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# Ample groupoids: Equivalence, homology, and Matui's HK conjecture

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**Abstract.** We investigate the homology of ample Hausdorff groupoids. We establish that a number of notions of equivalence of groupoids appearing in the literature coincide for ample Hausdorff groupoids, and deduce that they all preserve groupoid homology. We compute the homology of a Deaconu–Renault groupoid associated to $k$ pairwise-commuting local homeomorphisms of a zero-dimensional space, and show that Matui's HK conjecture holds for such a groupoid when $k$ is one or two. We specialize to $k$-graph groupoids, and show that their homology can be computed in terms of the adjacency matrices, using a chain complex developed by Evans. We show that Matui's HK conjecture holds for the groupoids of single vertex $k$-graphs which satisfy a mild joint-coprimality condition. We also prove that there is a natural homomorphism from the categorical homology of a $k$-graph to the homology of its groupoid.

## 1. Introduction

The purpose of this paper is to investigate the homology of ample Hausdorff groupoids, and to investigate Matui's HK-conjecture for groupoids associated to actions of $\mathbb{N}^k$ by local homeomorphisms on locally compact Hausdorff zero-dimensional spaces. Ample Hausdorff groupoids are an important source of examples of $C^*$-algebras. They provide models for the crossed-products associated to Cantor minimal systems [18], Cuntz–Krieger algebras and graph $C^*$-algebras and their higher-rank analogs [41, 26, 25], and recently models for large classes of classifiable $C^*$-algebras [6, 13, 38]. It is therefore very desirable to develop techniques for computing the $K$-theory of the $C^*$-algebra of an ample Hausdorff groupoid. Unfortunately, there are relatively few general techniques available.

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In a series of recent papers [30, 31, 32], Matui has advanced a conjecture that if $\mathcal{G}$ is a minimal effective ample Hausdorff groupoid with compact unit space, then $K_0(C_r^*(G))$ is isomorphic to the direct sum of the even homology groups $H_{2n}(\mathcal{G})$ of the groupoid as defined by Crainic and Moerdijk [10], and $K_1(C_r^*(\mathcal{G}))$ is isomorphic to the direct sum of the odd homology groups $H_{2n+1}(\mathcal{G})$. He has verified this conjecture for a number of key classes of groupoids, including finite cartesian products of groupoids associated to shifts of finite type, transformation groupoids for Cantor minimal systems, and AF groupoids with compact unit spaces. He has also developed tools for computing the homology of ample Hausdorff groupoids, including a spectral sequence that relates the homology of a groupoid $\mathcal{G}$ endowed with a cocycle $c$ taking values in a discrete abelian group $H$ with the homology of $H$ with values in the homology groups of the skew-product groupoid $\mathcal{G} \times_c H$. Other authors have subsequently verified Matui's conjecture for Exel–Pardo groupoids and certain graded ample Hausdorff groupoids (see [35, 19]).

A key ingredient in some of this work, particularly [30, 31, 32], is equivalence of groupoids. There are many notions of equivalence for groupoids in the literature, and in general, even for étale groupoids, they do not all coincide (see Example 3.13). However, previous work in this area has implicitly assumed that they do coincide, at least for ample étale groupoids with compact unit space; for example [30, 32] implicitly use that homological similarity implies strong Morita equivalence when appealing to results in [10], but this is only proved in [10] for Lie groupoids. Moreover, in this paper we will be interested in groupoids with non-compact unit spaces, which are not treated in Matui's work. So we begin Section 3 by discussing groupoid equivalence for arbitrary ample Hausdorff groupoids. Crainic and Moerdijk focus on the notion of Morita equivalence of groupoids (see [10, Subsect. 4.5]) while Matui employs the notions of similarity (see [30, Def. 3.4] and also [40, 41]) and Kakutani equivalence (see [30, Def. 4.1]). In the setting of ample Hausdorff groupoids with $\sigma$-compact unit spaces, it follows from [30, Thm. 3.6] that Kakutani equivalence implies similarity. We show that in fact similarity, Kakutani equivalence, Renault's notion of groupoid equivalence [42, 34], and the notion of groupoid Morita equivalence of Crainic and Moerdijk (as well as a number of other notions) all coincide for ample Hausdorff groupoids with $\sigma$-compact unit spaces (see Theorem 3.12). This provides a firm footing for our later results, and also fills in some details missing in earlier literature.

In Section 4 we recall the definition of homology for an arbitrary ample Hausdorff groupoid from [10, Subsect. 3.1] (see also [30, Def. 3.1]) and we appeal to a theorem of Crainic and Moerdijk to observe that groupoid Morita equivalence preserves groupoid homology for arbitrary ample Hausdorff groupoids (see [10, Cor. 4.6]). Matui also proved that similar Hausdorff étale groupoids have isomorphic homology groups (see [30, Prop. 3.5]), and this formulation allows us to give an explicit description of the isomorphism when the equivalence arises from a similarity. In Section 5 we introduce Matui's HK conjecture, and extend his proof that AF groupoids with compact unit space satisfy the HK conjecture to the case of non-compact unit spaces.

Our main computations of groupoid homology are in Section 6, where we investigate the homology of Deaconu–Renault groupoids $\mathcal{G}(X,\sigma)$ associated to actions $\sigma$ of $\mathbb{N}^k$ by local homeomorphisms on totally disconnected locally compact Hausdorff spaces $X$. We adapt techniques developed by Evans [14] in the context of $K$-theory for higher-rank graph $C^*$-algebras to construct a chain complex $A^\sigma$ in which the $n$-chains are elements of $\bigwedge^n \mathbb{Z}^k \otimes C_c(X,\mathbb{Z})$ and the boundary maps are built from the forward maps $\sigma^n_*$ on $C_c(X,\mathbb{Z})$ that satisfy $\sigma^n_*(1_U) = 1_{\sigma^n(U)}$ whenever $U \subseteq X$ is a compact open set on which $\sigma^n$ is injective. Our main result, Theorem 6.5, gives an explicit computation of the homology groups $H_n(\mathcal{G}(X,\sigma))$: we prove that $H_n(\mathcal{G}(X,\sigma))$ is canonically isomorphic to $H_n(A^\sigma_*)$. We then show that if $c : \mathcal{G}(X,\sigma) \to \mathbb{Z}^k$ is the canonical cocycle, then the homology groups $H_*(A^\sigma)$ also coincide with the homology groups $H_*(\mathbb{Z}^k, K_0(C^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)))$ appearing in Kasparov's spectral sequence for the double crossed product $C^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) \rtimes \mathbb{Z}^k$. Since this double crossed product is Morita equivalent to $C^*(\mathcal{G}(X,\sigma))$ by Takai duality, this provides a useful tool for calculating the $K$-theory of $C^*(\mathcal{G}(X,\sigma))$. In Theorems 6.7 and 6.10 we calculate both the $K$-groups of $C^*(\mathcal{G}(X,\sigma))$ and the homology groups of $\mathcal{G}(X,\sigma)$ explicitly for $k = 1, 2$, and in particular prove that ample Deaconu–Renault groupoids of rank at most 2 satisfy Matui's HK conjecture. We also discuss the differences between Kasparov's spectral sequence and Matui's for $k \geq 3$ and indicate where one might look for counterexamples to Matui's conjecture amongst such groupoids.

Finally, in Section 7, we specialize to $k$-graphs. The $k$-graph groupoid $\mathcal{G}_\Lambda$ of a row-finite $k$-graph $\Lambda$ with no sources is precisely the Deaconu–Renault groupoid $\mathcal{G}(\Lambda^\infty,\sigma)$ associated to the shift maps on the infinite-path space of $\Lambda$. We begin the section by linking the homology of the $k$-graph groupoid with the categorical homology of the $k$-graph by constructing a natural homomorphism from $H_*(\Lambda)$ to $H_*(\mathcal{G}_\Lambda)$. We then investigate how to apply the results of Section 6 in the specific setting of $k$-graph groupoids. We prove that the chain complex $A^\sigma_*$ associated to $(\Lambda^\infty,\sigma)$ as in Section 6 has the same homology as the much simpler chain complex $D^\Lambda_*$ described by Evans in [14]. This provides a very concrete calculation of the homology of a $k$-graph groupoid. It follows that the homology of $\mathcal{G}_\Lambda$ does not depend on the factorization rules in $\Lambda$. We use this and the preceding section to establish an explicit description of the homology of 1-graph groupoids and 2-graph groupoids and to see that these groupoids satisfy Matui's HK conjecture. We also prove that for arbitrary $k$, if $\Lambda$ is a $k$-graph with a single vertex such that the integers $|\Lambda^{e_1}|-1,\dots,|\Lambda^{e_k}|-1$ have no nontrivial common divisors, then both the homology of $\mathcal{G}_\Lambda$ and the $K$-theory of $C^*(\Lambda)$ are trivial, and in particular $\mathcal{G}_\Lambda$ satisfies the HK-conjecture.

## 2. Background

**2.1. Groupoids and their $C^*$-algebras.** We give some brief background on groupoids and their $C^*$-algebras and establish our notation; for details, see [15, 41, 47]. A groupoid is a small category $\mathcal{G}$ with inverses. We write $\mathcal{G}^{(0)}$ for the set of identity morphisms of $\mathcal{G}$, called the unit space, and we write $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ for the range and source maps. We write $\mathcal{G}^{(2)}$ for the set $\{(\gamma_1, \gamma_2) \in \mathcal{G} \times \mathcal{G} \mid s(\gamma_1) = r(\gamma_2)\}$ of composable pairs in $\mathcal{G}$. The groupoid $\mathcal{G}$ is a topological groupoid if it has a locally compact topology under which all operations in $\mathcal{G}$ are continuous and $\mathcal{G}^{(0)}$ is Hausdorff in the relative topology. If the topology on all of $\mathcal{G}$ is Hausdorff, we call $\mathcal{G}$ a Hausdorff groupoid. An étale groupoid is a topological groupoid in which $\mathcal{G}^{(0)}$ is open, and $r, s : \mathcal{G} \to \mathcal{G}^{(0)}$ are local homeomorphisms (in [41] such a groupoid is called $r$-discrete with Haar system). An open subset $U \subseteq \mathcal{G}$ is said to be an *open bisection* if both $r|_U$ and $s|_U$ are homeomorphisms onto their ranges. Given $u \in \mathcal{G}^{(0)}$, we write $\mathcal{G}^u$ for $\{\gamma \in \mathcal{G} \mid r(\gamma) = u\}$, $\mathcal{G}_u$ for $\{\gamma \in \mathcal{G} \mid s(\gamma) = u\}$ and $\mathcal{G}^u_u = \mathcal{G}^u \cap \mathcal{G}_u$.

A groupoid $\mathcal{G}$ is *ample* if it is étale and $\mathcal{G}^{(0)}$ is zero-dimensional; equivalently, $\mathcal{G}$ is ample if it has a basis of compact open bisections.

The *orbit* of a unit $u \in \mathcal{G}^{(0)}$ is the set $[u] := r(\mathcal{G}_u) = s(\mathcal{G}^u)$. A subset $U \subseteq \mathcal{G}^{(0)}$ is full if $U \cap [u] \neq \varnothing$ for every unit $u$. We say that $\mathcal{G}$ is *minimal* if the only nontrivial open invariant subset of $\mathcal{G}^{(0)}$ is $\mathcal{G}^{(0)}$; equivalently, $\mathcal{G}$ is minimal if the closure of $[u]$ is equal to $\mathcal{G}^{(0)}$ for every $u \in \mathcal{G}^{(0)}$.

The *isotropy* of $\mathcal{G}$ is the set $\bigcup_{u \in \mathcal{G}^{(0)}} \mathcal{G}^u_u$ of elements of $\mathcal{G}$ whose range and source coincide. A groupoid $\mathcal{G}$ is said to be *effective*[1] if the interior of its isotropy coincides with its unit space $\mathcal{G}^{(0)}$.

Let $A$ be an abelian group and let $c : \mathcal{G} \to A$ be a 1-cocycle. Then we may form the *skew product* groupoid $\mathcal{G} \times_c A$ which is the set $\mathcal{G} \times A$ with structure maps $r(\gamma, a) = (r(\gamma), a)$, $s(\gamma, a) = (s(\gamma), a + c(\gamma))$ and $(\gamma, a)(\eta, a + c(\gamma)) = (\gamma\eta, a)$ (see [41, Def. I.1.6]). There is a natural action $\alpha$ of $A$ on $\mathcal{G} \times_c A$ given by $\alpha_b(\gamma, a) = (\gamma, a + b)$.

Given a Hausdorff étale groupoid $\mathcal{G}$, the space $C_c(\mathcal{G})$ of continuous compactly supported functions from $\mathcal{G}$ to $\mathbb{C}$ becomes a $*$-algebra with operations given by

$$(f * g)(\gamma) = \sum_{\gamma = \gamma_1 \gamma_2} f(\gamma_1) g(\gamma_2) \quad \text{and} \quad f^*(\gamma) = \overline{f(\gamma^{-1})}.$$

Moreover, the *groupoid $C^*$-algebra* $C^*(\mathcal{G})$ is the universal $C^*$-algebra generated by a $^*$-representation of $C_c(\mathcal{G})$ (compare with [41, II.1.5]). For each unit $u \in \mathcal{G}^0$ there is a $^*$-representation $\pi_u : C_c(\mathcal{G}) \to \mathcal{B}(\ell^2(\mathcal{G}_u))$ given by

$$\pi_u(f)\delta_\eta = \sum_{\gamma \in \mathcal{G}_{r(\eta)}} f(\gamma)\delta_{\gamma\eta}.$$

[1]Matui uses the term *essentially principal*, and the term *topologically principal* has also been used elsewhere in the literature.

The reduced $C^*$-algebra $C^*_r(\mathcal{G})$ is the closure of the image of $C_c(\mathcal{G})$ under the representation $\bigoplus_{u\in\mathcal{G}^{(0)}} \pi_u$.

A cocycle $c : \mathcal{G} \to \mathbb{Z}^k$ determines an action of $\mathbb{T}^k$ by automorphisms of $C_c(\mathcal{G})$ given by $(z \cdot f)(\gamma) = z^{c(\gamma)} f(\gamma)$, and this extends to an action of $\mathbb{T}^k$ by automorphisms on each of $C^*(\mathcal{G})$ and $C^*_r(\mathcal{G})$. There is an isomorphism $C^*(\mathcal{G}\times_c\mathbb{Z}^k) \cong C^*(\mathcal{G})\rtimes\mathbb{T}^k$ that carries a function $f \in C_c(\mathcal{G}\times\{n\}) \subseteq C_c(\mathcal{G}\times_c\mathbb{Z}^k)$ to the function $z \mapsto (g \mapsto z^n f(g,n)) \in C(\mathbb{T}^k, C^*(\mathcal{G})) \subseteq C^*(\mathcal{G}) \rtimes \mathbb{T}^k$. This isomorphism descends to an isomorphism $C^*_r(\mathcal{G} \times_c \mathbb{Z}^k) \cong C^*_r(\mathcal{G}) \rtimes \mathbb{T}^k$.

We will be particularly interested in Deaconu–Renault groupoids associated to actions of $\mathbb{N}^k$, which are defined as follows: Let $X$ be a locally compact Hausdorff space, and let $\sigma$ be an action of $\mathbb{N}^k$ on $X$ by surjective local homeomorphisms. The associated Deaconu–Renault groupoid[2] $\mathcal{G} = \mathcal{G}(X,\sigma)$ is defined by (compare with [12, 16])

$$\mathcal{G} = \{(x, p-q, y) \in X \times \mathbb{Z}^k \times X \mid \sigma^p(x) = \sigma^q(y)\}.$$

We identify $X$ with the unit space via the map $x \mapsto (x, 0, x)$. The structure maps are given by

$$\begin{aligned} r(x,n,y) &= x,\\ s(x,n,y) &= y,\\ (x,m,y)(y,n,z) &= (x,m+n,z). \end{aligned}$$

A basis for the topology on $\mathcal{G}$ is given by the sets

$$\mathcal{Z}(U,p,q,V) := \{(x,p-q,y) \in U \times \{p-q\} \times V \mid \sigma^p(x) = \sigma^q(y)\}$$

indexed by pairs $U, V$ of open subsets of $\mathcal{G}$ and pairs $p, q$ of elements of $\mathbb{N}^k$. Indeed, since $\sigma^p$ and $\sigma^q$ are local homeomorphisms, the sets $\mathcal{Z}(U,p,q,V)$, where $U$, $V$ are open, $\sigma^p$ is injective on $U$, $\sigma^q$ is injective on $V$ and $\sigma^p(U) = \sigma^q(V)$, constitute a basis of open bisections for the same topology.

There is a natural cocycle $c : \mathcal{G}(X,\sigma) \to \mathbb{Z}^k$ given by $c(x,n,y) := n$. We can then form the skew-product groupoid $\mathcal{G} \times_c \mathbb{Z}^k$. With our conventions, in this groupoid we have $r((x,n,y),p) = (x,p)$, $s((x,n,y),p) = (y,p+n)$ and $((x,n,y),p)((y,m,z),p+n) = ((x,m+n,z),p)$.

There is an action $\widetilde{\sigma}$ of $\mathbb{N}^k$ on $\widetilde{X} = X \times \mathbb{Z}^k$ by surjective local homeomorphisms given by $\widetilde{\sigma}^q(x,p) = (\sigma^q(x), p+q)$. Moreover, there is an isomorphism $\mathcal{G} \times_c \mathbb{Z}^k \cong \mathcal{G}(\widetilde{X},\widetilde{\sigma})$ given by

$$((x,m,y),p) \mapsto ((x,p),m,(y,p+m)). \tag{1}$$

The full and reduced $C^*$-algebras of a Deaconu–Renault groupoid coincide (see, for example, [49, Lem. 3.5]).

[2]sometimes referred to as a transformation groupoid (see [16]).

2.2. **$k$-graphs, their path groupoids, and their $C^*$-algebras.** For $k \geq 1$, a *$k$-graph* is a non-empty countable small category equipped with a functor $d : \Lambda \to \mathbb{N}^k$ that satisfies the following *factorization property*: For all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$ there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$. When $d(\lambda) = n$, we say $\lambda$ has *degree* $n$, and we write $\Lambda^n = d^{-1}(n)$. The standard generators of $\mathbb{N}^k$ are denoted by $\varepsilon_1, \ldots, \varepsilon_k$, and we write $n_i$ for the $i$-th coordinate of $n \in \mathbb{N}^k$. We define a partial order on $\mathbb{N}^k$ by $m \leq n$ if $m_i \leq n_i$ for all $i \leq k$.

If $\Lambda$ is a $k$-graph, its *vertices* are the elements of $\Lambda^0$. The factorization property implies that these are precisely the identity morphisms, and so can be identified with the objects. For $\lambda \in \Lambda$ the *source* $s(\lambda)$ is the domain of $\lambda$, and the *range* $r(\lambda)$ is the codomain of $\lambda$ (strictly speaking, $s(\lambda)$ and $r(\lambda)$ are the identity morphisms associated to the domain and codomain of $\lambda$). Given $\lambda, \mu \in \Lambda$ and $E \subseteq \Lambda$, we define

$$\begin{aligned}\lambda E &= \{\lambda\nu \mid \nu \in E,\, r(\nu) = s(\lambda)\},\\ E\mu &= \{\nu\mu \mid \nu \in E,\, s(\nu) = r(\mu)\},\\ \lambda E\mu &= (\lambda E)\mu = \lambda(E\mu).\end{aligned}$$

In particular, for $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we have $v\Lambda^n = \{\lambda \in \Lambda^n \mid r(\lambda) = v\}$.

We say that the $k$-graph $\Lambda$ is *row-finite* if $|v\Lambda^n| < \infty$ for each $n \in \mathbb{N}^k$ and $v \in \Lambda^0$, and has *no sources* if $0 < |v\Lambda^m|$ for all $v \in \Lambda^0$ and $m \in \mathbb{N}^k$.

Let $A$ be an abelian group. Given a functor $c : \Lambda \to A$, we may form the skew-product $k$-graph $\Lambda \times_c A$ which is the set $\Lambda \times A$ endowed with structure maps given by (see [25, Def. 5.1])

$$\begin{aligned}r(\lambda, a) &= (r(\lambda), a),\\ s(\lambda, a) &= (s(\lambda), a + c(\lambda)),\\ (\lambda, a)(\mu, a + c(\lambda)) &= (\lambda\mu, a),\\ d(\lambda, a) &= d(\lambda).\end{aligned}$$

There is a natural $A$-action $\alpha$ on $\Lambda \times_c A$ given by $\alpha_b(\lambda, a) = (\lambda, a + b)$.

**Examples 2.3.** (a) A 1-graph is the path category of a directed graph [25].

(b) Let $\operatorname{Mor}\Omega_k = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k \mid m \leq n\}$, and $\operatorname{Obj}\Omega_k = \mathbb{N}^k$. Define $r, s : \operatorname{Mor}\Omega_k \to \operatorname{Obj}\Omega_k$ by $r(m, n) = m$, $s(m, n) = n$, and for $m \leq n \leq p \in \mathbb{N}^k$ define $(m, n)(n, p) = (m, p)$ and $d(m, n) = n - m$. Then $(\Omega_k, d)$ is a $k$-graph. We identify $\operatorname{Obj}\Omega_k$ with $\{(m, m) \mid m \in \mathbb{N}^k\} \subseteq \operatorname{Mor}\Omega_k$.

Let $\Lambda$ be a row-finite $k$-graph with no sources. The set

$$\Lambda^\infty = \{x : \Omega_k \to \Lambda \mid x \text{ is a degree-preserving functor}\}$$

is called the *infinite path space* of $\Lambda$. For $v \in \Lambda^0$, we put

$$v\Lambda^\infty = \{x \in \Lambda^\infty \mid x(0, 0) = v\}.$$

For $\lambda \in \Lambda$, let $Z(\lambda) = \{x \in \Lambda^\infty \mid x(0, d(\lambda)) = \lambda\}$. Then $\{Z(\lambda) \mid \lambda \in \Lambda\}$ forms a basis of compact open sets for a topology on $\Lambda^\infty$. For $p \in \mathbb{N}^k$, the shift map $\sigma^p : \Lambda^\infty \to \Lambda^\infty$ defined by $(\sigma^p(x))(m,n) = x(m+p, n+p)$ for $x \in \Lambda^\infty$ is a local homeomorphism (for more details see [25, Rem. 2.5, Lem. 2.6]).

Following [25, Def. 2.7], we define the $k$-graph groupoid of $\Lambda$ to be the Deaconu–Renault groupoid

$$\begin{aligned}\mathcal{G}_\Lambda &:= \mathcal{G}(\Lambda^\infty, \sigma)\\ &= \{(x, m-n, y) \in \Lambda^\infty \times \mathbb{Z}^k \times \Lambda^\infty \mid m, n \in \mathbb{N}^k,\ \sigma^m(x) = \sigma^n(y)\}. \end{aligned} \tag{2}$$

The sets $Z(\mu,\nu) := \{(\mu x, d(\mu) - d(\nu), \nu x) \mid x \in Z(s(\mu))\}$ indexed by pairs $\mu, \nu \in \Lambda$ with $s(\mu) = s(\nu)$ form a basis of compact open bisections for a locally compact Hausdorff topology on $\mathcal{G}_\Lambda$. With this topology $\mathcal{G}_\Lambda$ is an ample Hausdorff groupoid (see [25, Prop. 2.8]). The sets $Z(\lambda) = Z(\lambda,\lambda)$ form a basis for the relative topology on $\mathcal{G}_\Lambda^{(0)} \subseteq \mathcal{G}_\Lambda$. We identify $\mathcal{G}_\Lambda^{(0)} = \{(x, 0, x) \mid x \in \Lambda^\infty\}$ with $\Lambda^\infty$. The groupoid $\mathcal{G}_\Lambda$ is minimal if and only if $\Lambda$ is cofinal [25, Proof of Prop. 4.8].

As in the proof of [25, Thm. 5.2], there is a bijection between $\Lambda^\infty \times \mathbb{Z}^k$ and $(\Lambda \times_d \mathbb{Z}^k)^\infty$ given by $(x,p) \mapsto ((m,n) \mapsto (x(m,n), m+p))$. After making this identification, we obtain an isomorphism of the groupoid $\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k}$ of the skew-product $k$-graph with the skew-product groupoid $\mathcal{G}_\Lambda \times_c \mathbb{Z}^k$ corresponding to the canonical cocycle $c(x,n,y) = n$ via the map

$$((x,p), m-n, (y,q)) \mapsto ((x, m-n, y), p).$$

Let $\Lambda$ be a row-finite $k$-graph with no sources. A *Cuntz–Krieger $\Lambda$-family* in a $C^*$-algebra $B$ is a function $s : \lambda \mapsto s_\lambda$ from $\Lambda$ to $B$ such that

(CK1) $\{s_v \mid v \in \Lambda^0\}$ is a collection of mutually orthogonal projections;
(CK2) $s_\mu s_\nu = s_{\mu\nu}$ whenever $s(\mu) = r(\nu)$;
(CK3) $s_\lambda^* s_\lambda = s_{s(\lambda)}$ for all $\lambda \in \Lambda$;
(CK4) $s_v = \sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^*$ for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The *$k$-graph $C^*$-algebra* $C^*(\Lambda)$ is the universal $C^*$-algebra generated by a Cuntz–Krieger $\Lambda$-family. There is an isomorphism $C^*(\Lambda) \cong C^*(\mathcal{G}_\Lambda)$ satisfying $s_\lambda \mapsto 1_{Z(\lambda, s(\lambda))}$ (see [25, Cor. 3.5 (i)]).

As discussed above, we have $\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k} \cong \mathcal{G}_\Lambda \times_c \mathbb{Z}^k$, and so

$$C^*(\mathcal{G}_\Lambda \times_c \mathbb{Z}^k) \cong C^*(\mathcal{G}_{\Lambda \times_d \mathbb{Z}^k}) \cong C^*(\Lambda \times_d \mathbb{Z}^k).$$

These are approximately finite-dimensional (AF) algebras by [25, Lem. 5.4].

2.4. ***K*-theory for *C**-algebras.** This paper is concerned primarily with calculating groupoid homology, but it is motivated by the relationship between this and $K$-theory of groupoid $C^*$-algebras, and some of our key results concern $K$-theory. Readers unfamiliar with $C^*$-algebras and their $K$-theory, and who are primarily interested in groupoids and groupoid homology will not need to know more about $C^*$-algebraic $K$-theory than the following points: $C^*$-algebraic $K$-theory associates two abelian groups $K_0(A)$ and $K_1(A)$ to each $C^*$-algebra $A$; these groups are invariant under Morita equivalence, and continuous with respect to inductive limits; the $K_0$-group is the Grothendieck group of a semigroup consisting of equivalence classes of projections in matrix algebras over $A$; the $K$-groups of a crossed-product of a $C^*$-algebra $A$ by $\mathbb{Z}$ are related to those of $A$ by the Pimsner–Voiculescu exact sequence [37]; and the $K$-groups of a crossed-product of a $C^*$-algebra $A$ by $\mathbb{Z}^k$ fit into a spectral sequence, due to Kasparov [21], involving the homology groups of $\mathbb{Z}^k$ with values in the $K$-groups of $A$. For more background on $C^*$-algebraic $K$-theory, we refer the interested reader to [53, 43, 2].

2.5. ***c*-soft sheaves.** Let $X$ be a locally compact Hausdorff space. By a *sheaf of abelian groups*, or simply a sheaf, over $X$, we mean a (not necessarily Hausdorff) étale groupoid $\mathcal{F}$ with unit space $\mathcal{F}^{(0)} = X$ in which $r = s$, so every element belongs to the isotropy, and in which each isotropy group $\mathcal{F}_x = \mathcal{F}_x^x$ is abelian. We think of $\mathcal{F}$ as a group bundle over $X$ with bundle map $r = s$. Given a subset $W \subseteq X$, we write $\Gamma(W, \mathcal{F})$ for the set

$$\{t : W \to \mathcal{F} \mid t(w) \in \mathcal{F}_w \text{ and } t \text{ is continuous}\}$$

of continuous sections of $\mathcal{F}$ over $W$. A sheaf $\mathcal{F}$ over $X$ is said to be *c-soft* if the restriction map $\Gamma(X, \mathcal{F}) \to \Gamma(K, \mathcal{F})$ is surjective for any compact set $K \subseteq X$ (see, e.g., [20, Def. 2.5.5] or [4, Definition II.9.1]); that is, if every continuous section of $\Gamma$ over a compact subset of $X$ extends to a continuous section over all of $X$.

The property of $c$-softness is a key hypothesis for results of Crainic and Moerdijk (see [10]) that we will need in our study of the homology of ample Hausdorff groupoids.

**Lemma 2.6.** *Let $X$ be a zero-dimensional topological space, and let $K \subseteq X$ be compact. If $W$ is a compact open subset of $K$ (in the relative topology on $K$), then there exists a compact open set $V \subseteq X$ such that $W = V \cap K$.*

*Proof.* Since $W$ is open in the relative topology, $W = \widehat{V} \cap K$, with $\widehat{V}$ open in $X$. Let $\mathcal{U}$ be a basis of compact open sets for the topology on $X$; so $\mathcal{U}' := \{U \in \mathcal{U} \mid U \subseteq \widehat{V}\}$ satisfies $\widehat{V} = \bigcup_{U \in \mathcal{U}'} U$. Since $\mathcal{U}'_K := \{U \cap K \mid U \in \mathcal{U}'\}$ is an open cover of the compact set $W \subseteq K$ (in the relative topology), there is a finite subset $F \subseteq \mathcal{U}'$ such that

$$W = \bigcup_{U \in F} (U \cap W).$$

Now $V := \bigcup_{U \in F} U$ is compact open, and $W = V \cap K$. □

**Proposition 2.7.** *Let $X$ be a zero-dimensional topological space. Then the constant sheaf $\mathcal{F}$ on $X$ with values in a discrete abelian group $A$ is c-soft.*

*Proof.* Since $\mathcal{F}$ is the constant sheaf, for every $W \subseteq X$ we have $\Gamma(W, \mathcal{F}) \cong C(W, A)$. Let $K \subseteq X$ be compact and fix $f \in C(K, A)$. Then $f(K)$ is a compact subset of the discrete group $A$, and hence finite. For $a \in f(K)$ we let $U_a = f^{-1}(a)$. Since $f$ is continuous, each $U_a \subseteq K$ is clopen, and since $K$ is compact, $U_a$ is compact and open. By Lemma 2.6, for each $a \in A$ there exists a compact open set $V_a \subseteq X$ such that $U_a = V_a \cap K$. Fix a total order $\leq$ on $f(K)$, and for each $a \in f(K)$ define $V'_a := V_a \setminus \bigcup_{b<a} V_b$. Then each $V'_a$ is

a compact open subset of $X$ since the $V_a$ are compact open. Moreover, since the $U_a$ are mutually disjoint, we have $V_a' \cap K = V_a \cap K$ for all $a$. Hence the formula

$$\tilde{f}(x) := \begin{cases} a & \text{for } x \in V_a', \\ 0 & \text{otherwise,} \end{cases}$$

defines a continuous extension $\tilde{f}$ of $f$ to $X$. □

## 3. Equivalence of ample Hausdorff groupoids

There are a number of notions of equivalence of groupoids that are relevant to us here, and we need to know that they all coincide for ample Hausdorff groupoids. The notions that we consider are Morita equivalence [10], groupoid equivalence [34] (see also [42]), equivalence via a linking groupoid, equivalence via isomorphic ampliations, similarity [40, 41, 30], Kakutani equivalence [30], and stable isomorphism [8]. We show that the first four of these notions coincide for all Hausdorff étale groupoids (Proposition 3.10), and that they all coincide for ample Hausdorff groupoids with $\sigma$-compact unit spaces (Theorem 3.12). In addition to providing the basis for our own results, this furnishes details for some previous work on the subject, which has implicitly assumed that various of these notions of equivalence coincide for ample groupoids.

The following notion of similarity, recorded by Matui [30, Def. 3.4] and called homological similarity there, appears in the context of algebraic groupoids in Renault's thesis [41], and earlier still in [40], where Ramsay in turn attributes it to earlier work of Mackey. An adaptation of this notion to the situation of Lie groupoids also appears in [33] where it is called strong equivalence.

**Definition 3.1** ([41, Def. I.1.3], [30, Def. 3.4]). Let $\mathcal{G}, \mathcal{H}$ be Hausdorff étale groupoids and let $\rho, \sigma : \mathcal{G} \to \mathcal{H}$ be continuous homomorphisms. We say that $\rho$ is similar to $\sigma$ if there is a continuous map $\theta : \mathcal{G}^{(0)} \to \mathcal{H}$ such that

$$\theta(r(g))\rho(g) = \sigma(g)\theta(s(g)) \quad \text{for all } g \in \mathcal{G}.$$

We say that $\mathcal{G}$ and $\mathcal{H}$ are *similar* groupoids if there exist étale homomorphisms $\rho : \mathcal{G} \to \mathcal{H}$ and $\sigma : \mathcal{H} \to \mathcal{G}$ such that $\sigma \circ \rho$ is similar to $\mathrm{id}_{\mathcal{G}}$ and $\rho \circ \sigma$ is similar to $\mathrm{id}_{\mathcal{H}}$. In this case, each of the two maps, $\rho$ and $\sigma$, is called a *similarity.*

**Remark 3.2.** It is not stated in [41] or in [30] that similarity of groupoids is an equivalence relation, but this is standard (it is essentially the argument that category equivalence is an equivalence relation). It is also easy to give a direct argument: suppose that $\sigma : \mathcal{G} \to \mathcal{H}$ and $\rho : \mathcal{H} \to \mathcal{G}$ implement a similarity, and that $\alpha : \mathcal{H} \to \mathcal{K}$ and $\beta : \mathcal{K} \to \mathcal{H}$ also implement a similarity. We aim to show that $\alpha \circ \sigma$ and $\rho \circ \beta$ implement a similarity. By symmetry, it suffices to find $\kappa : \mathcal{G}^{(0)} \to \mathcal{G}$ such that

$$\kappa(r(g))(\rho \circ \beta \circ \alpha \circ \sigma)(g) = g\kappa(s(g)) \quad \text{for all } g.$$

Since $\rho \circ \sigma \sim \mathrm{id}_{\mathcal{G}}$ and $\beta \circ \alpha \sim \mathrm{id}_{\mathcal{H}}$, there are $\theta : \mathcal{G}^{(0)} \to \mathcal{G}$ and $\eta : \mathcal{H}^{(0)} \to \mathcal{H}$ such that

$$\theta(r(g))\rho(\sigma(g)) = g\theta(s(g)) \quad \text{and} \quad \eta(r(h))\beta(\alpha(h)) = h\eta(s(h)).$$

Define

$$\kappa(x) = \theta(x)\rho\big(\eta(r(\sigma(x)))\big).$$

Then, using that $r(\sigma(r(g))) = r(\sigma(g))$ and that $\rho$ is a homomorphism, we compute

$$\begin{aligned}\kappa(r(g))(\rho \circ \beta \circ \alpha \circ \sigma)(g) &= \theta(r(g))\rho\big(\eta(r(\sigma(g)))\beta(\alpha(\sigma(g)))\big)\\ &= \theta(r(g))\rho(\sigma(g))\rho\big(\eta(s(\sigma(g)))\big)\\ &= g\theta(s(g))\rho\big(\eta(s(\sigma(g)))\big) = g\kappa(s(g)).\end{aligned}$$

Thus $(\rho \circ \beta) \circ (\alpha \circ \sigma)$ is similar to $\mathrm{id}_{\mathcal{G}}$ and by symmetry $(\alpha \circ \sigma) \circ (\rho \circ \beta)$ is similar to $\mathrm{id}_{\mathcal{K}}$.

**Remark 3.3.** Let $\rho, \sigma : \mathcal{G} \to \mathcal{H}$ be continuous groupoid homomorphisms between Hausdorff étale groupoids. Then $\rho$ and $\sigma$ both induce well-defined orbit maps $[u] \mapsto [\rho(u)]$ and $[u] \mapsto [\sigma(u)]$. Suppose that $\rho$ and $\sigma$ are similar; then $[\rho(u)] = [\sigma(u)]$ for all $u \in \mathcal{G}^{(0)}$, and so the orbit maps induced by $\rho$ and $\sigma$ are equal. It follows that every similarity of groupoids induces a bijection between their orbit spaces.

**Definition 3.4** ([10, Sect. 4.5])**.** Let $\mathcal{G}$, $\mathcal{H}$ be groupoids. A continuous functor $\varphi : \mathcal{G} \to \mathcal{H}$ is a *weak equivalence* if

(i) the map from $\mathcal{G}^{(0)} *_{\mathcal{H}^{(0)}} \mathcal{H} := \{(u, \gamma) \in \mathcal{G}^{(0)} \times \mathcal{H} \mid \varphi(u) = r(\gamma)\}$ to $\mathcal{H}^{(0)}$ given by $(u, \gamma) \mapsto s(\gamma)$ is an étale surjection, and

(ii) the map $\varphi$ induces an isomorphism from $\mathcal{G}$ to the fibred product

$$\begin{aligned}&\mathcal{G}^{(0)} *_{\mathcal{H}^{(0)}} \mathcal{H} *_{\mathcal{H}^{(0)}} \mathcal{G}^{(0)}\\ &\qquad := \{(u, \gamma, v) \in \mathcal{G}^{(0)} \times \mathcal{H} \times \mathcal{G}^{(0)} \mid r(\gamma) = \varphi(u),\, s(\gamma) = \varphi(v)\}.\end{aligned}$$

If such a $\varphi$ exists, then we write $\mathcal{G} \xrightarrow{\sim} \mathcal{H}$. The groupoids $\mathcal{G}$ and $\mathcal{H}$ are *Morita equivalent* if there is a groupoid $\mathcal{K}$ that admits weak equivalences $\mathcal{G} \xleftarrow{\sim} \mathcal{K} \xrightarrow{\sim} \mathcal{H}$.

**Remark 3.5.** It is not stated explicitly in [10] that Morita equivalence is in fact an equivalence relation, though this is certainly standard. In any case it follows from Theorem 3.12 below.

In [33, Prop. 5.11] it is shown that the analog of similarity for Lie groupoids implies the analog of Morita equivalence. We briefly indicate how to modify their argument to obtain the same result for Hausdorff étale groupoids.

**Lemma 3.6.** *Let $\mathcal{G}$, $\mathcal{H}$ be Hausdorff étale groupoids. If $\mathcal{G}$ and $\mathcal{H}$ are similar, then they are Morita equivalent.*

*Proof.* Suppose that $\sigma : \mathcal{G} \to \mathcal{H}$ and $\rho : \mathcal{H} \to \mathcal{G}$ constitute a similarity of groupoids. We claim that the map $\sigma$ is a weak equivalence. It is easy to show that $(u, h) \mapsto s(h)$ is a surjection from $\mathcal{G}^{(0)} *_{\mathcal{G}^{(0)}} \mathcal{H}$ to $\mathcal{H}^{(0)}$: if $\theta : \mathcal{H}^{0} \to \mathcal{H}$

is the map that implements the similarity $\sigma \circ \rho \sim \mathrm{id}_{\mathcal{H}}$, then each $v \in \mathcal{H}^{(0)}$ is equal to the image of $(\rho(v), \theta(v)^{-1})$. We claim that it is an étale map. For this, fix $(u, h) \in \mathcal{G}^{(0)} *_{\mathcal{G}^{(0)}} \mathcal{H}$, and use that $\sigma$ is an étale map to choose a neighborhood $U$ of $u$ such that $\sigma|_U$ is a homeomorphism onto its range. Pick a bisection neighborhood $B$ of $h$. By shrinking if necessary, we can assume that $\sigma(U) = r_{\mathcal{H}}(B)$. Then $s \circ \pi_2$ is a homeomorphism of $U * B$ onto $B$.

Since $\rho, \sigma$ constitute a similarity of groupoids, [29, Thm. IV.4.1] implies that the map $g \mapsto (r(g), \sigma(g), s(g))$ is a bijection from $\mathcal{G}$ to $\mathcal{G}^{(0)} *_{\mathcal{H}^{(0)}} \mathcal{H} *_{\mathcal{H}^{(0)}} \mathcal{G}^{(0)}$. For $g \in G$ we use that $r$, $\sigma$ and $s$ are étale maps to find a neighborhood $U$ of $g$ on which they are all homeomorphisms, and observe that then $g \mapsto (r(g), \sigma(g), s(g))$ is a homeomorphism onto $r(U) *_{\mathcal{H}^{(0)}} \sigma(U) *_{\mathcal{H}^{(0)}} s(U)$. So $g \mapsto (r(g), \sigma(g), s(g))$ is continuous and open. □

The third notion of equivalence that we consider is the one formulated by Renault (see [42, Sect. 3]) and studied in [34]. Given a locally compact Hausdorff groupoid $\mathcal{G}$, we say that a locally compact Hausdorff space $Z$ is a *left $\mathcal{G}$-space* if it is equipped with a continuous open map $r : Z \to \mathcal{G}^{(0)}$ and a continuous pairing $(g, z) \mapsto g \cdot z$ from $\mathcal{G} * X$ to $X$ such that $r(g \cdot z) = r(g)$ and $(gh) \cdot z = g \cdot (h \cdot z)$ and such that $r(z) \cdot z = z$. We say that $Z$ is a free and proper left $\mathcal{G}$-space if the map $(g, x) \mapsto (g \cdot x, x)$ is a proper injection from $\mathcal{G} * X$ to $\mathcal{G} \times X$. Right $\mathcal{G}$-spaces are defined analogously.

**Definition 3.7** ([34, Def. 2.1], [42, Sect. 3])**.** The groupoids $\mathcal{G}$ and $\mathcal{H}$ are *equivalent* if there is a locally compact Hausdorff space $Z$ such that

(i) $Z$ is a free and proper left $\mathcal{G}$-space with anchor map $r : Z \to \mathcal{G}^{(0)}$,
(ii) $Z$ is a free and proper right $\mathcal{H}$-space with anchor map $s : Z \to \mathcal{H}^{(0)}$,
(iii) the actions of $\mathcal{G}$ and $\mathcal{H}$ on $Z$ commute,
(iv) $r : Z \to \mathcal{G}^{(0)}$ induces a homeomorphism $Z/\mathcal{H} \to \mathcal{G}^{(0)}$,
(v) $r : Z \to \mathcal{H}^{(0)}$ induces a homeomorphism $\mathcal{G} \setminus Z \to \mathcal{H}^{(0)}$.

The fourth notion of equivalence we need to discuss is the generalization of Kakutani equivalence developed by Matui [30, Def. 4.1] in the situation of ample Hausdorff groupoids with compact unit spaces, and extended to non-compact unit spaces in [8]. This notion has previously been discussed only for ample groupoids, but it makes sense for general Hausdorff étale groupoids, and in particular weak Kakutani equivalence is a fairly natural notion in this setting (though in this more general setting it is not an equivalence relation; see Example 3.13).

**Definition 3.8.** The Hausdorff étale groupoids $\mathcal{G}$ and $\mathcal{H}$ are *weakly Kakutani equivalent* if there are full open subsets $X \subseteq \mathcal{G}^{(0)}$ and $Y \subseteq \mathcal{H}^{(0)}$ such that $\mathcal{G}|_X \cong \mathcal{H}|_Y$. They are *Kakutani equivalent* if $X$ and $Y$ can be chosen to be clopen sets.

For ample Hausdorff groupoids with $\sigma$-compact unit spaces, [8, Thm. 3.2] shows that weak Kakutani equivalence and Kakutani equivalence both coincide with groupoid equivalence, and with a number of other notions of equivalence.

Our next two results show first that for Hausdorff étale groupoids, Morita equivalence and equivalence in the sense of [34] are equivalent to the existence of a linking groupoid, and to the existence of isomorphic ampliations of the two groupoids in the following sense.

If $\mathcal{G}$ is a Hausdorff étale groupoid, $X$ is a locally compact Hausdorff space, and $\psi : X \to \mathcal{G}^{(0)}$ is a local homeomorphism, then the *ampliation* (also known as the *blow-up* [55, Sect. 3.3]) $\mathcal{G}^\psi$ of $\mathcal{G}$ corresponding to $\psi$ is given by

$$\mathcal{G}^\psi = \{(x,\gamma,y) \in X \times \mathcal{G} \times X \mid \psi(x) = r(\gamma) \text{ and } \psi(y) = s(\gamma)\},$$

with $((x,\gamma,y),(w,\eta,z)) \in (\mathcal{G}^\psi)^{(2)}$ if and only if $y = w$, and composition and inverses given by $(x,\gamma,y)(y,\eta,z) = (x,\gamma\eta,z)$ and $(x,\gamma,y)^{-1} = (y,\gamma^{-1},x)$. This is a Hausdorff étale groupoid under the relative topology inherited from $X \times \mathcal{G} \times X$.

**Example 3.9.** Let $X$ and $Y$ be locally compact Hausdorff spaces and let $\psi : Y \to X$ be a local homeomorphism. Then we may regard

$$R(\psi) := \{(y_1,y_2) \in Y \times Y \mid \psi(y_1) = \psi(y_2)\}$$

as an Hausdorff étale groupoid (see [24]). Note that $R(\psi)$ is the ampliation of the trivial groupoid $X$ corresponding to $\psi$.

**Proposition 3.10.** *Let $\mathcal{G}$ and $\mathcal{H}$ be Hausdorff étale groupoids. The following are equivalent:*

(i) *$\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent;*
(ii) *there is a Hausdorff étale groupoid $\mathcal{L}$ and a decomposition $\mathcal{L}^{(0)} = X \sqcup Y$ of $\mathcal{L}^{(0)}$ into complementary full clopen subsets such that $\mathcal{L}|_X \cong \mathcal{G}$ and $\mathcal{L}|_Y \cong \mathcal{H}$;*
(iii) *$\mathcal{G}$ and $\mathcal{H}$ are equivalent in the sense of Renault;*
(iv) *$\mathcal{G}$ and $\mathcal{H}$ admit isomorphic ampliations.*

*If $\mathcal{G}$ and $\mathcal{H}$ are weakly Kakutani equivalent, then they satisfy* (i)–(iv).

*Proof.* For (i) $\Longrightarrow$ (ii), suppose that $\varphi : \mathcal{G} \to \mathcal{H}$ is a weak equivalence in the sense of Definition 3.4. Then $\mathcal{G}$ is isomorphic to the fibred product

$$\mathcal{G}^{(0)} * \mathcal{H} * \mathcal{G}^{(0)} := \{(x,\eta,y) \in \mathcal{G}^{(0)} \times \mathcal{H} \times \mathcal{G}^{(0)} \mid \varphi(x) = r(\eta) \text{ and } \varphi(y) = s(\eta)\}.$$

Using the first condition of Definition 3.4 it is straightforward to check that under the natural operations and topology, the disjoint union

$$\mathcal{L} := (\mathcal{G}^{(0)} * \mathcal{H} * \mathcal{G}^{(0)}) \sqcup (\mathcal{G}^{(0)} * \mathcal{H}) \sqcup (\mathcal{H} * \mathcal{G}^{(0)}) \sqcup \mathcal{H}$$

is a Hausdorff étale groupoid satisfying (ii) with respect to $X = \mathcal{G}^{(0)}$ and $Y := \mathcal{H}^{(0)}$.

For (ii) $\Longrightarrow$ (iii), one checks that given $\mathcal{L}$ as in (ii), the subspace

$$Z := \{z \in \mathcal{L} \mid r(z) \in X \text{ and } s(z) \in Y\},$$

under the actions of $\mathcal{G}$ and $\mathcal{H}$ by multiplication on either side, is a $\mathcal{G}$-$\mathcal{H}$-equivalence as in (iii).

For (iii) $\implies$ (iv), suppose that $Z$ is a $\mathcal{G}$–$\mathcal{H}$-equivalence; to avoid confusion, we will write $\rho : Z \to \mathcal{G}^{(0)}$ and $\sigma : Z \to \mathcal{H}^{(0)}$ for the anchor maps. Since $Z/\mathcal{H} \cong \mathcal{G}^{(0)}$ and since the right $\mathcal{H}$-action is free, if $x, y \in Z$ satisfy $\rho(x) = \rho(y)$, then there is a unique element $[x, y]_{\mathcal{H}}$ of $\mathcal{H}$ satisfying $x \cdot [x, y]_{\mathcal{H}} = y$. By [34, Sect. 2], the map $[\,\cdot\,,\cdot\,]_{\mathcal{H}}$ is continuous (see also [48, Lem. 2.1]). Similarly there is a continuous pairing $(x, y) \mapsto {}_{\mathcal{G}}[x, y]$ from $\{(x, y) \in Z^2 \mid \sigma(x) = \sigma(y)\}$ to $\mathcal{G}$ such that ${}_{\mathcal{G}}[x, y] \cdot y = x$. Consider the ampliations

$$\mathcal{G}^{\rho} := \{(x, \gamma, y) \mid x, y \in Z,\ \gamma \in \mathcal{G},\ \rho(x) = r(\gamma),\ \rho(y) = s(\gamma)\},$$
$$\mathcal{H}^{\sigma} := \{(x, \eta, y) \mid x, y \in Z,\ \eta \in \mathcal{H},\ \sigma(x) = r(\eta),\ \sigma(y) = s(\eta)\}.$$

If $(x, \gamma, y) \in \mathcal{G}^{\rho}$, then $\rho(\gamma \cdot y) = r(\gamma) = \rho(x)$, and so we can take the pairing $[x, \gamma \cdot y]_{\mathcal{H}}$ to obtain an element $\Theta(x, \gamma, y) := (x, [x, \gamma \cdot y]_{\mathcal{H}}, y) \in \mathcal{H}^{\sigma}$. It is routine to check that this is a continuous groupoid homomorphism. Symmetrically, we see that $\Theta' : (x, \eta, y) \mapsto (x, {}_{\mathcal{G}}[x \cdot \eta, y], y)$ is a continuous groupoid homomorphism from $\mathcal{H}^{\sigma}$ to $\mathcal{G}^{\rho}$. A simple calculation using the defining properties of ${}_{\mathcal{G}}[\,\cdot\,,\cdot\,]$ and $[\,\cdot\,,\cdot\,]_{\mathcal{H}}$ shows $\Theta$ and $\Theta'$ are mutually inverse. So $\Theta$ is an isomorphism, giving (iv).

For (iv) $\implies$ (i), fix ampliations $\mathcal{G}^{\varphi}$, $\mathcal{H}^{\psi}$ and an isomorphism $\Theta : \mathcal{G}^{\varphi} \to \mathcal{H}^{\psi}$. Write $\pi^{\mathcal{G}}$ for the canonical map $(x, \gamma, y) \mapsto \gamma$ from $\mathcal{G}^{\varphi}$ to $\mathcal{G}$, and $\pi^{\mathcal{H}}$ for the corresponding map from $\mathcal{H}^{\psi}$ to $\mathcal{H}$. We obtain continuous groupoid homomorphisms $\tilde{\varphi} : \mathcal{G}^{\varphi} \to \mathcal{G}$ and $\tilde{\psi} : \mathcal{G}^{\varphi} \to \mathcal{H}$ by $\tilde{\varphi} := \pi^{\mathcal{G}}$ and $\tilde{\psi} := \pi^{\mathcal{H}} \circ \Theta$, respectively. It is routine to check that this determines a Morita equivalence

$$\mathcal{G} \xleftarrow{\tilde{\varphi}} \mathcal{G}^{\varphi} \xrightarrow{\tilde{\psi}} \mathcal{H}.$$

For the final statement, observe that if $U$ is a full open subset of $\mathcal{G}^0$, then the argument of [9, Lem. 6.1] shows that $\mathcal{G}U = \{g \in \mathcal{G} \mid s(g) \in U\}$ is a $\mathcal{G}$–$\mathcal{G}|_U$-equivalence under the actions determined by multiplication in $\mathcal{G}$. So, writing $\sim_R$ for equivalence in the sense of Renault, if $\mathcal{G}$ and $\mathcal{H}$ are weakly Kakutani equivalent, say $\mathcal{G}|_U \cong \mathcal{H}|_V$, then we have $\mathcal{G} \sim_R \mathcal{G}|_U \cong \mathcal{H}|_V \sim_R \mathcal{H}$. Since $\sim_R$ is an equivalence relation, we deduce that $\mathcal{G} \sim_R \mathcal{H}$. □

**Remarks 3.11.** (i) Recall the definition of $R(\psi)$ from Example 3.9. Proposition 3.10 shows that $R(\psi)$ and $X$ are equivalent, and so $C^*(R(\psi))$ is Morita equivalent to $C_0(X)$.

(ii) It follows from the proof of Proposition 3.10 that if $\mathcal{G}$ and $\mathcal{H}$ admit isomorphic ampliations, then there exist a locally compact Hausdorff space $X$, local homeomorphisms $\varphi : X \to \mathcal{G}^{(0)}$ and $\psi : X \to \mathcal{H}^{(0)}$, and an isomorphism $\Theta : \mathcal{G}^{\varphi} \to \mathcal{H}^{\psi}$ such that $\Theta(x, \varphi(x), x) = (x, \psi(x), x)$ for all $x \in X$.

(iii) Let $\mathcal{G}$ and $\mathcal{H}$ be minimal Hausdorff étale groupoids which are equivalent in the sense of Renault. Then with notation as in Proposition 3.10 (ii), we may identify $\mathcal{G} = \mathcal{L}|_X$ and $\mathcal{H} = \mathcal{L}|_Y$ where $\mathcal{L}$ is a Hausdorff étale groupoid and $X$ and $Y$ are complementary full clopen subsets of $\mathcal{L}^{(0)}$. Let $U \subseteq \mathcal{L}$ be an open bisection such that $r(U) \subseteq X$ and $s(U) \subseteq Y$. Since $\mathcal{L}$ is minimal, both $r(U)$ and $s(U)$ are full open subsets. It follows that $\mathcal{G}|_{r(U)} \cong \mathcal{H}|_{s(U)}$, and so $\mathcal{G}$ and $\mathcal{H}$ are weakly Kakutani equivalent.

Our next result shows that the notions of equivalence in Proposition 3.10 are further equivalent to a number of additional conditions, including similarity, in the special case of ample Hausdorff groupoids with $\sigma$-compact unit spaces. We write $\mathcal{R}$ for the (discrete) full equivalence relation $\mathcal{R} = \mathbb{N} \times \mathbb{N}$.

**Theorem 3.12.** *Let $\mathcal{G}$ and $\mathcal{H}$ be ample Hausdorff groupoids with $\sigma$-compact unit spaces. Then the following are equivalent:*

(i) *$\mathcal{G}$ and $\mathcal{H}$ are similar;*
(ii) *$\mathcal{G}$ and $\mathcal{H}$ are Morita equivalent;*
(iii) *there exist an ample Hausdorff groupoid $\mathcal{L}$ and a decomposition $\mathcal{L}^{(0)} = X \sqcup Y$ of $\mathcal{L}^{(0)}$ into complementary full clopen subsets such that $\mathcal{L}|_X \cong \mathcal{G}$ and $\mathcal{L}|_Y \cong \mathcal{H}$;*
(iv) *$\mathcal{G}$ and $\mathcal{H}$ are equivalent in the sense of Renault;*
(v) *$\mathcal{G}$ and $\mathcal{H}$ admit isomorphic ampliations;*
(vi) *$\mathcal{G} \times \mathcal{R} \cong \mathcal{H} \times \mathcal{R}$;*
(vii) *$\mathcal{G}$ and $\mathcal{H}$ are Kakutani equivalent; and*
(viii) *$\mathcal{G}$ and $\mathcal{H}$ are weakly Kakutani equivalent.*

*Proof.* (i) $\Longrightarrow$ (ii) follows from Lemma 3.6.

Proposition 3.10 shows that (ii)–(v) are equivalent, and [8, Thm. 3.2] shows that (iv), (vi), (vii), and (viii) are equivalent. In particular, we have (i) $\Longrightarrow$ (ii) $\Longrightarrow$ $\cdots$ $\Longrightarrow$ (viii).

For (viii) $\Longrightarrow$ (i), suppose that $U \subseteq \mathcal{G}^{(0)}$ and $V \subseteq \mathcal{H}^{(0)}$ are full open sets with $\mathcal{G}|_U \cong \mathcal{H}|_V$. Matui proves in [30, Thm. 3.6 (2)] that $\mathcal{G}$ is similar to $\mathcal{G}|_U$ and $\mathcal{H}$ is similar to $\mathcal{H}|_V$. Since any isomorphism of groupoids is a similarity, and since similarity of groupoids is an equivalence relation (Remark 3.2), it follows that $\mathcal{G}$ and $\mathcal{H}$ are similar. □

To close the section, we present an example to show that groupoid equivalence does not imply either similarity or weak Kakutani equivalence in general. We also show that weak Kakutani equivalence is not an equivalence relation.

**Example 3.13.** Let $Y := \mathbb{R}$ and $X := S^1$ and define $\psi : Y \to X$ by $\psi(y) = e^{2\pi i y}$. Then $\psi$ is a local homeomorphism and the groupoid $R(\psi)$ (see Example 3.9) is equivalent to the trivial groupoid $X$ (see Remarks 3.11 (i)).

We claim that there is no similarity $\rho : X \to R(\psi)$. Indeed suppose that such a $\rho$ exists. Since $\rho$ is a groupoid map, we have $\rho(X^{(0)}) \subseteq R(\psi)^{(0)}$. Identifying $R(\psi)^{(0)} = \mathbb{R}$ and $X^{(0)} = S^1$, we obtain a continuous map $\rho : S^1 \to \mathbb{R}$. Since $\rho$ is a similarity, it induces a bijective map on orbits (see Remark 3.3). Since the orbits in $X$ are singletons, this implies that $\rho$ is injective, which is impossible.

We also claim that $X$ and $R(\psi)$ are not weakly Kakutani equivalent. To see this, suppose that $U$ is a full open subset of $X^{(0)}$. Since $X$ is trivial, we have $U = X^{(0)} = S^1$, which is not homeomorphic to any open subset of $\mathbb{R} = R(\psi)^{(0)}$. So there is no full open subset $V \subseteq R(\psi)^{(0)}$ such that $X|_U \cong R(\psi)|_V$, and so the two groupoids are not weakly Kakutani equivalent.

Consider the local homeomorphism $\varphi : X \sqcup Y \to X$ given by

$$\varphi(z) := \begin{cases} z & \text{if } z \in X, \\ \psi(z) & \text{if } z \in Y. \end{cases}$$

Then $X$ and $R(\psi)$ are each weakly Kakutani equivalent to $R(\varphi)$, but, as shown above, $X$ is not weakly Kakutani equivalent to $R(\psi)$.

## 4. Crainic–Moerdijk–Matui homology for ample Hausdorff groupoids

Crainic and Moerdijk introduced a compactly supported homology theory for Hausdorff étale groupoids in [10]. Matui reframed the theory for ample Hausdorff groupoids (though he did not explicitly require this; see [30, Def. 3.1]). To use the results of [10] we must ensure that the standing assumptions of [10, Sect. 2.5]) are satisfied. We therefore require that all groupoids we consider henceforth are locally compact, Hausdorff, second countable, and zero-dimensional.

For the reader's convenience we recall Matui's definition of homology for an ample Hausdorff groupoid $\mathcal{G}$ (see [30, Sect. 3.1]). Since a locally constant sheaf over such a groupoid with values in a discrete abelian group is $c$-soft (see Section 2.5), this agrees with the definition given by Crainic and Moerdijk [10, Sect. 3.1] under our standing assumptions.

We first need to establish some notation. Given a locally compact Hausdorff zero-dimensional space $X$ and a discrete abelian group $A$, let $C_c(X, A)$ denote the set of compactly supported $A$-valued continuous (equivalently, locally constant) functions on $X$. Then $C_c(X, A)$ is an abelian group under pointwise addition. Given a (not necessarily surjective) local homeomorphism $\psi : Y \to X$ between two such spaces, as in [30, Sect. 3.1] we define a homomorphism $\psi_* : C_c(Y, A) \to C_c(X, A)$ by

$$\psi_*(f)(x) := \sum_{\psi(y)=x} f(y) \quad \text{for all } f \in C_c(Y, A) \text{ and } x \in X. \tag{3}$$

If $U \subseteq Y$ is compact open and $\psi|_U$ is injective, then $\psi_*(1_U) = 1_{\psi(U)}$, where $1_U$ is the indicator function of $U$.

Recall that for $n > 0$ the space of composable $n$-tuples in a groupoid $\mathcal{G}$ is

$$\mathcal{G}^{(n)} = \{(g_1, \dots, g_n) \in \mathcal{G}^n \mid s(g_i) = r(g_{i+1}) \text{ for } 1 \leq i < n\}, \tag{4}$$

while $\mathcal{G}^{(0)}$ is the unit space. For $n \geq 2$ and $0 \leq i \leq n$, we define $d_i : \mathcal{G}^{(n)} \to \mathcal{G}^{(n-1)}$ by

$$d_i(g_1, \dots, g_n) := \begin{cases} (g_2, \dots, g_n), & i = 0, \\ (g_1, \dots, g_i g_{i+1}, \dots, g_n), & 1 \leq i \leq n-1, \\ (g_1, \dots, g_{n-1}), & i = n. \end{cases}$$

Note that $\mathcal{G}^{(n)}$ is zero-dimensional and each $d_i$ is a local homeomorphism.

**Definition 4.1.** Let $\mathcal{G}$ be a second-countable ample Hausdorff groupoid. For $n \geq 1$ define $\partial_n : C_c(\mathcal{G}^{(n)}, A) \to C_c(\mathcal{G}^{(n-1)}, A)$ by

$$\partial_1 = s_* - r_* \qquad \text{and} \qquad \partial_n := \sum_{i=0}^{n} (-1)^i (d_i)_* \quad \text{for } n \geq 2,$$

and define $\partial_0$ to be the zero map from $C_c(\mathcal{G}^{(0)}, A)$ to 0. Routine calculations show that this defines a chain complex $(C_c(\mathcal{G}^{(*)}, A), \partial_*)$. We define the homology of $\mathcal{G}$ with values in $A$ to be the homology of this complex, denoted by $H_*(\mathcal{G}, A)$. If $A = \mathbb{Z}$, we simply write $H_*(\mathcal{G})$.

**Remark 4.2.** An ample groupoid with one unit is just a discrete group. In this instance the groupoid homology just defined coincides with group homology; see [11, Sect. 2.22] and also [5].

Matui shows in [30, Prop. 3.5] that if $\mathcal{G}$ and $\mathcal{H}$ are similar, then $H_*(\mathcal{G}, A) \cong H_*(\mathcal{H}, A)$ for any discrete abelian group $A$. So Theorem 3.12 implies that if $\mathcal{G}$ and $\mathcal{H}$ are ample and have $\sigma$-compact unit spaces and are equivalent via any of the eight notions of equivalence listed in the statement of the theorem, then their homologies coincide. We give an explicit description of the isomorphism.

**Lemma 4.3.** *Let $\mathcal{G}$ and $\mathcal{H}$ be ample Hausdorff groupoids with $\sigma$-compact unit spaces. If the pair $\mathcal{G}$, $\mathcal{H}$ satisfies any of the eight equivalent conditions in Theorem 3.12, then $H_n(\mathcal{G}) \cong H_n(\mathcal{H})$. In particular, if $\rho : \mathcal{G} \to \mathcal{H}$ is a similarity (see Definition 3.1), then it induces an isomorphism $\rho_* : H_n(\mathcal{G}) \cong H_n(\mathcal{H})$. If $X$ is a full open subset of $\mathcal{G}^{(0)}$, then the inclusion $\mathcal{G}|_X \subseteq \mathcal{G}$ is a similarity and induces an isomorphism $H_*(\mathcal{G}|_X) \cong H_*(\mathcal{G})$.*

*Proof.* Crainic and Moerdijk show that a Morita equivalence between Hausdorff étale groupoids induces an isomorphism between their homology groups (see [10, Cor. 4.6]). The proof of [30, Prop. 3.5] shows that if $\rho : \mathcal{G} \to \mathcal{H}$ is a similarity, then $\rho$ induces an isomorphism $H_n(\mathcal{G}) \cong H_n(\mathcal{H})$ for all $n \geq 0$. Let $X$ be a full open subset of $\mathcal{G}^{(0)}$. Then the argument of [30, Thm. 3.6] proves that the inclusion $\mathcal{G}|_X \subseteq \mathcal{G}$ is a similarity. Hence, the inclusion map induces an isomorphism $H_*(\mathcal{G}|_X) \cong H_*(\mathcal{G})$. □

**Proposition 4.4.** *Let $X$ be a zero-dimensional space. If we regard $X$ as an ample groupoid with $X^{(0)} = X$ and with trivial multiplication, then*

$$H_n(X) = \begin{cases} C_c(X, \mathbb{Z}) & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* The boundary maps for the groupoid $X$ are all trivial and there are no nondegenerate $n$-chains for $n \geq 1$. □

**Remark 4.5.** (i) By following [30, Def. 3.1], if $\mathcal{G}$ is an ample Hausdorff groupoid, then there is a natural preorder on $H_0(\mathcal{G})$ determined by the cone

$$H_0(\mathcal{G})^+ := \{[f] \mid f \in C_c(\mathcal{G}^{(0)}, \mathbb{Z}) \text{ and } f(x) \geq 0 \text{ for all } x \in \mathcal{G}^{(0)}\}.$$

(ii) For any zero-dimensional space $X$ regarded as a groupoid as in Proposition 4.4, we have $C^*(X) \cong C_0(X)$, and

$$K_0(C_0(X)) \cong H_0(X) \cong C_c(X, \mathbb{Z}),$$

via an isomorphism that carries the positive cone of $K_0(C_0(X))$ to $H_0(X)^+$.

(iii) The notion of a type semigroup for the transformation group $(X, \Gamma)$, where $X$ is a Cantor set and $\Gamma$ is discrete, was introduced in [44]. This idea was generalized by Rainone and Sims in [39, Def. 5.4], and independently by Bönicke and Li [3], who introduced the type semigroup $S(\mathcal{G})$ of an ample Hausdorff groupoid $\mathcal{G}$. The map $[1_U]_{H_0(\mathcal{G})} \mapsto [1_U]_{G(S(\mathcal{G}))}$ induces an isomorphism of the homology group $H_0(\mathcal{G})$ onto the Grothendieck group of $S(\mathcal{G})$. This isomorphism carries $H_0(\mathcal{G})^+$ to the image of $S(\mathcal{G})$ in its Grothendieck group. In particular, the coboundary subgroup $H_{\mathcal{G}}$ of [39, Def. 6.4] is exactly $\operatorname{im} \partial_1$ as defined above.

**Remark 4.6.** Homology for ample Hausdorff groupoids is functorial in the following sense: Let $\mathcal{G}$ and $\mathcal{H}$ be ample Hausdorff groupoids and let $\varphi : \mathcal{G} \to \mathcal{H}$ be an étale groupoid homomorphism (so in particular, $\varphi$ is a local homeomorphism). Then, as Crainic and Moerdijk observe (see [10, Item 3.7.2]), the maps $\varphi_*^{(n)} : C_c(\mathcal{G}^{(n)}, \mathbb{Z}) \to C_c(\mathcal{H}^{(n)}, \mathbb{Z})$ induce homomorphisms on homology which we denote by $\varphi_* : H_n(\mathcal{G}) \to H_n(\mathcal{H})$, and $\varphi \mapsto \varphi_*$ preserves composition. If $\mathcal{G}$ is an open subgroupoid of an ample Hausdorff groupoid $\mathcal{H}$, then $\mathcal{G}$ is also an ample Hausdorff groupoid and the inclusion map $\iota : \mathcal{G} \to \mathcal{H}$ is an étale groupoid homomorphism. Hence $\iota$ induces a map $\iota_* : H_*(\mathcal{G}) \to H_*(\mathcal{H})$ satisfying $\iota_*[1_U]_{H_n(\mathcal{G})} = [1_U]_{H_n(\mathcal{H})}$.

One key point of the functoriality of homology described in the preceding remark is that it leads to the following notion of continuity for homology of ample Hausdorff groupoids.

**Proposition 4.7** (compare with [35, Lem. 1.5])**.** *Let $\mathcal{G}$ be an ample Hausdorff groupoid and let $\{\mathcal{G}_i\}$ be an increasing sequence of open subgroupoids of $\mathcal{G}$. Then*

$$H_*(\mathcal{G}) \cong \varinjlim H_*(\mathcal{G}_i).$$

*Proof.* For each $i$, by Remark 4.6, the inclusion map $\mathcal{G}_i \hookrightarrow \mathcal{G}$ induces a homomorphism $\iota_i : H_*(\mathcal{G}_i) \to H_*(\mathcal{G})$. So the universal property of the direct limit yields a homomorphism $\iota_\infty : \varinjlim H_*(\mathcal{G}_i) \to H_*(\mathcal{G})$. This homomorphism is injective because if $\iota_\infty(a)$ is a boundary, say $\iota_\infty(a) = \partial_n(f)$, then we have $f \in C_c(\mathcal{G}_i^{(n+1)})$ for large enough $i$, and then $a = \partial_n(f)$ belongs to $B_n(\mathcal{G}_i)$. It is surjective because if $U$ is a compact open subset of $\mathcal{G}^{(n)}$, then $U \subseteq \bigcup_i \mathcal{G}_i^{(n)}$, and since the $\mathcal{G}_i^{(n)}$ are open and nested, it follows that $U$ is a compact open subset of $\mathcal{G}_i^{(n)}$ for large $i$. Hence every generator of $H_*(\mathcal{G})$ belongs to the image of $\iota_\infty$. □

**Lemma 4.8.** *Let $X, Y$ be locally compact Hausdorff spaces, and let $\psi : Y \to X$ be a local homeomorphism. Suppose that there exists a continuous open section $\varphi : X \to Y$ of $\psi$. Then the groupoid maps $\rho : R(\psi) \to X$ given by $\rho(y_1, y_2) =$*

*$\psi(y_1)$ and $\sigma : X \to R(\psi)$ given by $\sigma(x) = (\varphi(x), \varphi(x))$ are both similarities. Indeed, $\sigma \circ \rho$ is similar to $\mathrm{id}_{R(\psi)}$ and $\rho \circ \sigma$ is similar to $\mathrm{id}_X$. Moreover, the induced maps $\rho_* : H_*(R(\psi)) \to H_*(X)$ and $\sigma_* : H_*(X) \to H_*(R(\psi))$ are inverse to each other.*

*Proof.* The first assertion follows from the second. To prove the second assertion, define $\theta : R(\psi)^{(0)} \to R(\psi)$ by $\theta(y, y) = (\varphi \circ \psi(y), y)$. Then

$$\begin{aligned}\sigma \circ \rho(y_1, y_2)\theta(y_2, y_2) &= (\varphi \circ \psi(y_1), \varphi \circ \psi(y_2))(\varphi \circ \psi(y_2), y_2) \\ &= (\varphi \circ \psi(y_1), y_2) \\ &= (\varphi \circ \psi(y_1), y_1)(y_1, y_2) \\ &= \theta(y_1, y_1)\,\mathrm{id}_{R(\psi)}(y_1, y_2).\end{aligned}$$

Hence, $\sigma \circ \rho$ is similar to $\mathrm{id}_{R(\psi)}$. Since $\rho \circ \sigma = \mathrm{id}_X$, it follows that $\rho \circ \sigma$ is similar to $\mathrm{id}_X$. The last assertion now follows from [30, Prop. 3.5]. □

**Definition 4.9** (compare with [30, Def. 2.2]). An ample groupoid $\mathcal{G}$ is said to be *elementary* if it is isomorphic to the groupoid $R(\psi)$ of Example 3.9 for some local homeomorphism $\psi : Y \to X$ between zero-dimensional spaces. An ample groupoid $\mathcal{G}$ is said to be *AF* if it can be expressed as a union of open elementary subgroupoids.

The only point of difference between Definition 4.9 and Matui's [30, Def. 2.2] is that we allow non-compact unit spaces.

For the following result, recall that if $\psi : Y \to X$ is a local homeomorphism, then the homomorphism $\psi_* : C_c(Y, \mathbb{Z}) \to C_c(X, \mathbb{Z})$ is given in (3). There is also an inclusion $\iota : C_0(Y) \hookrightarrow C^*(R(\psi))$ induced by the homeomorphism $Y \cong R(\psi)^{(0)}$, and this induces a homomorphism $\iota_* : C_c(Y, \mathbb{Z}) \to K_0(C^*(R(\psi))$.

**Theorem 4.10.** *Let $X$, $Y$ be locally compact Hausdorff spaces. Further, let $\psi : Y \to X$ be a local homeomorphism. Then $Y$ is an $R(\psi)$–$X$ equivalence with anchor maps $\mathrm{id} : Y \to R(\psi)^{(0)}$ and $\psi : Y \to X$, right action of $X$ given by $y \cdot \psi(y) = y$, and left action given by $(x, y) \cdot y = x$. Hence $H_*(R(\psi)) \cong H_*(X)$.*

*If $Y$ is $\sigma$-compact and totally disconnected, then the map $\psi$ admits a continuous open section and the map $\rho : R(\psi) \to X$ given by $\rho(y_1, y_2) = \psi(y_1)$ is a similarity and thus induces the isomorphism $H_0(R(\psi)) \cong H_0(X) = C_c(X, \mathbb{Z})$ determined by $[1_U] \mapsto \psi_*(1_U)$ for compact open $U \subseteq Y$. We have $H_n(R(\psi)) = 0$ for $n \geq 1$.*

*The groupoid $C^*$-algebra $C^*(R(\psi))$ is an AF algebra, the map $\psi_*$ induces an isomorphism $K_0(C^*(R(\psi))) \to C_c(X, \mathbb{Z})$ such that the diagram*

$$\begin{array}{ccc} C_c(Y,\mathbb{Z}) & & \\ \downarrow{\scriptstyle \iota_*} & \searrow^{\psi_*} & \\ K_0(C^*(R(\psi))) & \xrightarrow{\ \cong\ } & C_c(X,\mathbb{Z}) \end{array}$$

*commutes, and we have $K_1(C^*(R(\psi))) = \{0\}$.*

*Proof.* For the first statement, one just checks directly that the maps described satisfy the axioms for an equivalence of groupoids. The isomorphism

$$H_*(R(\psi)) \cong H_*(X)$$

follows from Lemma 4.3.

Now suppose that $Y$ is $\sigma$-compact and totally disconnected. Then $X$ is also $\sigma$-compact and totally disconnected. Choose a cover $Y = \bigcup_{i=1}^{\infty} U_i$ of $Y$ by countably many compact open sets. Since $\psi$ is a local homeomorphism and the $U_i$ are compact, each $U_i$ is a finite union of compact open sets on which $\psi$ is injective, so by relabelling, we may assume that the $U_i$ have this property. For each $i$, let

$$V_i := U_i \setminus \Big( \bigcup_{j=1}^{i-1} \psi^{-1}(\psi(U_j)) \Big).$$

Then $X = \bigsqcup_i \psi(V_i)$, and the $V_i$ are compact open sets on which $\psi$ restricts to a homeomorphism $\psi_i : V_i \to \psi(V_i)$. So we can define a continuous section $\varphi$ for $\psi$ by setting $\varphi|_{\psi(V_i)} = \psi_i^{-1} : \psi(V_i) \to V_i$. Hence Lemma 4.8 yields a similarity $\rho : R(\psi) \to X$ such that the restriction of $\rho_*$ to $C_c(Y, \mathbb{Z})$ coincides with $\psi_* : C_c(Y, \mathbb{Z}) \to C_c(X, \mathbb{Z})$.

The equivalence $Y$ of groupoids determines a Morita equivalence between $C^*(R(\psi))$ and $C_0(X)$ (see [34, Thm. 2.8]). Since approximate finite dimensionality is preserved by Morita equivalence and $C_0(X)$ is AF, we see that $C^*(R(\psi))$ is AF, and the Morita equivalence induces the desired isomorphisms in $K$-theory. To see that the diagram commutes, suppose that $U \subseteq Y$ is compact open and that $\psi|_U$ is injective. Then $\iota_*(1_U) = [1_U] \in K_0(C^*(R(\psi)))$, and this is carried to $1_{\psi(U)}$ by the isomorphism $K_0(C^*(R(\psi))) \to C_c(X, \mathbb{Z})$ just described. This is precisely $\psi_*(1_U)$. □

## 5. Matui's HK conjecture

In [32, Conjecture 2.6] Matui posed the HK conjecture for a certain class of ample Hausdorff groupoids. Recall that an étale groupoid $\mathcal{G}$ is said to be effective if the interior of its isotropy coincides with its unit space $\mathcal{G}^{(0)}$ and minimal if every orbit is dense.

**Matui's HK conjecture.** *Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid such that $\mathcal{G}^{(0)}$ is a Cantor set. Suppose that $\mathcal{G}$ is both effective and minimal. Then for $j = 0, 1$ we have*

$$K_j(C_r^*(\mathcal{G})) \cong \bigoplus_{i=0}^{\infty} H_{2i+j}(\mathcal{G}). \tag{5}$$

We are interested in the extent to which the isomorphism (5) holds for groupoids that do not necessarily have non-compact unit space and are not necessarily minimal or effective. To streamline our discussion, we make the following definition.

**Definition 5.1.** We define $\mathfrak{M}$ to be the class of ample Hausdorff groupoids for which the isomorphism (5) holds.

Matui proves in [30, Thm. 4.14] that the groupoids associated to shifts of finite type belong to $\mathfrak{M}$ and in [30, Thms. 4.10 and 4.11] that AF groupoids with compact unit space belong to $\mathfrak{M}$. He shows in [32, Prop. 2.7] that $\mathfrak{M}$ is closed under Kakutani equivalence, and he shows in [32, Thm. 5.5] that $\mathfrak{M}$ contains all finite cartesian products of groupoids associated to shifts of finite type. He proves in [30, Sect. 3.1] that $\mathfrak{M}$ contains the transformation groupoids of topologically free and minimal actions of $\mathbb{Z}$ on the Cantor set.

In [19] Hazrat and Li verify (5) for $j = 0$ in the setting of groupoids of row-finite 1-graphs with no sinks. We complete the analysis for $j = 1$ in Theorem 6.7. In [35] Ortega shows that the Katsura–Exel–Pardo groupoid $\mathcal{G}_{A,B}$ associated to square integer matrices with $A \geq 0$ belongs to $\mathfrak{M}$. Here we consider Deaconu–Renault groupoids and thereby study higher-dimensional aspects not present in other cases.

There are examples of ample Hausdorff groupoids $\mathcal{G}$ that belong to $\mathfrak{M}$ but either do not have compact unit spaces or which are not necessarily effective or minimal. For example, if $X$ is any noncompact totally disconnected space, then the groupoid $X \times \mathbb{Z}$ satisfies none of these conditions, but belongs to $\mathfrak{M}$. It is also easy to show that $\mathbb{Z}^n$ is in $\mathfrak{M}$ for all $n$. Let $\mathbb{F}_n$ denote the free group on $n$ letters. Then by [2, Theorem 10.8.1] and [5, Example II.4.2] we have

$$K_0(C^*_r(\mathbb{F}_n)) \cong H_0(\mathbb{F}_n) \cong \mathbb{Z}, \quad K_1(C^*_r(\mathbb{F}_n)) \cong H_1(\mathbb{F}_n) \cong \mathbb{Z}^n,$$

and $H_i(\mathbb{F}_n) = 0$ for all $i > 1$. Hence $\mathbb{F}_n$ lies in $\mathfrak{M}$. The integer Heisenberg group also belongs to $\mathfrak{M}$; see [22, Cor. 1] and [23, Example 8.24]. On the other hand, not every ample Hausdorff groupoid belongs to $\mathfrak{M}$: for example, $\mathfrak{M}$ contains no nontrivial finite cyclic group.

Here we expand the class of groupoids known to belong to $\mathfrak{M}$. We show that all AF groupoids, all Deaconu–Renault groupoids associated to actions of $\mathbb{N}$ or $\mathbb{N}^2$ on zero-dimensional spaces, and path groupoids associated to many one-vertex $k$-graphs belong to $\mathfrak{M}$.

By Proposition 4.4 and Remark 4.5, any zero-dimensional space $X$ regarded as a trivial groupoid belongs to $\mathfrak{M}$. More generally, the following corollary to Theorem 4.10 shows that all AF groupoids belong to $\mathfrak{M}$.

**Corollary 5.2.** *Let $\mathcal{G}$ be a groupoid that can be expressed as a direct limit $\mathcal{G} = \varinjlim \mathcal{G}_n$ of open subgroupoids each of which is isomorphic to $R(\psi_n)$ for some local homeomorphism $\psi_n : \mathcal{G}^{(0)} \to X_n$. Suppose that $\mathcal{G}^{(0)}$ is totally disconnected. Then there are maps $\varphi_n : X_n \to X_{n+1}$ such that $\varphi_n \circ \psi_n = \psi_{n+1}$ for all $n$. We have $H_n(\mathcal{G}) = 0$ for $n \geq 1$, and*

$$H_0(\mathcal{G}) \cong \varinjlim(H_0(X_n), (\varphi_n)_*).$$

*There is an isomorphism $K_0(C^*(\mathcal{G})) \cong H_0(\mathcal{G})$ that carries $[1_U]_0$ to $[1_U]$ for each compact open $U \subseteq \mathcal{G}^{(0)}$, and we have $K_1(C^*(\mathcal{G})) = \{0\}$. In particular, $\mathcal{G}$ belongs to $\mathfrak{M}$.*

*Proof.* Each $R(\psi_n)^{(0)}$ is totally disconnected because it is an open subspace of $\mathcal{G}^{(0)}$, and so Theorem 4.10 shows that $H_p(R(\psi_n)) = 0$ for $p \geq 1$ and all $n$, and that $H_0(R(\psi_n)) \cong C_c(X_n, \mathbb{Z}) \cong K_0(C^*(R(\psi_n)))$ with both isomorphisms induced by $(\psi_n)_*$. Since homology and $K$-theory are continuous with respect to inductive limits, the result follows. □

## 6. Deaconu–Renault groupoids

In this section we first show that the homology of an ample higher-rank Deaconu–Renault groupoid $\mathcal{G}(X,\sigma)$ is given by $H_*(\mathbb{Z}^k, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k))$ using spectral sequence arguments of Matui. In Theorem 6.5 we describe a complex which allows us to compute $H_*(\mathbb{Z}^k, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k))$ by adapting techniques from [14]. We then use this description and Kasparov's $K$-theory spectral sequence to prove that $\mathcal{G}(X,\sigma)$ belongs to $\mathfrak{M}$ when the rank is either one or two (see Theorems 6.7 and 6.10). Furthermore, we also give formulas for computing the $K$-theory of $C^*(\mathcal{G}(X,\sigma))$ in these cases.

Recall from Section 2 that if $\sigma$ is an action of $\mathbb{N}^k$ on a locally compact Hausdorff space $X$ by local homeomorphisms, then we write $\widetilde{X} := X \times \mathbb{Z}^k$, and there is an action $\tilde{\sigma}$ of $\mathbb{N}^k$ by local homeomorphisms on $\widetilde{X}$ given by $\tilde{\sigma}^q(x,p) = (\sigma^q(x), p+q)$. Equation (1) then defines an isomorphism of the skew-product groupoid $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k$ corresponding to the cocycle $c(x,m,y) = m$ onto the Deaconu–Renault groupoid $\mathcal{G}(\widetilde{X},\tilde{\sigma})$.

Our first result shows that $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k$ is equivalent to $c^{-1}(0)$; this in turn allows us to compute its homology.

**Lemma 6.1.** *Let $X$ be a locally compact Hausdorff totally disconnected space, and let $\sigma$ be an action of $\mathbb{N}^k$ on $X$ by local homeomorphisms. The set $X \times \{0\} \subseteq X \times \mathbb{Z}^k$ is a clopen $\mathcal{G}(\widetilde{X},\tilde{\sigma})$-full subspace of $\mathcal{G}(\widetilde{X},\tilde{\sigma})^{(0)}$. The map $(x,0,y) \mapsto ((x,0),0,(y,0))$ is an isomorphism of $c^{-1}(0) \subseteq \mathcal{G}(X,\sigma)$ onto $\mathcal{G}(\widetilde{X},\tilde{\sigma})|_{X\times\{0\}}$, and $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k$ is an AF groupoid.*

*There is an isomorphism of $H_*(c^{-1}(0))$ onto $H_*(\mathcal{G}(\widetilde{X},\tilde{\sigma}))$ that carries $[1_U]$ to $[1_{U\times\{0\}}]$ for every compact open $U \subseteq X$.*

*Proof.* It is clear that $X \times \{0\}$ is a clopen subset of $X \times \mathbb{Z}^k$, the unit space of $\mathcal{G}(\widetilde{X},\tilde{\sigma})$. We claim that it is full. Fix $(x,n) \in X \times \mathbb{Z}^k$ and write $n = n_+ - n_-$, where $n_+, n_- \in \mathbb{N}^k$. Then there exists $y \in X$ such that $\sigma^{n_+}(y) = \sigma^{n_-}(x)$. Set $\gamma = ((y,0), n_+ - n_-, (x,n))$. By construction we have $\tilde{\sigma}^{n_+}(y,0) = \tilde{\sigma}^{n_-}(x,n)$, and so $\gamma \in \mathcal{G}(\widetilde{X},\tilde{\sigma})$. Furthermore, $r(\gamma) = (y,0) \in X \times \{0\}$ and $s(\gamma) = (x,n)$, proving the claim.

The map $(x,0,y) \mapsto ((x,0),0,(y,0))$ is clearly an injective homomorphism. To show that it is surjective, let $\gamma = ((x,m), m-n, (y,n)) \in \mathcal{G}(\widetilde{X},\tilde{\sigma})|_{X\times\{0\}}$. Then $m = n = 0$, so $\gamma$ is in the range of $c^{-1}(0)$.

Since $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k \cong \mathcal{G}(\widetilde{X},\tilde{\sigma})$ via the isomorphism given in (1), and since $X \times \{0\}$ is $\mathcal{G}(\widetilde{X},\tilde{\sigma})$-full, it follows from Theorem 3.12 that $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k$ is equivalent to $\mathcal{G}(\widetilde{X}, \tilde{\sigma})|_{X \times \{0\}}$. By the preceding paragraph,

$$\mathcal{G}(\widetilde{X}, \tilde{\sigma})|_{X \times \{0\}} \cong c^{-1}(0).$$

Since $c^{-1}(0)$ can be written as an increasing union of the elementary groupoids $R(\sigma^n)$, it is AF (see Corollary 5.2), and so $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k$ is an AF groupoid, too.

The final statement follows from Lemma 4.3. □

To compute the homology of $c^{-1}(0)$, we decompose it as the increasing union of the subgroupoids $R(\sigma^n)$ as $n$ ranges over $\mathbb{N}^k$.

**Lemma 6.2.** *Let $X$ be a totally disconnected locally compact Hausdorff space, and let $\sigma$ be an action of $\mathbb{N}^k$ on $X$ by surjective local homeomorphisms. There is an isomorphism $\varinjlim(C_c(X,\mathbb{Z}), \sigma_*^n) \to H_0(c^{-1}(0))$ that takes $\sigma_*^{0,\infty}(1_U)$ to $[1_U]$ for every compact open $U \subseteq X$. We have*

$$H_q(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) \cong \begin{cases} \varinjlim_{n \in \mathbb{N}^k} (C_c(X,\mathbb{Z}), \sigma_*^n) & \text{if } q = 0, \\ 0 & \text{otherwise.} \end{cases}$$

*The isomorphism*

$$H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) \cong \varinjlim_{n \in \mathbb{N}^k} (C_c(X,\mathbb{Z}), \sigma_*^n)$$

*intertwines the action of $\mathbb{Z}^k$ on $H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)$ given by*

$$p \cdot ((x,m,y),n) = ((x,m,y), n+p)$$

*with the action of $\mathbb{Z}^k$ on $\varinjlim_{n \in \mathbb{N}^k}(C_c(X,\mathbb{Z}), \sigma_*^n)$ induced by the $\sigma_*^n$.*

*Proof.* If $U, V \subseteq X$ are compact open sets on which $\sigma^n$ is injective, and if we have $\sigma^n(U) = \sigma^n(V)$, then we have $[1_U] = [1_V]$ in $H_0(R(\sigma^n))$. Since every $W \subseteq X$ can be expressed as a finite disjoint union $W = \bigsqcup_j \sigma^n(U_j)$ where each $U_j$ is compact open and each $\sigma^n|_{U_j}$ is injective, it follows that there is a unique homomorphism $\varphi_n : C_c(X,\mathbb{Z}) \to H_0(R(\sigma^n))$ such that $\varphi_n(\sigma_*^n(1_U)) = [1_U]$ for all compact open $U$. This $\varphi_n$ is the map induced by the weak equivalence of groupoids $\varphi_n : R(\sigma^n) \to X$ (see Remark 3.11), so Proposition 3.10 and Lemma 4.3 imply that it is an isomorphism. For $m, n \in \mathbb{N}^k$, let $\iota_{m,n}$ be the inclusion map $R(\sigma^m) \hookrightarrow R(\sigma^{m+n})$, and let

$$(\iota_{m,n})_* : H_0(R(\sigma^m)) \to H_0(R(\sigma^{m+n}))$$

be the induced map in homology. Then we have a commuting diagram

$$\begin{array}{ccc} C_c(X,\mathbb{Z}) & \xrightarrow{\sigma_*^n} & C_c(X,\mathbb{Z}) \\ {\scriptstyle \varphi_m}\downarrow & & \downarrow{\scriptstyle \varphi_{m+n}} \\ H_0(R(\sigma^m)) & \xrightarrow[(\iota_{m,n})_*]{} & H_0(R(\sigma^{m+n})) \end{array}$$

Since $c^{-1}(0)$ is the increasing union of the closed subgroupoids $R(\sigma^n)$, continuity of homology gives $H_0(c^{-1}(0)) \cong \varinjlim(H_0(R(\sigma^m)), \iota_*)$. So the universal properties of the direct limits prove the first statement.

The isomorphism (1) of $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k$ with $\mathcal{G}(\widetilde{X}, \widetilde{\sigma})$ and Lemma 6.1 show that $H_*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) \cong H_*(c^{-1}(0))$. So the preceding paragraph proves that

$$H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) \cong \varinjlim_{n\in\mathbb{N}^k} (C_c(X,\mathbb{Z}), \sigma_*^n).$$

We saw in the preceding paragraph that $c^{-1}(0)$ is an AF groupoid, so Corollary 5.2 proves that $H_q(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) = 0$ for $q \geq 1$. The final statement follows from direct computation of the maps involved. □

Matui's spectral sequence [30, Thm. 3.8 (1)] relates $H_*(\mathcal{G}(X,\sigma))$ to the homology of $\mathbb{Z}^k$ with coefficients in $H_*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)$. Since

$$H_q(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) = 0$$

for $q \geq 1$ (it is AF by Lemma 6.2), the spectral sequence collapses: It follows that

$$H_q(\mathcal{G}(X,\sigma)) \cong H_q(\mathbb{Z}^k, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)) \quad \text{for } 0 \leq q \leq k.$$

The proof of the following lemma is based on the technique developed in [14, Sect. 3]. Let $R$ be a ring. Recall that for $1 \leq p \leq k$ the module $\bigwedge^p R^k$ is the free $R$-module generated by the elements $\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p}$ indexed by integer tuples

$$1 \leq i_1 < i_2 < \cdots < i_p \leq k.$$

We define $\bigwedge^0 R^k := R$.

**Lemma 6.3.** *Let $A$ be an abelian group, and suppose that $\sigma_1, \ldots, \sigma_k$ are pairwise commuting endomorphisms of $A$ (i.e. an action of $\mathbb{N}^k$). For $1 \leq p \leq k$, define*

$$\partial_p : \textstyle\bigwedge^p \mathbb{Z}^k \otimes A \to \bigwedge^{p-1} \mathbb{Z}^k \otimes A$$

*on spanning elements $(\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p}) \otimes a$ in which the $i_j$ are in strictly increasing order by*

$$\partial_p(\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p} \otimes a) = \begin{cases} \sum_j (-1)^{j+1} \varepsilon_{i_1} \wedge \cdots \wedge \widehat{\varepsilon}_{i_j} \wedge \cdots \wedge \varepsilon_{i_p} \otimes (\mathrm{id} - \sigma_{i_j})a & \text{for } p > 1, \\ 1 \otimes (\mathrm{id} - \sigma_{i_1})a & \text{for } p = 1. \end{cases}$$

*Then $\partial_{p-1} \circ \partial_p = 0$ for each $p$, so that $(\bigwedge^* \mathbb{Z}^k \otimes A, \partial_*)$ is a complex. For each $0 \leq i \leq k$, the homomorphism $\mathrm{id} \otimes \sigma_i : \bigwedge^* \mathbb{Z}^k \otimes A \to \bigwedge^* \mathbb{Z}^k \otimes A$ commutes with $\partial_*$, and the induced map $(\mathrm{id} \otimes \sigma_i)_*$ in homology is the identity map.*

*Proof.* Direct computation on a spanning element $(\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_{p+1}}) \otimes a$ of $\bigwedge^* \mathbb{Z}^k \otimes A$, using that the $\sigma_i$ commute, shows that $\partial_{p+1} \circ \partial_p = 0$. Clearly, the $\mathrm{id} \otimes \sigma_i$ commute with $\partial_*$.

For the final statement, we first claim that for $x \in \bigwedge^p \mathbb{Z}^k \otimes A$ and $1 \le l \le k$ we have

$$\partial_{p+1}(\varepsilon_l \wedge x) = -\varepsilon_l \wedge \partial_p(x) + (\mathrm{id} \otimes (\mathrm{id} - \sigma_l))(x).$$

To prove this, it suffices to consider

$$x = (\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p}) \otimes a,$$

where $1 \le i_1 < \cdots < i_p \le k$. So fix such an $x$ and fix $l \le k$. We consider two cases.

Case 1: $l \ne i_h$ for all $h$. Then there exists $0 \le j \le p$ such that $i_h < l$ for $h \le j$ and $i_h > l$ for $h > j$. Then, using at both the first and the third steps that $\varepsilon_l \wedge \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_n} = (-1)^n \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_n} \wedge \varepsilon_l$ for any $i_1 < i_2 < \cdots < i_n$, we calculate:

$$\begin{aligned}
\partial_{p+1}(\varepsilon_l \wedge x) &= (-1)^j \partial_{p+1}(\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_j} \wedge \varepsilon_l \wedge \varepsilon_{i_{j+1}} \wedge \cdots \wedge \varepsilon_{i_p} \otimes a) \\
&= (-1)^j \Big[ \sum_{h=1}^{j} (-1)^{h+1} \varepsilon_{i_1} \wedge \cdots \wedge \widehat{\varepsilon}_{i_h} \wedge \cdots \\
&\qquad \wedge \varepsilon_{i_j} \wedge \varepsilon_l \wedge \varepsilon_{i_{j+1}} \wedge \cdots \wedge \varepsilon_{i_p} \otimes (\mathrm{id} - \sigma_{i_h}) a \\
&\qquad + (-1)^{j+2} \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_j} \wedge \varepsilon_{i_{j+1}} \wedge \cdots \wedge \varepsilon_{i_p} \otimes (\mathrm{id} - \sigma_l) a \\
&\qquad + \sum_{h=j+1}^{p} (-1)^{h+2} \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_j} \wedge \varepsilon_l \wedge \varepsilon_{i_{j+1}} \wedge \cdots \\
&\qquad \wedge \widehat{\varepsilon}_{i_h} \wedge \cdots \wedge \varepsilon_{i_p} \otimes (\mathrm{id} - \sigma_{i_h}) a \Big] \\
&= \sum_{h=1}^{p} (-1)^h \varepsilon_l \wedge \varepsilon_{i_1} \wedge \cdots \wedge \widehat{\varepsilon}_{i_h} \wedge \cdots \wedge \varepsilon_{i_p} \otimes (\mathrm{id} - \sigma_{i_h}) a \\
&\qquad + \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p} \otimes (\mathrm{id} - \sigma_l) a \\
&= \varepsilon_l \wedge \Big( \sum_{h=1}^{p} (-1)^h \varepsilon_{i_1} \wedge \cdots \wedge \widehat{\varepsilon}_{i_h} \wedge \cdots \wedge \varepsilon_{i_p} \otimes (\mathrm{id} - \sigma_{i_h}) a \Big) \\
&\qquad + (\mathrm{id} \otimes (\mathrm{id} - \sigma_l))(\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p} \otimes a) \\
&= -\varepsilon_l \wedge \partial_p(x) + (\mathrm{id} \otimes (\mathrm{id} - \sigma_l))(x).
\end{aligned}$$

Case 2: $l = i_h$ for some $h$. Then $\partial_{p+1}(\varepsilon_l \wedge x) = \partial_{p+1}(0) = 0$. So we must show that $\varepsilon_l \wedge \partial_p(x) = (\mathrm{id} \otimes (\mathrm{id} - \sigma_l))(x)$. We have

$$\varepsilon_l \wedge \partial_p(x) = \sum_{j=1}^{p} \varepsilon_l \wedge (-1)^{j+1} \varepsilon_{i_1} \wedge \cdots \wedge \widehat{\varepsilon}_{i_j} \wedge \cdots \wedge \varepsilon_{i_p} \otimes (\mathrm{id} - \sigma_{i_j})(a).$$

The terms corresponding to $i_j \ne l$ are zero, so this collapses to

$$\begin{aligned}
\varepsilon_l \wedge \partial_p(x) &= (-1)^{h+1} \varepsilon_l \wedge \varepsilon_{i_1} \wedge \cdots \wedge \widehat{\varepsilon}_{i_h} \wedge \cdots \wedge \varepsilon_{i_p} \otimes (\mathrm{id} - \sigma_{i_h})(a) \\
&= \varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p} \otimes (\mathrm{id} - \sigma_l)(a) \\
&= (\mathrm{id} \otimes (\mathrm{id} - \sigma_l))(x).
\end{aligned}$$

This completes the proof of the claim.

We now prove the final statement. Fix $x \in \bigwedge^p \mathbb{Z}^k \otimes A$ such that $\partial_p(x) = 0$, and fix $l \leq p$. We just have to show that $(\mathrm{id} \otimes (\mathrm{id} - \sigma_l))(x) \in \mathrm{image}(\partial_{p+1})$. Since $\partial_p(x) = 0$ and by using the claim, we see that

$$(\mathrm{id} \otimes (\mathrm{id} - \sigma_l))(x) = (\mathrm{id} \otimes (\mathrm{id} - \sigma_l))(x) - \varepsilon_l \wedge \partial_p(x) = \partial_{p+1}(\varepsilon_l \wedge x),$$

and the result follows. □

**Lemma 6.4.** *Let $A$ be an abelian group, and suppose that $\sigma_1, \dots, \sigma_k$ are pairwise commuting endomorphisms of $A$ (i.e. an action of $\mathbb{N}^k$). Let $\partial_p : \bigwedge^* \mathbb{Z}^k \otimes A \to \bigwedge^* \mathbb{Z}^k \otimes A$ be as in Lemma 6.3. Let*

$$\widetilde{A} := \varinjlim_{\mathbb{N}^k} (A, \sigma^n).$$

*For $i \leq k$ let $\tilde{\sigma}_i$ be the automorphism of $\widetilde{A}$ induced by $\sigma_i$, and let $\tilde{\partial}_p : \bigwedge^* \mathbb{Z}^k \otimes \widetilde{A} \to \bigwedge^* \mathbb{Z}^k \otimes \widetilde{A}$ be the boundary map obtained from Lemma 6.3 applied to $\widetilde{A}$ and the $\tilde{\sigma}_i$. Then the canonical homomorphism $\sigma^{0,\infty} : A \to \widetilde{A}$ corresponding to the zeroth copy of $A$ induces an isomorphism $H_*(\bigwedge^* \mathbb{Z}^k \otimes A) \cong H_*(\bigwedge^* \mathbb{Z}^k \otimes \widetilde{A})$. Moreover, $\tilde{\sigma}$ extends to an action of $\mathbb{Z}^k$ on $\widetilde{A}$, and $H_*(\bigwedge^* \mathbb{Z}^k \otimes \widetilde{A}) \cong H_*(\mathbb{Z}^k, \widetilde{A})$.*

*Proof.* Since the homology functor is continuous (see [50, Thm. 4.1.7]), we have $H_*(\bigwedge^* \mathbb{Z}^k \otimes \widetilde{A}) \cong \varinjlim (H_*(\bigwedge^* \mathbb{Z}^k \otimes A), (\mathrm{id} \otimes \sigma^n)_*)$. Lemma 6.3 shows that this is equal to $\varinjlim (H_*(\bigwedge^* \mathbb{Z}^k \otimes A), \mathrm{id}) = H_*(\bigwedge^* \mathbb{Z}^k \otimes A)$.

For the second statement, we follow the argument of [14, Lem. 3.12] (see also [27, Thm. 5.5]). Let $G := \mathbb{Z}^k = \langle s_1, \dots, s_k \rangle$ and let $R := \mathbb{Z}G$. For $p \geq 2$, we define $\partial_p : \bigwedge^p R^k \to \bigwedge^{p-1} R^k$ by

$$\partial_p(\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p}) = \sum_j (-1)^{j+1} (1 - s_{i_j}) \varepsilon_{i_1} \wedge \cdots \wedge \widehat{\varepsilon}_{i_j} \wedge \cdots \wedge \varepsilon_{i_p}.$$

Define $\partial_1 : \bigwedge^1 R^k \to R$ by $\partial_1(\varepsilon_j) := 1 - s_j$, and let $\eta : R \to \mathbb{Z}$ be the augmentation homomorphism determined by $\eta(s_i) = 1$ for each $i$. Then

$$0 \longrightarrow \bigwedge^k R^k \xrightarrow{\partial_k} \cdots \xrightarrow{\partial_2} \bigwedge^1 R^k \xrightarrow{\partial_1} R \xrightarrow{\eta} \mathbb{Z}$$

is a free resolution of $\mathbb{Z}$. Hence, by the definition of homology with coefficients in the $\mathbb{Z}^k$-module $\widetilde{A}$ [5, Equation III(1.1)], we have

$$H_*(\mathbb{Z}^k, \widetilde{A}) \cong H_*(\bigwedge^* R^k \otimes_R \widetilde{A}).$$

As a group we have $\bigwedge^p R^k \otimes_R \widetilde{A} \cong \bigwedge^* \mathbb{Z}^k \otimes \widetilde{A}$, and this isomorphism intertwines the boundary maps in the complex defining $H_*(\bigwedge^* R^k \otimes_R \widetilde{A})$ with the maps $\widetilde{\partial}_p$. □

We can now state our main theorem for this section, which is a computation of the homology of the Deaconu–Renault groupoid $\mathcal{G}(X, \sigma)$ associated to an action of $\mathbb{N}^k$ by surjective local homeomorphisms of a totally disconnected locally compact space $X$.

**Theorem 6.5.** *Let $X$ be a second-countable totally disconnected locally compact space, and let $\sigma$ be an action of $\mathbb{N}^k$ by surjective local homeomorphisms $\sigma^n : X \to X$. For $1 \leq p \leq k$, let*

$$A_p^\sigma = \textstyle\bigwedge^p \mathbb{Z}^k \otimes C_c(X, \mathbb{Z})$$

*and let $A_p^\sigma = \{0\}$ for $p > k$. For $p \geq 1$, define $\partial_p : A_p^\sigma \to A_{p-1}^\sigma$ by $\partial_0 = 0$ and*

$$\partial_p(\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p} \otimes f) = \begin{cases} 1 \otimes (\mathrm{id} - \sigma_*^{e_{i_1}})f & \text{if } p = 1, \\ \sum_{j=1}^{p} (-1)^{j+1} \varepsilon_{i_1} \wedge \cdots \wedge \widehat{\varepsilon}_{i_j} \wedge \cdots \wedge \varepsilon_{i_p} \otimes (\mathrm{id} - \sigma_*^{e_{i_j}})f & \text{if } 2 \leq p \leq k, \\ 0 & \text{if } p \geq k+1. \end{cases}$$

*Then $(A_*^\sigma, \partial_*)$ is a complex, and*

$$H_*(\mathcal{G}(X,\sigma)) \cong H_*(\mathbb{Z}^k, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)) \cong H_*(A_*^\sigma, \partial_*).$$

*We have $H_p(\mathbb{Z}^k, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)) = 0$ for $p > k$.*

*Proof.* Lemma 6.3 implies that $(A_*^\sigma, \partial_*)$ is a complex.

The automorphisms $\alpha_p : ((x,m,y),n) \mapsto ((x,m,y),n+p)$ of $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k$ induce an action $\widetilde{\alpha}_p$ of $\mathbb{Z}^k$ on $H_*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)$. This action $\widetilde{\alpha}$ makes each $H_p(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)$ into a $\mathbb{Z}^k$-module, so it makes sense to discuss the homology groups $H_*(\mathbb{Z}^k, H_q(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k))$ of $\mathbb{Z}^k$ with coefficients in these modules. By [30, Thm. 3.8 (1)] applied to the cocycle $c : \mathcal{G}(X,\sigma) \to \mathbb{Z}^k$, there is a spectral sequence $E_{p,q}^r$ converging to $H_{p+q}(\mathcal{G}(X,\sigma))$ satisfying $E_{p,q}^2 = H_p(\mathbb{Z}^k, H_q(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k))$. Lemma 6.2 shows that $H_q(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)$ is zero for $q \neq 0$, and therefore the differential maps on the $E^2$-page and above of the spectral sequence are trivial. Hence $E_{p,q}^\infty = E_{p,q}^2$ for all $p, q$, and so [30, Thm. 3.8 (1)] shows that $H_p(\mathcal{G}(X,\sigma)) \cong E_{p,0}^2 \cong H_p(\mathbb{Z}^k, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k))$ for all $p$.

The last statement of Lemma 6.2 gives

$$H_*(\mathbb{Z}^k, H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)) \cong H_*(\mathbb{Z}^k, \varinjlim(X, \sigma_*^n)).$$

Lemma 6.4 shows that this is isomorphic to $H_*(A_*^\sigma, \partial_*)$. □

Our next two results show that $\mathcal{G}(X,\sigma)$ belongs to $\mathfrak{M}$ if $k \leq 2$. For $k = 1$ we use the Pimsner–Voiculescu sequence [37, Thm. 2.4]. To prove them, we need the following lemma, which follows from a standard argument.

**Lemma 6.6.** *Let $X$ be a second-countable locally compact totally disconnected space. Let $\sigma$ be an action of $\mathbb{N}^k$ on $X$ by surjective local homeomorphisms. Let $c : \mathcal{G}(X,\sigma) \to \mathbb{Z}^k$ be the cocycle $c(x,m,y) = m$. Then the action of $\mathbb{Z}^k$ on $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k$ given by $\alpha_p((x,m,y),n) = ((x,m,y),n+p)$ induces an action $\bar{\alpha}$ of $\mathbb{Z}^k$ on $C^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k)$. The crossed product $C^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) \rtimes_{\bar{\alpha}} \mathbb{Z}^k$ is stably isomorphic to $C^*(\mathcal{G}(X,\sigma))$.*

*Proof.* Every automorphism of a groupoid induces an automorphism of its $C^*$-algebra, and then simple calculations establish the first statement. Let $\gamma : \mathbb{T}^k \to \mathrm{Aut}(C^*(\mathcal{G}(X,\sigma)))$ be the gauge action $\gamma_z(f(x,m,y)) = z^m f(x,m,y)$. Then there is an isomorphism $\theta : C^*(\mathcal{G}(X,\sigma)\times_c \mathbb{Z}^k) \to C^*(\mathcal{G}(X,\sigma))\times_\gamma \mathbb{T}^k$ such that for $f \in C_c(\mathcal{G}(X,\sigma)\times\{m\}) \subseteq C_c(\mathcal{G}(X,\sigma)\times_c \mathbb{Z}^k)$ we have

$$(\theta(f)(z))(g) = z^m f(g,m).$$

This $\theta$ intertwines $\bar{\alpha}$ and the dual action $\widehat{\gamma}$ on $C^*(\mathcal{G}(X,\sigma)) \times_\gamma \mathbb{T}^k$, and so Takesaki–Takai duality implies that $C^*(\mathcal{G}(X,\sigma)\times_c \mathbb{Z}^k) \rtimes_{\bar{\alpha}} \mathbb{Z}^k$ is stably isomorphic to $C^*(\mathcal{G}(X,\sigma))$ (see [52, Thm. 4.5], [51, Thm. 3.4], and [7, Thm. 1.2]). □

**Theorem 6.7.** *Let $X$ be a second-countable locally compact totally disconnected locally compact space, let $\sigma : X \to X$ be a surjective local homeomorphism, and let $\sigma_* : C_c(X,\mathbb{Z}) \to C_c(X,\mathbb{Z})$ be the induced map. Then*

$$\begin{aligned} K_0(C^*(\mathcal{G}(X,\sigma)) &\cong H_0(\mathcal{G}(X,\sigma)) \cong \mathrm{coker}(\mathrm{id} - \sigma_*),\\ K_1(C^*(\mathcal{G}(X,\sigma)) &\cong H_1(\mathcal{G}(X,\sigma)) \cong \ker(\mathrm{id} - \sigma_*),\\ H_n(\mathcal{G}(X,\sigma)) &= 0 \quad \textit{for } n \geq 2. \end{aligned}$$

*In particular, $\mathcal{G}(X,\sigma)$ belongs to $\mathfrak{M}$.*

*Proof.* We first calculate the $K$-theory of $C^*(\mathcal{G}(X,\sigma))$. Lemma 6.6 applied with $k=1$ shows that $K_*(C^*(\mathcal{G}(X,\sigma))) \cong K_*(C^*(\mathcal{G}(X,\sigma)\times_c \mathbb{Z}) \rtimes_{\bar{\alpha}} \mathbb{Z})$. Corollary 5.2 shows that $K_1(C^*(\mathcal{G}(X,\sigma)\times_c \mathbb{Z})) = 0$, and so the exactness of the Pimsner–Voiculescu sequence [37, Thm. 2.4] implies that

$$\begin{aligned} K_0(C^*(\mathcal{G}(X,\sigma))) &\cong \mathrm{coker}(\mathrm{id} - \bar{\alpha}_*),\\ K_1(C^*(\mathcal{G}(X,\sigma))) &\cong \ker(\mathrm{id} - \bar{\alpha}_*). \end{aligned} \tag{6}$$

We now compute the homology groups of $\mathcal{G}(X,\sigma)$. Theorem 6.5 implies that $H_p(\mathcal{G}(X,\sigma)) \cong H_p(\mathbb{Z}, H_0(\mathcal{G}(X,\sigma)\times_c \mathbb{Z}))$ for all $p$. Moreover, Theorem 6.5 implies that $H_p(\mathbb{Z}, H_0(\mathcal{G}(X,\sigma)\times_c \mathbb{Z})) = 0$ for $p \geq 2$.

Let $\alpha_*$ be the action of $\mathbb{Z}$ on $H_0(\mathcal{G}(X,\sigma)\times_c \mathbb{Z})$ induced by $\alpha$. By [5, Example III.1.1],

$$\begin{aligned} H_0(\mathbb{Z}, H_0(\mathcal{G}(X,\sigma)\times_c \mathbb{Z})) &\cong \ker(\mathrm{id} - \alpha_*),\\ H_1(\mathbb{Z}, H_0(\mathcal{G}(X,\sigma)\times_c \mathbb{Z})) &\cong \mathrm{coker}(\mathrm{id} - \alpha_*). \end{aligned} \tag{7}$$

Recall from Lemma 6.1 that $\mathcal{G}(X,\sigma)\times_c \mathbb{Z}$ is an AF groupoid. The isomorphism between $H_0(\mathcal{G}(X,\sigma)\times_c \mathbb{Z})$ and $K_0(C^*(\mathcal{G}(X,\sigma)\times_c \mathbb{Z}))$ supplied by Corollary 5.2 intertwines the $\mathbb{Z}$-actions $\alpha_*$ and $\bar{\alpha}_*$. Thus $\mathrm{coker}(\mathrm{id} - \bar{\alpha}_*) \cong \mathrm{coker}(\mathrm{id} - \alpha_*)$ and $\ker(\mathrm{id} - \bar{\alpha}_*) \cong \ker(\mathrm{id} - \alpha_*)$. Hence by (6) and (7),

$$K_0(C^*(\mathcal{G}(X,\sigma))) \cong H_0(\mathcal{G}(X,\sigma)) \quad \text{and} \quad K_1(C^*(\mathcal{G}(X,\sigma))) \cong H_1(\mathcal{G}(X,\sigma)).$$

Since $k=1$ here, the complex $A_*^\sigma$ of Theorem 6.5 reduces to,

$$0 \longrightarrow C_c(X,\mathbb{Z}) \longrightarrow C_c(X,\mathbb{Z}) \longrightarrow 0$$

where the central map is $\mathrm{id} - \sigma_*$. Then we have

$$H_0(\mathcal{G}(X,\sigma)) \cong H_0(A^\sigma_*, \partial_*) \cong \operatorname{coker}(\mathrm{id} - \sigma_*),$$
$$H_1(\mathcal{G}(X,\sigma)) \cong H_1(A^\sigma_*, \partial_*) \cong \ker(\mathrm{id} - \sigma_*),$$

as required. □

**Remark 6.8.** Matui's paper [30], together with recent results by Hazrat and Li [19] and Ortega [35] suggest that the groupoids of (not necessarily finite) 1-graphs with no sources belong to $\mathfrak{M}$. This now follows from Theorem 6.7 since graph groupoids are, by definition, rank-1 Deaconu–Renault groupoids.

We now discuss Kasparov's $K$-theory spectral sequence for $C^*(\mathcal{G}(X,\sigma))$. Lemma 6.6 shows that $C^*(\mathcal{G}(X,\sigma))$ is stably isomorphic to the crossed product $C^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^k) \times_{\tilde{\alpha}} \mathbb{Z}^k$. Hence [21, Theorem 6.10] (see also [14, Sect. 3]) shows that there is a spectral sequence $E^r_{p,q}$ converging to $K_*(C^*(\mathcal{G}(X,\sigma)))$ with $E^2$-page given by

$$E^2_{p,q} = \begin{cases} H_p\big(\mathbb{Z}^k, K_q(C^*(\mathcal{G}(X,\sigma)) \times_c \mathbb{Z}^k)\big) & \text{if } q \text{ is even and } 0 \le p \le k,\\ 0 & \text{otherwise.}\end{cases}$$

The differentials in the spectral sequence are maps

$$d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}.$$

If $r > k$, then for any $p, q$ at least one of $E^r_{p,q}$ and $E^r_{p-r,q+r-1}$ is trivial, because $E^r_{p,q}$ is nontrivial only for $0 \le p \le k$. Hence $d^r_{p,q}$ is trivial for $r > k$. Thus

$$E^\infty_{p,q} = E^{k+1}_{p,q}.$$

If $k$ is even, we can improve on this: if $r = k$ is even, then at least one of $E^r_{p,q}$ and $E^r_{p-r,q+r-1}$ is trivial because $E^r_{p,q}$ is nontrivial only for $q$ even, and it follows that $E^\infty_{p,q} = E^k_{p,q}$ for all $p, q$. In particular, if $k = 2$, then we have

$$E^\infty_{p,q} = E^2_{p,q}.$$

For our next theorem, we need the well-known fact that if $X$ is locally compact Hausdorff space, then $C_c(X,\mathbb{Z})$ is a free abelian group. We provide a proof for completeness.

**Lemma 6.9.** *Let $X$ be a second-countable locally compact Hausdorff space. Then $C_c(X,\mathbb{Z})$ is a free abelian group.*

*Proof.* First note that if $X$ is not compact, then $C(X \cup \{\infty\}, \mathbb{Z}) \cong C_c(X,\mathbb{Z}) \oplus \mathbb{Z}$ via $f \mapsto (f - f(\infty)1, f(\infty))$, so it suffices to prove the result for $X$ compact.

So suppose that $X$ is compact. Since $X$ is metrizable by the Urysohn metrization theorem (see [54, Thms. 23.1 and 17.6 (a)]), the Alexandroff–Hausdorff theorem (see [54, Thm. 30.7]) shows that there is a continuous surjection $\varphi : \{0,1\}^\infty \to X$. Hence $\varphi^* : C(X,\mathbb{Z}) \to C(\{0,1\}^\infty,\mathbb{Z})$ is an injective group homomorphism. Since subgroups of free abelian groups are themselves free abelian, it therefore suffices to show that $C(\{0,1\}^\infty,\mathbb{Z})$ is free abelian.

For this, let $\{0,1\}^*$ denote the collection of all finite words in the symbols 0, 1, including the empty word $\varepsilon$. Let $I = \{\varepsilon\} \cup \{\omega 1 \mid \omega \in \{0,1\}^*\}$ denote the subset of $\{0,1\}^*$ consisting of the empty word and all nontrivial words that end with a 1. We claim that $B := \{1_{Z(\omega)} \mid \omega \in I\}$ is a family of free abelian generators of $C(\{0,1\}^\infty, \mathbb{Z})$. To see this, we first argue by induction on $n$ that $\operatorname{span}_{\mathbb{Z}}\{1_{Z(\omega)} \mid \omega \in I \text{ and } |\omega| \le n\} = \operatorname{span}_{\mathbb{Z}}\{1_{Z(\omega)} \mid \omega \in \{0,1\}^n\}$ for all $n$. The containment $\subseteq$ is trivial. The containment $\supseteq$ is also trivial for $n = 0$, and if it holds for $n = k$, then for each $\omega = \omega' 0 \in \{0,1\}^{k+1}$ that ends in a 0, we have $1_{Z(\omega)} = 1_{Z(\omega')} - 1_{Z(\omega' 1)}$. We have $\omega' 1 \in I$ with $|\omega' 1| = k+1$, and $1_{Z(\omega')} \in \operatorname{span}_{\mathbb{Z}}\{1_{Z(\omega)} \mid \omega \in I \text{ and } |\omega| \le k\}$ by the inductive hypothesis. So the containment $\supseteq$ also holds for $n = k+1$. Hence $B$ generates $C(\{0,1\}^\infty, \mathbb{Z})$ as a group. To see that $B$ is a family of free generators, suppose for contradiction that $F \subseteq I$ is a finite set and $\{a_\omega \mid \omega \in F\}$ are nonzero integers such that $\sum_{\omega \in F} a_\omega 1_{V_\omega} = 0$. Fix $\mu \in F$ of minimal length. Then $\mu 000 \cdots \in \{0,1\}^\infty$ belongs to $Z(\mu)$ but not to $Z(\omega)$ for any $\omega \in F \setminus \{\mu\}$. Hence $0 = (\sum_{\omega \in F} a_\omega 1_{V_\omega})(\mu 000 \cdots) = a_\mu$ contradicting the assumption that the $a_\omega$ are nonzero. □

**Theorem 6.10.** *Let $X$ be a second-countable locally compact totally disconnected space. Let $\sigma$ be an action of $\mathbb{N}^2$ on $X$ by surjective local homeomorphisms. Define $d_2 : C_c(X, \mathbb{Z}) \to C_c(X, \mathbb{Z}) \oplus C_c(X, \mathbb{Z})$ by*

$$d_2(f) = ((\sigma_*^{\varepsilon_2} - \mathrm{id}) f, (\mathrm{id} - \sigma_*^{\varepsilon_1}) f)$$

*and define $d_1 : C_c(X, \mathbb{Z}) \oplus C_c(X, \mathbb{Z}) \to C_c(X, \mathbb{Z})$ by*

$$d_1(f \oplus g) = (\mathrm{id} - \sigma_*^{\varepsilon_1}) f + (\mathrm{id} - \sigma_*^{\varepsilon_2}) g.$$

*Then*

$$\begin{aligned} K_0(C^*(\mathcal{G}(X,\sigma)) &\cong H_0(\mathcal{G}(X,\sigma)) \oplus H_2(\mathcal{G}(X,\sigma)) \cong \operatorname{coker}(d_1) \oplus \ker(d_2), \\ K_1(C^*(\mathcal{G}(X,\sigma)) &\cong H_1(\mathcal{G}(X,\sigma)) \cong \ker(d_1)/\operatorname{image}(d_2). \end{aligned}$$

*In particular, $\mathcal{G}(X,\sigma)$ belongs to $\mathfrak{M}$.*

*Proof.* Let

$$A := K_0(C^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2)) = H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2).$$

Lemma 6.6 applied with $k = 2$ shows that $C^*(\mathcal{G}(X,\sigma))$ is stably isomorphic to $C^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2) \times_{\bar{\alpha}} \mathbb{Z}^2$ for the action $\bar{\alpha}$ induced by translation in $\mathbb{Z}^2$ in the skew-product $\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2$.

We follow the argument of [14, 27]. As discussed above, Kasparov's spectral sequence [21, Thm. 6.10] for $K_*(C^*(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2) \times_{\bar{\alpha}} \mathbb{Z}^2)$ converges on the second page, and we deduce that $K_0(C^*(\mathcal{G}(X,\sigma))$ is an extension of $E^2_{0,2}$ by $E^2_{2,0}$ while $K_1(C^*(\mathcal{G}(X,\sigma))$ is isomorphic to $E^2_{0,1}$.

As discussed prior to the statement of the theorem, $E^2_{p,q}$ is isomorphic to the homology group $H_p(\mathbb{Z}^2, K_0(C^*(\mathcal{G}(X,\sigma)) \times_c \mathbb{Z}^k))$ for $q$ even and is zero for $q$ odd. Corollary 5.2 shows that $K_0(C^*(\mathcal{G}(X,\sigma)) \times_c \mathbb{Z}^2) \cong H_0(\mathcal{G}(X,\sigma) \times_c \mathbb{Z}^2)$, and that this isomorphism intertwines the actions of $\mathbb{Z}^2$ on the two groups induced by translation in the second coordinate. It therefore follows from Theorem 6.5 that for $q$ even we have $E^2_{p,q} \cong H_p(A^\sigma_*, \partial_*)$. Since $C_c(X, \mathbb{Z})$ is free abelian by Lemma 6.9, so is the subgroup $H_2(A^\sigma_*, \partial_*) = \ker(\partial_2)$. Hence the extension $K_0(C^*(\mathcal{G}(X,\sigma))$ of $E^2_{0,2}$ by $E^2_{2,0}$ splits, and we obtain $K_0(C^*(\mathcal{G}(X,\sigma)) \cong H_0(A^\sigma_*, \partial_*) \oplus H_2(A^\sigma_*, \partial_*)$ and $K_1(C^*(\mathcal{G}(X,\sigma))) \cong H_1(A^\sigma_*, \partial_*)$. The result then follows from Theorem 6.5 because the obvious identifications

$$\textstyle\bigwedge^j \mathbb{Z}^2 \otimes C(X,\mathbb{Z}) \cong C(X,\mathbb{Z})^{\binom{2}{j}}$$

for $0 \le j \le 2$ intertwine $\partial_*$ with $d_*$. □

It follows that the path groupoid of a 2-graph belongs to $\mathfrak{M}$ since it is a rank-2 Deaconu–Renault groupoid (see Corollary 7.7).

**Question 6.11.** Our proof Theorem 6.10 uses that $H_2(A^\sigma_*)$ is a free abelian group so that the extension

$$0 \to H_0(\mathcal{G}(X,\sigma)) \to K_0(C^*(\mathcal{G}(X,\sigma))) \to H_2(\mathcal{G}(X,\sigma)) \to 0$$

splits. Hence the map from $H_0(\mathcal{G}(X,\sigma)) \oplus H_2(\mathcal{G}(X,\sigma))$ to $K_0(C^*(\mathcal{G}(X,\sigma)))$ that we obtain is not natural. An interesting question arises: can the isomorphism (5) be chosen to be natural in some sense for elements of $\mathfrak{M}$ in general, and for rank-2 Deaconu–Renault groupoids in particular?

**Remark 6.12.** The proof of Theorem 6.10 is special to the situation $k = 2$, and issues arise already when $k = 3$. In this situation, the groups on the $E^3$-page of Kasparov's spectral sequence coincide with those on the $E^2$-page, but the $E^3$-page has potentially nontrivial differential maps, $d^3_{3,2l} : E^3_{3,2l} \to E^3_{0,2l+2}$. So $E^3_{p,q} = H_p(A^\sigma_*)$ if $q$ is even, and is 0 if $q$ is odd, and the $E^3$-page has the following form:

$$\begin{array}{cccccc}
\vdots & \vdots & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & 0 & \cdots \\
H_0(A^\sigma_*) & H_1(A^\sigma_*) & H_2(A^\sigma_*) & H_3(A^\sigma_*) & 0 & \cdots \\
 & & d^3_{3,0} & & & \\
0 & 0 & 0 & 0 & 0 & \cdots \\
H_0(A^\sigma_*) & H_1(A^\sigma_*) & H_2(A^\sigma_*) & H_3(A^\sigma_*) & 0 & \cdots
\end{array}$$

The sequence converges on the $E^4$-page, and so we have exact sequences

$$0 \to \operatorname{coker}(d^3_{3,0}) \to K_0(C^*(\mathcal{G}(X,\sigma))) \to H_2(A^\sigma_*) \to 0,$$
$$0 \to H_1(A^\sigma_*) \to K_1(C^*(\mathcal{G}(X,\sigma))) \to \ker(d^3_{3,0}) \to 0.$$

Unless $d^3_{3,0}$ is trivial, there is no reason to expect that $\operatorname{coker}(d^3_{3,0}) \cong H_0(A^\sigma_*)$; and even if $d^3_{3,0}$ is trivial, there is no reason to expect that $H_2(A^\sigma_*)$ is free abelian, so the extension defining $K_0(C^*(\mathcal{G}(X,\sigma)))$ need not split. This suggests rank-3 Deaconu–Renault groupoids as a potential source of counterexamples to Matui's HK-conjecture.

**Remark 6.13.** If the groups $H_*(A^\sigma_*)$ are finitely generated, and the natural homomorphism $H_0(\mathcal{G}(X,\sigma)) \to K_0(C^*(\mathcal{G}(X,\sigma)))$ is injective, then one would expect $d^3_{3,0}$ to be trivial, and then in the rank-3 case $\mathcal{G}(X,\sigma)$ would satisfy Matui's conjecture up to stabilization by $\mathbb{Q}$ (the so-called *rational HK-conjecture*). This suggests that it would be worthwhile to investigate when the homomorphism $H_0(\mathcal{G}(X,\sigma)) \to K_0(C^*(\mathcal{G}(X,\sigma)))$ is injective.

## 7. $k$-GRAPHS

In this section, we first establish the existence of a natural map from the homology of a $k$-graph to the homology of its groupoid. We show that this homomorphism is in general neither injective nor surjective. We then apply the results of Section 6 to see that all 1-graph and 2-graph groupoids belong to $\mathfrak{M}$. Finally, we restrict our attention to $k$-graphs with one vertex, and demonstrate that for any such $k$-graph in which $\gcd(|\Lambda^{\varepsilon_1}|-1,\dots,|\Lambda^{\varepsilon_k}|-1) = 1$, the corresponding $k$-graph groupoid belongs to $\mathfrak{M}$.

**A map from the categorical homology of a *k*-graph to the homology of its groupoid.** To define the *categorical*[3] homology groups $H_*(\Lambda)$ for a $k$-graph $\Lambda$, we use the following notation.

Given a $k$-graph $\Lambda$, let

$$\Lambda^{*n} := \begin{cases} \Lambda^0 & \text{if } n = 0, \\ \left\{ (\lambda_1,\dots,\lambda_n) \in \prod_{i=1}^n \Lambda \;\middle|\; s(\lambda_i) = r(\lambda_{i+1}) \right\} & \text{otherwise.} \end{cases}$$

For $n \geq 2$ and for $i \in \{0,\dots,n\}$, define a map $d_i : \Lambda^*_n \to \Lambda^*_{n-1}$ by

$$d_i(\lambda_1,\dots,\lambda_n) := \begin{cases} (\lambda_2,\dots,\lambda_n) & \text{if } i = 0, \\ (\lambda_1,\lambda_2,\dots,\lambda_i\lambda_{i+1},\dots,\lambda_n) & \text{if } 0 < i < n, \\ (\lambda_1,\dots,\lambda_{n-1}) & \text{if } i = n. \end{cases}$$

**Definition 7.1** (compare with [17, Rem. 2.14])**.** Let $\Lambda$ be a $k$-graph. For $n \geq 0$, let $C_n(\Lambda) := \mathbb{Z}\Lambda^{*n}$, the free abelian group with generators indexed by $\Lambda^{*n}$. Identifying elements of $\Lambda^{*n}$ with the corresponding generators of $C_n(\Lambda)$, we regard the boundary maps $d_i$ as homomorphisms $d_i : C_n(\Lambda) \to C_{n-1}(\Lambda)$. We obtain homomorphisms $\partial_n : C_n(\Lambda) \to C_{n-1}(\Lambda)$ by $\partial_n := \sum_{i=1}^n (-1)^i d_i$ for $n \geq 2$. Regarding $s$, $r$ as maps from $C_1(\Lambda)$ to $C_0(\Lambda)$, define $\partial_1 : C_1(\Lambda) \to C_0(\Lambda)$ by $\partial_1 := s - r$ and $\partial_0 : C_0(\Lambda) \to \{0\}$ to be the zero map. Standard calculations show that $(C_*(\Lambda),\partial_*)$ is a complex. The resulting groups

[3]So called because it matches with the categorical cohomology of $\Lambda$ defined in [28].

$H_n(\Lambda) := \ker(\partial_n)/\operatorname{image}(\partial_{n+1})$ are called the *categorical homology groups* of $\Lambda$.

Note that one may define a more general homology theory with coefficients in an abelian group $A$ as in [17], but we do not need this level of generality.

**Example 7.2.** Let $\Lambda$ be the 1-graph with vertex connectivity matrix $\left(\begin{smallmatrix} 5 & 2 \\ 2 & 3 \end{smallmatrix}\right)$. By [17, Thm. 6.2] the categorical homology of $\Lambda$ coincides with its cubical homology, which can computed as follows: Since $\Lambda$ is connected, we have $H_0(\Lambda) \cong \mathbb{Z}$; since $\Lambda$ is finite, its first homology group is free abelian with rank equal to its Betti number $p = |\Lambda^1| - |\Lambda^0| + 1 = 11$. Hence $H_1(\Lambda) \cong \mathbb{Z}^{11}$. Moreover, $H_n(\Lambda) = 0$ for all $n > 1$. We will return to this example in Remark 7.8 after establishing how to compute the homology of the associated groupoid.

**Remark 7.3.** One can check that $C_0(\Lambda)$ together with the subgroups of the $C_n(\Lambda)$ for $n \geq 1$ generated by elements $(\lambda_1, \dots, \lambda_n)$ in which each $\lambda_i \notin \Lambda^0$, form a subcomplex under the same boundary maps $\partial_n$, and that the homology of this subcomplex is isomorphic to $H_*(\Lambda)$.

Recall that if $\Lambda$ is a $k$-graph and $\lambda, \mu \in \Lambda$ satisfy $s(\lambda) = s(\mu)$, then the cylinder set $Z(\lambda, \mu)$ is defined by

$$Z(\lambda, \mu) = \{(\lambda x, d(\lambda) - d(\mu), \mu x) \mid x \in s(\lambda)\Lambda^\infty\} \subseteq \mathcal{G}_\Lambda,$$

and is a compact open subset of $\mathcal{G}_\Lambda$.

**Definition 7.4.** Let $\Lambda$ be a $k$-graph. Let $\mathcal{G}_\Lambda$ be the associated groupoid (see equation (2)), and for each $n$ let $\mathcal{G}_\Lambda^{(n)}$ be the collection of composable $n$-tuples in $\mathcal{G}_\Lambda$ as in (4). For $(\lambda_1, \dots, \lambda_n) \in \Lambda^{*\,n} \subseteq C_n(\Lambda)$, define

$$Y(\lambda_1, \dots, \lambda_n) := \left(\prod_{i=1}^{n} Z(\lambda_i, s(\lambda_i))\right) \cap \mathcal{G}_\Lambda^{(n)}.$$

Let $\Psi_* : C_*(\Lambda) \to C_c(\mathcal{G}_\Lambda^{(*)}, \mathbb{Z})$ be the homomorphism such that

$$\Psi_0(v) = 1_{Z(v)} \in C_c(\mathcal{G}_\Lambda^{(0)}, \mathbb{Z})$$

for $v \in \Lambda^0$, and

$$\Psi_n(\lambda_1, \dots, \lambda_n) = 1_{Y(\lambda_1, \dots, \lambda_n)}$$

for $n \geq 1$ and $(\lambda_1, \dots, \lambda_n) \in \Lambda^{*\,n}$.

**Theorem 7.5.** *Let $\Lambda$ be a $k$-graph. The maps*

$$\Psi_* : C_*(\Lambda) \to C_c(\mathcal{G}_\Lambda^{(*)}, \mathbb{Z})$$

*defined above comprise a chain map, and induce a homomorphism*

$$\Psi_* : H_*(\Lambda) \to H_*(\mathcal{G}_\Lambda).$$

*Proof.* It suffices to prove that $\Psi_*$ intertwines the boundary maps on generators of $C_*(\Lambda)$. Fix $\lambda \in \Lambda = \Lambda^{*1} \subseteq C_1(\Lambda)$. Then

$$\begin{aligned}\partial_1(\Psi_1(\lambda)) &= \partial_1(1_{Y(\lambda)})\\ &= s_*(1_{Y(\lambda)}) - r_*(1_{Y(\lambda)})\\ &= 1_{Z(s(\lambda))} - 1_{Z(r(\lambda))}\\ &= \Psi_0(s(\lambda)) - \Psi_0(r(\lambda))\\ &= \Psi_0(\partial_1(\lambda)).\end{aligned}$$

For $n \geq 2$, it suffices to prove that given an element $(\lambda_1, \dots, \lambda_n) \in \Lambda^{*n}$ and any $0 \leq i \leq n$, we have

$$(d_i)_*(\Psi_n(\lambda_1, \dots, \lambda_n)) = \Psi_n(d_i(\lambda_1, \dots, \lambda_n)). \tag{8}$$

In the following calculation, given sets $Z_1, \dots, Z_n \subseteq \mathcal{G}_\Lambda$, we define

$$Z_1 * Z_2 * \cdots * Z_n := (Z_1 \times Z_2 \times \cdots \times Z_n) \cap \mathcal{G}_\Lambda^{(n)}.$$

Note that $Y(\lambda_1, \dots, \lambda_n) = Z(\lambda_1, s(\lambda_1)) * Z(\lambda_2, s(\lambda_2)) * \cdots * Z(\lambda_n, s(\lambda_n))$. First suppose that $1 \leq i \leq n-1$. Then

$$\begin{aligned}(d_i)_*(\Psi_n(\lambda_1, \dots, \lambda_n)) &= (d_i)_*(1_{Y(\lambda_1, \dots, \lambda_n)})\\ &= (d_i)_*(1_{Z(\lambda_1, s(\lambda_1)) * Z(\lambda_2, s(\lambda_2)) * \cdots * Z(\lambda_n, s(\lambda_n))})\\ &= 1_{Z(\lambda_1, s(\lambda_1)) * \cdots * Z(\lambda_i, s(\lambda_i)) Z(\lambda_{i+1}, s(\lambda_{i+1})) * \cdots * Z(\lambda_n, s(\lambda_n))}\end{aligned}$$

and

$$\begin{aligned}\Psi_n(d_i(\lambda_1, \dots, \lambda_n)) &= \Psi_n(\lambda_1, \dots, \lambda_i\lambda_{i+1}, \dots, \lambda_n)\\ &= 1_{Y(\lambda_1, \dots, \lambda_i\lambda_{i+1}, \dots, \lambda_n)}\\ &= 1_{Z(\lambda_1, s(\lambda_1)) * \cdots * Z(\lambda_i\lambda_{i+1}, s(\lambda_{i+1})) * \cdots * Z(\lambda_n, s(\lambda_n))}.\end{aligned}$$

Since the cylinder sets $Z(\lambda_i, s(\lambda_i))$ and $Z(\lambda_{i+1}, s(\lambda_{i+1}))$ are both bisections and since $s(Z(\lambda_i, s(\lambda_i))) = Z(s(\lambda_i)) = r(Z(\lambda_{i+1}, s(\lambda_{i+1})))$, we have

$$Z(\lambda_i, s(\lambda_i)) Z(\lambda_{i+1}, s(\lambda_{i+1})) = Z(\lambda_i\lambda_{i+1}, s(\lambda_{i+1})),$$

and (8) follows.

Now consider $i = 0$ (the case $i = n$ is very similar). We have

$$\begin{aligned}(d_0)_*(\Psi_n(\lambda_1, \dots, \lambda_n)) &= (d_0)_*(1_{Y(\lambda_1, \dots, \lambda_n)})\\ &= (d_0)_*(Z(\lambda_1, s(\lambda_1)) * Z(\lambda_2, s(\lambda_2)) * \cdots * Z(\lambda_n, s(\lambda_n))))\\ &= 1_{Z(\lambda_2, s(\lambda_2)) * \cdots * Z(\lambda_n, s(\lambda_n))}\\ &= 1_{Y(\lambda_2, \dots, \lambda_n)}\\ &= \Psi_n(\lambda_2, \dots, \lambda_n)\\ &= \Psi_n(d_0(\lambda_1, \dots, \lambda_n)),\end{aligned}$$

as desired. □

In general the map $\Psi_*$ is neither injective nor surjective; see Remark 7.8.

**The HK conjecture for 1-graph and 2-graph groupoids.** In this subsection we apply the results of Section 6 to groupoids associated to 1-graphs and 2-graphs.

Recall from Section 2 that if $\Lambda$ is a $k$-graph, then there is an action $\sigma$ of $\mathbb{N}^k$ by endomorphisms on its infinite-path space $\Lambda^\infty$. Recall further that the $k$-graph $C^*$-algebra coincides with the $C^*$-algebra $C^*(\mathcal{G}(\Lambda^\infty, \sigma))$ of the associated Deaconu–Renault groupoid (see [25]). We begin this section by showing that the homology of $\mathcal{G}(\Lambda^\infty, \sigma)$ as computed in Theorem 6.5 coincides with the homology of the complex $D^\Lambda_*$ used by Evans [14] to compute the $K$-theory of $C^*(\Lambda)$.

The complex $(D^\Lambda_*, \partial_*)$ is given as follows: We continue to write $\varepsilon_1, \dots, \varepsilon_k$ for the generators of $\mathbb{Z}^k$, and we write $\varepsilon_v$ for the generators of $\mathbb{Z}\Lambda^0$. We write $M_j \in M_{\Lambda^0}(\mathbb{Z})$ for the vertex matrix given by $M_j(v, w) = |v\Lambda^{e_j} w|$, which we regard as an endomorphism of $\mathbb{Z}\Lambda^0$. For $p \geq 0$ we define $D^\Lambda_p = \bigwedge^p \mathbb{Z}^k \otimes \mathbb{Z}\Lambda^0$. We identify $D^\Lambda_0$ with $\mathbb{Z}\Lambda^0$, and observe that $D^\Lambda_p = \{0\}$ for $p > k$. We define $\partial_0 = 0$, and for $p \geq 1$ we define $\partial_p : D^\Lambda_p \to D^\Lambda_{p-1}$ by $\partial_p = 0$ if $p > k$,

$$\partial_p(\varepsilon_{i_1} \wedge \cdots \wedge \varepsilon_{i_p} \otimes \varepsilon_v) = \sum_{j=1}^{p} (-1)^{j+1} \varepsilon_{i_1} \wedge \cdots \wedge \widehat{\varepsilon}_{i_j} \wedge \cdots \wedge \varepsilon_{i_p} \otimes (I - M^t_{i_j})\varepsilon_v$$

if $2 \leq p \leq k$, and

$$\partial_1(\varepsilon_i \otimes \varepsilon_v) = (I - M^t_i)\varepsilon_v.$$

In the following proposition, we establish an isomorphism between the homology of Evans' complex $D^\Lambda_*$ and the homology of the complex $A^\sigma_*$ associated to the shift maps $\sigma^n$ on $\Lambda^\infty$. We could obtain isomorphisms $H_*(D^\Lambda_*) \cong H_*(A^\sigma_*)$ using that, by Evans' results, $H_*(D^\Lambda_*) \cong H_*(\mathbb{Z}^k, K_0(C^*(\Lambda \times_d \mathbb{Z}^k)))$, that by Matui's results, $H_*(A^\sigma_*) \cong H_*(\mathbb{Z}^k, H_0(\mathcal{G}_\Lambda \times_c \mathbb{Z}^k))$, and then by identifying $C^*(\Lambda \times_d \mathbb{Z}^k)$ with $C^*(\mathcal{G}_\Lambda \times_c \mathbb{Z}^k)$ and applying the HK conjecture for AF groupoids as stated in Corollary 5.2. However, we have chosen to present a more direct proof, which also has the advantage that it shows that the natural inclusion $\mathbb{Z}\Lambda^0 \hookrightarrow C_c(\Lambda^\infty, \mathbb{Z})$ induces the isomorphism.

**Proposition 7.6.** *Let $\Lambda$ be a row-finite $k$-graph with no sources. Let $\mathcal{G}_\Lambda = \mathcal{G}(\Lambda^\infty, \sigma)$ be the associated groupoid, and let $(A^\sigma_*)$ be the complex of Theorem 6.5. Then the homomorphism $\iota : \mathbb{Z}\Lambda^0 \to C_c(\Lambda^\infty, \mathbb{Z})$ determined by $\iota(1_v) = 1_{Z(v)}$ induces an isomorphism $H_*(A^\sigma_*) \to H_*(D^\Lambda_*)$. In particular, $H_*(\mathcal{G}(\Lambda^\infty, \sigma)) \cong H_*(D^\Lambda_*)$.*

*Proof.* Corollary 5.2 shows that we can express the complex $\bigwedge^* \mathbb{Z}^k \otimes H_0(c^{-1}(0))$ as the direct limit of the $A^\sigma_*$ under the maps induced by the $\sigma_*$. Lemma 6.4 shows that the inclusion of $A^\sigma_*$ in $\bigwedge^* \mathbb{Z}^k \otimes H_0(c^{-1}(0))$ induces an isomorphism in homology. Similarly, in [14, Thm. 3.14] Evans proves that one can express the complex $\bigwedge^* \mathbb{Z}^k \otimes K_0(C^*(\Lambda \times_d \mathbb{Z}^k))$ as the direct limit of $D^\Lambda_*$ under the maps induced by the $M^t_*$, and the inclusion of $D^\Lambda_*$ that takes $1_v$ to the class of $p_{v,0}$ in the direct limit induces an isomorphism in homology. By [25, Thm. 5.2], there is an isomorphism $C^*(\Lambda \times_d \mathbb{Z}^k) \cong C^*(\mathcal{G}(\Lambda^\infty, \sigma) \times_c \mathbb{Z}^k)$ that carries $p_{(v,0)}$ to

$1_{Z(v)\times\{0\}}$. So Corollary 5.2 shows that there is an isomorphism $K_0(C^*(\Lambda \times_d \mathbb{Z}^k)) \cong H_0(\mathcal{G}(\Lambda^\infty, \sigma) \times_c \mathbb{Z}^k)$ that takes $[p_{(v,0)}]$ to $[1_{Z(v)\times\{0\}}]$. Lemma 6.1 therefore implies that there is an isomorphism

$$K_0(C^*(\Lambda \times_d \mathbb{Z}^k)) \cong H_0(c^{-1}(0))$$

induced by the map that carries the class of $p_{v,0}$ to $1_{Z(v)}$.

Given $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, we have

$$\sigma_*^n(\iota(1_v)) = \sigma_*^n(1_{Z(v)}) = \sum_{\lambda \in v\Lambda^n} \sigma_*^n(1_{Z(\lambda)}) = \sum_{\lambda \in v\Lambda^n} 1_{Z(s(\lambda))} = \iota(M_n^t(1_v)).$$

Hence $\iota$ induces a map of complexes $\iota_* : D_*^\Lambda \to A_*^\sigma$ that intertwines the $\sigma_*^n$ with the $(M_n^t)_*$. The same computation combined with the universal property of the direct limit shows that we obtain the following commuting diagram:

$$\begin{array}{ccccccc}
D_*^\Lambda & \xrightarrow{M_*^t} & D_*^\Lambda & \longrightarrow & \cdots & \longrightarrow & \bigwedge^* \mathbb{Z}^k \otimes H_0(c^{-1}(0)) \\
\big\downarrow \iota_* & & \big\downarrow \iota_* & & & & \big\downarrow \iota_*^\infty \\
A_*^\sigma & \xrightarrow{\sigma_*^n} & A_*^\sigma & \longrightarrow & \cdots & \longrightarrow & \bigwedge^* \mathbb{Z}^k \otimes H_0(c^{-1}(0)).
\end{array}$$

By definition, $\iota_*([1_v]) = [1_{Z(v)}]$ for every $v \in \Lambda^0$. Write $(M_{n,\infty}^t)_*$ for the map from the $n$-th copy of $D_*^\Lambda$ into $\bigwedge^* \mathbb{Z}^k \otimes H_*(c^{-1}(0))$, and write $\sigma_*^{n,\infty}$ for the map from the $n$-th copy of $A_*^\sigma$ to $\bigwedge^* \mathbb{Z}^k \otimes H_*(c^{-1}(0))$. Then $(M_{n,\infty}^t)_*(1_v) = [1_{Z(\mu)}]$ for any $\mu \in \Lambda^n v$. Since

$$\sigma_*^{n,\infty}(\iota_*(1_v)) = \sigma_*^{n,\infty}(1_{Z(v)}) = \sigma_*^{n,\infty}(\sigma_*^n(1_{Z(\mu)})) = \sigma_*^{n,\infty}(1_{Z(\mu)}) = [1_{Z(\mu)}],$$

we see by the commutativity of the diagram that $\iota_*^\infty(1_{Z(\mu)}) = [1_{Z(\mu)}]$ for all $\mu$. Since the $1_{Z(\mu)}$ generate $C_c(\Lambda^\infty, \mathbb{Z})$, we deduce that $\iota_*^\infty$ is the identity map, and therefore induces the identity map in homology. Since the maps

$$H_*(D_*^\Lambda) \to H_*(\bigwedge^* \mathbb{Z}^k \otimes H_0(c^{-1}(0))), \quad H_*(A_*^\sigma) \to H_*(\bigwedge^* \mathbb{Z}^k \otimes H_0(c^{-1}(0)))$$

are isomorphisms, and the diagram above commutes, the functoriality of homology implies that $\iota_*$ induces an isomorphism $H_*(D_*^\Lambda) \to H_*(A_*^\sigma)$.

The final statement follows from Theorem 6.5. □

Though we already know that graph groupoids belong to $\mathfrak{M}$ by Remark 6.8, the following result goes a step further, computing the homology of the 1-graph and 2-graph groupoids in terms of the vertex matrices of the 1-graph or 2-graph. Recall that, given a $k$-graph $\Lambda$ and given $i \leq k$, we write $M_i$ for the $\Lambda^0 \times \Lambda^0$ integer matrix given by $M_i(v, w) = |v\Lambda^{e_i} w|$. If $\Lambda$ is the path category of a directed graph $E$, then $M_1$ is just the usual adjacency matrix $A_E$ of $E$.

**Corollary 7.7** ([14, Prop. 3.16]). (i) *Let $E$ be a row-finite graph with no sources. Then*

$$K_0(C^*(E)) \cong H_0(\mathcal{G}_E) \cong \operatorname{coker}(I - A_E^t),$$
$$K_1(C^*(E)) \cong H_1(\mathcal{G}_E) \cong \ker(I - A_E^t).$$

(ii) *Let $\Lambda$ be a row-finite 2-graph with no sources. Then*

$$K_0(C^*(\Lambda)) \cong H_0(\mathcal{G}_\Lambda) \cong \mathbb{Z}\Lambda^0/\big((I - M_1^t, I - M_2^t)(\mathbb{Z}\Lambda^0)^2\big) \oplus \ker(M_2^t - I) \cap \ker(I - M_1^t),$$

*and*

$$K_1(C^*(\Lambda)) \cong H_1(\mathcal{G}_\Lambda) \cong \ker(I - M_1^t, I - M_2^t)/\begin{pmatrix} (M_2^t - I) \\ (I - M_1^t) \end{pmatrix}\mathbb{Z}\Lambda^0.$$

*In particular, graph groupoids and 2-graph groupoids belong to $\mathfrak{M}$.*

*Proof.* Theorems 6.7 and 6.10 establish the isomorphisms between homology and $K$-theory. The descriptions of the homology groups follow from Proposition 7.6 and the definition of the complex $D_*^\Lambda$. □

**Remark 7.8.** The strongly connected 1-graph $\Lambda$ described in Example 7.2 has homology given by

$$H_n(\Lambda) = \begin{cases} \mathbb{Z} & \text{if } n = 0, \\ \mathbb{Z}^{11} & \text{if } n = 1, \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 7.7, the homology of $\mathcal{G}_\Lambda$ is

$$H_n(\mathcal{G}_\Lambda) = \begin{cases} \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z} & \text{if } n = 0, \\ 0 & \text{otherwise.} \end{cases}$$

So the map $\Psi_0 : H_0(\Lambda) \to H_0(\mathcal{G}_\Lambda)$ of Theorem 7.5 is neither surjective nor injective.

**Remark 7.9.** In [14, Prop. 3.18], Evans shows that if $\Lambda$ is a 3-graph and $\{(I - M_i^t)a \mid 1 \le i \le 3,\ a \in \mathbb{Z}\Lambda^0\}$ generates $\mathbb{Z}\Lambda^0$, then the $K$-theory of $C^*(\Lambda)$ is equal to the homology of $D^\Lambda$. So the groupoids of such $k$-graphs belong to $\mathfrak{M}$.

We also see that if $\Lambda$ is a 3-graph or a 4-graph for which the page 3 differentials in Kasparov's sequence are zero and $H_2(D^\Lambda)$ (and $H_3(D^\Lambda)$ in the case of a 4-graph) are free abelian, then $K_*(C^*(\Lambda))$ is determined by $H_*(D^\Lambda)$ and so $\mathcal{G}_\Lambda$ belongs to $\mathfrak{M}$; but of course the hypothesis on the differential maps in Kasparov's sequence are not checkable in practice.

**Remark 7.10.** Suppose that $\Lambda$ is a finite 3-graph. Then the homology groups $H_*(D_*^\Lambda)$ have finite rank, and $H_3(D_*^\Lambda)$ is free abelian. Consequently, if it were possible to construct an example of a finite 3-graph for which the page 3 differential $d_{3,0}^3$ in Kasparov's spectral sequence was nontrivial, then consideration of the ranks of the groups involved would show that the associated groupoid did not satisfy the HK conjecture, even up to stabilization by $\mathbb{Q}$.

**One vertex *k*-graphs.** In this section we will show that if $\Lambda$ is a 1-vertex $k$-graph in which each $|\Lambda^{\varepsilon_i}| \geq 2$ and in which $\gcd(|\Lambda^{\varepsilon_1}|-1, \dots, |\Lambda^{\varepsilon_k}|-1) = 1$, then $K_*(C^*(\Lambda)) = H_*(\mathcal{G}_\Lambda) = 0$. We work with row-finite $k$-graphs throughout, but we include a comment at the end of the section indicating how to extend our $K$-theory calculation to non-row-finite $k$-graphs. A similar result has been proved in [1, Thm. 6.4 (a)] under the assumption that the elements of $\{|\Lambda^{\varepsilon_i}| \mid 1 \leq i \leq k\}$ are pairwise relatively prime.

The key point is the following consequence of Matui's Künneth formula for the groupoid homology of an ample Hausdorff groupoid (see [32, Thm. 2.4]).

**Theorem 7.11.** *Let $\Lambda$ be a row-finite single-vertex $k$-graph with at least two edges of each color, and write $N_i := |\Lambda^{\varepsilon_i}| - 1$ for each $i \leq k$. Then*

$$H_n(\mathcal{G}_\Lambda) \cong \begin{cases} (\mathbb{Z}_{\gcd(N_1,\dots,N_k)})^{\binom{k-1}{n}} & \text{if } 0 \leq n \leq k-1, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* We proceed by induction. This follows from Corollary 7.7 (i) if $k = 1$. Suppose it holds for $k = K - 1$ and that $\Lambda$ is a $K$-graph with one vertex and at least two edges of each color. Since the complex $D_*^\Lambda$ is independent of the factorization rules in $\Lambda$, Proposition 7.6 shows that $H_*(\mathcal{G}_\Lambda)$ is independent of the factorization rules. So we can assume that $\Lambda = B_{N_1+1} \times \cdots \times B_{N_k+1}$ and so $\mathcal{G}_\Lambda = \prod_{i=1}^k \mathcal{G}_{B_{N_i+1}}$.

Write

$$\mathcal{G} = \prod_{i=1}^{K-1} \mathcal{G}_{B_{N_i+1}} \quad \text{and} \quad \mathcal{H} = \mathcal{G}_{B_{N_K+1}},$$

so that $\mathcal{G}_\Lambda \cong \mathcal{G} \times \mathcal{H}$. Matui's Künneth theorem gives a split exact sequence

$$0 \to \bigoplus_{i+j=n} H_i(\mathcal{G}) \otimes H_j(\mathcal{H}) \longrightarrow H_n(\mathcal{G} \times \mathcal{H}) \longrightarrow \bigoplus_{i+j=n-1} \operatorname{Tor}(H_i(\mathcal{G}), H_j(\mathcal{H})) \to 0,$$

and since $H_*(\mathcal{H}) = (\mathbb{Z}_{N_K}, 0, 0, \dots)$, this collapses to a split exact sequence

$$0 \to H_n(\mathcal{G}) \otimes \mathbb{Z}_{N_K} \longrightarrow H_n(\mathcal{G} \times \mathcal{H}) \longrightarrow \operatorname{Tor}(H_{n-1}(\mathcal{G}), \mathbb{Z}_{N_K}) \to 0.$$

Since the sequence splits, the middle term is the direct sum of the two ends. Write $\gamma := \gcd(N_1, \dots, N_{K-1})$. Then the inductive hypothesis gives

$$H_n(\mathcal{G}) = \mathbb{Z}_\gamma^{\binom{K-2}{n}} \quad \text{and} \quad H_{n-1}(\mathcal{G}) = \mathbb{Z}_\gamma^{\binom{K-2}{n-1}}.$$

Also, $\gcd(\gamma, N_K) = \gcd(N_1, \dots, N_K)$. As $\otimes$ and Tor are both additive in the first variable and $\mathbb{Z}_l \otimes \mathbb{Z}_m = \mathbb{Z}_{\gcd(l,m)} = \operatorname{Tor}(\mathbb{Z}_l, \mathbb{Z}_m)$, we have

$$\begin{aligned} H_n(\mathcal{G} \times \mathcal{H}) &= (\mathbb{Z}_{\gcd(\gamma, N_K)})^{\binom{K-2}{n}} \oplus (\mathbb{Z}_{\gcd(\gamma, N_K)})^{\binom{K-2}{n-1}} \\ &= (\mathbb{Z}_{\gcd(N_1,\dots,N_K)})^{\binom{K-2}{n} + \binom{K-2}{n-1}} \\ &= (\mathbb{Z}_{\gcd(N_1,\dots,N_K)})^{\binom{K-1}{n}}. \end{aligned}$$

□

We deduce that the groupoids of 1-vertex $k$-graphs in which there are at least two edges of each color, and in which $\gcd(|\Lambda^{\varepsilon_1}|-1,\dots,|\Lambda^{\varepsilon_k}|-1)=1$, belong to $\mathfrak{M}$.

**Corollary 7.12.** *If $\Lambda$ is a single-vertex $k$-graph with at least two edges of each color and $\gcd(|\Lambda^{\varepsilon_1}|-1,\dots,|\Lambda^{\varepsilon_k}|-1)=1$, then $K_*(C^*(\Lambda))=H_*(\mathcal{G}_\Lambda)=0$. In particular, if $C^*(\Lambda)$ is simple, then it is isomorphic to $\mathcal{O}_2$.*

*Proof.* Theorem 7.11 shows that $H_*(\mathcal{G}_\Lambda)=0$. It follows that the groups $E^2_{p,q}$ in Matui's spectral sequence are all zero. Since the terms $F^2_{p,q}$ in Kasparov's spectral sequence [21] (see also [14]) are given by $E^2_{p,0}$ if $q$ is even, and 0 if $q$ is odd, we deduce that the $F^2_{p,q}$ are all zero. So Evans' spectral sequence collapses, and we obtain $K_*(C^*(\Lambda))=0$ as well. The final statement follows from [46, Prop. 8.8 and Cor. 8.15]. □

**Remark 7.13.** Unfortunately, if $\gcd(N_1,\dots,N_k)>1$, we can conclude little new about the HK conjecture. The problem is that in $K$-theory, with the exception of tensor products, we only obtain from Evans' spectral sequence that the $K$-groups have filtrations of length at most $k-1$ with subquotients equal to direct sums of copies of $\mathbb{Z}_{\gcd(N_1,\dots,N_k)}$.

**Remark 7.14.** The above discussion deals only with row-finite $k$-graphs. We can extend the $K$-theory calculation to non-row-finite examples as follows: If $\Lambda$ is any 1-vertex $k$-graph, and $\Gamma$ is a 1-vertex $(k+1)$-graph such that $d_\Gamma^{-1}(\mathbb{N}^k)\cong\Lambda$ and $\Gamma^{e_{k+1}}$ is infinite, then as in [6] we can make the identification

$$C^*(\Gamma)\cong\mathcal{T}_{\ell^2(C^*(\Lambda))_{C^*(\Lambda)}},$$

and deduce from Pimsner's [36, Thm. 4.4] that the inclusion $C^*(\Lambda)\hookrightarrow C^*(\Gamma)$ determines a $KK$-equivalence, so $K_*(C^*(\Gamma))\cong K_*(C^*(\Lambda))$.

Applying this iteratively, we can compute the $K$-theory of the $C^*$-algebra of a 1-vertex, not-necessarily-row-finite $k$-graph as the $K$-theory of the $C^*$-algebra of the subgraph consisting only of those coordinates in which there are finitely many edges.

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## REFERENCES

[1] S. Barlak, T. Omland and N. Stammeier, On the $K$-theory of $C^*$-algebras arising from integral dynamics, Ergodic Theory Dynam. Systems **38** (2018), no. 3, 832–862. MR3784245

[2] B. Blackadar, $K$-theory for operator algebras, MSRI Publications, 5, Cambridge University Press, Cambridge, 1998. MR0859867

[3] C. Bönicke and K. Li, Ideal structure and pure infiniteness of ample groupoid $C^*$-algebras, Ergodic Theory Dynam. Systems, arXiv:1707.03740 [math.OA] (2019).

[4] G. E. Bredon, *Sheaf theory*, second edition, Graduate Texts in Mathematics, 170, Springer-Verlag, New York, 1997. MR1481706

[5] K. S. Brown, *Cohomology of groups*, corrected reprint of the 1982 original, Graduate Texts in Mathematics, 87, Springer-Verlag, New York, 1994. MR1324339

[6] J. H. Brown, L. O. Clark, A. Sierakowski and A. Sims, Purely infinite simple $C^*$-algebras that are principal groupoid $C^*$-algebras, J. Math. Anal. Appl. **439** (2016), no. 1, 213–234. MR3474359

[7] L. G. Brown, P. Green and M. A. Rieffel, Stable isomorphism and strong Morita equivalence of $C^*$-algebras, Pacific J. Math. **71** (1977), no. 2, 349–363. MR0463928

[8] T. M. Carlsen, E. Ruiz and A. Sims, Equivalence and stable isomorphism of groupoids, and diagonal-preserving stable isomorphisms of graph $C^*$-algebras and Leavitt path algebras, Proc. Amer. Math. Soc. **145** (2017), no. 4, 1581–1592. MR3601549

[9] L. O. Clark and A. Sims, Equivalent groupoids have Morita equivalent Steinberg algebras, J. Pure Appl. Algebra **219** (2015), no. 6, 2062–2075. MR3299719

[10] M. Crainic and I. Moerdijk, A homology theory for étale groupoids, J. Reine Angew. Math. **521** (2000), 25–46. MR1752294

[11] M. Crainic, Cyclic cohomology of étale groupoids: the general case, $K$-Theory **17** (1999), no. 4, 319–362. MR1706117

[12] V. Deaconu, Groupoids associated with endomorphisms, Trans. Amer. Math. Soc. **347** (1995), no. 5, 1779–1786. MR1233967

[13] R. J. Deeley, I. F. Putnam and K. R. Strung, Constructing minimal homeomorphisms on point-like spaces and a dynamical presentation of the Jiang-Su algebra, J. Reine Angew. Math. **742** (2018), 241–261. MR3849627

[14] D. G. Evans, On the $K$-theory of higher rank graph $C^*$-algebras, New York J. Math. **14** (2008), 1–31. MR2383584

[15] R. Exel, Inverse semigroups and combinatorial $C^*$-algebras, Bull. Braz. Math. Soc. (N.S.) **39** (2008), no. 2, 191–313. MR2419901

[16] R. Exel and J. Renault, Semigroups of local homeomorphisms and interaction groups, Ergodic Theory Dynam. Systems **27** (2007), no. 6, 1737–1771. MR2371594

[17] E. Gillaspy and J. Wu, Isomorphism of the cubical and categorical cohomology groups of a higher-rank graph, arXiv:1807.02245v3 [math.OA] (2019).

[18] T. Giordano, I. F. Putnam and C. F. Skau, Topological orbit equivalence and $C^*$-crossed products, J. Reine Angew. Math. **469** (1995), 51–111. MR1363826

[19] R. Hazrat and H. Li, Homology of étale groupoids a graded appropach, arXiv:1806.03398v2 [math.KT] (2019).

[20] M. Kashiwara and P. Schapira, *Sheaves on manifolds*, Grundlehren der Mathematischen Wissenschaften, 292, Springer-Verlag, Berlin, 1990. MR1074006

[21] G. G. Kasparov, Equivariant $KK$-theory and the Novikov conjecture, Invent. Math. **91** (1988), no. 1, 147–201. MR0918241

[22] K. Kodaka, $K$-theory for the $C^*$-algebras of the discrete Heisenberg group, Tokyo J. Math. **9** (1986), no. 2, 365–372. MR0875195

[23] K. P. Knudson, *Homology of linear groups*, Progress in Mathematics, 193, Birkhäuser Verlag, Basel, 2001. MR1807154

[24] A. Kumjian, Preliminary algebras arising from local homeomorphisms, Math. Scand. **52** (1983), no. 2, 269–278. MR0702958

[25] A. Kumjian and D. Pask, Higher rank graph $C^*$-algebras, New York J. Math. **6** (2000), 1–20. MR1745529

[26] A. Kumjian et al., Graphs, groupoids, and Cuntz–Krieger algebras, J. Funct. Anal. **144** (1997), no. 2, 505–541. MR1432596

[27] A. Kumjian, D. Pask and A. Sims, $C^*$-algebras associated to coverings of $k$-graphs, Doc. Math. **13** (2008), 161–205. MR2520478

[28] A. Kumjian, D. Pask and A. Sims, On twisted higher-rank graph $C^*$-algebras, Trans. Amer. Math. Soc. **367** (2015), no. 7, 5177–5216. MR3335414
[29] S. Mac Lane, *Categories for the working mathematician*, second edition, Graduate Texts in Mathematics, 5, Springer-Verlag, New York, 1998. MR1712872
[30] H. Matui, Homology and topological full groups of étale groupoids on totally disconnected spaces, Proc. Lond. Math. Soc. (3) **104** (2012), no. 1, 27–56. MR2876963
[31] H. Matui, Topological full groups of one-sided shifts of finite type, J. Reine Angew. Math. **705** (2015), 35–84. MR3377390
[32] H. Matui, Étale groupoids arising from products of shifts of finite type, Adv. Math. **303** (2016), 502–548. MR3552533
[33] I. Moerdijk and J. Mrčun, *Introduction to foliations and Lie groupoids*, Cambridge Studies in Advanced Mathematics, 91, Cambridge University Press, Cambridge, 2003. MR2012261
[34] P. S. Muhly, J. N. Renault and D. P. Williams, Equivalence and isomorphism for groupoid $C^*$-algebras, J. Operator Theory **17** (1987), no. 1, 3–22. MR0873460
[35] E. Ortega, Homology of the Katsura–Exel–Pardo groupoid, arXiv:1806.09297v2 [math.OA] (2018).
[36] M. V. Pimsner, A class of $C^*$-algebras generalizing both Cuntz-Krieger algebras and crossed products by **Z**, in *Free probability theory (Waterloo, ON, 1995)*, 189–212, Fields Inst. Commun., 12, Amer. Math. Soc., Providence, RI, 1997. MR1426840
[37] M. Pimsner and D. Voiculescu, Exact sequences for $K$-groups and Ext-groups of certain cross-product $C^*$-algebras, J. Operator Theory **4** (1980), no. 1, 93–118. MR0587369
[38] I. F. Putnam, Some classifiable groupoid $C^*$-algebras with prescribed $K$-theory, Math. Ann. **370** (2018), no. 3-4, 1361–1387. MR3770169
[39] T. Rainone and A. Sims, A dicothomy for groupoid $C^*$-algebras, Ergodic Theory Dynam. Systems, arXiv:1707.04516 [math.OA] (2019).
[40] A. Ramsay, Virtual groups and group actions, Advances in Math. **6** (1971), 253–322 (1971). MR0281876
[41] J. Renault, *A groupoid approach to $C^*$-algebras*, Lecture Notes in Mathematics, 793, Springer, Berlin, 1980. MR0584266
[42] J. N. Renault, $C^*$-algebras of groupoids and foliations, in *Operator algebras and applications, Part I (Kingston, Ont., 1980)*, 339–350, Proc. Sympos. Pure Math., 38, Amer. Math. Soc., Providence, RI, 1982. MR0679714
[43] M. Rørdam, F. Larsen and N. Laustsen, *An introduction to $K$-theory for $C^*$-algebras*, London Mathematical Society Student Texts, 49, Cambridge University Press, Cambridge, 2000. MR1783408
[44] M. Rørdam and A. Sierakowski, Purely infinite $C^*$-algebras arising from crossed products, Ergodic Theory Dynam. Systems **32** (2012), no. 1, 273–293. MR2873171
[45] E. Scarparo, Homology of odometers, arXiv:1811.05795 [math.OA] (2018).
[46] A. Sims, Gauge-invariant ideals in the $C^*$-algebras of finitely aligned higher-rank graphs, Canad. J. Math. **58** (2006), no. 6, 1268–1290. MR2270926
[47] A. Sims, Hausdorff étale groupoids and their $C^*$-algebras, in *Operator algebras and dynamics: groupoids, crossed products and Rokhlin dimension*, CRM Barcelona, Birkhäuser, to appear.
[48] A. Sims and D. P. Williams, Renault's equivalence theorem for reduced groupoid $C^*$-algebras, J. Operator Theory **68** (2012), no. 1, 223–239. MR2966043
[49] A. Sims and D. P. Williams, The primitive ideals of some étale groupoid $C^*$-algebras, Algebr. Represent. Theory **19** (2016), no. 2, 255–276. MR3489096
[50] E. H. Spanier, *Algebraic topology*, McGraw-Hill Book Co., New York, 1966. MR0210112
[51] H. Takai, On a duality for crossed products of $C^*$-algebras, J. Functional Analysis **19** (1975), 25–39. MR0365160
[52] M. Takesaki, Duality for crossed products and the structure of von Neumann algebras of type III, Acta Math. **131** (1973), 249–310. MR0438149

[53] N. E. Wegge-Olsen, *K-theory and $C^*$-algebras*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, 1993. MR1222415

[54] S. Willard, *General topology*, Reprint of the 1970 original [Addison-Wesley, Reading, MA; MR0264581], Dover Publications, Inc., Mineola, NY, 2004.

[55] D. P. Williams, Haar systems on equivalent groupoids, Proc. Amer. Math. Soc. Ser. B **3** (2016), 1–8. MR3478528

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