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Kirchberg–Wassermann exactness vs exactness: reduction to the unimodular totally disconnected case

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Abstract. We show that in order to prove that all locally compact groups with exact reduced group C^*-algebras are exact in the dynamical sense (i.e., KW-exact), it suffices to show this for totally disconnected locally compact groups.

1. Introduction

There are two natural notions of exactness for locally compact groups which to our knowledge were first mentioned by Kirchberg in [8]. A weak one, called here C^*-exactness which says that the reduced group algebra is an exact C^*-algebra, and a strong one called KW-exactness, which asserts that given any exact sequence of dynamical systems over the group, the corresponding sequence of reduced crossed products is exact. The stronger exactness property can thus be regarded as a dynamical form of exactness. Here KW stands for Kirchberg and Wassermann who introduced and studied these notions in [9] and [10]. Since the crossed product by trivial actions is just the tensor product by the reduced group algebra, it is evident that KW-exactness implies C^*-exactness. As announced in [8] and later proved in [9], the two concepts are equivalent for discrete groups but whether the same equivalence holds true in the case of general locally compact groups has been an open problem ever since.

Note that there are numerous other concepts related to exactness such as amenability at infinity or the non-existence of noncompact ghost operators, which have been studied and put forward in the past decades (see [1, 12, 14]).

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There has been considerable recent progress in the understanding of these conditions showing that they are equivalent to KW-exactness among second countable groups (see [3]). In view of those developments the question of equivalence of KW-exactness and C^*-exactness appears more pressing than ever.

In this note we do not answer this question but reduce the problem to the case when the group is unimodular and totally disconnected. Thus, if C^*-exact but non KW-exact groups exist, then there must also exist totally disconnected unimodular such groups. This had already been suspected by experts (see the introduction of [1]).

2. Preliminaries

2.1. **C^*-exactness and KW-exactness.** As is well known, a C^*-algebra A is exact if for any exact sequence

$$0 \to I \to E \to Q \to 0$$

of C^*-algebras, the sequence of minimal tensor products

$$0 \to A \otimes I \to A \otimes E \to A \otimes Q \to 0$$

is exact. Kirchberg and Wassermann proved that this property is equivalent to nuclear embeddability and passes to subalgebras and quotients (see [4, Chap. 10]). Exactness of the second sequence can only fail in the middle. That is, the kernel of the map onto $A \otimes Q$ maybe strictly larger than $A \otimes I$. It is easy to check directly from the definition that a minimal tensor product $A \otimes B$ of two C^*-algebras is exact if and only if A and B are exact (see [4, Prop. 10.2.7]).

Definition 2.2. Let G be a locally compact group. Then G is said to be *C^*-exact* if $C_r^*(G)$ is an exact C^*-algebra.

If A is a C^*-algebra and G is a locally compact group acting on A by $\alpha\colon G \to \mathrm{Aut}(A)$, then the action α is called *continuous* if for all $a \in A$, the map $g \mapsto \alpha_g(a)$ is norm continuous.

Definition 2.3. Let G be a locally compact group. G is said to be *KW-exact* (KW for Kirchberg and Wassermann) if for all C^*-algebras A and all continuous actions $\alpha\colon G \to \mathrm{Aut}(A)$ and for all closed two-sided ideals $I \trianglelefteq A$ such that $\alpha_g(I) = I$ for all $g \in G$, the sequence

$$0 \to I \rtimes_{\alpha,r} G \to A \rtimes_{\alpha,r} G \to A/I \rtimes_{\alpha,r} G \to 0$$

is exact.

By recent results in [12, 3] it is now known that it suffices, for second countable groups, to check exactness of only one such sequence. That is, a locally compact second countable group G is KW-exact if and only if

$$0 \to C_0(G) \rtimes_{L,r} G \to C_b^{lu}(G) \rtimes_{L,r} G \to (C_b^{lu}(G)/C_0(G)) \rtimes_{L,r} G \to 0$$

is exact, where L is the left translation action on the C^*-algebra of bounded left uniformly continuous functions $C_b^{lu}(G)$ on G.

As already mentioned, since $A \otimes C_r^*(G) \cong A \rtimes_{\tau,r} G$, where $\tau \colon G \to \operatorname{Aut}(A)$ is the trivial action, we have the following result.

Proposition 2.4. *If G is KW-exact, then it is C^*-exact.*

KW-exactness satisfies the following permanence properties.

Proposition 2.5. *Let G be a locally compact group.*

(1) *If G is amenable, then G is KW-exact* [10, Prop. 6.1].
(2) *If G is connected, then G is KW-exact* [10, Thm. 6.8].
(3) *Let $N \trianglelefteq G$ be a closed normal subgroup. If N and G/N are KW-exact, then G is KW-exact* [10, Thm. 5.1].

Given a subgroup H of a locally compact group G, elements in H and $C_r^*(H)$ only act as multipliers on $C_r^*(G)$. However, if H is open in G, then it is easy to see that $C_r^*(H) \subseteq C_r^*(G)$. Since exactness passes to subalgebras, we get the following proposition.

Proposition 2.6. *If G is a locally compact C^*-exact group and $H \leq G$ is an open subgroup, then $C_r^*(H) \hookrightarrow C_r^*(G)$ is an injective $*$-homomorphism and so $C_r^*(H)$ is also exact.*

2.7. **Structure of locally compact groups.** The following proposition follows from the closure properties of the class of amenable locally compact groups. We indicate the proof for the reader's convenience.

Proposition 2.8 ([15, Prop. 4.1.12])**.** *Every locally compact group G has a unique maximal amenable closed normal subgroup.*

Proof. Since unions of directed systems of amenable subgroups of G are again amenable one only needs to show that given two closed normal amenable subgroups H_1 and H_2, the closed subgroup H generated by them is amenable. Now the semi-direct product $H_1 \rtimes H_2$ is amenable and H is the closure of the continuous image of $H_1 \rtimes H_2$. This implies that H is also amenable. □

Definition 2.9. Let G be a locally compact group. Then the *amenable radical*, denoted by $\operatorname{Rad}(G)$, is the unique maximal amenable closed normal subgroup of G.

We have the following characterization of totally disconnected locally compact groups, which is a classical result by van Danzig.

Theorem 2.10 ([13])**.** *Let G be a locally compact group. Then G is totally disconnected if and only if it admits a neighborhood basis of the identity consisting of compact open subgroups.*

We use the following structure theorem of locally compact groups which is deduced from a solution to Hilbert's fifth problem [11, Thm. 4.6]. Recall that a subgroup $H \leq G$ is *characteristic* if it is preserved under every automorphism in $\operatorname{Aut}(G)$.

Theorem 2.11 ([5, Thm. 3.3.3], [6, Thm. 23]). *Let G be any locally compact group. The quotient group $G/\mathrm{Rad}(G)$ has a finite index open characteristic subgroup which splits as a direct product $S \times D$, where S is a connected semisimple Lie group and D is totally disconnected locally compact.*

3. Reduction to the unimodular totally disconnected locally compact case

The aim of this section is to prove the following theorem.

Theorem 3.1. *If KW-exactness and C^*-exactness are equivalent for all unimodular totally disconnected locally compact groups, then they are equivalent for all locally compact groups.*

3.2. **Induced representations and weak containment.**

3.2.1. *Induced representations.* Let G be a locally compact group and $H \leq G$ a closed subgroup. For a Borel measure ν on G/H and $g \in G$, denote by ν_g the measure defined as $\nu_g(E) = \nu(gE)$ for all Borel sets $E \subseteq G/H$. A regular Borel measure ν is *quasi-invariant* if $\nu_g \sim \nu$ for all $g \in G$, where $\sim$ denotes mutual absolute continuity of measures.

Let μ_H be a Haar measure on H and define a mapping $T_H\colon C_c(G) \to C_c(G/H)$, where

$$T_H(f)(xH) = \int_H f(xh)\, d\mu_H(h).$$

This map is surjective ([2, Lem. B.1.2]).

Lemma 3.3 ([2, Lem. B.1.3]). *Let $\rho\colon G \to \mathbb{R}^{>0}$ be a continuous function on G. Then the following are equivalent:*

(1) *For all $g \in G$ and $h \in H$, one has*

$$\rho(gh) = \frac{\Delta_H(h)}{\Delta_G(h)}\rho(g).$$

(2) *The functional $\lambda_\rho\colon C_c(G/H) \to \mathbb{C}$, defined by*

$$\lambda_\rho \circ T_H(f) = \int_G f(g)\rho(g)\, d\mu_G,$$

is well-defined and positive.

If the above conditions hold, then the regular Borel measure μ_ρ associated to the functional λ_ρ under the Riesz representation is quasi-invariant with Radon–Nikodym derivative

$$\frac{d(\mu_\rho)_y}{d\mu_\rho}(xH) = \frac{\rho(yx)}{\rho(x)} \quad \textit{for all } x, y \in G.$$

Such a function $\rho\colon G \to \mathbb{R}^{>0}$ is called a *rho-function for the pair* (G, H). For every pair (G, H), there always exists a rho-function for (G, H). Indeed if $f \in C_c(G)_+$, then

$$\rho_f(x) = \int_H \Delta_G(h)\Delta_H(h)^{-1} f(xh)\, d\mu_H(h)$$

is a continuous rho-function. Thus, there always exists a quasi-invariant regular Borel measure on G/H. In fact, every quasi-invariant regular Borel measure is associated to a rho-function for (G, H), see [2, Thm. B.1.4]. When H is a closed normal subgroup, then one takes $\rho = 1$, and the associated quasi-invariant regular Borel measure is the usual Haar measure on G/H.

Let $\pi\colon H \to \mathcal{U}(\mathcal{H})$ be a unitary representation and fix a quasi-invariant measure μ on G/H. Define a new Hilbert space

$$\mathcal{H}(\pi) = \Big\{\xi\colon G \to \mathcal{H} \;\Big|\; \xi(xh) = \pi(h^{-1})\xi(x) \text{ and } \int_{G/H} \|\xi(x)\|^2\, d\mu(x) < \infty\Big\},$$

with inner product given by

$$\langle \xi, \eta\rangle = \int_{G/H} \langle \xi(x), \eta(x)\rangle\, d\mu(x).$$

The *representation of* G *induced from* π or simply the *induced representation* is the representation $\mathrm{ind}_H^G\pi\colon G \to \mathcal{U}(\mathcal{H}(\pi))$ given by

$$\mathrm{ind}_H^G\pi(x)\xi(y) = \Big(\frac{d\mu_{x^{-1}}}{d\mu}(yH)\Big)^{1/2}\xi(x^{-1}y) \quad \text{for all } \xi \in \mathcal{H}(\pi) \text{ and } x, y \in G.$$

Given another quasi-invariant measure on G/H, the same construction gives a unitarily equivalent representation, so we can call $\mathrm{ind}_H^G\pi$ the induced representation of π to G without trepidation, see [2, Prop. E.1.5].

When $H = \{e\}$, then $\mathrm{ind}_H^G(1_H) = \lambda_G$. More generally, the representation $\mathrm{ind}_H^G(1_H)$ is called the *quasi-regular representation of* G/H. If H is normal in G, then $\mathrm{ind}_H^G(1_H)$ is unitarily equivalent to $\lambda_{G/H} \circ q$, where $q\colon G \to G/H$ is the natural surjection and $\lambda_{G/H}$ is the left regular representation of G/H, see [7, Prop. 2.38].

3.3.1. *Weak containment.* The following definition for weak containment of representations is not standard; however, it is sufficient for our applications.

Definition 3.4 ([2, Thm. F.4.4])**.** Let π and ρ be unitary representations of G. Then π *is weakly contained in* ρ, denoted by $\pi \prec \rho$, if $\|\pi(f)\| \le \|\rho(f)\|$ for all $f \in L^1(G)$.

We have the following properties of weak containment and induced representations.

Proposition 3.5. *Let* G *be a locally compact group and* $H \le G$ *a closed subgroup. Then*

(1) $\mathrm{ind}_H^G(\lambda_H)$ *is unitarily equivalent to* λ_G [7, Cor. 2.52],
(2) G *is amenable if and only if* $1_G \prec \lambda_G$ [2, Thm. G.3.2],

(3) *if σ and ρ are unitary representations on H and $\sigma \prec \rho$, then* $\mathrm{ind}_H^G(\sigma) \prec \mathrm{ind}_H^G(\rho)$ [2, Thm. F.3.5].

We believe the following is well known, but we provide a proof as we could not find a reference.

Lemma 3.6. *Let G be a locally compact group and suppose $H \leq G$ is a closed normal amenable subgroup. If G is C^*-exact, then G/H is C^*-exact.*

Proof. As H is amenable it follows that $1_H \prec \lambda_H$ and so $\mathrm{ind}_H^G(1_H) \prec \lambda_G$. However, $\mathrm{ind}_H^G(1_H)$ is unitarily equivalent to $\lambda_{G/H} \circ q$, where $q\colon G \to G/H$ is the quotient map and $\lambda_{G/H}$ is the left regular representation of G/H. Hence, $\lambda_{G/H} \circ q \prec \lambda_G$, and so it remains to show that the natural map T_H, defined by

$$T_H\colon C_c(G) \to C_c(G/H), \quad T_H(f)(gH) = \int_H f(gh)\, d\mu_H(h),$$

extends to a surjective $*$-homomorphism from $C_r^*(G) \to C_r^*(G/H)$. So let $\lambda_{G/H} \circ q\colon C_c(G) \to \mathcal{B}(L^2(G/H))$ be the natural extension of $\lambda_{G/H} \circ q$. That is,

$$\lambda_{G/H} \circ q(f) = \int_G f(g)\lambda_{G/H}(gH)\, dg$$

for all $f \in C_c(G)$. We will show that $\lambda_{G/H} \circ q(f) = \lambda_{G/H}(T_H(f))$ for all $f \in C_c(G)$. Then it will follow that

$$\|\lambda_{G/H}(T_H(f))\| = \|\lambda_{G/H} \circ q(f)\| \leq \|\lambda_G(f)\|,$$

as $\lambda_{G/H} \circ q \prec \lambda_G$, and so T_H extends to a surjective $*$-homomorphism from $C_r^*(G) \to C_r^*(G/H)$. So for all $f \in C_c(G)$, $\xi \in C_c(G/H)$ and $yH \in G/H$, we have

$$\begin{aligned}\lambda_{G/H} \circ q(f)\xi(yH) &= \int_G f(g)\xi(g^{-1}yH)\, dg \\ &= \int_{G/H}\int_H f(xh)\xi(h^{-1}x^{-1}yH)\, dh\, dx \\ &= \int_{G/H}\int_H f(xh)\xi(x^{-1}yH)\, dh\, dx,\end{aligned}$$

where the second equality follows from Weil's integration formula [7, Cor. 1.21], and the final equality follows from normality of H. Now

$$\begin{aligned}\lambda_{G/H}(T_H(f))\xi(yH) &= \int_{G/H} T_H f(xH)\lambda_{G/H}(xH)\xi(yH)\, d(xH) \\ &= \int_{G/H}\int_H f(xh)\xi(x^{-1}yH)\, dh\, dx.\end{aligned}$$

Hence, $\lambda_{G/H} \circ q(f) = \lambda_{G/H}(T_H(f))$ for all $f \in C_c(G)$, and so T_H extends to a surjection. As $C_r^*(G)$ is exact, it follows that $C_r^*(G/H)$ is also exact as exactness passes to quotients. □

3.7. **Reduction to totally disconnected case.**

Lemma 3.8. *If KW-exactness and C^*-exactness are equivalent in the class of unimodular totally disconnected locally compact groups, then they are equivalent in the class of totally disconnected locally compact groups.*

Proof. Let G be a totally disconnected locally compact group and suppose G is C^*-exact. Let $G_0 = \ker(\Delta)$. Then, in particular, G_0 is a closed normal unimodular subgroup of G. As G is totally disconnected, there exists a compact open subgroup $K \leq G$. Since compact groups are unimodular, it follows that $\Delta|_K = 1$. Hence, $K \leq G_0$ and so $\mu_G(G_0) \geq \mu_G(K) > 0$. Therefore G_0 is open. Thus, as G is C^*-exact, it follows that G_0 is C^*-exact.

By assumption, this implies that G_0 is KW-exact, and as G/G_0 is abelian, it follows that G is also KW-exact (Proposition 2.5 (3)). □

We are now ready to prove the main result of this section.

Proof of Theorem 3.1. Let G be a C^*-exact locally compact group. Let also $\mathrm{Rad}(G)$ be the amenable radical of G. Then, by Lemma 3.6, it follows that $G/\mathrm{Rad}(G)$ is C^*-exact. By Theorem 2.11, there exists an open normal finite index subgroup $N \leq G/\mathrm{Rad}(G)$ such that $N \cong S \times D$, where S is a connected semisimple Lie group and D is totally disconnected. We have the tensor decomposition where $C_r^*(N) \cong C_r^*(S) \otimes C_r^*(D)$. As $C_r^*(N)$ is exact, it follows that $C_r^*(D)$ is also exact, as we pointed out just before Definition 2.1.

By assumption and by Lemma 3.8, this implies that D is KW-exact. As connected locally compact groups are KW-exact (Proposition 2.5 (2)) and KW-exactness is preserved under extensions by closed normal subgroups (Proposition 2.5 (3)), it follows that N is KW-exact. We know N is open, so, in particular, it is closed in $G/\mathrm{Rad}(G)$. Further, N is cocompact in $G/\mathrm{Rad}(G)$, so $G/\mathrm{Rad}(G)$ is KW-exact. As $\mathrm{Rad}(G)$ is a closed normal and amenable subgroup of G and KW-exactness is preserved under extension, it follows that G is KW-exact (Proposition 2.5 (1) and (3)). □

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