Exact C*-algebras and $C_0(X)$-structure

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Abstract. We study tensor products of a $C_0(X)$-algebra A and a $C_0(Y)$-algebra B, and analyze the structure of their minimal tensor product $A \otimes_\alpha B$ as a $C_0(X \times Y)$-algebra. We show that when A and B define continuous C*-bundles, continuity of the bundle arising from the $C_0(X \times Y)$-algebra $A \otimes_\alpha B$ is a strictly weaker property than continuity of the 'fiberwise tensor products' studied by Kirchberg and Wassermann. For a fixed quasi-standard C*-algebra A, we show that $A \otimes_\alpha B$ is quasi-standard for all quasi-standard B precisely when A is exact, and exhibit some related equivalences.

1. Introduction

Bundles or fields of C*-algebras have long been an important aspect of the topological decomposition theory of C*-algebras. For instance the Gelfand–Naimark theorem may be seen as representing an arbitrary commutative C*-algebra as the section algebra of a bundle over its character space, with one-dimensional fibers. In the general (non-commutative) case, many analogous constructions have been studied. Often these constructions give rise to bundles of a very general type, and the structure of such bundles is in general difficult to determine. There are however many classes of C*-algebras for which a well-behaved bundle representation may be obtained. In this work we are concerned with the stability of such classes of C*-bundles under the operation of taking tensor products, in particular with respect to the minimal (or spatial) tensor norm.

A particular case of interest in the theory of C*-bundles is that where the base space coincides with the topological space of Glimm (or Minimal Primal) ideals of the bundle algebra. This approach has its origins in [16], where it was shown that a C*-algebra with Hausdorff primitive ideal space admits a representation as a continuous C*-bundle (or a maximal full algebra of operator fields, using Fell's terminology) over this space. The work of Dauns and

Hofmann in [14] gave a bundle representation theory that was valid for an arbitrary C*-algebra, at the possible expense of continuity of the norm function of a section.

The work of Archbold and Somerset [5] identified a class of C*-algebras with well-behaved Dauns–Hofmann representation, the so-called *quasi-standard* C*-algebras. This class represents a significant weakening of the condition that the C*-algebra has Hausdorff primitive spectrum, while preserving many of the nice properties associated with this stronger condition. Indeed, all von Neumann algebras are quasi-standard [18], as are the 'local multiplier algebras' studied extensively by Ara and Mathieu [2, §2.5] and many group C*-algebras [20].

Tensor products of continuous bundles of C*-algebras are known to exhibit pathological behavior. The earliest examples of this were given by Kirchberg and Wassermann in [23], who showed that continuity of a C*-bundle was in general not preserved by tensoring fiberwise with a fixed C*-algebra B. Moreover, it was shown that such an operation preserves continuity for all C*-bundles precisely when B is exact. Archbold later obtained a localization of this result in [4], where continuity at a point was characterized in terms of a weaker exactness-type condition. Similar questions have been studied extensively by Blanchard and coauthors in [7, 8, 9, 10, 12].

In [26], we constructed the Glimm ideal space of the minimal tensor product of two C*-algebras in terms of those of the factors. As a consequence, it is possible to construct the Dauns–Hofmann bundle of $A \otimes_\alpha B$ in terms of that of A and B, although the fiber algebras of this bundle remain difficult to describe without additional assumptions on A and B. In particular, it is not immediate from our results in [26, §5] whether or not this bundle agrees with the fiberwise tensor product of Kirchberg and Wassermann. Moreover, it remains difficult to show in general whether or not certain classes of C*-algebras with well-behaved Dauns–Hofmann representations, for example the quasi-standard C*-algebras, are stable under minimal tensor products.

The notion of a $C_0(X)$-algebra, introduced by Kasparov [21], is closely related to that of a C*-bundle. In some sense $C_0(X)$-algebras generalize the Dauns–Hofmann construction to give a bundle representation of a C*-algebra A over any locally compact Hausdorff space X that is a continuous image of the primitive ideal space of A. In Section 3 we study a natural construction which equips the minimal tensor product $A \otimes_\alpha B$ of a $C_0(X)$-algebra A and a $C_0(Y)$-algebra B with the structure of a $C_0(X \times Y)$-algebra. It has been observed previously that this bundle representation of $A \otimes_\alpha B$ may differ from the fiberwise tensor product [9, 23], and that exactness of A or B plays a decisive role in these considerations.

While the results of Kirchberg, Wassermann and Archbold give necessary and sufficient conditions for the continuity of the fiberwise minimal tensor product, less is known regarding the tensor product bundle that we study in this work. Indeed, we show in Section 4 that there are quasi-standard C*-algebras A and B such that $A \otimes_\alpha B$ is quasi-standard, while the fiberwise tensor product of A and B gives rise to a discontinuous C*-bundle.

Related work of Blanchard [9] (concerning the amalgamated $C_0(X)$-tensor product of two $C_0(X)$-algebras A and B) indicated that continuity may fail for the tensor product bundle that we define in Section 3 also. However, the argument used in [9, §3] relies on specific properties of the C*-algebras involved. We show in Section 6 that for an inexact continuous $C_0(X)$-algebra A, one can always construct a continuous $C_0(Y)$-algebra B such that $A \otimes_\alpha B$ is discontinuous as a $C_0(X \times Y)$-algebra. As a consequence it is shown in Theorem 6.3 that stability of continuity is in fact equivalent to exactness of A. Thus our tensor product construction identifies exactness in precisely the same way as the fiberwise tensor product of Kirchberg and Wassermann [23, Thm. 4.5].

In Section 6 we investigate the question of stability of the property of quasi-standardness under the operation of taking tensor products (in particular with respect to the minimal C*-norm). One consequence of this is the observation that, in general, the C*-bundle associated with the Dauns–Hofmann representation of such a tensor product is not given by the fiberwise tensor product of the corresponding bundles of the factors.

Until now, it appears that there were no known examples of a pair of quasi-standard C*-algebras whose minimal tensor product fails to be quasi-standard. It was shown by Kaniuth in [19] that if $A \otimes_\alpha B$ satisfies Tomiyama's property (F), then $A \otimes_\alpha B$ is quasi-standard if and only if A and B are quasi-standard. In particular, this is the case whenever either A or B is exact. The assumption of property (F) was weakened in [26] to an assumption involving exact sequences related to Glimm ideals.

We show in Theorem 6.6 that if A is a quasi-standard C*-algebra which is not exact, then one can always construct a quasi-standard C*-algebra B for which $A \otimes_\alpha B$ is not quasi-standard. In particular, it follows that a quasi-standard C*-algebra A is exact if and only if $A \otimes_\alpha B$ is quasi-standard for all quasi-standard B. This is consistent with the characterization of exactness obtained by Kirchberg and Wassermann in [23], though perhaps surprising in light of the results of Section 4. Similarly, in the unital case we show in Theorem 6.9 that stability of the property of quasi-standardness under taking maximal tensor products is equivalent to nuclearity.

2. Preliminaries on $C_0(X)$-algebras and C*-bundles

We begin with the basic definitions and properties of $C_0(X)$-algebras and C*-bundles, and list some well-known equivalences between the two. Further we will introduce the relevant (topological) spaces of ideals that arise as natural candidates for the base space X in this context, namely the spaces of Glimm and minimal primal ideals. A comprehensive introduction to the theory may be found in [33, Appendix C], see also [27].

Throughout this work, $C_0(X)$ will denote the algebra of continuous complex-valued functions vanishing at infinity on the locally compact Hausdorff space X. For a C*-algebra A, $\mathrm{Prim}(A)$ (respectively $\mathrm{Fac}(A)$) will denote the space of kernels of irreducible (respectively factorial) representations of A, and will be

considered as a topological space with its usual hull-kernel topology. Unless otherwise specified, an ideal of a C*-algebra will mean a proper, closed two-sided ideal. We will denote by $Z(A)$ the center of A and by $M(A)$ the multiplier algebra of A.

Definition 2.1. A C*-bundle is a triple $\mathscr{A} = (X, A, \pi_x : A \to A_x)$ where X is a locally compact Hausdorff space, A a C*-algebra, and $\pi_x : A \to A_x$ surjective *-homomorphisms for all $x \in X$ such that

(i) the family $\{\pi_x \mid x \in X\}$ is faithful, i.e., $\bigcap_{x \in X} \ker(\pi_x) = \{0\}$, and
(ii) for each $f \in C_0(X)$ and $a \in A$ there is an element $f \cdot a \in A$ with the property that

$$\pi_x(f \cdot a) = f(x)\pi_x(a) \quad \text{for all } x \in X.$$

If in addition the functions $N(a) : X \to \mathbb{R}_+$, $x \mapsto \|\pi_x(a)\|$, where $a \in A$, belong to $C_0(X)$ for all $a \in A$, then we say that $\mathscr{A}$ is a *continuous* C*-*bundle* over X. If for all $a \in A$ the functions $N(a)$ are upper semi-continuous (resp. lower semi-continuous) on X, and if for each $\varepsilon > 0$ the set $\{x \in X \mid N(a) \geq \varepsilon\}$ has compact closure in X, then we say that $\mathscr{A}$ is an *upper semi-continuous* C*-*bundle* (resp. lower semi-continuous C*-bundle).

Remark 2.2. There are several distinct (but often equivalent) definitions of C*-bundles throughout the literature. Our definition of a continuous C*-bundle was first used by Kirchberg and Wassermann in [23], and later by Archbold [4] and Blanchard and Wassermann [12]. This is in turn equivalent to the definition of a 'maximal algebra of operator fields' in the sense of Fell [16], also used in [1, 5, 15, 25].

While C*-bundles with semi-continuous norm functions are discussed in [4, 23], no precise definition of a semi-continuous C*-bundle is given there. In the upper semi-continuous case, our definition is easily seen to be equivalent to that of Rieffel [29], also considered by Nilsen [27] and Williams [33].

Definition 2.3. Let A be a C*-algebra and X a locally compact Hausdorff space. We say that A is a $C_0(X)$-*algebra* if there is a *-homomorphism $\mu_A : C_0(X) \to ZM(A)$ with the property that $\mu_A(C_0(X))A = A$.

It follows from the Dauns–Hofmann theorem [14] which we will discuss below (Theorem 2.6), that there is a *-isomorphism $\theta_A : C^b(\mathrm{Prim}(A)) \to ZM(A)$ with the property that

$$(1) \qquad\qquad \theta_A(f)a + P = f(P)(a + P)$$

for $a \in A$, $f \in C^b(\mathrm{Prim}(A))$, $P \in \mathrm{Prim}(A)$. This gives an equivalent definition of a $C_0(X)$-algebra: a C*-algebra A is a $C_0(X)$-algebra if and only if there exists a continuous map $\phi_A : \mathrm{Prim}(A) \to X$. The maps μ_A and ϕ_A are related via $\mu_A(f) = \theta_A(f \circ \phi_A)$ for all $f \in C_0(X)$; see [33, Prop. C.5].

For clarity we will denote any $C_0(X)$-algebra A by the triple (A, X, ϕ_A) or (A, X, μ_A). For $x \in X$ we define the ideal I_x via

$$I_x = \mu_A\big(\{f \in C_0(X) \mid f(x) = 0\}\big)A = \bigcap\{P \in \mathrm{Prim}(A) \mid \phi_A(P) = x\},$$

see [27, §2] for example.

The relationship between $C_0(X)$-algebras and C*-bundles is well known, see [27] or [33, App. C] for example. We give details in the following proposition, which will be used frequently in what follows.

Proposition 2.4. *Let A be a C*-algebra and X a locally compact Hausdorff space.*

(i) *If (A, X, μ_A) is a $C_0(X)$-algebra, then with $A_x = A/I_x$ and $\pi_x : A \to A_x$ the quotient $*$-homomorphism, the triple $(X, A, \pi_x : A \to A_x)$ is an upper semi-continuous C*-bundle [27, Thm. 2.3].*

(ii) *If $(X, A, \pi_x : A \to A_x)$ is a C*-bundle, then setting $\mu_A(f)a = f \cdot a$ for $f \in C_0(X)$, $a \in A$ defines a $*$-homomorphism $\mu_A : C_0(X) \to ZM(A)$ such that (A, X, μ_A) is a $C_0(X)$-algebra. Moreover, $(X, A, \pi_x : A \to A_x)$ is an upper semi-continuous C*-bundle if and only if $\ker(\pi_x) = I_x$ for all $x \in X$; see [23, Lem. 2.1, Lem. 2.3].*

(iii) *The $C_0(X)$-algebra (A, X, μ_A) gives rise to a continuous C*-bundle $(X, A, \pi_x : A \to A_x)$ if and only if the corresponding base map $\phi_A : \mathrm{Prim}(A) \to X$ is an open map [25, Thm. 5].*

As a consequence of Proposition 2.4, we will regard $C_0(X)$-algebras and upper semi-continuous C*-bundles as being (essentially) equivalent. Moreover, we may unambiguously speak of a $C_0(X)$-algebra (A, X, μ_A) being *continuous* if the corresponding C*-bundle $(X, A, \pi_x : A \to A_x)$ is continuous.

If A is a C*-algebra such that $\mathrm{Prim}(A)$ is Hausdorff, then it is clear that the triple $(A, \mathrm{Prim}(A), \theta_A)$, where θ_A is the Dauns–Hofmann $*$-isomorphism (1), is a continuous $C_0(\mathrm{Prim}(A))$-algebra. Moreover, the fiber algebras of the corresponding C*-bundle are necessarily simple in this case. This construction was first studied by Fell [16]. However, the class of C*-algebras A with $\mathrm{Prim}(A)$ Hausdorff is somewhat narrow. The Dauns–Hofmann theorem [14] shows that any C*-algebra A admits a representation as an upper semi-continuous C*-bundle over its space of *Glimm ideals* (or a closely related space), which we define below.

For a C*-algebra A, define an equivalence relation $\approx$ on $\mathrm{Prim}(A)$ as follows: for $P, Q \in \mathrm{Prim}(A)$, $P \approx Q$ if and only if $f(P) = f(Q)$ for all $f \in C^b(\mathrm{Prim}(A))$. As a set, we define $\mathrm{Glimm}(A)$ as the quotient space $\mathrm{Prim}(A)/\approx$, and we denote by $\rho_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ the quotient map.

For $f \in C^b(\mathrm{Prim}(A))$, define $f^\rho : \mathrm{Glimm}(A) \to \mathbb{C}$ as follows: $f^\rho \circ \rho_A(P) = f(P)$, which is well defined by definition of $\approx$. We equip $\mathrm{Glimm}(A)$ with the topology τ_{cr} induced by the functions $\{f^\rho \mid f \in C^b(\mathrm{Prim}(A))\}$, so that $(\mathrm{Glimm}(A), \tau_{cr})$ is a completely regular Hausdorff space, and $f \mapsto f^\rho$ is a $*$-isomorphism of $C^b(\mathrm{Prim}(A))$ onto $C^b(\mathrm{Glimm}(A))$, see [17, Thm. 3.9] or [5, p. 351]. Note that the map $\rho_A : \mathrm{Prim}(A) \to \mathrm{Glimm}(A)$ is continuous.

Definition 2.5. Let A be a C*-algebra. The topological space $(\mathrm{Glimm}(A), \tau_{cr})$ defined above is called the *Glimm space* of A. To each $\approx$-equivalence class $p = [P] \in \mathrm{Glimm}(A)$ we assign a closed two-sided ideal

$$G_p = \bigcap \{P' \mid P' \approx P\} = \bigcap \{P' \mid \rho_A(P') = p\},$$

the *Glimm ideal* of A corresponding to p.

We wish to represent A as a C*-bundle over $\mathrm{Glimm}(A)$, or a $C_0(\mathrm{Glimm}(A))$-algebra. In general however, it may happen that $\mathrm{Glimm}(A)$ is not locally compact. Being a completely regular Hausdorff space, $\mathrm{Glimm}(A)$ has a homeomorphic embedding into its Stone–Čech compactification $\beta \mathrm{Glimm}(A)$.

Theorem 2.6 (J. Dauns, K. H. Hofmann). *Let A be a C*-algebra. Then A is a $C_0(Y)$-algebra and hence the section algebra of an upper semi-continuous C*-bundle $(Y, A, \pi_p : A \to A_p)$ satisfying*

(i) *If $\mathrm{Glimm}(A)$ is locally compact, then $Y = \mathrm{Glimm}(A)$, $A_p = A/G_p$ and $\pi_p = q_p : A \to A/G_p$ is the quotient $*$-homomorphism for all $p \in \mathrm{Glimm}(A)$.*

(ii) *If $\mathrm{Glimm}(A)$ is not locally compact, then $Y = \beta \mathrm{Glimm}(A)$ and*
- *for $p \in \mathrm{Glimm}(A)$, $A_p = A/G_p$ and $\pi_p = q_p : A \to A/G_p$ is the quotient $*$-homomorphism, and*
- *for $p \in \beta \mathrm{Glimm}(A) \setminus \mathrm{Glimm}(A)$, $A_p = \{0\}$.*

If $\mathrm{Prim}(A)$ is Hausdorff, then being locally compact, it is necessarily completely regular. Thus $\mathrm{Prim}(A) = \mathrm{Glimm}(A)$, both as sets of ideals and topologically. In this case the Dauns–Hofmann bundle associated with A is precisely the continuous C*-bundle over $\mathrm{Prim}(A)$ obtained by Fell [16, Thm. 2.3]. More generally, Lee's theorem [25, Thm. 4] implies that the Dauns–Hofmann bundle of a C*-algebra A is a continuous C*-bundle if and only if the complete regularization map is open. Note that if this is the case, then necessarily $\mathrm{Glimm}(A)$ is locally compact.
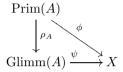
Definition 2.7. An ideal I of a C*-algebra A is said to be *primal* if given $n \geq 2$ and ideals $J_1, \ldots, J_n$ of A such that $J_1 J_2 \ldots J_n = \{0\}$, then there is an index $1 \leq i \leq n$ with $J_i \subseteq I$. A C*-algebra A is called *quasi-standard* if

(i) the Dauns–Hofmann representation of A is a continuous C*-bundle over $\mathrm{Glimm}(A)$, and

(ii) for each $p \in \mathrm{Glimm}(A)$, the Glimm ideal G_p is a primal ideal of A.

For a C*-algebra A we will denote by $\mathrm{Min\text{-}Primal}(A)$ its set of minimal (with respect to inclusion) primal ideals. The canonical topology τ on $\mathrm{Min\text{-}Primal}(A)$ is the weakest topology such that the norm functions $I \mapsto \|a + I\|$ on $\mathrm{Min\text{-}Primal}(A)$ are continuous for all $a \in A$. If A is a C*-algebra for which every $G \in \mathrm{Glimm}(A)$ is a primal ideal of A, then necessarily we have $\mathrm{Glimm}(A) = \mathrm{Min\text{-}Primal}(A)$ as sets. From [5, Thm. 3.3], a C*-algebra A is quasi-standard if and only if $(\mathrm{Glimm}(A), \tau_{cr}) = (\mathrm{Min\text{-}Primal}(A), \tau)$ (i.e., as sets of ideals and topologically).

There are other equivalent definitions of quasi-standard C*-algebras, see [5, §3] for example. Since primitive ideals are primal, the class of quasi-standard C*-algebras properly contains the class of C*-algebras A with $\mathrm{Prim}(A)$ Hausdorff. In fact, for separable A, quasi-standardness of A is equivalent to the condition that A is a continuous $C_0(X)$-algebra such that there exists a dense subset D of X with I_x primitive for all $x \in D$.

The Glimm space of a C*-algebra appears as an intermediate step in any representation of a C*-algebra as a $C_0(X)$-algebra, due to a certain universal property of the complete regularization of a topological space. If X is a completely regular space and $\phi : \mathrm{Prim}(A) \to X$ a continuous map, then ϕ induces a continuous map $\psi : \mathrm{Glimm}(A) \to X$ with $\phi = \psi \circ \rho_A$, i.e.,

$$
\begin{array}{c}
\mathrm{Prim}(A) \\
{\scriptstyle \rho_A} \downarrow \quad \searrow {\scriptstyle \phi} \\
\mathrm{Glimm}(A) \xrightarrow{\ \psi\ } X
\end{array}
$$

commutes. Conversely, starting with a continuous map $\psi : \mathrm{Glimm}(A) \to X$, we may set $\phi = \psi \circ \rho_A$, so that $\phi : \mathrm{Prim}(A) \to X$ is continuous.

If in addition X is locally compact, then A is a $C_0(X)$ algebra if and only if there is a continuous map $\psi_A : \mathrm{Glimm}(A) \to X$. This fact is useful when working with tensor products of C*-algebras, since by [26] we may always construct $\mathrm{Glimm}(A \otimes_\alpha B)$ in terms of $\mathrm{Glimm}(A)$ and $\mathrm{Glimm}(B)$. The same is not true in general for the spaces $\mathrm{Prim}(-)$ and $\mathrm{Fac}(-)$.

In the remainder of this section we give some technical results on the structure of $C_0(X)$-algebras and Glimm spaces of C*-algebras which we will make reference to in subsequent sections.

Lemma 2.8. *Let A be a $C_0(X)$-algebra with base map $\phi_A : \mathrm{Prim}(A) \to X$, and denote by $\psi_A : \mathrm{Glimm}(A) \to X$ the induced continuous map with the property that $\psi_A \circ \rho_A = \phi_A$. Then for each $x \in X$,*

$$
I_x = \bigcap \{ G \in \mathrm{Glimm}(A) \mid \psi_A(G) = x \}.
$$

Proof. Write $F = \psi_A^{-1}(\{x\}) \subseteq \mathrm{Glimm}(A)$, so that for $P \in \mathrm{Prim}(A)$, we have $\phi_A(P) = x$ if and only if $\rho_A(P) \in F$. Thus

$$
\begin{aligned}
I_x &= \bigcap \{ P \in \mathrm{Prim}(A) \mid \rho_A(P) \in F \} \\
&= \bigcap_{p \in F} \left\{ \bigcap \{ P \in \mathrm{Prim}(A) \mid \rho_A(P) = p \} \right\} \\
&= \bigcap \{ G \in \mathrm{Glimm}(A) \mid \psi_A(G) = x \}. \qquad \square
\end{aligned}
$$

As was the case in [19] and [26], we will make use of the fact that the space $\mathrm{Glimm}(A)$ may equivalently be constructed (both as a set of ideals and topologically) as complete regularization of $\mathrm{Fac}(A)$; see [19]. This is useful when working with tensor products, since there is a continuous retraction of the

canonical (homeomorphic) embedding of $\mathrm{Fac}(A) \times \mathrm{Fac}(B)$ into $\mathrm{Fac}(A \otimes_\alpha B)$. For $I, J \in \mathrm{Fac}(A)$ we will write $I \approx_f J$ to denote the equivalence relation $g(I) = g(J)$ for all $g \in C^b(\mathrm{Fac}(A))$, and denote by $\rho_A^f : \mathrm{Fac}(A) \to \mathrm{Glimm}(A)$ the complete regularization map, so that the restrictions of each to $\mathrm{Prim}(A)$ satisfy $\approx|_{\mathrm{Prim}(A)} = \approx$ and $\rho_A^f|_{\mathrm{Prim}(A)} = \rho_A$.

Lemma 2.9 and Proposition 2.10 below generalize parts of [19, Lem. 2.1, Lem. 2.2] from Glimm ideals to the ideals I_x defined in the $C_0(X)$-algebra case.

Lemma 2.9. *Let A be a C^*-algebra and X a locally compact Hausdorff space. Then any continuous map $\phi_A : \mathrm{Prim}(A) \to X$ has a unique extension to a continuous map $\phi_A^f : \mathrm{Fac}(A) \to X$. For $I \in \mathrm{Fac}(A)$ and $P \in \mathrm{hull}(I)$ we have $\phi_A^f(I) = \phi_A(P)$.*

Proof. By [24, Lem. 3.1] there is a unique continuous map $\tilde{\phi}_A : \mathrm{prime}(A) \to X$ extending ϕ_A. Set $\phi_A^f = \tilde{\phi}_A|_{\mathrm{Fac}(A)}$, then ϕ_A^f is continuous since $\tilde{\phi}_A$ is. Uniqueness follows from the fact that $\mathrm{Prim}(A)$ is dense in $\mathrm{Fac}(A)$.

Take $I \in \mathrm{Fac}(A)$ and $P \in \mathrm{hull}(I)$, so that $P \in \mathrm{cl}\{I\}$. Then since ϕ_A^f is continuous and extends ϕ_A, we necessarily have $\phi_A^f(I) = \phi_A^f(P) = \phi_A(P)$. $\square$

Proposition 2.10. *Let A be a C^*-algebra and X a locally compact Hausdorff space.*

(i) *A is a $C_0(X)$-algebra if and only if there exists a continuous map $\phi_A^f : \mathrm{Fac}(A) \to X$.*

(ii) *For $x \in \mathrm{Im}\,\phi_A$, $I_x = \bigcap\{I \in \mathrm{Fac}(A) \mid \phi_A^f(I) = x\}$.*

(iii) *For $I \in \mathrm{Fac}(A)$ and $x \in X$, we have $I \supseteq I_x$ if and only if $\phi_A^f(I) = x$.*

Proof. (i): If A is a $C_0(X)$-algebra with base map $\phi_A : \mathrm{Prim}(A) \to X$, then ϕ_A has a unique continuous extension to a map $\phi_A^f : \mathrm{Fac}(A) \to X$ by Lemma 2.9. Conversely, if such a map exists, then setting $\phi_A = \phi_A^f|_{\mathrm{Prim}(A)}$ defines a base map.

(ii): For $x \in \mathrm{Im}\,\phi_A$, we have

$$I_x = \bigcap\{P \in \mathrm{Prim}(A) \mid \phi_A(P) = x\} \supseteq \bigcap\{I \in \mathrm{Fac}(A) \mid \phi_A^f(I) = x\}$$

since ϕ_A^f extends ϕ_A. Take $I \in \mathrm{Fac}(A)$ such that $\phi_A^f(I) = x$. Then for all $P \in \mathrm{hull}(I)$, $\phi_A(P) = x$, hence $P \supseteq I_x$ for all such P. Hence $I_x \subseteq k(\mathrm{hull}(I)) = I$ for all $I \in \mathrm{Fac}(A)$ with $\phi_A^f(I) = x$, so that, as required,

$$I_x \subseteq \bigcap\{I \in \mathrm{Fac}(A) \mid \phi_A^f(I) = x\}.$$

(iii): It follows from (ii) that if $\phi_A^f(I) = x$ then $I \supseteq I_x$. Now suppose that $I \supseteq I_x$ and take $P \in \mathrm{hull}(I)$. Then $P \supseteq I_x$ and so $\phi_A(P) = x$ by [6, p. 74]. It then follows from Lemma 2.9 that

$$\phi_A^f(I) = \phi_A(P) = x. \qquad \square$$

The following lemma identifies when a subalgebra of a $C_0(X)$-algebra can be identified with a subbundle of the associated upper semi-continuous C^*-bundle.

Lemma 2.11. *Let (B, X, μ_B) be a $C_0(X)$-algebra, and $\iota : A \to B$ a $*$-monomorphism with the property that $\mu_B(f)\iota(a) \in \iota(A)$ for all $a \in A$ and $f \in C_0(X)$. Then the following assertions hold.*

(i) *There is a $*$-homomorphism $\mu_A : C_0(X) \to ZM(A)$ with the property that*

(2) $$\iota(\mu_A(f)a) = \mu_B(f)\iota(a).$$

Hence (A, X, μ_A) is a $C_0(X)$-algebra, and ι is a $C_0(X)$-module map.

(ii) *For $x \in X$ we let*

$$I_x = \mu_A(\{f \in C_0(X) \mid f(x) = 0\})A,$$
$$J_x = \mu_B(\{f \in C_0(X) \mid f(x) = 0\})B.$$

We have $\iota(I_x) = J_x \cap \iota(A)$.

(iii) *If B is a continuous $C_0(X)$-algebra, then so is A.*

Proof. (i): Denote by C the C*-subalgebra of $M(B)$ generated by $\iota(A)$ and $\mu_B(C_0(X))$. Since $\iota(A)$ is closed under multiplication by $\mu_B(C_0(X))$ by assumption, $\iota(A)$ is a closed two-sided ideal of C. Then by [13, Prop. 3.7 (i)], there is a $*$-homomorphism $\sigma : C \to M(A)$ extending ι^{-1} on $\iota(A)$. Moreover, for $f \in C_0(X)$ and $a \in A$, we have

$$(\sigma \circ \mu_B)(f)a = \sigma(\mu_B(f)\iota(a)) = \sigma(\iota(a)\mu_B(f)) = a(\sigma \circ \mu_B)(f),$$

since $\sigma \circ \iota$ is the identity on A. As in [26, (3.1)], we see that $(\sigma \circ \mu_B)(C_0(X)) \subseteq ZM(A)$. Thus we get a $*$-homomorphism $\mu_A = \sigma \circ \mu_B : C_0(X) \to ZM(A)$. To see that μ_A is non-degenerate, let (f_λ) be an approximate identity for $C_0(X)$. Then since μ_B is non-degenerate, $\mu_B(f_\lambda)b \to b$ for all $b \in B$. In particular, for all $a \in A$ we have

$$\mu_A(f_\lambda)a = \sigma(\mu_B(f_\lambda)\iota(a)) \to \sigma(\iota(a)) = a,$$

which shows that $\mu_A(C_0(X))A$ is dense in A. By the Cohen–Hewitt factorization theorem, $\mu_A(C_0(X))A = A$.

To see that (2) holds, note that for $f \in C_0(X)$ and $a \in A$, we have

$$\iota(\mu_A(f)a) = \iota\big((\sigma \circ \mu_B)(f)a\big) = \iota\big(\sigma(\mu_B(f)\iota(a))\big) = \mu_B(f)\iota(a).$$

Thus (A, X, μ_A) has the required properties.

(ii): The inclusion $\iota(I_x) \subseteq J_x \cap \iota(A)$ follows from (2). Now let $a \in J_x \cap \iota(A)$ and $\varepsilon > 0$. By upper semi-continuity there is a neighborhood U of x in X, with compact closure $\overline{U}$, such that $\|\iota(a) + J_y\| < \varepsilon$ for all $y \in U$. Choose a continuous function $f : X \to [0, 1]$ such that $f(x) = 1$ and $f(X \setminus U) \equiv 0$. Then $f \in C_0(X)$ since $\overline{U}$ is compact. Then

$$(1 - \mu_A(f))a + I_x = (a - f(x)a) + I_x = 0 + I_x,$$

so that $(1 - \mu_A(f))a \in I_x$.

Now we have

$$\|(1 - \mu_A(f))a - a\| = \|\mu_A(f)a\| = \|\sigma(\mu_B(f)\iota(a))\| = \|\mu_B(f)\iota(a)\|,$$

since σ is injective on $\iota(A)$. Moreover,

$$
\begin{aligned}
\|\mu_B(f)\iota(a)\| &= \sup_{y \in X} \|\mu_B(f)\iota(a) + J_y\| \\
&= \sup_{y \in X} \|f(y)(\iota(a) + J_y)\| \\
&= \sup_{y \in U} |f(y)| \cdot \|\iota(a) + J_y\| \\
&\leq \sup_{y \in U} \|\iota(a) + J_y\| \leq \varepsilon.
\end{aligned}
$$

Combining these facts, we see that

$$
\|a + I_x\| \leq \|(1 - \mu_A(f))a - a\| = \|\mu_B(f)\iota(a)\| \leq \varepsilon.
$$

Since $\varepsilon > 0$ was arbitrary and I_x closed, we have $a \in I_x$.

(iii): By [15, Cor. 1.8.4] and by part (ii) we may identify

$$
\frac{A}{I_x} = \frac{\iota(A)}{\iota(I_x)} = \frac{\iota(A)}{J_x \cap \iota(A)} \equiv \frac{\iota(A) + J_x}{J_x} \subseteq \frac{B}{J_x}.
$$

Hence for $a \in A$ and $x \in X$, $\|a + I_x\| = \|\iota(a) + J_x\|$. Since norm functions of elements of B are continuous on X, the same is true for A. $\qquad\square$

3. Tensor products of C*-bundles

For a C*-algebra A we let $\mathrm{Id}'(A)$ be the set of proper closed two-sided ideals of A. For two C*-algebras A and B let $A \otimes_\alpha B$ be their minimal tensor product. Define maps $\Phi, \Delta : \mathrm{Id}'(A) \times \mathrm{Id}'(B) \to \mathrm{Id}'(A \otimes_\alpha B)$ via

$$
\Phi(I, J) = \ker(q_I \otimes q_J), \quad \Delta(I, J) = I \otimes_\alpha B + A \otimes_\alpha J,
$$

where $q_I : A \to A/I$ and $q_J : B \to B/J$ are the quotient $*$-homomorphisms. We remark that by injectivity of the minimal tensor product, $\Delta(I, J)$ is precisely the closure in $A \otimes_\alpha B$ of the kernel of the algebraic $*$-homomorphism $q_I \odot q_J : A \odot B \to (A/I) \odot (B/J)$.

Starting with $K \in \mathrm{Id}'(A \otimes_\alpha B)$, we define $K^A \in \mathrm{Id}'(A)$ and $K^B \in \mathrm{Id}'(B)$ via

$$
K^A = \{a \in A \mid a \otimes_\alpha B \subseteq K\}, \quad K^B = \{b \in B \mid A \otimes_\alpha b \subseteq K\}.
$$

Defining $\Psi(K) = (K^A, K^B)$ gives a map $\Psi : \mathrm{Id}'(A \otimes_\alpha B) \to \mathrm{Id}'(A) \times \mathrm{Id}'(B)$. Moreover, we have

$$
(\Psi \circ \Phi)(I, J) = (\Psi \circ \Delta)(I, J) = (I, J)
$$

for all $(I, J) \in \mathrm{Id}'(A) \times \mathrm{Id}'(B)$ by [24, Thm. 2.6] and [26, Lem. 4.6 (ii)].

We remark that for all C*-algebras A and B and for all pairs of ideals $(I, J) \in \mathrm{Id}'(A) \times \mathrm{Id}'(B)$, the inclusion

$$
\Delta(I, J) \subseteq \Phi(I, J)
$$

holds. This inclusion may be strict in general, see for example [31, Prop. 2.7] or [32, Cor. 7].

Definition 3.1. Let A and B be C*-algebras. Their minimal tensor product $A \otimes_\alpha B$ is said to satisfy Tomiyama's property (F) if $\Delta(I, J) = \Phi(I, J)$ for all $(I, J) \in \mathrm{Id}'(A) \times \mathrm{Id}'(B)$.

Our definition of property (F) is equivalent to Tomiyama's original definition [30], i.e., the tensor product states separate the closed ideals of $A \otimes_\alpha B$, by [30, Thm. 5].

Definition 3.2. A sequence of $*$-homomorphisms between C*-algebras of the form

$$(3) \qquad 0 \to J \xrightarrow{\iota} B \xrightarrow{q} (B/J) \to 0,$$

where J is an ideal of B, ι the inclusion of J into B, and q the quotient $*$-homomorphism, is called a *short exact sequence of C*-algebras*. We say that a C*-algebra A is *exact* if the sequence

$$(4) \qquad 0 \to A \otimes_\alpha J \xrightarrow{\mathrm{id} \otimes \iota} A \otimes_\alpha B \xrightarrow{\mathrm{id} \otimes q} A \otimes_\alpha (B/J) \to 0$$

is exact for every short exact sequence of the form (3).

A C*-algebra A is exact if and only if $A \otimes_\alpha B$ has property (F) for all C*-algebras B. Clearly if A is such that $A \otimes_\alpha B$ satisfies (F) for all B, then for any short exact sequence of the form (3), exactness of (4) follows from the fact that $\Delta(\{0\}, J) = \Phi(\{0\}, J)$. The converse is shown in [11, Prop. 2.17]. We will make use of this equivalence repeatedly in the sequel.

The following theorem lists some of the known properties of tensor products of C*-bundles, and their relation to the maps Φ and Δ.

Theorem 3.3. *Let $\mathscr{A} = (X, A, \pi_x : A \to A_x)$ and $\mathscr{B} = (Y, B, \sigma_y : B \to B_y)$ be C*-bundles over locally compact spaces X and Y, respectively. Then*

(i) *the fiberwise tensor product $\mathscr{A} \otimes_\alpha \mathscr{B}$ of $\mathscr{A}$ and $\mathscr{B}$ defined via*

$$\mathscr{A} \otimes_\alpha \mathscr{B} = \left(X \times Y, A \otimes_\alpha B, \pi_x \otimes \sigma_y : A \otimes_\alpha B \to A_x \otimes_\alpha B_y \right)$$

is a C-bundle over $X \times Y$; see [4, pp. 136–137].*

If in addition $\mathscr{A}$ and $\mathscr{B}$ are continuous, then

(ii) *$\mathscr{A} \otimes_\alpha \mathscr{B}$ is lower semi-continuous over $X \times Y$; see [23, Prop. 4.9]; and*

(iii) *for $(x_0, y_0) \in X \times Y$, the function*

$$(x, y) \mapsto \|(\pi_x \otimes \sigma_y)(c)\|$$

is continuous for all $c \in A \otimes_\alpha B$ at (x_0, y_0) if and only if

$$\ker(\pi_{x_0} \otimes \sigma_{y_0}) = \ker(\pi_{x_0}) \otimes_\alpha B + A \otimes_\alpha \ker(\sigma_{y_0}),$$

that is, if and only if

$$\Phi(\ker(\pi_{x_0}), \ker(\sigma_{y_0})) = \Delta(\ker(\pi_{x_0}), \ker(\sigma_{y_0}));$$

see [4, Thm. 3.3].

We now introduce an alternative approach to defining a C*-bundle structure on the tensor product of two (upper semi-continuous) C*-bundles, based on the ideal structure of $A \otimes_\alpha B$ rather than the fiberwise tensor product. This construction was considered previously in [9], in the case where the base spaces are compact, however, we will need additional information on the interplay between the base and structure maps involved.

Suppose that (A, X, ϕ_A) is a $C_0(X)$-algebra and (B, Y, ϕ_B) a $C_0(Y)$-algebra, where $\phi_A : \mathrm{Prim}(A) \to X$ and $\phi_B : \mathrm{Prim}(B) \to Y$ are their base maps. Then we get a continuous map $\phi_A \times \phi_B : \mathrm{Prim}(A) \times \mathrm{Prim}(B) \to X \times Y$. By a theorem of Lazar [24, Thm. 3.2], we get a unique continuous map $\phi_\alpha : \mathrm{Prim}(A \otimes_\alpha B) \to X \times Y$ such that

$$
\begin{array}{ccc}
\mathrm{Prim}(A) \times \mathrm{Prim}(B) & \overset{\Phi}{\hookrightarrow} & \mathrm{Prim}(A \otimes_\alpha B) \\
& \phi_A \times \phi_B \searrow & \downarrow \phi_\alpha \\
& & X \times Y
\end{array}
$$

commutes, that is, $\phi_\alpha \circ \Phi = \phi_A \times \phi_B$. Thus, taking ϕ_α as the base map, $A \otimes_\alpha B$ becomes a $C_0(X \times Y)$-algebra, $(A \otimes_\alpha B, X \times Y, \phi_\alpha)$.

The structure maps $\mu_A : C_0(X) \to ZM(A)$, $\mu_B : C_0(Y) \to ZM(B)$ and $\mu_\alpha : C_0(X \times Y) \to ZM(A \otimes_\alpha B)$ are then uniquely determined by ϕ_A, ϕ_B and ϕ_α. We will show in Proposition 3.4 that in fact μ_α may be identified with the map $\mu_A \otimes \mu_B : C_0(X) \otimes C_0(Y) \to ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes_\alpha B)$.

For $x \in X, y \in Y$ define the ideals

$$
\begin{aligned}
I_x &= \mu_A\big(\{f \in C_0(X) \mid f(x) = 0\}\big)A \\
&= \bigcap\{P \in \mathrm{Prim}(A) \mid \phi_A(P) = x\}, \\
J_y &= \mu_B\big(\{g \in C_0(Y) \mid g(y) = 0\}\big)B \\
&= \bigcap\{Q \in \mathrm{Prim}(A) \mid \phi_B(Q) = y\}, \\
K_{x,y} &= \mu_\alpha\big(\{h \in C_0(X \times Y) \mid h(x,y) = 0\}\big)A \otimes_\alpha B \\
&= \bigcap\{M \in \mathrm{Prim}(A \otimes_\alpha B) \mid \phi_\alpha(M) = (x,y)\}.
\end{aligned}
$$

By [1], there is a canonical injective $*$-homomorphism $\iota : M(A) \otimes_\alpha M(B) \to M(A \otimes_\alpha B)$, and by [26, Lem. 3.1] this map satisfies

$$
\iota(x \otimes y)(a \otimes b) = (xa) \otimes (yb) \quad \text{and} \quad (a \otimes b)(\iota(x \otimes y)) = (ax) \otimes (by),
$$

for all elementary tensors $x \otimes y \in M(A) \otimes_\alpha M(B), a \otimes b \in A \otimes_\alpha B$, so that the image $\iota(ZM(A) \otimes ZM(B))$ is contained in $ZM(A \otimes_\alpha B)$. We will suppress mention of ι and simply consider $ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes_\alpha B)$.

Proposition 3.4. *Suppose (A, X, ϕ_A) is a $C_0(X)$-algebra and (B, Y, ϕ_B) a $C_0(Y)$ algebra. Then with the above notation:*

(i) *$A \otimes_\alpha B$ is a $C_0(X \times Y)$-algebra with base map $\phi_\alpha : \mathrm{Prim}(A \otimes_\alpha B) \to X \times Y$ satisfying $\phi_\alpha \circ \Phi = \phi_A \times \phi_B$.*

(ii) *The structure map $\mu_\alpha : C_0(X \times Y) \to ZM(A \otimes_\alpha B)$ corresponding to ϕ_α may be identified with the composition $(\mu_A \otimes \mu_B) \circ \eta$, where we denote by $\eta : C_0(X \times Y) \to C_0(X) \otimes C_0(Y)$ the canonical $*$-isomorphism and we identify $ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes_\alpha B)$.*

(iii) *Denoting by ϕ_A^f, ϕ_B^f and ϕ_α^f the extensions of ϕ_A, ϕ_B and ϕ_α to $\mathrm{Fac}(A)$, $\mathrm{Fac}(B)$ and $\mathrm{Fac}(A \otimes_\alpha B)$, respectively, we have $\phi_\alpha^f \circ \Phi = \phi_A^f \times \phi_B^f$ on $\mathrm{Fac}(A) \times \mathrm{Fac}(B)$.*

(iv) *For any $M \in \mathrm{Fac}(A \otimes_\alpha B)$ and $(x,y) \in X \times Y$, we have $\phi_\alpha^f(M) = (x,y)$ if and only if $(\phi_A^f \times \phi_B^f)(\Psi(M)) = (x,y)$,*

(v) *For all $(x,y) \in X \times Y$ we have $K_{x,y} = \Delta(I_x, J_y)$.*

Proof. Statement (i) is shown in the remarks preceding the proposition.

(ii): For $f \in C_0(X \times Y)$, we have

$$\mu_\alpha(f) = \theta_\alpha(f \circ \phi_\alpha),$$

where $\theta_\alpha : C^b(\mathrm{Prim}(A \otimes_\alpha B)) \to ZM(A \otimes_\alpha B)$ is the Dauns–Hofmann isomorphism of equation (1). For $f \otimes g \in C_0(X) \otimes C_0(Y)$ and $(x,y) \in X \times Y$, the $*$-isomorphism η satisfies $\eta^{-1}(f \otimes g)(x,y) = f(x)g(y)$. Thus by linearity and continuity, it suffices to show that for all $f \otimes g \in C_0(X) \otimes C_0(Y)$ and $a \otimes b \in A \otimes_\alpha B$ we have

$$\mu_\alpha(\eta^{-1}(f \otimes g))(a \otimes b) = (\mu_A \otimes \mu_B)(f \otimes g)(a \otimes b) = \mu_A(f)a \otimes \mu_B(g)b.$$

Take $(P,Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)$, then the Dauns–Hofmann $*$-isomorphism θ_α of (1) gives

$$(\mu_\alpha \circ \eta^{-1})(f \otimes g)(a \otimes b) + \Phi(P,Q)$$
$$= \theta_\alpha\big(\eta^{-1}(f \otimes g) \circ \phi_\alpha\big)(a \otimes b) + \Phi(P,Q)$$
$$= \eta^{-1}(f \otimes g) \circ \phi_\alpha(\Phi(P,Q))\big(a \otimes b + \Phi(P,Q)\big)$$
$$= (f \circ \phi_A)(P)(g \circ \phi_B)(Q)\big(a \otimes b + \Phi(P,Q)\big),$$

and since $A \otimes_\alpha B / \Phi(P,Q) \equiv (A/P) \otimes_\alpha (B/Q)$, the last line becomes

$$\big((f \circ \phi_A)(P)(a + P)\big) \otimes \big((g \circ \phi_B)(Q)(b + Q)\big).$$

On the other hand, applying the isomorphisms θ_A and θ_B of (1) associated with A and B, respectively, we get

$$(\mu_A \otimes \mu_B)(f \otimes g)(a \otimes b) + \Phi(P,Q)$$
$$= \big(\mu_A(f)a + P\big) \otimes \big(\mu_B(g)b + Q\big)$$
$$= \big(\theta_A(f \circ \phi_A)a + P\big) \otimes \big(\theta_B(g \circ \phi_B)b + Q\big)$$
$$= \big((f \circ \phi_A)(P)(a + P)\big) \otimes \big((g \circ \phi_B)(Q)(b + Q)\big).$$

Thus for all $(P,Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)$ we have

$$\big((\mu_\alpha \circ \eta^{-1})(f \otimes g) - (\mu_A \otimes \mu_B)(f \otimes g)\big)(a \otimes b) \in \Phi(P,Q),$$

and since $\bigcap \{\Phi(P,Q) \mid (P,Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)\} = \{0\}$, it follows that

$$(\mu_\alpha \circ \eta^{-1})(f \otimes g) = (\mu_A \otimes \mu_B)(f \otimes g).$$

(iii): Let $(I, J) \in \mathrm{Fac}(A) \times \mathrm{Fac}(B)$ and $(P, Q) \in \mathrm{hull}(I) \times \mathrm{hull}(J)$. Then by (i) we have $(\phi_A \times \phi_B)(P, Q) = (\phi_\alpha \circ \Phi)(P, Q)$. By [24, Cor. 2.3] we have $\Phi(P, Q) \in \mathrm{hull}(\Phi(I, J))$, so that Lemma 2.9 gives

$$(\phi_A^f \times \phi_B^f)(I, J) = (\phi_A \times \phi_B)(P, Q) = (\phi_\alpha \circ \Phi)(P, Q) = (\phi_\alpha^f \circ \Phi)(I, J).$$

(iv): By [26, Prop. 2.1, Prop. 4.1], we have $(\Phi \circ \Psi)(M) \in \mathrm{Fac}(A \otimes_\alpha B)$ with $M \supseteq (\Phi \circ \Psi)(M)$, so that $(\Phi \circ \Psi)(M) \in \mathrm{cl}\{M\}$ and hence

$$\phi_\alpha^f(M) = \phi_\alpha^f\big((\Phi \circ \Psi)(M)\big).$$

By (iii), the latter is precisely $(\phi_A^f \times \phi_B^f)(\Psi(M))$.

(v): By (iv) and Proposition 2.10 (iii), we have $M \in \mathrm{hull}_f(K_{x,y})$ if and only if $\Psi(M) \in \mathrm{hull}_f(I_x) \times \mathrm{hull}_f(J_y)$. But then since $\Psi^{-1}(\mathrm{hull}_f(I_x) \times \mathrm{hull}_f(J_y)) = \mathrm{hull}_f(\Delta(I_x, J_y))$ (see [26, Lem. 4.6 (i)]), it follows that $K_{x,y} = \Delta(I_x, J_y)$. $\qquad\square$

Definition 3.5. Let (X, A, μ_A) be a $C_0(X)$-algebra and (B, Y, μ_B) a $C_0(Y)$-algebra. We will denote by $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ the $C_0(X \times Y)$-algebra defined by Proposition 3.4, and we will consider this construction as the natural (minimal) tensor product in the category of $C_0(-)$-algebras,

$$(A, X, \mu_A) \otimes_\alpha (B, Y, \mu_B) \equiv (A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B).$$

The tensor product construction of Definition 3.5 does not agree in general with the fiberwise tensor product bundle studied by Kirchberg and Wassermann in [23]. This fact may be deduced from [23, Lem. 2.3, Prop. 4.3], and is demonstrated explicitly in [9, Prop. 3.1].

We now introduce some properties which characterize when these two notions of the tensor product of a pair of C*-bundles coincide. For C*-algebras A and B we define the properties $(\mathrm{F_{Gl}})$ and $(\mathrm{F_{MP}})$ on $A \otimes_\alpha B$ as follows:

$(\mathrm{F_{Gl}})$ $\Phi(G, H) = \Delta(G, H)$ for all $(G, H) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$,

$(\mathrm{F_{MP}})$ $\Phi(I, J) = \Delta(I, J)$ for all $(I, J) \in \mathrm{Min\text{-}Primal}(A) \times \mathrm{Min\text{-}Primal}(B)$.

If in addition (A, X, μ_A) is a $C_0(X)$-algebra and (B, Y, μ_B) a $C_0(Y)$-algebra, we will say that the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ satisfies *property* $(\mathrm{F_{X,Y}})$ if the equation

$$(\mathrm{F_{X,Y}}) \qquad \Phi(I_x, J_y) = \Delta(I_x, J_y) \quad \text{for all } (x, y) \in X \times Y$$

holds. For convenience, we will refer to $(\mathrm{F_{X,Y}})$ as a property of $A \otimes_\alpha B$, rather than $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$, when the context is clear.

We remark that if $A \otimes_\alpha B$ satisfies Tomiyama's property (F) then clearly $A \otimes_\alpha B$ satisfies properties $(\mathrm{F_{X,Y}})$, $(\mathrm{F_{Gl}})$ and $(\mathrm{F_{MP}})$; see [19, Thm. 1.1, Thm. 2.3]. The converse is not true in general; indeed, let A and B be C*-algebras such that $A \otimes_\alpha B$ does not satisfy property (F), and let $X = \{x\}$ and $Y = \{y\}$ be one-point spaces. Regarding A as a $C_0(X)$-algebra and B as a $C_0(Y)$-algebra in the obvious (trivial) way, we have $I_x = \{0\}$ and $J_y = \{0\}$. Then it is evident that $\Delta(I_x, J_y) = \Phi(I_x, J_y) = \{0\}$, so that $A \otimes_\alpha B$ satisfies property $(\mathrm{F_{X,Y}})$, hence this property does not imply (F).

To see that (F$_{\mathrm{MP}}$) and (F$_{\mathrm{Gl}}$) do not imply (F), let $A = B = B(H)$, where H is a separable infinite-dimensional Hilbert space. Then

$$\mathrm{Glimm}(B(H)) = \mathrm{Min\text{-}Primal}(B(H)) = \{0\},$$

so that as before, $B(H) \otimes_\alpha B(H)$ satisfies (F$_{\mathrm{MP}}$) and (F$_{\mathrm{Gl}}$), but does not satisfy (F) by [32]. Other examples are discussed in [4, pp. 140–141].

The following theorem relates the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$, its corresponding upper semi-continuous C*-bundle, and the fiberwise tensor product of the bundles associated with A and B.

Theorem 3.6. *Let (A, X, μ_A) be a $C_0(X)$-algebra and (B, Y, μ_B) a $C_0(Y)$-algebra, and denote by $\mathscr{A} = (X, A, \pi_x : A \to A_x)$ and $\mathscr{B} = (Y, B, \sigma_y : B \to B_y)$ the associated upper semi-continuous C*-bundles over X and Y, respectively. Then we have:*

(i) *The $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ defines an upper semi-continuous C*-bundle*

$$\big(X \times Y, A \otimes_\alpha B, \gamma_{(x,y)} : A \otimes_\alpha B \to (A \otimes_\alpha B)_{(x,y)}\big),$$

where $(A \otimes_\alpha B)_{(x,y)} = A \otimes_\alpha B / \Delta(I_x, J_y)$ for all $(x, y) \in X \times Y$.

(ii) *The bundle*

$$\big(X \times Y, A \otimes_\alpha B, \gamma_{(x,y)} : A \otimes_\alpha B \to (A \otimes_\alpha B)_{(x,y)}\big)$$

agrees with the fiberwise tensor product bundle $\mathscr{A} \otimes_\alpha \mathscr{B}$ if and only if $A \otimes_\alpha B$ satisfies property (F$_{X,Y}$).

(iii) *If $\mathscr{A}$ and $\mathscr{B}$ are continuous C*-bundles and $A \otimes_\alpha B$ satisfies property (F$_{X,Y}$), then $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is a continuous $C_0(X \times Y)$-algebra.*

Proof. Statement (i) is immediate from Proposition 3.4 (v) and the equivalence of $C_0(X)$-algebras (resp. $C_0(Y)$-, $C_0(X \times Y)$-algebras) and upper semi-continuous C*-bundles over X (resp. Y, $X \times Y$).

By definition of the maps Φ and Δ, for all $(x, y) \in X \times Y$ we have $\Phi(I_x, J_y) = \Delta(I_x, J_y)$ if and only if $(A \otimes_\alpha B)_{(x,y)} \equiv A_x \otimes_\alpha B_y$, from which (ii) follows.

If $\Phi(I_x, J_y) = \Delta(I_x, J_y)$ for all $(x, y) \in X \times Y$, then (iii) follows from (ii) and Theorem 3.3 (iii). $\qquad\square$

Remark 3.7. (i) Given continuous $C_0(X)$-algebras (A, X, μ_A) and (B, Y, μ_B), it is natural to ask whether or not the converse of Theorem 3.6 (iii) holds; that is, whether property (F$_{X,Y}$) is a necessary condition for the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ to be continuous. The analogous result for the fiberwise tensor product is true by Theorem 3.3 (iii). We will show in Section 4 that this is not the case; we can construct pairs (A, X, μ_A) and (B, Y, μ_B) such that $A \otimes_\alpha B$ does not satisfy property (F$_{X,Y}$) but $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is a continuous $C_0(X \times Y)$-algebra. One interesting consequence of this fact is that continuity of the associated fiberwise tensor product is a strictly stronger property than continuity of $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$.

(ii) A special case of Proposition 3.4 arises as follows; let (A, X, μ_A) be a $C_0(X)$-algebra and B a nonzero C*-algebra. Then we may regard B as a

$C(Y)$-algebra where $Y = \{y\}$ is a one-point space, so that $X \times Y = X$ and $A \otimes_\alpha B$ is also a $C_0(X)$-algebra. The base map $\phi_\alpha : \mathrm{Prim}(A \otimes_\alpha B) \to X$ is the extension of $\phi_A \circ p_1 : \mathrm{Prim}(A) \times \mathrm{Prim}(B) \to X$ to $\mathrm{Prim}(A \otimes_\alpha B)$, where p_1 is the projection onto the first factor. The corresponding structure map is given by

$$\mu_A \otimes 1 : C_0(X) \to ZM(A) \otimes ZM(B) \subseteq ZM(A \otimes_\alpha B),$$

where $\mu_A \otimes 1(f) = \mu_A(f) \otimes 1$ for all $f \in C_0(X)$.

Thus by Theorem 3.6 (i) we get an upper semi-continuous C*-bundle $(X, A \otimes_\alpha B, \gamma_x : A \otimes_\alpha B \to (A \otimes_\alpha B)_x)$, where $(A \otimes_\alpha B)_x = (A \otimes_\alpha B)/(I_x \otimes_\alpha B)$ for all $x \in X$. The analogous construction in the fiberwise tensor product case is as follows: for a C*-bundle $\mathscr{A} = (X, A, \pi_x : A \to A_x)$ and a C*-algebra B, we define the C*-bundle

$$\mathscr{A} \otimes_\alpha B = (X, A \otimes_\alpha B, \pi_x \otimes \mathrm{id} : A \otimes_\alpha B \to A_x \otimes_\alpha B).$$

The two bundles agree precisely when $\Phi(I_x, \{0\}) = \Delta(I_x, \{0\})$ for all $x \in X$, by Theorem 3.6 (ii). We will make use of this special case as an intermediate step in the construction of the tensor product of two C*-bundles in subsequent sections.

4. Comparison with the fiberwise tensor product

In this section we show that the assumption of property $(\mathrm{F}_{X,Y})$ in Theorem 3.6 (iii) is not necessary in general. More precisely, we show that for any inexact C*-algebra B, there is a continuous $C_0(X)$-algebra (A, X, μ_A) such that

(i) the fiberwise tensor product $\mathscr{A} \otimes_\alpha B$, where $\mathscr{A}$ is the continuous C*-bundle associated with (A, X, μ_A), is discontinuous, while
(ii) the $C_0(X)$-algebra $(A \otimes_\alpha B, X, \mu_A \otimes 1)$ is continuous.

This shows that the analog of Archbold's result [4, Thm. 3.3] for the bundles constructed in Section 3 is untrue. In particular, we deduce that for continuous $C_0(X)$-algebras (A, X, μ_A) and (B, Y, μ_B), the assumption that the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is equal to the fiberwise tensor product (i.e. $A \otimes_\alpha B$ satisfies property $(\mathrm{F}_{X,Y})$), is not a necessary condition for continuity of the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$.

Lemma 4.1. *Let (A, X, μ_A) be a $C_0(X)$-algebra. Denote by $\phi_A^f : \mathrm{Fac}(A) \to X$ the base map. For any closed subset $F \subseteq X$, set*

$$I_F = \bigcap \{I_x \mid x \in F\}.$$

Then the following assertions hold:

(i) *For $M \in \mathrm{Fac}(A)$, we have $M \supseteq I_F$ if and only if $\phi_A^f(M) \in F$.*
(ii) *For any C*-algebra B we have*

$$I_F \otimes_\alpha B = \bigcap_{x \in F} (I_x \otimes_\alpha B).$$

Proof. (i): Denote $m := \phi_A^f(M)$ and suppose first that $m \notin F$. Choose $a \in A$ with $\|a + M\| = 1$ and $f \in C_0(X)$ with $f(m) = 1$ and $f(F) = \{0\}$. Then

$$\mu_A(f)a + M = f(\phi_A^f(M))(a + M) = f(m)(a + M) = a + M,$$

so that $\mu_A(f)a \notin M$. On the other hand, for all $x \in F$,

$$\mu_A(f)a + I_x = f(x)(a + I_x) = 0,$$

so that $\mu_A(f)a \in I_F$. In particular, $M \not\supseteq I_F$.

Now suppose that $m \in F$. Then by Proposition 2.10 (iii), $M \supseteq I_m$ and so $M \supseteq \bigcap_{x \in F} I_x = I_F$.

(ii): We will regard $A \otimes_\alpha B$ as a $C_0(X)$-algebra as in Remark 3.7 (ii); the base map $\phi_\alpha : \operatorname{Prim}(A \otimes_\alpha B) \to X$ being the unique extension to $\operatorname{Prim}(A \otimes_\alpha B)$ of $\phi_A \circ p_1 : \operatorname{Prim}(A) \times \operatorname{Prim}(B) \to X$, with p_1 the projection onto the first factor.

Let $M \in \operatorname{Fac}(A \otimes_\alpha B)$ and let $(M^A, M^B) = \Psi(M)$. We first show that $M \supseteq I_F \otimes_\alpha B$ if and only if $M^A \supseteq I_F$. We have $\Psi(I_F \otimes_\alpha B) = (I_F, \{0\})$ by [26, Lem. 4.6 (ii)]. Since Ψ is order-preserving, it is clear that if $M \supseteq I_F \otimes_\alpha B$ then $M^A \supseteq I_F$. On the other hand, since Δ is also order-preserving, if $M^A \supseteq I_F$ then using [26, (4.2)], we see that

$$M \supseteq \Delta(M^A, M^B) \supseteq \Delta(I_F, \{0\}) = I_F \otimes_\alpha B.$$

By (i), we have $M^A \supseteq I_F$ if and only if $\phi_A^f(M^A) \in F$. But then by Proposition 3.4 (iv), we have

$$\phi_\alpha^f(M) = (\phi_A^f \times \phi_B^f)(M^A, M^B) = (\phi_A^f)(M^A)$$

and the conclusion follows.

$\square$

Lemma 4.2. *Let X be an extremally disconnected compact Hausdorff space. Then any $C(X)$-algebra (A, X, μ_A) is continuous.*

Proof. For each $x \in X$ we have

$$\|a + I_x\| = \inf\{\|(1 - \mu_A(f) + f(x))a\| \mid f \in C(X)\}$$

by [8, Lem. 1.10]. Moreover, it is easily seen that for a given $f \in C(X)$ and $a \in A$, the norm function $x \mapsto \|(1 - \mu_A(f) + f(x))a\|$ is continuous on X. Since X is extremally disconnected and compact, $C(X)$ is monotone complete, and so the above infimum belongs to $C(X)$. $\square$

Proposition 4.3. *Let B be a C*-algebra, M a von Neumann algebra and $(M, \operatorname{Glimm}(M), \theta_M)$ the $C(\operatorname{Glimm}(M))$-algebra associated with the Dauns–Hofmann representation of M. Then $(M \otimes_\alpha B, \operatorname{Glimm}(M), \theta_M \otimes 1)$ is a continuous $C(\operatorname{Glimm}(M))$-algebra.*

Proof. Since $Z(M)$ is a von Neumann algebra, $\operatorname{Glimm}(M) = \operatorname{Prim}(Z(M))$ is an extremally disconnected compact Hausdorff space. Continuity of $(M \otimes_\alpha B, \operatorname{Glimm}(M), \theta_M \otimes 1)$ then follows from Lemma 4.2. $\square$

Theorem 4.4. *Let B be an inexact C*-algebra. Then there is a von Neumann algebra M, whose Dauns–Hofmann representation $(M, \mathrm{Glimm}(M), \theta_M)$ and associated continuous C*-bundle $(\mathrm{Glimm}(M), M, \pi_p : M \to M_p)$ satisfy the following:*

(i) *$(M \otimes_\alpha B, \mathrm{Glimm}(M), \theta_M \otimes 1)$ is a continuous $C(\mathrm{Glimm}(M))$-algebra.*
(ii) *$(M \otimes_\alpha B, \mathrm{Glimm}(M), \pi_p \otimes \mathrm{id}_B : M \otimes_\alpha B \to M_p \otimes_\alpha B)$ is a discontinuous C*-bundle.*

If in addition B is a prime C*-algebra (e.g., if B primitive), then $M \otimes_\alpha B$ is quasi-standard.

Proof. Let $M = \prod_{n=1}^\infty M_n(\mathbb{C})$. Then M is a von Neumann algebra, hence it is quasi-standard by [3, §5]. Moreover, $Z(M)$ consists of the sequences $(\lambda_n 1_n)_{n=1}^\infty \in M$, where $\lambda_n \in \mathbb{C}$ and 1_n is the $n \times n$ identity matrix. It follows that $\mathrm{Glimm}(M) = \mathrm{Prim}(Z(M))$ is canonically homeomorphic to $\beta\mathbb{N}$.

Since B is inexact, the sequence

$$(5) \qquad 0 \to I_0 \otimes_\alpha B \to M \otimes_\alpha B \to (M/I_0) \otimes_\alpha B \to 0$$

is inexact by [22]. We claim that there is some $q \in \beta\mathbb{N}$ for which $G_q \otimes_\alpha B \subsetneq \ker(\pi_q \otimes \mathrm{id}_B)$. Suppose not. By [24, Lem. 2.2], we have

$$\ker(\pi_0 \otimes \mathrm{id}_B) = \bigcap_{q \in \beta\mathbb{N} \setminus \mathbb{N}} \ker(\pi_q \otimes \mathrm{id}_B),$$

and by Lemma 4.1

$$I_0 \otimes_\alpha B = \bigcap_{q \in \beta\mathbb{N} \setminus \mathbb{N}} I_q \otimes_\alpha B.$$

Thus if $I_q \otimes_\alpha B = \ker(\pi_q \otimes \mathrm{id}_B)$ for all $q \in \beta\mathbb{N} \setminus \mathbb{N}$, the above intersections would agree, which would imply that (5) were exact, which is not the case.

Thus $(M \otimes_\alpha B, \beta\mathbb{N}, \pi_q \otimes \mathrm{id}_B : M \otimes_\alpha B \to M_q \otimes_\alpha B)$ is discontinuous at some point $p \in \beta\mathbb{N} \setminus \mathbb{N}$ by [23, Prop. 2.7].

On the other hand, the $C(\mathrm{Glimm}(M))$-algebra $(M \otimes_\alpha B, \mathrm{Glimm}(M), \theta_M \otimes 1)$ is continuous by Proposition 4.3.

Under the additional assumption that the zero ideal of B is prime, necessarily we have $\mathrm{Glimm}(B) = \{0\}$ by [5, Lem. 2.2], and so $\mathrm{Glimm}(M \otimes_\alpha B)$ is homeomorphic to $\beta\mathbb{N}$ in the obvious way. In particular, $(M \otimes_\alpha B, \mathrm{Glimm}(M), \theta_M \otimes 1)$ corresponds to the Dauns–Hofmann representation of $M \otimes_\alpha B$, and the fiber algebras are prime throughout a dense subset of $\mathrm{Glimm}(M \otimes_\alpha B)$, namely the points of $\mathbb{N}$. By [5, Thm. 3.4 (iii)⇒(i)], $M \otimes_\alpha B$ is quasi-standard. $\square$

5. Continuity and exactness of the $C_0(X \times Y)$-algebra $A \otimes_\alpha B$

In this section we investigate the relationship between exactness of a continuous $C_0(X)$-algebra (A, X, μ_A) and its minimal tensor product with an arbitrary continuous $C_0(Y)$-algebra (B, Y, μ_B). The corresponding result regarding continuity of fiberwise tensor products of C*-bundles was obtained by Kirchberg and Wassermann in [23]:

Theorem 5.1 (E. Kirchberg, S. Wassermann [23, Thm. 4.5])**.** *The following conditions on a* C**-algebra B are equivalent:*

(i) *B is exact.*

(ii) *For every locally compact Hausdorff space X and continuous* C**-bundle $\mathscr{A} = (X, A, \pi_x : A \to A_x)$ over X, the fiberwise tensor product $\mathscr{A} \otimes_\alpha B$ is continuous.*

(iii) *For every separable, unital continuous* C**-bundle $\mathscr{A} = (\hat{\mathbb{N}}, A, \pi_n : A \to A_n)$ over $\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, the fiberwise tensor product $\mathscr{A} \otimes_\alpha B$ is continuous.*

While Theorem 4.4 shows that continuity of a $C_0(X \times Y)$-algebra of the form $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is a strictly weaker property than continuity of the corresponding fiberwise tensor product, a discontinuous example was already exhibited by Blanchard in [9]. The construction of this counterexample depends heavily on the specific properties of the algebras involved. Our main result of this section, Theorem 5.6, shows that this pathology is in some sense universal; more precisely, we show that (A, X, μ_A) is exact if and only if the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is continuous for each continuous $C_0(Y)$-algebra (B, Y, μ_B).

The following two lemmas are known, we include a proof for completness.

Lemma 5.2. *Let A and B be* C**-algebras and $(I, J) \in \mathrm{Id}'(A) \times \mathrm{Id}'(B)$. Then the quotient* C**-algebra $A \otimes_\alpha B / \Delta(I, J)$ may be naturally identified with $(A/I) \otimes_\gamma (B/J)$, where $\|\cdot\|_\gamma$ is a* C**-norm on $(A/I) \odot (B/J)$. Moreover, $\|\cdot\|_\gamma = \|\cdot\|_\alpha$ if and only if $\Phi(I, J) = \Delta(I, J)$.*

Proof. Let $\pi_I : A \to A/I$ and $\pi_J : B \to B/J$ be the quotient maps. We remark that if $(\pi_I \odot \pi_J) : A \odot B \to (A/I) \odot (B/J)$ denotes the canonical algebraic $*$-homomorphism, then the closure of its kernel in $A \otimes_\alpha B$ is

$$\overline{\ker(\pi_I \odot \pi_J)} = \Delta(I, J).$$

Take

$$z = \sum_{i=1}^n x_i \otimes y_i \in (A/I) \odot (B/J)$$

and choose $a_1, \dots, a_n \in A$ and $b_1, \dots, b_n \in B$ such that $\pi_I(a_i) = x_i$ and $\pi_J(b_i) = y_i$ for $1 \le i \le n$ and set $c = \sum_{i=1}^n a_i \otimes b_i$, so that $(\pi_I \odot \pi_J)(c) = z$. Define $\gamma : (A/I) \odot (B/J) \to [0, \infty)$ via $\gamma(z) = \|c + \Delta(I, J)\|$. Then γ is well-defined since if $c' \in A \odot B$ also satisfies $(\pi_I \odot \pi_J)(c') = z$, then $c - c' \in \ker(\pi_I \odot \pi_J) \subseteq \Delta(I, J)$, hence

$$\gamma(c') = \gamma(c' - c + c) = \|c' - c + c + \Delta(I, J)\| = \|c + \Delta(I, J)\| = \gamma(c).$$

Clearly γ is a semi-norm, and since $z^* z = (\pi_I \odot \pi_J)(c^* c)$, we have

$$\gamma(z^* z) = \|c^* c + \Delta(I, J)\|$$
$$= \|(c + \Delta(I, J))^*(c + \Delta(I, J))\|$$
$$= \|c + \Delta(I, J)\|^2 = \gamma(z)^2$$

(by the C*-condition on the quotient norm), so γ is a C*-semi-norm. Finally, if $\gamma(z) = 0$ then $c \in \Delta(I, J) \cap A \odot B = \ker(\pi_I \odot \pi_J)$, so that $z = 0$ in $(A/I) \odot (B/J)$. It follows that γ is a well-defined C*-norm on $(A/I) \odot (B/J)$.

It follows that $\pi_I \odot \pi_J : A \odot_\alpha B \to (A/I) \odot_\gamma (B/J)$ is a bounded, surjective *-homomorphism of normed *-algebras, and hence extends to a *-homomorphism $\pi_I \otimes_\gamma \pi_J : A \otimes_\alpha B \to (A/I) \otimes_\gamma (B/J)$. Since the range of $\pi_I \otimes_\gamma \pi_J$ is closed and contains the dense set $(A/I) \odot (B/J)$, it is surjective. We claim that $\ker(\pi_I \otimes_\gamma \pi_J) = \Delta(I, J)$.

Since $\ker(\pi_I \otimes_\gamma \pi_J)$ is closed and contains $I \odot B + A \odot J$, it must also contain $\Delta(I, J)$. To show the reverse inclusion, let $d \in \ker(\pi_I \otimes_\gamma \pi_J)$ and $\varepsilon > 0$. Then there is $c \in A \odot B$ with $\|c - d\| < \frac{\varepsilon}{2}$. By the definition of $\|\cdot\|_\gamma$,

$$\|c + \Delta(I, J)\| = \|(\pi_I \otimes_\gamma \pi_J)(c)\| = \|(\pi_I \otimes_\gamma \pi_J)(c - d)\| < \frac{\varepsilon}{2}$$

since $d \in \ker(\pi_I \otimes_\gamma \pi_J)$. It follows that

$$\|d + \Delta(I, J)\| = \|d - c + c + \Delta(I, J)\|$$
$$\leq \|d - c\| + \|c + \Delta(I, J)\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Since ε was arbitrary, $d \in \Delta(I, J)$. We have shown that $\ker(\pi_I \otimes_\gamma \pi_J) = \Delta(I, J)$, and hence we can conclude that $(A \otimes_\alpha B)/\Delta(I, J)$ is canonically *-isomorphic to $(A/I) \otimes_\gamma (B/J)$.

For the final assertion, note that $\|\cdot\|_\gamma = \|\cdot\|_\alpha$ if and only if $\pi_I \otimes_\gamma \pi_J$ is the canonical *-homomorphism $\pi_I \otimes \pi_J : A \otimes_\alpha B \to (A/I) \otimes_\alpha (B/J)$, whose kernel is by definition $\Phi(I, J)$. Hence the two norms are equal if and only if $\ker(\pi_I \otimes_\gamma \pi_J) \ (= \Delta(I, J))$ is equal to $\Phi(I, J)$. $\qquad\square$

Lemma 5.3. *Let* $\mathcal{A} = (X, A, \pi_x : A \to A_x)$ *be a C*-bundle and* $x_0 \in X$*. Then, for each* $a \in A$,

$$\|a + I_{x_0}\| = \inf_W \sup_{x \in W} \|\pi_x(a)\|,$$

as W *ranges over all open neighborhoods of* x_0 *in* X.

Proof. Fix an open neighborhood U of x_0 in X. We first claim that

$$\sup_{x \in U} \|a + I_x\| = \sup_{x \in U} \|\pi_x(a)\|.$$

It is clear that $\ker(\pi_x) \supseteq I_x$, hence we have $\|\pi_x(a)\| \leq \|a + I_x\|$ for all $x \in U$ and so the supremum on the left is always greater than or equal to that on the right.

Let $x_1 \in U$ and choose $f \in C_0(X)$, $0 \leq f \leq 1$, with $f(x_1) = 1$ and $f(X \setminus U) \equiv 0$, then $\|a + I_{x_1}\| = \|f \cdot a + I_{x_1}\|$. Moreover,

$$\|f \cdot a\| = \sup_{x \in X} \|\pi_x(f \cdot a)\|$$
$$= \sup_{x \in U} \|\pi_x(f \cdot a)\|$$

$$= \sup_{x \in U} |f(x)| \cdot \|\pi_x(a)\|$$
$$\leq \sup_{x \in U} \|\pi_x(a)\|,$$

whence

$$\|a + I_{x_1}\| = \|f \cdot a + I_{x_1}\| \leq \|f \cdot a\| \leq \sup_{x \in U} \|\pi_x(a)\|.$$

It follows that

$$\sup_{x \in U} \|a + I_x\| \leq \sup_{x \in U} \|\pi_x(a)\|,$$

and so

$$\sup_{x \in U} \|a + I_x\| = \sup_{x \in U} \|\pi_x(a)\|.$$

Suppose for a contradiction that

$$\alpha := \inf_{W} \sup_{x \in W} \|\pi_x(a)\| > \|a + I_{x_0}\|.$$

Then since $x \mapsto \|a + I_x\|$ is upper semi-continuous on X, we could find an open neighborhood U of x_0 such that

$$\|a + I_x\| < \left(\frac{\alpha + \|a + I_{x_0}\|}{2} \right) \quad \text{for all } x \in U.$$

But this would then imply that

$$\sup_{x \in U} \|a + I_x\| \leq \left(\frac{\alpha + \|a + I_{x_0}\|}{2} \right) < \alpha \leq \sup_{x \in U} \|\pi_x(a)\|,$$

contradicting the fact that these suprema must be equal for all open neighborhoods U of x_0. $\qquad \square$

Proposition 5.4. *Let B be an inexact C*-algebra. Denoting by $\hat{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ the one-point compactification of $\mathbb{N}$, there is a separable unital $C(\hat{\mathbb{N}})$-algebra $(A, \hat{\mathbb{N}}, \mu_A)$ such that the $C(\hat{\mathbb{N}})$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is discontinuous at ∞.*

Proof. Since B is inexact, by [23, Prop. 4.2] there is a separable, unital continuous C*-bundle $\mathcal{C} = (C, \hat{\mathbb{N}}, \sigma_n : C \to C_n)$ with the property that the minimal fiberwise tensor product $\mathcal{C} \otimes_\alpha B = (C \otimes_\alpha B, \hat{\mathbb{N}}, \sigma_n \otimes \text{id} : C \otimes_\alpha B \to C_n \otimes_\alpha B)$ is discontinuous at ∞.

Since C is continuous, for each $n \in \hat{\mathbb{N}}$ we have $I_n = \ker \sigma_n$. By [4, Thm. 3.3], we have $I_\infty \otimes_\alpha B \subsetneq \ker(\sigma_\infty \otimes \text{id})$, while $I_n \otimes_\alpha B = \ker(\sigma_n \otimes \text{id})$ for all $n \in \mathbb{N}$. It follows in particular that $(C \otimes_\alpha B)/(I_\infty \otimes_\alpha B)$ is canonically $*$-isomorphic to $C_\infty \otimes_\gamma B$, where $\|\cdot\|_\gamma$ is a C*-norm on $C_\infty \odot B$ distinct from $\|\cdot\|_\alpha$ by Lemma 5.2.

There is thus

$$y = \sum_{i=1}^{\ell} c_\infty^{(i)} \otimes b^{(i)} \in C_\infty \odot B$$

with the property that $\|y\|_\gamma > \|y\|_\alpha$. Choose $c^{(1)}, \ldots, c^{(\ell)} \in C$ such that $\sigma_\infty(c^{(i)}) = c^{(i)}_\infty$ for $1 \le i \le \ell$, and set

$$\overline{y} = \sum_{i=1}^{\ell} c^{(i)} \otimes b^{(i)} \in C \odot B.$$

Then

$$\|\overline{y} + I_\infty \otimes_\alpha B\| = \|y\|_\gamma > \|y\|_\alpha = \|(\sigma_\infty \otimes \mathrm{id})(y)\|,$$

by the definitions of $\|\cdot\|_\gamma$ and $\sigma_\infty \otimes \mathrm{id}$.

Now let $D = C(\hat{\mathbb{N}}) \otimes_\alpha C_\infty$ be the trivial (hence continuous) bundle on $\hat{\mathbb{N}}$ with fiber C_∞, and denote by ε_n the evaluation map at n, where $n \in \hat{\mathbb{N}}$. Let A be the pullback C*-algebra in the diagram

$$\begin{array}{ccc} A & \longrightarrow & C \\ \downarrow & & \downarrow {\scriptstyle \sigma_\infty} \\ D & \xrightarrow{\varepsilon_\infty} & C_\infty \end{array}$$

that is, the C*-subalgebra of $C \oplus D$ given by

$$A = \{c \oplus d \in C \oplus D \mid \sigma_\infty(c) = \varepsilon_\infty(d)\}.$$

Then we have a well-defined $*$-homomorphism $\pi_\infty : A \to C_\infty$ sending $c \oplus d$ to $\sigma_\infty(c) = \varepsilon_\infty(d)$; see [28, §2.2].

For $n \in \mathbb{N}$ we may extend (keeping the same notation) σ_n and ε_n to A by setting

$$\sigma_n(c \oplus d) = \sigma_n(c), \quad \varepsilon_n(c \oplus d) = \varepsilon_n(d).$$

A defines a continuous C*-bundle $(A, \hat{\mathbb{N}}, \pi_n : A \to A_n)$ as follows: for $n \in \mathbb{N}$ set

$$A_{2n-1} = C_n, \quad \pi_{2n-1} = \sigma_n, \quad A_{2n} = C_\infty, \quad \pi_{2n} = \varepsilon_n,$$

and $\pi_\infty : A \to A_\infty = C_\infty$ as above. Continuity of A follows easily from that of C and D: for $c \oplus d \in A$ we have that $\|\sigma_n(c)\|$ and $\|\varepsilon_n(d)\|$ both converge to $\|\pi_\infty(c \oplus d)\|$, it follows that $\|\pi_n(c \oplus d)\| \to \|\pi_\infty(c \oplus d)\|$.

Regarding A as a $C(\hat{\mathbb{N}})$-algebra, let $\mu_A : C(\hat{\mathbb{N}}) \to Z(A)$ be the structure map; where $f \in C(\hat{\mathbb{N}})$ acts by pointwise multiplication. Now $A \otimes B$ is a $C(\hat{\mathbb{N}})$-algebra with structure map $\mu_A \otimes 1 : C(\hat{\mathbb{N}}) \otimes \mathbb{C} \to ZM(A \otimes_\alpha B)$. For each $n \in \hat{\mathbb{N}}$ we let K_n be the ideal

$$(\mu \otimes 1)\big(\{f \in C(\hat{\mathbb{N}}) \mid f(n) = 0\} \otimes \mathbb{C}\big) A \otimes_\alpha B$$

of $A \otimes_\alpha B$, so that $n \mapsto \|x + K_n\|$ is upper semi-continuous on $\hat{\mathbb{N}}$. Again it follows from [4, Thm. 3.3] that for $n \in \mathbb{N}$ we have $K_n = \ker(\pi_n \otimes \mathrm{id})$, and by Proposition 3.4 (v) we have $K_\infty = I_\infty \otimes_\alpha B$.

On the other hand the (lower semi-continuous) C*-bundle

$$\mathcal{A} \otimes_\alpha B = (\hat{\mathbb{N}}, A \otimes_\alpha B, \pi_n \otimes \mathrm{id} : A \otimes_\alpha B \to A_n \otimes_\alpha B)$$

has fibers $A_n \otimes_\alpha B = C_{(n+1)/2} \otimes_\alpha B$ for n odd, $A_n \otimes_\alpha B = C_\infty \otimes_\alpha B$ for n even, and $A_\infty = C_\infty \otimes_\alpha B$.

For $1 \leq i \leq \ell$, let $d^{(i)} \in D$ be the constant section $\varepsilon_n(d^{(i)}) = c_\infty^{(i)}$, set $a^{(i)} = c^{(i)} \oplus d^{(i)} \in A$, and let $x = \sum_{i=1}^{\ell} a^{(i)} \otimes b^{(i)}$. Then for $n \in \mathbb{N}$ we have

$$(\pi_n \otimes \mathrm{id})(x) = (\pi_n \otimes \mathrm{id})\bigg(\sum_{i=1}^{\ell} a^{(i)} \otimes b^{(i)} \bigg)$$

$$= \sum_{i=1}^{\ell} \pi_n(a^{(i)}) \otimes b^{(i)}$$

$$= \begin{cases} \displaystyle\sum_{i=1}^{\ell} \sigma_{(n+1)/2}(c^{(i)}) \otimes b^{(i)} & \text{if } n \text{ is odd,} \\ \displaystyle\sum_{i=1}^{\ell} \varepsilon_{n/2}(d^{(i)}) \otimes b^{(i)} & \text{if } n \text{ is even} \end{cases}$$

$$= \begin{cases} (\sigma_{(n+1)/2} \otimes \mathrm{id})(\overline{y}) & \text{if } n \text{ is odd,} \\ \displaystyle\sum_{i=1}^{\ell} c_\infty^{(i)} \otimes b^{(i)} & \text{if } n \text{ is even.} \end{cases}$$

It then follows that

$$\|(\pi_n \otimes \mathrm{id})(x)\| = \begin{cases} \|(\sigma_{(n+1)/2} \otimes \mathrm{id})(\overline{y})\| & \text{if } n \text{ is odd,} \\ \left\| \displaystyle\sum_{i=1}^{\ell} c_\infty^{(i)} \otimes b^{(i)} \right\|_{C_\infty \otimes_\alpha B} = \|y\|_\alpha & \text{if } n \text{ is even.} \end{cases}$$

Finally, note that by Lemma 5.3 we have

$$\|x + K_\infty\| = \max\big(\limsup_n \|(\pi_n \otimes \mathrm{id})(x)\|, \|(\pi_\infty \otimes \mathrm{id})(x)\|\big),$$

$$\|y\|_\gamma = \|\overline{y} + I_\infty \otimes_\alpha B\| = \max\big(\limsup_n \|(\sigma_n \otimes \mathrm{id})(\overline{y})\|, \|(\sigma_\infty \otimes \mathrm{id})(\overline{y})\|\big).$$

Since we know that $\|(\sigma_\infty \otimes \mathrm{id})(\overline{y})\| = \|y\|_\alpha < \|y\|_\gamma$, the second equality becomes

$$\|y\|_\gamma = \limsup_n \|(\sigma_n \otimes \mathrm{id})(\overline{y})\|.$$

In particular, we conclude that

$$\|x + K_\infty\| \geq \limsup_n \|(\pi_n \otimes \mathrm{id})(x)\|$$

$$\geq \limsup\{\|(\pi_n \otimes \mathrm{id})(x)\| \mid n \text{ is odd}\}$$

$$= \limsup\{\|(\sigma_{(n+1)/2} \otimes \mathrm{id})(\overline{y})\| \mid n \text{ is odd}\}$$

$$= \|y\|_\gamma.$$

Thus for all even n (since $K_n = \ker(\pi_n \otimes \mathrm{id})$ for $n \in \mathbb{N}$ by continuity at these points), we have

$$\|x + K_n\| = \|(\pi_n \otimes \mathrm{id})(x)\| = \|y\|_\alpha < \|y\|_\gamma \leq \|x + K_\infty\|,$$

and it follows that $n \mapsto \|x + K_n\|$ is discontinuous at ∞. $\qquad\square$

Corollary 5.5. *Let B be an inexact* C*-*algebra. Then there is a separable unital, continuous $C(\hat{\mathbb{N}})$-algebra $(\tilde{A}, \hat{\mathbb{N}}, \mu_{\tilde{A}})$, with $\mathrm{Prim}(\tilde{A})$ homeomorphic to $\hat{\mathbb{N}}$, such that the $C(\hat{\mathbb{N}})$-algebra $(\tilde{A} \otimes_\alpha B, \hat{\mathbb{N}}, \mu_{\tilde{A}} \otimes 1)$ is discontinuous at ∞. Moreover, $\mathrm{Prim}(\tilde{A})$ is canonically homeomorphic to $\hat{\mathbb{N}}$, and $\mu_{\tilde{A}}$ agrees with the Dauns–Hofmann $*$-isomorphism $\theta_{\tilde{A}} : C(\mathrm{Prim}(\tilde{A})) \to Z(\tilde{A})$.*

Proof. Let $(A, \hat{\mathbb{N}}, \mu_A)$ be the separable, unital continuous $C(\hat{\mathbb{N}})$-algebra of Proposition 5.4. Then by [10, Cor. 4.7] there is a unital continuous $C(\hat{\mathbb{N}})$-algebra $(\tilde{A}, \hat{\mathbb{N}}, \mu_{\tilde{A}})$ with simple fibers and a $C(\hat{\mathbb{N}})$-module $*$-monomorphism $\iota : A \to \tilde{A}$. By injectivity of the minimal tensor product we get a $*$-monomorphism $\iota \otimes \mathrm{id} : A \otimes_\alpha B \to \tilde{A} \otimes_\alpha B$.

Now, $\tilde{A} \otimes_\alpha B$ is a $C(\hat{\mathbb{N}})$-algebra with base map $\mu_{\tilde{A}} \otimes 1$. To show that $\iota \otimes \mathrm{id}$ is a $C(\hat{\mathbb{N}})$-module map, take $a \otimes b \in A \otimes_\alpha B$ and $f \in C(\hat{\mathbb{N}})$. Then we have

$$(\mu_{\tilde{A}} \otimes 1)(f)(\iota \otimes \mathrm{id})(a \otimes b) = (\mu_{\tilde{A}}(f)\iota(a)) \otimes b$$
$$= (\iota \otimes \mathrm{id})(\mu_A(f)a \otimes b) \in (\iota \otimes \mathrm{id})(A \otimes_\alpha B).$$

In particular, it follows from Lemma 2.11 (iii) that $(\tilde{A} \otimes_\alpha B, \hat{\mathbb{N}}, \mu_{\tilde{A}} \otimes 1)$ is discontinuous, since it contains the discontinuous $C(\hat{\mathbb{N}})$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ as a $C(\hat{\mathbb{N}})$-submodule.

Denote by $\phi_{\tilde{A}} : \mathrm{Prim}(A) \to \hat{\mathbb{N}}$ the base map uniquely determined by $\mu_{\tilde{A}}$, and let

$$\tilde{I}_n = \mu_{\tilde{A}}\big(\{f \in C(\hat{\mathbb{N}}) \mid f(n) = 0\}\big)\tilde{A} = \bigcap\{P \in \mathrm{Prim}(\tilde{A}) \mid \phi_{\tilde{A}}(P) = n\}$$

for each $n \in \hat{\mathbb{N}}$. Then since each fiber $\tilde{A}/\tilde{I}_n$ is simple, it follows that $\tilde{I}_n$ is maximal (and in particular primitive) for all $n \in \hat{\mathbb{N}}$. Moreover, since every $P \in \mathrm{Prim}(\tilde{A})$ contains a unique $\tilde{I}_n$ for some $n \in \hat{\mathbb{N}}$, we see that $\tilde{I}_n \mapsto n$ is a bijection. The fact that this map is a homeomorphism then follows from Lee's theorem [25, Thm. 4].   □

Theorem 5.6. *The following conditions on a* C*-*algebra B are equivalent:*

  (i) *B is exact.*
 (ii) *For every separable, unital continuous $C(\hat{\mathbb{N}})$-algebra $(A, \hat{\mathbb{N}}, \mu_A)$, the $C(\hat{\mathbb{N}})$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is continuous.*
(iii) *For every separable, unital* C*-*algebra $\tilde{A}$ with $\mathrm{Prim}(\tilde{A})$ Hausdorff, the $C(\mathrm{Prim}(\tilde{A}))$-algebra $(\tilde{A} \otimes_\alpha B, \mathrm{Prim}(\tilde{A}), \theta_{\tilde{A}} \otimes 1)$ is continuous, where $\theta_{\tilde{A}} : C(\mathrm{Prim}(\tilde{A})) \to Z(\tilde{A})$ is the Dauns–Hofmann $*$-isomorphism.*
(iv) *For every locally compact Hausdorff space X and continuous $C_0(X)$-algebra (A, X, μ_A), the $C_0(X)$-algebra $(A \otimes_\alpha B, X, \mu_A \otimes 1)$ is continuous.*

Proof. (i) $\Rightarrow$ (iv): Suppose that B is exact and let (A, X, μ_A) be a continuous $C_0(X)$-algebra. Then since $A \otimes_\alpha B$ has property (F), we have $\Delta(I_x, \{0\}) = \Phi(I_x, \{0\})$ for all $x \in X$. It then follows from Theorem 3.6 (iii) that the $C_0(X)$-algebra $(A \otimes_\alpha B, X, \mu_A \otimes 1)$ is continuous.

The implications (iv) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (ii) are evident.

(ii) $\Rightarrow$ (i): Suppose that B is inexact. Then by Proposition 5.4 there is a separable, unital continuous $C(\hat{\mathbb{N}})$-algebra $(A, \hat{\mathbb{N}}, \mu_A)$ such that $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is discontinuous, so that (ii) fails.

(iii) $\Rightarrow$ (ii): This follows similarly from Corollary 5.5. $\qquad\square$

Corollary 5.7. *The following conditions on a $C_0(Y)$-algebra (B, Y, μ_B) are equivalent:*

(i) *B is exact.*
(ii) *For every separable unital $C_0(X)$-algebra (A, X, μ_A), with $X = \hat{\mathbb{N}}$, the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ satisfies property $(\mathrm{F}_{X,Y})$.*
(iii) *For every $C_0(X)$-algebra (A, X, μ_A), the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ satisfies property $(\mathrm{F}_{X,Y})$.*

If in addition (B, Y, μ_B) is a continuous $C_0(Y)$-algebra, then (i) *to* (iii) *are equivalent to:*

(iv) *For every separable, unital continuous $C(\hat{\mathbb{N}})$-algebra $(A, \hat{\mathbb{N}}, \mu_A)$, the $C(\hat{\mathbb{N}} \times Y)$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}} \times Y, \mu_A \otimes \mu_B)$ is continuous.*
(v) *For every continuous $C_0(X)$-algebra (A, X, μ_A), the $C_0(X \times Y)$-algebra $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is continuous.*

Proof. The equivalence of (i), (ii) and (iii) is shown in the proof of [9, Prop. 3.1].

To see that (iv) implies (i), we argue by contradiction. Indeed, suppose that B is inexact. Then by Proposition 5.4 there is a separable unital $C(\hat{\mathbb{N}})$-algebra $(A, \hat{\mathbb{N}}, \mu_A)$ with the property that the $C(\hat{\mathbb{N}})$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is discontinuous. We will show that the $C(\hat{\mathbb{N}} \times Y)$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}} \times Y, \mu_A \otimes \mu_B)$ must also be discontinuous.

Let $\phi_A : \mathrm{Prim}(A) \to \hat{\mathbb{N}}$ and $\phi_B : \mathrm{Prim}(B) \to Y$ be the base maps corresponding to μ_A and μ_B, respectively, which are open since both A and B are continuous. We will denote by
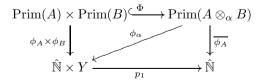
$$\phi_\alpha : \mathrm{Prim}(A \otimes_\alpha B) \to \hat{\mathbb{N}} \times Y \quad \text{and} \quad \overline{\phi_A} : \mathrm{Prim}(A \otimes_\alpha B) \to \hat{\mathbb{N}}$$

the base maps associated with $\mu_A \otimes \mu_B$ and $\mu_A \otimes 1$, respectively. Note that $\overline{\phi_A}$ is not an open mapping since the $C(\hat{\mathbb{N}})$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is not continuous, and that

$$\overline{\phi_A} \circ \Phi = p_1 \circ (\phi_A \times \phi_B) = p_1 \circ \phi_\alpha \circ \Phi$$

on $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ by Proposition 3.4 (i).

Consider now the diagram

where p_1 is the (open) projection onto the first factor. To see that the lower triangle of this diagram commutes, note that since $p_1 \circ \phi_\alpha$ agrees with $\overline{\phi_A}$

on $\Phi(\mathrm{Prim}(A) \times \mathrm{Prim}(B))$, both maps must agree on $\mathrm{Prim}(A \otimes_\alpha B)$ by the uniqueness part of [24, Thm. 3.2].

Then if ϕ_α were open, this would imply that $\overline{\phi_A} = p_1 \circ \phi_\alpha$ were open, which is impossible since $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is discontinuous. In particular, it follows that $(A \otimes_\alpha B, \hat{\mathbb{N}} \times Y, \mu_A \otimes \mu_B)$ is discontinuous.

The fact that (v) implies (iv) is evident. To see that (i) implies (v), note that if (i) holds then $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ satisfies property $(\mathrm{F}_{X,Y})$ by the equivalence of (i) and (iii). Then by Theorem 3.6 (iii), $(A \otimes_\alpha B, X \times Y, \mu_A \otimes \mu_B)$ is a continuous $C_0(X \times Y)$-algebra. $\square$

6. QUASI-STANDARD C*-ALGEBRAS AND HAUSDORFF PRIMITIVE IDEAL SPACES

This section is concerned with stability of the class of quasi-standard C*-algebras under tensor products. This question was first studied by Archbold in [4], where it was shown that if A and B are quasi-standard and $A \otimes_\alpha B$ satisfies property $(\mathrm{F}_{\mathrm{Gl}})$, then $A \otimes_\alpha B$ is quasi-standard. We gave a partial converse to this result in [26, Thm. 5.2]. However, it is clear from Theorem 4.4 that property $(\mathrm{F}_{\mathrm{Gl}})$ is not a necessary condition for $A \otimes_\alpha B$ to be quasi-standard.

We will show in Theorem 6.6 that a quasi-standard C*-algebra A is exact if and only if $A \otimes_\alpha B$ is quasi-standard for all quasi-standard B. As a related result, we show that a (not necessarily quasi-standard) C*-algebra A is exact if and only if $A \otimes_\alpha B$ satisfies property $(\mathrm{F}_{\mathrm{Gl}})$ for all C*-algebras B, if and only $A \otimes_\alpha B$ satisfies property $(\mathrm{F}_{\mathrm{MP}})$ for all C*-algebras B. The existence of C*-algebras A and B such that $A \otimes_\alpha B$ does not satisfy properties $(\mathrm{F}_{\mathrm{Gl}})$ and $(\mathrm{F}_{\mathrm{MP}})$ was previously unknown, thus our result answers a question posed by Archbold [4, p. 142] and Lazar [24, p. 250].

Proposition 6.1. *Let A and B be C*-algebras.*

(i) *If $A \otimes_\alpha B$ satisfies $(\mathrm{F}_{\mathrm{MP}})$, then $A \otimes_\alpha B$ satisfies $(\mathrm{F}_{\mathrm{Gl}})$.*

(ii) *If (A, X, μ_A) is a $C_0(X)$-algebra and (B, Y, μ_B) a $C_0(Y)$-algebra, then $A \otimes_\alpha B$ satisfies property $(\mathrm{F}_{\mathrm{Gl}})$ implies that $A \otimes_\alpha B$ satisfies property $(\mathrm{F}_{X,Y})$.*

Proof. (i): We first show that for any C*-algebra A and $G_1 \in \mathrm{Glimm}(A)$,

$$G_1 = \bigcap \{P \in \mathrm{Min\text{-}Primal}(A) \mid P \supseteq G_1\}.$$

Indeed, since each primitive ideal of A is primal, we necessarily have

$$G_1 = \bigcap \{P \in \mathrm{Primal}(A) \mid P \supseteq G_1\} \subseteq \bigcap \{P \in \mathrm{Min\text{-}Primal}(A) \mid P \supseteq G_1\}.$$

Denote by H the ideal on the right. Then if the above inclusion were strict, there would be some $Q \in \mathrm{Prim}(A)$ such that $Q \supseteq G_1$ but $Q \not\supseteq H$. Let R be a minimal primal ideal of A contained in Q, then by [5, Lem. 2.2] there is a unique Glimm ideal G_2 contained in R. Note $G_1 \neq G_2$ since otherwise R, and hence Q, would contain H. This in turn implies that $Q \not\supseteq G_1$, a contradiction.

Since $A \otimes_\alpha B$ satisfies property (F_{MP}), we have $\Phi = \Delta$ on Min-Primal(A) × Min-Primal(B), and so [4, Thm. 4.1] shows that Δ is a homeomorphism of Min-Primal(A) × Min-Primal(B) onto Min-Primal($A \otimes_\alpha B$). For $(G, H) \in$ Glimm(A)×Glimm(B) we have $\Delta(G, H) \in$ Glimm($A \otimes_\alpha B$) (see [26, Thm. 4.8]), which together with the above remarks gives

$$\Delta(G, H) = \bigcap \{ R \in \text{Min-Primal}(A \otimes_\alpha B) \mid R \supseteq \Delta(G, H) \}$$
$$= \bigcap \{ \Delta(I, J) \mid (I, J) \in \text{Min-Primal}(A) \times \text{Min-Primal}(B),$$
$$\Delta(I, J) \supseteq \Delta(G, H) \}$$

On the other hand by [24, Lem. 2.2] and the first part of the proof, $\Phi(G, H)$ equals

$$\Phi\left(\bigcap \{ I \in \text{Min-Primal}(A) \mid I \supseteq G \}, \bigcap \{ J \in \text{Min-Primal}(B) \mid J \supseteq H \} \right)$$
$$= \bigcap \{ \Phi(I, J) \mid (I, J) \in \text{Min-Primal}(A) \times \text{Min-Primal}(B), I \supseteq G, J \supseteq H \}$$
$$= \bigcap \{ \Delta(I, J) \mid (I, J) \in \text{Min-Primal}(A) \times \text{Min-Primal}(B), I \supseteq G, J \supseteq H \}.$$

Finally, since $\Psi \circ \Delta$ is the identity on $\text{Id}'(A) \times \text{Id}'(B)$ and since Ψ is order-preserving, we see that

$$\Delta(I, J) \supseteq \Delta(G, H) \quad \text{if and only if} \quad I \supseteq G \text{ and } J \supseteq H,$$

from which we conclude that $\Phi(G, H) = \Delta(G, H)$ for all $(G, H) \in$ Glimm(A)× Glimm(B). Hence $A \otimes_\alpha B$ satisfies property (F_{Gl}).

(ii): We will use the notation of Proposition 3.4. Note that for all $(x, y) \in X \times Y$, we have $K_{x,y} = \Delta(I_x, J_y)$ by Proposition 3.4 (iv), so we will show that $K_{x,y} = \Phi(I_x, J_y)$. Let ψ_A, ψ_B and ψ_α denote the continuous maps on the Glimm spaces of A, B and $A \otimes_\alpha B$ induced by the base maps ϕ_A, ϕ_B and ϕ_α, respectively. We first show that $\psi_\alpha \circ \Delta = \psi_A \times \psi_B$. Indeed, for all $(P, Q) \in \text{Prim}(A) \times \text{Prim}(B)$ we have

$$(\phi_A \times \phi_B)(P, Q) = (\phi_\alpha \circ \Phi)(P, Q)$$

by Proposition 3.4 (i). Hence by the definitions of ψ_A, ψ_B and ψ_α,

$$(\psi_A \times \psi_B) \circ (\rho_A \times \rho_B)(P, Q) = (\psi_\alpha \circ \rho_\alpha \circ \Phi)(P, Q).$$

Following the first paragraph of the proof of [26, Thm. 4.8], we see that $\rho_\alpha \circ \Phi = \Delta \circ (\rho_A \times \rho_B)$, which shows that

$$(\psi_A \times \psi_B)(\rho_A(P), \rho_B(Q)) = (\psi_\alpha \circ \Delta)(\rho_A(P), \rho_B(Q))$$

for all $(P, Q) \in \text{Prim}(A) \times \text{Prim}(B)$.

By Lemma 2.8,

$$K_{x,y} = \bigcap \{ G \in \text{Glimm}(A \otimes_\alpha B) \mid \psi_\alpha(G) = (x, y) \}.$$

Any such $G \in$ Glimm($A \otimes_\alpha B$) is the image $\Delta(G_p, G_q)$ of a pair of Glimm ideals of A and B by [26, Thm. 4.8]. Together with the fact that $\psi_\alpha \circ \Delta = \psi_A \times \psi_B$,

this gives

$$K_{x,y} = \bigcap\{\Delta(G_p, G_q) \mid (p,q) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B),$$
$$\psi_A(p) = x \text{ and } \psi_B(q) = y\}.$$

Using $\Phi = \Delta$ on $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$, [24, Lem. 2.2] shows that

$$K_{x,y} = \bigcap\{\Phi(G_p, G_q) \mid \psi_A(p) = x, \psi_B(q) = y\}$$
$$= \Phi\Big(\bigcap\{G_p \mid \psi_A(p) = x\}, \bigcap\{G_q \mid \psi_B(q) = y\}\Big)$$
$$= \Phi(I_x, J_y),$$

where the final equality follows from Lemma 2.8. Hence $A \otimes_\alpha B$ satisfies property $(\mathrm{F}_{X,Y})$. $\square$

Proposition 6.2. *Let B be an inexact C^*-algebra. Then there is a separable C^*-algebra A with $\mathrm{Prim}(A)$ Hausdorff such that*
- (i) *there is a pair $(G, H) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ with $\Delta(G, H) \subsetneq \Phi(G, H)$,*
- (ii) *there is a pair $(I, J) \in \mathrm{Min\text{-}Primal}(A) \times \mathrm{Min\text{-}Primal}(B)$ with $\Delta(I, J) \subsetneq \Phi(I, J)$,*
- (iii) *$\mathrm{Prim}(A \otimes_\alpha B)$ is non-Hausdorff,*
- (iv) *the complete regularization map $\rho_\alpha : \mathrm{Prim}(A \otimes_\alpha B) \to \mathrm{Glimm}(A \otimes_\alpha B)$ is not open.*

Proof. To prove (i), let $(A, \hat{\mathbb{N}}, \mu_A)$ be the continuous $C(\hat{\mathbb{N}})$-algebra constructed in Corollary 5.5, so that the $C(\hat{\mathbb{N}})$-algebra $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is discontinuous. We regard B as a (continuous) $C_0(Y)$-algebra over a one-point space $Y = \{y\}$ as in Remark 3.7 (ii). Then since $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ is discontinuous, it follows from Theorem 3.6 (iii) that $A \otimes_\alpha B$ does not satisfy property $(\mathrm{F}_{X,Y})$. Hence by Proposition 6.1 (ii), it must follow that $\Phi \neq \Delta$ on $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$.

Statement (ii) is immediate from (i) and Proposition 6.1 (i).

To prove (iii), note that if $\mathrm{Prim}(B)$ is non-Hausdorff, then the same is true of $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$. Since Φ maps $\mathrm{Prim}(A) \times \mathrm{Prim}(B)$ homeomorphically onto its image in $\mathrm{Prim}(A \otimes_\alpha B)$ (see [34, Lem. 16]), it then follows that the latter must also be non-Hausdorff.

Suppose now that $\mathrm{Prim}(B)$ is Hausdorff, then $\mathrm{Prim}(B) = \mathrm{Glimm}(B)$. Since $\mathrm{Prim}(A) = \mathrm{Glimm}(A)$ also, statement (i) implies that there are $(P, Q) \in \mathrm{Prim}(A) \times \mathrm{Prim}(B)$ such that $\Delta(P, Q) \subsetneq \Phi(P, Q)$. It follows that there is $R \in \mathrm{Prim}(A \otimes_\alpha B)$ such that $R \supseteq \Delta(P, Q)$ but $R \neq \Phi(P, Q)$. Since by [26, Thm. 4.8], $\Delta(P, Q)$ is a Glimm ideal of $A \otimes_\alpha B$, we have $R \approx \Phi(P, Q)$, so that $\mathrm{Prim}(A \otimes_\alpha B)$ is non-Hausdorff.

We finally prove (iv). If $\rho_B : \mathrm{Prim}(B) \to \mathrm{Glimm}(B)$ is not an open map, then since A is unital, ρ_α is not open by [26, Cor. 5.6]. Thus we will assume that ρ_B is open.

Since $A \otimes_\alpha B$ is a discontinuous $C(\hat{\mathbb{N}})$-algebra, the continuous mapping $\phi_\alpha : \mathrm{Prim}(A \otimes_\alpha B) \to \hat{\mathbb{N}}$ is not open. Moreover, ϕ_α is the unique extension of $\phi_A \circ p_1 : \mathrm{Prim}(A) \times \mathrm{Prim}(B) \to \hat{\mathbb{N}}$ to $\mathrm{Prim}(A \otimes_\alpha B)$, where p_1 is the

projection onto the first factor. We will denote by $\psi_A : \text{Glimm}(A) \to \hat{\mathbb{N}}$ the canonical homeomorphism, and by $\psi_\alpha : \text{Glimm}(A \otimes_\alpha B) \to \hat{\mathbb{N}}$ the map induced by ϕ_α given by the universal property of the complete regularization, so that $\phi_\alpha = \psi_\alpha \circ \rho_\alpha$.

Since $\text{Prim}(A)$ is compact, the complete regularization of $\text{Prim}(A) \times \text{Prim}(B)$ is canonically identified with $\text{Glimm}(A) \times \text{Glimm}(B)$ with the product topology [26, Prop. 1.9]. In particular, this implies that $\Psi : \text{Glimm}(A \otimes_\alpha B) \to \text{Glimm}(A) \times \text{Glimm}(B)$ is a homeomorphism by [26, Thm. 4.8].

Let $\tilde{p}_1 : \text{Glimm}(A) \times \text{Glimm}(B) \to \text{Glimm}(A)$ be the projection onto the first factor and consider now the diagram

$$
\begin{array}{ccccc}
\text{Prim}(A \otimes_\alpha B) & \xrightarrow{\rho_\alpha} & \text{Glimm}(A \otimes_\alpha B) & \xrightarrow{\Psi} & \text{Glimm}(A) \times \text{Glimm}(B) \\
\downarrow{\scriptstyle \phi_\alpha} & & & & \downarrow{\scriptstyle \tilde{p}_1} \\
\hat{\mathbb{N}} & \longleftarrow & & \hspace{-2cm}\underset{\psi_A}{\longleftarrow} & \text{Glimm}(A).
\end{array}
$$

Then by [26, Thm. 2.2, Thm. 4.8 (ii)] we have $\Psi \circ \rho_\alpha \circ \Phi = \rho_A \times \rho_B$ on $\text{Prim}(A) \times \text{Prim}(B)$, so that for all $(P, Q) \in \text{Prim}(A) \times \text{Prim}(B)$,

$$
\begin{aligned}
(\psi_A \circ \tilde{p}_1 \circ \Psi \circ \rho_\alpha)(\Phi(P, Q)) &= (\psi_A \circ \tilde{p}_1)(\rho_A(P), \rho_B(Q)) \\
&= (\psi_A \circ \rho_A)(P) \\
&= \phi_A(P) \\
&= (\phi_\alpha)(\Phi(P, Q)),
\end{aligned}
$$

the final equality holding by Proposition 3.4 (i). Since $\Phi(\text{Prim}(A) \times \text{Prim}(B))$ is dense in $\text{Prim}(A \otimes_\alpha B)$, it follows by continuity that the above diagram commutes.

Note that Ψ, $\tilde{p}_1$ and ψ_A are all open mappings. Thus if ρ_α were open, it would follow that ϕ_α were open, which would imply that $(A \otimes_\alpha B, \hat{\mathbb{N}}, \mu_A \otimes 1)$ were a continuous $C(\hat{\mathbb{N}})$-algebra, a contradiction. $\qquad \square$

Theorem 6.3. *The following conditions on a* C*-*algebra* B *are equivalent:*

(i) B *is exact.*
(ii) $A \otimes_\alpha B$ *satisfies property* (F_{Gl}) *for all* C*-*algebras* A.
(iii) $A \otimes_\alpha B$ *satisfies property* (F_{MP}) *for all* C*-*algebras* A.

Proof. (i) $\Rightarrow$ (ii) and (iii): If B is exact, then $A \otimes_\alpha B$ satisfies property (F) for all A, hence $A \otimes_\alpha B$ satisfies properties (F_{Gl}) and (F_{MP}) for all C*-algebras A.

To see that (ii) (resp. (iii)) implies (i), note that if B is inexact then there is by Proposition 6.2 (i) (resp. (ii)) a C*-algebra A for which $A \otimes_\alpha B$ does not satisfy property (F_{Gl}) (resp. (F_{MP})). $\qquad \square$

It was shown in [4] that if $A \otimes_\alpha B$ satisfies property (F_{MP}), then Δ (equivalently, Φ) maps $\text{Min-Primal}(A) \times \text{Min-Primal}(B)$ homeomorphically onto $\text{Min-Primal}(A \otimes_\alpha B)$. The following corollary shows that if B is inexact and quasi-standard then this map may fail to be a homeomorphism.

Corollary 6.4. *Let B be an inexact, quasi-standard* C**-algebra. Then there is a quasi-standard* C**-algebra A for which the restriction of the map Δ to* Min-Primal$(A) \times$ Min-Primal(B) *is not a homeomorphism of this space onto* Min-Primal$(A \otimes_\alpha B)$.

Proof. Again, let A be the C*-algebra constructed in Corollary 5.5, so that $A \otimes_\alpha B$ is not quasi-standard by Proposition 6.2 (iv). Since both A and B are quasi-standard, we have Glimm(A) = Min-Primal(A) and Glimm(B) = Min-Primal(B), both as sets and topologically [5, Thm. 3.3 (iii)]. By [26, Cor. 2.3, Thm. 4.8], Δ is a homeomorphism of Glimm$(A) \times$ Glimm(B) onto Glimm$(A \otimes_\alpha B)$. Thus if Δ were also a homeomorphism mapping the space Min-Primal$(A) \times$ Min-Primal(B) onto Min-Primal$(A \otimes_\alpha B)$, it would follow that Glimm$(A \otimes_\alpha B)$ = Min-Primal$(A \otimes_\alpha B)$, both as sets and topologically. This would imply that $A \otimes_\alpha B$ is quasi-standard by [5, Thm. 3.3 (iii)$\Rightarrow$(i)], which is a contradiction. $\square$

Example 6.5. Let B be a primitive, inexact C*-algebra, e.g., $B = B(H)$ (see [32]) or $B = $ C$^*(\mathbf{F}_2)$ (the full group C*-algebra of the free group on two generators; see [31]), so that Min-Primal$(B) = \{0\}$. Then Corollary 6.4 gives a C*-algebra A with Prim$(A) = $ Min-Primal$(A) = \{I_n \mid n \in \hat{\mathbb{N}}\}$ for which Δ is not a homeomorphism of Min-Primal$(A) \times$ Min-Primal(B) onto Min-Primal$(A \otimes_\alpha B)$.

We remark that by Theorem 3.3 (iii), $\Delta(I_n, \{0\}) = \Phi(I_n, \{0\})$ for all $n \in \mathbb{N}$, so that $\Delta(I_n, \{0\}) \in$ Prim$(A \otimes_\alpha B)$ for all such n. By [3, Prop. 4.5], we have $\Delta(I_n, \{0\}) \in$ Min-Primal$(A \otimes_\alpha B)$ for all $n \in \mathbb{N}$. On the other hand, it is not clear whether or not the Glimm ideal $\Delta(I_\infty, \{0\})$ is a primal ideal of $A \otimes_\alpha B$.

Theorem 6.6. *Let B be a* C**-algebra.*

(i) *If $\rho_B :$ Prim$(B) \to$ Glimm(B) is an open map, then B is exact if and only if $\rho_\alpha :$ Prim$(A \otimes_\alpha B) \to$ Glimm$(A \otimes_\alpha B)$ is open for every* C**-algebra A with $\rho_A :$ Prim$(A) \to$ Glimm(A) open.*

(ii) *If B is quasi-standard, then B is exact if and only if $A \otimes_\alpha B$ is quasi-standard for every quasi-standard* C**-algebra A.*

(iii) *If* Prim(B) *is Hausdorff, then B is exact if and only if* Prim$(A \otimes_\alpha B)$ *is Hausdorff for every* C**-algebra A with* Prim(A) *Hausdorff.*

Proof. (i): If B is exact, then $A \otimes_\alpha B$ satisfies property (F), hence property (F$_{\mathrm{Gl}}$), for all C*-algebras A. Hence ρ_α is open for all C*-algebras A with ρ_A open by [26, Thm. 5.2].

Conversely if B is inexact, then by Proposition 6.2 (iv) there is a C*-algebra A with Prim(A) Hausdorff, hence $\rho_A = $ id open, such that ρ_α is not open.

(ii): If B is exact, then it follows from [19, Cor. 2.5] that $A \otimes_\alpha B$ is quasi-standard for every quasi-standard A.

Conversely, if B is inexact, then by Proposition 6.2 (iv) there is a quasi-standard C*-algebra A for which $A \otimes_\alpha B$ is not quasi-standard.

(iii): We will make use of the fact that if A is a C*-algebra such that either $\mathrm{Prim}(A)$ or $\mathrm{Fac}(A)$ is Hausdorff, then $\mathrm{Prim}(A) = \mathrm{Fac}(A) = \mathrm{Prime}(A)$, see, e.g., [11, p. 474].

Suppose that B is exact and that A is a C*-algebra with $\mathrm{Prim}(A)$ Hausdorff. Then since $A \otimes_\alpha B$ has property (F), [24, Prop. 5.1] shows that Δ is a homeomorphism of $\mathrm{Fac}(A) \times \mathrm{Fac}(B) = \mathrm{Prim}(A) \times \mathrm{Prim}(B)$ onto $\mathrm{Fac}(A \otimes_\alpha B)$. Hence $\mathrm{Fac}(A \otimes_\alpha B)$ is Hausdorff, and thus the same is true of $\mathrm{Prim}(A \otimes_\alpha B)$.

Conversely, if B is inexact, then by Proposition 6.2 (iii) there is a separable C*-algebra A with $\mathrm{Prim}(A)$ Hausdorff for which $\mathrm{Prim}(A \otimes_\alpha B)$ is non-Hausdorff. $\qquad\square$

Example 6.7. Let $M = \prod_{n \geq 1} M_n(\mathbb{C})$, so that M is quasi-standard as in Theorem 4.4. Moreover, M is inexact by [22, Thm. 1.1]. Then by Proposition 6.2 (iv), there is a separable unital C*-algebra A with Hausdorff primitive ideal space $\mathrm{Prim}(A)$ homeomorphic to $\hat{\mathbb{N}}$ such that $A \otimes_\alpha M$ is not quasi-standard. In particular, the assumption that the zero ideal of the C*-algebra B of Theorem 4.4 is prime cannot be dropped.

We will give an analogous result for maximal tensor products of (unital) C*-algebras in Theorem 6.9. The following proposition gives some (known) properties of the Dauns–Hofmann representation of the maximal tensor product of two unital C*-algebras.

Proposition 6.8. *Let A and B be unital C*-algebras.*

(i) *The map $(G, H) \mapsto G \otimes_{\max} B + A \otimes_{\max} H$ is a homeomorphism of $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ onto $\mathrm{Glimm}(A \otimes_{\max} B)$.*

(ii) *The Glimm quotients of $A \otimes_{\max} B$ are canonically $*$-isomorphic to $(A/G) \otimes_{\max} (B/H)$ for $(G, H) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$.*

(iii) *The Dauns–Hofmann representation of $A \otimes_{\max} B$ defines canonically an upper semi-continuous C*-bundle*

$$\big(\mathrm{Glimm}(A) \times \mathrm{Glimm}(B), A \otimes_{\max} B,$$
$$\pi_G \otimes_{\max} \sigma_H : A \otimes_{\max} B \to (A/G) \otimes_{\max} (B/H)\big),$$

where $\pi_G : A \to A/G$ and $\sigma_H : B \to B/H$ are the quotient maps for $(G, H) \in \mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$.

Proof. Statement (i) is shown in [19, p. 304], and (ii) follows from (i) and [8, Prop. 3.15]. Since A and B are unital, $\mathrm{Glimm}(A) \times \mathrm{Glimm}(B)$ is compact, and so (iii) is immediate from the Dauns–Hofmann theorem (Theorem 2.6). $\qquad\square$

Theorem 6.9. *Let B be a unital quasi-standard C*-algebra. Then B is nuclear if and only if $A \otimes_{\max} B$ is quasi-standard for every quasi-standard C*-algebra A.*

Proof. If B is nuclear, then B is exact, so that for any C*-algebra A we have that $A \otimes_{\max} B = A \otimes_\alpha B$ has property (F). Thus for all quasi-standard A, $A \otimes_{\max} B$ is quasi-standard by [19, Cor. 2.5].

Conversely, suppose that B is non-nuclear. For each $q \in \mathrm{Glimm}(B)$ let G_q be the corresponding Glimm ideal of B and denote by $(\mathrm{Glimm}(B), B, \sigma_q : B \to B_q)$ the corresponding continuous C*-bundle over $\mathrm{Glimm}(B)$, where $B_q = B/G_q$ for all $q \in \mathrm{Glimm}(B)$. By [8, Prop. 3.23], there is $p \in \mathrm{Glimm}(A)$ for which B_p is non-nuclear. As in the proof of [23, Thm. 3.2], one can construct a Hilbert space H, a unital C*-subalgebra $C \subseteq B(H)$ and

$$t = \sum_{i=1}^{\ell} r_i \otimes s_i \in C \odot B_p$$

such that $\|t\|_{C \otimes_{\max} B_p} > \|t\|_{B(H) \otimes_{\max} B_p}$.

Denote by A the C*-algebra of sequences $(T_n) \subset B(H)$ such that T_n converges in norm to some element $T \in C$. Then, as in the proof of [5, Prop. 3.6], A is quasi-standard, $\mathrm{Glimm}(A)$ is homeomorphic to $\hat{\mathbb{N}}$, and the Glimm quotients A_n of A are given by

$$A_n = \begin{cases} B(H) & \text{if } n \in \mathbb{N}, \\ C & \text{if } n = \infty. \end{cases}$$

We denote by $(\hat{\mathbb{N}}, A, \pi_n : A \to A_n)$ the corresponding continuous C*-bundle over $\hat{\mathbb{N}}$. By Proposition 6.8 we may identify $\mathrm{Glimm}(A \otimes_{\max} B) = \hat{\mathbb{N}} \times \mathrm{Glimm}(B)$, and the Glimm quotients of $A \otimes_{\max} B$ are isomorphic to $A_n \otimes_{\max} B_p$ for $(n, p) \in \hat{\mathbb{N}} \times \mathrm{Glimm}(B)$.

For $1 \leq i \leq \ell$ choose $\overline{r_i} \in A$ and $\overline{s_i} \in B$ such that $\pi_n(\overline{r_i}) = r_i$ for all $n \in \hat{\mathbb{N}}$, and $\sigma_p(\overline{s_i}) = s_i$. Then setting

$$\overline{t} = \sum_{i=1}^{\ell} \overline{r_i} \otimes \overline{s_i} \in A \odot B,$$

we have $\pi_\infty \otimes \sigma_G(\overline{t}) = t$. Then we have $\|(\pi_n \otimes_{\max} \sigma_p)(\overline{t})\| = \|t\|_{B(H) \otimes_{\max} B}$ for $n \in \mathbb{N}$, while $\|(\pi_\infty \otimes_{\max} \sigma_p)(\overline{t})\| = \|t\|_{C \otimes_{\max} B}$. Since $(n, p) \to (\infty, p)$ in $\hat{\mathbb{N}} \times \mathrm{Glimm}(B)$, it follows that $(n, q) \mapsto \|(\pi_n \otimes_{\max} \sigma_q)(t)\|$ is discontinuous at (∞, p). In particular, $A \otimes_{\max} B$ is not quasi-standard. $\square$

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