha}_{f_k}\otimes\omega_{g_u}\otimes\omega_{h_u}).$$

According to the explicit reciprocity law Theorem A, this product is in turn equal to

$$(u-k/2-1)!\Bigl(1-\frac{\beta_{\boldsymbol{f}_k}\alpha_{\boldsymbol{g}_u}\alpha_{\boldsymbol{h}_u}}{p^{k/2+u-1}}\Bigr)^{-1}\cdot I_p(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u).$$

As $\alpha_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_u}\beta_{\boldsymbol{h}_u}$ is in $p^{k/2+u-1}\mathscr{O}_L$ for u in U^{bal}, it follows that the class

$$\kappa(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)=(-1)^{u-k/2-1}\Bigl(1-\frac{\alpha_{\boldsymbol{f}_k}\beta_{\boldsymbol{g}_u}\beta_{\boldsymbol{h}_u}}{p^{k/2+u-1}}\Bigr)\cdot\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)$$

belongs to $H^1(\mathbf{Q},\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))$ and that

$$\log_p\bigl(\mathrm{res}_p(\kappa(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))\bigr)(\eta^{\alpha}_{\boldsymbol{f}_k}\otimes\omega_{\boldsymbol{g}_u}\otimes\omega_{\boldsymbol{h}_u})$$

is equal to the expression displayed in Equation (51).

It remains to prove that there exists a nonnegative integer $A\geqslant 0$ such that $p^A\cdot\kappa(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)$ belongs to $H^1_{\mathrm{bal}}(\mathbf{Q},\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))$ for each u in U^{bal}. Because $\kappa(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)$ is an $\mathscr{O}_L$-multiple of $\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)$ and $\kappa^{\dagger}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)\otimes 1$ belongs to $H^1_{\mathrm{bal}}(\mathbf{Q},V(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u))$, it is sufficient to exhibit a constant $A\geqslant 0$ such that p^A kills the torsion subgroup of $H^1(\mathbf{Q}_p,\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{gh})/\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{gh})_{\mathrm{bal}}\otimes_u\mathscr{O}_L)$ for each balanced point u in U^{bal}.

Set $\mathtt{M}_u = \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh})/\mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh})_{\mathrm{bal}} \otimes_u \mathscr{O}_L$. There is then an exact sequence

$$0 \to \mathtt{M}_u^+ \to \mathtt{M}_u \to \mathtt{M}_u^- \to 0,$$

where (for some positive integer $a \geqslant 1$)

$$\mathtt{M}_u^+ = \mathscr{O}_L(\chi_{\mathrm{cyc}}^{k/2-u+1} \cdot \psi_{u,f})^a \oplus \mathscr{O}_L(\chi_{\mathrm{cyc}}^{1-k/2} \cdot \psi_{u,g})^a \oplus \mathscr{O}_L(\chi_{\mathrm{cyc}}^{1-k/2} \cdot \psi_{u,h})^a$$

and

$$\mathtt{M}_u^- = \mathscr{O}_L(\chi_{\mathrm{cyc}}^{2-u-k/2} \cdot \psi_u)^a,$$

and where the characters $\psi_\cdot$ are unramified and take on an arithmetic Frobenius σ in $G_{\mathbf{Q}_p}$ the values

$$\psi_{u,f}(\sigma) = \frac{\chi_f(p)\alpha_{\boldsymbol{g}_u}\alpha_{\boldsymbol{h}_u}}{\alpha_{\boldsymbol{f}_k}}, \quad \psi_{u,g}(\sigma) = \frac{\chi_g(p)\alpha_{\boldsymbol{f}_k}\alpha_{\boldsymbol{h}_u}}{\alpha_{\boldsymbol{g}_u}}, \quad \psi_{u,h}(\sigma) = \frac{\chi_h(p)\alpha_{\boldsymbol{f}_k}\alpha_{\boldsymbol{g}_u}}{\alpha_{\boldsymbol{h}_u}}$$

and

$$\psi_u(\sigma) = \alpha_{\boldsymbol{f}_k}\alpha_{\boldsymbol{g}_u}\alpha_{\boldsymbol{h}_u}.$$

It follows that the torsion subgroup of $H^1(\mathbf{Q}_p, \mathtt{M}_u)$ is killed by

$$\mu(u) = \prod_{\xi = f,g,h,\varnothing} (1 - \psi_{u,\xi}(\sigma)).$$

The values $\mu(u)$, for u in U^{cl}, are interpolated by an analytic function μ in Λ. Moreover, $\mu(1)$ is nonzero, as by assumption p does not divide the conductor of $\boldsymbol{f}_k$. Shrinking U if necessary, one can then assume that $\mathrm{ord}_p(\mu(u))$ equals the nonnegative integer $\mathrm{ord}_p(\mu(1))$ for all u in U. Taking $A = e(L/\mathbf{Q}_p) \cdot \mathrm{ord}_p(\mu(1))$ concludes the proof.

Step 2. There exist a *finite* subset $\mathscr{E}^{\mathrm{cl}}$ of U^{cl} and a constant $B \geqslant 0$ satisfying the following property: For each u in $\mathscr{U}^{\mathrm{cl}} = U^{\mathrm{cl}} - \mathscr{E}^{\mathrm{cl}}$, the isomorphism ρ_u (cp. Equation (48)) induces a short exact sequence of $\mathscr{O}_L$-modules

$$0 \to H^1_{\mathrm{bal}}(\mathbf{Q}, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{gh})) \otimes_u \mathscr{O}_L \to H^1_{\mathrm{bal}}(\mathbf{Q}, \mathtt{V}(\boldsymbol{f}_k, \boldsymbol{g}_u, \boldsymbol{h}_u)) \to \mathtt{Err}_u \to 0,$$

where $\mathtt{Err}_u$ is a finite $\mathscr{O}_L$-module killed by p^B.

Proof of Step 2. This follows from the general base-change results for Selmer complexes proved in [32, 37].

Step 3. One has $\mathcal{L}_{\boldsymbol{f}_k, \boldsymbol{gh}}(\mathrm{res}_p(\kappa(\boldsymbol{f}_k, \boldsymbol{gh}))) = \mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh})$ for a balanced class $\kappa(\boldsymbol{f}_k, \boldsymbol{gh})$ in $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{gh}))$.

Proof of Step 3. The statement is clear if $\mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh})$ is zero. Assume that $\mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh})$ is nonzero and let e_p be its order of vanishing at $\boldsymbol{u} = 1$. As $\mathcal{O}(U)$ is a principal ideal domain, the image of $\mathcal{L}_{\boldsymbol{f}_k, \boldsymbol{gh}}$ is a principal ideal, say generated by an analytic function $\mathscr{G}_{\mathrm{bal}}$ with order of vanishing e_{bal} at $\boldsymbol{u} = 1$. (By convention $e_{\mathrm{bal}} = +\infty$ if $\mathcal{L}_{\boldsymbol{f}_k, \boldsymbol{gh}}$ is the zero map.) According to the Weierstraß preparation theorem, shrinking U if necessary one can assume that $\mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh}) = (\boldsymbol{u} - 1)^{e_p} \cdot \mathscr{L}_p^*$ and $\mathscr{G}_{\mathrm{bal}} = (\boldsymbol{u} - 1)^{e_{\mathrm{bal}}} \cdot \mathscr{G}_{\mathrm{bal}}^*$, with $\mathscr{L}_p^*$ and $\mathscr{G}_{\mathrm{bal}}^*$

units in $\mathcal{O}(U)$ (and $(\boldsymbol{u}-1)^{e_{\mathrm{bal}}}$ equal to zero if $e_{\mathrm{bal}}=+\infty$). In order to prove the theorem, it is sufficient to show that

$$e_{\mathrm{bal}} \leqslant e_p. \tag{52}$$

Let $\mathscr{U}^{\mathrm{cl}}$ be as in Step 2. Without loss of generality, assume that $\mathscr{U}^{\mathrm{cl}}$ is contained in U^{bal} and that $\mathscr{L}_p(\boldsymbol{f}_k,\boldsymbol{gh})$ does not vanish at any point of $\mathscr{U}^{\mathrm{cl}}$. Let A and B be the constants which appear in Steps 1 and 2. Take $C \geqslant A+B$ such that $\|\mathcal{L}_{\boldsymbol{f}_k,\boldsymbol{gh}}(\mathfrak{z})\|_U \leqslant p^C$ for any class $\mathfrak{z}$ in $H^1_{\mathrm{bal}}(\mathbf{Q},\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{gh}))$. (This is possible since $H^1_{\mathrm{bal}}(\mathbf{Q},\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{gh}))$ is a finitely generated Λ-module.) According to Steps 1 and 2, for each u in $\mathscr{U}^{\mathrm{cl}}$, there exists a global balanced class

$$\tilde{\kappa}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u) \in H^1_{\mathrm{bal}}(\mathbf{Q},\mathtt{V}(\boldsymbol{f}_k,\boldsymbol{gh}))$$

such that (cp. Equations (50) and (51))

$$\mathcal{L}_{\boldsymbol{f}_k,\boldsymbol{gh}}\big(\mathrm{res}_p\big(\tilde{\kappa}(\boldsymbol{f}_k,\boldsymbol{g}_u,\boldsymbol{h}_u)\big),u\big) = p^C\cdot\mathscr{L}_p(\boldsymbol{f}_k,\boldsymbol{gh})(u)\neq 0. \tag{53}$$

In particular, $\mathscr{G}_{\mathrm{bal}}$ is nonzero, hence e_{bal} is a nonnegative integer.

Let $\{u_j\}_{j\geqslant 1}$ be a sequence in $\mathscr{U}^{\mathrm{cl}}$ which converges to 1. For each $j\geqslant 1$, define $\gamma_j\in\mathcal{O}(U)$ by the equation

$$\mathcal{L}_{\boldsymbol{f}_k,\boldsymbol{gh}}\big(\mathrm{res}_p\big(\tilde{\kappa}(\boldsymbol{f}_k,\boldsymbol{g}_{u_j},\boldsymbol{h}_{u_j})\big)\big) = \gamma_j\cdot\mathscr{G}_{\mathrm{bal}}.$$

Because $\|\gamma_j\cdot\mathscr{G}_{\mathrm{bal}}\|_U\leqslant p^C$ for any $j\geqslant 1$, the sequence $\|\gamma_j\|_U$ is bounded, say by p^D for some $D\geqslant 0$. Equation (53) and the Weierstraß preparation theorem show that for $j\gg 0$,

$$p^{-C}\xi_j\cdot|u_j-1|_p^{e_p} = |\gamma_j(u_j)|_p\cdot|u_j-1|_p^{e_{\mathrm{bal}}}\leqslant p^D\cdot|u_j-1|_p^{e_{\mathrm{bal}}},$$

where $\{\xi_j\}_{j\gg 0}$ converges to the *positive* rational number $|\mathscr{L}_p^*(1)|_p/|\mathscr{G}^*_{\mathrm{bal}}(1)|_p$. Equation (52) follows. □

5.4. **Conclusion of the proof.** This section concludes the proof of Theorem B. Write $H^1_{\mathrm{rel}}(\mathbf{Q},V(f,g,h))$ for the *relaxed* Selmer group of $V(f,g,h)$ over $\mathbf{Q}$, that is the set of global classes in $H^1(\mathbf{Q},V(f,g,h))$ which are unramified at every rational prime $\ell\neq p$. Let $\boldsymbol{g}^\sharp=\boldsymbol{g}^\sharp_\alpha$, $\boldsymbol{h}^\sharp=\boldsymbol{h}^\sharp_\alpha$, $\boldsymbol{g}$ and $\boldsymbol{h}$ be as in the previous sections.

Let ξ denote either g or h and let Frob_p be an arithmetic Frobenius in $G_{\mathbf{Q}_p}$. By Assumption 1.3, the restriction to $G_{\mathbf{Q}_p}$ of the Artin representation $V(\xi)$ is unramified and splits as the direct sum of the (distinct) Frob_p-eigenspaces

$$V(\xi)_\alpha = V(\xi)^{\mathrm{Frob}_p=\alpha_\xi/\chi_\xi(p)}\quad\text{and}\quad V(\xi)_\beta = V(\xi)^{\mathrm{Frob}_p=\beta_\xi/\chi_\xi(p)}.$$

As a consequence, the $G_{\mathbf{Q}_p}$-representation $V(f,g,h)$ decomposes as

$$V(f,g,h) = V(f)_{\alpha\alpha}\oplus V(f)_{\alpha\beta}\oplus V(f)_{\beta\alpha}\oplus V(f)_{\beta\beta}, \tag{54}$$

where $V(f)_{ij}=V(f)\otimes_L V(g)_i\otimes V(h)_j\otimes_{\mathbf{Q}_p}\mathbf{Q}_p(k/2)$. Similarly, for $\boldsymbol{\xi}=\boldsymbol{g},\boldsymbol{h}$, one has $V(\boldsymbol{\xi}_1)=V(\boldsymbol{\xi}_1)_\alpha\oplus V(\boldsymbol{\xi}_1)_\beta$ and $V(\boldsymbol{f}_k,\boldsymbol{g}_1,\boldsymbol{h}_1)=\bigoplus_{ij}V(\boldsymbol{f}_k)_{ij}$.

For each p-adic representation V of $G_{\mathbf{Q}_p}$, let V^+ be the submodule on which the inertia subgroup of $G_{\mathbf{Q}_p}$ acts via the $k/2$-th power of the p-adic cyclotomic character and set $V^-=V/V^+$. A class in $H^1_{\mathrm{rel}}(\mathbf{Q},V(f,g,h))$ belongs to the

Bloch–Kato Selmer group $\mathrm{Sel}(\mathbf{Q}, V(f,g,h))$ precisely if its restriction at p is in the kernel of

$$p^-\colon H^1(\mathbf{Q}_p, V(f,g,h)) \to H^1(\mathbf{Q}_p, V(f,g,h)^-), \tag{55}$$

and belongs to the balanced Selmer group $H^1_{\mathrm{bal}}(\mathbf{Q}, V(f,g,h))$ precisely if its restriction at p is in the kernel of the natural map

$$\begin{aligned} H^1(\mathbf{Q}_p, V(f,g,h) \to H^1(\mathbf{Q}_p, V(f)^-_{\alpha\beta}) \oplus H^1(\mathbf{Q}_p, V(f)^-_{\beta\alpha}) \\ \oplus H^1(\mathbf{Q}_p, V(f)_{\alpha\alpha}) \end{aligned} \tag{56}$$

(where $V(f)^-_{\cdot}$ is a shorthand for $(V(f)_{\cdot})^-$). A similar discussion applies with (f,g,h) replaced by $(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ everywhere. After these preliminaries, we can begin the actual proof of Theorem B, which is divided in three steps.

Step 1. There exist level-N test vectors $(\boldsymbol{f}_k, \boldsymbol{g}, \boldsymbol{h})$ for $(f, g^\sharp, h^\sharp)$ and a *nonzero* scalar $\mathcal{E}$ in L^* such that

$$\mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh})(1) = \mathcal{E} \cdot \frac{L(f\otimes g\otimes h, k/2)}{\pi^{2k-2}(f,f)_N}.$$

Proof. Under the running Assumption 1.3, this follows by the special value formulas proved by Garrett and Harris–Kudla [20, 21] (cp. [14, Section 4]). ☐

Step 2. Assume that $L(f\otimes g\otimes h, s)$ does not vanish at $s = k/2$. Then there exists a global class $\kappa(f,g,h)_{\alpha\alpha}$ in the relaxed Selmer group $H^1_{\mathrm{rel}}(\mathbf{Q}, V(f,g,h))$ such that (cp. Equations (54) and (55))

$$p^-\big(\mathrm{res}_p(\kappa(f,g,h)_{\alpha\alpha})\big) \quad \text{is a } \textit{nonzero} \text{ element in } H^1(\mathbf{Q}_p, V(f)^-_{\beta\beta}).$$

Proof. Step 1 implies that $\mathscr{L}_p(\boldsymbol{f}_k, \boldsymbol{gh})$ does not vanish at $\boldsymbol{u} = 1$ for some triple of level-N test vectors $(\boldsymbol{f}_k, \boldsymbol{g}, \boldsymbol{h})$. Theorem 5.3 then yields a global balanced class $\kappa(\boldsymbol{f}_k, \boldsymbol{gh})$ in $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{gh}))$ such that

$$\exp_p^*\big(\mathrm{res}_p(\kappa(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1))\big)(\eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_1} \otimes \omega_{\boldsymbol{h}_1}) \neq 0. \tag{57}$$

Here $\kappa(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ is the image of $\kappa(\boldsymbol{f}_k, \boldsymbol{gh})$ in $H^1_{\mathrm{bal}}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1))$ under the morphism induced in cohomology by ρ_1 (cp. Equation (48)) and one uses Assumption 1.3.2 to guarantee that the Euler factors which appear in Equation (50) are nonzero.

The projection p^- induces a canonical isomorphism

$$\mathrm{Fil}^0 V_{\mathrm{dR}}(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1) \cong D_{\mathrm{cris}}(V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)^-),$$

which we consider as an equality. Then $\exp_p^*$ is equal to the composition

$$\begin{aligned} H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)) \xrightarrow{p^-} H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_2)^-) \\ \xrightarrow{\exp^*} D_{\mathrm{cris}}(V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)^-), \end{aligned}$$

where $\exp^*$ is the dual exponential for $V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)^-$. Similarly, the inclusion $V(\boldsymbol{f}_k)(k/2)^+ \to V(\boldsymbol{f}_k)(k/2)$ induces a natural isomorphism

$$D_{\mathrm{cris}}(V(\boldsymbol{f}_k)(k/2)^+) \cong V_{\mathrm{dR}}(\boldsymbol{f}_k)^{\varphi=\alpha_f} \otimes_{\mathbf{Q}_p} \mathbf{Q}_p[k/2].$$

After recalling that $\omega_{\boldsymbol{\xi}_1}$, for $\boldsymbol{\xi} = \boldsymbol{g}, \boldsymbol{h}$, is a nonzero element of

$$D_{\mathrm{cris}}(V(\boldsymbol{\xi}_1))^{\varphi=\beta_{\boldsymbol{\xi}_1}} = D_{\mathrm{cris}}(V(\boldsymbol{\xi}_1)_\alpha),$$

we can then identify $\eta^\alpha_{\boldsymbol{f}_k} \otimes \omega_{\boldsymbol{g}_1} \otimes \omega_{\boldsymbol{h}_1}$ with an element $\mho_1$ of the crystalline Dieudonné module of the direct summand $V(\boldsymbol{f}_k)(k/2)^+ \otimes_L V(\boldsymbol{g}_1)_\alpha \otimes_L V(\boldsymbol{h}_1)_\alpha$ of $V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)^+$. Equation (57) can then be rewritten as

$$\exp^*\big(\kappa_p^-(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)_{\beta\beta}\big)(\mho_1) \neq 0,$$

where $\kappa_p^-(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)_{\beta\beta}$ is the $\beta\beta$-component of

$$\kappa_p^-(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1) = p^-\big(\mathrm{res}_p(\kappa(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1))\big).$$

On the other hand, since $\kappa(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ is the specialization of a *balanced* class, it follows that $\kappa_p^-(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1) = \kappa_p^-(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)_{\beta\beta}$ belongs to $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k)^-_{\beta\beta})$ (cp. the discussion around Equation (56)). In particular, $\kappa(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ is an element of the relaxed Selmer group $H^1_{\mathrm{rel}}(\mathbf{Q}, V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1))$ such that $\kappa_p^-(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ is a nonzero element of $H^1(\mathbf{Q}_p, V(\boldsymbol{f}_k)^-_{\beta\beta})$. Because the $G_{\mathbf{Q}}$-representation $V(\boldsymbol{f}_k, \boldsymbol{g}_1, \boldsymbol{h}_1)$ is the direct sum of a finite number of copies of $V(f, g, h)$, the statement follows. □

Step 3. Set $V = V(f, g, h)$. Then there is an exact sequence of L-modules

$$0 \to \mathrm{Sel}(\mathbf{Q}, V) \to H^1_{\mathrm{rel}}(\mathbf{Q}, V) \xrightarrow{\partial} H^1(\mathbf{Q}, V^-) \to \mathrm{Sel}(\mathbf{Q}, V)^{dual} \to H^1_{\mathrm{str}}(\mathbf{Q}, V)^{dual} \to 0,$$

where ∂ is the composition of p^- and res_p and $\cdot^{dual}$ denotes the L-linear dual.

Proof. As V is Kummer self-dual, this is an instance of global Poitou–Tate duality (cp. [30, Ch. 1]). □

Varying the choices of the roots α_g and α_h (cp. Assumption 1.3.3), Step 2 yields four classes (namely, $\kappa(f, g, h)_\cdot$ for $\cdot$ in $\{\alpha, \beta\}^2$) in $H^1_{\mathrm{rel}}(\mathbf{Q}, V)$, whose images under the morphism ∂ are linearly independent over L. Theorem B then follows from Step 3, after noting that $H^1(\mathbf{Q}_p, V^-)$ has dimension four over L under Assumption 1.3.2.

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