

Axiomatizability of the stable rank of C^* -algebras

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Abstract. We show that the class of C^* -algebras with stable rank greater than a given positive integer is axiomatizable in logic of metric structures. As a consequence we show that the stable rank is continuous with respect to forming ultrapowers of C^* -algebras, and that stable rank is Kadison–Kastler stable.

The notion of stable rank of a C^* -algebra was invented by Rieffel in [5] for the purpose of obtaining results on the non-stable K -theory of the C^* -algebra. The stable rank of a C^* -algebra, which can attain values in $\{1, 2, \dots, \infty\}$, can be viewed as a “non-commutative” dimension of the C^* -algebra; the stable rank of a C^* -algebra is one if and only if the invertible elements (in its unitization) are dense.

It was shown in [3, §3.8] that the class of C^* -algebras with stable rank equal to one is axiomatizable in logic of metric structures. In this note we improve this result, as stated in the abstract, to all values of the stable rank. In [2, Thm. 6.3] it was proved that if the Kadison–Kastler distance between two C^* -algebras is less than $1/8$, then if one of the two C^* -algebras has real rank zero, then so has the other. In [2, Quest.7.3] it was asked whether higher values of the real rank are stable under small perturbations and, likewise, what happens to the stable rank under small perturbations? Theorem 7(ii) provides the answer to the second part of this question; see [3, §3.9, §5.15] for more information on the first part of the question, stability of higher values of the real rank under small perturbations.

It was shown in [6] that if A is a unital C^* -algebra with $\text{sr}(A) > 1$, so that the invertible elements in A are not dense, then there is an element $a \in A$ of norm 1 such that the distance from a to the invertible elements of A is equal

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to 1. We extend this result to the situation where A is a unital C^* -algebra with $\text{sr}(A) > n$, for any integer $n \geq 1$ (Corollary 4). This is the ingredient in the proof of our main result, Theorem 7.

The results in [6] were extended by Brown and Pedersen in [1] to the situation of so-called “quasi-invertible” elements. We use the methods from [1] to obtain Corollary 4.

First we recall the definition of higher stable ranks from Rieffel’s paper [5]. Let A be a unital C^* -algebra. Let $\text{lg}_n(A)$ be the set of n -tuples $a = (a_1, \dots, a_n) \in A^n$ for which $A = Aa_1 + \dots + Aa_n$. Identify A^n with $M_{n,1}(A)$ equipped with the usual norm inherited from $M_n(A)$. With this convention, A^n is an $M_n(A)$ - A bimodule, and if $a \in A^n$, then $a^* \in M_{1,n}(A)$ (see, e.g., [4] for the theory of Hilbert C^* -modules). Rieffel defined $\text{sr}(A)$ to be the smallest integer $n \geq 1$ such that $\text{lg}_n(A)$ is dense in A^n (and if no such integer exists, then $\text{sr}(A) = \infty$); see [5] where stable rank was introduced as the ‘topological stable rank.’

If $a, b \in A^n$, then $a^*b \in A$ and $ab^* \in M_n(A)$. Note that $a \in \text{lg}_n(A)$ if and only if a^*a is invertible in A . In that case, $a = v|a|$, where $|a| = (a^*a)^{1/2} \in A$ and $v = a|a|^{-1} \in A^n$ is an isometry, i.e., $v^*v = 1$. Note also that $\|a\|^2 = \|a^*a\| = \|\sum_{i \leq n} a_i^*a_i\|$.

We will now assume that A is represented faithfully and non-degenerately on a Hilbert space H , so that $A \subseteq B(H)$. Then $A^n \subseteq B(H, H^n)$. Thus each $a \in A^n$ admits a polar decomposition $a = v|a| = |a^*|v$ with $|a|$ as above and with v a partial isometry in $B(H, H^n)$. This polar decomposition agrees with the one above when $a \in \text{lg}_n(A)$, but in general v is not an element of A^n . However, if $\psi \in C([0, 1])$ is such that $\psi(0) = 0$ and $a = (a_1, \dots, a_n) \in A^n$ has polar decomposition $a = v|a|$, then $v\psi(|a|)$ is a limit of $vP_n(|a|)$ for a sequence of polynomials $\{P_n\}$ vanishing at 0. Therefore the j th entry of $v\psi(|a|)$ belongs to $C^*(a_j, |a|)$, $1 \leq j \leq n$, and in particular, $v\psi(|a|)$ belongs to A^n . This standard fact will be used several times below.

The lemma below is an analog of [1, Prop. 1.7].

Lemma 1. *Let $a \in A^n$, $b \in \text{lg}_n(A)$, and $\beta > \|a - b\|$ be given. Write $b = w|b|$, with w an isometry in A^n . It follows that $a + \beta w \in \text{lg}_n(A)$.*

Proof. Since $|b| + \beta \cdot 1$ is bounded below by $\beta \cdot 1$, it is invertible with

$$\|(|b| + \beta \cdot 1)^{-1}\| \leq \beta^{-1}.$$

Next, $c = w^*(a - b) = w^*a - |b|$ satisfies $\|c\| < \beta$, so $\|c(|b| + \beta 1)^{-1}\| < 1$, which implies that $c(|b| + \beta)^{-1} + 1$ is invertible, being at distance less than 1 from the identity. Therefore

$$w^*a + \beta \cdot 1 = c + |b| + \beta \cdot 1 = (c(|b| + \beta \cdot 1)^{-1} + 1)(|b| + \beta \cdot 1)$$

is an invertible element of A . Now, $a + \beta w = w(w^*a + \beta \cdot 1)$ is the product of an element of $\text{lg}_n(A)$ and an invertible element of A , and therefore belongs to $\text{lg}_n(A)$. \square

The lemma below, and its proof, closely resembles [1, Thm. 2.2], which again was a refinement of [6, Lem. 2.1]. Let us fix some notation. Let $a \in A^n$ with polar decomposition $a = v|a|$ be given (with v a partial isometry in $B(H, H^n)$). For each $\lambda \geq 0$, let $e_\lambda \in B(H)$ and $f_\lambda \in B(H^n)$ denote the spectral projections corresponding to the interval $[0, \lambda]$ of $|a|$ and $|a^*|$, respectively, i.e., $e_\lambda = 1_{[0, \lambda]}(|a|)$ and $f_\lambda = 1_{[0, \lambda]}(|a^*|)$.

Lemma 2. *Let $n \geq 1$ be an integer, and let $a \in A^n$ with polar decomposition $a = v|a|$ be given. Let f_λ be as above, for $\lambda \geq 0$. Then, for each $\gamma > \text{dist}(a, \text{lg}_n(A))$, there exists $s \in \text{lg}_n(A)$ such that $(1 - f_\gamma)v = (1 - f_\gamma)s$.*

Proof. Choose $\text{dist}(a, \text{lg}_n(A)) < \beta < \gamma$. Find $b \in \text{lg}_n(A)$ such that $\|a - b\| < \beta$, and write $b = w|b| = |b^*|w$, with w an isometry in A^n . Let $\varphi, \psi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ be the continuous functions given by

$$\varphi(t) = \min\{\gamma^{-1}, t^{-1}\}, \quad \psi(t) = \min\{t\gamma^{-2}, t^{-1}\}, \quad t \in \mathbb{R}^+.$$

Then

$$\psi(|a|)v^* = (v\psi(|a|))^* \in M_{1,n}(A) \quad \text{and} \quad \|\beta\psi(|a|)v^*w\| < 1$$

because $\psi(0) = 0$ and $\|\beta\psi\|_\infty = \beta\gamma^{-1} < 1$, respectively. Since $\varphi(|a|)$ is bounded below by $\|a\|^{-1} \cdot 1$ and therefore invertible, it follows from Lemma 1 that

$$s = (a + \beta w)(1 + \beta\psi(|a|)v^*w)^{-1}\varphi(|a|)$$

belongs to $\text{lg}_n(A)$. With e_λ and f_λ , $\lambda \geq 0$, as defined above, note that

$$\begin{aligned} v(1 - e_\gamma) &= (1 - f_\gamma)v, \\ v(1 - e_\gamma)v^* &= (1 - f_\gamma), \\ (1 - e_\gamma)|a|\varphi(a) &= (1 - e_\gamma)|a|\psi(a) = (1 - e_\gamma). \end{aligned}$$

Hence,

$$\begin{aligned} (1 - f_\gamma)(a + \beta w) &= v(1 - e_\gamma)|a| + \beta v(1 - e_\gamma)v^*w \\ &= v(1 - e_\gamma)|a|(1 + \beta\psi(|a|)v^*w), \end{aligned}$$

so

$$(1 - f_\gamma)s = v(1 - e_\gamma)|a|\varphi(|a|) = v(1 - e_\gamma) = (1 - f_\gamma)v,$$

as required. □

As in [1, Thm. 2.2] and [6, Thm. 2.2] one can improve the lemma above as follows: if $a \in A^n$ and γ are as in Lemma 2, then there exists an isometry $u \in A^n$ such that $v(1 - e_\gamma) = u(1 - e_\gamma)$. However, we shall not need this stronger statement to obtain Corollary 4 and Theorem 7 below.

Recall that for a positive element $a \in A$ and $\lambda \geq 0$ we define $(a - \lambda)_+ = g(a)$, where $g(t) = \max\{t - \lambda, 0\}$.

Proposition 3. *Let $n \geq 1$ be an integer, and let $a \in A^n$ with polar decomposition $a = v|a|$ be given. Then*

$$\text{dist}(a, \text{lg}_n(A)) = \inf\{\lambda \geq 0 \mid \exists s \in \text{lg}_n(A) \text{ such that } (1 - f_\lambda)v = (1 - f_\lambda)s\}.$$

Proof. The inequality “ \geq ” follows from Lemma 2. To prove the other inequality, take $\lambda \geq 0$ for which there exists $s \in \text{lg}_n(A)$ with $(1 - f_\lambda)v = (1 - f_\lambda)s$. For any continuous function $h: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ which vanishes on $[0, \lambda]$ we have $h(1 - 1_{[0, \lambda]}) = h$. Hence

$$h(|a^*|)v = h(|a^*|)(1 - f_\lambda)v = h(|a^*|)(1 - f_\lambda)s = h(|a^*|)s.$$

In particular, $(|a^*| - \lambda)_+v = (|a^*| - \lambda)_+s$, and this element belongs to the closure of $\text{lg}_n(A)$. Indeed, if $s \in \text{lg}_n(A)$ and c is any positive element of $M_n(A)$, then cs belongs to the closure of $\text{lg}_n(A)$, because $c + \varepsilon \cdot 1$ is invertible in $M_n(A)$ for all $\varepsilon > 0$, whence $s(c + \varepsilon \cdot 1)$ belongs to $\text{lg}_n(A)$. This shows that

$$\begin{aligned} \text{dist}(a, \text{lg}_n(A)) &\leq \|a - (|a^*| - \lambda)_+v\| \\ &= \|(|a^*| - (|a^*| - \lambda)_+)v\| \\ &\leq \| |a^*| - (|a^*| - \lambda)_+ \| \leq \lambda, \end{aligned}$$

as required. □

Corollary 4. *Let A be a unital C^* -algebra with $\text{sr}(A) > n$, where $n \geq 1$ is an integer. It follows that there exists $b \in A^n$ such that*

$$\|b\| = \text{dist}(b, \text{lg}_n(A)) = 1.$$

Proof. Take any $a \in A^n$ with $\gamma = \text{dist}(a, \text{lg}_n(A)) > 0$. As above, write $a = v|a| = |a^*|v$ with v a partial isometry in $B(H, H^n)$. Consider the continuous function

$$h: \mathbb{R}^+ \rightarrow \mathbb{R}^+, \quad h(t) = \min\{\gamma^{-1}t, 1\}.$$

Set $b = vh(|a|) = h(|a^*|)v$. Then $b \in A^n$ because $h(0) = 0$, and $|b^*| = h(|a^*|)$. Also, $\|b\| = \|h(|a|)\| \leq 1$. Let f_λ and \tilde{f}_λ be the spectral projections corresponding to the interval $[0, \lambda]$ for $|a^*|$ and $|b^*| = h(|a^*|)$, respectively.

Suppose that $\alpha = \text{dist}(b, \text{lg}_n(A)) < 1$. Then, for any $\alpha < \beta < 1$, by Lemma 2, there exists $s \in \text{lg}_n(A)$ such that $(1 - \tilde{f}_\beta)v = (1 - \tilde{f}_\beta)s$. Since $1_{[0, \beta]} \circ h = 1_{[0, \gamma\beta]}$ we get that

$$\tilde{f}_\beta = 1_{[0, \beta]}(|b^*|) = (1_{[0, \beta]} \circ h)(|a^*|) = f_{\gamma\beta}.$$

Hence $(1 - f_{\gamma\beta})v = (1 - f_{\gamma\beta})s$. By Proposition 3 this would entail that $\text{dist}(a, \text{lg}_n(A)) \leq \gamma\beta < \gamma$, a contradiction. Hence $\text{dist}(b, \text{lg}_n(A)) \geq 1$. This completes the proof, since $\text{dist}(x, \text{lg}_n(A)) \leq \|x\|$ holds for all $x \in A^n$ (as $(0, \dots, 0)$ belongs to the closure of $\text{lg}_n(A)$). □

Prior to proving that $\text{sr}(A) \geq n$ is axiomatizable for every $n \geq 1$ we recall some facts and definitions from [3, §2.1]. Fix $k \geq 1$. The space of formulas of the language of C^* -algebras with free variables among $\bar{x} = (x_1, \dots, x_k)$ is denoted by $\mathfrak{F}^{\bar{x}}$ (see [3, Def. 2.1.1]). For $\varphi \in \mathfrak{F}^{\bar{x}}$, a C^* -algebra A , and a k -tuple \bar{a} in A of the same sort as \bar{x} (typically \bar{a} is a k -tuple in the unit ball of A), by $\varphi^A(\bar{a})$ we denote the interpretation of φ in A at \bar{a} . The space $\mathfrak{F}^{\bar{x}}$ has a natural algebra structure, and it is equipped with the semi-norm

$$\|\varphi\|_T = \sup \varphi^A(\bar{a}),$$

where A ranges over all C^* -algebras and \bar{a} ranges over all k -tuples of the same sort as \bar{x} . The completion of $\mathfrak{F}^{\bar{x}}$ with respect to this norm is a Banach algebra, denoted by $\mathfrak{W}^{\bar{x}}$ (see [3, §3.1 (a)]). Its elements are called *definable predicates*. Hence a definable predicate is an assignment of predicates (i.e., real-valued uniformly continuous functions on A^k) of a particular form to C^* -algebras. An assignment of closed subsets to C^* -algebras $A \mapsto \mathcal{S}^A \subseteq A^k$ is a *definable set* (see [3, Def. 3.2.1]) if for any formula $\psi(\bar{x}, y)$ of the language of C^* -algebras both $\sup_{y \in \mathcal{S}^A} \psi(\bar{x}, y)^A$ and $\inf_{y \in \mathcal{S}^A} \psi(\bar{x}, y)^A$ are definable predicates (inconveniently, being a definable set is a bit stronger than being the zero-set of a definable predicate). An extension of a language of metric structures is *conservative* if every model in the original language can be expanded uniquely to a model of the new language. In particular, an extension obtained by adding definable predicates and allowing quantification (i.e., taking sups and infs) over definable sets is conservative.

Lemma 5. *For any $n \geq 1$, the extension of the language of C^* -algebras by allowing quantification over the unit ball A_1^n of A^n and adding a predicate F_n for the standard norm on A^n is conservative.*

Proof. It suffices to prove that F_n is a definable predicate and the unit ball of A^n is a definable set. This proof is virtually identical to the proof that the operator norm on $M_n(A)$ is a definable predicate and the unit ball of $M_n(A)$ is a definable subset of A^{n^2} (see [3, Lem. 4.2.3]).

It is clear that function F_n commutes with taking ultraproducts, i.e., if A_j , for $j \in J$, are C^* -algebras, \mathcal{U} is an ultrafilter on J , and $A = \prod_{\mathcal{U}} A_j$, then for any $\bar{a} \in A$ and any representing sequence (\bar{a}_j) of $\bar{a} \in A$ we have

$$F_n^A(\bar{a}) = \lim_{j \rightarrow \mathcal{U}} F_n^{A_j}(\bar{a}_j).$$

Therefore the class \mathcal{C}' of all structures of the form (A, F_n^A) , where A is a C^* -algebra, is closed under taking ultraproducts and ultraroots, and by [3, Thm. 2.4.1] axiomatizable. The Beth definability theorem [3, Thm. 4.2.1] now implies that F_n is a definable predicate. Since every element in A_1^n has a representing sequence in $\prod_j (A_j)_1^n$, we have $A_1^n = \prod_{\mathcal{U}} (A_j)_1^n$. Therefore A_1^n is a definable set by [3, Thm. 3.2.5]. This completes the proof. \square

We shall also need the fact that the set A_+ of positive elements in a C^* -algebra A is definable (see [3, Ex. 3.2.6 (2)]) and therefore the language extended by allowing quantification over the positive part of the unit ball is a conservative extension of the language of C^* -algebras.

Lemma 6. *For $n \geq 1$, consider the sentence*

$$\varphi_n = \sup_{x \in A^n, \|x\| \leq 1} \inf_{v \in A^n, \|v\| \leq 1} \inf_{y \in A_+, \|y\| \leq 1} \max(\|x - vy\|, \|v^*v - 1\|)$$

in the extended language of unital C^ -algebras. For every unital C^* -algebra A we have $\text{sr}(A) \leq n$ if and only if $\varphi_n^A = 0$, and $\text{sr}(A) > n$ if and only if $\varphi_n^A \geq 1$.*

Proof. By a standard geometric series argument, $\|v^*v - 1\| < 1$ implies that v^*v is invertible and hence that $v = (v_1, \dots, v_n)$ is in $\text{lg}_n(A)$. It was proved in [3, Lem. 3.8.4] that $a = (a_1, \dots, a_n) \in A^n$ such that $\max_{i \leq n} \|a_i\| \leq 1$ is in the closure of $\text{lg}_n(A)$ if and only if it can be approximated arbitrarily well by elements of the form vy , for $v \in A^n$ and $y \in A_+$, such that $\|v^*v - 1\| < 1$ and $\|y\| \leq \sqrt{n}$. After renormalization of A^n , we have that $a \in A_1^n$ is in the closure of $\text{lg}_n(A)$ if and only if it can be approximated arbitrarily well by elements of the form vy , for $v \in A^n$ and $y \in A_+$, such that $\|v^*v - 1\| < 1$ and $\|y\| \leq 1$. Therefore $\varphi_n^A = 0$ is equivalent to $\text{sr}(A) \leq n$, and consequently $\varphi_n^A \geq 1$ implies $\text{sr}(A) > n$.

It remains to prove that $\text{sr}(A) > n$ implies $\varphi_n^A \geq 1$. Suppose otherwise, that $\text{sr}(A) > n$ and $\varphi_n^A < 1$. Let $b \in A^n$ be as in Corollary 4, so that $\|b\| = \text{dist}(b, \text{lg}_n(A)) = 1$. Since $\varphi_n^A < 1$, there exist $v \in A^n$ and $y \in A_+$ such that $\|y\| \leq 1$, $\|b - vy\| < 1$, and $\|v^*v - 1\| < 1$. The latter formula implies $v \in \text{lg}_n(A)$. With $\varepsilon > 0$ small enough to have $\|b - v(y + \varepsilon \cdot 1)\| < 1$, we have $v(y + \varepsilon \cdot 1) \in \text{lg}_n(A)$, contradicting the choice of b . \square

Recall that the *Kadison–Kastler distance* between subalgebras A and B of $B(H)$ for a fixed Hilbert space H is defined as

$$d_{\text{KK}}(A, B) = \max\left(\sup_{x \in A_1} \inf_{y \in B_1} \|x - y\|, \sup_{y \in B_1} \inf_{x \in A_1} \|x - y\|\right),$$

writing A_1 and B_1 for the unit balls of A and B , respectively. Thus $d_{\text{KK}}(A, B)$ is equal to the Hausdorff distance between A_1 and B_1 .

Theorem 7. *For every integer $n \geq 1$ the classes of C^* -algebras with stable rank greater than n , less than or equal to n , and equal to n , respectively, are axiomatizable in logic of metric structures. In particular:*

- (i) *If A_m , for $m \in \mathbb{N}$, are unital C^* -algebras and \mathcal{U} is an ultrafilter on \mathbb{N} , then*

$$\text{sr}\left(\prod_{\mathcal{U}} A_m\right) = \lim_{n \rightarrow \mathcal{U}} \text{sr}(A_n).$$

In particular, stable rank is preserved under ultrapowers.

- (ii) *For every $n \geq 1$, there exists $\varepsilon_n > 0$ such that for any two unital C^* -subalgebras A and B of $B(H)$ with $d_{\text{KK}}(A, B) < \varepsilon_n$, either $\text{sr}(A) = \text{sr}(B)$ or both of $\text{sr}(A)$ and $\text{sr}(B)$ are greater than n .*

Proof. In [3, Prop. 3.8.1] it was proved that having stable rank at most n is axiomatizable. By Lemma 6 and Lemma 5 having stable rank greater than n is also axiomatizable, and the axiomatizability of stable rank being equal to n follows.

By Łoś’s theorem (see [3, Thm. 2.3.1]) with φ_n as in Lemma 6 we have

$$\varphi_n^{\prod_{\mathcal{U}} A_m} = \lim_{m \rightarrow \mathcal{U}} \varphi_n^{A_m},$$

and therefore (i) follows. We infer from [3, Lem. 5.15.1] that the evaluation of φ_n is continuous with respect to d_{KK} for every integer $n \geq 1$. Choose $\varepsilon_n > 0$ small enough so that $d_{\text{KK}}(A, B) < \varepsilon_n$ implies $\|\varphi_j^A - \varphi_j^B\| < 1$ for all $j \leq n$ and

for all unital C^* -subalgebras A and B of $B(H)$. Then $\text{sr}(A) = j < n$ implies $\text{sr}(B) = j$, as required. \square

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