C*-algebras over topological spaces: the bootstrap class

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Dedicated to Joachim Cuntz on the occasion of his 60th birthday

Abstract. We carefully define and study C*-algebras over topological spaces, possibly non-Hausdorff, and review some relevant results from point-set topology along the way. We explain the triangulated category structure on the bivariant Kasparov theory over a topological space and study the analogue of the bootstrap class for C*-algebras over a finite topological space.

1. Introduction

If X is a locally compact Hausdorff space, then there are various equivalent characterizations of what it means for X to act on a C*-algebra A. The most common definition uses an essential *-homomorphism from $C_0(X)$ to the center of the multiplier algebra of A. An action of this kind is equivalent to a continuous map from the primitive ideal space $\mathrm{Prim}(A)$ of A to X. This makes sense in general: A C*-algebra over a topological space X, which may be non-Hausdorff, is a pair (A, ψ), where A is a C*-algebra and $\psi \colon \mathrm{Prim}(A) \to X$ is a continuous map. One of the purposes of this article is to discuss this definition and relate it to other notions due to Eberhard Kirchberg and Alexander Bonkat [10, 4].

An analogue of Kasparov theory for C*-algebras over locally compact Hausdorff spaces was defined already by Gennadi Kasparov in [9]. He used it in his proof of the Novikov conjecture for subgroups of Lie groups. Kasparov's definition was extended by Eberhard Kirchberg to the non-Hausdorff case in [10], in order to generalize classification results for simple, purely infinite, nuclear C*-algebras to the non-simple case. In his thesis [4], Alexander Bonkat studies

an even more general theory and extends the basic results of Kasparov theory to this setting.

This article is part of an ongoing project to compute the Kasparov groups $KK_*(X; A, B)$ for a topological space X and C*-algebras A and B over X. The aim is a Universal Coefficient Theorem in this context that is useful for the classification programme. At the moment, we can achieve this goal for *some* finite topological spaces (see [16]), but the general situation, even in the finite case, is unclear.

Here we describe an analogue of the bootstrap class for C*-algebras over a topological space. Although we also propose a definition for infinite spaces in §4.4, most of our results are limited to finite spaces.

Our first task is to clarify the definition of C*-algebras over X; this is the main point of Section 2. Our definition is quite natural, but more restrictive than the definitions in [10, 4]. The approach there is to use the map $\mathbb{O}(X) \to \mathbb{O}(\mathrm{Prim}\,A)$ induced by $\psi\colon \mathrm{Prim}(A) \to X$, where $\mathbb{O}(X)$ denotes the complete lattice of open subsets of X. If X is a sober space—this is a very mild assumption that is also made under a different name in [10, 4]—then we can recover it from the lattice $\mathbb{O}(X)$, and a continuous map $\mathrm{Prim}(A) \to X$ is equivalent to a map $\mathbb{O}(X) \to \mathbb{O}(\mathrm{Prim}\,A)$ that commutes with arbitrary unions and finite intersections.

The definition of the Kasparov groups $KK_*(X; A, B)$ still makes sense for *any* map $\mathbb{O}(X) \to \mathbb{O}(\mathrm{Prim}\,A)$ (in the category of sets), that is, even the restrictions imposed in [10, 4] can be removed. But such a map $\mathbb{O}(X) \to \mathbb{O}(\mathrm{Prim}\,A)$ corresponds to a continuous map $\mathrm{Prim}(A) \to Y$ for another, more complicated space Y that contains X as a subspace. Hence the definitions in [10, 4] are, in fact, not more general. But they complicate computations because the discontinuities add further input data which must be taken into account even for examples where they vanish because the action is continuous.

Since the relevant point-set topology is widely unknown among operator algebraists, we also recall some basic notions such as sober spaces and Alexandrov spaces. The latter are highly non-Hausdorff spaces—-Alexandrov T_1-spaces are necessarily discrete—which are essentially the same as preordered sets. Any finite topological space is an Alexandrov space, and their basic properties are crucial for this article. To get acquainted with the setup, we simplify the description of C*-algebras over Alexandrov spaces and discuss some small examples. These rather elementary considerations appeared previously in the theory of locales.

In Section 3, we briefly recall the definition and the basic properties of bivariant Kasparov theory for C*-algebras over a topological space. We omit most proofs because they are similar to the familiar arguments for ordinary Kasparov theory and because the technical details are already dealt with in [4]. We emphasize the triangulated category structure on the Kasparov category over X because it plays an important role in connection with the bootstrap class.

In Section 4, we define the bootstrap class over a topological space X. If X is finite, we give criteria for a C*-algebra over X to belong to the bootstrap class. These depend heavily on the relation between Alexandrov spaces and preordered sets and therefore do not extend directly to infinite spaces.

We define the X-equivariant bootstrap class $\mathcal{B}(X)$ as the localizing subcategory of the Kasparov category of C*-algebras over X that is generated by the basic objects $(\mathbb{C}, x)$ for $x \in X$, where we identify $x \in X$ with the corresponding constant map $\mathrm{Prim}(\mathbb{C}) \to X$. Notice that this is exactly the list of all C*-algebras over X with underlying C*-algebra $\mathbb{C}$.

We show that a *nuclear* C*-algebra (A, ψ) over X is in the X-equivariant bootstrap class if and only if its "fibers" $A(x)$ belong to the usual bootstrap class for all $x \in X$. These fibers are certain subquotients of A; if $\psi \colon \mathrm{Prim}(A) \to X$ is a homeomorphism, then they are exactly the *simple* subquotients of the C*-algebra A.

The bootstrap class we define is the class of objects where we expect a Universal Coefficient Theorem to hold. If A and B belong to the bootstrap class, then an element of $\mathrm{KK}_*(X; A, B)$ is invertible if and only if it is fiberwise invertible on K-theory, that is, the induced maps $\mathrm{K}_*\big(A(x)\big) \to \mathrm{K}_*\big(B(x)\big)$ are invertible for all $x \in X$. This follows easily from our definition of the bootstrap class. The proof of our criterion for a C*-algebra over X to belong to the bootstrap class already provides a spectral sequence that computes $\mathrm{KK}_*(X; A, B)$ in terms of nonequivariant Kasparov groups. Unfortunately, this spectral sequence is not useful for classification purposes because it rarely degenerates to an *exact* sequence.

We call a C*-algebra over X *tight* if the map $\mathrm{Prim}(A) \to X$ is a homeomorphism. This implies that its fibers are simple. We show in Section 5 that any separable nuclear C*-algebra over X is $\mathrm{KK}(X)$-equivalent to a tight, separable, nuclear, purely infinite, stable C*-algebra over X. The main issue is tightness. By Kirchberg's classification result, this model is unique up to X-equivariant *-isomorphism. In this sense, tight, separable, nuclear, purely infinite, stable C*-algebras over X are classified up to isomorphism by the isomorphism classes of objects in a certain triangulated category: the subcategory of nuclear C*-algebras over X in the Kasparov category. The difficulty is to replace this complete "invariant" by a more tractable one that classifies objects of the—-possibly smaller—bootstrap category $\mathcal{B}(X)$ by K-theoretic data.

If $\mathfrak{C}$ is a category, then we write $A \in\in \mathfrak{C}$ to denote that A is an object of $\mathfrak{C}$—as opposed to a morphism in $\mathfrak{C}$.

2. C*-ALGEBRAS OVER A TOPOLOGICAL SPACE

We define the category $\mathfrak{C}^*\mathfrak{alg}(X)$ of C*-algebras over a topological space X. In the Hausdorff case, this amounts to the familiar category of $\mathrm{C}_0(X)$-C*-algebras. For non-Hausdorff spaces, our notion is related to another one by Eberhard Kirchberg. For the Universal Coefficient Theorem, we must add some continuity conditions to Kirchberg's definition of $\mathfrak{C}^*\mathfrak{alg}(X)$. We explain in §2.9

why these conditions result in essentially no loss of generality. Furthermore, we explain briefly why it is allowed to restrict to the case where the underlying space X is sober, and we consider some examples, focusing on special properties of finite spaces and Alexandrov spaces.

2.1. **The Hausdorff case.** Let A be a C*-algebra and let X be a locally compact Hausdorff space. There are various equivalent additional structures on A that turn it into a C*-algebra over X (see [17] for the proofs of most of the following assertions). The most common definition is the following one from [9]:

Definition 2.1. A $C_0(X)$-C*-*algebra* is a C*-algebra A together with an essential *-homomorphism φ from $C_0(X)$ to the center of the multiplier algebra of A. We abbreviate $h \cdot a := \varphi(h) \cdot a$ for $h \in C_0(X)$.

A *-homomorphism $f\colon A \to B$ between two $C_0(X)$-C*-algebras is $C_0(X)$-*linear* if $f(h \cdot a) = h \cdot f(a)$ for all $h \in C_0(X)$, $a \in A$.

Let $\mathfrak{C}^*\mathfrak{alg}\big(C_0(X)\big)$ be the category of $C_0(X)$-C*-algebras, whose morphisms are the $C_0(X)$-linear *-homomorphisms.

A map φ as above is equivalent to an A-linear essential *-homomorphism

$$\bar{\varphi}\colon C_0(X, A) \cong C_0(X) \otimes_{\max} A \to A, \qquad f \otimes a \mapsto \varphi(f) \cdot a,$$

which exists by the universal property of the maximal tensor product; the centrality of φ ensures that $\bar{\varphi}$ is a *-homomorphism and well-defined. Conversely, we get φ back from $\bar{\varphi}$ by restricting to elementary tensors; the assumed A-linearity of $\bar{\varphi}$ ensures that $\varphi(h) \cdot a := \bar{\varphi}(h \otimes a)$ is a multiplier of A. The description via $\bar{\varphi}$ has two advantages: it requires no multipliers, and the resulting class in $KK_0(C_0(X, A), A)$ plays a role in connection with duality in bivariant Kasparov theory (see [8]).

Any $C_0(X)$-C*-algebra is isomorphic to the C*-algebra of $C_0(X)$-sections of an upper semi-continuous C*-algebra bundle over X (see [17]). Even more, this yields an equivalence of categories between $\mathfrak{C}^*\mathfrak{alg}\big(C_0(X)\big)$ and the category of upper semi-continuous C*-algebra bundles over X.

Definition 2.2. Let $\operatorname{Prim}(A)$ denote the *primitive ideal space* of A, equipped with the usual hull-kernel topology, also called Jacobson topology.

The Dauns-Hofmann Theorem identifies the center of the multiplier algebra of A with the C*-algebra $C_b\big(\operatorname{Prim}(A)\big)$ of bounded continuous functions on the primitive ideal space of A. Therefore, the map φ in Definition 2.1 is of the form

$$\psi^*\colon C_0(X) \to C_b(\operatorname{Prim} A), \qquad f \mapsto f \circ \psi,$$

for some continuous map $\psi\colon \operatorname{Prim}(A) \to X$ (see [17]). Thus φ and ψ are equivalent additional structures. We use such maps ψ to generalize Definition 2.1 to the non-Hausdorff case.

2.2. **The general definition.** Let X be an arbitrary topological space.

Definition 2.3. A C*-*algebra over* X is a pair (A, ψ) where A is a C*-algebra and is ψ a continuous map $\mathrm{Prim}(A) \to X$.

Our next task is to define morphisms between C*-algebras A and B over the same space X. This requires some care because the primitive ideal space is *not* functorial for arbitrary *-homomorphisms.

Definition 2.4. For a topological space X, let $\mathbb{O}(X)$ be the set of open subsets of X, partially ordered by $\subseteq$.

Definition 2.5. For a C*-algebra A, let $\mathbb{I}(A)$ be the set of all closed *-ideals in A, partially ordered by $\subseteq$.

The partially ordered sets $(\mathbb{O}(X), \subseteq)$ and $(\mathbb{I}(A), \subseteq)$ are *complete lattices*, that is, any subset in them has both an infimum $\bigwedge S$ and a supremum $\bigvee S$. Namely, in $\mathbb{O}(X)$, the supremum is $\bigcup S$, and the infimum is the *interior of* $\bigcap S$; in $\mathbb{I}(A)$, the infimum and supremum are

$$\bigwedge_{I \in S} I = \bigcap_{I \in S} I, \qquad \bigvee_{I \in S} I = \overline{\sum_{I \in S} I}.$$

We always identify $\mathbb{O}\big(\mathrm{Prim}(A)\big)$ and $\mathbb{I}(A)$ using the isomorphism

$$(2.6) \qquad \mathbb{O}\big(\mathrm{Prim}(A)\big) \cong \mathbb{I}(A), \qquad U \mapsto \bigcap_{\mathfrak{p} \in \mathrm{Prim}(A) \setminus U} \mathfrak{p}$$

(see [7, §3.2]). This is a lattice isomorphism and hence preserves infima and suprema.

Let (A, ψ) be a C*-algebra over X. We get a map

$$\psi^* \colon \mathbb{O}(X) \to \mathbb{O}(\mathrm{Prim}\, A) \cong \mathbb{I}(A), \qquad U \mapsto A(U) := \{\mathfrak{p} \in \mathrm{Prim}(A) \mid \psi(\mathfrak{p}) \in U\}.$$

We usually write $A(U) \in \mathbb{I}(A)$ for the ideal and $\psi^*(U)$ or $\psi^{-1}(U)$ for the corresponding open subset of $\mathrm{Prim}(A)$. If X is a locally compact Hausdorff space, then $A(U) := \mathrm{C}_0(U) \cdot A$ for all $U \in \mathbb{O}(X)$.

Example 2.7. For any C*-algebra A, the pair $(A, \mathrm{id}_{\mathrm{Prim}\, A})$ is a C*-algebra over $\mathrm{Prim}(A)$; the ideals $A(U)$ for $U \in \mathbb{O}(\mathrm{Prim}\, A)$ are given by (2.6). C*-algebras over topological spaces of this form play an important role in §5, where we call them *tight*.

Lemma 2.8. *The map* ψ^* *is compatible with arbitrary suprema* (*unions*) *and finite infima* (*intersections*), *so that*

$$A\bigg(\bigcup_{U \in S} U\bigg) = \overline{\sum_{U \in S} A(U)}, \qquad A\bigg(\bigcap_{U \in F} U\bigg) = \bigcap_{U \in F} A(U)$$

for any subset $S \subseteq \mathbb{O}(X)$ *and for any finite subset* $F \subseteq \mathbb{O}(X)$.

Proof. This is immediate from the definition. $\qquad\square$

Taking for S and F the empty set, this specializes to $A(\varnothing) = \{0\}$ and $A(X) = A$. Taking $S = \{U, V\}$ with $U \subseteq V$, this specializes to the monotonicity property

$$U \subseteq V \qquad \implies \qquad A(U) \subseteq A(V);$$

We will implicitly use later that these properties follow from compatibility with finite infima and suprema.

The following lemma clarifies when the map ψ^* is compatible with infinite infima.

Lemma 2.9. *If the map $\psi\colon \mathrm{Prim}(A) \to X$ is open or if X is finite, then the map $\psi^*\colon \mathbb{O}(X) \to \mathbb{I}(A)$ preserves infima—that is, it maps the interior of $\bigcap_{U \in S} U$ to the ideal $\bigcap_{U \in S} A(U)$ for any subset $S \subseteq \mathbb{O}(X)$. Conversely, if ψ^* preserves infima and X is a T_1-space, that is, points in X are closed, then ψ is open.*

Since preservation of infinite infima is automatic for finite X, the converse assertion cannot hold for general X.

Proof. If X is finite, then any subset of $\mathbb{O}(X)$ is finite, and there is nothing more to prove. Suppose that ψ is open. Let V be the interior of $\bigcap_{U \in S} U$. Let $W \subseteq \mathrm{Prim}(A)$ be the open subset that corresponds to the ideal $\bigcap_{U \in S} \psi^*(U)$. We must show $\psi^*(V) = W$. Monotonicity yields $\psi^*(V) \subseteq W$. Since ψ is open, $\psi(W)$ is an open subset of X. By construction, $\psi(W) \subseteq U$ for all $U \in S$ and hence $\psi(W) \subseteq V$. Thus $\psi^*(V) \supseteq \psi^*\big(\psi(W)\big) \supseteq W \supseteq \psi^*(V)$, so that $\psi^*(V) = W$.

Now suppose, conversely, that ψ^* preserves infima and that points in X are closed. Assume that ψ is not open. Then there is an open subset W in $\mathrm{Prim}(A)$ for which $\psi(W)$ is not open in X. Let $S := \{X \backslash \{x\} \mid x \in X \backslash \psi(W)\} \subseteq \mathbb{O}(X)$; this is where we need points to be closed. We have $\bigcap_{U \in S} U = \psi(W)$ and $\bigcap_{U \in S} \psi^*(U) = \psi^{-1}\big(\psi(W)\big)$. Since $\psi(W)$ is not open, the infimum V of S in $\mathbb{O}(X)$ is strictly smaller than $\psi(W)$. Hence $\psi^*(V)$ cannot contain W. But W is an open subset of $\psi^{-1}\big(\psi(W)\big)$ and hence contained in the infimum of $\psi^*(S)$ in $\mathbb{O}(\mathrm{Prim}\, A)$. Therefore, ψ^* does not preserve infima, contrary to our assumption. Hence ψ must be open. $\qquad\square$

For a locally compact Hausdorff space X, the map $\mathrm{Prim}(A) \to X$ is open if and only if A corresponds to a *continuous* C*-algebra bundle over X (see [17, Thm. 2.3]).

Definition 2.10. Let A and B be C*-algebras over a topological space X. A *-homomorphism $f\colon A \to B$ is *X-equivariant* if $f\big(A(U)\big) \subseteq B(U)$ for all $U \in \mathbb{O}(X)$.

For locally compact Hausdorff spaces, this is equivalent to $C_0(X)$-linearity by the following variant of [4, Prop. 5.4.7]:

Proposition 2.11. *Let A and B be C*-algebras over a locally compact Hausdorff space X, and let $f\colon A \to B$ be a *-homomorphism. The following assertions are equivalent:*

(1) f is $C_0(X)$-linear;
(2) f is X-equivariant, that is, $f\big(A(U)\big) \subseteq B(U)$ for all $U \in \mathbb{O}(X)$;
(3) f descends to the fibers, that is, $f\big(A(X \setminus \{x\})\big) \subseteq B(X \setminus \{x\})$ for all $x \in X$.

To understand the last condition, recall that the fibers of the C*-algebra bundle associated to A are $A_x := A \,/\, A(X \setminus \{x\})$. Condition (3) means that f descends to maps $f_x \colon A_x \to B_x$ for all $x \in X$.

Proof. It is clear that $(1) \Longrightarrow (2) \Longrightarrow (3)$. The equivalence $(3) \Longleftrightarrow (1)$ is the assertion of [4, Prop. 5.4.7]. To check that (3) implies (1), take $h \in C_0(X)$ and $a \in A$. We get $f(h \cdot a) = h \cdot f(a)$ provided both sides have the same values at all $x \in X$ because the map $A \to \prod_{x \in X} A_x$ is injective. Now (3) implies $f(h \cdot a)_x = h(x) \cdot f(a)_x$ because $\big(h - h(x)\big) \cdot a \in A(X \setminus \{x\})$. $\square$

Definition 2.12. Let $\mathfrak{C}^*\mathfrak{alg}(X)$ be the category whose objects are C*-algebras over X and whose morphisms are X-equivariant *-homomorphisms. We write $\mathrm{Hom}_X(A, B)$ for this set of morphisms.

Proposition 2.11 yields an isomorphism of categories

$$\mathfrak{C}^*\mathfrak{alg}\big(C_0(X)\big) \cong \mathfrak{C}^*\mathfrak{alg}(X).$$

In this sense, our theory for general spaces extends the more familiar theory of $C_0(X)$-C*-algebras.

2.3. Locally closed subsets and subquotients.

Definition 2.13. A subset C of a topological space X is called *locally closed* if it is the intersection of an open and a closed subset or, equivalently, of the form $C = U \setminus V$ with $U, V \in \mathbb{O}(X)$; we can also assume $V \subseteq U$ here. We let $\mathbb{LC}(X)$ be the set of locally closed subsets of X.

A subset is locally closed if and only if it is relatively open in its closure. Being locally closed is inherited by finite intersections, but *not* by unions or complements.

Definition 2.14. Let X be a topological space and let (A, ψ) be a C*-algebra over X. Write $C \in \mathbb{LC}(X)$ as $C = U \setminus V$ for open subsets $U, V \subseteq X$ with $V \subseteq U$. We define

$$A(C) := A(U) \,/\, A(V).$$

Lemma 2.15. *The subquotient $A(C)$ does not depend on U and V above.*

Proof. Let $U_1, V_1, U_2, V_2 \in \mathbb{O}(X)$ satisfy $V_1 \subseteq U_1$, $V_2 \subseteq U_2$, and $U_1 \setminus V_1 = U_2 \setminus V_2$. Then $V_1 \cup U_2 = U_1 \cup U_2 = U_1 \cup V_2$ and $V_1 \cap U_2 = V_1 \cap V_2 = U_1 \cap V_2$. Since $U \mapsto A(U)$ preserves unions, this implies

$$A(U_2) + A(V_1) = A(U_1) + A(V_2).$$

We divide this equation by $A(V_1 \cup V_2) = A(V_1) + A(V_2)$. This yields

$$\frac{A(U_2) + A(V_1)}{A(V_1 \cup V_2)} \cong \frac{A(U_2)}{A(U_2) \cap A(V_1 \cup V_2)} = \frac{A(U_2)}{A\big(U_2 \cap (V_1 \cup V_2)\big)} = \frac{A(U_2)}{A(V_2)}$$

on the left hand side and, similarly, $A(U_1) \,/\, A(V_1)$ on the right hand side. Hence $A(U_1) \,/\, A(V_1) \cong A(U_2) \,/\, A(V_2)$ as desired. $\square$

Now assume that $X = \mathrm{Prim}(A)$ and $\psi = \mathrm{id}_{\mathrm{Prim}(A)}$. Lemma 2.15 associates a subquotient $A(C)$ of A to each locally closed subset of $\mathrm{Prim}(A)$. Equation (2.6) shows that any subquotient of A arises in this fashion; here subquotient means: a quotient of one ideal in A by another ideal in A. Open subsets of X correspond to ideals, closed subsets to quotients of A. For any $C \in \mathbb{LC}(\mathrm{Prim}\, A)$, there is a canonical homeomorphism $\mathrm{Prim}\big(A(C)\big) \cong C$. This is well-known if C is open or closed, and the general case reduces to these special cases.

Example 2.16. If $\mathrm{Prim}(A)$ is a finite topological T_0-space, then any singleton $\{\mathfrak{p}\}$ in $\mathrm{Prim}(A)$ is locally closed (this holds more generally for the Alexandrov T_0-spaces introduced in §2.7 and follows from the description of closed subsets in terms of the specialization preorder).

Since $\mathrm{Prim}\, A(C) \cong C$, the subquotients $A_\mathfrak{p} := A(\{\mathfrak{p}\})$ for $\mathfrak{p} \in \mathrm{Prim}(A)$ are precisely the *simple subquotients* of A.

Example 2.17. Consider the interval $[0,1]$ with the topology where the nonempty closed subsets are the closed intervals $[a,1]$ for all $a \in [0,1]$. A nonempty subset is locally closed if and only if it is either of the form $[a,1]$ or $[a,b)$ for $a, b \in [0,1]$ with $a < b$. In this space, singletons are not locally closed. Hence a C*-algebra with this primitive ideal space has no simple subquotients.

2.4. **Functoriality and tensor products.**

Definition 2.18. Let X and Y be topological spaces. A continuous map $f \colon X \to Y$ induces a functor

$$f_* \colon \mathfrak{C}^*\mathfrak{alg}(X) \to \mathfrak{C}^*\mathfrak{alg}(Y), \qquad (A, \psi) \mapsto (A, f \circ \psi).$$

Thus $X \mapsto \mathfrak{C}^*\mathfrak{alg}(X)$ is a functor from the category of topological spaces to the category of categories (up to the usual issues with sets and classes).

Since $(f \circ \psi)^{-1} = \psi^{-1} \circ f^{-1}$, we have

$$(f_* A)(C) = A\big(f^{-1}(C)\big) \qquad \text{for all } C \in \mathbb{LC}(Y).$$

If $f \colon X \to Y$ is the embedding of a subset with the subspace topology, we also write

$$i_X^Y := f_* \colon \mathfrak{C}^*\mathfrak{alg}(X) \to \mathfrak{C}^*\mathfrak{alg}(Y)$$

and call this the *extension* functor from X to Y. We have $(i_X^Y A)(C) = A(C \cap X)$ for all $C \in \mathbb{LC}(Y)$.

Definition 2.19. Let X be a topological space and let Y be a *locally closed* subset of X, equipped with the subspace topology. Let (A, ψ) be a C*-algebra over X. Its *restriction to Y* is a C*-algebra $A|_Y$ over Y, consisting of the

C*-algebra $A(Y)$ defined as in Definition 2.14, equipped with the canonical map

$$\operatorname{Prim} A(Y) \xrightarrow{\cong} \psi^{-1}(Y) \xrightarrow{\psi} Y.$$

Thus $A|_Y(C) = A(C)$ for $C \in \mathbb{LC}(Y) \subseteq \mathbb{LC}(X)$.

It is clear that the restriction to Y provides a functor

$$r_X^Y \colon \mathfrak{C}^*\mathfrak{alg}(X) \to \mathfrak{C}^*\mathfrak{alg}(Y)$$

that satisfies $r_Y^Z \circ r_X^Y = r_X^Z$ if $Z \subseteq Y \subseteq X$ and $r_X^X = \operatorname{id}$.

If Y and X are Hausdorff and locally compact, then a continuous map $f \colon Y \to X$ also induces a pull-back functor

$$f^* \colon \mathfrak{C}^*\mathfrak{alg}(X) \cong \mathfrak{C}^*\mathfrak{alg}\big(\mathrm{C}_0(X)\big) \to \mathfrak{C}^*\mathfrak{alg}\big(\mathrm{C}_0(Y)\big) \cong \mathfrak{C}^*\mathfrak{alg}(Y),$$

$$A \mapsto \mathrm{C}_0(Y) \otimes_{\mathrm{C}_0(X)} A.$$

If f is the constant map $Y \to \star$, the functor f^* maps a C*-algebra A to $f^*(A) := \mathrm{C}_0(Y, A)$ with the obvious $\mathrm{C}_0(Y)$-C*-algebra structure. This functor has no analogue for a non-Hausdorff space Y. Therefore, a continuous map $f \colon Y \to X$ need not induce a functor $f^* \colon \mathfrak{C}^*\mathfrak{alg}(X) \to \mathfrak{C}^*\mathfrak{alg}(Y)$. For embeddings of locally closed subsets, the functor r_X^Y plays the role of f^*.

Lemma 2.20. *Let X be a topological space and let $Y \subseteq X$.*

(a) *If Y is* open, *then there are natural isomorphisms*

$$\operatorname{Hom}_X(i_Y^X(A), B) \cong \operatorname{Hom}_Y\big(A, r_X^Y(B)\big)$$

if A and B are C-algebras over Y and X, respectively.*
   In other words, i_Y^X is left adjoint to r_X^Y.

(b) *If Y is* closed, *then there are natural isomorphisms*

$$\operatorname{Hom}_Y(r_X^Y(A), B) \cong \operatorname{Hom}_X\big(A, i_Y^X(B)\big)$$

if A and B are C-algebras over X and Y, respectively.*
   In other words, i_Y^X is right adjoint to r_X^Y.

(c) *For any locally closed subset $Y \subseteq X$, we have $r_X^Y \circ i_Y^X(A) = A$ for all C*-algebras A over Y.*

Proof. We first prove (a). We have $i_Y^X(A)(U) = A(U \cap Y)$ for all $U \in \mathbb{O}(X)$, and this is an ideal in $A(U)$. A morphism $\varphi \colon i_Y^X(A) \to B$ is equivalent to a *-homomorphism $\varphi \colon A(Y) \to B(X)$ that maps $A(U \cap Y) \to B(U)$ for all $U \in \mathbb{O}(X)$. This holds for all $U \in \mathbb{O}(X)$ once it holds for $U \in \mathbb{O}(Y) \subseteq \mathbb{O}(X)$. Hence φ is equivalent to a *-homomorphism $\varphi' \colon A(Y) \to B(Y)$ that maps $A(U) \to B(U)$ for all $U \in \mathbb{O}(Y)$. The latter is nothing but a morphism $A \to r_X^Y(B)$. This proves (a).

Now we turn to (b). Again, we have $i_Y^X(B)(U) = B(U \cap Y)$ for all $U \in \mathbb{O}(X)$, but now this is a quotient of $B(U)$. A morphism $\varphi \colon A \to i_Y^X(B)$ is equivalent to a *-homomorphism $\varphi \colon A(X) \to B(Y)$ that maps $A(U) \to B(U \cap Y)$ for all $U \in \mathbb{O}(X)$. Hence $A(X \setminus Y)$ is mapped to $B(\varnothing) = 0$, so that φ descends to a map φ' from $A \,/\, A(X \setminus Y) \cong A(Y)$ to $B(Y)$ that maps $A(U \cap Y)$ to $B(U)$ for

all $U \in \mathbb{O}(X)$. The latter is equivalent to a morphism $r_X^Y(A) \to B$ as desired. This finishes the proof of (b).

Assertion (c) is trivial. $\square$

Example 2.21. For each $x \in X$, we get a map $i_x = i_x^X \colon \star \cong \{x\} \subseteq X$ from the one-point space to X. The resulting functor $\mathfrak{C}^*\mathfrak{alg} \to \mathfrak{C}^*\mathfrak{alg}(X)$ maps a C*-algebra A to the C*-algebra $i_x(A) = (A, x)$ over X, where x also denotes the constant map

$$x \colon \mathrm{Prim}(A) \to X, \qquad \mathfrak{p} \mapsto x \quad \text{for all } \mathfrak{p} \in \mathrm{Prim}(A).$$

If $C \in \mathbb{LC}(X)$, then

$$i_x(A)(C) = \begin{cases} A & \text{if } x \in C; \\ 0 & \text{otherwise.} \end{cases}$$

The functor i_x plays an important role if X is finite. The generators of the bootstrap class are of the form $i_x(\mathbb{C})$. Each C*-algebra over X carries a canonical filtration whose subquotients are of the form $i_x(A)$.

Lemma 2.22. *Let X be a topological space and let $x \in X$. Then*

$$\mathrm{Hom}_X\big(A, i_x^X(B)\big) \cong \mathrm{Hom}\big(A(\overline{\{x\}}), B\big)$$

for all $A \in\in \mathfrak{C}^*\mathfrak{alg}(X)$, $B \in\in \mathfrak{C}^*\mathfrak{alg}$, and

$$\mathrm{Hom}_X(i_x^X(A), B) \cong \mathrm{Hom}\Big(A, \bigcap_{U \in \mathcal{U}_x} B(U)\Big).$$

for all $A \in\in \mathfrak{C}^*\mathfrak{alg}$, $B \in\in \mathfrak{C}^*\mathfrak{alg}(X)$, where $\mathcal{U}_x$ denotes the open neighborhood filter of x in X. If x has a minimal open neighborhood U_x, then this becomes

$$\mathrm{Hom}_X(i_x^X(A), B) \cong \mathrm{Hom}\big(A, B(U_x)\big).$$

Recall that $A \in\in \mathfrak{C}$ means that A is an object of $\mathfrak{C}$.

Proof. Let $C := \overline{\{x\}}$. Then any nonempty open subset $V \subseteq C$ contains x, so that $i_x^C(B)(V) = B$. This implies $\mathrm{Hom}_C(A, i_x^C(B)) \cong \mathrm{Hom}\big(A(C), B\big)$. Combining this with $i_x^X = i_C^X \circ i_x^C$ and the adjointness relation in Lemma 2.20.(b) yields

$$\mathrm{Hom}_X\big(A, i_x^X(B)\big) \cong \mathrm{Hom}_C\big(r_X^C(A), i_x^C(B)\big) \cong \mathrm{Hom}(A(C), B).$$

An X-equivariant *-homomorphism $i_x^X A \to B$ restricts to a family of compatible maps $A = (i_x^X A)(U) \to B(U)$ for all $U \in \mathcal{U}_x$, so that we get a *-homomorphism from A to $\bigcap_{U \in \mathcal{U}_x} B(U)$. Conversely, such a *-homomorphism $A \to \bigcap_{U \in \mathcal{U}_x} B(U)$ provides an X-equivariant *-homomorphism $i_x^X A \to B$. This yields the second assertion. $\square$

Let A and B be C*-algebras and let $A \otimes B$ be their minimal (or spatial) C*-tensor product. Then there is a canonical continuous map

$$\mathrm{Prim}(A) \times \mathrm{Prim}(B) \to \mathrm{Prim}(A \otimes B).$$

Therefore, if A and B are C*-algebras over X and Y, respectively, then $A \otimes B$ is a C*-algebra over $X \times Y$. This defines a bifunctor

$$\otimes \colon \mathfrak{C}^*\mathfrak{alg}(X) \times \mathfrak{C}^*\mathfrak{alg}(Y) \to \mathfrak{C}^*\mathfrak{alg}(X \times Y).$$

In particular, if $Y = \star$ is the one-point space, then we get endofunctors $\_ \otimes B$ on $\mathfrak{C}^*\mathfrak{alg}(X)$ for $B \in \mathfrak{C}^*\mathfrak{alg}$ because $X \times \star \cong X$.

If X is a Hausdorff space, then the diagonal in $X \times X$ is closed and we get an internal tensor product functor $\otimes_X$ in $\mathfrak{C}^*\mathfrak{alg}(X)$ by restricting the external tensor product in $\mathfrak{C}^*\mathfrak{alg}(X \times X)$ to the diagonal. This operation has no analogue for general X.

2.5. **Restriction to sober spaces.** A space is sober if and only if it can be recovered from its lattice of open subsets. Any topological space can be completed to a sober space with the same lattice of open subsets. Therefore, it usually suffices to study C*-algebras over sober topological spaces.

Definition 2.23. A topological space is *sober* if each irreducible closed subset of X is the closure $\overline{\{x\}}$ of exactly one singleton of X. Here an *irreducible* closed subset of X is a nonempty closed subset of X which is not the union of two proper closed subsets of itself.

If X is not sober, let $\hat{X}$ be the set of all irreducible closed subsets of X. There is a canonical map $\iota \colon X \to \hat{X}$ which sends a point $x \in X$ to its closure. If $S \subseteq X$ is closed, let $\hat{S} \subseteq \hat{X}$ be the set of all $A \in \hat{X}$ with $A \subseteq S$. The map $S \mapsto \hat{S}$ commutes with finite unions and arbitrary intersections; in particular, it maps X itself to all of $\hat{X}$ and $\varnothing$ to $\hat{\varnothing} = \varnothing$. Hence the subsets of $\hat{X}$ of the form $\hat{S}$ for closed subsets $S \subseteq X$ form the closed subsets of a topology on $\hat{X}$.

The map ι induces a bijection between the families of closed subsets of X and $\hat{X}$. Hence ι is continuous, and it induces a bijection $\iota^* \colon \mathbb{O}(\hat{X}) \to \mathbb{O}(X)$. It also follows that $\hat{X}$ is a sober space because X and $\hat{X}$ have the same irreducible closed subsets.

Since the morphisms in $\mathfrak{C}^*\mathfrak{alg}(X)$ only use $\mathbb{O}(X)$, the functor

$$\iota_* \colon \mathfrak{C}^*\mathfrak{alg}(X) \to \mathfrak{C}^*\mathfrak{alg}(\hat{X})$$

is fully faithful. Therefore, we do not lose much if we assume our topological spaces to be sober.

The following example shows a pathology that can occur if the separation axiom T_0 fails:

Example 2.24. Let X carry the chaotic topology $\mathbb{O}(X) = \{\varnothing, X\}$. Then $\hat{X} = \star$ is the space with one point. By definition, an action of X on a C*-algebra A is a map $\mathrm{Prim}(A) \to X$. But for a *-homomorphism $A \to B$ between two C*-algebras over X, the X-equivariance condition imposes no restriction. Hence all maps $\mathrm{Prim}(A) \to X$ yield isomorphic objects of $\mathfrak{C}^*\mathfrak{alg}(X)$.

Lemma 2.25. *If X is a sober topological space, then there is a bijective correspondence between continuous maps $\mathrm{Prim}(A) \to X$ and maps $\mathbb{O}(X) \to \mathbb{I}(A)$*

that commute with arbitrary suprema and finite infima; it sends a continuous map $\psi\colon \mathrm{Prim}(A) \to X$ to the map

$$\psi^*\colon \mathbb{O}(X) \to \mathbb{O}\big(\mathrm{Prim}(A)\big) = \mathbb{I}(A).$$

Proof. We have already seen that a continuous map $\psi\colon \mathrm{Prim}(A) \to X$ generates a map ψ^* with the required properties for any space X.

Conversely, let $\psi^*\colon \mathbb{O}(X) \to \mathbb{I}(A)$ be a map that preserves arbitrary unions and finite intersections. Given $\mathfrak{p} \in \mathrm{Prim}(A)$, let $U_{\mathfrak{p}}$ be the union of all $U \in \mathbb{O}(X)$ with $\mathfrak{p} \notin \psi^*(U)$. Then $\mathfrak{p} \notin \psi^*(U_{\mathfrak{p}})$ because ψ^* preserves unions, and $U_{\mathfrak{p}}$ is the maximal open subset with this property. Thus $A_{\mathfrak{p}} := X \setminus U_{\mathfrak{p}}$ is the minimal closed subset with $\mathfrak{p} \notin \psi^*(X \setminus A_{\mathfrak{p}})$. This subset is nonempty because $\psi^*(X) = \mathrm{Prim}(A)$ contains $\mathfrak{p}$, and irreducible because ψ^* preserves finite intersections.

Since X is sober, there is a unique $\psi(\mathfrak{p}) \in X$ with $A_{\mathfrak{p}} = \overline{\{\psi(\mathfrak{p})\}}$. This defines a map $\psi\colon \mathrm{Prim}(A) \to X$. If $U \subseteq X$ is open, then $\psi(\mathfrak{p}) \notin U$ if and only if $A_{\mathfrak{p}} \cap U = \varnothing$, if and only if $\mathfrak{p} \notin \psi^*(U)$. Hence $\psi^*(U) = \psi^{-1}(U)$. This shows that ψ is continuous and generates ψ^*. Thus the map $\psi \to \psi^*$ is surjective.

Since sober spaces are T_0, two different continuous maps $\psi_1, \psi_2\colon \mathrm{Prim}(A) \to X$ generate different maps $\psi_1^*, \psi_2^*\colon \mathbb{O}(X) \to \mathbb{I}(A)$. Hence the map $\psi \to \psi^*$ is also injective. $\square$

2.6. Some very easy examples.
Here we describe C*-algebras over the three sober topological spaces with at most two points.

Example 2.26. If X is a single point, then $\mathfrak{C}^*\mathfrak{alg}(X)$ is isomorphic to the category of C*-algebras (without any extra structure).

Up to homeomorphism, there are two sober topological spaces with two points. The first one is the discrete space.

Example 2.27. The category of C*-algebras over the discrete two-point space is equivalent to the product category $\mathfrak{C}^*\mathfrak{alg} \times \mathfrak{C}^*\mathfrak{alg}$ of pairs of C*-algebras.

More generally, if $X = X_1 \sqcup X_2$ is a disjoint union of two subspaces, then

$$(2.28) \qquad \mathfrak{C}^*\mathfrak{alg}(X_1 \sqcup X_2) \simeq \mathfrak{C}^*\mathfrak{alg}(X_1) \times \mathfrak{C}^*\mathfrak{alg}(X_2).$$

Thus it usually suffices to study connected spaces.

Example 2.29. Another sober topological space with two points is $X = \{1, 2\}$ with

$$\mathbb{O}(X) = \big\{\varnothing, \{1\}, \{1, 2\}\big\}.$$

A C*-algebra over this space comes with a single distinguished ideal $A(1) \triangleleft A$, which is arbitrary. Thus we get the category of pairs (I, A) where I is an ideal in A. We may associate to this data the C*-algebra extension $I \rightarrowtail A \twoheadrightarrow A/I$. In fact, the morphisms in $\mathrm{Hom}_X(A, B)$ are the morphisms of extensions

$$\begin{array}{ccccc}
A(1) & \rightarrowtail & A & \twoheadrightarrow & A/A(1) \\
\downarrow & & \downarrow & & \downarrow \\
B(1) & \rightarrowtail & B & \twoheadrightarrow & B/B(1).
\end{array}$$

Thus $\mathfrak{C}^*\mathfrak{alg}(X)$ is equivalent to the category of C*-*algebra extensions*. This example is also studied in [4].

2.7. **Topologies and partial orders.** Certain non-Hausdorff spaces are very closely related to partially ordered sets. In particular, there is a bijection between sober topologies and partial orders on a finite set. Here we recall the relevant constructions.

Definition 2.30. Let X be a topological space. The *specialization preorder* $\preceq$ on X is defined by $x \preceq y$ if the closure of $\{x\}$ is contained in the closure of $\{y\}$ or, equivalently, if y is contained in all open subsets of X that contain x. Two points x and y are called *topologically indistinguishable* if $x \preceq y$ and $y \preceq x$, that is, the closures of $\{x\}$ and $\{y\}$ are equal.

The separation axiom T_0 means that topologically indistinguishable points are equal. Since this is automatic for sober spaces, $\preceq$ is a partial order on X in all cases we need. As usual, we write $x \prec y$ if $x \preceq y$ and $x \neq y$, and $x \succeq y$ and $x \succ y$ are equivalent to $y \preceq x$ and $y \prec x$, respectively.

The separation axiom T_1 requires points to be closed. This is equivalent to the partial order $\preceq$ being trivial, that is, $x \preceq y$ if and only if $x = y$. Thus our partial order is only meaningful for highly nonseparated spaces.

The following notion goes back to an article by Paul Alexandrov from 1937 ([1]); see also [2] for a more recent reference, or the English Wikipedia entry on the Alexandrov topology.

Definition 2.31. Let $(X, \leq)$ be a preordered set. A subset $S \subseteq X$ is called *Alexandrov-open* if $S \ni x \leq y$ implies $y \in S$. The Alexandrov-open subsets form a topology on X called the *Alexandrov topology*.

A subset of X is closed in the Alexandrov topology if and only if $S \ni x$ and $x \geq y$ imply $S \ni y$. It is locally closed if and only if it is *convex*, that is, $x \leq y \leq z$ and $x, z \in S$ imply $y \in S$. In particular, singletons are locally closed (compare Example 2.16).

The specialization preorder for the Alexandrov topology is the given preorder. Moreover, a map $(X, \leq) \to (Y, \leq)$ is continuous for the Alexandrov topology if and only if it is monotone. Thus we have identified the category of preordered sets with monotone maps with a full subcategory of the category of topological spaces.

It also follows that if a topological space carries an Alexandrov topology for some preorder, then this preorder must be the specialization preorder. In this case, we call the space an *Alexandrov space* or a *finitely generated space*. The following lemma provides some equivalent descriptions of Alexandrov spaces; the last two explain in what sense these spaces are finitely generated.

Lemma 2.32. *Let X be a topological space. The following are equivalent:*
- *X is an Alexandrov space;*
- *an arbitrary intersection of open subsets of X is open;*
- *an arbitrary union of closed subsets of X is closed;*

- *every point of X has a smallest neighborhood;*
- *a point x lies in the closure of a subset S of X if and only if $x \in \overline{\{y\}}$ for some $y \in S$;*
- *X is the inductive limit of the inductive system of its finite subspaces.*

Corollary 2.33. *Any finite topological space is an Alexandrov space. Thus the construction of Alexandrov topologies and specialization preorders provides a bijection between preorders and topologies on a finite set.*

Definition 2.34. Let X be an Alexandrov space. We denote the minimal open neighborhood of $x \in X$ by $U_x \in \mathbb{O}(X)$.

We have
$$U_x \subseteq U_y \iff x \in U_y \iff y \in \overline{\{x\}} \iff \overline{\{y\}} \subseteq \overline{\{x\}} \iff y \preceq x.$$

If X is a sober Alexandrov space, then we can simplify the data for a C*-algebra over X as follows:

Lemma 2.35. *A C*-algebra over a sober Alexandrov space X is determined uniquely by a C*-algebra A together with ideals $A(U_x) \triangleleft A$ for all $x \in X$, subject to the two conditions $\overline{\sum_{x \in X} A(U_x)} = A$ and*

$$(2.36) \qquad A(U_x) \cap A(U_y) = \overline{\sum_{z \in U_x \cap U_y} A(U_z)} \qquad \text{for all } x, y \in X.$$

Proof. A map $\mathbb{O}(X) \to \mathbb{I}(A)$ that preserves suprema and maps U_x to $A(U_x)$ for all $x \in X$ must map $U = \bigvee_{x \in U} U_x$ to $\bigvee_{x \in U} A(U_x) = \overline{\sum_{x \in U} A(U_x)}$. The map so defined preserves suprema by construction. The two hypotheses of the lemma ensure $A(X) = A$ and $A(U_x \cap U_y) = A(U_x) \cap A(U_y)$ for all $x, y \in X$. Hence they are necessary for preservation of finite infima.

Since the lattice $\mathbb{I}(A) \cong \mathbb{O}(\operatorname{Prim} A)$ is distributive, (2.36) implies

$$A(U) \wedge A(V) = \bigvee_{x \in U} A(U_x) \wedge \bigvee_{y \in V} A(V_y) = \bigvee_{(x,y) \in U \times V} A(U_x) \wedge A(V_y)$$
$$= \bigvee_{(x,y) \in U \times V} A(U_x \cap V_y) = A(U \cap V);$$

the last step uses that $U \mapsto A(U)$ commutes with suprema. We clearly have $A(\varnothing) = \{0\}$ as well, so that $U \mapsto A(U)$ preserves arbitrary finite intersections. Therefore, our map $\mathbb{O}(X) \to \mathbb{I}(A)$ satisfies the conditions in Lemma 2.25 and hence comes from a continuous map $\operatorname{Prim} A \to X$. $\qquad\square$

Of course, a *-homomorphism $A \to B$ between two C*-algebras over X is X-equivariant if and only if it maps $A(U_x) \to B(U_x)$ for all $x \in X$.

Equation (2.36) implies $A(U_x) \subseteq A(U_y)$ if $U_x \subseteq U_y$, that is, if $x \succeq y$. Thus the map $x \mapsto A(U_x)$ is order-reversing. It sometimes happens that $U_x \cap U_y = U_z$ for some $x, y, z \in X$. In this case, we may drop the ideal $A(U_z)$ from the description of a C*-algebra over X and replace the condition (2.36) for x, y by $A(U_w) \subseteq A(U_x) \cap A(U_y)$ for all $w \in U_x \cap U_y$.

2.8. **Some more examples.** A useful way to represent finite partially ordered sets and hence finite sober topological spaces is via finite *directed acyclic graphs*.

To a partial order $\preceq$ on X, we associate the finite directed acyclic graph with vertex set X and with an arrow $x \leftarrow y$ if and only if $x \prec y$ and there is no $z \in X$ with $x \prec z \prec y$. We can recover the partial order from this graph by letting $x \preceq y$ if and only if the graph contains a directed path $x \leftarrow x_1 \leftarrow \cdots \leftarrow x_n \leftarrow y$.

We have reversed arrows here because an arrow $x \to y$ means that $A(U_x) \subseteq A(U_y)$. Furthermore, $x \in U_y$ if and only if there is a directed path from x to y. Thus we can read the meaning of the relations (2.36) from the graph.

Example 2.37. Let $(X, \geq)$ be a set with a total order, such as $\{1, \ldots, n\}$ with the order $\geq$. The corresponding graph is

$$1 \longrightarrow 2 \longrightarrow 3 \longrightarrow \cdots \longrightarrow n.$$

For totally ordered X, (2.36) is equivalent to monotonicity of the map $x \mapsto A(U_x)$ with respect to the opposite order $\leq$ on X. As a consequence, a C*-algebra over X is nothing but a C*-algebra A together with a monotone map $(X, \leq) \to \mathbb{I}(A)$, $x \mapsto A(U_x)$, such that $\bigvee_{x \in X} A(U_x) = A$. For $X = (\{1, \ldots, n\}, \geq)$, the latter condition just means $A(U_n) = A$, so that we can drop this ideal. Thus we get C*-algebras with an increasing chain of $n - 1$ ideals $I_1 \triangleleft I_2 \triangleleft \cdots \triangleleft I_{n-1} \triangleleft A$. This situation is studied in detail in [16].

Using that any finite topological space is an Alexandrov space, we can easily list all homeomorphism classes of finite topological spaces with, say, three or four elements. We only consider sober spaces here, and we assume connectedness to further reduce the number of cases. Under these assumptions, Figure 1 contains a complete list. The first and fourth case are already contained in Example 2.37. Lemma 2.35 describes C*-algebras over the spaces in Figure 1 as C*-algebras equipped with three or four ideals $A(U_x)$ for $x \in X$, subject to some conditions, which often make some of the ideals redundant.

Example 2.38. The second graph in Figure 1 describes C*-algebras with three ideals $A(U_j)$, $j = 1, 2, 3$, subject to the conditions $A(U_2) \cap A(U_3) = A(U_1)$ and $A(U_2) + A(U_3) = A$. This is equivalent to prescribing only two ideals $A(U_2)$ and $A(U_3)$ subject to the single condition $A(U_2) + A(U_3) = A$.

Example 2.39. Similarly, the third graph in Figure 1 describes C*-algebras with two distinguished ideals $A(U_1)$ and $A(U_2)$ subject to the condition $A(U_1) \cap A(U_2) = \{0\}$; here $U_3 = X$ implies $A(U_3) = A$.

Example 2.40. The ninth graph in Figure 1 is more complicated. We label our points by $1, 2, 3, 4$ such that $1 \to 3 \leftarrow 2 \to 4$. Here we have a C*-algebra A with four ideals $I_j := A(U_j)$ for $j = 1, 2, 3, 4$, subject to the conditions

$$I_1 \subseteq I_3, \qquad I_1 \cap I_4 = \{0\}, \qquad I_2 = I_3 \cap I_4, \qquad I_3 + I_4 = A.$$

Thus the ideal I_2 is redundant, and we are left with three ideals I_1, I_3, I_4 subject to the conditions $I_1 \subseteq I_3$, $I_1 \cap I_4 = \{0\}$, and $I_3 + I_4 = A$.
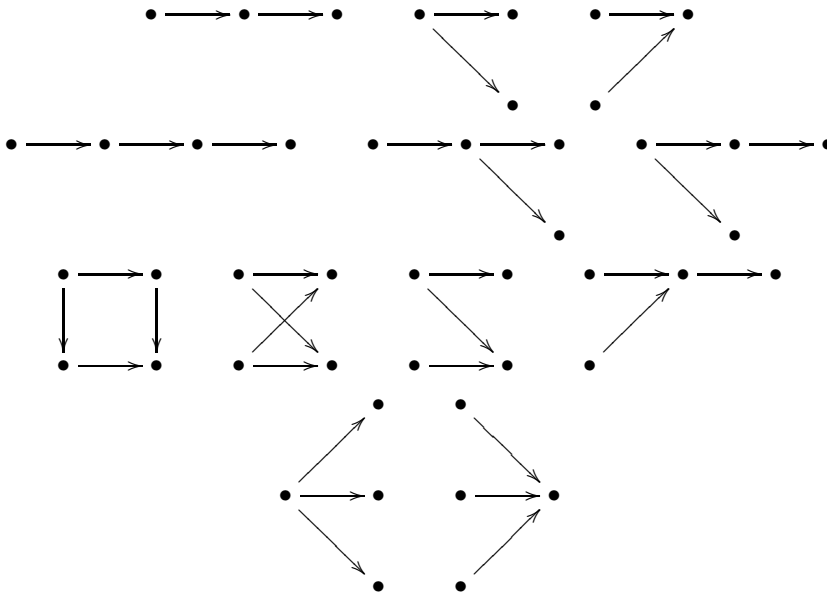
FIGURE 1. Connected directed acyclic graphs with three or
four vertices

2.9. **How to treat discontinuous bundles.** The X-equivariant Kasparov
theory constructed in [10, 4] works for *any* map $\psi^*\colon \mathbb{O}(X) \to \mathbb{I}(A)$, we do
not need the conditions in Lemma 2.25. Here we show how to reduce this
more general situation to the case considered above: discontinuous actions of
$\mathbb{O}(X)$ as in [10, 4] are equivalent to continuous actions of another space Y
that contains X as a subspace. The category $\mathfrak{C}^*\mathfrak{alg}(Y)$ contains $\mathfrak{C}^*\mathfrak{alg}(X)$ as
a full subcategory, and a similar statement holds for the associated Kasparov
categories. As a result, allowing general maps ψ^* merely amounts to replacing
the space X by the larger space Y. For C*-algebras that really live over
the subspace X, the extension to Y significantly complicates the computation
of the Kasparov groups. This is why we always require ψ^* to satisfy the
conditions in Lemma 2.25, which ensure that it comes from a continuous map
$\mathrm{Prim}(A) \to X$.

Example 2.41. Let $X = \{1, 2\}$ with the discrete topology. A monotone map
$\psi^*\colon \mathbb{O}(X) \to A$ with $\psi^*(\varnothing) = \{0\}$ and $\psi^*(X) = A$ as considered in [4, 10] is
equivalent to specifying two *arbitrary* ideals $A(1)$ and $A(2)$. This automatically
generates the ideals $A(1) \cap A(2)$ and $A(1) \cup A(2)$. We can encode these four
ideals in an action of a topological space Y with four points $\{1 \cap 2, 1, 2, 3\}$ and
open subsets

$$\varnothing, \quad \{1 \cap 2\}, \quad \{1 \cap 2, 1\}, \quad \{1 \cap 2, 2\}, \quad \{1 \cap 2, 1, 2\}, \quad \{1 \cap 2, 1, 2, 3\}.$$

The corresponding graph is the seventh one in Figure 1. The map ψ^* maps these open subsets to the ideals

$$\{0\}, \quad A(1) \cap A(2), \quad A(1), \quad A(2), \quad A(1) \cup A(2), \quad A,$$

respectively. This defines a complete lattice morphism $\mathbb{O}(Y) \to \mathbb{I}(A)$, and any complete lattice morphism is of this form for two ideals $A(1)$ and $A(2)$. Thus an action of $\mathbb{O}(\{1,2\})$ in the generalized sense considered in [4, 10] is equivalent to an action of Y in our sense.

Any X-equivariant *-homomorphism $A \to B$ between two such discontinuous C*-algebras over X will also preserve the ideals $A(1) \cap A(2)$ and $A(1) \cup A(2)$. Hence it is Y-equivariant as well. Therefore, the above construction provides an equivalence of categories between $\mathfrak{C}^*\mathfrak{alg}(Y)$ and the category of C*-algebras with an action of $\mathbb{O}(X)$ in the sense of [4, 10].

Whereas the computation of

$$\mathrm{KK}_*(X; A, B) \cong \mathrm{KK}_*\big(A(1), B(1)\big) \times \mathrm{KK}_*\big(A(2), B(2)\big)$$

for two C*-algebras A and B over X is trivial, the corresponding problem for C*-algebras over Y is an interesting problem: this is one of the small examples where filtrated K-theory does not yet suffice for classification.

This simple example generalizes as follows. Let

$$f \colon \mathbb{O}(X) \to \mathbb{I}(A) \cong \mathbb{O}(\mathrm{Prim}\, A)$$

be an arbitrary map. Let $Y := 2^{\mathbb{O}(X)}$ be the power set of $\mathbb{O}(X)$, partially ordered by inclusion. We describe the topology on Y below. We embed the original space X into Y by mapping $x \in X$ to its open neighborhood filter:

$$\mathcal{U} \colon X \to Y, \qquad x \mapsto \{U \in \mathbb{O}(X) \mid x \in U\}.$$

We define a map

$$\psi \colon \mathrm{Prim}(A) \to Y, \qquad \mathfrak{p} \mapsto \{U \in \mathbb{O}(X) \mid \mathfrak{p} \in f(U)\}.$$

For $y \in Y$, let $Y_{\supseteq y} := \{x \in Y \mid x \supseteq y\}$. For a singleton $\{U\}$ with $U \in \mathbb{O}(X)$, we easily compute

$$\psi^{-1}(Y_{\supseteq\{U\}}) = f(U) \in \mathbb{I}(A) \cong \mathbb{O}(\mathrm{Prim}\, A).$$

Moreover, $Y_{\supseteq y \cup z} = Y_{\supseteq y} \cap Y_{\supseteq z}$, so that we get

$$\psi^{-1}(Y_{\supseteq\{U_1,\ldots,U_n\}}) = f(U_1) \cap \cdots \cap f(U_n).$$

A similar argument shows that

$$\mathcal{U}^{-1}(Y_{\supseteq\{U_1,\ldots,U_n\}}) = \mathcal{U}^{-1}(Y_{\supseteq U_1}) \cap \cdots \cap \mathcal{U}^{-1}(Y_{\supseteq U_n}) = U_1 \cap \cdots \cap U_n.$$

We equip Y with the topology that has the sets $Y_{\supseteq F}$ for *finite* subsets F of $\mathbb{O}(X)$ as a basis. It is clear from the above computations that this makes the maps ψ and $\mathcal{U}$ continuous; even more, the subspace topology on the range of $\mathcal{U}$ is the given topology on X.

As a consequence, any map $f \colon \mathbb{O}(X) \to \mathbb{I}(A)$ turns A into a C*-algebra over the space $Y \supseteq X$. Conversely, given a C*-algebra over Y, we define

$f\colon \mathbb{O}(X) \to \mathbb{I}(A)$ by $f(U) := \psi^{-1}(Y_{\supseteq\{U\}})$. This construction is inverse to the one above. Furthermore, a *-homomorphism $A \to B$ that maps $f_A(U)$ to $f_B(U)$ for all $U \in \mathbb{O}(X)$ also maps $\psi_A^*(U)$ to $\psi_B^*(U)$ for all $U \in \mathbb{O}(Y)$. We can sum this up as follows:

Theorem 2.42. *The category of* C^**-algebras equipped with a map* $f\colon \mathbb{O}(X) \to \mathbb{I}(A)$ *is isomorphic to the category of* C^**-algebras over* Y.

If f has some additional properties like monotonicity, or is a lattice morphism, then this limits the range of the map ψ above and thus allows us to replace Y by a smaller subset. In [10, Def. 1.3] and [4, Def. 5.6.2], an action of a space X on a C*-algebra is defined to be a map $f\colon \mathbb{O}(X) \to \mathbb{I}(A)$ that is monotone and satisfies $f(\varnothing) = \{0\}$ and $f(X) = A$. These assumptions are equivalent to

$$U \in \psi(\mathfrak{p}), \quad U \subseteq V \qquad \Longrightarrow \qquad V \in \psi(\mathfrak{p})$$

and $\varnothing \notin \psi(\mathfrak{p})$ and $X \in \psi(\mathfrak{p})$ for all $\mathfrak{p} \in \mathrm{Prim}(A)$. Hence the category of C*-algebras with an action of X in the sense of [4, 10] is equivalent to the category of C*-algebras over the space

$$Y' := \{y \subseteq \mathbb{O}(X) \mid y \ni U \subseteq V \Longrightarrow V \in y, \ \varnothing \notin y, \ X \in y\},$$

equipped with the subspace topology from Y.

3. Bivariant K-theory for C*-algebras over topological spaces

Let X be a topological space. Eberhard Kirchberg [10] and Alexander Bonkat [4] define Kasparov groups $\mathrm{KK}_*(X; A, B)$ for separable C*-algebras A and B over X. More precisely, instead of a continuous map $\mathrm{Prim}(A) \to X$ they use a separable C*-algebra A with a monotone map $\psi^*\colon \mathbb{O}(X) \to \mathbb{I}(A)$ with $A(\varnothing) = \{0\}$ and $A(X) = A$. This is more general because any continuous map $\mathrm{Prim}(A) \to X$ generates such a map $\psi^*\colon \mathbb{O}(X) \to \mathbb{I}(A)$. Hence their definitions apply to C*-algebras over X in our sense. We have explained in §2.9 why the setting in [10, 4] is, despite appearances, not more general than our setting.

If X is Hausdorff and locally compact, $\mathrm{KK}_*(X; A, B)$ agrees with Gennadi Kasparov's theory $\mathcal{R}\mathrm{KK}_*(X; A, B)$ defined in [9]. In this section, we recall the definition and some basic properties of the functor $\mathrm{KK}_*(X; A, B)$ and the resulting category $\mathfrak{KK}(X)$, and we equip the latter with a triangulated category structure.

3.1. **The definition.** We assume from now on that the topology on X has a countable basis, and we restrict attention to separable C*-algebras.

Definition 3.1. A C*-algebra (A, ψ) over X is called *separable* if A is a separable C*-algebra. Let $\mathfrak{C}^*\mathfrak{sep}(X) \subseteq \mathfrak{C}^*\mathfrak{alg}(X)$ be the full subcategory of separable C*-algebras over X.

To describe the cycles for $\mathrm{KK}_*(X; A, B)$ recall that the usual Kasparov cycles for $\mathrm{KK}_*(A, B)$ are of the form $(\varphi, \mathcal{H}_B, F, \gamma)$ in the even case (for KK_0) and $(\varphi, \mathcal{H}_B, F)$ in the odd case (for KK_1), where

- $\mathcal{H}_B$ is a right Hilbert B-module;
- $\varphi \colon A \to \mathbb{B}(\mathcal{H}_B)$ is a *-representation;
- $F \in \mathbb{B}(\mathcal{H}_B)$;
- $\varphi(a)(F^2 - 1)$, $\varphi(a)(F - F^*)$, and $[\varphi(a), F]$ are compact for all $a \in A$;
- in the even case, γ is a $\mathbb{Z}/2$-grading on $\mathcal{H}_B$—that is, $\gamma^2 = 1$ and $\gamma = \gamma^*$—that commutes with $\varphi(A)$ and anti-commutes with F.

The following definition of X-equivariant bivariant K-theory is equivalent to the ones in [10, 4], see [10, Def. 4.1], and [4, Def. 5.6.11 and Satz 5.6.12].

Definition 3.2. Let A and B be C*-algebras over X (or, more generally, C*-algebras with a map $\mathbb{O}(X) \to \mathbb{I}(A)$). A Kasparov cycle $(\varphi, \mathcal{H}_B, F, \gamma)$ or $(\varphi, \mathcal{H}_B, F)$ for $\mathrm{KK}_*(A, B)$ is called X-*equivariant* if

$$\varphi\big(A(U)\big) \cdot \mathcal{H}_B \subseteq \mathcal{H}_B \cdot B(U) \qquad \text{for all } U \in \mathbb{O}(X).$$

Let $\mathrm{KK}_*(X; A, B)$ be the group of homotopy classes of such X-equivariant Kasparov cycles for $\mathrm{KK}_*(A, B)$; a homotopy is an X-equivariant Kasparov cycle for $\mathrm{KK}_*(A, C([0,1]) \otimes B)$, where we view $C([0,1]) \otimes B$ as a C*-algebra over X in the usual way (compare §2.4).

The subset $\mathcal{H}_B \cdot B(U) \subseteq \mathcal{H}_B$ is a closed linear subspace by the Cohen-Hewitt Factorization Theorem.

If X is Hausdorff, then the extra condition in Definition 3.2 is equivalent to $\mathrm{C}_0(X)$-linearity of φ (compare Proposition 2.11). Thus the above definition of $\mathrm{KK}_*(X; A, B)$ agrees with the more familiar definition of $\mathcal{R}\mathrm{KK}_*(X; A, B)$ in [9].

If $X = \star$ is the one-point space, the X-equivariance condition is empty and we get the plain Kasparov theory $\mathrm{KK}_*(\star; A, B) = \mathrm{KK}_*(A, B)$.

The same arguments as usual show that $\mathrm{KK}_*(X; A, B)$ remains unchanged if we strengthen the conditions for Kasparov cycles by requiring $F = F^*$ and $F^2 = 1$.

3.2. **Basic properties.** The Kasparov theory defined above has all the properties that we can expect from a bivariant K-theory.

(1) The groups $\mathrm{KK}_*(X; A, B)$ define a bifunctor from $\mathfrak{C}^*\mathfrak{sep}(X)$ to the category of $\mathbb{Z}/2$-graded abelian groups, contravariant in the first and covariant in the second variable.

(2) There is a natural, associative *Kasparov composition product*

$$\mathrm{KK}_i(X; A, B) \times \mathrm{KK}_j(X; B, C) \to \mathrm{KK}_{i+j}(X; A, C)$$

if A, B, C are C*-algebras over X.

Furthermore, there is a natural *exterior product*

$$\mathrm{KK}_i(X; A, B) \times \mathrm{KK}_j(Y; C, D) \to \mathrm{KK}_{i+j}(X \times Y; A \otimes C, B \otimes D)$$

for two spaces X and Y and C*-algebras A, B over X and C, D over Y.

The existence and properties of the Kasparov composition product and the exterior product are verified in a more general context in [4, §3.2].

Definition 3.3. Let $\mathfrak{K}\mathfrak{K}(X)$ be the category whose objects are the separable C*-algebras over X and whose morphism sets are $\mathrm{KK}_0(X; A, B)$.

  (3) The zero C*-algebra acts as a *zero object* in $\mathfrak{K}\mathfrak{K}(X)$, that is,

$$\mathrm{KK}_*(X; \{0\}, A) = 0 = \mathrm{KK}_*(X; A, \{0\}) \qquad \text{for all } A \in\in \mathfrak{K}\mathfrak{K}(X).$$

  (4) The C_0-direct sum of a sequence of C*-algebras behaves like a *coproduct*, that is,

$$\mathrm{KK}_*\left(X; \bigoplus_{n\in\mathbb{N}} A_n, B\right) \cong \prod_{n\in\mathbb{N}} \mathrm{KK}_*(X; A_n, B)$$

      if $A_n, B \in\in \mathfrak{K}\mathfrak{K}(X)$ for all $n \in \mathbb{N}$.

  (5) The direct sum $A \oplus B$ of two separable C*-algebras A and B over X is a *direct product* in $\mathfrak{K}\mathfrak{K}(X)$, that is,

$$\mathrm{KK}_*(X; D, A \oplus B) \cong \mathrm{KK}_*(X; D, A) \oplus \mathrm{KK}_*(X; D, B)$$

      for all $D \in\in \mathfrak{K}\mathfrak{K}(X)$ (see [4, Lem. 3.1.9]).

Properties (3)–(5) are summarized as follows:

Proposition 3.4. *The category $\mathfrak{K}\mathfrak{K}(X)$ is additive and has countable coproducts.*

  (6) The exterior product is compatible with the Kasparov product, C_0-direct sums, and addition, that is, it defines a countably additive bifunctor

$$\otimes \colon \mathfrak{K}\mathfrak{K}(X) \otimes \mathfrak{K}\mathfrak{K}(Y) \to \mathfrak{K}\mathfrak{K}(X \times Y).$$

      This operation is evidently associative.

  (7) In particular, $\mathfrak{K}\mathfrak{K}(X)$ is tensored over $\mathfrak{K}\mathfrak{K}(\star) \cong \mathfrak{K}\mathfrak{K}$, that is, $\otimes$ provides an associative bifunctor

$$\otimes \colon \mathfrak{K}\mathfrak{K}(X) \otimes \mathfrak{K}\mathfrak{K} \to \mathfrak{K}\mathfrak{K}(X).$$

  (8) The bifunctor $(A, B) \mapsto \mathrm{KK}_*(X; A, B)$ satisfies Bott periodicity, homotopy invariance, and C*-stability in each variable. This follows from the corresponding properties of $\mathfrak{K}\mathfrak{K}$ using the tensor structure in (7).

      For instance, the Bott periodicity isomorphism $C_0(\mathbb{R}^2) \cong \mathbb{C}$ in $\mathfrak{K}\mathfrak{K}$ yields $A \otimes C_0(\mathbb{R}^2) \cong A \otimes \mathbb{C} \cong A$ in $\mathfrak{K}\mathfrak{K}(X)$ for all $A \in\in \mathfrak{K}\mathfrak{K}(X)$.

  (9) The functor $f_* \colon \mathfrak{C}^*\mathfrak{alg}(X) \to \mathfrak{C}^*\mathfrak{alg}(Y)$ for a continuous map $f \colon X \to Y$ descends to a functor

$$f_* \colon \mathfrak{K}\mathfrak{K}(X) \to \mathfrak{K}\mathfrak{K}(Y).$$

      In particular, this covers the extension functors i_X^Y for a subspace $X \subseteq Y$.

 (10) The restriction functor r_X^Y for $Y \in \mathbb{LC}(X)$ also descends to a functor

$$r_X^Y \colon \mathfrak{K}\mathfrak{K}(X) \to \mathfrak{K}\mathfrak{K}(Y).$$

Definition 3.5 (see [4, Def. 5.6.6])**.** A diagram $I \to E \to Q$ in $\mathfrak{C}^*\mathfrak{alg}(X)$ is an *extension* if, for all $U \in \mathbb{O}(X)$, the diagrams $I(U) \to E(U) \to Q(U)$ are extensions of C*-algebras. We write $I \rightarrowtail E \twoheadrightarrow Q$ to denote extensions.

An extension is called *split* if it splits by an X-equivariant *-homomorphism.

An extension is called *semi-split* if there is a completely positive, contractive section $Q \to E$ that is *X-equivariant*, that is, it restricts to sections $Q(U) \to E(U)$ for all $U \in \mathbb{O}(X)$.

If $I \rightarrowtail E \twoheadrightarrow Q$ is an extension of C*-algebras over X, then we get C*-algebra extensions $I(Y) \rightarrowtail E(Y) \twoheadrightarrow Q(Y)$ for all locally closed subsets $Y \subseteq X$. If the original extension is semi-split, so are the extensions $I(Y) \rightarrowtail E(Y) \twoheadrightarrow Q(Y)$ for $Y \in \mathbb{LC}(X)$. Even more, the functor $r_X^Y \colon \mathfrak{C}^*\mathfrak{alg}(X) \to \mathfrak{C}^*\mathfrak{alg}(Y)$ maps extensions in $\mathfrak{C}^*\mathfrak{alg}(X)$ to extensions in $\mathfrak{C}^*\mathfrak{alg}(Y)$, and similarly for split and semi-split extensions.

Theorem 3.6. *Let $I \rightarrowtail E \twoheadrightarrow Q$ be a semi-split extension in $\mathfrak{C}^*\mathfrak{sep}(X)$ and let B be a separable* C**-algebra over X. There are six-term exact sequences*

$$
\begin{array}{ccccc}
\mathrm{KK}_0(X;Q,B) & \longrightarrow & \mathrm{KK}_0(X;E,B) & \longrightarrow & \mathrm{KK}_0(X;I,B) \\
\partial \uparrow & & & & \downarrow \partial \\
\mathrm{KK}_1(X;I,B) & \longleftarrow & \mathrm{KK}_1(X;E,B) & \longleftarrow & \mathrm{KK}_1(X;Q,B)
\end{array}
$$

and

$$
\begin{array}{ccccc}
\mathrm{KK}_0(X;B,I) & \longrightarrow & \mathrm{KK}_0(X;B,E) & \longrightarrow & \mathrm{KK}_0(X;B,Q) \\
\partial \uparrow & & & & \downarrow \partial \\
\mathrm{KK}_1(X;B,Q) & \longleftarrow & \mathrm{KK}_1(X;B,E) & \longleftarrow & \mathrm{KK}_1(X;B,I),
\end{array}
$$

where the horizontal maps in both exact sequences are induced by the given maps $I \to E \to Q$, and the vertical maps are, up to signs, Kasparov products with the class of our semi-split extension in $\mathrm{KK}_1(Q, I)$.

Furthermore, extensions with an X-equivariant completely positive section are semi-split.

Proof. The long exact sequences for semi-split extensions follow from [4, Satz 3.3.10] or from [4, Kor. 5.6.13].

The last statement of the theorem plays a technical role in the proof of Proposition 4.10. We have to replace an X-equivariant completely positive section $s \colon Q \to E$ by another section that is an X-equivarant completely positive *contraction*. Without X-equivariance, this is done in [6, Rem. 2.5]. We claim that the constructions during the proof yield X-equivariant maps if we start with X-equivariant maps.

Let Q^+ and E^+ be obtained by adjoining units to Q and E. Let $(u_n)_{n \in \mathbb{N}}$ be an approximate unit in Q and let $v_n := \sup(1, s(u_n))$ in E^+. Since $v_n \geq 1$, v_n is invertible. The maps

$$
s_n \colon Q^+ \to E^+, \qquad q \mapsto v_n^{-1/2} s(u_n^{1/2} q u_n^{1/2}) v_n^{-1/2}
$$

considered in [6] are completely positive and contractive because $s_n(1) \leq 1$. They are X-equivariant if s is X-equivariant. Since v_n lifts $1 \in Q^+$, the maps (s_n) lift the maps $q \mapsto u_n^{1/2} q u_n^{1/2}$, which converge pointwise to the identity map.

It remains to show that the space of maps $Q^+ \to Q^+$ that lift to an X-equivariant unital completely positive map $Q^+ \to E^+$ is closed in the topology of pointwise norm convergence. Without X-equivariance, this is [3, Thm. 6]. Its proof is based on the following construction. Let $\varphi, \psi \colon Q^+ \rightrightarrows E^+$ be two unital completely positive maps and let $(e_m)_{m \in \mathbb{N}}$ be a quasi-central approximate unit for I in Q^+. Then Arveson uses the unital completely positive maps

$$q \mapsto e_m^{1/2} \varphi(q) e_m^{1/2} + (1 - e_m)^{1/2} \psi(q) (1 - e_m)^{1/2}.$$

Clearly, this map is X-equivariant if φ and ψ are X-equivariant. Hence the argument in [3] produces a Cauchy sequence of X-equivariant unital completely positive maps $\hat{s}_n \colon Q^+ \to E^+$ lifting s_n. Its limit is an X-equivariant unital completely positive section $Q^+ \to E^+$. $\qquad\square$

Theorem 3.7. *The canonical functor $\mathfrak{C}^* \mathfrak{sep}(X) \to \mathfrak{KK}(X)$ is the universal split-exact C^*-stable (homotopy) functor.*

Proof. This follows from [4, Satz 3.5.10], compare also [4, Kor. 5.6.13]. The homotopy invariance assumption is redundant because, by a deep theorem of Nigel Higson, a split-exact, C^*-stable functor is automatically homotopy invariant. This holds for $\mathfrak{C}^* \mathfrak{sep}$ itself and is inherited by $\mathfrak{C}^* \mathfrak{sep}(X)$ because of the tensor product operation $\mathfrak{C}^* \mathfrak{sep}(X) \times \mathfrak{C}^* \mathfrak{sep} \to \mathfrak{C}^* \mathfrak{sep}(X)$. $\qquad\square$

3.3. **Triangulated category structure.** We are going to turn $\mathfrak{KK}(X)$ into a triangulated category as in [14]. We have already remarked that $\mathfrak{KK}(X)$ is additive. The *suspension functor* is $\Sigma(A) := \mathrm{C}_0(\mathbb{R}, A) = \mathrm{C}_0(\mathbb{R}) \otimes A$. This functor is an automorphism (up to natural isomorphisms) by Bott periodicity.

The *mapping cone triangle*

$$(3.8) \qquad \qquad A \xrightarrow{\ \varphi\ } B$$
$$C_\varphi$$

of a morphism $\varphi \colon A \to B$ in $\mathfrak{C}^* \mathfrak{sep}(X)$ is defined as in [14] and is a diagram in $\mathfrak{KK}(X)$. The circled arrow from B to C_φ means a *-homomorphism $\Sigma(B) \to C_\varphi$. A triangle in $\mathfrak{KK}(X)$ is called *exact* if it is isomorphic in $\mathfrak{KK}(X)$ to the mapping cone triangle of some morphism in $\mathfrak{C}^* \mathfrak{sep}(X)$.

As in [14], there is an equivalent description of the exact triangles using semi-split extensions in $\mathfrak{C}^* \mathfrak{sep}(X)$. An extension

$$(3.9) \qquad \qquad I \xrightarrow{\ i\ } E \xrightarrow{\ p\ } Q$$

gives rise to a commuting diagram

$$
\begin{array}{ccccccc}
\Sigma Q & & I & \xrightarrow{\ i\ } & E & \xrightarrow{\ p\ } & Q \\
\| & & \downarrow & & \| & & \| \\
\Sigma Q & \longrightarrow & C_p & \longrightarrow & E & \xrightarrow{\ p\ } & Q.
\end{array}
$$

Definition 3.10. We call the extension *admissible* if the map $I \to C_p$ is invertible in $\mathfrak{K}\mathfrak{K}(X)$.

The proof of the Excision Theorem 3.6 shows that this is the case if the extension is semi-split; but there are more admissible extensions than semi-split extensions. If the extension is admissible, then there is a unique map $\Sigma Q \to I$ so that the top row becomes isomorphic to the bottom row as a triangle in $\mathfrak{K}\mathfrak{K}(X)$. Thus any admissible extension in $\mathfrak{C}^*\mathfrak{sep}(X)$ yields an exact triangle $\Sigma Q \to I \to E \to Q$, called *extension triangle*.

Conversely, if $\varphi \colon A \to B$ is a morphism in $\mathfrak{C}^*\mathfrak{sep}(X)$, then its mapping cone triangle is isomorphic in $\mathfrak{K}\mathfrak{K}(X)$ to the extension triangle for the canonically semi-split extension $C_\varphi \rightarrowtail Z_\varphi \twoheadrightarrow B$, where Z_φ denotes the mapping cylinder of φ, which is homotopy equivalent to A. The above arguments work exactly as in the case of undecorated Kasparov theory discussed in [14].

As a result, a triangle in $\mathfrak{K}\mathfrak{K}(X)$ is isomorphic to a mapping cone triangle of some morphism in $\mathfrak{C}^*\mathfrak{sep}(X)$ if and only if it is isomorphic to the extension triangle of some semi-split extension in $\mathfrak{C}^*\mathfrak{sep}(X)$.

Proposition 3.11. *The category $\mathfrak{K}\mathfrak{K}(X)$ with the suspension automorphism and extension triangles specified above is a triangulated category.*

Proof. Most of the axioms amount to well-known properties of mapping cones and mapping cylinders, which are proven by translating corresponding arguments for the stable homotopy category of spaces, see [14].

The only axiom that requires a new argument in our case is (TR1), which asserts that any morphism in $\mathfrak{K}\mathfrak{K}(X)$ is part of some exact triangle. The argument in [14] uses the description of Kasparov theory via the universal algebra qA by Joachim Cuntz. This approach can be made to work in $\mathfrak{K}\mathfrak{K}(X)$, but it is rather inflexible because the primitive ideal space of qA is hard to control.

The following argument, which is inspired by [4], also applies to interesting subcategories of $\mathfrak{K}\mathfrak{K}(X)$ like the subcategory of nuclear C^*-algebras over X, which is studied in §5. Hence this is a triangulated category as well.

Let $f \in \mathrm{KK}_0(X; A, B)$. We identify $\mathrm{KK}_0(X; A, B) \cong \mathrm{KK}_1(X; A, \Sigma B)$. Represent the image of f in $\mathrm{KK}_1(X; A, \Sigma B)$ by a cycle $(\varphi, \mathcal{H}, F)$. Adding a degenerate cycle, if necessary, we can achieve that the map $\Phi \colon A \ni a \mapsto F^* \varphi(a) F \mod \mathbb{K}(\mathcal{H})$ is an injection from A into the Calkin algebra $\mathbb{B}(\mathcal{H})/\mathbb{K}(\mathcal{H})$ of $\mathcal{H}$ and that $\mathcal{H}$ is full, so that $\mathbb{K}(\mathcal{H})$ is $\mathrm{KK}(X)$-equivalent to ΣB. The properties of a Kasparov cycle mean that Φ is the Busby invariant of a semi-split extension $\mathbb{K}(\mathcal{H}) \rightarrowtail E \twoheadrightarrow A$ of C*-algebras over X. The composition product

of the map $\Sigma A \to \mathbb{K}(\mathcal{H})$ in the associated extension triangle and the canonical $\mathrm{KK}(X)$-equivalence $\mathbb{K}(\mathcal{H}) \simeq \Sigma B$ is the suspension of $f \in \mathrm{KK}_0(X; A, B)$. Hence we can embed f in an exact triangle. $\qquad\square$

3.4. **Adjointness relations.**

Proposition 3.12. *Let X be a topological space and let $Y \in \mathbb{LC}(X)$.*
  If $Y \subseteq X$ is open, then we have natural isomorphisms

$$\mathrm{KK}_*\big(X; i_Y^X(A), B\big) \cong \mathrm{KK}_*\big(Y; A, r_X^Y(B)\big)$$

for all $A \in\in \mathfrak{KK}(Y)$, $B \in\in \mathfrak{KK}(X)$, that is, i_Y^X is left adjoint to r_X^Y as functors $\mathfrak{KK}(Y) \leftrightarrow \mathfrak{KK}(X)$.
  If $Y \subseteq X$ is closed, then we have natural isomorphisms

$$\mathrm{KK}_*\big(Y; r_X^Y(A), B\big) \cong \mathrm{KK}_*\big(X; A, i_Y^X(B)\big)$$

for all $A \in\in \mathfrak{KK}(X)$, $B \in\in \mathfrak{KK}(Y)$, that is, i_Y^X is right adjoint to r_X^Y as functors $\mathfrak{KK}(Y) \leftrightarrow \mathfrak{KK}(X)$.

Proof. Since both i_Y^X and r_X^Y descend to functors between $\mathfrak{KK}(X)$ and $\mathfrak{KK}(Y)$, this follows from the adjointness on the level of $\mathfrak{C}^*\mathfrak{alg}(X)$ and $\mathfrak{C}^*\mathfrak{alg}(Y)$ in Lemma 2.20; an analogous assertion for induction and restriction functors for group actions on C^*-algebras is proven in [14, §3.2]. The point of the argument is that an adjointness relation is equivalent to the existence of certain natural transformations called unit and counit of the adjunction, subject to some conditions (see [12]). These natural transformations already exist on the level of *-homomorphisms, which induce morphisms in $\mathfrak{KK}(X)$ or $\mathfrak{KK}(Y)$. The necessary relations for unit and counit of adjunction hold in $\mathfrak{KK}(\dots)$ because they already hold in $\mathfrak{C}^*\mathfrak{alg}(\dots)$. The unit and counit are natural in $\mathfrak{KK}(\dots)$ and not just in $\mathfrak{C}^*\mathfrak{alg}(\dots)$ because of the uniqueness part of the universal property of $\mathfrak{KK}$. $\qquad\square$

Proposition 3.13. *Let X be a topological space and let $x \in X$. Then*

$$\mathrm{KK}_*\big(X; A, i_x(B)\big) \cong \mathrm{KK}_*\big(A(\overline{\{x\}}), B\big)$$

for all $A \in\in \mathfrak{C}^*\mathfrak{sep}(X)$, $B \in\in \mathfrak{C}^*\mathfrak{sep}$. That is, the functor $i_x \colon \mathfrak{KK} \to \mathfrak{KK}(X)$ is right adjoint to the functor $A \mapsto A(\overline{\{x\}})$. Moreover,

$$\mathrm{KK}_*\big(X; i_x(A), B\big) \cong \mathrm{KK}_*\bigg(A, \bigcap_{U \in \mathcal{U}_x} B(U)\bigg)$$

for all $A \in\in \mathfrak{C}^*\mathfrak{sep}$, $B \in\in \mathfrak{C}^*\mathfrak{sep}(X)$, where $\mathcal{U}_x$ denotes the open neighborhood filter of x in X, That is, the functor $i_x \colon \mathfrak{KK} \to \mathfrak{KK}(X)$ is left adjoint to the functor $B \mapsto \bigcap_{U \in \mathcal{U}_x} B(U)$. If x has a minimal open neighborhood U_x, then

$$\mathrm{KK}_*\big(X; i_x(A), B\big) \cong \mathrm{KK}_*\big(A, B(U_x)\big),$$

Proof. This follows from Lemma 2.22 in the same way as Proposition 3.12. Notice that $B \mapsto \bigcap_{U \in \mathcal{U}_x} B(U)$ commutes with C^*-stabilization and maps (semi)-split extensions in $\mathfrak{C}^*\mathfrak{alg}(X)$ again to (semi)-split extensions in $\mathfrak{C}^*\mathfrak{alg}$; therefore, it descends to a functor $\mathfrak{KK}(X) \to \mathfrak{KK}$. $\qquad\square$

4. The bootstrap class

Throughout this section, X denotes a *finite* and sober topological space. Finiteness is crucial here. First we construct a canonical filtration on any C*-algebra over X. We use this to study the analogue of the bootstrap class in $\mathfrak{KK}(X)$. Along the way, we also introduce the larger category of local C*-algebras over X. Roughly speaking, locality means that all the canonical C*-algebra extensions that we get from C*-algebras over X are admissible. Objects in the X-equivariant bootstrap category have the additional property that their fibers belong to the usual bootstrap category.

4.1. **The canonical filtration.** We recursively construct a canonical filtration

$$\varnothing = \mathfrak{F}_0 X \subset \mathfrak{F}_1 X \subset \cdots \subset \mathfrak{F}_\ell X = X$$

of X by open subsets $\mathfrak{F}_j X$, such that the differences

$$X_j := \mathfrak{F}_j X \setminus \mathfrak{F}_{j-1} X$$

are discrete for all $j = 1, \ldots, \ell$. In each step, we let X_j be the subset of all open points in $X \setminus \mathfrak{F}_{j-1} X$—so that X_j is discrete—and put $\mathfrak{F}_j X = \mathfrak{F}_{j-1} X \cup X_j$. Equivalently, X_j consists of all points of $X \setminus \mathfrak{F}_{j-1} X$ that are maximal for the specialization preorder $\prec$. Since X is finite, X_j is nonempty unless $\mathfrak{F}_{j-1} X = X$, and our recursion reaches X after finitely many steps.

Definition 4.1. The *length* ℓ of X is the length of the longest chain $x_1 \prec x_2 \prec \cdots \prec x_\ell$ in X.

We assume X finite to ensure that the above filtration can be constructed. It is easy to extend our arguments to *Alexandrov spaces of finite length*; the only difference is that the discrete spaces X_j may be infinite in this case, so that we need infinite direct sums in some places, forcing us in Proposition 4.7 to drop (2) and replace the words "triangulated" by "localizing" in the last sentence. It should be possible to treat Alexandrov spaces of infinite length in a similar way. Since such techniques cannot work for non-Alexandrov spaces, anyway, we do not pursue these generalizations here.

Definition 4.2. We shall use the functors

$$P_Y := i_Y^X \circ r_X^Y \colon \mathfrak{C}^* \mathfrak{alg}(X) \to \mathfrak{C}^* \mathfrak{alg}(X)$$

for $Y \in \mathbb{LC}(X)$. Thus $(P_Y A)(Z) \cong A(Y \cap Z)$ for all $Z \in \mathbb{LC}(X)$.

If $Y \in \mathbb{LC}(X)$, $U \in \mathbb{O}(Y)$, then we get an extension

(4.3) $$P_U(A) \rightarrowtail P_Y(A) \twoheadrightarrow P_{Y \setminus U}(A)$$

in $\mathfrak{C}^* \mathfrak{alg}(X)$ because of the extensions $A(Z \cap U) \rightarrowtail A(Z \cap Y) \twoheadrightarrow A(Z \cap Y \setminus U)$ for all $Z \in \mathbb{LC}(X)$.

Let A be a C*-algebra over X. We equip A with the canonical increasing filtration by the ideals

$$\mathfrak{F}_j A := P_{\mathfrak{F}_j X}(A), \qquad j = 0, \ldots, \ell,$$

so that

(4.4) $\mathfrak{F}_j A(Y) = A(Y \cap \mathfrak{F}_j X) = A(Y) \cap A(\mathfrak{F}_j X)$ for all $Y \in \mathbb{LC}(X)$.

Equation (4.3) shows that the subquotients of this filtration are
(4.5)
$$\mathfrak{F}_j A \,/\, \mathfrak{F}_{j-1} A \cong P_{\mathfrak{F}_j X \setminus \mathfrak{F}_{j-1} X}(A) = P_{X_j}(A) \cong \bigoplus_{x \in X_j} P_x(A) = \bigoplus_{x \in X_j} i_x\big(A(x)\big).$$

Here $i_x = i_x^X$ for $x \in X$ denotes the extension functor from the subset $\{x\} \subseteq X$:

$$i_x \colon \mathfrak{K}\mathfrak{K} \xrightarrow{\cong} \mathfrak{K}\mathfrak{K}(\{x\}) \xrightarrow{i_x} \mathfrak{K}\mathfrak{K}(X), \qquad (i_x B)(Y) = \begin{cases} B & \text{if } x \in Y, \\ 0 & \text{if } x \notin Y. \end{cases}$$

Example 4.6. Consider the space $X = \{1, 2\}$ with the nondiscrete topology described in Example 2.29. Here

$$\mathfrak{F}_0 X = \varnothing, \quad \mathfrak{F}_1 X = \{1\}, \quad \mathfrak{F}_2 X = \{1, 2\} = X, \qquad X_1 = \{1\}, \quad X_2 = \{2\}.$$

The filtration $\mathfrak{F}_j A$ on a C*-algebra over X has one nontrivial layer $\mathfrak{F}_1 A$ because $\mathfrak{F}_0 A = \{0\}$ and $\mathfrak{F}_2 A = A$. Recall that C*-algebras over X correspond to extensions of C*-algebras. For a C*-algebra extension $I \rightarrowtail A \twoheadrightarrow A/I$, the first filtration layer is simply the extension $I \rightarrowtail I \twoheadrightarrow 0$, so that the quotient $A/\mathfrak{F}_1 A$ is the extension $0 \rightarrowtail A/I \twoheadrightarrow A/I$. Our filtration decomposes $I \rightarrowtail A \twoheadrightarrow A/I$ into an extension of C*-algebra extensions as follows:

$$(I \rightarrowtail I \twoheadrightarrow 0) \rightarrowtail (I \rightarrowtail A \twoheadrightarrow A/I) \twoheadrightarrow (0 \rightarrowtail A/I \twoheadrightarrow A/I).$$

Proposition 4.7. *The following are equivalent for a separable C*-algebra A over X:*

(1) *The extensions $\mathfrak{F}_{j-1} A \rightarrowtail \mathfrak{F}_j A \twoheadrightarrow \mathfrak{F}_j A \,/\, \mathfrak{F}_{j-1} A$ in $\mathfrak{C}^*\mathfrak{sep}(X)$ are admissible for $j = 1, \ldots, \ell$.*
(2) *$A \in\in \mathfrak{K}\mathfrak{K}(X)$ belongs to the triangulated subcategory of $\mathfrak{K}\mathfrak{K}(X)$ generated by objects of the form $i_x(B)$ with $x \in X$, $B \in\in \mathfrak{K}\mathfrak{K}$.*
(3) *$A \in\in \mathfrak{K}\mathfrak{K}(X)$ belongs to the localizing subcategory of $\mathfrak{K}\mathfrak{K}(X)$ generated by objects of the form $i_x(B)$ with $x \in X$, $B \in\in \mathfrak{K}\mathfrak{K}$.*
(4) *For any $Y \in \mathbb{LC}(X)$, $U \in \mathbb{O}(Y)$, the extension*

$$P_U(A) \rightarrowtail P_Y(A) \twoheadrightarrow P_{Y \setminus U}(A)$$

in $\mathfrak{C}^*\mathfrak{sep}(X)$ described above is admissible.

Furthermore, if A satisfies these conditions, then it already belongs to the triangulated subcategory of $\mathfrak{K}\mathfrak{K}(X)$ generated by $i_x\big(A(x)\big)$ for $x \in X$.

Recall that the *localizing subcategory* generated by a family of objects in $\mathfrak{K}\mathfrak{K}(X)$ is the smallest subcategory that contains the given objects and is triangulated and closed under countable direct sums.

Proof. (2)$\Longrightarrow$(3) and (4)$\Longrightarrow$(1) are trivial. We will prove (1)$\Longrightarrow$(2) and (3)$\Longrightarrow$(4).

(1)$\Longrightarrow$(2): Since the extensions $\mathfrak{F}_{j-1}A \rightarrowtail \mathfrak{F}_j A \twoheadrightarrow \mathfrak{F}_j A/\mathfrak{F}_{j-1}A$ are admissible, they yield extension triangles in $\mathfrak{KK}(X)$. Thus $\mathfrak{F}_j A$ belongs to the triangulated subcategory of $\mathfrak{KK}(X)$ generated by $\mathfrak{F}_{j-1}A$ and $\mathfrak{F}_j A / \mathfrak{F}_{j-1}A$. Since $\mathfrak{F}_0 A = 0$, induction on j and (4.5) show that $\mathfrak{F}_j A$ belongs to the triangulated subcategory generated by $i_x A(x)$ with $x \in \mathfrak{F}_j X$. Thus $A = \mathfrak{F}_\ell A$ belongs to the triangulated subcategory of $\mathfrak{KK}(X)$ generated by $i_x\big(A(x)\big)$ for $x \in X$. This also yields the last statement in the proposition.

(3)$\Longrightarrow$(4): It is clear that (4) holds for objects of the form $i_x(B)$ because at least one of the three objects $P_U i_x(B)$, $P_Y i_x(B)$, or $P_{Y \setminus U} i_x(B)$ vanishes. The property (4) is inherited by (countable) direct sums, suspensions, and mapping cones. To prove the latter, we use the definition of admissibility as an isomorphism statement in $\mathfrak{KK}(X)$ and the Five Lemma in triangulated categories. Hence (4) holds for all objects of the localizing subcategory generated by $i_x(B)$ for $x \in X$, $B \in\in \mathfrak{KK}$. $\qquad\square$

Definition 4.8. Let $\mathfrak{KK}(X)_{\mathrm{loc}} \subseteq \mathfrak{KK}(X)$ be the full subcategory of all objects that satisfy the equivalent conditions of Proposition 4.7.

The functor $f_* \colon \mathfrak{KK}(X) \to \mathfrak{KK}(Y)$ for a continuous map $f \colon X \to Y$ restricts to a functor $\mathfrak{KK}(X)_{\mathrm{loc}} \to \mathfrak{KK}(Y)_{\mathrm{loc}}$ because $f_* \circ i_x^X = i_x^Y$ and f_* is an exact functor. Similarly, the restriction functor $r_X^Y \colon \mathfrak{KK}(X) \to \mathfrak{KK}(Y)$ for a locally closed subset $Y \subseteq X$ maps $\mathfrak{KK}(X)_{\mathrm{loc}}$ to $\mathfrak{KK}(Y)_{\mathrm{loc}}$ because it is exact and $r_X^Y \circ i_x^X$ is i_x^Y for $x \in Y$ and 0 otherwise.

Proposition 4.9. *Let X be a finite topological space. Let $A, B \in \mathfrak{KK}(X)_{\mathrm{loc}}$ and let $f \in \mathrm{KK}_*(X; A, B)$. If $f(x) \in \mathrm{KK}_*\big(A(x), B(x)\big)$ is invertible for all $x \in X$, then f is invertible in $\mathfrak{KK}(X)$. In particular, if $A(x) \cong 0$ in $\mathfrak{KK}$ for all $x \in X$, then $A \cong 0$ in $\mathfrak{KK}(X)$.*

Proof. The second assertion follows immediately from the last sentence in Proposition 4.7. It implies the first one by a well-known trick: embed α in an exact triangle by axiom (TR1) of a triangulated category, and use the long exact sequence to relate invertibility of α to the vanishing of its mapping cone. $\qquad\square$

Proposition 4.10. *Suppose that the C*-algebra extensions*

$$A(U_x \setminus \{x\}) \rightarrowtail A(U_x) \twoheadrightarrow A(x)$$

are semi-split for all $x \in X$. Then $A \in\in \mathfrak{KK}(X)_{\mathrm{loc}}$. In particular, this applies if the underlying C*-algebra of $A \in\in \mathfrak{KK}(X)$ is nuclear.

Proof. We claim that the extensions in Proposition 4.7.(1) are semi-split as extensions of C*-algebras over X, hence admissible in $\mathfrak{KK}(X)$. For this, we need a completely positive section $A(X_j) \to A(\mathfrak{F}_j X)$ that is X-equivariant, that is, restricts to maps $A(X_j \cap V) \to A(\mathfrak{F}_j X \cap V)$ for all $V \in \mathbb{O}(\mathfrak{F}_j X)$. We take the sum of the completely positive sections for the extensions $A(U_x \setminus \{x\}) \rightarrowtail$

$A(U_x) \twoheadrightarrow A(x)$ for $x \in X_j$. This map has the required property because any open subset containing x also contains U_x; it is irrelevant whether or not this section is contractive by the last sentence in Theorem 3.6. If A is nuclear, so are the ideals $A(U_x)$ and their quotients $A(x)$ for $x \in X$. Thus the above extensions have completely positive sections by the Choi-Effros Lifting Theorem (see [5]). □

It is not clear whether the mere admissibility in $\mathfrak{C}^*\mathfrak{sep}$ of the extensions

$$A(U_x \setminus \{x\}) \rightarrowtail A(U_x) \twoheadrightarrow A(x)$$

suffices to conclude that $A \in\in \mathfrak{KK}(X)_{\text{loc}}$. This condition is certainly necessary.

The constructions above yield spectral sequences as in [19]. These may be useful for spaces of length 1, where they degenerate to a short exact sequence. We only comment on this very briefly.

Let $A \in\in \mathfrak{KK}(X)_{\text{loc}}$. The admissible extensions $\mathfrak{F}_{j-1}A \rightarrowtail \mathfrak{F}_j A \twoheadrightarrow \mathfrak{F}_j A \, / \, \mathfrak{F}_{j-1}A$ for $j = 1, \ldots, \ell$ produce exact triangles in $\mathfrak{KK}(X)$. A homological or cohomological functor such as $\mathrm{KK}(X; D, \_)$ or $\mathrm{KK}(X; \_, D)$ maps these exact triangles to a sequence of exact chain complexes. These can be arranged in an *exact couple*, which generates a spectral sequence (see [11]). This spectral sequence could, in principle, be used to compute $\mathrm{KK}_*(X; A, B)$ in terms of

$$\mathrm{KK}_*(X; \mathfrak{F}_j A \, / \, \mathfrak{F}_{j-1}A, B) \cong \prod_{x \in X_j} \mathrm{KK}_*(X; i_x A(x), B)$$

$$\cong \prod_{x \in X_j} \mathrm{KK}_*\big(A(x), B(U_x)\big),$$

where we have used Proposition 3.13. These groups comprise the E_1-terms of the spectral sequence that we get from our exact couple for the functor $\mathrm{KK}(X; \_, B)$.

For instance, consider again the situation of Example 4.6. Let $I \triangleleft A$ and $J \triangleleft B$ be C*-algebras over X, corresponding to C*-algebra extensions $I \rightarrowtail A \twoheadrightarrow A/I$ and $J \rightarrowtail B \twoheadrightarrow B/J$. The above spectral sequence degenerates to a long exact sequence

$$
\begin{array}{ccccc}
\mathrm{KK}_0(A/I, B) & \longrightarrow & \mathrm{KK}_0(X; I \triangleleft A, J \triangleleft B) & \longrightarrow & \mathrm{KK}_0(I, J) \\
{\scriptstyle\delta}\uparrow & & & & \downarrow{\scriptstyle\delta} \\
\mathrm{KK}_1(I, J) & \longleftarrow & \mathrm{KK}_1(X; I \triangleleft A, J \triangleleft B) & \longleftarrow & \mathrm{KK}_1(A/I, B).
\end{array}
$$

The boundary map is the diagonal map in the following commuting diagram:

$$
\begin{array}{ccc}
\mathrm{KK}_0(I, J) & \longrightarrow & \mathrm{KK}_0(I, B) \\
\downarrow & {\scriptstyle\delta}\searrow & \downarrow \\
\mathrm{KK}_1(A/I, J) & \longrightarrow & \mathrm{KK}_1(A/I, B).
\end{array}
$$

We can rewrite the long exact sequence above as an extension:

$$\operatorname{coker}\delta \rightarrowtail \mathrm{KK}_*(X; I \lhd A, J \lhd B) \twoheadrightarrow \ker\delta.$$

But we lack a description of $\ker\delta$ and $\operatorname{coker}\delta$ as Hom- and Ext-groups. Therefore, the Universal Coefficient Theorem of Alexander Bonkat [4] seems more attractive.

4.2. **The bootstrap class.** The bootstrap class $\mathcal{B}$ in $\mathfrak{KK}$ is the localizing subcategory generated by the single object $\mathbb{C}$, that is, it is the smallest class of separable C*-algebras that contains $\mathbb{C}$ and is closed under KK-equivalence, countable direct sums, suspensions, and the formation of mapping cones (see [16]).

A localizing subcategory of $\mathfrak{KK}(X)$ or $\mathfrak{KK}$ is automatically closed under various other constructions, as explained in [14]. This includes admissible extensions, admissible inductive limits (the appropriate notion of admissibility is explained in [14]), and crossed products by $\mathbb{Z}$ and $\mathbb{R}$ and, more generally, by actions of torsion-free amenable groups.

The latter result uses the reformulation of the (strong) Baum-Connes property for such groups in [14]. This reformulation asserts that $\mathbb{C}$ with the trivial representation of an amenable group G belongs to the localizing subcategory of $\mathfrak{KK}(G)$ generated by $\mathrm{C}_0(G)$. Carrying this over to $\mathfrak{KK}(X)$, we conclude that $A \rtimes G$ for $A \in\in \mathfrak{KK}(X)$ belongs to the localizing subcategory of $\mathfrak{KK}(X)$ generated by $\big(A \otimes \mathrm{C}_0(G)\big) \rtimes G$, which is Morita-Riefel equivalent to A.

The following definition provides an analogue $\mathcal{B}(X) \subseteq \mathfrak{KK}(X)$ of the bootstrap class $\mathcal{B} \subseteq \mathfrak{KK}$ for a finite topological space X:

Definition 4.11. Let $\mathcal{B}(X)$ be the localizing subcategory of $\mathfrak{KK}(X)$ that is generated by $i_x(\mathbb{C})$ for $x \in X$.

Notice that $\{i_x(\mathbb{C}) \mid x \in X\}$ lists all possible ways to turn $\mathbb{C}$ into a C*-algebra over X.

Proposition 4.12. *Let X be a finite topological space and let $A \in\in \mathfrak{KK}(X)$. The following conditions are equivalent:*

1. *$A \in\in \mathcal{B}(X)$;*
2. *$A \in\in \mathfrak{KK}(X)_{\mathrm{loc}}$ and $A(x) \in\in \mathcal{B}$ for all $x \in X$;*
3. *the extensions $\mathfrak{F}_{j-1}A \rightarrowtail \mathfrak{F}_j A \twoheadrightarrow \mathfrak{F}_j A / \mathfrak{F}_{j-1} A$ are admissible for $j = 1, \ldots, \ell$, and $A(x) \in\in \mathcal{B}$ for all $x \in X$;*

In addition, in this case $A(Y) \in\in \mathcal{B}$ for all $Y \in \mathbb{LC}(X)$.

Proof. The equivalence of (2) and (3) is already contained in Proposition 4.7. Using the last sentence of Proposition 4.7, we also get the implication (3)$\Longrightarrow$(1) because i_x is exact and commutes with direct sums. The only assertion that is not yet contained in Proposition 4.7 is that $A \in\in \mathcal{B}(X)$ implies $A(Y) \in\in \mathcal{B}$ for all $Y \in \mathbb{LC}(X)$. The reason is that the functor $\mathfrak{KK}(X) \to \mathfrak{KK}$, $A \mapsto A(Y)$, is exact, preserves countable direct sums, and maps the generators $i_y(\mathbb{C})$ for $y \in X$ to either 0 or $\mathbb{C}$ and hence into $\mathcal{B}$. $\square$

Corollary 4.13. *If the underlying* C*-*algebra of A is nuclear, then $A \in\in \mathcal{B}(X)$ if and only if $A(x) \in\in \mathcal{B}$ for all $x \in X$.*

Proof. Combine Propositions 4.10 and 4.12. □

Example 4.14. View a separable nuclear C*-algebra A with only finitely many ideals as a C*-algebra over $\mathrm{Prim}(A)$. Example 2.16 and Corollary 4.13 show that A belongs to $\mathcal{B}(\mathrm{Prim}\,A)$ if and only if all its simple subquotients belong to the usual bootstrap class in $\mathfrak{KK}$.

Proposition 4.15. *Let X be a finite topological space. Let $A, B \in\in \mathcal{B}(X)$ and let $f \in \mathrm{KK}_*(X; A, B)$. If f induces invertible maps $\mathrm{K}_*\big(A(x)\big) \to \mathrm{K}_*\big(B(x)\big)$ for all $x \in X$, then f is invertible in $\mathfrak{KK}(X)$. In particular, if $\mathrm{K}_*\big(A(x)\big) = 0$ for all $x \in X$, then $A \cong 0$ in $\mathfrak{KK}(X)$.*

Proof. As in the proof of Proposition 4.9, it suffices to show the second assertion. Since $A(x) \in\in \mathcal{B}$ for all $x \in X$, vanishing of $\mathrm{K}_*\big(A(x)\big)$ implies vanishing of $A(x)$ in $\mathfrak{KK}$, so that Proposition 4.9 yields the assertion. □

4.3. **Complementary subcategories.** It is often useful to replace a given object of $\mathfrak{KK}(X)$ by one in the bootstrap class or $\mathfrak{KK}(X)_{\mathrm{loc}}$ that is as close to the original as possible. This is achieved by localization functors

$$L_{\mathcal{B}} \colon \mathfrak{KK}(X) \to \mathcal{B}(X), \qquad L \colon \mathfrak{KK}(X) \to \mathfrak{KK}(X)_{\mathrm{loc}}$$

that are right adjoint to the embeddings of these subcategories. That is, we want $\mathrm{KK}_*\big(X; A, L_{\mathcal{B}}(B)\big) \cong \mathrm{KK}_*(X; A, B)$ for all $A \in\in \mathcal{B}(X)$, $B \in \mathfrak{KK}(X)$ and similarly for L. These functors come with natural transformations $L_{\mathcal{B}} \Rightarrow L \Rightarrow \mathrm{id}$, and the defining property is equivalent to $L(A)_x \to A_x$ being a KK-equivalence and $\mathrm{K}_*(L_{\mathcal{B}}(A)_x) \to \mathrm{K}_*(A_x)$ being invertible for all $x \in X$, respectively.

The functors L and $L_{\mathcal{B}}$ exist because our two subcategories belong to complementary pairs of localizing subcategories in the notation of [14]. The existence of this complementary pair is straightforward to prove using the techniques of [13].

Definition 4.16. Let $\mathfrak{KK}(X)_{\mathrm{loc}}^{\dashv}$ be the class of all $A \in\in \mathfrak{KK}(X)$ for which $A(x)$ is KK-equivalent to 0 for all $x \in X$. Let $\mathcal{B}(X)^{\dashv}$ be the class of all $A \in\in \mathfrak{KK}(X)$ with $\mathrm{K}_*\big(A(x)\big) = 0$.

Theorem 4.17. *The pair of subcategories $(\mathcal{B}(X), \mathcal{B}(X)^{\dashv})$ is complementary. So is the pair $(\mathfrak{KK}(X)_{\mathrm{loc}}, \mathfrak{KK}(X)_{\mathrm{loc}}^{\dashv})$.*

Proof. We first prove the assertion for $\mathfrak{KK}(X)_{\mathrm{loc}}$. Consider the exact functor

$$F \colon \mathfrak{KK}(X) \to \prod_{x \in X} \mathfrak{KK}, \qquad A \mapsto (A_x)_{x \in X}.$$

Let $\mathcal{I}$ be the kernel of F on morphisms. Since F is an exact functor that commutes with countable direct sums, $\mathcal{I}$ is a stable homological ideal that is compatible with direct sums (see [15, 13]). The kernel of F on objects is

exactly $\mathfrak{K}\mathfrak{K}(X)^{\dashv}$. Proposition 3.13 shows that the functor F has a left adjoint, namely, the functor

$$F^{\vdash}\big((A_x)_{x\in X}\big) := \bigoplus_{x\in X} i_x(A_x).$$

Therefore, the ideal $\mathcal{I}$ has enough projective objects by [15, Prop. 3.37]; furthermore, the projective objects are retracts of direct sums of objects of the form $i_x(A_x)$. Hence the localizing subcategory generated by the $\mathcal{I}$-projective objects is $\mathfrak{K}\mathfrak{K}(X)_{\mathrm{loc}}$ by Proposition 4.7. Finally, [13, Thm. 4.6] shows that the pair of subcategories $(\mathfrak{K}\mathfrak{K}(X)_{\mathrm{loc}}, \mathfrak{K}\mathfrak{K}(X)_{\mathrm{loc}}^{\dashv})$ is complementary.

The argument for the bootstrap category is almost literally the same, but using the stable homological functor $\mathrm{K}_* \circ F \colon \mathfrak{K}\mathfrak{K}(X) \to \prod_{x\in X} \mathfrak{Ab}_{\mathrm{c}}^{\mathbb{Z}/2}$ instead of F, where $\mathfrak{Ab}_{\mathrm{c}}^{\mathbb{Z}/2}$ denotes the category of countable $\mathbb{Z}/2$-graded abelian groups. The adjoint of $\mathrm{K}_* \circ F$ is defined on families of countable free abelian groups, which is enough to conclude that $\ker(\mathrm{K}_* \circ F)$ has enough projective objects. This time, the projective objects generate the category $\mathcal{B}(X)$, and the kernel of F on objects is $\mathcal{B}(X)^{\dashv}$. Hence [13, Thm. 4.6] shows that the pair of subcategories $(\mathcal{B}(X), \mathcal{B}(X)^{\dashv})$ is complementary. $\qquad\square$

Lemma 4.18. *The following are equivalent for $A \in\in \mathfrak{K}\mathfrak{K}(X)$:*
   (1) $\mathrm{K}_*\big(A(x)\big) = 0$ *for all $x \in X$, that is, $A \in\in \mathcal{B}(X)^{\dashv}$;*
   (2) $\mathrm{K}_*\big(A(Y)\big) = 0$ *for all $Y \in \mathbb{LC}(X)$;*
   (3) $\mathrm{K}_*\big(A(U)\big) = 0$ *for all $U \in \mathbb{O}(X)$.*

Proof. It is clear that (2) implies both (1) and (3). Conversely, (3) implies (2): write $Y \in \mathbb{LC}(X)$ as $U \setminus V$ with $U, V \in \mathbb{O}(X)$, $V \subseteq U$, and use the K-theory long exact sequence for the extension $A(U) \rightarrowtail A(V) \twoheadrightarrow A(Y)$. It remains to check that (1) implies (2).

We prove by induction on j that (1) implies $\mathrm{K}_*\big(A(Y)\big) = 0$ for all $Y \in \mathbb{LC}(\mathfrak{F}_j X)$. This is trivial for $j = 0$. If $Y \subseteq \mathfrak{F}_{j+1} X$, then $\mathrm{K}_*\big(A(Y \cap \mathfrak{F}_j X)\big) = 0$ by the induction assumption. The K-theory long exact sequence for the extension

$$A(Y \cap \mathfrak{F}_j X) \rightarrowtail A(Y) \twoheadrightarrow \bigoplus_{x \in X_{j+1} \cap Y} A(x)$$

yields $\mathrm{K}_*\big(A(Y)\big) = 0$ as claimed. $\qquad\square$

We can also apply the machinery of [15, 13] to the ideal $\mathcal{I}$ to generate a spectral sequence that computes $\mathrm{KK}_*(X; A, B)$. This spectral sequence is more useful than the one from the canonical filtration because its second page involves derived functors. But this spectral sequence rarely degenerates to an exact sequence.

4.4. **A definition for infinite spaces.** The ideas in §4.3 suggest a definition of the bootstrap class for infinite spaces.

Definition 4.19. Let X be a topological space. Let $\mathcal{B}(X)^{\dashv} \subseteq \mathfrak{K}\mathfrak{K}(X)$ consist of all separable C*-algebras over X with $\mathrm{K}_*\big(A(U)\big) = 0$ for all $U \in \mathbb{O}(X)$.

Lemma 4.18 shows that this agrees with our previous definition for finite X. Furthermore, the same argument as in the proof of Lemma 4.18 yields $A \in\in \mathcal{B}(X)^{\dashv}$ if and only if $\mathrm{K}_*\big(A(Y)\big) = 0$ for all $Y \in \mathbb{LC}(X)$. The first condition in Lemma 4.18 has no analogue because of Example 2.17.

It is clear from the definition that $\mathcal{B}(X)^{\dashv}$ is a localizing subcategory of $\mathfrak{KK}(X)$.

Definition 4.20. Let X be a topological space. We let $\mathcal{B}(X)$ be the localization of $\mathfrak{KK}(X)$ at $\mathcal{B}(X)^{\dashv}$.

For finite X, we have seen that $\mathcal{B}(X)^{\dashv}$ is part of a complementary pair of localizing subcategories, with partner $\mathcal{B}(X)$. This shows that the localization of $\mathfrak{KK}(X)$ at $\mathcal{B}(X)^{\dashv}$ is canonically equivalent to $\mathcal{B}(X)$. For infinite X, it is unclear whether $\mathcal{B}(X)^{\dashv}$ is part of a complementary pair. If it is, the partner must be

$$\mathcal{D} := \{A \in\in \mathfrak{KK}(X) \mid \mathrm{KK}_*(X; A, B) = 0 \text{ for all } B \in\in \mathcal{B}(X)^{\dashv}\}.$$

Since $B \in\in \mathcal{B}(X)^{\dashv}$ implies nothing about the K-theory of $\bigcap_{U \in \mathcal{U}_x} B(U)$, in general, Proposition 3.13 shows that $i_x\mathbb{C}$ does not belong to $\mathcal{D}$ in general.

If X is Hausdorff, then $\mathrm{C}_0(U) \in\in \mathcal{B}(X)^{\dashv}$ for all $U \in \mathbb{O}(X)$. Nevertheless, it is not clear whether $(\mathcal{D}, \mathcal{B}(X)^{\dashv})$ is complementary.

5. Making the fibers simple

Definition 5.1. A C*-algebra (A, ψ) over X is called *tight* if $\psi \colon \mathrm{Prim}(A) \to X$ is a homeomorphism.

Tightness implies that the fibers $A_x = A(x)$ for $x \in X$ are simple C*-algebras. But the converse does not hold: the fibers are simple if and only if the map $\psi \colon \mathrm{Prim}(A) \to X$ is bijective.

To equip $\mathfrak{KK}(X)$ with a triangulated category structure, we must drop the tightness assumption because it is usually destroyed when we construct cylinders, mapping cones, or extensions of C*-algebras over X. Nevertheless, we show below that we may reinstall tightness by passing to a KK(X)-equivalent object, at least in the nuclear case.

The special case where the space X in question has only one point is already known:

Theorem 5.2 ([18, Prop. 8.4.5]). *Any separable nuclear C*-algebra is KK-equivalent to a C*-algebra that is separable, nuclear, purely infinite, C*-stable and simple.*

Stability is not part of the assertion in [18], but can be achieved by tensoring with the compact operators, without destroying the other properties. The main difficulty is to achieve simplicity. We are going to generalize this theorem as follows:

Theorem 5.3. *Let X be a finite topological space. Any separable nuclear C*-algebra over X is KK(X)-equivalent to a C*-algebra over X that is tight, separable, nuclear, purely infinite, and C*-stable.*

For the zero C*-algebra, viewed as a C*-algebra over X, this reproves the known statement that there is a separable, nuclear, purely infinite, and stable C*-algebra with spectrum X for any finite topological space X.

Proof. Since A is separable and nuclear, so are the subquotients A_x. Hence Theorem 5.2 provides simple, separable, nuclear, stable, purely infinite C*-algebras B_x and KK-equivalences $f_x \in \mathrm{KK}_0(A_x, B_x)$ for all $x \in X$.

We use the canonical filtration $\mathfrak{F}_j X$ of X and the resulting filtration $\mathfrak{F}_j A$ introduced in §4.1, see (4.4). The subquotients

$$A_j^0 := \mathfrak{F}_j A \,/\, \mathfrak{F}_{j-1} A$$

of the filtration are described in (4.5) in terms of the subquotients A_x for $x \in X$.

We will recursively construct a sequence B_j of C*-algebras over X that are supported on $\mathfrak{F}_j X$ and $\mathrm{KK}(X)$-equivalent to $\mathfrak{F}_j A$ for $j = 0, \dots, \ell$, such that $\mathfrak{F}_j B_k = B_j$ for $k \geq j$ and each B_j is tight over $\mathfrak{F}_j X$, separable, nuclear, purely infinite, and stable. The last object B_ℓ in this series is $\mathrm{KK}(X)$-equivalent to $\mathfrak{F}_\ell A = A$ and has all the required properties. Since $\mathfrak{F}_0 X = \varnothing$, the recursion must begin with $B_0 = A_0 = \{0\}$. We assume that B_j has been constructed. Let

$$B_{j+1}^0 := \bigoplus_{x \in X_{j+1}} i_x(B_x).$$

We will construct B_{j+1} as an extension of B_j by B_{j+1}^0. This ensures that the fibers of B_j are B_x for $x \in \mathfrak{F}_j X$ and 0 for $x \in X \setminus \mathfrak{F}_j X$.

First we construct, for each $x \in X_{j+1}$, a suitable extension of B_x by B_j. Let $U_x \subseteq \mathfrak{F}_{j+1} X$ be the minimal open subset containing x and let $U_x' := U_x \setminus \{x\}$. Since X_{j+1} is discrete, U_x' is an open subset of $\mathfrak{F}_j X$. The extension

$$A(U_x') \rightarrowtail A(U_x) \twoheadrightarrow A_x$$

is semi-split and thus provides a class δ_x^A in $\mathrm{KK}_1\big(A_x, A(U_x')\big)$ because A_x is nuclear. Since $B_x \simeq A_x$, $\mathfrak{F}_j A \simeq B_j$ and $\mathfrak{F}_j A(U_x') = A(U_x')$, we can transform this class to δ_x^B in $\mathrm{KK}_1\big(B_x, B_j(U_x')\big)$.

We abbreviate $B_{jx} := B_j(U_x')$ to simplify our notation. Represent δ_x^B by an odd Kasparov cycle $(\mathcal{H}, \varphi, F)$, where $\mathcal{H}$ is a Hilbert B_{jx}-module, $\varphi \colon B_x \to \mathbb{B}(\mathcal{H})$ is a *-homomorphism, and $F \in \mathbb{B}(\mathcal{H})$ satisfies $F^2 = 1$, $F = F^*$, and $[F, \varphi(b)] \in \mathbb{K}(\mathcal{H})$ for all $b \in B_x$. Now we apply the familiar correspondence between odd KK-elements and C*-algebra extensions. Let $P := \frac{1}{2}(1 + F)$, then

$$\psi \colon B_x \to \mathbb{B}(\mathcal{H}) \,/\, \mathbb{K}(\mathcal{H}), \qquad b \mapsto P\varphi(x)P$$

is a *-homomorphism and hence the Busby invariant of an extension of B_x by $\mathbb{K}(\mathcal{H})$. After adding a sufficiently big split extension, that is, a *-homomorphism $\psi_0 \colon B_x \to \mathbb{B}(\mathcal{H}')$, the map $\psi \colon B_x \to \mathbb{B}(\mathcal{H}) / \mathbb{K}(\mathcal{H})$ becomes injective and the ideal in $\mathbb{K}(\mathcal{H})$ generated by $\mathbb{K}(\mathcal{H})\psi(B_x)\mathbb{K}(\mathcal{H})$ is all of $\mathbb{K}(\mathcal{H})$. We assume these two extra properties from now on.

We also add to ψ the trivial extension $B_j \rightarrowtail B_j \oplus B_x \twoheadrightarrow B_x$, whose Busby invariant is the zero map. This produces an extension of B_x by $\mathbb{K}(B_j \oplus \mathcal{H}) \cong B_j$;

the last isomorphism holds because B_j is stable, so that $B_j \oplus \mathcal{H}' \cong B_j$ for any Hilbert B_j-module $\mathcal{H}'$. Since ψ is injective, the extension we get is of the form $B_j \rightarrowtail E_{jx} \twoheadrightarrow B_x$. This extension is still semi-split, and its class in $\mathrm{KK}_1(B_x, B_j)$ is the composite of δ_x^B with the embedding $B_{jx} \to B_j$. Our careful construction ensures that the ideal in B_j generated by $B_j \psi(B_x) B_j$ is equal to $B(U'_x)$.

Now we combine these extensions for all $x \in X$ by taking their external direct sum. This is an extension of $\bigoplus_{x \in X_{j+1}} B_x = B_{j+1}^0$ by the C*-algebra of compact operators on the Hilbert B_j-module $\bigoplus_{x \in X_{j+1}} B_j \cong B_j$, where we used the stability of B_j once more. Thus we obtain an extension $B_j \rightarrowtail B_{j+1} \twoheadrightarrow B_{j+1}^0$. We claim that the primitive ideal space of B_{j+1} identifies naturally with $\mathfrak{F}_{j+1} X$.

The extension $B_j \rightarrowtail B_{j+1} \twoheadrightarrow B_{j+1}^0$ decomposes $\mathrm{Prim}(B_{j+1})$ into an open subset homeomorphic to $\mathrm{Prim}(B_j) \cong \mathfrak{F}_j X$ and a closed subset homeomorphic to the discrete set $\mathrm{Prim}(B_{j+1}^0) = X_{j+1}$. This provides a canonical bijection between $\mathrm{Prim}(B_{j+1})$ and $\mathfrak{F}_{j+1} X$. We must check that it is a homeomorphism.

First let $U \subseteq \mathfrak{F}_{j+1} X$ be open in $\mathfrak{F}_{j+1} X$. Then $U \cap \mathfrak{F}_j X$ is open and contains U'_x for each $x \in U \cap X_{j+1}$. Our construction ensures that $\psi(B_x) \subseteq B_{j+1}$ multiplies B_j into $B_{jx} \subseteq B_j(U \cap \mathfrak{F}_j X)$. Hence

$$B_j(U \cap \mathfrak{F}_j X) + \sum_{x \in U \cap X_{j+1}} \psi(B_x)$$

is an ideal in B_{j+1}. This shows that U is open in $\mathrm{Prim}(B_{j+1})$.

Now let $U \subseteq \mathfrak{F}_{j+1} X$ be open in $\mathrm{Prim}(B_{j+1})$. Then $U \cap \mathfrak{F}_j$ must be open in $\mathfrak{F}_j X \cong \mathrm{Prim}(B_j)$. Furthermore, if $x \in U \cap X_{j+1}$, then the subset of $\mathrm{Prim}(B_j)$ corresponding to the ideal in B_j generated by $B_j \psi(B_x) B_j$ is contained in U. But our construction ensures that this subset is precisely U'_x. Hence

$$U = (U \cap \mathfrak{F}_j X) \cup \bigcup_{x \in U \cap X_{j+1}} U_x,$$

proving that U is open in the topology of $\mathfrak{F}_{j+1} X$. This establishes that our canonical map between $\mathrm{Prim}(B_{j+1})$ and $\mathfrak{F}_{j+1} X$ is a homeomorphism. Thus we may view B_{j+1} as a C*-algebra over X supported in $\mathfrak{F}_{j+1} X$. It is clear from our construction that $B_j \rightarrowtail B_{j+1} \twoheadrightarrow B_{j+1}^0$ is an extension of C*-algebras over X. Here we view B_{j+1}^0 as a C*-algebra over X in the obvious way, so that B_x is its fiber over x for $x \in X_{j+1}$.

There is no reason to expect B_{j+1} to be stable or purely infinite. But this is easily repaired by tensoring with $\mathbb{K} \otimes \mathcal{O}_\infty$. This does not change B_j and B_{j+1}^0, up to isomorphism, because these are already stable and purely infinite, and it has no effect on the primitive ideal space, nuclearity or separability. Thus we may achieve that B_{j+1} is stable and purely infinite.

By assumption, there is a KK(X)-equivalence $f_j \in \mathrm{KK}_0(X; \mathfrak{F}_j A, B_j)$. Furthermore, our construction of B_{j+1}^0 ensures a KK(X)-equivalence f_{j+1}^0 between

A^0_{j+1} and B^0_{j+1}. Due to the nuclearity of A, the arguments in §4.1 show that

$$\mathfrak{F}_j A \rightarrowtail \mathfrak{F}_{j+1} A \twoheadrightarrow A^0_{j+1}$$

is a semi-split extension of C*-algebras over X and hence provides an exact triangle in $\mathfrak{KK}(X)$. The same argument provides an extension triangle for the extension $B_j \rightarrowtail B_{j+1} \twoheadrightarrow B^0_{j+1}$. Let δ^A_j and δ^B_j be the classes in $\mathrm{KK}_1(X; A^0_{j+1}, \mathfrak{F}_j A)$ and $\mathrm{KK}_1(B^0_{j+1}, B_j)$ associated to these extension; they appear in the exact triangles described above.

Both classes δ^A_j and δ^B_j are, essentially, the sum of the classes δ^A_x and δ^B_x for $x \in X_{j+1}$, respectively. More precisely, we have to compose each δ^A_x with the embedding $A(U'_x) \to \mathfrak{F}_j A$. Hence the solid square in the diagram

$$
\begin{array}{ccccccc}
\Sigma A^0_{j+1} & \xrightarrow{\delta^A_j} & \mathfrak{F}_j A & \longrightarrow & \mathfrak{F}_{j+1} A & \longrightarrow & A^0_{j+1} \\
{\scriptstyle \Sigma f^0_{j+1}} \downarrow {\scriptstyle \cong} & & {\scriptstyle f_j} \downarrow {\scriptstyle \cong} & & {\scriptstyle f_{j+1}} \downarrow {\scriptstyle \cong} & & {\scriptstyle f^0_{j+1}} \downarrow {\scriptstyle \cong} \\
\Sigma B^0_{j+1} & \xrightarrow{\delta^B_j} & B_j & \longrightarrow & B_{j+1} & \longrightarrow & B^0_{j+1}
\end{array}
$$

commutes. By an axiom of triangulated categories, we can find the dotted arrow making the whole diagram commute. The Five Lemma for triangulated categories asserts that this arrow is invertible because f_j and f^0_{j+1} are. This shows that B_{j+1} has all required properties and completes the induction step. $\qquad\square$

Theorem 5.4. *Let X be a finite topological space and let A be a separable C*-algebra over X. The following are equivalent:*

- *$A \in\in \mathfrak{KK}(X)_{\mathrm{loc}}$ and A_x is KK-equivalent to a nuclear C*-algebra for each $x \in X$;*
- *A is KK(X)-equivalent to a C*-algebra over X that is tight, separable, nuclear, purely infinite, and C*-stable.*

Proof. The proof of Theorem 5.3 still works under the weaker assumption that $A \in\in \mathfrak{KK}(X)_{\mathrm{loc}}$ and A_x is KK-equivalent to a nuclear C*-algebra for each $x \in X$. The converse implication is trivial. $\qquad\square$

Corollary 5.5. *Let X be a finite topological space and let A be a separable C*-algebra over X. The following are equivalent:*

- *$A \in\in \mathcal{B}(X)$;*
- *A is KK(X)-equivalent to a C*-algebra over X that is tight, separable, nuclear, purely infinite, C*-stable, and has fibers A_x in the bootstrap class $\mathcal{B}$.*

Proof. Combine Theorem 5.4 and Proposition 4.12. $\qquad\square$

By a deep classification result by Eberhard Kirchberg (see [10]), two tight, separable, nuclear, purely infinite, stable C*-algebras over X are KK(X)-equivalent if and only if they are isomorphic as C*-algebras over X. Therefore, the representatives found in Theorems 5.3 and 5.4 are unique up to X-equivariant *-isomorphism.

Let $\mathfrak{KK}(X)_\mathrm{nuc}$ be the subcategory of $\mathfrak{KK}(X)$ whose objects are the separable nuclear C*-algebras over X. This is a triangulated category as well because the basic constructions like suspensions, mapping cones, and extensions never leave this subcategory. The subcategory of $\mathfrak{KK}(X)$ whose objects are the tight, separable, nuclear, purely infinite, stable C*-algebras over X is equivalent to $\mathfrak{KK}(X)_\mathrm{nuc}$ by Theorem 5.3 and hence inherits a triangulated category structure. It has the remarkable feature that isomorphisms in this triangulated category lift to X-equivariant *-isomorphisms.

Recall that a C*-algebra belongs to $\mathcal{B}$ if and only if it is KK-equivalent to a *commutative* C*-algebra. This probably remains the case at least for finite spaces X, but the authors do not know how to prove this. For infinite spaces, it is even less clear whether $\mathcal{B}(X)$ is equivalent to the KK(X)-category of commutative C*-algebras over X. We only have the following characterization:

Theorem 5.6. *A separable C*-algebra over X belongs to the bootstrap class $\mathcal{B}(X)$ if and only if it is* KK(X)-*equivalent to a* C*-*stable, separable* C*-*algebra over X of type I.*

Proof. Follow the proof of Theorem 5.3, but using stabilizations of commutative C*-algebras B_x instead of nuclear purely infinite ones. The proof shows that we can also achieve that the fibers B_x are all of the form $C_0(Y_x) \otimes \mathbb{K}$ for second countable locally compact spaces Y_x. $\qquad\square$

6. Outlook

We have defined a bootstrap class $\mathcal{B}(X) \subseteq \mathfrak{KK}(X)$ over a finite topological space X, which is the domain on which we should expect a Universal Coefficient Theorem to compute KK$_*(X; A, B)$. We have seen that any object of the bootstrap class is KK(X)-equivalent to a tight, purely infinite, stable, nuclear, separable C*-algebra over X, for which Kirchberg's classification results apply.

There are several spectral sequences that compute KK$_*(X; A, B)$, but applications to the classification programme require a short exact sequence. For some finite topological spaces, such a short exact sequence is constructed in [16] based on filtrated K-theory, so that filtrated K-theory is a complete invariant. This invariant comprises the K-theory $K_*\big(A(Y)\big)$ of all locally closed subsets Y of X together with the action of all natural transformations between them. This is a consequence of a Universal Coefficient Theorem in this case. It is also shown in [16] that there are finite topological spaces for which filtrated K-theory is not yet a complete invariant. At the moment, it is unclear whether there is a general, tractable complete invariant for objects of $\mathcal{B}(X)$.

Another issue is to treat infinite topological spaces. A promising approach is to approximate infinite spaces by finite non-Hausdorff spaces associated to open coverings of the space in question. In good cases, there should be a $\varprojlim^1$-sequence that relates $\mathrm{KK}_*(X; A, B)$ to Kasparov groups over such finite approximations to X, reducing computations from the infinite to the finite case. Such an exact sequence may be considerably easier for X-equivariant E-theory, where we do not have to worry about completely positive sections.

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