

Foliated manifolds, algebraic K -theory, and a secondary invariant

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(Communicated by Linus Kramer)

Abstract. We introduce a \mathbb{C}/\mathbb{Z} -valued invariant of a foliated manifold with a stable framing and with a partially flat vector bundle. This invariant can be expressed in terms of integration in differential K -theory or, alternatively, in terms of η -invariants of Dirac operators and local correction terms. Initially, the construction of the element in \mathbb{C}/\mathbb{Z} involves additional choices. But if the codimension of the foliation is sufficiently small, then this element is independent of these choices and therefore an invariant of the data listed above. We show that the invariant comprises various classical invariants like Adams' e -invariant, the ρ -invariant of twisted Dirac operators, or the Godbillon–Vey invariant from foliation theory. Using methods from differential cohomology theory, we construct a regulator map from the algebraic K -theory of smooth functions on a manifold to its connective K -theory with \mathbb{C}/\mathbb{Z} coefficients. Our main result is a formula for the invariant in terms of this regulator and integration in algebraic and topological K -theory.

1. INTRODUCTION

In this paper we introduce and analyze an invariant

$$\rho(M, \mathcal{F}, \nabla^I, s) \in \mathbb{C}/\mathbb{Z}$$

of an odd-dimensional closed spin manifold M , equipped with a real foliation \mathcal{F} , a complex vector bundle with flat partial connection ∇^I in the direction of the foliation, and a stable framing s of the foliation. In order to define this number we must choose in addition a Riemannian metric on M , an extension of the partial connection ∇^I to a connection and, similarly, an extension of the canonical flat partial connection on the normal bundle \mathcal{F}^\perp of the foliation.

Without any further conditions the number $\rho(M, \mathcal{F}, \nabla^I, s)$ may depend non-trivially on the additional geometric choices. But if the codimension of the

This work was partially supported by the SFB 1085 “Higher Invariants—Interactions between Arithmetic Geometry and Global Analysis” of the DFG..

foliation \mathcal{F} is sufficiently small, namely, if

$$(1) \quad 2\text{codim}(\mathcal{F}) < \dim(M),$$

then $\rho(M, \mathcal{F}, \nabla^I, s)$ does not depend on the additional choices.

The quickest way to define the invariant in Definition 4.2 is to use the integration in differential complex K -theory \widehat{KU}^* . Alternatively, $\rho(M, \mathcal{F}, \nabla^I, s)$ can also be expressed as a combination of η -invariants of twisted Dirac operators and correction terms involving integrals of characteristic forms and their transgressions, see Proposition 4.10.

The invariant $\rho(M, \mathcal{F}, \nabla^I, s)$ is very interesting since it combines various classical secondary invariants in spectral geometry, topology and foliation theory in one object. We will reveal these relations by analyzing special cases in Section 5. We will observe that $\rho(M, \mathcal{F}, \nabla^I, s)$ subsumes Adams' e -invariant for framed manifolds, the ρ -invariant, for Dirac operators twisted with flat bundles, and classical invariants from foliation theory like the Godbillon–Vey invariant.

While the construction of the invariant $\rho(M, \mathcal{F}, \nabla^I, s)$ and the verification of its basic properties are not very deep and based on well-known methods from differential geometry and local index theory, we think that its relation with algebraic K -theory is much less obvious. In the present paper we reveal this relation in the special case of a foliated manifold of the form

$$(M, \mathcal{F}) = (P \times X, T_{\mathbb{C}}P \boxplus \{0\}).$$

Here the manifold P is closed and stably framed. A complex vector bundle (V, ∇^I) with flat partial connection on M provides an algebraic K -theory class of the ring of complex-valued smooth functions $C^\infty(X)$. We will write this class as

$$f_!^{os}([V, \nabla^I]^{\text{alg}}) \in K_p(C^\infty(X))$$

with notation to be introduced in Section 6, where $p := \dim(P)$. If $p > \dim(X)$ (this is exactly condition (1)), then we can define a regulator transformation

$$\text{reg}_X : K_p(C^\infty(X)) \rightarrow \mathbf{ku}\mathbb{C}/\mathbb{Z}^{-p-1}(X),$$

see Definition 6.19. If we now assume that X is a closed spin manifold such that $p + \dim(X)$ is odd, then we have an integration map

$$\pi_!^o : \mathbf{ku}\mathbb{C}/\mathbb{Z}^{-p-1}(X) \rightarrow \mathbf{ku}\mathbb{C}/\mathbb{Z}^{-p-\dim(X)-1}(\ast) \cong \mathbb{C}/\mathbb{Z},$$

where $\pi : X \rightarrow \ast$ and o is the orientation of π for $\mathbf{ku}\mathbb{C}/\mathbb{Z}$ induced by the spin structure. Our main result is Theorem 6.2:

Theorem 1.1. *We have*

$$\rho(M, \mathcal{F}, \nabla^I, s) = \pi_!^o(\text{reg}_X(f_!^{os}([V, \nabla^I]^{\text{alg}}))).$$

The proof of this theorem will be finished in Section 6. It is based on the diagram (67) which comprises various Riemann–Roch type squares for integration in algebraic and topological K -theory and their differential refinements.

In Section 7 we put the regulator \mathbf{reg}_X into its natural general framework. We introduce the algebraic K -theory spectrum $\mathbf{K}(M, \mathcal{F})$ of a foliated manifold and the Hodge-filtered connective complex K -theory spectrum $\mathbf{ku}^{\text{flat}}(M, \mathcal{F})$. The regulator \mathbf{reg}_X used in the theorem above is then a special case of a regulator

$$\mathbf{reg} : \mathbf{K}(M, \mathcal{F}) \rightarrow \mathbf{ku}^{\text{flat}}(M, \mathcal{F}).$$

In order to justify calling this map a regulator, consider a complex manifold M as a real manifold with a complex foliation $\mathcal{F} := T^{0,1}M$. In this case $\mathbf{K}(M, T^{0,1}M)$ is the algebraic K -theory spectrum of M , defined by using holomorphic vector bundles. Moreover, the homotopy groups of $\mathbf{ku}^{\text{flat}}(M, T^{0,1}M)$ are the \mathbf{ku} -theory analogs of the integral Deligne cohomology groups. The regulator is an integral refinement of a version of Beilinson's regulator. We will explain all this in detail in Section 7.

In Section 2 we introduce basic definitions from the theory of foliated manifolds and characteristic classes. The experienced reader could skip this section in a first reading and use it as a reference for notation and normalization conventions. In Section 3 we give a quick introduction to the features of differential complex K -theory which are used in the construction of the invariant $\rho(M, \mathcal{F}, \nabla^I, s)$. The actual construction of this invariant will be given in Section 4.1. As mentioned above, in Section 4.7 we provide a spectral theoretic interpretation of the invariant, and in Section 5 relate it to various classical secondary invariants.

In Section 6 we develop the theory which is necessary to state and prove Theorem 1.1. Finally, Section 7 is devoted to the algebraic K -theory of foliated manifolds and the regulator in general. This section has a substantial overlap with the work of Karoubi [26, 24, 25]. In a certain sense, it reformulates his constructions using the new technology of the ∞ -categorical approach to K -theory and regulators developed in [5, 11, 12, 14].

2. BASIC NOTIONS

2.1. Foliated manifolds. We introduce the category of foliated manifolds $\mathbf{Mf}_{\mathbb{C}\text{-fol}}$ and its full subcategory \mathbf{Mf}_{fol} of manifolds with real foliations.

In the present paper we consider foliations from the infinitesimal point of view. Textbook references for real foliations (with emphasis on the decomposition of the manifold into leaves) are [17] or [31]. We further refer to [34], which also contains a comprehensive list of further references for foliation theory in general. For the complex foliation associated to a complex structure on a manifold, we refer to the textbook sections [28, Chap. IX.2] and [35, Chap. 1.3].

Let M be a smooth manifold and let $T_{\mathbb{C}}M := TM \otimes_{\mathbb{R}} \mathbb{C}$ be the complexified tangent bundle. A section of $T_{\mathbb{C}}M$ is called a complex vector field. A complex vector field $X \in \Gamma(M, T_{\mathbb{C}}M)$ acts as a derivation $(X, f) \mapsto X(f)$ on the algebra $C^{\infty}(M)$ of complex-valued smooth functions f . For a pair of complex vector fields X, Y we can consider the commutator $[X, Y] \in \Gamma(M, T_{\mathbb{C}}M)$. It is the

unique complex vector field such that

$$[X, Y](f) = X(Y(f)) - Y(X(f))$$

for all $f \in C^\infty(M)$.

If $\mathcal{F} \subseteq T_{\mathbb{C}}M$ is a subbundle, then we have an inclusion $\Gamma(M, \mathcal{F}) \subseteq \Gamma(M, T_{\mathbb{C}}M)$ of spaces of sections.

Definition 2.2. A subbundle $\mathcal{F} \subseteq T_{\mathbb{C}}M$ is called integrable if for any two sections $X, Y \in \Gamma(M, \mathcal{F})$, we also have $[X, Y] \in \Gamma(M, \mathcal{F})$.

Definition 2.3. A foliation of a smooth manifold M is an integrable subbundle $\mathcal{F} \subseteq T_{\mathbb{C}}M$. A foliated manifold is a pair (M, \mathcal{F}) of a manifold and a foliation.

Since $T_{\mathbb{C}}M$ is the complexification of the real vector bundle TM , we have a complex antilinear involution $X \mapsto \bar{X}$. For a subbundle $\mathcal{F} \subseteq T_{\mathbb{C}}M$, we let $\bar{\mathcal{F}} \subseteq T_{\mathbb{C}}M$ denote the subbundle obtained by applying this automorphism to the elements of \mathcal{F} .

Definition 2.4. A foliation is called real if $\bar{\mathcal{F}} = \mathcal{F}$. In this case we define the real integrable subbundle $\mathcal{F}_{\mathbb{R}} := \mathcal{F} \cap TM \subseteq TM$.

Let $f : M \rightarrow N$ be a smooth map between manifolds. Its differential is a map of bundles

$$df : T_{\mathbb{C}}M \rightarrow f^*T_{\mathbb{C}}N$$

over M .

Definition 2.5. We say that $f : (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$ is a foliated map if its differential preserves the foliations in the sense that $df(\mathcal{F}) \subseteq f^*\mathcal{F}'$.

The composition of two foliated maps is again a foliated map.

Definition 2.6. We let $\mathbf{Mf}_{\mathbb{C}\text{-fol}}$ denote the category of foliated manifolds and foliated maps. We further let $\mathbf{Mf}_{\text{fol}} \subset \mathbf{Mf}_{\mathbb{C}\text{-fol}}$ be the full subcategory of foliated manifolds with real foliations.

Let \mathbf{Mf} denote the category of smooth manifolds. Then we have functors

$$\mathbf{Mf}_{\text{fol}} \rightarrow \mathbf{Mf}_{\mathbb{C}\text{-fol}} \rightarrow \mathbf{Mf},$$

where the first is the inclusion of a full subcategory, and the second forgets the foliation. The category of foliated manifolds has a cartesian product. It is given by

$$(M, \mathcal{F}) \times (M', \mathcal{F}') \cong (M \times M', \mathcal{F} \boxplus \mathcal{F}').$$

Example 2.7. If M is a complex manifold, then the subbundle $\mathcal{F} := T^{0,1}M \subseteq T_{\mathbb{C}}M$ is a complex foliation. Vice versa, a complex foliation \mathcal{F} with the additional property that $\mathcal{F} \oplus \bar{\mathcal{F}} \cong T_{\mathbb{C}}M$ equips M with a complex structure such that $T^{0,1}M = \mathcal{F}$. Moreover, a foliated map between such foliated manifolds is the same as a holomorphic map.

Example 2.8. Every manifold M has a minimal foliation $\mathcal{F}_{\min} := \{0\}$ and a maximal foliation $\mathcal{F}_{\max} := T_{\mathbb{C}}M$. These foliations are real. If M is equipped with the minimal foliation and (M', \mathcal{F}') is a foliated manifold, then every smooth map $M \rightarrow M'$ is foliated. Similarly, if M' is equipped with the maximal foliation, then for a foliated manifold (M, \mathcal{F}) , every smooth map $M \rightarrow M'$ is foliated.

Example 2.9. Let $\pi : W \rightarrow B$ be a submersion. Then the complexification of the vertical bundle $T^v\pi := \ker(d\pi) \subseteq TW$ defines a real foliation \mathcal{F}^v , called the vertical foliation. The map π is foliated for any choice of a foliation on B .

Example 2.10. Let Γ be a discrete group which acts freely and properly on a manifold \tilde{B} from the right with quotient $B := \tilde{B}/\Gamma$. Furthermore, let X be a manifold with a left action of Γ . Then we consider the manifold $M := \tilde{B} \times_{\Gamma} X$. The vertical foliations $\tilde{\mathcal{F}}^v$ and $\tilde{\mathcal{F}}^H$ associated to the projections $\tilde{B} \times X \rightarrow \tilde{B}$ and $\tilde{B} \times X \rightarrow X$, respectively, descend to the quotient and define the vertical and horizontal foliations \mathcal{F}^v and \mathcal{F}^H on M . Note that \mathcal{F}^v is the vertical foliation of the submersion $M \rightarrow B$. We have $\mathcal{F}^H \oplus \mathcal{F}^v \cong T_{\mathbb{C}}M$.

Example 2.11. Let (B, \mathcal{F}) be a foliated manifold. We call a map $f : W \rightarrow B$ transversal to \mathcal{F} if for every $w \in W$, we have the relation

$$\mathcal{F}_{f(w)} + df(TW_x) = T_{f(x)}B.$$

If f is transversal to \mathcal{F} , then we can define a maximal foliation $f^{-1}\mathcal{F}$ on W such that f becomes a foliated map. We must set $f^{-1}\mathcal{F} := df^{-1}(f^*\mathcal{F})$.

In particular, if P is a manifold, then we can consider the projection $\pi : P \times B \rightarrow B$. In this case, $\pi^{-1}\mathcal{F} = T_{\mathbb{C}}P \boxplus \mathcal{F}$.

Remark 2.12. In the main part of the present paper from Section 4 to Section 6 we will be concerned with real foliations. The more general case of complex foliations will be relevant in Section 7, where we introduce the algebraic K -theory of foliations and the regulator. The general notion of a complex foliation interpolates between the more classical real foliations on the one end, and the complex foliation associated to a complex structure on a complex manifold on the other end.

2.13. Filtrations on the de Rham complex. A foliation on a manifold induces a decreasing multiplicative filtration of the de Rham complex, see, e.g., [25, §1.1] and [34, §4].

We consider a foliated manifold (M, \mathcal{F}) . By $(\Omega(M), d)$ we denote the complexified de Rham complex of M .

Definition 2.14. For $n, p \in \mathbb{N}$, we define the subspace

$$F^p\Omega^n(M) \subseteq \Omega^n(M)$$

of forms which vanish after the insertion of $n - p + 1$ sections of \mathcal{F} .

Roughly speaking, a form belongs to $F^p\Omega^n(M)$ if it has at least p legs in the direction normal to the foliation. The family of these subspaces for all p

forms a decreasing filtration of $\Omega^n(M)$. More precisely, we have the following chain of inclusions:

$$\begin{aligned} \Omega^n(M) &= F^0\Omega^n(M) \supseteq F^1\Omega^n(M) \\ &\supseteq \dots \supseteq F^{\text{codim}(\mathcal{F})}\Omega^n(M) \supseteq F^{\text{codim}(\mathcal{F})+1}\Omega^n(M) = 0. \end{aligned}$$

Combining these filtrations for all n together, we get a decreasing filtration $(F^p\Omega(M))_{p \in \mathbb{N}}$ of the graded commutative algebra $\Omega(M)$ which is multiplicative, i.e., the wedge product restricts to maps

$$(2) \quad \wedge : F^p\Omega^m(M) \otimes F^q\Omega^n(M) \rightarrow F^{p+q}\Omega^{m+n}(M).$$

These properties are in fact true for arbitrary subbundles \mathcal{F} of $T_{\mathbb{C}}M$. But as a consequence of the integrability of \mathcal{F} , this filtration is also preserved by the de Rham differential, i.e., $(F^p\Omega(M), d)$ is a subcomplex of $(\Omega(M), d)$ for every $p \in \mathbb{N}$.

Definition 2.15. For a foliated manifold (M, \mathcal{F}) , we write $\Omega(M, \mathcal{F})$ for the de Rham complex $\Omega(M)$ considered as a filtered commutative differential graded algebra.

Example 2.16. If (M, \mathcal{F}) is a complex manifold (Example 2.7), then the filtration on $\Omega(M, T^{0,1}M)$ is called the Hodge filtration.

If $f : (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$ is a foliated map, then $f^* : \Omega(M', \mathcal{F}') \rightarrow \Omega(M, \mathcal{F})$ is a morphism of filtered commutative differential graded algebras.

We let **CDGA** and **CDGA^{filt}** denote the categories of graded commutative differential graded algebras and filtered graded commutative differential graded algebras, respectively. For categories **C**, **D**, we can consider the functor category **Fun(C^{op}, D)**.¹ We can formalize the properties of the filtered de Rham complex discussed above by saying that we have a functor

$$\Omega \in \mathbf{Fun}(\mathbf{Mf}_{\mathbb{C}\text{-fol}}^{\text{op}}, \mathbf{CDGA}^{\text{filt}}).$$

Definition 2.17. We define the functors

$$DD^- \in \mathbf{Fun}(\mathbf{Mf}_{\mathbb{C}\text{-fol}}^{\text{op}}, \mathbf{CDGA}), \quad DD^{\text{per}} \in \mathbf{Fun}(\mathbf{Mf}^{\text{op}}, \mathbf{CDGA})$$

by

$$DD^-(M, \mathcal{F}) := \prod_{p \in \mathbb{Z}} F^p\Omega(M)[2p], \quad DD^{\text{per}} := \prod_{p \in \mathbb{Z}} \Omega(M)[2p].$$

We call DD^{per} the periodic and DD^- the negative de Rham complex. Note that DD^- has a decomposition into a product of components, i.e.,

$$DD^-(M, \mathcal{F}) \cong \prod_{p \in \mathbb{Z}} DD^-(M, \mathcal{F})(p), \quad DD^-(M, \mathcal{F})(p) := F^p\Omega(M)[2p],$$

¹Later in the paper we will also use the notation $\mathbf{PSh}_{\mathbf{D}}(\mathbf{C}) := \mathbf{Fun}(\mathbf{C}^{\text{op}}, \mathbf{D})$ and call these objects **D**-valued presheaves on **C**.

and the product on DD^- is induced by the wedge products of forms (2) componentwise as

$$DD^-(M, \mathcal{F})(p) \otimes DD^-(M, \mathcal{F})(q) \rightarrow DD^-(M, \mathcal{F})(p + q).$$

The description of the product for DD^{per} is similar.

Remark 2.18. The cohomology of $DD^{\text{per}}(M)$ is the two-periodic de Rham cohomology of M . It is the natural target of the Chern character from topological K -theory, see Definition 2.31. The complex $DD^-(M, \mathcal{F})$ will receive characteristic forms for vector bundles with connections which are flat in the direction of the foliation, see Definition 2.37.

2.19. Vector bundles with flat partial connections. We introduce the notion of a vector bundle with a flat partial connection on a foliated manifold. While a connection (see, e.g., [27, Chap. III]) allows to differentiate sections in all directions of the manifold, a partial connection only differentiates in the direction of the foliation. The typical example of a flat partial connection is a holomorphic structure $\bar{\partial}$ on a complex vector bundle on a complex manifold [35, Chap. III.2].

We consider a foliated manifold (M, \mathcal{F}) . Let $V \rightarrow M$ be a complex vector bundle.

Definition 2.20. A partial connection on V is a map

$$\nabla : \Gamma(M, V) \rightarrow \Gamma(M, \mathcal{F}^* \otimes V)$$

which satisfies the Leibniz rule.

Remark 2.21. For sections $X \in \Gamma(M, \mathcal{F})$ and $\phi \in \Gamma(M, V)$ we write as usual

$$\nabla_X \phi := i_X(\nabla(\phi)) \in \Gamma(M, V)$$

for the evaluation of $\nabla\phi$ at X . With this notation the Leibniz rule has the form

$$\nabla_X(f\phi) = X(f)\phi + f\nabla_X\phi$$

for all $f \in C^\infty(M)$, $\phi \in \Gamma(M, V)$ and $X \in \Gamma(M, \mathcal{F})$.

The foliation gives rise to a graded commutative differential graded algebra whose underlying commutative graded algebra is given by $\Omega(\mathcal{F}) := \Gamma(M, \Lambda^*\mathcal{F})$. Its differential $d^{\mathcal{F}}$ is fixed by the prescription

$$(3) \quad d^{\mathcal{F}} : \Omega^0(\mathcal{F}) \rightarrow \Omega^1(\mathcal{F}), \quad d^{\mathcal{F}}(\phi) := d\phi|_{\mathcal{F}},$$

where d is the usual de Rham differential and we use the identification $\Omega^0(\mathcal{F}) = C^\infty(M) = \Omega^0(M)$. We further write $\Omega(\mathcal{F}, V) := \Gamma(M, \Lambda^*\mathcal{F} \otimes V)$. As in the case of usual connections we can extend ∇ uniquely to a derivation on the $\Omega(\mathcal{F})$ -module $\Omega(\mathcal{F}, V)$. Its curvature, defined by

$$R^\nabla := \nabla^2 \in \text{End}(\Omega(\mathcal{F}, V)),$$

is $\Omega(\mathcal{F})$ -linear and hence a two-form on \mathcal{F} with values in $\text{End}(V)$, i.e., we have $R^\nabla \in \Omega^2(\mathcal{F}, \text{End}(V))$.

Definition 2.22. A partial connection ∇ on V is called flat if $R^\nabla = 0$.

We now consider a foliated map $f : (M', \mathcal{F}') \rightarrow (M, \mathcal{F})$. If $V \rightarrow M$ is a vector bundle with a partial connection ∇ , then f^*V has an induced partial connection $f^*\nabla$. It is characterized by

$$(4) \quad \nabla_X(f^*\phi) = f^*\nabla_{df(X)}\phi \quad \text{for all } m' \in M', X \in T_{m'}M' \text{ and } \phi \in \Gamma(M, V).$$

This formula has to be understood as an equality between elements in the fibre $(f^*V)_{m'}$. Because of the relation

$$f^*R^\nabla = R^{f^*\nabla},$$

the pullback of a flat partial connection is again flat.

Example 2.23. If M is a complex manifold with foliation $\mathcal{F} = T^{0,1}M$, then a flat partial connection on a complex vector bundle V is the same as a holomorphic structure. In this situation the flat partial connection is usually denoted by $\bar{\partial}$.

Example 2.24. If M is equipped with the minimal foliation, then a partial connection on a vector bundle is no additional data. In the opposite case, where M has the maximal foliation, a flat partial connection is the same as a flat connection.

Example 2.25. Let

$$\mathcal{F}^\perp := T_{\mathbb{C}}M/\mathcal{F}$$

be the normal bundle of a foliation. Then \mathcal{F}^\perp has a natural flat partial connection $\nabla^{\mathcal{F}^\perp}$. It is given by

$$\nabla_X[Y] := [[X, Y]],$$

where $X \in \Gamma(M, \mathcal{F})$ and the vector field $Y \in \Gamma(M, T_{\mathbb{C}}M)$ represents the section $[Y] \in \Gamma(M, \mathcal{F}^\perp)$ of the normal bundle.

Example 2.26. Let $\pi : W \rightarrow B$ be a submersion and consider the vertical foliation $T^v\pi$ on W , see Example 2.9. If $V \rightarrow B$ is any vector bundle, then $\pi^*V \rightarrow W$ has a canonical flat partial connection $\nabla^I = \pi^*\nabla$, where ∇ is the canonical flat partial connection on V in the direction of the trivial foliation. In view of (4), it is characterized by the condition that for $\phi \in \Gamma(B, V)$, we have $\nabla^I\pi^*\phi = 0$.

2.27. Connections and characteristic forms. We introduce the Chern character forms and Chern forms of complex vector bundles with connection (see, e.g., [6, §1.5] and [35, §III.3]). In the foliated case, we discuss the consequences of the fact that the connection extends a flat partial connection. These consequences are expressed in terms of the position of the characteristic forms with respect to the filtration of the de Rham complex introduced in Section 2.13, and they are the classical starting point for the construction of characteristic classes for foliations, see, e.g., [7, 23, 22, 25].

Let (M, \mathcal{F}) be a foliated manifold and let (V, ∇^I) be a complex vector bundle with a flat partial connection.

Definition 2.28. A connection ∇ on V is an extension of ∇^I , if the relation $\nabla_X \phi = \nabla_X^I \phi$ holds for all $\phi \in \Gamma(M, V)$ and $X \in \Gamma(M, \mathcal{F})$.

One can show that a flat partial connection admits extensions. Furthermore, the set of extensions of a flat partial connection is a torsor over the complex vector space

$$\Gamma(M, \mathcal{F}^{\perp,*} \otimes \text{End}(V)).$$

Example 2.29. A connection on \mathcal{F}^\perp which extends the flat partial connection $\nabla^{I, \mathcal{F}^\perp}$ of Example 2.25 is called a Bott connection.

Example 2.30. Let $f : W \rightarrow B$ be a submersion and let \mathcal{F}^v be the vertical foliation (Example 2.9). If $V \rightarrow B$ is a complex vector bundle, then $f^*V \rightarrow W$ has a canonical flat partial connection ∇^I , see Example 2.26. If ∇ is any connection on V , then $f^*\nabla$ extends ∇^I .

More generally, if $f : W \rightarrow B$ is transverse to a foliation \mathcal{F} on B and (V, ∇^I) is a vector bundle with flat partial connection on (B, \mathcal{F}) , then $(f^*V, f^*\nabla^I)$ is a vector bundle with flat partial connection on $(W, f^{-1}\mathcal{F})$, see Example 2.11. If ∇ is a connection on V extending ∇^I , then $f^*\nabla$ is a connection on f^*V extending $f^*\nabla^I$.

If ∇ is a connection on a complex vector bundle, then we consider its curvature

$$R^\nabla := \nabla^2 \in \Omega^2(M, \text{End}(V)).$$

The Chern character form of ∇ is the closed inhomogeneous complex-valued form

$$\text{ch}_0(\nabla) + \text{ch}_2(\nabla) + \text{ch}_4(\nabla) + \dots := \text{Tr} \exp\left(-\frac{R^\nabla}{2\pi i}\right),$$

with homogeneous components $\text{ch}_{2p}(\nabla) \in \Omega_{\text{cl}}^{2p}(M)$. We will consider the Chern character form as a zero cycle in the periodic complex $DD^{\text{per}}(M)$.

Definition 2.31. We define

$$\text{ch}(\nabla) := (\text{ch}_{2p}(\nabla))_{p \in \mathbb{Z}} \in Z^0(DD^{\text{per}}(M)).$$

Remark 2.32. In this remark we explain how the Chern character form behaves under complex conjugation and inserting adjoint connections. First of all, if we choose a hermitean metric h on the bundle V with connection ∇ , then we can form the adjoint connection ∇^* , which is characterized by the following relation in $\Omega^1(M)$:

$$dh(\phi, \psi) = h(\nabla\phi, \psi) + h(\phi, \nabla^*\psi) \quad \text{for all } \phi, \psi \in \Gamma(M, V).$$

Applying d to this equality again, we get

$$0 = h(R^\nabla\phi, \psi) + h(\phi, R^{\nabla^*}\psi).$$

In view of the $2\pi i$ -factor in the definition of the Chern character form this equality implies the relation

$$(5) \quad \overline{\text{ch}(\nabla)} = \text{ch}(\nabla^*).$$

The connection ∇ is called unitary if $\nabla^* = \nabla$. In this case, the Chern character form $\mathbf{ch}(\nabla)$ is real.

Let (V, ∇^I) be a complex vector bundle on (M, \mathcal{F}) with a flat partial connection ∇^I . We further choose a hermitean metric h on V .

Definition 2.33. A flat partial connection ∇^I is called unitary (with respect to h), if

$$d^{\mathcal{F}}h(\phi, \psi) = h(\nabla^I\phi, \psi) + h(\phi, \nabla^I\psi)$$

for all $\phi, \psi \in \Gamma(M, V)$. See (3) for the definition of $d^{\mathcal{F}}$.

Lemma 2.34. *If ∇^I is unitary (with respect to h), then it admits a unitary extension ∇ .*

Proof. Let ∇_0 be some extension of ∇^I . Then ∇_0^* is a second extension of ∇^I and

$$\nabla := \frac{1}{2}(\nabla_0 + \nabla_0^*)$$

is a unitary extension of ∇^I . □

There is a filtration on $\Omega(M, \mathbf{End}(V))$ defined similarly to the filtration of $\Omega(M, \mathcal{F})$ introduced in Definition 2.14. It is compatible with the $\Omega(M, \mathcal{F})$ -module structure.

Lemma 2.35. *If ∇ extends a flat partial connection ∇^I on V , then*

$$R^\nabla \in F^1\Omega^2(M, \mathbf{End}(V)).$$

Proof. We have

$$R_{|\Lambda^2\mathcal{F}}^\nabla = R^{\nabla^I} = 0. \quad \square$$

This lemma has consequences for the Chern character forms.

Corollary 2.36. *If ∇ extends a flat partial connection on V , then*

$$\mathbf{ch}_{2p}(\nabla) \in F^p\Omega_{\mathfrak{cl}}^{2p}(M, \mathcal{F}).$$

Definition 2.37. If ∇ extends a flat partial connection on V , then we define

$$\mathbf{ch}^-(\nabla) := (\mathbf{ch}_{2p}(\nabla))_{p \in \mathbb{Z}} \in Z^0(DD^-(M, \mathcal{F})).$$

We let $\mathbf{Vect}^{\text{flat}, \nabla}(M, \mathcal{F})$ and $\mathbf{Vect}^\nabla(M)$ denote the symmetric monoidal categories (with respect to the direct sum) of pairs (V, ∇) of complex vector bundles with connection, where in the first case ∇ extends a flat partial connection. In both cases, morphisms are connection preserving vector bundle morphisms.

If $f : M' \rightarrow M$ is a smooth map and $(V, \nabla) \in \mathbf{Vect}^\nabla(M)$, then we can define $(f^*V, f^*\nabla) \in \mathbf{Vect}^\nabla(M')$ and have the relation

$$(6) \quad f^*\mathbf{ch}(\nabla) = \mathbf{ch}(f^*\nabla).$$

Similarly, if $f : (M', \mathcal{F}') \rightarrow (M, \mathcal{F})$ is a foliated map and (V, ∇) belongs to $\mathbf{Vect}^{\text{flat}, \nabla}(M, \mathcal{F})$, then $(f^*V, f^*\nabla) \in \mathbf{Vect}^{\text{flat}, \nabla}(M', \mathcal{F}')$ and we have the relation

$$(7) \quad \mathbf{ch}^-(f^*\nabla) = f^*\mathbf{ch}^-(\nabla).$$

Thus, the Chern character forms are characteristic forms. In addition, they are additive, i.e., the Chern character form of a direct sum is the sum of the Chern character forms of the summands. These properties will be important for the construction of the regulator in Section 6.8.

Let (V, ∇) be a complex vector bundle with connection on a manifold M . Then we define the Chern forms

$$c_p(\nabla) \in \Omega^{2p}(M)$$

of ∇ as the homogeneous components of the following inhomogenous form:

$$1 - c_1(\nabla) + c_2(\nabla) - \dots = \det\left(1 - \frac{R^\nabla}{2\pi i}\right).$$

The Chern forms can be expressed as homogeneous polynomials in the Chern character forms. In particular, if (M, \mathcal{F}) is a foliated manifold and ∇ extends a flat partial connection, then we have

$$(8) \quad c_p(\nabla) \in F^p\Omega_{\mathbb{C}\mathbb{1}}^{2p}(M, \mathcal{F}).$$

2.38. Characteristic forms of real foliations. We introduce characteristic forms of real vector bundles on real foliated manifolds. We, in particular, discuss the $\hat{\mathbf{A}}$ -form (see, e.g., [6, §1.5]).

We consider a real foliated manifold (M, \mathcal{F}) . If $V \rightarrow M$ is a real vector bundle on a real foliated manifold (M, \mathcal{F}) , then in analogy with Definitions 2.20 and 2.22, we have the notion of a flat partial connection

$$\nabla^I : \Gamma(M, V) \rightarrow \Gamma(M, \mathcal{F}_{\mathbb{R}}^* \otimes V)$$

on V . We furthermore have the notion of a connection ∇ on V extending ∇^I (compare with Definition 2.28).

Let $V \rightarrow M$ be a real vector bundle with connection ∇ . We let $\nabla_{\mathbb{C}}$ denote the induced connection on the complexification $V \otimes \mathbb{C}$. We define the Pontryagin forms of ∇ by

$$p_k(\nabla) := (-1)^k c_{2k}(\nabla_{\mathbb{C}}) \in \Omega_{\mathbb{C}\mathbb{1}}^{4k}(M).$$

If \mathcal{F} is a real foliation and ∇ extends a flat partial connection, then by (8) we have

$$(9) \quad p_k(\nabla) \in F^{2k}\Omega_{\mathbb{C}\mathbb{1}}^{4k}(M, \mathcal{F}).$$

In order to define the $\hat{\mathbf{A}}$ -form, we consider the symmetric polynomials p_i of degree $4i$ in variables x_ℓ of degree 2, defined by the relation

$$1 + p_1 + p_2 + \dots = \prod_{\ell} (1 - x_\ell^2).$$

We define homogeneous polynomials $\hat{\mathbf{A}}_{4k}$ for $k \geq 1$ in the variables p_i by the relation

$$1 + \hat{\mathbf{A}}_4 + \hat{\mathbf{A}}_8 + \cdots = \prod_{\ell} \frac{\frac{x_{\ell}}{2}}{\sinh(\frac{x_{\ell}}{2})}.$$

Then the components of the $\hat{\mathbf{A}}$ -form are defined by

$$\hat{\mathbf{A}}_{4k}(\nabla) := \hat{\mathbf{A}}_{4k}(p_1(\nabla), p_2(\nabla), \dots) \in \Omega_{\text{cl}}^{4k}(M).$$

If (M, \mathcal{F}) is real foliated and the connection ∇ extends a flat partial connection, then by (9) we have

$$\hat{\mathbf{A}}_{4k}(\nabla) \in F^{2k} \Omega_{\text{cl}}^{4k}(M, \mathcal{F})$$

We again want to consider the $\hat{\mathbf{A}}$ -form as a zero cycle of $DD^{\text{per}}(M)$, or of $DD^-(M, \mathcal{F})$ in the foliated case. In order to simplify the notation, we set $\hat{\mathbf{A}}_{2k}(\nabla) := 0$ if k is odd.

Definition 2.39. If ∇ is a connection on a real vector bundle on M , then we define

$$\hat{\mathbf{A}}(\nabla) := (\hat{\mathbf{A}}_{2p}(\nabla))_{p \in \mathbb{Z}} \in Z^0(DD^{\text{per}}(M)).$$

If (M, \mathcal{F}) is a real foliated manifold and ∇ extends a flat partial connection, then we define

$$\hat{\mathbf{A}}^-(\nabla) := (\hat{\mathbf{A}}_{2p}(\nabla))_{p \in \mathbb{Z}} \in Z^0(DD^-(M, \mathcal{F})).$$

The $\hat{\mathbf{A}}$ -form is multiplicative, i.e., the $\hat{\mathbf{A}}$ -form of a direct sum of connections is the product of the $\hat{\mathbf{A}}$ -forms of the summands. Furthermore, the $\hat{\mathbf{A}}$ -form of a trivial connection is the multiplicative unit.

Example 2.40. If (M, \mathcal{F}) is a real foliated manifold, then the real normal bundle

$$\mathcal{F}_{\mathbb{R}}^{\perp} := TM/\mathcal{F}_{\mathbb{R}}$$

of the foliation has a flat partial connection $\nabla^{I, \mathcal{F}_{\mathbb{R}}^{\perp}}$, similar as in Example 2.25. If we choose a connection $\nabla^{\mathcal{F}_{\mathbb{R}}^{\perp}}$ extending $\nabla^{I, \mathcal{F}_{\mathbb{R}}^{\perp}}$, then we obtain a cycle

$$\hat{\mathbf{A}}^-(\nabla^{\mathcal{F}_{\mathbb{R}}^{\perp}}) \in Z^0(DD^-(M, \mathcal{F})).$$

2.41. Transgression. We introduce the transgression (see [6, §1.5]) of characteristic forms and discuss its basic properties. In the case of foliations, we discuss the consequences of the fact that the connections extend a fixed flat partial connection (see, e.g., [25, §4]). In this section we also introduce the notion of a stable framing and the conventions for the associated trivial connection.

We consider the unit interval $I := [0, 1]$ with coordinate t . For $i = 0, 1$, let $\iota_i : * \rightarrow I$ be the inclusions of the endpoints of the interval. Let M be a smooth manifold and let $V \rightarrow M$ be a vector bundle. Given two connections ∇_0 and ∇_1 , we can consider a connection $\tilde{\nabla}$ on $\pi^*V \rightarrow I \times M$ such that $(\iota_i \times \text{id}_M)^* \tilde{\nabla} = \nabla_i$ for $i = 0, 1$. For example, we could take the linear interpolation $t\pi^*\nabla_1 + (1-t)\pi^*\nabla_0$.

The integration of forms along the fibre of $\pi : I \times M \rightarrow M$ is a map of graded vector spaces

$$\int_{I \times M/M} : \Omega(I \times M) \rightarrow \Omega(M)[-1].$$

It induces a map

$$\int_{I \times M/M} : DD^{\text{per}}(I \times M) \rightarrow DD^{\text{per}}(M)[-1].$$

Since the interval I has a nonempty boundary, the integration is not a morphism of complexes. In fact, by Stokes' theorem, we have the relation

$$(10) \quad (\iota_1 \times \text{id}_M)^* - (\iota_0 \times \text{id}_M)^* = d \circ \int_{I \times M/M} + \int_{I \times M/M} \circ d.$$

Definition 2.42. The transgression of the Chern character form is defined by

$$\widetilde{\text{ch}}(\nabla_1, \nabla_0) := \int_{I \times M/M} \text{ch}(\widetilde{\nabla}) \in DD^{\text{per}}(M)^{-1}/\text{im}(d).$$

From (10) and the fact that the Chern character forms are closed and natural (see (6)), we immediately get the identity

$$(11) \quad d\widetilde{\text{ch}}(\nabla_1, \nabla_0) := \text{ch}(\nabla_1) - \text{ch}(\nabla_0).$$

One can check that the transgression is independent of the choice of the connection $\widetilde{\nabla}$ interpolating between ∇_0 and ∇_1 . At this point it is relevant that we consider the transgression as a class modulo exact forms. Furthermore, we have the identities

$$(12) \quad \begin{cases} \widetilde{\text{ch}}(\nabla_1, \nabla_0) + \text{ch}(\nabla_2, \nabla_1) + \widetilde{\text{ch}}(\nabla_0, \nabla_2) = 0, \\ \widetilde{\text{ch}}(\nabla_1, \nabla_0) + \widetilde{\text{ch}}(\nabla_0, \nabla_1) = 0. \end{cases}$$

In order to see e.g. the first equality in (12) one can integrate the Chern form of an interpolation between the three connections over a two-simplex.

Remark 2.43. If we choose a hermitean metric on V , then we can form the adjoint connections. From (5) we get the relation

$$(13) \quad \overline{\widetilde{\text{ch}}(\nabla_1, \nabla_0)} = \widetilde{\text{ch}}(\nabla_1^*, \nabla_0^*).$$

We now assume that (M, \mathcal{F}) is foliated and that the connections ∇_i for $i = 0, 1$ extend the same flat partial connection ∇^I . Then we can equip $I \times M$ with the foliation $T_{\mathbb{C}}I \boxplus \mathcal{F}$, introduced in Example 2.11. We can furthermore find an interpolation $\widetilde{\nabla}$ which extends the flat partial connection $\pi^*\nabla^I$, e.g., the linear interpolation.

We now observe that the integration preserves the filtration, i.e., that it induces a map

$$\int_{I \times M/M} : F^p\Omega^k(I \times M, T_{\mathbb{C}}I \boxplus \mathcal{F}) \rightarrow F^p\Omega^{k-1}(M, \mathcal{F}).$$

At this point it is crucial that we included the tangent bundle of the interval into the foliation. Hence, we get an induced map

$$\int_{I \times M/M} : DD^-(I \times M, T_{\mathbb{C}}I \boxplus \mathcal{F}) \rightarrow DD^-(M, \mathcal{F})[-1].$$

Definition 2.44. Let (M, \mathcal{F}) be a foliated manifold and let (V, ∇^I) be a complex vector bundle with a flat partial connection on M . If ∇_0 and ∇_1 are two connections on V extending ∇^I , then we define the transgression of the Chern character form by

$$\widetilde{\mathbf{ch}}^-(\nabla_1, \nabla_0) := \int_{I \times M/M} \mathbf{ch}^-(\widetilde{\nabla}) \in DD^-(M, \mathcal{F})^{-1}/\mathfrak{im}(d).$$

Note again that $\widetilde{\mathbf{ch}}^-(\nabla_1, \nabla_0)$ is independent of the choice of the interpolation $\widetilde{\nabla}$.

Example 2.45. We consider a foliated manifold (M, \mathcal{F}) and a complex vector bundle $V \rightarrow M$. If $p \in \mathbb{N}$ is such that $p > \text{codim}(\mathcal{F})$, then we have $F^p\Omega^{2p}(M) = 0$.

Assume that ∇_0^I and ∇_1^I are two flat partial connections on a complex vector bundle V and let ∇_0, ∇_1 be corresponding extensions. If $p > \text{codim}(\mathcal{F})$, then $\widetilde{\mathbf{ch}}_{2p}(\nabla_1, \nabla_0)$ is closed, since, by (10) and Corollary 2.36, its differential belongs to $F^p\Omega^{2p}(M) = 0$. Its cohomology class does not depend on the choice of the extensions ∇_1 and ∇_0 . We therefore get a secondary characteristic class

$$c(\nabla_1^I, \nabla_0^I) := \widetilde{\mathbf{ch}}_{2p}(\nabla_1, \nabla_0) \in H^{2p-1}(M; \mathbb{C}).$$

Example 2.46. The following construction generalizes the Kamber–Tondeur classes (introduced in this form in [8]) to the foliated case. Let ∇^I be a flat partial connection. If we choose a hermitean metric h^V on V , then, similarly as in the case of connections (see Remark 2.32), we can define an adjoint flat partial connection $\nabla^{I,*}$. It is characterized by the relation

$$d^{\mathcal{F}}h(\phi, \psi) = h(\nabla^I\phi, \psi) + h(\phi, \nabla^{I,*}\psi), \quad \phi, \psi \in \Gamma_c(M, V).$$

Let ∇ be an extension of ∇^I . Then ∇^* extends $\nabla^{I,*}$. We consider the form

$$\widetilde{\mathbf{ch}}_{2p}(\nabla, \nabla^*) \in F^p\Omega^{2p-1}(M, \mathcal{F})/\mathfrak{im}(d).$$

By (13) and (12), we have the relation

$$(14) \quad \overline{\widetilde{\mathbf{ch}}_{2p}(\nabla, \nabla^*)} = \widetilde{\mathbf{ch}}_{2p}(\nabla^*, \nabla) = -\widetilde{\mathbf{ch}}_{2p}(\nabla, \nabla^*),$$

i.e., the form $\widetilde{\mathbf{ch}}_{2p}(\nabla, \nabla^*)$ is imaginary.

For $p > \text{codim}(\mathcal{F})$, the class

$$c_{2p-1}(\nabla^I) := c(\nabla^I, \nabla^{I,*}) \in H^{2p-1}(M; \mathbb{C})$$

does not depend on the choice of the hermitean metric. By (14), it is imaginary, i.e., it belongs to the real subspace $iH^{2p-1}(M; \mathbb{R}) \subseteq H^{2p-1}(M; \mathbb{C})$.

We can apply this construction to the bundle \mathcal{F}^\perp with its canonical flat partial connection $\nabla^{I, \mathcal{F}^\perp}$, see Example 2.25. The class

$$(15) \quad c_{2p-1}(\nabla^{I, \mathcal{F}^\perp}) \in H^{2p-1}(M; \mathbb{C})$$

is closely related to the Godbillon–Vey class of the foliation. If the foliation \mathcal{F} is real, then we can explain the place of this invariant in the classification of characteristic classes for foliations. See Remark 5.21, in particular, (43).

Let $V \rightarrow M$ be a real vector bundle. Then, using a similar notation as above, we can define

$$(16) \quad \tilde{\mathbf{A}}(\nabla_1, \nabla_0) := \int_{I \times M/M} \hat{\mathbf{A}}(\tilde{\nabla}) \in DD^{\text{per}}(M)^{-1}/\text{im}(d).$$

We have

$$(17) \quad d\tilde{\mathbf{A}}(\nabla_1, \nabla_0) := \hat{\mathbf{A}}(\nabla_1) - \hat{\mathbf{A}}(\nabla_0).$$

The transgression is independent of the choice of the connection $\tilde{\nabla}$. Furthermore, we have the identities

$$\tilde{\mathbf{A}}(\nabla_1, \nabla_0) + \tilde{\mathbf{A}}(\nabla_2, \nabla_1) + \tilde{\mathbf{A}}(\nabla_0, \nabla_2) = 0, \quad \tilde{\mathbf{A}}(\nabla_1, \nabla_0) + \tilde{\mathbf{A}}(\nabla_0, \nabla_1) = 0.$$

Finally, if ∇_1 and ∇_0 extend the same flat partial connection, then we can define

$$\widetilde{\mathbf{A}}^-(\nabla_1, \nabla_0) := \int_{I \times M/M} \hat{\mathbf{A}}(\tilde{\nabla}) \in DD^-(M)^{-1}/\text{im}(d).$$

Let $V \rightarrow M$ be a real vector bundle. By $V \oplus \mathbb{R}^n$ we denote the sum of V with the trivial vector bundle of dimension n .

Definition 2.47. A stable framing of V is a trivialization of $V \oplus \mathbb{R}^n$ for some $n \in \mathbb{N}$.

A trivialization of $V \oplus \mathbb{R}^n$ induces a trivialization of $V \oplus \mathbb{R}^{n+k}$ in a natural way. We will consider the corresponding two stable framings as equivalent. In particular, we can always arrange that two stable framings of V trivialize the same bundle. A stable framing as above naturally induces a flat connection on $V \oplus \mathbb{R}^n$, which will be called the associated stable trivial connection and denoted by $\nabla^{V, \text{triv}}$ or ∇^s . If ∇^V is another connection on V and we want to consider the transgression of a characteristic form between ∇^V and $\nabla^{V, \text{triv}}$, then we secretly extend ∇^V to $V \oplus \mathbb{R}^n$ using the trivial connection on the second summand. The transgression will be invariant under further stabilization.

We now come back to the foliated manifold (M, \mathcal{F}) and assume that the foliation \mathcal{F} is real. Let g^{TM} be a Riemannian metric on M . We get a decomposition $TM \cong \mathcal{F}_\mathbb{R} \oplus \mathcal{F}_\mathbb{R}^\perp$ and choose a connection $\nabla^{\mathcal{F}_\mathbb{R}^\perp}$ extending $\nabla^{I, \mathcal{F}_\mathbb{R}^\perp}$. We

further assume that $\mathcal{F}_\mathbb{R}$ has a stable framing s . Let $\nabla^{\mathcal{F}_\mathbb{R},\text{triv}}$ be the associated stable trivial connection on $\mathcal{F}_\mathbb{R}$. We have forms

$$\widetilde{\mathbf{A}}(\nabla^{LC}, \nabla^{\mathcal{F}_\mathbb{R},\text{triv}} \oplus \nabla^{\mathcal{F}_\mathbb{R}^\perp}) \in DD^{\text{per}}(M)^{-1}/\text{im}(d)$$

and

$$\widehat{\mathbf{A}}(\nabla^{\mathcal{F}_\mathbb{R},\text{triv}} \oplus \nabla^{\mathcal{F}_\mathbb{R}^\perp}) = \widehat{\mathbf{A}}(\nabla^{\mathcal{F}_\mathbb{R}^\perp}) \in Z^0(DD^-(M)).$$

3. DIFFERENTIAL K -THEORY

In this subsection we recall some basic features of the Hopkins–Singer version of differential complex K -theory.

References for the following material are the foundational paper by Hopkins and Singer [20], but also [33] and [10]. For differential orientations and Umkehr maps, we refer to [10, 16, 13]

3.1. Basic structures. We describe the differential extension \widehat{KU}^* of the generalized cohomology theory KU^* . Here KU^* is the periodic topological complex K -theory which is represented by the spectrum KU . For every $p \in \mathbb{Z}$, we have a contravariant functor

$$\widehat{KU}^p : \mathbf{Mf}^{\text{op}} \rightarrow \mathbf{Ab},$$

from smooth manifolds to abelian groups. This functor is connected with the periodic topological complex K -theory KU^* via a transformation

$$I : \widehat{KU}^p \rightarrow KU^p$$

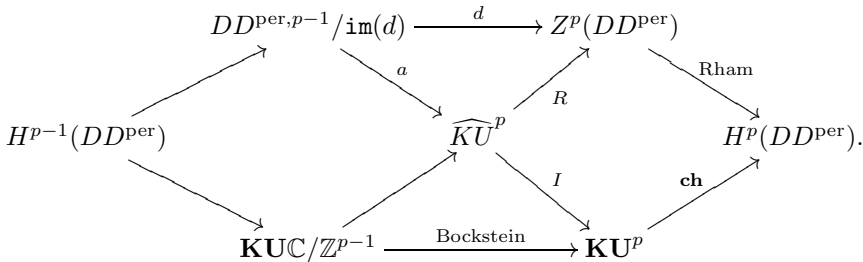
of abelian group valued functors. The transformation I maps differential K -theory classes to their underlying topological K -theory classes. Furthermore, differential K -theory is connected with differential forms through natural transformations R and a . The curvature R is a natural transformation

$$R : \widehat{KU}^p \rightarrow Z^p(DD^{\text{per}}(M)).$$

In particular, if $x \in \widehat{KU}^p(M)$, then $R(x) \in Z^p(DD^{\text{per}}(M))$ is a differential form representing the Chern character of the underlying topological class $I(x)$. The transformation

$$a : DD^{\text{per},p-1}/\text{im}(d) \rightarrow \widehat{KU}^p$$

encodes the secondary information contained in differential K -theory classes. All these structures and their compatibilities are nicely encoded in the following commutative diagram, also called the differential cohomology diagram [32]:



Its upper and lower parts are segments of long exact sequences, and the diagonals are exact at the center.

The flat part of differential K -theory is defined as the kernel of the curvature transformation R :

$$\widehat{KU}_{\text{flat}}^p := \ker(R : \widehat{KU}^p \rightarrow Z^p(DD^{\text{per}})).$$

It is canonically isomorphic to K -theory with coefficients in \mathbb{C}/\mathbb{Z} (with a shift):

$$\widehat{KU}_{\text{flat}}^p \cong \mathbf{KUC}/\mathbb{Z}^{p-1}.$$

Here \mathbf{KUC}/\mathbb{Z} is an abbreviation for the spectrum $\mathbf{KU} \wedge \mathbf{MC}/\mathbb{Z}$, where \mathbf{MC}/\mathbb{Z} is the Moore spectrum for the abelian group \mathbb{C}/\mathbb{Z} .

The sequence

$$\mathbf{KU}^{p-1} \xrightarrow{\text{ch}} DD^{\text{per}, p-1}/\text{im}(d) \xrightarrow{a} \widehat{KU}^p \xrightarrow{I} \mathbf{KU}^p \rightarrow 0$$

is exact.

The differential K -theory of a point is given by

$$(18) \quad \widehat{KU}^p(*) \cong \begin{cases} \mathbb{Z} & \text{if } p \text{ is even,} \\ \mathbb{C}/\mathbb{Z} & \text{if } p \text{ is odd.} \end{cases}$$

Differential K -theory is not homotopy invariant. The deviation from homotopy invariance is quantified by the homotopy formula. If $\hat{x} \in \widehat{KU}^p([0, 1] \times M)$, then the homotopy formula states that

$$(19) \quad (\iota_1 \times \text{id}_M)^*\hat{x} - (\iota_0 \times \text{id}_M)^*\hat{x} = a\left(\int_{[0,1] \times M/M} R(\hat{x})\right).$$

3.2. The cycle map. A complex vector bundle with connection (V, ∇) on a manifold M gives rise to a differential K -theory class $[V, \nabla] \in \widehat{KU}^0(M)$ such that

$$(20) \quad R([V, \nabla]) = \text{ch}(\nabla) \in Z^0(DD^{\text{per}}(M)), \quad I([V, \nabla]) = [V] \in \mathbf{KU}^0(M).$$

Let $M \mapsto \pi_0(\mathbf{Vect}^\nabla(M))$ denote the contravariant functor which associates to the manifold M the commutative monoid (induced by the direct sum) of isomorphism classes of pairs (V, ∇) of vector bundles with connection on M and to a smooth map between manifolds $f : M \rightarrow M'$, the pullback f^* . The additive natural transformation

$$[\dots] : \pi_0(\mathbf{Vect}^\nabla) \rightarrow \widehat{KU}^0$$

is called the cycle map and fits into the commuting diagram of natural transformations between monoid-valued functors

$$\begin{array}{ccc}
 & & Z^0(DD^{\text{per}}) \\
 & [V, \nabla] \mapsto \text{ch}(\nabla) \nearrow & \uparrow R \\
 \pi_0(\mathbf{Vect}^{\nabla}) & \xrightarrow{[\dots]} & \widehat{KU}^0 \\
 & [V, \nabla] \mapsto [V] \searrow & \downarrow I \\
 & & \mathbf{KU}^0.
 \end{array}$$

For compact manifolds M , the cycle map is known to be surjective [33]. Assume that ∇_0 and ∇_1 are two connections on the same complex vector bundle V . Then as a consequence of the homotopy formula (19), we have the equality

$$(21) \quad [V, \nabla_1] - [V, \nabla_0] = a(\tilde{\text{ch}}(\nabla_1, \nabla_0))$$

in $\widehat{KU}^0(M)$.

3.3. Differential orientations and Umkehr maps. Let $\pi : W \rightarrow B$ be a proper submersion such that the vertical bundle $T^v\pi$ has a Spin^c -structure. Then π is equipped with an orientation o (called the Atiyah–Bott–Shapiro orientation [1]) for the cohomology theory \mathbf{KU}^* and admits an Umkehr map

$$\pi_!^o : \mathbf{KU}^p(W) \rightarrow \mathbf{KU}^{p-d}(B),$$

where $d := \dim(W) - \dim(B)$ is the dimension of the fibre (assume for simplicity that B is connected). Since \mathbf{KUC}/\mathbb{Z} is a \mathbf{KU} -module spectrum, we also have an integration

$$(22) \quad \pi_!^o : \mathbf{KUC}/\mathbb{Z}^p(W) \rightarrow \mathbf{KUC}/\mathbb{Z}^{p-d}(B).$$

The \mathbf{KU} -orientation o determines a cohomology class

$$\hat{\mathbf{A}}(o) \in H^0(DD^{\text{per}}(W))$$

such that the Riemann–Roch theorem holds true:

$$\begin{array}{ccc}
 \mathbf{KU}^p(W) & \xrightarrow{\text{ch}} & H^p(DD^{\text{per}}(W)) \\
 \downarrow \pi_!^o & & \downarrow \int_{W/B} \hat{\mathbf{A}}(o) \cup \dots \\
 \mathbf{KU}^{p-d}(B) & \xrightarrow{\text{ch}} & H^{p-d}(DD^{\text{per}}(B)).
 \end{array}$$

Differential refinements of \mathbf{KU} -orientations have been studied in detail in [10, 16]; see also [13] for a more homotopy-theoretic approach. In order to refine the \mathbf{KU} -orientation o to a \widehat{KU} -orientation \hat{o} , we must choose additional geometric structures. First of all we choose a metric on the vertical tangent bundle $T^v\pi$ and a horizontal distribution $T^h\pi$, i.e., a complement of the vertical bundle in TW . These structures induce a vertical Levi-Civita connection $\nabla^{T^v\pi}$, see, e.g., [6, Prop. 10.2]. In order to fix \hat{o} , we must further choose a

Spin^c-extension $\tilde{\nabla}^{T^v}\pi$ of $\nabla^{T^v}\pi$. The \widehat{KU} -orientation \hat{o} gives rise to an Umkehr map (see [10, §3.2.3])

$$\pi_!^{\hat{o}} : \widehat{KU}^p(W) \rightarrow \widehat{KU}^{p-d}(B).$$

The \widehat{KU} -orientation further provides a form $\hat{\mathbf{A}}(\hat{o}) \in Z^0(DD^{\text{per}}(W))$ representing the class $\hat{\mathbf{A}}(o)$. The Umkehr map in \widehat{KU} -theory fits into the following commutative diagram:

$$\begin{array}{ccccc}
 & & & & I \\
 & \curvearrowright & & & \curvearrowright \\
 \mathbf{KUC}/\mathbb{Z}^{p-1}(W) & \xrightarrow{DD^{\text{per}}(W)^{p-1}/\text{im}(d)} & \xrightarrow{a} & \widehat{KU}^p(W) & \xrightarrow{R} & Z^p(DD^{\text{per}}(W)) & \mathbf{KU}^p(W) \\
 \downarrow \pi_!^{\hat{o}} & & \downarrow f_{W/B} \hat{\mathbf{A}}(\hat{o}) \wedge \dots & \downarrow \pi_!^{\hat{o}} & & \downarrow f_{W/B} \hat{\mathbf{A}}(\hat{o}) \cup \dots & \downarrow \pi_!^{\hat{o}} \\
 \mathbf{KUC}/\mathbb{Z}^{p-d-1}(B) & \xrightarrow{DD^{\text{per}}(B)^{p-d-1}/\text{im}(d)} & \xrightarrow{a} & \widehat{KU}^{p-d}(B) & \xrightarrow{\quad} & Z^{p-d}(DD^{\text{per}}(B)) & \mathbf{KU}^{p-d}(B). \\
 & \curvearrowleft & & & \curvearrowleft & & \\
 & & & & & & I
 \end{array}$$

The set of \widehat{KU} -orientations which refine an underlying topological \mathbf{KU} -orientation o is a torsor over $DD^{\text{per}}(W)^{-1}/\text{im}(d)$ such that

$$(24) \quad \hat{\mathbf{A}}(\hat{o} + \alpha) = \hat{\mathbf{A}}(\hat{o}) + d\alpha$$

for $\alpha \in DD^{\text{per}}(W)^{-1}/\text{im}(d)$.

In the case that B is a point and $p-d$ is odd, we have the rules (all following from [10, Eqn. (17)])

$$(25) \quad \pi_!^{\hat{o}+\alpha}(x) = \pi_!^{\hat{o}}(x) + \left[\int_W d\alpha \wedge R(x) \right]_{\mathbb{C}/\mathbb{Z}}, \quad \pi_!^{\hat{o}}(a(\omega)) = \left[\int_W \hat{\mathbf{A}}(\hat{o}) \wedge \omega \right]_{\mathbb{C}/\mathbb{Z}},$$

where we identify $\widehat{KU}^{p-d}(\ast)$ with \mathbb{C}/\mathbb{Z} , see (18).

If $\pi = \pi_1 \circ \pi_0$ is a composition of proper submersions and \hat{o}_i are \widehat{KU} -orientations of π_i , then we can define a composed orientation $\hat{o} = \hat{o}_1 \circ \hat{o}_0$ for π in a natural way such that

$$(26) \quad \pi_!^{\hat{o}} = \pi_{1,!}^{\hat{o}_1} \circ \pi_{0,!}^{\hat{o}_0}.$$

If

$$\begin{array}{ccc}
 W' & \xrightarrow{g} & W \\
 \downarrow \pi' & & \downarrow \pi \\
 B' & \xrightarrow{f} & B
 \end{array}$$

is a cartesian diagram and \hat{o} is a differential orientation of π , then we can define a \widehat{KU} -orientation \hat{o}' of π' such that

$$\pi_{1,!}^{\hat{o}'} \circ g^* = f^* \circ \pi_{0,!}^{\hat{o}}.$$

In order to avoid the additional complexity of the choice of Spin^c-extensions of connections on real vector bundles with Spin^c-structures, in the present

paper we will work with Spin-structures. If $T^v\pi$ has a spin structure, then it has an induced Spin^c -structure, and a connection $\widehat{\nabla}^{T^v\pi}$ has a canonical Spin^c -extension, which we take from now on. If the \widehat{KU} -orientation \hat{o} is defined using the vertical metric and the horizontal distribution as above, then we have

$$(27) \quad \hat{\mathbf{A}}(\hat{o}) = \hat{\mathbf{A}}(\widehat{\nabla}^{T^v\pi}),$$

where $\widehat{\nabla}^{T^v\pi}$ is the Levi-Civita connection.

Assume that $\pi : W \rightarrow B$ is a submersion with fibrewise boundary $\partial\pi : \partial W \rightarrow B$. If \hat{o} is a \widehat{KU} -orientation of π , then we can define an induced \widehat{KU} -orientation $\partial\hat{o}$ of $\partial\pi$. In this situation, we have the bordism formula [10, Prop. 5.18]. If $\hat{x} \in \widehat{KU}^p(W)$, then we have the equality

$$(28) \quad \partial\pi_!^{\partial\hat{o}}(\hat{x}|_{\partial W}) = a \left(\int_{W/B} \hat{\mathbf{A}}(\hat{o}) \wedge R(\hat{x}) \right).$$

4. THE INVARIANT

4.1. Construction. Given a closed odd-dimensional real foliated spin manifold (M, \mathcal{F}) such that

$$2\text{codim}(\mathcal{F}) < \dim(M),$$

with a stable framing s of $\mathcal{F}_{\mathbb{R}}$ and a complex vector bundle (V, ∇^I) with flat partial connection, we define an invariant

$$\rho(M, \mathcal{F}, \nabla^I, s) \in \mathbb{C}/\mathbb{Z}.$$

In order to define the invariant we first choose the following additional geometric data:

- (i) We choose a connection ∇ on V which extends ∇^I , see Definition 2.28.
- (ii) We choose an extension $\nabla^{\mathcal{F}_{\mathbb{R}}^{\perp}}$ of the flat partial connection $\nabla^{I, \mathcal{F}_{\mathbb{R}}^{\perp}}$, see Example 2.40.
- (iii) We choose a Riemannian metric g^{TM} .

By Definition 2.47, a stable framing s of $\mathcal{F}_{\mathbb{R}}$ is an isomorphism of real vector bundles

$$s : \mathcal{F}_{\mathbb{R}} \oplus \mathbb{R}^n \cong \mathbb{R}^m$$

for certain choices of $n, m \in \mathbb{N}$. The trivial flat connection ∇^s on $\mathcal{F}_{\mathbb{R}} \oplus \mathbb{R}^n$ associated to the stable framing is induced from the trivial connection on \mathbb{R}^m via the isomorphism s . The Riemannian metric on M further induces an orthogonal splitting $TM \cong \mathcal{F}_{\mathbb{R}} \oplus \mathcal{F}_{\mathbb{R}}^{\perp}$, so that we can consider both connections $\nabla^{LC} \oplus \nabla^{\mathbb{R}^n}$ and $\nabla^s \oplus \nabla^{\mathcal{F}_{\mathbb{R}}^{\perp}}$ on the same bundle

$$TM \oplus \mathbb{R}^n \cong (\mathcal{F}_{\mathbb{R}} \oplus \mathbb{R}^n) \oplus \mathcal{F}_{\mathbb{R}}^{\perp}.$$

In particular, we can define the transgression form (see (16))

$$\tilde{\hat{\mathbf{A}}}(LC, s) := \tilde{\hat{\mathbf{A}}}(\nabla^{LC} \oplus \nabla^{\mathbb{R}^n}, \nabla^s \oplus \nabla^{\mathcal{F}_{\mathbb{R}}^{\perp}}) \in DD^{\text{per}}(M)^{-1}/\text{im}(d),$$

where ∇^{LC} is the Levi-Civita connection on TM associated to the Riemannian metric g^{TM} .

We now consider the map $\pi : M \rightarrow *$. Since M is closed, this is a proper submersion. Since M is spin, this map has a KU -orientation o . The choice of a Riemannian metric refines the orientation o to a \widehat{KU} -orientation \hat{o} (note that the horizontal bundle is the zero bundle), see Section 3.3.

Definition 4.2. Let M be an odd-dimensional real foliated closed spin manifold, let s be a stable framing of $\mathcal{F}_{\mathbb{R}}$, and let ∇^I be a flat partial connection on a complex vector bundle on M . Assume further that we have fixed g^{TM} , $\nabla^{\mathcal{F}_{\mathbb{R}^\perp}}$ and ∇ . Then we define

$$\rho(M, \mathcal{F}, \nabla^I, s) := \pi_1^{\hat{o} - \widetilde{\hat{\mathbf{A}}}(LC, s)}([V, \nabla]) \in \widehat{KU}^{-\dim(M)}(*) \stackrel{(18)}{\cong} \mathbb{C}/\mathbb{Z}.$$

In general, this quantity depends on the additional choices g^{TM} , $\nabla^{\mathcal{F}_{\mathbb{R}^\perp}}$ and ∇ . It will be a consequence of the bordism invariance that $\rho(M, \mathcal{F}, \nabla^I, s)$ is actually independent of these choices, provided $2\text{codim}(\mathcal{F}) < \dim(M)$.

Proposition 4.3. Assume that (M, \mathcal{F}) , ∇^I , s , as well as g^{TM} , $\nabla^{\mathcal{F}_{\mathbb{R}^\perp}}$ and ∇ are as in Definition 4.2, with the exception that M is even-dimensional and has a boundary ∂M which is transversal to \mathcal{F} . We further assume that the geometric structures have a product structure near ∂M . Then we have the equality

$$\rho(\partial M, \mathcal{F}|_{\partial M}, \nabla^I|_{\partial M}, s|_{\partial M}) = \left[\int_M \hat{\mathbf{A}}^-(\nabla^{\mathcal{F}_{\mathbb{R}^\perp}}) \wedge \mathbf{ch}^-(\nabla) \right]_{\mathbb{C}/\mathbb{Z}}.$$

In particular, if $2\text{codim}(\mathcal{F}) < \dim(M)$, then $\rho(\partial M, \mathcal{F}|_{\partial M}, \nabla^I|_{\partial M}, s|_{\partial M}) = 0$.

Proof. By the bordism formula (28), we have

$$\rho(\partial M, \mathcal{F}|_{\partial M}, \nabla^I|_{\partial M}, s|_{\partial M}) = \left[\int_M \hat{\mathbf{A}}(\hat{o} - \widetilde{\hat{\mathbf{A}}}(LC, s)) \wedge R([V, \nabla]) \right]_{\mathbb{C}/\mathbb{Z}}.$$

Using (24), (27) and (20), we get

$$\rho(\partial M, \mathcal{F}|_{\partial M}, \nabla^I|_{\partial M}, s|_{\partial M}) = \left[\int_M (\hat{\mathbf{A}}(\nabla^{LC}) - d\widetilde{\hat{\mathbf{A}}}(LC, s)) \wedge \mathbf{ch}(\nabla) \right]_{\mathbb{C}/\mathbb{Z}}.$$

We apply (17) and the fact that $\hat{\mathbf{A}}$ is multiplicative in order to rewrite this as

$$\rho(\partial M, \mathcal{F}|_{\partial M}, \nabla^I|_{\partial M}, s|_{\partial M}) = \left[\int_M \hat{\mathbf{A}}(\nabla^{\mathcal{F}_{\mathbb{R}^\perp}}) \wedge \mathbf{ch}(\nabla) \right]_{\mathbb{C}/\mathbb{Z}}.$$

We now use that both $\nabla^{\mathcal{F}_{\mathbb{R}^\perp}}$ and ∇ extend flat partial connections. The associated characteristic forms therefore refine to cycles in $DD^-(M)$. Hence, by Example 2.40 and Definition 2.37, we have

$$\int_M \hat{\mathbf{A}}(\nabla^{\mathcal{F}_{\mathbb{R}^\perp}}) \wedge \mathbf{ch}(\nabla) = \int_M \hat{\mathbf{A}}^-(\nabla^{\mathcal{F}_{\mathbb{R}^\perp}}) \wedge \mathbf{ch}^-(\nabla).$$

This implies the first assertion.

The integral of \int_M factorizes over the component in

$$DD^-(M)(p)^0 = F^p \Omega^{2p}(M),$$

with $p = \dim(M)/2$. If $\text{codim}(\mathcal{F}) < p$, then we have $F^p\Omega^{2p}(M, \mathcal{F}) = 0$ and hence

$$(29) \quad Z^0(DD^-(M)(2p)) = 0.$$

This implies the second claim. □

In the following we define the opposite of a framing and a spin structure. Let (M, \mathcal{F}) and the stable framing s of $\mathcal{F}_{\mathbb{R}}$ be given. Then we can form the cylinder $I \times M$ with the foliation $T_{\mathbb{C}}I \boxplus \mathcal{F}$, see Example 2.11. We trivialize $TI \cong I \times \mathbb{R}$ using the section ∂_t , where t is the standard coordinate of the cylinder. Then we write $T(I \times M) \cong TI \boxplus TM \cong \underline{\mathbb{R}} \boxplus TM$ in order to define the induced spin structure on $I \times M$. Furthermore, the identification $TI \boxplus \mathcal{F}_{\mathbb{R}} \cong \underline{\mathbb{R}} \boxplus \mathcal{F}_{\mathbb{R}}$ provides the stable framing $I \times s$ of $TI \boxplus \mathcal{F}_{\mathbb{R}}$. These constructions are made such that (M, \mathcal{F}, s) is the boundary of $(I \times M, T_{\mathbb{C}}I \boxplus \mathcal{F}, I \times s)$ at the upper face of the cylinder corresponding to $1 \in I$.

Definition 4.4. We define $(M^{\text{op}}, \mathcal{F}, s^{\text{op}})$ to be the boundary of the cylinder at $0 \in I$.

Here M^{op} indicates that M is equipped with the opposite spin structure.

We adopt all the assumptions made in Definition 4.2 and fix choices for $\nabla, \nabla^{\mathcal{F}_{\mathbb{R}}^{\perp}}$ and g^{TM} . These can be extended constantly over the cylinder. In this case, $\hat{\mathbf{A}}^-(\nabla^{I \times \mathcal{F}_{\mathbb{R}}^{\perp}}) \wedge \mathbf{ch}^-(\mathbf{pr}^*\nabla)$ is pulled back from M and has no dt -component. Hence, its integral over $I \times M$ vanishes.

Corollary 4.5. *We have*

$$\rho(M, \mathcal{F}, \nabla^I, s) = -\rho(M^{\text{op}}, \mathcal{F}, \nabla^I, s^{\text{op}}).$$

Assume now that we have two choices for $\nabla, \nabla^{\mathcal{F}_{\mathbb{R}}^{\perp}}$ and g^{TM} . Then we can again consider the cylinder over M and interpolate between these choices. The second assertion of Proposition 4.3 and the vanishing (29) for $p = \frac{\dim(M)+1}{2}$ imply the following corollary.

Corollary 4.6. *If $2\text{codim}(\mathcal{F}) < \dim(M)$, then $\rho(M, \mathcal{F}, \nabla^I, s)$ is independent of the choices of $\nabla, \nabla^{\mathcal{F}_{\mathbb{R}}^{\perp}}$ and g^{TM} .*

4.7. A spectral geometric interpretation. In this section we express the invariant $\rho(M, \mathcal{F}, \nabla^I, s)$ in terms of spectral invariants of Dirac operators.

Let M be a closed spin manifold with a Riemannian metric g^{TM} and let $\mathbf{V} := (V, h^V, \nabla^u)$ be a hermitean vector bundle with a unitary connection. Then we can form the twisted Dirac operator $\not{D} \otimes \mathbf{V}$. It is a first order elliptic differential operator which acts on the space of sections of $\Gamma(M, S(TM) \otimes V)$, where $S(TM)$ is the spinor bundle of M . It is symmetric with respect to the natural L^2 -metric. Its spectrum is real and consists of eigenvalues of finite multiplicity accumulating at $\pm\infty$. By Weyl’s asymptotics, the number of eigenvalues with absolute value $\leq R$ (counted with multiplicity) grows as $R^{\dim(M)}$. The η -function of this operator was introduced by Atiyah, Patodi and Singer [3] and

is defined by

$$\eta(\not{D} \otimes \mathbf{V})(s) = \sum_{\lambda \neq 0} m_\lambda \text{sign}(\lambda) |\lambda|^{-s},$$

where the sum is taken over the nonzero eigenvalues of $\not{D} \otimes \mathbf{V}$ and m_λ denotes the multiplicity. The sum converges for $\text{Re}(s) > \dim(M)$. It has been further shown in [3] that the η -function has a meromorphic continuation to all of \mathbb{C} , which is regular at $s = 0$.

Definition 4.8. The η -invariant of $\not{D} \otimes \mathbf{V}$ is defined by

$$\eta(\not{D} \otimes \mathbf{V}) := \eta(\not{D} \otimes \mathbf{V})(0).$$

We further define the reduced η -invariant

$$\xi(\not{D} \otimes \mathbf{V}) := \left[\frac{\eta(\not{D} \otimes \mathbf{V}) + \dim(\ker(\not{D} \otimes \mathbf{V}))}{2} \right]_{\mathbb{C}/\mathbb{Z}} \in \mathbb{C}/\mathbb{Z}.$$

We consider the projection $\pi : M \rightarrow *$. The spin structure on M and the Riemannian metric g^{TM} induce a \widehat{KU} -orientation \hat{o} , see Section 3.3. The geometric bundle \mathbf{V} defines a class $[V, \nabla^u] \in \widehat{KU}^0(M)$. Using the identification (18), we get by [10, Cor. 5.5] the following result.

Proposition 4.9. *We have*

$$\pi_!^{\hat{o}}([V, \nabla^u]) = \xi(\not{D} \otimes \mathbf{V}).$$

We now adopt the assumptions of Definition 4.2. We further choose a hermitean metric h^V and a unitary connection ∇^u and set $\mathbf{V} := (V, h^V, \nabla^u)$.

Proposition 4.10. *We have the equality*

$$\begin{aligned} \rho(M, \mathcal{F}, \nabla^I, s) = \xi(\not{D} \otimes \mathbf{V}) - \left[\int_M \widetilde{\hat{\mathbf{A}}}(LC, s) \wedge \text{ch}(\nabla) \right. \\ \left. - \int_M \hat{\mathbf{A}}(\nabla^{LC}) \wedge \widetilde{\text{ch}}(\nabla, \nabla^u) \right]_{\mathbb{C}/\mathbb{Z}}. \end{aligned}$$

Proof. We use the rules (25) for $x = [V, \nabla]$, $\alpha = -\widetilde{\hat{\mathbf{A}}}(LC, s)$ and $\omega = \widetilde{\text{ch}}(\nabla, \nabla^u)$, Proposition 4.9, and equations (20), (27) and (21). \square

5. SPECIAL CASES

5.1. Adams e-invariant. We consider the case of the maximal foliation $\mathcal{F}_{\max} = T_{\mathbb{C}}M$ on a closed manifold M , see Example 2.8. Then a stable framing s of $\mathcal{F}_{\max, \mathbb{R}}$ is a stable framing (see Definition 2.47) of TM and (M, s) defines a framed bordism class $[M, s] \in \Omega_{\dim(M)}^{\text{fr}}$. The Pontryagin–Thom construction identifies the framed bordism theory Ω_*^{fr} with the homology theory represented by the sphere spectrum. In particular, its coefficients are the stable homotopy groups of the sphere $\Omega_*^{\text{fr}}(*) \cong \pi_*^s$. In his study of the j -homomorphism, Adams defined in [2] a homomorphism

$$e_{\mathbb{C}}^{\text{Adams}} : \pi_k^s \rightarrow \mathbb{C}/\mathbb{Z}$$

for odd $k \in \mathbb{N}$. A spectral geometric interpretation of $e_{\mathbb{C}}^{\text{Adams}}$ has been given by Atiyah, Patodi and Singer in [3]. In the following we describe $e_{\mathbb{C}}^{\text{Adams}}$ using differential KU -theory.

The stable framing s induces a spin structure on M . Given a Riemannian metric g^{TM} , we obtain a \widehat{KU} -orientation \hat{o} of $\pi : M \rightarrow *$. It has been observed in [10, Prop. 5.22] that the \widehat{KU} -orientation

$$\hat{o}_s := \hat{o} - \tilde{\mathbf{A}}(LC, s)$$

of π does not depend on the choice of the Riemannian metric.

Let $\mathbf{1} \in \widehat{KU}(M)$ be the class of the trivial one-dimensional bundle $(\mathbb{C}, \nabla^{\text{triv}})$. Then, by [10, Lem. 5.24], we have, using (18),

$$(30) \quad e_{\mathbb{C}}^{\text{Adams}}([M, s]) = \pi_1^{\hat{o}_s}(\mathbf{1}).$$

The following corollary immediately follows from Definition 4.2.

Corollary 5.2. *We have*

$$\rho(M, \mathcal{F}_{\max}, \nabla^{\text{triv}}, s) = e_{\mathbb{C}}^{\text{Adams}}([M, s]).$$

Using the first identity in (25), we get the expression (to be used later)

$$(31) \quad e_{\mathbb{C}}^{\text{Adams}}([M, s]) = \pi_1^{\hat{o}}(\mathbf{1}) - \left[\int_M \tilde{\mathbf{A}}(LC, s) \right]_{\mathbb{C}/\mathbb{Z}}$$

for the e -invariant.

5.3. The ρ -invariant of flat bundles. Assume now that (V, ∇) is a flat bundle on a closed odd-dimensional spin manifold M . The spin structure on M equips the map $\pi : M \rightarrow *$ with an orientation o for \mathbf{KU} . We observe that

$$[V, \nabla] - \dim(V)\mathbf{1} \in \widehat{KU}_{\text{flat}}^0(M) \cong \mathbf{KUC}/\mathbb{Z}^{-1}(M).$$

Hence, we can apply the integration map (22) to this difference.

Definition 5.4. We define the ρ -invariant of ∇ by

$$\rho(\nabla) := \pi_1^o([V, \nabla] - \dim(V)\mathbf{1}) \in \mathbb{C}/\mathbb{Z}.$$

If we choose a Riemannian metric, then we get a refinement of o to a \widehat{KU} -orientation \hat{o} of π . Using the integration in differential K -theory and (23), we can write

$$(32) \quad \rho(\nabla) := \pi_1^{\hat{o}}([V, \nabla]) - \dim(V)\pi_1^{\hat{o}}(\mathbf{1}).$$

Remark 5.5. Assume that h^V is a hermitean metric on V preserved by ∇ . Then we can form the geometric bundle $\mathbf{V} = (V, \nabla, h^V)$. As a consequence of Proposition 4.9, we have

$$(33) \quad \pi_1^{\hat{o}}([V, \nabla]) - \dim(V)\pi_1^{\hat{o}}(\mathbf{1}) = \xi(\not{D} \otimes \mathbf{V}) - \dim(V)\xi(\not{D}).$$

Combining this with (32), we get the statement of the index theorem for flat vector bundles by Atiyah, Patodi and Singer [4, Thm. 5.3], namely,

$$\xi(\not{D} \otimes \mathbf{V}) - \dim(V)\xi(\not{D}) = \rho(\nabla).$$

Observe that this is really a nontrivial statement. The left-hand side of this equality is the analytic index of the flat bundle, and the right-hand side is the topological index, since we have defined the ρ -invariant using the topological integration in \mathbf{KUC}/\mathbb{Z} -theory.

Let us now assume that the spin structure on M is induced by a stable framing s of TM .

Lemma 5.6. *We have*

$$(34) \quad \rho(M, \mathcal{F}_{\max}, \nabla, s) = \rho(\nabla) + \dim(V)e_{\mathbb{C}}^{\text{Adams}}([M, s]).$$

Proof. Since ∇ is flat, we have

$$(35) \quad \pi_1^{\hat{\sigma}-\tilde{\mathbf{A}}(LC,s)}([V, \nabla]) = \pi_1^{\hat{\sigma}}([V, \nabla]) - \dim(V) \left[\int_M \tilde{\mathbf{A}}(LC, s) \right]_{\mathbb{C}}.$$

We first use (31) in order to replace the second term in (35) and then apply (32). □

Remark 5.7. The decomposition (34) of the invariant $\rho(M, \mathcal{F}_{\max}, \nabla, s)$ is very interesting. A priori, this quantity depends on the isomorphism class of the flat bundle (V, ∇) . But we now observe that $\rho(M, \mathcal{F}_{\max}, \nabla, s)$ is actually an invariant of the class $[V, \nabla]^{\text{alg}} \in \mathbf{K}(\mathbb{C})^0(M)$, represented by (V, ∇) . This fact has already been shown in [21] as we will explain in the following.

Fix a base point $m \in M$, choose an identification $V_m \cong \mathbb{C}^{\dim(V)}$, and let $\alpha : \pi_1(M, m) \rightarrow GL(\dim(V), \mathbb{C})$ denote the holonomy representation associated to the flat connection ∇ on V . Then the quantity $e(M, \alpha) \in \mathbb{C}/\mathbb{Z}$ introduced in [21] (for M a homology sphere) can be written in the form (compare with [21, Thm. A])

$$e(M, \alpha) = \rho(\nabla).$$

The number $e(M, \alpha) \in \mathbb{C}/\mathbb{Z}$ only depends on the algebraic K -theory class of M determined by α , which in our notation is $[V, \nabla]^{\text{alg}} \in \mathbf{K}(\mathbb{C})^0(M)$. Since clearly $\dim(V)$ is an invariant of $[V, \nabla]^{\text{alg}}$ as well, the combination

$$\rho(M, \mathcal{F}_{\max}, \nabla, s) = e(M, \alpha) + \dim(V)e_{\mathbb{C}}^{\text{Adams}}([M, s])$$

only depends on the class $[V, \nabla]^{\text{alg}}$ of (V, ∇) .

In Section 6 we will show a much stronger result. We will see that the quantity $\rho(M, \mathcal{F}_{\max}, \nabla, s)$ only depends on the class

$$\pi_1^{o_s}([V, \nabla]^{\text{alg}}) \in \mathbf{K}(\mathbb{C})^{-\dim(M)}(*) \cong K_{\dim(M)}(\mathbb{C}),$$

where $\mathbf{K}(\mathbb{C})^*$ is the cohomology theory represented by the algebraic K -theory spectrum of \mathbb{C} , and o_s is the orientation of $\pi : M \rightarrow *$ for stable cohomotopy (and hence for every cohomology theory, since it is a module theory over stable cohomotopy) given by the framing s .

The formula (34) can be compared with the formulas in [15, Thm 5.5]. We conclude that $\rho(M, \mathcal{F}_{\max}, \nabla, s)$ can be expressed in terms of the universal η -invariant introduced in that reference.

5.8. **e -invariant for families.** Let

$$q : W \rightarrow B$$

be a proper submersion of relative dimension $p := \dim(W) - \dim(B) > 0$ and consider the vertical foliation $\mathcal{F}^v = T_{\mathbb{C}}^v q$ on W , see Example 2.9. This foliation is real. A framing s of $\mathcal{F}_{\mathbb{R}}^v$ induces an orientation o_q of the map q for the framed bordism cohomology theory $\Omega^{\text{fr},*}$. We get a class

$$[W \xrightarrow{q} B, s] = q_!^{o_q}(1_S) \in \Omega^{\text{fr},-p}(B),$$

where $1_S \in \Omega^{\text{fr},0}(W)$ is the unit. The construction (30) of Adams' e -invariant can be extended from $B = *$ to general B as a map

$$e_{\mathbb{C}}^{\text{Adams}} : \Omega^{\text{fr},-p}(B) \rightarrow \mathbf{KUC}/\mathbb{Z}^{-p-1}(B).$$

According to [10, Def. 5.23], its value on the class $[W \xrightarrow{q} B, s]$ is given by

$$(36) \quad e_{\mathbb{C}}^{\text{Adams}}([W \xrightarrow{q} B, s]) := q_!^{\hat{o}_s}(\mathbf{1}) \in \widehat{KU}_{\text{flat}}^{-p}(B) \cong \mathbf{KUC}/\mathbb{Z}^{-p-1}(B),$$

where \hat{o}_s is the \widehat{KU} -orientation of q induced by the framing, see Remark 5.9.

Remark 5.9. The construction of \hat{o}_s is along the same lines as in Section 5.1. The vertical framing induces a spin structure. We choose a fibrewise Riemannian metric and a horizontal distribution. Then we get a vertical Levi-Civita connection $\nabla^{T^v q}$. As explained in Section 3.3, we get a \widehat{KU} -orientation \hat{o} . Furthermore, using the trivial connection induced by the framing, we can define the transgression

$$\tilde{\mathbf{A}}(\nabla^{T^v q}, s) \in DD^{\text{per}}(W)^{-1}/\text{im}(d).$$

The \widehat{KU} -orientation

$$\hat{o}_s := \hat{o} - \tilde{\mathbf{A}}(\nabla^{T^v q}, s)$$

is then independent of the choice of the geometric structures.

In order to see that $e_{\mathbb{C}}^{\text{Adams}}([W \xrightarrow{q} B, s])$ is flat, we calculate its curvature using (23) and (24):

$$R(e_{\mathbb{C}}^{\text{Adams}}([W \xrightarrow{q} B, s])) = \int_{W/B} (\hat{\mathbf{A}}(\hat{o}_s) - d\tilde{\mathbf{A}}(\nabla^{T^v q}, s)) = 0.$$

Let us now assume that B is closed and has a spin structure. Then the projection $\pi : B \rightarrow *$ has a KU orientation o_{π} . We choose a Riemannian metric g^{TB} on B , a vertical metric $g^{T^v q}$ and a horizontal distribution $T^h q$. The metric g^{TB} lifts to a metric on the horizontal bundle $T^h q$ and induces, together with the vertical metric $g^{T^v q}$, a metric on W . Furthermore, the spin structure of B induces a spin structure on the horizontal bundle, which together with the framing of $T^v q$ provides a spin structure on W . Note that $\mathcal{F}_{\mathbb{R}}^{\perp} \cong T^h \pi$. The Levi-Civita connection of g^{TB} pulls back to the connection $\nabla_{\mathcal{F}_{\mathbb{R}}^{\perp}}$.

We consider a geometric vector bundle (V, ∇) on B . Then $(\pi^*V, \pi^*\nabla)$ is a bundle on W and the restriction of $\pi^*\nabla$ to \mathcal{F} is flat, see Example 2.30.

We now assume that $\dim(W)$ is odd.

Lemma 5.10. *We have the equality*

$$\rho(W, \mathcal{F}^v, \pi^* \nabla, s) = \pi_1^{o\pi} (e_{\mathbb{C}}^{\text{Adams}}([W \xrightarrow{q} B, s]) \cup [V]).$$

Proof. The geometry on B provides a \widehat{KU} -orientation \hat{o}_π . The geometry on W induces a \widehat{KU} -orientation $\hat{o}_{\pi \circ q}$. In the following calculation, we use [10, Def. 3.22] and $\hat{\mathbf{A}}(\hat{o}_\pi) = \hat{\mathbf{A}}(\nabla^{\mathcal{F}_{\mathbb{R}}^\perp})$ at the place marked by !:

$$\begin{aligned} \hat{o}_\pi \circ \hat{o}_s &= \hat{o}_\pi \circ (\hat{o} - \tilde{\mathbf{A}}(\nabla^{T^v q}, s)) \\ &= \hat{o}_\pi \circ \hat{o} - \hat{\mathbf{A}}(\hat{o}_\pi) \wedge \tilde{\mathbf{A}}(\nabla^{T^v q}, s) \\ &\stackrel{!}{=} \hat{o}_{\pi \circ q} - \hat{\mathbf{A}}(\nabla^{\mathcal{F}_{\mathbb{R}}^\perp}) \wedge \tilde{\mathbf{A}}(\nabla^{T^v q}, s) - \tilde{\mathbf{A}}(\nabla^{LC}, \nabla^{T^v q} \oplus \nabla^{\mathcal{F}_{\mathbb{R}}^\perp}) \\ &= \hat{o}_{\pi \circ q} - \tilde{\mathbf{A}}(\nabla^{T^v q} \oplus \nabla^{\mathcal{F}_{\mathbb{R}}^\perp}, \nabla^{\text{triv}, s} \oplus \nabla^{\mathcal{F}_{\mathbb{R}}^\perp}) - \tilde{\mathbf{A}}(\nabla^{LC}, \nabla^{T^v q} \oplus \nabla^{\mathcal{F}_{\mathbb{R}}^\perp}) \\ &= \hat{o}_{\pi \circ q} - \tilde{\mathbf{A}}(LC, s). \end{aligned}$$

We now use the fact that the integration is compatible with the identification

$$\mathbf{KUC}/\mathbb{Z}^{*-1} \cong \widehat{KU}_{\text{flat}}^*.$$

We get, using the projection formula for the integration in \widehat{KU} -theory,

$$\begin{aligned} \pi_1^{o\pi} (e_{\mathbb{C}}^{\text{Adams}}([W \rightarrow B, s]) \cup [V]) &= \pi_1^{\hat{o}\pi} (e_{\mathbb{C}}^{\text{Adams}}([W \rightarrow B, s]) \cup [V, \nabla]) \\ &\stackrel{(36)}{=} \pi_1^{\hat{o}\pi} (q_!^{\hat{o}s} (\pi^* [V, \nabla])) \\ &\stackrel{(26)}{=} (\pi \circ q)^{\hat{o}\pi \circ \hat{o}s} (\pi^* [V, \nabla]) \\ &= (\pi \circ q)_!^{\hat{o}\pi \circ q - \tilde{\mathbf{A}}(LC, s)} (\pi^* [V, \nabla]) \\ &= \rho(M, \mathcal{F}, \pi^* \nabla, s). \end{aligned} \quad \square$$

5.11. The dependence on the framing. Let s, s' be two stable framings of a foliation $\mathcal{F}_{\mathbb{R}}$. Then we get two connections ∇^s and $\nabla^{s'}$ on $\mathcal{F}_{\mathbb{R}} \oplus \mathbb{R}^n$. Since these connections are flat, by (17), we get a cohomology class

$$\tilde{\mathbf{A}}(\nabla^{s'}, \nabla^s) \in H^{-1}(DD^{\text{per}}(M)).$$

Definition 5.12. For every class $u \in \mathbf{KU}^0(M)$, we define the relative e -invariant of the pair (s', s) of stable framings of $\mathcal{F}_{\mathbb{R}}$ by

$$e_u(s', s) = \left[\int_M \tilde{\mathbf{A}}(\nabla^{s'}, \nabla^s) \cup \text{ch}(u) \right] \in \mathbb{C}/\mathbb{Z}.$$

Remark 5.13. If $\mathcal{F} = \mathcal{F}_{\text{max}}$, then

$$e_1(s', s) = e_{\mathbb{C}}^{\text{Adams}}([M, s']) - e_{\mathbb{C}}^{\text{Adams}}([M, s]).$$

Note that, in this case, $e_1(s', s)$ takes values in the well-known finite subgroup $\text{im}(e_{\mathbb{C}}^{\text{Adams}}) \subseteq \mathbb{C}/\mathbb{Z}$, calculated by Adams.

The proof of the following proposition is a straight-forward calculation.

Proposition 5.14. *We adopt the assumptions of Definition 4.2 and assume that s, s' are stable framings of $\mathcal{F}_{\mathbb{R}}$. Then we have*

$$\rho(M, \mathcal{F}, \nabla^I, s') - \rho(M, \mathcal{F}, \nabla^I, s) = e_{[V]}(s', s).$$

5.15. Real and imaginary parts.

5.15.1. *The decomposition.* In this subsection, we discuss the components $\rho(\dots)^{\mathbb{R}/\mathbb{Z}}$ and $\rho(\dots)^{i\mathbb{R}}$ of $\rho(M, \mathcal{F}, \nabla^I, s)$ associated to the decomposition of the target group

$$\mathbb{C}/\mathbb{Z} \cong \mathbb{R}/\mathbb{Z} \oplus i\mathbb{R}, \quad x = x^{\mathbb{R}/\mathbb{Z}} + x^{i\mathbb{R}},$$

into the real and the imaginary parts.

We adopt the assumptions made in Definition 4.2. In addition, we choose a hermitean metric h^V on the complex vector bundle V . Then we can define the adjoint connection ∇^* of ∇ (see Remark 2.32) and its unitarization

$$\nabla^u := \frac{1}{2}(\nabla + \nabla^*),$$

with respect to h^V .

We use (21) in order to write

$$(37) \quad [V, \nabla] = [V, \nabla^u] + a(\widetilde{\mathbf{ch}}(\nabla, \nabla^u)).$$

Then we calculate

$$\begin{aligned} (38) \quad \rho(M, \mathcal{F}, \nabla^I, s) &\stackrel{(25)}{=} \pi_1^{\hat{\rho}}([V, \nabla]) - \left[\int_M \widetilde{\hat{\mathbf{A}}}(LC, s) \wedge \mathbf{ch}(\nabla) \right]_{\mathbb{C}/\mathbb{Z}} \\ &\stackrel{(37)}{=} \pi_1^{\hat{\rho}}([V, \nabla^u]) + \left[\int_M \hat{\mathbf{A}}(\nabla^{LC}) \wedge \widetilde{\mathbf{ch}}(\nabla, \nabla^u) \right]_{\mathbb{C}/\mathbb{Z}} \\ &\quad - \left[\int_M \widetilde{\hat{\mathbf{A}}}(LC, s) \wedge \mathbf{ch}(\nabla) \right]_{\mathbb{C}/\mathbb{Z}} \\ &\stackrel{(11)}{=} \pi_1^{\hat{\rho}}([V, \nabla^u]) + \left[\int_M \hat{\mathbf{A}}(\nabla^{LC}) \wedge \widetilde{\mathbf{ch}}(\nabla, \nabla^u) \right]_{\mathbb{C}/\mathbb{Z}} \\ &\quad - \left[\int_M \widetilde{\hat{\mathbf{A}}}(LC, s) \wedge (\mathbf{ch}(\nabla^u) + d\widetilde{\mathbf{ch}}(\nabla, \nabla^u)) \right]_{\mathbb{C}/\mathbb{Z}} \\ &= \pi_1^{\hat{\rho}}([V, \nabla^u]) - \left[\int_M \widetilde{\hat{\mathbf{A}}}(LC, s) \wedge \mathbf{ch}(\nabla^u) \right]_{\mathbb{C}/\mathbb{Z}} \\ &\quad + \left[\int_M \hat{\mathbf{A}}(\nabla^{\mathcal{F}_{\mathbb{R}}^\perp}) \wedge \widetilde{\mathbf{ch}}(\nabla, \nabla^u) \right]_{\mathbb{C}/\mathbb{Z}}, \end{aligned}$$

using partial integration and Stokes' theorem in the last step.

The first two summands in (38) are real. The following is the decomposition of the transgression Chern form into the real and imaginary part (we use (12)

and (13)):

$$\widetilde{\mathbf{ch}}(\nabla, \nabla^u) = \frac{\widetilde{\mathbf{ch}}(\nabla, \nabla^u) + \widetilde{\mathbf{ch}}(\nabla^*, \nabla^u)}{2} + \frac{\widetilde{\mathbf{ch}}(\nabla, \nabla^*)}{2}.$$

We get

$$\begin{aligned} \rho(M, \mathcal{F}, \nabla^I, s)^{\mathbb{R}/\mathbb{Z}} &= \pi_1^\delta([V, \nabla^u]) - \left[\int_M \widetilde{\mathbf{A}}(LC, s) \wedge \mathbf{ch}(\nabla^u) \right]_{\mathbb{R}/\mathbb{Z}} \\ &\quad + \left[\int_M \hat{\mathbf{A}}(\nabla^{\mathcal{F}_\mathbb{R}^\perp}) \wedge \frac{\widetilde{\mathbf{ch}}(\nabla, \nabla^u) + \widetilde{\mathbf{ch}}(\nabla^*, \nabla^u)}{2} \right]_{\mathbb{R}/\mathbb{Z}}, \\ (39) \quad \rho(M, \mathcal{F}, \nabla^I, s)^{i\mathbb{R}} &= \int_M \hat{\mathbf{A}}(\nabla^{\mathcal{F}_\mathbb{R}^\perp}) \wedge \frac{\widetilde{\mathbf{ch}}(\nabla, \nabla^*)}{2}. \end{aligned}$$

5.15.2. *The imaginary part.* We see that the imaginary part $\rho(M, \mathcal{F}, \nabla^I, s)^{i\mathbb{R}}$ is just a characteristic number which can be calculated as an integral over locally computable quantities. It does not depend on the framing.

Example 5.16. We assume that ∇^I is unitary with respect to the metric h . Then we can take for ∇ the unitary extension constructed in Lemma 2.34. With this choice, we have $\nabla = \nabla^u$.

Corollary 5.17. *If ∇ is the unitary extension of ∇^I , then*

$$\rho(M, \mathcal{F}, \nabla^I, s)^{i\mathbb{R}} = 0.$$

In particular, if $2\text{codim}(\mathcal{F}) < \dim(M)$ and ∇^I is unitary, then

$$\rho(M, \mathcal{F}, \nabla^I, s)^{i\mathbb{R}} = 0.$$

Proof. The first assertion follows from (39), the fact that $\nabla = \nabla^*$ and the second equality in (12). The second assertion is then a consequence of the first and Corollary 4.6. \square

Example 5.18. If $\pi : \tilde{M} \rightarrow M$ is a finite covering of degree $[\tilde{M} : M] \in \mathbb{N}$, then we have the identity

$$(40) \quad \rho(\tilde{M}, \pi^*\mathcal{F}, \pi^*\nabla^I, \pi^*s)^{i\mathbb{R}} = [\tilde{M} : M]\rho(M, \mathcal{F}, \nabla^I, s)^{i\mathbb{R}}.$$

Given a foliated manifold (M, \mathcal{F}) , we have an associated bundle \mathcal{F}^\perp with a flat partial connection $\nabla^{I, \mathcal{F}}$. If we apply $\rho(\dots)^{i\mathbb{R}}$ to $(V, \nabla^I) = (\mathcal{F}^\perp, \nabla^{I, \mathcal{F}})$ or a bundle obtained from this by some operation of tensor calculus, we get an invariant of the foliation (M, \mathcal{F}) .

Example 5.19. In this example, for even $n \in \mathbb{Z}$, we consider a $(2n + 1)$ -dimensional closed oriented manifold M with a real foliation \mathcal{F} of codimension 1. We assume that $\mathcal{F}_\mathbb{R}$ is co-oriented.

We first recall the definition of the Godbillon–Vey class $\mathbf{GV}_{2k+1}(\mathcal{F}) \in H^{2k+1}(M; \mathbb{R})$ for $k \geq 1$. Since $\mathcal{F}_\mathbb{R}$ is co-oriented, there exists a real nowhere vanishing one-form $\kappa \in \Omega^1(M)$ such that $\mathcal{F}_\mathbb{R} = \ker(\kappa)$. The integrability of $\mathcal{F}_\mathbb{R}$ translates to the relation $\kappa \wedge d\kappa = 0$. We can choose a real one-form

$\omega \in \Omega^1(M)$ such that $d\kappa = \kappa \wedge \omega$. Note that ω is unique up to multiples of κ . Then the form $\omega \wedge (d\omega)^k \in \Omega^{2k+1}(M)$ is closed and represents the Godbillon–Vey class $\mathbf{GV}_{2k+1}(\mathcal{F})$.

We now assume that TM has a stable framing s_M and a Riemannian metric g^{TM} . The co-orientation of $\mathcal{F}_{\mathbb{R}}$ induces a framing s^\perp of $\mathcal{F}_{\mathbb{R}}^\perp$ by the positive normal unit vector field N . Then there is a unique stable framing s of $\mathcal{F}_{\mathbb{R}}$ (up to equivalence and homotopy) such that $s \oplus s^\perp \sim s_M$. For (V, ∇^I) , we take $(\mathcal{F}^\perp, \nabla^{I, \mathcal{F}^\perp})$.

Lemma 5.20. *Assume the following:*

- (i) M is a closed oriented $(2n + 1)$ -dimensional Riemannian manifold with a stable framing s_M .
- (ii) \mathcal{F} is a real, co-oriented codimension-one foliation on M with a stable framing s of $\mathcal{F}_{\mathbb{R}}$.
- (iii) $(V, \nabla^I) = (\mathcal{F}^\perp, \nabla^{I, \mathcal{F}^\perp})$.
- (iv) We have $s \oplus s^\perp \sim s_M$, where s^\perp is the stable framing of $\mathcal{F}_{\mathbb{R}}^\perp$ given by the positive unit normal vector field.

Then we have

$$\rho(M, \mathcal{F}, \nabla^I, s)^{i\mathbb{R}} = \frac{(-1)^{n+1}}{(2\pi i)^{n+1}n!} \int_M \mathbf{GV}_{2n+1}(\mathcal{F}).$$

Proof. Since $\dim(\mathcal{F}_{\mathbb{R}}^\perp) = 1$, we have $\hat{\mathbf{A}}_{4p}(\nabla^{\mathcal{F}_{\mathbb{R}}^\perp}) \in F^{2p}\Omega^{4p}(M, \mathcal{F}) = 0$ for all $p \geq 1$. Hence, (39) specializes to

$$\rho(M, \mathcal{F}, \nabla^{I, \mathcal{F}^\perp}, s)^{i\mathbb{R}} = \frac{1}{2} \int_M \widetilde{\mathbf{ch}}_{2n+2}(\nabla, \nabla^*).$$

So we must identify $\widetilde{\mathbf{ch}}_{2n+2}(\nabla, \nabla^*)$ with a multiple of $\mathbf{GV}_{2n+1}(\mathcal{F})$.

Using the unit normal vector field $N \in \Gamma(M, TM)$, we can normalize κ so that $\kappa(N) = 1$. Let ω be as above. We take ω as a connection one-form for a connection ∇ on $\mathcal{F}_{\mathbb{R}}^\perp$ with respect to the trivialization by N . For a section X of TM , we have, by definition,

$$\nabla_X N = \omega(X)N.$$

On the other hand, if X is a section of \mathcal{F} , then we have, by Cartan’s formula, $\omega(X) = (\kappa \wedge \omega)(N, X) = d\kappa(N, X) = N\kappa(X) - X\kappa(N) - \kappa([N, X]) = \kappa([X, N])$.

In view of the description of $\nabla^{I, \mathcal{F}^\perp}$ given in Example 2.25, this implies that the connection ∇ extends the flat partial connection $\nabla^{I, \mathcal{F}_{\mathbb{R}}^\perp}$.

We have

$$\frac{(-1)^{n+1}}{(2\pi i)^{n+1}n!} \omega \wedge (d\omega)^n = \widetilde{\mathbf{ch}}_{2n+2}(\nabla, \nabla^{\text{triv}}).$$

Similarly,

$$(-1)^{n+1} \frac{(-1)^{n+1}}{(2\pi i)^{n+1}n!} \omega \wedge (d\omega)^n = \widetilde{\mathbf{ch}}_{2n+2}(\nabla^*, \nabla^{\text{triv}}).$$

Hence, if n is even, then by taking the difference of these two equations, we get

$$\frac{2(-1)^{n+1}}{(2\pi i)^{n+1}n!} \mathbf{G}\mathbf{V}_{2n+1}(\mathcal{F}) = \widetilde{\mathbf{ch}}_{2n+2}(\nabla, \nabla^*). \quad \square$$

Remark 5.21. As noted above, we can take $(V, \nabla^I) := (\mathcal{F}^\perp, \nabla^{I, \mathcal{F}^\perp})$ in order to define an invariant which only depends on the foliation \mathcal{F} . In this example, assume that \mathcal{F} is real and that $\nabla^{\mathcal{F}^\perp}$ is the complexification of a connection $\nabla^{\mathcal{F}^\perp_{\mathbb{R}}}$ extending $\nabla^{I, \mathcal{F}^\perp_{\mathbb{R}}}$. We choose in addition a metric $h^{\mathcal{F}^\perp_{\mathbb{R}}}$ in order to define the adjoint $\nabla^{\mathcal{F}^\perp, *}$. In this remark, we explain the place of

$$\rho(M, \mathcal{F}, \nabla^{I, \mathcal{F}^\perp}, s)^{i\mathbb{R}} = \int_M \hat{\mathbf{A}}(\nabla^{\mathcal{F}^\perp_{\mathbb{R}}}) \wedge \frac{\widetilde{\mathbf{ch}}(\nabla^{\mathcal{F}^\perp}, \nabla^{\mathcal{F}^\perp, *})}{2}$$

in the classification of foliation invariants defined in terms of secondary characteristic classes of foliations.

We start with the classification of characteristic forms for foliations of codimension $q \in \mathbb{N}$ [7], see also [18]. Let $q' \in \mathbb{N}$ be the greatest odd integer $\leq q$. One defines the commutative graded algebra

$$WO_q := \mathbb{R}[\tilde{c}_1, \dots, \tilde{c}_{q'}] \otimes \mathbb{R}[c_1, \dots, c_q]^{\leq 2q},$$

where the degrees of the generators are given by

$$|\tilde{c}_j| = 2j - 1, \quad j \text{ odd}, \quad \text{and} \quad |c_j| = 2j,$$

and the superscript $[-]^{\leq 2q}$ indicates that we take only polynomials of degree less than $2q$.

On this ring we consider the differential d given by

$$d\tilde{c}_j := c_j, \quad dc_j = 0.$$

The cohomology $H^*(WO_q)$ of this DGA classifies secondary characteristic classes for foliations of codimension q . For a cohomology class $[U] \in H^q(WO_q)$, we let $\Delta([U]) \in H^*(M; \mathbb{R})$ denote the corresponding cohomology class.

In the following we describe Δ on the form level. Since $\nabla^{\mathcal{F}^\perp}$ and $\nabla^{\mathcal{F}^\perp, *}$ are complexifications of connections which are dual to each other on a real bundle, we have

$$\mathbf{ch}_{2n}(\nabla^{\mathcal{F}^\perp, *}) = (-1)^n \mathbf{ch}_{2n}(\nabla^{\mathcal{F}^\perp}).$$

By (11), we get, for odd n ,

$$d \frac{1}{2i^n} \widetilde{\mathbf{ch}}_{2n}(\nabla^{\mathcal{F}^\perp}, \nabla^{\mathcal{F}^\perp, *}) = \frac{1}{j^n} \mathbf{ch}_{2n}(\nabla^{\mathcal{F}^\perp}).$$

Therefore, the connection $\nabla^{\mathcal{F}^\perp}$, together with a choice of a metric $h^{\mathcal{F}^\perp_{\mathbb{R}}}$, induces a map of commutative differential graded algebras

$$\Delta_{(\nabla^{\mathcal{F}^\perp}, h^{\mathcal{F}^\perp_{\mathbb{R}}})} : WO_q \rightarrow \Omega(M)$$

by

$$\begin{aligned} \Delta_{(\nabla^{\mathcal{F}^\perp}, h^{\mathcal{F}^\perp}_{\mathbb{R}})}(\tilde{c}_n) &:= \frac{1}{2i^n} \widetilde{\mathbf{ch}}_{2n}(\nabla^{\mathcal{F}^\perp}, \nabla^{\mathcal{F}^\perp, *}), \\ \Delta_{(\nabla^{\mathcal{F}^\perp}, h^{\mathcal{F}^\perp}_{\mathbb{R}})}(c_n) &:= \frac{1}{i^n} \mathbf{ch}_{2n}(\nabla^{\mathcal{F}^\perp}). \end{aligned}$$

Then, for $[U] \in H^*(WO_q)$, the characteristic class $\Delta([U]) \in H^*(M; \mathbb{R})$ of the foliation \mathcal{F} is given by

$$(41) \quad \Delta([U]) := [\Delta_{(\nabla^{\mathcal{F}^\perp}, h^{\mathcal{F}^\perp}_{\mathbb{R}})}(U)].$$

There is a universal polynomial $A(c_1, \dots, c_q) \in \mathbb{R}[c_1, \dots, c_q]^{\leq 2q}$ such that

$$\hat{\mathbf{A}}(\nabla^{\mathcal{F}^\perp})^{\leq 2q} = A(\mathbf{ch}_2(\nabla^{\mathcal{F}^\perp}), \dots, \mathbf{ch}_{2q}(\nabla^{\mathcal{F}^\perp})).$$

We consider

$$(42) \quad U := \left[\left(\sum_{j=1, \text{odd}}^{q'} (-1)^{\frac{j+1}{2}} \tilde{c}_j \right) A(c_1, \dots, c_q) \right]_{\dim(M)} \in WO_q^{\dim(M)}.$$

If $2q < \dim(M)$, then U is a cycle.

Lemma 5.22. *Let \mathcal{F} be a real foliation of codimension q and assume that $2q < \dim(M)$. Then the class*

$$[U] \in H^{\dim(M)}(WO_q)$$

is the universal class classifying the imaginary part of $\rho(M, \mathcal{F}, \nabla^{I, \mathcal{F}^\perp}, s)$.

Proof. The relation

$$\rho(M, \mathcal{F}, \nabla^{I, \mathcal{F}^\perp}, s)^{i\mathbb{R}} = i \langle \Delta([U]), [M] \rangle$$

follows immediately from (42), the definition (41) of $\Delta([U])$ and (39). □

Let us assume that p is odd and $2p - 1 > q$. Then $d\tilde{c}_p = 0$ and we have the cohomology class $[\tilde{c}_p] \in H^{2p-1}(WO_q)$. If the foliation \mathcal{F} is real, then the characteristic class (15) is given by

$$(43) \quad [c_{2p-1}(\nabla^{I, \mathcal{F}^\perp})] = 2i^p \Delta[\tilde{c}_p].$$

5.22.1. *The real part.* The real part $\rho(M, \mathcal{F}, \nabla^I, s)^{\mathbb{R}/\mathbb{Z}}$ is more complicated and of global nature. A good case to look at is discussed in Example 5.1.

Example 5.23. The following example shows that $\rho(M, \mathcal{F}, \nabla^I, s)$ is not an integral over M of locally determined quantities. We consider the manifold $M := S^1$ with the maximal foliation $\mathcal{F}_{\max} = T_{\mathbb{C}}S^1$. The framing s of TS^1 is the bounding framing so that $[S^1, s] = 0$ in Ω_1^{fr} . Furthermore, we let $\mathbf{V}(r) := (V, h, \nabla(r))$ be the flat line bundle with holonomy $\exp(2\pi ir)$ for $r \in [0, 1)$. Then we can apply (34) and (33) to get

$$\rho(S^1, \mathcal{F}_{\max}, \nabla(r), s) = \rho(\nabla) = \xi(\not{D} \otimes \mathbf{V}(r)) - \xi(\not{D}).$$

In this case, the reduced η -invariant can be calculated explicitly. The result is

$$\xi(\not{D} \otimes \mathbf{V}(r)) = [-r]_{\mathbb{C}/\mathbb{Z}}.$$

Hence, we get

$$\rho(S^1, \mathcal{F}_{\max}, \nabla(r), s) = [-r]_{\mathbb{C}/\mathbb{Z}}.$$

In particular, our invariant depends nontrivially on r . The data

$$(S^1, \mathcal{F}_{\max}, \nabla(r), s)$$

for different r are locally isomorphic.

Note that in this example the analog of (40) nevertheless holds true.

6. FACTORIZATION OVER ALGEBRAIC K -THEORY OF SMOOTH FUNCTIONS

Let P be a closed p -dimensional manifold and let s be a stable framing of TP . For a manifold X , we consider a product of foliated manifolds

$$(44) \quad (M, \mathcal{F}) := (P \times X, T_{\mathbb{C}}P \boxplus 0) = (P, \mathcal{F}_{\max}) \times (X, \mathcal{F}_{\min}),$$

a pair (V, ∇^I) of a complex vector bundle and a flat partial connection on (M, \mathcal{F}) . We will show that the data represents an algebraic K -theory class

$$f_{\dagger}^{o_s}([V, \nabla^I]^{\text{alg}}) \in K_p(C^\infty(X))$$

of the ring $C^\infty(X)$. If we assume that X is closed, spin and that $\dim(X) < p$, then our main result is the equality

$$\rho(M, \mathcal{F}, \nabla^I, s) = \pi_{\dagger}^o(\text{reg}_X(f_{\dagger}^{o_s}([V, \nabla^I]^{\text{alg}}))),$$

where

$$\text{reg}_X : K_p(C^\infty(X)) \rightarrow \mathbf{ku}\mathbb{C}/\mathbb{Z}^{-p-1}(X)$$

is the regulator and the map

$$\pi : X \rightarrow *$$

has the \mathbf{ku} -orientation o from the spin structure of X .

6.1. Statement of the result. For manifolds X and P we consider the foliated manifold (44). From the point of view of foliation theory, it is trivial. The leaves of the foliation on the product $M = P \times X$ are just the submanifolds $P \times \{x\}$ for all $x \in X$.

We assume that P is closed and that the tangent bundle TP of P is equipped with a stable framing s . The framing s induces an orientation o_s of the map $f : P \rightarrow *$ for stable cohomotopy theory, the cohomology theory represented by the sphere spectrum \mathbf{S} (or equivalently, the framed bordism theory). Any spectrum \mathbf{E} is a module spectrum over \mathbf{S} . Consequently, f has an induced orientation for the cohomology theory \mathbf{E}^* which we denote by the same symbol o_s . We have an Umkehr or integration map between cohomology groups

$$f_{\dagger}^{o_s} : \mathbf{E}^*(P) \rightarrow \mathbf{E}^{*-p}(*),$$

where $p := \dim(P)$. We will apply this to the cohomology theory $\mathbf{K}(C^\infty(X))^*$, represented by the connective algebraic K -theory spectrum $\mathbf{K}(C^\infty(X))$ of the ring of complex-valued smooth functions on the manifold X .

We start with the class

$$[V, \nabla^I]^{\text{alg}} \in \mathbf{K}(C^\infty(X))^0(P)$$

(see Definition 6.14 for a technical description) represented by a pair (V, ∇^I) of a complex vector bundle on the foliated manifold (44) and a flat partial connection. We can form the algebraic K -theory class

$$(45) \quad f_1^{os}([V, \nabla^I]^{\text{alg}}) \in \mathbf{K}(C^\infty(X))^{-\dim(P)}(*) = K_{\dim(P)}(C^\infty(X)).$$

We now assume that X is closed and spin. We further assume that $\dim(P) + \dim(X)$ is odd and that $\dim(X) < \dim(P)$ or, equivalently,

$$2\text{codim}(\mathcal{F}) < \dim(M).$$

Then, by Corollary 4.6, the invariant $\rho(M, \mathcal{F}, \nabla^I, s) \in \mathbb{C}/\mathbb{Z}$ is well-defined and independent of additional geometric choices.

The main result of the present section shows that $\rho(M, \mathcal{F}, \nabla^I, s)$ can be expressed in terms of the class (45). In greater detail, for every $n \in \mathbb{N}$ with $n > \dim(X)$, we will construct, using methods from differential cohomology theory, a natural regulator

$$\text{reg}_X : K_n(C^\infty(X)) \rightarrow \mathbf{ku}\mathbb{C}/\mathbb{Z}^{-n-1}(X),$$

see Definition 6.19. Let $\pi : X \rightarrow *$ be the projection. The spin structure on X induces an orientation o for the periodic complex topological K -theory \mathbf{KU} , and hence for the \mathbf{KU} -modules \mathbf{ku} and $\mathbf{ku}\mathbb{C}/\mathbb{Z}$. We use the isomorphisms

$$\mathbf{ku}^k \cong \begin{cases} \mathbb{Z}, & k \in 2\mathbb{N}, \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \quad \mathbf{ku}\mathbb{C}/\mathbb{Z}^k \cong \begin{cases} \mathbb{C}/\mathbb{Z}, & k \in 2\mathbb{N}, \\ 0, & \text{otherwise,} \end{cases}$$

in order to interpret elements in $\mathbf{ku}\mathbb{C}/\mathbb{Z}^{2*}(*)$ (e.g., the left-hand side of (46)) as elements of \mathbb{C}/\mathbb{Z} .

Theorem 6.2. *We have the relation*

$$(46) \quad \pi_1^o(\text{reg}_X(f_1^{os}([V, \nabla^I]^{\text{alg}}))) = \rho(M, \mathcal{F}, \nabla^I, s).$$

The proof of this theorem will be finished in Section 6.21.

Remark 6.3. Every class $x \in K_*(C^\infty(X))$ can be presented in the form (45) for suitable stably framed manifolds P and pairs (V, ∇^I) . Indeed, the class x can be thought of as being represented by a map $x : S^n \rightarrow \text{BGL}(C^\infty(X))^+$, where we consider $\text{GL}(C^\infty(X))$ as a discrete group and $+$ stands for Quillen’s $+$ -construction.

Using the standard stable framing s_{can} of S^n , the triple (S^n, x, s_{can}) represents a framed bordism class $[S^n, x, s_{\text{can}}] \in \Omega_n^{\text{fr}}(\text{BGL}(C^\infty(X))^+)$. Since the $+$ -construction map

$$p : \text{BGL}(C^\infty(X)) \rightarrow \text{BGL}(C^\infty(X))^+$$

induces an isomorphism in generalized homology theories, there exists a unique class $[P, y, s] \in \Omega_n^{\text{fr}}(\text{BGL}(C^\infty(X)))$ such that $p_*([P, y, s]) = [S^n, x, s_{\text{can}}]$. Since P is compact, there exists a factorization of y as

$$P \xrightarrow{\tilde{y}} \text{BGL}(N, C^\infty(X)) \rightarrow \text{BGL}(C^\infty(X))$$

for a suitable $N \in \mathbb{N}$. The map \tilde{y} classifies a pair (V, ∇^I) over $P \times X$ of an N -dimensional complex vector bundle with a flat partial connection in the P -direction. We then have

$$f_1^{os}([V, \nabla^I]^{\text{alg}}) = x.$$

6.4. Algebraic K -theory sheaves. We consider the site $\mathbf{Mf}_{\mathcal{C}\text{-fol}}$ of pairs (M, \mathcal{F}) of manifolds M with a foliation \mathcal{F} and foliated maps (see Section 2.1 for definitions). The topology is given by open coverings. We have a morphism of sites

$$(47) \quad \mathbf{Mf}_{\mathcal{C}\text{-fol}} \rightarrow \mathbf{Mf},$$

which forgets the foliations.

In the following, we work in the framework of ∞ - or, more precisely, of $(\infty, 1)$ -categories developed by Joyal, Lurie and others, see [29, 30]. We refer to [14, §2.1], [9, §2] and [13, §4] for an introduction to the language as we will use it here and for further references. We will not discuss the size issues. They can be solved in the standard way for the examples used in the present paper.

For a presentable ∞ -category \mathbf{C} and a site \mathbf{M} , we consider the category

$$\mathbf{PSh}_{\mathbf{C}}(\mathbf{M}) := \mathbf{Fun}(\mathbf{M}^{op}, \mathbf{C})$$

of \mathbf{C} -values presheaves and its full subcategory of sheaves $\mathbf{Sh}_{\mathbf{C}}(\mathbf{M})$. They are related by an adjunction

$$(48) \quad L : \mathbf{PSh}_{\mathbf{C}}(\mathbf{M}) \rightleftarrows \mathbf{Sh}_{\mathbf{C}}(\mathbf{M}) : \text{inclusion},$$

where L is called the sheafification.

We consider the 1-category of categories \mathbf{Cat} with its cartesian symmetric monoidal structure. For the class W of categorical equivalences, we form the symmetric monoidal ∞ -category $\mathbf{Cat}[W^{-1}]$. By $\mathbf{CAlg}(\mathbf{Cat}[W^{-1}])$ we denote the category of commutative algebras in $\mathbf{Cat}[W^{-1}]$.

Remark 6.5. A commutative monoid can be considered as a symmetric monoidal category with only unit morphisms. It is an object of $\mathbf{CAlg}(\mathbf{Cat})$ and therefore represents one in $\mathbf{CAlg}(\mathbf{Cat}[W^{-1}])$. A general symmetric monoidal category has non-identity associator and commutativity constraints and is therefore not a commutative algebra in \mathbf{Cat} . But it naturally represents an object in $\mathbf{CAlg}(\mathbf{Cat}[W^{-1}])$.

The objects of $\mathbf{PSh}_{\mathbf{CAlg}(\mathbf{Cat}[W^{-1}])}(\mathbf{M})$ are called symmetric monoidal prestacks. Similarly, objects in $\mathbf{Sh}_{\mathbf{CAlg}(\mathbf{Cat}[W^{-1}])}(\mathbf{M})$ are called symmetric monoidal stacks.

We consider the following four symmetric monoidal stacks on \mathbf{Mf} or $\mathbf{Mf}_{\mathcal{C}\text{-fol}}$ of vector bundles with additional structures. The monoidal structure is always given by the direct sum.

- (i) For a manifold M , we let $\mathbf{Vect}(M)$ denote the category of vector bundles $V \rightarrow M$. A map $f : M \rightarrow M'$ induces a functor $f^* : \mathbf{Vect}(M') \rightarrow \mathbf{Vect}(M)$. We get a stack \mathbf{Vect} on the site \mathbf{Mf} with respect to the

topology of open coverings. We use the same symbol for its pullback to the site $\mathbf{Mf}_{\mathbb{C}\text{-fol}}$ along (47).

- (ii) We let $\mathbf{Vect}^{\nabla}(M)$ denote the category of pairs (V, ∇) of a vector bundle $V \rightarrow M$ and a connection. A map $f : M \rightarrow M'$ induces a functor $f^* : \mathbf{Vect}^{\nabla}(M') \rightarrow \mathbf{Vect}^{\nabla}(M)$. We get a symmetric monoidal stack \mathbf{Vect}^{∇} on the site \mathbf{Mf} . We use the same symbol for its pullback to the site $\mathbf{Mf}_{\mathbb{C}\text{-fol}}$ along (47).
- (iii) For a foliated manifold (M, \mathcal{F}) , we let $\mathbf{Vect}^{\text{flat}}(M, \mathcal{F})$ denote the category of pairs (V, ∇^I) of a vector bundles $V \rightarrow M$ and a flat partial connection ∇^I on V , see Section 2.19. A foliated map $f : (M, \mathcal{F}) \rightarrow (M', \mathcal{F}')$ induces a functor $f^* : \mathbf{Vect}^{\text{flat}}(M', \mathcal{F}') \rightarrow \mathbf{Vect}^{\text{flat}}(M, \mathcal{F})$. We get a stack $\mathbf{Vect}^{\text{flat}}$ on the site $\mathbf{Mf}_{\mathbb{C}\text{-fol}}$.
- (iv) We let $\mathbf{Vect}^{\text{flat}, \nabla}(M, \mathcal{F})$ denote the category of pairs (V, ∇) of a vector bundle $V \rightarrow M$ and a connection ∇ on V which is flat in the direction of the foliation. A foliated map f as above induces a functor $f^* : \mathbf{Vect}^{\text{flat}, \nabla}(M', \mathcal{F}') \rightarrow \mathbf{Vect}^{\text{flat}, \nabla}(M, \mathcal{F})$. We get a symmetric monoidal stack $\mathbf{Vect}^{\text{flat}, \nabla}$ on the site $\mathbf{Mf}_{\mathbb{C}\text{-fol}}$.

We will consider \mathbf{Vect} and \mathbf{Vect}^{∇} also as stacks on $\mathbf{Mf}_{\mathbb{C}\text{-fol}}$ via pullback along the forgetful morphism (47). There is a commutative diagram of forgetful maps

$$(49) \quad \begin{array}{ccc} \mathbf{Vect}^{\text{flat}, \nabla} & \longrightarrow & \mathbf{Vect}^{\text{flat}} \\ \downarrow & & \downarrow \\ \mathbf{Vect}^{\nabla} & \longrightarrow & \mathbf{Vect} \end{array}$$

in $\mathbf{Sh}_{\mathbf{CAlg}(\mathbf{Cat}^{[W-1]})}(\mathbf{Mf}_{\mathbb{C}\text{-fol}})$. We now apply the K -theory machine \mathcal{K} (see [9, Def. 6.1] and Remark 6.6) and get a commutative diagram of presheaves of spectra

$$(50) \quad \begin{array}{ccc} \mathcal{K}(\mathbf{Vect}^{\text{flat}, \nabla}) & \longrightarrow & \mathcal{K}(\mathbf{Vect}^{\text{flat}}) \\ \downarrow & & \downarrow \\ \mathcal{K}(\mathbf{Vect}^{\nabla}) & \longrightarrow & \mathcal{K}(\mathbf{Vect}) \\ \downarrow & & \downarrow \\ \widehat{\mathbf{ku}}^{\nabla} & \longrightarrow & \widehat{\mathbf{ku}} \end{array}$$

in $\mathbf{PSh}_{\mathbf{Sp}}(\mathbf{Mf}_{\mathbb{C}\text{-fol}})$. The upper square in (50) is by definition the image of (49) under \mathcal{K} . The lower horizontal map is defined by applying the sheafification L (see (48)) to the middle horizontal arrow, and the lower vertical arrows are the units of the sheafification. In particular, we use the notation

$$(51) \quad \widehat{\mathbf{ku}}^{\nabla} := L(\mathcal{K}(\mathbf{Vect}^{\nabla})), \quad \widehat{\mathbf{ku}} := L(\mathcal{K}(\mathbf{Vect})).$$

Remark 6.6. For the convenience of the reader, let us indicate some details on the K -theory machine \mathcal{K} . It is the composition

$$\mathbf{CAlg}(\mathbf{Cat}[W^{-1}]) \rightarrow \mathbf{CAlg}(\mathbf{Groupoids}[W^{-1}]) \rightarrow \mathbf{CommMon}(\mathbf{sSet}[W^{-1}]) \\ \rightarrow \mathbf{CommGroup}(\mathbf{sSet}[W^{-1}]) \simeq \mathbf{Sp}_{\geq 0} \rightarrow \mathbf{Sp}$$

of the following functorial constructions:

- (i) We first take the underlying symmetric monoidal groupoid.
- (ii) Then we apply the nerve in order to get a commutative monoid in the category of spaces $\mathbf{sSet}[W^{-1}]$, i.e., an E_∞ -space.
- (iii) Then we apply the group completion functor to obtain a commutative group in spaces, i.e., a grouplike E_∞ -space.
- (iv) Finally, we apply the functor which maps a commutative group in spaces to the corresponding connective spectrum whose ∞ -loop space is this group.

Remark 6.7. Note that the symmetric monoidal stacks \mathbf{Vect}^∇ and \mathbf{Vect} are pulled back from stacks on the site \mathbf{Mf} via the forgetful morphism (47). The same is true for the associated sheaves of K -theory spectra \mathbf{ku}^∇ and \mathbf{ku} . They represent differential versions of connective K -theory \mathbf{ku} and are studied in detail in [9, §6]

6.8. Characteristic cocycles. In order to construct the regulator, we use the method introduced in [11], based on the notion of characteristic cocycles. We consider the category of chain complexes \mathbf{Ch} . We have

$$DD^-, DD^{\text{per}} \in \mathbf{Sh}_{\mathbf{Ch}}(\mathbf{Mf}_{\mathbf{C}\text{-fol}}),$$

introduced in Definition 2.17, where here we forget the algebra structure, and we use [9, Lem. 7.12] for the sheaf condition.

Using the Chern character forms (Definitions 2.31 and 2.37) and their naturality (equations (6) and (7)), we get the characteristic cocycles (see [11, Def. 2.12])

$$\mathbf{ch}^- : \pi_0(\mathbf{Vect}^{\text{flat}, \nabla}) \rightarrow Z^0(DD^-), \quad \mathbf{ch} : \pi_0(\mathbf{Vect}^\nabla) \rightarrow Z^0(DD^{\text{per}}).$$

Here π_0 sends a symmetric monoidal category to its commutative monoid of isomorphism classes. We will consider commutative monoids as symmetric monoidal categories, see Remark 6.5.

The following diagram in $\mathbf{PSh}_{\mathbf{CAlg}(\mathbf{Cat}[W^{-1}])}(\mathbf{Mf}_{\mathbf{C}\text{-fol}})$ commutes:

$$\begin{array}{ccc} \pi_0(\mathbf{Vect}^{\text{flat}, \nabla}) & \xrightarrow{\mathbf{ch}^-} & Z^0(DD^-) \\ \downarrow & & \downarrow \\ \pi_0(\mathbf{Vect}^\nabla) & \xrightarrow{\mathbf{ch}} & Z^0(DD^{\text{per}}). \end{array}$$

We can now apply the algebraic K -theory machine described in Remark 6.6

and get the following commuting diagram in $\mathbf{PSh}_{\mathbf{Sp}}(\mathbf{Mf}_{\mathbb{C}\text{-fol}})$:

$$(52) \quad \begin{array}{ccc} \mathcal{K}(\pi_0(\mathbf{Vect}^{\text{flat}, \nabla})) & \xrightarrow{\mathcal{K}(\mathbf{ch}^-)} & \mathcal{K}(Z^0(DD^-)) \\ \downarrow & & \downarrow \\ \mathcal{K}(\pi_0(\mathbf{Vect}^\nabla)) & \xrightarrow{\mathcal{K}(\mathbf{ch})} & \mathcal{K}(Z^0(DD^{\text{per}})). \end{array}$$

Let

$$\mathbf{H} : \mathbf{Ch}[W^{-1}] \rightarrow \mathbf{Sp}$$

denote the Eilenberg–MacLane functor (see [11, (22)]). We will use the notation

$$(53) \quad \sigma^{\geq p} \mathbf{DD}^- := \mathbf{H}(\sigma^{\geq p} DD^-), \quad \sigma^{\geq p} \mathbf{DD}^{\text{per}} := \mathbf{H}(\sigma^{\geq p} DD^{\text{per}})$$

for $p \in \mathbb{Z}$, where $\sigma^{\geq p}$ (on the right-hand sides) is the so-called stupid truncation functor on chain complexes which sends a chain complex

$$\dots \rightarrow C^{p-2} \rightarrow C^{p-1} \rightarrow C^p \rightarrow C^{p+1} \rightarrow C^{p+2} \rightarrow \dots$$

to its part

$$\dots \rightarrow 0 \rightarrow 0 \rightarrow C^p \rightarrow C^{p+1} \rightarrow C^{p+2} \rightarrow \dots$$

of degree $\geq p$. Note that

$$\sigma^{\geq p} \mathbf{DD}^-, \sigma^{\geq p} \mathbf{DD}^{\text{per}} \in \mathbf{Sh}_{\mathbf{Sp}}(\mathbf{Mf}_{\mathbb{C}\text{-fol}}),$$

by [9, Lem. 7.12] and the fact that the Eilenberg–MacLane functor \mathbf{H} preserves limits, see also [14, §2.3].

The following construction is a case of the general construction of regulators given in [11, Def. 2.14]. If we consider an abelian group A as a symmetric monoidal category, then we have a natural equivalence $\mathcal{K}(A) \simeq \mathbf{H}(A)$, see [11, Rem 2.13]. Furthermore, for a chain complex C , we can view the inclusion $Z^0(C) \rightarrow C^0 \rightarrow \sigma^{\geq 0}C$ as a natural morphism of chain complexes. Using these observations, we can extend the diagram (52) to the right and obtain the following commuting diagram in $\mathbf{PSh}_{\mathbf{Sp}}(\mathbf{Mf}_{\mathbb{C}\text{-fol}})$:

$$\begin{array}{ccccccc} \mathcal{K}(\pi_0(\mathbf{Vect}^{\text{flat}, \nabla})) & \xrightarrow{\mathcal{K}(\mathbf{ch}^-)} & \mathcal{K}(Z^0(DD^-)) & \xrightarrow{\simeq} & \mathbf{H}(Z^0(DD^-)) & \longrightarrow & \sigma^{\geq 0} \mathbf{DD}^- \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathcal{K}(\pi_0(\mathbf{Vect}^\nabla)) & \xrightarrow{\mathcal{K}(\mathbf{ch})} & \mathcal{K}(Z^0(DD^{\text{per}})) & \xrightarrow{\simeq} & \mathbf{H}(Z^0(DD^{\text{per}})) & \longrightarrow & \sigma^{\geq 0} \mathbf{DD}^{\text{per}}. \end{array}$$

We keep the outer square and extend it further to the commuting diagram

$$(54) \quad \begin{array}{ccc} \mathcal{K}(\mathbf{Vect}^{\text{flat}, \nabla}) & \xrightarrow{r(\mathbf{ch}^-)} & \sigma^{\geq 0} \mathbf{DD}^- \\ \downarrow & & \downarrow \\ \mathcal{K}(\mathbf{Vect}^\nabla) & \xrightarrow{r(\mathbf{ch})} & \sigma^{\geq 0} \mathbf{DD}^{\text{per}} \\ \downarrow u & \nearrow r(\mathbf{ch}) & \\ \widehat{\mathbf{ku}}^\nabla & & \end{array}$$

in $\mathbf{PSh}_{\mathbf{Sp}}(\mathbf{Mf}_{\mathbb{C}\text{-fol}})$. In order to get the lower triangle, we use that $\sigma^{\geq 0}\mathbf{DD}^{\text{per}}$ is a sheaf and the universal property of the unit u of the sheafification involved in the definition (51) of $\widehat{\mathbf{ku}}^{\nabla}$.

6.9. The class $[\mathbf{V}, \nabla^I]^{\text{alg}}$. Let us fix a manifold X . We want to consider foliations whose space of leaves is X . Trivial foliations of this type are obtained by taking the product of the typical leaf with X . In this way, we actually obtain an inclusion of manifolds into foliations. More precisely, we consider the functor

$$(55) \quad j_X : \mathbf{Mf} \rightarrow \mathbf{Mf}_{\mathbb{C}\text{-fol}}, \quad j_X(P) := (P, \mathcal{F}_{\max}) \times (X, \mathcal{F}_{\min}).$$

A manifold Y also gives rise to endofunctors

$$(56) \quad i_Y : \mathbf{Mf} \rightarrow \mathbf{Mf}, \quad i_Y(P) := Y \times P,$$

and

$$(57) \quad i_Y : \mathbf{Mf}_{\mathbb{C}\text{-fol}} \rightarrow \mathbf{Mf}_{\mathbb{C}\text{-fol}}, \quad i_Y(P, \mathcal{F}) := (Y, \mathcal{F}_{\max}) \times (P, \mathcal{F}).$$

The projection $Y \rightarrow *$ induces a morphism $\text{id} \rightarrow i_Y^*$ on presheaves.

Let $I := [0, 1]$ denote the unit interval. Let \mathbf{C} be a presentable ∞ -category.

Definition 6.10. An object $A \in \mathbf{Sh}_{\mathbf{C}}(\mathbf{Mf})$ (or $\mathbf{PSh}_{\mathbf{C}}(\mathbf{Mf})$, $\mathbf{Sh}_{\mathbf{C}}(\mathbf{Mf}_{\mathbb{C}\text{-fol}})$ or $\mathbf{PSh}_{\mathbf{C}}(\mathbf{Mf}_{\mathbb{C}\text{-fol}})$) is called homotopy invariant if the natural morphism

$$A \rightarrow i_I^* A$$

is an equivalence.

We indicate the full subcategories of homotopy invariant (pre)sheaves by an upper index h . Note that in the foliated case we include the tangent bundle of the interval into the foliation direction.

Example 6.11. By [9, Prop. 2.6, 1.] (see also Lemma 6.13 below), for a homotopy invariant sheaf $\mathbf{E} \in \mathbf{Sh}_{\mathbf{C}}^h(\mathbf{Mf})$, we have a natural equivalence

$$\underline{\mathbf{E}(*)} \simeq \mathbf{E},$$

where $\underline{\mathbf{E}(*)}$ denotes the sheaf obtained from the constant presheaf with value $\mathbf{E}(*)$ by sheafification. If $\mathbf{C} = \mathbf{Sp}$, then for $M \in \mathbf{Mf}$ and $k \in \mathbb{Z}$, we have a natural isomorphism of abelian groups

$$(58) \quad \pi_k(\mathbf{E}(M)) \cong \mathbf{E}(*)^{-k}(M).$$

Observe that a similar statement is not true for homotopy invariant sheaves on $\mathbf{Mf}_{\mathbb{C}\text{-fol}}$.

Lemma 6.12. *The sheaf $\mathbf{Vect}^{\text{flat}}$ is homotopy invariant.*

Proof. The reason is that the foliation of $i_I(M, \mathcal{F}) = (I \times M, T_{\mathbb{C}}I \boxplus \mathcal{F})$ contains the I -direction. For $(V, \nabla^I) \in \mathbf{Vect}^{\text{flat}}(i_X(M, \mathcal{F}))$, we can use the flat connection ∇^I in order to define a parallel transport in the I -direction.

Hence, a vector bundle (V, ∇^I) with a flat partial connection or a morphism between two such objects over $I \times M$ is uniquely determined by the restriction to $\{0\} \times M$. □

We now use the fact that on the site \mathbf{Mf} sheafification preserves homotopy invariance.

Lemma 6.13. *If $\mathbf{F} \in \mathbf{PSh}_{\mathbf{C}}^h(M)$, then $L(\mathbf{F}) \simeq \underline{\mathbf{F}}(*)$. In particular,*

$$L(\mathbf{F}) \in \mathbf{Sh}_{\mathbf{C}}^h(\mathbf{Mf}).$$

Proof. This is [9, Prop. 2.6.2] □

We define

$$(59) \quad \mathbf{K}_X := L(j_X^* \mathcal{K}(\mathbf{Vect}^{\text{flat}})) \in \mathbf{Sh}_{\mathbf{Sp}}^h(\mathbf{Mf}).$$

Note that j_X^* preserves homotopy invariance and the sheaf condition. By Lemmas 6.12 and 6.13 we see that \mathbf{K}_X is indeed a homotopy invariant sheaf.

We have a chain of equivalences of symmetric monoidal categories:

$$j_X^* \mathbf{Vect}^{\text{flat}}(*) \simeq \mathbf{Vect}^{\text{flat}}(X, \mathcal{F}_{\min}) \simeq \mathbf{Vect}(X) \simeq \mathbf{Proj}(C^\infty(X)),$$

where the first three are obtained by specializing definitions, and the last is Swan’s theorem. This implies

$$(60) \quad \mathbf{K}_X(*) \cong \mathcal{K}(\mathbf{Proj}(C^\infty(X))) \stackrel{\text{def}}{=} \mathbf{K}(C^\infty(X)),$$

where the last equality is our definition of the connective algebraic K -theory spectrum of the ring $C^\infty(X)$.

We can now give the technical definition of the class

$$[V, \nabla^I]^{\text{alg}} \in \mathbf{K}(C^\infty(X))^0(P)$$

for a pair

$$(V, \nabla^I) \in \mathbf{Vect}^{\text{flat}}(P \times X, T_{\mathbf{C}}P \oplus 0).$$

Indeed, we have $(V, \nabla^I) \in j_X^* \mathbf{Vect}^{\text{flat}}(P)$. This object naturally represents a point in $\Omega^\infty \mathbf{K}_X(P)$.²

Definition 6.14. We define

$$[V, \nabla^I]^{\text{alg}} \in \pi_0(\mathbf{K}_X(P)) \stackrel{(58)}{\cong} \mathbf{K}_X(*)^0(P) \stackrel{(60)}{\cong} \mathbf{K}(C^\infty(X))^0(P).$$

to be the connected component represented by the point $[V, \nabla^I]$.

6.15. Differential K -theory and the regulator map. We assume that \mathbf{C} is a stable presentable ∞ -category like spectra \mathbf{Sp} or chain complexes $\mathbf{Ch}[W^{-1}]$. We have an adjunction

$$\mathcal{H} : \mathbf{Sh}_{\mathbf{C}}(\mathbf{Mf}) \rightleftarrows \mathbf{Sh}_{\mathbf{C}}^h(\mathbf{Mf}) : \text{inclusion},$$

where \mathcal{H} is called the homotopification. By [9, Prop. 7.6 (2)], it is given by a composition $\mathcal{H} \simeq L \circ \mathcal{H}^{\text{pre}}$, where $\mathcal{H}^{\text{pre}} : \mathbf{Sh}_{\mathbf{C}}(\mathbf{Mf}) \rightarrow \mathbf{PSh}_{\mathbf{C}}^h(\mathbf{Mf})$ is given by

$$(61) \quad \mathcal{H}^{\text{pre}} \simeq \text{colim}_{\Delta^{\text{op}}} i_{\Delta^{\bullet}}^*,$$

²Note that an object in a symmetric monoidal category \mathbf{C} naturally represents a point in the nerve $\mathbf{N}(\mathbf{C})$ of \mathbf{C} and therefore a point (up to contractible choice, i.e., a component) in its group completion. The latter is, by definition, $\Omega^\infty \mathcal{K}(\mathbf{C})$, see Remark 6.6.

using the notation (56). Similarly, for the site $\mathbf{Mf}_{\mathbb{C}\text{-fol}}$, we have an adjunction

$$(62) \quad \mathcal{H}^{\text{flat}} : \mathbf{Sh}_{\mathbb{C}}(\mathbf{Mf}_{\mathbb{C}\text{-fol}}) \rightleftarrows \mathbf{Sh}_{\mathbb{C}}^h(\mathbf{Mf}_{\mathbb{C}\text{-fol}}) : \text{inclusion},$$

where $\mathcal{H}^{\text{flat}} = L \circ \mathcal{H}^{\text{flat,pre}}$, with $\mathcal{H}^{\text{flat,pre}}$ given again by (61), but now using (57). For a manifold X , the functor j_X^* (see (55)) preserves homotopy invariant sheaves. Moreover, if X is compact, then we have

$$(63) \quad j_X^* \circ \mathcal{H}^{\text{flat}} \simeq \mathcal{H} \circ j_X^*$$

(compare with [14, Lem. 2.4 (4)] for a proof of a similar statement).

Lemma 6.16. *The sheaves \mathbf{DD}^{per} and \mathbf{DD}^- are homotopy invariant. Moreover, for every $p \in \mathbb{Z}$, the inclusions*

$$\sigma^{\geq p} \mathbf{DD}^- \rightarrow \mathbf{DD}^-, \quad \sigma^{\geq p} \mathbf{DD}^{\text{per}} \rightarrow \mathbf{DD}^{\text{per}}$$

are equivalent to the units of the homotopification.

Proof. We start with the case of the map $\sigma^{\geq p} \mathbf{DD}^{\text{per}} \rightarrow \mathbf{DD}^{\text{per}}$ between sheaves on \mathbf{Mf} . Recall definition (53). We let

$$\iota : \mathbf{Ch} \rightarrow \mathbf{Ch}[W^{-1}]$$

be the canonical localization map. We have

$$\sigma^{\geq p} DD^{\text{per}} \cong \prod_{q \in \mathbb{Z}} (\sigma^{\geq p+2q} \Omega)[2q].$$

We discuss the factors separately. By [9, Lem. 7.15], the map

$$\iota(\sigma^{\geq p+2q} \Omega)[2q] \rightarrow \iota(\Omega)[2q]$$

is the unit of the homotopification. This implies the assertion for \mathbf{DD}^{per} after applying the Eilenberg–MacLane functor \mathbf{H} .

We now discuss DD^- . We first observe that $\iota(DD^-)$ is a homotopy invariant sheaf on the site $\mathbf{Mf}_{\mathbb{C}\text{-fol}}$ with values in $\mathbf{Ch}[W^{-1}]$. We again consider one factor of

$$DD^- \cong \prod_{q \in \mathbb{Z}} F^q \Omega[2q]$$

at a time. For a foliated manifold (M, \mathcal{F}) the integration $\int_{I \times M/M}$ preserves the filtration and induces a map

$$\int_{I \times M/M} : F^q \Omega(I \times M, T_{\mathbb{C}} I \boxplus \mathcal{F}) \rightarrow F^q \Omega(M, \mathcal{F})[-1]$$

such that

$$d \int_{I \times M/M} x = x_{\{1\} \times M} - x_{\{0\} \times M}.$$

This implies that $\iota(F^q \Omega)$ is homotopy invariant.

Remark 6.17. The point here is that we define homotopy invariance along the leaf direction. If we would include transverse directions, then the integral would not preserve the filtration. In this case, we only have

$$\int_{I \times M/M} F^p \Omega(I \times M, \{0\} \boxplus \mathcal{F}) \rightarrow F^{p-1} \Omega(M, \mathcal{F})[-1],$$

and the integration would not be defined on DD^- .

Once we know that $\iota(F^q \Omega) \in \mathbf{Sh}_{\mathbf{Ch}[W^{-1}]}(\mathbf{Mf}_{\mathbb{C}\text{-fol}})$ is homotopy invariant, we shall show that

$$\iota(\sigma^{\geq p} F^q \Omega) \rightarrow \iota(F^q \Omega)$$

is the unit of the homotopification exactly as in [9, Lem. 7.15]. Note that by (the analog of) [9, Lem. 7.13], $\mathcal{H}^{\text{flat}}(\iota(F^q \Omega)^\ell) = 0$ for every $\ell \in \mathbb{Z}$. This implies, as in the proof of [9, Lem. 7.15], that

$$\mathcal{H}^{\text{flat}}(\iota(\sigma^{< p} F^q \Omega)) = 0.$$

The claim now follows from an application of $\mathcal{H}^{\text{flat}} \circ \iota$ to the exact sequence of \mathbf{Ch} -valued sheaves

$$0 \rightarrow \sigma^{< p} F^q \Omega \rightarrow F^q \Omega \rightarrow \sigma^{\geq p} F^q \Omega \rightarrow 0. \quad \square$$

Recall the definition (59) of the sheaf of spectra \mathbf{K}_X . We define

$$\mathbf{K}_X^\nabla := L(j_X^* \mathcal{K}(\mathbf{Vect}^{\text{flat}, \nabla})).$$

Lemma 6.18. *The morphisms*

$$\mathbf{K}_X^\nabla \rightarrow \mathbf{K}_X, \quad \widehat{\mathbf{ku}} \rightarrow \underline{\mathbf{ku}}, \quad \widehat{\mathbf{ku}}^\nabla \rightarrow \underline{\mathbf{ku}}$$

are equivalent to the units of the homotopification.

Proof. The second and the third cases are consequences of [9, Lem. 6.3] and [9, Lem. 6.5]. It remains to discuss the first case. We know that \mathbf{K}_X is homotopy invariant. Then the assertion follows from the analog of [9, Lem. 6.4] for $\mathbf{Vect}^{\text{flat}, \nabla} \rightarrow \mathbf{Vect}^{\text{flat}}$. \square

From (54) and the fact that the two objects on the right and the lower left corner are sheaves, we get the diagram

$$(64) \quad \begin{array}{ccc} \mathbf{K}_X^\nabla & \xrightarrow{j_X^* r(\mathbf{ch}^-)} & j_X^* \sigma^{\geq 0} \mathbf{DD}^- \\ \downarrow & & \downarrow \\ j_X^* \widehat{\mathbf{ku}}^\nabla & \xrightarrow{j_X^* r(\mathbf{ch})} & j_X^* \sigma^{\geq 0} \mathbf{DD}^{\text{per}}. \end{array}$$

We now assume that X is compact. Then, by (63), homotopification commutes with j_X^* . Applying homotopification to this square and using Lemmas 6.16

and 6.18, we get the square

$$(65) \quad \begin{array}{ccc} \mathbf{K}_X & \xrightarrow{\omega_X^-} & j_X^* \mathbf{DD}^- \\ \downarrow & & \downarrow \\ j_X^* \underline{\mathbf{ku}} & \xrightarrow{\omega_X} & j_X^* \mathbf{DD}^{\text{per}}. \end{array}$$

We consider the following three versions of Hopkins–Singer type (see [20] for the original definition and [9] for more information) differential algebraic and differential K -theories for $p \in \mathbb{Z}$:

$$\begin{array}{ccc} \hat{\mathbf{K}}_X^p \longrightarrow j_X^* \sigma^{\geq p} \mathbf{DD}^- & \widehat{\mathbf{ku}}_X^{\text{flat},p} \xrightarrow{R} j_X^* \sigma^{\geq p} \mathbf{DD}^- & \widehat{\mathbf{ku}}^p \longrightarrow \sigma^{\geq p} \mathbf{DD}^{\text{per}} \\ \downarrow I & \downarrow & \downarrow \\ \mathbf{K}_X \xrightarrow{\omega_X^-} j_X^* \mathbf{DD}^- & j_X^* \underline{\mathbf{ku}} \xrightarrow{\omega_X} j_X^* \mathbf{DD}^{\text{per}}, & \underline{\mathbf{ku}} \longrightarrow \mathbf{DD}^{\text{per}}, \end{array}$$

defined by the respective pullback square in $\mathbf{Sh}_{\mathbf{Sp}}(\mathbf{Mf})$. We define the corresponding differential cohomology groups by

$$\begin{aligned} \hat{K}_X^p(P) &:= \pi_{-p}(\hat{\mathbf{K}}_X^p(P)), \\ \widehat{ku}_X^{\text{flat},p}(P) &:= \pi_{-p}(\widehat{\mathbf{ku}}_X^{\text{flat},p}(P)), \\ \widehat{ku}^p(P) &:= \pi_{-p}(\widehat{\mathbf{ku}}^p(P)). \end{aligned}$$

The square (65), together with the obvious commutative square

$$(66) \quad \begin{array}{ccc} \sigma^{\geq p} \mathbf{DD}^- & \longrightarrow & \mathbf{DD}^- \\ \downarrow & & \downarrow \\ \sigma^{\geq p} \mathbf{DD}^{\text{per}} & \longrightarrow & \mathbf{DD}^{\text{per}}, \end{array}$$

induces a chain of morphisms

$$\hat{\mathbf{K}}_X^p \rightarrow \widehat{\mathbf{ku}}_X^{\text{flat},p} \rightarrow j_X^* \widehat{\mathbf{ku}}^p.$$

Using (64), we finally get the square

$$\begin{array}{ccc} \mathbf{K}_X^\nabla & \xrightarrow{\text{cycl}} & \hat{\mathbf{K}}_X^0 \\ \downarrow & & \downarrow \\ j_X^* \mathbf{ku}^\nabla & \xrightarrow{\text{cycl}} & j_X^* \widehat{\mathbf{ku}}^0, \end{array}$$

where the horizontal maps are the differential cycle maps.

The following exact sequences are part of the general features of a Hopkins–Singer differential cohomology, see, e.g., [5, Prop. 2.3.1] or [9, Rem. 4.9]. The

sequence

$$\dots \rightarrow DD^-(P \times X, T_{\mathbb{C}}P \boxplus \{0\})^{\ell-1} / \mathbf{im}(d) \xrightarrow{a} \hat{K}_X^{\ell}(P) \xrightarrow{I} \mathbf{K}_X(*)^{\ell}(P) \rightarrow 0$$

describes the set of possible differential lifts of topological classes. The second sequence

$$0 \rightarrow \widehat{ku}_{X,\text{flat}}^{\text{flat},\ell}(P) \rightarrow \widehat{ku}_X^{\text{flat},\ell}(P) \xrightarrow{R} Z^{\ell}(DD^-(X \times P, T_{\mathbb{C}}P \boxplus \{0\})) \rightarrow \dots$$

reflects the definition of the flat subgroup.

We consider the case $P = *$, $\ell := -p$ and assume that $\dim(X) < p$. Then it is straight-forward to see that $DD^-(X, \mathcal{F}_{\min})^{-p-1} = 0$ and $DD^-(X, \mathcal{F}_{\min})^{-p} = 0$.

This implies the isomorphisms

$$I : \hat{K}_X^{-p}(*) \xrightarrow{\cong} \mathbf{K}_X(*)^{-p}, \quad \widehat{ku}_X^{\text{flat},-p}(*) \xrightarrow{\cong} \widehat{ku}_{X,\text{flat}}^{\text{flat},-p}(*)$$

Definition 6.19. For $p \in \mathbb{N}$ such that $\dim(X) < p$, we define the regulator map \mathbf{reg}_X as the composition

$$\begin{aligned} K_p(C^{\infty}(X)) * &\cong \mathbf{K}_X(*)^{-p} \xleftarrow{\cong} \hat{K}_X^{-p}(*) \rightarrow \widehat{ku}_X^{\text{flat},-p}(*) \\ &\xrightarrow{\cong} \widehat{ku}_{X,\text{flat}}^{\text{flat},-p}(*) \rightarrow \widehat{ku}_{\text{flat}}^{-p}(X, \mathcal{F}_{\min}) \stackrel{!}{\cong} \mathbf{ku}\mathbb{C}/\mathbb{Z}^{-p-1}(X). \end{aligned}$$

For the natural isomorphism marked by ! in the formula above we refer to [5, Prop. 2.3.2] or [9, Rem. 4.9]. In Remark 7.18, we will explain how this regulator can be obtained by specializing a more basic regulator.

Remark 6.20. In [14, Thm 1.1], we defined a similar regulator map,

$$\sigma_p : K_p(C^{\infty}(X)) \rightarrow \mathbf{ku}\mathbb{C}/\mathbb{Z}^{-p-1}(X),$$

using different methods. While here, in order to define the Chern character, we use characteristic forms associated to connections, in [14], we use the Goodwillie–Jones Chern character. The two Chern characters are equivalent as primary invariants [14, Lem. 2.27]. In order to compare the two regulator maps σ_p and \mathbf{reg}_X , we would need to compare the two Chern characters on the space level. So at the moment it remains an open question whether $\sigma_p = \mathbf{reg}_X$.

6.21. Integration and proof of Theorem 6.2. We now assume that P is closed and has a stable framing s . Then $f : P \rightarrow *$ has a natural differential orientation \hat{o}_s (see [13, Ex. 4.230]) and we have an associated Umkehr map in every Hopkins–Singer differential cohomology theory. We further assume that X is closed, spin and equipped with a Riemannian metric. This induces a differential \mathbf{ku} -orientation \hat{o} of the projection $\pi : X \rightarrow *$, see Section 3.3.

Let $p := \dim(P)$ and $d := \dim(X)$. We have the commutative diagram (67)

$$\begin{array}{ccccccc}
 & & & & & & \xrightarrow{(V, \nabla^V) \mapsto \rho(M, \mathcal{F}, \nabla^I, s)} \\
 & & & & & & \searrow \\
 & & & & & & \widehat{KU}^0(P \times X) \xrightarrow{(\pi \circ f)_!^{\hat{o} \circ \hat{o}_s}} \widehat{KU}^{-p-d}(\ast) \xrightarrow{\cong} \mathbb{C}/\mathbb{Z} \\
 & & & & \uparrow \textcircled{2} & & \uparrow \\
 \pi_0(\mathbf{K}_X^\nabla) & \xrightarrow{\text{cycl}} \widehat{K}_X^0(P) \longrightarrow \widehat{ku}^0(P \times X) & \xrightarrow{(\pi \circ f)_!^{\hat{o} \circ \hat{o}_s}} \widehat{ku}^{-p-d}(\ast) & \xrightarrow{\cong} \mathbb{C}/\mathbb{Z} \\
 \downarrow \# & \swarrow & \downarrow f_!^{\hat{o}_s} \textcircled{2} & \downarrow \hat{f}_!^{\hat{o}_s \times X} \textcircled{1} & \downarrow \cong & & \downarrow \cong \\
 \mathbf{K}_X(\ast)^0(P) \textcircled{3} & \widehat{K}_X^{-p}(\ast) \longrightarrow \widehat{ku}^{-p}(X) & \xrightarrow{\pi_1^{\hat{o}}} \widehat{ku}^{-d-p}(\ast) & \xrightarrow{\cong} \mathbb{C}/\mathbb{Z} \\
 \downarrow f_!^{o_s} & \swarrow \cong & \downarrow \textcircled{4} & \downarrow \textcircled{5} & \downarrow \cong & & \downarrow \cong \\
 \mathbf{K}_X(\ast)^{-p} & \xrightarrow{\text{reg}_X} \mathbf{ku}\mathbb{C}/\mathbb{Z}^{-p-1}(X) & \xrightarrow{\pi_1^o} \mathbf{ku}\mathbb{C}/\mathbb{Z}^{-d-p-1}(\ast)
 \end{array}$$

The map marked by $\#$ sends (V, ∇^V) to the class $[V, \nabla^I]^{\text{alg}}$. The square $\textcircled{1}$ commutes by the \mathbf{ku} -analog of (26). For the squares $\textcircled{2}$, we use the fact that that integration commutes with transformations between Hopkins–Singer differential cohomology theories, provided the orientations are related correspondingly. For the square $\textcircled{3}$, we use the right-most square of the \mathbf{ku} -analog of (23). The square $\textcircled{4}$ commutes by the Definition 6.19 of the regulator. Here we also use the square

$$\begin{array}{ccc}
 \mathbf{ku}\mathbb{C}/\mathbb{Z}^{-p-1}(X) & \xrightarrow{!!} & \widehat{ku}^{-p}(X) \\
 \uparrow \cong & & \uparrow \subseteq \\
 \widehat{ku}_{\text{flat}}^{-p}(X, \mathcal{F}_{\text{min}}) & \longrightarrow & \widehat{ku}_{\text{flat}}^{-p}(X),
 \end{array}$$

explaining the arrows marked by $!!$ above and in (67). For $\textcircled{5}$, we use that the identification of the flat subgroup in a Hopkins–Singer differential cohomology with the \mathbb{C}/\mathbb{Z} -version of the underlying cohomology theory is compatible with integration, i.e., the left-most square in the \mathbf{ku} -analog of (23).

The upper composition in (67) maps (V, ∇^V) , essentially by definition, to $\rho(M, \mathcal{F}, \nabla^I, s)$, as indicated. The down-right composition sends (V, ∇^V) to

$$\pi_1^o(\text{reg}_X(f_!^{o_s}([V, \nabla^I]^{\text{alg}}))).$$

Thus, Theorem 6.2 follows from the commutativity of (67). □

7. ALGEBRAIC K -THEORY OF FOLIATIONS

In this section we define the algebraic K -theory sheaf \mathbf{K} on $\mathbf{Mf}_{\mathbb{C}\text{-fol}}$. Its homotopy groups

$$K^*(M, \mathcal{F}) := \pi_{-*}(\mathbf{K}(M, \mathcal{F}))$$

can be considered as the algebraic K -theory groups of the foliation (M, \mathcal{F}) . We further introduce the Hodge-filtered connective K -theory sheaf $\mathbf{ku}^{\text{flat}}$ and define a regulator

$$\text{reg} : \mathbf{K} \rightarrow \mathbf{ku}^{\text{flat}}.$$

For $p > \text{codim}(\mathcal{F})$, it induces a map

$$\widetilde{\text{reg}} : K^{-p}(M, \mathcal{F}) \rightarrow \mathbf{ku}\mathbb{C}/\mathbb{Z}^{-p-1}(M),$$

which generalizes the regulator introduced in Definition 6.19.

Remark 7.1. This section has a considerable overlap with the work of Karoubi [25, 26]. We add this section to the present paper, since it fits well with the set-up developed here and puts the regulator in its natural framework. We will study this regulator and examples elsewhere.

We will use the notation introduced in Section 6.4. In particular, $\mathbf{Vect}^{\text{flat}}$ and $\mathbf{Vect}^{\text{flat}, \nabla}$ denote the symmetric monoidal stacks of pairs (V, ∇^I) and (V, ∇) of complex vector bundles and flat partial connections, or complex vector bundles and connections whose restriction to the foliation is flat, respectively. The symbols L and $\mathcal{H}^{\text{flat}}$ denote the sheafification and the homotopification operations.

Definition 7.2. We define the sheaves of spectra

$$\begin{aligned} \mathbf{K} &:= \mathcal{H}^{\text{flat}}(L(\mathcal{K}(\mathbf{Vect}^{\text{flat}}))) \in \mathbf{Sh}_{\mathbf{Sp}}^h(\mathbf{Mf}_{\mathbb{C}\text{-fol}}), \\ \mathbf{K}^{\nabla} &:= L(\mathcal{K}(\mathbf{Vect}^{\text{flat}, \nabla})) \in \mathbf{Sh}_{\mathbf{Sp}}(\mathbf{Mf}_{\mathbb{C}\text{-fol}}). \end{aligned}$$

For $p \in \mathbb{Z}$, we define the algebraic K -theory of a foliated manifold (M, \mathcal{F}) by

$$K^p(M, \mathcal{F}) := \pi_{-p}(\mathbf{K}(M, \mathcal{F})).$$

Remark 7.3. Note that $\mathcal{K}(\mathbf{Vect}^{\text{flat}})$ is homotopy invariant. We expect that the sheafification preserves homotopy invariance so that the homotopification is not really necessary in this definition.

In order to motivate this definition let us discuss some special cases.

Example 7.4. Recall the functor $j := j_* : \mathbf{Mf} \rightarrow \mathbf{Mf}_{\mathbb{C}\text{-fol}}$ given by

$$j(M) := (M, \mathcal{F}_{\text{max}}),$$

see (55). Let $\mathbf{K}(\mathbb{C})$ denote the connective algebraic K -theory spectrum of the field \mathbb{C} .

Lemma 7.5. *We have an equivalence $j^*\mathbf{K} \simeq \underline{\mathbf{K}(\mathbb{C})}$.*

Proof. Since $j(I \times M) \cong (I, T_{\mathbb{C}}I) \times j(M)$, we conclude that j^* preserves homotopy invariant sheaves. Since \mathbf{K} is homotopy invariant, the sheaf $j^*\mathbf{K}$ is homotopy invariant. Therefore (see [9, Prop. 2.6.1]), we have an equivalence

$$j^*\mathbf{K} \simeq \underline{(j^*\mathbf{K})}.$$

If \mathbf{E} is a presheaf of spectra on $\mathbf{Mf}_{\mathbb{C}\text{-fol}}$ and L is the sheafification (48), then we have a natural equivalence of spectra $L(\mathbf{E})(*) \simeq \mathbf{E}(*)$. Consequently,

$$(j^*\mathbf{K})(*) \simeq \mathcal{K}(\mathbf{Vect}^{\text{flat}}(*, \mathcal{F}_{\text{max}})).$$

The category $\mathbf{Vect}^{\text{flat}}(*, \mathcal{F}_{\text{max}})$ is the category of finite-dimensional complex vector spaces. Consequently, we have an equivalence of spectra

$$\mathcal{K}(\mathbf{Vect}^{\text{flat}}(*, \mathcal{F}_{\text{max}})) \simeq \mathbf{K}(\mathbb{C}).$$

The combination of these equivalences gives the assertion of the lemma. \square

As a consequence of Lemma 7.5, for a manifold M , we have

$$(68) \quad K^*(M, \mathcal{F}_{\text{max}}) \cong \mathbf{K}(\mathbb{C})^*(M).$$

Example 7.6. We have a natural functor $\kappa : \mathbf{Mf} \rightarrow \mathbf{Mf}_{\mathbb{C}\text{-fol}}$, which is given by $M \mapsto (M, \mathcal{F}_{\text{min}})$. On the site \mathbf{Mf} , we have the differential cohomology theory $\widehat{\mathbf{ku}}$, see (51) and [9]. We have an equivalence of sheaves of spectra on $\mathbf{Mf}_{\mathbb{C}\text{-fol}}$

$$\kappa^*\mathbf{K} \simeq \widehat{\mathbf{ku}}.$$

Consequently,

$$K^*(M, \mathcal{F}_{\text{min}}) \cong \widehat{\mathbf{ku}}^*(M).$$

Example 7.7. For a fixed manifold P , there is a natural map

$$(69) \quad \mathbf{K}_X(P) \rightarrow \mathbf{K}(P \times X, T_{\mathbb{C}}P \boxplus \{0\}),$$

which is natural in X . It is essentially the sheafification morphism in the direction of X . The spectrum-valued functor

$$P \mapsto \mathbf{K}(P \times X, T_{\mathbb{C}}P \boxplus 0)$$

is a homotopy invariant sheaf on \mathbf{Mf} . Since

$$\mathbf{K}(\{*\} \times X, T_{\mathbb{C}}\{0\} \boxplus 0) \simeq \widehat{\mathbf{ku}}(X),$$

by [9, Prop. 2.6.1], it is therefore equivalent to $\widehat{\mathbf{ku}}(X)$. We thus get a map

$$K_X^*(P) \rightarrow K^*(P \times X, T_{\mathbb{C}}P \boxplus 0) \cong \widehat{\mathbf{ku}}(X)^*(P).$$

Example 7.8. Assume that X is a smooth complex algebraic variety and let X^{an} be its associated complex manifold with the foliation $\mathcal{F} := T^{0,1}X^{\text{an}}$. Then we can consider the algebraic K -theory of $\mathbf{K}^{\text{alg}}(X)$. It is defined like $\mathbf{K}(M, \mathcal{F})$ as the sheafification of the presheaf $X \supset U \mapsto \mathcal{K}(\mathbf{Vect}^{\text{alg}}(U))$, where $\mathbf{Vect}^{\text{alg}}(U)$ is the symmetric monoidal category of algebraic vector bundles on the Zariski-open subset U . Since the analytic topology of M refines the Zariski topology of X , the transformations $\mathbf{Vect}^{\text{alg}}(U) \rightarrow \mathbf{Vect}(U^{\text{an}})$ induce a map

$$\mathbf{K}^{\text{alg}}(X) \rightarrow \mathbf{K}(X^{\text{an}}, T^{0,1}X).$$

This example justifies to call $\mathbf{K}(M, \mathcal{F})$ the algebraic K -theory spectrum of the foliated manifold (M, \mathcal{F}) .

Example 7.9. If (V, ∇^I) is a complex vector bundle with flat partial connection on a foliated manifold (M, \mathcal{F}) , then we get a class

$$[V, \nabla^I]^{\text{alg}} \in K^0(M, \mathcal{F}).$$

Similarly, if ∇ is a connection which extends ∇^I , then we get a class

$$[V, \nabla]^{\text{alg}} \in \pi_0(\mathbf{K}^\nabla(M, \mathcal{F})).$$

From (54) and the fact that the objects on the right and the lower left corner are sheaves, we get the diagram

$$\begin{array}{ccc} \mathbf{K}^\nabla & \xrightarrow{r(\text{ch}^-)} & \sigma^{\geq 0} \mathbf{DD}^- \\ \downarrow & & \downarrow \\ \widehat{\mathbf{ku}}^\nabla & \xrightarrow{r(\text{ch})} & \sigma^{\geq 0} \mathbf{DD}^{\text{per}}. \end{array}$$

Applying homotopification to this square and using the Lemmas 6.16 and 6.18, we get the square

$$(70) \quad \begin{array}{ccc} \mathbf{K} & \xrightarrow{\omega^-} & \mathbf{DD}^- \\ \downarrow & & \downarrow \\ \underline{\mathbf{ku}} & \xrightarrow{\omega} & \mathbf{DD}^{\text{per}}. \end{array}$$

Definition 7.10. We define the Hodge-filtered connective complex \mathbf{ku} -theory sheaf $\mathbf{ku}^{\text{flat}}$ on $\mathbf{Mf}_{\mathbb{C}\text{-fol}}$ by the pullback square

$$(71) \quad \begin{array}{ccc} \mathbf{ku}^{\text{flat}} & \longrightarrow & \mathbf{DD}^- \\ \downarrow & & \downarrow \\ \underline{\mathbf{ku}} & \xrightarrow{\omega} & \mathbf{DD}^{\text{per}}. \end{array}$$

We let

$$ku^{\text{flat},p}(M, \mathcal{F}) := \pi_{-p}(\mathbf{ku}^{\text{flat}}(M, \mathcal{F}))$$

be the corresponding Hodge-filtered \mathbf{ku} -theory groups of (M, \mathcal{F}) .

Remark 7.11. In [25, 26] Karoubi introduced, starting from a filtration of the de Rham complex, the multiplicative K -theory \mathbf{MK} . Applied to the filtration (Definition 2.14) coming from a foliation the multiplicative K -theory groups $\mathbf{MK}^*(M, \mathcal{F})$ are the Hodge-filtered \mathbf{KU} -theory groups of (M, \mathcal{F}) . In other words, $ku^{\text{flat},*}$ is the connective K -theory analog of Karoubi’s multiplicative K -theory. If one applies the functor Ω^∞ to (71), then one obtains a pullback square of sheaves of spaces which is the analog of the square just before the statement of Theorem 7.3 in [26]. The fact that $\mathbf{ku}^{\text{flat}}$ is a sheaf of spectra implies a Mayer–Vietoris type sequence for an open decomposition of a foliated manifold. This is Karoubi’s theorem [26, Thm. 7.7].

For a justification to use the term *Hodge-filtered...* instead of *multiplicative...*, see Remark 7.12.

Remark 7.12. The Hodge-filtered connective complex **ku**-theory $\mathbf{ku}^{\text{flat}}$ is the **ku**-theory analog of the integral Deligne cohomology, which would be the Hodge filtered version of $H\mathbb{Z}$. While the integral Deligne cohomology is the natural target for cycle maps from Chow groups of algebraic cycles, $\mathbf{ku}^{\text{flat}}$ is the natural target of the regulator from algebraic K -theory. In [19] Hopkins and Quick defined for every spectrum over \mathbf{HZ} a Hodge-filtered version. In an analogous manner, replacing the integral Deligne cohomology by $\mathbf{ku}^{\text{flat}}$, one could construct Hodge filtered cohomology theories for spectra over **ku**. Observe that $\mathbf{ku}^{\text{flat}}$ is the Hodge-filtered version associated to the identity $\mathbf{ku} \rightarrow \mathbf{ku}$. This fact motivates the name.

In view of Lemma 6.16, the sheaf $\mathbf{ku}^{\text{flat}}$ is homotopy invariant (compare with [25, Thm. 4.8]). This fact is reflected in our notation by not using a $(\hat{\cdot})$ -decoration.

Definition 7.13. We define the regulator $\mathbf{reg} : \mathbf{K} \rightarrow \mathbf{ku}^{\text{flat}}$ to be the morphism induced by the square (70) and the universal property of the pullback square (71).

Remark 7.14. Such a regulator has first been defined in [26, §4]. Karoubi’s regulator provides a factorization

$$\mathbf{K} \rightarrow \mathbf{MK} \rightarrow \mathbf{KU}$$

of the map from algebraic to topological K -theory. Our analog is

$$\mathbf{K} \rightarrow \mathbf{ku}^{\text{flat}} \rightarrow \mathbf{ku}.$$

Remark 7.15. The map

$$\mathbf{reg} : \mathbf{K} \rightarrow \mathbf{ku}^{\text{flat}}$$

could be considered as a foliated and integral analog of Beilinson’s regulator. In order to see this, we show that the classical Beilinson regulator can be factored over the regulator \mathbf{reg} defined above.

We first interpret real Deligne cohomology as Hodge-filtered $\mathbf{ku}\mathbb{R}$ -theory. Here, as usual, we write $\mathbf{ku}\mathbb{R} := \mathbf{ku} \wedge \mathbf{M}\mathbb{R}$ for the product of **ku** with the Moore spectrum of \mathbb{R} . The Chern character induces an equivalence of spectra

$$\mathbf{ku}\mathbb{R} \simeq \coprod_{p \geq 0} \mathbf{H}\mathbb{R}[2p].$$

The de Rham equivalence $\mathbf{H}\mathbb{R} \simeq \mathbf{H}\Omega_{\mathbb{R}}$ provides the second equivalence in the composition

$$\underline{\mathbf{ku}\mathbb{R}} \simeq \coprod_{p \geq 0} \mathbf{H}\mathbb{R}[2p] \simeq \coprod_{p \geq 0} \mathbf{H}\Omega_{\mathbb{R}}[2p] \rightarrow \mathbf{DD}^{\text{per}},$$

where the last map is the natural inclusion. This composition provides the lower horizontal map in the pullback square in $\mathbf{Sh}_{\mathbf{Sp}}(\mathbf{Mf}_{\mathbb{C}\text{-fol}})$

$$(72) \quad \begin{array}{ccc} \mathbf{H}_{\mathbb{R},\text{Del}} & \longrightarrow & \mathbf{DD}^- \\ \downarrow & & \downarrow \\ \mathbf{ku}\mathbb{R} & \longrightarrow & \mathbf{DD}^{\text{per}}, \end{array}$$

which defines

$$\mathbf{H}_{\mathbb{R},\text{Del}} \in \mathbf{Sh}_{\mathbf{Sp}}^h(\mathbf{Mf}_{\mathbb{C}\text{-fol}}).$$

On the one hand, this is the Hodge-filtered version of $\mathbf{ku}\mathbb{R}$ -theory. On the other hand, it is a generalization of real Deligne cohomology to foliated manifolds. In fact, for a smooth complex algebraic variety X , we have a natural isomorphism

$$(73) \quad \pi_*(\mathbf{H}_{\mathbb{R},\text{Del}}(X^{\text{an}}, T^{0,1}X)) \cong \prod_{p \in \mathbb{N}} H_{\text{Del},\text{an}}^{2p-*}(X^{\text{an}}, \mathbb{R}(p)).$$

Note that $\mathbf{H}_{\mathbb{R},\text{Del}}$ does not involve the weight-filtration and therefore reflects the “wrong” Hodge filtration on $H^*(X^{\text{an}}; \mathbb{C})$ for nonproper X .

The natural map $\mathbf{ku} \rightarrow \mathbf{ku}\mathbb{R}$ induces a morphism of pullback squares (71) \rightarrow (72) and therefore a morphism

$$\mathbf{ch}_{\mathbb{R},\text{Del}} : \mathbf{ku}^{\text{flat}} \rightarrow \mathbf{H}_{\mathbb{R},\text{Del}}.$$

The composition

$$\mathbf{K} \xrightarrow{\text{reg}} \mathbf{ku}^{\text{flat}} \xrightarrow{\mathbf{ch}_{\mathbb{R},\text{Del}}} \mathbf{H}_{\mathbb{R},\text{Del}}$$

yields indeed Beilinson’s regulator if one applies this to the foliated manifolds $(X^{\text{an}}, T^{0,1}X)$, precomposes with $\mathbf{K}^{\text{alg}}(X) \rightarrow \mathbf{K}(X^{\text{an}}, T^{0,1}X)$, see Example 7.8, and uses the identification (73). This easily follows from the description of Beilinson’s regulator given in [11, 12].

Let (M, \mathcal{F}) be a foliated manifold.

Lemma 7.16. *If $\text{codim}(\mathcal{F}) < p$, then we have a natural isomorphism*

$$\mathbf{ku}\mathbb{C}/\mathbb{Z}^{-p-1}(M) \cong \mathbf{ku}^{\text{flat}, -p}(M, \mathcal{F}).$$

Proof. This easily follows from

$$\pi_p(\mathbf{DD}^-(M, \mathcal{F})) \cong 0 \cong \pi_{p+1}(\mathbf{DD}^-(M, \mathcal{F})). \quad \square$$

Corollary 7.17. *If $\text{codim}(\mathcal{F}) < p$, then the regulator (Definition 7.13) induces a map*

$$\widetilde{\text{reg}} : K^{-p}(M, \mathcal{F}) \rightarrow \mathbf{ku}\mathbb{C}/\mathbb{Z}^{-p-1}(M).$$

Remark 7.18. We have a factorization of reg_X defined in Definition 6.19 as

$$\begin{array}{ccc} K_X^{-p}(\ast) & \xrightarrow{\text{reg}_X} & \mathbf{ku}\mathbb{C}/\mathbb{Z}^{-p-1}(X) \\ \downarrow (69) & \nearrow \widetilde{\text{reg}} & \uparrow \sigma \\ K^{-p}(X, \mathcal{F}_{\text{max}}) & \xrightarrow[\cong]{(68)} & \mathbf{K}(\mathbb{C})^{-p}(X) \end{array}$$

for every $p \geq 1$. Here $\sigma : \mathbf{K}(\mathbb{C}) \rightarrow \mathbf{ku}\mathbb{C}/\mathbb{Z}$ is the morphism discussed, e.g., in [24, §7.21], [9, Ex. 6.9].

Remark 7.19. In [14], we asked whether the map

$$(74) \quad K_p(C^\infty(X)) \rightarrow K_p^{\text{top}}(C^\infty(X))$$

can be nontrivial for $p > \dim(X)$. This question has an analog in the foliated case.

Note that

$$\kappa^* \mathbf{K} \rightarrow \mathbf{ku}$$

(see Example 7.6 for κ) is the homotopification morphism. The question is now as follows.

Question 7.20. *Let (M, \mathcal{F}) be a foliated manifold and $p \in \mathbb{N}$ be such that $\text{codim}(\mathcal{F}) < p$. Is the map*

$$K^{-p}(M, \mathcal{F}) \rightarrow \mathbf{ku}^{-p}(M)$$

trivial?

In the special case of a minimal foliation we ask whether

$$K^{-p}(X, \mathcal{F}_{\min}) \rightarrow \mathbf{ku}^{-p}(X)$$

can be nontrivial for $p > \dim(X)$. The difference to (74) can best be explained by the commutative diagram

$$\begin{array}{ccc} K_p(C^\infty(X)) & \longrightarrow & K_p^{\text{top}}(C^\infty(X)) \\ \downarrow & & \downarrow \\ K^{-p}(X, \mathcal{F}_{\min}) & \longrightarrow & \mathbf{ku}^{-p}(X), \end{array}$$

where the vertical maps are induced by sheafification in the X -direction.

In the foliation case we can answer Question 7.20 affirmatively at least rationally.

Proposition 7.21. *Let (M, \mathcal{F}) be a foliated manifold, assume that $p \in \mathbb{N}$ satisfies $\text{codim}(\mathcal{F}) < p$, and let $x \in K^{-p}(M, \mathcal{F})$. Then the image $x_{\mathbb{Q}} \in \mathbf{ku}\mathbb{Q}^{-p}(M)$ vanishes.*

Proof. Note that the natural map $\mathbf{ku}\mathbb{Q}^*(M) \rightarrow \mathbf{ku}\mathbb{C}^*(M)$ is injective and that the Bockstein sequence

$$\mathbf{ku}\mathbb{C}/\mathbb{Z}^{-p-1}(M) \xrightarrow{\beta} \mathbf{ku}^{-p}(M) \xrightarrow{c} \mathbf{ku}\mathbb{C}^{-p}(M)$$

is exact. We write $x_{\mathbb{C}}$ for the image of $x_{\mathbb{Q}}$ in $\mathbf{ku}\mathbb{C}^{-p}(M)$. We have $x_{\mathbb{C}} = c(\beta(\mathbf{r}\hat{\mathbf{e}}\mathbf{g}(x))) = 0$. □

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Received July 7, 2017; accepted June 19, 2018

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