Cobordism category of manifolds with Baas–Sullivan singularities

Nathan Perlmutter

(Communicated by Michael Weiss)

Abstract. For a fixed closed manifold P, we construct a cobordism category of embedded manifolds with Baas–Sullivan singularities modeled on P. Our main theorem identifies the homotopy type of the classifying space of this cobordism category with that of the infinite loop-space of a certain spectrum $\mathsf{MT}_P(d)$, related to the spectrum $\mathsf{MT}(d)$ studied by Galatius, Madsen, Tillmann, and Weiss in [6]. We obtain an analog of the Bockstein–Sullivan exact couple that arises between the classical bordism theories MO and MO_P on the level of cobordism categories and their classifying spaces.
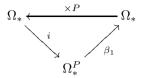
1. Introduction and statement of main results

Fix a closed, smooth manifold P. Following [1, 4], a manifold with Baas–Sullivan singularities modeled on P is a smooth manifold W equipped with the following:

(i) The boundary is equipped with a decomposition $\partial W = \partial_0 W \cup \partial_1 W$ such that $\partial(\partial_1 W) = \partial_0 W \cap \partial_1 W = \partial(\partial_0 W)$ is a closed manifold.
(ii) The face $\partial_1 W$ has the factorization $\partial_1 W = \beta_1 W \times P$ for some manifold $\beta_1 W$.

We will call such manifolds P-manifolds. The face $\partial_0 W$ is said to be the *boundary* of W. If W is compact and $\partial_0 W = \varnothing$ then W is said to be a closed P-manifold. Two closed d-dimensional P-manifolds M_a and M_b are said to be cobordant if there is a $(d+1)$-dimensional P-manifold W such that $\partial_0 W = M_a \sqcup M_b$.

We are interested in the cobordism theory of P-manifolds. To simplify our presentation we will assume that all manifolds are unoriented. However, the same constructions work in the same way for an arbitrary tangential structure, $\theta : B \to BO$. We denote by Ω_* the graded cobordism group of unoriented manifolds. Using the above definitions of P-manifolds and cobordism of P-manifolds, one can define the graded cobordism group Ω_*^P of unoriented P-manifolds.

The group Ω_*^P is related to Ω_* by means of the well-known Bockstein–Sullivan exact couple:

$$
\begin{array}{ccc}
\Omega_* & \xleftarrow{\;\times P\;} & \Omega_* \\
& \searrow{\scriptstyle i} \quad \nearrow{\scriptstyle \beta_1} & \\
& \Omega_*^P &
\end{array}
$$

The map $\times P$ is the homomorphism of degree $\dim(P)$ given by multiplication by P. The map i is given by inclusion and β_1 is the degree -1 homomorphism given by $M \mapsto \beta_1 M$. This exact couple arises from a cofiber sequence of spectra, $\Sigma^p \mathsf{MO} \to \mathsf{MO} \to \mathsf{MO}_P$, where MO_P is the classifying spectrum for Ω_*^P. Details on the construction of this exact couple can be found in [1, 4].

Motivated by the ideas in [2], we construct a cobordism category of manifolds with Baas–Sullivan singularities which generalizes the cobordism category of [6], and then determine the homotopy type of its classifying space. In [6], Galatius et al. construct a topological category $\mathbf{Cob}_{d+1}$ whose morphisms are $(d+1)$-dimensional submanifolds $W \subseteq [a,b] \times \mathbb{R}^{d+\infty}$ that intersect the walls $\{a,b\} \times \mathbb{R}^{d+\infty}$ orthogonally in ∂W. This category is topologized in such a way so that there are weak homotopy equivalences

$$
\mathcal{O}b(\mathbf{Cob}_{d+1}) \simeq \bigsqcup_M \mathrm{BDiff}(M),
$$

$$
\mathcal{M}or(\mathbf{Cob}_{d+1}) \simeq \Big(\bigsqcup_W \mathrm{BDiff}(W) \Big) \sqcup \mathcal{O}b(\mathbf{Cob}_{d+1}),
$$

where M varies over diffeomorphism classes of d-dimensional closed manifolds and W varies over diffeomorphism classes of cobordisms. Above, the space of identity morphisms is identified with the space of objects. In [6], Galatius et al. determine the homotopy type of the classifying space of $\mathbf{Cob}_{d+1}$, namely they prove that there is a weak homotopy equivalence

$$
B\mathbf{Cob}_{d+1} \simeq \Omega^{\infty-1} \mathsf{MT}(d+1).
$$

On the right-hand side, $\mathsf{MT}(d+1)$ is the spectrum whose $(n+d+1)$-st space is the Thom-space $\mathsf{Th}(U_{d+1,n}^\perp)$, where $U_{d+1,n}^\perp$ is the orthogonal complement to the canonical $(d+1)$-plane bundle over the Grassmannian $G(d+1,n)$ of $(d+1)$-dimensional vector subspaces of $\mathbb{R}^{d+1+n}$.

Following [6], we construct an analogous cobordism category of P-manifolds. We fix once and for all an embedding

$$
\tag{1} i_P : P \hookrightarrow \mathbb{R}^{p+m}
$$

with $p = \dim(P)$ and $m \gg p$. We construct a topological category $\mathbf{Cob}_{d+1}^P$ whose morphisms are given by $(d+1)$-dimensional embedded P-manifolds

$$
W \subseteq [a,b] \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}
$$

such that

$$W \cap (\{a, b\} \times \mathbb{R}_+ \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}) = \partial_0 W,$$
$$W \cap ([a, b] \times \{0\} \times \mathbb{R}^{d-1+\infty} \times \mathbb{R}^{p+m}) = \partial_1 W,$$

and $\partial_1 W$ has the factorziation

$$\partial_1 W = \beta_1 W \times i_P(P),$$

where $\beta_1 W \subset [a, b] \times \{0\} \times \mathbb{R}^{d-1+\infty}$ is a submanifold and $i_P(P) \subset \mathbb{R}^{p+m}$ is the submanifold given by the embedding specified in (1). Here and throughout this paper, $\mathbb{R}_+$ denotes the half-open interval $[0, \infty)$. We topologize this category in a way similar to [6] so that there are homotopy equivalences

$$\mathcal{O}b(\mathbf{Cob}_{d+1}^P) \simeq \bigsqcup_M \mathrm{BDiff}_P(M),$$
$$\mathcal{M}or(\mathbf{Cob}_{d+1}^P) \simeq \Big(\bigsqcup_W \mathrm{BDiff}_P(W) \Big) \sqcup \mathcal{O}b(\mathbf{Cob}_{d+1}^P),$$

where M varies over diffeomorphism classes of closed d-dimensional P-manifolds and W varies over diffeomorphism classes of $(d+1)$-dimensional P-manifold cobordisms. For a P-manifold W, $\mathrm{Diff}_P(W)$ is defined to be the group of diffeomorphisms $g : W \to W$ such that the restriction $g|_{\partial_1 W}$ is equal to the product $g_{\beta_1 W} \times \mathrm{Id}_P$ where $g_{\beta_1 W}$ is a diffeomorphism of $\beta_1 W$.

The main goal of this paper is to determine the homotopy type of the classifying space $B\mathbf{Cob}_{d+1}^P$. To do so we construct a new spectrum $\mathsf{MT}_P(d+1)$ as follows.

From the embedding in (1) used to construct $\mathbf{Cob}_{d+1}^P$, we obtain a Pontryagin–Thom map $c_P : S^{p+m} \to \mathsf{Th}(U_{p,m}^\perp)$. The natural multiplication map given by sending a pair of vector sub-spaces to their product,

$$
\begin{array}{ccc}
U_{d-p,n-m}^\perp \times U_{p,m}^\perp & \xrightarrow{\hat{\mu}} & U_{d,n}^\perp \\
\downarrow & & \downarrow \\
G(d-p, n-m) \times G(p, m) & \xrightarrow{\mu} & G(d, n),
\end{array}
$$

yields a map of Thom-spaces,

$$\mathsf{Th}(\hat{\mu}) : \mathsf{Th}(U_{d-p,n-m}^\perp) \wedge \mathsf{Th}(U_{p,m}^\perp) \to \mathsf{Th}(U_{d,n}^\perp).$$

The composition

$$\mathsf{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m} \xrightarrow{\mathrm{Id} \wedge c_P} \mathsf{Th}(U_{d-p,n-m}^\perp) \wedge \mathsf{Th}(U_{p,m}^\perp) \xrightarrow{\mathsf{Th}(\hat{\mu})} \mathsf{Th}(U_{d,n}^\perp)$$

then induces a map of spectra which we denote by $\tau_P : \mathsf{MT}(d-p) \to \mathsf{MT}(d)$.

There is another map of spectra $\hat{j}_d : \Sigma^{-1}\mathsf{MT}(d) \to \mathsf{MT}(d+1)$, induced by the bundle map covering the standard embedding $G(d, n) \hookrightarrow G(d+1, n)$ of Grassmannians. We define $\mathsf{MT}_P(d+1)$ to be the cofiber of the composition of spectrum maps,

$$\Sigma^{-1}\mathsf{MT}(d-p) \xrightarrow{\Sigma^{-1}\tau_P} \Sigma^{-1}\mathsf{MT}(d) \xrightarrow{\hat{j}_d} \mathsf{MT}(d+1).$$

Details of this construction are covered in Section 6. We now state our main result.

Theorem 1.1 (Main Theorem). *There is a weak homotopy equivalence*

$$B\mathbf{Cob}_{d+1}^P \simeq \Omega^{\infty-1}\mathsf{MT}_P(d+1).$$

The spectrum $\mathsf{MT}_P(d+1)$ was constructed using the Pontryagin–Thom map for a particular embedding of our manifold P. The homotopy class of this Pontryagin–Thom map only depends on the cobordism class of P. This observation leads to the following corollary.

Corollary 1.2. *Let P_1 and P_2 be smooth closed manifolds of the same dimension. Suppose that P_1 and P_2 are cobordant. Then there is a weak homotopy equivalence*

$$B\mathbf{Cob}_{d+1}^{P_1} \simeq B\mathbf{Cob}_{d+1}^{P_2}.$$

We consider the functors

$$\mathbf{Cob}_{d+1} \xrightarrow{i} \mathbf{Cob}_{d+1}^P \xrightarrow{\beta_1} \mathbf{Cob}_{d-p},$$

where i is given by inclusion and β_1 sends a $(d+1)$-dimensional P-manifold W to the $(d-p)$-dimensional manifold $\beta_1 W$. We have the following theorem.

Theorem 1.3. *Passing to classifying spaces, the sequence of functors given above induces a homotopy fiber sequence*

$$B\mathbf{Cob}_{d+1} \xrightarrow{B(i)} B\mathbf{Cob}_{d+1}^P \xrightarrow{B(\beta_1)} B\mathbf{Cob}_{d-p}.$$

1.4. **Outline of paper.** This paper is structured as follows. Sections 2 and 3 are devoted to carefully defining P-manifolds and the different mapping spaces associated to them which include diffeomorphism groups and certain spaces of embeddings. In Section 4, we give a rigorous definition of the cobordism category. In Section 5, we give a recollection of sheaves and define the main sheaf $\mathbf{D}_{d+1}^P$ whose representing space is later shown to be weakly equivalent to $B\mathbf{Cob}_{d+1}^P$. In Section 6, we construct the spectrum $\mathsf{MT}_P(d+1)$. Section 7 is devoted to proving the weak homotopy equivalence $|\mathbf{D}_{d+1}^P| \simeq \Omega^{\infty-1}\mathsf{MT}_P(d+1)$. In Section 8, we complete the proofs of Theorems 1.1 and 1.3. Sections 9, 10, 11 and the appendix are devoted to the proofs of technical results used earlier in the paper.

To simplify the exposition we will only treat unoriented manifolds with Baas–Sullivan singularities modeled on a single fixed manifold P. One could easily adapt our proofs to derive a corresponding theorem for P-manifolds with arbitrary tangential structure.

2. Manifolds with singularities

We begin with a definition of manifolds with Baas–Sullivan singularities modeled on a fixed manifold P. Fix once and for all a closed, smooth manifold P and let p denote the dimension of P. Throughout the paper, we will let $\mathbb{R}_+$ denote the half-open interval $[0, \infty)$.

Definition 2.1. Let M be a d-dimensional smooth manifold with corners, equipped with the following extra structure:

(i) The boundary of M is given a decomposition $\partial M = \partial_0 M \cup \partial_1 M$ into a union of $(d-1)$-dimensional manifolds such that

$$\partial_0 M \cap \partial_1 M = \partial(\partial_0 M) = \partial(\partial_1 M)$$

is a closed $(d-2)$-dimensional manifold. We write $\partial_{0,1} M := \partial_0 M \cap \partial_1 M$.

(ii) There are embeddings

$$h_0 : \partial_0 M \times \mathbb{R}_+ \to M \quad \text{and} \quad h_1 : \partial_1 M \times \mathbb{R}_+ \to M$$

which satisfy:
(a) $h_0^{-1}(\partial_0 M) = \partial_0 M \times \{0\}$ and $h_1^{-1}(\partial_1 M) = \partial_1 M \times \{0\}$.
(b) $h_0(\partial_{0,1} M \times [0, \infty)) \subset \partial_1 M$ and $h_1(\partial_{0,1} M \times [0, \infty)) \subset \partial_0 M$.
(c) $h_0(h_1(x, t_1), t_0) = h_1(h_0(x, t_0), t_1)$ for all $(x, t_0, t_1) \in \partial_{0,1} M \times \mathbb{R}_+^2$.

(iii) There is a manifold $\beta_1 M$ and a diffeomorphism

$$\phi_1 : \partial_1 M \xrightarrow{\cong} \beta_1 M \times P.$$

We let $\beta_{0,1} M$ denote the boundary $\partial(\beta_1 M)$ and let

$$\phi_{0,1} : \partial_{0,1} M \xrightarrow{\cong} \beta_{0,1} M \times P$$

denote the diffeomorphism obtained by restricting ϕ_1 to $\partial_{0,1} M = \partial(\partial_1 M)$.

With the above conditions satisfied, the triple $(M, (\phi_1, \phi_{0,1}), (h_0, h_1))$ is called a *P-manifold.* The manifold $\beta_1 M$ is called the *Bockstein*, the pair of diffeomorphisms $(\phi_1, \phi_{0,1})$ is called the *structure map*, and the pair of embeddings (h_0, h_1) is called the *collar*.

Notation 2.2. When denoting a P-manifold we will usually drop the structure maps and collar from the notation and denote the P-manifold by its underlying manifold. We will denote $M := (M, (\phi_1, \phi_{0,1}), (h_0, h_1))$.

Let M be a P-manifold as described in the above definition. By setting

$$\partial_1(\partial_0 M) = \partial_{0,1} M, \quad \partial_0(\partial_0 M) = \varnothing, \quad \text{and} \quad \beta_1(\partial_0 M) = \beta_{0,1} M,$$

and restricting the structure map and collar associated to M, we obtain for $\partial_0 M$ the structure of a P-manifold. We call $\partial_0 M$ the *boundary* of M. If $\partial_0 M = \varnothing$, then M is said to be a P-manifold without boundary. If M is compact and $\partial_0 M = \varnothing$, then M is said to be a *closed P-manifold.*

We will need to consider maps from P-manifolds to arbitrary topological spaces.

Definition 2.3. If M is a P-manifold and X is a topological space, then a continuous map $f : M \to X$ is said to be a *P-map* if there exists a map $f_{\beta_1} : \beta_1 M \to X$ such that the restriction of f to $\partial_1 M$ factors as

(2) $$\partial_1 M \xrightarrow{\Phi} \beta_1 M \times P \xrightarrow{\text{proj}_{\beta_1 M}} \beta_1 M \xrightarrow{f_{\beta_1}} X.$$

If X is a smooth manifold, then a P-map $f : M \to X$ is said to be smooth if f is a smooth map when considering M as a smooth manifold with corners.

We will have to consider vector bundles over P-manifolds.

Definition 2.4. Let M be a P-manifold. Let $\pi : E \to M$, $\pi_{\beta_1} : E_{\beta_1} \to \beta_1 M$, and $\pi_P : E_P \to P$ be vector bundles and let

$$\hat{\phi}_E : E|_{\partial_1 M} \xrightarrow{\cong} (E_{\beta_1} \times E_P) \oplus \epsilon^1$$

be a vector bundle isomorphism that covers the structure map

$$\phi_1 : \partial_1 M \xrightarrow{\cong} \beta_1 M \times P$$

(the bundle on the right-hand side is assumed to be over $\beta_1 M \times P$). The pair $(E, \hat{\phi}_E)$ is said to be a P-*vector* bundle over M. We refer to $\hat{\phi}_E$ as the structure map.

Notation 2.5. When working with a P-vector bundle as in the previous definition, we will drop the bundle isomorphism $\hat{\phi}_1 : E|_{\partial_1 M} \to (E_{\beta_1} \times E_P) \oplus \epsilon^1$ from the notation and denote $E := (E, \hat{\phi}_1)$. We will always use the same Greek letter to denote the structure map and will always use the same notational convention to denote the auxiliary bundles $E_{\beta_1 M}$ and E_P.

Example 2.6. For any P-manifold, the tangent bundle $TM \to M$ naturally has the structure of a P-vector bundle as follows. The collar embedding $h_1 : \partial_1 M \times \mathbb{R}_+ \to M$ induces a bundle isomorphism

$$TM|_{\partial_1 M} \xrightarrow{\cong} T\partial_1 M \oplus \epsilon^1$$

which covers the identity on M. Using this bundle isomorphism, we obtain the bundle isomorphism

$$(3) \qquad TM|_{\partial_1 M} \xrightarrow{\cong} T\partial_1 M \oplus \epsilon^1 \xrightarrow[\cong]{d\phi_1 \oplus \mathrm{Id}_{\epsilon^1}} (T\beta_1 M \times TP) \oplus \epsilon^1,$$

where $d\phi_1$ denotes the differential of the structure map. In this way the bundle isomorphism (3) endows $TM \to M$ with the structure of a P-vector bundle.

Transversality will play an important role in the constructions used to prove the main theorem.

Definition 2.7. Let U be a smooth manifold, let $K \subset U$ be a submanifold, and let M be a P-manifold. A smooth P-map $f : M \to U$ is said to be *transverse* to the submanifold K if both f and the map $f_{\beta_1} : \beta_1 M \to U$ are transverse as smooth maps to K. In this case we write $f \pitchfork K$ to indicate transversality.

The following proposition is easy to verify.

Proposition 2.8. *Let U be a manifold of dimension r, $K \subset U$ a submanifold of dimension k, and let M be a P-manifold of dimension d. Let $f : M \to U$ be a smooth P-map transverse to K. Then the space $f^{-1}(K)$ is P-manifold of dimension $d + k - r$ with Bockstein given by $\beta_1(f^{-1}(K)) = f_{\beta_1}^{-1}(K)$.*

On a similar note, submersions will play an important role in the constructions used in the proof of the main theorem. If U is a smooth manifold and M is a P-manifold, a smooth P-map $f : M \to U$ is said to be a P-submersion if both f and f_{β_1} are submersions when treating M as a smooth manifold. It follows immediately from Proposition 2.8 that for $x \in U$, the space $f^{-1}(x)$ is a P-manifold of dimension $\dim(M) - \dim(U)$ with $\beta_1(f^{-1}(x)) = f_{\beta_1}^{-1}(x)$.

Example 2.9. Let M be a P-manifold and let X be a smooth manifold. Let $\pi : M \to X$ be a P-submersion. Denote by $T^{\pi}M \to M$ the sub-vector bundle of the tangent bundle TM given by the kernel of the differential of the submersion π. Denote by $T^{\pi}\beta_1 M \to \beta_1 M$ the sub-bundle of $T\beta_1 M$ given by the kernel of the submersion $\pi_{\beta_1} : \beta_1 M \to X$. The factorization from (2) of the restriction of π to $\partial_1 M$ implies that there is an isomorphism

$$T^{\pi}M|_{\partial_1 M} \xrightarrow{\cong} (T^{\pi}\beta_1 M \times TP) \oplus \epsilon^1$$

that covers the structure map $\phi_1 : \partial_1 M \xrightarrow{\cong} \beta_1 M \times P$. It follows that the kernel bundle of any P-submersion has the structure of a P-vector bundle.

We are interested in the cobordism theory of P-manifolds. For this we make the following definition.

Definition 2.10. Let M_a and M_b be closed P-manifolds of dimension d and let W be a compact P-manifold of dimension $d + 1$. If $\partial_0 W = M_a \sqcup M_b$ then the triple $(W; M_a, M_b)$ is said to be a P-manifold cobordism triple. Two closed P-manifolds M_a and M_b of the same dimension are said to be cobordant if there exists a $(d + 1)$-dimensional P-manifold W such that $\partial_0 W = M_a \sqcup M_b$.

3. Mapping spaces

We will need to consider certain spaces of maps between P-manifolds.

3.1. **Diffeomorphisms.** For what follows, let M_a and M_b be P-manifolds. For $i = 0, 1$, we denote by h_i^a and h_i^b the collar embeddings associated to M_a and M_b. We denote by ϕ_1^a and ϕ_1^b the structure maps.

Definition 3.2. A smooth map $f : M_a \to M_b$ is said to be a P-morphism if the following conditions are satisfied:

(i) $f(\partial_0 M_a) \subset \partial_0 M_b$ and $f(\partial_1 M_a) \subset \partial_1 M_b$.
(ii) There exists a real number $\varepsilon > 0$ such that

$$f(h_0^a(x, t)) = h_0^b(f(x), t) \quad \text{for } (x, t) \in \partial_0 M_a \times [0, \varepsilon),$$
$$f(h_1^a(y, s)) = h_1^b(f(y), s) \quad \text{for } (y, s) \in \partial_1 M_a \times [0, \varepsilon).$$

(iii) There exists a smooth map $f_{\beta_1} : \beta_1 M_a \to \beta_1 M_b$ such that the restriction of f to $\partial_1 M_a$ has the factorization

$$\partial_1 M_a \xrightarrow[\cong]{\phi_1^a} \beta_1 M_a \times P \xrightarrow{f_{\beta_1} \times \mathrm{Id}_P} \beta_1 M_b \times P \xrightarrow[\cong]{(\phi^b)^{-1}} \partial_1 M_b.$$

We denote by $C^\infty_P(M_a, M_b)$ the space of P-morphisms $M_a \to M_b$, topologized as a subspace of the space of smooth maps $M_a \to M_b$, in the C^∞-topology. For a P-manifold M, we let $C^\infty_P(M)$ denote the space $C^\infty_P(M, M)$. We will need to consider diffeomorphisms of P-manifolds as well.

Definition 3.3. A smooth map between P-manifolds $f : M_a \to M_b$ is said to be a *P-diffeomorphism* if it is both a diffeomorphism as a map of smooth manifolds and a P-morphism, i.e., it satisfies all conditions of Definition 3.2.

We denote by $\mathrm{Diff}_P(M_a, M_b)$ the space of P-diffeomorphisms from M_a to M_b, where $\mathrm{Diff}_P(M_a, M_b)$ is topologized as a subspace of $C^\infty_P(M_a, M_b)$. For a P-manifold M, we let $\mathrm{Diff}_P(M)$ denote the space $\mathrm{Diff}_P(M, M)$ of self-P-diffeomorphisms $M \to M$. The space $\mathrm{Diff}_P(M)$ has the structure of a topological group with product given by composition.

Proposition 3.4. *Let M_a and M_b be two compact P-manifolds. Then $\mathrm{Diff}_P(M_a, M_b)$ is an open subset of $C^\infty_P(M_a, M_b)$.*

Proof. Denote by $C^\infty_\partial(M_a, M_b)$ the space of smooth maps $M_a \to M_b$ that satisfy conditions (i) and (ii) of Definition 3.2 but which may fail to satisfy condition (iii). The space $C^\infty_\partial(M_a, M_b)$ is topologized in the C^∞-topology. Similarly, we denote by $\mathrm{Diff}_\partial(M_a, M_b)$ the subspace of $C^\infty_\partial(M_a, M_b)$ which consists of all smooth maps $f : M_a \to M_b$ satisfying conditions (i) and (ii) of Definition 3.2 such that f is also a diffeomorphism of smooth manifolds. It follows from [9, Thm. 1.7] that $\mathrm{Diff}_\partial(M_a, M_b) \subset C^\infty_\partial(M_a, M_b)$ is an open subset. It follows from

$$\mathrm{Diff}_P(M_a, M_b) = \mathrm{Diff}_\partial(M_a, M_b) \cap C^\infty_P(M_a, M_b)$$

that $\mathrm{Diff}_P(M_a, M_b)$ is an open subset of $C^\infty_P(M_a, M_b)$. $\qquad\square$

3.5. **Embeddings.** We will need to consider certain spaces of embeddings of P-manifolds. Fix once and for all a smooth embedding

$$i_P : P \to \mathbb{R}^{p+m} \tag{4}$$

with $m > p$. We will use this choice of embedding in all of our constructions to come. Here and subsequently, we will use the following notational convention. For $n \in \mathbb{N}$, we will denote

$$\bar{n} := n - p - m - 1.$$

For what follows, let X be a smooth manifold without boundary. For a positive integer n, let

$$q : \mathbb{R}_+ \times X \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m} \to X \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$$

be the "permutation" map defined by $q(t, x, y, z) = (x, t, y, z)$.

Definition 3.6. Let M be a P-manifold with $\partial_0 M = \varnothing$ (we allow for M to be noncompact). We define $\mathcal{E}_{P,n}(M, X)$ to be the space of smooth embeddings

$$\varphi : M \to X \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$$

which satisfy the following conditions:

(i) $\varphi(\partial_1 M) \subset X \times \{0\} \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$.

(ii) There exists a real number $\varepsilon > 0$ such that if $(y, s) \in \partial_1 M \times [0, \varepsilon)$, then
$$\varphi(h_1(y, s)) = q(\varphi(y), s).$$

(iii) There exists a map $\varphi_{\beta_1} : \beta_1 M \to X \times \mathbb{R}^{\bar{n}}$ such that the restriction of φ to $\partial_1 M$ has the factorization
$$\partial_1 M \xrightarrow[\cong]{\phi_1} \beta_1 M \times P \xrightarrow{\varphi_{\beta_1} \times i_P} (X \times \mathbb{R}^{\bar{n}}) \times \mathbb{R}^{p+m}.$$

The space $\mathcal{E}_{P,n}(M, X)$ is topologized as a subspace of the space of smooth maps from M to $X \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$ in the C^∞-topology. We let $\mathcal{E}_{P,n}(M)$ denote the space $\mathcal{E}_{P,n}(M, \mathrm{pt.})$, i.e. the space of embeddings $M \to \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$ that satisfy the conditions given above.

By the following proposition, we are justified in excluding the embedding (4) from the notation.

Proposition 3.7. *For any P-manifold M with $\partial_0 M = \varnothing$, smooth manifold X, and positive integer n, the homeomorphism type of the space $\mathcal{E}_{P,n}(M, X)$ does not depend on the embedding $P \hookrightarrow \mathbb{R}^{p+m}$ used to define it.*

Proof. Let $i'_P : P \hookrightarrow \mathbb{R}^{p+m}$ be another embedding and let $\mathcal{E}_{P',n}(M, X)$ denote the space of embeddings $M \to X \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$ that satisfy all conditions from Definition 3.6 with respect to the embedding i'_P. Since $m > p$, there exists an isotopy $\psi_t : P \to \mathbb{R}^{p+m}$ through embeddings such that $\psi_0 = i_P$ and $\psi_1 = i'_P$. By the *isotopy extension theorem* [9, Thm. 1.3] there exists a diffeotopy (isotopy through diffeomorphisms) $\Psi_t : \mathbb{R}^{p+m} \to \mathbb{R}^{p+m}$ such that $\Psi_0 = \mathrm{Id}_{\mathbb{R}^{p+m}}$ and $\Psi_t \circ i_P = \psi_t$ for all $t \in [0, 1]$. Denote by Φ the diffeomorphism of $X \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$ given by the formula
$$\Phi(x, t, y, z) = (x, t, y, \Psi_1(z)) \quad \text{for } x \in X, \, t \in \mathbb{R}_+, \, y \in \mathbb{R}^{\bar{n}}, \, z \in \mathbb{R}^{p+m}.$$
We define a map

(5) $$\mathcal{E}_{P,n}(M, X) \to \mathcal{E}_{P',n}(M, X), \quad \varphi \mapsto \Phi \circ \varphi.$$

The inverse to (5) is given by the formula $\varphi \mapsto \Phi^{-1} \circ \varphi$. Thus (5) is a homeomorphism. This concludes the proof. $\qquad \square$

For each n there is a natural embedding $\mathcal{E}_{P,n}(M) \hookrightarrow \mathcal{E}_{P,n+1}(M)$. We then define
$$\mathcal{E}_P(M) := \operatorname*{colim}_{n \to \infty} \mathcal{E}_{P,n}(M).$$

Theorem 3.8. *Let M be a closed P-manifold, that is, M is compact and $\partial_0 M = \varnothing$. Then the space $\mathcal{E}_P(M)$ is weakly contractible.*

Proof. For each $n \in \mathbb{N}$, denote by $\mathcal{E}_{\partial,n}(M)$ the space of embeddings $M \to \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$ that satisfy conditions (i) and (ii) of Definition 3.6 but which may fail to factor as a product on $\partial_1 M$ as in condition (iii) of Definition 3.6. We then denote $\mathcal{E}_\partial(M) := \operatorname{colim}_{n \to \infty} \mathcal{E}_{\partial,n}(M)$. Let
$$r_{\partial_1} : \mathcal{E}_\partial(M) \to \mathrm{Emb}(\partial_1 M, \mathbb{R}^\infty \times \mathbb{R}^{p+m})$$

be the map defined by restricting embeddings to the boundary. Define

$$r_{\beta_1} : \mathcal{E}_P(M) \to \mathrm{Emb}(\beta_1 M, \mathbb{R}^\infty), \quad \varphi \mapsto \varphi_{\beta_1}.$$

Consider the map

$$T_{i_P} : \mathrm{Emb}(\beta_1 M, \mathbb{R}^\infty) \to \mathrm{Emb}(\partial_1 M, \mathbb{R}^\infty \times \mathbb{R}^{p+m})$$

defined by sending an embedding $\varphi : \beta_1 M \to \mathbb{R}^\infty$ to the embedding given by the composition

$$\partial_1 M \xrightarrow[\cong]{\phi_1} \beta_1 M \times P \xrightarrow{\varphi \times i_P} \mathbb{R}^\infty \times \mathbb{R}^{p+m}.$$

It follows immediately from Definition 3.6 that the diagram

$$(6) \qquad \begin{array}{ccc} \mathcal{E}_P(M) & \longrightarrow & \mathcal{E}_\partial(M) \\ \Big\downarrow{\scriptstyle r_{\beta_1}} & & \Big\downarrow{\scriptstyle r_{\partial_1}} \\ \mathrm{Emb}(\beta_1 M, \mathbb{R}^\infty) & \xrightarrow{T_{i_P}} & \mathrm{Emb}(\partial_1 M, \mathbb{R}^\infty \times \mathbb{R}^{p+m}) \end{array}$$

is cartesian, where the top horizontal map is the inclusion. By the main theorem of [10], the restriction map r_{∂_1} is a locally trivial fiber bundle. It follows from this that diagram (6) is *homotopy cartesian*. By [7, Thm. 2.7] the spaces $\mathcal{E}_\partial(M)$, $\mathrm{Emb}(\beta_1 M, \mathbb{R}^\infty)$, and $\mathrm{Emb}(\partial_1 M, \mathbb{R}^\infty \times \mathbb{R}^{p+m})$ are all weakly contractible. This together with the fact that (6) is homotopy cartesian implies that $\mathcal{E}_P(M)$ is homotopy cartesian as well. This concludes the proof. $\qquad \square$

We now define similar mapping spaces for P-manifold cobordism triples. Let X be a smooth manifold without boundary as above. Let

$$q_0 : [0,1] \times X \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m} \to X \times [0,1] \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{p+m},$$
$$q_1 : \mathbb{R}_+ \times X \times [0,1] \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m} \to X \times [0,1] \times \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{p+m}$$

be the "permutation" maps defined by

$$q_0(t, x, s, y, z) = (x, t, s, y, z) \quad \text{and} \quad q_1(s, x, t, y, z) = (x, t, s, y, z)$$

where $x \in X$, $t \in [0,1]$, $s \in \mathbb{R}_+$, $y \in \mathbb{R}^n$, and $z \in \mathbb{R}^{p+m}$.

Definition 3.9. Let $(W; M_a, M_b)$ be a P-manifold bordism triple and let n be a positive integer. We define $\mathcal{E}_{P,n}((W; M_a, M_b), X)$ to be the space of smooth embeddings

$$\varphi : W \to X \times [0,1] \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$$

that satisfy the following conditions:

(i) The following containments hold:

$$\varphi(M_a) \subset X \times \{0\} \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m},$$
$$\varphi(M_b) \subset X \times \{1\} \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m},$$
$$\varphi(\partial_1 W) \subset X \times [0,1] \times \{0\} \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}.$$

(ii) There exists a positive real number ε such that

$$\varphi(h_0(x,t)) = q_0(t,\varphi(x)) \qquad \text{if } (x,t) \in M_a \times [0,\varepsilon),$$
$$\varphi(h_0(x,t)) = q_0(1-t,\varphi(x)) \quad \text{if } (x,t) \in M_b \times [0,\varepsilon),$$
$$\varphi(h_1(y,s)) = q_1(s,\varphi(y)) \qquad \text{if } (y,s) \in \partial_1 W \times [0,\varepsilon).$$

(iii) There exists a map $\varphi_{\beta_1} : \beta_1 W \to X \times [0,1] \times \{0\} \times \mathbb{R}^{\bar{n}}$ such that the restriction of φ to $\partial_1 W$ has the factorization

$$\partial_1 W \xrightarrow[\cong]{\phi_1} \beta_1 W \times P \xrightarrow{\varphi_{\beta_1} \times i_P} (X \times [0,1] \times \{0\} \times \mathbb{R}^{\bar{n}}) \times \mathbb{R}^{p+m}.$$

The space $\mathcal{E}_{P,n}((W; M_a, M_b), X)$ is topologized in the C^∞-topology. We let $\mathcal{E}_{P,n}(W; M_a, M_b)$ denote the space $\mathcal{E}_{P,n}((W; M_a, M_b), \mathrm{pt.})$. We then set

$$\mathcal{E}_P(W; M_a, M_b) := \operatorname*{colim}_{n \to \infty} \mathcal{E}_{P,n}(W; M_a, M_b).$$

Remark 3.10. There are analogs of Proposition 3.7 and Theorem 3.8 for the spaces $\mathcal{E}_{P,n}((W; M_a, M_b), X)$ of embeddings of P-bordisms. They are proven in the same way.

The terminology given in the next definition will be useful later on when we define the cobordism category of P-manifolds and related constructions.

Definition 3.11. Let X be a smooth manifold without boundary. Let M be a P-manifold with $\partial_0 M = \varnothing$ that is embedded as a submanifold of $X \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$ for some n, such that the inclusion map

$$M \hookrightarrow X \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$$

is an element of the space $\mathcal{E}_{P,n}(M, X)$. Then M is called a *P-submanifold*. Similarly, let $(W; M_a, M_b)$ be a P-manifold bordism triple with W embedded as a submanifold of $X \times [0,1] \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$ such that the inclusion

$$W \hookrightarrow X \times [0,1] \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$$

is an element of the space $\mathcal{E}_{P,n}((W; M_a, M_b), X)$. Then W is called a *P-sub-cobordism*.

Remark 3.12 (Normal bundles)**.** Let $M \subset X \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$ be a closed P-submanifold. Let $\pi : M \to X$ denote the restriction of the projection $X \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m} \to X$ onto M. It follows immediately from condition (iii) of Definition 3.6 that π is a smooth P-map (see Definition 2.3). Denote by $N \to M$ the normal bundle. The factorization $\partial_1 M = \beta_1 M \times i_P(P)$ for $\beta_1 M \subset X \times \{0\} \times \mathbb{R}^{\bar{n}}$ and $i_P(P) \subset \mathbb{R}^{p+m}$ implies that the restriction of N to $\partial_1 M$ has the factorization

$$N|_{\partial_1 M} = (N_{\beta_1} \times N_P) \oplus \epsilon^1,$$

where $N_{\beta_1} \to \beta_1 M$ and $N_P \to i_P(P)$ are the normal bundles for $\beta_1 M$ and $i_P(P)$, respectively. It follows that the normal bundle of a P-submanifold has the structure of a P-vector bundle as in Definition 2.4.

3.13. *P*-manifold fiber bundles. Let M be a closed P-manifold. Consider the space $\mathcal{E}_P(M)$. There is a continuous group action

$$\operatorname{Diff}_P(M) \times \mathcal{E}_P(M) \to \mathcal{E}_P(M), \quad (g, \varphi) \mapsto \varphi \circ g.$$

It is clear that this action is a free action. We let $\mathcal{M}_P(M)$ denote the orbit space $\mathcal{E}_P(M)/\operatorname{Diff}_P(M)$. Similarly, if $(W; M_a, M_b)$ is a P-manifold bordism triple, there is a continuous group action

$$\operatorname{Diff}_P(W; M_a, M_b) \times \mathcal{E}(W; M_a, M_b) \to \mathcal{E}(W; M_a, M_b), \quad (g, \varphi) \mapsto \varphi \circ g$$

which is clearly a free action. We let $\mathcal{M}_P(W; M_a, M_b)$ denote the orbit space $\mathcal{E}(W; M_a, M_b)/\operatorname{Diff}_P(W; M_a, M_b)$. We have the following theorem whose proof we defer to Appendix A.

Theorem 3.14. *The quotient maps*

$$\mathcal{E}_P(M) \to \mathcal{M}_P(M) \quad and \quad \mathcal{E}(W; M_a, M_b) \to \mathcal{M}_P(W; M_a, M_b)$$

are locally trivial fiber bundles.

Remark 3.15. Combining the above theorem with Theorem 3.8 (and the corresponding version of Theorem 3.8 of P-cobordisms), we have weak homotopy equivalences

$$\operatorname{BDiff}_P(M) \simeq \mathcal{M}_P(M), \quad \operatorname{BDiff}_P(W; M_a, M_b) \simeq \mathcal{M}_P(W; M_a, M_b),$$

where $\operatorname{BDiff}_P(M)$ and $\operatorname{BDiff}_P(W; M_a, M_b)$ are the *classifying spaces* of the topological groups $\operatorname{Diff}_P(M)$ and $\operatorname{Diff}_P(W; M_a, M_b)$.

Using the Borel construction, we define

$$\hat{\mathcal{E}}_P(M) := \mathcal{E}_P(M) \times_{\operatorname{Diff}_P(M)} M.$$

The standard projection yields a fiber bundle $\hat{\mathcal{E}}_P(M) \to \mathcal{M}_P(M)$ with fiber M and structure group $\operatorname{Diff}_P(M)$. This fiber bundle comes with a natural embedding

$$\hat{\mathcal{E}}_P(M) \hookrightarrow \mathcal{M}_P(M) \times \mathbb{R}_+ \times \mathbb{R}^\infty \times \mathbb{R}^{p+m}$$

defined by the formula $(\varphi, x) \mapsto ([\varphi], \varphi(x))$. Now, if $f : X \to \mathcal{M}_P(M)$ is a smooth map (when treating $\mathcal{M}_P(M)$ as a *Banach manifold*) from a smooth manifold X, then the pullback

$$f^* \hat{\mathcal{E}}_P(M) = \left\{ (x, v) \in X \times (\mathbb{R}_+ \times \mathbb{R}^\infty \times \mathbb{R}^{p+m}) \mid (f(x), v) \in \hat{\mathcal{E}}_P(M) \right\}$$

is a smooth $\dim(M) + \dim(X)$-dimensional P-submanifold

$$E \subseteq X \times \mathbb{R}_+ \times \mathbb{R}^\infty \times \mathbb{R}^{p+m}$$

such that the projection $\pi : E \to X$ is a fiber bundle with structure group $\operatorname{Diff}_P(M)$ and fiber M. The fiber over x, denoted by E_x, has the structure of P-manifold diffeomorphic to M such that the inclusion map $E_x \hookrightarrow \mathbb{R}_+ \times \mathbb{R}^\infty \times \mathbb{R}^{p+m}$ is an element of the space $\mathcal{E}_P(M)$. We have the following lemma.

Lemma 3.16. *Let X be a smooth manifold without boundary and let M be a closed P-manifold. There is a one-to-one correspondence between smooth maps $X \to \mathcal{M}_P(M)$ and closed P-submanifolds $E \subset X \times \mathbb{R}_+ \times \mathbb{R}^\infty \times \mathbb{R}^{p+m}$, such that the projection $\pi : E \to X$ is a smooth fiber bundle with fiber M and structure group $\mathrm{Diff}_P(M)$.*

Proof. Let $f : X \to \mathcal{M}_P(M)$ be a smooth map. By the discussion above, the pullback $f^*\hat{\mathcal{E}}_P(M)$ comes with a canonical embedding into $X \times \mathbb{R}_+ \times \mathbb{R}^\infty \times \mathbb{R}^{p+m}$ such that the projection onto X is a fiber bundle with structure group $\mathrm{Diff}_P(M)$ and fiber M. This describes one direction of the correspondence.

In the other direction, let $E \subset X \times \mathbb{R}_+ \times \mathbb{R}^\infty \times \mathbb{R}^{p+m}$ be a P-submanifold such that the projection $\pi : E \to X$ onto X is a fiber bundle with fiber M and structure group $\mathrm{Diff}_P(M)$. We obtain a map

$$(7) \qquad X \to \mathcal{M}_P(M), \quad x \mapsto E_x,$$

where $E_x \subset \{x\} \times \mathbb{R}_+ \times \mathbb{R}^{\bar{n}} \times \mathbb{R}^{p+m}$ is the fiber of the projection π over $x \in X$. It follows easily that (7) is the inverse to the correspondence given by $f \mapsto f^*(\hat{\mathcal{E}}_P(M))$ described above. $\qquad\square$

We have a similar lemma for P-manifold cobordism triples which is proven in the same way.

Lemma 3.17. *Let X be a smooth manifold without boundary. Let $(W; M_a, M_b)$ be a P-manifold cobordism triple. There is a one-to-one correspondence between smooth maps $X \to \mathcal{M}_P(W; M_a, M_b)$ and P-subcobordisms $E \subset X \times [0, 1] \times \mathbb{R}_+ \times \mathbb{R}^\infty \times \mathbb{R}^{p+m}$ such that the projection $\pi : E \to X$ is a smooth fiber bundle with fiber W and structure group $\mathrm{Diff}_P(W; M_a, M_b)$.*

4. The cobordism category of P-manifolds

4.1. The cobordism category $\mathbf{Cob}^P_{d+1}$.
Let $i_P : P \to \mathbb{R}^{p+m}$ be the embedding specified in (4) used to construct the spaces of embeddings in the previous section. We now give a rigorous construction of the category $\mathbf{Cob}^P_{d+1}$ that was discussed in the introduction.

An object of $\mathbf{Cob}^P_{d+1}$ is a pair (M, a) where $a \in \mathbb{R}$ and

$$M \subseteq \{a\} \times \mathbb{R}_+ \times \mathbb{R}^\infty \times \mathbb{R}^{p+m}$$

is a closed d-dimensional P-submanifold. The space of objects is topologized by the identification

$$(8) \qquad \mathcal{O}b(\mathbf{Cob}^P_{d+1}) = \bigsqcup_M (\mathcal{M}_P(M) \times \mathbb{R}),$$

where the disjoint union is taken over the diffeomorphism classes of closed P-manifolds of dimension d.

A non-identity morphism of $\mathbf{Cob}^P_{d+1}$ from (M_a, a) to (M_b, b) is a triple $(W; a, b)$ with

$$(a, b) \in \mathbb{R}^2_< := \{(a, b) \in \mathbb{R}^2 \mid a < b\}$$

and $W \subseteq [a,b] \times \mathbb{R}_+ \times \mathbb{R}^\infty \times \mathbb{R}^{p+m}$ is a $(d+1)$-dimensional, compact P-submanifold such that

$$M_a = W \cap (\{a\} \times \mathbb{R}_+ \times \mathbb{R}^\infty \times \mathbb{R}^{p+m})$$

and

$$M_b = W \cap (\{b\} \times \mathbb{R}_+ \times \mathbb{R}^\infty \times \mathbb{R}^{p+m}).$$

It follows that $(W; M_a, M_b)$ is a P-manifold bordism triple. The morphisms $(W_1; a, b)$ and $(W_2; b, c)$ can be composed if

$$W_1 \cap (\{b\} \times \mathbb{R}^\infty \times \mathbb{R}_+ \times \mathbb{R}^{p+m}) = W_2 \cap (\{b\} \times \mathbb{R}^\infty \times \mathbb{R}_+ \times \mathbb{R}^{p+m}).$$

In this case the composition is given by $(W_1 \cup W_2; a, c)$. The collars from condition (ii) of Definition 3.6 ensure that this union is a smooth submanifold with a canonical smooth structure induced by the smooth structure on the ambient space. The identity morphisms are identified with the space of objects. The space of morphisms is topologized by the identification

$$(9) \quad \mathcal{M}or(\mathbf{Cob}_{d+1}^P) = \mathcal{O}b(\mathbf{Cob}_{d+1}^P) \sqcup \Big(\bigsqcup_{(W; M_a, M_b)} \big(\mathbb{R}_<^2 \times \mathcal{M}_P(W; M_a, M_b)\big) \Big).$$

The disjoint union (on the right) is taken over the diffeomorphism classes of $(d+1)$-dimensional P-manifold bordism triples. It follows easily from (8) and (9) that composition and the source and target maps are continuous. Thus $\mathbf{Cob}_{d+1}^P$ defined in this way is a topological category.

5. Sheaf models

5.1. A recollection from [11] of sheaves. Let $\mathcal{X}$ denote the category of smooth manifolds without boundary with morphisms given by smooth maps. By a sheaf on $\mathcal{X}$ we mean a contravariant functor $\mathcal{F}$ from $\mathcal{X}$ to the category of sets which satisfies the following condition. For any open covering $\{U_i \mid i \in \Lambda\}$ of some $X \in \mathcal{O}b(\mathcal{X})$, and every collection $s_i \in \mathcal{F}(U_i)$ satisfying

$$s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j} \quad \text{for all } i, j \in \Lambda,$$

there is a unique $s \in \mathcal{F}(X)$ such that $s|_{U_i} = s_i$ for all $i \in \Lambda$.

Definition 5.2. Let $\mathcal{F}$ be a sheaf on $\mathcal{X}$. Two elements s_0 and s_1 of $\mathcal{F}(X)$ are said to be concordant if there exists $s \in \mathcal{F}(X \times \mathbb{R})$ that agrees with $\mathrm{pr}^*(s_0)$ in an open neighborhood of $X \times (-\infty, 0]$ and agrees with $\mathrm{pr}^*(s_1)$ in an open neighborhood of $X \times [1, \infty)$, where $\mathrm{pr} : X \times \mathbb{R} \to X$ is the projection onto the first factor.

We denote the set of concordance classes of $\mathcal{F}(X)$ by $\mathcal{F}[X]$. The correspondence $X \mapsto \mathcal{F}[X]$ is clearly functorial in X.

Definition 5.3. For a sheaf $\mathcal{F}$ we define the *representing space* $|\mathcal{F}|$ to be the geometric realization of the simplicial set given by the formula $k \mapsto \mathcal{F}(\triangle_e^k)$ where

$$\triangle_e^k := \big\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} \mid \sum x_i = 1\big\}$$

is the standard extended k-simplex.

From this definition it is easy to see that any map of sheaves $\mathcal{F} \to \mathcal{G}$ induces a map $|\mathcal{F}| \to |\mathcal{G}|$ between the representing spaces.

Definition 5.4. Let $\mathcal{F}$ be a sheaf on $\mathcal{X}$. Let $A \subset X$ be a closed subset, and let s be a germ near A, i.e., $s \in \operatorname{colim}_U \mathcal{F}(U)$ with U ranging over all open sets containing A in X. Then we define $\mathcal{F}(X, A; s)$ to be the set of all $t \in \mathcal{F}(X)$ whose germ near A coincides with s. Then two elements t_0 and t_1 are concordant *relative* to A and s, if they are related by a concordance whose germ near A is the constant concordance equal to s. The set of such relative concordance classes is denoted by $\mathcal{F}[X, A; s]$.

Any element $z \in \mathcal{F}(\text{pt.})$ determines a point in $|\mathcal{F}|$ which we also denote by z. For any $X \in \mathcal{Ob}(\mathcal{X})$, such an element $z \in \mathcal{F}(\text{pt.})$ determines an element, which we give the same name, $z \in \mathcal{F}(X)$, by pulling back by the constant map. In [11, Prop. 2.17] it is proven that there is a natural bijection of sets

$$(10) \qquad\qquad [(X, A), (|\mathcal{F}|, z)] \cong \mathcal{F}[X, A; z].$$

Here the set on the left-hand side is the set of homotopy classes of maps of pairs. The non-relative case of this isomorphism with A the empty set holds as well. Using these observations, we define the homotopy groups of a sheaf by setting

$$\pi_n(\mathcal{F}, z) := \mathcal{F}[S^n, \text{pt.}; z].$$

By (10) we get $\pi_n(\mathcal{F}, z) \cong \pi_n(|\mathcal{F}|, z)$ for any choice of $z \in \mathcal{F}(\text{pt.})$. Using this definition of homotopy group, a map of sheaves is said to be a *weak equivalence* if it induces an isomorphisms on all homotopy groups.

5.5. The sheaf $\mathbf{D}_{d+1}^P$. In this section we define a sheaf $\mathbf{D}_{d+1}^P$ on $\mathcal{X}$. It will be seen in Section 8 that the representing space $|\mathbf{D}_{d+1}^P|$ is weakly homotopy equivalent to the classifying space $B\mathbf{Cob}_{d+1}^P$.

For what follows, let $i_p : P \to \mathbb{R}^{p+m}$ be the embedding specified in (4) that was used in the construction of $\mathbf{Cob}_{d+1}^P$ and the mapping spaces of Section 3.5. For an integer n we will use the same notation

$$\bar{n} = n - p - m - 1$$

as was used in the previous sections. Before defining $\mathbf{D}_{d+1}^P$ we must fix some more terminology and notation.

Let d and n be nonnegative integers and let $X \in \mathcal{X}$. We will need to consider $(d + 1)$-dimensional, P-submanifolds

$$W \subset X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$$

with $\partial_0 W = \varnothing$, where W is not assumed to be compact. Recall that this means the following:

(i) $\partial_1 W$ is embedded in $X \times \mathbb{R} \times \{0\} \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$ with a collar as in condition (ii) of Definition 3.6.

(ii) There is the factorization $\partial_1 W = \beta_1 W \times i_P(P)$ where $\beta_1 W \subset X \times \mathbb{R} \times \{0\} \times \mathbb{R}^{d+\bar{n}}$ is a submanifold.

In other words, the inclusion map $W \hookrightarrow X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$ is an element of the space $\mathcal{E}_{P,d+n}(X \times \mathbb{R})$.

Denote by

$$(\pi, f) : W \to X \times \mathbb{R}$$

the restriction of the projection $X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m} \to X \times \mathbb{R}$ to W. It follows from Definition 2.3 that

$$\pi : W \to X, \quad f : W \to \mathbb{R} \quad \text{and} \quad (\pi, f) : W \to X \times \mathbb{R}$$

are all smooth P-maps. Notice that if $K \subset X$ is a submanifold and π is transverse to K, then

$$\pi^{-1}(K) \subset K \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$$

is a P-submanifold. We are now ready to define $\mathbf{D}_{d+1}^P$.

Definition 5.6. Let $X \in \mathcal{Ob}(\mathcal{X})$. For nonnegative integers n and d we define $\mathbf{D}_{d+1,n}^P(X)$ to be the set of $(d + 1 + \dim(X))$-dimensional P-submanifolds

$$W \subset X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m},$$

with $\partial_0 W = \varnothing$, which satisfy the following:

(i) The map $\pi : W \to X$ is a P-submersion.
(ii) The map $(\pi, f) : W \to X \times \mathbb{R}$ is a proper P-map (recall that a map is proper if the preimage of any compact subset is compact).

Let $X, Y \in \mathcal{X}$ and let $f : X \to Y$ be a smooth map. If $W \in \mathbf{D}_{d+1,n}^P(Y)$, then the pullback

$$f^*(W) = \{(x, w) \in X \times W \mid f(x) = \pi(w)\}$$

naturally embeds in $X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$ so that the projection onto X is a submersion, and thus the pullback $f^*(W)$ determines an element of $\mathbf{D}_{d+1,n}^P(X)$ (see the discussion of pullbacks in [6, §2.2] for details). The correspondence $W \mapsto f^*W$ yields a map $\mathbf{D}_{d+1,n}^P(Y) \to \mathbf{D}_{d+1,n}^P(X)$. In this way, it follows that the assignment $X \mapsto \mathbf{D}_{d+1,n}^P(X)$ is a contravariant functor from $\mathcal{X}$ to the category of sets. It is easy to verify that this functor satisfies the sheaf condition from Section 5.1.

To eliminate dependence on n, we define

$$\mathbf{D}_{d+1}^P := \operatorname*{colim}_{n \to \infty} \mathbf{D}_{d+1,n}^P,$$

where the colimit is understood to be taken in the category of sheaves on $\mathcal{X}$. Concretely, $\mathbf{D}_{d+1}^P(X)$ is the set of all $(d+1)$-dimensional P-submanifolds $W \subset X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\infty} \times \mathbb{R}^{p+m}$ satisfying conditions (i) and (ii) of Definition 5.6, such that for any compact subset $K \subset X$ there exists $n \in \mathbb{N}$ such that

$$\pi^{-1}(K) \subset K \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$$

(cp. [6, Def. 3.3]). It follows from this characterization that for each nonnegative integer k, the natural map $\mathrm{colim}_{n \to \infty} \mathbf{D}^P_{d+1,n} \to \mathbf{D}^P_{d+1}$ induces an isomorphism on all homotopy groups and thus there is a homotopy equivalence

$$|\mathbf{D}^P_{d+1}| \simeq \mathrm{colim}_{n \to \infty} |\mathbf{D}^P_{d+1,n}|.$$

The following lemma is proven in the same way as [11, Lem. 2.20].

Lemma 5.7. *For all $X \in \mathcal{X}$, every concordance class in $\mathbf{D}^P_{d+1,n}(X)$ has a representative*

$$W \subset X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$$

such that the map $f : W \to \mathbb{R}$ is a bundle projection. Thus there is a diffeomorphism $W \cong f^{-1}(0) \times \mathbb{R}$.

6. A cofiber of Thom-spectra

6.1. **The spectrum $\mathsf{MT}_P(d+1)$.** In this section we define a spectrum $\mathsf{MT}_P(d+1)$ as the cofiber of a map between $\Sigma^{-1}\mathsf{MT}(d-p)$ and $\mathsf{MT}(d+1)$, where $\mathsf{MT}(d+1)$ is the spectrum defined in [6]. We use the same notation for Grassmannian manifolds and their canonical bundles as in [6].

Let $i_P : P \hookrightarrow \mathbb{R}^{p+m}$ be the embedding from (4). Denote by $G(p,m)$ the Grassmannian manifold of p-dimensional vector subspaces of $\mathbb{R}^{p+m}$. Denote by $U_{p,m} \to G(p,m)$ the canonical vector bundle (which has fiber dimension p) and denote by $U^\perp_{p,m} \to G(p,m)$ the orthogonal complement to $U_{p,m}$, which has fiber dimension m. The normal bundle $N_P \to P$ associated to the embedding $i_P : P \hookrightarrow \mathbb{R}^{p+m}$ has Gauss map

$$(11) \qquad \begin{array}{ccc} N_P & \xrightarrow{\hat{\gamma}} & U^\perp_{p,m} \\ \downarrow & & \downarrow \\ P & \xrightarrow{\gamma} & G(p,m) \end{array}$$

which induces a map of the Thom spaces, $\mathsf{Th}(\hat{\gamma}) : \mathsf{Th}(N_p) \to \mathsf{Th}(U^\perp_{p,m})$. Fix an embedding of the normal bundle N_P as a tubular neighborhood,

$$(12) \qquad e_P : N_P \hookrightarrow \mathbb{R}^{p+m}.$$

This tubular neighborhood together with $\mathsf{Th}(\hat{\gamma})$ yields the Pontryagin–Thom map

$$(13) \qquad S^{p+m} \xrightarrow{c_P} \mathsf{Th}(U^\perp_{p,m}).$$

We now consider the standard multiplication-map

$$\mu : G(d-p,n) \times G(p,m) \to G(d,n+m)$$

given by $(V, W) \mapsto V \times W$. The multiplication map μ is covered by a bundle map

$$
\begin{array}{ccc}
U_{d-p,n-m}^{\perp} \times U_{p,m}^{\perp} & \xrightarrow{\hat{\mu}} & U_{d,n}^{\perp} \\
\downarrow & & \downarrow \\
G(d-p, n-m) \times G(p, m) & \xrightarrow{\mu} & G(d, n)
\end{array}
$$

which induces

$$
\mathsf{Th}(U_{d-p,n-m}^{\perp}) \wedge \mathsf{Th}(U_{p,m}^{\perp}) \xrightarrow{\mathsf{Th}(\hat{\mu})} \mathsf{Th}(U_{d,n}^{\perp}).
$$

Putting this together with c_P from (13), we define

$$
(14) \qquad \tau_{P,n} := \mathsf{Th}(\hat{\mu}) \circ (\mathrm{Id} \wedge c_P) : \mathsf{Th}(U_{d-p,n-m}^{\perp}) \wedge S^{p+m} \to \mathsf{Th}(U_{d,n}^{\perp}).
$$

As defined in [6], $\mathsf{Th}(U_{d,n}^{\perp})$ is the $(d+n)$-th space of the spectrum $\mathsf{MT}(d)$. The structure maps in this spectrum $\mathsf{MT}(d)$ come from applying $\mathsf{Th}(\_)$ to the bundle map

$$
\begin{array}{ccc}
U_{d,n}^{\perp} \oplus \epsilon^1 & \xrightarrow{\hat{i}_n} & U_{d,n+1}^{\perp} \\
\downarrow & & \downarrow \\
G(d, n) & \xrightarrow{i_n} & G(d, n+1)
\end{array}
$$

where the map i_n is the standard embedding and ϵ^1 is the trivial line bundle. The map from (14) yields a map of spectra

$$
\tau_P : \mathsf{MT}(d-p) \to \mathsf{MT}(d).
$$

Here we take into account that the spectrum with $(d+p)$-th space

$$
\mathsf{Th}(U_{d-p,n-m}^{\perp}) \wedge S^{p+m}
$$

is homotopy equivalent to $\mathsf{MT}(d-p)$. Now, consider the map $G(d, n) \to G(d+1, n)$ given by sending a d-dimensional vector subspace $V \subset \mathbb{R}^{p+n}$ to the subspace $\mathbb{R} \times V \subset \mathbb{R} \times \mathbb{R}^{d+n}$. This is covered by a bundle map $U_{d,n}^{\perp} \to U_{d+1,n}^{\perp}$. This induces a map on Thom-spaces

$$
(15) \qquad j_{d,n} : \mathsf{Th}(U_{d,n}^{\perp}) \to \mathsf{Th}(U_{d+1,n}^{\perp})
$$

and in turn a map of spectra which we denote by

$$
(16) \qquad \hat{j}_d : \Sigma^{-1} \mathsf{MT}(d) \to \mathsf{MT}(d+1).
$$

Consider the map of spectra given by the composition

$$
\Sigma^{-1} \mathsf{MT}(d-p) \xrightarrow{\Sigma^{-1}\tau_P} \Sigma^{-1} \mathsf{MT}(d) \xrightarrow{\hat{j}_d} \mathsf{MT}(d+1).
$$

Finally, we define

$$
\mathsf{MT}_P(d+1) := \mathrm{Cofiber}(\hat{j}_d \circ (\Sigma^{-1}\tau_P)).
$$

The spectrum $\mathsf{MT}_P(d+1)$ defined above is our primary spectrum of interest and is the spectrum that appears in the statement of Theorem 1.1 from the introduction. Applying the *infinite loop-space* functor to the cofiber sequence

$$\Sigma^{-1}\mathsf{MT}(d-p) \to \mathsf{MT}(d+1) \to \mathsf{MT}_P(d+1)$$

yields a homotopy fiber sequence

$$\Omega^\infty \mathsf{MT}(d+1) \to \Omega^\infty \mathsf{MT}_P(d+1) \to \Omega^\infty \mathsf{MT}(d-p).$$

Remark 6.2. Since the map of spectra $\tau_P : \mathsf{MT}(d-p) \to \mathsf{MT}(d)$ is induced by the Pontryagin–Thom construction applied to the embedding $i_P : P \hookrightarrow \mathbb{R}^{p+m}$, it follows that the homotopy class of τ_P is an invariant of the cobordism class of the manifold P. It follows that if P' is a manifold which is cobordant to P then the spectrum $\mathsf{MT}_{P'}(d+1)$ is homotopy equivalent to $\mathsf{MT}_P(d+1)$.

6.3. A filtration of MO_P. We now describe how the spectrum $\mathsf{MT}_P(d+1)$ constructed in the previous section relates to the spectrum MO_P, which classifies the homology theory associated to the cobordism groups of P-manifolds. There is a direct system of spectra

$$(17) \qquad \cdots \to \Sigma^{(d-1)}\mathsf{MT}(d-1) \to \Sigma^d \mathsf{MT}(d) \to \Sigma^{(d+1)}\mathsf{MT}(d+1) \to \cdots,$$

where the d-th map is the d-th suspension of the map defined in (16). The direct limit is a spectrum which we denote by MTO. The following lemma is proven in [6, p. 14]. We provide the proof here for completeness.

Lemma 6.4. *There is a homotopy equivalence $\mathsf{MTO} \simeq \mathsf{MO}$.*

Proof. There is a homeomorphism $G(d,n) \to G(n,d)$ given by $V \mapsto V^\perp$. This map is covered by a bundle isomorphism $U_{d,n}^\perp \to U_{n,d}$ and thus yields maps

$$\mathsf{Th}(U_{d,n}^\perp) \xrightarrow[\cong]{\perp} \mathsf{Th}(U_{n,d}) \xrightarrow{i} \mathsf{Th}(U_{n,\infty}),$$

where $\mathsf{Th}(U_{n,\infty}) := \mathrm{colim}_{d\to\infty} \mathsf{Th}(U_{n,d})$. The space $\mathsf{Th}(U_{n,\infty})$ is the n-th space of the spectrum MO, thus the above map induces a map of spectra $\Sigma^d \mathsf{MT}(d) \to \mathsf{MO}$ (or a degree d map $\mathsf{MT}(d) \to \mathsf{MO}$). The space $\mathsf{Th}(U_{n,\infty})$ is the n-th space in the spectrum MO. Now, $\mathsf{Th}(U_{n,\infty})$ can be built out of $\mathsf{Th}(U_{n,d})$ by attaching cells of dimension greater than dimension $n+d$. This implies that $\Sigma^d \mathsf{MT}(d) \to \mathsf{MO}$ induces an isomorphism on π_k for $k < d$ and a surjection for $k = d$. This proves that $\mathsf{MTO} \simeq \mathsf{MO}$. $\qquad\square$

Since $\pi_{d-1}\mathsf{MO} \cong \Omega_{d-1}$ where Ω_{d-1} is the cobordism group of unoriented $(d-1)$-dimensional manifolds, the above lemma implies that $\pi_{-1}\mathsf{MT}(d) \cong \Omega_{d-1}$. For each d, the diagram

$$
\begin{array}{ccc}
\Sigma^{(d+p)}\mathsf{MT}(d) & \xrightarrow{\ \Sigma^{(d+p)}\hat{j}_d\ } & \Sigma^{(d+1+p)}\mathsf{MT}(d+1) \\[2pt]
{\scriptstyle \Sigma^{(d+1+p)}[\hat{j}_d \circ (\Sigma^{-1}\tau_P)]}\Big\downarrow & & \Big\downarrow{\scriptstyle \Sigma^{(d+2+p)}[\hat{j}_{d+1} \circ (\Sigma^{-1}\tau_P)]} \\[2pt]
\Sigma^{(d+1+p)}\mathsf{MT}(d+1+p) & \xrightarrow[\ \Sigma^{(d+1+p)}\hat{j}_{d+1+p}\ ]{} & \Sigma^{(d+2+p)}\mathsf{MT}(d+2+p)
\end{array}
$$

commutes up to homotopy. Passing to the cofibers of the vertical maps induces a map of spectra

$$\Sigma^{d+1+p}\mathsf{MT}_P(d+1+p) \to \Sigma^{d+2+p}\mathsf{MT}_P(d+2+p)$$

for each d. These maps form a direct system similar to (17). We denote the direct limit of this direct system by MTO_P.

Lemma 6.5. *There is a homotopy equivalence $\mathsf{MTO}_P \simeq \mathsf{MO}_P$ where MO_P is the classifying spectrum for the cobordism theory Ω_*^P for manifolds with type P-singularities.*

Proof. The spectrum MO_P is given as the cofiber of the map $\times P : \Sigma^p\mathsf{MO} \to \mathsf{MO}$ which is induced by the degree p homomorphism $\times P : \Omega_* \to \Omega_*$. On the level of spectra, this map is defined concretely as follows. The map

$$\mu : G(n,d) \times G(m,p) \to G(n+m,d+p), \quad (V,W) \mapsto V \times W$$

induces

$$\mu' : G(n,\infty) \times G(m,p) \to G(n+m,\infty)$$

in the limit as $d \to \infty$. The map μ' is covered by a bundle map $U_{n,\infty} \times U_{m,p} \to U_{n+m,\infty}$ which induces a map of Thom spaces

$$\mathsf{Th}(U_{n,\infty}) \wedge \mathsf{Th}(U_{m,p}) \to \mathsf{Th}(U_{n+m,\infty}).$$

The normal bundle N_P for $i_P(P) \subset \mathbb{R}^{p+m}$ has Gauss map

$$
\begin{array}{ccc}
N_P & \longrightarrow & U_{m,p} \\
\downarrow & & \downarrow \\
P & \longrightarrow & G(m,p).
\end{array}
$$

We emphasize that this map is different than the map (11) where the target space was $G(p,m)$ with bundle $U_{p,m}^\perp$. The Pontryagin–Thom map $S^{p+m} \to \mathsf{Th}(U_{m,p})$ associated to the Gauss map for $i_P(P)$ yields

$$\mathsf{Th}(U_{n,\infty}) \wedge S^{p+m} \to \mathsf{Th}(U_{n,\infty}) \wedge \mathsf{Th}(U_{m,p}) \to \mathsf{Th}(U_{n+m,\infty}).$$

Since the spectrum with $(n+m)$-th space equal to $\mathsf{Th}(U_{n,\infty}) \wedge S^m$ is equivalent to MO, the map above induces a map of spectra, $\Sigma^p\mathsf{MO} \to \mathsf{MO}$, which defines $\times P$. Upon inspection, it can be seen that for all d the diagram

$$
\text{(18)} \qquad
\begin{array}{ccc}
\Sigma^{d+p}\mathsf{MT}(d) & \longrightarrow & \Sigma^p\mathsf{MO} \\
{\scriptstyle \hat{j}_d \circ \tau_P} \downarrow & & \downarrow {\scriptstyle \times P} \\
\Sigma^{d+p+1}\mathsf{MT}(d+p+1) & \longrightarrow & \mathsf{MO}
\end{array}
$$

commutes, where the horizontal maps are induced by the composition

$$\mathsf{Th}(U_{d,n}^\perp) \xrightarrow[\cong]{\perp} \mathsf{Th}(U_{n,d}) \xrightarrow{i} \mathsf{Th}(U_{n,\infty}).$$

As was used in the proof of Lemma 6.4, the Thom space $\mathsf{Th}(U_{n,\infty})$ can be built out of $\mathsf{Th}(U_{n,d})$ by attaching cells of dimension greater than $n+d$. This implies

that the lower and upper horizontal maps above in (18) induce isomorphisms on π_k for $k < d + p$ and surjections on π_k for $k = d + p$. By applying the *five lemma* to the long exact sequence on homotopy groups associated to the cofibers of the vertical maps, we see that the induced map

$$\Sigma^{d+p+1}\mathsf{MT}_P(d + p + 1) \to \mathsf{MO}_P$$

induces an isomorphism on π_k for $k < d + p$ and a surjection on π_k for $k = d + p$. Taking the direct limit as $d \to \infty$, we see that $\mathsf{MTO}_P \simeq \mathsf{MO}_P$. $\square$

Corollary 6.6. *There is an isomorphism $\pi_{-1}\mathsf{MT}_P(d+1) \cong \Omega_d^P$, where Ω_d^P is the classical cobordism-group of d-dimensional manifolds with singularities of type P.*

6.7. **Infinite loop spaces.** Our main result, Theorem 1.1, establishes a weak homotopy equivalence, $B\mathbf{Cob}_{d+1}^P \simeq \Omega^{\infty-1}\mathsf{MT}_P(d + 1)$. It is difficult to construct a map from a space directly into the infinite loop-space $\Omega^{\infty-1}\mathsf{MT}_P(d+1)$. It will be useful for us to construct certain auxiliary models for the homotopy type of $\Omega^{\infty-1}\mathsf{MT}_P(d + 1)$ which will be easier to map into. Recall the maps

$$\mathsf{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m} \xrightarrow{\tau_{P,n}} \mathsf{Th}(U_{d,n}^\perp) \xrightarrow{j_{d,n}} \mathsf{Th}(U_{d+1,n}^\perp)$$

from (14) and (15).

Definition 6.8. Let n and d be nonnegative integers. Define $\Omega_\partial^{d+n}\mathsf{Th}(U_{d+1,n}^\perp)^\wedge$ to be the space of pairs $(\hat{f}, f)$ of based maps

$$\hat{f} : D^{d+n} \to \mathsf{Th}(U_{d+1,n}^\perp),$$
$$f : S^{d+n-1} \to \mathsf{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m},$$

which make the diagram

(19)
$$
\begin{array}{ccc}
D^{d+n} & \xrightarrow{\tilde{f}} & \mathsf{Th}(U_{d+1,n}^\perp) \\
\big\uparrow & & \big\uparrow{\scriptstyle j_{d,n}\circ\tau_{P,n}} \\
S^{d+n-1} & \xrightarrow{f} & \mathsf{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m}
\end{array}
$$

commute, where the left vertical map is the standard inclusion. Now, let

$$\alpha : S^{d+n-1} \xrightarrow{\cong} S^{d+\bar{n}} \wedge S^{p+m}$$

be the standard identification (where $\bar{n} = n - p - m - 1$ as in the previous section). The space $\Omega_P^{d+n}\mathsf{Th}(U_{d+1,n}^\perp)^\wedge$ is defined to be the subspace of $\Omega_\partial^{d+n}\mathsf{Th}(U_{d+1,n}^\perp)^\wedge$ consisting of all pairs $(\hat{f}, f)$ for which there exists a map

$$f_0 : S^{d+\bar{n}} \to \mathsf{Th}(U_{d-p,n-m}^\perp)$$

such that $f : S^{d-1+n} \to \mathsf{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m}$ has the factorization

$$S^{d+n-1} \xrightarrow[\cong]{\alpha} S^{d+\bar{n}} \wedge S^{p+m} \xrightarrow{f_0 \wedge \mathrm{Id}_{S^{p+m}}} \mathsf{Th}(U_{d-p,n-m}^\perp) \wedge S^{p+m}.$$

It follows that the map f_0 is uniquely determined.

We then define

$$\Omega_P^{\infty-1}\mathsf{Th}(U_{d+1,\infty}^{\perp})^{\wedge} := \operatorname*{colim}_{n\to\infty} \Omega_P^{d+n}\mathsf{Th}(U_{d+1,n}^{\perp})^{\wedge},$$

$$\Omega_{\partial}^{\infty-1}\mathsf{Th}(U_{d+1,\infty}^{\perp})^{\wedge} := \operatorname*{colim}_{n\to\infty} \Omega_{\partial}^{d+n}\mathsf{Th}(U_{d+1,n}^{\perp})^{\wedge}.$$

Proposition 6.9. *The following natural embedding is a homotopy equivalence:*

$$\Omega_P^{\infty-1}\mathsf{Th}(U_{d+1,\infty}^{\perp})^{\wedge} \to \Omega_{\partial}^{\infty-1}\mathsf{Th}(U_{d+1,\infty}^{\perp})^{\wedge}.$$

Proof. For each n, the space $\Omega_P^{d+n}\mathsf{Th}(U_{d+1,n}^{\perp})^{\wedge}$ can be realized as the pullback

$$(20) \quad \begin{array}{ccc} \Omega_P^{d+n}\mathsf{Th}(U_{d+1,n}^{\perp})^{\wedge} & \hookrightarrow & \Omega_{\partial}^{d+n}\mathsf{Th}(U_{d+1,n}^{\perp})^{\wedge} \\ {\scriptstyle r_0}\big\downarrow & & \big\downarrow{\scriptstyle r} \\ \Omega^{d+\bar{n}}\mathsf{Th}(U_{d-p,n-m}^{\perp}) & \xrightarrow{\;\_\wedge\mathrm{Id}_{S^{p+m}}\;} & \Omega^{d+n-1}(\mathsf{Th}(U_{d-p,n-m}^{\perp})\wedge S^{p+m}), \end{array}$$

where $r(\hat{f}, f) = f$ and $r_0(\hat{f}, f) = f_0$. The bottom horizontal map is the standard $(p+m)$-fold suspension map. The top horizontal map in the diagram is the inclusion. It will suffice to show that this upper-horizontal map is highly connected and that its connectivity approaches infinity as $n \to \infty$.

Now, the map r is a Serre-fibration. It follows from this that the pullback square (20) is *homotopy cartesian*. The Thom-space $\mathsf{Th}(U_{d-p,n-m}^{\perp})$ is $(n-m-1)$-connected and so its connectivity approaches ∞ with n. The *Freudenthal suspension theorem* implies that the connectivity of the bottom horizontal map $\_ \wedge \mathrm{Id}_{S^{p+m}}$ of (20) approaches ∞ (notice that as n increases without bound, the integers p and m are held fixed). Since the diagram is a homotopy pullback square, it follows from this that the connectivity of the upper horizontal map tends to ∞ with n. This implies the proposition and completes the proof. $\qquad\square$

We now compare the space $\Omega_P^{\infty-1}\mathsf{Th}(U_{d+1,\infty}^{\perp})^{\wedge}$ to the infinite loop-space $\Omega^{\infty-1}\mathsf{MT}_P(d+1)$. For each n there is a map

$$\sigma_n : \Omega_{\partial}^{d+n}\mathsf{Th}(U_{d+1,n}^{\perp})^{\wedge} \to \Omega^{d+n}\mathrm{Cofiber}(j_{d,n}\circ\tau_{P,n})$$

defined by sending a pair of maps

$$\hat{f} : D^{d+n} \to \mathsf{Th}(U_{d+1,n}^{\perp}), \quad f : S^{d+n-1} \to \mathsf{Th}(U_{d-p,n-m}^{\perp}) \wedge S^{p+m}$$

which make diagram (19) commute to its induced map

$$D^{d+n}/S^{d+n-1} \to \mathrm{Cofiber}(j_{d,n}\circ\tau_{P,n}).$$

In the limit $n \to \infty$, the maps σ_n induce

$$(21) \qquad\qquad \sigma : \Omega_{\partial}^{\infty-1}\mathsf{Th}(U_{d+1,\infty}^{\perp})^{\wedge} \to \Omega^{\infty-1}\mathsf{MT}_P(d+1).$$

Proposition 6.10. *The map σ of (21) is a homotopy equivalence.*

Proof. For each k, we have a commutative diagram

$$\begin{CD}
\pi_k(\Omega_\partial^{d+n}\mathsf{Th}(U_{d+1,n}^\perp)^\wedge) @>>> \pi_k(\Omega^{d+n}\,\mathrm{Cofiber}(j_{d,n}\circ\tau_{P,n})) \\
@VVV @VVV \\
\pi_{k+d+n}(\mathsf{Th}(U_{d+1,n}^\perp),\mathsf{Th}(U_{d-p,n-m}^\perp)\wedge S^{p+m}) @>>> \pi_{k+d+n}(\mathrm{Cofiber}(j_{d,n}\circ\tau_{P,n}))
\end{CD}$$

where the vertical maps are given by adjunction, the top horizontal map is induced by σ_n, and the bottom horizontal map σ_n^k is induced by sending a pair of maps

$$\hat{f}: D^{k+d+n} \to \mathsf{Th}(U_{d+1,n}^\perp), \quad f: S^{k+d+n-1} \to \mathsf{Th}(U_{d-p,n-m}^\perp)\wedge S^{p+m}$$

which make diagram (19) commute to its induced map

$$S^{k+d+n} \cong D^{k+d+n}/S^{k+d+n-1} \to \mathrm{Cofiber}(j_{d,n}\circ\tau_{P,n}).$$

The space $\mathsf{Th}(U_{d-p,n-m}^\perp)\wedge S^{p+m}$ is $(n+p-1)$-connected and the map

$$j_{d,n}\circ\tau_{P,n}: \mathsf{Th}(U_{d-p,n-m}^\perp)\wedge S^{p+m} \to \mathsf{Th}(U_{d+1,n}^\perp)^\wedge$$

is at least $(n-1)$-connected. It follows from [8, Prop. 4.28] that the bottom horizontal homomorphism σ_n^k is an isomorphism when $k \le n+p-d-2$. Commutativity of the above diagram then implies that $(\sigma_n)_*$ is an isomorphism when $k \le n+p-d-2$ as well. Passing to the limit $n \to \infty$ then yields the result. This completes the proof of the proposition. $\square$

Combining Propositions 6.9 and 6.10 yields:

Corollary 6.11. *The natural map* $\Omega_P^{\infty-1}\mathsf{Th}(U_{d+1,\infty}^\perp)^\wedge \to \Omega^{\infty-1}\mathsf{MT}_P(d+1)$ *is a homotopy equivalence.*

7. The main theorem

This section is devoted to proving the following theorem.

Theorem 7.1. *There is a homotopy equivalence* $|\mathbf{D}_{d+1}^P| \simeq \Omega^{\infty-1}\mathsf{MT}_P(d+1)$.

This theorem is proven in a way similar to [6, Thm. 3.4].

7.2. Isomorphism of concordance class functors. First we prove the following lemma.

Lemma 7.3. *For* $X \in \mathcal{O}b(\mathcal{X})$ *and* $n \in \mathbb{N}$, *there is a natural map*

$$T_n: \mathbf{D}_{d+1,n}^P[X] \to [X, \Omega_P^{d+n}\mathsf{Th}(U_{d+1,n}^\perp)^\wedge].$$

Proof. Let $X \in \mathcal{O}b(\mathcal{X})$. We construct the map T_n as follows. Let

$$W \subset X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$$

be an element of $\mathbf{D}_{d+1,n}^P(X)$. We may assume that f is transverse as a smooth P-map to $0 \in \mathbb{R}$, in the sense of Definition 2.7. This means that both f and

f_β are transverse to 0. It follows from Proposition 2.8 that $f^{-1}(0)$ has the structure of a P-submanifold of $X \times \{0\} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$. We have

$$M := f^{-1}(0), \quad \partial_1 M := (f|_{\partial_1 W})^{-1}(0), \quad \beta_1 M := (f|_{\beta_1 W \times \{0\}})^{-1}(0)$$

and

$$\partial_1 M = \beta_1 M \times i_P(P).$$

Denote by N_M and $N_{\beta_1 M}$ the normal bundles of M and $\beta_1 M$ in

$$X \times \{0\} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m} \quad \text{and} \quad X \times \{0\} \times \{0\} \times \mathbb{R}^{d+\bar{n}},$$

respectively. It follows from Remark 3.12 that there is a bundle isomorphism

(22) $$N_M|_{\partial_1 M} \xrightarrow{\cong} (N_{\beta_1 M} \times N_P) \oplus \epsilon^1,$$

where $N_P \to P$ is the normal bundle for $i_P(P) \subset \mathbb{R}^{p+m}$. These normal bundles yield Gauss maps

$$
\begin{array}{ccc}
N_M & \xrightarrow{\hat{\gamma}_M} & U^\perp_{d+1,n} \\
\downarrow & & \downarrow \\
M & \xrightarrow{\gamma_M} & G(d+1,n)
\end{array}
$$

and

$$
\begin{array}{ccccc}
N_{\beta M} \times N_P & \xrightarrow{\hat{\gamma}_{\beta M} \times \hat{\gamma}_P} & U^\perp_{d-p,n-m} \times U^\perp_{p,m} & \xrightarrow{\hat{\mu}} & U^\perp_{d,n} \\
\downarrow & & \downarrow & & \downarrow \\
\beta M \times P & \xrightarrow{\gamma_{\beta M} \times \gamma_P} & G(d-p,n-m) \times G(p,m) & \xrightarrow{\mu} & G(d,n)
\end{array}
$$

where μ and $\hat{\mu}$ are the maps defined in Section 6. These bundle maps induce the following maps on Thom spaces:

$$\mathsf{Th}(N_M) \xrightarrow{\mathsf{Th}(\hat{\gamma}_M)} \mathsf{Th}(U^\perp_{d+1,n})$$

and

$$\mathsf{Th}(N_{\beta M}) \wedge \mathsf{Th}(N_P) \xrightarrow{\mathsf{Th}(\hat{\gamma}_{\beta M}) \wedge \mathsf{Th}(\hat{\gamma}_P)} \mathsf{Th}(U^\perp_{d-p,n-m}) \wedge \mathsf{Th}(U^\perp_{p,m}).$$

There are tubular neighborhood embeddings of the normal bundles N_M and $N_{\beta_1 M}$ into

$$X \times \{0\} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m} \quad \text{and} \quad X \times \{0\} \times \{0\} \times \mathbb{R}^{d+\bar{n}},$$

respectively, which yield collapsing maps

$$c_M : X_+ \wedge D^{d+n} \to \mathsf{Th}(N_M),$$
$$c_{\beta_1 M} : X_+ \wedge S^{d+\bar{n}} \to \mathsf{Th}(N_{\beta_1 M}),$$

where X_+ denotes the *one-point compactification* of X. Composing $\mathsf{Th}(\hat{\gamma}_M)$ and $\mathsf{Th}(\hat{\gamma}_{\beta M})$ with the above collapsing maps yields the diagram

$$
\begin{array}{ccc}
X_+ \wedge D^{d+n} & \xrightarrow{\ \mathsf{Th}(\hat{\gamma}_{N_M}) \circ c_M\ } & \mathsf{Th}(U_{d+1,n}^{\perp}) \\
\big\uparrow & & \big\uparrow{\scriptstyle j_{d,n} \circ \tau_{P,n}} \\
X_+ \wedge S^{d+\bar{n}} \wedge S^{p+m} & \xrightarrow{[\mathsf{Th}(\hat{\gamma}_{N_{\beta M}}) \circ c_{\beta M}] \wedge \mathrm{Id}_{S^{p+m}}} & \mathsf{Th}(U_{d-p,n-p}^{\perp}) \wedge S^{p+m}.
\end{array}
$$

It follows from the bundle factorization of (22) that this diagram does indeed commute. By adjunction this commutative diagram yields

$$
f : X \to \Omega_P^{d+n} \mathsf{Th}(U_{d+1,n}^{\perp})^{\wedge}.
$$

By standard Pontryagin–Thom theory (see [14, §2]) it follows that choosing a different representative of the concordance class of W yields a map homotopic to the one which we just produced; just run the same process on a concordance. We then define $T_n([W]) := [f]$. It is easy to check that this definition is natural in the variable X. $\qquad\square$

For $X \in \mathcal{X}$, denote by

$$
T : \mathbf{D}_{d+1}^{P}[X] \to [X, \Omega_P^{\infty-1} \mathsf{Th}(U_{d+1,\infty}^{\perp})^{\wedge}]
$$

the map induced in the limit $n \to \infty$ by the maps T_n constructed in the previous lemma.

Lemma 7.4. *For compact X, the map $T : \mathbf{D}_{d+1}^{P}[X] \to [X, \Omega_P^{\infty-1} \mathsf{Th}(U_{d+1,\infty}^{\perp})^{\wedge}]$ is an isomorphism of sets.*

Proof. We now make the assumption that X is compact. We construct an inverse to

$$
T : \mathbf{D}_{d+1}^{P}[X] \to [X, \Omega_P^{\infty-1} \mathsf{Th}(U_{d+1,\infty}^{\perp})^{\wedge}]
$$

which we will denote by H. Let

$$
(23) \qquad
\begin{array}{ccc}
X_+ \wedge D^{d+n} & \xrightarrow{\quad\tilde{f}\quad} & \mathsf{Th}(U_{d+1,n}^{\perp}) \\
\big\uparrow & & \big\uparrow{\scriptstyle j_{d,n} \circ \tau_{P,n}} \\
(X_+ \wedge S^{\bar{d}-1+\bar{n}}) \wedge S^{p+m} & \xrightarrow{f := f_0 \wedge \mathrm{Id}_{S^{p+m}}} & \mathsf{Th}(U_{d-p,n-m}^{\perp}) \wedge S^{p+m}
\end{array}
$$

represent an element $[(\tilde{f}, f)] \in [X, \Omega_P^{d+n} \mathsf{Th}(U_{d+1,n}^{\perp})^{\wedge}]$.

By applying an appropriate homotopy, we may assume the following about $(\tilde{f}, f)$:

(i) The maps $\tilde{f}$ and f are both smooth away from the preimage of the basepoint. Furthermore, both $\tilde{f}$ and f are transverse to $G(d+1, n)$ and $G(d-p, n-m)$ as submanifolds of $U_{d+1,n}^{\perp}$ and $U_{d-p,n-m}^{\perp}$, respectively.

(ii) By transversality in (i) we have a pair of submanifolds

$$(M, \partial_1 M) \subseteq (X \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}, X \times \{0\} \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m})$$

where

$$M := \tilde{f}^{-1}(G(d+1, n)) \quad \text{and} \quad \partial_1 M := (\tau_{P,n} \circ f)^{-1}(G(d, n)).$$

Furthermore, $\partial_1 M$ factors as $\partial_1 M = \beta_1 M \times i_P(P)$ where

$$\beta_1 M := f_0^{-1}(G(d-p, n-m)) \subset X \times \{0\} \times \mathbb{R}^{d+\bar{p}}$$

is a closed submanifold.

(iii) There exists $\varepsilon > 0$ such that

$$M \cap (X \times [0, \varepsilon) \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}) = \partial_1 M \times [0, \varepsilon).$$

The above conditions imply that $M \subset X \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$ is a closed P-submanifold in the sense of Definition 3.11.

The submanifolds M, $\partial_1 M$, and $\beta_1 M$ have normal bundles given by the pullbacks,

$$N_M = \tilde{f}^*(U_{d+1,n}^\perp), \quad N_{\partial_1 M} = (\tau_{P,n} \circ f)^*(U_{d,n}^\perp), \quad N_{\beta_1 M} = f_0^*(U_{d-p,n-m}^\perp).$$

Furthermore, the normal bundle $N_{\partial_1 M}$ has the factorization $N_{\partial_1 M} = N_{\beta_1 M} \times N_P$, where N_P is the normal bundle for $P \subset \mathbb{R}^{p+m}$. We define vector bundles

$$T^\pi M = \tilde{f}^*(U_{d+1,n}), \quad T^\pi \partial_1 M = (\tau_{P,n} \circ f)^*(U_{d,n}),$$
$$T^\pi \beta_1 M = f_0^*(U_{d-p,n-m})$$

over M, $\partial_1 M$, and $\beta_1 M$, respectively. Below we construct bundle epimorphisms for which these bundles are the kernels. By the definition of these bundles and the factorization of f, it follows that there are bundle splittings

$$T^\pi M|_{\partial_1 M} = T^\pi \partial_1 M \oplus \epsilon^1, \qquad T^\pi \partial_1 M = T^\pi \beta_1 M \times TP,$$

and bundle isomorphisms

$$(24) \qquad N_M \oplus T^\pi M \cong \epsilon^{d+n+1}, \quad N_{\partial_1 M} \oplus T^\pi \partial_1 M \cong \epsilon^{d+n},$$
$$N_{\beta_1 M} \oplus T^\pi \beta_1 M \cong \epsilon^{d+\bar{n}}.$$

Denote by

$$(i_M, i_{\partial_1 M}) : (M, \partial_1 M) \to (X \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}, X \times \{0\} \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m})$$

the inclusion map and let

$$\pi_0 : M \to X \quad \text{and} \quad \pi_0^\partial : \partial_1 M \to X$$

denote the projections onto X. Pulling back the tangent bundle of $X \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$ by $(i_M, i_{\partial_1 M})$ yields isomorphisms

$$i_M^*(TX) \oplus \epsilon^{d+n} \cong TM \oplus N_M,$$
$$i_{\partial_1 M}^*(TX) \oplus \epsilon^{d+n-1} \cong T\partial_1 M \oplus N_{\partial M},$$
$$i_M^*(TX) \oplus \epsilon^{d+n} \cong \pi_0^*(TX) \oplus \epsilon^{d+n},$$
$$i_{\partial_1 M}^*(TX) \oplus \epsilon^{d+n-1} \cong (\pi_0^\partial)^*(TX) \oplus \epsilon^{d+n-1}.$$

Combining these isomorphisms yields

$$(25) \qquad \begin{aligned} \pi_0^*(TX) \oplus \epsilon^{d+n} &\cong TM \oplus N_M, \\ (\pi_0^\partial)^*(TX) \oplus \epsilon^{d+n-1} &\cong T\partial_1 M \oplus N_{\partial_1 M}, \end{aligned}$$

both of which cover the identity on M and $\partial_1 M$. By adding $T^\pi M$ and $T^\pi \partial_1 M$ via Whitney-sum to both sides of the equations in (25) and using the isomorphisms of (24), we obtain a commutative diagram

$$(26) \qquad \begin{array}{ccc} TM \oplus \epsilon^{n+d+1} & \xrightarrow[\cong]{\hat{\pi}_0} & \pi_0^* TX \oplus T^\pi M \oplus \epsilon^{d+n} \\ \big\uparrow & & \big\uparrow \\ (T\partial_1 M \oplus \epsilon^{n+d}) \oplus \epsilon^1 & \xrightarrow[\cong]{\hat{\pi}_0^\partial \oplus \mathrm{Id}_{\epsilon^1}} & (\pi_0^* TX \oplus T^\pi \partial_1 M \oplus \epsilon^{d+n-1}) \oplus \epsilon^1 \end{array}$$

of bundle isomorphisms where the horizontal maps cover the identity on M and the vertical maps cover the inclusion of $\partial_1 M$ into M. Furthermore the bundle isomorphism (coming from the bottom horizontal arrow)

$$\hat{\pi}_0^\partial : T\partial_1 M \oplus \epsilon^{n+d} \to \pi_0^* TX \oplus T^\pi \partial_1 M \oplus \epsilon^{d+n-1}$$

has the factorization

$$\hat{\pi}_0^\partial = \hat{\pi}_0^\beta \times \mathrm{Id}_{TP},$$

where

$$\hat{\pi}_0^\beta : T\beta_1 M \oplus \epsilon^{d+n} \to \pi_0^* TX \oplus T^\pi \beta_1 M \oplus \epsilon^{d+n-1}$$

is a bundle isomorphism which covers the identity on $\beta_1 M$. We will need to use the following *destabilization*. We postpone the proof of the following result to Section 10.

Claim 7.5. *The bundle isomorphism pair* $(\hat{\pi}_0, \hat{\pi}_0^\partial \oplus \mathrm{Id}_{\epsilon^1})$ *from* (26) *is induced by a pair of bundle isomorphisms*

$$\begin{array}{ccc} TM \oplus \epsilon^1 & \xrightarrow[\cong]{\hat{\pi}_1} & \pi_0^* TX \oplus T^\pi M \\ \big\uparrow & & \big\uparrow \\ (T\partial_1 M \oplus \epsilon^1) \oplus \epsilon^1 & \xrightarrow[\cong]{\hat{\pi}_1^\partial \oplus \mathrm{Id}_{\epsilon^1}} & (\pi_0^* TX \oplus T^\pi \partial_1 M) \oplus \epsilon^1 \end{array}$$

with factorization

$$\hat{\pi}_1^\partial = \hat{\pi}_1^\beta \times \mathrm{Id}_{TP},$$

where

$$\hat{\pi}_1^\beta : T\beta_1 M \oplus \epsilon^1 \to \pi_0^* TX \oplus T^\pi \beta_1 M$$

is a bundle isomorphism covering the identity on $\beta_1 M$. *Furthermore, the bundle map* $(\hat{\pi}_1, \hat{\pi}_1^\partial \oplus \mathrm{Id}_{\epsilon^1})$ *is unique up to homotopy through bundle map pairs with the factorization specified above.*

We now define spaces

$$W := M \times \mathbb{R}, \quad \partial_1 W := \partial_1 M \times \mathbb{R} \quad \text{and} \quad \beta_1 W := \beta_1 M \times \mathbb{R}.$$

We define the bundles $T^\pi W$, $T^\pi \partial_1 W$, $T^\pi \beta_1 W$ to be the pullbacks of the bundles $T^\pi M$, $T^\pi \partial_1 M$, $T^\pi \beta_1 M$, over the projections of W, ∂W, βW onto M, $\partial_1 M$, $\beta_1 M$, respectively. Denote by

$$i_W : W \hookrightarrow X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$$

the inclusion map. Let s_0, s_0^∂, and s_0^β denote the projections of W, $\partial_1 W$, and $\beta_1 W$ onto the factor X (the reason for the notation s_0 used for this projection onto X will become clear momentarily). The result of Claim 7.5 yields bundle isomorphisms

$$(27) \qquad \begin{cases} TW \cong s_0^*(TX) \oplus T^\pi W, \\ T\partial_1 W \cong (s_0^\partial)^*(TX) \oplus T^\pi \partial_1 W, \\ T\beta_1 W \cong (s_0^\beta)^*(TX) \oplus T^\pi \beta_1 W, \end{cases}$$

all which cover the identity. Using (27), we obtain a bundle epimorphism with kernel $T^\pi W$,

$$(28) \qquad \begin{array}{ccc} TW & \xrightarrow{\hat{s}_0} & TX \\ \downarrow & & \downarrow \\ W & \xrightarrow{s_0} & X \end{array}$$

such that the restriction $(\hat{s}_0, s_0)|_{\partial_1 W}$ has the factorization

$$(29) \qquad \begin{array}{ccccc} T\partial_1 W & \xrightarrow{pr} & T\beta_1 W & \xrightarrow{\hat{s}_0^\beta} & TX \\ \downarrow & & \downarrow & & \downarrow \\ \partial_1 W & \xrightarrow{pr} & \beta_1 W & \xrightarrow{s_0^\beta} & X, \end{array}$$

where $(\hat{s}_0^\beta, s_0^\beta)$ is a bundle-epimorphism covering s_0^β which is the projection onto X.

Claim 7.6. *There exists a homotopy $(\hat{s}_t, s_t)$ through bundle-epimorphisms such that:*

(i) *At $t = 0$, $(\hat{s}_0, s_0)$ is equal to the bundle epimorphism given in (28).*

(ii) *The bundle epimorphism $(\hat{s}_1, s_1)$ is integrable, i.e., $Ds_1 = \hat{s}_1$ and thus s_1 is a submersion.*

(iii) *For all t, $(\hat{s}_t, s_t)$ has the factorization given in (29).*

Moreover, the integrable bundle-epimorphism $(\hat{s}_1, s_1)$ is unique up to homotopy though integrable bundle-epimorphisms.

We provide a proof of this claim in Section 9. This is essentially a relative version of *Phillips' submersion theorem* [13] adapted for P-manifolds.

Let $(\hat{s}_t, s_t)$ be the desired family of bundle epimorphisms with the above stated properties. The map s_1 is now a submersion of W onto X. In order to obtain an element of $\mathbf{D}_{d+1}^P[X]$, we need to realize s_1 as the composition of some embedding

$$W \hookrightarrow X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$$

followed by projection onto X. Recall that $W = \mathbb{R} \times M$ where M is a closed P-manifold. By Theorem 3.8 it follows that for some integer n' (possibly larger than n) there exists a P-embedding

$$j : M \to \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}'} \times \mathbb{R}^{p+m}$$

(by P-embedding here we mean an element of the space $\mathcal{E}_{P,n'+d}(M)$, see Definition 3.6). Furthermore, it follows from Theorem 3.8 again that if n' is large enough then any two choices of embeddings j are isotopic through P-embeddings. Now consider the embedding

$$W = \mathbb{R} \times M \to X \times \mathbb{R} \times (\mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}'} \times \mathbb{R}^{p+m}), \quad (t,x) \mapsto (s_1(t,x), t, j(x)),$$

where $s_1(t,x) \in X$, $t \in \mathbb{R}$, and $j(x) \in \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}'} \times \mathbb{R}^{p+m}$. Denote by W' the image of the above embedding. It follows that $W' \in \mathbf{D}_{d,n'}^P(X)$.

We define

$$H : [X, \Omega_P^{\infty-1}\mathsf{Th}(U_{d+1,\infty}^\perp)^\wedge] \to \mathbf{D}_{d+1}^P[X]$$

by sending $[\bar{f}] \in [X, \Omega_P^{\infty-1}\mathsf{Th}(U_{d+1,\infty}^\perp)^\wedge]$ (the class that we started with from (23)) to the concordance class containing the image of W' in $\mathbf{D}_{d+1}^P(X)$. This map is well defined because all choices made in the construction of W' were shown to be unique up to homotopy (namely the submersion found in Claim 7.6 and the embedding constructed above). One can verify directly that $T \circ H = \mathrm{Id}$. To see that $H \circ T = \mathrm{Id}$, recall Lemma 5.7 which states that any concordance class $\mathbf{D}_{d+1,n}^P[X]$ has a representative

$$W \subset X \times \mathbb{R} \times (\mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m})$$

such that $f : W \to \mathbb{R}$ is a bundle projection and thus W is diffeomorphic to the product $f^{-1}(0) \times \mathbb{R}$. It is easy to check that $H \circ T$ acts as the identity on such elements. This concludes the proof of the lemma. $\qquad\square$

7.7. A parametrized Pontryagin–Thom construction.
Lemma 7.3 establishes a bijection $\mathbf{D}_{d+1}^P[X] \cong [X, \Omega_P^{\infty-1}\mathsf{Th}(U_{d+1,\infty}^\perp)^\wedge]$ for any closed manifold X. In order to obtain the weak equivalence $|\mathbf{D}_{d+1}^P| \simeq \Omega_P^{\infty-1}\mathsf{Th}(U_{d+1,\infty}^\perp)^\wedge$ we need to show that this bijection is induced by an actual natural transformation of sheaves, $\mathbf{D}_{d+1}^P \to \mathrm{Maps}(\_, \Omega_P^{\infty-1}\mathsf{Th}(U_{d+1,\infty}^\perp)^\wedge)$. We proceed in a way very similar to [11, pp. 868–869]. We start with a definition.

Definition 7.8. Let $\pi : Y \to X$ be a submersion. Let $i_C : C \hookrightarrow Y$ be a smooth submanifold and suppose that $\pi|_C$ is still a submersion. A *vertical tubular neighborhood* for C in Y consists of an open embedding $e : N \to Y$ of the normal bundle of $i_C(C) \subset Y$, such that $e \circ s = i_C$ (where s is the zero-section) and $\pi \circ e = \pi \circ i_C \circ q$.

Using this definition, we define a variant of the sheaf $\mathbf{D}^P_{d+1,n}$ which we will denote by $\hat{\mathbf{D}}^P_{d+1,n}(X)$.

Definition 7.9. For $X \in \mathcal{O}b(\mathcal{X})$ we define $\hat{\mathbf{D}}^P_{d+1,n}(X)$ to be the set of pairs (W, e) where $W \in \mathbf{D}^P_{d+1,n}(X)$ and

$$e : N_W \to X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$$

is a vertical tubular neighborhood for W with respect to the submersion $\pi : W \to X$, subject to the following extra condition. The restriction of e to $N_{\partial_1 W} = N_{\beta_1 W} \times N_P$ (which is the normal bundle of $\partial_1 W$ in $X \times \mathbb{R} \times \{0\} \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$) is equal to the product $e_\beta \times e_P$ where

$$e_\beta : N_{\beta_W} \to X \times \mathbb{R} \times \{0\} \times \mathbb{R}^{d+\bar{n}}$$

is a vertical tubular neighborhood for $\beta_1 W$ and $e_P : N_P \to \mathbb{R}^{p+m}$ is the tubular neighborhood embedding, specified in (12) that was used in our construction of the spectrum $\mathsf{MT}_P(d+1)$.

We define

$$\hat{\mathbf{D}}^P_{d+1} := \operatorname*{colim}_{n \to \infty} \hat{\mathbf{D}}^P_{d+1,n},$$

where the colimit is taken in the category of sheaves on $\mathcal{X}$. It follows easily that $\hat{\mathbf{D}}^P_{d+1} = \operatorname{colim}_{n \to \infty}(\hat{\mathbf{D}}^P_{d+1,n}(X))$ for any closed manifold X, and thus there is a weak homotopy equivalence

$$|\hat{\mathbf{D}}^P_{d+1}| \simeq \operatorname*{colim}_{n \to \infty} |\hat{\mathbf{D}}^P_{d+1,n}|.$$

For each n there is a forgetful map $F_n : \hat{\mathbf{D}}^P_{d+1,n} \to \mathbf{D}^P_{d+1,n}$ defined by sending an element (W, e) to W. Passing to the direct limit as $n \to \infty$ yields a natural transformation

$$F : \hat{\mathbf{D}}^P_{d+1} \to \mathbf{D}^P_{d+1}$$

which by the existence and uniqueness up to isotopy of tubular neighborhoods induces a homotopy equivalence $|\hat{\mathbf{D}}^P_{d+1}| \simeq |\mathbf{D}^P_{d+1}|$.

Definition 7.10. For positive integers n and d, we define a sheaf $\mathcal{Z}^P_{d+1,n}$ on $\mathcal{X}$ by setting

$$\mathcal{Z}^P_{d+1,n}(X) := \operatorname{Maps}(X \times \mathbb{R}, \Omega^{d+n}_P \mathsf{Th}(U^\perp_{d+1,n})^\wedge)$$

for $X \in \mathcal{O}b(\mathcal{X})$. On the right-hand side, $\operatorname{Maps}(\_,\_)$ simply means here the set of maps with no topology given. These are strictly set-valued sheaves.

For each n the natural map

$$\Omega^{d+n}_P \mathsf{Th}(U^\perp_{d+1,n}) \to \Omega^{d+n+1}_P \mathsf{Th}(U^\perp_{d+1,n+1})$$

induces the natural transformation $\mathcal{Z}^P_{d+1,n} \to \mathcal{Z}^P_{d+1,n+1}$. We define

$$\mathcal{Z}^P_{d+1} := \operatorname*{colim}_{n \to \infty} \mathcal{Z}^P_{d+1,n}.$$

For each n, the map

$$j_0 : \mathcal{Z}^P_{d+1,n} \to \operatorname{Maps}(X, \Omega^{d+n}_P \mathsf{Th}(U^\perp_{d+1,n})^\wedge)$$

given by restricting a map in $\mathcal{Z}^P_{d+1,n}(X)$ to $X \times \{0\}$ is a weak equivalence of sheaves. In the limit $n \to \infty$, the maps j_0 induce a homotopy equivalence

$$|\mathcal{Z}^P_{d+1}| \simeq \Omega^{\infty-1}_P \mathsf{Th}(U^\perp_{d+1,\infty})^\wedge.$$

The *Pontryagin–Thom* construction yields a map of sheaves

$$\hat{T}_n : \hat{\mathbf{D}}^P_{d+1,n} \to \mathcal{Z}^P_{d+1,n}$$

for each n which we describe in detail. Let (W,e) be an element of $\hat{\mathbf{D}}^P_{d+1,n}(X)$. Since $(\pi, f) : W \to X \times \mathbb{R}$ is a proper map, for each $(x,t) \in X \times \mathbb{R}$ there exists a positive real number denoted by $\lambda(x,t) > 0$, such that the element

$$(x,t,z) \in X \times \mathbb{R} \times (\mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m})$$

lies in the complement of $W \subset X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$ whenever $|z| \geq \lambda(x,t)$ (where $|z|$ denotes the length of the vector z in the Euclidean metric). The numbers $\lambda(x,t)$ can be chosen to make $(x,t) \mapsto \lambda(x,t)$ a continuous function. It follows that the collapsing map

$$X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m} \to \mathsf{Th}(N_W)$$

induced by the vertical tubular embedding e extends to

$$X \times \mathbb{R} \times D^{d-1+n} \to \mathsf{Th}(N_W),$$

where D^{d-1+n} is viewed as the one-point-compactification of $\mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$. Putting together the above collapsing map and the functor $\mathsf{Th}(\underline{})$ applied to the Gauss maps of the normal bundles N_W and $N_{\partial_1 W} = N_{\beta_1 W} \times N_P$, we obtain the commutative diagram

$$
\begin{array}{ccccc}
X \times \mathbb{R} \times D^{d+n} & \longrightarrow & \mathsf{Th}(N_W) & \longrightarrow & \mathsf{Th}(U^\perp_{d+1,n}) \\
\big\uparrow & & & & \big\uparrow{\scriptstyle j_{d,n} \circ \tau_{P,n}} \\
X \times \mathbb{R} \times S^{d+\bar{n}} \wedge S^{m+p} & \longrightarrow & \mathsf{Th}(N_{\beta W}) \wedge S^{m+p} & \longrightarrow & \mathsf{Th}(U^\perp_{d-p,n-m}) \wedge S^{m+p}
\end{array}
$$

which yields an element of $\mathcal{Z}^P_{d+1,n}$ via adjunction. Let $\hat{T} : \hat{\mathbf{D}}^P_{d+1} \to \mathcal{Z}^P_{d+1}$ denote the induced map in the direct limit as $n \to \infty$. The concordance class functors associated to our newly defined sheaves fit into the commutative diagram

$$
\begin{array}{ccc}
\hat{\mathbf{D}}^P_{d+1}[\underline{}] & \xrightarrow{\hat{T}_*} & \mathcal{Z}^P_{d+1}[\underline{}] \\
{\scriptstyle \cong}\big\downarrow & & {\scriptstyle \cong}\big\downarrow \\
\mathbf{D}^P_{d+1}[\underline{}] & \xrightarrow{T} & [\underline{}, \Omega^{\infty-1}_P \mathsf{Th}(U^\perp_{d+1,\infty})^\wedge]
\end{array}
$$

where T is the natural transformation defined in Section 7.2, the vertical maps are induced by F and j_0. The bottom row is an isomorphism whenever applied to a compact manifold, and so, by commutativity, the top map is as well. It follows that for each $k \geq 0$, the map $|\hat{T}|$ induces a bijection

$$|\hat{T}| : [S^k, |\hat{\mathbf{D}}^P_{d+1}|] \xrightarrow{\cong} [S^k, \Omega^{\infty-1}_P \mathsf{Th}(U^\perp_{d+1,\infty})^\wedge].$$

However, these are isomorphisms of sets of homotopy classes of unbased maps and not isomorphisms of the actual homotopy groups. In order to prove that the map $|\hat{T}|$ is a homotopy equivalence, we need the following proposition whose proof is similar to [11, Thm. 3.8]. We give the proof in Section 11.

Proposition 7.11. *The map $|\hat{T}|$ induces an isomorphism on all homotopy groups with respect to any choice of basepoint.*

Proposition 7.11 together with Corollary 6.11 implies Theorem 7.1 which states that there is a weak homotopy equivalence

$$|\mathbf{D}_{d+1}^{P}| \simeq \Omega^{\infty-1}\mathsf{MT}_P(d+1).$$

By Remark 6.2 we have that if P is cobordant to a closed manifold P' then there is a homotopy equivalence $\Omega^{\infty-1}\mathsf{MT}_P(d+1) \simeq \Omega^{\infty-1}\mathsf{MT}_{P'}(d+1)$. This implies the following corollary.

Corollary 7.12. *If the closed manifolds P and P' are cobordant, then there is a weak homotopy equivalence $|\mathbf{D}_{d+1}^{P}| \simeq |\mathbf{D}_{d+1}^{P'}|$.*

8. The classifying space of $\mathbf{Cob}_{d+1}^{P}$

In this section we construct a weak homotopy equivalence

$$|\mathbf{D}_{d+1}^{P}| \simeq \mathbf{BCob}_{d+1}^{P}.$$

Combining this weak equivalence with the results of the previous section implies Theorem 1.1.

8.1. **Category-valued sheaves.** We will need to consider sheaves on $\mathcal{X}$ that are valued in the category of small categories, which is denoted by **CAT**.

Definition 8.2. A contravariant functor $\mathcal{F} : \mathcal{X} \to \mathbf{CAT}$ is said to be a **CAT**-valued sheaf if for any $X \in \mathcal{O}b(\mathcal{X})$, $\mathcal{F}(X)$ satisfies the same gluing condition described in Section 5.1 for set-valued sheaves, with respect to any open cover of X.

We now recall some important facts about **CAT**-valued sheaves. Let $\mathcal{F}$ be a **CAT**-valued sheaf. For each nonnegative integer k, there is an auxiliary set-valued sheaf denoted by $N_k\mathcal{F}$ which is defined by setting $N_k\mathcal{F}(X)$ equal to the k-th level of the nerve of the category $\mathcal{F}(X)$. The representing space $|\mathcal{F}|$ naturally obtains the structure of a topological category by setting

$$\mathcal{O}b(|\mathcal{F}|) = |N_0\mathcal{F}| \quad \text{and} \quad \mathcal{M}or(|\mathcal{F}|) = |N_1\mathcal{F}|.$$

The classifying space $B|\mathcal{F}|$ can be constructed by taking the geometric realization of the *diagonal* simplicial set, $k \mapsto N_k\mathcal{F}(\triangle_e^k)$.

If $\mathcal{C}$ is a topological category with a smooth structure (i.e., $\mathcal{C}$ is a smooth manifold or an infinite-dimensional *Banach manifold* like in the case with $\mathbf{Cob}_{d+1}^{P}$), then for $X \in \mathcal{O}b(\mathcal{X})$, the set of smooth maps $C^\infty(X,\mathcal{C})$ has the structure of a small category by pointwise composition. The contravariant functor $C^\infty(\_\,,\mathcal{C})$ defines a **CAT**-valued sheaf on $\mathcal{X}$. As with all **CAT**-valued

sheaves, the representing space $|C^\infty(\_,\mathcal{C})|$ has the structure of a topological category. In this way, the natural map

$$|C^\infty(\_,\mathcal{C})| \to \mathcal{C} \qquad (30)$$

defined between the geometric realization of the singular complex of a space and the space itself (see [12]) is a continuous functor. It follows from [6] that (30) induces a weak-homotopy equivalence

$$B|C^\infty(\_,\mathcal{C})| \to B\mathcal{C}. \qquad (31)$$

Note that the above construction and weak homotopy equivalence hold for the **CAT**-valued sheaf $\mathrm{Maps}(\_,\mathcal{C})$ as well.

8.3. **Cocycle sheaves.** We will need to use another important construction from [6, 11] relating to **CAT**-valued sheaves. For the next definition, fix once and for all an uncountable set J. For elements $X \in \mathcal{O}b(\mathcal{X})$ we will need to consider open covers $\mathcal{U}$ of X indexed by the set J. For subsets $S \subset J$, we denote $U_S := \bigcap_{i \in S} U_i$.

Definition 8.4. Let $\mathcal{F}$ be a **CAT**-valued sheaf on $\mathcal{X}$. For each $X \in \mathcal{O}b(\mathcal{X})$, $\bar{\beta}\mathcal{F}(X)$ is defined to be the set of pairs $(\mathcal{U},\Phi)$ where $\mathcal{U} = \{U_i \mid j \in J\}$ is an open cover of $\mathcal{X}$ indexed by J, and Φ is collection of morphisms, $\varphi_{RS} \in N_1\mathcal{F}(U_S)$, indexed by the pairs $R \subseteq S$ of nonempty finite subsets of J, subject to the following conditions:

(i) $\varphi_{RR} = \mathrm{Id}_{C_R}$ for some object $C_R \in N_0\mathcal{F}(U_R)$.
(ii) For each nonempty finite $R \subseteq S$, φ_{RS} is a morphism from C_S to $C_R|_{U_S}$.
(iii) For all triples $R \subseteq S \subseteq T$ of finite nonempty subsets of J, we have

$$\varphi_{RT} = (\varphi_{RS}|_{U_T}) \circ \varphi_{ST}.$$

It can be verified that for any **CAT**-valued sheaf $\mathcal{F}$ on $\mathcal{X}$, the assignment $X \mapsto \bar{\beta}\mathcal{F}(X)$ defines a set-valued sheaf on $\mathcal{X}$. The sheaf $\bar{\beta}\mathcal{F}$ is called the *cocycle-sheaf* associated to $\mathcal{F}$.

In [11] it is proven that for any **CAT**-valued sheaf $\mathcal{F}$ on $\mathcal{X}$, there is a weak homotopy equivalence $|\bar{\beta}\mathcal{F}| \simeq B|\mathcal{F}|$, where $|\bar{\beta}\mathcal{F}|$ is the representing space of the set-valued sheaf $\bar{\beta}\mathcal{F}$ and $B|\mathcal{F}|$ is the classifying space of the topological category $|\mathcal{F}|$. This homotopy equivalence is natural in the following sense.

Remark 8.5. Since any set may by considered a category with only identity morphisms, a set-valued sheaf $\mathcal{F}$ on $\mathcal{X}$ may be considered a **CAT**-valued sheaf by considering $\mathcal{F}(X)$ a category with only identity morphisms, for $X \in \mathcal{X}$. In this way, we may consider $\bar{\beta}\mathcal{F}$. For any $X \in \mathcal{O}b(\mathcal{X})$, $\bar{\beta}\mathcal{F}(X)$ reduces to the set of pairs $(\mathcal{U},\Phi)$ where $\mathcal{U}$ is a cover of X (indexed by the designated uncountable set J from Definition 8.4) and Φ is a collection of elements $\varphi_S \in \mathcal{F}(U_S)$ for $S \subset J$, which are compatible under restrictions. Using the sheaf gluing condition, any element $(\mathcal{U},\Phi) \in \bar{\beta}\mathcal{F}(X)$ induces a unique element $\varphi \in \mathcal{F}(X)$ such that $\varphi|_{U_S} = \varphi_S$ for all subsets $S \subset J$. This correspondence $(\mathcal{U},\Phi) \mapsto \varphi$ yields a natural isomorphism

$$\bar{\beta}\mathcal{F} \xrightarrow{\cong} \mathcal{F}.$$

Notation 8.6. In [6, 11], the notation $\beta\mathcal{F}$ is used for the above cocycle-sheaf construction. Since in this paper, the Greek letter β is already used so heavily, we denote the cocycle-sheaves by $\bar{\beta}\mathcal{F}$ so as to not have any conflict of notation.

8.7. **Sheaf models for $\mathbf{Cob}_{d+1}^P$.** We will need to consider the **CAT**-valued sheaf $C^\infty(\_, \mathbf{Cob}_{d+1}^P)$ with composition defined pointwise. By (31) there is a weak homotopy equivalence

$$B|C^\infty(\_, \mathbf{Cob}_{d+1}^P)| \simeq B\mathbf{Cob}_{d+1}^P$$

induced by the natural map $|C^\infty(\_, \mathbf{Cob}_{d+1}^P)| \to \mathbf{Cob}_{d+1}^P$. In this section we will define three new **CAT**-valued sheaves $\mathbf{C}_{d+1}^P$, $\mathbf{C}_{d+1}^{P,\pitchfork}$, and $\mathbf{D}_{d+1}^{P,\pitchfork}$, along with a zig-zag of natural transformations

$$C^\infty(\_, \mathbf{Cob}_{d+1}^P) \xrightarrow{\eta} \mathbf{C}_{d+1}^P \xrightarrow{i} \mathbf{C}_{d+1}^{P,\pitchfork} \xleftarrow{\alpha} \mathbf{D}_{d+1}^{P,\pitchfork} \xrightarrow{\gamma} \mathbf{D}_{d+1}^P$$

(on the right, $\mathbf{D}_{d+1}^P$ is considered a **CAT**-valued sheaf with only identity morphisms) which induce weak homotopy equivalences

$$B\mathbf{Cob}_{d+1}^P \xrightarrow{\simeq} B|\mathbf{C}_{d+1}^P| \xrightarrow{\simeq} B|\mathbf{C}_{d+1}^{P,\pitchfork}| \xleftarrow{\simeq} B|\mathbf{D}_{d+1}^{P,\pitchfork}| \xrightarrow{\simeq} |\mathbf{D}_{d+1}^P|.$$

These weak equivalences together with Theorem 7.1 will imply Theorem 1.1. The constructions of this section closely follow [6, 7].

Remark 8.8. The **CAT**-valued sheaves $\mathbf{C}_{d+1}^P$, $\mathbf{C}_{d+1}^{P,\pitchfork}$, and $\mathbf{D}_{d+1}^{P,\pitchfork}$ correspond directly to the **CAT**-valued sheaves from [6] denoted by $\mathbf{C}_{d+1}$, $\mathbf{C}_{d+1}^{\pitchfork}$, and $\mathbf{D}_{d+1}^{\pitchfork}$. In particular, these sheaves are isomorphic to the ones defined in this section in the case that $P = \varnothing$.

Let $i_P : P \hookrightarrow \mathbb{R}^{p+m}$ be the embedding used in the construction of $\mathbf{Cob}_{d+1}^P$ and throughout the paper.

Notation 8.9. For $X \in \mathcal{O}b(\mathcal{X})$ and smooth functions $a, b : X \to \mathbb{R}$ with $a(x) \leq b(x)$ for all $x \in X$, we denote

$$X \times [a, b] := \{(x, u) \in X \times \mathbb{R} \mid a(x) < u < b(x)\},$$
$$X \times (a, b) := \{(x, u) \in X \times \mathbb{R} \mid a(x) \leq u \leq b(x)\}.$$

Definition 8.10. Let $\varepsilon > 0$ be a real number, $X \in \mathcal{O}b(\mathcal{X})$, and $a, b : X \to \mathbb{R}$ smooth functions with $a(x) \leq b(x)$ for all $x \in X$. We define $\mathbf{C}_{d+1}^{P,\pitchfork}(X; a, b, \varepsilon)$ to be the set of $(d+1+\dim(X))$-dimensional P-submanifolds (see Definition 3.11)

$$W \subset X \times (a - \varepsilon, b + \varepsilon) \times \mathbb{R}_+ \times \mathbb{R}^{d+\infty} \times \mathbb{R}^{p+m}$$

which satisfy the following conditions:

(i) The projection of $\pi : W \to X$ is a P-submersion with $(d+1)$-dimensional fibers.

(ii) The projection of W onto $X \times (a - \varepsilon, b + \varepsilon)$, denoted by $(\pi, f) : W \to X \times (a - \varepsilon, b + \varepsilon)$, is a proper P-map.

(iii) The restriction of (π, f) to $(\pi, f)^{-1}(X \times (\nu - \varepsilon, \nu + \varepsilon))$ for $\nu = a, b$ is a P-submersion.

Remark 8.11. Condition (iii) of the above definition implies that the map π is a local-trivial fiber bundle and not just a submersion. The manifold $\hat{W} := (\pi, f)^{-1}(X \times [a, b])$ is a P-manifold of dimension $d+1$ with boundary given by

$$\partial_0 \hat{W} = (\pi, f)^{-1}(X \times \{a\}) \sqcup (\pi, f)^{-1}(X \times \{b\}).$$

The restriction of π to $\partial_0 \hat{W}$ is a fiber bundle with closed P-manifold fibers.

We eliminate dependence on ε by setting

$$\mathbf{C}_{d+1}^{P,\pitchfork}(X; a, b) := \operatorname*{colim}_{\varepsilon \to 0} \mathbf{C}_{d+1}^{P,\pitchfork}(X; a, b, \varepsilon).$$

Definition 8.12. We define

$$\mathbf{C}_{d+1}^{P,\pitchfork}(X) := \bigsqcup \mathbf{C}_{d+1}^{P,\pitchfork}(X; a, b)$$

with union ranging over all pairs of smooth real-valued functions (a, b) with $a \leq b$.

This definition makes $\mathbf{C}_{d+1}^{P,\pitchfork}$ into a **CAT**-valued sheaf.

Definition 8.13. For $X \in \mathcal{X}$, smooth functions $a, b : X \to \mathbb{R}$ with $a(x) \leq b(x)$ for all $x \in X$, and a real number $\varepsilon > 0$, we define

$$\mathbf{C}_{d+1}^{P}(X, a, b, \varepsilon) \subset \mathbf{C}_{d+1}^{P,\pitchfork}(X, a, b, \varepsilon)$$

to be the subset consisting of all elements $W \in \mathbf{C}_{d+1}^{P,\pitchfork}(X, a, b, \varepsilon)$ which satisfy the further condition:

(iv) For $\nu = a, b$ and $x \in X$, let J_ν be the interval $((\nu - \epsilon_0)(x), (\nu + \epsilon_0)(x)) \subseteq \mathbb{R}$ and let

$$V_\nu = (\pi, f)^{-1}(\{x\} \times J_\nu) \subset \{x\} \times J_\nu \times \mathbb{R}_+ \times \mathbb{R}^{d+\infty} \times \mathbb{R}^{p+m}.$$

Then

$$V_\nu = \{x\} \times J_\nu \times M \subset \{x\} \times J_\nu \times \mathbb{R}_+ \times \mathbb{R}^{d+\infty} \times \mathbb{R}^{p+m}$$

holds for some d-dimensional submanifold $M \subset \mathbb{R}_+ \times \mathbb{R}^{d+\infty} \times \mathbb{R}^{p+m}$.

It follows that the boundary $\partial M = M \cap (\{x\} \times J_a \times \mathbb{R}_+ \times \mathbb{R}^{d+\infty} \times \mathbb{R}^{p+m})$ has the factorization $\partial_1 M = \beta_1 M \times i_P(P)$.

The above definition should be compared to [6, Def. 2.8]. We then may define $\mathbf{C}_{d+1}^{P}$ in the same way as for $\mathbf{C}_{d+1}^{P,\pitchfork}$ by taking the limits as $\epsilon_0, \epsilon_1 \to 0$ and taking the disjoint union over all pairs of real-valued functions $a, b : X \to \mathbb{R}$ such that $a(x) \leq b(x)$.

Using the topological structure on $\mathbf{Cob}_{d+1}^{P}$ given in (8) together with Lemmas 3.16 and 3.17 regarding fiber bundles with P-manifold fibers, there is a natural isomorphism

$$\beta : C^\infty(\_, \mathbf{Cob}_{d+1}^{P}) \xrightarrow{\cong} \mathbf{C}_{d+1}^{P},$$

given by sending a smooth map $f : X \to \mathbf{Cob}_{d+1}^{P}$ to the fiber bundle of P-manifold cobordisms over X that f induces by pullback. By (31), this isomorphism induces a weak homotopy equivalence $B\mathbf{Cob}_{d+1}^{P} \simeq B|\mathbf{C}_{d+1}^{P}|$.

Denote by $i : \mathbf{C}_{d+1}^P \to \mathbf{C}_{d+1}^{P,\pitchfork}$ the natural transformation induced by inclusion. The following proposition is proven in exactly the same way as [6, Prop. 4.4].

Proposition 8.14. *The inclusion map* $i : \mathbf{C}_{d+1}^P \to \mathbf{C}_{d+1}^{P,\pitchfork}$ *induces a weak homotopy equivalence*

$$B|\mathbf{C}_{d+1}^P| \simeq B|\mathbf{C}_{d+1}^{P,\pitchfork}|.$$

We now define a new sheaf which can be compared directly to $\mathbf{D}_{d+1}^P$.

Definition 8.15. For $X \in \mathcal{O}b(\mathcal{X})$, we define $\mathbf{D}_{d+1}^{P,\pitchfork}(X)$ to be the set of pairs $(W, a) \in \mathbf{D}_{d+1}^P(X) \times C^\infty(X, \mathbb{R})$ such that for all $x \in X$, the restriction map

$$f|_{\pi^{-1}(x)} : \pi^{-1}(x) \to \mathbb{R}$$

is transverse to $a(x) \in \mathbb{R}$. We say that f is *fiberwise transverse* to $a : X \to \mathbb{R}$ with respect to the submersion π. The set $\mathbf{D}_{d+1}^{P,\pitchfork}(X)$ is then given the structure of a partially ordered set by declaring $(W, a) \leq (V, b)$ if $W = V$ and $a(x) \leq b(x)$ for all $x \in X$. By considering the partially ordered set $\mathbf{D}_{d+1}^{P,\pitchfork}(X)$, a category $\mathbf{D}_{d+1}^{P,\pitchfork}$ is a **CAT**-valued sheaf of $\mathcal{X}$.

There is a natural transformation

$$\alpha : \mathbf{D}_{d+1}^{P,\pitchfork} \to \mathbf{C}_{d+1}^{P,\pitchfork}$$

which is defined in exactly the same way as the map $\mathbf{D}_{d+1}^{\pitchfork} \to \mathbf{C}_{d+1}^{\pitchfork}$ on [6, p. 17] for the corresponding sheaves for non-singular manifolds. The following proposition is proved in exactly the same way as [6, Prop. 4.4].

Proposition 8.16. *The natural transformation* $\alpha : \mathbf{D}_{d+1}^{P,\pitchfork} \to \mathbf{C}_{d+1}^{P,\pitchfork}$ *induces a weak homotopy equivalence* $B|\mathbf{D}_{d+1}^{P,\pitchfork}| \simeq B|\mathbf{C}_{d+1}^{P,\pitchfork}|$.

We now compare $\mathbf{D}_{d+1}^{P,\pitchfork}$ to $\mathbf{D}_{d+1}^P$. By considering $\mathbf{D}_{d+1}^P$ as a **CAT**-valued sheaf with only identity morphisms, there is a forgetful functor $\gamma : \mathbf{D}_{d+1}^{P,\pitchfork} \to \mathbf{D}_{d+1}^P$ defined by sending $(W, a) \in \mathbf{D}_{d+1}^{P,\pitchfork}(X)$ to $W \in \mathbf{D}_{d+1}^P(X)$. This induces a natural transformation

$$\bar{\beta}\gamma : \bar{\beta}\mathbf{D}_{d+1}^{P,\pitchfork} \to \bar{\beta}\mathbf{D}_{d+1}^P \cong \mathbf{D}_{d+1}^P,$$

where $\bar{\beta}\mathbf{D}_{d+1}^P \cong \mathbf{D}_{d+1}^P$ is the isomorphism from Remark 8.5. The next proposition is proven in the same way as [6, Prop. 4.2].

Proposition 8.17. *The map* $\bar{\beta}\gamma : \bar{\beta}\mathbf{D}_{d+1}^{P,\pitchfork} \to \mathbf{D}_{d+1}^P$ *induces a weak homotopy equivalence* $B|\mathbf{D}_{d+1}^{P,\pitchfork}| \simeq |\mathbf{D}_{d+1}^P|$.

The last four propositions imply that there is a weak homotopy equivalence

$$|\mathbf{D}_{d+1}^P| \simeq B\mathbf{Cob}_{d+1}^P.$$

Combining this with Theorem 7.1 yields the weak homotopy equivalence

$$B\mathbf{Cob}_{d+1}^P \simeq \Omega_P^{\infty-1}\mathsf{MT}(d+1),$$

thus completing the proof of Theorem 1.1.

8.18. **The Bockstein functor.** Recall the cobordism category $\mathbf{Cob}_{d+1}$ from [6]. Setting P equal to $\varnothing$, we have an isomorphism of topological categories

$$\mathbf{Cob}_{d+1}^{\varnothing} \cong \mathbf{Cob}_{d+1}.$$

We now consider the functor $\beta_1 : \mathbf{Cob}_{d+1}^{P} \to \mathbf{Cob}_{d-p}$ defined by sending a P-subcobordism

$$W \subset [a,b] \times \mathbb{R}_+ \times \mathbb{R}^{d+\infty} \times \mathbb{R}^{p+m}$$

to the embedded (non-singular) cobordism given by $\beta_1 W \subset [a,b] \times \{0\} \times \mathbb{R}^{d+\infty}$. The functor β_1 is defined similarly on objects. Furthermore, the category $\mathbf{Cob}_{d+1}$ is isomorphic to the subcategory of $\mathbf{Cob}_{d+1}^{P}$ consisting of all submanifolds

$$W \subset [a,b] \times \mathbb{R}_+ \times \mathbb{R}^{d+\infty} \times \mathbb{R}^{p+m}$$

such that

$$W \cap ([a,b] \times \{0\} \times \mathbb{R}^{d+\infty} \times \mathbb{R}^{p+m}) = \varnothing.$$

We denote by

$$i : \mathbf{Cob}_{d+1} \to \mathbf{Cob}_{d+1}^{P}$$

the inclusion functor. Theorem 1.3 states that the composition of functors

$$\mathbf{Cob}_{d+1} \xrightarrow{i} \mathbf{Cob}_{d+1}^{P} \xrightarrow{\beta_1} \mathbf{Cob}_{d-p}$$

induces a homotopy fiber sequence on the level of classifying spaces,

$$B\mathbf{Cob}_{d+1} \xrightarrow{B(i)} B\mathbf{Cob}_{d+1}^{P} \xrightarrow{B(\beta_1)} B\mathbf{Cob}_{d-p}.$$

We now give a proof of Theorem 1.3.

Proof of Theorem 1.3. There are versions of the functors i_d and β_1 defined on the sheaf level which yield a commutative diagram of natural transformations:

$$
\begin{array}{ccccccccc}
C^\infty(\cdot, \mathbf{Cob}_{d+1}) & \xrightarrow{\eta} & \mathbf{C}_{d+1} & \xrightarrow{i} & \mathbf{C}_{d+1}^{\pitchfork} & \xleftarrow{\alpha} & \mathbf{D}_{d+1}^{\pitchfork} & \xrightarrow{\gamma} & \mathbf{D}_{d+1} \\
\downarrow{\scriptstyle i_d} & & \downarrow{\scriptstyle i_d} & & \downarrow{\scriptstyle i_d} & & \downarrow{\scriptstyle i_d} & & \downarrow{\scriptstyle i_d} \\
C^\infty(\cdot, \mathbf{Cob}_{d+1}^{P}) & \xrightarrow{\eta} & \mathbf{C}_{d+1}^{P} & \xrightarrow{i} & \mathbf{C}_{d+1}^{P,\pitchfork} & \xleftarrow{\alpha} & \mathbf{D}_{d+1}^{P,\pitchfork} & \xrightarrow{\gamma} & \mathbf{D}_{d+1}^{P} \\
\downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_1} \\
C^\infty(\cdot, \mathbf{Cob}_{d-p}) & \xrightarrow{\eta} & \mathbf{C}_{d-p} & \xrightarrow{i} & \mathbf{C}_{d-p}^{\pitchfork} & \xleftarrow{\alpha} & \mathbf{D}_{d-p}^{\pitchfork} & \xrightarrow{\gamma} & \mathbf{D}_{d-p}.
\end{array}
$$

This diagram of natural transformations then induces a commutative diagram of maps of spaces,

$$(32) \quad \begin{array}{ccccccccc}
B\mathbf{Cob}_{d+1} & \xrightarrow[\simeq]{B\eta} & B|\mathbf{C}_{d+1}| & \xrightarrow[\simeq]{Bi} & B|\mathbf{C}_{d+1}^{\pitchfork}| & \xleftarrow[\simeq]{B\alpha} & B|\mathbf{D}_{d+1}^{\pitchfork}| & \xrightarrow[\simeq]{B\gamma} & |\mathbf{D}_{d+1}| \\
\downarrow{\scriptstyle Bi_d} & & \downarrow{\scriptstyle Bi_d} & & \downarrow{\scriptstyle Bi_d} & & \downarrow{\scriptstyle Bi_d} & & \downarrow{\scriptstyle Bi_d} \\
B\mathbf{Cob}_{d+1}^P & \xrightarrow[\simeq]{B\eta} & B|\mathbf{C}_{d+1}^P| & \xrightarrow[\simeq]{Bi} & B|\mathbf{C}_{d+1}^{P,\pitchfork}| & \xleftarrow[\simeq]{B\alpha} & B|\mathbf{D}_{d+1}^{P,\pitchfork}| & \xrightarrow[\simeq]{B\gamma} & |\mathbf{D}_{d+1}^P| \\
\downarrow{\scriptstyle B\beta_1} & & \downarrow{\scriptstyle B\beta_1} & & \downarrow{\scriptstyle B\beta_1} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle B\beta_1} \\
B\mathbf{Cob}_{d-p} & \xrightarrow[\simeq]{B\eta} & B|\mathbf{C}_{d-p}| & \xrightarrow[\simeq]{Bi} & B|\mathbf{C}_{d-p}^{\pitchfork}| & \xleftarrow[\simeq]{B\alpha} & B|\mathbf{D}_{d-p}^{\pitchfork}| & \xrightarrow[\simeq]{B\gamma} & |\mathbf{D}_{d-p}|
\end{array}$$

such that all horizontal maps are weak homotopy equivalences. Now consider the fiber sequence of infinite loop-spaces

$$\Omega^{\infty-1}\mathsf{MT}(d+1) \to \Omega^{\infty-1}\mathsf{MT}_P(d+1) \to \Omega^{\infty-1}\mathsf{MT}(d-p)$$

which is induced by the cofiber sequence of spectra

$$\Sigma^{-1}\mathsf{MT}(d-p) \to \mathsf{MT}(d+1) \to \mathsf{MT}_P(d+1).$$

The weak homotopy equivalences constructed in the proof of Theorem 7.1 yield a homotopy commutative diagram,

$$(33) \quad \begin{array}{ccc}
|\mathbf{D}_{d+1}| & \xrightarrow{\simeq} & \Omega^{\infty-1}\mathsf{MT}(d+1) \\
\downarrow{\scriptstyle |i|} & & \downarrow \\
|\mathbf{D}_{d+1}^P| & \xrightarrow{\simeq} & \Omega^{\infty-1}\mathsf{MT}_P(d+1) \\
\downarrow{\scriptstyle |\beta_1|} & & \downarrow \\
|\mathbf{D}_{d-p}| & \xrightarrow{\simeq} & \Omega^{\infty-1}\mathsf{MT}(d-p)
\end{array}$$

where the horizontal maps are weak equivalences. Homotopy commutativity of (33), together with the fact that the right column is a fiber sequence, implies that

$$|\mathbf{D}_{d+1}| \xrightarrow{|i_d|} |\mathbf{D}_{d+1}^P| \xrightarrow{|\beta_1|} |\mathbf{D}_{d-p}|$$

is a homotopy fiber sequence. Then, commutativity of (32) implies that

$$B\mathbf{Cob}_{d+1} \xrightarrow{B|i_d|} B\mathbf{Cob}_{d+1}^P \xrightarrow{B|\beta_1|} B\mathbf{Cob}_{d-p}$$

is a homotopy fiber sequence. This completes the proof of Theorem 1.3. $\square$

9. The space of P-submersions

In this section we prove a result which implies Claim 7.6 used in the proof of Lemma 7.4. Our result is a version of *Phillips' submersion theorem* [13] for P-manifolds.

Before stating our main result, we must review some submersion theory. Let X and Y be smooth manifolds. We denote by $\mathrm{Sub}(X,Y)$ the space of submersions $X \to Y$, topologized in the C^∞-topology. We denote by $\mathrm{Sub}^f(X,Y)$ the space of fiberwise surjective bundle maps $TX \to TY$. We will express elements of $\mathrm{Sub}^f(X,Y)$ as pairs $(\hat{f}, f)$ where $\hat{f}: TX \to TY$ is a fiberwise surjective bundle map and $f: X \to Y$ is the continuous map which underlies $\hat{f}$, i.e., f is the unique map from X to Y such that the diagram

$$
\begin{array}{ccc}
TX & \xrightarrow{\ \hat{f}\ } & TY \\
\downarrow & & \downarrow \\
X & \xrightarrow{\ f\ } & Y
\end{array}
$$

commutes, where the vertical maps are the bundle projections. Elements of $\mathrm{Sub}^f(X,Y)$ are referred to as *formal submersions*. Since f is uniquely determined by $\hat{f}$, there is redundancy in this notation, however it will be useful to keep track of the underlying map. There is a map

$$ D : \mathrm{Sub}(X,Y) \to \mathrm{Sub}^f(X,Y) $$

given by sending a submersion $f : X \to Y$ to the pair (df, f) where $df : TX \to Y$ is the differential of f. Clearly, the map D is an embedding. Now, a smooth manifold is said to be an *open manifold* if it has no compact components. The main theorem from [13] is the following.

Theorem 9.1 (Phillips 1967). *Let X be an open smooth manifold and let Y be a smooth manifold without boundary. Then the embedding $D : \mathrm{Sub}(X,Y) \to \mathrm{Sub}^f(X,Y)$ is a weak homotopy equivalence.*

For what follows, let W be a P-manifold and let X be a smooth manifold. Recall from Section 2 that a smooth P-map $f : W \to X$ is said to be a *P-submersion* if both f and $f_{\beta_1} : \beta_1 W \to X$ are submersions. We denote by $\mathrm{Sub}_P(W,X)$ the space of P-submersions $W \to X$, topologized as a subspace of the space of smooth maps from W to X. Below we define the space of *formal P-submersions*.

Definition 9.2. We denote by $\mathrm{Sub}_P^f(W,X)$ the subspace of $\mathrm{Sub}^f(W,X)$ consisting of all formal submersions $(\hat{f}, f)$ which satisfy the following:

(i) The underlying map $f : W \to X$ is a P-map.
(ii) There exists a fiberwise surjective bundle map $\hat{f}_{\beta_1} : T\beta_1 W \to TX$ such that the restriction of the bundle map $\hat{f} : TW \to TX$ to the sub-bundle $T\partial_1 W \subset TW|_{\partial_1 W}$ (defined over $\partial_1 W$) has the factorization

$$
T\partial_1 W \xrightarrow{\ d\phi_1\ } T\beta_1 W \times TP \xrightarrow{\ \mathrm{proj.}\ } T\beta_1 W \xrightarrow{\ \hat{f}_\beta\ } TX,
$$

where $d\phi_1$ is the differential of the structure map

$$ \phi_1 : \partial_1 W \xrightarrow{\ \cong\ } \beta_1 W \times P. $$

Notice that condition (ii) implies that $\hat{f}|_{T\partial_1 W} : T\partial_1 W \to TX$ is fiberwise surjective. Notice also that the correspondence $(\hat{f}, f) \mapsto (\hat{f}_{\beta_1}, f_{\beta_1})$ defines a continuous map

$$\beta_1 : \operatorname{Sub}_P^f(W, X) \to \operatorname{Sub}^f(\beta_1 W, X).$$

The embedding $D : \operatorname{Sub}(W, X) \to \operatorname{Sub}^f(W, X)$ from Theorem 9.1 restricts to an embedding

$$D_P : \operatorname{Sub}_P(W, X) \to \operatorname{Sub}_P^f(W, X).$$

Definition 9.3. A P-manifold W is said to be an *open P-manifold* if $\partial_0 W = \varnothing$ and both $\beta_1 W$ and W have no compact components.

We have the following generalization of Theorem 9.1.

Theorem 9.4. *Let W be an open P-manifold and let X be a smooth manifold without boundary. Then the embedding $D_P : \operatorname{Sub}_P(W, X) \to \operatorname{Sub}_P^f(W, X)$ is a weak homotopy equivalence.*

The above theorem is proven in stages. We must first derive some intermediate results.

Lemma 9.5. *The following map is a Serre-fibration:*

$$\beta_1 : \operatorname{Sub}_P^f(W, X) \to \operatorname{Sub}^f(\beta_1 W, X), \quad (\hat{f}, f) \mapsto (\hat{f}_{\beta_1}, f_{\beta_1}).$$

Proof. Denote by $\operatorname{Sub}^f(W, X)|_{\partial_1 W}$ the space of pairs $(\hat{g}, g)$ where

$$\hat{g} : TW|_{\partial_1 W} \to TX$$

is a fiberwise surjective bundle map and $g : \partial_1 W \to X$ is map which makes the diagram

$$
\begin{array}{ccc}
TW|_{\partial_1 W} & \xrightarrow{\hat{g}} & TX \\
\downarrow & & \downarrow \\
\partial_1 W & \xrightarrow{g} & X
\end{array}
$$

commute. We have a restriction map

$$(34) \qquad r|_{\partial_1} : \operatorname{Sub}^f(W, X) \to \operatorname{Sub}^f(W, X)|_{\partial_1 W}, \quad (\hat{f}, f) \mapsto (\hat{f}|_{\partial_1 W}, f|_{\partial_1 W})$$

which is a Serre-fibration by [13, Lem. 5.3].

From Example 2.6 we see that TW has the structure of a P-vector bundle. The collar embedding $h_1 : \partial_1 W \times \mathbb{R}_+ \to W$, together with the differential of the structure map

$$\phi_1 : \partial_1 W \xrightarrow{\cong} \beta_1 W \times P,$$

induces a bundle isomorphism

$$(35) \qquad\qquad \hat{\phi}_1 : TW|_{\partial_1 W} \xrightarrow{\cong} (T\beta_1 W \oplus \epsilon^1) \times TP,$$

which covers ϕ_1 (above, ϵ^1 is the one-dimensional trivial bundle over $\beta_1 W$). Using (35), we define a map

$$(36) \qquad\qquad T_P : \operatorname{Sub}^f(\beta_1 W, X) \to \operatorname{Sub}^f(W, X)|_{\partial_1 W}$$

by sending a fiberwise surjective bundle map $\hat{g} : T\beta_1 W \to TX$ to the fiberwise surjective bundle map $TW|_{\partial_1 W} \to TX$ given by the composition

$$TW|_{\partial_1 W} \xrightarrow[\cong]{\hat{\phi}_1} (T\beta_1 W \oplus \epsilon^1) \times TP \xrightarrow{\text{proj.}} T\beta_1 W \xrightarrow{\hat{g}} TX.$$

It follows immediately from the definition of (36) that the diagram

$$
\begin{array}{ccc}
\mathrm{Sub}_P^f(W, X) & \hookrightarrow & \mathrm{Sub}^f(W, X) \\
\downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle r|_{\partial_1}} \\
\mathrm{Sub}^f(\beta_1 W, X) & \xrightarrow{T_P} & \mathrm{Sub}^f(W, X)|_{\partial_1 W}
\end{array}
$$

is a cartesian square. Since (34) is a Serre-fibration, the fact that the above diagram is cartesian implies that $\beta_1 : \mathrm{Sub}_P^f(W, X) \to \mathrm{Sub}^f(\beta_1 W, X)$ is a Serre-fibration as well. This concludes the proof of the lemma. □

We need to define an intermediate space which can be compared to both $\mathrm{Sub}_P(W, X)$ and $\mathrm{Sub}_P^f(W, X)$.

Definition 9.6. For a P-manifold W and smooth manifold X, denote by $\mathrm{Sub}_{P,\beta}^f(W, X)$ the subspace of $\mathrm{Sub}_P^f(W, X)$ consisting of all formal submersions $(\hat{f}, f)$ such that $\hat{f}_{\beta_1} = df_{\beta_1}$ where df_{β_1} denotes the differential of f_{β_1}.

Lemma 9.7. *Let W be an open P-manifold and let X be a smooth manifold. The inclusion $\mathrm{Sub}_{P,\beta}^f(W, X) \hookrightarrow \mathrm{Sub}_P^f(W, X)$ is a weak homotopy equivalence.*

Proof. We may consider $\mathrm{Sub}(\beta W, X)$ as the subspace of $\mathrm{Sub}^f(\beta W, X)$ which consists of all formal submersions $(\hat{g}, g)$ such that g is smooth and $\hat{g} = dg$ (here dg is the differential of g). Recall the restriction map

$$\beta_1 : \mathrm{Sub}_{P,\beta_1}^f(W, X) \to \mathrm{Sub}^f(\beta_1 W, X)$$

which is a Serre-fibration by Lemma 9.5. By definition we have

$$\mathrm{Sub}_{P,\beta}^f(W, X) = \beta_1^{-1}(\mathrm{Sub}(\beta W, X)),$$

thus the diagram

$$
\begin{array}{ccc}
\mathrm{Sub}_{P,\beta}^f(W, X) & \hookrightarrow & \mathrm{Sub}_P^f(W, X) \\
\downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_1} \\
\mathrm{Sub}(\beta_1 W, X) & \hookrightarrow & \mathrm{Sub}^f(\beta_1 W, X)
\end{array}
$$

is cartesian. Since the right vertical map is a Serre-fibration, it follows that the above diagram is homotopy-cartesian. Since W is an open P-manifold, it follows that $\beta_1 W$ is an open manifold. By Theorem 9.1 the bottom horizontal map in the above diagram is a weak equivalence. Since it is homotopy cartesian, it follows that the upper-horizontal map is a weak equivalence as well. This completes the proof. □

Proof of Theorem 9.4. The space $\mathrm{Sub}_P(W, X)$ may be realized as the subspace of $\mathrm{Sub}^f_{P,\beta}(W, X)$ which consists of all formal P-submersions $(\hat{f}, f)$ such that f is a smooth P-map and $df = \hat{f}$. To prove the theorem it will suffice to show that the relative homotopy group $\pi_n(\mathrm{Sub}^f_{P,\beta}(W, X), \mathrm{Sub}_P(W, X))$ is zero for all n. Let

$$F : (D^n, S^{n-1}) \to (\mathrm{Sub}^f_{P,\beta}(W, X), \mathrm{Sub}_P(W, X))$$

be a map of pairs. By definition of the space $\mathrm{Sub}^f_{P,\beta}(W, X)$, for all $x \in D^n$ we have

$$\beta_1 F(x) \in \mathrm{Sub}(\beta_1 W, X) \subset \mathrm{Sub}^f(\beta_1 W, X).$$

In other words, the formal submersion $\beta_1 F(x)$ is integrable for all $x \in D^n$. Since W is an open P-manifold, it follows automatically that $\partial_1 W$ is an open manifold, as well as W. We then may apply the *relative parametric H-principle* (see [5, §6.2 C] for the definition and [5, §7] for the statement of the relevant theorem needed to apply it) to obtain a homotopy

$$F_t : (D^n, S^{n-1}) \to (\mathrm{Sub}^f_{P,\beta}(W, X), \mathrm{Sub}_P(W, X)) \quad \text{for } t \in [0, 1]$$

such that:

  (i) $F_0 = F$,
  (ii) $F_t|_{\partial_1 W} = F|_{\partial_1 W}$ for all $t \in [0, 1]$, and
 (iii) $F_1(D^n) \subset \mathrm{Sub}_P(W, X)$.

This completes the proof of the theorem. $\qquad\square$

10. Stabilization of sections of vector bundles

In this section we prove a lemma that implies Claim 7.5 used in the proof of Lemma 7.4. This result is essentially a relative version of [11, Lem. 6].

For any space X and vector bundles V_1 and V_2 over X, let $\mathrm{Iso}(E^1, E^2)$ be the space of bundle-isomorphisms covering the identity map. We have a stabilization map

$$\sigma : \mathrm{Iso}(E^1, E^2) \to \mathrm{Iso}(E_1 \oplus \epsilon^1, E^2 \oplus \epsilon^1), \quad f \mapsto f \oplus \mathrm{Id}_{\epsilon^1}.$$

From [11, Lem. 6] we have the following result.

Lemma 10.1. *Let X be a manifold and let $E^1, E^2 \to V$ be vector bundles of fiber dimension k. The stabilization map*

$$\sigma : \mathrm{Iso}(E^1, E^2) \to \mathrm{Iso}(E^1 \oplus \epsilon^1, E^2 \oplus \epsilon^1)$$

is $(k - \dim(X) - 1)$-connected.

We will need to use a relative version of the above lemma. Let $A \subset X$ be a submanifold and let $E^1, E^2 \to X$ be vector bundles. For $g \in \mathrm{Iso}(E^1|_A, E^2|_A)$, denote by $\mathrm{Iso}(E^1, E^2)_g$ the subspace of $\mathrm{Iso}(E^1, E^2)$ consisting of all bundle isomorphisms

$$f : E^1 \xrightarrow{\cong} E^2$$

such that the restriction of f to $E^1|_A$ is equal to g.

Proposition 10.2. *Let $A \subset X$ be a submanifold of positive codimension and let $E^1, E^2 \to X$ be vector bundles of fiber dimension k and let $g \in \mathrm{Iso}(E^1|_A, E^2|_A)$. Then the stabilization map*

$$\sigma : \mathrm{Iso}(E^1, E^2)_g \to \mathrm{Iso}(E^1 \oplus \epsilon^1, E^2 \oplus \epsilon^1)_{g \oplus \mathrm{Id}_{\epsilon^1}}$$

is $(k - \dim(X) - 1)$-connected.

Proof. Consider the restriction map

$$r : \mathrm{Iso}(E^1, E^2) \to \mathrm{Iso}(E^1|_A, E^2|_A), \quad f \mapsto f|_A.$$

The space $\mathrm{Iso}(E^1, E^2)_g$ is equal to the subspace $r^{-1}(g) \subset \mathrm{Iso}(E^1, E^2)$. Furthermore it follows from [13, Lem. 5.3] that the map r is a Serre-fibration. The stabilization map induces a map of fiber sequences

$$
\begin{array}{ccc}
\mathrm{Iso}(E^1, E^2)_g & \longrightarrow & \mathrm{Iso}(E^1 \oplus \epsilon^1, E^2 \oplus \epsilon^1)_{g \oplus \mathrm{Id}_{\epsilon^1}} \\
\downarrow & & \downarrow \\
\mathrm{Iso}(E^1, E^2) & \longrightarrow & \mathrm{Iso}(E^1 \oplus \epsilon^1, E^2 \oplus \epsilon^1) \\
\downarrow & & \downarrow \\
\mathrm{Iso}(E^1|_A, E^2|_A) & \longrightarrow & \mathrm{Iso}(E^1|_A \oplus \epsilon^1, E^2|_A \oplus \epsilon^1).
\end{array}
$$

By Lemma 10.1 the middle horizontal map is $(k - \dim(X) - 1)$-connected. Since $\dim(A) < \dim(X)$, again by Lemma 10.1 the bottom horizontal map is at least $(k - \dim(X))$-connected. It then follows by application of the *five lemma* to the long exact sequence in homotopy groups that the top horizontal map is $(k - \dim(X) - 1)$-connected as well. This completes the proof. $\square$

To prove Claim 7.5 we will need a version of Lemma 10.1 adapted for P-vector bundles over P-manifolds. For what follows, let M be a P-manifold. Let $E_P \to P$ be a vector bundle and let $E^1, E^2 \to M$ be P-vector bundles of the same fiber dimension, equipped with a specified identification $E_P = E_P^1 = E_P^2$. We denote by $\mathrm{Iso}_{E_P}(E^1, E^2)$ the space of pairs (f, f_β) where

$$f : E^1 \xrightarrow{\cong} E^2 \quad \text{and} \quad f_\beta : E_{\beta_1}^1 \xrightarrow{\cong} E_{\beta_1}^2$$

are bundle isomorphisms that cover the identity maps such that the diagram

$$
\begin{array}{ccc}
E^1|_{\partial_1 M} & \xrightarrow{\;\; f|_{\partial_1 M} \;\;} & E^2|_{\partial_1 M} \\
{\scriptstyle \hat\phi_{E^1}} \downarrow & & \downarrow {\scriptstyle \hat\phi_{E^1}} \\
E_{\beta_1}^1 \times E_P \oplus \epsilon^1 & \xrightarrow{\;\; f_{\beta_1} \times \mathrm{Id}_{E_P} \oplus \mathrm{Id}_{\epsilon^1} \;\;} & E_{\beta_1}^2 \times E_P \oplus \epsilon^1
\end{array}
$$

commutes. There is a stabilization map

$$(37) \qquad \begin{aligned} \sigma_{E_P} : \mathrm{Iso}_{E_P}(E^1, E^2) &\to \mathrm{Iso}_{E_P}(E^1 \oplus \epsilon^1, E^2 \oplus \epsilon^1), \\ (f, f_\beta) &\mapsto (f \oplus \epsilon^1, f_\beta \oplus \epsilon^1). \end{aligned}$$

The main result of this section is the following.

Theorem 10.3. *Let M, E_P, E^1 and E^2 be as above and denote by k the fiber dimension of E^1 and E^2. Suppose that the fiber dimension of E_P is equal to $p = \dim(P)$. Then the stabilization map given in (37) is $(k - \dim(M) - 1)$-connected.*

Proof. Consider the map

$$T_{E_P} : \mathrm{Iso}(E^1_{\beta_1}, E^2_{\beta_2}) \to \mathrm{Iso}(E^1|_{\partial_1 M}, E^2|_{\partial_1 M})$$

defined by sending a bundle isomorphism $f : E^1_{\beta_1} \to E^2_{\beta_1}$ to the bundle isomorphism given by the composition

$$E^1|_{\partial_1 M} \xrightarrow{\cong} (E^1_{\beta_1} \times E_P) \oplus \epsilon^1 \xrightarrow{f_\beta \times \mathrm{Id}_{E_P} \oplus \mathrm{Id}_{\epsilon^1}} (E^1_{\beta_1} \times E_P) \oplus \epsilon^1 \xrightarrow{\cong} E^2|_{\partial_1 M}.$$

We also have the map

$$\beta_1 : \mathrm{Iso}(E^1_{\beta_1}, E^2_{\beta_2}) \to \mathrm{Iso}(E^1_{\beta_1}, E^2_{\beta_1}), \quad (f, f_\beta) \mapsto f_\beta.$$

It follows immediately from the definition of the space $\mathrm{Iso}_{E_P}(E^1, E^2)$ that the diagram

$$(38) \qquad \begin{array}{ccc} \mathrm{Iso}_{E_P}(E^1, E^2) & \lhook\joinrel\longrightarrow & \mathrm{Iso}(E^1, E^2) \\ \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle r} \\ \mathrm{Iso}(E^1_{\beta_1}, E^2_{\beta_1}) & \xrightarrow{\ T_{E_P}\ } & \mathrm{Iso}(E^1|_{\partial_1 M}, E^2|_{\partial_1 M}) \end{array}$$

is cartesian, where the top horizontal map is the inclusion map and the right vertical map r is the restriction map. By [13, Lem. 5.3] the restriction map r is a Serre-fibration with fiber over $g \in \mathrm{Iso}(E^1|_{\partial_1 M}, E^2|_{\partial_1 M})$ equal to the space $\mathrm{Iso}(E^1, E^2)_g$. It follows that diagram (38) is homotopy cartesian and that the left vertical map β_1 is a Serre-fibration as well. For $g \in \mathrm{Iso}(E^1_{\beta_1}, E^2_{\beta_1})$, the fiber over the map β_1 over g equals the space $\mathrm{Iso}(E^1|_{\partial_1 M}, E^2|_{\partial_1 M})_{T_{E_P}(g)}$. We then have a map of fiber sequences

$$\begin{array}{ccc} \mathrm{Iso}(E^1, E^2)_{T_{E_P}(g)} & \longrightarrow & \mathrm{Iso}(E^1 \oplus \epsilon^1, E^2 \oplus \epsilon^1)_{T_{E_P}(g) \oplus \mathrm{Id}_{\epsilon^1}} \\ \downarrow & & \downarrow \\ \mathrm{Iso}_{E_P}(E^1, E^2) & \longrightarrow & \mathrm{Iso}_{E_P}(E^1 \oplus \epsilon^1, E^2 \oplus \epsilon^1) \\ \downarrow & & \downarrow \\ \mathrm{Iso}(E^1_{\beta_1}, E^2_{\beta_1}) & \longrightarrow & \mathrm{Iso}(E^1_{\beta_1} \oplus \epsilon^1, E^2_{\beta_1} \oplus \epsilon^1). \end{array}$$

By Lemma 10.1 and Proposition 10.2 the top horizontal map in the above diagram is $(k - \dim(M) - 1)$-connected and the degree of connectivity of the bottom horizontal map is equal to

$$[(k - p - 1) - \dim(\beta_1 M) - 1] = [(k - p - 1) - (\dim(M) - p - 1) - 1]$$
$$= (k - \dim(M) - 1)$$

as well. It then follows by application of the *five lemma* to the long-exact sequence in homotopy groups associated to the above fibrations that the middle map is $(k - \dim(X) - 1)$-connected. □

11. Proof of Proposition 7.11

In Section 7 we proved that for all k there is an isomorphism of sets

$$[S^k, |\mathbf{D}_{d+1}^P|] \cong [S^k, \Omega^{\infty-1}\mathsf{MT}_P(d+1)]$$

which is induced by a zig-zag

$$|\mathbf{D}_{d+1}^P| \xleftarrow{\simeq} |\hat{\mathbf{D}}_{d+1}^P| \xrightarrow{T} \Omega^{\infty-1}\mathsf{MT}_P(d+1),$$

where the first map is a weak homotopy equivalence. In order to prove Proposition 7.11, we need to show that we have an isomorphism of homotopy groups for any choice of base-point on any path component. In this section we resolve this issue.

First note that since $\Omega^{\infty-1}\mathsf{MT}_P(d+1)$ is an infinite loop-space, it has the structure of a topological monoid. In particular, this implies that the identity component of $\Omega^{\infty-1}\mathsf{MT}_P(d+1)$ is a connected H-space. It then follows from [8, Ex. 4A.3] that for all $k \in \mathbb{N}$ and any point x_0 in the identity component $\Omega_0^{\infty-1}\mathsf{MT}_P(d+1) \subset \Omega^{\infty-1}\mathsf{MT}_P(d+1)$, there is a bijection between the homotopy group $\pi_k(\Omega^{\infty-1}\mathsf{MT}_P(d+1), x_0)$ and the set $[S^k, \Omega_0^{\infty-1}\mathsf{MT}_P(d+1)]$ induced by the natural map defined by forgetting the base point. Now, the monoid structure on $\pi_0(\Omega^{\infty-1}\mathsf{MT}_P(d+1))$ is a group. This implies that all path components of $\Omega^{\infty-1}\mathsf{MT}_P(d+1)$ are homotopy equivalent. It follows that for all $x \in \Omega^{\infty-1}\mathsf{MT}_P(d+1)$ in any path component and all $k \in \mathbb{N}$, there is an isomorphism

$$\pi_k(\Omega^{\infty-1}\mathsf{MT}_P(d+1), x) \xrightarrow{\cong} [S^k, \Omega_x^{\infty-1}\mathsf{MT}_P(d+1)],$$

where $\Omega_x^{\infty-1}\mathsf{MT}_P(d+1) \subset \Omega^{\infty-1}\mathsf{MT}_P(d+1)$ is the path component that contains the element $x \in \Omega^{\infty-1}\mathsf{MT}_P(d+1)$.

We will need to show that $|\mathbf{D}_{d+1}^P|$ has the structure of a monoid (with product defined up to homotopy) and that the map $|T| : |\mathbf{D}_{d+1}^P| \to \Omega^{\infty-1}\mathsf{MT}_P(d+1)$ is a monoid homomorphism inducing isomorphism on π_0. The method of this section is very similar to the proof of [11, Thm. 3.8].

Proposition 11.1. *The space $|\hat{\mathbf{D}}_{d+1}^P|$ has the structure of topological monoid up to homotopy.*

Proof. The monoid (up to homotopy) structure on $|\hat{\mathbf{D}}_{d+1}^P|$ is defined as follows: Let $\hat{\mathbf{D}}_{d+1}^P \bar{\times} \hat{\mathbf{D}}_{d+1}^P$ be the sheaf defined by letting $(\hat{\mathbf{D}}_{d+1}^P \bar{\times} \hat{\mathbf{D}}_{d+1}^P)(X)$ consist of all pairs

$$((W_1, e_1), (W_2, e_2)) \in \hat{\mathbf{D}}_{d+1}^P(X) \times \hat{\mathbf{D}}_{d+1}^P(X)$$

such that the images of e^1 and e^2 are disjoint in $X \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$. There is a natural map

$$(39) \qquad \mu : (\hat{\mathbf{D}}_{d+1}^P \bar{\times} \hat{\mathbf{D}}_{d+1}^P)(X) \to \hat{\mathbf{D}}_{d+1}^P(X)$$

defined by sending a pair $((W_1, e_1), (W_2, e_2))$ to the element $(W_1 \sqcup W_2, e_1 \sqcup e_2)$. This map yields a partially defined product on $\hat{\mathbf{D}}_{d+1}^P(X)$ which is clearly associative and commutative. The identity element is given by the empty set. The inclusion map

$$j : \hat{\mathbf{D}}_{d+1}^P \,\bar{\times}\, \hat{\mathbf{D}}_{d+1}^P \to \hat{\mathbf{D}}_{d+1}^P \times \hat{\mathbf{D}}_{d+1}^P$$

is a weak equivalence of sheaves. Roughly, given

$$((W_1, e_1), (W_2, e_2)) \in \hat{\mathbf{D}}_{d+1}^P(X) \times \hat{\mathbf{D}}_{d+1}^P(X)$$

such that the images of e_1 and e_2 intersect, after increasing the dimension of the ambient space, one can find an isotopy of embeddings which pulls W_1 away from W_2. Letting $|k|$ be a pseudo-inverse for $|j|$, the product described above yields a homotopy monoid structure on the representing space $|\hat{\mathbf{D}}_{d+1}^P|$ with product given by

$$(40) \quad |\hat{\mathbf{D}}_{d+1}^P| \times |\hat{\mathbf{D}}_{d+1}^P| \xrightarrow{\simeq} |\hat{\mathbf{D}}_{d+1}^P \times \hat{\mathbf{D}}_{d+1}^P| \xrightarrow{|k|} |\hat{\mathbf{D}}_{d+1}^P \,\bar{\times}\, \hat{\mathbf{D}}_{d+1}^P| \xrightarrow{|\mu|} |\hat{\mathbf{D}}_{d+1}^P|,$$

where the left-most map is some choice of homotopy equivalence. The empty-set element in $\hat{\mathbf{D}}_{d+1}^P(\mathrm{pt.})$ (which induces the empty-set element in $\hat{\mathbf{D}}_{d+1}^P(X)$ for any X by pulling back over the constant map) determines an element $e \in |\hat{\mathbf{D}}_{d+1}^P|$. This is easy to see by examining the construction of $|\hat{\mathbf{D}}_{d+1}^P|$ as the geometric realization of the simplicial set $(l \mapsto \hat{\mathbf{D}}_{d+1}^P(\triangle^l))$. From the fact that the empty set is the identity for the partially defined product in (39) it follows that e is the identity (up to homotopy) for the product defined in (40). Associativity also follows from associativity of (39). $\qquad\square$

Since $|\hat{\mathbf{D}}_{d+1}^P|$ has the structure of a monoid up to homotopy, it follows from [8, Ex. 4A.3] that for each $k \in \mathbb{N}$ and any point x_0 on the identity component $|\hat{\mathbf{D}}_{d+1}^P|_0 \subset |\hat{\mathbf{D}}_{d+1}^P|$, there is an isomorphism

$$\pi_k(|\hat{\mathbf{D}}_{d+1}^P|, x_0) \xrightarrow{\cong} [S^k, |\hat{\mathbf{D}}_{d+1}^P|_0]$$

induced by the map defined by forgetting base points. Now consider the map

$$|H| \circ |\hat{T}| : |\hat{\mathbf{D}}_{d+1}^P| \to \Omega^{\infty-1}\mathsf{MT}_P(d+1)$$

from Section 7.7.

Proposition 11.2. *The map $|H| \circ |\hat{T}|$ induces a monoid isomorphism*

$$\pi_0(|\hat{\mathbf{D}}_{d+1}^P|) \xrightarrow{\cong} \pi_0(\Omega^{\infty-1}\mathsf{MT}_P(d+1)).$$

Proof. This proposition is proven by examining the Pontryagin–Thom map

$$\hat{T} : \hat{\mathbf{D}}_{d+1}^P(\mathrm{pt.}) \to \mathcal{Z}_{d+1}^P(\mathrm{pt.})$$

and checking that it sends a disjoint union $(W_1 \sqcup W_2, e_1 \sqcup e_2) \in \hat{\mathbf{D}}_{d+1}^P(\mathrm{pt.})$ to a sum of "loops" in $\mathcal{Z}_{d+1}^P(\mathrm{pt.})$. This follows the exact same argument as in the proof of the classical *Pontryagin–Thom theorem* from [14]. We refer the reader there for details. $\qquad\square$

Now, since $\pi_0(\Omega^{\infty-1}\mathsf{MT}_P(d+1))$ is a group, it follows from Proposition 11.2 that $\pi_0(|\hat{\mathbf{D}}_{d+1}^P|)$ is a group as well.

From this group structure it follows that all path components of $|\hat{\mathbf{D}}_{d+1}^P|$ are homotopy equivalent. We then have that for every $x \in |\hat{\mathbf{D}}_{d+1}^P|$ in any path component and for all $k \in \mathbb{N}$, there is an isomorphism

$$\pi_k(|\hat{\mathbf{D}}_{d+1}^P|, x) \xrightarrow{\cong} [S^k, |\hat{\mathbf{D}}_{d+1}^P|_x],$$

where $|\hat{\mathbf{D}}_{d+1}^P|_x \subset |\hat{\mathbf{D}}_{d+1}^P|$ is the path component containing x. Consider the commutative diagram

$$
\begin{array}{ccc}
\pi_k(|\hat{\mathbf{D}}_{d+1}^P|, x) & \xrightarrow{\ |H|\circ|\hat{T}|\ } & \pi_k(\Omega^{\infty-1}\mathsf{MT}_P(d+1), y) \\
\Big\downarrow{\cong} & & \Big\downarrow{\cong} \\
[S^k, |\hat{\mathbf{D}}_{d+1}^P|_x] & \xrightarrow[\cong]{\ |H|\circ|\hat{T}|\ } & [S^k, \Omega_y^{\infty-1}\mathsf{MT}_P(d+1)],
\end{array}
$$

where $y = |H| \circ |\hat{T}|(x)$. It follows that the top horizontal map is an isomorphism. This concludes the proof of Proposition 7.11.

Appendix A

In this appendix we prove a result which implies Theorem 3.14. This result is a slight modification of the main theorem from [3] and our proof is similar.

Lemma A.1. *Let* $(W; M_a, M_b)$ *be a P-manifold cobordism triple. For any positive integer n, the quotient map*

$$q : \mathcal{E}_{P,n}(W; M_a, M_b) \to \frac{\mathcal{E}_{P,n}(W; M_a, M_b)}{\mathrm{Diff}_P(W; M_a, M_b)}$$

is a locally trivial fiber bundle.

Proof. Let $f \in \mathcal{E}_{P,n}(W; M_a, M_b)$ and let $[f]$ denote the class of f in the orbit space $\mathcal{E}_{P,n}(W; M_a, M_b)/\mathrm{Diff}_P(W; M_a, M_b)$. By definition of the action we have

$$g(W) = f(W)$$

for any $g \in q^{-1}([f])$. By definition of the space $\mathcal{E}_{P,n}(W; M_a, M_b)$ we have

$$f(\partial_1 W) \subset [0,1] \times \{0\} \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$$

and there exists a real number $\varepsilon > 0$ such that

$$[0,\varepsilon] \times f(\partial_1 W) \subset f(W),$$

where $[0,\varepsilon] \times f(\partial_1 W)$ is understood to be the set of all

$$(t, s, x, y) \in [0,1] \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$$

such that $(t, 0, x, y) \in f(\partial_1 W)$ and $s \in [0,\varepsilon] \subset \mathbb{R}_+$. Let

$$N \subset [0,1] \times \mathbb{R}_+ \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}$$

be a geodesic neighborhood for $f(W)$. Denote by $\pi : N \to f(W)$ the projection map. We have

(41) $$N \cap ([0,1] \times \{0\} \times \mathbb{R}^{d+\bar{n}} \times \mathbb{R}^{p+m}) = N_{\beta_1} \times N_P,$$

where $N_{\beta_1} \subset \mathbb{R}^{d+\bar{n}}$ is a geodesic neighborhood for $f_{\beta_1}(\beta_1 W) \subset \mathbb{R}^{d+\bar{n}}$ and $N_P \subset \mathbb{R}^{p+m}$ is a geodesic neighborhood for $i_P(P) \subset \mathbb{R}^{p+m}$. We denote by

$$\pi_{\beta_1} : N_{\beta_1 W} \to \beta_1 W \quad \text{and} \quad \pi_P : N_P \to P$$

the projection maps. Now let

$$z \in ([0,1] \times [0,\varepsilon) \times \mathbb{R}^{d+\bar{n}} \times i_P(P)) \cap N.$$

We write $z = (s,t,x,y)$ with $s \in [0,\varepsilon)$, $(t,x) \in [0,1] \times \mathbb{R}^{\bar{d}-1+\bar{n}}$, and $y \in i_P(P)$ (we permute the factors of $[0,\varepsilon)$ and $[0,1]$ to make for more convenient notation for the constructions ahead). It follows from the factorization of (41) that

(42) $$\pi(s,t,x,y) = (s, \pi_{\beta_1}(t,x), y).$$

Let

$$U \subset \mathcal{E}_{P,n}(W; M_a, M_b)$$

be an open neighborhood of f with the property that $g(W) \subset N$ for all $g \in U$. By definition of the C^∞-topology, such a subset does indeed exist. Now let q be the quotient map from the statement of the theorem. Let $\bar{U}$ denote the image $q(U)$. For any such $g \in q^{-1}(\bar{U})$, we obtain a smooth map $W \to W$ given by the formula $x \mapsto f^{-1} \circ \pi \circ g(x)$. We will abuse notation and denote this map by

$$f^{-1} \circ \pi \circ g : W \to W.$$

It follows from (42) that for all $g \in q^{-1}(\bar{U})$, the map $f^{-1} \circ \pi \circ g$ is an element of the mapping space $C_P^\infty(W; M_a, M_b)$ introduced in Section 3. We have a map

$$\alpha : q^{-1}(\bar{U}) \to C_P^\infty(W; M_a, M_b), \quad g \mapsto f^{-1} \circ \pi \circ g.$$

Notice that $\alpha(f) = \mathrm{Id}_W$, which is of course an element of $\mathrm{Diff}_P(W; M_a, M_b)$. By Proposition 3.4,

$$\mathrm{Diff}_P(W; M_a, M_b) \subset C_P^\infty(W; M_a, M_b)$$

is an open subset. We may then choose a small neighborhood $\bar{U}' \subset \bar{U}$ of $[f]$ such that

$$\alpha(q^{-1}(\bar{U}')) \subset \mathrm{Diff}_P(W; M_a, M_b).$$

Using α, we define a map

$$\Phi : q^{-1}(\bar{U}') \to \bar{U}' \times \mathrm{Diff}_P(W; M_a, M_b), \quad g \mapsto ([g], \alpha(g)).$$

It follows easily that this map Φ is a local trivialization of the projection q. This concludes the proof of the lemma. $\square$

References

[1] N. A. Baas, On bordism theory of manifolds with singularities, Math. Scand. **33** (1973), 279–302 (1974). MR0346824

[2] N. A. Baas, A note on cobordism categories, preprint, NTNU, 2009.

[3] E. Binz and H. R. Fischer, The manifold of embeddings of a closed manifold, in *Differential geometric methods in mathematical physics (Proc. Internat. Conf., Tech. Univ. Clausthal, Clausthal-Zellerfeld, 1978)*, 310–329, Lecture Notes in Phys., 139, Springer, Berlin, 1981. MR0613004

[4] B. I. Botvinnik, *Manifolds with singularities and the Adams–Novikov spectral sequence*, London Mathematical Society Lecture Note Series, 170, Cambridge Univ. Press, Cambridge, 1992. MR1192127

[5] Y. Eliashberg and N. Mishachev, *Introduction to the h-principle*, Graduate Studies in Mathematics, 48, Amer. Math. Soc., Providence, RI, 2002. MR1909245

[6] S. Galatius, I. Madsen, U. Tillmann and M. Weiss, The homotopy type of the cobordism category, Acta Math. **202** (2009), no. 2, 195–239. MR2506750

[7] J. Genauer, Cobordism categories of manifolds with corners, Trans. Amer. Math. Soc. **364** (2012), no. 1, 519–550. MR2833590

[8] A. Hatcher, *Algebraic topology*, Cambridge Univ. Press, Cambridge, 2002. MR1867354

[9] M. W. Hirsch, *Differential topology*, Springer, New York, 1976. MR0448362

[10] E. L. Lima, On the local triviality of the restriction map for embeddings, Comment. Math. Helv. **38** (1964), 163–164. MR0161343

[11] I. Madsen and M. Weiss, The stable moduli space of Riemann surfaces: Mumford's conjecture, Ann. of Math. (2) **165** (2007), no. 3, 843–941. MR2335797

[12] J. Milnor, The geometric realization of a semi-simplicial complex, Ann. of Math. (2) **65** (1957), 357–362. MR0084138

[13] A. Phillips, Submersions of open manifolds, Topology **6** (1967), 171–206. MR0208611

[14] R. E. Stong, *Notes on cobordism theory*, Mathematical notes, Princeton Univ. Press, Princeton, NJ, 1968. MR0248858

Nathan Perlmutter
Department of Mathematics, Stanford University,
Stanford, CA 94305, USA
E-mail: `nperlmut@stanford.edu`