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Smooth and rough modules over self-induced algebras

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Abstract. A nonunital algebra in a closed monoidal category is called self-induced if the multiplication induces an isomorphism $A \otimes_A A \cong A$. For such an algebra, we define smoothening and roughening functors that retract the category of modules onto two equivalent subcategories of smooth and rough modules, respectively. These functors generalize previous constructions for group representations on bornological vector spaces. We also study the pairs of adjoint functors between categories of smooth and rough modules that are induced by bimodules and Morita equivalence.

1. Introduction

Many algebras that are considered in noncommutative geometry are nonunital. Typical examples are the convolution algebra $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$ of smooth compactly supported functions on a locally compact group G (see [7]) or the algebras $\mathbb{M}_\infty$ and $\mathcal{K}$ of finite matrices and of infinite matrices with rapidly decreasing entries (see [2]). Both algebras carry additional structure: both $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$ and $\mathcal{K}$ are complete convex bornological algebras. We may also view $\mathcal{K}$ as a Fréchet algebra, but this structure is less relevant here.

When dealing with nonunital algebras, the usual unitality condition for modules makes no sense. But simply dropping this condition would give too many modules. On the one hand, the bornological algebras $\mathbb{M}_\infty$ and $\mathcal{K}$ are Morita equivalent to $\mathbb{C}$, so that we expect an equivalence of module categories. On the other hand, the categories of nonunital (bornological) modules over $\mathbb{M}_\infty$ and $\mathcal{K}$ are not equivalent to the category of $\mathbb{C}$-modules.

This article grew out of the manuscript [8], which will not be published any more because there are too many small things that I want changed. One of them is that, while [8] only considers bornological algebras, it is sometimes necessary to consider other categories instead of bornological vector spaces, such

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as the category of inductive systems of Banach spaces (see [9]). Therefore, we discuss smoothening and roughening functors and the functors induced by bimodules in much greater generality here. We work with algebras in an arbitrary monoidal category, and we replace the quasi-unitality assumption in [8] by the much weaker assumption of being self-induced. A monoidal category is an additive category $\mathcal{C}$ with an associative tensor product functor $\otimes$ and a tensor unit $\mathbb{1}$ that satisfies suitable coherence laws (see [6, 11]).

Following Niels Grønbæk [3], we call an algebra A in such a tensor category *self-induced* if the multiplication map $A \otimes A \to A$ induces an isomorphism $A \otimes_A A \cong A$. If A is self-induced, Grønbæk calls a left A-module X *A-induced* if the multiplication map $A \otimes X \to X$ induces an isomorphism $A \otimes_A X \cong X$.

For instance, let $\mathcal{C}$ be the symmetric monoidal category of complete convex bornological vector spaces with the complete projective bornological tensor product and let $A = \mathrm{C}_{\mathrm{c}}^{\infty}(G)$ be the convolution algebra of smooth functions with compact support on a locally compact group G (in the sense of François Bruhat [1]), viewed as an algebra in $\mathcal{C}$. Then A is self-induced, and the category of A-induced modules is isomorphic to the category of smooth representations of G on complete convex bornological vector spaces (see [7]). Therefore, we call A-induced modules *smooth*. Smooth modules over self-induced algebras also appear in the work of Christian Voigt on bornological quantum groups [12].

An A-module X over a self-induced algebra A is called *rough* if the adjoint $X \to (A \Rightarrow X)$ of the module multiplication map $A \otimes X \to X$ induces an isomorphism $X \cong A \Rightarrow_A X$. Here $(A \Rightarrow X) = \mathrm{Hom}(A, X)$ denotes the internal Hom functor and $A \Rightarrow_A X = \mathrm{Hom}_A(A, X)$ denotes the subfunctor of "A-linear maps." The existence of such internal Hom functors is the defining property of a *closed* monoidal category. In the category of complete convex bornological vector spaces, $A \Rightarrow_A X$ is the space of bounded A-module homomorphisms $A \to X$.

Rough and smooth modules and smoothening and roughening functors for group convolution algebras are already studied in [7]. Here we extend some of the properties observed in [7] to the general setting explained above. The smoothening and the roughening of a module X are defined by

$$\mathrm{S}(A) := A \otimes_A X \qquad \text{and} \qquad \mathrm{R}(A) := A \Rightarrow_A X,$$

respectively. As the name suggests, these A-modules are smooth and rough, respectively. There are natural maps $\mathrm{S}(X) \to X \to \mathrm{R}(X)$, the first is an isomorphism if and only if X is smooth, the second if and only if X is rough. Thus S and R are retractions from the category of all modules onto the subcategories of smooth and rough modules, respectively. We show also that S is the right adjoint of the embedding of the subcategory of smooth modules, while R is left adjoint to the embedding of the subcategory of rough modules. And R is right adjoint to S. Finally, $\mathrm{S} \circ \mathrm{R} = \mathrm{S}$ and $\mathrm{R} \circ \mathrm{S} = \mathrm{R}$, so that the functors S and R provide an equivalence of categories between the categories of smooth and rough A-modules.

This is useful when we want to turn bimodules into functors between categories of smooth or rough modules. Of course, an A, B-bimodule M induces a functor $X \mapsto M \otimes_B X$ from left B- to left A-modules. If M is smooth as a left A-module, this maps smooth modules again to smooth modules. The functor $Y \mapsto M \Rightarrow_B Y$ in the opposite direction is defined between the categories of rough modules, that is, $M \Rightarrow_A Y$ is a rough B-module if M is a smooth B-module. Using the smoothening functor, we may turn this into a functor between categories of smooth modules as well. The resulting functor $Y \mapsto \mathrm{S}(M \Rightarrow_A Y)$ is right adjoint to the functor $X \mapsto M \otimes_B X$ in the opposite direction. In particular, the functor $X \mapsto M \otimes_B X$ between smooth module categories always has a right adjoint functor.

An algebra homomorphism $f\colon A \to B$ allows us to view B as an A, B-bimodule or as a B, A-bimodule. These two bimodules provide two pairs of adjoint functors between the categories of smooth modules over A and B.

As an example of our general theory, we consider the biprojective algebras of the form $W \otimes V$ for a sufficiently nondegenerate map $b\colon V \otimes W \to \mathbb{1}$. This includes the bornological algebras $\mathbb{M}_\infty$ and $\mathcal{K}$ of finite and rapidly decreasing matrices. This construction also provides examples of self-induced bornological algebras where the canonical map $\mathrm{S}(X) \to X$ is not always a monomorphism. This should be contrasted with [7, Lemma 4.4], which asserts that this map is always injective provided A is a bornological algebra with an approximate identity in a suitable sense.

We also consider the functors that relate Lie group and Lie algebra representations for a Lie group G. Let $\mathrm{U}(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g}$ of G. Thus the category of unital $\mathrm{U}(\mathfrak{g})$-modules is equivalent to the category of Lie algebra representations of $\mathfrak{g}$. We may view $\mathrm{C}_\mathrm{c}^\infty(G)$ as a $\mathrm{C}_\mathrm{c}^\infty(G), \mathrm{U}(\mathfrak{g})$- or $\mathrm{U}(\mathfrak{g}), \mathrm{C}_\mathrm{c}^\infty(G)$-bimodule. This provides two functors from smooth representations of G to Lie algebra representations of $\mathfrak{g}$. The first equips a smooth representation with the induced representation of $\mathfrak{g}$, the second takes the induced representation of $\mathfrak{g}$ on the roughening. In the opposite direction, we get two functors that integrate representations of $\mathfrak{g}$ to smooth representations of G.

2. Preliminaries

Additive monoidal categories provide the categorical framework to define algebras and modules. In the same generality, we may define self-induced algebras and smooth modules. We need a *closed* monoidal category, that is, an internal Hom functor, to define rough modules as well. Here we briefly recall these basic category theoretic definitions. Then we turn the categories of Banach spaces, of complete convex bornological vector spaces, and of inductive systems of Banach spaces into closed monoidal categories. We also discuss the monoidal category of complete locally convex topological vector spaces and why it is not closed.

Readers who are only interested in bornological and topological algebras need not read this section in detail because everything we explain here is fairly obvious in those cases. They mainly have to remember the functors $X \otimes_A Y$ and $X \Rightarrow_A Y$ described concretely in Example 2.11 and the basic adjointness relation (3).

A monoidal category is a category $\mathcal{C}$ with a bifunctor $\otimes\colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ called *tensor product* and an object $\mathbb{1}$ called (tensor) *unit*, and natural isomorphisms

$$\begin{aligned} \alpha_{A,B,C}\colon (A \otimes B) \otimes C &\xrightarrow{\cong} A \otimes (B \otimes C), \\ \lambda_A\colon \mathbb{1} \otimes A &\xrightarrow{\cong} A, \\ \rho_A\colon A \otimes \mathbb{1} &\xrightarrow{\cong} A, \end{aligned}$$

called *associator*, *left unitor* and *right unitor*, subject to two coherence conditions: for all objects A, B, C and D in $\mathcal{C}$, the pentagon diagram

$$\begin{tikzcd} & \big(A \otimes (B \otimes C)\big) \otimes D \arrow[dr, "\alpha_{A,B\otimes C,D}"] & \\ \big((A \otimes B) \otimes C\big) \otimes D \arrow[ur, "\alpha_{A,B,C}\otimes D"] \arrow[d, "\alpha_{A\otimes B,C,D}"'] & & A \otimes \big((B \otimes C) \otimes D\big) \arrow[d, "A\otimes\alpha_{B,C,D}"] \\ (A \otimes B) \otimes (C \otimes D) \arrow[rr, "\alpha_{A,B,C\otimes D}"'] & & A \otimes \big(B \otimes (C \otimes D)\big) \end{tikzcd}$$

commutes, and for all objects A, B and C of $\mathcal{C}$, the diagram

$$\begin{tikzcd} (A \otimes \mathbb{1}) \otimes B \arrow[rr, "\alpha_{A,\mathbb{1},B}"] \arrow[dr, "\rho_A\otimes B"'] & & A \otimes (\mathbb{1} \otimes B) \arrow[dl, "A\otimes\lambda_B"] \\ & A \otimes B & \end{tikzcd}$$

commutes. By Mac Lane's Coherence Theorem [6], these two coherence conditions imply that any diagram constructed using only associators and unitors commutes.

A *braided monoidal category* [5] is a monoidal category together with *braiding automorphisms*

$$\gamma_{A,B}\colon A \otimes B \to B \otimes A$$

that are compatible with the associators in the sense that the following hexagons commute:

$$\begin{tikzcd}
A \otimes (B \otimes C) \arrow[r, "\gamma_{A,B\otimes C}"] & (B \otimes C) \otimes A \arrow[d, "\alpha_{B,C,A}"] \\
(A \otimes B) \otimes C \arrow[u, "\alpha_{A,B,C}"] \arrow[d, "\gamma_{A,B}\otimes C"'] & B \otimes (C \otimes A) \\
(B \otimes A) \otimes C \arrow[r, "\alpha_{B,A,C}"'] & B \otimes (A \otimes C) \arrow[u, "B\otimes\gamma_{A,C}"']
\end{tikzcd}$$

$$\begin{tikzcd}
(A \otimes B) \otimes C \arrow[r, "\gamma_{A\otimes B,C}"] & C \otimes (A \otimes B) \arrow[d, "\alpha^{-1}_{C,A,B}"] \\
A \otimes (B \otimes C) \arrow[u, "\alpha^{-1}_{A,B,C}"] \arrow[d, "A\otimes\gamma_{B,C}"'] & (C \otimes A) \otimes B. \\
A \otimes (C \otimes B) \arrow[r, "\alpha^{-1}_{A,C,B}"'] & (A \otimes C) \otimes B \arrow[u, "\gamma_{A,C}\otimes B"']
\end{tikzcd}$$

This implies compatibility with unitors, that is, a commuting diagram

$$\begin{tikzcd}
A \otimes \mathbb{1} \arrow[rr, "\gamma_{A,\mathbb{1}}"] \arrow[dr, "\rho_A"'] & & \mathbb{1} \otimes A \arrow[dl, "\lambda_A"] \\
& A. &
\end{tikzcd}$$

A *symmetric monoidal category* is a braided monoidal category that, in addition, satisfies $\gamma_{A,B}\gamma_{B,A} = \mathrm{Id}_{A\otimes B}$ for all objects A and B.

An additive (braided) monoidal category is a category that is at the same time additive and (braided) monoidal, and such that the bifunctor $\otimes$ is additive. We will only consider additive monoidal categories.

Example 2.1. The basic example of an additive symmetric monoidal category is the category of Abelian groups with the usual tensor product, $\mathbb{1} = \mathbb{Z}$, and the obvious associator, unitors, and brading.

Example 2.2. Let $\mathfrak{Bor}$ be the category of complete convex bornological vector spaces. Let $\otimes$ be the complete projective bornological tensor product, usually denoted $\hat{\otimes}$, and let $\mathbb{1}$ be $\mathbb{C}$, assuming we are dealing with complex vector spaces. Then the obvious associator, unitors, and braiding on the algebraic tensor product induce maps on the completions and provide the corresponding data in $\mathfrak{Bor}$. This turns $\mathfrak{Bor}$ into a symmetric monoidal category.

Our examples will all be in this symmetric monoidal category.

Example 2.3. Let $\overrightarrow{\mathfrak{Ban}}$ be the category of inductive systems of Banach spaces. The projective Banach space tensor product has a unique extension $\otimes$ to $\overrightarrow{\mathfrak{Ban}}$

that commutes with inductive limits. Let $\mathbb{1}$ be $\mathbb{C}$, assuming we are dealing with complex vector spaces. There are an obvious associator, unitors, and braiding that turn this into a symmetric monoidal category. We refer to [9] for more details and an explanation why it is useful to replace $\mathfrak{Bor}$ by $\overrightarrow{\mathfrak{Ban}}$.

Example 2.4. Let $\mathfrak{Tvs}$ be the category of complete locally convex topological vector spaces. Let $\otimes$ be the complete projective topological tensor product, usually denoted $\hat{\otimes}_\pi$, and let $\mathbb{1}$ be $\mathbb{C}$, assuming we are dealing with complex vector spaces. Then the obvious associators, unitors, and braidings on the algebraic tensor products extend to the completions and provide the corresponding data in $\mathfrak{Tvs}$. This turns $\mathfrak{Tvs}$ into a symmetric monoidal category.

Definition 2.5. A monoidal category $\mathcal{C}$ is called (left) *closed* if the tensor product functor $B \mapsto A \otimes B$ has a right adjoint for each object A. In this case, the adjoints define a bifunctor $\mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathcal{C}$, $(A, B) \mapsto A \Rightarrow B$ with natural isomorphisms

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(B, A \Rightarrow C) \tag{1}$$

for all objects A, B and C of $\mathcal{C}$. The isomorphisms in (1) provide natural transformations $\mathrm{ev}_{AB} \colon A \otimes (A \Rightarrow B) \to B$, called *evaluation map*, and $B \to A \Rightarrow (A \otimes B)$.

Example 2.6. The symmetric monoidal category $\mathfrak{Bor}$ in Example 2.2 is closed. The internal Hom space $A \Rightarrow C$ is the space of bounded linear maps $A \to C$ equipped with the bornology of equibounded sets of linear maps. This bornology is complete and convex if C is, and the defining isomorphism (1) is well-known.

Banach spaces form an additive subcategory of $\mathfrak{Bor}$ that is closed both under $\otimes$ and $\Rightarrow$. Hence they form a closed symmetric monoidal category in their own right. The category of inductive systems $\overrightarrow{\mathfrak{Ban}}$ is closed symmetric monoidal as well, see [9] for the construction of the internal Hom functor in $\overrightarrow{\mathfrak{Ban}}$.

Example 2.7. The symmetric monoidal category $\mathfrak{Tvs}$ in Example 2.4 is not closed. The complete projective topological tensor product functor cannot have a right adjoint because this would force it to commute with arbitrary colimits. But it does not even commute with direct sums.

The complete inductive tensor product of [4] does commute with direct sums. It is defined by a universal property for separately continuous bilinear maps. But since separately continuous bilinear maps need not extend from dense subspaces, it seems likely that the *completed* inductive tensor product is not associative in complete generality. It is, therefore, unclear how to turn the category of all complete locally convex topological vector spaces into a closed monoidal category.

The internal Hom functor of a closed monoidal category comes with several canonical maps (see also [11]). The most important ones are:

- a lifting of the adjointness isomorphism (1) to internal Homs:

$$(A \otimes B) \Rightarrow C \cong B \Rightarrow (A \Rightarrow C); \tag{2}$$

- the canonical *composition map*

$$(X \Rightarrow Y) \otimes (Y \Rightarrow Z) \to X \Rightarrow Z, \qquad f \otimes g \mapsto g \circ f,$$

  for three objects X, Y and Z, which is adjoint to the composition

$$X \otimes \big((X \Rightarrow Y) \otimes (Y \Rightarrow Z)\big) \cong \big(X \otimes (X \Rightarrow Y)\big) \otimes (Y \Rightarrow Z) \to Y \otimes (Y \Rightarrow Z) \to Z;$$

- and the *inflation map*

$$X \Rightarrow Y \to (Z \otimes X) \Rightarrow (Z \otimes Y), \qquad f \mapsto Z \otimes f = \mathrm{Id}_Z \otimes f,$$

  for three objects X, Y, and Z, which is adjoint to the map $(Z \otimes X) \otimes (X \Rightarrow Y) \cong Z \otimes \big(X \otimes (X \Rightarrow Y)\big) \to Z \otimes Y$.

Let $\mathcal{C}$ be an additive monoidal category. An *algebra* in $\mathcal{C}$ is simply a semigroup object in $\mathcal{C}$, that is, an object A with a map $\mu\colon A \otimes A \to A$ called *multiplication map*, such that the usual associativity diagram

$$\begin{tikzcd}
(A \otimes A) \otimes A \arrow[r, "\alpha"] \arrow[d, "\mu\otimes A"'] & A \otimes (A \otimes A) \arrow[r, "A\otimes\mu"] & A \otimes A \arrow[d, "\mu"] \\
A \otimes A \arrow[rr, "\mu"'] & & A
\end{tikzcd}$$

commutes. A *unital algebra* in $\mathcal{C}$ is a monoid object in $\mathcal{C}$, that is, it is an algebra together with a morphism $\eta\colon \mathbb{1} \to A$ called *unit* such that the diagram

$$\begin{tikzcd}
\mathbb{1} \otimes A \arrow[r, "\eta\otimes A"] \arrow[dr, "\lambda_A"'] & A \otimes A \arrow[d, "\mu"] & A \otimes \mathbb{1} \arrow[l, "A\otimes\eta"'] \arrow[dl, "\rho_A"] \\
& A &
\end{tikzcd}$$

commutes. The usual trick shows that if an algebra has a left and a right unit, then both coincide and provide a two-sided unit. In particular, the unit of an algebra is unique if it exists.

Let (A, μ) be an algebra in $\mathcal{C}$. A *left A-module* is an object X of $\mathcal{C}$ with a map $\mu_X\colon A \otimes X \to X$, also called *multiplication*, such that the usual associativity diagram commutes:

$$\begin{tikzcd}
(A \otimes A) \otimes X \arrow[r, "\alpha"] \arrow[d, "\mu\otimes A"'] & A \otimes (A \otimes X) \arrow[r, "A\otimes\mu_X"] & A \otimes X \arrow[d, "\mu_X"] \\
A \otimes X \arrow[rr, "\mu"'] & & X
\end{tikzcd}$$

A module over a unital algebra is *unital* if the following diagram commutes:

$$\begin{tikzcd} \mathbb{1} \otimes X \arrow[r, "\eta\otimes X"] \arrow[dr, "\lambda_X"'] & A \otimes X \arrow[d, "\mu_X"] \\ & X \end{tikzcd}$$

Right modules and bimodules are defined similarly. In a braided monoidal category, any algebra A has an opposite algebra A^{op} with multiplication

$$A \otimes A \xrightarrow{\gamma_{A,A}} A \otimes A \xrightarrow{\mu} A,$$

where μ and γ are the multiplication of A and the braiding. The assumptions of a braided monoidal category imply that this is again an algebra.

If $\mu_X \colon X \otimes A \to X$ is a right A-module structure, then

$$A \otimes X \xrightarrow{\gamma_{A,X}} X \otimes A \xrightarrow{\mu_X} X$$

is a left A^{op}-module structure on X, and vice versa; once again we need the assumptions of a braided monoidal category here. Hence right A-modules are equivalent to left A^{op}-modules. Thus there is no significant difference between left and right modules in braided monoidal categories.

Example 2.8. (Unital) algebras in the symmetric monoidal category of Abelian groups are (unital) rings, and (unital) modules over such algebras are (unital) modules over rings in the usual sense.

(Unital) algebras in $\mathfrak{Tvs}$ (Example 2.4) are complete locally convex topological (unital) algebras with jointly continuous multiplication $A \times A \to A$. (Unital) modules over them are complete locally convex topological (unital) modules with jointly continuous multiplication map $A \times X \to X$.

Similarly, (unital) algebras in $\mathfrak{Bor}$ (Example 2.2) are (unital) complete convex bornological algebras, and modules also have their usual meaning.

Next we define $X \otimes_A Y$ and $X \Rightarrow_A Y$ for an algebra A and A-modules X and Y.

Definition 2.9. Let $\mathcal{C}$ be a monoidal category in which each morphism has a cokernel. Let A be an algebra in $\mathcal{C}$, let X be a right A-module, and let Y be a left A-module, with multiplication maps $\mu_X \colon X \otimes A \to X$ and $\mu_Y \colon A \otimes Y \to Y$. We define the *balanced tensor product* $X \otimes_A Y$ to be the cokernel of the map

$$\mu_X \otimes Y - X \otimes \mu_Y \colon X \otimes A \otimes Y \to X \otimes Y.$$

Roughly speaking, $\mu_X \otimes Y - X \otimes \mu_A$ corresponds to the formula $x \cdot a \otimes y - x \otimes a \cdot y$.

Definition 2.10. Let $\mathcal{C}$ be a closed monoidal category in which each morphism has a kernel. Let A be an algebra in $\mathcal{C}$, and let X and Y be left A-modules with multiplication maps $\mu_X \colon A \otimes X \to X$ and $\mu_Y \colon A \otimes Y \to Y$. We define the *balanced internal Hom* $X \Rightarrow_A Y$ to be the kernel of the map

$$X \Rightarrow Y \to (A \otimes X) \Rightarrow Y, \qquad f \mapsto f \circ \mu_X - \mu_Y \circ (A \otimes f).$$

Roughly speaking, this corresponds to the map $a \otimes x \mapsto f(a \cdot x) - a \cdot f(x)$.

Example 2.11 ([9]). If $\mathcal{C} = \mathfrak{Bor}$, then $X \otimes_A Y$ is the quotient of $X \hat{\otimes} Y$ by the *closed* linear span of $xa \otimes y - x \otimes ay$ for $x \in X$, $a \in A$, $y \in Y$. Taking the closure ensures that the quotient is again separated, even complete. And $X \Rightarrow_A Y$ is the space of bounded A-module homomorphisms $X \to Y$ with the equibounded bornology.

These constructions in general monoidal categories by and large have the same formal properties as for rings and modules. We will indicate some of them now, see [11] for more details.

Let A and B be algebras in $\mathcal{C}$, let X be a B, A-bimodule and Y a left A-module. Assume that the tensor product functor commutes with cokernels in both variables. Then $B \otimes (X \otimes_A Y)$ is the cokernel of the natural map

$$B \otimes \mu_X \otimes Y - B \otimes X \otimes \mu_Y \colon B \otimes X \otimes A \otimes Y \to B \otimes X \otimes Y.$$

Hence the multiplication map $\mu_{BX} \otimes Y \colon B \otimes X \otimes Y \to X \otimes Y$ descends to a map $B \otimes (X \otimes_A Y) \to X \otimes_A Y$, which turns $X \otimes_A Y$ into a left B-module. Similarly, an A, C-module structure on Y induces a right C-module structure on $X \otimes_A Y$, and if X is a B, A-bimodule and Y is an A, C-bimodule, then $X \otimes_A Y$ is a B, C-bimodule.

This allows us to form triple balanced tensor products $X \otimes_A Y \otimes_C Z$. This is a B, D-bimodule if X, Y, and Z are bimodules over B, A, A, C, and C, D respectively. More precisely, we get two such bimodules, $(X \otimes_A Y) \otimes_C Z$ and $X \otimes_A (Y \otimes_C Z)$, which are related by a canonical isomorphism that satisfies coherence laws similar to those for $\otimes$. Therefore, it is legitimate to drop brackets in such tensor product expressions.

The internal Hom $X \Rightarrow_A Y$ inherits a B, C-bimodule structure if X is an A, B-bimodule and Y is an A, C-bimodule. The B-module structure in

$$\mathcal{C}(B \otimes (X \Rightarrow_A Y), X \Rightarrow_A Y) \cong \mathcal{C}_A(X \otimes B \otimes (X \Rightarrow_A Y), Y)$$

is adjoint to the A-module map

$$X \otimes B \otimes (X \Rightarrow_A Y) \xrightarrow{\mu_{XB} \otimes (X \Rightarrow_A Y)} X \otimes (X \Rightarrow_A Y) \xrightarrow{\mathrm{ev}_{XY}} Y,$$

while the right C-module structure in

$$\mathcal{C}\big((X \Rightarrow_A Y) \otimes C, X \Rightarrow_A Y\big) \cong \mathcal{C}_A(X \otimes (X \Rightarrow_A Y) \otimes C, Y)$$

is adjoint to the composition

$$X \otimes (X \Rightarrow_A Y) \otimes C \xrightarrow{\mathrm{ev}_{XY} \otimes C} Y \otimes C \xrightarrow{\mu_{YC}} Y.$$

In examples, these definitions reproduce the usual bimodule structure on spaces of linear maps, $b \cdot f \cdot c(x) := f(x \cdot b) \cdot c$. Routine diagram chases show that these maps define a B, C-bimodule structure.

The functors $\otimes_A$ and $\Rightarrow_A$ are related by the expected adjointness relation:

$$\mathcal{C}_{B,C}(X \otimes_A Y, Z) \cong \mathcal{C}_{A,C}(Y, X \Rightarrow_B Z), \tag{3}$$

where X is a B, A-module, Y is an A, C-module, Z is a B, C-module, and $\mathcal{C}_{B,C}$ denotes B, C-module homomorphisms. Of course, we use the canonical bimodule structures on $X \otimes_A Y$ and $X \Rightarrow_B Z$ here. To prove (3), identify both sides

with subspaces of $\mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(Y, X \Rightarrow Z)$ by (1) and check that the additional conditions involving A, B and C correspond to each other.

3. Self-induced algebras, smooth and rough modules

From now on, we fix a closed monoidal category $\mathcal{C}$ with tensor product functor $\otimes$, tensor unit $\mathbb{1}$, and internal Hom functor $\Rightarrow$. We also assume that all morphisms in $\mathcal{C}$ have a kernel and a cokernel, so that we may form the balanced tensor product $X \otimes_A Y$ and the balanced internal Hom $X \Rightarrow_A Y$.

Algebras and modules are understood to be algebras and modules in $\mathcal{C}$. Readers may think about the category $\mathcal{C} = \mathfrak{Bor}$ of complete convex bornological vector spaces (Example 2.2), where our constructions become much more concrete. The functors $X \otimes_A Y$ and $X \Rightarrow_A Y$ in this case are described in Example 2.11.

Let A be an algebra with multiplication $\mu\colon A \otimes A \to A$, and let X be a left A-module with multiplication $\mu_X\colon A \otimes X \to X$. The associativity relation

$$\mu_X \circ (A \otimes \mu_X) = \mu_X \circ (\mu \otimes X)$$

means that μ_X descends to a canonical map $A \otimes_A X \to X$ by the definition of the balanced tensor product. We denote this induced map by $\bar{\mu}_X\colon A \otimes_A X \to X$. This is an A-module homomorphism with respect to the canonical A-module structure on $A \otimes_A X$ by left multiplication. In particular, for $X = A$ the multiplication on A induces a map $\bar{\mu}\colon A \otimes_A A \to A$. This is an A-bimodule homomorphism with respect to the canonical A-bimodule structures on A and $A \otimes_A A$ from left and right multiplication.

The associativity of μ_X also implies that $\bar{\mu}_X$ is an A-module homomorphism $A \otimes_A X \to X$. Hence the adjointness isomorphism

$$\mathcal{C}_A(A \otimes_A X, X) \cong \mathcal{C}_A(X, A \Rightarrow_A X)$$

in (3) turns $\bar{\mu}_X$ into an A-module homomorphism $\bar{\mu}_X^\dagger\colon X \to (A \Rightarrow_A X)$.

Example 3.1. In the category of bornological vector spaces, $\bar{\mu}_X$ is the map on the quotient $A \otimes_A X$ of $A \hat{\otimes} X$ induced by the map $A \hat{\otimes} X \to X$, $a \otimes x \mapsto a \cdot x$, and $\bar{\mu}_X^\dagger$ maps $x \in X$ to the A-module map $A \to X$, $a \mapsto a \cdot x$.

The natural maps $\bar{\mu}$, $\bar{\mu}_X$, and $\bar{\mu}_X^\dagger$ are needed for our main definitions:

Definition 3.2. An algebra A is *self-induced* (see [3]) if $\bar{\mu}\colon A \otimes_A A \to A$ is an isomorphism. Let A be a self-induced algebra.

A left A-module X is *smooth* if $\bar{\mu}_X\colon A \otimes_A X \to X$ is an isomorphism, and *rough* if $\bar{\mu}_X^\dagger\colon X \to (A \Rightarrow_A X)$ is an isomorphism.

We may define smoothness for right A-modules by requiring the analogous map $X \otimes_A A \to X$ to be invertible.

To correctly define roughness for right modules, we must use the *right* internal Hom functor, that is, the right adjoint to $A \mapsto B \otimes A$. In a braided monoidal category, the right and left internal Hom functors are naturally isomorphic; we may even view right A-modules as left A^{op}-modules, and A is self-induced if

and only if A^{op} is self-induced. Therefore, in the tensor categories of greatest interest, there is no need for a separate definition of roughness for right modules. Since we will not use rough right modules in the following, we do not examine this technical issue any further here.

Definition 3.3. Let A be a self-induced algebra. The *smoothening* and *roughening* of a left A-module X are defined by

$$\mathrm{S}_A(X) = \mathrm{S}(X) := A \otimes_A X, \qquad \mathrm{R}_A(X) = \mathrm{R}(X) := A \Rightarrow_A X.$$

This defines functors on the category of A-modules. The maps $\bar{\mu}_X$ and $\bar{\mu}_X^\dagger$ provide natural transformations

$$\bar{\mu}_X \colon \mathrm{S}(X) \to X \quad \text{and} \quad \bar{\mu}_X^\dagger \colon X \to \mathrm{R}(X).$$

If $\mathcal{C}$ is the symmetric monoidal category of Banach spaces with the projective Banach space tensor product, then the self-induced algebras are exactly those of Niels Grønbæk [3], and the smooth modules are the A-induced modules of [3].

Notice that we only defined smooth and rough modules and the smoothening and roughening functors for a self-induced algebra A. If A is self-induced, then A is smooth as an A-bimodule.

Of course, self-induced algebras, smooth modules, and the smoothening functor make sense without an internal Hom functor. Thus we may still speak of self-induced complete locally convex topological algebras, smooth topological modules over them, and smoothenings of such modules (Example 2.4). But here we are mainly interested in the interplay between smooth and rough modules.

Example 3.4. Let $\mathcal{C}$ be the category $\mathfrak{Bor}$ of complete convex bornological vector spaces with the complete projective bornological tensor product (Example 2.2). François Bruhat [1] used the Montgomery-Zippin structure theory for locally compact groups to define a space of smooth functions on G for any locally compact group G. For Lie groups, smoothness has the usual meaning, for totally disconnected groups, smooth functions are locally constant. The smooth, compactly supported functions on G form an algebra under convolution. This is a complete convex bornological algebra $\mathrm{C}_{\mathrm{c}}^\infty(G)$, see [7] for the definition of the bornology.

It is proved in [7] that the algebra $\mathrm{C}_{\mathrm{c}}^\infty(G)$ in $\mathfrak{Bor}$ is self-induced and that the category of smooth $\mathrm{C}_{\mathrm{c}}^\infty(G)$-modules is isomorphic to the category of smooth representations of G on complete convex bornological vector spaces. Furthermore, [7] introduces smoothening and roughening functors for $\mathrm{C}_{\mathrm{c}}^\infty(G)$. These constructions in [7] are special cases of Definition 3.3. In the following, we will generalize some of the results of [7] to arbitrary self-induced algebras.

Proposition 3.5. *Let A be a unital algebra. Then A is self-induced, and a left A-module is smooth if and only if it is rough if and only if it is unital.*

Conversely, if A is rough as a left A-module, then A is a unital algebra.

Since A is always a smooth A-module, it follows that unital algebras are the only ones for which smooth and rough modules are the same.

Proof. Assume first that A has a unit. Then the A-modules $A \otimes X$ and $A \Rightarrow X$ are unital for all X. Hence so are $A \otimes_A X$ as a quotient of $A \otimes X$, and $A \Rightarrow_A X$ as a submodule of $A \Rightarrow X$. Therefore, smooth modules and rough modules are unital. Conversely, let X be a unital left or right module.

We define a map $s_X \colon X \to A \otimes X$ by composing the unitor $X \to \mathbb{1} \otimes X$ and the unit map $\mathbb{1} \to A$ tensored with X. The induced map $X \to A \otimes_A X$ is a section for $\bar{\mu}_X$. We also get a map $s_{A\otimes X} \colon A \otimes X \to A \otimes A \otimes X$. Let

$$b' := \mu \otimes X - A \otimes \mu_X \colon A \otimes A \otimes X \to A \otimes X$$

be the map whose cokernel is the balanced tensor product $A \otimes_A X$. We compute $b' \circ s_{A\otimes X} + s_X \circ \mu_X = \mathrm{Id}_{A\otimes X}$. This implies that $\bar{\mu}_X$ is invertible. Thus unital modules are smooth. In particular, A is self-induced.

We also define a map $s'_X \colon (A \Rightarrow X) \to X$ by composing with the unit map $\mathbb{1} \to A$ and identifying $(\mathbb{1} \Rightarrow X) \cong X$. There is a similar map

$$s''_X = s'_{A\Rightarrow X} \colon (A \otimes A) \Rightarrow X \cong A \Rightarrow (A \Rightarrow X) \to A \Rightarrow X$$

that composes with $A \otimes \mathbb{1} \colon A \to A \otimes A$. We compute that s'_X is a section for $\mu_X^\dagger$ and that $s''_X \circ (b')^\dagger + \mu_X^\dagger \circ s'_X = \mathrm{Id}_{A\Rightarrow X}$, where $(b')^\dagger$ is the map whose kernel is $A \Rightarrow_A X$ and $\mu_X^\dagger \colon X \to A \Rightarrow X$ is adjoint to μ_X. This implies that $\bar{\mu}_X^\dagger$ is invertible, that is, unital modules are rough.

Now let A be an arbitrary self-induced algebra and assume that A is rough as a left A-module. That is, the canonical map $A \to A \Rightarrow_A A$ is invertible. Adjoint associativity yields $\mathcal{C}(\mathbb{1}, A \Rightarrow_A A) \cong \mathcal{C}_A(A, A)$, and this always contains a canonical element: the identity map on A. If the map $A \to A \Rightarrow_A A$ is invertible, then we get a unique $\eta \in \mathcal{C}(\mathbb{1}, A)$ with $\mu \circ (A \otimes \eta) = \mathrm{Id}_A$, that is, η is a right unit element. Consider the map $\mu \circ (\eta \otimes A) \colon A \to A$. When we compose it with the isomorphism $A \to A \Rightarrow_A A$, we get again the canonical map $A \to A \Rightarrow_A A$ because η is a right unit. Since the map $A \to A \Rightarrow_A A$ is invertible, it follows that $\mu \circ (\eta \otimes A)$ is equal to the identity map, that is, η is a left unit as well. □

For a unital algebra, any module X decomposes naturally as a direct sum $X_0 \oplus X_1$, where X_0 carries the zero module structure and X_1 is a unital module. Both the smoothening and the roughening functors map X to X_1, and the natural maps $\mathrm{S}(X) \to X \to \mathrm{R}(X)$ are the maps $X_1 \to X \to X_1$ from the direct sum decomposition $X \cong X_0 \oplus X_1$.

The following proposition summarizes the formal properties of the smoothening and roughening functors:

Theorem 3.6. *The following diagram commutes, and the indicated maps are isomorphisms:*

$$\begin{tikzcd}
\mathrm{S}\,\mathrm{S}(X) \arrow[r, "\bar\mu_{\mathrm{S}(X)}", "\cong"'] \arrow[d, "\mathrm{S}(\bar\mu_X)"', "\cong"] & \mathrm{S}(X) \arrow[r, "\bar\mu^\dagger_{\mathrm{S}(X)}"] \arrow[d, "\bar\mu_X"] \arrow[dl, equal] & \mathrm{R}\,\mathrm{S}(X) \arrow[d, "\mathrm{R}(\bar\mu_X)", "\cong"'] \\
\mathrm{S}(X) \arrow[r, "\bar\mu_X"] \arrow[d, "\mathrm{S}(\bar\mu^\dagger_X)"', "\cong"] & X \arrow[r, "\bar\mu^\dagger_X"] \arrow[d, "\bar\mu^\dagger_X"'] & \mathrm{R}(X) \arrow[d, "\mathrm{R}(\bar\mu^\dagger_X)", "\cong"'] \arrow[dl, equal] \\
\mathrm{S}\,\mathrm{R}(X) \arrow[r, "\bar\mu_{\mathrm{R}(X)}"'] & \mathrm{R}(X) \arrow[r, "\bar\mu^\dagger_{\mathrm{R}(X)}", "\cong"'] & \mathrm{R}\,\mathrm{R}(X)
\end{tikzcd}$$

That is, the two canonical maps $\mathrm{S}\,\mathrm{S}(X) \to \mathrm{S}(X)$ *and* $\mathrm{R}(X) \to \mathrm{R}\,\mathrm{R}(X)$ *are equal, and there are natural isomorphisms* $\mathrm{S}\,\mathrm{S} \cong \mathrm{S} \cong \mathrm{S}\,\mathrm{R}$ *and* $\mathrm{R}\,\mathrm{S} \cong \mathrm{R} \cong \mathrm{R}\,\mathrm{R}$. *In particular, modules of the form* $\mathrm{S}(X)$ *are always smooth and modules of the form* $\mathrm{R}(X)$ *are always rough.*

The functor S *is left adjoint to* R, *that is, there is a natural isomorphism*

$$\mathcal{C}_A\big(\mathrm{S}(X), Y\big) \cong \mathcal{C}_A\big(X, \mathrm{R}(Y)\big)$$

for all A*-modules* X *and* Y.

The functor $X \mapsto \mathrm{S}(X)$ *is right adjoint to the embedding of the category of smooth modules: composition with* $\bar\mu_X$ *induces an isomorphism*

$$\mathcal{C}_A\big(X, \mathrm{S}(Y)\big) \cong \mathcal{C}_A(X, Y)$$

if X *is a smooth* A*-module and* Y *is any* A*-module.*

The functor $X \mapsto \mathrm{R}(X)$ *is left adjoint to the embedding of the category of rough modules: composition with* $\bar\mu^\dagger_X$ *induces an isomorphism*

$$\mathcal{C}_A(\mathrm{R}(X), Y) \cong \mathcal{C}_A(X, Y)$$

if X *is any* A*-module and* Y *is a rough* A*-module.*

Proof. Since A is self-induced, we have $A \otimes_A A \cong A$. Using the associativity of the balanced tensor product, this implies

$$\mathrm{S}\,\mathrm{S}(X) := A \otimes_A (A \otimes_A X) \cong (A \otimes_A A) \otimes_A X \cong A \otimes_A X =: \mathrm{S}(X).$$

This isomorphism $\mathrm{S}\,\mathrm{S}(X) \to \mathrm{S}(X)$ is induced by $\mu \otimes X \colon A \otimes A \otimes X \to A \otimes_A X$, that is, it is equal to $\bar\mu_{\mathrm{S}(X)}$; thus $\mathrm{S}(X)$ is a smooth module. Moreover, $\mu \otimes X = A \otimes \mu_X$ as maps to $A \otimes_A X$, so that $\bar\mu_{\mathrm{S}(X)} = \mathrm{S}(\bar\mu_X)$.

The adjointness of S and R is a special case of the adjointness between balanced tensor products and internal homs:

$$\mathcal{C}_A(\mathrm{S}(X), Y) := \mathcal{C}_A(A \otimes_A X, Y) \cong \mathcal{C}_A(X, A \Rightarrow_A Y) =: \mathcal{C}_A\big(X, \mathrm{R}(Y)\big).$$

The natural isomorphism $\mathrm{S} \circ \mathrm{S} \cong \mathrm{S}$ induces a natural isomorphism $\mathrm{R} \cong \mathrm{R} \circ \mathrm{R}$ for the right adjoint functors. A routine computation, which we omit, shows that this isomorphism is induced by $\bar\mu^\dagger_{\mathrm{R}(X)} = \mathrm{R}(\bar\mu^\dagger_X)$.

Next we show that composition with $\bar{\mu}_Y$ is an isomorphism $\mathcal{C}_A\big(X, \mathrm{S}(Y)\big) \cong \mathcal{C}_A(X, Y)$ if X is a smooth A-module and Y is any A-module. We claim that its inverse is the composite

$$\mathcal{C}_A(X, Y) \xrightarrow{\mathrm{S}} \mathcal{C}_A\big(\mathrm{S}(X), \mathrm{S}(Y)\big) \xleftarrow[\cong]{\bar{\mu}_X^*} \mathcal{C}_A\big(X, \mathrm{S}(Y)\big),$$

where we use that X is smooth, so that $\bar{\mu}_X^*$ is invertible.

The naturality of the transformation $\bar{\mu}$ yields commuting diagrams

$$\begin{tikzcd}
\mathrm{S}(X) \arrow[r, "\mathrm{S}(f)"] \arrow[d, "\bar{\mu}_X", "\cong"'] & \mathrm{S}(Y) \arrow[d, "\bar{\mu}_Y"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

for all $f \in \mathcal{C}_A(X, Y)$. Thus the composition $\mathcal{C}_A(X, Y) \to \mathcal{C}_A(X, \mathrm{S}(Y)) \to \mathcal{C}_A(X, Y)$ is the identity map. If $f \in \mathcal{C}_A\big(X, \mathrm{S}(Y)\big)$, then the diagram

$$\begin{tikzcd}
\mathrm{S}(X) \arrow[r, equal] \arrow[d, "\mathrm{S}(f)"'] & A \otimes_A X \arrow[r, "\bar{\mu}_X", "\cong"'] \arrow[d, "A \otimes_A f"'] & X \arrow[d, "f"'] \arrow[dr, "\bar{\mu}_Y \circ f"] & \\
\mathrm{S}(\mathrm{S}\, Y) \arrow[r, equal] & A \otimes_A A \otimes_A Y \arrow[r, "\bar{\mu} \otimes_A Y"'] & A \otimes_A Y \arrow[r, "\bar{\mu}_Y"'] & Y
\end{tikzcd}$$

commutes. Since $\bar{\mu} \otimes_A Y = A \otimes_A \bar{\mu}_Y$, we get $f = \big(\mathrm{S}(\bar{\mu}_Y f)\big) \circ (\bar{\mu}_X)^{-1}$. This means that the composite map $\mathcal{C}_A(X, \mathrm{S}(Y)) \to \mathcal{C}_A(X, Y) \to \mathcal{C}_A(X, \mathrm{S}(Y))$ is the identity map as well.

The adjointness relations already established imply

$$\mathcal{C}_A\big(X, \mathrm{R} \circ \mathrm{S}(Y)\big) \cong \mathcal{C}_A\big(\mathrm{S}(X), \mathrm{S}(Y)\big) \cong \mathcal{C}_A(\mathrm{S}(X), Y) \cong \mathcal{C}_A\big(X, \mathrm{R}(Y)\big)$$

for all A-modules X and Y. Hence $\mathrm{R} \circ \mathrm{S}(Y) \cong \mathrm{R}(Y)$ by the Yoneda Lemma.

Let Y be an A-module and let X be a smooth A-module. Then

$$\mathcal{C}_A\big(X, \mathrm{S}\,\mathrm{R}(Y)\big) \cong \mathcal{C}_A\big(X, \mathrm{R}(Y)\big) \cong \mathcal{C}_A(\mathrm{S}(X), Y) \cong \mathcal{C}_A(X, Y) \cong \mathcal{C}_A\big(X, \mathrm{S}(Y)\big).$$

Hence the Yoneda Lemma yields $\mathrm{S} \circ \mathrm{R}(Y) \cong \mathrm{S}(Y)$.

It is routine to check that these isomorphisms $\mathrm{R}\,\mathrm{S}(Y) \cong \mathrm{R}(Y)$ and $\mathrm{S}(Y) \cong \mathrm{S}\,\mathrm{R}(Y)$ are the canonical maps $\mathrm{R}(\bar{\mu}_Y)$ and $\mathrm{S}(\bar{\mu}_Y^\dagger)$.

If Y is rough, that is, $Y \cong \mathrm{R}(Y)$, then we compute

$$\begin{aligned}
\mathcal{C}_A(\mathrm{R}(X), Y) &\cong \mathcal{C}_A\big(\mathrm{R}(X), \mathrm{R}(Y)\big) \cong \mathcal{C}_A(\mathrm{S}\,\mathrm{R}(X), Y) \\
&\cong \mathcal{C}_A(\mathrm{S}(X), Y) \cong \mathcal{C}_A\big(X, \mathrm{R}(Y)\big) \cong \mathcal{C}_A(X, Y),
\end{aligned}$$

that is, R is left adjoint to the embedding of the category of rough modules. $\square$

4. Rough modules and unital modules over multiplier algebras

Let A be a self-induced algebra in $\mathcal{C}$. We view A as a left A-module and let

$$\mathcal{M}_{\mathrm{l}}(A) := A \Rightarrow_A A$$

be the algebra of left A-module endomorphisms on A. This is a unital algebra in $\mathcal{C}$. It comes with a canonical algebra homomorphism $A \to \mathcal{M}_{\mathrm{l}}(A)$ by right multiplication.

We may also view $\mathcal{M}_{\mathrm{l}}(A)$ as the roughening of the left A-module structure on A, and the map $A \to \mathcal{M}_{\mathrm{l}}(A)$ as the canonical map $\bar{\mu}^{\dagger}\colon A \to \mathrm{R}(A)$. Theorem 3.6 implies $A \otimes_A \mathcal{M}_{\mathrm{l}}(A) \cong A$. Roughly speaking, this means that A is a left ideal in $\mathcal{M}_{\mathrm{l}}(A)$ (but the map $A \to \mathcal{M}_{\mathrm{l}}(A)$ need not be monic, see Section 6.1).

If X is a left A-module, then the A-module structure on $\mathrm{R}(X) := A \Rightarrow_A X$ extends canonically to a unital left $\mathcal{M}_{\mathrm{l}}(A)$-module structure because A is an $A, \mathcal{M}_{\mathrm{l}}(A)$-bimodule by construction. Thus rough A-modules become unital $\mathcal{M}_{\mathrm{l}}(A)$-modules, and this provides a fully faithful functor from the category of rough A-modules to the category of unital $\mathcal{M}_{\mathrm{l}}(A)$-modules. Conversely, any unital $\mathcal{M}_{\mathrm{l}}(A)$-module becomes an A-module by restricting the action. But such restricted modules need not be rough, and the restriction functor need not be fully faithful.

To see this, consider free modules. Free unital $\mathcal{M}_{\mathrm{l}}(A)$-modules have the form $\mathcal{M}_{\mathrm{l}}(A) \otimes V = (A \Rightarrow_A A) \otimes V$ for some object V of $\mathcal{C}$. We view this as a left A-module and simplify its roughening using Theorem 3.6 and the associativity of $\otimes$:

$$\begin{aligned}\mathrm{R}(\mathcal{M}_{\mathrm{l}}(A) \otimes V) &\cong \mathrm{R}\,\mathrm{S}(\mathcal{M}_{\mathrm{l}}(A) \otimes V) \cong \mathrm{R}\big(\mathrm{S}\big(\mathcal{M}_{\mathrm{l}}(A)\big) \otimes V\big)\\ &\cong \mathrm{R}\big(\mathrm{S}\,\mathrm{R}(A) \otimes V\big) \cong \mathrm{R}(A \otimes V) = A \Rightarrow_A (A \otimes V).\end{aligned}$$

In general, $(A \Rightarrow_A A) \otimes V$ is different from $A \Rightarrow_A (A \otimes V)$.

We may also view smooth modules as modules over a suitable *right* multiplier algebra $\mathcal{M}_{\mathrm{r}}(A)$. This is a unital algebra such that A is an $\mathcal{M}_{\mathrm{r}}(A), A$-bimodule. Since this involves a left module structure $\mathcal{M}_{\mathrm{r}}(A) \otimes A \to A$, we need the *right* internal Hom functor defined by $\mathcal{C}(X \otimes Y, Z) \cong \mathcal{C}(X, Z \Leftarrow Y)$. This functor is similar to $X \Rightarrow Y$, but the evaluation maps are of the form $X \Leftarrow Y \otimes X \to Y$, and the composition maps are of the form $(Y \Leftarrow Z) \otimes (X \Leftarrow Y) \to X \Leftarrow Z$. Thus

$$\mathcal{M}_{\mathrm{r}}(A) := A \Leftarrow_A A$$

becomes a unital algebra such that A is an $\mathcal{M}_{\mathrm{r}}(A), A$-bimodule. The A-module structure on $\mathrm{S}(X) := A \otimes_A X$ extends canonically to a left unital $\mathcal{M}_{\mathrm{r}}(A)$-module structure for any A-module X. This provides a fully faithful embedding from the category of smooth left A-modules to the category of unital left $\mathcal{M}_{\mathrm{r}}(A)$-modules. Once again, this functor is not an isomorphism of categories.

In general, $\mathcal{M}_{\mathrm{l}}(A)$ and $\mathcal{M}_{\mathrm{r}}(A)$ are different, even in the symmetric monoidal category $\mathfrak{Bor}$. For instance, this happens for the biprojective algebras $V \otimes W$ studied in Section 6.1.

5. Functoriality for homomorphisms and bimodules

Let A and B be algebras in an additive closed monoidal category.

Definition 5.1. Let $\mathfrak{Mod}_A$ denote the category of smooth modules over a self-induced algebra A.

An A, B-bimodule M induces a functor $M \otimes_B \sqcup$ from the category of B-modules to the category of A-modules and a functor $M \Rightarrow_A \sqcup$ from the category of A-modules to the category of B-modules. These two functors are adjoint to each other by (3):

$$\mathcal{C}_A(M \otimes_B X, Y) \cong \mathcal{C}_B(X, M \Rightarrow_A Y) \tag{4}$$

if X and Y are a B-module and an A-module, respectively. When do these functors preserve smooth or rough modules?

The module $M \Rightarrow_A Y$ is usually not smooth, even if Y is, and $M \otimes_B X$ is usually not rough, even if X is. But we have the following positive results:

Proposition 5.2. *Let A and B be algebras, assume that A is self-induced. Let Y be any left B-module. If M is an A, B-bimodule that is smooth as a left A-module, then $M \otimes_B Y$ is a smooth A-module.*

If M is a B, A-bimodule that is smooth as a right A-module, then $M \Rightarrow_B Y$ is a rough A-module.

Proof. The first assertion follows from the associativity of balanced tensor products:

$$\mathrm{S}(M \otimes_B Y) := A \otimes_A (M \otimes_B Y) \cong (A \otimes_A M) \otimes_B Y \cong M \otimes_B Y.$$

The second assertion uses a strengthening of the adjointness relation (3) as in (2) with internal Hom functors instead of morphism sets. Thus

$$\mathrm{R}(M \Rightarrow_B Y) \cong A \Rightarrow_A (M \Rightarrow_B Y) \cong (M \otimes_A A) \Rightarrow_B Y \cong M \Rightarrow_B Y. \qquad \square$$

For general M, we get smooth or rough modules if we compose the two functors above with the smoothening or roughening functors. The functor $\mathrm{S}(M \Rightarrow_B \sqcup)$ maps B-modules to smooth A-modules, and $\mathrm{R}(M \otimes_B \sqcup)$ maps B-modules to rough A-modules. The other two combinations of our functors are not worth considering because the computations above show

$$\mathrm{S}_A(M \otimes_B X) \cong \mathrm{S}_A(M) \otimes_B X, \qquad \mathrm{R}_A(M \Rightarrow_B Y) \cong (\mathrm{S}_A M) \Rightarrow_B Y.$$

Proposition 5.3. *Let M be a smooth A, B-bimodule. Then the functors*

$$\begin{aligned} \mathfrak{Mod}_B &\to \mathfrak{Mod}_A, & X &\mapsto M \otimes_B X, \\ \mathfrak{Mod}_A &\to \mathfrak{Mod}_B, & Y &\mapsto \mathrm{S}_B(M \Rightarrow_A Y), \end{aligned}$$

are adjoint to each other, that is, $\mathcal{C}_A(M \otimes_B X, Y) \cong \mathcal{C}_B\big(X, \mathrm{S}_B(M \Rightarrow_A Y)\big)$ if X is a smooth B-module and Y a smooth A-module.

Proof. Theorem 3.6 implies $\mathcal{C}_B\big(X, \mathrm{S}_B(M \Rightarrow_A Y)\big) \cong \mathcal{C}_B(X, M \Rightarrow_A Y)$, and this is isomorphic to $\mathcal{C}_A(M \otimes_B X, Y)$ by (4). $\square$

We may define Morita equivalence for self-induced algebras as in [3]:

Definition 5.4. Two self-induced algebras A and B are called *Morita equivalent* if there exist a smooth A, B-bimodule P, a smooth B, A-bimodule Q, and natural isomorphisms $P \otimes_B Q \cong A$ and $Q \otimes_A P \cong B$.

Proposition 5.5. *If A and B are Morita equivalent via the bimodules P and Q, then the categories of smooth A- and B-modules are equivalent via the functors*

$$\begin{aligned} \mathfrak{Mod}_A &\to \mathfrak{Mod}_B, \qquad & X &\mapsto Q \otimes_A X, \\ \mathfrak{Mod}_B &\to \mathfrak{Mod}_A, \qquad & Y &\mapsto P \otimes_B Y. \end{aligned}$$

The categories of rough A- and B-modules are equivalent via the functors $X \mapsto P \Rightarrow_A X$ and $Y \mapsto Q \Rightarrow_B Y$.

Proof. The equivalence $\mathfrak{Mod}_A \cong \mathfrak{Mod}_B$ follows from the associativity of tensor products and the assumed isomorphisms $P \otimes_B Q \cong A$, $Q \otimes_A P \cong B$, and from the definition of smooth modules: $A \otimes_A X \cong X$ and $B \otimes_B Y \cong Y$. The corresponding assertions about rough modules also use the adjointness relation (4). □

Since the categories of smooth and rough modules are equivalent, we may also construct the equivalence between rough module categories from the equivalence between the smooth module categories by first smoothening, then applying the equivalence, and then roughening. A straightforward computation shows that this sequence of steps produces the functor described above:

$$\mathrm{R}_A\big(P \otimes_B \mathrm{S}_B(X)\big) \cong Q \Rightarrow_B X \tag{5}$$

for all rough B-modules X. First, since P is smooth, we compute

$$P \otimes_B \mathrm{S}_B(X) \cong P \otimes_B (B \otimes_B X) \cong (P \otimes_B B) \otimes_B X \cong P \otimes_B X.$$

The argument that shows that $Q \otimes_A \sqcup$ is an equivalence of categories shows more: tensoring with Q induces an isomorphism

$$X \Rightarrow_A Y \cong (Q \otimes_A X) \Rightarrow_B (Q \otimes_A Y)$$

for all smooth A-modules X and Y. Hence

$$A \Rightarrow_A (P \otimes_B X) \cong (Q \otimes_A A) \Rightarrow_B \big(Q \otimes_A (P \otimes_B X)\big) \cong Q \Rightarrow_B (B \otimes_B X) \cong Q \Rightarrow_B X.$$

Although the author is not aware of an example, it seems likely that there exist equivalences $\mathfrak{Mod}_A \cong \mathfrak{Mod}_B$ (even for unital A and B) that are not induced by a bimodule as above. To get a bimodule from an equivalence of categories, we assume that further structure is preserved. There is a tensor product operation $X, Y \mapsto X \otimes Y$ for a smooth A-module X and an object Y of $\mathcal{C}$, which turns the category of smooth A-modules into a right $\mathcal{C}$-category in the notation of [10]. The functor $M \otimes_B \sqcup$ for a bimodule M is a $\mathcal{C}$-functor in the notation of [10], that is, there are natural isomorphisms $M \otimes_B (X \otimes Y) \cong (M \otimes_B X) \otimes Y$ satisfying appropriate coherence laws.

Proposition 5.6. *Let A and B be self-induced algebras. A functor*

$$F\colon \mathfrak{Mod}_A \to \mathfrak{Mod}_B$$

is of the form $M \otimes_B _$ for a smooth B, A-bimodule M if and only if it preserves cokernels and is a $\mathcal{C}$-functor. The bimodule M is determined uniquely up to isomorphism.

Proof. The underlying left B-module of M must be $F(A)$, of course. To get the right A-module structure on $M := F(A)$, we use the multiplication map $\mu\colon A \otimes A \to A$. This module homomorphism induces a natural B-module map

$$M \otimes A := F(A) \otimes A \cong F(A \otimes A) \to F(A) =: M.$$

Now let X be any smooth A-module. Then $X \cong A \otimes_A X$ is the cokernel of the canonical map $A \otimes A \otimes X \to A \otimes X$ that defines $A \otimes_A X$. Since F is compatible with cokernels and tensor products, $F(X)$ is naturally isomorphic to the cokernel of an induced map $F(A) \otimes A \otimes X \to F(A) \otimes X$. But this is exactly the map that defines $F(A) \otimes_A X$, so that $F(X) \cong F(A) \otimes_A X$ for all smooth A-modules X. In particular, $F(A)$ is smooth as a right A-module. It is easy to see that $F(A)$ with the bimodule structure described above is the only one that may induce the functor F. □

We may use an algebra homomorphism $f\colon A \to B$ to turn B-modules into A-modules. But when does this functor f^* preserve smoothness or roughness of bimodules? To analyze this, we use f to view B as an A, B-bimodule or as a B, A-bimodule. Then $B \otimes_B X \cong X$ for smooth B-modules X, so that f^* on smooth modules is the tensor product functor for the A, B-bimodule B. By Proposition 5.2, this maps smooth B-modules to smooth A-modules provided B is smooth as a left A-module. And $X \cong (B \Rightarrow_B X)$ for a rough B-module X, so that f^* on rough modules is the internal Hom functor for the B, A-bimodule B. By Proposition 5.2, this maps rough B-modules to rough A-modules provided B is smooth as a right A-module. Summing up:

Lemma 5.7. *Let $f\colon A \to B$ be an algebra homomorphism. Assume that B is smooth both as a left and as a right A-module. Then the induced functor f^* from B-modules to A-modules maps smooth modules to smooth modules and rough modules to rough modules.*

More generally, the above construction only used compatible A, B- and B, A-bimodule structures on B. These still exist if we replace f by an algebra homomorphism into the multiplier algebra (also called double centralizer algebra) of B.

Even if B is not smooth as a left or right A-module, the above discussion shows how to get functors between the smooth and rough module categories: simply replace the A, B- or B, A-bimodule B by the appropriate smoothening and argue exactly as above. Furthermore, we may turn the functor f^* on rough modules into one on smooth modules by composing with the smoothening:

$$X \mapsto \mathrm{S}_A(B \otimes_B X) \cong \mathrm{S}_A(f^* X), \qquad X \mapsto \mathrm{S}_A(B \Rightarrow_B X) \cong \mathrm{S}_A(f^* \mathrm{R}_B X)$$

for a smooth left B-module X.

As a consequence, any algebra homomorphism from A to the multiplier algebra of B yields two pairs of adjoint functors between the categories of smooth modules over A and B. The first pair consists of the functors

$$\begin{aligned} \mathfrak{Mod}_B &\to \mathfrak{Mod}_A, & X &\mapsto \mathrm{S}_A(f^*X),\\ \mathfrak{Mod}_A &\to \mathfrak{Mod}_B, & Y &\mapsto \mathrm{S}_B((A\otimes_A B)\Rightarrow_A Y), \end{aligned}$$

the second pair of the functors

$$\begin{aligned} \mathfrak{Mod}_A &\to \mathfrak{Mod}_B, & Y &\mapsto B\otimes_A Y,\\ \mathfrak{Mod}_B &\to \mathfrak{Mod}_A, & X &\mapsto \mathrm{S}_A((B\otimes_A A)\Rightarrow_B X). \end{aligned}$$

6. Applications

6.1. **A simple biprojective example.** First we consider a very simple and well-known class of examples. Let V and W be objects of $\mathcal{C}$ and let $b\colon W\otimes V\to \mathbb{1}$ be a map. Then $A := W\otimes V$ becomes an associative nonunital algebra for the product

$$V\otimes W\otimes V\otimes W \xrightarrow{V\otimes b\otimes W} V\otimes W.$$

Similar maps define a left A-module structure on V and a right A-module structure on W. We may also view V and W as an $A,\mathbb{1}$-bimodule and a $\mathbb{1},A$-bimodule because any object of $\mathcal{C}$ carries a canonical unital $\mathbb{1}$-bimodule structure given by the left and right unitors.

From now on, we assume also that b is nondegenerate in the sense that there exist maps $v\colon \mathbb{1}\to V$ and $w\colon \mathbb{1}\to W$ for which $b\circ(w\otimes v)$ is the canonical isomorphism $\mathbb{1}\otimes\mathbb{1}\to\mathbb{1}$. Then the map $V\otimes w\otimes v\otimes W\colon A\to A\otimes A$ is a bimodule section for the multiplication map $A\otimes A\to A$. This implies that A is biprojective, that is, A is projective as an A-bimodule. A straightforward computation, which we omit, shows that A is self-induced. The bimodules V and W are smooth and implement a Morita equivalence between $\mathbb{1}$ and A.

We may use this to describe the categories of smooth and rough A-modules. First, Proposition 3.5 identifies smooth and rough $\mathbb{1}$-modules with unital $\mathbb{1}$-modules. Since any object of $\mathcal{C}$ carries a unique unital $\mathbb{1}$-module structure, it follows that the categories of smooth and rough $\mathbb{1}$-modules are both equivalent to $\mathcal{C}$. Due to the Morita equivalence, the categories of smooth and rough A-modules are equivalent to $\mathcal{C}$ as well (Proposition 5.5). More precisely, the equivalences map an object X of $\mathcal{C}$ to $V\otimes X$ and $W\Rightarrow X$, respectively, where we use the left A-module structure on V and the right A-module structure on W.

For instance, if $\mathcal{C}$ is the category of complete convex bornological vector spaces (Example 2.2), then we may take $V = W = \bigoplus_{\mathbb{N}}\mathbb{C}$ with the obvious pairing $b(x,y) := \sum_{n\in\mathbb{N}} x_n y_n$. Then A is the algebra $\mathbb{M}_\infty$ of finite matrices. Our results show that the categories of smooth and rough $\mathbb{M}_\infty$-modules are both equivalent to the category of complete convex bornological vector spaces, where a bornological vector space X corresponds to the smooth $\mathbb{M}_\infty$-module

$V \otimes X \cong \bigoplus_{\mathbb{N}} X$ and the rough $\mathbb{M}_\infty$-module $W \Rightarrow X \cong \prod_{\mathbb{N}} X$, with finite matrices acting by the usual matrix-vector multiplication.

We may also take $V_{\mathcal{S}} = W_{\mathcal{S}} = \mathcal{S}(\mathbb{N})$ with the same formula for b. The resulting algebra is $\mathcal{K}$, the algebra of rapidly decreasing matrices. Once again, we get a complete description of the categories of smooth and rough $\mathcal{K}$-modules. This time, the tensor product $V_{\mathcal{S}} \otimes X$ and the space $W_{\mathcal{S}} \Rightarrow X$ are spaces of sequences in X with certain growth conditions: $V_{\mathcal{S}} \otimes X$ consists of *rapidly decreasing* sequences, $W_{\mathcal{S}} \Rightarrow X$ of sequences of *polynomial growth*.

A sequence (x_n) has rapid decay if there is a sequence of scalars (ε_n) with rapid decay and a bounded subset $S \subseteq X$ with $x_n \in \varepsilon_n \cdot S$ for all $n \in \mathbb{N}$. A subset T of $V_{\mathcal{S}} \otimes X$ is bounded if it has uniformly rapid decay: the same ε_n and S work for all sequences in T. This bornological vector space of rapidly decreasing sequences is isomorphic to $V_{\mathcal{S}} \otimes X$.

A sequence (x_n) has polynomial growth if $\{\varepsilon_n \cdot x_n \mid n \in \mathbb{N}\}$ is bounded for each rapidly decreasing sequence of scalars (ε_n). A set T of polynomial growth sequences is bounded if it has uniform polynomial growth: the set $\bigcup_{(x_n)\in T}\{\varepsilon_n \cdot x_n \mid n \in \mathbb{N}\}$ is bounded. This bornological vector space of polynomial growth sequences is isomorphic to $W_{\mathcal{S}} \Rightarrow X$.

The categories of *all* $\mathbb{M}_\infty$-modules and of all $\mathcal{K}$-modules are not equivalent to the category of complete convex bornological vector spaces: smooth and rough modules are different for $\mathbb{M}_\infty$ and $\mathcal{K}$, while they are the same for $\mathbb{C}$.

The left and right multiplier algebras of A are the roughenings of the canonical left and right module structures on A. Using the Morita equivalence to $\mathbb{C}$, we get

$$\mathcal{M}_{\mathrm{r}}(A) \cong (V \Rightarrow V), \qquad \mathcal{M}_{\mathrm{l}}(A) \cong (W \Rightarrow W).$$

These obviously act on $A := V \otimes W$ on the left and right by multiplication. If we let V be finite-dimensional and W infinite-dimensional, then the two algebras are obviously quite different.

The *multiplier algebra* (or double centralizer algebra) in this case is

$$\{(L, R) \in (V \Rightarrow V) \times (W \Rightarrow W) \mid b \circ (\mathrm{Id}_W \otimes L) = b \circ (R \otimes \mathrm{Id}_V)\},$$

where the multiplication uses the opposite multiplication on $W \Rightarrow W$.

The natural map $\mathrm{S}(X) \to X$ for an A-module X need not be monic (injective) for the algebras considered above. Thus it is necessary to assume approximate identities in [7, Lemma 4.4] even if the algebra in question is self-induced. For instance, take X to be the right multiplier algebra $\mathcal{M}_{\mathrm{l}}(A) \cong W \Rightarrow W$. Since $\mathcal{M}_{\mathrm{l}}(A) \cong \mathrm{R}(A)$, we have $\mathrm{S}\,\mathcal{M}_{\mathrm{l}}(A) \cong A := V \otimes W$, so that we are dealing with the question whether the map $V \otimes W \to W \Rightarrow W$ induced by b is monic. This fails if $b \circ (\mathrm{Id}_V \otimes f) = 0$ for some map $f\colon W_0 \to W$, which is still allowed by our rather weak nondegeneracy assumption on b. Even if b is nondegenerate, say, if we work in the category of Banach spaces and $V = W^*$ is the dual Banach space of W, the map $V \otimes W \to W \Rightarrow W$ may fail to be injective: this is related to a failure of Grothendieck's Approximation Property for W.

6.2. **Lie group and Lie algebra representations.** Let $\mathcal{C}$ be the tensor category of complete convex bornological vector spaces. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$ be the space of smooth, compactly supported functions on G with the convolution product and the natural bornology, where a subset is bounded if its functions are all supported in the same compact subset and have uniformly bounded derivatives of all orders. This is a complete convex bornological algebra. It is shown in [7] that the category of smooth group representations of G on bornological vector spaces is equivalent to the category of smooth $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$-modules in $\mathcal{C}$.

Let $\mathrm{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, equipped with the fine bornology. The category of bounded Lie algebra representations of $\mathfrak{g}$ on complete convex bornological vector spaces is equivalent to the category of unital $\mathrm{U}(\mathfrak{g})$-modules in $\mathcal{C}$.

A smooth group representation of G may be differentiated to a Lie algebra representation of $\mathfrak{g}$, that is, to a unital $\mathrm{U}(\mathfrak{g})$-module structure. This provides a functor

$$\mathfrak{Mod}_{\mathrm{C}_{\mathrm{c}}^{\infty}(G)} \to \mathfrak{Mod}_{\mathrm{U}(\mathfrak{g})},$$

the differentiation functor. It is fully faithful if and only if G is connected.

The left regular representation of G on $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$ yields a Lie algebra representation of $\mathfrak{g}$ on $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$. If V is a smooth representation of G or, equivalently, a smooth $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$-module, then the induced $\mathrm{U}(\mathfrak{g})$-module structure on V is the natural module structure on $V \cong \mathrm{C}_{\mathrm{c}}^{\infty}(G) \otimes_{\mathrm{C}_{\mathrm{c}}^{\infty}(G)} V$ induced by the $\mathrm{U}(\mathfrak{g})$-module structure on $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$. As a consequence, the differentiation functor

$$\mathrm{d}\colon \mathfrak{Mod}_{\mathrm{C}_{\mathrm{c}}^{\infty}(G)} \to \mathfrak{Mod}_{\mathrm{U}(\mathfrak{g})}$$

is naturally isomorphic to the tensor product functor $V \mapsto \mathrm{C}_{\mathrm{c}}^{\infty}(G) \otimes_{\mathrm{C}_{\mathrm{c}}^{\infty}(G)} V$ with the canonical $\mathrm{U}(\mathfrak{g}), \mathrm{C}_{\mathrm{c}}^{\infty}(G)$-bimodule structure on $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$.

More explicitly, the representation of $\mathfrak{g}$ on $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$ identifies $\mathfrak{g}$ with the space of right-invariant vector fields on G and lets the latter act on $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$ as derivations. The induced action of $\mathrm{U}(\mathfrak{g})$ proceeds by identifying $\mathrm{U}(\mathfrak{g})$ with the algebra of right-invariant differential operators on G. Equivalently, we may identify $\mathrm{U}(\mathfrak{g})$ with the algebra of distributions on G supported at the identity element. Since compactly supported distributions on G act on smooth functions by convolution on the left and right, this provides left and right $\mathrm{U}(\mathfrak{g})$-module structures on $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$. These commute with the $\mathrm{U}(\mathfrak{g})$- and $\mathrm{C}_{\mathrm{c}}^{\infty}(G)$-module structures on the other side because convolution is associative.

By our general theory, the differentiation functor comes together with three other functors. First, it has a right adjoint functor

$$\begin{aligned} \mathrm{d}^*\colon \mathfrak{Mod}_{\mathrm{U}(\mathfrak{g})} &\to \mathfrak{Mod}_{\mathrm{C}_{\mathrm{c}}^{\infty}(G)}, \\ W &\mapsto \mathrm{S}_{\mathrm{C}_{\mathrm{c}}^{\infty}(G)}\bigl(\mathrm{C}_{\mathrm{c}}^{\infty}(G) \Rightarrow_{\mathrm{U}(\mathfrak{g})} W\bigr) = \mathrm{S}_{\mathrm{C}_{\mathrm{c}}^{\infty}(G)} \operatorname{Hom}_{\mathrm{U}(\mathfrak{g})}(\mathrm{C}_{\mathrm{c}}^{\infty}(G), W). \end{aligned}$$

Since G is connected, the differentiation functor is fully faithful. Equivalently, $\mathrm{d}^* \circ \mathrm{d}(V) \cong V$ for any smooth group representation V of G.

Secondly, we may map smooth $\mathrm{C}^\infty(G)$-modules to rough $\mathrm{C}^\infty(G)$-modules by the roughening functor, and then equip the latter with a canonical $\mathrm{U}(\mathfrak{g})$-module structure—rough modules are sufficiently differentiable for such a $\mathrm{U}(\mathfrak{g})$-module structure to exist. We may rewrite this alternative differentiation functor as

$$\bar{\mathrm{d}}\colon \mathfrak{Mod}_{\mathrm{C}_\mathrm{c}^\infty(G)} \to \mathfrak{Mod}_{\mathrm{U}(\mathfrak{g})},$$
$$V \mapsto \mathrm{R}(W) = \mathrm{Hom}_{\mathrm{C}_\mathrm{c}^\infty(G)}(\mathrm{C}_\mathrm{c}^\infty(G), W) = \bigl(\mathrm{C}_\mathrm{c}^\infty(G) \Rightarrow_{\mathrm{C}_\mathrm{c}^\infty(G)} W\bigr),$$

where we view $\mathrm{C}_\mathrm{c}^\infty(G)$ as a $\mathrm{C}_\mathrm{c}^\infty(G), \mathrm{U}(\mathfrak{g})$-bimodule by letting $\mathrm{U}(\mathfrak{g})$ act by right convolution. Finally, $\bar{\mathrm{d}}$ has a left adjoint functor

$$\bar{\mathrm{d}}^*\colon \mathfrak{Mod}_{\mathrm{U}(\mathfrak{g})} \to \mathfrak{Mod}_{\mathrm{C}_\mathrm{c}^\infty(G)}, \qquad W \mapsto \mathrm{C}_\mathrm{c}^\infty(G) \otimes_{\mathrm{U}(\mathfrak{g})} W.$$

Since G is connected and roughening is fully faithful, the functor $\bar{\mathrm{d}}$ is fully faithful. Equivalently, $\bar{\mathrm{d}}^* \circ \bar{\mathrm{d}}(V) \cong V$ for any smooth group representation V of G.

Thus the two differentiation functors d and $\bar{\mathrm{d}}$ from smooth representations of G to Lie algebra representations of $\mathfrak{g}$ come together with two integration functors d^* and $\bar{\mathrm{d}}^*$ that map Lie algebra representations of $\mathfrak{g}$ to group representations of G.

The integration functor d^* is right adjoint to d. That is, G-equivariant bounded linear maps $V \to \mathrm{d}^*(W)$ for smooth G-representations V correspond bijectively to bounded $\mathrm{U}(\mathfrak{g})$-module homomorphisms from $\mathrm{d}(V)$ to W.

The integration functor $\bar{\mathrm{d}}^*$ is left adjoint to $\bar{\mathrm{d}}$. That is, bounded G-equivariant linear maps $\bar{\mathrm{d}}^*(W) \to V$ for a smooth G-representation V correspond bijectively to bounded $\mathrm{U}(\mathfrak{g})$-module homomorphisms from W to the roughening of V.

Thus our two integration functors both satisfy a universal property, meaning that they are, in some sense, optimal ways to integrate a $\mathrm{U}(\mathfrak{g})$-module. The integration $\mathrm{d}^*(W)$ is the maximal smooth G-representation equipped with a $\mathrm{U}(\mathfrak{g})$-module map $W \to V$ in the sense that any other such V maps to $\mathrm{d}^*(W)$. And the integration $\bar{\mathrm{d}}^*(W)$ is the minimal smooth G-representation equipped with a $\mathrm{U}(\mathfrak{g})$-module map $W \to \mathrm{R}(V)$ in the sense that it maps to any other such V.

However, these two universal properties are not compatible. There is usually no canonical map between $\mathrm{d}^*(W)$ and $\bar{\mathrm{d}}^*(W)$ in either direction.

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