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Weak Admissibility of Hodge-Pink Lattices in Terms of Geometric Invariant Theory

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Weak Admissibility of Hodge-Pink Lattices in Terms of Geometric Invariant Theory

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Zusammenfassung

In dieser Arbeit werden z-Isokristalle mit Hodge-Pink Gittern untersucht. Diese treten beim Studium von Anderson t-Motiven und lokalen Shtuka als kristalline Realisierungen auf und erfüllen in diesem Fall die numerische Bedingung der schwachen Zulässigkeit.

Um bei fixiertem Isokristall einen Modulraum für variierende Hodge-Pink Gitter zu konstruieren, werden Familien von Hodge-Pink Gittern definiert und eine Beschränktheitsbedingung für diese eingeführt. Dieser Modulraum, dessen Konstruktion als Untervarietät einer Graßmannschen beschrieben wird, parametrisiert Hodge-Pink Gitter, die kleiner oder gleich einem vorgegeben Hodge-Pink Gewicht sind. Auf diesem Modulraum der beschränkten Hodge-Pink Gitter operiert eine algebraische Gruppe, die durch Anwenden eines Funktors aus der algebraischen Gruppe der Automorphismen des Isokristalls entsteht.

Um einen Zusammenhang zur geometrischen Invariantentheorie herzustellen, der im analogen Fall von Hodge-Filtrierungen existiert, wird ein linearisiertes Geradenbündel auf dem Modulraum der beschränkten Hodge-Pink Gitter definiert. Dieses Geradenbündel erhält man durch Zurückziehen des Bündels $\mathscr{O}(1)$ unter einer Einbettung der Graßmannschen in einen projektiven Raum $\mathbb{P}^{N-1}$. Es werden zwei verschiedene Möglichkeiten dieser Einbettung präsentiert und es wird analysiert, inwiefern sich diese beiden Linearisierungen des Geradenbündels unterscheiden. Der Zusammenhang zur geometrischen Invariantentheorie besteht darin, dass ein Hodge-Pink Gitter über einem Körper genau dann schwach zulässig ist, wenn der zugehörige Punkt im Modulraum das Hilbert-Mumford-Kriterium für Semistabilität bezüglich dieses linearisierten Geradenbündels und gewisser 1-Parameter-Untergruppen erfüllt. Diese 1-Parameter-Untergruppen entstehen aus den 1-Paramter-Untergruppen der algebraischen Gruppe der Automorphismen des zugrunde liegenden Isokristalls.

Zuletzt wird das funktorielle Verhalten der Moduläume, Einbettungen und linearisierten Geradenbündel untersucht, das auftritt, wenn man zwei verschiedene Hodge-Pink Gewichte betrachtet, von denen eins kleiner als das andere ist.

Contents

Terminology

Let $\varphi\colon A \to B$ be a morphism of rings and let M (resp. N) be an A-modules (resp. a B-module). If we consider N via φ as an A-module we denote it by $N_{[A]}$. Furthermore $M \otimes_A B$ considered as a B-module is denoted by M_B.

By $\mathrm{GL}_A(M)$ we mean the group of invertible endomorphisms of M. They are the A-valued points of the corresponding algebraic group $\mathrm{GL}(M)$. Thus in our notation we have $\mathrm{GL}(M)(B) = \mathrm{GL}_B(M_B)$.

Let S be a scheme. Furthermore let X be a scheme over S and let $\mathscr{M}$ be an $\mathscr{O}_X$-module. If $f\colon T \to S$ is a morphism of schemes we denote the base change $X \times_S T$ by X_T and by $\mathscr{M}_T$ we mean the $\mathscr{O}_{X_T}$-module that is induced by $\mathscr{M}$.

Introduction

Let $\mathscr{O}_L$ be a complete discrete valuation ring with perfect residue field κ of characteristic $p > 0$ such that the fraction field L of $\mathscr{O}_L$ is of characteristic 0. The ring of p-typical Witt vectors of κ is denoted by $\mathrm{W}(\kappa)$ and L is a totally ramified extension of $L_0 = \mathrm{W}(\kappa)[\frac{1}{p}]$. On $\mathrm{W}(\kappa)$ we have the Frobenius lift σ of the morphism $\kappa \to \kappa,\ x \mapsto x^p$. We consider a smooth proper scheme X over $\mathscr{O}_L$ and denote $X \otimes \kappa$ by $\overline{X}$. Associated to X we have different cohomology groups. For example the étale cohomology $\mathrm{H}^i_{\text{ét}}(\mathrm{X} \times_{\mathscr{O}_\mathrm{L}} \operatorname{Spec} \mathrm{L}^{\mathrm{alg}}, \mathbb{Z}_\mathrm{p})$ which is a $\mathbb{Z}_p$-module, the crystalline cohomology $\mathrm{H}^i_{\mathrm{cris}}(\overline{\mathrm{X}}/\mathrm{W}(\kappa))$ which is a $\mathrm{W}(\kappa)$-module and the de Rham cohomology $\mathrm{H}^i_{\mathrm{dR}}(\mathrm{X}/\mathscr{O}_\mathrm{L})$ which is a module over $\mathscr{O}_L$. These are related via comparison-isomorphisms

$$\mathrm{H}^i_{\mathrm{dR}}(\mathrm{X}/\mathscr{O}_\mathrm{L}) \otimes_{\mathscr{O}_\mathrm{L}} \mathrm{L} \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{cris}}(\overline{\mathrm{X}}/\mathrm{W}(\kappa)) \otimes_{\mathrm{W}(\kappa)} \mathrm{L},$$

$$\mathrm{H}^i_{\text{ét}}(\mathrm{X} \times_{\mathscr{O}_\mathrm{L}} \operatorname{Spec} \mathrm{L}^{\mathrm{alg}}, \mathbb{Z}_\mathrm{p}) \otimes_{\mathbb{Z}_\mathrm{p}} \mathbf{B}_{\mathrm{cris}} \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{cris}}(\overline{\mathrm{X}}/\mathrm{W}(\kappa)) \otimes_{\mathrm{W}(\kappa)} \mathbf{B}_{\mathrm{cris}},$$

$$\mathrm{H}^i_{\text{ét}}(\mathrm{X} \times_{\mathscr{O}_\mathrm{L}} \operatorname{Spec} \mathrm{L}^{\mathrm{alg}}, \mathbb{Z}_\mathrm{p}) \otimes_{\mathbb{Z}_\mathrm{p}} \mathbf{B}_{\mathrm{dR}} \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{dR}}(\mathrm{X}/\mathscr{O}_\mathrm{L}) \otimes_{\mathscr{O}_\mathrm{L}} \mathbf{B}_{\mathrm{dR}}.$$

Beside being modules, the cohomology groups carry more structure. First note that $\mathrm{H}^i_{\mathrm{cris}}(\overline{\mathrm{X}}/\mathrm{L}_0) = \mathrm{H}^i_{\mathrm{cris}}(\overline{\mathrm{X}}/\mathrm{W}(\kappa)) \otimes_{\mathrm{W}(\kappa)} \mathrm{L}_0$ is what is called an F-isocrystal, i.e. a finite dimensional L_0-vector space together with a σ-linear automorphism, and $\mathrm{H}^i_{\mathrm{dR}}(\mathrm{X}/\mathscr{O}_\mathrm{L}) \otimes_{\mathscr{O}_\mathrm{L}} \mathrm{L}$ provides a filtration on $\mathrm{H}^i_{\mathrm{cris}}(\overline{\mathrm{X}}/\mathrm{L}_0)$. Such objects are called filtered isocrystals. For the other two comparison isomorphisms we need the rings $\mathbf{B}_{\mathrm{cris}}$ and $\mathbf{B}_{\mathrm{dR}}$ which were constructed by Fontaine [Fon2]. Actually he associated with a p-adic Galois representation V such as $\mathrm{H}^i_{\text{ét}}(\mathrm{X} \times_{\mathscr{O}_\mathrm{L}} \mathrm{L}^{\mathrm{alg}}, \mathbb{Q}_\mathrm{p}) = \mathrm{H}^i_{\text{ét}}(\mathrm{X} \times_{\mathscr{O}_\mathrm{L}} \mathrm{L}^{\mathrm{alg}}, \mathbb{Z}_\mathrm{p}) \otimes_{\mathbb{Z}_\mathrm{p}} \mathbb{Q}_\mathrm{p}$ a filtered isocrystal $\mathbf{D}_{\mathrm{cris}}(V) = (V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}})^{G_L}$ and calls V crystalline if $\dim_{L_0} \mathbf{D}_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p}(V)$. Here a p-adic Galois representation is a finite dimensional $\mathbb{Q}_p$-vector space together with a continuous group homomorphism $G_L = \mathrm{Gal}(L^{\mathrm{alg}}/L) \to \mathrm{Aut}_{\mathbb{Q}_p}(V)$. The ring $\mathbf{B}_{\mathrm{cris}}$ is an L_0-algebra provided with additional structure such as a Frobenius-morphism and an action of the Galois group G_L. It is a subring of $\mathbf{B}_{\mathrm{dR}}$ which has a natural filtration and we have $(\mathbf{B}_{\mathrm{cris}})^{G_L} = L_0$ and $(\mathbf{B}_{\mathrm{dR}})^{G_L} = L$. Thus $\mathbf{D}_{\mathrm{cris}}(V)$ is an F-isocrystal and $\mathbf{D}_{\mathrm{cris}}(V) \otimes_{L_0} L = (V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{dR}})^{G_L} = \mathbf{D}_{\mathrm{dR}}(V)$ is provided with a filtration. If moreover V is crystalline we have a comparison isomorphism $V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\mathrm{cris}} \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris}}(V) \otimes_{L_0} \mathbf{B}_{\mathrm{cris}}$. Fontaine conjectured [Fon1] and Faltings [Fal] proved that $\mathrm{H}^i_{\text{ét}}(\mathrm{X} \times_{\mathscr{O}_\mathrm{L}} \mathrm{L}^{\mathrm{alg}}, \mathbb{Q}_\mathrm{p})$ is crystalline and $\mathbf{D}_{\mathrm{cris}}(\mathrm{H}^i_{\text{ét}}(\mathrm{X} \times_{\mathscr{O}_\mathrm{L}} \mathrm{L}^{\mathrm{alg}}, \mathbb{Q}_\mathrm{p})) = \mathrm{H}^i_{\mathrm{cris}}(\overline{\mathrm{X}}/\mathrm{L}_0)$ thus obtaining the comparison isomorphism

$\mathrm{H}^{\mathrm{i}}_{\mathrm{\acute{e}t}}(\mathrm{X} \times_{\mathscr{O}_{\mathrm{L}}} \operatorname{Spec} \mathrm{L}^{\mathrm{alg}}, \mathbb{Z}_{\mathrm{p}}) \otimes_{\mathbb{Z}_{\mathrm{p}}} \mathbf{B}_{\mathrm{cris}} \xrightarrow{\sim} \mathrm{H}^{\mathrm{i}}_{\mathrm{cris}}(\overline{\mathrm{X}}/\,\mathrm{W}(\kappa)) \otimes_{\mathrm{W}(\kappa)} \mathbf{B}_{\mathrm{cris}}$. In view of the various cohomology modules of X and their comparison isomorphisms Grothendieck had the idea of a universal cohomology theory which he called "motives". The above cohomology theories then are called the realizations of the motive associated to X. Unfortunately, so far this category of motives could not be constructed. The filtered isocrystals coming from crystalline Galois representations are called admissible. Filtered isocrystals satisfying a certain numerical criterion are called weakly admissible. It is easy to see that every admissible filtered isocrystal is also weakly admissible. The criterion is also sufficient which was shown by Colmez and Fontaine [CF].

We will work in the analogous case of function fields. Note that there are no varieties like X that yield cohomology theories. On the other hand in the function field setup we have a category playing the role of Grothendieck's motives. These are called Anderson A-motives [And]. We fix a field $\mathbb{F}_q$ with q elements. Let $A = \mathbb{F}_q[z]$, let S be a complete discrete valuation ring and let $\gamma\colon A \hookrightarrow S$ be a ring homomorphism. We denote the maximal ideal of S by $\mathfrak{m}_S$ and its residue field by k. We suppose that $\zeta := \gamma(z) \in \mathfrak{m}_S - \{0\}$ and in this case we see that the kernel of the induced homomorphism $A \to k$ is equal to (z) the ideal generated by z. We let σ be the endomorphism of the polynomial ring $S[z]$ with $\sum b_i z^i \mapsto \sum b_i^q z^i$ for $b_i \in S$. An A-motive over S is a pair $\underline{M} = (M, F_M)$ consisting of a locally free $S[z]$-module M and an injective $S[z]$-homomorphism $F_M\colon \sigma^*M := M \otimes_{S[z],\sigma} S[z] \hookrightarrow M$ whose cokernel is a finite free S-module and annihilated by a power of the ideal $(z-\zeta) = \ker(\gamma \otimes \mathrm{id}_S\colon S[z] = A \otimes_{\mathbb{F}_q} S \to S) \subseteq S[z]$. Now we consider the z-adic completion $S[\![z]\!]$ of $S[z]$. A local shtuka over S is a pair $\underline{\hat{M}} = (\hat{M}, F_{\hat{M}})$ consisting of a locally free $S[\![z]\!]$-module $\hat{M}$ and an isomorphism $F_{\hat{M}}\colon \sigma^*\hat{M}[\frac{1}{z-\zeta}] \xrightarrow{\sim} \hat{M}[\frac{1}{z-\zeta}]$. To our A-motive we have an associated local shtuka $\underline{\hat{M}}_z(\underline{M}) = \underline{M} \otimes_{S[z]} S[\![z]\!] = (M \otimes_{S[z]} S[\![z]\!], F_M \otimes \mathrm{id})$. Note that in this case $F_{\hat{M}}(\sigma^*\hat{M}) \subseteq \hat{M}$ and we call this type of local shtuka effective.

Like in the number field case, where we associate a filtered isocrystal to a crystalline p-adic Galois representation, we can do something similar in the function field case. We assume that there is a section $k \hookrightarrow S$ and fix one. There is a functor $\mathbb{H}\colon \underline{\hat{M}} \mapsto \mathbb{H}(\underline{\hat{M}}) = (D, F_D, \mathfrak{q})$, called the "mysterious functor", from the category of local shtukas to the category of z-isocrystals with Hodge-Pink lattice (for the definition of the second category see below and Definition 2.2.1). The z-isocrystal is given by $(D, F_D) = \underline{\hat{M}} \otimes_{S[\![z]\!]} k(\!(z)\!)$. It is a finite dimensional $k(\!(z)\!)$-vector space D together with an isomorphism $F_D\colon \sigma^*D \xrightarrow{\sim} D$. For the construction of the Hodge-Pink lattice $\mathfrak{q}$ we need the section $k \hookrightarrow S$ in order to get a morphism $k(\!(z)\!) \hookrightarrow L[\![z-\zeta]\!]$, $z \mapsto \zeta + (z-\zeta)$, where L is the fraction field of S. Then $\mathfrak{q}$ is an $L[\![z-\zeta]\!]$-lattice of full rank in $\sigma^*D \otimes_{k(\!(z)\!)} L(\!(z-\zeta)\!)$, where $L(\!(z-\zeta)\!)$ is the fraction field of the complete discrete valuation ring $L[\![z-\zeta]\!]$ in the "variable" $z-\zeta$. This construction is described in [GL] and [Har]. Like in the number field case there is a numerical condition that a z-isocrystal with Hodge-Pink lattice can satisfy and we call these weakly admissible. All the $\mathbb{H}(\underline{\hat{M}})$ are weakly admissible and since we are in the situation that our ring is discretely valued we have that $(D, F_D, \mathfrak{q}_D)$ is weakly admissible if and only if $(D, F_D, \mathfrak{q}_D) = \mathbb{H}(\underline{\hat{M}})$ for some $\underline{\hat{M}}$, i.e. weakly admissible is equivalent to admissible [Har]. Because of this we have to work with local shtukas and cannot restrict to effective local shtukas such as the ones coming from A-motives.

Now we change our point of view. So far we have fixed a ring S with $k = S/\mathfrak{m}_S$ and to one $\underline{\hat{M}}$ we have associated a z-isocrystal with Hodge-Pink lattice. Now we start with the fixed field k which is a field extension of $\mathbb{F}_q$ and we also fix a z-isocrystal over k. We consider varying Hodge-Pink lattices $\mathfrak{q}$ over various (field) extensions R of $K = k((\zeta))$. For example we consider $R = L$ a valued field whose valuation extends the ζ-adic valuation of K. Thus its valuation ring $\mathscr{O}_L$ contains $k[\![\zeta]\!]$ and $\mathscr{O}_L/\mathfrak{m}_{\mathscr{O}_L} \supseteq k$. Since we want to define a moduli space for the varying $\mathfrak{q}$ we do not restrict ourselves to these cases but work with arbitrary K-algebras R.

In order to state the main results, we now give a summary of the ideas used to formulate these results. We use the notations summarized in Section 1.1. In the first chapter we discuss z-isocrystals over k. We are especially interested in z-isocrystals that are of the form $\underline{D} = \bigoplus_{\nu\in\mathbb{Q}} \underline{D}_\nu$, where each $\underline{D}_\nu$ is isoclinic of slope ν (c.f. Definition 1.2.3). A very important object is the algebraic group of automorphism $J_{\underline{D}}$ of the z-isocrystal which is an algebraic group over $\mathbb{F}_q((z))$ and has $\mathrm{Aut}(\underline{D})$ as its $\mathbb{F}_q((z))$-valued points. If we have a decomposition $\underline{D} = \bigoplus_{\nu\in\mathbb{Q}} \underline{D}_\nu$ of the z-isocrystal the algebraic group of automorphisms decomposes as $J_{\underline{D}} = \prod_{\nu\in\mathbb{Q}} J_{\underline{D}_\nu}$. For every $\nu \in \mathbb{Q}$ we define a character $\chi_\nu \colon J_{\underline{D}_\nu} \to \mathbb{G}_m$. These characters will play an important role. If the z-isocrystal is of a certain type, that is called split semi-simple, this is automatic for example if the field k is algebraically closed, we link these characters to the general concept of the reduced norm on $\mathrm{End}(\underline{D})$. This is possible since in this case the z-isocrystal is a direct sum of standard simple objects $\underline{E}_\nu^k$ (c.f. Definition 1.2.6) and their endomorphism rings are central $\mathbb{F}_q((z))$-division algebras. For these algebras there exists a kind of norm map called the reduced norm and we show that this intrinsic concept, that does not use the fact that the division algebra comes from a z-isocrystal, coincides with our character χ_ν.

In the second chapter we deal with Hodge-Pink lattices and construct a moduli space for them. As described above we work with a fixed field k and a fixed z-isocrystal over k and let the Hodge-Pink lattices vary. Therefore we define $K = k((\zeta))$ as the field of formal Laurent series over k in the variable ζ and consider Hodge-Pink lattices defined over any K-algebra R. We further need the ring $R[\![z-\zeta]\!]$ of formal power series in the "variable" $z-\zeta$

and the ring $R(\!(z-\zeta)\!) = R[\![z-\zeta]\!][\frac{1}{z-\zeta}]$ of formal Laurent series in $z-\zeta$. With this setup we get a morphism $k(\!(z)\!) \to R[\![z-\zeta]\!]$ sending z to $\zeta + z - \zeta$. A Hodge-Pink lattice over R of a z-isocrystal (D, F_D) is a finitely generated $R[\![z-\zeta]\!]$-submodule $\mathfrak{q}$ of $\sigma^* D \otimes_{k(\!(z)\!)} R(\!(z-\zeta)\!)$ with $R(\!(z-\zeta)\!)\mathfrak{q} = \sigma^* D \otimes_{k(\!(z)\!)} R(\!(z-\zeta)\!)$ which is a direct summand as an R-module. One example is the special Hodge-Pink lattice $\mathfrak{p}_R = \sigma^* D \otimes_{k(\!(z)\!)} R[\![z-\zeta]\!]$. In the study of Hodge-Pink lattices the following result plays an important role: *Zariski locally on* $\operatorname{Spec} R$ *the* $R[\![z-\zeta]\!]$*-module* $\mathfrak{q}$ *is free of the same rank as* $\mathfrak{p}_R$. We prove this in Proposition 2.2.5. One should view Hodge-Pink lattices over a ring as a family parameterizing Hodge-Pink lattices over a field. In the case of a Hodge-Pink lattice $\mathfrak{q}$ being defined over a field we can associate to $\mathfrak{q}$ a tuple of ordered integers $w_1 \geq \ldots \geq w_r$ which is called the Hodge-Pink weights of $\mathfrak{q}$ such that $(z-\zeta)^{w_i}$ are the elementary divisors of $\mathfrak{q}$ with respect to $\mathfrak{p}$ (Definition 2.2.1.(vi)). The set of such tuples can be equipped with the Bruhat order defined by $(v_1 \geq \ldots \geq v_r) \preceq (w_1 \geq \ldots \geq w_r)$ if and only if $v_1 + \ldots + v_i \leq w_1 + \ldots + w_i$ for all $1 \leq i \leq r$ with equality for $i = r$. In order to define a moduli space we define a boundedness condition for Hodge-Pink lattices using these ordered tuples of integers. This condition is studied in Section 2.3. It turns out that a Hodge-Pink lattice $\mathfrak{q}$ defined over a ring R and being bounded by a tuple w parameterizes Hodge-Pink lattices over fields that have Hodge-Pink weights smaller or equal to w for the Bruhat order. Hence it makes sense to consider the space $Q_{D,\leq w}$ of Hodge-Pink lattices bounded by w. In Section 2.4 we show that this space is representable by a projective scheme by embedding it into a Grassmannian. This is also desirable in view of Geometric Invariant Theory. In the number field case this situation is different. There we consider Hodge-filtrations instead of Hodge-Pink lattices. A good family of filtrations has constant Hodge-Tate weights. These families have a partial flag variety as moduli space. Note that this space is projective. This is not the case in the function field setup where one could study a space like $Q_{D,=w}$ having constant Hodge-Pink weights. This subspace of $Q_{D,\leq w}$ is not projective but open and dense in $Q_{D,\leq w}$ hence only quasi-projective. It is thoroughly studied in [Har]. In the last part of Chapter 2 we define the Newton slope $\mathrm{t_N}(D, F_D)$ for a z-isocrystal (D, F_D) as $\mathrm{ord}_z(\det M_{F_D,\mathcal{B}})$, where $M_{F_D,\mathcal{B}}$ is the matrix corresponding to F_D with respect to a $k(\!(z)\!)$-basis of D. For a z-isocrystal with Hodge-Pink lattice $(D, F_D, \mathfrak{q}_D)$ over a field the Hodge slope $\mathrm{t_H}(D, F_D, D)$ is defined as the negative of the sum of the Hodge-Pink weights. With these two numbers it is possible to state the numerical condition of weak admissibility, i.e. $(D, F_D, \mathfrak{q}_D)$ is weakly admissible if $\mathrm{t_H}(D', F_{D'}, \mathfrak{q}_{D'}) \leq \mathrm{t_N}(D', F_{D'})$ for any sub-z-isocrystal with Hodge-Pink lattice $(D', F_{D'}, \mathfrak{q}_{D'})$ of $(D, F_D, \mathfrak{q}_D)$ with equality for $(D', F_{D'}, \mathfrak{q}_{D'}) = (D, F_D, \mathfrak{q}_D)$. This is exactly the numerical condition that all admissible Hodge-Pink lattices, i.e. the ones coming from local shtukas, satisfy. Our aim is to find a connection of this condition to Geometric Invariant Theory.

In the number field case such a connection was established by Totaro [Tot], answering a question of Rapoport and Zink [RZ] (see also [DOR]).

In order to describe this connection in the function field case, we start in Chapter 3 with a summary of the concepts of Geometric Invariant Theory that we will need later. Most importantly we define the GIT-slope $\mu^{\mathscr{L}}(x,\lambda)$ for a 1-parameter subgroup (1-PS) λ of an algebraic group G acting algebraically on a proper scheme X over a field F and an F-valued point x. Here $\mathscr{L}$ is a G-linearized invertible sheaf on X, i.e. a line bundle on X equipped with a G-action on $\mathscr{L}$ which is compatible with the G-action on X. This GIT-slope is helpful in Geometric Invariant Theory in order to describe semi-stable points which are used to construct quotients on open subsets of X by the group G. In fact the point x is semi-stable if and only if $\mu^{\mathscr{L}}(x,\lambda) \geq 0$ for all 1-PS's λ of G. Next we make a thorough discussion of how one should equip the Grassmannian which contains $Q_{D,\leq w}$ with a G-linearized invertible sheaf by embedding it into projective space. We present two closed embeddings into $\mathbb{P}^{N-1}$ which are isomorphic, but it turns out that we get two different GL_N-linearizations on the line bundle $\mathscr{O}_{\mathbb{P}^{N-1}}(1)$ on $\mathbb{P}^{N-1}$. They correspond to the two interpretations of $\mathbb{P}^{N-1}$ as a moduli space parameterizing 1-dimensional subspaces, respectively quotients of F^N. The two GL_N-linearizations differ by multiplying with the determinant. In order to apply Geometric Invariant Theory in our case, we first look at 1-PS's of the algebraic group $J = J_{\underline{D}}$. After a base change from $\mathbb{F}_q(\!(z)\!)$ to $k(\!(z)\!)$ the group $J_{k(\!(z)\!)}$ acts naturally on D and σ^*D and every 1-PS leads to a decomposition on D, respectively on σ^*D, into eigenspaces. We show that, on D, this is a decomposition into sub-z-isocrystals and how it is related to the decomposition on σ^*D. The next thing we need is an action of the algebraic group $J_{\underline{D}}$ on the scheme $Q_{D,\leq w}$. Note that $J_{\underline{D}}$ is defined over $\mathbb{F}_q(\!(z)\!)$ whereas $Q_{D,\leq w}$ is an object over K. In order to fix this problem, we set $|w| = w_1 - w_r$ and consider the induced action of $J^{\sim w} = \mathrm{Res}_{K[\![z-\zeta]\!]/(z-\zeta)^{|w|}|K}(J \times_{\mathbb{F}_q(\!(z)\!)} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})$ on $(z-\zeta)^{w_r}\mathfrak{p}_K/(z-\zeta)^{w_1}\mathfrak{p}_K$. In Section 3.3.1 we see that this induces an action of $J^{\sim w}$ on $Q_{D,\leq w}$. If $\lambda\colon \mathbb{G}_m \to J$ is a morphism of algebraic groups, the morphism $\lambda^{\sim w}\colon {\mathbb{G}_m}^{\sim w} \to J^{\sim w}$ is not a 1-PS. Therefore we consider $\lambda_0\colon \mathbb{G}_{m,K} \to J^{\sim w}$ which is defined as the composition $\lambda^{\sim w} \circ i_0$, where i_0 is the canonical section $a_0 \mapsto a_0(z-\zeta)^0$ to the projection morphism $(\_)_0\colon {\mathbb{G}_m}^{\sim w} \to \mathbb{G}_{m,K}$, $\sum_{i=0}^{|w|-1} a_i(z-\zeta)^i \mapsto a_0$. If we have a decomposition $\underline{D} = \bigoplus_{\nu\in\mathbb{Q}} \underline{D}_\nu$ into isoclinic components of slope ν, we get a decomposition $J^{\sim w} = \prod_{\nu\in\mathbb{Q}} (J_{\underline{D}_\nu})^{\sim w}$. We let $\chi_{\nu,0}\colon (J_{\underline{D}_\nu})^{\sim w} \to \mathbb{G}_{m,K}$ be the composition $(\_)_0 \circ (\chi_\nu)^{\sim w}$, where χ_ν was defined above. The last thing we need in order to apply GIT is a $J^{\sim w}$-linearized invertible sheaf on $Q_{D,\leq w}$. We have already seen that we have two embeddings $\iota\colon Q_{D,\leq w} \to \mathbb{P}^{N-1}$. Before we pull back $\mathscr{O}_{\mathbb{P}^{N-1}}(1)$ to $Q_{D,\leq w}$, we modify its linearization by a product of certain powers of the characters $\chi_{\nu,0}$. This is on the one hand responsible for handling the Newton slope and on the other hand a kind of normalization which is due to the fact that the linearization of $\mathscr{O}_{\mathbb{P}^{N-1}}(1)$ on $\mathbb{P}^{N-1}$ is not canonical. In this way both $J^{\sim w}$-linearizations obtained via the two embeddings of $Q_{D,\leq w}$ into $\mathbb{P}^{N-1}$ become equal. We denote this modified $J^{\sim w}$-linearized invertible sheaf by $\mathscr{L}_w$. Now we are ready to state our main result:

Theorem 3.3.1. *Let L be a field extension of K and let $\mathfrak{q} \in \mathrm{Q}_{D,\leq w}(L)$ be a Hodge-Pink lattice over L of $\underline{D}$. Then $\mathfrak{q}$ is weakly admissible if and only if*

$$\mu^{\iota^*\mathscr{L}_w}(\mathfrak{q}, \lambda_0) \geq 0$$

for all 1-PS λ of J defined over $\mathbb{F}_q(\!(z)\!)$.

In the proof we use all the preparation we have done so far. It is mainly a matter of carefully calculating $\mu^{\iota^*\mathscr{L}_w}(\mathfrak{q}, \lambda_0)$ and describing it in such a way that we can handle the difference between the Newton slope and the Hodge slope of the sub-z-isocrystals induced by the 1-PS λ. In the last section we look at the case of two different Hodge-Pink weights v and w with $v \preceq w$ for the Bruhat order. In this case $Q_{D,\leq v}$ is a subset of $Q_{D,\leq w}$ and we extend this inclusion to a morphism $F^v_w \colon \mathbb{P}^{N_v-1} \to \mathbb{P}^{N_w-1}$ of the projective spaces in which we embed $Q_{D,\leq v}$ and $Q_{D,\leq w}$. We further analyze how this morphism is compatible with the $J^{\sim w}$-linearized invertible sheaves on these spaces. The precise statement is our second main result:

Theorem 3.4.1. *Let $v = (v_1, \ldots, v_r) \in \mathbb{Z}^r$ and $w = (w_1, \ldots, w_r) \in \mathbb{Z}^r$ with $v_1 \geq \ldots \geq v_r$ and $w_1 \geq \ldots \geq w_r$ such that $v \preceq w$ for the Bruhat-order. Let $F^v_w \colon \mathbb{P}^{N_v-1} \to \mathbb{P}^{N_w-1}$ be the above morphism and let $\mathscr{L}_v$ on $\mathbb{P}^{N_v-1}$ and $\mathscr{L}_w$ on $\mathbb{P}^{N_w-1}$ be the invertible sheaves together with their linearization from above. In this case we have*

$$\mathscr{L}_v \cong (F^v_w)^*(\mathscr{L}_w)$$

as $J^{\sim w}$-linearized line bundles.

This result especially tells us that it does not matter whether we view a Hodge-Pink lattice as an element of $Q_{D,\leq v}$ or of $Q_{D,\leq w}$ in order to calculate the GIT-slope. This is in accordance to the fact that weak admissibility is of course independent of the chosen bound. Moreover the proof of this Theorem helps us to better understand the part of the change of the linearization that is not responsible for handling the Newton slope. This understanding comes from the fact that in the definition of the morphism F^v_w we need to consider both embeddings into projective space that induce different linearizations of the invertible sheaf.

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Chapter 1

z-isocrystals and the reduced norm

1.1 Notations

Let $q \in \mathbb{N}$ be a power of a prime number. We fix the following notations:

$\mathbb{F}_q$	is a finite field with q elements,
k	a field extension of $\mathbb{F}_q$,
$k[\![z]\!]$	the ring of formal power series over k in the variable z,
$k(\!(z)\!)$	the field of fractions of $k[\![z]\!]$.

The endomorphism

$$\begin{aligned} \sigma\colon k(\!(z)\!) &\to k(\!(z)\!) \\ \sum a_i z^i &\mapsto \sum a_i^q z^i \end{aligned}$$

is called the *Frobenius lift*. If M is a $k(\!(z)\!)$-vector space we write $\sigma^* M = M \otimes_{k(\!(z)\!),\sigma} k(\!(z)\!)$ and similar for morphisms of $k(\!(z)\!)$-vector spaces. We have a canonical morphism

$$\begin{aligned} \sigma_M^*\colon M &\to \sigma^* M. \\ m &\mapsto m \otimes 1 \end{aligned}$$

For $t \in \mathbb{N}$ and $m \in M$ we abbreviate $(\sigma^*)^t M = \underbrace{\sigma^* \ldots \sigma^*}_{t\text{-times}} M$ and $(\sigma_M^*)^t(m) = \sigma^*_{(\sigma^*)^{t-1}M} \circ \ldots \circ \sigma^*_{\sigma^* M} \circ \sigma^*_M(m) \in (\sigma^*)^t M$.

1.2 z-isocrystals

Definition 1.2.1.

i) A *z-isocrystal over k* is a pair $\underline{D} = (D, F_D)$ consisting of a finite dimensional $k(\!(z)\!)$-vector space D and an isomorphism $F_D\colon \sigma^* D \to D$.

ii) For a z-isocrystal $\underline{D} = (D, F_D)$ over k, $\operatorname{rk} \underline{D} = \dim_{k(\!(z)\!)} D$ is called the *rank of* $\underline{D}$.

iii) A *morphism* between z-isocrystals $\underline{D} = (D, F_D)$, $\underline{E} = (E, F_E)$ over k is a $k(\!(z)\!)$-linear homomorphism $f\colon D \to E$ with $F_E \circ \sigma^* f = f \circ F_D$.

iv) A *sub-z-isocrystal of* a z-isocrystal $\underline{D} = (D, F_D)$ over k is a z-isocrystal $\underline{D}' = (D', F_{D'})$ over k such that $D' \subseteq D$ and the inclusion is a morphism of z-isocrystals, i.e. $F_D|\sigma^*D'$ factors as in the commutative diagram

$$\begin{tikzcd} \sigma^*D' \arrow[r, "F_D|\sigma^*D'"] \arrow[dr, "F_{D'}"'] & D \\ & D' \arrow[u, hook, "i"'] \end{tikzcd}$$

(i the inclusion morphism). We say that a subset $D' \subseteq D$ is *F_D-invariant* if $F_D(\sigma^*D') \subseteq D'$. Thus a $k(\!(z)\!)$-subspace of D gives rise to a unique sub-z-isocrystal of $\underline{D}$ if and only it is F_D-invariant.

v) A *quotient-z-isocrystal of* a z-isocrystal $\underline{D} = (D, F_D)$ over k is a z-isocrystal $\underline{D}' = (D', F_{D'})$ over k such that D' is a quotient $\pi\colon D \to D'$ of the $k(\!(z)\!)$-vector space D with π a morphism of z-isocrystals.

Let $k'|k$ be a field extension. There is a base change functor from the category of z-isocrystals over k to the category of z-isocrystals over k'. Namely it is given by

(1.2.1) $$\underline{D} = (D, F_D) \mapsto \underline{D}_{k'} = (D \otimes_{k(\!(z)\!)} k'(\!(z)\!), F_{D'})$$

where $F_{D'}$ is the morphism $F_D \otimes \mathrm{id}_{k'(\!(z)\!)}\colon \sigma^*D \otimes_{k(\!(z)\!)} k'(\!(z)\!) \to D \otimes_{k(\!(z)\!)} k'(\!(z)\!) = D'$ with the canonical identification $\sigma^*D' = \sigma^*D \otimes_{k(\!(z)\!)} k'(\!(z)\!)$.

Notation 1.2.2.

i) Let $r \in \mathbb{N}$. Let D be an r-dimensional $k(\!(z)\!)$-vector space and let $A \in \mathrm{GL}_r(k(\!(z)\!))$. If $\mathcal{B} = (b_1, \ldots, b_r)$ is a basis of D we denote the induced basis $(b_1 \otimes 1, \ldots, b_r \otimes 1)$ on σ^*D by $\sigma^*\mathcal{B}$. Let $F_D\colon \sigma^*D \to D$ be the $k(\!(z)\!)$-linear morphism given by the matrix A with respect to the bases $\sigma^*\mathcal{B}$ and $\mathcal{B}$. (D, F_D) is a z-isocrystal over k which we denote by $(D, A_{\mathcal{B}})$. If $D = k(\!(z)\!)^{\oplus r}$ and $\mathcal{B}$ is the canonical basis we just write $(k(\!(z)\!)^{\oplus r}, A)$ for $(D, A_{\mathcal{B}})$.

ii) On the other hand let (D, F_D) be a z-isocrystal over k. If $\mathcal{B}$ is a basis of D we denote the matrix corresponding to F_D with respect to $\sigma^*\mathcal{B}$ and $\mathcal{B}$ by $M_{F_D,\mathcal{B}}$.

iii) Let $r, r' \in \mathbb{N}$. Let D (resp. D') be a $k(\!(z)\!)$-vector space of dimension r (resp. r') and let $\mathcal{B}$ (resp. $\mathcal{B}'$) be a basis of D (resp. D'). Moreover let $A \in \mathrm{GL}_r(k(\!(z)\!))$ and let $A' \in \mathrm{GL}_{r'}(k(\!(z)\!))$. For a matrix $B \in \mathrm{M}_{r' \times r}(k(\!(z)\!))$ we denote by $\sigma(B)$ the matrix $(\sigma(B_{ij}))_{i,j} \in \mathrm{M}_{r' \times r}(k(\!(z)\!))$. If f is the morphism given by B with respect to $\mathcal{B}$ and $\mathcal{B}'$ then $\sigma(B)$ is the matrix corresponding to σ^*f with respect to $\sigma^*\mathcal{B}$ and $\sigma^*\mathcal{B}'$. Moreover f is a morphism between the z-isocrystals $(D, A_{\mathcal{B}})$ and $(D', A'_{\mathcal{B}'})$ if and only if $A'\sigma(B) = BA$.

iv) Let $\underline{D} = (D, F_D)$ be a z-isocrystal over k. For $t \in \mathbb{N}$ we abbreviate

$$(F_D)^t := F_D \circ \sigma^*F_D \circ \ldots \circ (\sigma^*)^t F_D$$

which is a morphism from $(\sigma^*)^t M$ to M.

1.2.1 Dieudonné-Manin decomposition and split semi-simple z-isocrystals

Definition 1.2.3. Let $\underline{D} = (D, F_D)$ be a z-isocrystal over k and let $\lambda \in \mathbb{Q}$. $\underline{D}$ is called *isoclinic of slope* λ if there exist integers $s, t \in \mathbb{Z}$ with $t > 0$, $(s,t) = 1$, $\lambda = s/t$ and a $k[\![z]\!]$-lattice $M \subseteq D$ such that

$$(F_D)^t((\sigma^*)^t M) = z^s M.$$

Remark 1.2.4. Let $\underline{D}$ (resp. $\underline{D}'$) be a z-isocrystal over k which is isoclinic of slope $\lambda \in \mathbb{Q}$ (resp. $\lambda' \in \mathbb{Q}$). If $\lambda \neq \lambda'$ then $\mathrm{Hom}(\underline{D}, \underline{D}') = 0$ (c.f. [Zin, 6.20 Korollar]).

The next Lemma can be proved in the same way as [Zin, Satz 6.22].

Lemma 1.2.5. *Let $\underline{D} = (D, F_D)$ be a z-isocrystal over k. If k is a perfect field we have a unique decomposition*

$$\underline{D} = \bigoplus_{\lambda \in \mathbb{Q}} \underline{D}_\lambda$$

of $\underline{D}$ into isoclinic sub-z-isocrystals $\underline{D}_\lambda$ of slope $\lambda \in \mathbb{Q}$.

For each $\lambda = s/t \in \mathbb{Q}$ with $t > 0$ and $(s,t) = 1$ let $\underline{E}^k_\lambda$ be the z-isocrystal $(k(\!(z)\!)^{\oplus t}, A_\lambda)$ with

$$A_\lambda = \begin{pmatrix} 0 & \cdots & \cdots & 0 & z^s \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix} \in \mathrm{M}_{t \times t}(k(\!(z)\!)).$$

$\underline{E}^k_\lambda$ is a simple object in the category of z-isocrystals over k and it is isoclinic of slope λ.

Definition 1.2.6. Let $\underline{D} = (D, F_D)$ be a z-isocrystal over k. We call $\underline{D}$ *split semi-simple* if $\underline{D}$ can be written as a direct sum

$$\underline{D} = \bigoplus_{\lambda \in \mathbb{Q}} \left(\underline{E}^k_\lambda\right)^{\oplus n_\lambda} \quad (n_\lambda \in \mathbb{N}_0)$$

with $\mathbb{F}_{q^t} \subseteq k$ whenever $n_\lambda \neq 0$ with $\lambda = s/t$, $t > 0$, $(s,t) = 1$.

Remark. Definition 1.2.6 can be found in [DOR, Definition 8.1.2] but note that in general it is not true that $\underline{D}$ is split semi-simple if the base change map

$$\mathrm{End}(\underline{D}) \to \mathrm{End}(\underline{D}_{k'})$$

is an isomorphism for an algebraically closed field extension $k'|k$. In order to see this, consider the z-isocrystal $\underline{D} = (\mathbb{F}_q(\!(z)\!), (1-z))$ over $\mathbb{F}_q$. For any field extension $l|\mathbb{F}_q$ we have

$$\mathrm{End}(\underline{D}_l) = \{a \in l(\!(z)\!) \mid (1-z)\sigma(a) = a(1-z)\} = \{a \in l(\!(z)\!) \mid \sigma(a) = a\} = \mathbb{F}_q(\!(z)\!).$$

Assume that there is a morphism $f\colon \underline{D} \xrightarrow{\sim} \underline{E}_n^{\mathbb{F}_q} = (\mathbb{F}_q((z)), z^n)$ $(n \in \mathbb{Z})$. Since f is a morphism of z-isocrystals over $\mathbb{F}_q$ and since $f(1) \in \mathbb{F}_q((z))$ we have

$$(1-z)f(1) = f(1)(1-z) = z^n\sigma(f(1)) = z^n f(1).$$

Therefore $\underbrace{(z^n - (1-z))}_{\neq 0} f(1) = 0$ which implies $f(1) = 0$.

Let $\underline{D} = \bigoplus_{\lambda\in\mathbb{Q}} \left(\underline{E}_\lambda^k\right)^{n_\lambda}$ be a split semi-simple z-isocrystal over k such that $\mathbb{F}_{q^t} \subseteq k$ whenever $n_\lambda \neq 0$ with $\lambda = s/t$, $t > 0$, $(s,t) = 1$. Let $\tilde{t}$ be the lowest common multiple for all these t. In this case we have a canonical model of $\underline{D}$ over the field $\mathbb{F}_{q^{\tilde{t}}}$. It is given by

$$\underline{\tilde{D}} = \bigoplus_{\lambda\in\mathbb{Q}} \left(\underline{E}_\lambda^{\mathbb{F}_{q^{\tilde{t}}}}\right)^{\oplus n_\lambda}$$

and is a quasi-inverse to the base change functor (1.2.1) (c.f. [DOR, p. 191]). Thus, when considering split semi-simple z-isocrystals, we can often restrict to the case where $k = \mathbb{F}_{q^t}$ is a finite extension of $\mathbb{F}_q$.

1.3 The algebraic group of automorphisms of a z-isocrystal

Let $\underline{D} = (D, F_D)$ be a z-isocrystal over k. We define $J(\mathbb{F}_q((z))) = \mathrm{Aut}(\underline{D})$ to be the automorphism group of the z-isocrystal. Let A be an $\mathbb{F}_q((z))$-algebra. We denote by σ_A the morphism $\sigma \otimes \mathrm{id}_A\colon k((z)) \otimes_{\mathbb{F}_q((z))} A \to k((z)) \otimes_{\mathbb{F}_q((z))} A$. If M is a $(k((z)) \otimes_{\mathbb{F}_q((z))} A)$-module we abbreviate $\sigma_A^* M = M \otimes_{k((z))\otimes_{\mathbb{F}_q((z))}A, \sigma_A} (k((z)) \otimes_{\mathbb{F}_q((z))} A)$. Note that, in the case $M = N \otimes_{\mathbb{F}_q((z))} A$ for a $k((z))$-module N, $\sigma_A^* M$ is canonically isomorphic to $\sigma^* N \otimes_{\mathbb{F}_q((z))} A$ via

$$\begin{aligned}
&\sigma_A^*(N \otimes_{\mathbb{F}_q((z))} A) = N \otimes_{\mathbb{F}_q((z))} A \otimes_{(k((z))\otimes_{\mathbb{F}_q((z))}A),\sigma_A} k((z)) \otimes_{\mathbb{F}_q((z))} A \\
&\cong N \otimes_{k((z))} (k((z)) \otimes_{\mathbb{F}_q((z))} A) \otimes_{(k((z))\otimes_{\mathbb{F}_q((z))}A),\sigma_A} k((z)) \otimes_{\mathbb{F}_q((z))} A \\
&\cong N \otimes_{k((z)),\sigma} k((z)) \otimes_{\mathbb{F}_q((z))} A = \sigma^* N \otimes_{\mathbb{F}_q((z))} A.
\end{aligned} \tag{1.3.1}$$

With these notations it makes sense to define more generally for an $\mathbb{F}_q((z))$-algebra A

$$J_{\underline{D}}(A) = J(A) = \left\{ g \in \mathrm{GL}_{k((z))\otimes_{\mathbb{F}_q((z))}A}(D \otimes_{\mathbb{F}_q((z))} A) \;\middle|\; (F_D \otimes \mathrm{id}_A) \circ \sigma_A^* g = g \circ (F_D \otimes \mathrm{id}_A) \right\}.$$

The functor J is representable by an algebraic group [Har, Proposition 3.1.12].

We will now summarize some canonical identifications which we will use over and over again. First we remark that for a finite dimensional $k((z))$-vector space M (e.g. D or $\sigma^* D$) and for every $\mathbb{F}_q((z))$-algebra A the canonical morphism

$$\begin{aligned}
\mathrm{End}_{k((z))}(M) \otimes_{\mathbb{F}_q((z))} A &\to \mathrm{End}_{k((z))\otimes_{\mathbb{F}_q((z))}A}(M \otimes_{\mathbb{F}_q((z))} A) \\
g \otimes a &\mapsto (d \otimes b \mapsto g(d) \otimes ab)
\end{aligned} \tag{1.3.2}$$

is an isomorphism. In order to make the diagram

$$\begin{tikzcd}
\mathrm{End}_{k(\!(z)\!)}(D) \otimes_{\mathbb{F}_q(\!(z)\!)} A \arrow[r] \arrow[dd, "\cong"] & \mathrm{End}_{k(\!(z)\!)}(\sigma^* D) \otimes_{\mathbb{F}_q(\!(z)\!)} A \arrow[d, "\cong"] \\
 & \mathrm{End}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)} A}(\sigma^* D \otimes_{\mathbb{F}_q(\!(z)\!)} A) \arrow[d, "\cong"] \\
\mathrm{End}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)} A}(D \otimes_{\mathbb{F}_q(\!(z)\!)} A) \arrow[r] & \mathrm{End}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)} A}(\sigma_A^*(D \otimes_{\mathbb{F}_q(\!(z)\!)} A))
\end{tikzcd}$$

commutative, the morphism in the top row must be given by $g \otimes a \mapsto \sigma^* g \otimes a$. Let A be an $\mathbb{F}_q(\!(z)\!)$-algebra and for $g \in \mathrm{GL}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)} A}(D \otimes_{\mathbb{F}_q(\!(z)\!)} A)$ consider $(F_D \otimes \mathrm{id}_A) \circ \sigma_A^* g$ (resp. $g \circ (F_D \otimes \mathrm{id}_A)) \in \mathrm{Hom}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)} A}(\sigma^* D \otimes_{\mathbb{F}_q(\!(z)\!)} A, D \otimes_{\mathbb{F}_q(\!(z)\!)} A)$. The element g corresponds to $\tilde{g} = \sum g_i \otimes a_i \in (\mathrm{End}_{k(\!(z)\!)}(D) \otimes_{\mathbb{F}_q(\!(z)\!)} A)^*$ where we can choose the a_i to be linearly independent over $\mathbb{F}_q(\!(z)\!)$. With these notations $(F_D \otimes \mathrm{id}_A) \circ \sigma_A^* g$ (resp. $g \circ (F_D \otimes \mathrm{id}_A))$ can be viewed as the element $\sum (F_D \circ \sigma^* g_i) \otimes a_i$ (resp. $\sum (g_i \circ F_D) \otimes a_i$) in $\mathrm{Hom}_{k(\!(z)\!)}(\sigma^* D, D) \otimes_{\mathbb{F}_q(\!(z)\!)} A$. We claim that $g \in J(A)$ if and only if $g_i \in \mathrm{End}(\underline{D})$ for each i. By the above $\tilde{g} \in J(A)$ if and only if $\sum (F_D \circ \sigma^* g_i) \otimes a_i = \sum (g_i \circ F_D) \otimes a_i$ and therefore $F_D \circ \sigma^* g_i = g_i \circ F_D$ for all i since we have chosen the a_i to be linearly independent. Thus it follows that $g_i \in \mathrm{End}(\underline{D})$ for all i. Using these identifications we can describe J as

$$J(A) = (\mathrm{End}(\underline{D}) \otimes A)^*.$$

Now let $\underline{D} = (D, F_D)$ be a z-isocrystal over k such that we have a decomposition $\underline{D} = \bigoplus_{\lambda \in \mathbb{Q}} \underline{D}_\lambda$, where $\underline{D}_\lambda = (D_\lambda, F_{D_\lambda})$ are isoclinic sub-z-isocrystals of slope $\lambda \in \mathbb{Q}$, and let A be an $\mathbb{F}_q(\!(z)\!)$-algebra. Using the second description of J and Remark 1.2.4 we see that also J decomposes as

$$J_{\underline{D}} = \prod_{\lambda \in \mathbb{Q}} J_{\underline{D}_\lambda}.$$

Since every $g \in J_{\underline{D}_\lambda}(A)$ lies in $\mathrm{End}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)} A}(D_\lambda \otimes_{\mathbb{F}_q(\!(z)\!)} A)$ it makes sense to define a morphism

$$\chi_\lambda(A) \colon J_{\underline{D}_\lambda}(A) \to k(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} A$$

as

$$\left(\det \colon \mathrm{End}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)} A}(D_\lambda \otimes_{\mathbb{F}_q(\!(z)\!)} A) \to k(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} A\right)\Big|_{J_{\underline{D}_\lambda}(A)}.$$

Lemma 1.3.1. *With the above notation we have*

$$\chi_\lambda(A)(g) \in A,$$

for every $g \in J_{\underline{D}_\lambda}(A)$.

Proof. Let $r_\lambda = \dim D_\lambda$ and let $\mathcal{B}$ be a basis of D_λ. It induces a basis of the $(k(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} A)$-module $D_\lambda \otimes_{\mathbb{F}_q(\!(z)\!)} A$ which we denote by $\mathcal{B}_A$. Moreover $\mathcal{B}_A$ induces a basis $\sigma_A^*(\mathcal{B}_A)$ of the $(k(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} A)$-module space $\sigma_A^*(D_\lambda \otimes_{\mathbb{F}_q(\!(z)\!)} A)$. We denote by $M_g \in \mathrm{M}_{r_\lambda \times r_\lambda}\big(k(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} A\big)$ the matrix corresponding to g with respect to $\mathcal{B}_A$ and by $M_{\sigma_A^* g} \in \mathrm{M}_{r_\lambda \times r_\lambda}\big(k(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} A\big)$ the matrix corresponding to $\sigma_A^* g$ with respect to $\sigma_A^*(\mathcal{B}_A)$. The matrix $M_{\sigma_A^* g}$ is actually the matrix $\sigma_A^*(M_g)$ which is obtained from M_g by applying σ_A to every entry of the matrix. This shows that

$$\det(\sigma_A^* g) = \det(M_{\sigma_A^* g}) = \sigma_A(\det(M_g)) = \sigma_A(\det(g)).$$

Since $g \in J_{\underline{D}_\lambda}(A)$ we know that $\sigma_A^* g = (F_D \otimes \mathrm{id}_A)^{-1} \circ g \circ (F_D \otimes \mathrm{id}_A)$ and therefore

$$\det(\sigma_A^* g) = \det(g).$$

Combining these equalities we get that $\sigma_A(\det(g)) = \det(g)$ and if we write $\det(g) = \sum f_i \otimes a_i$ with a_i linearly independent over $\mathbb{F}_q(\!(z)\!)$ we see that $\sum \sigma(f_i) \otimes a_i = \sum f_i \otimes a_i$ and hence $\sigma(f_i) = f_i$ which means that $f_i \in \mathbb{F}_q(\!(z)\!)$. Therefore $\det(g)$ lies in A. $\square$

This lemma shows that for every $\lambda \in \mathbb{Q}$ we get a character

$$\chi_\lambda \colon J_{\underline{D}_\lambda} \to \mathbb{G}_m.$$

In the case of $\underline{D} = \bigoplus_{\lambda \in \mathbb{Q}} \big(\underline{E}_\lambda^k\big)^{\oplus n_\lambda}$ being a split semi-simple z-isocrystal over k and $\underline{D}_\lambda = \big(\underline{E}_\lambda^k\big)^{\oplus n_\lambda}$ we are able to give an intrinsic definition of the morphisms χ_λ which does not use that $J_{\underline{D}_\lambda}$ is a subset of the endomorphisms of a module. The reason for this lies in the fact that the endomorphism rings of the z-isocrystals $\underline{E}_\lambda^k$ are of the right kind to apply the general concept of the reduced norm which we will discuss in the next section.

1.4 The reduced norm

In this section we analyze the endomorphism rings of the standard simple z-isocrystals $\underline{E}_\lambda^k$ in the case when $\mathbb{F}_{q^t}(\!(z)\!) \subseteq k$, where $\lambda = s/t$ with $t > 0$ and $(s,t) = 1$. This enables us to link the morphisms χ_λ defined in Section 1.3 to the reduced norm in the case when we are working with split semi-simple z-isocrystals. First we give a brief review of the construction of the reduced norm.

Let K be a field. A *division ring* is a (not necessarily commutative) ring R such that every nonzero element is invertible. A K-algebra R is called a *division ring over* K if it is a division ring with K contained in its center R^{c} and we say that R is a *central K-division algebra* if $R^{\mathrm{c}} = K$ and $(R : K) < \infty$. Moreover a *central simple K-algebra* is a (not necessarily commutative) simple K-algebra A such that $A^{\mathrm{c}} = K$ and $(A : K) < \infty$. By Wedderburn's

Theorem, a central simple K-algebra A is of the form $\mathrm{M}_{n\times n}(R)$ for some division ring R over K. We see that $R^{\mathrm{c}} = K$ and $(A:K) = n^2(R:K)$. Now let $E|K$ be a field extension. We say that E is a *splitting field for* A, or that E *splits* A, if $E\otimes_K A \cong \mathrm{M}_{r\times r}(E)$. It is clear that E splits A if and only if E splits R. As a first result we know that R contains splitting fields:

Proposition 1.4.1 ([Rei, (7.15) Theorem])**.** *Let R be a central K-division algebra. Every maximal subfield E of R contains K and is a splitting field for R. If $m = (E:K)$ then $(R:K) = m^2$ and $E\otimes_K R \cong \mathrm{M}_{m\times m}(E)$. There exists a maximal subfield of R which is separable over K.*

Example 1.4.2. Suppose that K is a complete discretely valued field such that the residue class field is finite with q elements. By [Rei, p. 145] there exists an unramified splitting field for R. More precisely, if $(R:K) = n^2$ then every subfield $W = K(\omega)$, where $\omega\in R$ is a primitive (q^n-1)-th root of unity, is a maximal subfield and hence a splitting field for R.

Now we turn to reduced norms. Let A be a central simple K-algebra. Take a field extension E of K which splits A, i.e. there is an isomorphism $\gamma\colon E\otimes_K A \xrightarrow{\sim} \mathrm{M}_{n\times n}(E)$ ($n^2 = (A:K)$). For $a\in A$ we define

$$\mathrm{Nred}(a) = \mathrm{Nred}_{A/K}(a) = \det(\gamma(1\otimes a)).$$

In [Rei, §9] it is shown that this definition is independent of the chosen isomorphism γ and the splitting field E (also see Section 1.4.1 where we summarize these arguments in a relative situation). Using Proposition 1.4.1 one can show

Lemma & Definition 1.4.3. Let A and K be as above. For every $a\in A$, $\mathrm{Nred}(a)$ lies in K and is called the *reduced norm of* a.

Proposition 1.4.4. *Let $\lambda = s/t\in\mathbb{Q}$ with $t>0$ and $(s,t)=1$. Suppose that $\mathbb{F}_{q^t}\subseteq k$. In this case* $\mathrm{End}(\underline{E}^k_\lambda)$ *is the central $\mathbb{F}_q(\!(z)\!)$-division algebra with Hasse invariant $-\lambda$ mod $\mathbb{Z}$.*

Remark. For the notion of Hasse invariant see [Rei, (31.7) p. 266].

We follow the argumentation given in [Dem, Chapter IV] and Proposition 1.4.4 will follow from Lemma 1.4.5. In order to state this Lemma, we need a few more notations. Write $at+bs = 1$ with $a,b\in\mathbb{Z}$. Let G^λ be the associative $\mathbb{F}_{q^t}(\!(z)\!)$-algebra $G^\lambda = \mathbb{F}_{q^t}(\!(z)\!)[u]/(u^t-z)$ with $ux = \sigma^b(x)u$ for all $x\in\mathbb{F}_{q^t}(\!(z)\!)$. In [Rei, §30] G^λ is denoted by $(\mathbb{F}_{q^t}(\!(z)\!)/\mathbb{F}_q(\!(z)\!),\sigma^b,z)$ and [Rei, (30.7) Corollary] shows that this is a central $\mathbb{F}_q(\!(z)\!)$-division algebra (c.f. [Dem, p.77]). Now let $\mathbb{F}_{q^t}\subseteq k$ and consider $G^k_\lambda = k(\!(z)\!)\otimes_{\mathbb{F}_{q^t}(\!(z)\!)} G^\lambda$ which is a $k(\!(z)\!)$-vector space with basis $1\otimes u^i$ ($i = 0,\dots,t-1$). We equip it with a z-isocrystal structure by setting

$$F_{G^k_\lambda}\colon \sigma^*(k(\!(z)\!)\otimes_{\mathbb{F}_{q^t}(\!(z)\!)} G^\lambda) \to k(\!(z)\!)\otimes_{\mathbb{F}_{q^t}(\!(z)\!)} G^\lambda.$$
$$\sigma^*_{G^k_\lambda}(1\otimes u^i) \mapsto 1\otimes u^{i+s}$$

Lemma 1.4.5.

i) The z-isocrystals $\underline{G}^k_\lambda = (G^k_\lambda = k(\!(z)\!) \otimes_{\mathbb{F}_{q^t}(\!(z)\!)} G^\lambda, F_{G^k_\lambda})$ and $\underline{E}^k_\lambda$ are isomorphic.

ii) Let $\underline{H} = (H, F_H)$ be a z-isocrystal over k and let $e_0 \in k(\!(z)\!)^{\oplus t}$ be the first canonical basis vector. The map

$$\begin{aligned} \mathrm{Hom}(\underline{E}^k_\lambda, \underline{H}) &\to \left\{x \in H \mid (F_H)^t((\sigma_H^*)^t(x)) = z^s x\right\} \\ f &\mapsto f(e_0) \end{aligned}$$

is an isomorphism.

iii) The set of all $x \in G^k_\lambda$ such that $(F_{G^k_\lambda})^t((\sigma^*_{G^k_\lambda})^t(x)) = z^s x$ agrees with $\left\{1 \otimes y \mid y \in G^\lambda\right\}$.

iv) The endomorphisms of $\underline{G}^k_\lambda$ are exactly the right multiplications by elements of G^λ on $G^k_\lambda = k(\!(z)\!) \otimes_{\mathbb{F}_{q^t}(\!(z)\!)} G^\lambda$.

Proof.

i) For $i \in \{0, \dots, t-1\}$ let e_i be the $(i+1)$-th basis vector of $k(\!(z)\!)^{\oplus t}$ and for $i \in \mathbb{Z}$ let $e_i = z^{cs} e_d$, where $i = ct + d$ with $d \in \{0, \dots, t-1\}$. With these notations $F_{E^k_\lambda}(\sigma^*_{E^k_\lambda} e_i) = e_{i+1}$ and it is easy to check that

$$\begin{aligned} G^k_\lambda &\to k(\!(z)\!)^{\oplus t} \\ 1 \otimes u^i &\mapsto z^{ai} e_{bi} \end{aligned}$$

is an isomorphism of z-isocrystals.

ii) An inverse for this morphism is given by

$$\begin{aligned} \left\{x \in H \mid (F_H)^t((\sigma_H^*)^t(x)) = z^s x\right\} &\to \mathrm{Hom}(\underline{E}^k_\lambda, \underline{H}). \\ x &\mapsto (f_x \colon E^k_\lambda \to H, e_i \mapsto (F_H)^i((\sigma_H^*)^i x)) \end{aligned}$$

iii) Let $x = \sum_{i=0}^{t-1} \alpha_i \otimes u^i$ with $\alpha_i \in k(\!(z)\!)$. Since $(F_{G^k_\lambda})^t((\sigma^*_{G^k_\lambda})^t(x)) = \sum_{i=0}^{t-1} z^s \sigma^t(\alpha_i) \otimes u^i$ the assumption $(F_{G^k_\lambda})^t((\sigma^*_{G^k_\lambda})^t(x)) = z^s x$ implies that $\sigma^t(\alpha_i) = \alpha_i$ and hence $\alpha_i \in \mathbb{F}_{q^t}(\!(z)\!)$. Therefore we can write $x = 1 \otimes \sum_{i=0}^{t-1} \alpha_i u^i$.

iv) Note that the right multiplications by elements of G^λ on G^k_λ are morphisms of z-isocrystals: For every $\alpha \in \mathbb{F}_{q^t}(\!(z)\!)$ we calculate

$$\begin{aligned} F_{G^k_\lambda}(\sigma^*_{G^k_\lambda}((1 \otimes u^i) \cdot \alpha)) &= F_{G^k_\lambda}(\sigma^*_{G^k_\lambda}(1 \otimes \sigma^{bi}(\alpha) u^i)) \\ &= F_{G^k_\lambda}(\sigma^{bi+1}(\alpha) \sigma^*_{G^k_\lambda}(1 \otimes u^i)) = \sigma^{bi+1}(\alpha)(1 \otimes u^{i+s}) \end{aligned}$$

and

$$F_{G^k_\lambda}(\sigma^*_{G^k_\lambda}(1 \otimes u^i)) \cdot \alpha = (1 \otimes u^{i+s}) \cdot \alpha = \sigma^{bi+bs}(\alpha)(1 \otimes u^{i+s})$$

which is the same since for $\alpha \in \mathbb{F}_{q^t}(\!(z)\!)$ we have $\sigma^{bi+1}(\alpha) = \sigma^{bi+bs+at}(\alpha) = \sigma^{bi+bs}(\alpha)$. Moreover we have

$$\begin{aligned} F_{G^k_\lambda}(\sigma^*_{G^k_\lambda}(1 \otimes u^i) \cdot u) &= F_{G^k_\lambda}(\sigma^*_{G^k_\lambda}(1 \otimes u^{i+1})) = 1 \otimes u^{i+1+s} \\ &= (1 \otimes u^{i+s}) \cdot u = F_{G^k_\lambda}(\sigma^*_{G^k_\lambda}(1 \otimes u^i)) \cdot u \end{aligned}$$

which shows that right multiplication with u is a morphism of z-isocrystals. Therefore the mapping

$$\begin{aligned}\Phi\colon G^{\lambda} &\to \mathrm{End}(\underline{G}^k_{\lambda})\\ x &\mapsto r_x\colon y \mapsto y\cdot x\end{aligned}$$

is well defined. We will now give an inverse Ψ to this map. Let $f \in \mathrm{End}(\underline{G}^k_{\lambda})$ and let $\tilde{f} \in \mathrm{Hom}(\underline{E}^k_{\lambda}, \underline{G}^k_{\lambda})$ be the composition of f with the inverse of the isomorphism described in the proof of (i). By (ii) this morphism corresponds to $\tilde{f}(e_0) \in \{x \in G^k_{\lambda} \,|\, (F_{G^k_{\lambda}})^t((\sigma^*_{G^k_{\lambda}})(x)) = z^s x\}$ and by (iii) this is an element $y_f \in G^{\lambda}$. Let

$$\begin{aligned}\Psi\colon\ \mathrm{End}(\underline{G}^k_{\lambda}) &\to G^{\lambda}.\\ f &\mapsto y_f\end{aligned}$$

If $\tilde{f}(e_0) = f(1\otimes u^0) = \sum_{j=0}^{t-1} \alpha_j \otimes u^j$, with $\alpha_j \in \mathbb{F}_{q^t}(\!(z)\!)$, we calculate:

$$\begin{aligned}f(1\otimes u^i) = \tilde{f}(z^{ai}e_{bi}) = z^{ai}\tilde{f}(e_{bi}) &= z^{ai}(F_{G^k_{\lambda}})^{bi}((\sigma^*_{G^k_{\lambda}})^{bi}(\tilde{f}(e_0)))\\ &= z^{ai}(F_{G^k_{\lambda}})^{bi}((\sigma^*_{G^k_{\lambda}})^{bi}(f(1\otimes u^0))) = z^{ai}\sum_{j=0}^{t-1}\alpha_j\otimes u^{j+bis}.\end{aligned}$$

On the other hand we also have:

$$\begin{aligned}r_{\tilde{f}(e_0)}(1\otimes u^i) = r_{f(1\otimes u^0)}(1\otimes u^i) &= (1\otimes u^i)\cdot f(1\otimes u^0)\\ &= (1\otimes u^i)\cdot\sum_{j=0}^{t-1}\alpha_j\otimes u^j = \sum_{j=0}^{t-1}\alpha_j\otimes u^{j+i}.\end{aligned}$$

Now $u^{j+i} = u^{j+i(at+bs)} = u^{ait}\cdot u^{j+bis} = z^{ai}u^{j+bis}$ and hence $f(1\otimes u^i) = r_{\tilde{f}(e_0)}(1\otimes u^i)$. This shows that $\Phi\circ\Psi = \mathrm{id}_{\mathrm{End}(\underline{G}^k_{\lambda})}$ and $\Psi\circ\Phi = \mathrm{id}_{G^{\lambda}}$ is also true since $(1\otimes u^0)\cdot x = 1\otimes x$ for every $x \in G^{\lambda}$.

$\square$

Remark 1.4.6. By Lemma 1.4.5, in the case when $\mathbb{F}_{q^t} \subseteq k$, $\mathrm{End}(\underline{E}^k_{\lambda}) = (G^{\lambda})^{\mathrm{opp}}$ since the order of multiplication is reversed if we concatenate right multiplications. As in [Rei, §31] and with the same notation we get that

$$G^{\lambda} = (\mathbb{F}_{q^t}(\!(z)\!)/\mathbb{F}_q(\!(z)\!), \sigma^b, z) \cong (\mathbb{F}_{q^t}(\!(z)\!)/\mathbb{F}_q(\!(z)\!), \sigma, z^s) \cong \bigoplus_{i=0}^{t-1}\mathbb{F}_{q^t}(\!(z)\!)u^i$$

with $u\alpha = \sigma(\alpha)u$ ($\alpha \in \mathbb{F}_{q^t}(\!(z)\!)$) and $u^t = z^s$. Therefore we can write $(G^{\lambda})^{\mathrm{opp}} \cong \bigoplus_{i=0}^{t-1}\mathbb{F}_{q^t}(\!(z)\!)u^i$ with $\alpha u = u\sigma(\alpha)$ and $u^t = z^s$. We can rephrase this to $(G^{\lambda})^{\mathrm{opp}} \cong \bigoplus_{i=0}^{t-1}\mathbb{F}_{q^t}(\!(z)\!)(u^{-1})^i$ with $u^{-1}\alpha = \sigma(\alpha)u^{-1}$ and $(u^{-1})^t = z^{-s}$. This shows that $(G^{\lambda})^{\mathrm{opp}}$ has Hasse invariant $-\lambda \mod \mathbb{Z}$.

Now we know that in the split semi-simple case $\mathrm{End}(\underline{E}_\lambda^k)$ is a central $\mathbb{F}_q((z))$-division algebra and by Example 1.4.2 we are also able to provide a splitting field for it. Therefore we can write down an explicit isomorphism to a matrix algebra. This is done in the following

Example 1.4.7. Let $\mathbb{F}_{q^t} \subseteq k$. In this example we identify $\mathrm{End}(\underline{E}_\lambda^k)$ with a subset of $\mathrm{M}_{t\times t}(k((z)))$ via the canonical basis on $k((z))^{\oplus t}$. From the above it is clear that we can identify $\mathbb{F}_{q^t}((z))[u]/(u^t - z^s)$, where $\alpha u = u\sigma(\alpha)$ for all $\alpha \in \mathbb{F}_{q^t}((z))$, with $\mathrm{End}(\underline{E}_\lambda^k)$ via

$$\alpha \mapsto \begin{pmatrix} \alpha & 0 & \cdots & 0 \\ 0 & \sigma(\alpha) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \sigma^{t-1}(\alpha) \end{pmatrix}$$

and

$$u \mapsto \begin{pmatrix} 0 & \cdots & 0 & z^s \\ 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0\ \ 1 & 0 \end{pmatrix}.$$

This especially shows that $\mathrm{End}(\underline{E}_\lambda^k)$ is actually a subset of $\mathrm{M}_{t\times t}(\mathbb{F}_{q^t}((z)))$. By Example 1.4.2, $\mathbb{F}_{q^t}((z))$ is a splitting field for $\mathbb{F}_{q^t}((z))[u]/(u^t - z^s)$. We have a canonical identification

$$\mathbb{F}_{q^t}((z)) \otimes_{\mathbb{F}_q((z))} \mathbb{F}_{q^t}((z))[u]/(u^t - z^s) \cong \left(\prod_{i\in\mathbb{Z}/t\mathbb{Z}} \mathbb{F}_{q^t}((z))\right)[u]/(u^t - z^s).$$

$$\alpha \otimes \sum \beta_j u^j \mapsto \sum \left(\alpha \cdot \sigma^i(\beta_j)\right)_i u^j$$

With the help of this identification we write down a morphism from $\mathbb{F}_{q^t}((z)) \otimes_{\mathbb{F}_q((z))} \mathbb{F}_{q^t}((z))[u]/(u^t - z^s)$ to the matrix algebra over $\mathbb{F}_{q^t}((z))$:

$$\gamma\colon \mathbb{F}_{q^t}((z)) \otimes_{\mathbb{F}_q((z))} \mathbb{F}_{q^t}((z))[u]/(u^t - z^s) \to \mathrm{M}_{t\times t}(\mathbb{F}_{q^t}((z)))$$

by sending $(\alpha_1, \ldots, \alpha_t) \in \prod_{i\in\mathbb{Z}/t\mathbb{Z}} \mathbb{F}_{q^t}((z))$ to the matrix

$$\begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_t \end{pmatrix}$$

and u to the matrix

$$\begin{pmatrix} 0 & \cdots & \cdots & 0 & z^s \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

Since both algebras are simple $\mathbb{F}_{q^t}(\!(z)\!)$-algebras it is clear that this morphism is an isomorphism. Thus if we view $\operatorname{End}(\underline{E}_\lambda^k) \subseteq \mathrm{M}_{t\times t}(\mathbb{F}_{q^t}(\!(z)\!))$ the map

$$\begin{aligned} \operatorname{End}(\underline{E}_\lambda^k) &\to \mathrm{M}_{t\times t}(\mathbb{F}_{q^t}(\!(z)\!)) \\ f &\mapsto \gamma(1\otimes f) \end{aligned}$$

is induced by the identity. Hence if we take determinants we arrive at the following result: If $\mathbb{F}_{q^t}(\!(z)\!) \subseteq k$

$$\operatorname{Nred}(f) = \det(f) \tag{1.4.1}$$

for every $f \in \operatorname{End}(\underline{E}_\lambda^k) \subseteq \operatorname{End}_{k(\!(z)\!)}\big(k(\!(z)\!)^{\oplus t}\big)$. By Lemma 1.4.3 this determinant lies in $\mathbb{F}_q(\!(z)\!)$.

1.4.1 The reduced norm for a split semi-simple z-isocrystal

In the case of a split semi-simple z-isocrystal we want to define the reduced norm in a relative situation in order to get a morphism of algebraic groups. As a first step we do this for the standard simple z-isocrystals $\underline{E}_\lambda^k$. Hence let $\mathcal{E}_\lambda = \operatorname{End}(\underline{E}_\lambda^k)$ with $\mathbb{F}_{q^t}(\!(z)\!) \subseteq k$ where $\lambda = s/t \in \mathbb{Q}$ with $t > 0$ and $(s,t) = 1$. We let F be a splitting field for $\mathcal{E}_\lambda$, i.e. there is an isomorphism

$$\gamma\colon F \otimes_{\mathbb{F}_q(\!(z)\!)} \mathcal{E}_\lambda \xrightarrow{\sim} \mathrm{M}_{t\times t}(F).$$

and for each $\mathbb{F}_q(\!(z)\!)$-algebra A we define

$$\gamma_A\colon F \otimes_{\mathbb{F}_q(\!(z)\!)} \mathcal{E}_\lambda \otimes_{\mathbb{F}_q(\!(z)\!)} A \xrightarrow{\sim} \mathrm{M}_{t\times t}(F) \otimes_{\mathbb{F}_q(\!(z)\!)} A \cong \mathrm{M}_{t\times t}\big(F \otimes_{\mathbb{F}_q(\!(z)\!)} A\big)$$

as $\gamma \otimes \mathrm{id}_A$. For $g \in \mathcal{E}_\lambda \otimes_{\mathbb{F}_q(\!(z)\!)} A$ we set

$$\operatorname{Nred}_A(g) = \det(\gamma_A(1\otimes g)).$$

In order to see that Nred_A is well defined, we have to check that it does not depend on the chosen isomorphism γ and the splitting field F. Therefore let $\delta\colon F \otimes_{\mathbb{F}_q(\!(z)\!)} \mathcal{E}_\lambda \xrightarrow{\sim} \mathrm{M}_{t\times t}(F)$ be another isomorphism. By the Skolem-Noether Theorem the automorphism $\psi = \gamma \circ \delta^{-1}$ is inner ([Rei, (7.23) Corollary]) and therefore there exists an invertible matrix $T = (t_{ij})_{i,j} \in \mathrm{M}_{t\times t}(F)$ such that

$$\gamma(u) = T \cdot \delta(u) \cdot T^{-1}$$

for every $u \in F \otimes_{\mathbb{F}_q((z))} \mathcal{E}_\lambda$. From this it follows that the automorphism $\tilde{\psi}\colon \mathrm{M}_{t\times t}\big(F \otimes_{\mathbb{F}_q((z))} A\big) \to \mathrm{M}_{t\times t}\big(F \otimes_{\mathbb{F}_q((z))} A\big)$ with $\tilde{\psi} = \gamma_A \circ \delta_A^{-1}$ is also inner and we get that

$$\gamma_A(u) = \tilde{T} \cdot \delta_A(u) \cdot \tilde{T}^{-1}$$

with $\tilde{T} = (t_{ij} \otimes 1)_{i,j} \in \mathrm{M}_{t\times t}\big(F \otimes_{\mathbb{F}_q((z))} A\big)$ and hence the definition of Nred_A is independent of the chosen isomorphism γ. Now with the same idea as in [Rei, (9.3) Theorem] one can show that the definition of Nred_A does not depend on the splitting field F.

Lemma 1.4.8. *For every $g \in \mathcal{E}_\lambda \otimes_{\mathbb{F}_q((z))} A$, $\mathrm{Nred}_A(g)$ lies in A and*

$$\mathrm{Nred}_A(g) = \det(g)$$

where we view g as an element in $\mathrm{End}_{k((z))\otimes_{\mathbb{F}_q((z))} A}(E_\lambda^k \otimes_{\mathbb{F}_q((z))} A)$ via $\mathcal{E}_\lambda \otimes_{\mathbb{F}_q((z))} A \subseteq \mathrm{End}_{k((z))}(E_\lambda^k) \otimes_{\mathbb{F}_q((z))} A \cong \mathrm{End}_{k((z))\otimes_{\mathbb{F}_q((z))} A}(E_\lambda^k \otimes_{\mathbb{F}_q((z))} A)$.

Proof. By Proposition 1.4.1 we can assume that the splitting field F is a Galois extension of $\mathbb{F}_q((z))$ with Galois group G. Every $\xi \in G$ acts on $F \otimes_{\mathbb{F}_q((z))} A$ via the first factor. Using the fact that the definition of Nred_A is independent of γ one can show in the same way as in [Rei, (9.3) Theorem] that $\mathrm{Nred}_A(g)$ is stable under the action of ξ on $F \otimes_{\mathbb{F}_q((z))} A$. We write $\mathrm{Nred}_A(g) = \sum g_i \otimes a_i$ such that the a_i are linearly independent over $\mathbb{F}_q((z))$. We get that $\sum g_i \otimes a_i = \sum \xi(g_i) \otimes a_i$ and therefore $g_i = \xi(g_i)$ for every i. This being true for every $\xi \in G$ we see that g_i lies in $\mathbb{F}_q((z))$ and hence $\mathrm{Nred}_A(g)$ lies in A. For the second claim we take $F = \mathbb{F}_{q^t}((z))$ and note that in Example 1.4.2 we have seen that the morphism

$$\begin{aligned} \mathcal{E}_\lambda &\to \mathrm{M}_{t\times t}(\mathbb{F}_{q^t}((z))) \\ f &\mapsto \gamma(1 \otimes f) \end{aligned}$$

is induced by the identity if we view an element $f \in \mathcal{E}_\lambda$ as a matrix with entries in $\mathbb{F}_{q^t}((z))$. Tensoring this morphism with A over $\mathbb{F}_q((z))$ we get that

$$\begin{aligned} \mathcal{E}_\lambda \otimes_{\mathbb{F}_q((z))} A &\to \mathrm{M}_{t\times t}(\mathbb{F}_{q^t}((z))) \otimes_{\mathbb{F}_q((z))} A \cong \mathrm{M}_{t\times t}\big(\mathbb{F}_{q^t}((z)) \otimes_{\mathbb{F}_q((z))} A\big) \\ g &\mapsto \gamma_A(1 \otimes g) \end{aligned}$$

is induced by the identity if we consider $g \in \mathrm{End}_{k((z))\otimes_{\mathbb{F}_q((z))} A}(E_\lambda^k \otimes_{\mathbb{F}_q((z))} A)$ as a matrix with entries in $\mathbb{F}_{q^t}((z)) \otimes_{\mathbb{F}_q((z))} A$ via the canonical basis. Taking the determinant on both sides proves the second claim of the Lemma. □

Now let $\underline{D}$ be a split semi-simple z-isocrystal over k, i.e. we have

$$\underline{D} = \bigoplus_{\lambda \in \mathbb{Q}} (\underline{E}_\lambda^k)^{\oplus n_\lambda} = \bigoplus_{\lambda \in \mathbb{Q}} \underline{D}_\lambda$$

with $\underline{D}_\lambda = (\underline{E}_\lambda^k)^{\oplus n_\lambda}$. Its automorphism group is given by

$$J(\mathbb{F}_q(\!(z)\!)) = \prod_{\lambda \in \mathbb{Q}} \mathrm{GL}_{n_\lambda}(\mathcal{E}_\lambda)$$

where $\mathcal{E}_\lambda = \mathrm{End}(\underline{E}_\lambda^k)$ is the central division algebra over $\mathbb{F}_q(\!(z)\!)$ with Hasse invariant $-\lambda \bmod \mathbb{Z}$. If furthermore A is an $\mathbb{F}_q(\!(z)\!)$-algebra we have seen in Section 1.3 that we can identify $J_{\underline{E}_\lambda^k}(A)$ with $(\mathcal{E}_\lambda \otimes_{\mathbb{F}_q(\!(z)\!)} A)^* = \mathbb{G}_m(\mathcal{E}_\lambda \otimes_{\mathbb{F}_q(\!(z)\!)} A)$ and hence for the split semi-simple z-isocrystal $\underline{D} = \bigoplus_{\lambda \in \mathbb{Q}} \left(\underline{E}_\lambda^k\right)^{\oplus n_\lambda}$ we see that

$$J(A) \cong \prod_{\lambda \in \mathbb{Q}} \mathrm{GL}_{n_\lambda}(\mathcal{E}_\lambda \otimes_{\mathbb{F}_q(\!(z)\!)} A).$$

Therefore the group J decomposes as

$$J = J_{\underline{D}} = \prod_{\lambda \in \mathbb{Q}} J_{\underline{D}_\lambda}$$

where $J_{\underline{D}_\lambda}(A) = \mathrm{GL}_{n_\lambda}(\mathcal{E}_\lambda \otimes_{\mathbb{F}_q(\!(z)\!)} A)$. For every $\lambda \in \mathbb{Q}$ we define a morphism

$$\tilde{\chi}_\lambda \colon J_{\underline{D}_\lambda} \to \mathbb{G}_m$$

in the following way: Let F be a splitting field for $\mathcal{E}_\lambda$. As we have remarked, this is also a splitting field for $\mathrm{End}(\underline{D}_\lambda)$. We have an isomorphism

$$\eta_A \colon F \otimes_{\mathbb{F}_q(\!(z)\!)} \mathrm{End}(\underline{D}_\lambda) \otimes_{\mathbb{F}_q(\!(z)\!)} A \xrightarrow{\sim} \mathrm{M}_{(n_\lambda \cdot t) \times (n_\lambda \cdot t)}(F) \otimes_{\mathbb{F}_q(\!(z)\!)} A \cong \mathrm{M}_{(n_\lambda \cdot t) \times (n_\lambda \cdot t)}\big(F \otimes_{\mathbb{F}_q(\!(z)\!)} A\big)$$

and if $g \in J_{\underline{D}_\lambda}(A) \subseteq \mathrm{End}(\underline{D}_\lambda) \otimes_{\mathbb{F}_q(\!(z)\!)} A$ we set $\tilde{\chi}_\lambda(A)(g) = \det(\eta_A(1 \otimes g))$. Everything that we have said about Nred_A is also true for $\tilde{\chi}_\lambda(A)$ with the same argumentation. Therefore its definition is well defined and summarizing our other results we arrive at the following

Proposition 1.4.9. *Let A be an $\mathbb{F}_q(\!(z)\!)$-algebra. With the above notations for $g \in J_{\underline{D}_\lambda}(A)$ we have*

$$\tilde{\chi}_\lambda(A)(g) = \chi_\lambda(A)(g).$$

Chapter 2

Hodge-Pink lattices

2.1 Notations

From here on we denote by $K = k(\!(\zeta)\!)$ the field of formal Laurent series over k in the variable ζ. By $K[\![z-\zeta]\!]$ we mean the ring of formal power series in the "variable" $z-\zeta$ and by $K(\!(z-\zeta)\!)$ its field of fractions. Note that there is a homomorphism

$$k(\!(z)\!) \to K[\![z-\zeta]\!] \tag{2.1.1}$$

sending z to $z = \zeta \cdot (z-\zeta)^0 + 1 \cdot (z-\zeta)$. If R is a K-algebra we also denote by $R[\![z-\zeta]\!]$ the ring of formal power series in $z-\zeta$ but here $R(\!(z-\zeta)\!)$ should mean the ring of formal Laurent series in $z-\zeta$ with finite principle part. Note that $R[\![z-\zeta]\!]$ and $R(\!(z-\zeta)\!)$ are $K[\![z-\zeta]\!]$-algebras and therefore also algebras over $k(\!(z)\!)$ via (2.1.1). For later use we mention:

Lemma 2.1.1. *If $R \to R'$ is a morphism of K-algebras and M is an $R[\![z-\zeta]\!]/(z-\zeta)^c$-module ($c \in \mathbb{N}$) then*

$$\mathrm{Tor}_1^{R[\![z-\zeta]\!]}(M, R'[\![z-\zeta]\!]) \cong \mathrm{Tor}_1^{R[\![z-\zeta]\!]/(z-\zeta)^c}(M, R'[\![z-\zeta]\!]/(z-\zeta)^c) \cong \mathrm{Tor}_1^R(M, R').$$

Proof. The functor $M \otimes_{R[\![z-\zeta]\!]} \_$ equals the composition of the functors $(R[\![z-\zeta]\!]/(z-\zeta)^c) \otimes_{R[\![z-\zeta]\!]} \_$ followed by $M \otimes_{R[\![z-\zeta]\!]/(z-\zeta)^c} \_$. Therefore $\mathrm{Tor}_1^{R[\![z-\zeta]\!]}(M, R'[\![z-\zeta]\!])$ can be computed from a change of rings spectral sequence and its associated sequence of low degrees

$$\begin{aligned} \ldots \to\ & \mathrm{Tor}_1^{R[\![z-\zeta]\!]}(R[\![z-\zeta]\!]/(z-\zeta)^c, R'[\![z-\zeta]\!]) \otimes_{R[\![z-\zeta]\!]/(z-\zeta)^c} M \\ & \to \mathrm{Tor}_1^{R[\![z-\zeta]\!]}(M, R'[\![z-\zeta]\!]) \to \mathrm{Tor}_1^{R[\![z-\zeta]\!]/(z-\zeta)^c}(M, R'[\![z-\zeta]\!]/(z-\zeta)^c) \to 0. \end{aligned}$$

The left term is zero since $(z-\zeta)^c$ is a nonzero-divisor in both $R[\![z-\zeta]\!]$ and $R'[\![z-\zeta]\!]$ which gives the first isomorphism. The second is [Rot, Theorem 11.64]. $\square$

2.2 Definition and general properties

Definition 2.2.1. Let R be a K-algebra.

i) Let (D, F_D) be a z-isocrystal over k. A *Hodge-Pink lattice over R of* (D, F_D) is a finitely generated $R[\![z-\zeta]\!]$-submodule $\mathfrak{q} \subseteq \sigma^* D \otimes_{k(\!(z)\!)} R(\!(z-\zeta)\!)$ with $R(\!(z-\zeta)\!)\mathfrak{q} = \sigma^* D \otimes_{k(\!(z)\!)} R(\!(z-\zeta)\!)$ which is a direct summand as an R-module. We always have the special Hodge-Pink lattice $\mathfrak{p}_R := \mathfrak{p}_{D,R} := \sigma^* D \otimes_{k(\!(z)\!)} R[\![z-\zeta]\!]$ over R of (D, F_D).

ii) A triple $(D, F_D, \mathfrak{q}_D)$ consisting of a z-isocrystal (D, F_D) over k and a Hodge-Pink lattice $\mathfrak{q}_D$ over R of (D, F_D) is called a *z-isocrystal with Hodge-Pink lattice over R.*

iii) Let $(D, F_D, \mathfrak{q}_D)$ and $(D', F_{D'}, \mathfrak{q}_{D'})$ be z-isocrystals with Hodge-Pink lattice over R. A *morphism* from $(D, F_D, \mathfrak{q}_D)$ to $(D', F_{D'}, \mathfrak{q}_{D'})$ is a morphism of z-isocrystals $f\colon D \to D'$ such that $(\sigma^* f)_{R(\!(z-\zeta)\!)}\colon \sigma^* D \otimes_{k(\!(z)\!)} R(\!(z-\zeta)\!) \to \sigma^* D' \otimes_{k(\!(z)\!)} R(\!(z-\zeta)\!)$ satisfies $(\sigma^* f)_{R(\!(z-\zeta)\!)}(\mathfrak{q}_D) \subseteq \mathfrak{q}_{D'}$. The morphism f is called *strict* if

$$(\sigma^* f)_{R(\!(z-\zeta)\!)}(\mathfrak{q}_D) = \mathfrak{q}_{D'} \cap (\sigma^* f)(\sigma^* D) \otimes_{k(\!(z)\!)} R(\!(z-\zeta)\!).$$

iv) Let $(D, F_D, \mathfrak{q}_D)$ be a z-isocrystal with Hodge-Pink lattice over R. A *sub-z-isocrystal with Hodge-Pink lattice of* $(D, F_D, \mathfrak{q}_D)$ is a z-isocrystal with Hodge-Pink lattice $(D', F_{D'}, \mathfrak{q}_{D'})$ over R such that the inclusion is a morphism of z-isocrystals with Hodge-Pink lattice, i.e. $(D', F_{D'})$ is a sub-z-isocrystal of the z-isocrystal (D, F_D) with

$$\mathfrak{q}_{D'} \subseteq \mathfrak{q}_D \cap \sigma^* D' \otimes_{k(\!(z)\!)} R(\!(z-\zeta)\!). \tag{2.2.1}$$

We call a sub-z-isocrystal with Hodge-Pink lattice $(D', F_{D'}, \mathfrak{q}_{D'})$ of $(D, F_D, \mathfrak{q}_D)$ *strict* if the inclusion morphism $D' \hookrightarrow D$ is a strict morphism of z-isocrystals with Hodge-Pink lattice, i.e. if there is equality in (2.2.1).

v) Let $(D, F_D, \mathfrak{q}_D)$ be a z-isocrystal with Hodge-Pink lattice over R. A *quotient-z-isocrystal with Hodge-Pink lattice of* $(D, F_D, \mathfrak{q}_D)$ is a z-isocrystal with Hodge-Pink lattice $(D', F_{D'}, \mathfrak{q}_{D'})$ over R such that $(D', F_{D'})$ is a quotient-z-isocrystal of the z-isocrystal (D, F_D) with the projection morphism $\pi\colon D \to D'$ being a morphism of z-isocrystals with Hodge-Pink lattice. The quotient-z-isocrystal with Hodge-Pink lattice $(D', F_{D'}, \mathfrak{q}_{D'})$ is called *strict* if the morphism π is a strict morphism of z-isocrystals with Hodge-Pink lattice.

vi) Let (D, F_D) be a z-isocrystal over k of rank r ($r \in \mathbb{N}$) and let $R = L$ be a field extension of K. If $\mathfrak{q}$ is a Hodge-Pink lattice over L of (D, F_D) there exists (since $L[\![z-\zeta]\!]$ is a principal ideal domain) a $L[\![z-\zeta]\!]$-basis $(x_1, \ldots, x_r)$ of $\mathfrak{p}_L$ such that the lattice $\mathfrak{q}$ has an $L[\![z-\zeta]\!]$-basis $((z-\zeta)^{w_1} x_1, \ldots, (z-\zeta)^{w_r} x_r)$ for some integers $w_1, \ldots, w_r \in \mathbb{Z}$ which we can assume to be ordered $w_1 \geq \ldots \geq w_r$. In this case we call $(w_1, \ldots, w_r)$ the *Hodge-Pink weights of* $\mathfrak{q}$.

Remark 2.2.2. Let L be a field extension of K. If (D, F_D) is a z-isocrystal over k of rank r and $\mathfrak{q}$ is a Hodge-Pink lattice over L of (D, F_D) with Hodge-Pink weights $(w_1, \ldots, w_r)$ then for every integer c such that $\mathfrak{q} \subseteq (z-\zeta)^c \mathfrak{p}_L$ (resp. $(z-\zeta)^c \mathfrak{p}_L \subseteq \mathfrak{q}$) we have

$$(z-\zeta)^c \mathfrak{p}_L / \mathfrak{q} \simeq \bigoplus_{i=1}^{r} L[\![z-\zeta]\!]/(z-\zeta)^{-c+w_i}$$

$$(\text{resp. } \mathfrak{q}/(z-\zeta)^c \mathfrak{p}_L \simeq \bigoplus_{i=1}^{r} L[\![z-\zeta]\!]/(z-\zeta)^{c-w_i}).$$

For any K-algebra R we denote $\sigma^* D \otimes_{k(\!(z)\!)} R(\!(z-\zeta)\!)$ by V_R.

Lemma 2.2.3. *Let (D, F_D) be a z-isocrystal over k. Let R be a K-algebra and let $\mathfrak{q}$ be an $R[\![z-\zeta]\!]$-submodule of $\sigma^* D \otimes_{k(\!(z)\!)} R(\!(z-\zeta)\!)$. $\mathfrak{q}$ is a Hodge-Pink lattice over R of (D, F_D) if and only if there exists integers $d, e \in \mathbb{Z}$ ($d \leq e$) such that $(z-\zeta)^d \mathfrak{p}_R \supseteq \mathfrak{q} \supseteq (z-\zeta)^e \mathfrak{p}_R$ and $(z-\zeta)^d \mathfrak{p}_R/\mathfrak{q}$, $\mathfrak{q}/(z-\zeta)^e \mathfrak{p}_R$ are finite locally free R-modules (i.e. finitely generated projective).*

Proof. "$\Rightarrow$": Let $\langle x_1, \ldots, x_n \rangle_{R[\![z-\zeta]\!]} = \mathfrak{q}$. As $x_i \in V_R$ we can find $d \in \mathbb{Z}$ such that $(z-\zeta)^{-d} x_i \in \mathfrak{p}_R$ for all $i = 1, \ldots, n$. Therefore $x_i \in (z-\zeta)^d \mathfrak{p}_R$ and hence $\mathfrak{q} \subseteq (z-\zeta)^d \mathfrak{p}_R$. On the other hand, if we have $\mathfrak{p}_R = \langle y_1, \ldots, y_m \rangle_{R[\![z-\zeta]\!]}$ then we get $y_i \in V_R = R(\!(z-\zeta)\!)\mathfrak{q}$. Therefore we find for all i a common $e \in \mathbb{Z}$ such that $(z-\zeta)^e y_i \in \mathfrak{q}$ which means that $(z-\zeta)^e \mathfrak{p}_R \subseteq \mathfrak{q}$. Let

$$s \colon V_R/(z-\zeta)^e \mathfrak{p}_R \twoheadrightarrow \mathfrak{q}/(z-\zeta)^e \mathfrak{p}_R$$

be a morphism such that $s \circ i = \mathrm{id}_{\mathfrak{q}/(z-\zeta)^e \mathfrak{p}_R}$ where i is the inclusion

$$\mathfrak{q}/(z-\zeta)^e \mathfrak{p}_R \subseteq V_R/(z-\zeta)^e \mathfrak{p}_R.$$

Then i factors as

$$\mathfrak{q}/(z-\zeta)^e \mathfrak{p}_R \xrightarrow{i'} (z-\zeta)^d \mathfrak{p}_R/(z-\zeta)^e \mathfrak{p}_R \xrightarrow{j} V_R/(z-\zeta)^e \mathfrak{p}_R.$$

Let

$$s' = s \circ j \colon (z-\zeta)^d \mathfrak{p}_R/(z-\zeta)^e \mathfrak{p}_R \to \mathfrak{q}/(z-\zeta)^e \mathfrak{p}_R.$$

Then $s' \circ i' = s \circ j \circ i' = s \circ i = \mathrm{id}_{\mathfrak{q}/(z-\zeta)^e \mathfrak{p}_R}$. This realizes $\mathfrak{q}/(z-\zeta)^e \mathfrak{p}_R$ and $(z-\zeta)^d \mathfrak{p}_R/\mathfrak{q}$ as direct summands of the finite free R-module $(z-\zeta)^d \mathfrak{p}_R/(z-\zeta)^e \mathfrak{p}_R$.

"$\Leftarrow$": We have $V_R = R(\!(z-\zeta)\!)(z-\zeta)^d \mathfrak{p}_R \supseteq R(\!(z-\zeta)\!)\mathfrak{q} \supseteq R(\!(z-\zeta)\!)(z-\zeta)^e \mathfrak{p}_R = V_R$ which implies $R(\!(z-\zeta)\!)\mathfrak{q} = V_R = \sigma^* D \otimes_{k(\!(z)\!)} R(\!(z-\zeta)\!)$ and $\mathfrak{q}$ is a finitely generated $R[\![z-\zeta]\!]$-module since $(z-\zeta)^e \mathfrak{p}_R$ and $\mathfrak{q}/(z-\zeta)^e \mathfrak{p}_R$ are finitely generated $R[\![z-\zeta]\!]$-modules. The projectivity of $(z-\zeta)^d \mathfrak{p}_R/\mathfrak{q}$ yields the following decomposition

$$(z-\zeta)^d \mathfrak{p}_R/(z-\zeta)^e \mathfrak{p}_R = \mathfrak{q}/(z-\zeta)^e \mathfrak{p}_R \oplus (z-\zeta)^d \mathfrak{p}_R/\mathfrak{q}.$$

Combining this with the decomposition

$$V_R = (z-\zeta)^e \mathfrak{p}_R \oplus (z-\zeta)^d \mathfrak{p}_R/(z-\zeta)^e \mathfrak{p}_R \oplus V_R/(z-\zeta)^d \mathfrak{p}_R$$

we can realize $\mathfrak{q}$ as a direct summand of $\sigma^* D \otimes_{k(\!(z)\!)} R(\!(z-\zeta)\!)$. $\square$

Proposition 2.2.4. *Let (D, F_D) be a z-isocrystal over k and let R, R' be K-algebras such that R' is an R-algebra. If $\mathfrak{q}$ is a Hodge-Pink lattice over R of (D, F_D) then $\mathfrak{q} \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!]$ is a Hodge-Pink lattice over R' of (D, F_D).*

Proof. We will show that $\mathfrak{q}' = \mathfrak{q} \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!]$ is an $R'[\![z-\zeta]\!]$-submodule of $V_{R'}$ which satisfies the conditions of Lemma 2.2.3. First note that $V_R \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!] = V_{R'}$ and $(z-\zeta)^c \mathfrak{p}_R \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!] = (z-\zeta)^c \mathfrak{p}_{R'}$ for all $c \in \mathbb{Z}$. Let $d, e \in \mathbb{Z}$ be as in Lemma 2.2.3. From the inclusions $(z-\zeta)^e \mathfrak{p}_R \subseteq \mathfrak{q} \subseteq (z-\zeta)^d \mathfrak{p}_R$ we get the following commutative diagram

$$
\begin{tikzcd}
 & (z-\zeta)^e \mathfrak{p}_{R'} \arrow[dl, hook'] \arrow[d, hook] \\
\mathfrak{q}' \arrow[r] & (z-\zeta)^d \mathfrak{p}_{R'}.
\end{tikzcd}
$$

In order to show that $\mathfrak{q}' \to (z-\zeta)^d \mathfrak{p}_{R'}$ is also injective, we tensor the exact sequence

$$
0 \to \mathfrak{q} \to (z-\zeta)^d \mathfrak{p}_R \to \underbrace{(z-\zeta)^d \mathfrak{p}_R / \mathfrak{q}}_{=:M} \to 0
$$

over $R[\![z-\zeta]\!]$ with $R'[\![z-\zeta]\!]$ and get the following exact sequence

$$
\mathrm{Tor}_1^{R[\![z-\zeta]\!]}(M, R'[\![z-\zeta]\!]) \to \mathfrak{q}' \to (z-\zeta)^d \mathfrak{p}_{R'} \to M \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!] \to 0
$$

where $\mathrm{Tor}_1^{R[\![z-\zeta]\!]}(M, R'[\![z-\zeta]\!]) = \mathrm{Tor}_1^R(M, R')$ by Lemma 2.1.1 which is zero because M is a projective R-module by Lemma 2.2.3. Therefore

$$
(z-\zeta)^d \mathfrak{p}_{R'} / (\mathfrak{q} \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!]) \cong ((z-\zeta)^d \mathfrak{p}_R / \mathfrak{q}) \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!]
$$

and

$$
(\mathfrak{q} \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!]) / (z-\zeta)^e \mathfrak{p}_{R'} \cong (\mathfrak{q} / (z-\zeta)^e \mathfrak{p}_R) \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!].
$$

If $M = (z-\zeta)^d \mathfrak{p}_R / \mathfrak{q}$ or $M = \mathfrak{q} / (z-\zeta)^e \mathfrak{p}_R$ then

$$
\begin{aligned}
M \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!] &\cong M \otimes_{R[\![z-\zeta]\!]/(z-\zeta)^{e-d}} R'[\![z-\zeta]\!]/(z-\zeta)^{e-d} \\
&\cong M \otimes_{R[\![z-\zeta]\!]/(z-\zeta)^{e-d}} R[\![z-\zeta]\!]/(z-\zeta)^{e-d} \otimes_R R' \cong M \otimes_R R'.
\end{aligned}
$$

This shows that $(z-\zeta)^d \mathfrak{p}_{R'} / (\mathfrak{q} \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!])$ and $(\mathfrak{q} \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!]) / (z-\zeta)^e \mathfrak{p}_{R'}$ are finite locally free R'-modules. □

Let (D, F_D) be a z-isocrystal over k and let $d, e \in \mathbb{Z}$ with $d \leq e$. For each K-algebra R we denote by $Q_D(R)$ the set of all Hodge-Pink lattices $\mathfrak{q}$ over R of (D, F_D) and by $Q_{D,d,e}(R)$ the subset of $\mathfrak{q} \in Q_D(R)$ such that $(z-\zeta)^d \mathfrak{p}_R \supseteq \mathfrak{q} \supseteq (z-\zeta)^e \mathfrak{p}_R$. By Proposition 2.2.4 and its proof $Q_D(\_)$ and $Q_{D,d,e}(\_)$ define functors from the category of K-algebras to the category of sets and actually both are sheaves on the category of K-algebras. In order to see this, we look at the functor

$$\begin{aligned} \tilde{Q}_{D,d,e}\colon (K\text{-algebras}) &\to (\text{sets}) \\ R &\mapsto \tilde{Q}_{D,d,e}(R) \\ (R \to R') &\mapsto (\tilde{\mathfrak{q}} \mapsto \tilde{\mathfrak{q}} \otimes_R R') \end{aligned}$$

where, for a K-algebra R, $\tilde{Q}_{D,d,e}(R)$ is the set of $R[\![z-\zeta]\!]/(z-\zeta)^{e-d}$-submodules $\tilde{\mathfrak{q}}$ of $P_R^{(d,e)} = (z-\zeta)^d\mathfrak{p}_R/(z-\zeta)^e\mathfrak{p}_R$ such that $\tilde{\mathfrak{q}}$ and $P_R^{(d,e)}/\tilde{\mathfrak{q}}$ are finite locally free as R-modules. The projection $V_R \to V_R/(z-\zeta)^e\mathfrak{p}_R$ induces a bijection $\mathfrak{q} \mapsto \tilde{\mathfrak{q}}$ between $Q_{D,d,e}(R)$ and $\tilde{Q}_{D,d,e}(R)$. If $R \to \mathrm{R}'$ is a K-morphism the proof of Proposition 2.2.4 shows that $\tilde{\mathfrak{q}} \otimes_R R'$ is the $R'[\![z-\zeta]\!]/(z-\zeta)^{e-d}$-submodule of $P_{R'}^{(d,e)}$ associated to $\mathfrak{q} \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!]$. Clearly $\tilde{Q}_{D,d,e}$ and hence $Q_{D,d,e}$ is a sheaf. In order to test that Q_D is a sheaf, we can also work with $\tilde{Q}_{D,d,e}$ since every $\mathrm{Spec}(R)$ is quasi-compact. Q_D (resp. $Q_{D,d,e}$, resp. $\tilde{Q}_{D,d,e}$) extends to a sheaf on the category of K-schemes which we again denote by Q_D (resp. $Q_{D,d,e}$, resp. $\tilde{Q}_{D,d,e}$). In the case of $\tilde{Q}_{D,d,e}$ we will give an explicit description of this functor. Let S be a K-scheme. We have the following sheaves on S:

- For $c \geq 0$ the sheaf $U \mapsto \mathscr{O}_S(U)[\![z-\zeta]\!]/(z-\zeta)^c$ is denoted by $\mathscr{O}_S[\![z-\zeta]\!]/(z-\zeta)^c$ and
- for $d, e \in \mathbb{Z}$ with $d \leq e$ the sheaf $U \mapsto P_{\mathscr{O}_S(U)}^{(d,e)}$ is denoted by $\mathscr{P}_{(S)}^{(d,e)}$.

Now for every K-scheme S, we define $\tilde{Q}_{D,d,e}(S)$ to be the set of all $\mathscr{O}_S[\![z-\zeta]\!]/(z-\zeta)^{e-d}$-submodules $\mathscr{Q}$ of $\mathscr{P}_{(S)}^{(d,e)}$ such that $\mathscr{Q}$ and $\mathscr{P}_{(S)}^{(d,e)}/\mathscr{Q}$ are finite locally free $\mathscr{O}_S$-modules.

Proposition 2.2.5. *Let (D, F_D) be a z-isocrystal over k. If R is a K-algebra and if $\mathfrak{q}$ is a Hodge-Pink lattice over R of (D, F_D) then Zariski locally on* $\mathrm{Spec}\, R$ *the $R[\![z-\zeta]\!]$-module $\mathfrak{q}$ is free of the same rank as $\mathfrak{p}_R$.*

Proof. Let $d, e \in \mathbb{Z}$ be as in Lemma 2.2.3. The finitely generated K-subalgebras of R form a filtered inductive system having R as its inductive limit. Hence by [EGA IV, Théorème (8.5.2), Proposition (8.5.5) and Corollaire (8.5.7)] there exists a finitely generated K-subalgebra $\tilde{R}$ of R and an exact sequence of $\tilde{R}$-modules

$$\tilde{P} \to \tilde{M} \to 0 \tag{2.2.2}$$

such that $\tilde{P}, \tilde{M}$ are finite locally free and (2.2.2) tensored over $\tilde{R}$ with R is isomorphic to

$$(z-\zeta)^d\mathfrak{p}_R/(z-\zeta)^e\mathfrak{p}_R \to (z-\zeta)^d\mathfrak{p}_R/\mathfrak{q} \to 0. \tag{2.2.3}$$

In the following we will change the K-subalgebra $\tilde{R}$ (which will remain finitely generated) without mentioning it. Since multiplication with $(z-\zeta)$ induces endomorphisms of

$(z-\zeta)^d \mathfrak{p}_R/(z-\zeta)^e \mathfrak{p}_R$ and $(z-\zeta)^d \mathfrak{p}_R/\mathfrak{q}$ such that $(z-\zeta)^{e-d}$ is zero we can view (2.2.2) as an exact sequence of $\tilde{R}[\![z-\zeta]\!]/(z-\zeta)^{e-d}$-modules. Furthermore since

$$(z-\zeta)^d \mathfrak{p}_{\tilde{R}}/(z-\zeta)^e \mathfrak{p}_{\tilde{R}} \otimes_{\tilde{R}} R \cong (z-\zeta)^d \mathfrak{p}_R/(z-\zeta)^e \mathfrak{p}_R \simeq \tilde{P} \otimes_{\tilde{R}} R$$

by [EGA IV, Corollaire (8.5.2.5)] we can assume that we have an isomorphism $(z-\zeta)^d \mathfrak{p}_{\tilde{R}}/(z-\zeta)^e \mathfrak{p}_{\tilde{R}} \xrightarrow{\sim} \tilde{P}$. Let $\tilde{\mathfrak{q}}$ be the kernel of the composition

$$(z-\zeta)^d \mathfrak{p}_{\tilde{R}} \to (z-\zeta)^d \mathfrak{p}_{\tilde{R}}/(z-\zeta)^e \mathfrak{p}_{\tilde{R}} \to \tilde{M}.$$

Since $\tilde{R}[\![z-\zeta]\!]$ is noetherian $\tilde{\mathfrak{q}}$ is a finitely generated $\tilde{R}[\![z-\zeta]\!]$-module. Let $\mathfrak{m} \subseteq \tilde{R}[\![z-\zeta]\!]$ be a maximal ideal. It satisfies $(z-\zeta) \in \mathfrak{m}$ because otherwise $1 + a(z-\zeta) \in \mathfrak{m}$ for some $a \in \tilde{R}[\![z-\zeta]\!]$ which is impossible since $1 + a(z-\zeta)$ is invertible. Therefore $\mathfrak{n} := \tilde{R} \cap \mathfrak{m}$ is maximal in $\tilde{R}$.

Claim: $\tilde{\mathfrak{q}}_{\mathfrak{m}}$ is a flat $\tilde{R}[\![z-\zeta]\!]_{\mathfrak{m}}$-module.

In order to prove the claim, we note that $\tilde{\mathfrak{q}}$ is a direct summand of $(z-\zeta)^d \mathfrak{p}_{\tilde{R}}$ as an $\tilde{R}$-module since $\tilde{M}$ is a locally free $\tilde{R}$-module. Thus $\tilde{\mathfrak{q}}$ is flat over $\tilde{R}$. Moreover we have $\tilde{R}[\![z-\zeta]\!] \otimes_{\tilde{R}} \tilde{R}/\mathfrak{n} \cong \tilde{R}/\mathfrak{n}[\![z-\zeta]\!]$. Clearly the natural morphism $\tilde{R}[\![z-\zeta]\!] \otimes_{\tilde{R}} \tilde{R}/\mathfrak{n} \to \tilde{R}/\mathfrak{n}[\![z-\zeta]\!]$ is surjective. It is also injective; for this we can write $\sum f_i \otimes \bar{a}_i \in \tilde{R}[\![z-\zeta]\!] \otimes_{\tilde{R}} \tilde{R}/\mathfrak{n}$ as $(\sum_{\nu \in \mathbb{N}_0} r_\nu (z-\zeta)^\nu) \otimes \bar{1}$ with $r_\nu \in \mathfrak{n}$ if it is mapped to zero in $\tilde{R}/\mathfrak{n}[\![z-\zeta]\!]$. Since $\mathfrak{n}$ is finitely generated, $\mathfrak{n} = (e_j)_{j=1,\ldots,n}$ with $n \in \mathbb{N}$ and we write $r_\nu = \sum_{j=1}^n b_{\nu,j} e_j$. Combining this, we get

$$\begin{aligned} \sum f_i \otimes \bar{a}_i &= \left(\sum_{\nu \in \mathbb{N}_0} r_\nu (z-\zeta)^\nu \right) \otimes \bar{1} = \left(\sum_{j=1}^n (\sum_{\nu \in \mathbb{N}_0} b_{\nu,j} (z-\zeta)^\nu) e_j \right) \otimes \bar{1} \\ &= \sum_{j=1}^n (\sum_{\nu \in \mathbb{N}_0} b_{\nu,j} (z-\zeta)^\nu) \otimes \bar{e}_j = 0. \end{aligned}$$

As $\mathfrak{n}$ is a maximal ideal $\tilde{\mathfrak{q}} \otimes_{\tilde{R}} \tilde{R}/\mathfrak{n} \cong \tilde{\mathfrak{q}} \otimes_{\tilde{R}[\![z-\zeta]\!]} \tilde{R}/\mathfrak{n}[\![z-\zeta]\!]$ as a submodule of $(z-\zeta)^d \mathfrak{p}_{\tilde{R}} \otimes_{\tilde{R}} \tilde{R}/\mathfrak{n} \cong (z-\zeta)^d \mathfrak{p}_{\tilde{R}} \otimes_{\tilde{R}[\![z-\zeta]\!]} \tilde{R}/\mathfrak{n}[\![z-\zeta]\!]$ is a free module of rank r over the principal ideal domain $\tilde{R}/\mathfrak{n}[\![z-\zeta]\!]$. Now the claim follows from [EGA IV, Théorème (11.3.10)].

This being true for all maximal ideals of $\tilde{R}[\![z-\zeta]\!]$, we see that $\tilde{\mathfrak{q}}$ is a finitely generated flat hence projective module over the noetherian ring $\tilde{R}[\![z-\zeta]\!]$. By [EGA Inew, Proposition (10.10.8.6)] it is locally on $\operatorname{Spec} \tilde{R}$ free over $\tilde{R}[\![z-\zeta]\!]$.

It remains to show that $\tilde{\mathfrak{q}} \otimes_{\tilde{R}[\![z-\zeta]\!]} R[\![z-\zeta]\!] \simeq \mathfrak{q}$. Consider the following diagram ($A = \tilde{R}[\![z-\zeta]\!]$, $B = R[\![z-\zeta]\!]$):

$$\begin{tikzcd} \operatorname{Tor}_1^A(\tilde{M}, B) \arrow[r] & \tilde{\mathfrak{q}} \otimes_A B \arrow[r] \arrow[d] & (z-\zeta)^d \mathfrak{p}_{\tilde{R}} \otimes_A B \arrow[r] \arrow[d, equal] & \tilde{M} \otimes_A B \arrow[r] \arrow[d, "\cong"] & 0 \\ 0 \arrow[r] & \mathfrak{q} \arrow[r] & (z-\zeta)^d \mathfrak{p}_R \arrow[r] & (z-\zeta)^d \mathfrak{p}_R/\mathfrak{q} \arrow[r] & 0. \end{tikzcd}$$

The top row is obtained from the exact sequence $0 \to \tilde{\mathfrak{q}} \to (z-\zeta)^d \mathfrak{p}_{\tilde{R}} \to \tilde{M} \to 0$ by tensoring over $\tilde{R}[\![z-\zeta]\!]$ with $R[\![z-\zeta]\!]$. By Lemma 2.1.1 we have $\mathrm{Tor}_1^{\tilde{R}[\![z-\zeta]\!]}(\tilde{M}, R[\![z-\zeta]\!]) = \mathrm{Tor}_1^{\tilde{R}}(\tilde{M}, R)$ which is zero since $\tilde{M}$ is flat over $\tilde{R}$. □

2.3 Bounded Hodge-Pink lattices

Let $r \in \mathbb{N}$ and $i \in \{1, \dots, r\}$. Let (D, F_D) be a z-isocrystal over k of rank r, R be a K-algebra and let $\mathfrak{q}$ be a Hodge-Pink lattice over R of (D, F_D) such that $(z-\zeta)^d \mathfrak{p}_R \supseteq \mathfrak{q} \supseteq (z-\zeta)^e \mathfrak{p}_R$. By the above Proposition 2.2.5, $\mathfrak{q}$ is a projective $R[\![z-\zeta]\!]$-module. Thus the natural maps

$$\bigwedge^i_{R[\![z-\zeta]\!]} (z-\zeta)^e \mathfrak{p}_R \to \bigwedge^i_{R[\![z-\zeta]\!]} \mathfrak{q} \to \bigwedge^i_{R[\![z-\zeta]\!]} (z-\zeta)^d \mathfrak{p}_R$$

are injective ([BouAlg, III.7.9 Corollary to Proposition 12]). We remark that we can view $\bigwedge^i_{R[\![z-\zeta]\!]} \mathfrak{q}$ and $\bigwedge^i_{R[\![z-\zeta]\!]} \mathfrak{p}_R$ as submodules of $\bigwedge^i_{R[\![z-\zeta]\!]} V_R \cong \bigwedge^i_{R(\!(z-\zeta)\!)} V_R$. In order to see this, notice that $\bigwedge^i_{R[\![z-\zeta]\!]} V_R = (\bigwedge^i_{R[\![z-\zeta]\!]} (z-\zeta)^d \mathfrak{p}_R)[\frac{1}{z-\zeta}]$ and since $\bigwedge^i_{R[\![z-\zeta]\!]} (z-\zeta)^d \mathfrak{p}_R$ is free over $R[\![z-\zeta]\!]$ also the morphism into the localization $\bigwedge^i_{R[\![z-\zeta]\!]} V_R$ is injective. Together with the injection $\bigwedge^i_{R[\![z-\zeta]\!]} \mathfrak{q} \hookrightarrow \bigwedge^i_{R[\![z-\zeta]\!]} (z-\zeta)^d \mathfrak{p}_R$ we get the injectivity of $\bigwedge^i_{R[\![z-\zeta]\!]} \mathfrak{q} \hookrightarrow \bigwedge^i_{R[\![z-\zeta]\!]} V_R$. The inclusion $\bigwedge^i_{R[\![z-\zeta]\!]} \mathfrak{p}_R \subseteq \bigwedge^i_{R[\![z-\zeta]\!]} V_R$ is clear. Also note that in the $R(\!(z-\zeta)\!)$-module $\bigwedge^i_{R[\![z-\zeta]\!]} V_R$ we have $\bigwedge^i_{R[\![z-\zeta]\!]} (z-\zeta)^d \mathfrak{p}_R = (z-\zeta)^{id} \bigwedge^i_{R[\![z-\zeta]\!]} \mathfrak{p}_R$.

Now we want to define a boundedness condition for Hodge-Pink lattices. Let $r \in \mathbb{N}$. We fix a z-isocrystal (D, F_D) over k of rank r and integers $w_1, \dots, w_r \in \mathbb{Z}$ with $w_1 \geq \ldots \geq w_r$. Set $w = (w_1, \dots, w_r)$ and $|w| = w_1 - w_r$. For each K-algebra R let $Q_{D,\leq w}(R)$ be the set of Hodge-Pink lattices $\mathfrak{q}$ over R of (D, F_D) which satisfy the following conditions:

$$\bigwedge^r_{R[\![z-\zeta]\!]} \mathfrak{q} = (z-\zeta)^{w_1+\ldots+w_r} \bigwedge^r_{R[\![z-\zeta]\!]} \mathfrak{p}_R \tag{2.3.1}$$

$$\bigwedge^i_{R[\![z-\zeta]\!]} \mathfrak{q} \supseteq (z-\zeta)^{w_1+\ldots+w_i} \bigwedge^i_{R[\![z-\zeta]\!]} \mathfrak{p}_R \qquad i = 1, \dots, r \tag{2.3.2}$$

$$\bigwedge^i_{R[\![z-\zeta]\!]} \mathfrak{q} \subseteq (z-\zeta)^{w_{r+1-i}+\ldots+w_r} \bigwedge^i_{R[\![z-\zeta]\!]} \mathfrak{p}_R \qquad i = 1, \dots, r \tag{2.3.3}$$

Remark 2.3.1. Since $\left(\bigwedge^i_{R[\![z-\zeta]\!]} \mathfrak{q}\right) \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!] = \bigwedge^i_{R'[\![z-\zeta]\!]} \left(\mathfrak{q} \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!]\right)$ for each R-algebra R', $Q_{D,\leq w}(\_)$ defines a subfunctor of the functor Q_{D,w_r,w_1} which we denote by $Q_{D,\leq w}$.

Lemma 2.3.2. *Conditions* (2.3.1) *and* (2.3.2) *are equivalent to conditions* (2.3.1) *and* (2.3.3).

Proof. Set $I = \{1, \ldots, r\}$. By Proposition 2.2.5 we can assume that $\mathfrak{q}$ is a free $R[\![z-\zeta]\!]$-module. Let $(e_j)_{1\leq j\leq r}$ (resp. $(f_j)_{1\leq j\leq r}$) be a basis of $\mathfrak{p}_R$ (resp. $\mathfrak{q}$) as an $R[\![z-\zeta]\!]$-module. Let A be the matrix corresponding to the identity morphism with respect to the bases (e_j) and (f_j), where we consider $\mathrm{id}\colon V_R \to V_R$ as an $R(\!(z-\zeta)\!)$-linear map. Now $\bigwedge^i(A)$ is the matrix corresponding to the morphism $\bigwedge^i(\mathrm{id})\colon \bigwedge^i V_R \to \bigwedge^i V_R$ with respect to the bases $(e_G)_{G\in\mathfrak{F}_i(I)}$ and $(f_G)_{G\in\mathfrak{F}_i(I)}$ where $\mathfrak{F}_i(I) = \{G \subseteq I \mid \sharp G = i\}$ and $e_G = e_{g_1} \wedge \ldots \wedge e_{g_i}$ with $\{g_1, \ldots, g_i\} = G$ and $g_1 < \ldots < g_i$. With these notations we can reformulate (2.3.1), (2.3.2) and (2.3.3) in the following way:

$$\det(A) = c(z-\zeta)^{-w_1-\ldots-w_r}, \quad c \in R[\![z-\zeta]\!]^* \tag{2.3.4}$$

$$\bigwedge^i(A) \in \mathrm{M}_{\binom{r}{i}\times\binom{r}{i}}\big((z-\zeta)^{-w_1-\ldots-w_i} R[\![z-\zeta]\!]\big) \tag{2.3.5}$$

$$\left(\bigwedge^i(A)\right)^{-1} \in \mathrm{M}_{\binom{r}{i}\times\binom{r}{i}}\big((z-\zeta)^{w_{r+1-i}+\ldots+w_r} R[\![z-\zeta]\!]\big) \tag{2.3.6}$$

If we define $B_i = (\rho_{H,H'}\rho_{G,G'}\det(A_{G',H'}))_{(G,H)\in\mathfrak{F}_i(I)\times\mathfrak{F}_i(I)}$ (notation from [BouAlg, III.7.8 (19) p.519]) we get

$$\det(A)\cdot \mathrm{E}_{\binom{r}{i}} = {}^{\mathrm{T}}B_i \cdot \bigwedge^i(A). \tag{2.3.7}$$

Proof of equation (2.3.7):

$$\begin{aligned}\left({}^{\mathrm{T}}B_i \cdot \bigwedge^i(A)\right)_{G,H} &= \sum_{P\in\mathfrak{F}_i(I)} \underbrace{({}^{\mathrm{T}}B_i)_{G,P}}_{=(B_i)_{P,G}} \cdot \bigwedge^i(A)_{P,H} = \sum_{P\in\mathfrak{F}_i(I)} \rho_{G,G'}\rho_{P,P'}\det(A_{P',G'})\det(A_{P,H}) \\ &= \begin{cases}\det(A) & \text{if } G = H,\\ 0 & \text{if } G \neq H.\end{cases}\end{aligned}$$

The last "$=$" is exactly [BouAlg, III.8.6 (21) and (22)]. By the definition of B_i we see that

$$\begin{aligned}&{}^{\mathrm{T}}B_i \in \mathrm{M}_{\binom{r}{i}\times\binom{r}{i}}\big((z-\zeta)^{-w_1-\ldots-w_{r-i}} R[\![z-\zeta]\!]\big)\\ \Leftrightarrow &\bigwedge^{r-i}(A) \in \mathrm{M}_{\binom{r}{r-i}\times\binom{r}{r-i}}\big((z-\zeta)^{-w_1-\ldots-w_{r-i}} R[\![z-\zeta]\!]\big)\end{aligned} \tag{2.3.8}$$

The claim follows from (2.3.7) and (2.3.8). □

Remark 2.3.3. With the notation of Lemma 2.3.2 and in the situation that $R = L$ is a field we can take the bases $(e_j)_{1\leq j\leq r}$ and $(f_j)_{1\leq j\leq r}$ to be the bases from Definition 2.2.1.(vi). If $(v_1, \ldots, v_r)$ are the Hodge-Pink weights of $\mathfrak{q}$ then $A = \mathrm{diag}((z-\zeta)^{-v_1}, \ldots, (z-\zeta)^{-v_r})$. From (2.3.4) and (2.3.5) we see that $\mathfrak{q} \in Q_{D,\leq w}(L)$ if and only if

$$v_1 + \ldots + v_i \leq w_1 + \ldots + w_i \qquad i = 1, \ldots, r \qquad \text{with “$=$” for } i = r. \tag{2.3.9}$$

Thus $Q_{D,\preceq w}(L)$ is the set of all Hodge-Pink lattices over L of (D, F_D) with Hodge-Pink weights $v = (v_1, \ldots, v_r)$ such that $v \preceq w$ in the Bruhat order, i.e.

$$v \preceq w \quad :\Leftrightarrow \quad w - v = \sum_{i=1}^{r-1} n_i \alpha_i \quad \text{with } n_i \in \mathbb{N}_0$$

where $\alpha_i = ((\alpha_i)_j)_{j \in \{1,\ldots,r\}} \in \mathbb{Z}^r$ with

$$(\alpha_i)_j = \begin{cases} 1 & j = i, \\ -1 & j = i+1, \\ 0 & \text{else} \end{cases}$$

for $i = 1, \ldots, r-1$.
Also this especially shows that in the situation when R is an arbitrary K-algebra and $\mathfrak{q} \in Q_{D,\preceq w}(R)$ the locally free R-module $(z-\zeta)^{w_r}\mathfrak{p}_R/\mathfrak{q}$ has rank $-rw_r + \sum_{i=1}^{r} w_i$. Namely, by the proof of Proposition 2.2.4, for an R-field L, $((z-\zeta)^{w_r}\mathfrak{p}_R/\mathfrak{q}) \otimes_R L$ is the quotient $(z-\zeta)^c\mathfrak{p}_L/\mathfrak{q}$ in Remark 2.2.2 with $c = w_r$ and $\mathfrak{q} = \mathfrak{q} \otimes_{R[\![z-\zeta]\!]} L[\![z-\zeta]\!]$.

Let R be a K-algebra. The next Lemma shows how one can interpret conditions (2.3.1) - (2.3.3), that $\mathfrak{q}$ has to satisfy to be an element of $Q_{D,\preceq w}(R)$ as closed conditions.

Lemma 2.3.4. *Let $j \in \{1, \ldots, r\}$. Let $\mathfrak{q} \in Q_{D,w_r,w_1}(R)$ be a Hodge-Pink lattice over R. We denote by $\alpha^{(j)}_{\mathfrak{q},w}$ the natural morphism (see the proof)*

$$\bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{q} \Big/ \bigwedge^j_{R[\![z-\zeta]\!]} (z-\zeta)^{w_1}\mathfrak{p}_R \to \bigwedge^j_{R[\![z-\zeta]\!]} (z-\zeta)^{w_r}\mathfrak{p}_R \Big/ (z-\zeta)^{w_{r+1-j}+\ldots+w_r} \bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{p}_R$$

and by $\beta_{\mathfrak{q},w}$ the morphism

$$(z-\zeta)^{w_1+\ldots+w_r} \bigwedge^r_{R[\![z-\zeta]\!]} \mathfrak{p}_R \Big/ \bigwedge^r_{R[\![z-\zeta]\!]} \mathfrak{q} \to \bigwedge^r_{R[\![z-\zeta]\!]} (z-\zeta)^{w_r-w_1}\mathfrak{q} \Big/ \bigwedge^r_{R[\![z-\zeta]\!]} \mathfrak{q}.$$

Then $\alpha^{(j)}_{\mathfrak{q},w}$ is zero if and only if the Hodge-Pink lattice $\mathfrak{q}$ satisfies (2.3.3) for $i = j$ and $\mathfrak{q}$ satisfies (2.3.1) if and only if $\alpha^{(r)}_{\mathfrak{q},w}$ and $\beta_{\mathfrak{q},w}$ are both zero.

Proof.

$$\begin{aligned}
&\bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{q} \subseteq (z-\zeta)^{w_{r+1-j}+\ldots+w_r} \bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{p}_R \\
\Leftrightarrow &\left(\bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{q} \hookrightarrow \bigwedge^j_{R[\![z-\zeta]\!]} (z-\zeta)^{w_r}\mathfrak{p}_R \twoheadrightarrow \bigwedge^j_{R[\![z-\zeta]\!]} (z-\zeta)^{w_r}\mathfrak{p}_R \Big/ (z-\zeta)^{w_{r+1-j}+\ldots+w_r} \bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{p}_R \right) = 0 \\
\Leftrightarrow &\left(\bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{q} \Big/ \bigwedge^j_{R[\![z-\zeta]\!]} (z-\zeta)^{w_1}\mathfrak{p}_R \to \bigwedge^j_{R[\![z-\zeta]\!]} (z-\zeta)^{w_r}\mathfrak{p}_R \Big/ (z-\zeta)^{w_{r+1-j}+\ldots+w_r} \bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{p}_R \right) = 0
\end{aligned}$$

Hence our claim about condition (2.3.3). In order to show the second claim, first note that, since $(z-\zeta)^{w_1}\mathfrak{p}_R \subseteq \mathfrak{q}$, we get $(z-\zeta)^{w_r}\mathfrak{p}_R \subseteq (z-\zeta)^{w_r-w_1}\mathfrak{q}$. Therefore we have the inclusions

$$(z-\zeta)^{w_1+\ldots+w_r} \bigwedge_{R[\![z-\zeta]\!]}^{r} \mathfrak{p}_R \subseteq \bigwedge_{R[\![z-\zeta]\!]}^{r} (z-\zeta)^{w_r}\mathfrak{p}_R \subseteq \bigwedge^{r} (z-\zeta)^{w_r-w_1}\mathfrak{q}$$

and hence it follows analogously that $(z-\zeta)^{w_1+\ldots+w_r} \bigwedge_{R[\![z-\zeta]\!]}^{r} \mathfrak{p}_R \subseteq \bigwedge_{R[\![z-\zeta]\!]}^{r} \mathfrak{q}$ if and only if $(z-\zeta)^{w_1+\ldots+w_r} \bigwedge_{R[\![z-\zeta]\!]}^{r} \mathfrak{p}_R / \bigwedge_{R[\![z-\zeta]\!]}^{r} \mathfrak{q} \to \bigwedge_{R[\![z-\zeta]\!]}^{r} (z-\zeta)^{w_r-w_1}\mathfrak{q} / \bigwedge_{R[\![z-\zeta]\!]}^{r} \mathfrak{q}$ is zero. □

Remark 2.3.5. The modules

$$\bigwedge_{R[\![z-\zeta]\!]}^{j} (z-\zeta)^{w_r}\mathfrak{p}_R \Big/ (z-\zeta)^{w_{r+1-j}+\ldots+w_r} \bigwedge_{R[\![z-\zeta]\!]}^{j} \mathfrak{p}_R \qquad \text{and} \qquad \bigwedge_{R[\![z-\zeta]\!]}^{r} (z-\zeta)^{w_r-w_1}\mathfrak{q} \Big/ \bigwedge_{R[\![z-\zeta]\!]}^{r} \mathfrak{q}$$

are finite locally free as R-modules. This is due to the fact that $\mathfrak{p}_R$ is a free $R[\![z-\zeta]\!]$-module and, by Proposition 2.2.5, $\mathfrak{q}$ is locally on $\operatorname{Spec} R$ a free $R[\![z-\zeta]\!]$-module. By working locally on $\operatorname{Spec} R$, we can assume that both $\mathfrak{p}_R$ and $\mathfrak{q}$ are free $R[\![z-\zeta]\!]$-modules and hence $\mathfrak{p}_R$ and $\mathfrak{q}$ are isomorphic to $R[\![z-\zeta]\!]^{\oplus r}$. Therefore $\bigwedge_{R[\![z-\zeta]\!]}^{j} \mathfrak{p}_R$ (resp. $\bigwedge_{R[\![z-\zeta]\!]}^{j} \mathfrak{q}$) is isomorphic to $R[\![z-\zeta]\!]^{\oplus\binom{r}{j}}$ (resp. $R[\![z-\zeta]\!]$). We get that (locally on $\operatorname{Spec} R$)

$$\bigwedge_{R[\![z-\zeta]\!]}^{j} (z-\zeta)^{w_r}\mathfrak{p}_R \Big/ (z-\zeta)^{w_{r+1-j}+\ldots+w_r} \bigwedge_{R[\![z-\zeta]\!]}^{j} \mathfrak{p}_R \simeq R^{\oplus\binom{r}{j}\cdot(w_{r+1-j}+\ldots+w_{r-1})}$$

and

$$\bigwedge_{R[\![z-\zeta]\!]}^{r} (z-\zeta)^{w_r-w_1}\mathfrak{q} \Big/ \bigwedge_{R[\![z-\zeta]\!]}^{r} \mathfrak{q} \simeq R^{\oplus w_1-w_r}.$$

Now, with the help of the morphisms of Lemma 2.3.4, we want to extend the functor $Q_{D,\leq w}$ to the category of schemes. Let M be one of the sources or targets of the morphisms $\alpha_{\mathfrak{q},w}^{(j)}$ or $\beta_{\mathfrak{q},w}$, i.e. M is one of the following modules:

$$\bigwedge_{R[\![z-\zeta]\!]}^{j} \mathfrak{q} \Big/ \bigwedge_{R[\![z-\zeta]\!]}^{j} (z-\zeta)^{w_1}\mathfrak{p}_R,$$

$$\bigwedge_{R[\![z-\zeta]\!]}^{j} (z-\zeta)^{w_r}\mathfrak{p}_R \Big/ (z-\zeta)^{w_{r+1-j}+\ldots+w_r} \bigwedge_{R[\![z-\zeta]\!]}^{j} \mathfrak{p}_R,$$

$$(z-\zeta)^{w_1+\ldots+w_r} \bigwedge_{R[\![z-\zeta]\!]}^{j} \mathfrak{p}_R \Big/ \bigwedge_{R[\![z-\zeta]\!]}^{r} \mathfrak{q},$$

$$\bigwedge_{R[\![z-\zeta]\!]}^{r} (z-\zeta)^{w_r-w_1}\mathfrak{q} \Big/ \bigwedge_{R[\![z-\zeta]\!]}^{r} \mathfrak{q}.$$

In any case M is actually an $R[\![z-\zeta]\!]/(z-\zeta)^c$-module for some $c \in \mathbb{N}$. Therefore we get that

$$\begin{aligned} M \otimes_R R' &\cong M \otimes_{R[\![z-\zeta]\!]/(z-\zeta)^c} R[\![z-\zeta]\!]/(z-\zeta)^c \otimes_R R' \\ &\cong M \otimes_{R[\![z-\zeta]\!]/(z-\zeta)^c} R'[\![z-\zeta]\!]/(z-\zeta)^c \cong M \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!]. \end{aligned}$$

Moreover we have already seen that $\bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{p}_R$ and $\bigwedge^r_{R[\![z-\zeta]\!]} \mathfrak{q}$ are finite locally free and hence, for example in the case when $M = \bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{q} / \bigwedge^j_{R[\![z-\zeta]\!]} (z-\zeta)^{w_1}\mathfrak{p}_R$,

$$M \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!] \cong \bigwedge^j_{R'[\![z-\zeta]\!]} (\mathfrak{q} \otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!]) \Big/ \bigwedge^j_{R'[\![z-\zeta]\!]} (z-\zeta)^{w_1}\mathfrak{p}_{R'}$$

and similar in the other cases. This shows that we can identify $\alpha^{(j)}_{\mathfrak{q},w} \otimes_R \mathrm{id}_{R'}$ with $\alpha^{(j)}_{(\mathfrak{q}\otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!]),w}$ and $\beta_{\mathfrak{q},w} \otimes_R \mathrm{id}_{R'}$ with $\beta_{(\mathfrak{q}\otimes_{R[\![z-\zeta]\!]} R'[\![z-\zeta]\!]),w}$. Hence we can identify $Q_{D,\leq w}$ with the following subfunctor of $\tilde{Q}_{D,w_r,w_1}$:

$$\begin{aligned} \tilde{Q}_{D,\leq w} \colon (K\text{-algebras}) &\to (\text{sets}), \\ R &\mapsto \tilde{Q}_{D,\leq w}(R) \end{aligned}$$

where $\tilde{Q}_{D,\leq w}(R) = \{\tilde{\mathfrak{q}} \in \tilde{Q}_{D,w_r,w_1}(R) \mid \alpha^{(i)}_{\mathfrak{q},w} = 0,\ i = 1, \dots, r \text{ and } \beta_{\mathfrak{q},w} = 0\}$. The functor $\tilde{Q}_{D,\leq w}$ is a sheaf on the category of K-algebras. We also denote the extension of $Q_{D,\leq w}$ (resp. $\tilde{Q}_{D,\leq w}$) to the category of K-schemes by $Q_{D,\leq w}$ (resp. $\tilde{Q}_{D,\leq w}$). If $F \in \tilde{Q}_{D,w_r,w_1}(S)$, where S is a K-scheme, and if $S = \bigcup_{i \in I} \operatorname{Spec} R_i$ is an open affine covering of S we define (for each $j \in \{1, \dots, r\}$) $A^{(j)}_{F|\operatorname{Spec} R_i,w}$ (resp. $B_{F|\operatorname{Spec} R_i,w}$) to be $(\alpha^{(j)}_{\mathfrak{q}_i,w})^{\sim \operatorname{Spec} R_i}$ (resp. $(\beta_{\mathfrak{q}_i,w})^{\sim \operatorname{Spec} R_i}$) with $\tilde{\mathfrak{q}}_i = \Gamma(\operatorname{Spec} R_i, F|\operatorname{Spec} R_i) \in \tilde{Q}_{D,w_r,w_1}(R_i)$. We have just seen that these morphisms glue to morphisms $A^{(j)}_{F,w}$ (resp. $B_{F,w}$) of $\mathscr{O}_S$-modules. With these notations we get

$$\tilde{Q}_{D,\leq w}(S) = \{F \in \tilde{Q}_{D,w_r,w_1}(S) \mid A^{(j)}_{F,w} = 0,\ B_{F,w} = 0\}.$$

Furthermore if T is an S-scheme and $F \in \tilde{Q}_{D,w_r,w_1}(S)$ we have $(A^{(j)}_{F,w})_T = A^{(j)}_{F_T,w}$ and $(B_{F,w})_T = B^{(j)}_{F_T,w}$.

2.4 $Q_{D,\leq w}$ as a projective scheme

Let $d, e \in \mathbb{Z}$, $d \leq e$, $\Phi \in \mathbb{Z}[n]$. Set $W_{d,e} = \operatorname{Spec} K[\![z-\zeta]\!]/(z-\zeta)^{e-d}$, $P^{(d,e)} = (z-\zeta)^d\mathfrak{p}_K/(z-\zeta)^e\mathfrak{p}_K$ and $\mathscr{P}^{(d,e)} = (P^{(d,e)})^{\sim W_{d,e}} = P^{(d,e)} \otimes \mathscr{O}_{W_{d,e}}$. If there is no matter of confusion we write W (resp. $\mathscr{P}$) instead of $W_{d,e}$ (resp. $\mathscr{P}^{(d,e)}$) and if $w = (w_1, \dots, w_r) \in \mathbb{Z}^r$ with $w_1 \geq \ldots \geq w_r$ we also denote W_{w_r,w_1} (resp. $\mathscr{P}^{(w_r,w_1)}$) by W_w (resp. $\mathscr{P}^{(w)}$). For each K-scheme S, let $\operatorname{Quot}^{\Phi}_{\mathscr{P}/W/K}(S)$ be the set of finitely presented S-flat quotients of $\mathscr{P}_S$ on W_S with Hilbert polynomial Φ on each fiber.

Remark. This defines a variant of Grothendieck's Quot-functor. In [FGA, no. 221] it is shown that this functor is representable in the category of locally noetherian K-schemes. Altman and Kleiman have shown [AK, (2.6) Theorem] that one can even drop the noetherian hypothesis under mild finiteness conditions. These are fulfilled in our situation. Namely:

• The finiteness of $W = \operatorname{Spec} K[\![z-\zeta]\!]/(z-\zeta)^{e-d}$ over K implies that W is strongly projective in the sense of [AK] and that for each K-scheme S the support of an element of $\mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}(S)$ is proper and finitely presented over S.

• $\mathscr{P}$ is isomorphic to $f^*\mathscr{O}_K^{\oplus r}$ if $f\colon W \to \operatorname{Spec} K$ is the structure morphism.

Thus we see that both functors coincide and all extra conditions of [AK, (2.6) Theorem] are satisfied.

Let $n \geq 1$, S a scheme and $\mathscr{E}$ a quasi-coherent $\mathscr{O}_S$-module. Recall that, if we define

$$\mathrm{Grass}_n(\mathscr{E}) = \{\mathscr{E}/\mathscr{N} = \mathscr{F} \mid \mathscr{F} \text{ finite locally free of rank } n\}$$

then for each S-scheme T the assignment

$$T \mapsto \mathrm{Grass}_n(\mathscr{E})(T) := \mathrm{Grass}_n(\mathscr{E}_T)$$

defines a functor from the category of S-schemes to the category of sets which is representable by a projective S-scheme that is also denoted by $\mathrm{Grass}_n(\mathscr{E})$ (c.f. [EGA Inew, §9.7]). We want to describe an inclusion of functors $\mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}(S) \hookrightarrow \mathrm{Grass}_n(\mathscr{E})$ for suitable n and $\mathscr{E}$ and show that this morphism of functors is representable by a closed immersion. Thus we have an explicit proof of the representability of $\mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}$ in our situation.

2.4.1 Representability of $\mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}$

Let $f\colon W \to \operatorname{Spec} K$ denote the structure morphism and let $g\colon S \to \operatorname{Spec} K$ be a K-scheme. $f_S\colon W_S \to S$ is affine with $W_S = \operatorname{Spec} \mathscr{O}_S[\![z-\zeta]\!]/(z-\zeta)^{e-d}$ because $g^*((K[\![z-\zeta]\!]/(z-\zeta)^{e-d})^{\sim_{\operatorname{Spec} K}}) = g^*(\mathscr{O}_{\operatorname{Spec} K}[\![z-\zeta]\!]/(z-\zeta)^{e-d}) = \mathscr{O}_S[\![z-\zeta]\!]/(z-\zeta)^{e-d}$.

Lemma 2.4.1. *Let $\mathscr{F}, \mathscr{G}, \mathscr{H}$ be $\mathscr{O}_{W_S}$-modules.*

i) $0 \to \mathscr{F} \to \mathscr{G} \to \mathscr{H} \to 0$ is exact if and only if $0 \to (f_S)_*\mathscr{F} \to (f_S)_*\mathscr{G} \to (f_S)_*\mathscr{H} \to 0$ is exact.

ii) If $(f_S)_*\mathscr{F} = (f_S)_*\mathscr{G}$ as $(f_S)_*\mathscr{O}_{W_S}$-modules then $\mathscr{F} = \mathscr{G}$.

iii) If $a\colon T \to S$ is a morphism of schemes and if we denote the projections $W_T \to W_S$ (resp. $W_T \to T$) by b (resp. f_T) we have a canonical isomorphism $a^*(f_S)_*\mathscr{F} \cong (f_T)_*b^*\mathscr{F}$.

Even more precisely we have:

$$\begin{pmatrix} \text{quasi-coherent} \\ \mathscr{O}_{W_S}\text{-modules} \end{pmatrix} \to \begin{pmatrix} \text{quasi-coherent} \\ \mathscr{O}_S[\![z-\zeta]\!]/(z-\zeta)^{e-d}\text{-modules} \end{pmatrix}$$
$$\mathscr{F} \mapsto (f_S)_*\mathscr{F}$$

is an exact functor which is an equivalence of categories such that it is compatible with base change [EGA Inew, Théorème (9.2.1), Corollaire (9.2.5), Corollaire (9.3.3)]. $\mathscr{F}$ is an $\mathscr{O}_{W_S}$-module of finite presentation if and only if $(f_S)_*\mathscr{F}$ is an $(f_S)_*\mathscr{O}_{W_S}$-module of finite presentation [EGA Inew, Corollaire (9.2.6)].

Remark 2.4.2.

i) Note that in our situation $\mathscr{O}_S[\![z-\zeta]\!]/(z-\zeta)^{e-d}$ is a finite free $\mathscr{O}_S$-module. Therefore, if $\mathscr{M}$ is an $\mathscr{O}_S[\![z-\zeta]\!]/(z-\zeta)^{e-d}$-module of finite presentation then $\mathscr{M}$ is of finite presentation as an $\mathscr{O}_S$-module.

ii) In the situation of Lemma 2.4.1.(iii) let $\mathscr{G}$ be an element of $\mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}(S)$ such that we have the exact sequence $0 \to \mathscr{N} \to \mathscr{P}_S \to \mathscr{G} \to 0$ and therefore the exact sequence $0 \to (f_S)_*\mathscr{N} \to (f_S)_*\mathscr{P}_S \to (f_S)_*\mathscr{G} \to 0$. By (i) $(f_S)_*\mathscr{G}$ is finite locally free and therefore $0 \to \underbrace{a^*(f_S)_*\mathscr{N}}_{\cong (f_T)_*b^*\mathscr{N}} \to \underbrace{a^*(f_S)_*\mathscr{P}_S}_{\cong (f_T)_*b^*\mathscr{P}_S} \to \underbrace{a^*(f_S)_*\mathscr{G}}_{\cong (f_T)_*b^*\mathscr{G}} \to 0$ is exact [GW, Proposition 8.10 (Remark 8.11)]. This implies the exactness of $0 \to \mathscr{N}_T \to \mathscr{P}_T \to \mathscr{G}_T \to 0$ and hence $\mathscr{G}_T = \mathscr{P}_T/\mathscr{N}_T$.

We will also need the following converse to Remark 2.4.2.(i).

Lemma 2.4.3. *Let A be a ring and B a finitely generated A-algebra. Suppose that M is a B-module such that M is of finite presentation as an A-module. Then M is of finite presentation as a B-module.*

Proof. We can assume that $B = A[T]$ is generated by one element and even that it is a polynomial ring over A. Let $f\colon A^n \twoheadrightarrow M$ $(n \in \mathbb{N})$ be a surjective homomorphism with $\ker(f) = \langle x_1, \dots, x_m\rangle_A$ $(m \in \mathbb{N},\ x_i \in A^n)$. f induces by tensoring up with B a morphism $(A[T])^n = B^n \twoheadrightarrow M \otimes_A B$. Let $\tilde{f}\colon (A[T])^n \twoheadrightarrow M$ be the composition of this morphism with the canonical morphism $M \otimes_A B \twoheadrightarrow M$. For $i = 1, \dots, n$ let $e_i \in A^n$ be the i-th standard basis vector and choose $y_i \in A^n$ such that $f(y_i) = \tilde{f}(Te_i)$. Then clearly $\tilde{f}(Te_i - y_i) = \tilde{f}(Te_i) - f(y_i) = 0$. We claim that

$$N := \langle x_1, \dots, x_m, Te_1 - y_1, \dots, Te_n - y_n\rangle_{A[T]} = \ker(\tilde{f}).$$

Proof of the claim: We have already seen "$\subseteq$".
"$\supseteq$": Let $P = (P_1, \dots, P_n) \in (A[T])^n$ with $\tilde{f}(P) = 0$. We can write $P = (P_{1,0} + TP'_1, \dots, P_{n,0} + TP'_n)$ with $P_{i,0} \in A$ and $\deg(P'_i) < \deg(P_i)$ if $P_i \notin A$ and $P'_i = 0$ if $P_i \in A$. Thus

$$P = (P_{1,0}, \dots, P_{n,0}) + \sum_{i=1}^{n} P'_i Te_i = (P_{1,0}, \dots, P_{n,0}) + \sum_{i=1}^{n} P'_i y_i + \underbrace{\sum_{i=1}^{n} P'_i(Te_i - y_i)}_{\in N}.$$

As long as one of the $P_i \notin A$ write $Q = (Q_1, \ldots, \mathrm{Q}_n)$ where $Q := (P_{1,0}, \ldots, P_{n,0}) + \sum_{i=1}^{n} P_i' y_i$ and then we get

$$\max_{i=1,\ldots,n} \deg(Q_i) \leq \max_{i=1,\ldots,n} \deg(P_i') < \max_{i=1,\ldots,n} \deg(P_i).$$

Thus by replacing P with Q and continuing this way, we can achieve after finitely many steps that all $P_i \in A$. But then we have $P \in \langle x_1, \ldots, x_m \rangle$. Thus we have proved the claim.

By the claim $\ker(\tilde{f})$ is finitely generated and therefore M is of finite presentation as a B-module. $\square$

Lemma 2.4.4. *Let S be a K-scheme, $s \in S$ and $\mathscr{G} \in \mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}(S)$. Then $(f_S)_*\mathscr{G}$ is finite locally free and $\chi(\mathscr{G}_s)(n) = c_s$ where c_s is the rank of $(f_S)_*\mathscr{G}$ (at s).*

Proof. By Remark 2.4.2.(i) $(f_S)_*\mathscr{G}$ is of finite presentation as an $\mathscr{O}_S$-module and it is flat by definition hence finite locally free. Since f and therefore f_S is affine we have

$$\begin{aligned}\chi(\mathscr{G}_s)(n) &= \sum_{i=0}^{\infty} (-1)^i \dim_{\kappa(s)} \underbrace{\mathrm{H}^i(W_s, \underbrace{\mathscr{G}_s(n)}_{=\mathscr{G}_s})}_{=0\ \forall i>0} = \dim_{\kappa(s)} \Gamma(W_s, \mathscr{G}_s) \\ &= \dim_{\kappa(s)} \Gamma(f^{-1}_{\kappa(s)}(\operatorname{Spec} \kappa(s)), \mathscr{G}_s) = \dim_{\kappa(s)} \Gamma(\operatorname{Spec} \kappa(s), (f_{\kappa(s)})_*\mathscr{G}_s) \\ &= \dim_{\kappa(s)} \Gamma(\operatorname{Spec} \kappa(s), ((f_S)_*\mathscr{G}) \otimes_S \kappa(s)).\end{aligned}$$

The last "=" is Lemma 2.4.1.(iii). $\square$

With the notation of Lemma 2.4.4, since $\mathscr{P}_S \to \mathscr{G} \to 0$ is exact also $(f_S)_*\mathscr{P}_S \to (f_S)_*G \to 0$ is exact and $(f_S)_*\mathscr{P}_S \cong (f_*\mathscr{P})_S$ (Lemma 2.4.1 (i) and (iii)). This defines by Lemma 2.4.4 a morphism $\mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}(S) \to \mathrm{Grass}_{\Phi}(f_*\mathscr{P})(S)$ (Φ constant) which is injective by Lemma 2.4.1.(ii). If further $T \to S$ is a morphism of schemes Lemma 2.4.1.(iii) shows that the diagram

$$\begin{array}{ccc} \mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}(S) & \longrightarrow & \mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}(T) \\ \downarrow & & \downarrow \\ \mathrm{Grass}_{\Phi}(f_*\mathscr{P})(S) & \longrightarrow & \mathrm{Grass}_{\Phi}(f_*\mathscr{P})(T) \end{array}$$

commutes (where the vertical morphisms are $\mathscr{G} \mapsto (f_S)_*\mathscr{G}$ resp. $\mathscr{G} \mapsto (f_T)_*\mathscr{G}$). Hence we get a morphism of functors

$$\mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K} \to \mathrm{Grass}_{\Phi}(f_*\mathscr{P}).$$

Proposition 2.4.5. *The morphism of functors*

$$\mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K} \to \mathrm{Grass}_{\Phi}(f_*\mathscr{P})$$

is representable by a closed immersion.

Proof. $\mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K} \to \mathrm{Grass}_{\Phi}(f_*\mathscr{P})$ being representable by a closed immersion means that for all K-schemes S and all morphisms of functors $\mathrm{Hom}_K(\_\,, S) \to \mathrm{Grass}_{\Phi}(f_*\mathscr{P})$ the functor

$$T \mapsto \mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}(T) \times_{\mathrm{Grass}_{\Phi}(f_*\mathscr{P})(T)} \mathrm{Hom}_K(T, S)$$

is representable by a closed subscheme of S. By the Yoneda Lemma a morphism of functors $\mathrm{Hom}_K(\_\,, S) \to \mathrm{Grass}_{\Phi}(f_*\mathscr{P})$ corresponds to an element $u\colon (f_*\mathscr{P})_S \to \mathscr{G} \in \mathrm{Grass}_{\Phi}(f_*\mathscr{P})(S)$ and we have to show that there exists a closed subscheme S_0 of S such that a morphism $T \to S$ factors through S_0 if and only if $\mathscr{G}_T$ comes from an element of $\mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}(T)$ which is the case if $\mathscr{G}_T$ is an $\mathscr{O}_T[\![z-\zeta]\!]/(z-\zeta)^{e-d}$-module of finite presentation since it is already flat as a finite locally free $\mathscr{O}_T$-module. By Lemma 2.4.3 we just have to check whether $\mathscr{G}_T$ is an $\mathscr{O}_T[\![z-\zeta]\!]/(z-\zeta)^{e-d}$-module. Let $t\colon (f_*\mathscr{P})_S \to (f_*\mathscr{P})_S$ be multiplication with $(z-\zeta) + (z-\zeta)^{e-d}\mathscr{O}_S(S)[\![z-\zeta]\!] \in \Gamma(S, \mathscr{O}_S[\![z-\zeta]\!]/(z-\zeta)^{e-d})$ and let φ be the composition $u \circ t\colon (f_*\mathscr{P})_S \to \mathscr{G}$. If now T is an S-scheme we have $\varphi_T = u_T \circ t_T$ and t_T is multiplication with $(z-\zeta) + (z-\zeta)^{e-d}\mathscr{O}_T(T)[\![z-\zeta]\!]$ in $(f_*\mathscr{P})_T$. This shows that $\mathscr{G}_T$ comes from an element of $\mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}(T)$ if and only if φ_T factors through $\mathscr{G}_T$. Now let $\mathscr{N} = \ker(u)$. We have an exact sequence

$$\mathscr{N} \xrightarrow{v} (f_*\mathscr{P})_S \xrightarrow{u} \mathscr{G} \to 0$$

and therefore

$$\mathscr{N}_T \xrightarrow{v_T} (f_*\mathscr{P})_T \xrightarrow{u_T} \mathscr{G}_T \to 0$$

is exact. With these notations φ_T factors through $\mathscr{G}_T$ if and only if $\varphi_T \circ v_T = (\varphi \circ v)_T \colon \mathscr{N}_T \to \mathscr{G}_T$ is zero. We are now in the situation to apply [EGA Inew, Lemma (9.7.9.1)] from which the Proposition follows. $\square$

2.4.2 Representability of $Q_{D,\leq w}$

We want to identify $Q_{D,\leq w}$ with a subfunctor of $\mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}$. Since $\tilde{Q}_{D,\leq w}$ is a sheaf it suffices to identify $\tilde{Q}_{D,\leq w}(R)$ with a subset of $\mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}(\mathrm{Spec}\, R)$ for every K-algebra R and show that every induced morphism $\mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}(\mathrm{Spec}\, R) \to \mathrm{Quot}^{\Phi}_{\mathscr{P}/W/K}(\mathrm{Spec}\, R')$, coming from a K-algebra morphism $R \to R'$, coincides with $\tilde{Q}_{D,\leq w}(R) \to \tilde{Q}_{D,\leq w}(R')$ on $\tilde{Q}_{D,\leq w}(R)$. Let R be a K-algebra. For $\mathfrak{q} \in Q_{D,\leq w}(R)$ we write $\tilde{\mathfrak{q}}$ for the corresponding element in $\tilde{Q}_{D,\leq w}(R)$. Such a $\tilde{\mathfrak{q}}$ corresponds to a quotient

$$u_{\mathfrak{q}}\colon P_R^{(w)} \to P_R^{(w)}/\tilde{\mathfrak{q}}$$

Since $P_R^{(w)}/\tilde{\mathfrak{q}} \cong (z-\zeta)^{w_r}\mathfrak{p}_R/\mathfrak{q}$ is finite locally free as an R-module $u_{\mathfrak{q}}$ gives rise to an element of $\mathrm{Grass}_{\Phi_w}(f_*\mathscr{P}^{(w)})(R)$ where $f\colon W \to \mathrm{Spec}\, K$ is the structure morphism and $\Phi_w = -rw_r+$

$\sum_{i=1}^{r} w_r$ is the rank of $(z-\zeta)^{w_r}\mathfrak{p}_R/\mathfrak{q}$ by Remark 2.3.3. As we have seen in the proof of Proposition 2.4.5 the fact that $(z-\zeta)^{w_r}\mathfrak{p}_R/\mathfrak{q}$ is also an $R[\![z-\zeta]\!]/(z-\zeta)^{|w|}$-module shows that $(u_\mathfrak{q})^{\sim_{\operatorname{Spec} R[\![z-\zeta]\!]/(z-\zeta)^{|w|}}}$ lies in $\operatorname{Quot}^{\Phi_w}_{\mathscr{P}^{(w)}/W_w/K}$. Remark 2.4.2.(ii) shows the compatibility of this inclusion with morphisms $R \to R'$. This identification is compatible with the following identification of $\operatorname{Quot}^{\Phi_w}_{\mathscr{P}^{(w)}/W_w/K}$ as a subfunctor of $\tilde{Q}_{D,w_r,w_1}$. For each K-scheme S and $u\colon \mathscr{P}^{(w)}_S \to \mathscr{G} \in \operatorname{Quot}^{\Phi_w}_{\mathscr{P}^{(w)}/W_w/K}(S)$ we set $\tilde{Q}(u) = \ker(u)$ which is an object of $\tilde{Q}_{D,w_r,w_1}(S)$. In order to see this, first note that $(f_S)_*\tilde{Q}(u) = \ker((f_S)_*u\colon (f_S)_*\mathscr{P}^{(w)}_S \to (f_S)_*\mathscr{G})$ and we have already seen that $(f_S)_*\mathscr{G}$ is finite locally free. Since $(f_S)_*\mathscr{P}^{(w)}_S$ is finite free we also get that $(f_S)_*\tilde{Q}(u)$ is finite locally free. The compatibility of this identification with morphisms $T \to S$ follows from Remark 2.4.2.(ii) and Lemma 2.4.1.(iii).

Proposition 2.4.6. *The inclusion $Q_{D,\le w} \subseteq \operatorname{Quot}^{\Phi_w}_{\mathscr{P}^{(w)}/W_w/K}$ is representable by a closed immersion.*

Proof. Let S be a K-scheme and let $u\colon \mathscr{P}^{(w)}_S \to \mathscr{G} \in \operatorname{Quot}^{\Phi_w}_{\mathscr{P}^{(w)}/W_w/K}(S)$. We have to show that there exists a closed subscheme S_0 of S such that a morphism $T \to S$ factors through S_0 if and only if $u_T\colon \mathscr{P}^{(w)}_T \to \mathscr{G}_T$ lies in $Q_{D,\le w}(T)$.

We associate to $u\colon \mathscr{P}^{(w)}_S \to \mathscr{G}$ the object $\tilde{Q}(u) \in \tilde{Q}_{D,w_r,w_1}(S)$ and consider the morphisms $A^{(j)}_{\tilde{Q}(u),w}$ for $j = 1,\dots,r$ and $B_{\tilde{Q}(u),w}$. In this case u_T lies in $Q_{D,\le w}(T)$ if and only if

$$A^{(j)}_{\tilde{Q}(u_T),w} \cong A^{(j)}_{\tilde{Q}(u)_T,w} \cong (A^{(j)}_{\tilde{Q}(u),w})_T$$

and

$$B_{\tilde{Q}(u_T),w} \cong B_{\tilde{Q}(u)_T,w} \cong (B_{\tilde{Q}(u),w})_T$$

are all zero ($j = 1,\dots,r$). In order to apply [EGA Inew, Lemma (9.7.9.1)], we have to check that the targets of $A^{(j)}_{\tilde{Q}(u),w}$ and $B_{\tilde{Q}(u),w}$ are finite locally free $\mathscr{O}_S$-modules. This follows by their local description in Lemma 2.3.4 and by Remark 2.3.5. □

2.5 Weak admissibility

2.5.1 Filtered vector spaces

Definition 2.5.1. Let L be a field.

i) Let V be a finite-dimensional L-vector space. A family $\mathcal{F} = (\mathcal{F}^i)_{i\in\mathbb{R}}$ of L-subspaces of V with index set $\mathbb{R}$ is called an $\mathbb{R}$-*filtration of* V if

a) $\mathcal{F}^i \subseteq \mathcal{F}^j$ for $i > j$,

b) there exist $i, j \in \mathbb{R}$ such that $\mathcal{F}^i = V$ and $\mathcal{F}^j = 0$ and

c) for every $i \in \mathbb{R}$ $\mathcal{F}^i = \bigcap_{j<i} \mathcal{F}^j$.

In this case, for every $i \in \mathbb{R}$, we denote $\mathcal{F}^i/(\sum_{j>i} \mathcal{F}^j)$ by $\mathrm{gr}^i_{\mathcal{F}}(V)$. The finite set J where $\mathrm{gr}^j_{\mathcal{F}}(V) \neq 0$ for $j \in J$ and $\mathrm{gr}^j_{\mathcal{F}}(V) = 0$ for $j \notin J$ is called the *jumps of* $\mathcal{F}$. If I is a subset of $\mathbb{R}$ and $\mathrm{gr}^i_{\mathcal{F}}(V) = 0$ for every $x \in \mathbb{R} - I$ we say that $\mathcal{F}$ is an *I-filtration*.

ii) A *filtered vector space over* L is a tuple $(V, \mathrm{Fil}^\bullet V)$ where V is a finite-dimensional L-vector space and $\mathrm{Fil}^\bullet V = (\mathrm{Fil}^i V)_{i\in\mathbb{R}}$ is an $\mathbb{R}$-filtration of V. The sub-quotients $\mathrm{gr}^i_{\mathrm{Fil}^\bullet V}(V)$ are denoted by $\mathrm{gr}^i(V)$.

Let V be a finite-dimensional vector space over a field L and let $\mathcal{F}^\bullet = (\mathcal{F}^i)_{i\in\mathbb{R}}$ be an $\mathbb{R}$-filtration of V. If $U \subseteq V$ is a subspace we get an induced filtration on U denoted by $\mathcal{F}^\bullet|U = (\mathcal{F}^i|U)_{i\in\mathbb{R}}$ which is given by

$$\mathcal{F}^i|U = U \cap \mathcal{F}^i.$$

The subquotients $\mathrm{gr}^i_{\mathcal{F}^\bullet|U}(U)$ of U coming from the filtration $\mathcal{F}^\bullet|U$ are also denoted by $\mathrm{gr}^i_{\mathcal{F}^\bullet}(V)|U$.

Definition 2.5.2. Let $(V, \mathrm{Fil}^\bullet V)$ be a filtered vector space over a field L. We call

$$\deg(V, \mathrm{Fil}^\bullet V) = \sum_{i\in\mathbb{R}} i \cdot \dim \mathrm{gr}^i(V)$$

the *degree of* $(V, \mathrm{Fil}^\bullet V)$.

2.5.2 Newton slope

Definition 2.5.3. Let (D, F_D) be a z-isocrystal over k and let $\mathcal{B}$ be a basis of D. If $M_{F_D,\mathcal{B}}$ is the matrix corresponding to F_D with respect to $\mathcal{B}$ as defined in Remark 1.2.2.(ii), we denote $\mathrm{ord}_z(\det M_{F_D,\mathcal{B}})$ by $\mathrm{t_N}(D, F_D)$ and call it the *Newton slope of* (D, F_D). Note that this definition does not depend on the chosen basis.

Remark 2.5.4. Let $\underline{D} = (D, F_D)$ be a z-isocrystal over k where either k is a perfect field or the z-isocrystal is split semi-simple. In both cases we have a decomposition $\underline{D} = \bigoplus_{\lambda\in\mathbb{Q}} \underline{D}_\lambda$ of $\underline{D}$ into isoclinic sub-z-isocrystals $\underline{D}_\lambda = (D_\lambda, F_{D_\lambda})$ of slope λ (Lemma 1.2.5 and Definition 1.2.6). This decomposition gives rise to a $\mathbb{Q}$-filtration of D

$$\mathcal{F}^i D = \bigoplus_{-j\geq i} D_\lambda$$

and we get that

$$\mathrm{t_N}(D, F_D) = \deg(D, \mathcal{F}^\bullet D).$$

2.5.3 Hodge slope

Definition 2.5.5. Let $L|K$ be a field extension and let $(D, F_D, \mathfrak{q}_D)$ be a z-isocrystal with Hodge-Pink lattice over L. If $(w_1, \dots, w_r) \in \mathbb{Z}^r$ are the Hodge-Pink weights of $\mathfrak{q}_D$ we set

$$\mathrm{t_H}(D, F_D, \mathfrak{q}_D) = -\sum_{i=1}^{r} w_i$$

and call this the *Hodge slope of* $(D, F_D, \mathfrak{q}_D)$. It is also the integer $n \in \mathbb{Z}$ such that $\bigwedge^r \mathfrak{p}_L = (z-\zeta)^n \bigwedge^r \mathfrak{q}_D$ which follows from (2.3.9) in Remark 2.3.3 where n is exactly $\mathrm{ord}_{(z-\zeta)}(\det(A))$.

Lemma 2.5.6. *Let L be a field extension of K and let $(D, F_D, \mathfrak{q}_D)$ be a z-isocrystal with Hodge-Pink lattice over L. If $c \in \mathbb{Z}$ with $(z-\zeta)^c \mathfrak{p}_L \subseteq \mathfrak{q}_D$ and $(z-\zeta)^c \mathfrak{p}_L \subseteq \mathfrak{p}_L)$ then*

$$\mathrm{t_H}(D, F_D, \mathfrak{q}_D) = \dim_K \mathfrak{q}_D/(z-\zeta)^c \mathfrak{p}_L - \dim_K \mathfrak{p}_L/(z-\zeta)^c \mathfrak{p}_L$$

and if $(w_1, \dots, w_r) \in \mathbb{Z}$ are the Hodge-Pink weights of $\mathfrak{q}_D$ then

$$\dim_k(\mathfrak{q}_D/(z-\zeta)^{w_1} \mathfrak{p}_L) = \mathrm{t_H}(D, F_D, \mathfrak{q}_D) + w_1 \cdot \dim_{k(\!(z)\!)} D.$$

Proof. Both formulas follow from the description of $\mathfrak{q}_D/(z-\zeta)^c \mathfrak{p}_L$ given in Remark 2.2.2. For example:

$$\dim_K(\mathfrak{q}_D/(z-\zeta)^{w_1} \mathfrak{p}_L) = \underbrace{r \cdot w_1}_{=w_1 \cdot \dim_{k(\!(z)\!)} D} \underbrace{-\sum_{i=1}^{r} w_i}_{=\mathrm{t_H}(D, F_D, \mathfrak{q}_D)} .$$

□

Hodge filtration

Let $L|K$ be a field extension. If $(D, F_D, \mathfrak{q}_D)$ is a z-isocrystal with Hodge-Pink lattice over L we denote the L-vector space $\sigma^* D \otimes_{k(\!(z)\!)} L[\![z-\zeta]\!]/(z-\zeta) = \mathfrak{p}_L/(z-\zeta)\mathfrak{p}_L$ by D_L. The Hodge-Pink lattice $\mathfrak{q}_D$ gives rise to a $\mathbb{Z}$-filtration on D_L by setting

$$\mathrm{Fil}^i D_L = ((z-\zeta)^i \mathfrak{q}_D \cap \mathfrak{p}_L)/((z-\zeta)^i \mathfrak{q}_D \cap (z-\zeta)\mathfrak{p}_L).$$

This filtration is called the *Hodge filtration of* $(D, F_D, \mathfrak{q}_D)$. Now let $(w_1, \dots, w_r) \in \mathbb{Z}^r$ be the Hodge-Pink weights of $\mathfrak{q}_D$. By choosing an $L[\![z-\zeta]\!]$-basis $(x_1, \dots, x_r)$ of $\mathfrak{p}_L$ as in Definition 2.2.1.(vi) such that $((z-\zeta)^{w_1} x_1, \dots, (z-\zeta)^{w_r} x_r)$ is an $L[\![z-\zeta]\!]$-basis for $\mathfrak{q}_D$ we see that

$$(z-\zeta)^i \mathfrak{q}_D \cap \mathfrak{p}_L = \langle (z-\zeta)^{\max\{0, i+w_1\}} x_1, \dots, (z-\zeta)^{\max\{0, i+w_r\}} \rangle$$

and

$$(z-\zeta)^i \mathfrak{q}_D \cap (z-\zeta)\mathfrak{p}_L = \langle (z-\zeta)^{\max\{1, i+w_1\}} x_1, \dots, (z-\zeta)^{\max\{1, i+w_r\}} x_r \rangle$$

for every $i \in \mathbb{Z}$. Hence we get that

$$\dim_L(\operatorname{Fil}^i D_L) = \#\{j \mid i \leq -w_j\}$$

and therefore

$$\dim_L(\operatorname{gr}^i_{\operatorname{Fil}^\bullet}(D_L)) = \#\{j \mid i = -w_j\}.$$

This shows that the jumps of the Hodge filtration are the negative of the Hodge-Pink weights. Altogether we have seen that

$$\mathrm{t_H}(D, F_D, \mathfrak{q}_D) = \deg(D_L, \operatorname{Fil}^\bullet D_L).$$

2.5.4 Weakly admissible z-isocrystals with Hodge-Pink lattice

Definition 2.5.7. Let $L|K$ be a field extension. A z-isocrystal with Hodge-Pink lattice $(D, F_D, \mathfrak{q}_D)$ over L is called *weakly admissible* if $\mathrm{t_H}(D, F_D, \mathfrak{q}_D) = \mathrm{t_N}(D, F_D)$ and the following equivalent conditions hold:

a) $\mathrm{t_H}(D', F_{D'}, \mathfrak{q}_{D'}) \leq \mathrm{t_N}(D', F_{D'})$ for any sub-z-isocrystal with Hodge-Pink lattice $(D', F_{D'}, \mathfrak{q}_{D'})$ of $(D, F_D, \mathfrak{q}_D)$,

b) $\mathrm{t_H}(D', F_{D'}, \mathfrak{q}_{D'}) \leq \mathrm{t_N}(D', F_{D'})$ for any strict sub-z-isocrystal with Hodge-Pink lattice $(D', F_{D'}, \mathfrak{q}_{D'})$ of $(D, F_D, \mathfrak{q}_D)$,

c) $\mathrm{t_H}(D', F_{D'}, \mathfrak{q}_{D'}) \geq \mathrm{t_N}(D', F_{D'})$ for any quotient-z-isocrystal with Hodge-Pink lattice $(D', F_{D'}, \mathfrak{q}_{D'})$ of $(D, F_D, \mathfrak{q}_D)$,

d) $\mathrm{t_H}(D', F_{D'}, \mathfrak{q}_{D'}) \geq \mathrm{t_N}(D', F_{D'})$ for any strict quotient-z-isocrystal with Hodge-Pink lattice $(D', F_{D'}, \mathfrak{q}_{D'})$ of $(D, F_D, \mathfrak{q}_D)$.

If (D, F_D) is a z-isocrystal over k and $\mathfrak{q}$ is a Hodge-Pink lattice over L of (D, F_D) we say that $\mathfrak{q}$ is *weakly admissible* if the z-isocrystal with Hodge-Pink lattice $(D, F_D, \mathfrak{q})$ is weakly admissible.

For the proof of the equivalence of these conditions see [Pin, Proposition 4.4]. Our definition of weakly admissible is a modification of what is called semistable in [Pin].

Remark 2.5.8. Since it is enough to test weak admissibility for strict sub-objects (resp. quotient-objects) we only have to look at sub- (resp. quotient-) z-isocrystals of (D, F_D) because every such object induces a strict sub- (resp. quotient) z-isocrystal with Hodge-Pink lattice and every strict sub- (resp. quotient) z-isocrystal with Hodge-Pink lattice arises in this way.

Chapter 3

The relation of weak admissibility to GIT

3.1 Reminder on Geometric Invariant Theory

In this section we summarize some concepts and results from Geometric Invariant Theory which we will need later. The main reference for this is [GIT]. We consider schemes over a fixed field denoted by F and products and morphism are always defined over F if not otherwise stated.

3.1.1 Actions of an algebraic group

Definition 3.1.1. Let G be a group scheme over F. An *action of G on* a scheme X is a morphism of schemes $\sigma\colon G \times X \to X$ over F such that the diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{\mathrm{id}_G \times \sigma} & G \times X \\ \downarrow{\scriptstyle \mu \times \mathrm{id}_X} & & \downarrow{\scriptstyle \sigma} \\ G \times X & \xrightarrow{\quad\sigma\quad} & X \end{array}$$

commutes (μ denotes the multiplication morphism of G).

Definition 3.1.2. Let G be a linear algebraic group, let $A = \Gamma(G, \mathscr{O}_G)$, and let $\alpha\colon A \to A \otimes_F A$ (resp. $\beta\colon A \to F$) be the homomorphism defining the multiplication (resp. the identity). Let V be a vector space over F.

i) A *dual action of G on V* is a homomorphism of vector spaces

$$\hat{\sigma}\colon V \to A \otimes_F V$$

such that the diagram

$$\begin{array}{ccc} V & \xrightarrow{\quad\hat{\sigma}\quad} & A \otimes_F V \\ \downarrow{\scriptstyle \hat{\sigma}} & & \downarrow{\scriptstyle \alpha \otimes \mathrm{id}_V} \\ A \otimes_F V & \xrightarrow{\mathrm{id}_A \otimes \hat{\sigma}} & A \otimes_F A \otimes_F V \end{array}$$

commutes and

$$V \xrightarrow{\hat{\sigma}} A \otimes_F V \xrightarrow{\beta \otimes \mathrm{id}_V} V$$

is the identity.

ii) Let $\hat{\sigma}$ be a dual action of G on V. We say that $v \in V$ is *invariant under the action of* G if $\hat{\sigma}(v) = 1 \otimes v$.

3.1.2 Linearization of an invertible sheaf

We denote the projection morphism $G \times X \to X$ (resp. $G \times G \times X \to G \times X$) by pr_2 (resp. pr_{23}).

Definition 3.1.3. Let G be a linear algebraic group, X a scheme, $\sigma\colon G \times X \to X$ an action of G on X and $\mathscr{L}$ an invertible sheaf on X. A *G-linearization of $\mathscr{L}$* is an isomorphism of sheaves $\phi\colon \sigma^*\mathscr{L} \xrightarrow{\sim} \mathrm{pr}_2^*\mathscr{L}$ on $G \times X$ which satisfies the *cocycle condition*, i.e. the commutativity of the following diagram of morphisms of sheaves on $G \times G \times X$.

$$
\begin{tikzcd}
(\sigma \circ (\mathrm{id}_G \times \sigma))^*\mathscr{L} \arrow[r, "(\mathrm{id}_G \times \sigma)^*\phi"] \arrow[d, equal] & (\mathrm{pr}_2 \circ (\mathrm{id}_G \times \sigma))^*\mathscr{L} \arrow[r, equal] & (\sigma \circ \mathrm{pr}_{23})^*\mathscr{L} \arrow[d, "\mathrm{pr}_{23}^*\phi"] \\
(\sigma \circ (\mu \times \mathrm{id}_X))^*\mathscr{L} \arrow[r, "(\mu \times \mathrm{id}_X)^*\phi"] & (\mathrm{pr}_2 \circ (\mu \times \mathrm{id}_X))^*\mathscr{L} \arrow[r, equal] & (\mathrm{pr}_2 \circ \mathrm{pr}_{23})^*\mathscr{L}
\end{tikzcd}
\tag{3.1.1}
$$

The following relation between G-linearizations of invertible sheaves and G-actions on their corresponding geometric line bundles which are compatible with the action of G on X is taken from [GIT, Chapter 1 §3 p. 31]. With the notation of Definition 3.1.3 let $\pi\colon L \to X$ be the geometric line bundle corresponding to $\mathscr{L}$. The isomorphism ϕ corresponds to an isomorphism Φ of line bundles over $G \times X$:

$$(G \times X) \times_{\sigma,X} L \xleftarrow{\Phi} (G \times X) \times_{\mathrm{pr}_2,X} L = G \times L.$$

By composing Φ with the projection morphism $(G \times X) \times_{\sigma,X} L \to L$ we see that ϕ corresponds to a morphism of line bundles $\Sigma\colon G \times L \to L$ such that

$$
\begin{tikzcd}
G \times L \arrow[r, "\Sigma"] \arrow[d, "\mathrm{id}_G \times \pi"] & L \arrow[d, "\pi"] \\
G \times X \arrow[r, "\sigma"] & X
\end{tikzcd}
$$

commutes. The translation of the cocycle condition to this setup is given by the commutativity of the following cube:

$$\begin{tikzcd}
G \times G \times L \arrow[rr, "\mu\times \mathrm{id}_L"] \arrow[dr, "\mathrm{id}_G \times \Sigma"] \arrow[dd] & & G \times L \arrow[dr, "\Sigma"] \arrow[dd] & \\
& G \times L \arrow[rr, "\Sigma"] \arrow[dd] & & L \arrow[dd] \\
G \times G \times X \arrow[rr, "\mu\times \mathrm{id}_X"] \arrow[dr, "\mathrm{id}_G \times \sigma"] & & G \times X \arrow[dr, "\sigma"] & \\
& G \times X \arrow[rr, "\sigma"] & & X
\end{tikzcd}$$

Thus it is equivalent to consider G-linearizations of invertible sheaves on X or actions of G on their corresponding geometric line bundles which are compatible with the action of G on X.

Given an algebraic group G acting on a scheme X we denote by $\mathrm{Pic}^G(X)$ the group of isomorphism classes of invertible sheaves on X together with a G-linearization. If $f\colon X \to Y$ is a G-linear morphism of schemes on which G acts we get a morphism

$$f^*\colon \mathrm{Pic}^G(Y) \to \mathrm{Pic}^G(X).$$

Example 3.1.4 ([GIT, Chapter 1 §3 p. 32]). Let E denote a field and let $X = \operatorname{Spec} E$, $\mathscr{L} = \mathscr{O}_X$ and hence $L = \mathbb{A}^1$. Note that the group of automorphisms of $\mathbb{A}^1$ as a line bundle over X is $\mathbb{G}_m$ and therefore in this case G-linearizations are just characters $\chi\colon G \to \mathbb{G}_m$. The action Σ of G on $\mathbb{A}^1$ corresponding to χ is given by $\Sigma(a,z) = \chi(a)z$. Furthermore, if $\bar{\chi} \in \Gamma(G, \mathscr{O}_G^*)$ is the global section representing χ, we get an isomorphism $\phi\colon \mathscr{O}_G \otimes_F E \xrightarrow{\sim} \mathscr{O}_G \otimes_F E, g \mapsto \bar{\chi}^{-1} g$ which is the corresponding G-linearization.

For a scheme X equipped with an action of an algebraic group G we also get that every G-linearization ϕ of an invertible sheaf $\mathscr{L}$ on X induces a dual action on $\Gamma(X,\mathscr{L})$. It is given by the following morphism:

$$\Gamma(X,\mathscr{L}) \xrightarrow{\sigma^*} \Gamma(G \times X, \sigma^*\mathscr{L}) \xrightarrow{\phi} \Gamma(G \times X, \mathrm{pr}_2^*\mathscr{L}) \cong \Gamma(G,\mathscr{O}_G) \otimes \Gamma(X,\mathscr{L}).$$

The last isomorphism follows from the Künneth formula and the commutativity conditions for a dual action follow from the cocycle condition. By invariant sections of ϕ we mean sections $s \in \Gamma(X,\mathscr{L})$ which are invariant under this induced dual action of ϕ, i.e. s is mapped to $1 \otimes s$.

Concerning the uniqueness of G-linearizations we have the following

Proposition 3.1.5 ([GIT, Proposition 1.4])**.** *Let a connected algebraic group G act on a scheme X. Assume that there is no homomorphism of $G \times \Omega$ onto $\mathbb{G}_m \times \Omega$ (Ω an algebraic* *closure of F) and that X is geometrically reduced. Then each invertible sheaf $\mathscr{L}$ on X has at most one G-linearization.*

Later we will exploit the fact that in our case the assumption of this Proposition is not fulfilled. Thus for us it is possible to change the G-linearization to our needs.

3.1.3 Semi-stability

Definition 3.1.6. Let G be an algebraic group acting on a scheme X, $\mathscr{L}$ an invertible sheaf on X and ϕ a G-linearization of $\mathscr{L}$. Moreover let x be a geometric point of X. In this case x is called *semi-stable* if there exists a section $s \in \Gamma(X, \mathscr{L}^n)$ for some n, such that $s(x) \neq 0$, $X_s = \{x \in X \mid s(x) \neq 0\}$ is affine and s is an invariant section of $\phi_n \colon \sigma^*(\mathscr{L}^n) \to \mathrm{pr}_2^*(\mathscr{L}^n)$ induced by ϕ, i.e. $\phi_n(\sigma^*(s)) = \mathrm{pr}_2^*(s)$.

A numerical criterion

Definition 3.1.7. Let G be an algebraic group. A *1-PS* (*1-parameter subgroup*) *of* G is a homomorphism $\mathbb{G}_m \to G$.

Now suppose we are given an action σ of an algebraic group G on a scheme X which is proper over F. Let $\lambda \colon \mathbb{G}_m \to G$ be a 1-PS and let $x \in X(F)$ be an F-rational point. We consider the morphism

$$\mathbb{G}_m \to X, a \mapsto \lambda(a) \cdot x.$$

By identifying $\mathbb{G}_m$ with $\operatorname{Spec} F[\alpha, \alpha^{-1}]$ we can embed it into $\mathbb{A}^1 = \operatorname{Spec} F[\alpha]$:

$$\begin{tikzcd} \mathbb{G}_m \arrow[r] \arrow[d, hook] & X \\ \mathbb{A}^1 \arrow[ur, dashed, "\psi"'] & \end{tikzcd}$$

The dashed arrow is the unique morphism ψ making the diagram commutative. Its existence and uniqueness follows from the fact that the scheme X is proper over F and that the local ring of $\mathbb{A}^1$ at the origin (0) is a valuation ring. The point $\psi(0)$ is a fixed point under the action of $\mathbb{G}_m$ on X induced by λ. It is called the *specialization of* $\sigma(\lambda(\alpha), x)$ *when* $\alpha \to 0$. Next suppose we are given an invertible sheaf $\mathscr{L}$ on X together with a G-linearization. As we have seen in Example 3.1.4 the restriction of $\mathscr{L}$ to the fixed point $\psi(0)$ together with the induced $\mathbb{G}_m$-linearization is given by a character $t \in \mathbb{Z}$ of $\mathbb{G}_m$ ([GIT, Chapter 2 §1 p. 49]).

Definition 3.1.8. Let G be an algebraic group acting on a scheme X which is proper over F. Furthermore let $\mathscr{L}$ be an invertible sheaf on X together with a G-linearization. If $x \in X(F)$ and λ is a 1-PS of G we set (with the notation above)

$$\mu^{\mathscr{L}}(x, \lambda) = -t$$

and call it the *GIT-slope of λ in x*.

Remark 3.1.9. We have the following functorial properties of $\mu^{\mathscr{L}}(x, \lambda)$ ([GIT, Chapter 2 §1 p. 49], [DOR, p. 37]):

- $\mu^{\mathscr{L}}(\sigma(a, x), \lambda) = \mu^{\mathscr{L}}(x, a^{-1}\lambda a)$ for $a \in G(F)$.
- For fixed x and λ, $\mu^{\mathscr{L}}(x, \lambda)$ defines a homomorphism from $\mathrm{Pic}^G(X)$ to $\mathbb{Z}$.
- If $f : X \to Y$ is a G-linear morphism of schemes on which G acts, $\mathscr{L} \in \mathrm{Pic}^G(Y)$ and $x \in X(F)$ then
$$\mu^{f^*\mathscr{L}}(x, \lambda) = \mu^{\mathscr{L}}(f(x), \lambda).$$
- If $\sigma(\lambda(\alpha), x) \to y$ as $\alpha \to 0$ then $\mu^{\mathscr{L}}(x, \lambda) = \mu^{\mathscr{L}}(y, \lambda)$.

The connection of this concept to semi-stability is established by the Hilbert-Mumford criterion:

Theorem 3.1.10 ([GIT, Theorem 2.1])**.** *Let G be a reductive group acting on a scheme X which is proper over F and let $\mathscr{L} \in \mathrm{Pic}^G(X)$ such that $\mathscr{L}$ is ample. If F is algebraically closed and $x \in X(F)$ then*

$$x \textit{ is semi-stable } \Leftrightarrow \mu^{\mathscr{L}}(x, \lambda) \geq 0 \textit{ for all 1-PS's } \lambda.$$

Remark 3.1.11. In accordance to [GIT] we have defined $\mu^{\mathscr{L}}(x, \lambda)$ for F-valued points of X. We actually want to calculate the GIT-slope also for L-valued points, where L is a field extension of F. As one can check, there is no need to restrict this definition to F-valued points but one can also do this for L-valued points. By going through the examples below one sees that Proposition 3.1.13 and 3.1.16 stay valid in this extended context. Another way to handle L-valued points is to do base change to L and work with the given definition by considering rational points on X_L.

3.1.4 Example: The Grassmannian

Let $n \in \mathbb{N}$ and let V be an n-dimensional vector space over F. We are interested in the question of how we can define a GIT-slope for the Grassmannian. Since we have an action of $\mathrm{GL}(V)$ on the Grassmannian (induced by the action of $\mathrm{GL}(V)$ on V) our first task is to provide it with an invertible sheaf which is $\mathrm{GL}(V)$-linearized. We can embed the Grassmannian into projective space and therefore it is natural to search there for an obvious candidate and take the pullback. It turns out that there is no canonical choice for this construction.

We will proceed as follows: First we look at the case of projective space $\mathbb{P}^{n-1}$ and calculate the GIT-slope there. Then we describe two embeddings of the Grassmannian into projective space and their relation. We fix one of these embeddings but since there is no canonical choice we have to take care of this in our later applications.

A linearization of $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$

On $\mathbb{A}^n$ we have an action of the algebraic group GL_n. If x is an R-valued point of $\mathbb{A}^n$ (R an F-algebra), a matrix M acts on x as $M \cdot x$, where we view x as a column vector in $\mathrm{M}_{n\times 1}(R)$. There is also a natural action of the algebraic group PGL_n on $\mathbb{P}^{n-1}$ and we have a canonical morphism $\mathrm{GL}_n \to \mathrm{PGL}_n$. Thus we get an action of GL_n on $\mathbb{P}^{n-1}$. One way to view this action is as follows: The operation of GL_n on the affine cone $\mathbb{A}^n$ of $\mathbb{P}^{n-1}$ leaves the zero-section $(0) \subseteq \mathbb{A}^n$ invariant. Thus GL_n acts on $\mathbb{A}^n - (0)$ and the action of GL_n on $\mathbb{P}^{n-1}$ is the same as the one induced by the projection

$$\pi\colon \mathbb{A}^n - (0) \to \mathbb{P}^{n-1}.$$

Let L be the geometric line bundle corresponding to $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$. By [EGA II, Remarque (8.7.8)] L can be viewed as the blowing up of $\mathbb{A}^n$ in (0). As we have just mentioned GL_n acts on $\mathbb{A}^n$ and therefore we get a GL_n-action on L. It is compatible with the action of GL_n on $\mathbb{P}^{n-1}$ since the canonical morphism $L \to \mathbb{P}^{n-1}$ is obtained from π by [EGA II, Proposition (8.6.2)]. Altogether this shows that we have a GL_n-action on projective space $\mathbb{P}^{n-1}$ together with a GL_n-linearization of the invertible sheaf $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$.

Remark 3.1.12. We can change the GL_n-linearization by characters $\chi\colon \mathrm{GL}_n \to \mathbb{G}_m$ without changing the action of the algebraic group GL_n on $\mathbb{P}^{n-1}$ by letting a matrix M operate as $\chi(M) \cdot M$. This is due to the fact that the action of GL_n on $\mathbb{P}^{n-1}$ factors through PGL_n and is thus not affected, but on the other hand this modified action on $\mathbb{A}^n$ is different from the original one (c.f. Proposition 3.1.5).

The GIT-slope in the case of $\mathbb{P}^{n-1}$

Now let $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_n$ be a 1-PS. This induces a linear action of $\mathbb{G}_m$ on $\mathbb{P}^{n-1}$ plus a $\mathbb{G}_m$-linearization of $\mathscr{O}_{\mathbb{P}^{n-1}}(1)$. We can choose coordinates such that we can assume that we have a linear action of $\mathbb{G}_m$ on the affine cone $\mathbb{A}^n$ of $\mathbb{P}^{n-1}$ that is given by

$$\alpha \mapsto \left(\alpha^{t_i} \cdot \delta_{ij}\right)_{1 \leq i,j \leq n}$$

with $t_1, \ldots, t_n \in \mathbb{Z}$. With these notations we can formulate the following

Proposition 3.1.13 ([GIT, Proposition 2.3]). *Let $x \in \mathbb{P}^{n-1}(F)$ and let $\tilde{x}$ be a representative in $\mathbb{A}^n(F)$ of x. Assume moreover that we have fixed coordinates such that the action of λ is diagonalized as above and $\tilde{x} = (\tilde{x}_1, \ldots, \tilde{x}_n)$. In this situation we get that*

$$\mu^{\mathscr{O}_{\mathbb{P}^{n-1}}(1)}(x, \lambda) = \max\{-t_i \mid \tilde{x}_i \neq 0\}.$$

An action of $\mathrm{GL}(V)$ on the Grassmannian

Let $e \in \mathbb{N}$ and let $\mathscr{V} = V^{\sim \operatorname{Spec} F} = V \otimes \mathscr{O}_{\operatorname{Spec} F}$. If we denote $f^*\mathscr{V}$ by $\mathscr{V}_T$ for an F-scheme $f\colon T \to \operatorname{Spec} F$, recall that the T-valued points of $\mathrm{Grass}_{n-e}(\mathscr{V})$ are given by

$$\mathrm{Grass}_{n-e}(\mathscr{V})(T) = \{\mathscr{V}_T \to \underbrace{\mathscr{V}_T/\mathscr{U}}_{=\mathscr{M}} \mid \mathscr{M} \text{ finite locally free of rank } n-e\}.$$

Therefore, if R is an F-algebra and $V_R = V \otimes R$, we can view $\mathrm{Grass}_{n-e}(\mathscr{V})(R)$ as the set of surjective morphisms $V_R \to M$ such that M is Zariski locally isomorphic to $R^{\oplus n-e}$. On this set the group $\mathrm{GL}_R(V \otimes R)$ operates in the following way: An element $g \in \mathrm{GL}_R(V \otimes R)$ is mapped to the morphism

$$\begin{gathered} \mathrm{Grass}_{n-e}(\mathscr{V})(R) \to \mathrm{Grass}_{n-e}(\mathscr{V})(R). \\ (\pi\colon V_R \to M) \mapsto (\pi \circ g^{-1}\colon V_R \to M) \end{gathered}$$

There is also another way to describe this action. If $M = V_R/U$, the element g maps π to the quotient $V_R \to V_R/g(U)$. Indeed both theses actions coincide since $\ker(\pi \circ g^{-1}) = g(U)$. Thus we see that we have an action of the algebraic group $\mathrm{GL}(V)$ on $\mathrm{Grass}_{n-e}(\mathscr{V})$.

Now consider the space $\mathrm{Grass}_e(\mathscr{V}^\vee)$. Applying what we have just said to the case $V = V^\vee$, we see that there is an action of the algebraic group $\mathrm{GL}(V^\vee)$ on $\mathrm{Grass}_e(\mathscr{V}^\vee)$. Since we have a morphism of algebraic groups, given on R-valued points (R an F-algebra) by

$$\begin{gathered} \mathrm{GL}(V)(R) = \mathrm{GL}_R(V \otimes R) \to \mathrm{GL}_R((V \otimes R)^\vee) \cong \mathrm{GL}(V^\vee)(R), \\ g \mapsto (g^\vee)^{-1} \end{gathered} \tag{3.1.2}$$

we also have an action of $\mathrm{GL}(V)$ on $\mathrm{Grass}_e(\mathscr{V}^\vee)$ and for $g \in \mathrm{GL}(V)(R)$ it is explicitly given by the morphism

$$\begin{gathered} \mathrm{Grass}_e(\mathscr{V}^\vee)(R) \to \mathrm{Grass}_e(\mathscr{V}^\vee)(R). \\ (\phi\colon V_R^\vee \to M) \mapsto (\phi \circ g^\vee\colon V_R^\vee \to M) \end{gathered} \tag{3.1.3}$$

We will see that there is an isomorphism between $\mathrm{Grass}_{n-e}(\mathscr{V})$ and $\mathrm{Grass}_e(\mathscr{V}^\vee)$ that is compatible with the $\mathrm{GL}(V)$-actions on these schemes. Let T be an F-scheme. An element of $\mathrm{Grass}_{n-e}(\mathscr{V})$ gives rise to an exact sequence

$$0 \to \mathscr{U} \to \mathscr{V}_T \to \mathscr{V}_T/\mathscr{U} \to 0.$$

Taking the dual provides us with the exact sequence

$$0 \to (\mathscr{V}_T/\mathscr{U})^\vee \to \mathscr{V}_T^\vee \to \mathscr{U}^\vee \to 0.$$

Since $\mathscr{U}$ and hence $\mathscr{U}^\vee$ are locally free of rank e this exact sequence corresponds to an element of $\mathrm{Grass}_e(\mathscr{V}^\vee)(T)$ and hence we get a morphism

$$\delta\colon\ \mathrm{Grass}_{n-e}(\mathscr{V}) \to \mathrm{Grass}_e(\mathscr{V}^\vee).$$

We claim that this morphism is equivariant for the actions of the algebraic group $\mathrm{GL}(V)$ described above. In order to see this, we look at an F-algebra R, an element $g \in \mathrm{GL}(V)(R)$ and an exact sequence

$$0 \to U \xrightarrow{i} V_R \to V_R/U \to 0$$

of R-modules corresponding to an element $x \in \mathrm{Grass}_{n-e}(\mathscr{V})(R)$. By the second description of the $\mathrm{GL}(V)$-action on $\mathrm{Grass}_{n-e}(\mathscr{V})$, we know that $\delta(g\cdot x)$ corresponds to the exact sequence

$$0 \to (V_R/g(U))^\vee \to V_R^\vee \to g(U)^\vee \to 0$$

and on the other hand we know that $g \cdot \delta(x)$ corresponds to the exact sequence

$$0 \to \ker(i^\vee \circ g^\vee) \to V_R^\vee \xrightarrow{i^\vee \circ g^\vee} U^\vee \to 0.$$

These two sequences correspond to the same element in $\mathrm{Grass}_e(\mathscr{V}^\vee)$ since $\ker(i^\vee \circ g^\vee) = \ker((g \circ i)^\vee) = (\mathrm{coker}(g \circ i))^\vee = (V_R/g(U))^\vee$.

Linearizations on $\mathbb{P}(V)$ and $\mathbb{P}(V^\vee)$

We recall that $\mathbb{P}(V^\vee)$ is defined as $\mathrm{Grass}_1(\mathscr{V}^\vee)$ with the action of $\mathrm{GL}(V)$ given by (3.1.3). On the other hand $\mathbb{P}(V^\vee)$ can also be described as $\mathrm{Proj}(\mathrm{Sym}\, V^\vee)$. Its affine cone $\mathbb{V}(V^\vee) = \mathrm{Spec}(\mathrm{Sym}\, V^\vee)$ is also the scheme representing the functor $T \mapsto \mathrm{Hom}(\mathscr{V}_T^\vee, \mathscr{O}_T) = \Gamma(T, \mathscr{V}_T) = \mathscr{V} \otimes_F \Gamma(T, \mathscr{O}_T)$ and therefore its R-valued points, for an F-algebra R, correspond to $V \otimes R$ on which the R-valued points of $\mathrm{GL}(V)$ operate. This action of $\mathrm{GL}(V)$ leaves invariant the zero section Z of $\mathbb{V}(V^\vee)$. Therefore the projection morphism $\mathbb{V}(V^\vee) - Z \to \mathbb{P}(V^\vee)$ induces a second $\mathrm{GL}(V)$-action on $\mathbb{P}(V^\vee)$. Both actions are in fact equal as can be seen from (3.1.3).

By choosing a basis for V we identify $\mathrm{GL}(V)$ with GL_n. Moreover the dual basis induces isomorphisms $V^\vee \cong F^{\oplus n}$ and $\operatorname{Sym} V^\vee \xrightarrow{\sim} F[T_1, \dots, T_n]$. Thus we have chosen coordinates for $\mathbb{P}(V^\vee)$ (resp. $\mathbb{V}(V^\vee)$) and can identify it with projective space $\mathbb{P}^{n-1} = \operatorname{Proj}(F[T_1, \dots, T_n])$ (resp. affine space $\mathbb{A}^n = \operatorname{Spec}(F[T_1, \dots, T_n])$. The actions of $\mathrm{GL}(V)$ on $\mathbb{V}(V^\vee)$ and $\mathbb{P}(V^\vee)$ described above correspond exactly to the actions of GL_n on $\mathbb{A}^n$ and $\mathbb{P}^{n-1}$. The diagonalization of a morphism $\lambda\colon \mathbb{G}_m \to \mathrm{GL}_n$ corresponds under these isomorphisms to the decomposition

$$V = \bigoplus_{i \in \mathbb{Z}} V(i)$$

of V into weight spaces induced by the action of $\mathbb{G}_m$ on V.

Thus providing a representation of $\mathrm{GL}(V)$ on V gives rise to an action of $\mathrm{GL}(V)$ on $\mathbb{P}(V^\vee)$ plus a $\mathrm{GL}(V)$-linearization of $\mathscr{O}_{\mathbb{P}(V^\vee)}(1)$; see the description before Remark 3.1.12. Conversely we have seen that a $\mathrm{GL}(V)$-linearization of $\mathscr{O}_{\mathbb{P}(V^\vee)}(1)$ gives rise to a dual action

$$\underbrace{\Gamma(\mathbb{P}(V^\vee), \mathscr{O}_{\mathbb{P}(V^\vee)}(1))}_{=V^\vee} \to \Gamma(\mathrm{GL}(V), \mathscr{O}_{\mathrm{GL}(V)}) \otimes \underbrace{\Gamma(\mathbb{P}(V^\vee), \mathscr{O}_{\mathbb{P}(V^\vee)}(1))}_{=V^\vee}.$$

Actually this defines on $V^\vee$ a left co-module structure which corresponds to a right co-module structure on $V^\vee$ with respect to $\Gamma(\mathrm{GL}(V^\vee), \mathscr{O}_{\mathrm{GL}(V^\vee)})$ and this gives a representation of $\mathrm{GL}(V^\vee)$ on $V^\vee$ and its transpose gives back the action of $\mathrm{GL}(V)$ on V.

Similarly we can consider the projective space $\mathbb{P}(V)$ and its affine cone $\mathbb{V}(V)$. In this case we have $\mathbb{V}(V)(R) = V^\vee \otimes R$ and hence we see that we have an action of $\mathrm{GL}(V^\vee)$ on $\mathbb{P}(V)$ plus a $\mathrm{GL}(V^\vee)$-linearization of $\mathscr{O}_{\mathbb{P}(V)}(1)$. It is also possible to define an action of $\mathrm{GL}(V)$ on $\mathbb{P}(V)$ and a $\mathrm{GL}(V)$-linearization of $\mathscr{O}_{\mathbb{P}(V)}(1)$ via the isomorphism (3.1.2). This $\mathrm{GL}(V)$-linearization is induced by the action of $\mathrm{GL}(V)$ on $\mathbb{V}(V)$ that is given by

$$\begin{aligned} \underbrace{\mathbb{V}(V)(R)}_{=(V\otimes R)^\vee} &\to \underbrace{\mathbb{V}(V)(R)}_{=(V\otimes R)^\vee} \\ x &\mapsto (g^\vee)^{-1}(x) \end{aligned}$$

for every $g \in \mathrm{GL}(V)(R)$. In the following we will always mean this linearization when we speak of a $\mathrm{GL}(V)$-linearization of $\mathscr{O}_{\mathbb{P}(V)}(1)$.

Embeddings of the Grassmannian into projective space

Let $e \in \mathbb{N}$. We have the following morphisms of algebraic groups

$$\mathrm{GL}(V) \to \mathrm{GL}(V^\vee)$$

described in (3.1.2) and

$$\mathrm{GL}(V) \to \mathrm{GL}(\bigwedge^e V)$$

which sends $g \in GL(V)(R)$ to $\wedge^e g$. Hence we get actions of $\mathrm{GL}(V)$ on $V^\vee$ and $\bigwedge^e V$. Combining these we can, for example, define an action of $\mathrm{GL}(V)$ on $\bigwedge^e V^\vee$ as $x \mapsto g \cdot x = (\wedge^e (g^\vee)^{-1})(x)$ or on $(\bigwedge^e V)^\vee$ as $x \mapsto g \cdot x = ((\wedge^e g)^\vee)^{-1}(x)$.

Before we look at embeddings of the Grassmannian into projective space we summarize some canonical identifications about exterior products of vector spaces and their duals. We make sure that these identifications respect the corresponding $\mathrm{GL}(V)$-actions. We have a canonical pairing

$$\begin{aligned} \Psi\colon \bigwedge^{n-e} V^\vee \times \bigwedge^{n-e} V &\to F \\ (\lambda_1 \wedge \ldots \wedge \lambda_{n-e}, v_1 \wedge \ldots v_{n-e}) &\mapsto \det(\lambda_i(v_j))_{i,j} \end{aligned}$$

which gives rise to an isomorphism

$$\psi\colon \bigwedge^{n-e} V^\vee \xrightarrow{\sim} \left(\bigwedge^{n-e} V\right)^\vee .$$

Since $\Psi(g \cdot x, g \cdot y) = \Psi(x, y)$ the morphism ψ becomes $\mathrm{GL}(V)$-equivariant with respect to the induced $\mathrm{GL}(V)$-actions described above. In the following we will therefore identify $\bigwedge^{n-e} V^\vee$ and $(\bigwedge^{n-e} V)^\vee$ together with their actions of $\mathrm{GL}(V)$.

Moreover we also have a pairing

$$\begin{aligned} \Phi\colon \bigwedge^{e} V \times \bigwedge^{n-e} V &\to \bigwedge^{n} V \\ (x, y) &\mapsto x \wedge y \end{aligned}$$

and this leads to an isomorphism

$$\phi\colon \bigwedge^{e} V \xrightarrow{\sim} \underbrace{\left(\bigwedge^{n-e} V\right)^\vee}_{=\bigwedge^{n-e} V^\vee} \otimes \bigwedge^{n} V. \tag{3.1.4}$$

With the $\mathrm{GL}(V)$-actions described above we have $\Phi(g\cdot x, g\cdot y) = g \cdot \Phi(x,y) = \det(g)\Phi(x,y)$. Therefore the morphism ϕ becomes $\mathrm{GL}(V)$-equivariant if we let an element g of $\mathrm{GL}(V)$ act on $\bigwedge^{n-e} V^\vee \otimes \bigwedge^n V$ as $x \otimes y \mapsto g \cdot x \otimes g \cdot y = \wedge^{n-e}(g^\vee)^{-1}(x) \otimes \det(g) y$.

The embedding $\iota_1\colon \mathrm{Grass}_{n-e}(\mathscr{V}) \to \mathbb{P}(\bigwedge^e V^\vee)$: We define ι_1 as the composition of the isomorphism $\mathrm{Grass}_{n-e}(\mathscr{V}) \xrightarrow{\sim} \mathrm{Grass}_e(\mathscr{V}^\vee)$ with the Plücker-morphism $\mathrm{Grass}_e(\mathscr{V}^\vee) \to \mathbb{P}(\bigwedge^e V^\vee)$ as it is described in [EGA Inew, §9.8]. Hence for an F-scheme T, an element in $\mathrm{Grass}_{n-e}(\mathscr{V})(T)$, corresponding to an exact sequence

$$0 \to \mathscr{U} \to \mathscr{V}_T \to \mathscr{V}_T/\mathscr{U} \to 0 \tag{3.1.5}$$

is mapped to the quotient

$$\bigwedge^{e}(\mathscr{V}_T)^\vee \to \bigwedge^{e}\mathscr{U}^\vee.$$

The embedding ι_2: $\mathrm{Grass}_{n-e}(\mathscr{V}) \to \mathbb{P}(\bigwedge^{n-e} V)$: The second embedding of $\mathrm{Grass}_{n-e}(\mathscr{V})$ into projective space is just the usual Plücker-morphism. Hence with the same notation as above the element (3.1.5) is mapped to the quotient

$$\bigwedge^{n-e}\mathscr{V}_T \to \bigwedge^{n-e}\mathscr{V}_T/\mathscr{U}.$$

Comparison of ι_1 and ι_2: Let $\mathscr{A}$ be a basis of $\bigwedge^n V$. This choice of a basis for $\bigwedge^n V$ induces, for every F-algebra R, an isomorphism $\bigwedge^{n-e} V^\vee \otimes \bigwedge^n V \otimes R \xrightarrow{\sim} \bigwedge^{n-e} V^\vee \otimes R$ and hence we get, by composing with $\phi \otimes \mathrm{id}_R$: $\bigwedge^e V \otimes R \to \bigwedge^{n-e} V^\vee \otimes \bigwedge^n V \otimes R$, an isomorphism

$$\alpha_R\colon \underbrace{\bigwedge^{e} V \otimes R}_{=\mathbb{V}(\bigwedge^e V^\vee)(R)} \xrightarrow{\sim} \underbrace{\bigwedge^{n-e} V^\vee \otimes R}_{=\mathbb{V}(\bigwedge^{n-e} V)(R)}.$$

If we choose a different basis $\mathscr{A}'$ of $\bigwedge^n V$ we get an isomorphism α'_R that differs from α_R by multiplication with a unit in F. The morphisms α_R induce an isomorphism

$$\alpha\colon \mathbb{V}(\bigwedge^{e} V^\vee) \xrightarrow{\sim} \mathbb{V}(\bigwedge^{n-e} V). \tag{3.1.6}$$

On the other hand, consider the dual $\phi^\vee$: $\bigwedge^{n-e} V \otimes \bigwedge^n V^\vee \xrightarrow{\sim} \bigwedge^e V^\vee$ of the morphism ϕ. We compose it with the isomorphism $\bigwedge^{n-e} V \xrightarrow{\sim} \bigwedge^{n-e} V \otimes \bigwedge^n V^\vee$ which is again induced by the basis $\mathscr{A}$. Tensoring with an F-algebra R we obtain isomorphisms β_R: $\bigwedge^{n-e} V \otimes R \xrightarrow{\sim} \bigwedge^e V^\vee \otimes R$. We define an isomorphism $\beta\colon \mathbb{P}(\bigwedge^e V^\vee) \xrightarrow{\sim} \mathbb{P}(\bigwedge^{n-e} V)$ on R-valued points (R an F-algebra) as

$$\begin{aligned} \beta(R)\colon \mathbb{P}(\bigwedge^{e} V^\vee)(R) &\to \mathbb{P}(\bigwedge^{n-e} V)(R). \\ (\pi\colon \bigwedge^{e} V^\vee \otimes R \to M) &\mapsto (\pi\circ\beta_R\colon \bigwedge^{n-e} V \otimes R \to M) \end{aligned} \tag{3.1.7}$$

As above, a different choice of a basis $\mathscr{A}'$ of $\bigwedge^n V$ would induce isomorphisms β'_R: $\bigwedge^{n-e} V \otimes R \to \bigwedge^e V^\vee \otimes R$ that differ from β_R by multiplication with a unit in F. Therefore $\ker(\pi\circ\beta_R) = \ker(\pi\circ\beta'_R)$ and hence $\pi\circ\beta_R$ and $\pi\circ\beta'_R$ are the same in $\mathbb{P}(\bigwedge^{n-e} V)(R)$. This shows that the isomorphism $\beta\colon \mathbb{P}(\bigwedge^e V^\vee) \xrightarrow{\sim} \mathbb{P}(\bigwedge^{n-e} V)$ is independent of the chosen basis $\mathscr{A}$.

The morphisms α and β are related in the following way: If we denote the zero section of $\mathbb{V}(\bigwedge^e V^\vee)$ (resp. $\mathbb{V}(\bigwedge^{n-e} V)$) by Z (resp. $\tilde{Z}$) the morphism α maps $\mathbb{V}(\bigwedge^e V^\vee) - Z$

isomorphically onto $\mathbb{V}(\bigwedge^{n-e} V) - \tilde{Z}$ and, by the definition of the projection maps $\mathbb{V}(\bigwedge^{e} V^\vee) - Z \to \mathbb{P}(\bigwedge^{e} V^\vee)$ and $\mathbb{V}(\bigwedge^{n-e}) - \tilde{Z} \to \mathbb{P}(\bigwedge^{n-e})$, the diagram

$$\begin{tikzcd} \mathbb{V}(\bigwedge^{e} V^\vee) - Z \arrow[r, "\alpha"] \arrow[d] & \mathbb{V}(\bigwedge^{n-e} V) - \tilde{Z} \arrow[d] \\ \mathbb{P}(\bigwedge^{e} V^\vee) \arrow[r, "\beta"] & \mathbb{P}(\bigwedge^{n-e} V) \end{tikzcd} \tag{3.1.8}$$

commutes. Moreover we claim that $\iota_2 = \beta \circ \iota_1$, i.e. that the diagram

$$\begin{tikzcd} & \mathbb{P}(\bigwedge^{e} V^\vee) \arrow[d, "\beta"] \\ \mathrm{Grass}_{n-e}(\mathscr{V}) \arrow[ur, "\iota_1"] \arrow[r, "\iota_2"'] & \mathbb{P}(\bigwedge^{n-e} V) \end{tikzcd} \tag{3.1.9}$$

commutes. In order to show this, let T be an F-scheme and let $x = \mathscr{V}_T \to \mathscr{V}_T/\mathscr{U} \in \mathrm{Grass}_{n-e}(\mathscr{V})(T)$. We have seen that

$$\iota_1(x) = (\bigwedge^{e} \mathscr{V}_T^\vee \to \bigwedge^{e} \mathscr{U}^\vee)$$

and

$$\iota_2(x) = (\bigwedge^{n-e} \mathscr{V}_T \to \bigwedge^{n-e} \mathscr{V}_T/\mathscr{U}).$$

The element $\beta(\iota_1(x))$ can also be constructed in the following way: The isomorphism $\phi^\vee$ induces a map $\mathbb{P}(\bigwedge^{e} V^\vee) \xrightarrow{\sim} \mathbb{P}(\bigwedge^{n-e} V \otimes \bigwedge^{n} V^\vee)$ and the element $\iota_1(x)$ is send to $\bigwedge^{n-e} \mathscr{V}_T \otimes \bigwedge^{n} \mathscr{V}_T^\vee \to \bigwedge^{e} \mathscr{U}^\vee \in \mathbb{P}(\bigwedge^{n-e} V \otimes \bigwedge^{n} V^\vee)(T)$. This element corresponds under the canonical isomorphism $\mathbb{P}(\bigwedge^{n-e} V \otimes \bigwedge^{n} V^\vee) \xrightarrow{\sim} \mathbb{P}(\bigwedge^{n-e} V)$ to

$$(\bigwedge^{n-e} \mathscr{V}_T \to \bigwedge^{e} \mathscr{U}^\vee \otimes \bigwedge^{n} \mathscr{V}_T) = \beta(\iota_1(x)).$$

In order to show that the diagram (3.1.9) commutes, we have to find an isomorphism $\xi\colon \bigwedge^{n-e} \mathscr{V}_T/\mathscr{U} \xrightarrow{\sim} \bigwedge^{e} \mathscr{U}^\vee \otimes \bigwedge^{n} \mathscr{V}_T$ such that $\beta(\iota_1(x)) = \xi \circ \iota_2(x)$. By tensoring everything with the invertible module $\bigwedge^{e} \mathscr{U}$ it remains to show the following statement: There is an isomorphism $\tilde{\xi}\colon \bigwedge^{e} \mathscr{U} \otimes \bigwedge^{n-e} \mathscr{V}_T/\mathscr{U} \xrightarrow{\sim} \bigwedge^{n} \mathscr{V}_T$ making the diagram

$$\begin{tikzcd} \bigwedge^{e} \mathscr{U} \otimes \bigwedge^{n-e} \mathscr{V}_T \arrow[r, "\rho_1"] \arrow[d, "\rho_2"'] & \bigwedge^{n} \mathscr{V}_T \\ \bigwedge^{e} \mathscr{U} \otimes \bigwedge^{n-e} \mathscr{V}_T/\mathscr{U} \arrow[ur, "\tilde{\xi}"'] & \end{tikzcd} \tag{3.1.10}$$

commutative, where ρ_1 and ρ_2 are induced by $\beta(\iota_1(x))$ and $\iota_2(x)$. In order to show this statement, we use the following Lemma 3.1.14:

Let R be a ring and let L (resp. M, resp. N) be a finitely generated projective R-module of rank e (resp. n, resp. $n-e$) such that we have an exact sequence

$$0 \to L \to M \xrightarrow{\varepsilon} N \to 0.$$

Let

$$\begin{aligned} r_1\colon \bigwedge^{e} L \otimes \bigwedge^{n-e} M &\to \bigwedge^{n} M \\ x \otimes y &\mapsto x \wedge y \end{aligned}$$

and let $(r_2\colon \bigwedge^e L \otimes \bigwedge^{n-e} M \to \bigwedge^e L \otimes \bigwedge^{n-e} N) = \mathrm{id}_{\bigwedge^e L} \otimes \wedge^{n-e}\varepsilon$. Every section $s\colon N \to M$ of ε induces a morphism $r_s\colon \bigwedge^e L \otimes \bigwedge^{n-e} N \to \bigwedge^n M$ making the diagram

$$\begin{tikzcd} \bigwedge^e L \otimes \bigwedge^{n-e} M \arrow[r, "r_1"] \arrow[d, "r_2"'] & \bigwedge^n M \\ \bigwedge^e L \otimes \bigwedge^{n-e} N \arrow[ur, "r_s"'] & \end{tikzcd}$$

commutative.

Lemma 3.1.14. *With the above notations the morphism $r_s\colon \bigwedge^e L \otimes \bigwedge^{n-e} N \to \bigwedge^n M$ is independent of the section s and it is an isomorphism.*

Proof. The morphism $r_s\colon \bigwedge^e L \otimes \bigwedge^{n-e} N \to \bigwedge^n M$ is defined as $r_1 \circ (\mathrm{id}_{\bigwedge^e L} \otimes \wedge^{n-e} s)$. If $s'\colon N \to M$ is a second section of ε, we know that $\mathrm{im}(s-s') \subseteq L$ and hence $\mathrm{im}(\wedge^{n-e}(s-s')) \subseteq \bigwedge^{n-e} L$. Since $\mathrm{rk}\, L = e$ we see that $r_s - r_{s'} = r_{s-s'} = 0$, by the definition of r_1. We define an inverse of the morphism r_s. Let $x_1 \wedge \ldots \wedge x_n \in \bigwedge^n M$. Note that each $x_i \in M$ can be written uniquely as $x_i = y_i + s(z_i)$, with $y_i \in L$ and $z_i \in N$. Furthermore, since $\mathrm{rk}\, L = e$ and $\mathrm{rk}\, N = n-e$ we get that

$$x_1 \wedge \ldots \wedge x_n = \sum_I (-1)^{\nu(I)} y_{i_1} \wedge \ldots \wedge y_{i_e} \wedge s(z_{j_1}) \wedge \ldots \wedge s(z_{j_{n-e}}),$$

where $I = \{i_1, \ldots, i_e\}$ with $i_1 < \ldots < i_e$ runs over all subsets of $\{1, \ldots, n\}$ such that $\sharp I = e$ and $J = \{j_1, \ldots, j_{n-e}\}$ with $j_1 < \ldots < j_{n-e}$ such that $I \cup J = \{1, \ldots, n\}$. Moreover $\nu(I) \in \{0,1\}$ is defined such that $(-1)^{\nu(I)} y_{i_1} \wedge \ldots \wedge y_{i_e} \wedge s(z_{j_1}) \wedge \ldots \wedge s(z_{j_{n-e}}) = w_1 \wedge \ldots \wedge w_n$ with $w_i = y_i$ if $i \in I$ and $w_i = s(z_i)$ if $i \in J$. With these notations it is now easy to see that there exists a map

$$\begin{aligned} q_s\colon \bigwedge^{n} M &\to \bigwedge^{e} L \otimes \bigwedge^{n-e} N \\ x_1 \wedge \ldots \wedge x_n &\mapsto \sum_I (-1)^{\nu(I)} y_{i_1} \wedge \ldots \wedge y_{i_e} \otimes z_{j_1} \wedge \ldots \wedge z_{j_{n-e}} \end{aligned}$$

satisfying $r_s \circ q_s = \mathrm{id}_{\bigwedge^n M}$ and $q_s \circ r_s = \mathrm{id}_{\bigwedge^e L \otimes \bigwedge^{n-e} N}$. □

We return to the problem of defining an isomorphism $\tilde{\xi}\colon \bigwedge^e \mathscr{U} \otimes \bigwedge^{n-e} \mathscr{V}_T/\mathscr{U} \xrightarrow{\sim} \bigwedge^n \mathscr{V}_T$. Working locally, we see that the morphisms ρ_1 and ρ_2 correspond to r_1 and r_2 of the above Lemma 3.1.14. Hence this Lemma shows that the isomorphism exists locally and since it is independent of the chosen section these morphisms glue to the desired isomorphism $\tilde{\xi}\colon \bigwedge^e \mathscr{U} \otimes \bigwedge^{n-e} \mathscr{V}_T/\mathscr{U} \xrightarrow{\sim} \bigwedge^n \mathscr{V}_T$ making the diagram (3.1.10) commutative.

Now we compare the two induced $\mathrm{GL}(V)$-linearized invertible sheaves, obtained from the embeddings ι_1 and ι_2 on $\mathrm{Grass}_{n-e}(\mathscr{V})$. We denote these sheaves by $\mathscr{L}_1 = \iota_1^* \mathscr{O}_{\mathbb{P}(\bigwedge^e V^\vee)}(1)$ and $\mathscr{L}_2 = \iota_2^* \mathscr{O}_{\mathbb{P}(\bigwedge^{n-e} V)}(1)$. From the isomorphisms α and β and the commutativity of the diagrams (3.1.8) and (3.1.9) it follows that $\mathscr{L}_1$ and $\mathscr{L}_2$ are isomorphic as sheaves. Moreover the isomorphism $\beta\colon \mathbb{P}(\bigwedge^e V^\vee) \xrightarrow{\sim} \mathbb{P}(\bigwedge^{n-e} V)$ is $\mathrm{GL}(V)$-equivariant, but the $\mathrm{GL}(V)$-linearizations of $\mathscr{L}_1$ and $\mathscr{L}_2$ are different. This can be seen by comparing the $\mathrm{GL}(V)$-actions on the affine cones $\mathbb{V}(\bigwedge^e V^\vee)$ and $\mathbb{V}(\bigwedge^{n-e} V)$. First note that the action on $\mathbb{V}(\bigwedge^e V^\vee)$ induces, via $\alpha\colon \mathbb{V}(\bigwedge^e V^\vee) \xrightarrow{\sim} \mathbb{V}(\bigwedge^{n-e} V)$, an action on $\mathbb{V}(\bigwedge^{n-e} V)$. This induced action is independent of the chosen basis for $\bigwedge^n V$ since α and α' differ by multiplication with a unit in F and this unit multiplication commutes with the $\mathrm{GL}(V)$-action. The induced action is explicitly given, for an F-algebra R and $g \in \mathrm{GL}(V)(R)$, by the morphisms

$$\underbrace{\mathbb{V}(\bigwedge^{n-e} V)(R)}_{=\bigwedge^{n-e}(V\otimes R)^\vee} \to \underbrace{\mathbb{V}(\bigwedge^{n-e} V)(R)}_{=\bigwedge^{n-e}(V\otimes R)^\vee}.$$
$$x \mapsto \det(g) \cdot \wedge^{n-e}(g^\vee)^{-1}(x)$$

This follows from the definition of the morphism α given before (3.1.6) and by the description of $\mathrm{GL}(V)$-equivariant morphism $\phi\colon \bigwedge^e V \xrightarrow{\sim} \bigwedge^{n-e} V^\vee \otimes \bigwedge^n V$ after (3.1.4). This action differs from the canonical action which is given by

$$\mathbb{V}(\bigwedge^{n-e} V)(R) \to \mathbb{V}(\bigwedge^{n-e} V)(R).$$
$$x \mapsto \wedge^{n-e}(g^\vee)^{-1}(x)$$

Thus we see that the isomorphism $\alpha\colon \mathbb{V}(\bigwedge^e V^\vee) \xrightarrow{\sim} \mathbb{V}(\bigwedge^{n-e} V)$ is not $\mathrm{GL}(V)$-equivariant and hence the induced invertible sheaves $\mathscr{L}_1$ and $\mathscr{L}_2$ are not isomorphic as $\mathrm{GL}(V)$-linearized invertible sheaves.

Convention 3.1.15. We have just seen that there is no canonical invertible sheaf together with a $\mathrm{GL}(V)$-linearization on $\mathrm{Grass}_{n-e}(\mathscr{V})$. Therefore we will make the following choice: In the following we will work with the embedding $\iota_1\colon \mathrm{Grass}_{n-e}(\mathscr{V}) \to \mathbb{P}(\bigwedge^e V^\vee)$ together with the $\mathrm{GL}(V)$-linearization of $\mathscr{O}_{\mathbb{P}(\bigwedge^e V^\vee)}(1)$ induced by the canonical $\mathrm{GL}(V)$-action on the affine cone $\mathbb{V}(\bigwedge^e V^\vee)$. Since in our applications we will also use the embedding $\iota_2\colon \operatorname{Grass}_{n-e}(\mathscr{V}) \hookrightarrow \mathbb{P}(\bigwedge^{n-e} V)$ and hence the affine cone $\mathbb{V}(\bigwedge^{n-e} V)$, we will make these objects compatible with the above choice by changing the $\operatorname{GL}(V)$-action on $\mathbb{V}(\bigwedge^{n-e} V)$ to the one induced by the isomorphism $\alpha\colon \mathbb{V}(\bigwedge^e V^\vee) \xrightarrow{\sim} \mathbb{V}(\bigwedge^{n-e} V)$ and similarly changing the $\operatorname{GL}(V)$-linearization of $\mathscr{O}_{\mathbb{P}(\bigwedge^{n-e} V)}(1)$. In other words, we can say that this change of linearization makes $\mathscr{L}_1$ and $\mathscr{L}_2$ isomorphic as $\operatorname{GL}(V)$-linearized invertible sheaves on $\operatorname{Grass}_{n-e}(\mathscr{V})$.

The GIT-slope for linear subspaces

By choosing a basis for V we get coordinates such that $\mathbb{P}(\bigwedge^e V^\vee) \simeq \mathbb{P}^{N-1}$ and $\mathbb{V}(\bigwedge^e V^\vee) \simeq \mathbb{A}^N$, where $N = \binom{n}{e}$. Using Convention 3.1.15, we see that we have an embedding $\iota\colon \operatorname{Grass}_{n-e}(\mathscr{V}) \hookrightarrow \mathbb{P}^{N-1}$ and a $\operatorname{GL}(V)$-linearized invertible sheaf $\mathscr{L} = \iota^*\mathscr{O}_{\mathbb{P}^{N-1}}(1)$. Moreover with these coordinates, if $g \in \operatorname{GL}(V)(F)$ acts on V via a matrix A, then the action of g on the affine cone $\mathbb{A}^N$ of $\mathbb{P}^{N-1}$ is given by the matrix $\bigwedge^e(A)$. We write

$$\mathbb{P}^{N-1} = \operatorname{Proj} F[T_{i_1,\dots,i_e}]$$

where $i_1, \dots, i_e \in \{1, \dots, n\}$ with $i_1 < \dots < i_e$. Now let $U \in \operatorname{Grass}_{n-e}(\mathscr{V})(F)$ be a linear subspace of V of dimension e and let us denote by $T_{i_1,\dots,i_e}(U)$ its homogeneous coordinates. Suppose that we are given a 1-PS subgroup of $\operatorname{GL}(V)$ which is given by the matrices

$$\alpha \mapsto (\alpha^{t_i} \cdot \delta_{ij})_{1 \le i,j \le n}.$$

With these notations it follows from Proposition 3.1.13 that

$$\mu^{\mathscr{L}}(U, \lambda) = \max\{-t_{i_1} - \dots - t_{i_e} \mid T_{i_1,\dots,i_e}(U) \neq 0\}.$$

A further analysis, which is done in [GIT, Chapter 4 §4 p.87] for SL_n but is also true for GL_n with the same arguments, leads to the following description of $\mu^{\mathscr{L}}(U, \lambda)$: Let

$$V = \bigoplus_{i \in \mathbb{Z}} V(i)$$

be the decomposition induced by λ into weight spaces. This grading gives rise to a filtration of V, which we denote by $\operatorname{Fil}^\bullet_\lambda V$, in the following way:

$$\operatorname{Fil}^i_\lambda V = \bigoplus_{j \ge i} V(j).$$

We denote $\operatorname{gr}^i_{\operatorname{Fil}^\bullet_\lambda}(V)$ by $\operatorname{gr}^i_\lambda(V)$.

Proposition 3.1.16. *Let $\mathscr{L}$, U and λ be as above. With the same notation we have*

$$-\mu^{\mathscr{L}}(U, \lambda) = \sum_{i=1}^{n} i \cdot \dim(\operatorname{gr}^i_\lambda(V)|U) = \deg(U, \operatorname{Fil}^\bullet_\lambda(V)|U). \tag{3.1.11}$$

Remark 3.1.17. In [GIT] the above Proposition 3.1.16 is proved by first considering only 1-PS λ which induce decompositions of V into 1-dimensional weight spaces. That is λ is given by matrices

$$\lambda(\alpha) = (\alpha^{t_i} \cdot \delta_{ij})_{1 \leq i,j \leq n}$$

with $t_1 > \ldots > t_n$. After obtaining the result for such λ it is remarked ([GIT, p. 88]) that the same result is also true for general λ as one could also see by modifying the proof.

3.2 Actions of the algebraic group J

Let $r \in \mathbb{N}$. We fix a z-isocrystal $\underline{D} = (D, F_D)$ of rank r. Consider the linear algebraic group J over $\mathbb{F}_q(\!(z)\!)$ whose group of $\mathbb{F}_q(\!(z)\!)$-valued points is the automorphism group of the z-isocrystal $\underline{D}$. It was defined in Section 1.3 where we already discussed it.

3.2.1 A representation of $J_{k(\!(z)\!)}$ on D

There is a natural action ρ of $J_{k(\!(z)\!)} = J \times_{\mathbb{F}_q(\!(z)\!)} k(\!(z)\!)$ on the $k(\!(z)\!)$-vector space D, namely: Let B be a $k(\!(z)\!)$-algebra. We have a natural morphism

$$\begin{aligned}\rho(B)\colon J_{k(\!(z)\!)}(B) &\subseteq \mathrm{GL}_{k(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} B}\left(D \otimes_{\mathbb{F}_q(\!(z)\!)} B\right) \\ &\to \mathrm{GL}_B\left(D \otimes_{k(\!(z)\!)} B\right) = \mathrm{GL}(D)(B)\end{aligned} \tag{3.2.1}$$

which is induced by the multiplication map

$$\begin{aligned} k(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} B &\to B. \\ f \otimes b &\mapsto fb \end{aligned}$$

This morphism $J_{k(\!(z)\!)} \to \mathrm{GL}(D)$ of algebraic groups is injective. We will give a direct argument in the case that the z-isocrystal is defined over some finite field extension of $\mathbb{F}_q$ which we can assume if the z-isocrystal is a split semi-simple z-isocrystal since in this case we can find $t \in \mathbb{N}$ such that there exists a canonical model over $\mathbb{F}_{q^t}$. For the general case see [Kot, Appendix A]. So let us assume that the z-isocrystal $\underline{D}$ is defined over $k = \mathbb{F}_{q^t}$ in our argumentation. The isomorphism

$$\begin{aligned} \mathbb{F}_{q^t}(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} \mathbb{F}_{q^t}(\!(z)\!) &\xrightarrow{\sim} \prod_{i \in \mathbb{Z}/t\mathbb{Z}} \mathbb{F}_{q^t}(\!(z)\!) \\ f \otimes g &\mapsto (\sigma^i(f) g)_i \end{aligned}$$

gives rise to an isomorphism

$$D \otimes_{\mathbb{F}_q(\!(z)\!)} \mathbb{F}_{q^t}(\!(z)\!) \xrightarrow{\sim} \prod_{i \in \mathbb{Z}/t\mathbb{Z}} D_i$$

where $D_i = D$ with $\mathbb{F}_{q^t}(\!(z)\!)$-module structure $\mathbb{F}_{q^t}(\!(z)\!) \times D_i \to D_i$, $(f,d) \mapsto \sigma^{-i}(f)d$. Even more generally for every $\mathbb{F}_{q^t}(\!(z)\!)$-algebra B we get

$$\begin{aligned}\mathbb{F}_{q^t}(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} B &\cong \mathbb{F}_{q^t}(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} \mathbb{F}_{q^t}(\!(z)\!) \otimes_{\mathbb{F}_{q^t}(\!(z)\!)} B \\ &\cong \left(\prod_{i\in\mathbb{Z}/t\mathbb{Z}} \mathbb{F}_{q^t}(\!(z)\!)\right) \otimes_{\mathbb{F}_{q^t}(\!(z)\!)} B \cong \prod_{i\in\mathbb{Z}/t\mathbb{Z}} B_i,\end{aligned}$$

where each $B_i = B$ is and $\mathbb{F}_{q^t}(\!(z)\!)$-algebra via $\mathbb{F}_{q^t}(\!(z)\!) \times B_i \to B_i$, $(f,b) \mapsto \sigma^i(f)b$ and

$$D \otimes_{\mathbb{F}_q(\!(z)\!)} B \cong \prod_{i\in\mathbb{Z}/t\mathbb{Z}} \left(D_i \otimes_{\mathbb{F}_{q^t}(\!(z)\!)} B\right) = \prod_{i\in\mathbb{Z}/t\mathbb{Z}} (D_i)_B = \prod_{i\in\mathbb{Z}/t\mathbb{Z}} D_{B_i}.$$

Also note that we have canonical isomorphisms

$$\begin{aligned}&\mathrm{End}_{\mathbb{F}_{q^t}(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)}B}(D \otimes_{\mathbb{F}_q(\!(z)\!)} B) \cong \mathrm{End}_{\mathbb{F}_{q^t}(\!(z)\!)}(D) \otimes_{\mathbb{F}_{q^t}(\!(z)\!)} (\mathbb{F}_{q^t}(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} B) \\ &\cong \mathrm{End}_{\mathbb{F}_{q^t}(\!(z)\!)}(D) \otimes_{\mathbb{F}_{q^t}(\!(z)\!)} \prod_{i\in\mathbb{Z}/t\mathbb{Z}} B_i \cong \prod_{i\in\mathbb{Z}/t\mathbb{Z}} \left(\mathrm{End}_{\mathbb{F}_{q^t}(\!(z)\!)}(D) \otimes_{\mathbb{F}_{q^t}(\!(z)\!)} B_i\right) \\ &\cong \prod_{i\in\mathbb{Z}/t\mathbb{Z}} \mathrm{End}_B(D \otimes_{\mathbb{F}_{q^t}(\!(z)\!)} B_i) \cong \prod_{i\in\mathbb{Z}/t\mathbb{Z}} \mathrm{End}_B((D_i)_B).\end{aligned}$$

With these notations the map $\rho(B)$ is

$$\begin{aligned}J_{\mathbb{F}_{q^t}(\!(z)\!)}(B) &\subseteq \mathrm{GL}_{\mathbb{F}_{q^t}(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)}B}\left(D \otimes_{\mathbb{F}_q(\!(z)\!)} B\right) \\ &\xrightarrow{\sim} \prod_{i\in\mathbb{Z}/t\mathbb{Z}} \mathrm{GL}_B\left((D_i)_B\right) \xrightarrow{\mathrm{pr}_0} \mathrm{GL}_B\left((D_0)_B\right)\end{aligned}$$

where pr_0 is the projection on the 0-th component. In the commutative diagram

$$\begin{tikzcd} a \otimes b \arrow[rrr, mapsto] \arrow[ddd, mapsto] & & & \sigma(a) \otimes b \arrow[ddd, mapsto] \\ & \mathbb{F}_{q^t}(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} B \arrow[r, "\sigma_B"] \arrow[dd, "\sim"] & \mathbb{F}_{q^t}(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} B \arrow[dd, "\sim"] & \\ & & & \\ (\sigma^i(a)b)_i & \prod_{\mathbb{Z}/t\mathbb{Z}} B \arrow[r, dashed] & \prod_{\mathbb{Z}/t\mathbb{Z}} B & (\sigma^{i+1}(a)b)_i \end{tikzcd}$$

the dashed arrow is given by $(b_i)_i \mapsto (b_{i+1})_i$. Hence the morphism $\sigma_B = \sigma \otimes \mathrm{id}_B$ corresponds to a right shift under the above identification and therefore for every $\mathbb{F}_{q^t}(\!(z)\!)$-algebra B we get that

$$\begin{aligned}\sigma_B^*(D \otimes_{\mathbb{F}_q(\!(z)\!)} B) &= D \otimes_{\mathbb{F}_q(\!(z)\!)} B \otimes_{(\mathbb{F}_{q^t}(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)}B),\sigma_B} (\mathbb{F}_{q^t}(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} B) \\ &= \left(\prod_{i\in\mathbb{Z}/t\mathbb{Z}} (D_i)_B\right) \otimes_{(\prod_{i\in\mathbb{Z}/t\mathbb{Z}} B),\sigma_B} \prod_{i\in\mathbb{Z}/t\mathbb{Z}} B = \prod_{i\in\mathbb{Z}/t\mathbb{Z}} (D_{i-1})_B.\end{aligned}$$

We can rephrase this by saying that

$$\big(\sigma_B^*(D \otimes_{\mathbb{F}_q((z))} B)\big)_i = (D_{i-1})_B.$$

Now the isomorphism

$$F_D \otimes \mathrm{id}_B \colon \underbrace{\sigma^* D \otimes_{\mathbb{F}_q((z))} B}_{=\sigma_B^*(D \otimes_{\mathbb{F}_q((z))} B)} \xrightarrow{\sim} D \otimes_{\mathbb{F}_q((z))} B$$

induces isomorphisms

$$(F_D \otimes \mathrm{id}_B)_i \colon \underbrace{\big(\sigma_B^*(D \otimes_{\mathbb{F}_q((z))} B)\big)_i}_{=(D_{i-1})_B} \xrightarrow{\sim} \underbrace{\big(D \otimes_{\mathbb{F}_q((z))} B\big)_i}_{=(D_i)_B}.$$

In order to see that the morphism (3.2.1) is injective, we start with an element $g \in J_{\mathbb{F}_{q^t}((z))}(B)$ (B an $\mathbb{F}_{q^t}((z))$-algebra) and consider its image $(g_i)_{i \in \mathbb{Z}/t\mathbb{Z}}$ under the above identification. Note that $(\sigma_B^* g)_i = g_{i-1}$. Since $(F_D \otimes \mathrm{id}_B) \circ \sigma_B^* g = g \circ (F_D \otimes \mathrm{id}_B)$ we get that

$$(F_D \otimes \mathrm{id}_B)_i \circ \underbrace{(\sigma_B^* g)_i}_{=g_{i-1}} = g_i \circ (F_D \otimes \mathrm{id}_B)_i.$$

Therefore we can reconstruct all components of $(g_i)_{i \in \mathbb{Z}/t\mathbb{Z}}$ from g_0 since the $(F_D \otimes \mathrm{id}_B)_i$ are isomorphisms. So $g_0 = 0$ implies $g_i = 0$ for all i and hence $g = 0$. This proves the injectivity.

3.2.2 A representation of $J_{k((z))}$ on $\sigma^* D$

There is also a second action of $J_{k((z))}$; this time it acts on the $k((z))$-vector space $\sigma^* D$. For every $\mathbb{F}_q((z))$-algebra A we have the morphism

$$\begin{aligned} \mathrm{GL}_{k((z)) \otimes_{\mathbb{F}_q((z))} A}\,(D \otimes_{\mathbb{F}_q((z))} A) &\to \mathrm{GL}_{k((z)) \otimes \mathbb{F}_q((z)) A}\,(\sigma^* D \otimes_{\mathbb{F}_q((z))} A) \\ g &\mapsto \sigma_A^* g \end{aligned} \tag{3.2.2}$$

via the identification (1.3.1). This leads to an action, denoted by ${}^\sigma\rho$, of $J_{k((z))}$ on the $k((z))$-vector space $\sigma^* D$. Namely if B is a $k((z))$-algebra it is given on B-valued points by the morphism

$$\begin{aligned} {}^\sigma\rho(B) \colon J_{k((z))}(B) &\subseteq \mathrm{GL}_{k((z)) \otimes_{\mathbb{F}_q((z))} B}\,(D \otimes_{\mathbb{F}_q((z))} B) \\ &\to \mathrm{GL}_{k((z)) \otimes_{\mathbb{F}_q((z))} B}\,(\sigma^* D \otimes_{\mathbb{F}_q((z))} B) \\ &\to \mathrm{GL}_B\,(\sigma^* D \otimes_{k((z))} B) = \mathrm{GL}(\sigma^* D)(B) \end{aligned}$$

which is also induced by the multiplication map $k((z)) \otimes_{\mathbb{F}_q((z))} B \to B$.

3.2.3 1-parameter subgroups and decompositions into sub-z-isocrystals

There is a correspondence between direct sum decompositions of D into sub-z-isocrystals (indexed by $\mathbb{Z}$) and morphisms $\mathbb{G}_m \to J$. First we describe how to associate a decomposition to a morphism; so let $\lambda\colon \mathbb{G}_m \to J$ be a morphism of algebraic groups. Base change to $k(\!(z)\!)$ leads to a morphism $\lambda_{k(\!(z)\!)}\colon \mathbb{G}_{m,k(\!(z)\!)} \to J_{k(\!(z)\!)}$ which we can compose with ρ to get a representation of $\mathbb{G}_{m,k(\!(z)\!)}$ over $k(\!(z)\!)$:

$$\rho \circ \lambda_{k(\!(z)\!)}\colon \mathbb{G}_{m,k(\!(z)\!)} \to \mathrm{GL}(D).$$

This representation provides us with a decomposition of D into weight spaces

$$D = \bigoplus_{i \in \mathbb{Z}} D(i)$$

where $d \in D$ lies in $D(i)$ if for every $k(\!(z)\!)$-algebra B and every $b \in B$ we have $\rho(\lambda_{k(\!(z)\!)}(b))(d \otimes 1) = b^i(d \otimes 1) = d \otimes b^i$. We claim that these $k(\!(z)\!)$-sub-vector spaces are actually sub-z-isocrystals. More precisely we will show that

$$D(i) = \{d \in D \mid \rho(\lambda_{k(\!(z)\!)}(x))(d) = x^i d \text{ for all } x \in \mathbb{F}_q(\!(z)\!) \subseteq k(\!(z)\!)\} =: \tilde{D(i)}$$

and that the $\tilde{D(i)}$ are sub-z-isocrystals of $\underline{D}$. In order to see this, note that, for $x \in \mathbb{F}_q(\!(z)\!) \subset k(\!(z)\!)$, the element $\rho(\lambda_{k(\!(z)\!)}(x))$ lies in $J(\mathbb{F}_q(\!(z)\!)) \subseteq \mathrm{GL}(D)(k(\!(z)\!))$ since the following diagram

$$\begin{tikzcd}
\mathbb{G}_m(\mathbb{F}_q(\!(z)\!)) \arrow[r] \arrow[d, hook] & \mathrm{GL}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)}\mathbb{F}_q(\!(z)\!)}(D \otimes_{\mathbb{F}_q(\!(z)\!)} \mathbb{F}_q(\!(z)\!)) \arrow[r, "\cong"] \arrow[d] & \mathrm{GL}_{k(\!(z)\!)}(D) \arrow[d, equal] \\
\mathbb{G}_m(k(\!(z)\!)) \arrow[r] & \mathrm{GL}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)}k(\!(z)\!)}(D \otimes_{\mathbb{F}_q(\!(z)\!)} k(\!(z)\!)) \arrow[r, "\tilde{m}"] & \mathrm{GL}_{k(\!(z)\!)}(D)
\end{tikzcd}$$

commutes ($\tilde{m}$ is induced by the multiplication map). In other words we see that $\rho(\lambda_{k(\!(z)\!)}(x)) = \lambda(x)$ commutes with F_D and hence the $\tilde{D(i)}$ are sub-z-isocrystals. Now it is clear that $D(i) \subseteq \tilde{D(i)}$ for every $i \in \mathbb{Z}$, but moreover the $\tilde{D(i)}$ are $k(\!(z)\!)$-sub-vector spaces with

$$\tilde{D(i)} \cap \sum_{j \neq i} \tilde{D(j)} = 0$$

since $z^i \neq z^j$ for $i \neq j$. Therefore we get $D = \bigoplus_{i \in \mathbb{Z}} D(i) = \bigoplus_{i \in \mathbb{Z}} \tilde{D(i)}$ and $D(i) = \tilde{D(i)}$.

On the other hand let $D = \bigoplus_{i \in \mathbb{Z}} D_i$ be a decomposition of D into sub-z-isocrystals and let A be an $\mathbb{F}_q(\!(z)\!)$-algebra. We define a morphism

$$\lambda(A)\colon \mathbb{G}_m(A) \to \mathrm{GL}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)} A}(D \otimes_{\mathbb{F}_q(\!(z)\!)} A)$$

by letting $a \in \mathbb{G}_m(A)$ act on an element of $D_i \otimes_{\mathbb{F}_q((z))} A$ as multiplication by $(1 \otimes a)^i$. We claim that $\lambda(A)(a)$ actually lies in $J(A) \subseteq \mathrm{GL}_{k((z))\otimes_{\mathbb{F}_q((z))} A}(D \otimes_{\mathbb{F}_q((z))} A)$. Since $D = \bigoplus_{i\in\mathbb{Z}} D_i$ is a decomposition into sub-z-isocrystals, i.e. every D_i is F_D-stable, it is enough to check the condition $(F_D \otimes \mathrm{id}_A) \circ \sigma_A^*(\lambda(A)(a)) = \lambda(A)(a) \circ (F_D \otimes \mathrm{id}_A)$ on each $D_i \otimes_{\mathbb{F}_q((z))} A$. This is clear since $\sigma_A^*(\lambda(A)(a)) = \sigma_A^*((1 \otimes a)^i \,\mathrm{id}_{D_i}) = (1 \otimes a)^i \,\mathrm{id}_{D_i}$ is multiplication with a scalar.

A comparison of two decompositions into weight spaces

Now let $\lambda\colon \mathbb{G}_m \to J$ be a morphism. Beside the representation $\rho \circ \lambda_{k((z))}$ of $\mathbb{G}_{m,k((z))}$ on D described above, we get in the same way the representation ${}^{\sigma}\rho \circ \lambda_{k((z))}$ of $\mathbb{G}_{m,k((z))}$ on σ^*D. We denote the decomposition of $\rho \circ \lambda_{k((z))}$ into weight spaces by $D = \bigoplus D(i)$ and the decomposition of ${}^{\sigma}\rho \circ \lambda_{k((z))}$ by $\sigma^*D = \bigoplus(\sigma^*D)(i)$. We want to see how these two decompositions are related.

Lemma 3.2.1. *With the above notations we have*

$$(\sigma^*D)(i) = \sigma^*(D(i)).$$

Proof. Let $d \in D(i)$ and let $b \in B$ where B is a $k((z))$-algebra. We denote the multiplication morphism $k((z)) \otimes_{\mathbb{F}_q((z))} B \to B$ by m. We use the canonical identifications to describe $\lambda_{k((z))}(b) = \sum_\nu f_\nu \otimes b_\nu$ as an element in $\mathrm{End}_{k((z))}(D) \otimes_{\mathbb{F}_q((z))} B$. As we have seen in Section 1.3 we have more precisely $f_\nu \in \mathrm{End}(\underline{D})$ and we know that $F_D \circ \sigma^* f_\nu = f_\nu \circ F_D$. Since $F_D\colon \sigma^*D \xrightarrow{\sim} D$ is an isomorphism we see that $\sigma^* f_\nu = F_D^{-1} \circ f_\nu \circ F_D$. Therefore we get that

$$\sigma_B^*(\lambda_{k((z))}(b)) = \sum_\nu \sigma^* f_\nu \otimes b_\nu = \sum_\nu F_D^{-1} \circ f_\nu \circ F_D \otimes b_\nu.$$

We claim that $F_D^{-1}(d) \in \sigma^*D$ lies in $(\sigma^*D)(i)$. In order to show this, consider the element

$$\begin{aligned}\sigma_B^*(\lambda_{k((z))}(b)) \otimes 1 &\in \mathrm{End}_{k((z))\otimes_{\mathbb{F}_q((z))} B}(\sigma_B^*(D \otimes_{\mathbb{F}_q((z))} B)) \otimes_{(k((z))\otimes_{\mathbb{F}_q((z))} B),m} B \\ &\cong \mathrm{End}_B(\sigma^*D \otimes_{k((z))} B).\end{aligned}$$

With the above notations we can describe it as

$$\sum_\nu F_D^{-1} \circ f_\nu \circ F_D \otimes b_\nu \otimes 1$$

which is an element in

$$\mathrm{End}_{k((z))}(\sigma^*D) \otimes_{\mathbb{F}_q((z))} B \otimes_{(k((z))\otimes_{\mathbb{F}_q((z))} B),m} B.$$

We calculate

$$
\begin{aligned}
&(\sum_\nu F_D^{-1} \circ f_\nu \circ F_D \otimes b_\nu \otimes 1)(F_D^{-1}(d) \otimes 1) \\
=& \left((\sum_\nu F_D^{-1} \circ f_\nu \circ F_D \otimes b_\nu)(F_D^{-1}(d) \otimes 1) \right) \otimes 1 \\
=& \left((F_D^{-1} \otimes 1) \cdot \underbrace{\sum_\nu f_\nu(d) \otimes b_\nu}_{=d \otimes b^i} \right) \otimes 1 = F_D^{-1}(d) \otimes b^i \otimes 1
\end{aligned}
$$

and this proves the claim. By the claim we see that $F_D^{-1}(D(i)) \subseteq (\sigma^* D)(i)$ and since

$$\bigoplus_{i \in \mathbb{Z}} (\sigma^* D)(i) = \sigma^* D = \bigoplus_{i \in \mathbb{Z}} F_D^{-1}(D(i))$$

it follows that $F_D^{-1}(D(i)) = (\sigma^* D)(i)$. We have already seen that the $D(i)$ are sub-z-isocrystals and therefore we know that F_D induces isomorphisms $F_D|\sigma^*(D(i)) \colon \sigma^*(D(i)) \xrightarrow{\sim} D(i)$ and hence

$$(\sigma^* D)(i) = F_D^{-1}(D(i)) = \sigma^*(D(i)).$$

$\square$

Another way to look at Lemma 3.2.1 is the following: Let again $\lambda \colon \mathbb{G}_m \to J$ be a 1-PS. We can choose a basis $\mathcal{B}$ of D such that the composition

$$\mathbb{G}_{m,k((z))} \xrightarrow{\rho \circ \lambda_{k((z))}} \mathrm{GL}(D) \to \mathrm{GL}_r\,,$$

where the second morphism is induced by the basis $\mathcal{B}$, is given by

$$\alpha \mapsto \left(\alpha^{t_i} \cdot \delta_{ij}\right)_{1 \le i,j \le r}\,,$$

with $t_1, \ldots, t_r \in \mathbb{Z}$. Since $\mathcal{B}$ induces a basis $\sigma^*\mathcal{B}$ on $\sigma^* D$ we also get a second morphism from $\mathbb{G}_{m,k((z))}$ to GL_r as the composition

$$\mathbb{G}_{m,k((z))} \xrightarrow{{}^\sigma \rho \circ \lambda_{k((z))}} \mathrm{GL}(\sigma^* D) \to \mathrm{GL}_r\,,$$

with the second morphism being induced by $\sigma^*\mathcal{B}$. Now Lemma 3.2.1 says that again this morphism is given by

$$\alpha \mapsto \left(\alpha^{t_i} \cdot \delta_{ij}\right)_{1 \le i,j \le r}\,.$$

This is not clear a priori since, for a $k((z))$-Algebra B, in the following diagram, where the horizontal arrows are induced by the multiplication map $k((z)) \otimes_{\mathbb{F}_q((z))} B \to B$, there is no obvious morphism from $\mathrm{GL}_B(D \otimes_{k(\!(z)\!)} B)$ to $\mathrm{GL}_B(\sigma^* D \otimes_{k(\!(z)\!)} B)$ making it commutative.

$$\begin{array}{ccc} g & \mathrm{GL}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)} B}(D \otimes_{\mathbb{F}_q(\!(z)\!)} B) \longrightarrow \mathrm{GL}_B(D \otimes_{k(\!(z)\!)} B) \\ \big\downarrow & \big\downarrow \qquad\qquad\qquad\qquad \\ \sigma_B^* g & \mathrm{GL}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)} B}(\sigma^* D \otimes_{\mathbb{F}_q(\!(z)\!)} B) \longrightarrow \mathrm{GL}_B(\sigma^* D \otimes_{k(\!(z)\!)} B) \end{array}$$

3.2.4 The functor $\_^{\sim w}$

We fix integers $w_1, \ldots, w_r \in \mathbb{Z}$ with $w_1 \geq \ldots \geq w_r$. We set $w = (w_1, \ldots, w_r)$ and $|w| = w_1 - w_r$. Since we have a morphism

$$\begin{aligned} \mathbb{F}_q(\!(z)\!) &\to K[\![z-\zeta]\!] \\ z &\mapsto \zeta + (z-\zeta) \end{aligned}$$

we can make the following

Definition 3.2.2. For a linear algebraic group G over $\mathbb{F}_q(\!(z)\!)$ we set

$$G^{\sim w} = \mathrm{Res}_{K[\![z-\zeta]\!]/(z-\zeta)^{|w|}|K}(G \times_{\mathbb{F}_q(\!(z)\!)} K[\![z-\zeta]\!]/(z-\zeta)^{|w|}).$$

This is a linear algebraic group over K and for any K-algebra R we get that

$$G^{\sim w}(R) = G(R[\![z-\zeta]\!]/(z-\zeta)^{|w|}).$$

We are especially interested in $G^{\sim w}$ when G is J or $\mathbb{G}_m$. Since we have seen that we have a representation ${}^\sigma\rho$ of J on $\sigma^* D$ we get that $J(R[\![z-\zeta]\!]/(z-\zeta)^{|w|})$ operates on $\sigma^* D \otimes_{k(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{|w|}$. Thus we have a morphism

$$\begin{aligned} J^{\sim w}(R) &\to \mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{|w|}}\left(\sigma^* D \otimes_{k(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{|w|}\right) \\ &\subseteq \underbrace{\mathrm{GL}_R\left((\sigma^* D \otimes_{k(\!(z)\!)} K[\![z-\zeta]\!]/(z-\zeta)^{|w|}) \otimes_K R\right)}_{=\mathrm{GL}\left((\sigma^* D \otimes_{k(\!(z)\!)} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})_{[K]}\right)(R)}. \end{aligned}$$

In other words, there is a representation of $J^{\sim w}$ on the K-vector space $(\sigma^* D \otimes_{k(\!(z)\!)} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})_{[K]}$ denoted by

$${}^\sigma\rho_w \colon J^{\sim w} \to \mathrm{GL}\left((\sigma^* D \otimes_{k(\!(z)\!)} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})_{[K]}\right).$$

The zero-component of ${\mathbb{G}_m}^{\sim w}$

In order to apply the general concepts of Geometric Invariant Theory (more precisely we want to calculate GIT-slopes), we need 1-parameter subgroups. We will start with a morphism $\lambda\colon \mathbb{G}_m \to J$ over $\mathbb{F}_q(\!(z)\!)$ and apply the functor $\_^{\sim w}$ which leads to a morphism $\lambda^{\sim w}\colon {\mathbb{G}_m}^{\sim w} \to J^{\sim w}$. Of course this is not a 1-PS of $J^{\sim w}$ anymore but we are able to fix this problem by only considering the constant part of ${\mathbb{G}_m}^{\sim w}$. We let $(\_)_0\colon {\mathbb{G}_m}^{\sim w} \to \mathbb{G}_{m,K}$ be the morphism

$$a = \sum_{i=0}^{|w|-1} a_i(z-\zeta)^i \mapsto (a)_0 = a_0.$$

Moreover let

$$U(R) = \{1 + \sum_{i=1}^{|w|-1} a_i(z-\zeta)^i \mid a_i \in R\} \subseteq {\mathbb{G}_m}^{\sim w}(R).$$

This defines a subgroup of ${\mathbb{G}_m}^{\sim w}$ on which $\mathbb{G}_{m,K}$ acts. We have an exact sequence

$$0 \to U \to {\mathbb{G}_m}^{\sim w} \xrightarrow{(\_)_0} \mathbb{G}_{m,K} \to 0$$

and the morphism $(\_)_0$ has a canonical section i_0 given by

$$a \mapsto a(z-\zeta)^0.$$

Altogether this shows that

$${\mathbb{G}_m}^{\sim w} = \mathbb{G}_{m,K} \ltimes U.$$

In the following we will denote the composition $\lambda^{\sim w} \circ i_0$ by λ_0 which is a morphism from $\mathbb{G}_{m,K}$ to $J^{\sim w}$.

1-parameter-subgroups of $J^{\sim w}$

Let $\lambda\colon \mathbb{G}_m \to J$ be a 1-PS over $\mathbb{F}_q(\!(z)\!)$. As we have seen, it gives rise to a representation of $\mathbb{G}_{m,k(\!(z)\!)}$ on the $k(\!(z)\!)$-vector space D. We can choose a basis $\mathcal{B} = (b_1, \dots, b_r)$ of D such that the morphism

$$\lambda_{k(\!(z)\!)}\colon \mathbb{G}_{m,k(\!(z)\!)} \to J_{k(\!(z)\!)} \to \mathrm{GL}(D) \to \mathrm{GL}_r$$

is of the form

$$\alpha \mapsto \left(\alpha^{t_i} \cdot \delta_{ij}\right)_{1\le i,j\le r}$$

with $t_1, \dots, t_r \in \mathbb{Z}$. For every K-algebra R the induced basis $\sigma^*\mathcal{B}$ gives rise to a morphism

$${\mathbb{G}_m}^{\sim w}(R) \to \underbrace{J^{\sim w}(R)}_{=J(R[\![z-\zeta]\!]/(z-\zeta)^{|w|})} \to \mathrm{GL}(\sigma^*D)(R[\![z-\zeta]\!]/(z-\zeta)^{|w|}) \to \mathrm{GL}_r(R[\![z-\zeta]\!]/(z-\zeta)^{|w|})$$

which again is $\alpha \mapsto (\alpha^{t_i} \cdot \delta_{ij})_{1\leq i,j\leq r}$ by Lemma 3.2.1. Now we let $(\sigma^*\mathcal{B})_{[K]}$ be the following K-basis on $(\sigma^*D \otimes_{k(\!(z)\!)} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})_{[K]}$:

$$\begin{aligned}(\sigma^*\mathcal{B})_{[K]} = \big(&\sigma^*b_1 \otimes 1, \sigma^*b_1 \otimes (z-\zeta), \ldots, \sigma^*b_1 \otimes (z-\zeta)^{|w|-1},\\ &\vdots\\ &\sigma^*b_r \otimes 1, \sigma^*b_r \otimes (z-\zeta), \ldots, \sigma^*b_r \otimes (z-\zeta)^{|w|-1}\big).\end{aligned}$$

If an element in $\mathrm{GL}_{K[\![z-\zeta]\!]/(z-\zeta)^{|w|}}(\sigma^*D \otimes K[\![z-\zeta]\!]/(z-\zeta)^{|w|})$ corresponds via the basis $\sigma^*\mathcal{B}$ to a matrix $(a_i \cdot \delta_{ij})_{1\leq i,j\leq r}$, with $a_i = \sum_{j=0}^{|w|-1} a_{ij}(z-\zeta)^j$ the corresponding element via the basis $(\sigma^*\mathcal{B})_{[K]}$ in $\mathrm{GL}_K((\sigma^*D \otimes_{k(\!(z)\!)} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})_{[K]})$ is the block matrix

$$\left(\begin{array}{cccc} \boxed{\begin{array}{ccc} a_{1,0} & 0 \cdots & 0 \\ a_{1,1} & a_{1,0} & \vdots \\ \vdots & \vdots & 0 \\ a_{1,|w|-1} & a_{1,|w|-2} \cdots & a_{1,0} \end{array}} & |w| & 0 & \\ |w| & \ddots & & \\ & & & \\ 0 & |w| & \boxed{\begin{array}{ccc} a_{r,0} & 0 \cdots & 0 \\ a_{r,1} & a_{r,0} & \vdots \\ \vdots & \vdots & 0 \\ a_{r,|w|-1} & a_{|w|-2} \cdots & a_{r,0} \end{array}} & \\ & & |w| & \end{array}\right).$$

Hence we see that the morphism

$$\mathbb{G}_{m,K} \xrightarrow{i_0} \mathbb{G}_m{}^{\sim w} \to \mathrm{GL}\left((\sigma^*D \otimes_{k(\!(z)\!)} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})_{[K]}\right) \to \mathrm{GL}_{r\cdot|w|}\,,$$

where the last arrow is induced by the basis $(\sigma^*\mathcal{B})_{[K]}$, is given by

$$\alpha \mapsto \left(\begin{array}{cccc} \boxed{\begin{array}{ccc} \alpha^{t_1} & 0 \cdots & 0 \\ 0 & \ddots & \vdots \\ \vdots & \ddots & 0 \\ 0 \cdots & 0 & \alpha^{t_1} \end{array}} & |w| & 0 & \\ |w| & \ddots & & \\ & & & \\ 0 & |w| & \boxed{\begin{array}{ccc} \alpha^{t_r} & 0 \cdots & 0 \\ 0 & \ddots & \vdots \\ \vdots & \ddots & 0 \\ 0 \cdots & 0 & \alpha^{t_r} \end{array}} & \\ & & |w| & \end{array}\right),$$

where each α^{t_i} occurs $|w|$ times on the diagonal. We can summarize this result in terms of weight spaces and this is done in the following

Proposition 3.2.3. *Let λ be a 1-PS of J which provides us with a decomposition into sub-z-isocrystals*

$$D = \bigoplus_{i=1}^{r} D(t_i),$$

with $t_1, \ldots, t_r \in \mathbb{Z}$. This decomposition induces a decomposition of the K-vector space $(\sigma^* D \otimes_{k((z))} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})_{[K]}$ which coincides with the one associated to the morphism

$$^{\sigma}\rho_w \circ \lambda_0 \colon \mathbb{G}_{m,K} \to \mathrm{GL}\left((\sigma^* D \otimes_{k((z))} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})_{[K]}\right).$$

In other words we have

$$\left(\sigma^*(D(t_i)) \otimes_{k((z))} K[\![z-\zeta]\!]/(z-\zeta)^{|w|}\right)_{[K]} \cong \left(\sigma^* D \otimes_{k((z))} K[\![z-\zeta]\!]/(z-\zeta)^{|w|}\right)_{[K]}(t_i).$$

A quasi-character for $J^{\sim w}$

In this section we assume that the fixed z-isocrystal $\underline{D}$ is of the form

$$\underline{D} = \bigoplus_{\nu \in \mathbb{Q}} \underline{D}_\nu$$

with $\underline{D}_\nu$ isoclinic of slope ν. Therefore the group J decomposes as

$$J = J_{\underline{D}} = \prod_{\nu \in \mathbb{Q}} J_{\underline{D}_\nu}.$$

For example this is the case if the ground field k is perfect or if the z-isocrystal is split semi-simple. We get a decomposition of the group $J^{\sim w}$ as

$$J^{\sim w} = \prod_{\nu \in \mathbb{Q}} (J_{\underline{D}_\nu})^{\sim w}.$$

For every $\nu \in \mathbb{Q}$ we have a morphism $\chi_\nu \colon J_{\underline{D}_\nu} \to \mathbb{G}_m$ defined in Section 1.3 before Lemma 1.3.1 and hence we get a morphism $(\chi_\nu)^{\sim w} \colon (J_{\underline{D}_\nu})^{\sim w} \to {\mathbb{G}_m}^{\sim w}$. We denote the composition

$$(J_{\underline{D}_\nu})^{\sim w} \xrightarrow{(\chi_\nu)^{\sim w}} {\mathbb{G}_m}^{\sim w} \xrightarrow{(\_)_0} \mathbb{G}_{m,K}$$

by $\chi_{\nu,0}$.

If $g \in J^{\sim w}$ we write $g = (g_\nu)_{\nu \in \mathbb{Q}}$ with $g_\nu \in (J_{\underline{D}_\nu})^{\sim w}$. With these notations we define a quasi-character ψ_w of $J^{\sim w}$ as

$$\begin{aligned} \psi_w \colon J^{\sim w} &\to \mathbb{G}_{m,K}. \\ g &\mapsto \prod_{\nu \in \mathbb{Q}} \chi_{\nu,0}(g_\nu)^{-w_1-\nu} \end{aligned}$$

Here a quasi-character is an element of $X(J^{\sim w})\otimes_{\mathbb{Z}}\mathbb{Q}$. It is not a morphism from $J^{\sim w}$ to $\mathbb{G}_{m,K}$ but we should think of the elements of $X(J^{\sim w})\otimes_{\mathbb{Z}}\mathbb{Q}$ as objects that become characters after being multiplied by a suitable integer. Since in our application there is no difference between working with quasi-characters instead of characters we also do not make a difference in our notations and write them down as if they were morphisms from $J^{\sim w}$ to $\mathbb{G}_{m,K}$. In the next Lemma we will analyze what this quasi-character looks like, if we compose it with a 1-PS λ_0 of $J^{\sim w}$, coming from a 1-PS λ of J. Hence let $\lambda\colon \mathbb{G}_m \to J$ be such a 1-PS. It induces a decomposition of $D = \bigoplus_{i=1}^n D(s_i)$ with $s_1, \ldots, s_n \in \mathbb{Z}$ and $s_1 > \ldots > s_n$. Moreover we have seen that the $D(s_i)$ are sub-z-isocrystals. Let $d_i = \dim_{k(\!(z)\!)} D(s_i)$. Denote the corresponding filtration $\mathrm{Fil}^\bullet_{\rho\circ\lambda_{k(\!(z)\!)}} D$ by $\mathrm{Fil}^\bullet_\lambda D$ and the sub-quotients $\mathrm{gr}^i_{\rho\circ\lambda_{k(\!(z)\!)}}(D)$ by $\mathrm{gr}^i_\lambda(D)$.

Lemma 3.2.4. *With the above notations the quasi-character*

$$\mathbb{G}_{m,K} \xrightarrow{\lambda_0} J^{\sim w} \xrightarrow{\psi_w} \mathbb{G}_{m,K}$$

of $\mathbb{G}_{m,K}$ *is given by*

$$\sum_{i=1}^n s_i\left(-w_1 \cdot \dim_{k(\!(z)\!)} \mathrm{gr}^{s_i}_\lambda(D) - \mathrm{t_N}(\mathrm{gr}^{s_i}_\lambda(D))\right) \in \mathbb{Q}.$$

Proof. By abuse of notation we formally calculate this quasi-character as if it were a character of $\mathbb{G}_{m,K}$. Let R be a K-algebra and let $a \in \mathbb{G}_{m,K}(R)$. By the definition of χ_ν, if we view $\lambda_0(a) \in J^{\sim w}(R)$ as an element of $\mathrm{GL}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{|w|}}(D \otimes_{\mathbb{F}_q(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{|w|})$, in order to calculate $\chi_\nu(\lambda_0(a)_\nu)$, we have to take the determinant. Moreover note that the diagram (with $B = R[\![z-\zeta]\!]/(z-\zeta)^{|w|}$)

$$\begin{tikzcd}
\mathrm{GL}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)} B}(D_\nu \otimes_{\mathbb{F}_q(\!(z)\!)} B) \arrow[r, "\det"] \arrow[d, "\tilde{m}"] & k(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} B \arrow[d, "m"] \\
\mathrm{GL}_B(D_\nu \otimes_{k(\!(z)\!)} B) \arrow[r, "\det"] & B
\end{tikzcd}$$

commutes, where $m\colon k(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{|w|} \to R[\![z-\zeta]\!]/(z-\zeta)^{|w|}$ is the multiplication map and $\tilde{m}$ is induced by m. But as we have seen in Section 1.3, $\det(\lambda_0(a)_\nu)$ lies in $\mathbb{F}_q(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{|w|} = R[\![z-\zeta]\!]/(z-\zeta)^{|w|}$ and therefore $\det(\lambda_0(a)_\nu) = \det(\tilde{m}(\lambda_0(a)_\nu)) = \det(\rho(\lambda_0(a)_\nu))$. Thus we can choose a basis of D_ν such that $\lambda_0(a)_\nu$ is a diagonal matrix $(a^{t_i} \cdot \delta_{ij})_{ij}$ with $t_i \in \{s_1, \ldots, s_n\}$. Now if we calculate ψ_w we see that each $\chi_{\nu,0}(\lambda_0(a)_\nu)^{-w_1-\nu}$ is given by

$$\prod_i a^{t_i(-w_1-\nu)}.$$

We take the product over all of these and look at the exponent of a. This exponent can be written in the form $\sum_{i=1}^n s_i \cdot x_i$ where x_i is a sum of $-w_1$ and different $-\nu$. Counting them we see that $-w_1$ occurs $\dim_{k((z))} \mathrm{gr}_\lambda^{s_i}(D)$ times and the different $-\nu$ sum up to the Newton slope of $\mathrm{gr}_\lambda^{s_i}(D)$. Namely, we can write $\mathrm{gr}_\lambda^{s_i}(D) \cong \bigoplus_{\nu\in\mathbb{Q}}(\mathrm{gr}_\lambda^{s_i}(D))_\nu$ with each $(\mathrm{gr}_\lambda^{s_i}(D))_\nu$ isoclinic of slope ν. Then the summand $s_i \cdot (-\nu)$ occurs with multiplicity $\dim_{k((z))}(\mathrm{gr}_\lambda^{s_i}(D))_\nu$ and we compute

$$\sum_\nu (-\nu) \cdot \dim_{k((z))}(\mathrm{gr}_\lambda^{s_i}(D))_\nu = \sum_\nu \mathrm{t_N}((\mathrm{gr}_\lambda^{s_i}(D))_\nu) = \mathrm{t_N}(\mathrm{gr}_\lambda^{s_i}(D)).$$

□

3.3 The relation to Geometric Invariant Theory

Let $r \in \mathbb{N}$. In the following we suppose that $\underline{D} = (D, F_D)$ is a z-isocrystal over k of rank r such that we have a decomposition $\underline{D} = \bigoplus_{\nu\in\mathbb{Q}} \underline{D}_\nu$, where $\underline{D}_\nu = (D_\nu, F_{D_\nu})$ are isoclinic sub-z-isocrystals of slope $\nu \in \mathbb{Q}$. We fix integers $w_1, \dots, w_r \in \mathbb{Z}$ with $w_1 \geq \dots \geq w_r$ and set $w = (w_1, \dots, w_r)$. We use the notations of Section 2.4 and in addition we denote the K-vector space $(P^{(w)})_{[K]} = ((z-\zeta)^{w_r}\mathfrak{p}_K/(z-\zeta)^{w_1}\mathfrak{p}_K)_{[K]}$ by $V^{(w)}$ and its dimension $\dim_K V^{(w)} = r \cdot |w|$ by n_w. If $f\colon W_w \to \operatorname{Spec} K$ denotes the structure morphism we get that $f_*\mathscr{P}^{(w)} = (V^{(w)})^{\sim_{\operatorname{Spec} K}}$ which we denote by $\mathscr{V}^{(w)}$. Moreover let $e_w = r\cdot w_1 - \sum_{i=1}^r w_r$ and hence we have $\Phi_w = n_w - e_w$. As seen in Section 2.4, there is a closed embedding $Q_{D,\leq w} \hookrightarrow \mathrm{Grass}_{\Phi_w}(\mathscr{V}^{(w)})$ and in accordance with Convention 3.1.15 we embed this Grassmannian into $\mathbb{P}(\bigwedge^{e_w} V^{(w)^\vee})$ and consider also its affine cone $\mathbb{V}(\bigwedge^{e_w} V^{(w)^\vee})$. As a first step we define actions of $J^{\sim w}$ on all these spaces.

3.3.1 $J^{\sim w}$-actions

The action on $Q_{D,\leq w}$

For every K-algebra R we have a canonical isomorphism $\sigma^*D \otimes_{k((z))} R[\![z-\zeta]\!]/(z-\zeta)^{|w|} \cong P_R^{(w)}$. Since we have a morphism $J^{\sim w}(R) \hookrightarrow \mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{|w|}}(\sigma^*D \otimes_{k((z))} R[\![z-\zeta]\!]/(z-\zeta)^{|w|})$ this isomorphism induces an action of $J^{\sim w}(R)$ on $P_R^{(w)}$. We define a $J^{\sim w}$-action on $Q_{D,\leq w}$ in the following way: Let R be a K-algebra and let $g \in J^{\sim w}(R)$ which we view as an automorphism of $P_R^{(w)}$. Furthermore let $\mathfrak{q} \in Q_{D,\leq w}(R)$ which we can identify with an element $\tilde{\mathfrak{q}}$ of $\tilde{Q}_{D,\leq w}(R)$. We claim that $g(\tilde{\mathfrak{q}})$ lies in $\tilde{Q}_{D,\leq w}(R)$ and therefore we have to check that the element $\mathfrak{q}' \in Q_{D,w_r,w_1}(R)$ corresponding to $g(\tilde{\mathfrak{q}}) \in \tilde{Q}_{D,w_r,w_1}(R)$ satisfies the boundedness conditions (2.3.1) - (2.3.3). In order to do so, we will give another description of $\mathfrak{q}'$. Via a $k((z))$-basis of D we can identify $\mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{|w|}}(\sigma^*D \otimes_{k((z))} R[\![z-\zeta]\!]/(z-\zeta)^{|w|})$ with $\mathrm{GL}_r(R[\![z-\zeta]\!]/(z-\zeta)^{|w|})$. Note that the morphism $\mathrm{GL}_r(R[\![z-\zeta]\!]) \to \mathrm{GL}_r(R[\![z-\zeta]\!]/(z-\zeta)^{|w|})$, induced by the homomorphism $R[\![z-\zeta]\!] \to R[\![z-\zeta]\!]/(z-\zeta)^{|w|}$, is surjective. Actually there is a section of this morphism that is given by

$$\begin{aligned} \mathrm{GL}_r(R[\![z-\zeta]\!]/(z-\zeta)^{|w|}) &\to \mathrm{GL}_r(R[\![z-\zeta]\!]). \\ \sum_{i=0}^{|w|-1} A_i(z-\zeta)^i &\mapsto \sum_{i=0}^{|w|-1} A_i(z-\zeta)^i \end{aligned}$$

Every $h \in \mathrm{GL}_r(R[\![z-\zeta]\!])$ induces an automorphism of $\mathfrak{p}_R$ and hence an automorphism of $(z-\zeta)^{w_r}\mathfrak{p}_R$ which leaves $(z-\zeta)^{w_1}\mathfrak{p}_R$ invariant. Now choose $h \in \mathrm{GL}_r(R[\![z-\zeta]\!])$ mapping to $g \in \mathrm{GL}_r(R[\![z-\zeta]\!]/(z-\zeta)^{|w|})$. We claim that the morphism

$$\begin{aligned} J^{\sim w} \times Q_{D,w_r,w_1}(R) &\to Q_{D,w_r,w_1}(R) \\ (g,\mathfrak{q}) &\mapsto h(\mathfrak{q}) \end{aligned}$$

is well-defined. Let $h' \in \mathrm{GL}_r(R[\![z-\zeta]\!])$ be another element mapping to g. We have $h'h^{-1} = 1+u$ with $u \in (z-\zeta)^{|w|}\mathrm{M}_{r\times r}(R[\![z-\zeta]\!])$ and we get that $h'(\mathfrak{q}) = h'h^{-1}h(\mathfrak{q}) = h(\mathfrak{q}) + uh(\mathfrak{q})$. Since $uh(\mathfrak{q}) \subseteq (z-\zeta)^{|w|}h((z-\zeta)^{w_r}\mathfrak{p}_R) \subseteq h((z-\zeta)^{w_1}\mathfrak{p}_R) \subseteq h(\mathfrak{q})$ we see that $h'(\mathfrak{q}) = h(\mathfrak{q})$. Moreover this shows that $\mathfrak{q}' = h(\mathfrak{q})$. For every $j \in \{1,\dots,r\}$ we consider the morphism

$$\alpha^{(j)}_{\mathfrak{q},w}\colon \bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{q} \Big/ \bigwedge^j_{R[\![z-\zeta]\!]} (z-\zeta)^{w_1}\mathfrak{p}_R \to \bigwedge^j_{R[\![z-\zeta]\!]} (z-\zeta)^{w_r}\mathfrak{p}_R \Big/ (z-\zeta)^{w_{r+1-j}+\ldots+w_r} \bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{p}_R$$

from Lemma 2.3.4 and the following commutative diagram defines a morphism $h(\alpha^{(j)}_{\mathfrak{q},w})$.

$$\begin{tikzcd} \bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{q} \Big/ \bigwedge^j_{R[\![z-\zeta]\!]} (z-\zeta)^{w_1}\mathfrak{p}_R \arrow[r, "\alpha^{(j)}_{\mathfrak{q},w}"] \arrow[d, "\cong"] & \bigwedge^j_{R[\![z-\zeta]\!]} (z-\zeta)^{w_r}\mathfrak{p}_R \Big/ (z-\zeta)^{w_{r+1-j}+\ldots+w_r} \bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{p}_R \arrow[d, "\cong"] \\ \bigwedge^j_{R[\![z-\zeta]\!]} h(\mathfrak{q}) \Big/ \bigwedge^j_{R[\![z-\zeta]\!]} (z-\zeta)^{w_1}\mathfrak{p}_R \arrow[r, "h(\alpha^{(j)}_{\mathfrak{q},w})"] & \bigwedge^j_{R[\![z-\zeta]\!]} (z-\zeta)^{w_r}\mathfrak{p}_R \Big/ (z-\zeta)^{w_{r+1-j}+\ldots+w_r} \bigwedge^j_{R[\![z-\zeta]\!]} \mathfrak{p}_R \end{tikzcd}$$

Here the vertical morphisms are induced by h. Now $h(\alpha^{(j)}_{\mathfrak{q},w})$ is zero if and only if $\alpha^{(j)}_{h(\mathfrak{q}),w}$ is zero. This is the case by the commutativity of the diagram since $\mathfrak{q} \in Q_{D,\le w}(R)$ and therefore $\alpha^{(j)}_{\mathfrak{q},w}$ is zero by Lemma 2.3.4. In the same way we see that the morphism

$$\beta_{h(\mathfrak{q}),w}\colon (z-\zeta)^{w_1+\ldots+w_r} \bigwedge^r_{R[\![z-\zeta]\!]} \mathfrak{p}_R \Big/ \bigwedge^r_{R[\![z-\zeta]\!]} h(\mathfrak{q}) \to \bigwedge^r_{R[\![z-\zeta]\!]} (z-\zeta)^{w_r-w_1}h(\mathfrak{q}) \Big/ \bigwedge^r_{R[\![z-\zeta]\!]} h(\mathfrak{q})$$

is zero. By Lemma 2.3.4 the element $h(\mathfrak{q}) \in Q_{D,w_r,w_1}(R)$ lies in $Q_{D,\le w}(R)$. Therefore it makes sense to define $g\cdot\mathfrak{q}$ as $\mathfrak{q}' = h(\mathfrak{q}) \in Q_{D,\le w}(R)$.

The action on the Grassmannian and projective space

The isomorphism of K-vector spaces $(\sigma^* D \otimes_{k((z))} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})_{[K]} \cong V^{(w)}$ induces an action of $J^{\sim w}$ on $V^{(w)}$ by composing ${}^{\sigma}\rho_w \colon J^{\sim w} \to \mathrm{GL}((\sigma^* D \otimes_{k((z))} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})_{[K]})$ (defined in Section 3.2.4) with $\mathrm{GL}((\sigma^* D \otimes_{k((z))} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})_{[K]}) \xrightarrow{\sim} \mathrm{GL}(V^{(w)})$. By abuse of notation, we also denote this representation by ${}^{\sigma}\rho_w$. As described in Section 3.1.4, this representation gives rise to an action of $J^{\sim w}$ on $\mathrm{Grass}_{\Phi_w}(\mathscr{V}^{(w)})$ and on $\mathbb{P}(\bigwedge^{e_w} V^{(w)\vee})$. The inclusion $Q_{D,\leq w} \hookrightarrow \mathrm{Grass}_{\Phi_w}(\mathscr{V}^{(w)})$ is compatible with respect to these $J^{\sim w}$-actions.

Modification of the linearization

The representation ${}^{\sigma}\rho_w \colon J^{\sim w} \to \mathrm{GL}(V^{(w)})$ also induces an action of $J^{\sim w}$ on $\mathbb{V}(\bigwedge^{e_w} V^{(w)\vee})$ and this action gives rise to a $J^{\sim w}$-linearization of $\mathscr{O}_{\mathbb{P}(\bigwedge^{e_w} V^{(w)\vee})}(1)$. Actually the resulting sheaf together with its linearization is not the one we need. Before we pull it back to $\mathrm{Grass}_{\Phi_w}(\mathscr{V}^{(w)})$ and later to $Q_{D,\leq w}$, we need to modify the linearization of $\mathscr{O}_{\mathbb{P}(\bigwedge^{e_w} V^{(w)\vee})}(1)$. We have already remarked that we can do this by changing the linearization by characters of the group $\mathrm{GL}(V^{(w)})$. Actually what we really need is a modification by the quasi-character ψ_w. The resulting object is no longer an element of $\mathrm{Pic}^{J^{\sim w}}(\mathbb{P}(\bigwedge^{e_w} V^{(w)\vee}))$ but an element in $\mathrm{Pic}^{J^{\sim w}}(\mathbb{P}(\bigwedge^{e_w} V^{(w)\vee})) \otimes_{\mathbb{Z}} \mathbb{Q}$. From the properties of the morphism $\mu^{\bullet}(x,\lambda)$ one observes that it makes sense to define $\mu^{\mathscr{L}}(x,\lambda)$ also for elements $\mathscr{L}$ of $\mathrm{Pic}^{J^{\sim w}}(\mathbb{P}(\bigwedge^{e_w} V^{(w)\vee})) \otimes_{\mathbb{Z}} \mathbb{Q}$. It is possible to avoid this by multiplying the modified element of $\mathrm{Pic}^{J^{\sim w}}(\mathbb{P}(\bigwedge^{e_w} V^{(w)\vee})) \otimes_{\mathbb{Z}} \mathbb{Q}$ with a suitable integer m to obtain an element of $\mathrm{Pic}^{J^{\sim w}}(\mathbb{P}(\bigwedge^{e_w} V^{(w)\vee}))$. Here multiplication of $\mathscr{L}$ with an integer equals taking the tensor product, i.e. $\mathscr{L} \cdot (\mathscr{O} \otimes m) = \mathscr{L}^{\otimes m}$. Since we can apply the result of the calculation of $\mu^{\mathscr{L}}(x,\lambda)$ for $\mathscr{L} = \mathscr{O}_{\mathbb{P}(\bigwedge^{e_w} V^{(w)\vee})}(1)$ from [GIT] it is more convenient to work with $\mathrm{Pic}^{J^{\sim w}}(\mathbb{P}(\bigwedge^{e_w} V^{(w)\vee})) \otimes_{\mathbb{Z}} \mathbb{Q}$ as it is done in [Tot] and [DOR]. Therefore, if we denote the induced representation of ${}^{\sigma}\rho_w$ on $\bigwedge^{e_w} V^{(w)}$ by $\hat{\rho}$, we get an element called $\mathscr{L}_w \in \mathrm{Pic}^{J^{\sim w}}(\mathbb{P}(\bigwedge^{e_w} V^{(w)\vee})) \otimes_{\mathbb{Z}} \mathbb{Q}$ as $\mathscr{O}_{\mathbb{P}(\bigwedge^{e_w} V^{(w)\vee})}(1)$ together with the $J^{\sim w}$-quasi-linearization provided by the quasi-action of $J^{\sim w}$ on $\mathbb{V}(\bigwedge^{e_w} V^{(w)\vee})$ that is induced by the modification of the representation $\hat{\rho}$ by ψ_w:

$$g \mapsto \psi_w(g) \cdot \hat{\rho}(g)$$

3.3.2 The Main Theorem

Now we can formulate and prove our main result. It relates the concept of weak admissibility to Geometric Invariant Theory. More precisely we give a criterion in terms of GIT-slopes whether a point in $Q_{D,\leq w}$ is weakly admissible. Note that we work with L-valued points of $Q_{D,\leq w}$, where L is a field extension of K. This can be done although we actually defined the GIT-slope only for K-rational points of $Q_{D,\leq w}$ (see Remark 3.1.11). We denote the inclusion $Q_{D,\leq w} \hookrightarrow \mathbb{P}(\bigwedge^{e_w} V^{(w)\vee})$ by ι.

Theorem 3.3.1. *Let L be a field extension of K and let $\mathfrak{q} \in \mathrm{Q}_{D,\leq w}(L)$ be a Hodge-Pink lattice over L of $\underline{D}$. Then $\mathfrak{q}$ is weakly admissible if and only if*

$$\mu^{\iota^* \mathscr{L}_w}(\mathfrak{q}, \lambda_0) \geq 0$$

for all 1-PS λ of J defined over $\mathbb{F}_q(\!(z)\!)$.

Proof. With all the preparation we have done so far, it is only a matter of calculating $\mu^{\iota^* \mathscr{L}_w}(\mathfrak{q}, \lambda_0)$. Let $\mathcal{B}$ be a $k(\!(z)\!)$-basis of D such that the representation $\rho \circ \lambda$ on D is given by matrices:

$$\alpha \mapsto \left(\alpha^{t_i} \cdot \delta_{ij}\right)_{1 \leq i,j \leq r},$$

with $t_1, \ldots, t_r \in \mathbb{Z}$ and $t_1 \geq \ldots \geq t_r$. Let $n \in \mathbb{N}$ and $s_1, \ldots, s_n \in \{t_1, \ldots, t_r\}$ with $s_1 > \ldots > s_n$ such that

$$D = \bigoplus_{i=1}^{n} D(s_i)$$

is the decomposition of D associated to λ into weight spaces. The $D(s_i)$ are F_D-stable and therefore are actually sub-z-isocrystal of $\underline{D}$. In order to calculate the GIT-slope, we view $\mathfrak{q}$ as the element $V^{(w)} \to V^{(w)}/\tilde{\mathfrak{q}}$ of the Grassmannian $\mathrm{Grass}_{\Phi_w}(\mathscr{V}^{(w)})(L)$, where $\tilde{\mathfrak{q}} = \mathfrak{q}/(z-\zeta)^{w_1}\mathfrak{p}_L$, and use Proposition 3.1.16. Since the change of the linearization by the quasi-character ψ_w is not affecting the calculation done in Proposition 3.1.16 we can simply use Proposition 3.1.13 to observe the effect of the change of the linearization. Therefore we get

$$\begin{aligned}\mu^{\iota^* \mathscr{L}_w}(\mathfrak{q}, \lambda_0) = &-\sum_{i=1}^{n} s_i \cdot \dim_L \left(\mathrm{gr}^{s_i}_{\sigma_{\rho_w \circ \lambda_0}}(V_L^{(w)})\big|\tilde{\mathfrak{q}}\right) \\ &+ \sum_{i=1}^{n} s_i \cdot \left(w_1 \cdot \dim_{k(\!(z)\!)} \mathrm{gr}^{s_i}_{\lambda}(D) + \mathrm{t_N}(\mathrm{gr}^{s_i}_{\lambda}(D))\right),\end{aligned}$$

with the second line coming from Lemma 3.2.4. This is equal to

$$\begin{aligned}&-\sum_{i=1}^{n} s_i \cdot \left(\dim_L(\mathrm{Fil}^{s_i}_{\sigma_{\rho_w \circ \lambda_0}} V_L^{(w)}\big|\tilde{\mathfrak{q}}) - \dim_L(\mathrm{Fil}^{s_{i-1}}_{\sigma_{\rho_w \circ \lambda_0}} V_L^{(w)}\big|\tilde{\mathfrak{q}})\right) \\ &+\sum_{i=1}^{n} s_i \cdot \left(w_1 \cdot (\dim_{k(\!(z)\!)}(\mathrm{Fil}^{s_i}_{\lambda} D) - \dim_{k(\!(z)\!)}(\mathrm{Fil}^{s_{i-1}}_{\lambda} D)) \right. \\ &\qquad \left. + \mathrm{t_N}(\mathrm{Fil}^{s_i}_{\lambda} D, F_{\mathrm{Fil}^{s_i}_{\lambda} D}) - \mathrm{t_N}(\mathrm{Fil}^{s_{i-1}}_{\lambda} D, F_{\mathrm{Fil}^{s_{i-1}}_{\lambda} D})\right),\end{aligned} \tag{3.3.1}$$

with $s_0 = s_1 + 1$. Under the isomorphism $V^{(w)} \cong (\sigma^* D \otimes_{k(\!(z)\!)} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})_{[K]}$ the subspace $V^{(w)}(s_i)$ corresponds to $(\sigma^* D \otimes_{k(\!(z)\!)} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})_{[K]}(s_i)$ which is, by

Proposition 3.2.3, isomorphic to $(\sigma^*(D(s_i)) \otimes_{k((z))} K[\![z-\zeta]\!]/(z-\zeta)^{|w|})_{[K]}$. Hence we see that $\mathrm{Fil}^{s_i}_{\sigma_{\rho_w}\circ\lambda_0} V^{(w)} = (z-\zeta)^{w_r}\mathfrak{p}_{\mathrm{Fil}^{s_i}_\lambda D,K}/(z-\zeta)^{w_1}\mathfrak{p}_{\mathrm{Fil}^{s_i}_\lambda D,K}$. If we denote $\mathfrak{q}_{\mathrm{Fil}^{s_i}_\lambda D} = \mathfrak{q} \cap \sigma^*(\mathrm{Fil}^{s_i}_\lambda D) \otimes_{k((z))} L((z-\zeta))$ we get that

$$\mathrm{Fil}^{s_i}_{\sigma_{\rho_w}\circ\lambda_0} V^{(w)}_L\big|\tilde{\mathfrak{q}} = \tilde{\mathfrak{q}} \cap \mathrm{Fil}^{s_i}_{\sigma_{\rho_w}\circ\lambda_0} V^{(w)}_L = \mathfrak{q}_{\mathrm{Fil}^{s_i}_\lambda D}/(z-\zeta)^{w_1}\mathfrak{p}_{\mathrm{Fil}^{s_i}_\lambda D,L}$$

and its dimension is, by Lemma 2.5.6, equal to

$$\mathrm{t_H}(\mathrm{Fil}^{s_i}_\lambda D, F_{\mathrm{Fil}^{s_i}_\lambda D}, \mathfrak{q}_{\mathrm{Fil}^{s_i}_\lambda D}) + w_1 \cdot \dim_{k((z))} \mathrm{Fil}^{s_i}_\lambda D.$$

Therefore (3.3.1) is equal to

$$\begin{aligned}
&-\sum_{i=1}^{n} s_i \cdot \big(\mathrm{t_H}(\mathrm{Fil}^{s_i}_\lambda D, F_{\mathrm{Fil}^{s_i}_\lambda D}, \mathfrak{q}_{\mathrm{Fil}^{s_i}_\lambda D}) - \mathrm{t_H}(\mathrm{Fil}^{s_{i-1}}_\lambda D, F_{\mathrm{Fil}^{s_{i-1}}_\lambda D}, \mathfrak{q}_{\mathrm{Fil}^{s_{i-1}}_\lambda D})\big)\\
&+\sum_{i=1}^{n} s_i \cdot \big(\mathrm{t_N}(\mathrm{Fil}^{s_i}_\lambda D, F_{\mathrm{Fil}^{s_i}_\lambda D}) - \mathrm{t_N}(\mathrm{Fil}^{s_{i-1}}_\lambda D, F_{\mathrm{Fil}^{s_{i-1}}_\lambda D})\big).
\end{aligned}$$

Since $\mathrm{Fil}^{s_0}_\lambda D = 0$ and $\mathrm{Fil}^{s_n}_\lambda D = D$ this is equal to

$$\begin{aligned}
&\sum_{i=1}^{n-1} \underbrace{(s_i - s_{i+1})}_{>0} \cdot \Big(\mathrm{t_N}(\mathrm{Fil}^{s_i}_\lambda D, F_{\mathrm{Fil}^{s_i}_\lambda D}) - \mathrm{t_H}(\mathrm{Fil}^{s_i}_\lambda D, F_{\mathrm{Fil}^{s_i}_\lambda D}, \mathfrak{q}_{\mathrm{Fil}^{s_i}_\lambda D})\Big)\\
&+s_n \cdot (\mathrm{t_N}(D, F_D) - \mathrm{t_H}(D, F_D, \mathfrak{q}))\,.
\end{aligned}$$

Now if $\mathfrak{q}$ is weakly admissible $\mathrm{t_N}(D, F_D) - \mathrm{t_H}(D, F_D, \mathfrak{q}) = 0$ and $\mathrm{t_N}(\mathrm{Fil}^{s_i}_\lambda D, F_{\mathrm{Fil}^{s_i}_\lambda D}) - \mathrm{t_H}(\mathrm{Fil}^{s_i}_\lambda D, F_{\mathrm{Fil}^{s_i}_\lambda D}, \mathfrak{q}_{\mathrm{Fil}^{s_i}_\lambda D}) \geq 0$ for $i \in \{1, \ldots, n-1\}$. Hence we get that $\mu^{\iota^*\mathscr{L}_w}(\mathfrak{q}, \lambda_0) \geq 0$. On the other hand let $\mathfrak{q}$ be a Hodge-Pink lattice over L of $\underline{D}$ which is not weakly admissible; say for example $\mathrm{t_N}(D', F_{D'}) - \mathrm{t_H}(D', F_{D'}, \mathfrak{q}_{D'}) < 0$ for a sub-z-isocrystal $\underline{D}' = (D', F_{D'})$ of $\underline{D}$. As we have seen in Section 3.2.3 we can find a 1-PS λ of J such that the associated filtration $\mathrm{Fil}^0_\lambda D = D$, $\mathrm{Fil}^1_\lambda D = D'$, $\mathrm{Fil}^2_\lambda D = 0$ has two jumps $s_1 = 1$ and $s_2 = 0$. With this choice for λ we get that $\mu^{\iota^*\mathscr{L}_w}(\mathfrak{q}, \lambda_0) < 0$ and this proves the Theorem. □

3.4 Functorial behavior

In the last section we have seen that we have to modify the linearization of $\mathscr{O}_{\mathbb{P}(\bigwedge^{e_w} V^{(w)\vee})}(1)$ by the quasi-character ψ_w which was defined before Lemma 3.2.4 as

$$\begin{aligned}
\psi_w \colon J^{\sim w} &\to \mathbb{G}_{m,K}.\\
g &\mapsto \prod_{\nu\in\mathbb{Q}} \chi_{\nu,0}(g_\nu)^{-w_1-\nu}
\end{aligned}$$

The component $-\nu$ in the exponent is needed since we have to compare the Hodge slope with the Newton slope in order to test weak admissibility, but the component $-w_1$ may seem somewhat artificial. It turns out that this part of the exponent of the quasi-character is due to our Convention 3.1.15. In this section we will analyze what happens in the case of two different Hodge-Pink weights, with one of them being smaller than the other one. The observation of this case also gives a good explanation of the question why we have to modify the linearization in this way.

Let $\underline{D} = (D, F_D)$ be a z-isocrystal over k with rank of $\underline{D}$ being equal to $r \in \mathbb{N}$ such that we have a decomposition $\underline{D} = \bigoplus_{\nu \in \mathbb{Q}} \underline{D}_\nu$, where $\underline{D}_\nu = (D_\nu, F_{D_\nu})$ are isoclinic sub-z-isocrystals of slope $\nu \in \mathbb{Q}$. Let $v_1, \ldots, v_r \in \mathbb{Z}$ (resp. $w_1, \ldots, w_r \in \mathbb{Z}$) with $v_1 \geq \ldots \geq v_r$ (resp. $w_1 \geq \ldots \geq w_r$). Set $v = (v_1, \ldots, v_r)$ and $w = (w_1, \ldots, w_r)$ and assume that $v \preceq w$ for the Bruhat-order, i.e.

$$v_1 + \ldots + v_i \leq w_1 + \ldots + w_i$$

for all $i \in \{1, \ldots, r\}$ and

$$v_1 + \ldots + v_r = w_1 + \ldots + w_r.$$

We set $u = v_1 + \ldots + v_r = w_1 + \ldots + w_r$. Since $v \preceq w$ we especially get that $w_1 \geq v_1 \geq v_r \geq w_r$ and $|w| \geq |v|$. Moreover we define $n_v = r \cdot |v|$ and $e_v = r \cdot v_1 - u$ and similar n_w and e_w for w. With these notations we have $\Phi_v = n_v - e_v$ and $\Phi_w = n_w - e_w$. Let R be a K-algebra. The morphism $R[\![z-\zeta]\!]/(z-\zeta)^{|w|} \to R[\![z-\zeta]\!]/(z-\zeta)^{|v|}$ induces a morphism

$$\underbrace{J(R[\![z-\zeta]\!]/(z-\zeta)^{|w|})}_{J^{\sim w}(R)} \to \underbrace{J(R[\![z-\zeta]\!]/(z-\zeta)^{|v|})}_{J^{\sim v}(R)}$$

and hence a morphism $J^{\sim w} \to J^{\sim v}$ which gives rise to an action of $J^{\sim w}$ on $Q_{D,\leq v}$. Since every $\mathfrak{q} \in Q_{D,\leq v}(R)$ also satisfies the boundedness conditions (2.3.1) - (2.3.3) for w we get a map $Q_{D,\leq v} \to Q_{D,\leq w}$ that is induced by the identity. It is clear that this morphism is equivariant for the action of the group $J^{\sim w}$ by the description of the action on both spaces that we gave in Section 3.3.1 by choosing a lift in $\mathrm{GL}_{R[\![z-\zeta]\!]}(\sigma^* D \otimes_{k((z))} R[\![z-\zeta]\!])$. Moreover we have seen in Section 3.3.1 that $Q_{D,\leq v}$ (resp. $Q_{D,\leq w}$) may be embedded into projective space and that the affine cone induces an invertible sheaf together with a $J^{\sim v}$-linearization (resp. $J^{\sim w}$-linearization). Our aim is to extend the morphism $Q_{D,\leq v} \to Q_{D,\leq w}$ to these spaces. In the case of projective space the resulting morphism will be equivariant for the $J^{\sim w}$-action but on the affine cone we have to change the actions in order to get an equivariant morphism.

Recall that for $c, d \in \mathbb{Z}$ with $c \leq d$ we have defined $P^{(c,d)} = (z-\zeta)^c \mathfrak{p}_K / (z-\zeta)^d \mathfrak{p}_K$. If we consider $P^{(c,d)}$ as a K-vector space we denote it by $V^{(c,d)}$ and hence, with the notation of Section 3.3.2, we have $V^{(v)} = V^{(v_r, v_1)}$ and $V^{(w)} = V^{(w_r, w_1)}$. For every K-algebra R the group $\mathrm{GL}_{R[\![z-\zeta]\!]}(\sigma^* D \otimes_{k((z))} R[\![z-\zeta]\!])$ operates on $V_R^{(c,d)}$. By choosing a basis we can describe the part of the operation that is trivial by

$$\left\{ E_r + M \mid M \in \mathrm{M}_{r \times r}\Big((z-\zeta)^{d-c} R[\![z-\zeta]\!]\Big) \right\}.$$

It follows that the group $\mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^b}(\sigma^*D \otimes_{k(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^b)$ operates on $V_R^{(c,d)}$ for every $b \geq d-c$ and this operation factors through $\mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{d-c}}(\sigma^*D \otimes R[\![z-\zeta]\!]/(z-\zeta)^{d-c})$ that acts on $V_R^{(c,d)}$ since it is a subgroup of $\mathrm{GL}((\sigma^*D \otimes_{k(\!(z)\!)} K[\![z-\zeta]\!]/(z-\zeta)^{d-c})_{[K]})(R)$. The canonical morphisms

$$\underbrace{(z-\zeta)^{v_r}\mathfrak{p}_R/(z-\zeta)^{w_1}\mathfrak{p}_R}_{=P_R^{(v_r,w_1)}} \twoheadrightarrow \underbrace{(z-\zeta)^{v_r}\mathfrak{p}_R/(z-\zeta)^{v_1}\mathfrak{p}_R}_{=P_R^{(v)}}$$

and

$$\underbrace{(z-\zeta)^{v_r}\mathfrak{p}_R/(z-\zeta)^{w_1}\mathfrak{p}_R}_{=P_R^{(v_r,w_1)}} \hookrightarrow \underbrace{(z-\zeta)^{w_r}\mathfrak{p}_R/(z-\zeta)^{w_1}\mathfrak{p}_R}_{=P_R^{(w)}}$$

give rise to morphisms

$$V_R^{(v)} \xleftarrow{(p_w^v)_R} V_R^{(v_r,w_1)} \xrightarrow{(i_w^v)_R} V_R^{(w)}.$$

Since $w_1 - v_r \geq |v|$ the group $\mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{w_1-v_r}}(\sigma^*D \otimes_{k(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{w_1-v_r})$ acts on $V_R^{(v)}$ and this action factors through $\mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{|v|}}(\sigma^*D \otimes_{k(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{|v|})$. Therefore the morphism $(p_w^v)_R \colon V_R^{(v_r,w_1)} \to V_R^{(v)}$ is equivariant for the action of $\mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{w_1-v_r}}(\sigma^*D \otimes_{k(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{w_1-v_r})$. In the same way we see that the morphism $(i_w^v)_R \colon V_R^{(v_r,w_1)} \to V_R^{(w)}$ is equivariant for the action of $\mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{|w|}}(\sigma^*D \otimes_{k(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{|w|})$ since $|w| \geq w_1 - v_r$. The morphism $p_w^v \colon V^{(v_r,w_1)} \to V^{(v)}$ and hence the morphism $\wedge^{n_v-e_v} p_w^v \colon \bigwedge^{n_v-e_v} V^{(v_r,w_1)} \to \bigwedge^{n_v-e_v} V^{(v)}$ is surjective. Therefore we get morphisms

$$a_{\mathbb{P}} \colon \mathbb{P}(\bigwedge^{n_v-e_v} V^{(v)}) \to \mathbb{P}(\bigwedge^{n_v-e_v} V^{(v_r,w_1)})$$

and

$$a_{\mathbb{V}} \colon \mathbb{V}(\bigwedge^{n_v-e_v} V^{(v)}) \to \mathbb{V}(\bigwedge^{n_v-e_v} V^{(v_r,w_1)}).$$

On the other hand the morphism $i_w^v \colon V^{(v_r,w_1)} \to V^{(w)}$ is injective and hence the morphisms $i_w^{v\,\vee} \colon V^{(w)^\vee} \to V^{(v_r,w_1)^\vee}$ and $\wedge^{e_w} i_w^{v\,\vee} \colon \bigwedge^{e_w} V^{(w)^\vee} \to \bigwedge^{e_w} V^{(v_r,w_1)^\vee}$ are surjective. Therefore we get morphisms

$$b_{\mathbb{P}} \colon \mathbb{P}(\bigwedge^{e_w} V^{(v_r,w_1)^\vee}) \to \mathbb{P}(\bigwedge^{e_w} V^{(w)^\vee})$$

and

$$b_{\mathbb{V}} \colon \mathbb{V}(\bigwedge^{e_w} V^{(v_r,w_1)^\vee}) \to \mathbb{V}(\bigwedge^{e_w} V^{(w)^\vee}).$$

Since the morphisms p_w^v and i_w^v are equivariant also these four morphisms are equivariant with respect to the natural actions induced by the actions described above. We define a morphism $F_w^v\colon \mathbb{P}(\bigwedge^{e_v} V^{(v)^\vee}) \to \mathbb{P}(\bigwedge^{e_w} V^{(w)^\vee})$ as the composition

$$\begin{tikzcd}
\mathbb{P}(\bigwedge^{n_v-e_v} V^{(v)}) \arrow[r, "a_{\mathbb{P}}"] & \mathbb{P}(\bigwedge^{n_v-e_v} V^{(v_r,w_1)}) \arrow[d, "\cong"] & \\
\mathbb{P}(\bigwedge^{e_v} V^{(v)^\vee}) \arrow[u, "\cong"] & \mathbb{P}(\bigwedge^{e_w} V^{(v_r,w_1)^\vee}) \arrow[r, "b_{\mathbb{P}}"] & \mathbb{P}(\bigwedge^{e_w} V^{(w)^\vee}),
\end{tikzcd}$$

where the vertical isomorphisms are described in (3.1.7) in Section 3.1.4. For the second vertical isomorphism note that

$$r(w_1 - v_r) - \underbrace{(n_v - e_v)}_{\Phi_v} = rw_1 - u = e_w.$$

Since every single morphism is equivariant on R-valued points, where R is a K-algebra, the morphism $F_w^v\colon \mathbb{P}(\bigwedge^{e_v} V^{(v)^\vee}) \to \mathbb{P}(\bigwedge^{e_w} V^{(w)^\vee})$ is equivariant on R-valued points with respect to the action of the group $\mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{|w|}}(\sigma^* D \otimes_{k(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{|w|})$. Moreover, since we have a commutative diagram (recall that the horizontal morphisms are induced by the multiplication map $k(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{|w|} \to R[\![z-\zeta]\!]/(z-\zeta)^{|w|}$)

$$\begin{tikzcd}
J^{\sim w}(R) = J(R[\![z-\zeta]\!]/(z-\zeta)^{|w|}) \arrow[r] \arrow[d] & \mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{|w|}}(\sigma^* D \otimes_{k(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{|w|}) \arrow[d] \\
J^{\sim v}(R) = J(R[\![z-\zeta]\!]/(z-\zeta)^{|v|}) \arrow[r] & \mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{|v|}}(\sigma^* D \otimes_{k(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{|v|}),
\end{tikzcd}$$

the morphism F_w^v is also $J^{\sim w}$-equivariant. The morphism F_w^v is compatible with our inclusion $Q_{D,\leq v} \to Q_{D,\leq w}$, i.e. the diagram

$$\begin{tikzcd}
\mathbb{P}(\bigwedge^{e_v} V^{(v)^\vee}) \arrow[r, "F_w^v"] & \mathbb{P}(\bigwedge^{e_w} V^{(w)^\vee}) \\
Q_{D,\leq v} \arrow[r] \arrow[u, "\iota_v"] & Q_{D,\leq w} \arrow[u, "\iota_w"']
\end{tikzcd}$$

commutes. On the other hand we define the morphism $G_w^v\colon \mathbb{V}(\bigwedge^{e_v} V^{(v)^\vee}) \to \mathbb{V}(\bigwedge^{e_w} V^{(w)^\vee})$ as the composition

$$\begin{tikzcd}
\mathbb{V}(\bigwedge^{n_v-e_v} V^{(v)}) \arrow[r, "a_{\mathbb{V}}"] & \mathbb{V}(\bigwedge^{n_v-e_v} V^{(v_r,w_1)}) \arrow[d, "\cong"] & \\
\mathbb{V}(\bigwedge^{e_v} V^{(v)^\vee}) \arrow[u, "\cong"] & \mathbb{V}(\bigwedge^{e_w} V^{(v_r,w_1)^\vee}) \arrow[r, "b_{\mathbb{V}}"] & \mathbb{V}(\bigwedge^{e_w} V^{(w)^\vee}),
\end{tikzcd} \tag{3.4.1}$$

where we use (3.1.6). As opposed to the case of the morphism F^v_w, the answer to the question whether the morphism is equivariant is different. The morphism $a_{\mathbb{V}}\colon \mathbb{V}(\bigwedge^{n_v-e_v} V^{(v)}) \to \mathbb{V}(\bigwedge^{n_v-e_v} V^{(v_r,w_1)})$ is equivariant on R-valued points only with respect to the natural action induced by the action of $\mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{w_1-v_r}}(\sigma^*D \otimes_{k((z))} R[\![z-\zeta]\!]/(z-\zeta)^{w_1-v_r})$, but in order to make the vertical isomorphisms equivariant, we need to change the actions on $\mathbb{V}(\bigwedge^{n_v-e_v} V^{(v)})$ and $\mathbb{V}(\bigwedge^{n_v-e_v} V^{(v_r,w_1)})$ as described in Convention 3.1.15. Therefore we are in the following situation: Let $g \in J^{\sim w}(R) = J(R[\![z-\zeta]\!]/(z-\zeta)^{|w|})$ which we can view as an element in $\mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{|w|}}(\sigma^*D \otimes_{k((z))} R[\![z-\zeta]\!]/(z-\zeta)^{|w|})$ and suppose it maps under the natural morphisms to $\overline{g} \in \mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{w_1-v_r}}(\sigma^*D \otimes_{k((z))} R[\![z-\zeta]\!]/(z-\zeta)^{w_1-v_r})$ and to $\overline{\overline{g}} \in \mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{|v|}}(\sigma^*D \otimes_{k((z))} R[\![z-\zeta]\!]/(z-\zeta)^{|v|})$. With these notations the action of g on R-valued points of (3.4.1), where R is a K-algebra, is given in the following diagram.

$$\begin{tikzcd}
\bigwedge^{n_v-e_v}(\overline{\overline{g}}^{\vee})^{-1} \cdot \bigwedge^{n_v}\overline{\overline{g}} \arrow[r, rightsquigarrow] & \bigwedge^{n_v-e_v}(\overline{g}^{\vee})^{-1} \cdot \bigwedge^{n_v}\overline{\overline{g}} \arrow[d, rightsquigarrow] & \\
\bigwedge^{e_v}\overline{\overline{g}} \arrow[u, rightsquigarrow] & \bigwedge^{e_w}\overline{g} \cdot (\bigwedge^{r(w_1-v_r)}\overline{g})^{-1} \cdot \bigwedge^{n_v}\overline{\overline{g}} \arrow[r, rightsquigarrow] & \bigwedge^{e_w} g \cdot \underbrace{(\bigwedge^{r(w_1-v_r)}\overline{g})^{-1} \cdot \bigwedge^{n_v}\overline{\overline{g}}}_{=(\det_R \overline{g})^{-1}\cdot \det_R \overline{\overline{g}}}
\end{tikzcd}$$

This shows that the morphism $G^v_w\colon \mathbb{V}(\bigwedge^{e_v} V^{(v)^\vee}) \to \mathbb{V}(\bigwedge^{e_w} V^{(w)^\vee})$ is not equivariant for the action of the group $J^{\sim w}$ since we have the extra factor $(\det_R \overline{g})^{-1} \cdot \det_R \overline{\overline{g}}$. In order to fix this issue, we will now calculate what this factor looks like. Let $c, d \in \mathbb{Z}$ with $c \leq d$. If $\mathcal{B} = (b_1, \ldots, b_r)$ is a basis of D we get an induced basis $\sigma^*\mathcal{B} = (\sigma^*b_1, \ldots, \sigma^*b_r)$ on σ^*D. For every K-algebra R the basis $\sigma^*\mathcal{B}$ gives rise to an $R[\![z-\zeta]\!]/(z-\zeta)^{d-c}$-basis of $P^{(c,d)}$. Furthermore we denote the K-basis

$$\begin{aligned}
\big(&\sigma^*b_1 \otimes (z-\zeta)^c, \ldots, \sigma^*b_r \otimes (z-\zeta)^c, \\
&\vdots \\
&\sigma^*b_1 \otimes (z-\zeta)^{d-1}, \ldots, \sigma^*b_r \otimes (z-\zeta)^{d-1}\big).
\end{aligned}$$

of $V^{(c,d)}$ by $(\sigma^*\mathcal{B})^{(c,d)}$. Let $h \in \mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{d-c}}(\sigma^*D \otimes_{k((z))} R[\![z-\zeta]\!]/(z-\zeta)^{d-c})$ and suppose that h corresponds, with respect to the $R[\![z-\zeta]\!]/(z-\zeta)^{d-c}$-basis induced by $\sigma^*\mathcal{B}$, to the matrix $h = (h_{ij})_{1\leq i,j\leq r}$ with $h_{ij} = \sum_{l=0}^{d-c-1} h_{ijl}(z-\zeta)^l$ $(h_{ijl} \in R)$. If we consider h as an element of $\mathrm{GL}_R(\sigma^*D \otimes_{k((z))} R[\![z-\zeta]\!]/(z-\zeta)^{d-c})$ the matrix corresponding to h with respect to the basis induced by $(\sigma^*\mathcal{B})^{(c,d)}$ is the block matrix

$$\begin{pmatrix}
H_0 & 0 & \cdots & 0 \\
H_1 & H_0 & \ddots & \vdots \\
\vdots & \vdots & \ddots & 0 \\
H_{d-c-1} & H_{d-c-2} & \cdots & H_0
\end{pmatrix},$$

with $H_l = (h_{ijl})_{1\leq i,j\leq r}$. Hence we see that $\det_R h = (\det_R H_0)^{d-c}$. The morphism

$$\begin{aligned}(\_)_0\colon R[\![z-\zeta]\!]/(z-\zeta)^{d-c} &\to R\\ \sum_{i=0}^{d-c-1} a_i(z-\zeta)^i &\mapsto a_0\end{aligned}$$

induces a morphism $\mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{d-c}}(\sigma^*D \otimes_{k(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{d-c}) \to \mathrm{GL}_R(\sigma^*D \otimes_{k(\!(z)\!)} R)$ which we again denote by $(\_)_0$ and we get that

$$\det_R H_0 = \det_R(h)_0 = (\det_{R[\![z-\zeta]\!]/(z-\zeta)^{d-c}} h)_0.$$

Therefore we see that

$$\begin{aligned}(\det_R \overline{g})^{-1} \cdot \det_R \overline{\overline{g}} &= (\det_{R[\![z-\zeta]\!]/(z-\zeta)^{w_1-v_r}} \overline{g})_0^{v_r-w_1} \cdot (\det_{R[\![z-\zeta]\!]/(z-\zeta)^{|v|}} \overline{\overline{g}})_0^{v_1-v_r}\\ &= (\det_{R[\![z-\zeta]\!]/(z-\zeta)^{|w|}} g)_0^{v_1-w_1}.\end{aligned}$$

This observation makes it possible to change the action of $J^{\sim w}$, in order to get an equivariant morphism $G_w^v\colon \mathbb{V}(\bigwedge^{e_v} V^{(v)\vee}) \to \mathbb{V}(\bigwedge^{e_w} V^{(w)\vee})$. For every Hodge-Pink weight w we change the $\mathrm{GL}_{R[\![z-\zeta]\!]/(z-\zeta)^{|w|}}(\sigma^*D \otimes_{k(\!(z)\!)} R[\![z-\zeta]\!]/(z-\zeta)^{|w|})$ action on $\mathbb{V}(\bigwedge^{e_w} V^{(w)\vee})(R)$ to

$$g \mapsto \wedge^{e_w} g \cdot (\det_{R[\![z-\zeta]\!]/(z-\zeta)^{|w|}} g)_0^{-w_1}.$$

This extra factor $(\det_{R[\![z-\zeta]\!]/(z-\zeta)^{|w|}} g)_0^{-w_1}$ takes care of the choice of the embedding of the Grassmannian into projective space we have done in Convention 3.1.15. We need this normalization since the definition of the morphism $G_w^v\colon \mathbb{V}(\bigwedge^{e_v} V^{(v)\vee}) \to \mathbb{V}(\bigwedge^{e_w} V^{(w)\vee})$ uses both embeddings we have discussed there and hence is not independent of this choice.

This also explains the change of the linearization done in Section 3.3.1. We define $\chi\colon J \to \mathbb{G}_m$ on A-valued points (A an $\mathbb{F}_q(\!(z)\!)$-algebra) as the restriction of $\det\colon \mathrm{End}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)}A}(D \otimes_{\mathbb{F}_q(\!(z)\!)} A) \to k(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} A$ to $J(A)$, which has values in A by the same argument as in Lemma 1.3.1. Again by the argumentation in the proof of Lemma 1.3.1, we know that this restriction of the determinant is invariant under σ. Therefore it coincides with the restriction of $\det\colon \mathrm{End}_{k(\!(z)\!)\otimes_{\mathbb{F}_q(\!(z)\!)}A}(\sigma^*D \otimes_{\mathbb{F}_q(\!(z)\!)} A) \to k(\!(z)\!) \otimes_{\mathbb{F}_q(\!(z)\!)} A$ and moreover we have seen in the proof of Lemma 3.2.4 that this is equal to $\det_B g$ if we consider $g \in J(B)$ as an element of $\mathrm{GL}_B(\sigma^*D \otimes_{k(\!(z)\!)} B)$ via the the morphism induced by the multiplication map for every $k(\!(z)\!)$-algebra B. We denote the composition

$$J^{\sim w} \xrightarrow{\chi^{\sim w}} {\mathbb{G}_m}^{\sim w} \xrightarrow{(\_)_0} \mathbb{G}_{m,K}$$

by χ_0. Thus for a K-algebra R and $g \in J^{\sim w}$ we get that $\chi_0(g) = (\det_{R[\![z-\zeta]\!]/(z-\zeta)^{|w|}} g)_0$. With these notations we can rewrite the quasi-character $\psi_w\colon J^{\sim w} \to \mathbb{G}_{m,K}$ as

$$g \mapsto \chi_0(g)^{-w_1} \cdot \prod_{\nu\in\mathbb{Q}} \chi_{\nu,0}(g)^{-\nu}.$$

The factor $\chi_0(g)^{-w_1}$ is responsible for the normalization described above and the factor $\prod_{\nu\in\mathbb{Q}} \chi_{\nu,0}(g)^{-\nu}$ handles the Newton slope and is independent of the fixed Hodge-Pink weight. Summarizing this we have shown

Theorem 3.4.1. *Let $v = (v_1, \dots, v_r) \in \mathbb{Z}^r$ and $w = (w_1, \dots, w_r) \in \mathbb{Z}^r$ with $v_1 \geq \dots \geq v_r$ and $w_1 \geq \dots \geq w_r$ such that $v \preceq w$ for the Bruhat-order. Let $F_w^v \colon \mathbb{P}(\bigwedge^{e_v} V^{(v)^\vee}) \to \mathbb{P}(\bigwedge^{e_w} V^{(w)^\vee})$ be the morphism defined above and let $\mathscr{L}_v \in \operatorname{Pic}^{J^{\sim v}}(\mathbb{P}(\bigwedge^{e_v} V^{(v)^\vee})) \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathscr{L}_w \in \operatorname{Pic}^{J^{\sim w}}(\mathbb{P}(\bigwedge^{e_w} V^{(w)^\vee})) \otimes_{\mathbb{Z}} \mathbb{Q}$ be the invertible sheaves together with their quasi-linearization as being defined at the end of Section 3.3.1. In this case we have*

$$\mathscr{L}_v \cong (F_w^v)^*(\mathscr{L}_w)$$

as elements of $\operatorname{Pic}^{J^{\sim v}}(\mathbb{P}(\bigwedge^{e_v} V^{(v)^\vee})) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Corollary 3.4.2. *If we denote the embedding of $Q_{D,\leq v}$ (resp. $Q_{D,\leq w}$) into $\mathbb{P}(\bigwedge^{e_v} V^{(v)^\vee})$ (resp. $\mathbb{P}(\bigwedge^{e_w} V^{(w)^\vee})$) by ι_v (resp. ι_w) and if we have $\mathfrak{q} \in Q_{D,\leq v}(L) \subseteq Q_{D,\leq w}(L)$ for a field extension L of K, we get $\mu^{\iota_v^*\mathscr{L}_v}(\mathfrak{q}, \lambda_0) = \mu^{\iota_w^*\mathscr{L}_w}(\mathfrak{q}, \lambda_0)$ for every 1-PS λ of J.*

This Corollary tells us that it does not matter whether we view a Hodge-Pink lattice as an element of $Q_{D,\leq v}$ or of $Q_{D,\leq w}$ in order to calculate the GIT-slope. This makes perfect sense since the property of being weakly admissible is of course independent of the chosen bound.

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