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A real Jiang–Su algebra

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Abstract. A real C^*-algebra with complexification the Jiang–Su algebra is constructed and some basic properties of the algebra are established.

1. Introduction

In [13], Jiang and Su constructed and established many fundamental properties of a simple, unital, projectionless, infinite-dimensional, nuclear C^*-algebra $\mathcal{Z}$ with a unique tracial state and the same K-theory as the complex numbers. If A is a simple C^*-algebra and $K_0(A)$ is weakly unperforated as an ordered group, it was shown in [9] that A and $A \otimes \mathcal{Z}$ have the same Elliott invariant, leading to intensive work on $\mathcal{Z}$-stable algebras, i.e. those for which A is isomorphic to $A \otimes \mathcal{Z}$. This work has culminated in the result [29, Cor. D] that if A and B are separable, unital, simple and infinite-dimensional C^*-algebras with finite nuclear dimension which satisfy the UCT, then A is isomorphic to B if and only if A and B have isomorphic Elliott invariants. The proof, by many hands but notably including [35, Thm. 7.1] and [10, Thm. 29.8], relies heavily on $\mathcal{Z}$-stability and its relation to the regularity properties of having finite nuclear dimension and having the strict comparison property for positive elements.

Although, as shown by Rosenberg in [22], real C^*-algebras have a significant number of applications, their classification is much less developed than the complex case, with the only major results being in the AF and simple nuclear purely infinite cases. It seems likely that further progress will require an appropriate real version of $\mathcal{Z}$-stability and its relation to real versions of finite nuclear dimension and the strict comparison property for positive elements. The purpose of the present paper is to start this work.

Recent work has revealed analogies between the Jiang–Su algebra and the hyperfinite II_1 factor. The latter algebra has a unique real structure up to isomorphism, as was shown in [8] and [27], so it can be hoped that the same is true for the Jiang–Su algebra. However, not only is this not known but even the weaker form of uniqueness established in [13] for suitable inductive limits cannot be proved by the methods of that paper. As a result, different proofs are given here for results which in the complex case have been derived from Jiang and Su's uniqueness theorem.

The construction of a real analog $\mathcal{Z}_R$ of the Jiang–Su algebra $\mathcal{Z}$, given in Section 2 of this paper, follows exactly as in [13] but is presented here in detail to show that the relevant unitaries can be chosen to be real and can be adjusted if necessary to lie in a given connected component of the orthogonal group. A real form of the Jiang–Su algebra has also been introduced in [6, Ex. 3.1], although without an extensive investigation of its properties.

It is checked in Section 3 that the local existence results from [13, §4] can be adapted without trouble to the real case. However the local uniqueness results from [13, §5] do not carry over because the orthogonal group is not connected: an example is presented in Section 4 of two homomorphisms failing to satisfy [13, Thm. 5.3], which is a key tool in the classification theorem [13, Thm. 6.2] and in the results, from [13, §8], that $\mathcal{Z} \cong \mathcal{Z} \otimes \mathcal{Z}$ and then that $\mathcal{Z} \cong \bigotimes_{i=1}^{\infty} \mathcal{Z}$. An alternative approach, based on [23], is therefore presented in Section 4, leading to the result $\mathcal{Z}_R \cong \mathcal{Z}_R \otimes \mathcal{Z}_R$ for a particular algebra $\mathcal{Z}_R$ with complexification $\mathcal{Z}$.

In Section 5, it is checked that appropriate minor changes can be made to results about weak stability and semiprojectivity, leading to a real analog of the key strongly self-absorbing property of $\mathcal{Z}$. In Section 6, two basic examples having a real version, $\mathcal{Z}_R$-stability, of $\mathcal{Z}$-stability are presented and then, in Sections 7 and 8, a relation is obtained between $\mathcal{Z}_R$-stability and strict comparison for positive elements. In [21], the proof that $\mathcal{Z}$-stable algebras have the strict comparison property for positive elements uses the uniqueness property from [13] and so an alternative approach is given here in Section 7, using properties of the Cuntz semigroup of $\mathcal{Z}$ and analogs for Cuntz semigroups of the complexification and realification maps in K-theory. The partial converse, from [16] and [26], relies on a property known as excision of pure states, which does not in general hold in the real case. However appropriate modifications are made in Section 8 to this and various other arguments from [16] and [26], to establish Theorem 8.12, which is the real counterpart of [26, Cor. 1.2].

2. CONSTRUCTION

Following [13, §2], let $M_n(\mathbb{R})$ be the algebra of all real $n \times n$ matrices and identify $M_m(\mathbb{R}) \otimes M_n(\mathbb{R})$ with $M_{mn}(\mathbb{R})$ by means of

$$a \otimes b = \begin{pmatrix} b_{11}a & b_{12}a & \cdots & b_{1n}a \\ b_{21}a & b_{22}a & \cdots & b_{2n}a \\ \vdots & \vdots & & \vdots \\ b_{n1}a & b_{n2}a & \cdots & b_{nn}a \end{pmatrix} \in M_{mn}(\mathbb{R}).$$

Further let the real dimension drop algebra $I_R[p,m,q]$ be defined by

$$I_R[p,m,q] = \{ f \in C([0,1], M_m(\mathbb{R})) \mid f(0) \in M_p \otimes I_{m/p}, f(1) \in I_{m/q} \otimes M_q(\mathbb{R}) \}.$$

The proof of [13, Lem. 2.2] applies unchanged to show that $I_R[p, m, q]$ is projectionless when p, q are relatively prime, in which case $I_R[p, pq, q]$ is said to be prime. In this paper the focus will be only on prime dimension drop algebras.

Lemma 2.3 of [13] gives the K-theory of the complex dimension drop algebra $A = I[m_0, m, m_1]$, namely $(K_0(A), K_0(A)^+, [1_A]) \cong (\mathbb{Z}, \mathbb{N}, r)$, where $r = (m_0, m_1)$, and $K_1(A) \cong \mathbb{Z}_p$ where $p = mr/(m_0 m_1)$. In particular, when m_0 and m_1 are relatively prime and $m = m_0 m_1$, then

$$(K_0(A), K_0(A)^+, [1_A]) \cong (\mathbb{Z}, \mathbb{N}, 1) \quad \text{and} \quad K_1(A) = 0,$$

so that the unital injection of $\mathbb{C}$ into $I[m_0, m, m_1]$ gives an isomorphism in K-theory. As noted in [3, Prop. 1.14], the unital injection from $\mathbb{R}$ into $I_R[m_0, m, m_1]$ therefore gives an isomorphism from the united K-theory $K^{CRT}(\mathbb{R})$ of $\mathbb{R}$ to the united K-theory $K^{CRT}(I_R[m_0, m, m_1])$ of $I_R[m_0, m, m_1]$. Similarly any unital homomorphism between non-commutative prime real dimension drop algebras (which, as observed in the proof of [13, Prop. 2.8], must be injective) gives rise to an isomorphism in united K-theory.

As in [13, Lem. 2.4], the induced map from the center defines an isomorphism from the tracial state space $T(I_R[p, m, q])$ of $I_R[p, m, q]$ to that of $C([0, 1], \mathbb{R})$ and from $\mathrm{Aff}(T(I_R[p, m, q]))$ to $C([0, 1], \mathbb{R})$. Furthermore each real trace τ on $I_R[p, m, q]$ extends uniquely to a complex-linear trace on $I[p, m, q]$ and each complex trace on $I[p, m, q]$ restricts to a real-valued trace on $I_R[p, m, q]$ (which is zero on skew-adjoint elements).

Proposition 2.5 of [13] gives the construction of the Jiang–Su algebra $\mathcal{Z}$. The real version is almost identical.

Proposition 2.1. *There exists an inductive sequence*

$$A_1 \xrightarrow{\varphi_1} A_2 \xrightarrow{\varphi_2} A_3 \xrightarrow{\varphi_3} \cdots,$$

where each $A_n = I_R[p_n, d_n, q_n]$ is a prime real dimension drop algebra, such that each connecting map $\varphi_{m,n} = \varphi_{n-1} \circ \cdots \circ \varphi_{m+1} \circ \varphi_m : A_m \to A_n$ is an injective morphism of the form

$$\varphi_{m,n}(f) = u^* \begin{pmatrix} f \circ \xi_1 & 0 & \cdots & 0 \\ 0 & f \circ \xi_2 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f \circ \xi_k \end{pmatrix} u$$

for all $f \in A_m$, where u is a continuous path of unitaries in $M_{d_n}(\mathbb{R})$ and $\{\xi_i\}$ is a sequence of continuous paths in $[0, 1]$, each one of which satisfies $|\xi_i(x) - \xi_i(y)| \le (\frac{1}{2})^{n-m}$ for all $x, y \in [0, 1]$.

Proof. Proceed exactly as in [13] to define dimension drop algebras $A_m = I_R[p_m, d_m, q_m]$ and paths $\{\xi_i\}$. The construction of the connecting maps requires the unitary elements u_0 and u_1 to be chosen to be real and with equal determinants. The following argument explains why this can be done.

For each $1 \leq i \leq r_0$, where r_0 is defined in [13], $\xi_i(0) = 0$ and therefore each matrix $f(\xi_i(0))$ is of the form

$$\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \in M_{d_m}(\mathbb{R}),$$

where $a \in M_{p_m}(\mathbb{R})$ and is repeated q_m times. The remaining $k - r_0 = sq_{m+1}$ blocks in the block diagonal matrix

$$\begin{pmatrix} f(\xi_1(0)) & 0 & \cdots & 0 \\ 0 & f(\xi_2(0)) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f(\xi_k(0)) \end{pmatrix} \in M_{d_{m+1}}(\mathbb{R})$$

are of the form $b = f(\frac{1}{2}) \in M_{d_m}(\mathbb{R})$.

There are therefore $q_m r_0 = tq_{m+1}$ diagonal blocks a and sq_{m+1} diagonal blocks b, which can be permuted to give q_{m+1} diagonal blocks

$$\begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & b \end{pmatrix}$$

of size $sp_m + td_m = p_{m+1}$, where a occurs t times and b occurs s times. Thus there exists a real unitary permutation matrix u_0 with

$$u_0^* \begin{pmatrix} f(\xi_1(0)) & 0 & \cdots & 0 \\ 0 & f(\xi_2(0)) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f(\xi_k(0)) \end{pmatrix} u_0 \in M_{p_{m+1}}(\mathbb{R}) \otimes I_{q_{m+1}}.$$

The unitary u_1 with

$$u_1^* \begin{pmatrix} f(\xi_1(1)) & 0 & \cdots & 0 \\ 0 & f(\xi_2(1)) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f(\xi_k(1)) \end{pmatrix} u_1 \in I_{p_{m+1}} \otimes M_{q_{m+1}}(\mathbb{R})$$

can be chosen to be real by a similar argument. Firstly note that for any p, q the unitary

$$u = \begin{pmatrix} e_{11} & e_{21} & \cdots & e_{p1} \\ e_{12} & e_{22} & \cdots & e_{p2} \\ \vdots & \vdots & & \vdots \\ e_{1q} & e_{2q} & \cdots & e_{pq} \end{pmatrix},$$

where each matrix unit e_{ij} is a $p \times q$ matrix, satisfies

$$\begin{aligned}
u(a \otimes I_p) &= \begin{pmatrix} e_{11} & e_{21} & \cdots & e_{p1} \\ e_{12} & e_{22} & \cdots & e_{p2} \\ \vdots & \vdots & & \vdots \\ e_{1q} & e_{2q} & \cdots & e_{pq} \end{pmatrix} \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & a \end{pmatrix} \\
&= \begin{pmatrix} e_{11}a & e_{21}a & \cdots & e_{p1}a \\ e_{12}a & e_{22}a & \cdots & e_{p2}a \\ \vdots & \vdots & & \vdots \\ e_{1q}a & e_{2q}a & \cdots & e_{pq}a \end{pmatrix} \\
&= \begin{pmatrix} a_{11}I_p & a_{12}I_p & \cdots & a_{1q}I_p \\ a_{21}I_p & a_{22}I_p & \cdots & a_{2q}I_p \\ \vdots & \vdots & & \vdots \\ a_{q1}I_p & a_{q2}I_p & \cdots & a_{qq}I_p \end{pmatrix} \begin{pmatrix} e_{11} & e_{21} & \cdots & e_{p1} \\ e_{12} & e_{22} & \cdots & e_{p2} \\ \vdots & \vdots & & \vdots \\ e_{1q} & e_{2q} & \cdots & e_{pq} \end{pmatrix} \\
&= (I_p \otimes a)u
\end{aligned}$$

for each $a \in M_q(\mathbb{R})$. Therefore each diagonal block $f(1)$ is of the form $v(a \otimes I_{p_m})v^*$ for some real unitary matrix v and some $a \in M_{q_m}(\mathbb{R})$. As in the construction of u_0, a real unitary w can then be found so that

$$w^* \begin{pmatrix} f(\xi_1(1)) & 0 & \cdots & 0 \\ 0 & f(\xi_2(1)) & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & f(\xi_k(1)) \end{pmatrix} w \in M_{q_{m+1}}(\mathbb{R}) \otimes I_{p_{m+1}}$$

and then u_1 can be found by repeating the argument used to obtain $f(1) = v(a \otimes I_{p_m})v^*$.

If p_m is odd, then

$$\operatorname{diag}(-I_{p_m}, I_{d_m - p_m}) f(\xi_1(0)) \operatorname{diag}(-I_{p_m}, I_{d_m - p_m})^* = f(\xi_1(0)),$$

so the sign of $\det(u_0)$ can be adjusted if necessary to make $\det(u_0) = \det(u_1)$ by premultiplying u_0 by the matrix $\operatorname{diag}(-I_{p_m}, I_{d_{m+1}-p_m})$. If p_m is even then q_m is odd and the sign of $\det(u_1)$ can then be adjusted as for $\det(u_0)$ after obtaining $v^* f(\xi_1(1))v = a \otimes I_{p_m}$. Then a path u of unitaries in $M_{d_{m+1}}(\mathbb{R})$ can be found connecting u_0 and u_1. □

In summary, a real version of the construction of [13, Prop. 2.5] produces an inductive sequence which, on complexification, gives the sequence of that proposition. It follows that the inductive limit has complexification $\mathcal{Z}$. Using [13, Prop. 2.8], the inductive limit is therefore a unital simple real C^*-algebra with a unique real tracial state. Furthermore, the united K-theory of any algebra A with complexification $\mathcal{Z}$ is isomorphic to the united K-theory of $\mathbb{R}$, using the result from [3, Prop. 1.14] that the K-theory homomorphism induced by the unital injection from $\mathbb{R}$ into A is an isomorphism because it is an isomorphism on its complex part.

3. Local existence

The irreducible representations V_y of $A = I_R[m_0, m, m_1]$ are defined exactly as in [13, §3] and the real analog of [13, Lem. 3.3] holds without change of proof to describe the morphisms from A to $M_n(\mathbb{R})$.

The use of the KK-theory developed in [13] can be avoided when restricting to the prime case, when all unital homomorphisms give rise to the identity map on $\mathbb{Z}$ under the identifications of [13, §3]. Instead, simpler versions of the arguments of that section can be used, such as the following, based on [13, Lem. 3.5].

Lemma 3.1. *Let $A = I_R[m_0, m, m_1]$ and $B = I_R[n_0, n, n_1]$ be prime and satisfy $n_0 \geq m$ and $n_1 \geq m$. Then there is a unital homomorphism from A to B.*

Proof. The condition $n_i \geq m$ guarantees that $a > 0$ when n_i is written in the form $am_0 + bm_1$ with $0 \leq b < m_0$. This enables the construction of morphisms ρ_i from A to $M_{n_i}(\mathbb{R})$ for $i \in \{0, 1\}$ and hence of morphisms $\varphi_0 = \rho_0 \otimes I_{n_1}$ and $\varphi_1 = I_{n_0} \otimes \rho_1$ from A to $M_n(\mathbb{R})$. As in [13, Lem. 3.3] these maps have compressed diagonalizations of the form

$$u_i^* \begin{pmatrix} V_0(f) \otimes \mathrm{id}_{\mu_0} & 0 & \cdots & 0 & 0 \\ 0 & f(\lambda_1(i)) & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & f(\lambda_\mu(i)) & 0 \\ 0 & 0 & \cdots & 0 & V_1(f) \otimes \mathrm{id}_{\mu_1} \end{pmatrix} u_i.$$

These compressed diagonalizations both extend by complexification to the corresponding complex dimension drop algebras and hence, by [13, Lem. 3.1, 3.4], μ_0, μ_1, μ are uniquely determined by $n = (m_1\mu + \mu_0)m_0 + \mu_1 m_1$ with $0 \leq \mu_0 < m_1$ and $0 \leq \mu_1 < m_0$. If $\det(u_0) \neq \det(u_1)$ then, if n_1 is odd, φ_0 can be replaced by $w_0\varphi_0 w_0^*$ where $w_0 = v_0 \otimes I_{n_1}$ for some unitary v_0 with $\det(v_0) = -1$. If instead n_0 is odd then φ_1 can be replaced by $w_1\varphi_1 w_1^*$ where $w_1 = I_{n_0} \otimes v_1$ for some unitary v_1 with $\det(v_1) = -1$. Then, as in the proof of [13, Lem. 3.5], a morphism φ can be constructed by connecting u_0 to u_1. □

Using the notation Δ^φ introduced in [13, Not. 5.2], the morphism φ constructed in Lemma 3.1 is of the form $u^*\Delta^\varphi u$ where $u \in C([0,1], M_n(\mathbb{R}))$; such a morphism will here be called *standard*.

The proof of [13, Thm. 4.1] can now be used to obtain the following existence theorem.

Theorem 3.2. *Let $A = I_R[m_0, m, m_1]$ be a prime dimension drop algebra, $F \subseteq A$ a finite subset and $\varepsilon > 0$ a constant. Then there exists a number $N > 0$ such that if*

(i) *$B = I_R[n_0, n, n_1]$ is prime with $n_0 > N$, $n_1 > N$, and*
(ii) *$\theta : T(B) \to T(A)$ is a continuous affine map,*

then there exists an injective standard morphism $\varphi : A \to B$ *such that*

$$|\langle f, (\varphi^* - \theta)(\tau)\rangle| < \varepsilon \quad \text{for all } f \in F,\ \tau \in T(B).$$

The proof proceeds exactly as in [13] until the definition of φ_i. (The restriction to selfadjoint elements of F is possible here because real traces are zero on skew-adjoint elements.) There needs to be a continuous unitary path connecting u_0 and u_1 but this can be arranged by replacing φ_0 by $v^*\varphi_0 v$ or φ_1 by $v^*\varphi_1 v$ for a suitable unitary v, as was done above when adapting the proof of [13, Lem. 3.5].

4. Limited uniqueness

In [13], the reader is referred to [28, Cor. 1.5] for a proof of [13, Thm. 5.3], which states that if A, B are dimension drop algebras, then two morphisms $\varphi, \psi : A \to B$ with $\Delta^\varphi = \Delta^\psi$ are approximately unitarily equivalent. This result does not hold in the real situation in general. For a counterexample, let $A = B = I_R(2,6,3)$, let $\varphi = \mathrm{id}$ and let $\psi(f) = ufu^*$ where $u \in C([0,1], M_6(\mathbb{R}))$ is a unitary with $u_0 = \mathrm{diag}(1,-1,1,-1,1,-1)$ and

$$u_1 = v \begin{pmatrix} 0_3 & I_3 \\ I_3 & 0_3 \end{pmatrix} v^*,$$

where $I_2 \otimes M_3(\mathbb{R}) = v(M_3(\mathbb{R}) \otimes I_2)v^*$, so that $\det(u_0) = \det(u_1) = -1$ and u_1 commutes with $I_2 \otimes M_3(\mathbb{R})$. Then $\Delta^\varphi(f) = \Delta^\psi(f) = (f)$. Note that if w is any unitary in A then, for some 3×3 matrix w_1',

$$\det(w_1) = \det\left(v \begin{pmatrix} w_1' & 0_3 \\ 0_3 & w_1' \end{pmatrix} v^* \right) = \det(w_1')^2 = 1$$

and therefore $\det(w_0) = 1$. Thus $w_0 = \mathrm{diag}(w_0', w_0', w_0')$ where $\det(w_0') = 1$.

If $f \in A$ with $\|\varphi(f) - w\psi(f)w^*\| < \varepsilon$ then, putting $f(0) = \mathrm{diag}(f_0', f_0', f_0')$, we have

$$\left\| f_0' w_0' \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - w_0' \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} f_0' \right\| < \varepsilon.$$

So let F be a finite set in A such that every 2×2 matrix unit e_{ij} occurs as f_0' for some $f \in F$. If

$$w_0' \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = v = v_{11}e_{11} + v_{12}e_{12} + v_{21}e_{21} + v_{22}e_{22},$$

then, using $f_0' = e_{11}$ and $f_0' = e_{12}$,

$$\|v_{11}e_{11} + v_{12}e_{12} - v_{11}e_{11} - v_{21}e_{21}\| = \|v_{12}e_{12} - v_{21}e_{21}\| < \varepsilon$$

and

$$\|v_{21}e_{11} + v_{22}e_{12} - v_{11}e_{12} - v_{21}e_{22}\| < \varepsilon.$$

It follows that $|v_{12}| < \varepsilon$, $|v_{21}| < \varepsilon$ and $|v_{11} - v_{22}| < \varepsilon$. So $v_{11}v_{22} - \varepsilon^2 < \det(v) = -1$ and then, using $|v_{11}| \le 1$, we deduce $v_{11}^2 - \varepsilon < v_{11}v_{22} < -1 + \varepsilon^2$. For $\varepsilon < \frac{1}{2}$ this gives a contradiction and hence a contradiction to the real analog of [13, Thm. 5.3].

Lemma 4.1. *Let p,q and P,Q be pairs of coprime natural numbers. Let $\xi_1,\dots,\xi_{PQ}$ be paths in $[0,1]$ with $\xi_i(0)\in[0,\frac{1}{2}]$ and $\xi_i(1)\in[\frac{1}{2},1]$ for each $1\le i\le PQ$ and let $\varphi: I_R[p,pq,q]\to I_R[Pp,PpQq,Qq]$ be defined by $\varphi=u^*\Delta^\varphi u$ where*

$$\Delta^\varphi(f)=\begin{pmatrix} f\circ\xi_1 & 0 & \cdots & 0\\ 0 & f\circ\xi_2 & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & f\circ\xi_{PQ}\end{pmatrix}$$

and u is a unitary in $C([0,1],M_{PpQq}(\mathbb{R}))$. Then there exists a unital embedding ψ of $I_R[P,PQ,Q]$ into $I_R[Pp,PpQq,Qq]$ such that, for every $f\in I_R[p,pq,q]$ with $\|\Delta^\varphi(f)(x)-\Delta^\varphi(f)(y)\|<\varepsilon$ for all $x,y\in[0,1]$,

$$\|\psi(g)\varphi(f)-\varphi(f)\psi(g)\|<2\varepsilon$$

for all $g\in I_R[P,PQ,Q]$ with $\|g\|\le 1$.

Proof. This is just the real version of a special case of [23, Lem. 4.1]. To check the real version, the details are presented here. Let

$$\psi': I_R[P,PQ,Q]\to C([0,1],M_{PpQq}(\mathbb{R}))$$

with $\psi'(g)=I_{pq}\otimes g$ for each g. Also let

$$A_L=\{f\in I_R[p,pq,q]\mid f(x)=f(0)\text{ for }0\le x\le\tfrac{1}{2}\},$$
$$A_R=\{f\in I_R[p,pq,q]\mid f(x)=f(1)\text{ for }\tfrac{1}{2}\le x\le 1\},$$

and note that

$$\Delta^\varphi(A_L)(0)=M_p(\mathbb{R})\otimes I_{PQq},$$
$$\Delta^\varphi(A_R)(1)=I_p\otimes M_q(\mathbb{R})\otimes I_{PQ}.$$

Thus, for each $x\in[0,1]$, the elements of $(\psi'(I_R[P,PQ,Q]))(x)$ commute with the elements of $\Delta^\varphi(A_L)(0)$ and with the elements of $\Delta^\varphi(A_R)(1)$.

Note also that

$$\Delta^\varphi(A_L)(0)\subset u(0)I_R[Pp,PpQq,Qq](0)u(0)^*\cong M_{Pp}(\mathbb{R}),$$
$$\Delta^\varphi(A_R)(1)\subset u(1)I_R[Pp,PpQq,Qq](1)u(1)^*\cong M_{Qq}(\mathbb{R}),$$

so that, regarding $\Delta^\varphi(A_L)(0)$ and $\Delta^\varphi(A_R)(1)$ as subalgebras of $M_{PpQq}(\mathbb{R})$,

$$\Delta^\varphi(A_L)(0)'\cap u(0)I_R[Pp,PpQq,Qq](0)u(0)^*\cong M_P(\mathbb{R}),$$
$$\Delta^\varphi(A_R)(1)'\cap u(1)I_R[Pp,PpQq,Qq](1)u(1)^*\cong M_Q(\mathbb{R}).$$

Within $\Delta^\varphi(A_L)(0)'\cong M_{PQq}(\mathbb{R})$ there exists a unitary w_L with

$$w_L\psi'(I_R[P,PQ,Q])(0)w_L^*=\Delta^\varphi(A_L)(0)'\cap u(0)I_R[Pp,PpQq,Qq](0)u(0)^*.$$

After multiplying, if necessary, by an element of $\psi'(I_R[P,PQ,Q])(0)$, if Qq is odd, or an element of its commutant in $\Delta^\varphi(A_L)(0)'$, if P is odd, w_L can be connected to 1 by a path of unitaries in $\Delta^\varphi(A_L)(0)'$. Similarly within

$\Delta^\varphi(A_R)(1)' \cong M_{PpQ}(\mathbb{R})$ there exists a unitary w_R, connected to 1 by a path of unitaries in $\Delta^\varphi(A_R)(1)'$, with

$$w_R\psi'(I_R[P,PQ,Q])(1)w_R^* = \Delta^\varphi(A_R)(1)' \cap u(1)I_R[Pp,PpQq,Qq](1)u(1)^*.$$

Let W be a unitary in $C([0,1],M_{PpQq}(\mathbb{R}))$ with

$$\begin{aligned} &W(0) = w_L,\\ &W(x) \in \Delta^\varphi(A_L)(0)' \quad \text{for } 0 \le x \le \tfrac{1}{2},\\ &W(\tfrac{1}{2}) = 1,\\ &W(x) \in \Delta^\varphi(A_R)(1)' \quad \text{for } \tfrac{1}{2} \le x \le 1,\\ &W(1) = w_R. \end{aligned}$$

If $\psi = \operatorname{Ad} u^*W \circ \psi'$ then ψ is a unital embedding of $I_R[P,PQ,Q]$ into $I_R[Pp,PpQq,Qq]$. If $f \in I_R[p,pq,q]$ with $\|\Delta^\varphi(f)(x) - \Delta^\varphi(f)(y)\| < \varepsilon$ for all $x,y \in [0,1]$ and $g \in I_R[P,PQ,Q]$ with $\|g\| \le 1$, let $f_L \in A_L$ and $f_R \in A_R$ with $f_L(0) = f(0)$ and $f_R(1) = f(1)$. Then for $0 \le x \le \frac{1}{2}$,

$$\begin{aligned} \psi(g)(x)\varphi(f)(x) &= \psi(g)(x)u(x)^*\Delta^\varphi(f)(x)u(x)\\ &\approx_\varepsilon \psi(g)(x)u(x)^*\Delta^\varphi(f_L)(0)u(x)\\ &= u(x)^*W(x)\psi'(g)(x)W(x)^*\Delta^\varphi(f_L)(0)u(x)\\ &= u(x)^*W(x)\psi'(g)(x)\Delta^\varphi(f_L)(0)W(x)^*u(x)\\ &= u(x)^*W(x)\Delta^\varphi(f_L)(0)\psi'(g)(x)W(x)^*u(x)\\ &= u(x)^*\Delta^\varphi(f_L)(0)W(x)^*\psi'(g)(x)W(x)^*u(x)\\ &\approx_\varepsilon u(x)^*\Delta^\varphi(f)(x)W(x)^*\psi'(g)(x)W(x)^*u(x)\\ &= \varphi(f)(x)\psi(g)(x). \end{aligned}$$

Similarly, for $\frac{1}{2} \le x \le 1$,

$$\psi(g)(x)\varphi(f)(x) \approx_{2\varepsilon} \varphi(f)(x)\psi(g)(x),$$

as required. □

In order to apply the argument used to prove [23, Thm. 4.3] the following proposition is needed.

Proposition 4.2. *Let $\mathcal{S}$ be an inductive system, of the type constructed in Proposition 2.1, using prime real dimension drop algebras $A_n = I_R(p_n, p_nq_n, q_n)$ of odd order p_nq_n or order $p_nq_n \equiv 0 \pmod 4$ and let $\mathcal{Z}_R^{\mathcal{S}}$ be the inductive limit. Then the maps ψ_1 and ψ_2, defined from $\mathcal{Z}_R^{\mathcal{S}}$ to $\mathcal{Z}_R^{\mathcal{S}} \otimes \mathcal{Z}_R^{\mathcal{S}}$ by $\psi_1(a) = a \otimes 1$ and $\psi_2(a) = 1 \otimes a$, are approximately unitarily equivalent.*

Proof. The following argument shows that the proof of [13, Prop. 8.3] applies to show this.

Given $F \subset A_m$ and $\varepsilon > 0$, pick n and ω as in the proof of [13, Prop. 8.3] and, as in the complex case, define $\varphi : A_m \to A_n$ by $\varphi(a)(x) = \varphi_{m,n}(a)[\omega(x)]$ for all $x \in [0,1]$ and $a \in A_m$. To simplify notation let $p_n = p$ and $q_n = q$. The construction in the real case still produces a continuous unitary path u in U_{pq} with $\varphi = u^*\Delta^\varphi u$, $u_x = u_0$ for $0 \le x \le c$ and $u_x = u_1$ if $1 - c \le x \le 1$.

The symmetry $S_p \in (M_p(\mathbb{R}) \otimes I_q) \otimes (M_p(\mathbb{R}) \otimes I_q)$ implementing the flip automorphism can be chosen to be in the real matrix algebra: note that the formula

$$S_p = \sum_{1 \le i,j \le p} (e_{ij} \otimes I_q) \otimes (e_{ji} \otimes I_q)$$

is a suitable choice. An analogous result holds for the symmetry

$$S_q \in (I_p \otimes M_q(\mathbb{R})) \otimes (I_p \otimes M_q(\mathbb{R})).$$

With $S = S_p S_q$, the continuous $U : [0,1] \times [0,1] \to U_{p^2q^2}$ defined in [13] by $U_{x,y} = [(u_x^* u_y) \otimes I_{pq}]S$ is real-valued and satisfies

$$\|U^*(\varphi(f) \otimes 1)U - 1 \otimes \varphi(f)\| \le \frac{\varepsilon}{3}$$

for all f in F. However the formula in [13] defining the unitary path v in $U_{p^2q^2}$ does not apply to the real case. Note instead that a path v can be constructed with $v_0 = S_q$, $v_x \in (I_p \otimes M_q) \otimes (I_p \otimes M_q(\mathbb{R})) \cap \{S_q\}'$ for $0 \le x \le c$, $v_x = I_{p^2q^2}$ for $c \le x \le 1 - c$, $v_x \in (M_p(\mathbb{R}) \otimes I_q) \otimes (M_p(\mathbb{R}) \otimes I_q) \cap \{S_p\}'$ for $1 - c \le x \le 1$ and $v_1 = S_p$. To see this, note firstly that if $pq \equiv 0 \pmod 4$ then either p is odd and $q \equiv 0 \pmod 4$ or q is odd and $p \equiv 0 \pmod 4$. When p is even then $\det(S_q) = 1$ and $\det(S_p) = 1$ if $p \equiv 0 \pmod 4$; when q is even then $\det(S_p) = 1$ and $\det(S_q) = 1$ if $q \equiv 0 \pmod 4$: to see this note that if $T = \sum_{i,j=1}^n e_{i,j} \otimes e_{j,i} \in M_n \otimes M_n$ then the trace of T is n and so T has $n(n-1)/2$ eigenvalues -1. When pq is odd and $\det(S_q) = -1$ then $\det(-S_q) = 1$ and $-S_q$ is still a symmetry in $(I_p \otimes M_q(\mathbb{R})) \otimes (I_p \otimes M_q(\mathbb{R}))$ implementing the flip; similarly $\det(S_p)$ can be chosen to be 1.

If $Q_q = \frac{1}{2}(1 - S_q)$, then Q_q is a projection of even dimension (onto the eigenspace of S_q with eigenvalue -1). Rotating $-Q_q$ to Q_q and adding the resultant path to $1 - Q_q$ produces a unitary path in $(I_p \otimes M_q(\mathbb{R})) \otimes (I_p \otimes M_q(\mathbb{R})) \cap \{S_q\}'$ from S_q to $I_{p^2q^2}$. The path for $1 - c \le x \le 1$ is constructed similarly.

For any path v as above, W can be constructed by the same formula as in [13]. By construction $v_x S = S v_x$ for each $0 \le x \le 1$ which, as in [13], implies that W is continuous. Then, just as in [13], checking the boundary conditions shows that $W \in A_n \otimes A_n$, completing the proof. □

Theorem 4.3. *Let $\mathcal{S} = \{(A_n, \alpha_n) \mid n \in \mathbb{N}\}$ and $\mathcal{T} = \{(B_n, \beta_n) \mid n \in \mathbb{N}\}$ be inductive sequences of prime dimension drop algebras of odd order or order a multiple of 4, with connecting maps of the form specified in Proposition 2.1. Then*

$$\mathcal{Z}_R^{\mathcal{S}} \cong \mathcal{Z}_R^{\mathcal{S}} \otimes \mathcal{Z}_R^{\mathcal{T}} \cong \mathcal{Z}_R^{\mathcal{T}}.$$

Proof. $\mathcal{Z}_R^{\mathcal{S}}$ can be written as an inductive limit of prime dimension drop algebras $I_R[p_n, p_n q_n, q_n]$ with $p_{n+1}/p_n > 2p_n q_n$ and $q_{n+1}/q_n > 2p_n q_n$ for each n. Applying Lemma 4.1 with $P = p_{n+1}/p_n$, $Q = q_{n+1}/q_n$, $p = p_n$, $q = q_n$ and

$\varphi = \alpha_n$, there exists

$$\psi_n : I_R\Big[\frac{p_{n+1}}{p_n}, \frac{p_{n+1}q_{n+1}}{p_nq_n}, \frac{q_{n+1}}{q_n}\Big] \to I_R[p_{n+1}, p_{n+1}q_{n+1}, q_{n+1}]$$

with the properties specified in the lemma.

Let p, q be relatively prime numbers. By Lemma 3.1 there exists N such that, for all $n \geq N$, there are morphisms

$$\varphi_n : I_R[p, pq, q] \to I_R\Big[\frac{p_{n+1}}{p_n}, \frac{p_{n+1}q_{n+1}}{p_nq_n}, \frac{q_{n+1}}{q_n}\Big]$$

and thus morphisms $\gamma_n : I_R[p, pq, q] \to \mathcal{Z}_R^{\mathcal{S}}$, defined by $\gamma_n = \alpha_{n+1,\infty} \circ \psi_n \circ \varphi_n$. These give rise to $\gamma : I_R[p, pq, q] \to \ell_\infty(\mathcal{Z}_R^{\mathcal{S}})$ with $\gamma(g)_m = \gamma_m(g)$ for all $m \geq N$.

Given $f \in I_R[p_n, p_nq_n, q_n]$, with $n \geq N$, and $\varepsilon > 0$, there exists $M \geq n$ such that $\|\Delta^{\alpha_{n,m+1}}(f)(x) - \Delta^{\alpha_{n,m+1}}(f)(y)\| < \varepsilon$ for all $x, y \in [0, 1]$ and all $m \geq M$. Applying Lemma 4.1, $\|\gamma_m(g)\alpha_{n,\infty}(f) - \alpha_{n,\infty}(f)\gamma_m(g)\| < 2\varepsilon$ whenever $\|g\| \leq 1$, leading to γ giving rise to a unital embedding of $I_R[p, pq, q]$ into the central sequence algebra $(\mathcal{Z}_R^{\mathcal{S}})_\infty$. Then as in the proof of [32, Prop. 2.2], starting with an application of the real version of the Choi–Effros theorem from [12], there exists a unital embedding of $\mathcal{Z}_R^{\mathcal{T}}$ into $(\mathcal{Z}_R^{\mathcal{S}})_\infty$.

The maps ψ_1 and ψ_2, defined from $\mathcal{Z}_R^{\mathcal{T}}$ to $\mathcal{Z}_R^{\mathcal{T}} \otimes \mathcal{Z}_R^{\mathcal{T}}$ by $\psi_1(a) = a \otimes 1$ and $\psi_2(a) = 1 \otimes a$, are approximately unitarily equivalent by Proposition 4.2. Therefore, as in the proof of [32, Prop. 2.2], the real version of [20, Thm. 7.2.2] gives $\mathcal{Z}_R^{\mathcal{S}} \cong \mathcal{Z}_R^{\mathcal{S}} \otimes \mathcal{Z}_R^{\mathcal{T}}$. Interchanging $\mathcal{S}$ and $\mathcal{T}$ then gives $\mathcal{Z}_R^{\mathcal{T}} \cong \mathcal{Z}_R^{\mathcal{S}} \otimes \mathcal{Z}_R^{\mathcal{T}}$. □

The notation $\mathcal{Z}_R$ will be used for the limit of a sequence of prime dimension drop algebras of odd order or order a multiple of 4, with connecting maps of the form in the previous theorem. The following corollary is therefore an immediate consequence of the theorem.

Corollary 4.4. $\mathcal{Z}_R \cong \mathcal{Z}_R \otimes \mathcal{Z}_R$.

5. Strong self-absorption

In [31], a separable unital C^*-algebra D is defined to be strongly self-absorbing if it is not isomorphic to $\mathbb{C}$ and there is an isomorphism φ from D to $D \otimes D$ which is approximately unitarily equivalent to $\mathrm{id}_D \otimes 1_D$. Replacing $\mathbb{C}$ by $\mathbb{R}$ gives the corresponding definition for real C^*-algebras. The main step in showing that $\mathcal{Z}_R$ is strongly self-absorbing is to show that $\mathcal{Z}_R$ is isomorphic to $\bigotimes_{i=1}^{\infty} \mathcal{Z}_R$.

The proof of [13, Prop. 7.3], that $I[p, pq, q]$ is weakly stable, uses the results from [13, Ex. 7.2] that $I[1, p, p]$ and $I[1, p, 1]$ are both weakly stable. To check the real analog of these results, firstly note that, when $p \geq 2$, the *-isomorphism from $C^*[G_p|R_p]$ onto $I[p, p, 1]$ given in [13] maps each generator a_j in G_p into $I_R[p, p, 1]$, so that $I_R[p, p, 1]$ is also the universal (real) C^*-algebra generated by a finite set of relations. Although the isomorphism from $C^*[G'', R'']$ onto $I[1, p, 1]$ given in [14, Prop. 2.9] does not map the generators $v, x_1, \ldots, x_p$ into

$I_R[1,p,1]$ it does map each of

$$\frac{1}{2}(v+v^*), \frac{1}{2i}(v-v^*), \frac{1}{2}(x_1+x_1^*), \frac{1}{2i}(x_1-x_1^*), \ldots, \frac{1}{2}(x_p+x_p^*), \frac{1}{2i}(x_p-x_p^*)$$

into $I_R[1,p,1]$. Recasting the relations in terms of the generators a_j (which do map into $I_R[1,p,1]$) and the alternative generators above, $I_R[1,p,1]$ is also the universal (real) C^*-algebra generated by a finite set of relations. The proof of [15, Thm. 4.1.4] also holds in the real case, so it suffices to show that both $I_R[p,p,1]$ and $I_R[1,p,1]$ are weakly semiprojective. As in the complex case, by adjoining units to $C_0((0,1],M_p(\mathbb{R}))\cong C_0([0,1),M_p(\mathbb{R}))$ and

$$\mathbb{I}_p^R=\{f\in C_0((0,1],M_p(\mathbb{R}))\mid f(1)\in\mathbb{R}\},$$

it suffices to consider these two algebras.

The proofs of [15, Thm. 10.2.1] and the preliminary results also hold for the real case, to demonstrate that $C_0((0,1],M_p(\mathbb{R}))$ is projective and hence semiprojective. For $\mathbb{I}_p^R$ as above, it suffices to show that $C_0((0,1),M_p(\mathbb{R}))$ is semiprojective because the argument on [15, p. 127], the proof of [15, Thm. 16.1.1] and the proofs of the preliminary results from earlier chapters also hold in the real case, as does the proof that the map γ defined on [15, p. 126] is corona extendible. Furthermore, the proof of [15, Thm. 14.2.2] and the proofs of the preliminary results from earlier chapters hold in the real case to show that $M_p(A)$ is semiprojective for any semiprojective σ-unital real C^*-algebra A, so it suffices to consider $C_0((0,1),\mathbb{R})$.

The proof that $C_0((0,1),\mathbb{R})$ is semiprojective holds by adjusting the proof of [15, Lem. 14.1.8] as follows. If u is the canonical unitary generator of $C(S^1,\mathbb{C})$, then a homomorphism α from $C(S^1,\mathbb{C})$ into the complexification $B^{\mathbb{C}}$ of a real C^*-algebra B will map $C(S^1,\mathbb{R})$ into B precisely when $\Phi(\alpha(u))=\alpha(u)$, where Φ is the involutory *-antiautomorphism of $B^{\mathbb{C}}$ associated with B. Thus to show $C(S^1,\mathbb{R})$ is semiprojective it suffices to show that the unitary lift w obtained in [15, Lem. 14.1.8] can be chosen to satisfy $\Phi(w)=w$ for the appropriate antiautomorphism Φ. However the invertible lift x can be chosen to satisfy $\Phi(x)=x$ and then the polar decomposition of x satisfies

$$w|x|=x=\Phi(x)=\Phi(w)[\Phi(w)^*\Phi(|x|)\Phi(w)].$$

The uniqueness of the polar decomposition then gives the required result.

The argument above shows that $I_R[1,p,p]$ and $I_R[1,p,1]$ are both weakly stable. Using these facts and that the generators and relations given for $I[p,pq,q]$ in the proof of [13, Prop. 7.3] are also real generators and relations for $I_R[p,pq,q]$, the proof of that proposition shows that $I_R[p,pq,q]$ is weakly stable.

Theorem 5.1. *$\mathcal{Z}_R$ is strongly self-absorbing.*

Proof. As in the proof of [13, Cor. 8.8], the weak stability of real dimension drop algebras, together with Theorem 4.3, implies that $\mathcal{D}=\bigotimes_{i=1}^{\infty}\mathcal{Z}_R$ is the closure of an increasing union of prime dimension drop algebras of odd order. As in the proof of Theorem 4.3 there is a unital embedding of each of these dimension drop algebras into $(\mathcal{Z}_R)_\infty$ and therefore, again using the real version of [32, Prop. 2.2], there is a unital embedding of D in $(\mathcal{Z}_R)_\infty$. From Proposition 4.2 it follows that the homomorphisms $a \mapsto a \otimes 1$ and $a \mapsto 1 \otimes a$ from D to $D \otimes D$ are approximately unitarily equivalent. The real version of [20, Thm. 7.2.2] thus gives an isomorphism from $\mathcal{Z}_R$ onto $\mathcal{Z}_R \otimes \mathcal{D} \cong \mathcal{D}$. The result then follows from the real version of [31, Prop. 1.9], using the approximately inner half flip property of $\mathcal{Z}_R$ established in Proposition 4.2. □

6. $\mathcal{Z}_R$-stability

The following two results give simple examples of $\mathcal{Z}_R$-stability. The proof of the first result is a minor variant of [32, Thm. 2.3].

Theorem 6.1. *Let A be the closure of an increasing sequence A_n of finite-dimensional real C^*-algebras. Let A be unital, simple and infinite-dimensional. Then A is isomorphic to $A \otimes \mathcal{Z}_R$.*

Proof. Write $\mathcal{Z}_R$ as the closure of an increasing sequence B_i of prime real dimension drop algebras $B_i \cong I_R(p_i, p_iq_i, q_i)$ and, as in the proof of Lemma 3.1 note that there is a unital *-homomorphism from B_i to $M_k(\mathbb{R})$ for each $k > p_iq_i$. From [7, Cor. III.4.3] it follows, by omitting terms of the sequence, that, for each n, each simple summand of A_n embeds with multiplicity at least $p_iq_i + 1$ into A_{n+1} and hence that there is a unital subalgebra of $A_{n+1} \cap A_n'$ isomorphic to $M_{k_1}(\mathbb{R}) \oplus \cdots \oplus M_{k_t}(\mathbb{R})$ where $k_j \geq p_iq_i$ for each j. Thus there is a unital homomorphism $\psi_{i,n}$ from B_i into $A_{n+1} \cap A_n'$.

Let $\gamma : B_i \to \ell_\infty(A)/c_0(A)$ be defined by $\gamma(f) = (\psi_{i,n}(f))_{n\in\mathbb{N}} + c_0(A)$. Then $\gamma(B_i)$ commutes with A, so γ is a *-homomorphism from B_i to A_∞. By the real version of [32, Prop. 2.2] there exists a *-homomorphism from $\mathcal{Z}_R$ into A_∞ and by Proposition 4.2, the maps ψ_1 and ψ_2 from $\mathcal{Z}_R$ to $\mathcal{Z}_R \otimes \mathcal{Z}_R$ defined by $\psi_1(a) = a \otimes 1$ and $\psi_2(a) = 1 \otimes a$ are approximately unitarily equivalent. The real version of [20, Thm. 7.2.2] therefore applies to show that A is isomorphic to $A \otimes \mathcal{Z}_R$. □

As in the complex case, the following result follows immediately from the appropriate classification theorem.

Theorem 6.2. *Let A be a unital separable nuclear purely infinite real C^*-algebra with simple complexification and which satisfies the universal coefficient theorem. Then A is isomorphic to $A \otimes \mathcal{Z}_R$.*

Proof. $K^{CRT}(\mathcal{Z}_R)$ is a free CRT-module with generator [1] in the real part and therefore, by [3, Prop. 3.5, 4.4],

$$K^{CRT}(A \otimes \mathcal{Z}_R) \cong K^{CRT}(A) \otimes_{CRT} K^{CRT}(\mathcal{Z}_R) \cong K^{CRT}(A),$$

where the isomorphism takes $[1_{A\otimes\mathcal{Z}_R}] \in K_0(A \otimes \mathcal{Z}_R)$ to $[1_A] \in K_0(A)$. The result follows by [5, Thm. 10.2 (2)]. □

7. The Cuntz semigroup of $\mathcal{Z}_R$

As for the complex case, described in [21], the Cuntz semigroup $W(A)$ of a real C^*-algebra A is defined as follows. Let $M_\infty(A)^+$ denote the disjoint union $\bigcup_{n=1}^\infty M_n(A)^+$. For $a \in M_n(A)^+$ and $b \in M_m(A)^+$ set $a \oplus b = \operatorname{diag}(a,b) \in M_{n+m}(A)^+$ and write $a \precsim b$ if there is a sequence $\{x_k\}$ in $M_{m,n}(A)$ such that $x_k^* b x_k \to a$. Write $a \sim b$ if $a \precsim b$ and $b \precsim a$ and let $W(A) = M_\infty(A)^+/\sim$. Then $W(A)$ is a partially ordered abelian semigroup when equipped with the relations $\langle a\rangle + \langle b\rangle = \langle a \oplus b\rangle$ and $\langle a\rangle \le \langle b\rangle \Leftrightarrow a \precsim b$, where $\langle a\rangle$, $\langle b\rangle$ are the equivalence classes containing $a, b \in M_\infty(A)^+$. As in the complex case the set $V(A)$ of Murray–von Neumann equivalence classes $[p]$ of projections p in $M_\infty(A)$ is an abelian semigroup under the operation $[p] + [q] = [p \oplus q]$ and $[p] \mapsto \langle p\rangle$ is a semigroup homomorphism φ from $V(A)$ to $W(A)$. The arguments in [19, Prop. 2.1] and [2, Lem. 2.20] apply also to the real case so φ is an injection when A is stably finite, in which case $\varphi(V(A))$ will be identified with $V(A)$. The remaining elements of $W(A)$ will be denoted by $W(A)_+$.

In [21], it is shown that for any $\mathcal{Z}$-stable C^*-algebra A, its Cuntz semigroup $W(A)$ is almost unperforated, i.e. whenever $n\langle a\rangle \le m\langle b\rangle$ with $n > m$ then $\langle a\rangle \le \langle b\rangle$. The proof relies on the construction, for each $n \in \mathbb{N}$, of a positive element e_n of $\mathcal{Z}$ with $n\langle e_n\rangle \le \langle 1\rangle \le (n+1)\langle e_n\rangle$ but the proof of this result relies on the connectedness of a unitary group and cannot be applied to the real case. Instead the corresponding result for $\mathcal{Z}_R$ will be deduced from known results about $W(\mathcal{Z})$ and the following general facts about Cuntz semigroups of real and complex C^*-algebras.

If $A^{\mathbb{C}} = A \otimes \mathbb{C}$ is the complexification of A and $\alpha : a + ib \mapsto a - ib$ is the involutory automorphism of $A^{\mathbb{C}}$ associated with A, let $c : W(A) \to W(A^{\mathbb{C}})$ be the map arising from the embedding of A in $A^{\mathbb{C}}$, let $r : W(A^{\mathbb{C}}) \to W(A)$ be the map arising from the embedding

$$a + ib \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

of $A^{\mathbb{C}}$ in $M_2(A)$ and let $\alpha : W(A^{\mathbb{C}}) \to W(A^{\mathbb{C}})$ arise from $\alpha : A^{\mathbb{C}} \to A^{\mathbb{C}}$. Then r and c are semigroup homomorphisms with $r \circ c = 2\,\mathrm{id}$ and, using the identity

$$\frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} = \begin{pmatrix} a + ib & 0 \\ 0 & a - ib \end{pmatrix},$$

with $c \circ r = \mathrm{id} + \alpha$.

The proof of the real analog of [21, Lem. 4.2] also uses the strict comparison property of $\mathcal{Z}$. To formulate this and the corresponding notion for real C^*-algebras, recall that a dimension function on a C^*-algebra A is an additive order-preserving function $d : W(A) \to [0, \infty]$ with $d(\langle 1\rangle) = 1$ when A is unital. The set of dimension functions is denoted by $\mathrm{DF}(A)$. The same definition can be used when A is a real C^*-algebra and there are natural maps $c_* : d \mapsto d \circ c$ from $\mathrm{DF}(A^{\mathbb{C}})$ to $\mathrm{DF}(A)$ and $r_* : d \mapsto \frac{1}{2} d \circ r$ from $\mathrm{DF}(A)$ to $\mathrm{DF}(A^{\mathbb{C}})$, with $c_* r_* = \mathrm{id}$ and $r_* c_*(d) = \frac{1}{2}(d + d \circ \alpha)$. The image $r_* d$ satisfies $(r_* d) \circ \alpha = r_* d$ and r_* gives a bijection between $\mathrm{DF}(A)$ and $\mathrm{DF}_\alpha(A^{\mathbb{C}})$, the set of α-invariant dimension functions on $A^{\mathbb{C}}$. The set $\mathrm{LDF}(A^{\mathbb{C}})$ of lower semi-continuous dimension functions on $A^{\mathbb{C}}$ consists of the elements d of $\mathrm{DF}(A^{\mathbb{C}})$ satisfying $d(\langle x\rangle) \leq \liminf_n d(\langle x_n\rangle)$ whenever $x_n \to x$ in norm. If $\mathrm{LDF}(A)$ is defined in the same way for a real C^*-algebra A then

$$r_*(\mathrm{LDF}(A)) \subseteq \mathrm{LDF}(A^{\mathbb{C}}), \quad c_*(\mathrm{LDF}(A^{\mathbb{C}})) = \mathrm{LDF}(A),$$

and r_* gives a bijection between

$$\mathrm{LDF}(A) \quad \text{and} \quad \mathrm{LDF}_\alpha(A^{\mathbb{C}}) = \mathrm{LDF}(A^{\mathbb{C}}) \cap \mathrm{DF}_\alpha(A^{\mathbb{C}}).$$

As in the complex case, a real C^*-algebra is said to have strict comparison if, for all $\langle x\rangle, \langle y\rangle \in W(A)$, $\langle x\rangle \leq \langle y\rangle$ whenever $d(\langle x\rangle) < d(\langle y\rangle)$ for all $d \in \mathrm{LDF}(A)$.

The following real analog of [21, Lem. 4.2] can now be established.

Lemma 7.1. *For all natural numbers n there exists a positive element e_n in $\mathcal{Z}_R$ such that $n\langle e_n\rangle \leq \langle 1\rangle \leq (n+1)\langle e_n\rangle$.*

Proof. For $0 < \lambda \leq 1$ and $0 \leq t \leq 1$ let $g_\lambda(t) = \max((t+\lambda-1)/\lambda, 0)$ and let $h(t) = 1-t$. Let τ be the unique trace on $\mathcal{Z}_R$ and let τ_0 be the normalized trace on $I = I_R[2n,2n,1]$ arising from Lebesgue measure on $[0,1]$, so $\tau_0(g_{1/(2n+1)}1) = 1/(4n+2)$. By Theorem 3.2 there is a unital embedding $\psi : I_R[2n,2n,1] \to \mathcal{Z}_R$ with

$$\left|(\tau \circ \psi)(g_{1/(2n+1)}1) - \frac{1}{4n+2}\right| < \frac{1}{(4n+1)(4n+2)}$$

and therefore $(\tau\circ\psi)(g_{1/(2n+1)}1) < 1/(4n+1)$. Let $e_n = \psi(he_{11} + he_{22})$, so $n\langle e_n\rangle = \langle \psi(h1)\rangle \leq \langle 1\rangle$.

The trace $\tau\circ\psi$ on $I_R[2n,2n,1]$ defines a probability measure μ on $[0,1]$ with $\tau\circ\psi(f1) = \int_{[0,1]} f\, d\mu$ for all $f \in C([0,1],\mathbb{R})$ and with the corresponding lower semi-continuous dimension function $d_{\tau\circ\psi}$ satisfying

$$d_{\tau\circ\psi}(f1) = \mu(\{x \mid f(x) \neq 0\})$$

for each $f \in C([0,1],\mathbb{R})$.

Suppose firstly that $\mu(\{1\}) = 0$. Then $\mu([1-\lambda,1]) \to 0$ as $\lambda \to 0$, so there exists $\lambda > 0$ with

$$d_{\tau\circ\psi}(g_\lambda 1) < \frac{1}{2n} d_{\tau\circ\psi}(h1) = d_{\tau\circ\psi}(he_{11}).$$

Next let $\mu = k\delta_1 + (1-k)\mu'$ where $\mu'(\{1\}) = 0$ and $k > 0$. Then, for each $\lambda \leq 1/(2n+1)$,

$$k \leq (\tau\circ\psi)(g_\lambda 1) \leq (\tau\circ\psi)(g_{1/(2n+1)}1) < \frac{1}{4n+1}$$

and so

$$2k < \frac{1-k}{2n}.$$

Since $\mu'([1-\lambda,1)) \to 0$ as $\lambda \to 0$, there exists λ with $(1-k)\mu'([1-\lambda,1)) < k$. It follows that

$$d_{\tau\circ\psi}(g_\lambda 1) < 2k < \frac{1-k}{2n} = d_{\tau\circ\psi}(he_{11}).$$

In both cases for k, using strict comparison in $\mathcal{Z}$, $c\langle\psi(g_\lambda 1)\rangle \leq c\langle\psi(he_{11})\rangle$ and then

$$2\langle\psi(g_\lambda 1)\rangle = rc\langle\psi(g_\lambda 1)\rangle \leq rc\langle\psi(he_{11})\rangle = 2\langle\psi(he_{11})\rangle = \langle e_n\rangle.$$

Therefore

$$\begin{aligned}(n+1)\langle e_n\rangle &= \langle\psi(h1)\rangle + \langle e_n\rangle \\ &\geq \langle\psi(h1)\rangle + 2\langle\psi(g_\lambda 1)\rangle \\ &\geq \langle\psi(h1)\rangle + \langle\psi(g_\lambda 1)\rangle \\ &\geq \langle\psi(h1 + g_\lambda 1)\rangle \\ &= \langle 1\rangle,\end{aligned}$$

where the last equality follows from the fact that $h + g_\lambda$ is invertible, so $\langle(h + g_\lambda)1\rangle = \langle 1\rangle$. □

The following real analog of [21, Thm. 4.5] now follows with the same proof.

Proposition 7.2. *Let A be a $\mathcal{Z}_R$-absorbing real C^*-algebra. Then $W(A)$ is almost unperforated.*

As with [21, Cor. 4.6], the proof of [19, Thm. 5.2] can also be applied in the real case to give the following result.

Proposition 7.3. *Let A be a simple unital $\mathcal{Z}_R$-absorbing real C^*-algebra. Then A has strict comparison.*

In [18, Thm. 3.1], it is shown that $W(\mathcal{Z})$ is the disjoint union of $V(\mathcal{Z}) \cong \mathbb{Z}^+$ and $W(\mathcal{Z})_+ \cong \mathbb{R}^{++}$ with $\langle I_n\rangle$ corresponding to $n \in \mathbb{Z}^+$, $\langle z_\lambda\rangle$ to $\lambda \in (0,1]$ and $\langle I_n\rangle + \langle z_\lambda\rangle$ to $n + \lambda \in (n, n+1]$, whenever $z_\lambda \in \mathcal{Z}$ satisfies $d_\tau(z_\lambda) = \lambda$ and $z_1 \not\approx 1$. The arguments used in [18] can now be used to establish the structure of $W(\mathcal{Z}_R)$.

Proposition 7.4. *Let $A = \mathcal{Z}_R$. The map $c : W(A) \to W(\mathcal{Z})$ is an isomorphism, so $W(A) \cong \mathbb{Z}^+ \sqcup \mathbb{R}^{++}$.*

Proof. From $K_0(A) \cong \mathbb{Z}$ with generator $[1]$ it follows that $V(A) \cong \mathbb{Z}^+$ with generator $\langle 1\rangle$, so c restricts to an isomorphism from $V(A)$ onto $V(\mathcal{Z})$. The only involutory automorphism of $W(\mathcal{Z}) \cong \mathbb{Z}^+ \sqcup \mathbb{R}^+$ is the identity, so

$$c \circ r = 2\,\mathrm{id} : W(\mathcal{Z}) \to W(\mathcal{Z}),$$

from which it follows that $cW(A)_+ \supseteq 2W(\mathcal{Z})_+ = W(\mathcal{Z})_+$. To show that $cW(A)_+ \subseteq W(\mathcal{Z})_+$, suppose that $\langle a\rangle \in W(A)$ with $c\langle a\rangle \in V(\mathcal{Z})$. By [18, Prop. 2.8], 0 is an isolated point of $\sigma(a)$ or $0 \notin \sigma(a)$, from which it follows using the functional calculus that $\langle a\rangle \in V(A)$, as required. It therefore remains to show that $c : W(A)_+ \to W(\mathcal{Z})$ is injective.

Let $\langle a\rangle, \langle b\rangle \in W(A)_+$, with $c\langle a\rangle = c\langle b\rangle$ and chose n so that a, b are positive elements of $M_n(A)$ with $0 \in \sigma(a) \cap \sigma(b)$. Then 0 is not an isolated point in $\sigma(a)$ because otherwise, using the functional calculus, $\langle a\rangle = \langle p\rangle$ for a projection p.

So let $\varepsilon_n \in \sigma(a)\backslash\{0\}$ converge to 0. Let $d^{\mathbb{C}}$ be the unique lower semi-continuous dimension function on $\mathcal{Z}$ so that $d = c_* d^{\mathbb{C}}$ is the unique lower semi-continuous dimension function on A. By [18, Lem. 2.4],

$$d(\langle (a-\varepsilon_n)_+\rangle) = d^{\mathbb{C}}(c\langle (a-\varepsilon_n)_+\rangle) < d^{\mathbb{C}}(c\langle a\rangle) = d^{\mathbb{C}}(c\langle b\rangle) = d(\langle b\rangle).$$

Using Proposition 7.3 it follows that $\langle (a-\varepsilon_n)_+\rangle \leq \langle b\rangle$ for each n and hence that $\langle a\rangle \leq \langle b\rangle$. Repeating the argument with a and b interchanged gives $\langle b\rangle \leq \langle a\rangle$ and hence $\langle a\rangle = \langle b\rangle$, showing that c is injective. □

8. Strict comparison and $\mathcal{Z}_R$-stability

In Proposition 7.3 above, it was shown that a simple unital $\mathcal{Z}_R$-absorbing real C^*-algebra has strict comparison. In this section we obtain a real counterpart to the partial converse from [16] and [26]. We use $T(A)$ for the set of real traces on a real C^*-algebra and note that the restriction of a trace on $A^{\mathbb{C}}$ need not restrict to a real trace on A because the restriction may not be real-valued. However, if $\Phi : a + ib \to a^* + ib^*$ is the involutory antiautomorphism of $A^{\mathbb{C}}$ associated with A, for which $A = \{x \in A^{\mathbb{C}} \mid \Phi(x) = x^*\}$, then Φ-invariant traces on $A^{\mathbb{C}}$ do restrict to real-valued traces and the restriction is a bijection from the set $T_\Phi(A^{\mathbb{C}})$ of Φ-invariant traces onto $T(A)$. The involutory antiautomorphism Φ gives rise to an affine homeomorphism Φ^* of $T(A^{\mathbb{C}})$ with $\Phi^*(\tau) = \tau \circ \Phi$, which maps $\partial_e T(A^{\mathbb{C}})$ onto itself.

The first step is to obtain a real analog of [16, Prop. 2.2]. This relates to the property of excision in small central sequences which can be defined for real algebras in exactly the same way as in [16, Def. 2.1], as follows.

Definition 8.1. Let A be a separable real C^*-algebra with nonempty real tracial state space $T(A)$. A completely positive map $\varphi : A \to A$ can be excised in small central sequences when, for any central sequences $(e_n)_n$ and $(f_n)_n$ of positive contractions in A satisfying

$$\lim_{n\to\infty} \max_{\tau\in T(A)} \tau(e_n) = 0, \qquad \lim_{m\to\infty} \liminf_{n\to\infty} \min_{\tau\in T(A)} \tau(f_n^m) > 0,$$

there exist $s_n \in A$, $n \in \mathbb{N}$ such that, for any $a \in A$,

$$\lim_{n\to\infty} \|s_n^* a s_n - \varphi(a) e_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|f_n s_n - s_n\| = 0.$$

In the following proposition a pure state ω of A is said to be of *real type* if it satisfies the equivalent conditions of [4, Thm. 4], for example $\pi_\omega(A)' = \mathbb{R}1$, where π_ω is the GNS representation associated with ω. Another equivalent condition from that theorem is that the canonical Φ-invariant extension $\omega^{\mathbb{C}}$ of ω to the complexification $A^{\mathbb{C}}$ is pure, where $\Phi : a + ib \to a^* + ib^*$ is the involutory antiautomorphism of $A^{\mathbb{C}}$ associated with A. It is shown in [11, Cor. 3.6] that when $A^{\mathbb{C}}$ is non-type I, separable, simple and unital then it has a Φ-invariant pure state; this will restrict to a pure state of A of real type.

Proposition 8.2. *Let A be a unital separable simple infinite-dimensional C^*-algebra with a nonempty real tracial state space $T(A)$ and with $A^{\mathbb{C}}$ also simple. Let ω be a pure state of A of real type and let $c_i, d_i \in A$ for $1 \leq i \leq N$. If A has strict comparison then the completely positive map $\varphi : A \to A$ defined by*

$$\varphi(a) = \sum_{i,j=1}^{N} \omega(d_i^* a d_j) c_i^* c_j$$

can be excised in small central sequences.

Proof. The proof is a direct adaptation of that in [16]. When [16, Lem. 2.3] is applied to $A^{\mathbb{C}}$, starting with a central sequence $(f_n)_n$ of positive contractions with $\Phi(f_n) = f_n$, the constructed elements $\tilde{f_n}$ belong to A. Furthermore, for each $\tau \in T(A^{\mathbb{C}})$, τ and $\frac{1}{2}(\tau + \tau \circ \Phi)$ agree on selfadjoint elements of A so that, in the conclusion of the lemma, the minimum over $T(A)$ is equal to the minimum over $T(A^{\mathbb{C}})$. Lemma 2.4 of [16] applies directly to $A^{\mathbb{C}}$ as does the proof of [16, Lem. 2.5], noting that the nonzero positive contraction a_0 can be chosen to belong to A and that the strict comparison property of A leads to $q_n \in A$. The proof of [16, Prop. 2.2] can now be followed because ω has been chosen to be a pure state of real type and so, by [4, Thm. 4], $\ker \omega = L + L^*$, as required for the proof of [1, Prop. 2.2] to apply in the real case. ☐

As in [26], for a trace τ on a separable simple real C^*-algebra A and $a \in A$, let

$$\|a\|_\tau = \tau(a^* a)^{1/2}, \quad \|a\|_2 = \sup_{\tau \in T(A)} \|a\|_\tau,$$
$$c_0 = \{(a_n)_n \in l^\infty(\mathbb{N}, A) \mid \lim_{n\to\infty} \|a_n\| = 0\},$$
$$c_{t0} = \{(a_n)_n \in l^\infty(\mathbb{N}, A) \mid \lim_{n\to\infty} \|a_n\|_2 = 0\},$$
$$A^\infty = l^\infty(\mathbb{N}, A)/c_0, \quad A_t^\infty = l^\infty(\mathbb{N}, A)/c_{t0},$$
$$A_\infty = A^\infty \cap A', \quad A_{t\infty} = A_t^\infty \cap A'.$$

The following few results are adapted directly from [26]. The first uses the notion of a completely positive map of order zero from a finite-dimensional real C^*-algebra. The definition in the complex case, from [33], is that $\varphi(e) \perp \varphi(f)$ for each pair e, f of mutually orthogonal projections. Applying this definition directly in the real case results in the completely positive map $\varphi : \mathbb{H} \to \mathbb{R}$ with $\varphi(a+bi+cj+dk) = a$ being of order 0, although it does not satisfy the structure properties in [33, Prop. 3.2] and [34, Prop. 1.2.1] and its complexification $\varphi^{\mathbb{C}} = \text{trace} : M_2(\mathbb{C}) \to \mathbb{C}$ is not of order 0. We therefore choose to define a completely positive map $\varphi : F \to A$ between real C^*-algebras to be of order zero if its complexification $\varphi^{\mathbb{C}}$ is of order zero. When F is complex, this agrees with the usual definition.

Lemma 8.3. *Let A be a real C^*-algebra for which $A^{\mathbb{C}}$ is unital, separable, simple and infinite-dimensional and for which $\partial_e(T(A^{\mathbb{C}}))$ is compact.*

(i) *For any central sequence* $(f_n)_n \in A_\infty$ *and* $a \in A$,

$$\lim_{n\to\infty} \max_{\tau\in\partial_e T(A^{\mathbb{C}})} |\tau(f_n a) - \tau(f_n)\tau(a)| = 0.$$

(ii) *Moreover, if* A *is nuclear, for mutually orthogonal positive functions* $f_i \in C(\partial_e(T(A^{\mathbb{C}})))$ *with* $f_i = f_i \circ \Phi^*$ *for* $1 \le i \le N$ *there exist central sequences* $(a_{i,n})_n$ *of positive elements of* A, *for* $1 \le i \le N$, *such that, for* $1 \le i \le N$,

$$\lim_{n\to\infty} \max_{\tau\in\partial_e T(A^{\mathbb{C}})} |\tau(a_{i,n}) - f_i(\tau)| = 0$$

and

$$\lim_{n\to\infty} \|a_{i,n}a_{j,n}\| = 0 \quad \textit{for } i \neq j.$$

Proof. The only change required in the proof of [26, Lem. 4.2] is to ensure that $\Phi(a_{i,n}) = a_{i,n}$, which follows from $\Phi(b_{i,n}) = b_{i,n}$. In [26, Prop. 4.1], if $f = f \circ \Phi^*$ then, replacing a_n by $\frac{1}{2}(a_n + \Phi(a_n))$, we can arrange that $\Phi(a_n) = a_n$. Therefore, returning to the proof of [26, Lem. 4.2], $\Phi(b'_{i,n}) = b'_{i,n}$. The required result $\Phi(b_{i,n}) = b_{i,n}$ now follows by noting that when $\Phi(a) = a$ in [26, Cor. 3.3], then a_n can be replaced by $\frac{1}{2}(a_n + \Phi(a_n))$. □

Lemma 8.4. *Let* A *be a real* C^**-algebra for which* $A^{\mathbb{C}}$ *is unital, separable, simple, infinite-dimensional and nuclear, let* $\tau \in \partial_e(T(A^{\mathbb{C}}))$, *let* $\varepsilon > 0$ *and let* F *be a finite set of contractions in* $A^{\mathbb{C}}$. *Then, for any odd* $k \in \mathbb{N}$, *there exists a completely positive map* $\varphi_\tau : M_k(\mathbb{C}) \to A^{\mathbb{C}}$, *mapping* $M_k(\mathbb{R})$ *to* A *such that*

$$\begin{aligned} &\|\varphi_\tau(x)\varphi_\tau(y)\| < \varepsilon\|x\|\,\|y\| && \textit{for any } x, y \in M_k(\mathbb{C})^+ \textit{ with } xy = 0,\\ &\|[\varphi_\tau(x), a]\| < \varepsilon\|x\| && \textit{for all } x \in M_k(\mathbb{C}) \textit{ and } a \in F,\\ &\|1 - \varphi_\tau(1_k)\|_\tau \le \tau(1 - \varphi_\tau(1_k))^2 < \varepsilon. \end{aligned}$$

Proof. As in the proof of [16, Lem. 3.3], let π be the GNS representation of $A^{\mathbb{C}}$ associated with $\sigma = \frac{1}{2}(\tau + \tau \circ \Phi)$ and note that Φ and τ extend to the weak closure $\pi(A^{\mathbb{C}})''$ of $\pi(A^{\mathbb{C}})$. When $\tau = \tau \circ \Phi$ then $\pi(A^{\mathbb{C}})''$ is isomorphic to the unique hyperfinite II_1-factor N and when $\tau \neq \tau \circ \Phi$ then $\pi(A^{\mathbb{C}})''$ is isomorphic to the direct sum of two copies of N. The real form $\pi(A)''$ of $\pi(A^{\mathbb{C}})''$ associated with Φ is isomorphic to the unique real hyperfinite II_1-factor R, from [27] and [8], in the first case and to N in the second. Thus in both cases $\pi(A)'' \otimes_{\mathbb{R}} R \cong \pi(A)''$, so that, for each $k \in \mathbb{N}$, $\pi(A)''$ contains a sequence of matrix units $E_{i,j,n}$ for $M_k(\mathbb{R})$ such that

$$\lim_{n\to\infty} \|[E_{i,j,n}, x]\|_\sigma = 0$$

for any $x \in \pi(A)''$. Since $E_{1,1,n} = \Phi(E_{1,1,n})$, the central sequence $(e_n)_n$ of positive contractions in $\pi(A^{\mathbb{C}})$ obtained from [25, Lem. 2.1] can be replaced by $(\frac{1}{2}(e_n + \Phi(e_n))_n$ and thus can be taken to belong to $\pi(A)$.

As in [16] let

$$U_n = \sum_{i=1}^{k} E_{i,i+1,n}$$

and, as in [25, Lem. 2.1], let $H_n = \frac{1}{i\pi}\log(U_n)$, where the standard branch of log is taken. When $k = 2r+1$ is odd, from $\Phi(U_n) = U_n^*$, the spectral decomposition of U_n has the form

$$U_n = \sum_{j=1}^{2r+1} \lambda_j P_j = \sum_{j=1}^{2r+1} \overline{\lambda_j}\Phi(P_j),$$

where λ_j are the $(2r+1)$st roots of unity and then

$$H_n = \frac{1}{i\pi}\log(U_n) = \sum_{j=1}^{r} \frac{2j}{2r+1}(P_j - \Phi(P_j))$$

so that $H_n = -\Phi(H_n)$. It follows that the central sequence $(h_n)_n$ of positive contractions in $\pi(A^{\mathbb{C}})$ obtained from [25, Lem. 2.1] can be replaced by $(\frac{1}{2}(h_n - \Phi(h_n)))_n$ and thus satisfy $h_n = -\Phi(h_n)$. The corresponding unitary $u_n = e^{i\pi h_n}$ then satisfies $\Phi(u_n) = u_n^*$ and so belongs to $\pi(A)$. The elements $\mathrm{Ad}\, u_n^j(e_n)e_n$ of $\pi(A^{\mathbb{C}})$ thus belong to $\pi(A)$, as do the modifications made in [16, Lem. 3.2], and so the completely positive contractive order zero map $\psi : M_k(\mathbb{C}) \to A_\infty^{\mathbb{C}}$ from [16] maps $M_k(\mathbb{R})$ to A_∞. The argument in the proof of [26, Prop. 5.1], together with the final calculation in the proof of [16, Lem. 3.3] now gives the required result. □

Proposition 8.5. *Let A be a real C^*-algebra for which $A^{\mathbb{C}}$ is unital, separable, simple, infinite-dimensional and nuclear. Suppose that $\partial_e(T(A^{\mathbb{C}}))$ is compact and $d = \dim(\partial_e(T(A^{\mathbb{C}}))) < \infty$. Then, for any odd $k \in \mathbb{N}$ there exist order zero completely positive maps $\varphi_l : M_k(\mathbb{R}) \to A_{t\infty}$ for $0 \le l \le d$ such that*

$$\sum_{l=0}^{d} \varphi_l(1_k) = 1$$

and

$$[\varphi_l(a), \varphi_m(b)] = 0 \quad \textit{for all } l \neq m, \textit{ and } a, b \in M_k(\mathbb{R}).$$

Proof. It needs to be shown that the maps φ_l constructed in [26, Prop. 5.1] can be chosen to restrict to maps from $M_k(\mathbb{R})$ into $A_{t\infty}$. This will entail showing that the elements $a_{i,n,m}$ of $A^{\mathbb{C}}$ can be chosen to belong to A and that the maps φ_{τ_i} can be chosen to map $M_k(\mathbb{R})$ into A. This will be achieved by varying the argument from [26] to apply to the quotient $[\partial_e T(A^{\mathbb{C}})]$ of $\partial_e T(A^{\mathbb{C}})$ under the action $\tau \mapsto \Phi^*\tau = \tau \circ \Phi$; since the inverse image of each point in $[\partial_e T(A^{\mathbb{C}})]$ under the quotient map q has either one or two elements, it follows from [17, Prop. 9.2.16] that $\dim([\partial_e T(A^{\mathbb{C}})]) = \dim(\partial_e T(A^{\mathbb{C}}))$.

For a Φ^*-invariant subset B_0 of dimension less than c modify the inductive assumption of [26] to assume that the maps $\psi_{l,n}$ restrict to maps from $M_k(\mathbb{R})$ into A. To carry out the inductive step, take a Φ^*-invariant closed subset B of dimension c and proceed as in [26]. The existence of appropriate maps φ_τ follows from Lemma 8.4. Now apply the argument of [26] to the quotient space to obtain a partition of unity $\{f_{0,n}\} \cup \{f_{i,n}\}_{i=1}^N \subset C([\partial_e(T(A^{\mathbb{C}}))])$ such that

$\mathrm{supp}(f_{0,n}) \subset [W_{0,n}]$ and $\mathrm{supp}(f_{i,n}) \subset [W_i]$. Applying Lemma 8.3 to the maps $f_{i,n} \circ q \in C(\partial_e T(A^{\mathbb{C}}))$ gives $a_{i,n,m} \in A$, as required. □

The real counterpart of the main theorem of [26] can now be established.

Theorem 8.6. *Let A be a real C^*-algebra for which $A^{\mathbb{C}}$ is unital, separable, simple, infinite-dimensional and nuclear and for which the extreme boundary of the nonempty trace space $T(A^{\mathbb{C}})$ is a compact finite-dimensional space. Then, for any odd $k \in \mathbb{N}$, there exists a unital embedding of $M_k(\mathbb{R})$ into $A_{t\infty}$.*

Proof. The isomorphism in [26, Lem. 2.1 (i)] from $\tilde{I}(A_0^{\mathbb{C}}, A_1^{\mathbb{C}})$ to $I(A_0^{\mathbb{C}}, A_1^{\mathbb{C}})$ restricts to an isomorphism from $\tilde{I}(A_0, A_1)$ to $I(A_0, A_1)$. Then, replacing u by $-u$ if necessary, when k is odd the proof of [26, Lem. 2.1 (ii)] shows that if A_0 and A_1 both contain $M_k(\mathbb{R})$ unitally, then so does $I(A_0, A_1)$. If $\mathcal{U}_{d,k}$ is the universal real C^*-algebra generated by the relations of [26, Cor. 2.3], then the isomorphism of that corollary from $\mathcal{U}_{d,k}^{\mathbb{C}}$ to $\tilde{I}(\mathcal{U}_{d-1,k}^{\mathbb{C}}, M_k(\mathbb{C}))$ restricts to an isomorphism from $\mathcal{U}_{d,k}$ to $\tilde{I}(\mathcal{U}_{d-1,k}, M_k(\mathbb{R}))$. Then, defining $\Delta_{d,k}$ inductively by $\Delta_{0,k} = M_k(\mathbb{R})$ and $\Delta_{d,k} = I(\Delta_{d-1,k}, M_k(\mathbb{R}))$, the isomorphism of [26] from $\Delta_{d,k}^{\mathbb{C}}$ to $\mathcal{U}_{d,k}^{\mathbb{C}}$ restricts to an isomorphism from $\Delta_{d,k}$ to $\mathcal{U}_{d,k}$. Thus, as in [26], $\mathcal{U}_{d,k}$ contains $M_k(\mathbb{R})$ unitally and the result follows from Proposition 8.5. □

Corollary 8.7. *Let A be a real C^*-algebra for which $A^{\mathbb{C}}$ is unital, separable, simple, infinite-dimensional and nuclear and for which the extreme boundary of the nonempty trace space $T(A^{\mathbb{C}})$ is a compact finite-dimensional space.*

(i) *For any odd $k \in \mathbb{N}$, there exists a completely positive contractive order zero map $\psi : M_k(\mathbb{R}) \to A_\infty$ such that*

$$\lim_{n\to\infty} \max_{\tau \in T(A^{\mathbb{C}})} \left|\tau(c_n^m) - \frac{1}{k}\right| = 0$$

for any $m \in \mathbb{N}$, where $[(c_n)_n] = \psi(e)$ for e a minimal projection in $M_k(\mathbb{R})$.

(ii) *For any central sequence $(f_n)_n$ of positive contractions in A and any odd $k \in \mathbb{N}$ there exist central sequences $(f_{i,n})_n$, for $1 \le i \le k$, of positive contractions in A such that $(f_n f_{i,n})_n = (f_{i,n})_n$, $(f_{i,n} f_{j,n})_n = 0$ for $i \ne j$ and*

$$\lim_{m\to\infty} \liminf_{n\to\infty} \min_{\tau \in T(A^{\mathbb{C}})} \tau(f_{i,n}^m) = \lim_{m\to\infty} \liminf_{n\to\infty} \min_{\tau \in T(A^{\mathbb{C}})} \frac{\tau(f_n^m)}{k}.$$

Proof. (i) As in [34, Prop. 1.2,4] or [30, Lem. 2.1], lift the unital embedding φ of $M_k(\mathbb{R})$ into $A_{t\infty}$ given by Theorem 8.6 to a completely positive order zero map ψ from $M_k(\mathbb{R})$ to A_∞. As in the proof of [30, Lem. 2.5], if $(\psi_n)_n$ is a lifting of ψ to a sequence of completely positive order zero maps from $M_k(\mathbb{R})$ to A, then for each $\tau \in T(A^{\mathbb{C}})$, $\tau \circ \psi_n^m$ is a trace on $M_k(\mathbb{R})$, so $\tau(c_n^m) = \tau(\psi_n^m(1_k))/k$, from which the result follows.

(ii) Let ψ be the map of (i), let $E_{i,j}$ be the standard matrix units for $M_k(\mathbb{R})$ and let $\psi(E_{i,i}) = (e_{i,n})_n$ for $1 \le i \le k$. In [16, Lem. 3.4], $f_{i,n}$ is defined to be $f_n^l e_{i,N} f_n^l$ for some sufficiently large N, so $f_{i,n} \in A$. □

Lemma 8.8. *Let A be a real C^*-algebra for which $A^{\mathbb{C}}$ is unital, separable, simple, infinite-dimensional and nuclear and let ω be a Φ-invariant pure state of A^C. Then any completely positive map from A to A can be approximated in the pointwise norm topology by completely positive maps φ of the form*

$$\varphi(a) = \sum_{i,j=1}^{N} \omega(d_i^* a d_j) c_i^* c_j,$$

where $c_i, d_i \in A$ for $1 \leq i \leq N$, where N is odd.

Proof. As in the proof of [16, Lem. 3.1], note that, by nuclearity of A, the given map can be approximated by completely positive maps of the form $\sigma \circ \rho$ where $\rho : A \to M_N(\mathbb{R})$ and $\sigma : M_N(\mathbb{R}) \to A$ are completely positive maps, where ρ can be taken to be unital. If N is even, replace ρ by $\gamma \circ \rho$ and σ by $\sigma \circ \theta$ where $\gamma : M_N(\mathbb{R}) \to M_{N+1}(\mathbb{R})$ and $\theta : M_{N+1}(\mathbb{R}) \to M_N(\mathbb{R})$ are defined by

$$\gamma(A) = \begin{pmatrix} A_{NN} & 0 \\ 0 & A \end{pmatrix} \quad \text{and} \quad \theta \begin{pmatrix} A & B \\ C & D \end{pmatrix} = D,$$

so that γ is unital and $\theta \circ \gamma = \mathrm{id}$. Let $(\pi, \mathcal{H}, \xi)$ be the GNS representation of A on the real Hilbert space $\mathcal{H}$. The proofs given in [7, Lem. II.5.1, II.5.2] of a result of Voiculescu apply also to real C^*-algebras, yielding a sequence $V_n : \mathbb{R}^N \to \mathcal{H}$ of isometries with $\lim_{n\to\infty} \|\rho(a) - V_n^* \pi(a) V_n\| = 0$ for any $a \in A$. The state ω has been chosen so that Kadison's transitivity theorem applies to π, using the results from [4], so the proof of [16, Lem. 3.1] gives the required result. □

The next step is to consider the real version of the property (SI) used in [16].

Definition 8.9. A separable real C^*-algebra A with $T(A) \neq \varnothing$ has the property (SI) when, for any central sequences $(e_n)_n$ and $(f_n)_n$ of positive contractions in A satisfying

$$\lim_{n\to\infty} \max_{\tau \in T(A)} \tau(e_n) = 0, \qquad \lim_{m\to\infty} \liminf_{n\to\infty} \min_{\tau \in T(A)} \tau(f_n^m) > 0,$$

there exists a central sequence $(s_n)_n$ in A such that

$$\lim_{n\to\infty} \|s_n^* s_n - e_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|f_n s_n - s_n\| = 0.$$

Proposition 8.10. *Let A be a real C^*-algebra for which $A^{\mathbb{C}}$ is unital, separable, simple, infinite-dimensional and nuclear and for which the extreme boundary of the nonempty trace space $T(A^{\mathbb{C}})$ is a compact finite-dimensional space. If A has strict comparison then it has the property* (SI).

Proof. As in the proof of [16, Thm. 1.1 (ii) ⇒ (iii)], using Lemma 8.8, Corollary 8.7 (ii) and Proposition 8.2 instead of [16, Lem. 3.1], [16, Lem. 3.4] and [16, Prop. 2.2], the identity map from A to A can be excised in small central sequences. The proof of [16, Thm. 1.1 (iii) ⇒ (iv)] completes the proof. □

To obtain the converse of Proposition 7.3 the following lemma is required.

Lemma 8.11. (i) *If* $k \not\equiv 2 \pmod 4$*, then* $I_R[k, k(k+1), k+1]$ *is isomorphic to the universal real* C^**-algebra generated by elements* c_j *for* $1 \le j \le k$ *and* s*, subject to the relations* $c_1 \ge 0$*,* $c_i c_j^* = \delta_{i,j} c_1^2$*,* $c_1 s = s$ *and*

$$s^* s + \sum_{j=1}^{k} c_j^* c_j = 1.$$

(ii) $\mathcal{Z}_R$ *is an inductive limit of algebras* $I_R[k_n, k_n(k_n+1), k_n+1]$ *where* $k_n = 3^{3^n}$*, for which* $k_n + 1 \not\equiv 2 \pmod 4$.

Proof. (i) The proof follows that in [24, §2]. The selfadjoint unitary u_1 defined at the bottom of [24, p. 455] (also used, but called T, in the proof of Proposition 4.2) has trace k and therefore determinant $(-1)^{k(k-1)/2}$. Thus $\det(u_1) = 1$ when $k \equiv 0, 1 \pmod 4$. When $k \equiv 3 \pmod 4$, u_1 can be replaced by $-u_1$, so in all these cases u_1 can be connected to 1 by a unitary path in $M_k(\mathbb{R}) \otimes M_k(\mathbb{R})$. The elements c_j, s of $I[k, k(k+1), k+1]$ defined in [24] belong to $I_R[k, k(k+1), k+1]$ and therefore the isomorphism of [24, Prop. 2.1] maps the universal real C^*-algebra with the given generators and relations onto $I_R[k, k(k+1), k+1]$.

(ii) If $p_n = 3^{3^n}$ and $q_n = 3^{3^n} + 1$ then $p_{n+1} = k_0 p_n$ and $q_{n+1} = k_1 q_n$ where $k_0 = 3^{2.3^n}$ and $k_1 = (1 + 3^{2.3^n} - 3^{3^n})$ satisfy $k_0 > 2q_n$ and $k_1 > 2p_n$. The real version of the construction in [13, Prop. 2.5] can therefore be applied. From the factorization $3^k - 1 = 2(1 + 3 + \cdots + 3^{k-1})$ it follows that when k is odd then $3^k - 1 \equiv 2 \pmod 4$ and thus $q_n \equiv 0 \pmod 4$. The limited uniqueness result, Theorem 4.3, shows that the limit is isomorphic to $\mathcal{Z}_R$. □

The required partial converse to Proposition 7.3 can now be obtained.

Theorem 8.12. *Let* A *be a real* C^**-algebra for which* $A^{\mathbb{C}}$ *is unital, separable, simple, infinite-dimensional and nuclear and for which the extreme boundary of the nonempty trace space* $T(A^{\mathbb{C}})$ *is a compact finite-dimensional space. If* A *has strict comparison, then* $A \cong A \otimes \mathcal{Z}_R$.

Proof. By Proposition 8.10, A has property (SI). The previous lemma then ensures that the proof of [16, Thm. 1.1 (iv) ⇒ (i)] applies directly, replacing [16, Lem. 3.3] by Theorem 8.6 and Corollary 8.7 above. □

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