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The group of automorphisms of the algebra of one-sided inverses of a polynomial algebra

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Dedicated to T. Lenagan on the occasion of his 60th birthday

Abstract. The algebra $\mathbb{S}_n$ in the title is obtained from a polynomial algebra P_n in n variables by adding commuting, *left* (but not two-sided) inverses of the canonical generators of P_n. Ignoring the non-Noetherian property, the algebra $\mathbb{S}_n$ belongs to a family of algebras like the Weyl algebra A_n and the polynomial algebra P_{2n}. The group of automorphisms G_n of the algebra $\mathbb{S}_n$ is found:

$$G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n) \supseteq S_n \ltimes \mathbb{T}^n \ltimes \underbrace{\mathrm{GL}_\infty(K) \ltimes \cdots \ltimes \mathrm{GL}_\infty(K)}_{2^n-1 \text{ times}} =: G'_n,$$

where S_n is the symmetric group, $\mathbb{T}^n$ is the n-dimensional algebraic torus, $\mathrm{Inn}(\mathbb{S}_n)$ is the group of inner automorphisms of $\mathbb{S}_n$ (which is huge), and $\mathrm{GL}_\infty(K)$ is the group of invertible infinite dimensional matrices. This result may help in understanding of the structure of the groups of automorphisms of the Weyl algebra A_n and the polynomial algebra P_{2n}. An analog of the *Jacobian homomorphism*: $\mathrm{Aut}_{K-\mathrm{alg}}(P_{2n}) \to K^*$, the so-called *global determinant* is introduced for the group G'_n (notice that the algebra $\mathbb{S}_n$ is *noncommutative* and neither left nor right Noetherian).

Contents

1. Introduction

Throughout, ring means an associative ring with 1; module means a left module; $\mathbb{N} := \{0, 1, \dots\}$ is the set of natural numbers; K is a field and K^* is its group of units; $P_n := K[x_1, \dots, x_n]$ is a polynomial algebra over K; $\partial_1 := \frac{\partial}{\partial x_1}, \dots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives (K-linear derivations) of P_n.

Definition ([4])**.** The *algebra* $\mathbb{S}_n$ *of one-sided inverses* of P_n is an algebra generated over a field K by $2n$ elements $x_1, \dots, x_n, y_n, \dots, y_n$ that satisfy the defining relations:

$$y_1x_1 = \cdots = y_nx_n = 1, \ [x_i, y_j] = [x_i, x_j] = [y_i, y_j] = 0 \text{ for all } i \neq j,$$

where $[a, b] := ab - ba$, the commutator of elements a and b.

By the very definition, the algebra $\mathbb{S}_n$ is obtained from the polynomial algebra P_n by adding commuting, left (or right) inverses of its canonical generators. The algebra $\mathbb{S}_1$ is a well-known primitive algebra [7, p. 35, Ex. 2]. Over the field $\mathbb{C}$ of complex numbers, the completion of the algebra $\mathbb{S}_1$ is the *Toeplitz algebra* which is the C^*-algebra generated by a unilateral shift on the Hilbert space $l^2(\mathbb{N})$ (note that $y_1 = x_1^*$). The Toeplitz algebra is the universal C^*-algebra generated by a proper isometry.

Example ([4]). Consider a vector space $V = \bigoplus_{i \in \mathbb{N}} Ke_i$ and two shift operators on V, $X : e_i \mapsto e_{i+1}$ and $Y : e_i \mapsto e_{i-1}$ for all $i \geq 0$, where $e_{-1} := 0$. The subalgebra of $\mathrm{End}_K(V)$ generated by the operators X and Y is isomorphic to the algebra $\mathbb{S}_1$ ($X \mapsto x$, $Y \mapsto y$). By taking the n'th tensor power $V^{\otimes n} = \bigoplus_{\alpha \in \mathbb{N}^n} Ke_\alpha$ of V we see that the algebra $\mathbb{S}_n$ is isomorphic to the subalgebra of $\mathrm{End}_K(V^{\otimes n})$ generated by the $2n$ shifts $X_1, Y_1, \dots, X_n, Y_n$ that act in different directions.

It is an experimental fact ([4]) that the algebra $\mathbb{S}_1$ has properties that are a mixture of the properties of the polynomial algebra P_2 in two variable and the *first Weyl* algebra A_1, which is not surprising when we look at their defining relations:

$$P_2 = K\langle x, y\rangle : yx - xy = 0;$$
$$A_1 = K\langle x, y\rangle : yx - xy = 1;$$
$$\mathbb{S}_1 = K\langle x, y\rangle : yx = 1.$$

The same is true for their higher analogs: $P_{2n} = P_2^{\otimes n}$, $A_n := A_1^{\otimes n}$ (the n'th *Weyl* algebra), and $\mathbb{S}_n = \mathbb{S}_1^{\otimes n}$. For example,

$$\mathrm{cl.Kdim}(\mathbb{S}_n) \overset{[4]}{=} 2n = \mathrm{cl.Kdim}(P_{2n}),$$

$$\mathrm{gldim}(\mathbb{S}_n) \overset{[4]}{=} n = \mathrm{gldim}(A_n), \ (\mathrm{char}(K) = 0)$$

$$\mathrm{GK}(\mathbb{S}_n) \overset{[4]}{=} 2n = \mathrm{GK}(A_n) = \mathrm{GK}(P_{2n}),$$

where cl.Kdim, gldim, and GK stand for the classical Krull dimension, the global homological dimension, and the Gelfand-Kirillov dimension respectively. The big difference between the algebra $\mathbb{S}_n$ and the algebras P_{2n} and A_n is that $\mathbb{S}_n$ is neither left nor right Noetherian and is not a domain either.

The algebras $\mathbb{S}_n$ are fundamental non-Noetherian algebras, they are universal non-Noetherian algebras of their own kind in a similar way as the polynomial algebras are universal in the class of all the commutative algebras and the Weyl algebras are universal in the class of algebras of differential operators.

The algebra $\mathbb{S}_n$ often appears as a subalgebra or a factor algebra of many non-Noetherian algebras. For example, $\mathbb{S}_1$ is a factor algebra of certain non-Noetherian down-up algebras as was shown by Jordan [8] (see also Benkart and Roby [5]; Kirkman, Musson, and Passman [11]; Kirkman and Kuzmanovich [10]); and $\mathbb{S}_n$ is a subalgebra of the Jacobian algebra $\mathbb{A}_n$ (see below) [1].

The aim of this paper is to find the group $G_n := \mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{S}_n)$ of automorphisms of the algebra $\mathbb{S}_n$.

- (Theorem 5.1) $G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$.
- (Lemma 7.8) $G_n \supseteq G'_n := S_n \ltimes \mathbb{T}^n \ltimes \underbrace{\mathrm{GL}_\infty(K) \ltimes \cdots \ltimes \mathrm{GL}_\infty(K)}_{2^n-1 \text{ times}}$,

where S_n is the symmetric group, $\mathbb{T}^n$ is the n-dimensional algebraic torus, $\mathrm{Inn}(\mathbb{S}_n)$ is the group of inner automorphisms of the algebra $\mathbb{S}_n$, and $\mathrm{GL}_\infty(K)$ is the group of all the invertible infinite dimensional matrices of the type $1 + M_\infty(K)$, where the algebra (without 1) of infinite dimensional matrices $M_\infty(K) := \varinjlim M_d(K) = \bigcup_{d\geq 1} M_d(K)$ is the injective limit of matrix algebras. A semidirect product $H_1 \ltimes H_2 \ltimes \cdots \ltimes H_m$ of several groups means that $H_1 \ltimes (H_2 \ltimes (\cdots \ltimes (H_{m-1} \ltimes H_m) \cdots))$.

The proof of Theorem 5.1 is rather long (and nontrivial) and based upon several results proved in this paper (and in [4]) which are interesting on their own. Let me explain briefly the logical structure of the proof. There are two cases to consider when $n = 1$ and $n > 1$. The proofs of both cases are based on different ideas. The case $n = 1$ is a kind of a degeneration of the second case and is much more easier. The key point in finding the group G_1 is to use the *index* of linear maps in infinite dimensional vector spaces and the fact that each automorphism of the algebra $\mathbb{S}_n$ is determined by its action on the set $\{x_1, \dots, x_n\}$ (or $\{y_1, \dots, y_n\}$):

- (Theorem 3.7) (Rigidity of the group G_n) Let $\sigma, \tau \in G_n$. Then the following statements are equivalent.
  1. $\sigma = \tau$.
  2. $\sigma(x_1) = \tau(x_1), \dots, \sigma(x_n) = \tau(x_n)$.
  3. $\sigma(y_1) = \tau(y_1), \dots, \sigma(y_n) = \tau(y_n)$.

For $n > 1$, one of the key ideas in finding the group G_n is to use the action of the group G_n on the set $\mathcal{H}_1$ of all the height 1 prime ideals of the algebra $\mathbb{S}_n$. The set $\mathcal{H}_1 = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ is finite and is found in [4]. It follows that the group

$$G_n = S_n \ltimes \mathrm{St}_{G_n}(\mathcal{H}_1)$$

is the semidirect product of the symmetric group S_n and the stabilizer of the set $\mathcal{H}_1$ in G_n,

$$\mathrm{St}_{G_n}(\mathcal{H}_1) := \{\sigma \in G_n \mid \sigma(\mathfrak{p}_1) = \mathfrak{p}_1, \ldots, \sigma(\mathfrak{p}_n) = \mathfrak{p}_n\}.$$

The group $\mathrm{St}_{G_n}(\mathcal{H}_1)$ contains the n-dimensional torus $\mathbb{T}^n$. Using a Membership Criterion (Theorem 6.2) for elements of the algebra $\mathbb{S}_n$, it follows that

$$\mathrm{St}_{G_n}(\mathcal{H}_1) = \mathbb{T}^n \ltimes \mathrm{st}_{G_n}(\mathcal{H}_1),$$

where

$$(1) \quad \mathrm{st}_{G_n}(\mathcal{H}_1) = \{\sigma \in \mathrm{St}_{G_n}(\mathcal{H}_1) \mid \sigma(x_i) \equiv x_i \bmod \mathfrak{p}_i,\ \sigma(y_i) \equiv y_i \bmod \mathfrak{p}_i,\ i = 1, \ldots, n\}.$$

Moreover,

- (Corollary 5.5) $\mathrm{st}_{G_n}(\mathcal{H}_1) = \mathrm{Inn}(\mathbb{S}_n)$.

One of the key points of the proof of Theorem 5.1 and Corollary 5.5 is the fact that

- ([4, Cor. 3.3]): P_n is the only simple, faithful $\mathbb{S}_n$-module (up to isomorphism),

and so the algebra $\mathbb{S}_n$ can be seen as a subalgebra of the endomorphism algebra $E_n := \mathrm{End}_K(P_n)$ of all the linear maps from the vector space P_n to itself and we can visualize the group G_n via the group $\mathrm{Aut}_K(P_n)$ of units of the algebra E_n as follows:

- (Theorem 3.2) $G_n = \{\sigma_\varphi \mid \varphi \in \mathrm{Aut}_K(P_n)$ such that $\varphi \mathbb{S}_n \varphi^{-1} = \mathbb{S}_n\}$, where $\sigma_\varphi(a) := \varphi a \varphi^{-1}$, $a \in \mathbb{S}_n$.

To represent the group G_n via linear maps in an infinite dimensional space helps not much unless we have a criterion of when a linear map belongs to the group G_n (or to the algebra $\mathbb{S}_n$). Several membership criteria are proved in Section 6 which are used at the final stage of the proof of Theorem 5.1:

- (Theorem 6.2) Let $\varphi \in \mathrm{End}_K(P_n)$. Then $\varphi \in \mathbb{S}_n$ if and only if $[x_1, \varphi] \in \varphi\mathfrak{p}_1 + \mathfrak{p}_1, \ldots, [x_n, \varphi] \in \varphi\mathfrak{p}_n + \mathfrak{p}_n$.
- (Corollary 6.7) Let $F_n := \bigcap_{i=1}^n \mathfrak{p}_i$. Then

$$\{\varphi \in \mathrm{End}_K(P_n) \mid [x_i, \varphi] \in F_n,\ [y_i, \varphi] \in F_n,\ i = 1, \ldots, n\} = \begin{cases} \mathbb{S}_1, & \text{if } n = 1, \\ K + F_n, & \text{if } n > 1. \end{cases}$$

The structure of the group $G_1 = \mathbb{T}^1 \ltimes \mathrm{GL}_\infty(K)$ is yet another confirmation of "similarity" of the algebras P_2, A_1, and $\mathbb{S}_1$. The groups of automorphisms of the polynomial algebra P_2 and the Weyl algebra A_1 were found by Jung [9], van der Kulk [14], and Dixmier [6] respectively. These two groups have almost identical structure, they are "infinite GL-groups" in the sense that they are generated by the torus $\mathbb{T}^1$ and by the obvious automorphisms: $x \mapsto x + \lambda y^i$, $y \mapsto y$; $x \mapsto x$, $y \mapsto y + \lambda x^i$, where $i \in \mathbb{N}$ and $\lambda \in K$; which are sort of "elementary infinite dimensional matrices" (i.e. "infinite dimensional transvections").

The same picture as for the group G_1. In prime characteristic, the group of automorphism of the Weyl algebra A_1 was found by Makar-Limanov [12] (see also Bavula [3] for a different approach and for further developments). More on polynomial automorphisms the reader can find in the book of van den Essen [13].

There is an important homomorphism from the group $\mathrm{Aut}_{K-\mathrm{alg}}(P_{2n})$ of automorphisms of the polynomial algebra P_{2n} to the group K^*, the so-called *Jacobian* (map or homomorphism):

$$\mathcal{J} : \mathrm{Aut}_{K-\mathrm{alg}}(P_{2n}) \to K^*, \ \sigma \mapsto \det\left(\frac{\partial \sigma(x_i)}{\partial x_j}\right).$$

Note that the Jacobian homomorphism is a determinant. In this paper (Section 8), its analog is introduced for the algebra $\mathbb{S}_n$ which is called the *global determinant*:

$$\det : G'_n \to K^*, \ \sigma \mapsto \det(\sigma).$$

It is a group homomorphism (Corollary 8.7) which is defined as follows. By Lemma 7.8, each element σ of G'_n is a unique product $\sigma = \tau t_\lambda \sigma_1 \cdots \sigma_{2^n-1}$, where $\tau \in S_n$, $t_\lambda \in \mathbb{T}^n$, $\lambda = (\lambda_1, \ldots, \lambda_n) \in K^{*n}$, and $\sigma_i \in \mathrm{GL}_\infty(K)$. Then

$$\det(\sigma) := \mathrm{sgn}(\tau) \cdot \prod_{i=1}^{n} \lambda_i \cdot \prod_{j=1}^{2^n-1} \det(\sigma_j), \tag{2}$$

where $\mathrm{sgn}(\tau)$ is the parity of the permutation τ and $\det(\sigma_j)$ is the "usual" determinant of the element σ_j of the group $\mathrm{GL}_\infty(K)$. It is an interesting question of whether it is possible to extend the global determinant to the group G_n.

The paper is organized as follows. In Section 2, some useful results from [4] are collected which are used later.

In Section 3, several subgroups of the group G_n are introduced, a useful description (Theorem 3.2) of the group G_n is given, and a criterion of equality of two elements of the group G_n is proved (Theorem 3.7).

In Section 4, the group G_1 is found (Theorem 4.1).

In Section 5, the group G_n is found (Theorem 5.1). Several corollaries are obtained. It is proved that the groups G_n and $\mathrm{Inn}(\mathbb{S}_n)$ have trivial center (Corollary 5.6).

In Section 6, several Membership Criteria are proved for the algebras $\mathbb{S}_n$, $P_n + F_n$ and $K + F_n$ (Theorem 6.2, Corollaries 6.6 and 6.7).

In Section 8, the global determinant is extended to a certain monoid $S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n$, the group of units of which is isomorphic to the group G'_n (Corollary 8.12.(1)). Moreover,

- (Corollary 8.12.(2)) $G'_n \simeq \{a \in S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n \mid \det(a) \neq 0\}$.

Intuitively, the pair $(S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n, G'_n)$, a monoid and its group of units, is an infinite dimensional analog of the pair $(M_d(K), \mathrm{GL}_d(K))$. Theorem 8.6.(3) shows that the global determinant can be computed effectively (in finitely many steps).

In Section 9, the stabilizers in the group G_n of several classes of ideals of the algebra $\mathbb{S}_n$ are computed. In particular, the stabilizers of all the prime ideals of $\mathbb{S}_n$ are found (Corollary 9.2.(2) and Corollary 9.9).

The ideal $\mathfrak{a}_n := \mathfrak{p}_1 + \cdots + \mathfrak{p}_n$ is a prime idempotent ideal of the algebra $\mathbb{S}_n$ of height n, [4].

- (Theorem 9.7) The ideal $\mathfrak{a}_n$ is the only nonzero, prime, G_n-invariant ideal of the algebra $\mathbb{S}_n$.
- (Theorem 9.10) Let $\mathfrak{p}$ be a prime ideal of $\mathbb{S}_n$. Then its stabilizer $\mathrm{St}_{G_n}(\mathfrak{p})$ is a maximal subgroup of the group G_n if and only if $n > 1$ and $\mathfrak{p}$ is of height 1, and, in this case, $[G_n : \mathrm{St}_{G_n}(\mathfrak{p})] = n$.
- (Corollary 9.12) Let $\mathfrak{a}$ be a proper ideal of $\mathbb{S}_n$. Then its stabilizer $\mathrm{St}_{G_n}(\mathfrak{a})$ has finite index in the group G_n if and only if $\mathfrak{a}^2 = \mathfrak{a}$.
- (Corollary 9.4) If $\mathfrak{a}$ is a generic idempotent ideal of $\mathbb{S}_n$ then its stabilizer is written via the wreath products of the symmetric groups:

$$\mathrm{St}_{G_n}(\mathfrak{a}) = (S_m \times \prod_{i=1}^{t}(S_{h_i} \wr S_{n_i})) \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n).$$

In Section 10, we classify all the algebra endomorphisms of $\mathbb{S}_n$ that stabilize the elements $x_1, \dots, x_n$ and show that each such endomorphism is a *monomorphism* but *not* an isomorphism provided it is not the identity map (Corollary 10.1). Therefore, an analogous question to the Question of Dixmier, namely, *is a monomorphism of the algebra* $\mathbb{S}_n$ *is an automorphism?* has a negative answer. The original Question/Problem of Dixmier states [6]: *is every homomorphism of the Weyl algebra* A_n *an automorphism?* The Weyl algebra A_n is a simple algebra, so any homomorphism is automatically a monomorphism. In [6], Dixmier poses this question only for the first Weyl algebra A_1.

2. Preliminaries on the algebras $\mathbb{S}_n$

In this section, we collect some results without proofs on the algebras $\mathbb{S}_n$ from [4] that will be used in this paper, their proofs can be found in [4].

Clearly, $\mathbb{S}_n = \mathbb{S}_1(1) \otimes \cdots \otimes \mathbb{S}_1(n) \simeq \mathbb{S}_1^{\otimes n}$, where $\mathbb{S}_1(i) := K\langle x_i, y_i \mid y_i x_i = 1\rangle \simeq \mathbb{S}_1$ and

$$\mathbb{S}_n = \bigoplus_{\alpha,\beta\in\mathbb{N}^n} K x^\alpha y^\beta,$$

where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, $\alpha = (\alpha_1, \dots, \alpha_n)$, $y^\beta := y_1^{\beta_1} \cdots y_n^{\beta_n}$, $\beta = (\beta_1, \dots, \beta_n)$. In particular, the algebra $\mathbb{S}_n$ contains two polynomial subalgebras P_n and $Q_n := K[y_1, \dots, y_n]$ and is equal, as a vector space, to their tensor product $P_n \otimes Q_n$. Note that the Weyl algebra A_n is also the tensor product (as a vector space) $P_n \otimes K[\partial_1, \dots, \partial_n]$ of its two polynomial subalgebras.

When $n = 1$, we usually drop the subscript "1" if this does not lead to confusion. So, $\mathbb{S}_1 = K\langle x, y \mid yx = 1\rangle = \bigoplus_{i,j\geq 0} K x^i y^j$. For each natural number $d \geq 1$, let $M_d(K) := \bigoplus_{i,j=0}^{d-1} K E_{ij}$ be the algebra of d-dimensional matrices, where $\{E_{ij}\}$ are the matrix units, and

$$M_\infty(K) := \varinjlim M_d(K) = \bigoplus_{i,j\in\mathbb{N}} KE_{ij}$$

be the algebra (without 1) of infinite dimensional matrices. The algebra $\mathbb{S}_1$ contains the ideal $F := \bigoplus_{i,j\in\mathbb{N}} KE_{ij}$, where

$$E_{ij} := x^i y^j - x^{i+1}y^{j+1},\ i,j \geq 0. \tag{3}$$

For all natural numbers i, j, k, and l, $E_{ij}E_{kl} = \delta_{jk}E_{il}$, where δ_{jk} is the Kronecker delta function. The ideal F is an algebra (without 1) isomorphic to the algebra $M_\infty(K)$ via $E_{ij} \mapsto E_{ij}$. For all $i,j \geq 0$,

$$xE_{ij} = E_{i+1,j},\ yE_{ij} = E_{i-1,j}\ (E_{-1,j} := 0), \tag{4}$$

$$E_{ij}x = E_{i,j-1},\ E_{ij}y = E_{i,j+1}\ (E_{i,-1} := 0). \tag{5}$$

$$\mathbb{S}_1 = K \oplus xK[x] \oplus yK[y] \oplus F, \tag{6}$$

the direct sum of vector spaces. Then

$$\mathbb{S}_1/F \simeq K[x,x^{-1}] =: L_1,\ x \mapsto x,\ y \mapsto x^{-1}, \tag{7}$$

since $yx = 1$, $xy = 1 - E_{00}$ and $E_{00} \in F$.

The algebra $\mathbb{S}_n = \bigotimes_{i=1}^n \mathbb{S}_1(i)$ contains the ideal

$$F_n := F^{\otimes n} = \bigotimes_{i=1}^n F(i) = \bigoplus_{\alpha,\beta\in\mathbb{N}^n} KE_{\alpha\beta},$$

where

$$E_{\alpha\beta} := \prod_{i=1}^n E_{\alpha_i\beta_i}(i), \quad E_{st}(i) = x_i^s y_i^t - x_i^{s+1}y_i^{t+1},$$

and $F(i) := \bigoplus_{s,t\in\mathbb{N}} KE_{st}(i)$. Note that $E_{\alpha\beta}E_{\gamma\rho} = \delta_{\beta\gamma}E_{\alpha\rho}$ for all elements $\alpha,\beta,\gamma,\rho \in \mathbb{N}^n$, where $\delta_{\beta\gamma}$ is the Kronecker delta function.

- *$F_n a \neq 0$ and $aF_n \neq 0$ for all nonzero elements a of the algebra $\mathbb{S}_n$.*
- *F_n is the smallest (with respect to inclusion) nonzero ideal of the algebra $\mathbb{S}_n$ (i.e. F_n is contained in all nonzero ideals of $\mathbb{S}_n$); $F_n^2 = F_n$; F_n is an essential left and right submodule of $\mathbb{S}_n$; F_n is the socle of the left and right $\mathbb{S}_n$-module $\mathbb{S}_n$; F_n is the socle of the $\mathbb{S}_n$-bimodule $\mathbb{S}_n$ and F_n is a simple $\mathbb{S}_n$-bimodule.*

The involution η on $\mathbb{S}_n$. The algebra $\mathbb{S}_n$ admits the *involution*

$$\eta : \mathbb{S}_n \to \mathbb{S}_n,\ x_i \mapsto y_i,\ y_i \mapsto x_i,\ i = 1,\dots,n,$$

i.e. it is a K-algebra anti-isomorphism ($\eta(ab) = \eta(b)\eta(a)$ for all $a,b \in \mathbb{S}_n$) such that $\eta^2 = \mathrm{id}_{\mathbb{S}_n}$, the identity map on $\mathbb{S}_n$. So, the algebra $\mathbb{S}_n$ is *self-dual* (i.e. it is isomorphic to its opposite algebra, $\eta : \mathbb{S}_n \simeq \mathbb{S}_n^{op}$). The involution η acts on the "matrix" ring F_n as the transposition,

$$\eta(E_{\alpha\beta}) = E_{\beta\alpha}. \tag{8}$$

The canonical generators x_i, y_j ($1 \leq i, j \leq n$) determine the ascending filtration $\{\mathbb{S}_{n,\leq i}\}_{i\in\mathbb{N}}$ on the algebra $\mathbb{S}_n$ in the obvious way (i.e. by the total degree of the generators): $\mathbb{S}_{n,\leq i} := \bigoplus_{|\alpha|+|\beta|\leq i} Kx^\alpha y^\beta$, where $|\alpha| = \alpha_1 + \cdots + \alpha_n$ ($\mathbb{S}_{n,\leq i}\mathbb{S}_{n,\leq j} \subseteq \mathbb{S}_{n,\leq i+j}$ for all $i, j \geq 0$). Then $\dim(\mathbb{S}_{n,\leq i}) = \binom{i+2n}{2n}$ for $i \geq 0$, and so the Gelfand–Kirillov dimension $\mathrm{GK}(\mathbb{S}_n)$ of the algebra $\mathbb{S}_n$ is equal to $2n$. It is not difficult to show that the algebra $\mathbb{S}_n$ is neither left nor right Noetherian. Moreover, it contains infinite direct sums of left and right ideals (see [4]).

- *The algebra $\mathbb{S}_n$ is central, prime, and catenary. Every nonzero ideal of $\mathbb{S}_n$ is an essential left and right submodule of $\mathbb{S}_n$.*
- *The ideals of $\mathbb{S}_n$ commute ($IJ = JI$); and the set of ideals of $\mathbb{S}_n$ satisfy the a.c.c.*
- *The classical Krull dimension* $\mathrm{cl.Kdim}(\mathbb{S}_n)$ *of $\mathbb{S}_n$ is $2n$.*
- *Let I be an ideal of $\mathbb{S}_n$. Then the factor algebra $\mathbb{S}_n/I$ is left (or right) Noetherian if and only if the ideal I contains all the height one primes of $\mathbb{S}_n$.*

The set of height 1 primes of $\mathbb{S}_n$. Consider the ideals of the algebra $\mathbb{S}_n$:

$$\mathfrak{p}_1 := F \otimes \mathbb{S}_{n-1},\ \mathfrak{p}_2 := \mathbb{S}_1 \otimes F \otimes \mathbb{S}_{n-2}, \ldots, \mathfrak{p}_n := \mathbb{S}_{n-1} \otimes F.$$

Then $\mathbb{S}_n/\mathfrak{p}_i \simeq \mathbb{S}_{n-1} \otimes (\mathbb{S}_1/F) \simeq \mathbb{S}_{n-1} \otimes K[x_i, x_i^{-1}]$ and $\bigcap_{i=1}^n \mathfrak{p}_i = \prod_{i=1}^n \mathfrak{p}_i = F^{\otimes n}$. Clearly, $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for all $i \neq j$.

- *The set $\mathcal{H}_1$ of height 1 prime ideals of the algebra $\mathbb{S}_n$ is $\{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$.*

Let $\mathfrak{a}_n := \mathfrak{p}_1 + \cdots + \mathfrak{p}_n$. Then the factor algebra

$$\mathbb{S}_n/\mathfrak{a}_n \simeq (\mathbb{S}_1/F)^{\otimes n} \simeq \bigotimes_{i=1}^n K[x_i, x_i^{-1}] = K[x_1, x_1^{-1}, \ldots, x_n, x_n^{-1}] =: L_n \tag{9}$$

is a skew Laurent polynomial algebra in n variables, and so $\mathfrak{a}_n$ is a prime ideal of height and co-height n of the algebra $\mathbb{S}_n$. The algebra L_n is commutative, and so

$$[a, b] \in \mathfrak{a}_n \text{ for all } a, b \in \mathbb{S}_n. \tag{10}$$

That is $[\mathbb{S}_n, \mathbb{S}_n] \subseteq \mathfrak{a}_n$. In particular, $[\mathbb{S}_1, \mathbb{S}_1] \subseteq F$. Since $\eta(\mathfrak{a}_n) = \mathfrak{a}_n$, the involution of the algebra $\mathbb{S}_n$ induces the *automorphism* $\overline{\eta}$ of the factor algebra $\mathbb{S}_n/\mathfrak{a}_n$ by the rule:

$$\overline{\eta} : L_n \to L_n,\ x_i \mapsto x_i^{-1},\ i = 1, \ldots, n. \tag{11}$$

It follows that $\eta(ab) - \eta(a)\eta(b) \in \mathfrak{a}_n$ for all elements $a, b \in \mathbb{S}_n$.

3. Certain subgroups of $\mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{S}_n)$

Recall that $G_n := \mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{S}_n)$ is the group of automorphisms of the algebra $\mathbb{S}_n$. In this section, a useful description of the group G_n is given (Theorem 3.2), an important (rather peculiar) criterion of equality of two elements of G_n (Theorem 3.7) is found, and several subgroups of G_n are introduced that are building blocks of the group G_n. These results are important in finding the group G_n.

Proposition 3.1 ([4]). *The polynomial algebra P_n is the only (up to isomorphism) faithful simple $\mathbb{S}_n$-module.*

In more detail, ${}_{\mathbb{S}_n}P_n \simeq \mathbb{S}_n/(\sum_{i=0}^n \mathbb{S}_n y_i) = \bigoplus_{\alpha\in\mathbb{N}^n} Kx^\alpha\overline{1}$, $\overline{1} := 1+\sum_{i=1}^n \mathbb{S}_n y_i$; and the action of the canonical generators of the algebra $\mathbb{S}_n$ on the polynomial algebra P_n is given by the rule:

$$x_i * x^\alpha = x^{\alpha+e_i},\ y_i * x^\alpha = \begin{cases} x^{\alpha-e_i}, & \text{if } \alpha_i > 0,\\ 0, & \text{if } \alpha_i = 0,\end{cases} \text{ and } E_{\beta\gamma} * x^\alpha = \delta_{\gamma\alpha}x^\beta,$$

where $e_1 := (1,0,\dots,0),\dots,e_n := (0,\dots,0,1)$ is the canonical basis for the free $\mathbb{Z}$-module $\mathbb{Z}^n = \bigoplus_{i=1}^n \mathbb{Z}e_i$. We identify the algebra $\mathbb{S}_n$ with its image in the algebra $\mathrm{End}_K(P_n)$ of all the K-linear maps from the vector space P_n to itself, i.e. $\mathbb{S}_n \subset \mathrm{End}_K(P_n)$. Let $\mathrm{Aut}_K(P_n)$ be the group of units of the algebra $\mathrm{End}_K(P_n)$. $\mathrm{Aut}_K(P_n)$ is the group of all the *invertible* K-linear maps from P_n to itself. Each element $\varphi \in \mathrm{Aut}_K(P_n)$ yields an inner automorphism $\omega_\varphi : f \mapsto \varphi f\varphi^{-1}$ of the algebra $\mathrm{End}_K(P_n)$. Suppose that the automorphism ω_φ respects the subalgebra $\mathbb{S}_n$, that is $\omega_\varphi(\mathbb{S}_n) = \mathbb{S}_n$, then its restriction $\sigma_\varphi : \omega_\varphi|_{\mathbb{S}_n} : a \mapsto \varphi a\varphi^{-1}$ is an automorphism of the algebra $\mathbb{S}_n$.

The next result shows that all the automorphisms of the algebra $\mathbb{S}_n$ can be obtained in this way.

Theorem 3.2. $G_n = \{\sigma_\varphi \mid \varphi \in \mathrm{Aut}_K(P_n)$ *such that* $\varphi\mathbb{S}_n\varphi^{-1} = \mathbb{S}_n\}$, *where* $\sigma_\varphi(a) := \varphi a\varphi^{-1}$, $a \in \mathbb{S}_n$.

Proof. Let $\sigma \in G_n$. The twisted by the automorphism σ module P_n, denoted ${}^\sigma P_n$, is simple and faithful. Recall that as a vector space the module ${}^\sigma P_n$ coincides with the module P_n but the action of the algebra $\mathbb{S}_n$ is given by the rule: $a \cdot p := \sigma(a) * p$, where $a \in \mathbb{S}_n$ and $p \in P_n$. By Proposition 3.1, the $\mathbb{S}_n$-modules P_n and ${}^\sigma P_n$ are isomorphic. So, there exists an element $\varphi \in \mathrm{Aut}_K(P_n)$ such that $\varphi a = \sigma(a)\varphi$ for all $a \in \mathbb{S}_n$, and so $\sigma(a) = \varphi a\varphi^{-1}$, as required. □

Theorem 3.3 ([4]). *The ideal $\mathfrak{a}_n$ is the smallest ideal of the algebra $\mathbb{S}_n$ such that the factor algebra $\mathbb{S}_n/\mathfrak{a}_n$ is commutative.*

Lemma 3.4. $\sigma(\mathfrak{a}_n) = \mathfrak{a}_n$ *for all* $\sigma \in G_n$.

Remark. We will see that the ideal $\mathfrak{a}_n$ is the *only* nonzero, prime, G_n-invariant ideal of the algebra $\mathbb{S}_n$ (Theorem 9.7).

Proof. For each element $\sigma \in G_n$ the map

$$\mathbb{S}_n/\mathfrak{a}_n \to \mathbb{S}_n/\sigma(\mathfrak{a}_n),\ a + \mathfrak{a}_n \mapsto \sigma(a) + \sigma(\mathfrak{a}_n),$$

is an isomorphism of algebras. By Theorem 3.3, $\sigma(\mathfrak{a}_n) = \mathfrak{a}_n$ for all $\sigma \in G_n$ since $\mathbb{S}_n/\mathfrak{a}_n$ is a commutative algebra. □

The automorphism $\widehat{\eta} \in \mathrm{Aut}(G_n)$. The involution η of the algebra $\mathbb{S}_n$ yields the automorphism $\widehat{\eta} \in \mathrm{Aut}(G_n)$ of the group G_n:

$$\widehat{\eta} : G_n \to G_n,\ \sigma \mapsto \eta\sigma\eta^{-1}. \tag{12}$$

Clearly, $\widehat{\eta}^2 = e$ and $\widehat{\eta}(\sigma) = \eta\sigma\eta$ since $\eta^2 = e$. By Lemma 3.4, we have the group homomorphism (recall that $L_n = \mathbb{S}_n/\mathfrak{a}_n$):

$$\xi : G_n \to \mathrm{Aut}_{K-\mathrm{alg}}(L_n),\ \sigma \mapsto (\overline{\sigma} : a + \mathfrak{a}_n \mapsto \sigma(a) + \mathfrak{a}_n). \tag{13}$$

The homomorphisms $\widehat{\eta}$ and ξ will be used often in the study of the group G_n. We can easily find the group $\mathrm{Aut}_{K-\mathrm{alg}}(L_n)$ of algebra automorphisms of the Laurent polynomial algebra L_n. We are interested in finding the image and the kernel of the homomorphism ξ (Corollary 5.3). We will see that the image of ξ is small (and the homomorphism ξ is far from being surjective).

Next, several important subgroups of G_n are introduced, they are building blocks of the group G_n (Theorem 5.1).

The group $\mathrm{Inn}(\mathbb{S}_n)$ of inner automorphism of $\mathbb{S}_n$. Let $\mathbb{S}_n^*$ be the group of units of the algebra $\mathbb{S}_n$. The center $Z(\mathbb{S}_n)$ of the algebra $\mathbb{S}_n$ is K, [4]. For each element $u \in \mathbb{S}_n^*$, let $\omega_u : \mathbb{S}_n \to \mathbb{S}_n$, $a \mapsto uau^{-1}$, be the inner automorphism associated with the element u. Then the group of inner automorphisms of the algebra $\mathbb{S}_n$,

$$\mathrm{Inn}(\mathbb{S}_n) = \{\omega_u \mid u \in \mathbb{S}_n^*\} \simeq \mathbb{S}_n^*/K^*,$$

is a normal subgroup of G_n. It follows from

$$\widehat{\eta}(\omega_u) = \omega_{\eta(u)^{-1}},\ u \in \mathbb{S}_n^*, \tag{14}$$

that $\widehat{\eta}(\mathrm{Inn}(\mathbb{S}_n)) = \mathrm{Inn}(\mathbb{S}_n)$. The factor algebra $\mathbb{S}_n/\mathfrak{a}_n = L_n$ is commutative, and so $\xi(\mathrm{Inn}(\mathbb{S}_n)) = \{e\}$.

The torus $\mathbb{T}^n$. The n-dimensional algebraic torus $\mathbb{T}^n := \{t_\lambda \mid \lambda = (\lambda_1, \dots, \lambda_n) \in K^{*n}\}$ is a subgroup of G_n, where

$$t_\lambda(x_i) = \lambda_i x_i,\ t_\lambda(y_i) = \lambda_i^{-1} y_i,\ i = 1, \dots, n.$$

The algebraic torus $\mathbb{T}^n := \{t_\lambda \mid \lambda \in K^{*n}\}$ is also a subgroup of the group $\mathrm{Aut}_{K-\mathrm{alg}}(L_n)$, where

$$t_\lambda(x_i) = \lambda_i x_i,\ i = 1, \dots, n.$$

Then $\widehat{\eta}(\mathbb{T}^n) = \mathbb{T}^n$ and $\widehat{\eta}(t_\lambda) = t_\lambda^{-1} = t_{\lambda^{-1}}$, where $\lambda^{-1} := (\lambda_1^{-1}, \dots, \lambda_n^{-1})$; $\xi(\mathbb{T}^n) = \mathbb{T}^n$ and $\xi(t_\lambda) = t_\lambda$. So, the maps $\widehat{\eta} : \mathbb{T}^n \to \mathbb{T}^n$ and $\xi : \mathbb{T}^n \to \mathbb{T}^n$ are group isomorphisms. Note that

$$t_\lambda(E_{\alpha\beta}) = \lambda^{\alpha-\beta} E_{\alpha,\beta}, \tag{15}$$

where $\lambda^{\alpha-\beta} := \prod_{i=1}^n \lambda_i^{\alpha_i - \beta_i}$.

The symmetric group S_n. The group G_n contains the symmetric group S_n where each elements τ of S_n is identified with the automorphism of the algebra $\mathbb{S}_n$ given by the rule:

$$\tau(x_i) = x_{\tau(i)},\ \tau(y_i) = y_{\tau(i)},\ i = 1, \dots, n.$$

The group S_n is also a subgroup of the group $\mathrm{Aut}_{K-\mathrm{alg}}(L_n)$, where

$$\tau(x_i) = x_{\tau(i)},\ i = 1, \dots, n.$$

Clearly, $\widehat{\eta}(S_n) = S_n$ and $\widehat{\eta}(\tau) = \tau$ for all $\tau \in S_n$; $\xi(S_n) = S_n$ and $\xi(\tau) = \tau$ for all $\tau \in S_n$. Note that

$$\tau(E_{\alpha\beta}) = E_{\tau(\alpha)\tau(\beta)}, \tag{16}$$

where $\tau(\alpha) := (\alpha_{\tau^{-1}(1)}, \dots, \alpha_{\tau^{-1}(n)})$.

The groups G_n and $\mathrm{Aut}_{K-\mathrm{alg}}(L_n)$ contain the semidirect product $S_n \ltimes \mathbb{T}^n$ since $\mathbb{T}^n \cap S_n = \{e\}$ and

$$\tau t_\lambda \tau^{-1} = t_{\tau(\lambda)},\ \text{where } \tau(\lambda) := (\lambda_{\tau^{-1}(1)}, \dots, \lambda_{\tau^{-1}(n)}), \tag{17}$$

for all $\tau \in S_n$ and $t_\lambda \in \mathbb{T}^n$. Clearly, the maps

$$\widehat{\eta} : S_n \ltimes \mathbb{T}^n \to S_n \ltimes \mathbb{T}^n, \tau t_\lambda \mapsto \tau t_\lambda^{-1},$$
$$\xi : S_n \ltimes \mathbb{T}^n \to S_n \ltimes \mathbb{T}^n, \tau t_\lambda \mapsto \tau t_\lambda,$$

are group isomorphisms.

Lemma 3.5. *$S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n) \subseteq G_n$.*

Proof. We know already that $\mathrm{Inn}(\mathbb{S}_n)$ and $S_n \ltimes \mathbb{T}^n$ are subgroups of G_n. Since $\mathrm{Inn}(\mathbb{S}_n) \subseteq \ker(\xi)$ and $\xi : S_n \ltimes \mathbb{T}^n \simeq S_n \ltimes \mathbb{T}^n$, we see that $\mathrm{Inn}(\mathbb{S}_n) \cap (S_n \ltimes \mathbb{T}^n) = \{e\}$, and the result follows. □

Let r be an element of a ring R. The element r is called *regular* if $\mathrm{l.ann}_R(r) = 0$ and $\mathrm{r.ann}_r(r) = 0$, where $\mathrm{l.ann}_R(r) := \{s \in R \mid sr = 0\}$ is the *left annihilator* of r and $\mathrm{r.ann}_R(r) := \{s \in R \mid rs = 0\}$ is the *right annihilator* of r.

The next lemma shows that the elements x and y of the algebra $\mathbb{S}_1$ are not regular.

Lemma 3.6 ([4]).

1. $\mathrm{l.ann}_{\mathbb{S}_1}(x) = \mathbb{S}_1 E_{00} = \bigoplus_{i \geq 0} K E_{i,0} = \bigoplus_{i \geq 0} K x^i (1 - xy)$ *and* $\mathrm{r.ann}_{\mathbb{S}_1}(x) = 0$.
2. $\mathrm{r.ann}_{\mathbb{S}_1}(y) = E_{00}\mathbb{S}_1 = \bigoplus_{i \geq 0} K E_{0,i} = \bigoplus_{i \geq 0} K(1 - xy) y^i$ *and* $\mathrm{l.ann}_{\mathbb{S}_1}(y) = 0$.

It follows from Lemma 3.6 that, for each $i = 1, \dots, n$,

$$\mathrm{l.ann}_{\mathbb{S}_n}(x_i) = \mathbb{S}_{n-1} \otimes \mathrm{l.ann}_{\mathbb{S}_1(i)}(x_i) = \bigoplus_{j \geq 0} \mathbb{S}_{n-1} E_{j,0}(i) = \bigoplus_{j \geq 0} \mathbb{S}_{n-1} x_i^j E_{00}(i), \tag{18}$$

$$\mathrm{r.ann}_{\mathbb{S}_n}(y_i) = \mathbb{S}_{n-1} \otimes \mathrm{r.ann}_{\mathbb{S}_1(i)}(y_i) = \bigoplus_{j \geq 0} E_{0,j}(i) \mathbb{S}_{n-1} = \bigoplus_{j \geq 0} E_{00}(i) y_i^j \mathbb{S}_{n-1}, \tag{19}$$

where $\mathbb{S}_{n-1}$ stands for $\bigotimes_{k\neq i}\mathbb{S}_1(k)$.

For an algebra A and a subset $S\subseteq A$, $\mathrm{Cen}_A(S) := \{a\in A \mid as = sa$ for all $s\in S\}$ is the *centralizer* of the set S in A. It is a subalgebra of A. It was proved in [4] that

(20) $\mathrm{Cen}_{\mathbb{S}_n}(x_1,\dots,x_n) = K[x_1,\dots,x_n],\ \mathrm{Cen}_{\mathbb{S}_n}(y_1,\dots,y_n) = K[y_1,\dots,y_n].$

Let $E_n := \mathrm{End}_{K-\mathrm{alg}}(\mathbb{S}_n)$ be the monoid of all the K-algebra endomorphisms of $\mathbb{S}_n$. The group of units of this monoid is G_n. The automorphism $\widehat{\eta}\in \mathrm{Aut}(G_n)$ can be extended to an automorphism $\widehat{\eta}\in\mathrm{Aut}(E_n)$ of the monoid E_n:

$$\widehat{\eta}: E_n\to E_n,\ \sigma\mapsto \eta\sigma\eta^{-1}. \tag{21}$$

The next (curious) result is instrumental in finding the group of automorphisms of the algebra $\mathbb{S}_n$.

Theorem 3.7. *Let $\sigma,\tau\in G_n$. Then the following statements are equivalent.*
1. $\sigma=\tau$.
2. $\sigma(x_1)=\tau(x_1),\dots,\sigma(x_n)=\tau(x_n)$.
3. $\sigma(y_1)=\tau(y_1),\dots,\sigma(y_n)=\tau(y_n)$.

Proof. Without loss of generality we may assume that $\tau = e$, the identity automorphism. Consider the following two subgroup of G_n, the stabilizers of the sets $\{x_1,\dots,x_n\}$ and $\{y_1,\dots,y_n\}$:

$$\mathrm{St}(x_1,\dots,x_n) :=\{g\in G_n \mid g(x_1)=x_1,\dots,g(x_n)=x_n\},$$
$$\mathrm{St}(y_1,\dots,y_n) :=\{g\in G_n \mid g(y_1)=y_1,\dots,g(y_n)=y_n\}.$$

Then

$$\widehat{\eta}(\mathrm{St}(x_1,\dots,x_n)) = \mathrm{St}(y_1,\dots,y_n),\ \widehat{\eta}(\mathrm{St}(y_1,\dots,y_n)) = \mathrm{St}(x_1,\dots,x_n).$$

Therefore, the theorem (where $\tau=e$) is equivalent to the single statement that $\mathrm{St}(x_1,\dots,x_n)=\{e\}$. So, let $\sigma\in\mathrm{St}(x_1,\dots,x_n)$. We have to show that $\sigma=e$. For each $i=1,\dots,n$, $1=\sigma(y_ix_i)=\sigma(y_i)x_i$ and $1=y_ix_i$. By taking the difference of these equalities we see that $a_i:=\sigma(y_i)-y_i\in \mathrm{l.ann}_{\mathbb{S}_n}(x_i)$. By (18), $a_i=\sum_{j\geq 0}\lambda_{ij}E_{j0}(i)$ for some elements $\lambda_{ij}\in\bigotimes_{k\neq i}\mathbb{S}_1(k)$, and so

$$\sigma(y_i)=y_i+\sum_{j\geq 0}\lambda_{ij}E_{j0}(i).$$

The element $\sigma(y_i)$ commutes with the elements $\sigma(x_k)=x_k$, $k\neq i$, hence all $\lambda_{ij}\in K[x_1,\dots,\widehat{x}_i,\dots,x_n]$, by (20). Since $E_{j0}(i)=x_i^jE_{00}(i)$, we can write

$$\sigma(y_i)=y_i+p_iE_{00}(i) \text{ for some } p_i\in P_n.$$

We have to show that all $p_i=0$. Suppose that this is not the case. Then $p_i\neq 0$ for some i. We seek a contradiction. Note that $\sigma^{-1}\in\mathrm{St}(x_1,\dots,x_n)$, and so $\sigma^{-1}(y_i)=y_i+q_iE_{00}(i)$ for some $q_i\in P_n$. Recall that $E_{00}(i)=1-x_iy_i$. Then $\sigma^{-1}(E_{00}(i))=1-x_i(y_i+q_iE_{00}(i))=(1-x_iq_i)E_{00}(i)$, and

$$y_i=\sigma^{-1}\sigma(y_i)=\sigma^{-1}(y_i+p_iE_{00}(i))=y_i+(q_i+p_i(1-x_iq_i))E_{00}(i),$$

and so $q_i+p_i = x_ip_iq_i$ since the map $P_n \to P_nE_{00}$, $p \mapsto pE_{00}$, is an isomorphism of P_n-modules as it follows from (4). This is impossible by comparing the degrees of the polynomials on both sides of the equality. □

Theorem 3.7 states that each automorphism of the noncommutative, finitely generated, non-Noetherian algebra $\mathbb{S}_n$ is uniquely determined by its action on its commutative, finitely generated subalgebra P_n. A similar result is true for the ring $\mathcal{D}(P_n)$ of differential operators on the polynomial algebra P_n over a field of *prime* characteristic. The algebra $\mathcal{D}(P_n)$ is a noncommutative, *non-finitely generated*, non-Noetherian algebra.

Theorem 3.8 ([2], Rigidity of the group $\mathrm{Aut}_{K-\mathrm{alg}}(\mathcal{D}(P_n))$)**.** *Let K be a field of prime characteristic, and $\sigma, \tau \in \mathrm{Aut}_{K-\mathrm{alg}}(\mathcal{D}(P_n))$. Then $\sigma = \tau$ if and only if $\sigma(x_1) = \tau(x_1), \dots, \sigma(x_n) = \tau(x_n)$.*

The above theorem does not hold in characteristic zero and does not hold in prime characteristic for the ring of differential operators on a Laurent polynomial algebra [2].

4. The groups $\mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{S}_1)$ and $\mathbb{S}_1^*$

In this section, the groups $\mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{S}_1)$ and $\mathbb{S}_1^*$ are found (Theorems 4.1 and 4.6). The case $n = 1$ is rather special and much more simpler than the general case. It is a sort of a degeneration of the general case. Briefly, the key idea in finding the group of automorphisms of the algebra $\mathbb{S}_1$ is to use Theorem 3.7 and some properties of the index of linear maps in the vector space $P_1 = K[x]$. We start this section with a sketch of the proof of Theorem 4.1. Then we prove necessary results about the index of certain elements of the algebra $\mathbb{S}_1$, and using them we find the group $\mathbb{S}_1^*$ of units of the algebra $\mathbb{S}_1$ and the group $\mathrm{Inn}(\mathbb{S}_1)$ of inner automorphisms of $\mathbb{S}_1$; and finally we give the proof of Theorem 4.1. The proof is constructive in the sense that for each automorphism σ of the algebra $\mathbb{S}_1$ it gives explicitly the presentation $\sigma = t_\lambda\omega_\varphi$ of σ as the product of an inner automorphism ω_φ and and element t_λ of the torus $\mathbb{T}^1$ (Corollary 4.7).

Theorem 4.1. $\mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{S}_1) = \mathbb{T}^1 \ltimes \mathrm{Inn}(\mathbb{S}_1) \simeq \mathbb{T}^1 \ltimes \mathrm{GL}_\infty(K)$.

Sketch of the Proof. Step 1. Let $\sigma \in G_1$. By Lemma 3.5, we have to show that $\sigma \in \mathbb{T}^1 \ltimes \mathrm{Inn}(\mathbb{S}_1)$. Using some properties of the index of linear maps of $\mathrm{End}_K(P_1)$ that have finite dimensional kernel and cokernel, we show that

$$\sigma(x) \equiv \lambda x \bmod F,$$
$$\sigma(y) \equiv \lambda^{-1} y \bmod F,$$

for some element $\lambda \in K^*$.

Step 2. Changing σ for $t_{\lambda^{-1}}\sigma$ we may assume that $\lambda = 1$.

Step 3. Changing σ for $\omega_\varphi\sigma$ for a suitable choice of a unit φ of the algebra $\mathbb{S}_1$ we may assume that $\sigma(y) = y$.

Step 4. Then, by Theorem 3.7, $\sigma = e$. □

Remark. The multiplication in the skew product $\mathbb{T}^1 \ltimes \mathrm{GL}_\infty(K)$ is given by the rule:

$$\varphi t_\lambda \cdot \psi t_\mu = \varphi t_\lambda(\psi) t_{\lambda\mu}, \tag{22}$$

where $t_\lambda, t_\mu \in \mathbb{T}^1$; $\varphi, \psi \in \mathrm{GL}_\infty(K)$; and $t_\lambda(\psi)$ is defined in (15).

The index ind of linear maps and its properties. Let $\mathcal{C} = \mathcal{C}(K)$ be the family of all K-linear maps with finite dimensional kernel and cokernel.

Definition. For a linear map $\varphi \in \mathcal{C}$, the integer

$$\mathrm{ind}(\varphi) := \dim \ker(\varphi) - \dim \mathrm{coker}(\varphi)$$

is called the *index* of the map φ.

Example. Note that $\mathbb{S}_1 \subset \mathrm{End}_K(P_1)$. One can easily prove that

$$\mathrm{ind}(x^i) = -i \text{ and } \mathrm{ind}(y^i) = i, \; i \geq 1. \tag{23}$$

Lemma 4.2 shows that $\mathcal{C}$ is a multiplicative semigroup with zero element (if the composition of two elements of C is undefined we set their product to be zero).

Lemma 4.2. *Let $\psi : M \to N$ and $\varphi : N \to L$ be K-linear maps. If two of the following three maps: ψ, φ, and $\varphi\psi$, belong to the set $\mathcal{C}$ then so does the third; and in this case,*

$$\mathrm{ind}(\varphi\psi) = \mathrm{ind}(\varphi) + \mathrm{ind}(\psi).$$

Proof. For an arbitrary K-linear map $f : V \to U$, we use the following notation: ${}_fV := \ker(f)$ and $U_f := \mathrm{coker}(f)$. The result follows from the long exact sequence of K-linear maps (where all the maps are natural):

$$0 \to {}_\psi M \to {}_{\varphi\psi}M \xrightarrow{\psi} {}_\varphi N \to N_\psi \xrightarrow{\varphi} L_{\varphi\psi} \to L_\varphi \to 0. \tag{24}$$

In particular, taking the Euler characteristic of the long exact sequence (24) gives the identity $\mathrm{ind}(\psi) - \mathrm{ind}(\varphi\psi) + \mathrm{ind}(\varphi) = 0$. $\square$

Lemma 4.3. *Let*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_1 & \longrightarrow & V_2 & \longrightarrow & V_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \\ 0 & \longrightarrow & U_1 & \longrightarrow & U_2 & \longrightarrow & U_3 & \longrightarrow & 0 \end{array}$$

be a commutative diagram of K-linear maps with exact rows. Suppose that $\varphi_1, \varphi_2, \varphi_3 \in \mathcal{C}$. Then

$$\mathrm{ind}(\varphi_2) = \mathrm{ind}(\varphi_1) + \mathrm{ind}(\varphi_3).$$

Proof. The Snake Lemma yields the long exact sequence:

$$0 \to \ker(\varphi_1) \to \ker(\varphi_2) \to \ker(\varphi_3) \to \mathrm{coker}(\varphi_1) \to \mathrm{coker}(\varphi_2) \to \mathrm{coker}(\varphi_3) \to 0$$

Taking its Euler characteristic gives $\mathrm{ind}(\varphi_1) - \mathrm{ind}(\varphi_2) + \mathrm{ind}(\varphi_3) = 0$. $\square$

Each nonzero element u of the Laurent polynomial algebra $L_1 = K[x, x^{-1}]$ is a unique sum $u = \lambda_s x^s + \lambda_{s+1}x^{s+1} + \cdots + \lambda_d x^d$, where all $\lambda_i \in K$, $\lambda_d \neq 0$, and $\lambda_d x^d$ is the *leading term* of the element u. The integer $\deg_x(u) = d$ is called the *degree* of the element u. It is an extension to L_1 of the usual degree of polynomials in $K[x]$. The next lemma explains how to compute the indices of the elements $\mathbb{S}_1 \setminus F$ using the degree function $\deg_x$ and shows that the index is a G_1-invariant concept. Note that $F \cap \mathcal{C} = \varnothing$.

Lemma 4.4.
1. $\mathbb{S}_1 \setminus F \subseteq \mathcal{C}$ *(recall that* $\mathbb{S}_1 \subset \mathrm{End}_K(P_1)$*) and, for each element* $a \in \mathbb{S}_1 \setminus F$,

$$\mathrm{ind}(a) = -\deg_x(\overline{a}),$$

where $\overline{a} = a + F \in \mathbb{S}_1/F = L_1$.

2. $\mathrm{ind}(\sigma(a)) = \mathrm{ind}(a)$ *for all* $\sigma \in G_1$ *and* $a \in \mathbb{S}_1 \setminus F$.

Proof. 1. Let $a \in \mathbb{S}_1 \setminus F$ and $d := \deg_x(\overline{a})$. The element of the algebra $\mathbb{S}_1$,

$$b := \begin{cases} y^d a, & \text{if } d \geq 0, \\ ax^{-d}, & \text{if } d < 0, \end{cases}$$

does not belong to the ideal F (since $\overline{b} = x^{-d}\overline{a} \neq 0$), and $\deg_x(\overline{b}) = 0$. By Lemma 4.2 and (23), it suffices to prove that $\mathrm{ind}(b) = 0$ since then

$$0 = \mathrm{ind}(b) = d + \mathrm{ind}(a),$$

that is $\mathrm{ind}(a) = -\deg_x(\overline{a})$. The element b can be written as a sum $b = \lambda + \sum_{i\geq 1} \lambda_i y^i + f$ for some elements $\lambda \in K^*$, $\lambda_i \in K$, and $f \in F$. Fix a natural number m such that $f \in M_{m+1}(K)$ (recall that $F = \cup_{i\geq 1} M_i(K)$). Abusing notation, let $K[b]$ be the K-subalgebra of $\mathrm{End}_K(P_1)$ generated by the element b. Then $V := \bigoplus_{i=0}^m Kx^i$ is a $K[b]$-submodule of P_1, and $U := P_1/V$ is the factor module. Let b_1 and b_2 be the linear maps that are determined by the action of the element b on the vector spaces V and U respectively. Then $\mathrm{ind}(b_1) = 0$ since $\dim(V) < \infty$; and $\mathrm{ind}(b_2) = 0$ since $b_2 = \lambda + \sum_{i\geq 1} \lambda_i y^i$ is a bijection. Applying Lemma 4.3 to the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] \arrow[d, "b_1"] & P_1 \arrow[r] \arrow[d, "b"] & U \arrow[r] \arrow[d, "b_2"] & 0 \\ 0 \arrow[r] & V \arrow[r] & P_1 \arrow[r] & U \arrow[r] & 0 \end{tikzcd}$$

we have the result: $\mathrm{ind}(b) = \mathrm{ind}(b_1) + \mathrm{ind}(b_2) = 0$.

2. By Theorem 3.2, $\mathrm{ind}(\sigma(a)) = \mathrm{ind}(\varphi a \varphi^{-1}) = \mathrm{ind}(a)$, where $\sigma = \sigma_\varphi$. □

The group of units $(1 + F)^*$ and $\mathbb{S}_1^*$. Recall that the algebra (without 1) $F = \bigoplus_{i,j\in\mathbb{N}} KE_{ij}$ is the union $M_\infty(K) := \bigcup_{d\geq 1} M_d(K) = \varinjlim M_d(K)$ of the matrix algebras $M_d(K) := \bigoplus_{0\leq i,j\leq d-1} KE_{ij}$, i.e. $F = M_\infty(K)$.

For each $d \geq 1$, consider the (usual) determinant $\det_d = \det : 1 + M_d(K) \to K$, $u \mapsto \det(u)$. These determinants determine the (global) *determinant*,

$$\det : 1 + M_\infty(K) = 1 + F \to K, \ u \mapsto \det(u), \tag{25}$$

where $\det(u)$ is the common value of all determinants $\det_d(u)$, $d \gg 1$. The (global) determinant has usual properties of the determinant. In particular, for all $u, v \in 1 + M_\infty(K)$,

$$\det(uv) = \det(u) \cdot \det(v).$$

It follows from this equality and the Cramer's formula for the inverse of a matrix that the group $\mathrm{GL}_\infty(K) := (1 + M_\infty(K))^*$ of units of the monoid $1 + M_\infty(K)$ is equal to

$$\mathrm{GL}_\infty(K) = \{u \in 1 + M_\infty(K) \mid \det(u) \neq 0\}. \tag{26}$$

Therefore,

$$(1 + F)^* = \{u \in 1 + F \mid \det(u) \neq 0\} = \mathrm{GL}_\infty(K). \tag{27}$$

The kernel

$$\mathrm{SL}_\infty(K) := \{u \in \mathrm{GL}_\infty(K) \mid \det(u) = 1\}$$

of the group epimorphism $\det : \mathrm{GL}_\infty(K) \to K^*$ is a *normal* subgroup of $\mathrm{GL}_\infty(K)$.

Let V be an infinite dimensional vector space that has countable basis. A sequence $\mathcal{V}$ of finite dimensional vector spaces in V, $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_i \subseteq \cdots$, such that $V = \bigcup_{i \geq 0} V_i$ is called a *finite dimensional vector space filtration* on V. The next result reveals an invariant nature of the (global) determinant.

Lemma 4.5. *Let $\mathcal{V} = \{V_i\}_{i \geq 0}$ be a finite dimensional vector space filtration on the polynomial algebra $P_1 = K[x]$ and $a \in \mathbb{M}_1 := 1 + M_\infty(K)$. Then $a(V_i) \subseteq V_i$ for all $i \gg 0$, and $\det(a|_{V_i}) = \det(a|_{V_j})$ for all $i, j \gg 0$, where $\det(a|_{V_i})$ is the determinant of the linear map $a|_{V_i} : V_i \to V_i$. Moreover, this common value of the determinants, $\det(a) = \det_{\mathcal{V}}(a)$, does not depend on the filtration $\mathcal{V}$ and, therefore, coincides with the determinant defined in* (25).

Proof. Let $a \in \mathbb{M}_1$. Then $a = 1 + \sum_{i,j=0}^{d} \lambda_{ij} E_{ij}$ for some scalars $\lambda_{ij} \in K$ and $d \in \mathbb{N}$. Note that the global determinant $\det(a)$, as defined in (25), is equal to the usual determinant $\det(a|_{P_{1,\leq i}})$ for all $i \geq d$, where $\{P_{1,\leq i} := \sum_{j=0}^{i} Kx^i\}_{i \in \mathbb{N}}$ is the *degree filtration* on P_1. Then $\mathrm{im}(a - 1) \subseteq P_{1,\leq d} \subseteq V_e$ for some $e \in \mathbb{N}$. Since $a = 1 + (a - 1)$, we have $a(V_i) \subseteq V_i$ and $\det(a|_{V_i}) = \det(a|_{V_e})$ for all $i \geq e$. Note that this is true for an arbitrary finite dimensional vector space filtration $\mathcal{V}$. Consider the following finite dimensional vector space filtration

$$\mathcal{V}' := \{V_i' := P_{1,\leq d},\ i = 0, \ldots, e - 1;\ V_j' := V_j,\ j \geq e\}.$$

Then

$$\det(a) = \det(a|_{P_{1,\leq d}}) = \det(a|_{V'_{e-1}}) = \det(a|_{V'_j}) = \det(a|_{V_j}),\ j \geq e.$$

This completes the proof of the lemma. $\square$

The *center* of a group G is denoted $Z(G)$.

Theorem 4.6.

1. $\mathbb{S}_1^* = K^* \times (1 + F)^* \simeq K^* \times \mathrm{GL}_\infty(K)$.

2. $Z(\mathbb{S}_1^*) = K^*$ *and* $Z((1+F)^*) = \{1\}$.
3. $\mathrm{Inn}(\mathbb{S}_1) \simeq \mathrm{GL}_\infty(K)$, $\omega_u \leftrightarrow u$.

Proof. 1. Note that $\mathbb{S}_1^* \supseteq K(1+F)^* \simeq K^* \times (1+F)^* \simeq K^* \times \mathrm{GL}_\infty(K)$ since $K^* \cap (1+F)^* = \{1\}$. It remains to prove the reverse inclusion. If an element u is a unit of the algebra $\mathbb{S}_1$ then the element $\overline{u} = u + F$ is a unit of the factor algebra $L_1 = \mathbb{S}_1/F$, and so $\overline{u} = \lambda x^i$ for some $\lambda \in K^*$ and $i \in \mathbb{Z}$. Therefore, either $u = \lambda x^i + f$ or $u = \lambda y^i + f$ for some $\lambda \in K^*$, $f \in F$ and $i \in \mathbb{N}$. The element $u \in \mathbb{S}_1 \setminus F$ is a unit, hence $u \in \mathrm{End}_K(P_1)$ is an invertible linear map (recall that $\mathbb{S}_1 \subset \mathrm{End}_K(P_1)$), and so $\mathrm{ind}(u) = 0$. By Lemma 4.4.(1) and (23), $i = 0$, and so $u \in K^*(1+F)^*$.

2. Note that $Z(\mathbb{S}_1^*) = K^* Z((1+F)^*)$. It suffices to show that $Z((1+F)^*) = \{1\}$. Let $z = 1 + \sum \lambda_{ij} E_{ij} \in Z((1+F)^*)$, where $\lambda_{ij} \in K$. For all $k \neq l$, $1 + E_{kl} \in (1+F)^*$ since $\det(1+E_{kl}) = 1$. Now, $z(1+E_{kl}) = (1+E_{kl})z$ for all $k \neq l$ if and only if $\sum_i \lambda_{ik} E_{il} = \sum_j \lambda_{lj} E_{kj}$ for all $k \neq l$ if and only if all $\lambda_{ij} = 0$ if and only if $z = 1$.

3. $\mathrm{Inn}(\mathbb{S}_1) \simeq \mathbb{S}_1^*/Z(\mathbb{S}_1^*) \simeq (K^* \times \mathrm{GL}_\infty(K))/K^* \simeq \mathrm{GL}_\infty(K)$. □

Proof of Theorem 4.1. By Theorem 4.6.(3), $\mathbb{T}^1 \ltimes \mathrm{Inn}(\mathbb{S}_1) = \mathbb{T}^1 \ltimes \mathrm{GL}_\infty(K)$.

Let $\sigma \in G_1$. By Lemma 3.5, in order to finish the proof of the theorem we have to show that $\sigma \in \mathbb{T}^1 \ltimes \mathrm{Inn}(\mathbb{S}_1)$. By Lemma 3.4, $\sigma(F) = F$, and so the map

$$\overline{\sigma} : L_1 = \mathbb{S}_1/F \to L_1 = \mathbb{S}_1/F,\ \overline{a} = a + F \mapsto \sigma(a) + F,$$

is an isomorphism of the Laurent polynomial algebra $L_1 = K[x, x^{-1}]$. Therefore, either $\overline{\sigma}(y) = \lambda x^{-1}$ or, otherwise, $\overline{\sigma}(y) = \lambda x$ for some scalar $\lambda \in K^*$. Equivalently, either $\sigma(y) = \lambda y + f$ or $\sigma(y) = \lambda x + f$ for some element $f \in F$. By Lemma 4.4, the second case is impossible since, by (23),

$$1 = \mathrm{ind}(y) = \mathrm{ind}(\sigma(y)) = \mathrm{ind}(\lambda x + f) = -\deg_x(\lambda x) = -1.$$

Therefore, $\sigma(y) = \lambda y + f$. Then $t_\lambda \sigma(y) = y + g$, where $g := t_\lambda(f) \in F$ since $t_\lambda(F) = F$ (Lemma 3.5). Fix a natural number m such that $g \in M_{m+1}(K)$. Then the finite dimensional vector spaces

$$V := \bigoplus_{i=0}^{m} K x^i \subset V' := \bigoplus_{i=0}^{m+1} K x^i$$

are y'-invariant, where $y' := t_\lambda \sigma(y) = y + g$. Note that $y' * x^{m+1} = y * x^{m+1} = x^m$ since $g * x^{m+1} = 0$. Note that $P_1 = \bigcup_{i \geq 1} \ker(y^i)$ and $\dim \ker_{P_1}(y) = 1$. Since the $\mathbb{S}_1$-modules P_1 and ${}^{t_\lambda \sigma}P_1$ are isomorphic, $P_1 = \bigcup_{i \geq 1} \ker(y'^i)$ and $\dim \ker_{P_1}(y') = 1$. This implies that the elements $x'_0, x'_1, \dots, x'_m, x^{m+1}$ are a K-basis for the vector space V', where

$$x'_i := y'^{m+1-i} * x^{m+1},\ i = 0, 1, \dots, m;$$

and the elements $x'_0, x'_1, \dots, x'_m$ are a K-basis for the vector space V. Then the elements

$$x'_0, x'_1, \dots, x'_m, x^{m+1}, x^{m+2}, \dots$$

are a K-basis for the vector space P_1. The K-linear map

$$\varphi : P_1 \to P_1,\ x^i \mapsto x_i' \ (i = 0, 1, \dots, m),\ x^j \mapsto x^j \ (j > m), \tag{28}$$

belongs to the group $(1 + F)^* = \mathrm{GL}_\infty(K)$ and satisfies the property that

$$y'\varphi = \varphi y,$$

the equality is in $\mathrm{End}_K(P_1)$. This equality can be rewritten as follows:

$$\omega_{\varphi^{-1}} t_\lambda \sigma(y) = y,\ \text{ where } \omega_{\varphi^{-1}} \in \mathrm{Inn}(\mathbb{S}_1).$$

By Theorem 3.7, $\sigma = t_{\lambda^{-1}} \omega_\varphi \in \mathbb{T}^1 \ltimes \mathrm{Inn}(\mathbb{S}_1)$, as required. □

Corollary 4.7. *Each automorphism σ of the algebra $\mathbb{S}_1$ is a unique product $\sigma = t_{\lambda^{-1}} \omega_\varphi$, where $\sigma(y) \equiv \lambda y \mod F$ and $\varphi \in (1 + F)^* = \mathrm{GL}_\infty(K)$ is defined as in* (28).

Proof. The result was established in the proof of Theorem 4.1 apart from the uniqueness of φ which follows from the fact that the center of the group $(1 + F)^* = \mathrm{GL}_\infty(K)$ is $\{1\}$ (Theorem 4.6.(3)). □

Proposition 4.8. *Each algebra endomorphism of $\mathbb{S}_1$ is either a monomorphism or, otherwise, its image is a commutative finite dimensional algebra. In the second case, all positive integers occur as the dimension of the image.*

Proof. Recall that F is the smallest nonzero ideal of the algebra $\mathbb{S}_1$, and $\mathbb{S}_1/F \simeq K[x, x^{-1}]$ (see (7)). If an algebra endomorphism σ of $\mathbb{S}_1$ is not a monomorphism then $F \subseteq \ker(\sigma)$, and so $\sigma(x) \in \mathbb{S}_1^* = K^*(1 + F)^*$ (Theorem 4.6.(1)) since the equalities $yx = 1$ and $xy = 1 - E_{00}$ imply the equalities $\sigma(y)\sigma(x) = 1$ and $\sigma(x)\sigma(y) = 1$; and $\mathrm{im}(\sigma) = K\langle \sigma(x), \sigma(x^{-1})\rangle$. Therefore, the image of σ is a commutative finite dimensional algebra since the algebra $K\langle \sigma(x), \sigma(x^{-1})\rangle$ can be seen as a subalgebra of the matrix algebra $M_d(K)$ for some d. The image of the endomorphism $\mathbb{S}_1 \to \mathbb{S}_1$, $x \mapsto 1$, $y \mapsto 1$, is K, hence one-dimensional. For each natural number $n \geq 2$, the image of the endomorphism

$$\sigma_n : \mathbb{S}_1 \to \mathbb{S}_1,\ x \mapsto 1 + \mathfrak{n},\ y \mapsto (1 + \mathfrak{n})^{-1},\ \mathfrak{n} := \sum_{i=0}^{n-2} E_{i,i+1},$$

has dimension n since the set $1, \mathfrak{n}, \mathfrak{n}^2, \dots, \mathfrak{n}^{n-1}$ is a K-basis of the image of σ_n. □

5. The group of automorphisms of the algebra $\mathbb{S}_n$

In this section, the group G_n is found (Theorem 5.1). It is shown that the groups G_n and $\mathrm{Inn}(\mathbb{S}_n)$ have trivial center (Corollary 5.6).

By the very definition, the subset $\mathrm{st}_{G_n}(\mathcal{H}_1)$ of $\mathrm{St}_{G_n}(\mathcal{H}_1)$ (see (1)) is a subgroup of $\mathrm{St}_{G_n}(\mathcal{H}_1)$.

Theorem 5.1. $G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$.

Proof. The group G_n acts in the obvious way, $(\sigma, \mathfrak{p}_i) \mapsto \sigma(\mathfrak{p}_i)$, on the set $\mathcal{H}_1 := \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}$ of all the height 1 prime ideals of the algebra $\mathbb{S}_n$. In particular, the symmetric group S_n, which is a subgroup of G_n, permutes the ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$, i.e. $\tau(\mathfrak{p}_i) = \mathfrak{p}_{\tau(i)}$ for $\tau \in S_n$. The stabilizer

$$\mathrm{St}_{G_n}(\mathcal{H}_1) = \{\sigma \in G_n \mid \sigma(\mathfrak{p}_1) = \mathfrak{p}_1, \ldots, \sigma(\mathfrak{p}_n) = \mathfrak{p}_n\}$$

is a *normal* subgroup of G_n such that $G_n = S_n \, \mathrm{St}_{G_n}(\mathcal{H}_1)$ and $S_n \cap \mathrm{St}_{G_n}(\mathcal{H}_1) = \{e\}$, and so

$$G_n = S_n \ltimes \mathrm{St}_{G_n}(\mathcal{H}_1). \tag{29}$$

Clearly, $\mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n) \subseteq \mathrm{St}_{G_n}(\mathcal{H}_1)$. So, in order to finish the proof of the theorem we have to prove that the inverse inclusion holds.

Let $\sigma \in \mathrm{St}_{G_n}(\mathcal{H}_1)$. We have to show that $\sigma \in \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$. Since $\sigma(\mathfrak{p}_n) = \mathfrak{p}_n$, the automorphism σ induces the automorphism

$$\sigma_n : \mathbb{S}_n/\mathfrak{p}_n = \mathbb{S}_{n-1} \otimes L_1 \to \mathbb{S}_n/\mathfrak{p}_n = \mathbb{S}_{n-1} \otimes L_1, \ a + \mathfrak{p}_n \mapsto \sigma(a) + \mathfrak{p}_n.$$

The restriction of the automorphism σ_n to the center $Z(\mathbb{S}_{n-1} \otimes L_1) = K[x_n, x_n^{-1}]$ of the algebra $\mathbb{S}_n/\mathfrak{p}_n$ yields its automorphism, and so either $\sigma_n(x_n) = \lambda x_n$ or $\sigma_n(x_n) = \lambda x_n^{-1}$ for some scalar $\lambda \in K^*$. Therefore, there are two options:

$$\text{(i) } \sigma(x_n) = \lambda_n x_n + p_n, \ \sigma(y_n) = \lambda_n^{-1} y_n + q_n;$$
$$\text{(ii) } \sigma(x_n) = \lambda_n y_n + p_n, \ \sigma(y_n) = \lambda_n^{-1} x_n + q_n;$$

for some $\lambda_n \in K^*$ and $p_n, q_n \in \mathfrak{p}_n$. We aim to show that the second case is impossible. This is true for $n = 1$, by Theorem 4.1. So, let $n > 1$. Suppose that $\sigma(x_n) = \lambda_n y_n + p_n$, wee seek a contradiction. By symmetry of the indices, for each $i = 1, \ldots, n$, there are two options:

$$\text{(i) } \sigma(x_i) = \lambda_i x_i + p_i, \ \sigma(y_i) = \lambda_i^{-1} y_i + q_i;$$
$$\text{(ii) } \sigma(x_i) = \lambda_i y_i + p_i, \ \sigma(y_i) = \lambda_i^{-1} x_i + q_i;$$

for some $\lambda_i \in K^*$ and $p_i, q_i \in \mathfrak{p}_i$. Since $\sigma(\mathfrak{p}_1 + \cdots + \mathfrak{p}_{n-1}) = \mathfrak{p}_1 + \cdots + \mathfrak{p}_{n-1}$ and $\mathbb{S}_n/(\mathfrak{p}_1 + \cdots + \mathfrak{p}_{n-1}) \simeq L_{n-1} \otimes \mathbb{S}_1(n)$, where $L_{n-1} = K[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$, the automorphism σ of the algebra $\mathbb{S}_n$ induces an automorphism, say $\overline{\sigma}$, of the algebra $L_{n-1} \otimes \mathbb{S}_1(n)$ such that either $\overline{\sigma}(x_i) = \lambda_i x_i$ or $\overline{\sigma}(x_i) = \lambda_i x_i^{-1}$ for all $i = 1, \ldots, n-1$. We see that $\overline{\sigma}(L_{n-1}) = L_{n-1}$. Let γ be the restriction of the automorphism $\overline{\sigma}$ to the algebra L_{n-1}. Then $\gamma \otimes \mathrm{id}_{\mathbb{S}_1(n)}$ is the automorphism of the algebra $L_{n-1} \otimes \mathbb{S}_1(n)$. Then $\widetilde{\sigma} := (\gamma \otimes \mathrm{id}_{\mathbb{S}_1(n)})^{-1} \overline{\sigma}$ is the L_{n-1}-algebra automorphism of the algebra $L_{n-1} \otimes \mathbb{S}_1(n)$ which can be uniquely extended to a $\mathrm{Frac}(L_{n-1})$-automorphism of the algebra $\mathrm{Frac}(L_{n-1}) \otimes \mathbb{S}_1(n)$ over the field of fractions $\mathrm{Frac}(L_{n-1}) = K(x_1, \ldots, x_{n-1})$ of the algebra L_{n-1}. By Theorem 4.1 (or Corollary 4.7), we must have the case (i) for x_n and y_n.

By symmetry of the indices, it follows from the case (i) that

$$\sigma(x_i) = \lambda_i x_i + p_i, \ \sigma(y_i) = \lambda_i^{-1} y_i + q_i, \ i = 1, \ldots, n, \tag{30}$$

for some scalars $\lambda_i \in K^*$ and some elements $p_i, q_i \in \mathfrak{p}_i$.

Changing σ for $t_{\lambda^{-1}}\sigma$, where $\lambda = (\lambda_1, \dots, \lambda_n)$, we may assume that $\lambda_1 = \cdots = \lambda_n = 1$, that is, $\sigma \in \mathrm{st}_{G_n}(\mathcal{H}_1)$. It follows that $G_n = S_n \mathbb{T}^n \, \mathrm{st}_{G_n}(\mathcal{H}_1)$. To finish the proof of the theorem it suffices to show that $\mathrm{st}_{G_n}(\mathcal{H}_1) \subseteq \mathrm{Inn}(\mathbb{S}_n)$ since then, by Lemma 3.5, $G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$ and also

$$\mathrm{st}_{G_n}(\mathcal{H}_1) = \mathrm{Inn}(\mathbb{S}_n). \tag{31}$$

Let $\sigma \in \mathrm{st}_{G_n}(\mathcal{H}_1)$. Then $\sigma^{-1} \in \mathrm{st}_{G_n}(\mathcal{H}_1)$ since $\mathrm{st}_{G_n}(\mathcal{H}_1)$ is a group. By Theorem 3.2, $\sigma = \sigma_\varphi$ for some element $\varphi \in \mathrm{Aut}_K(P_n)$ such that $\varphi \mathbb{S}_n \varphi^{-1} = \mathbb{S}_n$. For each number $i = 1, \dots, n$, $p_i := \sigma(x_i) - x_i \in \mathfrak{p}_i$ since $\sigma \in \mathrm{st}_{G_n}(\mathcal{H}_1)$. By multiplying this equality on the left by φ^{-1}, we obtain the equality $x_i \varphi^{-1} = \varphi^{-1}(x_i + p_i)$ for each $i = 1, \dots, n$. By Theorem 6.2, $\varphi^{-1} \in \mathbb{S}_n$. Repeating the same arguments for the automorphism $\sigma^{-1} = \sigma_{\varphi^{-1}} \in \mathrm{st}_{G_n}(\mathcal{H}_1)$, we have $\varphi \in \mathbb{S}_n$, that is $\varphi \in \mathbb{S}_n^*$, and so σ is an inner automorphism of the algebra $\mathbb{S}_n$. $\square$

Corollary 5.2. *The group* $\mathrm{Out}(\mathbb{S}_n) := G_n / \mathrm{Inn}(\mathbb{S}_n)$ *of outer automorphisms of the algebra* $\mathbb{S}_n$ *is isomorphic to the group* $S_n \ltimes \mathbb{T}^n$.

Proof. By Theorem 5.1, $\mathrm{Out}(\mathbb{S}_n) = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)/\mathrm{Inn}(\mathbb{S}_n) \simeq S_n \ltimes \mathbb{T}^n$. $\square$

The next corollary describes the image and the kernel of the group homomorphism $\xi : G_n \to \mathrm{Aut}_{K-\mathrm{alg}}(L_n)$, see (13).

Corollary 5.3.
1. $\mathrm{im}(\xi) = S_n \ltimes \mathbb{T}^n$.
2. $\ker(\xi) = \mathrm{Inn}(\mathbb{S}_n)$.

Proof. By Theorem 5.1, $G_n = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$; $\mathrm{Inn}(\mathbb{S}_n) \subseteq \ker(\xi)$ since L_n is a commutative algebra. Now, the results follow from the fact that the homomorphism ξ maps isomorphically the subgroup $S_n \ltimes \mathbb{T}^n$ of G_n to the subgroup $S_n \ltimes \mathbb{T}^n$ of $\mathrm{Aut}_{K-\mathrm{alg}}(L_n)$. $\square$

Corollary 5.4. *The group G_n contains an isomorphic copy of each linear algebraic group over K. In particular, G_n contains an isomorphic copy of each finite group.*

Proof. The result is obvious since the group G_n contains the group $\mathrm{GL}_\infty(K)$ and any linear algebraic group can be embedded in $\mathrm{GL}_\infty(K)$. $\square$

Corollary 5.5.
1. $\mathrm{st}_{G_n}(\mathcal{H}_1) = \mathrm{Inn}(\mathbb{S}_n)$.
2. (Characterization of the inner automorphisms $\mathrm{Inn}(\mathbb{S}_n)$ *via the height 1 primes of* $\mathbb{S}_n$*) An automorphism $\sigma \in G_n$ is an inner automorphism if and only if $\sigma(\mathfrak{p}_1) = \mathfrak{p}_1, \dots, \sigma(\mathfrak{p}_n) = \mathfrak{p}_n$ and*

$$\sigma(x_1) \equiv x_i \ \mathit{mod}\ \mathfrak{p}_i, \ \sigma(y_i) \equiv y_i \ \mathit{mod}\ \mathfrak{p}_i, \ i = 1, \dots, n.$$

3. If $\sigma \in \mathrm{Inn}(\mathbb{S}_n)$ then $\sigma = \omega_\varphi$ for a unique element $\varphi \in \mathbb{S}_n^*/K^*$ and $\sigma(x_i) = x_i + p_i$, $\sigma(y_i) = y_i + q_i$, where $p_i = [\varphi, x_i]\varphi^{-1}$ and $q_i = [\varphi, y_i]\varphi^{-1}$ for $i = 1, \dots, n$.

Proof. 1. See (31).

2. Statement 2 is equivalent to statement 1.

3.

$$\varphi x_i \varphi^{-1} = \sigma(x_i) = x_i + p_i \Leftrightarrow p_i = [\varphi, x_i]\varphi^{-1},$$
$$\varphi y_i \varphi^{-1} = \sigma(y_i) = y_i + q_i \Leftrightarrow q_i = [\varphi, y_i]\varphi^{-1}. \qquad \square$$

The inner automorphism $\sigma \in \mathrm{Inn}(\mathbb{S}_n)$ can be defined in two different ways:

(i) $\sigma = \omega_\varphi$ for a unique element $\varphi \in \mathbb{S}_n^*/K^*$; or

(ii) by the elements $p_i := \sigma(x_i) - x_i$, $q_i := \sigma(y_i) - y_i$, $i = 1, \dots, n$.

Corollary 5.5.(3) explains how to pass from (i) to (ii). The reverse passage, i.e. from (ii) to (i), is more subtle. Suppose that the elements $\{p_i, q_i \mid i = 1, \dots, n\}$ are given. Below, it is explained how to construct the element $\varphi \in \mathbb{S}_n^* \subseteq E_n$ which is unique up to K^*. By Theorem 3.2, the map $\varphi : P_n \to {}^\sigma P_n$ is an isomorphism of the $\mathbb{S}_n$-modules P_n and ${}^\sigma P_n$ (which is unique up to K^* since $\mathrm{End}_{\mathbb{S}_n}(P_n) \simeq K$, [4]). The isomorphism φ is determined by the polynomial $v := \varphi(1) \in P_n$ which is unique up to K^*:

$$Kv = \bigcap_{i=1}^n \ker_{P_n}(\sigma(y_i)) = \bigcap_{i=1}^n \ker_{P_n}(y_i + q_i).$$

Then φ is the change-of-the-basis matrix

$$x^\alpha \mapsto \prod_{i=1}^n (x_i + p_i)^{\alpha_i} * v.$$

Note that $\{x^\alpha\}_{\alpha \in \mathbb{N}^n}$ and $\{\sigma(x^\alpha) * v = \prod_{i=1}^n (x_i + p_i)^{\alpha_i} * v\}_{\alpha \in \mathbb{N}^n}$ are two bases for the vector space P_n.

The next corollary shows that the groups G_n and $\mathrm{Inn}(\mathbb{S}_n)$ have trivial center as well as some of the subgroups of G_n.

Corollary 5.6.

1. $Z(G_n) = \{e\}$.

2. $Z(\mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)) = \{e\}$.

3. $Z(\mathrm{Inn}(\mathbb{S}_n)) = \{e\}$.

4. $Z(S_n \ltimes \mathbb{T}^n) = \{t_{(\lambda,\dots,\lambda)} \mid \lambda \in K^*\} \simeq \mathbb{T}^1$.

5. $Z(S_n \ltimes \mathrm{Inn}(\mathbb{S}_n)) = \{e\}$.

Proof. 3. To prove statement 3 we use induction on n. The case $n = 1$ is true (Theorem 4.6). So, let $n > 1$ and we assume that the statement holds for all $n' < n$. Since $\mathrm{Inn}(\mathbb{S}_n) \simeq \mathbb{S}_n^*/K^*$, we have show that $Z(\mathbb{S}_n^*) = K^*$. Let $z \in Z(\mathbb{S}_n^*)$. For each $i = 1, \dots, n$, let $\mathbb{S}_{n-1,i} := \bigotimes_{j \neq i} \mathbb{S}_1(j)$ and consider the obvious algebra homomorphisms:

$$\mathbb{S}_n \to \mathbb{S}_n/\mathfrak{p}_i \simeq K[x_i, x_i^{-1}] \otimes \mathbb{S}_{n-1,i} \to K(x_i) \otimes \mathbb{S}_{n-1,i}.$$

By induction, the center of the group of units of the algebra $K(x_i) \otimes \mathbb{S}_{n-1,i}$ is $K(x_i)^*$, hence the image of the element z under the first map $(a \mapsto a + \mathfrak{p}_i)$

belongs to the Laurent polynomial algebra $K[x_i, x_i^{-1}]$. This implies that $z \in \mathcal{L}_1(i) + \mathfrak{p}_i$, where $\mathcal{L}_1(i) := (\bigoplus_{j\geq 1} Ky_i^j) \bigoplus K \bigoplus (\bigoplus_{j\geq 1} Kx_i^j)$, and so

$$z \in \bigcap_{i=1}^{n} (\mathcal{L}_1(i) + \mathfrak{p}_i) \subseteq \bigcap_{i=1}^{n} (K + \mathfrak{p}_i) \subseteq K + F_n.$$

In particular, $z \in Z((K + F_n)^*) = K^*$ since $K + F_n \simeq K + M_\infty(K)$ and $Z((K + M_\infty(K))^*) = K$ (see Theorem 4.6).

4. This is obvious.

2. Let $z = t_\lambda \omega_u \in Z(\mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}))$, where $t_\lambda \in \mathbb{T}^n$ and $\omega_u \in \mathrm{Inn}(\mathbb{S}_n)$. For $\alpha \in \mathbb{N}^n$, we write $\alpha \gg 0$ if $\alpha_i \gg 0$ for all $i = 1, \dots, n$. By Corollary 5.3.(2), for all elements $\alpha, \beta \in \mathbb{N}^n$ such that $\alpha, \beta \gg 0$, the elements u and $v(\alpha, \beta) := 1 + E_{\alpha\beta}$ commute. Therefore, the elements t_λ and $\omega_{v(\alpha,\beta)}$ commute. By (15), $t_\lambda = e$, and so $z = \omega_u \in Z(\mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S})) \cap \mathrm{Inn}(\mathbb{S}_n) \subseteq Z(\mathrm{Inn}(G_n)) = \{e\}$ (by statement 3), hence $z = e$.

1. Let $z \in Z(G_n)$. Then $z = \tau t_\lambda \omega_u$ for some elements $\tau \in S_n$, $t_\lambda \in \mathbb{T}^n$ and $\omega_u \in \mathrm{Inn}(G_n)$. The element τ is the image of the element z under the group epimorphism $G_n \to G_n/\mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n) \simeq S_n$. The element τ belongs to the center of the group S_n which is equal to

$$Z(S_n) = \begin{cases} S_2, & \text{if } n = 2, \\ e, & \text{if } n \neq 2. \end{cases}$$

Therefore, $\tau = e$ if $n \neq 2$. If $n = 2$ then the element τt_λ is the image of the element z under the group epimorphism $G_2 \to G_2/\mathrm{Inn}(\mathbb{S}_2) \simeq S_2 \ltimes \mathbb{T}^2$, and so it belongs to the center of the group $S_2 \ltimes \mathrm{Inn}(\mathbb{S}_2)$, and so $\tau = e$, by statement 4. Therefore, in general, $\tau = e$, and so $z \in Z(G_n) \cap \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n) \subseteq Z(\mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)) = \{e\}$ (by statement 2), hence $z = e$.

5. Let $z = \tau\omega_u \in Z(S_n \ltimes \mathrm{Inn}(\mathbb{S}_n))$. Using the same arguments as in the proof of statement 2, the elements τ and $\omega_{v(\alpha,\beta)}$ commute for all elements $\alpha, \beta \in \mathbb{N}^n$ such that $\alpha, \beta \gg 0$. Then $\tau = e$, by (16), and so $z = \omega_u \in Z(S_n \ltimes \mathrm{Inn}(\mathbb{S}_n)) \cap \mathrm{Inn}(\mathbb{S}_n) \subseteq Z(\mathrm{Inn}(\mathbb{S}_n)) = \{e\}$ (by statement 3), hence $z = e$. $\square$

6. A membership criterion for elements of the algebra $\mathbb{S}_n$

This section is independent of Section 5. In this section, membership criteria for the algebras $\mathbb{S}_n$, $P_n + F_n$ and $K + F_n$ are found in terms of commutators (Theorem 6.2, Corollaries 6.6 and 6.7). The most difficult result of this section is Theorem 6.2 which is used in the proof of Theorem 5.1. Corollary 6.7 is used in the proof of Theorem 7.7. A general result of constructing algebras using commutators is proved (Theorem 6.3) which shows that the obtained criteria are rather special (and tight).

For each $i = 1, \dots, n$, equality (6) can be written as follows

$$\mathbb{S}_1(i) = \mathcal{L}_1(i) \bigoplus F(i), \tag{32}$$

where

$$\mathcal{L}_1(i) := \Big(\bigoplus_{j\geq 1} Ky_i^j\Big)\bigoplus K \bigoplus \Big(\sum_{j\geq 1} Kx_i^j\Big) = \bigoplus_{j\in\mathbb{Z}} Kv_j(i),$$

where

$$v_j(i) := \begin{cases} x_i^j & \text{if } j\geq 0,\\ y_i^{-j} & \text{if } j<0.\end{cases}$$

So, each element $a\in\mathbb{S}_1(i)$ can be uniquely written as a sum

$$a = \sum_{j\geq 1}\lambda_{-j}y_i^j + \lambda_0 + \sum_{j\geq 1}\lambda_j x_i^j + \sum_{k,l\in\mathbb{N}}\lambda_{kl}E_{kl}(i) = \sum_{j\in\mathbb{Z}}\lambda_j v_j(i) + \sum_{k,l\in\mathbb{N}}\lambda_{kl}E_{kl}(i)$$

where the coefficients are scalars. On the other hand, each element $a\in\mathbb{S}_1(i)$ is a unique sum $a=\sum_{k,l\in\mathbb{N}}\mu_{kl}x_i^k y_i^l$, where $\mu_{kl}\in K$. Using the formula (3) the second presentation of the element a can be easily obtained from the first one; and the other way round can be done using the formula (33) below.

For all $i,j\in\mathbb{N}$,

$$x^iy^j = \begin{cases} x^{i-j} - \sum_{k=0}^{j-1}E_{i-j+k,k}, & \text{if } i\geq j,\\ y^{j-i} - \sum_{k=0}^{i-1}E_{k,j-i+k}, & \text{if } i<j.\end{cases} \tag{33}$$

It suffices to prove the equality (33) in the case when $i\geq j$ since then the second case can be obtained from the first case: indeed, for $i<j$,

$$x^iy^j = x^iy^iy^{j-i} = \Big(1-\sum_{k=0}^{i-1}E_{kk}\Big)y^{j-i} = y^{j-i} - \sum_{k=0}^{i-1}E_{k,j-i+k}.$$

To prove the first case we use induction on j. The result is obvious for $j=0$. So, let $j>0$ and we assume that the formula (33) holds for all $j'<j$. Using induction and the equality $xy=1-E_{00}$, we have the result:

$$\begin{aligned} x^iy^j &= x^iy^{j-1}y = \Big(x^{i-j+1} - \sum_{k=0}^{j-2}E_{i-(j-1)+k,k}\Big)y\\ &= x^{i-j}(1-E_{00}) - \sum_{k=0}^{j-2}E_{i-j+k+1,k+1} = x^{i-j} - \sum_{k=0}^{j-1}E_{i-j+k,k}.\end{aligned}$$

Let $\mathcal{B}_n$ be the set of all functions $f:\{1,2,\dots,n\}\to\mathbb{F}_2 := \{0,1\}$, where $\mathbb{F}_2 := \mathbb{Z}/2\mathbb{Z}$ is the field that contains two elements. $\mathcal{B}_n$ is a commutative ring with respect to addition and multiplication of functions. For $f,g\in\mathcal{B}_n$, we write $f\geq g$ if and only if $f(i)\geq g(i)$ for all $i=1,\dots,n$ where $1>0$. Then $(\mathcal{B}_n,\geq)$ is a partially ordered set. For each function $f\in\mathcal{B}_n$, let $|f| := \sum_{i=1}^n f_i = \#\{i\mid f_i=1\}$ and $\mathbb{S}_{n,f} := \bigotimes_{i=1}^n \mathbb{S}_{1,f_i}(i)$, where

$$\mathbb{S}_{1,f_i}(i) := \begin{cases}\mathcal{L}_1(i), & \text{if } f_i=1,\\ F(i), & \text{if } f_i=0.\end{cases}$$

By (32) and $\mathbb{S}_n = \bigotimes_{i=1}^n \mathbb{S}_1(i)$, we have the direct sum

$$\mathbb{S}_n = \bigoplus_{f\in\mathcal{B}_n} \mathbb{S}_{n,f}, \tag{34}$$

and so each element $a \in \mathbb{S}_n$ is a unique sum

$$a = \sum_{f\in\mathcal{B}_n} a_f, \tag{35}$$

where $a_f \in \mathbb{S}_{n,f}$. The vector space $\mathcal{L}_n := \bigotimes_{i=1}^n \mathcal{L}_1(i) = \bigoplus_{\alpha\in\mathbb{Z}^n} Kv_\alpha$, where $v_\alpha := \prod_{i=1}^n v_{\alpha_i}(i)$, is not an algebra but it is an algebra modulo the ideal $\mathfrak{a}_n$ which is canonically isomorphic to the Laurent polynomial algebra L_n (via $v_\alpha + \mathfrak{a}_n \leftrightarrow x^\alpha$): $(\mathcal{L}_n + \mathfrak{a}_n)/\mathfrak{a}_n = \mathbb{S}_n/\mathfrak{a}_n = L_n$. The elements $\{v_\alpha\}_{\alpha\in\mathbb{Z}^n}$ have remarkable properties which are used in the proof of the Membership Criterion for the elements of the algebra $\mathbb{S}_n$ (Theorem 6.2).

$$v_\alpha * x^\beta = \begin{cases} x^{\alpha+\beta}, & \text{if } \alpha+\beta \in \mathbb{N}^n, \\ 0, & \text{if } \alpha+\beta \notin \mathbb{N}^n, \end{cases} \tag{36}$$

$$v_\alpha * x^\beta x^\gamma = x^\beta v_\alpha * x^\gamma, \text{ if } \alpha+\gamma \in \mathbb{N}^n. \tag{37}$$

There is an obvious (useful) criterion of when an element of the algebra $\mathbb{S}_n$ belongs to the ideal F_n. It is used in the proof of Theorem 6.2.

Lemma 6.1. *Let $a \in \mathbb{S}_n$. Then $a \in F_n$ if and only if $a * (\sum_{i=1}^n P_n x_i^d) = 0$ for some $d \in \mathbb{N}$.*

Proof. ($\Rightarrow$) Trivial.

($\Leftarrow$) Let $C_n(d-1) := \{\alpha \in \mathbb{N}^n \mid \alpha_1 < d, \dots, \alpha_n < d\}$ and, for each element $\alpha \in C_n(d)$,

$$a * x^\alpha = \sum_{\beta\in\mathbb{N}^n} \lambda_{\alpha\beta} x^\beta = \Big(\sum_{\beta\in\mathbb{N}^n} \lambda_{\alpha\beta} E_{\beta\alpha}\Big) * x^\alpha$$

for some elements $\lambda_{\alpha\beta} \in K$, and so $a = \sum_{\beta\in\mathbb{N}^n}\sum_{\alpha\in C_n(d)} \lambda_{\alpha\beta}E_{\beta\alpha} \in F_n$ since we have the equality $P_n = C_n(d-1) \bigoplus (\sum_{i=1}^n P_n x_i^d)$. $\square$

The next theorem is a criterion of when a linear map $\varphi \in \mathrm{End}_K(P_n)$ belongs to the algebra $\mathbb{S}_n$ in terms of commutators. This result is tight when we compare it with general results of that sort, see Theorem 6.3 and Corollary 6.4. It is not obvious from the outset that the linear maps that satisfy the commutator conditions of Theorem 6.2 form an algebra.

Theorem 6.2 (A Membership Criterion)**.** *Let $\varphi \in \mathrm{End}_K(P_n)$. Then the following statements are equivalent.*

1. *$\varphi \in \mathbb{S}_n$.*
2. *$[x_1, \varphi] \in \mathfrak{p}_1, \dots, [x_n, \varphi] \in \mathfrak{p}_n$.*
3. *$x_i\varphi = \varphi \cdot (x_i + p_i) + q_i$, $i = 1, \dots, n$, for some elements $p_i, q_i \in \mathfrak{p}_i$.*

Proof. (1 $\Rightarrow$ 2) Let $\mathbb{S}_{n-1,i} := \bigotimes_{j\neq i}\mathbb{S}_1(j)$. Recall that $[x_i, \mathbb{S}_1(i)] \subseteq F(i)$, by (10), for $n=1$. Then, for each $i=1,\dots,n$,

$$[x_i, \mathbb{S}_n] \subseteq [x_i, \mathbb{S}_1(i)] \otimes \mathbb{S}_{n-1,i} \subseteq F(i) \otimes \mathbb{S}_{n-1,i} = \mathfrak{p}_i.$$

(2 $\Rightarrow$ 3) Trivial.

(3 $\Rightarrow$ 1) Suppose that a map φ satisfies the conditions of statement 3. The key idea of the proof of the fact that $\varphi \in \mathbb{S}_n$ is to use a downward induction on a natural number s starting with $s=n$ and $\varphi := \varphi_{n+1}$ to construct elements $a_f \in \mathbb{S}_{n,f}$ $(0 \neq f \in \mathcal{B}_n)$, elements $q_{i,s+1} \in \mathfrak{p}_i$ $(i=1,\dots,n;\ s=1,\dots,n)$, and natural numbers $d_n \leq d_{n-1} \leq \cdots \leq d_1$ such that the maps $\varphi_s := \varphi - \sum_{|f|\geq s} a_f$ satisfy the following conditions: for all $s=1,\dots,n$,

(38)

$$x_i\varphi_{s+1} = \varphi_{s+1}\cdot(x_i+p_i) + q_{i,s+1}, \quad p_i, q_{i,s+1} \in \mathbb{S}_{n-1,i} \bigotimes \left(\bigoplus_{k,l=0}^{d_s-1} KE_{kl}(i)\right), \quad i=1,\dots,n,$$

(39)
$$\varphi_s * \left(\sum_{0\leq i_1<\dots<i_s\leq n} P_n(x_{i_1}\cdots x_{i_s})^{d_s}\right) = 0.$$

Note that $\varphi_{n+1} = \varphi$ and all the maps φ_s satisfy the assumptions of statement 3. Suppose that we have proved this fact then, for $s=1$, the condition (39) is

$$\left(\varphi - \sum_{|f|\geq 1} a_f\right) * \left(\sum_{i=1}^n P_n x_i^{d_1}\right) = 0.$$

Then, by Lemma 6.1, $a_0 := \varphi - \sum_{|f|\geq 1} a_f \in F_n$, and so $\varphi = \sum_{f\in\mathcal{B}_n} a_f \in \mathbb{S}_n$, as required.

For $s=n$, by the assumption, we can fix a natural number d_n such that (38) holds, that is

$$x_i\varphi_{n+1} = \varphi_{n+1}\cdot(x_i+p_i) + q_{i,n+1}; \quad p_i, q_{i,n+1} \in \mathbb{S}_{n-1,i} \bigotimes \left(\bigoplus_{k,l=0}^{d_n-1} KE_{kl}(i)\right), \quad i=1,\dots,n,$$

where $\varphi = \varphi_{n+1}$ and $q_{i,n+1} = q_i$. We have to construct the element $a_f \in \mathbb{S}_{n,f} = \mathcal{L}_n$, where $f=(1,\dots,1)$ such that (39) holds. Let $\underline{d}_n = (d_n,\dots,d_n) \in \mathbb{N}^n$. Then

$$\varphi * x^{\underline{d}_n} = \sum_{\beta\in\mathbb{N}^n} \lambda_\beta x^\beta = \left(\sum_{\beta\in\mathbb{N}^n} \lambda_\beta v_{\beta-\underline{d}_n}\right) * x^{\underline{d}_n}$$

for some scalars $\lambda_\beta \in K$. Let $a_f := \sum_{\beta\in\mathbb{N}^n} \lambda_\beta v_{\beta-\underline{d}_n}$. Since $p_i * x^{\alpha+\underline{d}_n} = 0$ and $q_{i,s+1} * x^{\alpha+\underline{d}_n} = 0$, we have (using the equalities in statement 3)

$$\varphi * x^{\alpha+\underline{d}_n} = x^\alpha \varphi * x^{\underline{d}_n} \text{ for all } \alpha \in \mathbb{N}^n.$$

Using these equalities and (37), we see that

$$\varphi_n * x^{\alpha+\underline{d}_n} = x^\alpha \varphi_n * x^{\underline{d}_n} = x^\alpha(\varphi * x^{\underline{d}_n} - a_f * x^{\underline{d}_n}) = 0 \text{ for all } \alpha \in \mathbb{N}^n,$$

and so the equality (39) holds for $s = n$ and d_n.

Suppose that $s < n$ and we have found elements $a_f \in \mathbb{S}_{n,f}$ ($|f| \geq s+1$), elements $q_{i,t} \in \mathfrak{p}_i$ ($t = s+1, \dots, n+1$), and natural numbers $d_n \leq d_{n-1} \leq \cdots \leq d_s$ that satisfy the conditions (38) and (39). Note that (38) holds automatically for all natural numbers larger than d_s. To prove the inductive step at s, it remains to find the maps φ_s that satisfies (38) and (39). We may increase the number d_s. For each element $f \in \mathcal{B}_n$ with $|f| = s$, the element a_f is defined as follows. The set $\{1, \dots, n\}$ is the disjoint union of its two subsets $\{i_1, \dots, i_s\}$ and $\{i_{s+1}, \dots, i_n\}$, where $f(i_1) = \cdots = f(i_s) = 1$ and $f(i_{s+1}) = \cdots = f(i_n) = 0$. For each vector $\nu = (\nu_{s+1}, \dots, \nu_n) \in \mathbb{N}^{n-s}$ with all $\nu_k < d_s$,

$$(40)\quad \varphi_{s+1} * \big((x_{i_1}\cdots x_{i_s})^{d_s} x_{i_{s+1}}^{\nu_{s+1}} \cdots x_{i_n}^{\nu_n}\big) = \sum_{\alpha\in\mathbb{N}^n} \lambda_{\alpha\nu} x^\alpha$$
$$= a_f * \big((x_{i_1}\cdots x_{i_s})^{d_s} x_{i_{s+1}}^{\nu_{s+1}} \cdots x_{i_n}^{\nu_n}\big),$$

where $\lambda_{\alpha\nu} \in K$ and
(41)

$$a_f := \sum_{\alpha\in\mathbb{N}^n} \lambda_{\alpha\nu} v_{\alpha_{i_1}-d_s}(i_1)\cdots v_{\alpha_{i_s}-d_s}(i_s) E_{\alpha_{i_{s+1}},\nu_{s+1}}(i_{s+1})\cdots E_{\alpha_{i_n},\nu_n}(i_n).$$

By (38), for all elements $\alpha = (\alpha_1, \dots, \alpha_s) \in \mathbb{N}^s$,

$$(42)\quad \varphi_{s+1} * (x_{i_1}^{\alpha_1}\cdots x_{i_s}^{\alpha_s}(x_{i_1}\cdots x_{i_s})^{d_s} x_{i_{s+1}}^{\nu_{s+1}} \cdots x_{i_n}^{\nu_n})$$
$$= x_{i_1}^{\alpha_1}\cdots x_{i_s}^{\alpha_s} \varphi_{s+1} * \big((x_{i_1}\cdots x_{i_s})^{d_s} x_{i_{s+1}}^{\nu_{s+1}} \cdots x_{i_n}^{\nu_n}\big).$$

This equalities hold for any new d_s which is not smaller than the old d_s.

Define $\varphi_s := \varphi_{s+1} - \sum_{|f|=s} a_f$ and choose a new number d_s which is not smaller than the old d_s and such that (38) holds for the map φ_s. Using the equalities (42) (for all possible choices of f with $|f| = s$) and for the new choice of d_s together with (37), the equality (39) follows at once: the ideal $\sum_{0\leq i_1<\cdots<i_{s+1}\leq n} P_n(x_{i_1}\cdots x_{i_{s+1}})^{d_s}$ is annihilated both by the map φ_{s+1} (due to (39) for $s+1$ and $d_s \geq d_{s+1}$) and by the element $\sum_{|f|=s} a_f$, by the choice of d_s, hence it is annihilated by the map φ_s (each element a_f, where $|f| = s$, annihilates this ideal). In order to prove (39) it is sufficient to show that the map φ_s annihilates the monomials of the type $u = (x_{i_1}\cdots x_{i_s})^{d_s} x_{i_{s+1}}^{\nu_{s+1}} \cdots x_{i_n}^{\nu_n}$, but this is obvious since

$$\varphi_s * u = (\varphi_{s+1} - \sum_{|g|=s} a_g) * u = (\varphi_{s+1} - a_f) * u = 0,$$

by (40), since $a_g(u) = 0$ for all $g \neq f$. □

Theorem 6.3. *Let $A \subseteq B$ be K-algebras and M be a faithful B-module (and so $A \subseteq B \subseteq \mathrm{End}_K(M)$). Suppose that I is a left ideal of the algebra B such that $I \subseteq A$. Then*

1. the set $A' := \{b \in B \mid [b, A] \subseteq I\}$ *is a subalgebra of* B*. If* $[A, A] \subseteq I$ *then* $A \subseteq A'$*.*
2. If I *is also an ideal of the algebra* A*, and* $\{a_s\}_{s\in S}$ *is a set of* K*-algebra generators for* A *then* $A' = \{b \in B \mid [b, a_s] \in I$ *for all* $s \in S\}$*.*

Proof. 1. The set A' is a vector space over the field K, to prove that A' is an algebra we have to show that $A'A' \subseteq A'$. Let $b, c \in A'$. Then

$$\begin{aligned}[bc, A] &\subseteq [b, A]c + b[c, A] \subseteq Ic + bI \\ &\subseteq [I, c] + cI + I \subseteq I.\end{aligned}$$

If $[A, A] \subseteq I$ then, obviously, $A \subseteq A'$.

2. Let $A'' := \{b \in B \mid [b, a_s] \in I \text{ for all } s \in S\}$. Then $A' \subseteq A''$. To prove the reverse inclusion it is enough to show that $[b, a_{s_1} \cdots a_{s_m}] \in I$ for all products $u = a_{s_1} \cdots a_{s_m}$ of the generators $\{a_s\}_{s\in S}$. We use induction on m to prove this fact. The case $m = 1$ is obvious. So, let $m > 1$ and we assume that the result is true for all $m' < m$. Then

$$[b, a_{s_1} \cdots a_{s_m}] = [b, a_{s_1} \cdots a_{s_{m-1}}]a_{s_m} + a_{s_1} \cdots a_{s_{m-1}}[b, a_{s_m}] \in Ia_{s_m} + I \subseteq I.$$

□

Corollary 6.4. *The set* $\mathbb{S}'_1 := \{\varphi \in \mathrm{End}_K(P_1) \mid [x, \varphi] \in F, [y, \varphi] \in F\}$ *is a subalgebra of* $\mathrm{End}_K(P_n)$ *such that* $\mathbb{S}_1 \subseteq \mathbb{S}'_1$*. In fact,* $\mathbb{S}_1 = \mathbb{S}'_1$*, by Theorem 6.2.*

Proof. This is a direct consequence of Theorem 6.3 where $A = \mathbb{S}_1 = K\langle x, y\rangle$, $M = P_1$, $B = \mathrm{End}_K(P_1)$, and $I = F$ is an ideal of $\mathbb{S}_1$ such that $[\mathbb{S}_1, \mathbb{S}_1] \subseteq F$. It is obvious that the ideal F of the algebra $\mathbb{S}_1$ is a left ideal of the endomorphism algebra $\mathrm{End}_K(P_1)$ since an element $f \in \mathrm{End}_k(P_1)$ belongs to F if and only if $f * P_1x^d = 0$ for some $d \in \mathbb{N}$. □

For all integers $i, j \in \mathbb{N}$ (where $E_{i,-1} := 0$ and $E_{-1,j} := 0$)

$$[x, y^i] = -E_{0,i-1},\ \ [y, x^i] = E_{i-1,0}, \tag{43}$$

$$[x, E_{ij}] = E_{i+1,j} - E_{i,j-1},\ \ [y, E_{ij}] = E_{i-1,j} - E_{i,j+1}. \tag{44}$$

For an algebra A and an element $a \in A$, let $\mathrm{ad}(a) := [a, \cdot] : b \mapsto [a, b] = ab - ba$ be the *inner derivation* of the algebra A determined by the element a. The kernel $\ker \mathrm{ad}(a)$ of the inner derivation $\mathrm{ad}(a)$ is a subalgebra of A.

Lemma 6.5.
1. $\bigcap_{i=1}^n \ker \mathrm{ad}(x_i) = K[x_1, \dots, x_n]$.
2. $\bigcap_{i=1}^n \ker \mathrm{ad}(y_i) = K[y_1, \dots, y_n]$.

Proof. 1. We use induction on n. Let $n = 1$ and $a \in \ker \mathrm{ad}(x_1)$. By (11), $a = a_1 + a_0$ for unique elements $a_0 \in F$ and $a_1 = \sum_{i\geq1} \lambda_{-i}y_1^i + p$, $p \in K[x_1]$. Using the expressions for the commutators $[x_1, y_1^i]$ and $[x_1, E_{ij}]$ given by (43) and (44), we deduce that $a_1 = p$ and $a_0 = 0$, and so $a \in K[x_1]$. This proves the equality in the case $n = 1$. Let $n > 1$ and we assume that the result holds for all $n' < n$. By induction, $\bigcap_{i=1}^{n-1} \ker_{\mathbb{S}_{n-1}} \mathrm{ad}(x_i) = P_{n-1}$. Since $\mathbb{S}_n = \mathbb{S}_{n-1} \otimes \mathbb{S}_1$, we have $\bigcap_{i=1}^{n-1} \ker_{\mathbb{S}_n} \mathrm{ad}(x_i) = P_{n-1} \otimes \mathbb{S}_1(n)$, and finally $\bigcap_{i=1}^{n} \ker \mathrm{ad}(x_i) = P_n$ since $\ker_{\mathbb{S}_1(n)} \mathrm{ad}(x_n) = K[x_n]$.

2. Applying the involution η to statement 1 we obtain statement 2. $\square$

Corollary 6.6.

$$\{\varphi \in \mathrm{End}_K(P_n) \mid [x_1, \varphi] \in F_n, \dots, [x_n, \varphi] \in F_n\} = \begin{cases} \mathbb{S}_1, & \text{if } n = 1, \\ P_n + F_n, & \text{if } n > 1. \end{cases}$$

Proof. For $n = 1$, the result follows from Theorem 6.2. Let $n > 1$. Let L and R denote the LHS and the RHS of the equality. Then $\mathbb{S}_n \supseteq L \supseteq R$, by Theorem 6.2. Let $a \in L$, it remains to show that $a \in R$. For each $i = 1, \dots, n$, let $\mathbb{S}_{n-1,i} := \bigotimes_{j \neq i} \mathbb{S}_1(j)$ and $F_{n-1,i} := \bigotimes_{j \neq i} F(j)$.

Note that $\mathbb{S}_n = \mathbb{S}_1 \otimes \mathbb{S}_{n-1,1}$ and $[x_1, \mathbb{S}_1] \subseteq F$ (see (10) for $n = 1$). The inclusion $[x_1, a] \in F_n$ implies that $a \in K[x_1] \otimes \mathbb{S}_{n-1,1} + \mathbb{S}_1 \otimes F_{n-1,1}$. The conditions $[x_j, a] \in F_n$ for $j = 2, \dots, n$, imply that $a \in K[x_1] \otimes \mathbb{S}_{n-1,1} + F_n$ (see (44)). Then $a \in K[x_i] \otimes \mathbb{S}_{n-1,i} + F_n$ for all i (by symmetry of the indices), and

$$a \in \bigcap_{i=1}^{n} (K[x_i] \otimes \mathbb{S}_{n-1,i} + F_n) = P_n + F_n.$$

$\square$

Corollary 6.7 (A Membership Criterion for F_n)**.**

$$\{\varphi \in \mathrm{End}_K(P_n) \mid [x_i, \varphi] \in F_n, [y_i, \varphi] \in F_n, i = 1, \dots, n\}$$
$$= \begin{cases} \mathbb{S}_1, & \text{if } n = 1, \\ K + F_n, & \text{if } n > 1. \end{cases}$$

Proof. This follows from Corollary 6.6 and (43). $\square$

Remarks. 1. The set in Corollary 6.7 is, in fact, an algebra which is not obvious from the outset. This fact can be deduced from Theorems 6.2 and 6.3: let L be the LHS of the equality in Corollary 6.7. Since $F_n \subseteq \mathfrak{p}_i$ for all i, $L \subseteq \mathbb{S}_n$, by Theorem 6.2. Then L is a subalgebra of $\mathbb{S}_n$ by applying Theorem 6.3 in the case $A = B = \mathbb{S}_n$ and $I = F_n$.

2. Corollaries 6.4 and 6.7 also show that in order to have the inclusion $A \subseteq A'$ in Theorem 6.3.(1), the condition $[A, A] \subseteq I$ cannot be dropped: for $n > 1$, let L be as above. By Theorem 6.2, $L \subseteq \mathbb{S}_n$, and so $L = \{b \in \mathbb{S}_n \mid [b, x_i] \in F_n, [b, y_i] \in F_n, i = 1, \dots, n\}$, $I = F_n$ is an ideal of $A = B = \mathbb{S}_n$. Since $[\mathbb{S}_n, \mathbb{S}_n] \not\subseteq F_n$ and $L = K + F_n \not\supseteq A$, we see that in Theorem 6.3 the condition $[A, A] \subseteq I$ cannot be dropped and still have the inclusion $A \subseteq A'$.

7. The groups $\mathbb{M}_n^*$ and G_n'

In this section, the subgroups $\mathbb{M}_n^*$ and G_n' of the groups $\mathbb{S}_n^*$ and G_n respectively are introduced. It is proved that the group $\mathbb{M}_n^*$ has trivial center (Corollary 7.6) and is a skew direct product of $2^n - 1$ copies of the group

$\mathrm{GL}_\infty(K)$ (Theorem 7.2). An analog of the polynomial Jacobian homomorphism, the so-called global determinant, is introduced for the group $\mathbb{M}_n^*$. In Section 8, the global determinant is extended to the group G_n'.

For each nonempty subset I of the set of indices $\{1, \dots, n\}$, define the K-algebra without 1,

$$F(I) := \bigotimes_{i\in I} F(i) = \bigoplus_{\alpha,\beta\in\mathbb{N}^I} KE_{\alpha\beta}(I) \subseteq \mathbb{S}_n, \ E_{\alpha\beta}(I) := \prod_{i\in I} E_{\alpha_i\beta_i}(i),$$

where $\alpha = (\alpha_i)_{i\in I}$ and $\beta = (\beta_i)_{i\in I}$. The algebra $F(I)$ is isomorphic *non-canonically* to the matrix algebra (without 1) $M_\infty(K) = \bigcup_{d\geq 1} M_d(K)$ when we fix a bijection $b : \mathbb{N}^m \to \mathbb{N}$. Then the matrix unit $E_{\alpha\beta}(I)$ becomes the usual matrix unit $E_{b(\alpha)b(\beta)}$ of the matrix algebra $M_\infty(K)$. The function b determines the finite dimensional monomial vector space filtration $\mathcal{V}_b := \{V_{b,i} := \sum_{b(\alpha)\leq i} Kx^\alpha\}_{i\in\mathbb{N}}$ on P_n. The algebra (without 1) $F(I)$ is an ideal of the following algebra with 1,

$$\mathbb{F}_I := K + F(I) \subseteq \mathbb{S}_n.$$

The algebra $\mathbb{F}_I$ contains the multiplicative monoid $\mathbb{M}_I := 1 + F(I) \simeq 1 + M_\infty(K)$. We define the (global) determinant on $\mathbb{M}_I$ as in (25):

$$\det = \det_{I,b} : \mathbb{M}_I \to K, \ u \mapsto \det(u). \tag{45}$$

We will see that the determinant $\det_{I,b}$ does not depend on the bijection b. The (global) determinant has usual properties of the determinant. In particular, for all elements $u, v \in \mathbb{M}_I$,

$$\det(uv) = \det(u) \cdot \det(v).$$

The group of units $\mathbb{M}_I^*$ of the monoid $\mathbb{M}_I$ is

$$\mathbb{M}_I^* = \{u \in \mathbb{M}_I \mid \det(u) \neq 0\} \simeq \mathrm{GL}_\infty(K). \tag{46}$$

It contains the normal subgroup $S\mathbb{M}_I^* = \{u \in \mathbb{M}_I \mid \det(u) = 1\} \simeq \mathrm{SL}_\infty(K)$ which is the kernel of the group epimorphism $\det : \mathbb{M}_I^* \to K^*$. The inversion formula for u^{-1} is, basically, the Cramer's formula for the inverse of a matrix of finite size. The group of units $\mathbb{F}_I^*$ of the algebra $\mathbb{F}_I$ is

$$\mathbb{F}_I^* = K^*\mathbb{M}_I^* \simeq K^* \times \mathbb{M}_I^* \simeq K^* \times \mathrm{GL}_\infty(K).$$

Corollary 7.1. *Let I be a nonempty subset of $\{1, \dots, n\}$. Then $\mathbb{M}_I^* = \{u \in \mathbb{M}_I \mid \det(u) \neq 0\} \simeq \mathrm{GL}_\infty(K)$ and $Z(\mathbb{M}_I^*) = \{1\}$.*

Proof. This follows from Theorem 4.6. □

Definition. Let $\mathbb{F}_n := \bigotimes_{i=1}^n \mathbb{F}_{\{i\}} = K \bigoplus \left(\bigoplus_{\varnothing\neq I\subseteq\{1,\dots,n\}} F(I)\right) \subseteq \mathbb{S}_n$ (this is a subalgebra of $\mathbb{S}_n$) and $\mathbb{M}_n := 1 + \sum_{\varnothing\neq I\subseteq\{1,\dots,n\}} F(I)$, this is a multiplicative submonoid of the algebra $\mathbb{F}_n$.

The group of units $\mathbb{F}_n^*$ of the algebra $\mathbb{F}_n$ is

$$\mathbb{F}_n^* = K^*\mathbb{M}_n^* \simeq K^* \times \mathbb{M}_n^*,$$

where $\mathbb{M}_n^*$ is the group of units of the monoid $\mathbb{M}_n$. The algebra $\mathbb{F}_n$ contains all the algebras $\mathbb{F}_I$, the monoid $\mathbb{M}_n$ contains all the monoids $\mathbb{M}_I$, and the group $\mathbb{M}_n^*$ contains all the groups $\mathbb{M}_I^*$.

Let $X_1, \ldots, X_m$ be nonempty subsets of a group G and $X_1 \cdots X_n := \{x_1 \cdots x_n \mid x_i \in X_i\}$ be their *ordered* product. We sometime write ${}^{\mathrm{set}}\prod_{i=1}^n X_i$ for this product in order to distinguish it from the direct product of groups. In general, $X_1 \cdots X_n$ is not a subgroup of G. If each element of the product $X_1 \cdots X_n$ has a unique presentation $x_1 \cdots x_n$, where $x_i \in X_i$, then we say that the product is *exact* and write $X = {}^{\mathrm{exact}}\prod_{i=1}^n X_i$.

Theorem 7.2. $\mathbb{M}_n^* \simeq \underbrace{\mathrm{GL}_\infty(K) \ltimes \cdots \ltimes \mathrm{GL}_\infty(K)}_{2^n-1 \text{ times}}$.

Proof. The theorem follows from the fact that there is a chain of normal subgroups of the group $\mathbb{M}_n^*$:

$$\mathbb{M}_n^* = \mathbb{M}_{n,1}^* \supset \mathbb{M}_{n,2}^* \supset \cdots \supset \mathbb{M}_{n,i}^* \supset \cdots \supset \mathbb{M}_{n,n}^* \supset \mathbb{M}_{n,n+1}^* = \{1\} \tag{47}$$

such that, for each number $s = 1, \ldots, n$,

(48)
$$\mathbb{M}_{n,s} = {}^{\mathrm{set}}\prod_{|I|=s} \mathbb{M}_I^* \cdot \mathbb{M}_{n,s+1}^* \quad \text{and} \quad \mathbb{M}_{n,s}^*/\mathbb{M}_{n,s+1}^* \simeq \prod_{|I|=s} \mathbb{M}_I^* \simeq \mathrm{GL}_\infty(K)^{\binom{n}{s}},$$

where the first product is the product of subsets in the group $\mathbb{M}_{n,s}^*$ in arbitrary order, and the second product is the direct product of groups (in particular, the product of sets ${}^{\mathrm{set}}\prod_{|I|=s} \mathbb{M}_I^*$ has trivial intersection with the group $\mathbb{M}_{n,s+1}^*$, i.e. $\{1\}$). The groups $\mathbb{M}_{n,s}^*$ are constructed below, see (49). □

In their construction the following two lemmas are used repeatedly.

Lemma 7.3. *Let R be a ring and $I_1, \ldots, I_n$ be ideals of the ring R such that $I_iI_j = 0$ for all $i \neq j$. Let $a = 1 + a_1 + \cdots + a_n \in R$, where $a_1 \in I_1, \ldots, a_n \in I_n$. The element a is a unit of the ring R if and only if all the elements $1 + a_i$ are units; and, in this case, $a^{-1} = (1 + a_i)^{-1}(1 + a_2)^{-1} \cdots (1 + a_n)^{-1}$.*

Proof. Note that the elements $1 + a_i$ commute, and $a = \prod_{i=1}^n (1 + a_i)$. Now, the statement is obvious. □

Let R be a ring, R^* be its group of units, I be an ideal of R such that $I \neq R$, and let $(1 + I)^*$ be the group of units of the multiplicative monoid $1 + I$.

Lemma 7.4. *Let R and I be as above. Then*

1. $R^* \cap (1 + I) = (1 + I)^*$.
2. $(1 + I)^*$ *is a normal subgroup of* R^*.

Proof. 1. The inclusion $R^*\cap(1+I)\supseteq(1+I)^*$ is obvious. To prove the reverse inclusion, let $1+a\in R^*\cap(1+I)$, where $a\in I$, and let $(1+a)^{-1}=1+b$ for some $b\in R$. The equality $1=(1+a)(1+b)$ can be written as $b=-a(1+b)\in I$, i.e. $1+a\in(1+I)^*$. This proves the reverse inclusion.

2. For all $a\in R^*$, $a(1+I)a^{-1}=a(R^*\cap(1+I))a^{-1}=aR^*a^{-1}\cap a(1+I)a^{-1}=R^*\cap(1+I)=(1+I)^*$. Therefore, $(1+I)^*$ is a normal subgroup of R^*. □

The set $\mathcal{F}:=\bigoplus_{\varnothing\neq I\subseteq\{1,\dots,n\}}F(I)$ is an ideal of the algebra $\mathbb{F}_n=K+\mathcal{F}$. There is the strictly descending chain of ideals of the algebra $\mathbb{F}_n$,

$$\mathcal{F}\supset\mathcal{F}^2\supset\cdots\supset\mathcal{F}^s\supset\cdots\supset\mathcal{F}^n=F_n,$$

where $\mathcal{F}^s:=\bigoplus_{|I|\geq s}F(I)$. The subalgebra $K+\mathcal{F}^s$ of $\mathbb{F}_n$ contains the multiplicative monoid $\mathbb{M}_{n,s}:=1+\mathcal{F}^s$. For each number $s=1,\dots,n$, let

$$\mathbb{M}^*_{n,s}:=(1+\mathcal{F}^s)^* \tag{49}$$

be the group of units of the monoid $\mathbb{M}_{n,s}$, and so we have the chain of normal subgroups (47) of the group $\mathbb{M}^*_n$.

For each number $s=1,\dots,n$, consider the factor algebra $(K+\mathcal{F}^s)/\mathcal{F}^{s+1}=K\bigoplus\bigoplus_{|I|=s}J_I$, where

$$J_I:=(F(I)+\mathcal{F}^{s+1})/\mathcal{F}^{s+1}\simeq F(I)/F(I)\cap\mathcal{F}^{s+1}\simeq F(I)/0\simeq F(I)$$

are ideals of the factor algebra such that $J_IJ_{I'}=0$ if $I\neq I'$. By Lemma 7.3, the group of units of the factor algebra $(K+\mathcal{F}^s)/\mathcal{F}^{s+1}$ is

$$K^*\cdot\prod_{|I|=s}(1+J_I)^*\simeq K^*\times\prod_{|I|=s}(1+J_I)^*.$$

Then the group $\mathbb{M}^*_{n,s+1}$ is the kernel of the group homomorphism

$$\mathbb{M}^*_{n,s}\to\prod_{|I|=s}(1+J_I)^*,\ 1+f\mapsto 1+f+\mathcal{F}^{s+1}. \tag{50}$$

Note that $\mathbb{M}^*_I\subseteq\mathbb{M}^*_{n,s}$ (where $|I|=s$), and the composition of the group homomorphisms

$$\mathbb{M}^*_I\to\mathbb{M}^*_{n,s}\to\prod_{|I'|=s}(1+J_{I'})^*\to(1+J_{I'})^*$$

is an isomorphism if $I'=I$ and is the trivial homomorphism if $I'\neq I$ (i.e. $\mathbb{M}^*_I\to 1$). Therefore, the image of the homomorphism (50) is isomorphic to the direct product of groups $\prod_{|I|=s}\mathbb{M}^*_I\simeq\mathrm{GL}_\infty(K)^{\binom{n}{s}}$, and (48) follows. This completes the proof of Theorem 7.2.

For each number $s=1,\dots,n$, let $\mathbb{M}^*_{n,[s]}:={}^{\mathrm{set}}\prod_{|I|=s}\mathbb{M}^*_I$ be the product of the sets $\mathbb{M}^*_I$, $|I|=s$, in the group $\mathbb{M}^*_n$ in an *arbitrary* but *fixed* order. By (48), there is a natural *bijection* between the sets

$$\mathbb{M}^*_{n,[s]}\to\prod_{|I|=s}\mathbb{M}^*_I,\ u\mapsto\prod_{|I|=s}u_i, \tag{51}$$

where the RHS is the direct product of groups. So, each element v of the set $\mathbb{M}^*_{n,[s]}$ is a *unique* product $\prod_{|I|=s} v_I$ (in the fixed order) of elements v_I of the groups $\mathbb{M}^*_I$.

Corollary 7.5. $\mathbb{M}^*_n = \mathbb{M}^*_{n,[1]}\mathbb{M}^*_{n,[2]}\cdots\mathbb{M}^*_{n,[n]}$ *and there is a natural bijection (determined by* (51)*),*

$$\mathbb{M}^*_n \to {}^{\text{exact}}\prod_{s=1}^{n}\prod_{|I_s|=s}\mathbb{M}^*_{I_s},\ u\mapsto \prod_{s=1}^{n}\prod_{|I_s|=s} u_{I_s},$$

where $u_{I_s}\in\mathbb{M}^*_{I_s}$. *So, each element* u *of* $\mathbb{M}^*_n$ *is a unique product* $u=\prod_{s=1}^n\prod_{|I_s|=s} u_{I_s}$, *where* $u_{I_s}\in\mathbb{M}^*_{I_s}$.

Proof. The result follows from (48) and (50). □

For a group G, let $Z(G)$ denote its center. The next corollary shows that the group $\mathbb{M}^*_n$ has trivial center.

Corollary 7.6. $Z(\mathbb{M}^*_n)=\{1\}$.

Proof. This follows from (47), (48) and the fact that $Z(\mathrm{GL}_\infty(K))=\{1\}$. □

The next theorem gives a characterization of the subgroup $\mathcal{M}_n := \{\omega_u \mid u\in\mathbb{M}^*_n\}\simeq\mathbb{M}^*_n$, $\omega_u\leftrightarrow u$, of G_n. Clearly, $\mathcal{M}_n\subseteq\mathrm{Inn}(\mathbb{S}_n)$.

Theorem 7.7. *The subgroup* $\mathcal{M}_n := \{\omega_u \mid u\in\mathbb{M}^*_n\}$ *of* G_n *is equal to* $\{\sigma\in G_n \mid \sigma(x_i)-x_i,\ \sigma(y_i)-y_i\in\mathbb{F}_n,\ i=1,\dots,n\}$. *Moreover, for each element* $\sigma\in\mathcal{M}_n$,

$$\sigma=\prod_{|I_1|=1}\omega_{u(I_1)}\cdot\prod_{|I_2|=2}\omega_{u(I_2)}\cdots\prod_{|I_s|=s}\omega_{u(I_s)}\cdots\prod_{|I_n|=n}\omega_{u(I_n)}$$

for unique elements $u(I_s)\in\mathbb{M}^*_{I_s}$ *where the orders in the products are arbitrary but fixed.*

Proof. The inclusion $\{\omega_u\mid u\in\mathbb{M}^*_n\}\subseteq W_n := \{\sigma\in G_n\mid \sigma(x_i)-x_i,\ \sigma(y_i)-y_i\in\mathbb{F}_n,\ i=1,\dots,n\}$ is obvious since

$$\omega_u(x_i)-x_i=[u,x_i]u^{-1}\in\mathbb{F}_n,\ \omega_u(y_i)-y_i=[u,y_i]u^{-1}\in\mathbb{F}_n,\ i=1,\dots,n.$$

To prove the reverse inclusion it suffices to show existence of the product for each element $\sigma\in W_n$.

Uniqueness follows from Corollaries 7.5 and 7.6 since the RHS is equal to ω_u, where

$$u=\prod_{|I_1|=1}u(I_1)\cdot\prod_{|I_2|=2}u(I_2)\cdots\prod_{|I_s|=s}u(I_s)\cdots\prod_{|I_n|=n}u(I_n).$$

It follows from the explicit action of the group $S_n\ltimes\mathbb{T}^n$ on the elements x_i and y_i $(i=1,\dots,n)$ and the equalities $G_n=S_n\ltimes\mathbb{T}^n\ltimes\mathrm{Inn}(\mathbb{S}_n)$ and $\mathrm{Inn}(\mathbb{S}_n)=\mathrm{st}_{G_n}(\mathcal{H}_1)$, that $W_n=\{\sigma\in\mathrm{Inn}(\mathbb{S}_n)\mid\sigma(x_i)-x_i,\ \sigma(y_i)-y_i\in\mathbb{F}_n,\ i=1,\dots,n\}$.

Since $\mathrm{Inn}(\mathbb{S}_n) = \mathrm{st}_{G_n}(\mathcal{H}_1)$ and $\sigma \in W_n$, we have the inclusions (see Corollary 5.5.(2))

$$\sigma(x_i) \in x_i + F(i) + F(i)\mathcal{F}, \ \sigma(y_i) \in y_i + F(i) + F(i)\mathcal{F}, \ i = 1, \dots, n. \tag{52}$$

It remains to prove existence of the elements $u(I_s)$. We use induction on n. The case $n = 1$ is obvious (Theorem 4.1). Let $n > 1$ and we assume that the statement holds for all $n' < n$. Let us find the elements $u(I_1)$, $|I_1| = 1$, i.e. the elements $u(i)$, $i = 1, \dots, n$. Since $\sigma \in \mathrm{Inn}(\mathbb{S}_n) = \mathrm{st}_{G_n}(\mathcal{H}_1)$, $\sigma(\sum_{j\neq i} \mathfrak{p}_j) = \sum_{j\neq i} \mathfrak{p}_j$ for each number $i = 1, \dots, n$. Therefore, the automorphism σ induces an automorphism, say σ_i, of the factor algebra

$$\mathbb{S}_n / \sum_{j\neq i} \mathfrak{p}_j \simeq L_{n,i} \otimes \mathbb{S}_1(i),$$

where $L_{n,i} := \bigotimes_{j\neq i} L_1(j)$, such that $\sigma_i(x_j) = x_j$ for all $j \neq i$, and $\sigma_i(\mathbb{S}_1(i) \subseteq \mathbb{S}_1(i)$, by (52). Then

$$\sigma_i(\mathbb{S}_1(i)) = \mathbb{S}_1(i).$$

By induction, there exists an element $u(i) \in (1 + F(i))^*$ such that the inner automorphism $\omega_{u(i)}$ of the algebra $\mathbb{S}_n$ induces on the factor algebra $\mathbb{S}_n / \sum_{j\neq i} \mathfrak{p}_j$ the automorphism σ_i. Let $\omega_{[1]} := \prod_{i=1}^n \omega_{u(i)}$, where the order is fixed as in the theorem, and let $\sigma_{[2]} := \omega_{[1]}^{-1}\sigma$. Then

$$\sigma_{[2]}(x_i) - x_i, \ \sigma_{[2]}(y_i) - y_i \in \bigoplus_{i\in I, |I|\geq 2} F(I), \ i = 1, \dots, n.$$

Suppose that $s > 1$ and we have already found the elements $u(I)$, $|I| < s$, that satisfy the following conditions: for all $t = 2, \dots, s$,

$$\sigma_{[t]}(x_i) - x_i, \ \sigma_{[t]}(y_i) - y_i \in \bigoplus_{i\in I, |I|\geq t} F(I), \ i = 1, \dots, n, \tag{53}$$

where $\sigma_{[t]} := \omega_{[t-1]}^{-1} \cdots \omega_{[1]}^{-1}\sigma$ and $\omega_{[r]} := \prod_{|I_r|=r} \omega_{u(I_r)}$. To finish the proof of the theorem by induction on s we have to find the elements $u(I_s)$, $|I_s| = s$, such that the automorphism $\sigma_{[s+1]} := \omega_{[s]}^{-1}\sigma_{[s]}$ satisfy (53) for $t = s + 1$, where $\omega_{[s]} := \prod_{|I|=s} \omega_{u(I)}$, the order as in the theorem.

Case (i): $s < n$. For each subset I of $\{1, \dots, n\}$, let CI denote its complement. Let $|I| = s$ and $\mathfrak{p}_{CI} := \prod_{j\in CI} \mathfrak{p}_j$. Then $\sigma_{[s]}(\mathfrak{p}_{CI}) = \mathfrak{p}_{CI}$. Therefore, the automorphism $\sigma_{[s]}$ induces an automorphism $\sigma_{[s],I}$ of the factor algebra

$$\mathbb{S}_n/\mathfrak{p}_{CI} \simeq L_{CI} \otimes \mathbb{S}_I,$$

where $L_{CI} := \bigotimes_{j\in CI} L_1(j)$ and $\mathbb{S}_I := \bigotimes_{j\in I} \mathbb{S}_1(j)$, such that $\sigma_{[s],I}(x_j) = x_j$ for all $j \in CI$, and $\sigma_{[s],I}(\mathbb{S}_I) \subseteq \mathbb{S}_I$, by (53). Therefore,

$$\sigma_{[s],I}(\mathbb{S}_I) = \mathbb{S}_I.$$

Moreover,

$$\sigma_{[s],I}(x_i) - x_i, \ \sigma_{[s],I}(y_i) - y_i \in F(I) = \bigotimes_{j\in I} F(j) \subseteq \mathbb{S}_I, \ i = 1, \dots, n.$$

Since $|I| = s < n$, by induction on n, there is an element $u(I) \in \mathbb{M}_I^*$ such that the inner automorphism $\omega_{u(I)}$ of the algebra $\mathbb{S}_n$ induces the automorphism $\sigma_{[s],I}$. The automorphism $\sigma_{[s+1]} = \omega_{[s]}^{-1}\sigma_{[s]}$ satisfies the condition (53) for $t = s+1$, where $\omega_{[s]} = \prod_{|I|=s}\omega_{u(I)}$, the order as in the theorem.

Case (ii): $s = n$. In this case, we cannot use the induction on n as we did in the previous case. Instead, we are going to use the Membership Criterion (Corollary 6.7) in the case $n > 1$. For $s = n$, the condition (53) states that

$$p_i := \sigma_{[n]}(x_i) - x_i,\ q_i := \sigma_{[n]}(y_i) - y_i \in F_n,\ i = 1, \dots, n.$$

Notice that $\sigma_{[n]}(a) = \varphi a \varphi^{-1}$ (where $a \in \mathbb{S}_n$) for some element $\varphi \in \mathbb{S}_n^*$. Then $\varphi x_i = (x_i + p_i)\varphi$ and $\varphi y_i = (y_i + q_i)\varphi$, and so

$$[\varphi, x_i] = p_i\varphi = \varphi\varphi^{-1}p_i\varphi = \varphi\sigma_{[n]}^{-1}(p_i) \in \varphi\sigma_{[n]}^{-1}(F_n) = \varphi F_n \subseteq F_n$$

since $\sigma_{[n]}^{-1}(F_n) = F_n$ (as F_n is the least nonzero ideal of the algebra $\mathbb{S}_n$) and $\varphi F_n \subseteq F_n$. Similarly,

$$[\varphi, y_i] = q_i\varphi = \varphi\varphi^{-1}q_i\varphi = \varphi\sigma_{[n]}^{-1}(q_i) \in \varphi\sigma_{[n]}^{-1}(F_n) = \varphi F_n \subseteq F_n.$$

By Corollary 6.7, $\varphi \in (K + F_n)^* = K^* \times (1 + F_n)^*$, and so the element φ can be taken from the group $\mathbb{M}_{\{1,\dots,n\}} = (1 + F_n)^*$. Then $\sigma_{[n]} = \omega_\varphi$, and the automorphism $\sigma_{[n+1]} := \omega_\varphi^{-1}\sigma_{[n]} = e$ satisfies the condition (53) for $t = n+1$ which states that $\sigma_{[n+1]} = e$. The proof of the theorem is complete. □

The group G'_n and its generators. The monoid $\mathbb{M}_n$ is stable under the action of the subgroup $S_n \ltimes \mathbb{T}^n$ of G_n, hence so is its group $\mathbb{M}_n^*$ of units. Therefore, $G'_n := S_n \ltimes \mathbb{T}^n \ltimes \mathcal{M}_n$ is a subgroup of G_n.

Lemma 7.8. $G'_n \simeq S_n \ltimes \mathbb{T}^n \ltimes \underbrace{\mathrm{GL}_\infty(K) \ltimes \cdots \ltimes \mathrm{GL}_\infty(K)}_{2^n-1 \text{ times}}$.

Proof. $G'_n \simeq S_n \ltimes \mathbb{T}^n \ltimes (\mathbb{M}_n^*/Z(\mathbb{M}_n^*)) \simeq S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n^*$ (Corollary 7.6) and the statement follows from Theorem 7.2. □

For each element $u \in \mathbb{M}_n^*$, let $\omega_u : a \mapsto uau^{-1}$ be the inner automorphism of $\mathbb{S}_n$ determined by the element u. It follows from Lemma 7.8 that the group G'_n admits the following set of generators (in the cases (i) and (ii) only nontrivial action of automorphisms on the canonical generators is shown):

(i) for each pair $i \neq j$, where $i, j \in \{1, \dots, n\}$,

$$s_{ij} : x_i \mapsto x_j,\ y_i \mapsto y_j,\ x_j \mapsto x_i,\ y_j \mapsto y_i;$$

(ii) for each $i = 1, \dots, n$ and $\lambda \in K^*$,

$$t_\lambda(i) : x_i \mapsto \lambda x_i,\ y_i \mapsto \lambda^{-1}y_i;$$

(iii) for each nonempty subset I of $\{1, \dots, n\}$, elements $k = (k_i)_{i \in I}$, $l = (l_i)_{i \in I} \in \mathbb{N}^I$ such that $k \neq l$, and a scalar $\lambda \in K$, the inner automorphism ω_u, where

$$u = u(I; k, l; \lambda) := 1 + \lambda \prod_{i \in I}(x_i^{k_i}y_i^{l_i} - x_i^{k_i+1}y_i^{l_i+1});$$

(iv) for each nonempty subset I of $\{1,\dots,n\}$ and a scalar $\lambda \in K \setminus \{-1\}$, the inner automorphism ω_v, where

$$v = v(I,\lambda) := 1 + \lambda \prod_{i\in I}(1 - x_i y_i).$$

8. An analog of the Jacobian map - the global determinant

The aim of this section is to introduce an analog of the polynomial Jacobian homomorphism, the so-called global determinant on G_n' and to prove that it is a group homomorphism from G_n' to K^* (Corollary 8.7).

The determinant det on the group $\mathbb{M}_n^*$. By Corollary 7.5, each element $u \in \mathbb{M}_n^*$ is a unique ordered product (i.e. for fixed orders of the multiples in each set $\mathbb{M}_{[n],i}^*$)

$$u = \prod_{s=1}^{n} \prod_{|I_s|=s} u_{I_s}, \ u_{I_s} \in \mathbb{M}_{I_s}^*,$$

and $\det_{I_s, b(I_s)}(u_{I_s}) \neq 0$.

Definition. The scalar $\det(u) := \prod_{s=1}^n \prod_{|I_s|=s} \det_{I_s,b(I_s)}(u_{I_s}) \in K^*$ is called the *global determinant* of the element u (we will often drop the adjective "global").

We are going to prove that the determinant (map):

$$\det : \mathbb{M}_n^* \to K^*, \ u \mapsto \det(u) \tag{54}$$

is well-defined (i.e. it does not depend on the orders of the multiples in the product for u, and the functions $b(I_s)$), moreover, it is a group homomorphism (Theorem 8.6).

The group $\mathrm{GL}_n(K)$ is the semidirect product $U_n(K) \ltimes E_n(K)$ of its two subgroups: $U_n(K) := \{\lambda E_{11} + E - E_{11} \mid \lambda \in K^*\} \simeq K^*$, $\lambda E_{11} + E - E_{11} \leftrightarrow \lambda$, where E is the $n \times n$ identity matrix, and $E_n(K)$ is the subgroup of $\mathrm{GL}_n(K)$ generated by the elementary matrices $\{E + \lambda E_{ij} \mid \lambda \in K, i \neq j\}$. The group $E_n(K)$ is the commutant $[\mathrm{GL}_n(K), \mathrm{GL}_n(K)]$ of the group $\mathrm{GL}_n(K)$. Apart from the usual definition, the determinant $\det : \mathrm{GL}_n(K) \to K^*$ can be defined as the group epimorphism $\det : \mathrm{GL}_n(K) \to \mathrm{GL}_n(K)/[\mathrm{GL}_n(K), \mathrm{GL}_n(K)] \simeq U_n(K) \simeq K^*$. Similarly, the determinant map (54) can be defined in this way (see Theorem 8.6), and using this second presentation it is easy to prove that the determinant map (54) is a group homomorphism.

The polynomial algebra P_n is equipped with the *cubic* filtration $\mathcal{C} := \{\mathcal{C}_m := \sum_{\alpha \in C_m} K x^\alpha\}_{m\in\mathbb{N}}$, where $C_m := \{\alpha \in \mathbb{N}^n \mid \text{ all } \alpha_i \leq m\}$. The filtration $\mathcal{C}$ is an ascending, finite dimensional filtration such that $P_n = \bigcup_{m\geq 0} \mathcal{C}_m$ and $\mathcal{C}_m \mathcal{C}_l \subseteq \mathcal{C}_{m+l}$ for all $m, l \geq 0$. In the case when $I = \{1,\dots,n\}$, the next result shows that the determinant det, defined in (45), does not depend on the bijection b.

Theorem 8.1. *Let* $\mathcal{V} = \{V_i\}_{i\in\mathbb{N}}$ *be a finite dimensional vector space filtration on* P_n *and* $a \in \mathbb{M}_{\{1,\dots,n\}} = 1 + F_n$. *Then* $a(V_i) \subseteq V_i$ *and* $\det(a|_{V_i}) = \det(a|_{V_j})$ *for all* $i, j \gg 0$. *Moreover, this common value of the determinants does not depend on the filtration* $\mathcal{V}$ *and, therefore, coincides with the determinant in* (45) *for* $I = \{1, \dots, n\}$.

Proof. Let $a \in 1 + F_n$. Then $a = 1 + \sum_{\alpha,\beta\in C_d} \lambda_{\alpha\beta} E_{\alpha\beta}$ for some $\lambda_{\alpha\beta} \in K$ and $d \in \mathbb{N}$. Then $a(\mathcal{C}_i) \subseteq \mathcal{C}_i$ for all $i \geq d$. Note that the global determinant in (45), for $I = \{1, \dots, n\}$, is equal to the usual determinant $\det(a|_{\mathcal{C}_i})$ for $i \geq d$; then $\mathrm{im}(a-1) \subseteq \mathcal{C}_d \subseteq V_e$ for some $e \in \mathbb{N}$. Since $a = 1 + (a-1)$, we have $a(V_i) \subseteq V_i$ and $\det(a|_{V_i}) = \det(a|_{V_e})$ for all $i \geq e$. Note that this is true for an arbitrary filtration $\mathcal{V}$, where $e = e(\mathcal{V})$. Consider the following finite dimensional vector space filtration $\mathcal{V}' := \{V_i' := \mathcal{C}_d, i = 0.\dots, e-1; V_j' := V_j, j \geq e\}$. Then

$$\det(a) = \det(a|_{\mathcal{C}_d}) = \det(a|_{V'_{e-1}}) = \det(a|_{V'_j}) = \det(a|_{V_j}),\ j \geq e.$$

This completes the proof of the theorem. □

Corollary 8.2. *For each nonempty subset* I *of the set* $\{1, \dots, n\}$*, the determinant defined in* (45) *does not depend on the function* b.

Proof. This is simply Theorem 8.1 where the polynomial algebra P_n is replaced by the polynomial algebra $P_I := K[x_{i_1}, \dots, x_{i_s}]$, where $I = \{i_1, \dots, i_s\}$. □

Corollary 8.2 shows that the global determinant det, defined in (54), does not depend on the choices of the functions $b(I_s)$.

Each element $u \in \mathbb{M}_n$ is a unique finite sum

$$u = 1 + \sum_I \sum_{\alpha,\beta\in\mathbb{N}^I} \lambda_{\alpha\beta}(I) E_{\alpha\beta}(I),\ \lambda_{\alpha\beta} \in K,$$

where I runs through all the nonempty subsets of the set $\{1, \dots, n\}$.

Definition. The *size* $s(u)$ of the element u is the maximal value of all the coordinates of the vectors α and β in the sum above for the element u with $\lambda_{\alpha\beta}(I) \neq 0$.

For all elements $u, v \in \mathbb{M}_n$, $s(uv) \leq \max\{s(u), s(v)\}$.

Lemma 8.3. *Let* $u \in \mathbb{M}_n^*$ *and* $u = \prod_{s=1}^n \prod_{|I_s|=s} u_{I_s}$ *be its unique ordered product, where* $u_{I_s} \in \mathbb{M}^*_{I_s}$. *Then the size* $s(u)$ *of the element* u *is the maximum of the sizes* $s(u_{I_s})$ *of the elements* u_{I_s}.

Proof. Let $u_{[s]} := \prod_{|I_s|=s} u_{I_s}$. Then $u = u_{[1]} \cdots u_{[n]}$. The statement is obvious if $u = u_{[i]}$ for some i (multiply out the elements in the product). Moreover, by the Cramer's formula for the inverse of a matrix, $s(u_{I_s}^{-1}) = s(u_{I_s})$ for all I_s (indeed, it is obvious that $s(u_{I_s}^{-1}) \leq s(u_{I_s})$ but then $s(u_{I_s}) = s((u_{I_s}^{-1})^{-1}) \leq s(u_{I_s}^{-1})$, and the claim follows). This implies that $s(u_{[i]}^{-1}) = s(u_{[i]})$ since $u_{[i]}^{-1} = \prod_{|I_i|=i} u_{I_i}^{-1}$ (in the reverse order to the original order) and $u_{I_i}^{-1} \in \mathbb{M}_{I_i}$. Clearly,

$$s(u_{[i]} u_{[i+1]} \cdots u_{[n]}) \geq s(u_{[i]}) \text{ for all } i.$$

We use a downward induction on i starting with $i = n$ to prove that if $u = u_{[i]} \cdots u_{[n]}$ then the statement of the lemma holds. The statement is obvious for $i = n$, i.e. when $u = u_{[n]} = u_{\{1,\dots,n\}}$. Suppose that $i < n$, $u = u_{[i]} \cdots u_{[n]}$ and the statement is true for all $i' > i$. Suppose that the statement is not true for the element u, we seek a contradiction. Then, $s(u_{[i]}) \leq s(u) < s(u_{[i+1]} \cdots u_{[n]})$, by induction. On the other hand, $s(u_{[i+1]} \cdots u_{[n]}) = s(u_{[i]}^{-1} u) \leq \max\{s(u_{[i]}^{-1}), s(u)\} = \max\{s(u_{[i]}), s(u)\} < s(u_{[i+1]} \cdots u_{[n]})$, a contradiction. □

Corollary 8.4. *Let $u \in \mathbb{M}_n^*$. Then $s(u^{-1}) = s(u)$.*

Proof. Let $u = \prod_{s=1}^n \prod_{|I_s|=s} u_{I_s}$, where $u_{I_s} \in \mathbb{M}_{I_s}^*$. Then $s(u_{I_s}^{-1}) \leq s(u_{I_s})$, hence $s(u^{-1}) = s(\prod_{s=1}^n \prod_{|I_s|=s} u_{I_s}^{-1})$ [in the reverse order] $\leq \max\{s(u_{I_s}^{-1}) \mid I_s\} \leq \max\{s(u_{I_s}) \mid I_s\} = s(u)$, by Lemma 8.3. Now, $s(u^{-1}) \leq s(u) = s((u^{-1})^{-1}) \leq s(u)$, and so $s(u^{-1}) = s(u)$. □

Lemma 8.5. *Let $u \in \mathbb{M}_I$, where I is a nonempty subset of $\{1, \dots, n\}$. Then $u(\mathcal{C}_i) \subseteq \mathcal{C}_i$ and $u(\mathcal{C}_i(I)) \subseteq \mathcal{C}_i(I)$ for all $i \geq s(u)$ (where $\mathcal{C}_i(I)$ is defined in the proof).*

Proof. For $I = \{1, \dots, n\}$, this is simply Theorem 8.1 (see the proof of Theorem 8.1, where if $\mathcal{V} = \mathcal{C}$ the elements d and e can be set to be equal to $s(u)$). The case when $I \neq \{1, \dots, n\}$ follows from the previous one when we observe that $P_n = P_I \otimes P_{CI}$, where $P_I := K[x_{i_1}, \dots, x_{i_s}]$, $I = \{i_1, \dots, i_s\}$, and CI is the complement of I. Then $\mathcal{C}_i = \mathcal{C}_i(I) \otimes \mathcal{C}_i(CI)$, where $\{\mathcal{C}_i(I)\}_{i \in \mathbb{N}}$ and $\{\mathcal{C}_i(CI)\}_{i \in \mathbb{N}}$ are the cubic filtrations for the polynomial algebras P_I and P_{CI} respectively. Note that $u|_{\mathcal{C}_i} = u|_{\mathcal{C}_i(I) \otimes \mathcal{C}_i(CI)} = u|_{\mathcal{C}_i(I)} \otimes \mathrm{id}_{\mathcal{C}_i(CI)}$ for all $i \geq s(u)$. □

The group $\mathrm{GL}_\infty(K)$ is the semidirect product $U(K) \ltimes E_\infty(K)$ of its two subgroups: $U(K) := \{\lambda E_{00} + 1 - E_{00} \mid \lambda \in K^*\} \simeq K^*$, $\lambda E_{00} + 1 - E_{00} \leftrightarrow \lambda$, and $E_\infty(K)$ is the subgroup of $\mathrm{GL}_\infty(K)$ generated by the elementary matrices $\{1 + \lambda E_{ij} \mid \lambda \in K, i \neq j\}$. The group $E_\infty(K)$ coincides with the commutant $[\mathrm{GL}_\infty(K), \mathrm{GL}_\infty(K)]$ of the group $\mathrm{GL}_\infty(K)$.

For each nonempty subset I of $\{1, \dots, n\}$, the group $\mathbb{M}_I^*$ is isomorphic to the group $\mathrm{GL}_\infty(K)$. Therefore, $\mathbb{M}_I^* = U_I(K) \ltimes E_I(K)$ is the semidirect product of its subgroups: $U_I(K) := \{\lambda E_{00}(I) + 1 - E_{00}(I) \mid \lambda \in K^*\} \simeq K^*$, $\lambda E_{00}(I) + 1 - E_{00}(I) \leftrightarrow \lambda$, and $E_I(K)$ is the subgroup of $\mathbb{M}_I^*(K)$ generated by the elementary matrices $\{1 + \lambda E_{\alpha\beta}(I) \mid \lambda \in K, \alpha, \beta \in \mathbb{N}^I, \alpha \neq \beta\}$. The group $E_I(K)$ coincides with the commutant $[\mathbb{M}_I^*, \mathbb{M}_I^*]$ of the group $\mathbb{M}_I^*$.

For $u \in U_I(K)$ and $u' \in U_{I'}(K)$, $uu' = u'u$ as follows from

$$(\lambda E_{00}(I) + 1 - E_{00}(I)) * x^\alpha = \begin{cases} \lambda x^\alpha, & \text{if } \forall i \in I : \alpha_i = 0, \\ x^\alpha, & \text{otherwise.} \end{cases}$$

So, the elements u and u' are diagonal matrices in the monomial basis for P_n. By Corollary 7.5, the subgroup $\mathbb{U}_n$ of $\mathbb{M}_n^*$ generated by the groups $U_I(K)$ is equal to their direct product, $\mathbb{U}_n = \prod_{I\neq\varnothing} U_I(K) \simeq K^{*(2^n-1)}$. Consider the group epimorphism

$$\mu : \mathbb{U}_n \to K^*, \ \prod_{I\neq\varnothing} (\lambda_I E_{00}(I) + 1 - E_{00}(I)) \mapsto \prod_{I\neq\varnothing} \lambda_I. \tag{55}$$

For each number $s = 1, \dots, n$, let $\mathbb{U}_{n,[s]} := \prod_{|I|=s} U_I(K)$ and $\mathbb{U}_{n,s} := \mathbb{U}_{n,[s]} \times \mathbb{U}_{n,[s+1]} \times \cdots \times \mathbb{U}_{n,[n]}$. By Corollary 7.5, for each $s = 1, \dots, n$, the set $E_{n,[s]} := \prod_{|I|=s} E_I(K)$ is an exact product of groups in arbitrary but fixed order, and $E_{n,s} := E_{n,[s]} E_{n,[s+1]} \cdots E_{n,[n]}$ is the exact product of sets. We will see that the set $E_{n,s}$ is a group.

Theorem 8.6.

1. *$\mathbb{M}_n^* = \mathbb{U}_n \ltimes [\mathbb{M}_n^*, \mathbb{M}_n^*]$ and $[\mathbb{M}_n^*, \mathbb{M}_n^*] = E_{n,1}$.*
2. *$\mathbb{M}_{n,s}^* = \mathbb{U}_{n,s} \ltimes [\mathbb{M}_{n,s}^*, \mathbb{M}_{n,s}^*]$ and $[\mathbb{M}_{n,s}^*, \mathbb{M}_{n,s}^*] = E_{n,s}$ for all $s = 1, \dots, n$.*
3. *The determinant map* det *(see* (54)*) is the composition of the group homomorphisms (see* (55)*):*

$$\det : \mathbb{M}_n^* \to \mathbb{M}_n^*/[\mathbb{M}_n^*, \mathbb{M}_n^*] \simeq \mathbb{U}_n \xrightarrow{\mu} K^*.$$

In particular, $\det(uv) = \det(u)\det(v)$ *for all* $u, v \in \mathbb{M}_n^*$.

Proof. 1. Statement 1 is a part of statement 2 when $s = 1$.

2. To prove statement 2 we use a downward induction on s starting with $s = n$. In this case, both statements follow at once from the fact that $\mathbb{M}_{n,n}^* = (1+F_n)^* \simeq \mathrm{GL}_\infty(K) = U(K) \ltimes E_\infty(K)$ and $E_\infty(K) = [\mathrm{GL}_\infty(K), \mathrm{GL}_\infty(K)]$ is the subgroup of $\mathrm{GL}_\infty(K)$ generated by the elementary matrices. Suppose that $s < n$ and the statements hold for all $s' = s+1, \dots, n$. By the uniqueness of the product in Corollary 7.5, $\mathbb{U}_n \cap E_{n,s} = \{1\}$. It is obvious that $E_{n,s} \subseteq [\mathbb{M}_n^*, \mathbb{M}_n^*]$ and $\mathbb{M}_n^* \supseteq \mathbb{U}_n E_{n,s}$. Recall that the groups $\mathbb{M}_{n,t}^*$ are normal subgroups of the group $\mathbb{M}_n^*$. It follows that the set $E_{n,s} = E_{n,[s]} E_{n,s+1} = E_{n,[s]}[\mathbb{M}_{n,s+1}^*, \mathbb{M}_{n,s+1}^*]$ is a subgroup of $\mathbb{M}_{n,s}^*$. Using elementary matrices and the generators for the group $\mathbb{U}_{n,s}$ it is easy to verify that

$$uE_{n,[s]}u^{-1} \subseteq E_{n,s} \text{ for all } u \in \mathbb{U}_{n,s} \text{ and all } s. \tag{56}$$

Note that each element $u \in \mathbb{U}_{n,s}$ is a diagonal matrix in the monomial basis for P_n. This implies that $E_{n,[s]}\mathbb{U}_{n,n+1} \subseteq \mathbb{U}_{n,n+1}E_{n,s}$. Now,

$$\begin{aligned} \mathbb{M}_{n,s}^* &= \mathbb{U}_{n,[s]} E_{n,[s]} \mathbb{M}_{n,s+1}^* = \mathbb{U}_{n,[s]} E_{n,[s]} \mathbb{U}_{n,s+1} E_{n,s+1} \\ &\subseteq \mathbb{U}_{n,[s]} \mathbb{U}_{n,s+1} E_{n,s} = \mathbb{U}_{n,s} E_{n,s}, \end{aligned}$$

and so $\mathbb{M}_{n,s}^* = \mathbb{U}_{n,s}E_{n,s}$. Since $E_{n,s} = E_{n,[s]}E_{n,s+1} = E_{n,[s]}[\mathbb{M}_{n,s+1}^*, \mathbb{M}_{n,s+1}^*]$ and $\mathbb{M}_{n,s+1}^*$ is a normal subgroup of $\mathbb{M}_n^*$, we see that $uE_{n,s}u^{-1} \subseteq E_{n,s}$ for all elements $u \in \mathbb{U}_{n,s}$, by (56), i.e. $E_{n,s}$ is a normal subgroup of $\mathbb{M}_{n,s}^*$. Hence, $\mathbb{M}_{n,s}^* = \mathbb{U}_{n,s} \ltimes E_{n,s}$. Then $[\mathbb{M}_{n,s}^*, \mathbb{M}_{n,s}^*] \subseteq E_{n,s}$ since the group $\mathbb{U}_{n,s}$ is abelian. The opposite inclusion is obvious. Therefore, $E_{n,s} = [\mathbb{M}_{n,s}^*, \mathbb{M}_{n,s}^*]$. By induction, statement 2 holds.

3. By Corollary 7.5, each element u of the group $\mathbb{M}_n^*$ is the unique product $\prod_{s=1}^n \prod_{|I_s|=s} u_{I_s}$, where each element $u_{I_s} \in \mathbb{M}_{I_s}^*$ is a unique product

$u_{I_s}(\lambda_{I_s})e_{I_s}$, where $u_{I_s}(\lambda_{I_s}) := \lambda_{I_s}E_{00}(I_s) + 1 - E_{00}(I_s)$ and $e_{I_s} \in E_{I_s}(K)$. Then $\det(u) = \prod_{s=1}^n \prod_{|I_s|=s} \lambda_{I_s}$. By statement 2, the element u is a unique product $\prod_{s=1}^n \prod_{|I_s|=s} u_{I_s}(\lambda_{I_s}) \cdot e$, where $e \in E_{n,1}$, and statement 3 follows. □

The global determinant det on the group G_n'. Recall that $G_n' \simeq S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n^*$, it is convenient to identify these two groups via the isomorphism. Each element σ of G_n' is a unique product $\sigma = \tau t_\lambda u$, where $\tau \in S_n$, $t_\lambda \in \mathbb{T}^n$, and $u \in \mathbb{M}_n^*$.

Definition. The scalar $\det(\sigma) := \mathrm{sgn}(\tau) \cdot \prod_{i=1}^n \lambda_i \cdot \det(u) \in K^*$ is called the *global determinant* of the element σ (we often drop the adjective "global"), where $\mathrm{sgn}(\tau)$ is the parity of τ.

Our next goal is to prove that the determinant map

$$\det : G_n' \to K^*, \ \sigma \mapsto \det(\sigma),$$

is a group homomorphism (Corollary 8.7).

The group $S_n \ltimes \mathbb{T}^n$ can be seen as a subgroup of the general linear group $\mathrm{GL}(V)$, where $V = \bigoplus_{i=1}^n Kx_i \subseteq P_n$ ($\tau(x_i) = x_{\tau(i)}$ and $t_\lambda(x_i) = \lambda_i x_i$). The global determinant $\det(\tau t_\lambda)$ of the element $\tau t_\lambda \in S_n \ltimes \mathbb{T}^n$ is simply the usual determinant of the element $\tau t_\lambda \in \mathrm{GL}(V)$. So, in order to prove Corollary 8.7 it suffices to show that $\det(\tau t_\lambda u (\tau t_\lambda)^{-1}) = \det(u)$ for all $u \in \mathbb{M}_n^*$ and $\tau t_\lambda \in S_n \ltimes \mathbb{T}^n$. This follows from Theorem 8.6.(1) and the fact that the element τt_λ respects the groups $\mathbb{U}_n$ and $[\mathbb{M}_n^*, \mathbb{M}_n^*]$, and, for each element $u = \prod_{I \neq \varnothing} u_I \in \mathbb{U}_n$, the conjugation $\tau t_\lambda u (\tau t_\lambda)^{-1}$ permutes the components $u_I \in U_I(K)$.

Corollary 8.7. $\det(ab) = \det(a)\det(b)$ *for all* $a, b \in G_n'$.

The global determinant det on the monoids $\mathbb{M}_n$ and $S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n$. Lemma 8.5 and Theorem 8.6 give an idea of how to extend the global determinant from the group $\mathbb{M}_n^*$ to the monoid $\mathbb{M}_n$. Let $u \in \mathbb{M}_n$ and $s(u)$ be its size. Then $u(\mathcal{C}_i) \subseteq \mathcal{C}_i$ for all $i \geq s(u)$. If the map $u \in \mathrm{End}_K(P_n)$ is a bijection then, by Theorem 8.8, $u \in \mathbb{M}_n^*$. If the map u is not a bijection then $\det(u|_{\mathcal{C}_i}) = 0$ for all $i \gg 0$. Hence, if $u, v \in \mathbb{M}_n$ and $uv \in \mathbb{M}_n^*$ then $u, v \in \mathbb{M}_n^*$ (this proves the first statement of Theorem 8.9).

Definition. We can extend the (global) determinant det to the map

$$\det : \mathbb{M}_n \to K, \ u \mapsto \begin{cases} \det(u), & \text{if } u \in \mathbb{M}_n^*, \\ 0, & \text{otherwise.} \end{cases}$$

This common value $\det(u)$ of the determinants is called the *global determinant* of the element $u \in \mathbb{M}_n$ (we often drop the adjective "global").

The global determinant is a homomorphism from the monoid $\mathbb{M}_n$ to the multiplicative monoid $(K, \cdot)$ (Theorem 8.9.(2)), and the group $\mathbb{M}_n^*$ of units of the monoid $\mathbb{M}_n$ is the set of all the elements of $\mathbb{M}_n$ with nonzero global determinant (Corollary 8.10). These results are based on Theorem 8.8. We keep the notation of Section 5. The monoid $\mathbb{M}_n = 1+\mathcal{F}$ has the descending monoid filtration

$$\mathbb{M}_n = 1+\mathcal{F} \supset 1+\mathcal{F}^2 \supset \cdots \supset 1+\mathcal{F}^n = 1+F_n.$$

For each element $u \in \mathbb{M}_n$, there is a unique number i such that $u \in (1+\mathcal{F}^i)\setminus(1+\mathcal{F}^{i+1})$. The number i is called the *degree* of the element u, denoted $\deg(u)$.

For each nonempty subset I of $\{1,\dots,n\}$, let $\mathcal{C}(I) := \{\mathcal{C}_i(I)\}_{i\in\mathbb{N}}$ be the cubic filtration for the polynomial algebra $P_I := K[x_j]_{j\in I}$.

Theorem 8.8. *$\mathbb{M}_n^* = \mathbb{M}_n \cap \mathrm{Aut}_K(P_n)$ but $\mathbb{S}_n^* \subsetneqq \mathbb{S}_n \cap \mathrm{Aut}_K(P_n)$.*

Proof. Let $u \in \mathbb{M}_n \cap \mathrm{Aut}_K(P_n)$. We have to show that $u \in \mathbb{M}_n^*$ since the inclusion $\mathbb{M}_n^* \subseteq \mathbb{M}_n \cap \mathrm{Aut}_K(P_n)$ is obvious. We prove this fact by a downward induction on the degree $i = \deg(u)$. If $i = n$, that is $u \in (1+F_n) \cap \mathrm{Aut}_K(P_n) = (1+F_n)^*$, the statement is obvious. Suppose that $i < n$, and the statement holds for all elements u' with $\deg(u') > i$. In particular, $(1+\mathcal{F}^{i+1}) \cap \mathrm{Aut}_K(P_n) \subseteq \mathbb{M}_n^*$. Note that $u = 1+\sum_{|I|=i} a_I + \sum_{|I|>i} a_I$ for unique elements $a_I \in F(I)$. Let $u_I := 1+a_I$ and $u' := \prod_{|I|=i} u_I$ (in arbitrary order). Note that $s(u_I) \le s(u)$ for all I such that $|I| = i$. For each natural number $m > s(u)$, let $B_m(I) := \mathcal{C}_m(I) \otimes (\prod_{j\in CI} x_j^m \cdot P_{CI})$. By the choice of m,

$$u|_{B_m(I)} = u_I|_{B_m(I)}, \tag{57}$$

and so the linear map $u_I : \mathcal{C}_m(I) \to \mathcal{C}_m(I)$ is an injection, hence a bijection (since $\dim_K(\mathcal{C}_m(I)) < \infty$) for all $m > s(u)$. Now,

$$u_I \in (1+F(I)) \cap \mathrm{Aut}_K(P_I) = (1+F(I))^* = \mathbb{M}_I^* \subseteq \mathbb{M}_n^*.$$

Then $u' \in \mathbb{M}_n^*$, and

$$u(u')^{-1} \in (1+\mathcal{F}^{i+1}) \cap \mathrm{Aut}_K(P_n) \subseteq \mathbb{M}_n^*,$$

therefore, $u = u(u')^{-1}\cdot u' \in \mathbb{M}_n^*$.

$\mathbb{S}_n^* \subsetneqq \mathbb{S}_n \cap \mathrm{Aut}_K(P_n)$ since the element $u := \prod_{i=1}^n (1-y_i)$ of the algebra $\mathbb{S}_n$ belongs to the set $\mathrm{Aut}_K(P_n)\setminus \mathbb{S}_n^*$. The element u is not a unit of the algebra $\mathbb{S}_n$ since the element $u+\mathfrak{a}_n$ is not a unit of the algebra $\mathbb{S}_n/\mathfrak{a}_n$. To show the inclusion $u \in \mathrm{Aut}_K(P_n)$ we may assume that $n = 1$ since $P_n = \bigotimes_{i=1}^n K[x_i]$. The kernel of the linear map u is equal to zero since $(1-y)*p = 0$ for an element $p \in K[x]$ implies that $p = y*p = y^2*p = \cdots = y^s*p = 0$ for all $s \gg 0$ (y is a locally nilpotent map). The map u is surjective since for each element $q \in K[x]$ there exists a natural number, say t, such that $y^t * q = 0$, and so $q = (1-y^t)*q = u(1+y+\cdots+y^{t-1})*q$. Therefore, $u \in \mathrm{Aut}_K(P_n)$. □

Theorem 8.9.

1. *If $u, v \in \mathbb{M}_n$ and $uv \in \mathbb{M}_n^*$ then $u, v \in \mathbb{M}_n^*$.*
2. *$\det(uv) = \det(u)\det(v)$ for all elements $u, v \in \mathbb{M}_n$.*

Proof. 2. The second statement follows from the first. □

Corollary 8.10.

1. $\mathbb{M}_n^* = \{u \in \mathbb{M}_n \mid \det(u) \neq 0\}$, *i.e. an element* $u \in \mathbb{M}_n$ *is a unit if and only if* $\det(u) \neq 0$.
2. *Let* $u \in \mathbb{M}_n$. *Then the following statements are equivalent.*
   (a) *The element* u *has left inverse in* $\mathbb{S}_n$ ($vu = 1$ *for some* $v \in \mathbb{S}_n$).
   (b) *The element* u *has right inverse in* $\mathbb{S}_n$ ($uv = 1$ *for some* $v \in \mathbb{S}_n$).
   (c) *The element* u *is invertible in* $\mathbb{S}_n$.
   (d) $\det(u) \neq 0$.

Proof. 1. Trivial.

2. Statement 2 follows from statement 1 (using the facts that $vu = 1$ implies $\det(u)\det(u) = 1$, and $uv = 1$ implies $\det(u)\det(v) = 1$). □

We can extend the global determinant to the monoid $S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n$ by the rule:

$$\det : S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n \to K, \ \tau t_\lambda u \mapsto \det(\tau t_\lambda)\det(u),$$

where $\tau \in S_n$, $t_\lambda \in \mathbb{T}^n$, and $u \in \mathbb{M}_n$. It follows from Corollary 8.11 that this is a well-defined monoid homomorphism.

We define the *size* $s(a)$ of an element $a = \tau t_\lambda u \in S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n$ as $s(u)$. Then $s(ab) \leq \max\{s(a), s(b)\}$ for all $a, b \in S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n$ and $s(a^{-1}) = s(a)$ for all $a \in S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n^*$, by Lemma 8.4.

Corollary 8.11.

1. *Let* $a \in S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n$. *Then* $u(\mathcal{C}_i) \subseteq \mathcal{C}_i$ *for all* $i, j > s(a)$.
2. $\det(ab) = \det(a)\det(b)$ *for all elements* $a, b \in S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n$.

Corollary 8.12.

1. *The group of units of the monoid* $S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n$ *is* $S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n^* \simeq G_n'$.
2. $S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n^* = \{a \in S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n \mid \det(a) \neq 0\}$.
3. $S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n^* = (S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n) \cap \mathrm{Aut}_K(P_n)$.
4. *Let* $a \in S_n \ltimes \mathbb{T}^n \ltimes \mathbb{M}_n$. *Then the following statements are equivalent.*
   (a) *The element* u *has left inverse.*
   (b) *The element* u *has right inverse.*
   (c) *The element* u *is invertible.*
   (d) $\det(u) \neq 0$.

9. STABILIZERS IN $\mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{S}_n)$ OF THE PRIME OR IDEMPOTENT IDEALS OF THE ALGEBRA $\mathbb{S}_n$

In this section, for each nonzero idempotent ideal $\mathfrak{a}$ of the algebra $\mathbb{S}_n$ its stabilizer $\mathrm{St}_{G_n}(\mathfrak{a}) := \{\sigma \in G_n \mid \sigma(\mathfrak{a}) = \mathfrak{a}\}$ is found (Theorem 9.3). If, in addition, the ideal $\mathfrak{a}$ is generic this result can be refined even further (Corollary 9.4) where the wreath product of groups appears. The stabilizers of all the prime ideals of the algebra $\mathbb{S}_n$ are found (Corollary 9.2.(2) and Corollary 9.9). In particular, when $n > 1$ the stabilizer of each height 1 prime of $\mathbb{S}_n$ is a maximal subgroup of G_n of index n (Corollary 9.2.(1)). It is proved that the ideal $\mathfrak{a}_n$ is the only nonzero, prime, G_n-invariant ideal of the algebra $\mathbb{S}_n$ (Theorem 9.7).

Idempotent ideals of the algebra $\mathbb{S}_n$. An ideal $\mathfrak{a}$ of a ring R is called an *idempotent* ideal (resp. a *proper* ideal) if $\mathfrak{a}^2 = \mathfrak{a}$ (resp. $\mathfrak{a} \neq 0, R$). For an ideal $\mathfrak{a}$, $\mathrm{Min}(\mathfrak{a})$ is the set of all the minimal primes over $\mathfrak{a}$. Two ideals $\mathfrak{a}$ and $\mathfrak{b}$ are called *incomparable* if neither $\mathfrak{a} \subseteq \mathfrak{b}$ nor $\mathfrak{b} \subseteq \mathfrak{a}$. The idempotent ideals of the algebra $\mathbb{S}_n$ are studied in detail in [4]. Below (Theorem 9.1), we collect results on the idempotent ideals of $\mathbb{S}_n$ that are used in the proofs of this section. For the proof of Theorem 9.1 and for more information on the idempotent ideals of $\mathbb{S}_n$ the interested reader is referred to [4].

Theorem 9.1 ([4, Thm. 7.2, Cor. 4.9, Thm. 4.13])**.**

1. *Let $\mathfrak{a}$ be a proper, idempotent ideal of the algebra $\mathbb{S}_n$. Then $\mathrm{Min}(\mathfrak{a})$ is a finite nonempty set each element of which is an idempotent, prime ideal of $\mathbb{S}_n$. The ideal $\mathfrak{a}$ is a unique product and a unique intersection of incomparable, idempotent, prime ideals of $\mathbb{S}_n$. Moreover,*
$$\mathfrak{a} = \prod_{\mathfrak{p}\in \mathrm{Min}(\mathfrak{a})} \mathfrak{p} = \bigcap_{\mathfrak{p}\in \mathrm{Min}(\mathfrak{a})} \mathfrak{p}.$$
2. *Each nonzero, idempotent, prime ideal $\mathfrak{p}$ of the algebra $\mathbb{S}_n$ is equal to $\mathfrak{p}_I := \sum_{i\in I} \mathfrak{p}_i$ for some nonempty subset of $\{1,\dots,n\}$ and vice versa; and this presentation is unique.*
3. *The height of the prime ideal $\mathfrak{p}_I$ is $|I|$.*

Corollary 9.2.

1. $\mathrm{St}_{G_n}(\mathfrak{p}_i) \simeq S_{n-1} \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$*, for $i = 1,\dots,n$. Moreover, if $n > 1$ then the groups $\mathrm{St}_{G_n}(\mathfrak{p}_i)$ are maximal subgroups of G_n (if $n = 1$ then $\mathrm{St}_{G_1}(\mathfrak{p}_1) = G_1$, by Theorem 9.7).*
2. *Let $\mathfrak{p}$ be a nonzero, idempotent, prime ideal of the algebra $\mathbb{S}_n$ and $h = \mathrm{ht}(\mathfrak{p})$ be its height. Then $\mathrm{St}_{G_n}(\mathfrak{p}) \simeq (S_h \times S_{n-h}) \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$.*
3. $\mathrm{St}_{G_n}(\mathcal{H}_1) = \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)$.

Proof. 1. Note that $\mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n) \subseteq \mathrm{St}_{G_n}(\mathfrak{p}_i)$ and $\mathrm{St}_{G_n}(\mathfrak{p}_i) \cap S_n = \{\tau \in S_n \mid \tau(\mathfrak{p}_i) = \mathfrak{p}_i\} \simeq S_{n-1}$. Then
$$\begin{aligned}\mathrm{St}_{G_n}(\mathfrak{p}_i) &= \mathrm{St}_{G_n}(\mathfrak{p}_i) \cap G_n = \mathrm{St}_{G_n}(\mathfrak{p}_i) \cap (S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n))\\ &= (\mathrm{St}_{G_n}(\mathfrak{p}_i) \cap S_n) \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n) \simeq S_{n-1} \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n).\end{aligned}$$
When $n > 1$, the group $\mathrm{St}_{G_n}(\mathfrak{p}_i)$ is a maximal subgroup of G_n since
$$S_{n-1} \simeq \mathrm{St}_{G_n}(\mathfrak{p}_i)/(\mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)) \subseteq G_n/(\mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n)) \simeq S_n$$
and $S_{n-1} = \{\sigma \in S_n \mid \sigma(i) = i\}$ is a maximal subgroup of S_n.

2. By Theorem 9.1.(2), $\mathfrak{p} = \mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_h}$ for some distinct indices $i_1,\dots,i_h \in \{1,\dots,n\}$. Let $I = \{i_1,\dots,i_h\}$ and CI be its complement. Since $\mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{S}_n) \subseteq \mathrm{St}_{G_n}(\mathfrak{p})$ and
$$\mathrm{St}_{G_n}(\mathfrak{p}) \cap S_n = \{\sigma \in S_n \mid \sigma(I) = I,\ \sigma(CI) = CI\} \simeq S_h \times S_{n-h},$$
the result follows using the same arguments as in the previous case.

3. Statement 3 follows from statement 1. □

Let Sub_n be the set of all subsets of $\{1,\dots,n\}$. Sub_n is a partially ordered set with respect to "$\subseteq$". Let SSub_n be the set of all subsets of Sub_n. An element $\{X_1,\dots,X_s\}$ of SSub_n is called *incomparable* if for all $i\neq j$ such that $1\leq i,j\leq s$ neither $X_i\subseteq X_j$ nor $X_i\supseteq X_j$. An empty set and one element set are called incomparable by definition. Let Inc_n be the subset of SSub_n of all incomparable elements of SSub_n. The symmetric group S_n acts in the obvious way on the set SSub_n $(\sigma\cdot\{X_1,\dots,X_s\}=\{\sigma(X_1),\dots,\sigma(X_s)\})$.

Theorem 9.3. *Let $\mathfrak{a}$ be a proper idempotent ideal of the algebra $\mathbb{S}_n$. Then*

$$\mathrm{St}_{G_n}(\mathfrak{a})=\mathrm{St}_{S_n}(\mathrm{Min}(\mathfrak{a}))\ltimes\mathbb{T}^n\ltimes\mathrm{Inn}(\mathbb{S}_n),$$

where $\mathrm{St}_{S_n}(\mathrm{Min}(\mathfrak{a}))=\{\sigma\in S_n\mid\sigma(\mathfrak{q})\in\mathrm{Min}(\mathfrak{a})$ *for all* $\mathfrak{q}\in\mathrm{Min}(\mathfrak{a})\}$. *Moreover, if* $\mathrm{Min}(\mathfrak{a})=\{\mathfrak{q}_1,\dots,\mathfrak{q}_s\}$ *and, for each number $t=1,\dots,s$, $\mathfrak{q}_t=\sum_{i\in I_t}\mathfrak{p}_i$ for some subset I_t of $\{1,\dots,n\}$. Then the group* $\mathrm{St}_{S_n}(\mathrm{Min}(\mathfrak{a}))$ *is the stabilizer in the group S_n of the element $\{I_1,\dots,I_s\}$ of* SSub_n.

Remark. Note that the group

$$\mathrm{St}_{S_n}(\mathrm{Min}(\mathfrak{a}))=\mathrm{St}_{S_n}(\{I_1,\dots,I_s\})$$

(and also the group $\mathrm{St}_{G_n}(\mathfrak{a})$) can be effectively computed in finitely many steps.

Proof. By Theorem 9.1.(1,2), and Corollary 9.2, $\mathbb{T}^n\ltimes\mathrm{Inn}(\mathbb{S}_n)\subseteq\mathrm{St}_{G_n}(\mathfrak{a})$. Note that $\mathrm{St}_{G_n}(\mathfrak{a})\cap S_n=\mathrm{St}_{S_n}(\mathrm{Min}(\mathfrak{a}))$. Now,

$$\mathrm{St}_{G_n}(\mathfrak{a})=(\mathrm{St}_{G_n}(\mathfrak{a})\cap S_n)\ltimes\mathbb{T}^n\ltimes\mathrm{Inn}(\mathbb{S}_n)=\mathrm{St}_{S_n}(\mathrm{Min}(\mathfrak{a}))\ltimes\mathbb{T}^n\ltimes\mathrm{Inn}(\mathbb{S}_n).$$

By Theorem 9.1.(1), $\mathrm{St}_{S_n}(\mathrm{Min}(\mathfrak{a}))=\mathrm{St}_{S_n}(\{I_1,\dots,I_s\})$. $\square$

We are going to apply Theorem 9.3 to find the stabilizers of the generic idempotent ideals (see Corollary 9.4) but first we recall the definition of the *wreath product* $A\wr B$ of finite groups A and B. The set $\mathrm{Fun}(B,A)$ of all functions $f:B\to A$ is a group: $(fg)(b):=f(b)g(b)$ for all $b\in B$, where $g\in\mathrm{Fun}(B,A)$. There is a group homomorphism

$$B\to\mathrm{Aut}(\mathrm{Fun}(B,A)),\ b_1\mapsto(f\mapsto b_1(f):b\mapsto f(b_1^{-1}b)).$$

Then the semidirect product $\mathrm{Fun}(B,A)\rtimes B$ Is called the *wreath product* of the groups A and B denoted $A\wr B$, and so the product in $A\wr B$ is given by the rule:

$$f_1b_1\cdot f_2b_2=f_1b_1(f_2)b_1b_2,\ \text{where } f_1,f_2\in\mathrm{Fun}(B,A),\ b_1,b_2\in B.$$

By Theorem 9.1.(2), each nonzero, idempotent, prime ideal $\mathfrak{p}$ of $\mathbb{S}_n$ is a unique sum $\mathfrak{p}=\sum_{i\in I}\mathfrak{p}_i$ of height 1 prime ideals. The set $\mathrm{Supp}(\mathfrak{p}):=\{\mathfrak{p}_i\mid i\in I\}$ is called the *support* of $\mathfrak{p}$.

Definition. We say that a proper, idempotent ideal $\mathfrak{a}$ of $\mathbb{S}_n$ is *generic* if $\mathrm{Supp}(\mathfrak{p})\cap\mathrm{Supp}(\mathfrak{q})=\varnothing$ for all $\mathfrak{p},\mathfrak{q}\in\mathrm{Min}(\mathfrak{a})$ such that $\mathfrak{p}\neq\mathfrak{q}$.

Corollary 9.4. *Let $\mathfrak{a}$ be a generic idempotent ideal of the algebra $\mathbb{S}_n$, the set* $\mathrm{Min}(\mathfrak{a})$ *of minimal primes over $\mathfrak{a}$ is the disjoint union of nonempty subsets* $\mathrm{Min}_{h_1}(\mathfrak{a})\bigcup\cdots\bigcup\mathrm{Min}_{h_t}(\mathfrak{a})$, *where $1\leq h_1<\cdots<h_t\leq n$ and the set* $\mathrm{Min}_{h_i}(\mathfrak{a})$ *contains all the minimal primes over $\mathfrak{a}$ of height h_i. Let $n_i:=|\mathrm{Min}_{h_i}(\mathfrak{a})|$. Then*

$$\mathrm{St}_{G_n}(\mathfrak{a})=(S_m\times\prod_{i=1}^t(S_{h_i}\wr S_{n_i}))\ltimes\mathbb{T}^n\ltimes\mathrm{Inn}(\mathbb{S}_n),$$

where $m=n-\sum_{i=1}^t n_ih_i$.

Proof. Suppose that $\mathrm{Min}(\mathfrak{a})=\{\mathfrak{q}_1,\dots,\mathfrak{q}_s\}$ and the sets $I_1,\dots,I_s$ are defined in Theorem 9.3. Since the ideal $\mathfrak{a}$ is generic, the sets $I_1,\dots,I_s$ are disjoint. By Theorem 9.3, we have to show that

$$\mathrm{St}_{S_n}(\{I_1,\dots,I_s\})\simeq S_m\times\prod_{i=1}^t(S_{h_i}\wr S_{n_i}). \tag{58}$$

The ideal $\mathfrak{a}$ is generic, and so the set $\{1,\dots,n\}$ is the disjoint union $\bigcup_{i=0}^t M_i$ of its subsets, where $M_i:=\bigcup_{|I_j|=h_i}I_j$, $i=1,\dots,t$, and M_0 is the complement of the set $\bigcup_{i=1}^t M_i$. Let $S(M_i)$ be the symmetric group corresponding to the set M_i (i.e. the set of all bijections $M_i\to M_i$). Then each element $\sigma\in\mathrm{St}_{S_n}(\{I_1,\dots,I_s\})$ is a unique product $\sigma=\sigma_0\sigma_1\cdots\sigma_t$, where $\sigma_i\in S(M_i)$. Moreover, σ_0 can be an arbitrary element of $S(M_0)\simeq S_m$, and, for $i\neq 0$, the element σ_i permutes the sets $\{I_j\mid |I_j|=h_i\}$ and simultaneously permutes the elements inside each of the sets I_j, i.e. $\sigma_i\in S_{h_i}\wr S_{n_i}$. Now, (58) is obvious. □

Corollary 9.5. *For each number $s=1,\dots,n$, let $\mathfrak{b}_s:=\prod_{|I|=s}(\sum_{i\in I}\mathfrak{p}_i)$, where I runs through all the subsets of the index set $\{1,\dots,n\}$ that contain exactly s elements. The ideals $\mathfrak{b}_s$ are the only proper, idempotent, G_n-invariant ideals of the algebra $\mathbb{S}_n$.*

Proof. By Theorem 5.1 and Corollary 9.2.(3), the ideals $\mathfrak{b}_s$ are G_n-invariant, and they are proper and idempotent. The converse follows at once from the classification of proper idempotent ideals (Theorem 9.1.(1)). □

The prime ideals of the algebra $\mathbb{S}_n$. In order to prove Theorem 9.7, we recall a classification of prime ideals for the algebra $\mathbb{S}_n$ which is obtained in [4]. For a subset $\mathcal{N}=\{i_1,\dots,i_s\}$ of the set of indices $\{1,\dots,n\}$, let $C\mathcal{N}$ be its complement, $|\mathcal{N}|=s$, $\mathbb{S}_{\mathcal{N}}:=\mathbb{S}_1(i_1)\otimes\cdots\otimes\mathbb{S}_1(i_s)$,

$$\mathfrak{a}_{\mathcal{N}}:=F\otimes\mathbb{S}_1(i_2)\otimes\cdots\otimes\mathbb{S}_1(i_s)+\cdots+\mathbb{S}_1(i_1)\otimes\cdots\otimes\mathbb{S}_1(i_{s-1})\otimes F, \tag{59}$$

$P_{\mathcal{N}}:=K[x_{i_1},\dots,x_{i_s}]$. Clearly, $\mathbb{S}_n=\mathbb{S}_{\mathcal{N}}\otimes\mathbb{S}_{C\mathcal{N}}$. Let $L_{\mathcal{N}}:=K[x_{i_1},x_{i_1}^{-1},\dots,x_{i_s},x_{i_s}^{-1}]$. Then $\mathbb{S}_{\mathcal{N}}/\mathfrak{a}_{\mathcal{N}}\simeq L_{\mathcal{N}}$. Consider the epimorphism

$$\pi_{\mathcal{N}}:\mathbb{S}_{\mathcal{N}}\to\mathbb{S}_{\mathcal{N}}/\mathfrak{a}_{\mathcal{N}}\simeq L_{\mathcal{N}},\ a\mapsto a+\mathfrak{a}_{\mathcal{N}}. \tag{60}$$

By [4, Prop. 4.3.(2)], there is the injection

$$\mathrm{spec}(L_{C\mathcal{N}})\to\mathrm{spec}(\mathbb{S}_n),\ \mathfrak{q}\mapsto\mathbb{S}_{\mathcal{N}}\otimes\pi_{C\mathcal{N}}^{-1}(\mathfrak{q}).$$

The image of this injection is denoted by

$$\operatorname{spec}(\mathbb{S}_n,\mathcal{N}) := \{\mathbb{S}_\mathcal{N}\otimes \pi_{C\mathcal{N}}^{-1}(\mathfrak{q}) \mid \mathfrak{q}\in \operatorname{spec}(L_{C\mathcal{N}})\}.$$

Note that $\operatorname{spec}(\mathbb{S}_n,\varnothing) = \{\pi_{\{1,\dots,n\}}^{-1}(\mathfrak{q}) \mid \mathfrak{q}\in\operatorname{spec}(L_n)\}\simeq \operatorname{spec}(L_n)$ and $\operatorname{spec}(\mathbb{S}_n, \{1,\dots,n\}) = \{0\}$ since $\pi_\varnothing : K\to K$, $\lambda\mapsto\lambda$.

The next theorem shows that all the prime ideals of the algebra $\mathbb{S}_n$ can be obtained in this way.

Theorem 9.6 ([4, Thm. 4.4])**.**

1. $\operatorname{spec}(\mathbb{S}_n) = \coprod_{\mathcal{N}\subseteq\{1,\dots,n\}}\operatorname{spec}(\mathbb{S}_n,\mathcal{N})$*, the disjoint union.*
2. *Each prime ideal* $\mathfrak{p}$ *of the algebra* $\mathbb{S}_n$ *can be uniquely written as* $\mathbb{S}_\mathcal{N}\otimes\pi_{C\mathcal{N}}^{-1}(\mathfrak{q})$ *for some subset* $\mathcal{N}$ *of the set* $\{1,\dots,n\}$ *and some prime ideal* $\mathfrak{q}$ *of the algebra* $L_{C\mathcal{N}}$*.*

Theorem 9.7. *The ideal* $\mathfrak{a}_n$ *is the only nonzero, prime,* G_n*-invariant ideal of the algebra* $\mathbb{S}_n$*.*

Proof. By Lemma 3.4 (or by Corollary 9.2.(2)), the ideal $\mathfrak{a}_n$ is G_n-invariant. Conversely, let $\mathfrak{p}$ be a nonzero, prime, G_n-invariant ideal of the algebra $\mathbb{S}_n$. By Theorem 9.6.(2) and the fact that $\mathfrak{p}$ is also S_n-invariant, the ideal $\mathfrak{p}$ contains the sum $\mathfrak{p}_1+\cdots+\mathfrak{p}_n = \mathfrak{a}_n$. Suppose that $\mathfrak{p}\neq\mathfrak{a}_n$, we seek a contradiction. In this case, the ideal $\mathfrak{p}/\mathfrak{a}_n$ of the algebra $\mathbb{S}_n/\mathfrak{a}_n = L_n$ is $\mathbb{T}^n$-invariant, hence $\mathfrak{p} = L_n$, a contradiction. □

The classical Krull dimension of the algebra $\mathbb{S}_n$ is $2n$ ([4, Thm. 4.11]). For each natural number $i = 0,1,\dots,2n$, let

$$\mathcal{H}_i := \{\mathfrak{p}\in\operatorname{Spec}(\mathbb{S}_n) \mid \operatorname{ht}(\mathfrak{p}) = i\},$$
$$\operatorname{St}_{G_n}(\mathcal{H}_i) := \{\sigma\in G_n \mid \sigma(\mathfrak{p}) = \mathfrak{p} \text{ for all } \mathfrak{p}\in\mathcal{H}_i\}.$$

Corollary 9.8. $\operatorname{St}_{G_n}(\mathcal{H}_i) = \begin{cases} G_n, & \text{if } i=0,\\ \mathbb{T}^n\ltimes\operatorname{Inn}(\mathbb{S}_n), & \text{if } i=1,\\ \operatorname{Inn}(\mathbb{S}_n), & \text{if } i=2,\dots,2n.\end{cases}$

Proof. The statement is obvious for $i=0$ (since $\mathcal{H}_0 = \{0\}$) and for $i=1$ (Corollary 9.2.(3)). So, let $i\geq 2$. Briefly, the statement follows from the fact that in the algebra L_n there is no proper $\mathbb{T}^n$-invariant ideals (since any such an ideal would have contained a monomial in x_i, x_i^{-1}, $i=1,\dots,n$; but all of them are units). Fix a presentation $i = m+l$, where $1\leq l\leq m\leq n$. For each subset $\mathcal{N}$ of $\{1,\dots,n\}$ such that $|C\mathcal{N}| = m$ and, for each prime ideal $\mathfrak{q}$ of $L_{C\mathcal{N}}$ of height l,

$$\operatorname{St}_{G_n}(\mathbb{S}_\mathcal{N}\otimes\pi_{C\mathcal{N}}^{-1}(\mathfrak{q})) = S(\mathcal{N})\ltimes\mathbb{T}^{|\mathcal{N}|}(\mathcal{N})\ltimes\operatorname{St}_{S(C\mathcal{N})\ltimes\mathbb{T}^{|C\mathcal{N}|}(C\mathcal{N})}(\mathfrak{q})\ltimes\operatorname{Inn}(\mathbb{S}_n),$$

where $S(\mathcal{N})$ is the symmetric group on $\mathcal{N}$ and $\mathbb{T}^{|\mathcal{N}|}(\mathcal{N})$ is the torus in the group of automorphisms of the algebra $\mathbb{S}_\mathcal{N}$. It is obvious that $\operatorname{Inn}(\mathbb{S}_n)\subseteq\operatorname{St}_{G_n}(\mathcal{H}_i)$. For $i = 2,\dots,2n-1$,

$$\bigcap_{\mathcal{N},\mathfrak{q}}\operatorname{St}_{G_n}(S_\mathcal{N}\otimes\pi_{C\mathcal{N}}^{-1}(\mathfrak{q})) = \operatorname{Inn}(S_n),$$

and so $\mathrm{St}_{G_n}(\mathcal{H}_i) = \mathrm{Inn}(\mathbb{S}_n)$. For $i = 2n$, the statement is obvious. $\square$

Let $\mathfrak{p}$ be a prime ideal of the algebra $\mathbb{S}_n$. When, in addition, $\mathfrak{p}$ is an idempotent ideal its stabilizer is found in Corollary 9.2.(2). The next corollary, which is obtained in the proof of Corollary 9.8, gives the stabilizer of $\mathfrak{p}$ when the prime ideal $\mathfrak{p}$ is not an idempotent ideal.

Corollary 9.9. *Let $\mathfrak{p}$ be a prime ideal of the algebra $\mathbb{S}_n$ which is not an idempotent ideal, i.e. $\mathfrak{p} = \mathbb{S}_{\mathcal{N}} \otimes \pi_{C\mathcal{N}}^{-1}(\mathfrak{q})$ for some subset $\mathcal{N}$ of $\{1, \dots, n\}$ and a nonzero prime ideal $\mathfrak{q}$ of the Laurent polynomial algebra $L_{C\mathcal{N}}$. Then $\mathrm{St}_{G_n}(\mathfrak{p}) = S(\mathcal{N}) \ltimes \mathbb{T}^{|\mathcal{N}|}(\mathcal{N}) \ltimes \mathrm{St}_{S(C\mathcal{N})\ltimes\mathbb{T}^{|C\mathcal{N}|}(C\mathcal{N})}(\mathfrak{q}) \ltimes \mathrm{Inn}(\mathbb{S}_n)$ (see the proof of Corollary 9.8 for details).*

Theorem 9.10.

1. *Let $n > 1$ and let $\mathfrak{p}$ be a prime ideal of the algebra $\mathbb{S}_n$. Then the stabilizer $\mathrm{St}_{G_n}(\mathfrak{p})$ is a maximal subgroup of G_n if and only if the ideal $\mathfrak{p}$ has height 1, and in this case the index $[G_n : \mathrm{St}_{G_n}(\mathfrak{p})] = n$.*
2. *Let $n = 1$ and $\mathfrak{p}$ be a prime ideal of the algebra $\mathbb{S}_n$. Then the stabilizer $\mathrm{St}_{G_n}(\mathfrak{p})$ is not a maximal subgroup of G_n.*

Proof. The theorem follows from Corollary 9.2 and Corollary 9.9. $\square$

Corollary 9.11. $\mathrm{St}_{G_n}(\mathrm{Spec}(\mathbb{S}_n)) = \mathrm{St}_{G_n}(\mathrm{Max}(\mathbb{S}_n)) = \mathrm{Inn}(\mathbb{S}_n)$.

Proof. By Corollary 9.8,

$$\mathrm{Inn}(\mathbb{S}_n) \subseteq \mathrm{St}_{G_n}(\mathrm{Spec}(\mathbb{S}_n)) \subseteq \mathrm{St}_{G_n}(\mathrm{Max}(\mathbb{S}_n)) \subseteq \mathrm{St}_{G_n}(\mathcal{H}_{2n}) = \mathrm{Inn}(\mathbb{S}_n),$$

and so the result. $\square$

The algebra $\mathbb{S}_n$ is $\mathbb{Z}^n$-graded. The algebra $\mathbb{S}_n = \bigoplus_{\alpha\in\mathbb{Z}_n} \mathbb{S}_{n,\alpha}$ is a $\mathbb{Z}^n$-graded algebra, where $\mathbb{S}_{n,\alpha} := \mathbb{S}_{1,\alpha_1} \otimes \cdots \otimes \mathbb{S}_{1,\alpha_n}$, $\alpha = (\alpha_1, \dots, \alpha_n)$,

$$\mathbb{S}_{1,i} := \begin{cases} x^i\mathbb{S}_{1,0} = \mathbb{S}_{1,0}x^i, & \text{if } i \geq 1, \\ \mathbb{S}_{1,0}, & \text{if } i = 0, \\ y^{|i|}\mathbb{S}_{1,0} = \mathbb{S}_{1,0}y^{|i|}, & \text{if } i \leq -1, \end{cases}$$

$\mathbb{S}_{1,0} := K\langle E_{00}, E_{11}, \dots\rangle = K \oplus KE_{00} \oplus KE_{11} \oplus \cdots$ is a commutative non-Noetherian algebra ($KE_{00} \subset KE_{00} \oplus KE_{11} \subset \cdots$ is an ascending chain of ideals of the algebra $\mathbb{S}_{1,0}$). For each $i = 1, \dots, n$, and $j \in \mathbb{N}$, let

$$v_j(i) := \begin{cases} x_i^j, & \text{if } j \geq 0, \\ y_i^{|j|}, & \text{if } j < 0, \end{cases}$$

and, for $\alpha \in \mathbb{Z}^n$, let $v_\alpha := \prod_{i=1}^n v_{\alpha_i}(i)$. Then $\mathbb{S}_{n,\alpha} = v_\alpha\mathbb{S}_{n,0} = \mathbb{S}_{n,0}v_\alpha$, where

$$\mathbb{S}_{n,0} := \bigotimes_{i=1}^n \mathbb{S}_{1,0}(i) = \bigotimes_{i=1}^n K\langle E_{00}(i), E_{11}(i), \dots\rangle = K \bigoplus \bigoplus_I \bigoplus_{\alpha\in\mathbb{N}^{|I|}} KE_{\alpha\alpha}(I),$$

where I runs through all the nonempty subsets of $\{1,\dots,n\}$, and $E_{\alpha\alpha}(I) := E_{\alpha_1\alpha_1}(i_1)\cdots E_{\alpha_s\alpha_s}(i_s)$ for $I=\{i_1,\dots,i_s\}$. Each element a of the algebra $\mathbb{S}_{n,0}$ is a unique finite sum

$$a = a_0 + \sum_I \sum_{\alpha\in\mathbb{N}^{|I|}} \lambda_{\alpha,I} E_{\alpha\alpha}(I), \tag{61}$$

where $a_0,\ \lambda_{\alpha,I}\in K$. The set of elements $\{v_\gamma, v_\delta(CI)E_{\alpha\beta}(I)\}$ is a K-basis for the algebra $\mathbb{S}_n$, where $E_{\alpha\beta} := E_{\alpha_1\beta_1}(i_1)\cdots E_{\alpha_s\beta_s}(i_s)$ and, for the complement $CI=\{j_1,\dots,j_t\}$ of the set I, $v_\delta(CI) := v_{\delta_1}(j_1)\cdots v_{\delta_t}(j_t)$. Each nonzero element u of $\mathbb{S}_n$ is a finite linear combination of the basis elements, and each nonzero summands is called a *component* of u.

Definition. The *volume* $\mathrm{vol}(u)$ of a nonzero element u of $\mathbb{S}_n$ is the number of nonzero coordinates of the element u with respect to the basis $\{v_\gamma, v_\delta(CI)\ E_{\alpha\beta}(I)\}$, or, equivalently, the number of its nonzero components. We set $\mathrm{vol}(0)=0$.

Note that $\mathrm{vol}(\sigma(u)) = \mathrm{vol}(u)$ for all $\sigma\in S_n \ltimes \mathbb{T}^n$.

Let G be a group and H be its subgroup. Then $[G:H]$ denotes the index of H in G.

Corollary 9.12. *Let $\mathfrak{a}$ be a proper ideal of the algebra $\mathbb{S}_n$. Then $[G_n : \mathrm{St}_{G_n}(\mathfrak{a})]<\infty$ if and only if $\mathfrak{a}^2=\mathfrak{a}$.*

Proof. ($\Leftarrow$) This implication follows from Theorem 9.3.

($\Rightarrow$) Suppose that $[G_n:\mathrm{St}_{G_n}(\mathfrak{a})]<\infty$ for a proper ideal $\mathfrak{a}$ of $\mathbb{S}_n$. Note that $\mathbb{T}^n=\prod_{i=1}^n \mathbb{T}^1(i)$. For each $i=1,\dots,n$, let $T_i := \mathbb{T}^1(i)\cap \mathrm{St}_{G_n}(\mathfrak{a})$. Then $[\mathbb{T}^1(i):T_i]\le [G_n:\mathrm{St}_{G_n}(\mathfrak{a})]<\infty$, and so the group T_i contains *infinitely many* elements. Consider the subgroup $T' := T_1\times\cdots\times T_n$ of $\mathbb{T}^n\cap \mathrm{St}_{G_n}(\mathfrak{a})$. We have to show that $\mathfrak{a}^2=\mathfrak{a}$. It suffices to show that the ideal $\mathfrak{a}$ is generated (as an ideal) by elements of volume 1. Suppose that this is not the case for the ideal $\mathfrak{a}$, we seek a contradiction. Let v be the minimum of the volumes of all the nonzero elements of the ideal $\mathfrak{a}$ such that all their components do not belong to $\mathfrak{a}$. Fix one such an element $u\in\mathfrak{a}$ with $\mathrm{vol}(u)=v$. Since $T'\subseteq \mathrm{St}_{G_n}(\mathfrak{a})$, the element u has to be of the type $v_\beta a$ for some $\beta\in\mathbb{Z}^n$ and a nonzero element a of the algebra $\mathbb{S}_{n,0}$. The element a is a unique sum as in (61). To get a contradiction we use an induction on n. Suppose that $n=1$, and so $u=v_\beta(\lambda+\sum_{\nu=1}^s a_\nu E_{i_\nu i_\nu})$ for some scalars λ and $a_\nu\in K^*$, $\nu\ge 1$.

If $\lambda\neq 0$ then the ideal of $\mathbb{S}_1$ generated by the element u is $\mathbb{S}_1$. This implies that $u=v_\beta\lambda$ and so $\mathrm{vol}(u)=1$, a contradiction.

If $\lambda=0$ then $uE_{i_\nu i_\nu} = a_\nu v_\beta E_{i_\nu i_\nu}\in\mathfrak{a}$ for all ν, a contradiction.

Suppose that $n>1$. Then, up to action of the symmetric group S_n, we may assume that

$$u = v_\beta\left(\lambda + \sum_{\nu=1}^s a_\nu E_{i_\nu i_\nu}(n)\right)$$

for some scalar $\lambda\in K$ and nonzero elements $a_\nu\in\mathbb{S}_{n-1}$. If $\lambda\neq 0$ and all $a_\nu\in K$ then the ideal of the algebra $\mathbb{S}_1(n)$ generated by the element $v_{\beta_n}(\lambda+$

$\sum_{\nu=1}^{s} a_\nu E_{i_\nu i_\nu}(n)) \in \mathbb{S}_1(n)$ is equal to $\mathbb{S}_1(n)$. Then all the summands of the element u belongs to the ideal $\mathfrak{a}$, a contradiction.

If $\lambda \neq 0$ and not all the elements a_ν belong to the field K, say $a_1 \notin K$, then the volume of the following nonzero element of $\mathfrak{a}$, $uE_{i_1 i_1}(n) = v_\beta(\lambda + a_1)E_{i_1 i_1}(n)$, is not 1 and does not exceed $\mathrm{vol}(u)$. Therefore, $a_2 = \cdots = a_s = 0$ and $\mathrm{vol}(uE_{i_1 i_1}) = \mathrm{vol}(u)$. Repeating the same argument several times we obtain an element of the ideal $\mathfrak{a}$,

$$uE_{ii}(k)E_{jj}(k+1)\cdots E_{i_1 i_1}(n) = v_\beta(\lambda + b)E_{ii}(k)E_{jj}(k+1)\cdots E_{i_1 i_1}(n),$$

having volume $\mathrm{vol}(u)$ but $b \in F_1(k-1)$ (up to action of the group S_n). Since the ideal of the algebra $\mathbb{S}_1(k-1)$ generated by its element $v_{\beta_{k-1}}(\lambda+b)$ is equal to $\mathbb{S}_1(k-1)$, we have a contradiction.

If $\lambda = 0$ then all the elements $uE_{i_\nu i_\nu}(n) = v_\beta a_\nu E_{i_\nu i_\nu}(n)$ belong to the ideal $\mathfrak{a}$. Therefore, $u = v_\beta a_1 E_{i_1 i_1}(n)$ for some nonzero element $a_1 \in \mathbb{S}_{n-1}$ of volume $\mathrm{vol}(u)$. Now, repeating the same argument as above or use induction on n, we come to a contradiction. The proof of the corollary is complete. $\square$

10. Endomorphisms of the algebra $\mathbb{S}_n$

In this section, we classify all the algebra endomorphisms of $\mathbb{S}_n$ that stabilize the elements $x_1, \ldots, x_n$ and show that each such endomorphism is a *monomorphism* but *not* an isomorphism provided it is not the identity map (Corollary 10.1).

Let

$$\mathrm{st}(x_1, \ldots, x_n) := \{g \in E_n \mid g(x_1) = x_1, \ldots, g(x_n) = x_n\},$$
$$\mathrm{st}(y_1, \ldots, y_n) := \{g \in E_n \mid g(y_1) = y_1, \ldots, g(y_n) = y_n\}.$$

These monoids are the stabilizers of the sets $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$ in $E_n = \mathrm{End}_{K-\mathrm{alg}}(\mathbb{S}_n)$. Note that

$$\widehat{\eta}(\mathrm{st}(x_1, \ldots, x_n)) = \mathrm{st}(y_1, \ldots, y_n), \ \widehat{\eta}(\mathrm{st}(y_1, \ldots, y_n)) = \mathrm{st}(x_1, \ldots, x_n).$$

By Theorem 3.7,

$$G_n \cap (\mathrm{st}(x_1, \ldots, x_n) = G_n \cap \mathrm{st}(y_1, \ldots, y_n) = \{e\},$$

i.e. if an algebra endomorphism of $\mathbb{S}_n$ which is not the identity map stabilizers either the set $\{x_1, \ldots, x_n\}$ or $\{y_1, \ldots, y_n\}$ then necessarily it is *not* an automorphism of $\mathbb{S}_n$. Our next step is to describe all such endomorphisms and to show that all of them are *monomorphisms*. Note that the algebra $\mathbb{S}_n$ has plenty of ideals (see [4]) and contains the ring of infinite dimensional matrices, so there is no problem in producing an algebra endomorphism which is *not* a monomorphism, e.g. $\mathbb{S}_n \to \mathbb{S}_n/(\mathfrak{a}_n + \sum_{i=1}^{n} \mathbb{S}_n(x_i - 1)\mathbb{S}_n) \simeq K \to \mathbb{S}_n$.

In the proof of Corollary 10.1, the following identities are used. For $i = 1, \ldots, n$ and $p \in K[x_1, \ldots, x_n]$,

$$[y_i, p] = x_i^{-1}(p - p|_{x_i=0})E_{00}(i), \tag{62}$$

$$[p, E_{00}(i)] = (p - p|_{x_i=0})E_{00}(i). \tag{63}$$

In more detail, it suffices to prove the identities in the case when $p = x_i^m$, $m \geq 1$. Then $[y_i, x_i^m] = x_i^{m-1} - x_i^m y_i = x_i^{m-1}(1 - x_i y_i) = x_i^{m-1} E_{00}(i)$, and $[x_i^m, E_{00}(i)] = x_i^m E_{00}(i) - E_{00}(i) x_i^m = x_i^m E_{00}(i)$.

Corollary 10.1.

1. *The monoid* $\mathrm{st}(x_1, \ldots, x_n)$ *is an abelian monoid each nonidentity element of which is a monomorphism of the algebra* $\mathbb{S}_n$ *but not an automorphism. Moreover, it contains precisely the following endomorphisms of* $\mathbb{S}_n$*:*

$$\sigma_p : y_i \mapsto y_i + p_i E_{00}(i), \ i = 1, \ldots, n,$$

where the n*-tuple* $p = (p_1, \ldots, p_n) \in K[x_1, \ldots, x_n]^n$ *satisfies the following conditions: for each pair of indices* $i \neq j$*,*

$$-x_j^{-1}(p_i - p_{i,j}) + x_i^{-1}(p_j - p_{j,i}) + p_i p_{j,i} - p_j p_{i,j} = 0, \tag{64}$$

where $p_{i,j} := p_i|_{x_j=0}$*.*

2. *The monoid* $\mathrm{st}(y_1, \ldots, y_n)$ *is an abelian monoid each nonidentity element of which is a monomorphism of the algebra* $\mathbb{S}_n$ *but not an automorphism. Moreover, it contains precisely the following endomorphisms of* $\mathbb{S}_n$*:*

$$\tau_p : y_i \mapsto y_i + E_{00}(i) q_i, \ i = 1, \ldots, n,$$

where the n*-tuple* $q = (q_1, \ldots, q_n) \in K[y_1, \ldots, y_n]^n$ *satisfies the following conditions: for each pair of indices* $i \neq j$*,*

$$-y_j^{-1}(q_i - q_{i,j}) + y_i^{-1}(q_j - q_{j,i}) + q_i q_{j,i} - q_j q_{i,j} = 0, \tag{65}$$

where $q_{i,j} := q_i|_{y_j=0}$*.*

Proof. 1. In fact, at the beginning of the proof of Theorem 3.7, we proved that each element $\sigma \in \mathrm{st}(x_1, \ldots, x_n)$ has the form $\sigma = \sigma_p$ for *some* n-tuple $p = (p_1, \ldots, p_n) \in K[x_1, \ldots, x_n]^n$ (there, in proving this, we did not use the fact the σ is an automorphism). The endomorphism σ_p is well-defined if and only if the elements $\sigma_p(y_1), \ldots, \sigma_p(y_n)$ commute (since $[\sigma_p(y_i), \sigma_p(x_j)] = [\sigma_p(y_i), x_j] = 0$ for all $i \neq j$). Let us show that the elements $\sigma_p(y_1), \ldots, \sigma_p(y_n)$ commute if and only if the conditions (64) hold. Moreover, we will prove that for each pair $i \neq j$ the condition (64) is equivalent to the condition that the elements $\sigma_p(y_i)$ and $\sigma_p(y_j)$ commute. Indeed, using (62) and (63), we have

$$\begin{aligned} 0 &= [\sigma_p(y_i), \sigma_p(y_j)] = [y_i + p_i E_{00}(i), y_j + p_j E_{00}(j)] \\ &= [p_i, y_j] E_{00}(i) + [y_i, p_j] E_{00}(j) + p_i [E_{00}(i), p_j] E_{00}(j) + p_j [p_i, E_{00}(j)] E_{00}(i) \\ &= (-x_j^{-1}(p_i - p_{i,j}) + x_i^{-1}(p_j - p_{j,i}) + p_i p_{j,i} - p_j p_{i,j}) E_{00}(i) E_{00}(j), \end{aligned}$$

and so (64) holds, and vice versa.

Given $\sigma_p, \sigma_{p'} \in \mathrm{st}(x_1, \ldots, x_n)$. Then

$$\sigma_p \sigma_{p'}(y_i) = y_i + (p_i + p_i' - x_i p_i p_i') E_{00}(i), \ i = 1, \ldots, n.$$

Hence, $\sigma_p \sigma_{p'} = \sigma_{p'} \sigma_p$, and so the monoid $\mathrm{st}(x_1, \ldots, x_n)$ is abelian.

It remains to show that each endomorphism σ_p is a *monomorphism*, i.e. $\ker(\sigma_p) = 0$. Suppose that $\ker(\sigma_p) \neq 0$ for some p, we seek a contradiction.

Then $F_n \subseteq \ker(\sigma_p)$, since F_n is the least nonzero ideal of the algebra $\mathbb{S}_n$, [4]; but

$$\sigma_p(E_{00}(1)) = 1 - x_1(y_1 + p_1 E_{00}(1)) = (1 - x_1 p_1)E_{00}(1) \neq 0,$$

a contradiction.

2. Note that $\widehat{\eta}(\mathrm{st}(x_1, \dots, x_n)) = \mathrm{st}(y_1, \dots, y_n)$ and $\widehat{\eta}(\sigma_p) = \tau_{\eta(p)}$, where $\eta(p) := (\eta(p_1), \dots, \eta(p_n))$ (since $\widehat{\eta}(\sigma_p)(x_i) = \eta\sigma_p\eta(x_i) = \eta(y_i + p_i E_{00}(i)) = x_i + E_{00}(i)\eta(p_i)$). $\square$

For $n = 1$, the conditions (64) and (65) are vacuous, and so Corollary 10.1 takes a simpler form.

Corollary 10.2.

1. $\mathrm{st}(x) = \{\sigma_p : y \mapsto pE_{00} \mid p \in K[x]\}$.
2. $\mathrm{st}(y) = \{\sigma_p : x \mapsto E_{00}q \mid q \in K[y]\}$.

For each $i = 1, \dots, n$, let $G_1(i) := \mathrm{Aut}_{K-\mathrm{alg}}(\mathbb{S}_1(i))$ and $E_1(i) := \mathrm{End}_{K-\mathrm{alg}}(\mathbb{S}_1(i))$. There is a natural inclusion of groups $\prod_{i=1}^n G_1(i) \subset G_n$. Similarly, there is a natural inclusion of monoids $\prod_{i=1}^n E_1(i) \subset E_n$ which yields the inclusions of submonoids:

$$\prod_{i=1}^n \mathrm{st}(x_i) \subset \mathrm{st}(x_1, \dots, x_n) \text{ and } \prod_{i=1}^n \mathrm{st}(y_i) \subset \mathrm{st}(y_1, \dots, y_n).$$

These inclusions are *not* equalities as the following example shows.

Example. Fix an *arbitrary* polynomial p_i from the ideal $(x_1 \cdots x_n)$ of the polynomial algebra $K[x_1, \dots, x_n]$, and put $p_j := x_j^{-1} x_i p_i$ for all $j \neq i$. Then the conditions (64) hold, and so $\sigma_p \in E_n$, where $p = (p_1, \dots, p_n)$. An element $\sigma_{p'} \in \mathrm{st}(x_1, \dots, x_n)$ belongs to the submonoid $\prod_{i=1}^n \mathrm{st}(x_i)$ if and only if $p'_1 \in K[x_1], \dots, p'_n \in K[x_n]$. Now, it is obvious that $\prod_{i=1}^n \mathrm{st}(x_i) \neq \mathrm{st}(x_1, \dots, x_n)$. By applying $\widehat{\eta}$, we see that $\prod_{i=1}^n \mathrm{st}(y_i) \neq \mathrm{st}(y_1, \dots, y_n)$.

References

[1] V. V. Bavula, The Jacobian algebras, J. Pure Appl. Algebra **213** (2009), no. 5, 664–685. MR2494359 (2010c:16031)

[2] V. V. Bavula, The group of order preserving automorphisms of the ring of differential operators on a Laurent polynomial algebra in prime characteristic, Proc. Amer. Math. Soc. **137** (2009), no. 6, 1891–1898. MR2480268 (2010h:16057)

[3] V. V. Bavula, The group of automorphisms of the first Weyl algebra in prime characteristic and the restriction map, Glasg. Math. J. **51** (2009), no. 2, 263–274. MR2500750 (2010j:16053)

[4] V. V. Bavula, The algebra of one-sided inverses of a polynomial algebra, J. Pure Appl. Algebra **214** (2010), no. 10, 1874–1897. MR2608115 (2011f:16069)

[5] G. Benkart and T. Roby, Down-up algebras, J. Algebra **209** (1998), no. 1, 305–344. MR1652138 (2000e:06001a)

[6] J. Dixmier, Sur les algèbres de Weyl, Bull. Soc. Math. France **96** (1968), 209–242. MR0242897 (39 #4224)

[7] N. Jacobson, *Structure of rings*, American Mathematical Society Colloquium Publications, Vol. 37. Revised edition American Mathematical Society, Providence, R.I. 1964. MR0222106 (36 #5158)

[8] D. A. Jordan, Down-up algebras and ambiskew polynomial rings, J. Algebra **228** (2000), no. 1, 311–346. MR1760967 (2001c:16057)
[9] H. W. E. Jung, Über ganze birationale Transformationen der Ebene, J. Reine Angew. Math. **184** (1942), 161–174. MR0008915 (5,74f)
[10] E. Kirkman and J. Kuzmanovich, Non-Noetherian down-up algebras, Comm. Algebra **28** (2000), no. 11, 5255–5268. MR1785498 (2001j:16043)
[11] E. Kirkman, I. M. Musson and D. S. Passman, Noetherian down-up algebras, Proc. Amer. Math. Soc. **127** (1999), no. 11, 3161–3167. MR1610796 (2000b:16042)
[12] L. Makar-Limanov, On automorphisms of Weyl algebra, Bull. Soc. Math. France **112** (1984), no. 3, 359–363. MR0794737 (87a:16034)
[13] A. van den Essen, *Polynomial automorphisms and the Jacobian conjecture*, Progress in Mathematics, 190, Birkhäuser, Basel, 2000. MR1790619 (2001j:14082)
[14] W. van der Kulk, On polynomial rings in two variables, Nieuw Arch. Wiskunde (3) **1** (1953), 33–41. MR0054574 (14,941f)

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