Westfälische
Wilhelms-Universität
Münster

› Existence of Typical Scales for Manifolds with Lower Ricci Curvature Bound

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2016

wissen.leben
WWU Münster

Fachbereich 10
Mathematik und
Informatik

Mathematik

Existence of Typical Scales for Manifolds with Lower Ricci Curvature Bound

Inaugural-Dissertation
zur Erlangung des Doktorgrads
der Naturwissenschaften im Fachbereich
Mathematik und Informatik
der Mathematisch-Naturwissenschaftlichen Fakultät
der Westfälischen Wilhelms-Universität Münster

vorgelegt von
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aus Münster
– 2016 –

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Erster Gutachter:	Prof. Dr. Burkhard Wilking
Zweiter Gutachter:	Prof. Dr. Christoph Böhm
Tag der mündlichen Prüfung:	05. Juli 2016
Tag der Promotion:	05. Juli 2016

Abstract

This thesis investigates collapsing sequences of Riemannian manifolds which satisfy a uniform lower Ricci curvature bound. It is shown that in this situation there exists a sequence of rescaling factors (scales) such that for a set of good base points of large measure the pointed rescaled manifolds subconverge to a product of a Euclidean and a compact space. Moreover, all possible Euclidean factors have the same dimension and all possible compact factors satisfy the same diameter bounds. Further, the dimension of the compact factor does not depend on the choice of the base point (along a fixed subsequence).

Contents

Introduction

In recent years, the focus of Riemannian geometry turned from investigating a single Riemannian manifold to examining classes of Riemannian manifolds. In [Gro81], Gromov introduced the notion of *Gromov-Hausdorff convergence*. For the class of (compact or pointed) Riemannian manifolds of a fixed dimension satisfying a uniform lower Ricci curvature bound, Gromov's Pre-compactness Theorem asserts that each sequence of such manifolds has a convergent subsequence. Notice that the limit need not be a manifold, but is a length space. Examining the limit and the manifolds close to it can give insights about the topological information that is different in the limit.

If such a sequence of manifolds has even a uniform lower sectional curvature bound, this bound carries over to the possibly non-smooth Alexandrov limit. If the limit is actually a smooth manifold, by Yamaguchi's Fibration Theorem, [Yam91], the manifolds fibre over the limit in the following way: If M_i is a sequence of n-dimensional manifolds with a uniform lower sectional curvature bound and a uniform upper diameter bound converging to a compact manifold N of lower dimension, then for sufficiently large $i \in \mathbb{N}$ there are fibrations $M_i \to N$ which are close to Riemannian submersions. Moreover, up to a finite covering, each fibre is the total space of a fibration over a torus.

For proving the latter, a crucial argument is the following: Consider the pre-image of some ball under the fibration $M_i \to N$. After rescaling the metric up, this pre-image converges to a product of a Euclidean and a compact space. In fact, it is possible to replace the rescaling factors by larger ones such that the limit again has the form of such a product, but the Euclidean factor has higher dimension. This can be iterated until, finally, the compact factor vanishes. Such scaling factors are called *typical scales*. Similar techniques have been used by e.g. Shioya and Yamaguchi in [SY00] and Kapovitch, Petrunin and Tuschmann in [KPT10].

If a sequence of manifolds only satisfies a uniform lower Ricci curvature bound, Yamaguchi's Fibration Theorem might fail. This was proven by Anderson in [And92]. However, in recent years Cheeger and Colding obtained deep structure results for limits of such sequences, [CC97, CC00a, CC00b], by using *measured Gromov-Hausdorff convergence*: After renormalizing the measure of the manifolds and passing to a subsequence, the manifolds converge to a metric measure space such that the (renormalised) measures converge to the limit measure. The uniform lower Ricci curvature bound carries over to the limit in the sense that the limit measure still satisfies the Bishop-Gromov Theorem.

Another difficulty occurring with only a lower Ricci curvature bound is the following:

Unlike for lower section curvature bounds, tangent cones of the limit space need not be metric cones. In fact, in the case of a collapsing sequence, the tangent cones at some point may depend on the choice of the rescaling sequence, cf. [CC97]. However, Colding and Naber [CN12] proved that any limit of a sequence of manifolds with uniform lower Ricci curvature bound contains a connected subset of full measure such that for each point in this subset the tangent cone is unique and a Euclidean space of a fixed dimension $k \in \mathbb{N}$. This k is called the *dimension* of the limit space. Notice that this dimension is at most the Hausdorff-dimension of the space. In particular, $k < n$.

If a collapsing sequence of manifolds M_i satisfies the lower Ricci curvature bound $-\varepsilon_i$, where $\varepsilon_i \to 0$, and if this sequence converges to a Euclidean space, then the Rescaling Theorem of Kapovitch and Wilking in [KW11] already provides the existence of one sequence of typical scales. For this sequence, the blow-ups of the manifold split into products of this Euclidean space and a compact factor. The main theorem of this thesis generalises this statement by allowing arbitrary limit spaces and a uniform lower Ricci curvature bound.

Main Theorem. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a collapsing sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$ and converge to a limit (X, p) of dimension $k < n$ in the measured Gromov-Hausdorff sense. Then for every $\varepsilon \in (0,1)$ there exist a subset of good points $G_1(p_i) \subseteq B_1(p_i)$ satisfying*

$$\mathrm{vol}(G_1(p_i)) \geq (1-\varepsilon) \cdot \mathrm{vol}(B_1(p_i)),$$

a sequence $\lambda_i \to \infty$ and a constant $D > 0$ such that the following holds: For any choice of base points $q_i \in G_1(p_i)$ and every sublimit (Y, q) of $(\lambda_i M_i, q_i)_{i\in\mathbb{N}}$ there is a compact metric space K of dimension $l \leq n-k$ with $\frac{1}{D} \leq \mathrm{diam}(K) \leq D$ such that Y splits isometrically as a product

$$Y \cong \mathbb{R}^k \times K.$$

Moreover, for any base points $q_i, q_i' \in G_1(p_i)$ such that, after passing to a subsequence, both $(\lambda_i M_i, q_i) \to (\mathbb{R}^k \times K, \cdot)$ and $(\lambda_i M_i, q_i') \to (\mathbb{R}^k \times K', \cdot)$ as before, $\dim(K) = \dim(K')$.

Note that the theorem does not prove that all possible compact spaces need to have the same dimension, but that the dimension only depends on the regarded subsequence and not on the choice of the base points.

Furthermore, observe that $\dim(K) < n-k$ might occur in the situation of the theorem: Consider the sequence of flat tori $M_i := S^1 \times S^1(\frac{1}{i}) \times S^1(\frac{1}{i^2})$ where $S^1(r)$ denotes a circle of radius $r > 0$. In this example, it is easy to imagine the last two factors 'collapsing to a point' in the limit, although the very last factor collapses faster than central one. Hence, M_i converges to S^1. For $\lambda_i = i$, the rescaled manifolds $\lambda_i M_i$ converge to $\mathbb{R} \times S^1$. Using the notation of the main theorem, one has $n = 3$, $k = 1$ and $l = 1 < 2 = n - k$.

This example also illustrates Yamaguchi's fibration theorem and typical scales: As a product, M_i obviously fibres over the first factor S^1 with fibre $F_i := S^1(\frac{1}{i}) \times S^1(\frac{1}{i^2})$, which is a torus. For the scales $\lambda_i^1 = i$, $\lambda_i^2 = i^2$ and $\lambda_i^3 = i^3$ one gets convergence $\lambda_i^1 M_i \to \mathbb{R} \times S^1$, $\lambda_i^2 M_i \to \mathbb{R}^2 \times S^1$ and $\lambda_i^3 M_i \to \mathbb{R}^3$.

Now turn to the proof of the main theorem. Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a collapsing sequence as in the main theorem and let (X, p) denote its limit. First will be proven that for points $q_i \in M_i$ and a small radius $r > 0$ the conclusion of the main theorem holds correspondingly on $B_r(q_i)$ for a subset of good points $G_r(q_i)$ and a sequence of scales $\lambda_i \to \infty$, i.e. a 'local' version of the main theorem will be established.

Recall that the set of good points $G_r(q_i) \in B_r(q_i)$ and the scales $\lambda_i \to \infty$ have to satisfy the following: Any point $x_i \in G_r(q_i)$ needs to have the property that each sublimit of $(\lambda_i M_i, x_i)$ is the product of $\mathbb{R}^k$ with a compact metric space where the compact spaces (essentially) all have the same dimension. This is achieved in two steps.

First, let $G_r^1(q_i) \subseteq B_r(q_i)$ denote the set of all points x_i such that all sublimits of $(\mu_i M_i, x_i)$ split off an $\mathbb{R}^k$-factor where $\mu_i \to \infty$ is an arbitrary sequence. In particular (once $\lambda_i \to \infty$ is constructed), for any $x_i \in G_r^1(q_i)$, any limit of the sequence $(\lambda_i M_i, x_i)$ splits off an $\mathbb{R}^k$-factor. That these sets $G_r^1(q_i)$ have large volume inside $B_r(q_i)$ is obtained by generalising results of Cheeger, Colding [CC00a] and Kapovitch, Wilking [KW11] involving modified distance functions coming from splitting theorems. For more details, see section 2.1.

Next, define scales $\lambda_i \to \infty$ and another subset $G_r^2(q_i) \subseteq B_r(q_i)$ of large volume as the set of points x_i such that $(\lambda_i M_i, x_i)$ has small distance to a product of $\mathbb{R}^k$ and a compact metric space of diameter 1. The existence of such $\lambda_i \to \infty$ and $G_r^2(q_i)$ is obtained by using the Rescaling Theorem of Kapovitch and Wilking in [KW11]. For further explanations, see section 2.2.

These scales $\lambda_i \to \infty$ together with the intersection $G_r(q_i) = G_r^1(q_i) \cap G_r^2(q_i)$ give the splitting result in the local version of the main theorem.

In order to finish the local version, prove that two such limits have the same dimension. First, the following special case is investigated: Suppose that two points $x_i, y_i \in G_r(q_i)$ are connected by an integral curve of a vector field whose flow is measure preserving and bi-Lipschitz (on a set of large enough volume). In this situation, via Gromov-Hausdorff convergence one obtains a bi-Lipschitz map between subsets of positive volume inside of the limit spaces of $(\lambda_i M_i, x_i)$ and $(\lambda_i M_i, y_i)$, respectively. In particular, these limits need to have the same dimension.

For the general situation, recall that the flow of any divergence-free vector field is measure preserving. Moreover, using (slight generalisations of) results [KW11], it is bi-Lipschitz if its derivative is small—in some L_α-norm, $\alpha > 1$, and taking some average value (in volume sense). In fact every two points of the set $G_r^2(q_i)$ are connected by the curve of such a vector field (or are at least sufficiently close to its start and end point). For more details, see section 2.3.

For verifying the main theorem, i.e. defining $G_1(p_i)$ and $\lambda_i \to \infty$, fix $r > 0$ and finitely many sequences $(q_i)_{i\in\mathbb{N}}$ such that the union of the subsets of good points $G_r(q_i) \subseteq B_r(q_i)$ has sufficiently large volume in $B_1(p_i)$. Define $G_1(p_i)$ as the union of these $G_r(q_i)$. Now difficulties arise since each sequence $(q_i)_{i\in\mathbb{N}}$ provides its own sequence of scales $\lambda_i \to \infty$ and these sequences might be pairwise different, but the main theorem requires only a single sequence of scales. This problem can be solved if, given two such sequences $(q_i^1)_{i\in\mathbb{N}}$

and $(q_i^2)_{i\in\mathbb{N}}$ and their corresponding scales $\lambda_i^1 \to \infty$ and $\lambda_i^2 \to \infty$, the local version of the main theorem still holds for $(q_i^2)_{i\in\mathbb{N}}$ and $\lambda_i^1 \to \infty$ (instead of $\lambda_i^2 \to \infty$). Indeed, this is achieved by a clever choice of the finitely many $(q_i)_{i\in\mathbb{N}}$ utilizing the Hölder continuity result of Colding and Naber [CN12]. For more intuitive details on this approach, see the introduction to chapter 3.

This thesis is structured as follows: Since Gromov-Hausdorff convergence is a main tool, chapter 1 gives an introduction into this topic. The subsequent chapters deal with the proof of the main theorem: chapter 2 proves the above mentioned local version of the main theorem: If points q_i and some small $r > 0$ satisfy that the rescaled $(\frac{1}{r}M_i, q_i)$ are sufficiently close to $\mathbb{R}^k$, then the statement of the main theorem holds on the ball $B_r(q_i)$ analogously. As explained before, using finitely many of such sequences q_i and taking the union of the obtained subsets $G_r(q_i) \subseteq B_r(q_i)$ will be used in chapter 3 to prove the main theorem. The appendix covers (in greater detail) Gromov-Hausdorff convergence (Appendix A), the behaviour of geometric notions under rescaling of metrics (Appendix B) and, for the purpose of estimating volumes, the Bishop-Gromov Theorem and closely related statements (Appendix C).

Acknowledgements

First of all, I would like to thank my advisor Prof. Dr. Burkhard Wilking for his guidance and expertise. I am very grateful to all former and present members of the Arbeitsgruppe Differentialgeometrie and the second floor for interesting and helpful discussions and the great time we had. I thank Prof. Dr. Wilderich Tuschmann for pointing out helpful literature to me. Sincere thanks go to everyone who read parts of preliminary versions of this thesis, especially to those who read the whole act: Svenja, Wolfgang and—most of all—Sebastian.

Chapter 1

Foundations

The purpose of this chapter is to introduce the so called Gromov-Hausdorff convergence of metric spaces. Beforehand, some notation and other results needed throughout this thesis are provided.

1.1 Bishop-Gromov volume comparison

Theorem 1.1 (Bishop-Gromov Theorem, [Pet06, Chapter 9, Lemma 1.6])**.** *Let M be a complete n-dimensional Riemannian manifold with $\mathrm{Ric}_M \geq (n-1)\cdot\kappa$ for some $\kappa \in \mathbb{R}$ and let $p \in M$. Then the map*

$$r \mapsto \frac{\mathrm{vol}_M(B_r(p))}{V_\kappa^n(r)}$$

is monotonically decreasing with limit 1 as $r \to 0$, where $V_\kappa^n(r)$ is the volume of an r-ball in the n-dimensional space form of sectional curvature κ.

In particular, for $R \geq r > 0$,

$$\mathrm{vol}_M(B_R(p)) \leq \frac{V_\kappa^n(R)}{V_\kappa^n(r)} \cdot \mathrm{vol}_M(B_r(p)).$$

This factor is independent of M and denoted by

$$C_{BG}(n,\kappa,r,R) := \frac{V_\kappa^n(R)}{V_\kappa^n(r)}.$$

In the following, this theorem will always be applied using the notion $C_{BG}(n,\kappa,r,R)$. As a function of radii or curvature bound, C_{BG} has the following properties. For more information and the proof of this lemma, see Appendix C.

Lemma 1.2. *Let $n \in \mathbb{N}$, $-1 \leq \kappa \leq 0$, $R \geq r > 0$, $c \geq 1$ and $y > 0$. Then*

a) $C_{BG}(n,-1,r,cr) \leq C_{BG}(n,-1,R,cR)$,

b) $C_{BG}(n,\kappa,r,R) \leq C_{BG}(n,1,r,R)$ *and*

c) $\lim_{x\to\infty} C_{BG}(n,-1,x,x+y) = e^{(n-1)y}$.

1.2 Rescaling of metrics

Throughout this thesis, an important tool is rescaling of manifolds: For any Riemannian manifold (M, g) let αM denote the Riemannian manifold $(M, \alpha^2 g)$, where $\alpha > 0$. Basic properties of this rescaling are straight-forward; for an overview, see Appendix B. Among others, the following statement is proven there.

Often, functions will be given on some manifold M but needed on the rescaled manifold αM. In this case, it is useful to 'rescale' these functions by the same factor. In fact, given those functions, some specific terms using their gradients and Hessians will be used. The following lemma states how these terms behave under rescaling. First, the notion of the average integral needs to be introduced.

Definition 1.3. For a Riemannian manifold M, a measurable subset $U \subseteq M$ and an integrable function $f : U \to \mathbb{R}$ let

$$\fint_U f \, dV_M := \frac{1}{\mathrm{vol}_M(U)} \cdot \int_U f \, dV_M$$

denote the *average integral*.

Lemma 1.4. *Let (M, g) be a Riemannian manifold, $\alpha > 0$ and $\tilde{g} = \alpha^2 g$. For smooth $f_i : M \to \mathbb{R}$, $1 \le i \le k$, let $\tilde{f}_i := \alpha \cdot f_i$, $f := (f_i)_{1 \le i \le k}$ and $\tilde{f} = (\tilde{f}_i)_{1 \le i \le k} = \alpha f$, respectively. Furthermore, let*

$$\psi_\nabla^g(f) := \sum_{i,j=1}^{k} |g(\nabla^g f_i, \nabla^g f_j) - \delta_{ij}| \quad \textit{and} \quad \psi_H^g(f) := \sum_{i=1}^{k} \| \operatorname{Hess}_g(f) \|_g^2.$$

Then, using the analogous definitions for $\tilde{g}$,

$$\psi_\nabla^{\tilde{g}}(\tilde{f}) = \psi_\nabla^g(f) \quad \textit{and} \quad \psi_H^{\tilde{g}}(\tilde{f}) = \frac{1}{\alpha} \cdot \psi_H^g(f).$$

In particular, for $r > 0$ and $p \in M$,

$$\fint_{B_r^{\alpha M}(p)} \psi_\nabla^{\tilde{g}}(\tilde{f}) \, dV_{\alpha M} = \fint_{B_{r/\alpha}^M(p)} \psi_\nabla^g(f) \, dV_M \quad \textit{and}$$

$$\fint_{B_r^{\alpha M}(p)} \psi_H^{\tilde{g}}(\tilde{f})^2 \, dV_{\alpha M} = \frac{1}{\alpha^2} \cdot \fint_{B_{r/\alpha}^M(p)} \psi_H^g(f)^2 \, dV_M \,.$$

1.3 Gromov-Hausdorff convergence

For the proofs of all the statements given in the remaining chapter and more information on Gromov-Hausdorff convergence, see Appendix A.

In order to define Gromov-Hausdorff convergence of proper metric spaces, Gromov-Hausdorff distance of compact metric spaces is needed.

Definition 1.5. For bounded subsets A and B of a metric space (X, d), the *Hausdorff distance of A and B* is defined as

$$d_H^d(A, B) := \inf\{\varepsilon > 0 \mid A \subseteq B_\varepsilon^X(B) \text{ and } B \subseteq B_\varepsilon^X(A)\}.$$

For base points $a \in A$, $b \in B$ the *pointed Hausdorff distance of (A, a) and (B, b)* is given by

$$d_H^d((A, a), (B, b)) := d_H^d(A, B) + d(a, b).$$

For compact metric spaces $(X, d_X), (Y, d_Y)$ the *Gromov-Hausdorff distance of X and Y* is defined as

$$d_{GH}(X, Y) := \inf\{d_H^d(X, Y) \mid d \text{ admissible metric on } X \amalg Y\},$$

and the *pointed Gromov-Hausdorff distance between* the pointed compact metric spaces (X, d_X, x_0) and (Y, d_Y, y_0) is defined as

$$d_{GH}((X, x_0), (Y, y_0)) := \inf\{d_H^d((X, x_0), (Y, y_0)) \mid d \text{ admissible metric on } X \amalg Y\}$$

where a metric d on the disjoint union $X \amalg Y$ is called *admissible* if $d_{|X \times X} = d_X$ and $d_{|Y \times Y} = d_Y$.

Using this, Gromov-Hausdorff convergence for non-compact proper metric spaces can be defined. A metric space is called *proper* if all closed balls are compact. In the following, for a metric space (X, d_X), $p \in X$ and $r > 0$, the open and closed ball, respectively, of radius r around p will be denoted by $B_r(p)$ and $\bar{B}_r(p)$, respectively.

Definition 1.6. Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed proper metric spaces. The sequence (X_i, p_i) *converges to* (X, p) *(in the pointed Gromov-Hausdorff sense)* if for all $r > 0$,

$$d_{GH}((\bar{B}_r^{X_i}(p_i), p_i), (\bar{B}_r^X(p), p)) \to 0 \text{ as } i \to \infty$$

where the balls are equipped with the restricted metric. If (X_i, p_i) converges to (X, p), this is denoted by $(X_i, p_i) \to (X, p)$ and (X, p) is called the *(pointed Gromov-Hausdorff) limit* of (X_i, p_i).

Frequently, a sequence (X_i, p_i) does not converge itself but has a converging subsequence. The limit of such a subsequence is called *sublimit* of (X_i, p_i), and (X_i, p_i) is said to *subconverge* to this limit.

From now on, assume all metric spaces to be proper. Recall that a metric space (X, d_X) is called *length space* if

$$d(x, y) = \inf\{L(c) \mid c \text{ continuous curve from } x \text{ to } y\}$$

for any $x, y \in X$, where $L(c)$ denotes the length of c.

Proposition 1.7. *Let (X, d_X, p), (Y, d_Y, q) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed length spaces. Assume X and Y to be complete. If $(X_i, p_i) \to (X, p)$ and $(X_i, p_i) \to (Y, q)$, then there exists a pointed isometry between (X, p) and (Y, q).*

Notation. Following common practice, the compactness of the balls and the dependence on the base point will be suppressed, and the pointed distance

$$d_{GH}((\bar{B}_r^X(p), p), (\bar{B}_r^Y(q), q))$$

will be denoted by

$$d_{GH}(B_r^X(p), B_r^Y(q)).$$

Moreover, if $d_{GH}(B_{1/\varepsilon}^X(p), B_{1/\varepsilon}^Y(q)) \leq \varepsilon$ for some $\varepsilon > 0$, this will be denoted by

$$d_{GH}((X, p), (Y, q)) \leq \varepsilon\,.$$

Proposition 1.8. *Let (X, d_X, p), (Y, d_Y, q) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed metric spaces.*

a) If X and Y are compact with $d_{GH}((X, p), (Y, q)) < \frac{\varepsilon}{2}$, then there are maps

$$f : X \to Y \quad \text{and} \quad g : Y \to X$$

with $f(p) = q$ and $g(q) = p$ such that the following holds for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$:

$$\begin{aligned} |d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \varepsilon, &\qquad d_X(g \circ f(x), x) < \varepsilon,\\ |d_Y(y_1, y_2) - d_X(g(y_1), g(y_2))| < \varepsilon, &\qquad d_Y(f \circ g(y), y) < \varepsilon\,. \end{aligned}$$

Such (f, g) are called (ε-)Gromov-Hausdorff approximations *or* ε-approximations between (X, p) and (Y, q).

b) If the X_i and X are length spaces, the following are equivalent:

(i) $(X_i, p_i) \to (X, p)$.

(ii) There is a sequence $\varepsilon_i \to 0$ and ε_i-approximations (f_i, g_i) between the balls $(\bar{B}_{1/\varepsilon_i}^{X_i}(p_i), p_i)$ and $(\bar{B}_{1/\varepsilon_i}^X(p), p)$.

(iii) There is a sequence $\varepsilon_i \to 0$ such that $d_{GH}((X_i, p_i), (X, p)) \leq \varepsilon_i$ for all i.

From now on, for converging length spaces $(X_i, p_i) \to (X, p)$, such ε_i-approximations (f_i, g_i) as in Proposition 1.8 b)(ii) will be implicitly fixed.

Definition 1.9. Let $(X_i, p_i) \to (X, p)$ be converging length spaces. For $q_i \in \bar{B}_{1/\varepsilon_i}^{X_i}(p_i)$ and $q \in X$, q_i *converges to* q, denoted by $q_i \to q$, if $f_i(q_i)$ converges to q (in X).

Given $(X_i, p_i) \to (X, p)$ as above, $p_i \to p$ due to $f_i(p_i) = p$. Moreover, for each $x \in X$ there exists such a sequence p_i^x satisfying $p_i^x \to x$, namely $p_i^x := g_i(x)$. From now on, let $p_i^x := g_i(x)$ denote this sequence.

Remark. For $(X_i, p_i) \to (X, p)$ and $x \in X$, one has $(X_i, p_i^x) \to (X, x)$ as well.

The following three lemmata state several properties of Gromov-Hausdorff distance and convergence: The first lemma deals with several conditions ensuring convergence, the second one provides estimates for the distance of balls. The last lemma is used to construct maps between limit spaces given maps between two convergent sequences.

As for Riemannian manifolds, for a metric space (X, d) let αX denote the rescaled metric space $(X, \alpha\, d)$.

Lemma 1.10. *Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed length spaces.*

a) If $(X_i, p_i) \to (X, p)$ and $g : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ satisfies $\lim_{x\to 0} g(x) = 0$, then there exists $\varepsilon_i \to 0$ with $d_{GH}(B^{X_i}_{1/\varepsilon_i}(p_i), B^X_{1/\varepsilon_i}(p)) \le g(\varepsilon_i)$.

b) Let $C > 0$ and $q_i \in X_i$ such that $d_{X_i}(p_i, q_i) \le C$. If $(X_i, p_i) \to (X, p)$, then there exists $q \in X$ such that (X_i, q_i) subconverges to (X, q).

c) If $(X_i, p_i) \to (X, p)$ and $\alpha_i \to \alpha$ for some $\alpha > 0$, then $(\alpha_i X_i, p_i) \to (\alpha X, p)$.

Lemma 1.11. *Let (X, d_X, p) and (Y, d_Y, q) be pointed metric spaces.*

a) Let Y be compact. Then $d_{GH}(B^X_r(p), B^{X\times Y}_r((p, q))) \le \mathrm{diam}(Y)$ for all $r > 0$.

b) If X and Y are length spaces and $R \ge r > 0$, then

$$d_{GH}(B^X_r(p), B^Y_r(q)) \le 16 \cdot d_{GH}(B^X_R(p), B^Y_R(q)).$$

In particular: If $d_{GH}((X, p), (Y, q)) \le \varepsilon$ for an $\varepsilon > 0$, then $d_{GH}((X, p), (Y, q)) \le \varepsilon'$ for all $\varepsilon' \ge 16\, \varepsilon$.

Lemma 1.12. *Let (X, d_X), (Y, d_Y), (X_i, d_{X_i}) and (Y_i, d_{Y_i}), $i \in \mathbb{N}$, be compact length spaces such that $X_i \to X$ and $Y_i \to Y$. Moreover, let $\alpha > 0$, $K_i \subseteq X_i$ be compact subsets and $f_i : K_i \to Y_i$ be α-bi-Lipschitz. After passing to a subsequence, the following holds:*

a) There exist compact subsets $K \subseteq X$ and $K' \subseteq Y$ which are Gromov-Hausdorff limits of K_i and $f_i(K_i)$, respectively, and an α-bi-Lipschitz map $f : K \to K'$ with $f(K) = K'$.

b) For any compact subset $L \subseteq K \subseteq X$ there exist compact subsets $L_i \subseteq K_i$ such that $L_i \to L$ and $f_i(L_i) \to f(L)$ in the Gromov-Hausdorff sense.

For manifolds, the following theorem by Gromov states that in some cases at least a (Gromov-Hausdorff) sublimit exists.

Theorem 1.13 (Gromov's Pre-compactness Theorem, [Pet06, Cor. 1.11])**.** *For $n \ge 2$, $\kappa \in \mathbb{R}$ and $D > 0$, the following classes are pre-compact, i.e. every sequence in the class has a convergent subsequence whose limit lies in the closure of this class:*

a) The collection of closed Riemannian manifolds with $\mathrm{Ric} \ge (n-1)\cdot\kappa$ and $\mathrm{diam} \le D$.

b) The collection of pointed complete Riemannian manifolds with $\mathrm{Ric} \ge (n-1)\cdot\kappa$.

1.3.1 Ultralimits

Since sequences of pointed metric spaces do not necessarily converge in the pointed Gromov-Hausdorff sense, the notions of ultrafilters and ultralimits are useful: The ultralimit of a sequence of pointed metric spaces is always a sublimit of this sequence in the pointed Gromov Hausdorff sense. In order to compare (Gromov-Hausdorff) sublimits of two different sequences of metric spaces, it is useful to know which subsequences (of either sequence) converge to the given sublimits. This will be investigated in the lemma below.

Definition and Lemma 1.14. A *non-principal ultrafilter on* $\mathbb{N}$ is a finitely additive probability measure ω on $\mathbb{N}$ such that all subsets $S \subseteq \mathbb{N}$ are ω-measurable with value $\omega(S) \in \{0, 1\}$ and $\omega(S) = 0$ if S is finite.

Let ω be a non-principal ultrafilter on $\mathbb{N}$. For every bounded sequence $(a_i)_{i\in\mathbb{N}}$ there exists a unique real number $l \in \mathbb{R}$ such that

$$\omega(\{i \in \mathbb{N} \mid |a_i - l| < \varepsilon\}) = 1$$

for every $\varepsilon > 0$. Let $\lim_\omega a_i := l$ denote this real number.

Let ω be a non-principal ultrafilter on $\mathbb{N}$, (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed metric spaces and

$$X_\omega := \{[(x_i)_{i\in\mathbb{N}}] \mid x_i \in X_i \text{ and } \sup_{i\in\mathbb{N}} d_{X_i}(x_i, p_i) < \infty\}$$

where

$$(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}} \quad \text{if and only if} \quad \lim_\omega d_{X_i}(x_i, y_i) = 0.$$

Furthermore, let $d_\omega([(x_i)_{i\in\mathbb{N}}], [(y_i)_{i\in\mathbb{N}}]) := \lim_\omega d_{X_i}(x_i, y_i)$. Then (X_ω, d_ω) is a metric space, called *ultralimit* of (X_i, d_{X_i}, p_i) and denoted by $\lim_\omega(X_i, d_{X_i}, p_i)$.

Remark. As is customary for Gromov-Hausdorff convergence, the dependence on the metric often will be suppressed. Then X_ω is said to be the ultralimit of (X_i, p_i).

Essentially, the subsequent lemma states the following: Given two sequences of metric spaces and a non-principal ultrafilter, by passing to the same subsequence of indices for both sequences, the ultralimits (of the original sequences) can be realised as (sub)limits in the pointed Gromov-Hausdorff sense. Conversely, if two sequences of metric spaces are convergent after passing to the same subsequence of indices, there is a non-principal ultrafilter realising the limits of the converging subsequences as ultralimits.

Thus, in the following chapters, instead of investigating common subsequences of indices, often ultralimits are used.

Lemma 1.15. *Let (X_i, d_{X_i}, p_i) and (Y_i, d_{Y_i}, q_i), $i \in \mathbb{N}$, be pointed length spaces.*

a) Let ω be a non-principal ultrafilter on $\mathbb{N}$. Then $\lim_\omega(X_i, p_i)$ is a sublimit in the pointed Gromov-Hausdorff sense. Concretely, there exists a subsequence $(i_j)_{j\in\mathbb{N}}$ such that both

$$(X_{i_j}, p_{i_j}) \to \lim_\omega(X_i, p_i) \quad \textit{and} \quad (Y_{i_j}, q_{i_j}) \to \lim_\omega(Y_i, q_i) \quad \textit{as } j \to \infty$$

in the pointed Gromov-Hausdorff sense.

b) *The sublimit of a sequence of pointed length spaces in the pointed Gromov-Hausdorff sense is the ultralimit with respect to a non-principal ultrafilter. To be more precise: If* (X, d_X, p) *and* (Y, d_Y, q) *are pointed length spaces and* $(i_j)_{j\in\mathbb{N}}$ *is a subsequence such that both*

$$(X_{i_j}, p_{i_j}) \to (X, p) \quad \text{and} \quad (Y_{i_j}, q_{i_j}) \to (Y, q) \qquad \text{as } j \to \infty$$

in the pointed Gromov-Hausdorff sense, then there exists a non-principal ultrafilter ω *on* $\mathbb{N}$ *such that there are isometries*

$$\lim_\omega (X_i, p_i) \cong (X, p) \quad \text{and} \quad \lim_\omega (Y_i, q_i) \cong (Y, q).$$

1.3.2 Measured Gromov-Hausdorff convergence

Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$. If $\mathrm{vol}_{M_i}(B_1(p_i)) \to 0$ as $i \to \infty$, this sequence is said to be *collapsing*.

In this situation, renormalised limit measures are used, cf. [CC97, section 1]: Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a collapsing sequence as above. Then (M_i, p_i) subconverges to a metric space (X, p) such that a 'renormalisation' of the measures vol_{M_i} converges to a limit measure vol_X. In fact, the following is true.

Theorem 1.16 ([CC97, Theorem 1.6, Theorem 1.10])**.** *Let* $(M_i, p_i)_{i\in\mathbb{N}}$ *be a sequence of pointed complete connected* n*-dimensional Riemannian manifolds satisfying the uniform lower Ricci curvature bound* $\mathrm{Ric}_{M_i} \geq -(n-1)$*. Then* (M_i, p_i) *subconverges to a metric space* (X, p) *in the pointed Gromov-Hausdorff sense and there exists a Radon measure* vol_X *on* X *such that for all* $x \in X$*,* $x_i \to x$ *and* $r > 0$*,*

$$\frac{\mathrm{vol}_{M_i}(B_r^{M_i}(x_i))}{\mathrm{vol}_{M_i}(B_1^{M_i}(p_i))} \to \mathrm{vol}_X(B_r^X(x)) \text{ as } i \to \infty.$$

Moreover, for any $R \geq r > 0$ *and* $x \in X$*,*

$$\frac{\mathrm{vol}_X(B_R^X(x))}{\mathrm{vol}_X(B_r^X(x))} \leq C_{BG}(n, -1, r, R).$$

This vol_X *is called* (renormalised) limit measure*.*

Observe that the limit measure of a sequence (M_i, p_i) depends on the choice of the base points and the considered subsequence, cf. again [CC97, section 1]. Moreover, observe the following: Gromov's Pre-compactness Theorem ensures subconvergence (for pointed Riemannian manifolds of the same dimension with a lower Ricci curvature bound), but the above theorem guarantees more, namely subconvergence (for the same class) including convergence of the (renormalised) measures. Throughout this thesis, only measured Gromov-Hausdorff convergence will be used, i.e. whenever a sequence $(M_i, p_i)_{i\in\mathbb{N}}$ converges to a limit space (X, p), this limit is equipped with a measure vol_X as in the above theorem.

The following propositions provide informations about sets whose measures converge. For more information and the proofs, see section A.4.

Proposition 1.17. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a sequence of pointed complete connected n-dimensional Riemannian manifolds converging to a metric space (X, p) in the pointed Gromov-Hausdorff sense and let $\varepsilon_i \to 0$ and*

$$(f_i, g_i) \in Appr_{\varepsilon_i}((\bar{B}^{M_i}_{1/\varepsilon_i}(p_i), p_i), (\bar{B}^X_{1/\varepsilon_i}(p), p))$$

be as in Corollary A.27. For $x \in X$, let p_i^x denote $g_i(x)$.

a) Let $A_i \subseteq M_i$ be compact with $A_i \subseteq \bar{B}_R(p_i)$ for some $R > 0$. After passing to a subsequence, there exists $A \subseteq X$ with $A_i \to A$ and $\mu_i(A_i) \to \mathrm{vol}_X(A)$.

b) Let $0 < R < 1$, $x_1, \dots, x_l \in B^X_{1-R}(p)$ and

$$A_i := B^{M_i}_1(p_i) \setminus \bigcup_{j=1}^{l} B^{M_i}_R(p_i^{x_j}) \quad \text{and} \quad A := B^X_1(p) \setminus \bigcup_{j=1}^{l} B^X_R(x_j).$$

Then $\mu_i(A_i) \to \mathrm{vol}_X(A)$.

Proposition 1.18. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$, let $x_i, y_i \in M_i$ with $d_{M_i}(x_i, y_i) \leq 2$ and assume (M_i, x_i) and (M_i, y_i), respectively, to converge to metric spaces (X, x_∞) and (Y, y_∞), respectively. Moreover, let $r > 0$, $K_i \subseteq \bar{B}^{M_i}_r(x_i)$ be compact and $f_i : K_i \to M_i$ be α-bi-Lipschitz and measure preserving for some $\alpha > 0$.*

After passing to a subsequence, there exist a compact subset $K \subseteq \bar{B}^X_r(x_\infty)$, an α-bi-Lipschitz map $f : K \to Y$ and a constant $C > 0$ such that $\mathrm{vol}_Y(f(A)) = C \cdot \mathrm{vol}_X(A)$ for any measurable subset $A \subseteq K$.

1.3.3 Generic points

Let the complete pointed metric space (X, p) be the pointed Gromov-Hausdorff limit of a sequence of pointed connected n-dimensional Riemannian manifolds (M_i, p_i) with uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$. As introduced in [CC97, p. 408], a *tangent cone* at $x \in X$ is a Gromov-Hausdorff limit of $(\lambda_i X, x)$ where $\lambda_i \to \infty$ as $i \to \infty$. In general, this limit depends on the choice of $x \in X$ and the sequence $\lambda_i \to \infty$. If the limit is independent of the choice of $\lambda_i \to \infty$, it is denoted by $C_x X$. If $C_x X = \mathbb{R}^k$, this point x is called *k-regular* and the set of all k-regular points is denoted by $\mathcal{R}_k$. Furthermore, $\mathcal{R} = \bigcup_k \mathcal{R}_k$ denotes the set of all regular points.

Moreover, Cheeger and Colding proved that there are points such that non-unique tangent cones of different dimensions occur, cf. [CC97, Example 8.80]. In particular, there are points that are not regular. However, they proved that for any renormalised limit measure the complement $X \setminus \mathcal{R}$ has measure 0, i.e. almost all points are regular, cf. [CC97, Theorem 2.1]. Even more, it was conjectured that there is some k such that $\mathcal{R} \setminus \mathcal{R}_k$ has measure 0 as well, i.e. almost all points are k-regular. This conjecture was proven by Colding and Naber in [CN12].

Theorem 1.19 ([CN12, Theorem 1.18 and p. 1185]). *Let* $(M_i, p_i)_{i\in\mathbb{N}}$ *be a sequence of pointed complete connected* n*-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound* $\mathrm{Ric}_{M_i} \geq -(n-1)$ *and converge to a limit* (X, p)*. Then there is* $k = k(X) \in \mathbb{N}$ *such that* $\mathcal{R}_k$ *has full measure and is connected.*

This k *is called the* dimension *of* X*, a* k*-regular point is called* generic *and* $X_{\mathrm{gen}} := \mathcal{R}_k$ *denotes the set of all generic points.*

Note that $k < n$ if the sequence is collapsing.

Chapter 2

Local construction

For a collapsing sequence of pointed complete connected n-dimensional Riemannian manifolds (M_i, p_i) satisfying the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$ and converging to a limit space (X, p) of dimension $k = \dim(X)$, the main proposition of this chapter provides a condition on points $q_i \in M_i$ such that on balls around these points with sufficiently small radius a 'local version' of the main theorem holds: In fact, the statement of the main theorem holds on $B_r(q_i)$ if the rescaled manifolds $(\frac{1}{r}M_i, q_i)$ are sufficiently close to the Euclidean space. Applying this result to finitely many sequences of such points q_i and radii r will prove the main theorem in chapter 3.

This local result follows from generalising several theorems of Cheeger and Colding in [CC96, CC00a] and Kapovitch and Wilking in [KW11]. Those results make statements assuming a sequence of manifolds to converge to a Euclidean space $\mathbb{R}^k$. The generalisations do not assume such a convergence but that the manifolds are sufficiently close to $\mathbb{R}^k$, and then make similar statements as the mentioned theorems.

In the situation of a sequence (M_i, p_i) converging to a limit (X, p) as in the main proposition, there is no reason why the manifolds should already be sufficiently close to $\mathbb{R}^k$. On the other hand, there is hope that this is true after rescaling all manifolds with (the same) factor: For a generic point $x \in X$, the rescaled limit space $(\lambda X, x)$ converges to $\mathbb{R}^k$ as $\lambda \to \infty$. In particular, $(\lambda X, x)$ is close to $\mathbb{R}^k$ for sufficiently large $\lambda > 0$. Moreover, given any sequence of points $x_i \in M_i$ converging to $x \in X$, the equally rescaled manifolds $(\lambda M_i, x_i)$ converge to $(\lambda X, x)$. Hence, the $(\lambda M_i, x_i)$ are close to $\mathbb{R}^k$ for sufficiently large λ and i.

So, one can expect to be able to use the above explained generalisations for the rescaled manifolds. In fact, those (generalised) theorems make statements about balls of radius 1. Applied to the rescaled manifolds λM_i, this corresponds to balls of radius $\frac{1}{\lambda}$ in the unscaled manifolds M_i. Thus, in the following, instead of λ the notation $\frac{1}{r}$ will be used, where $r > 0$ is sufficiently small, and statements about balls of radius r will be obtained. This leads to the following local version of the main theorem, where the choice of notation $\hat{\varepsilon}$ and $\hat{\delta}$—while seemingly artificial—will turn out to be helpful when proving the main theorem by applying the 'local version'.

Proposition 2.1. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a collapsing sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$ and converge to a limit (X, p) of dimension $k < n$. Given $\hat{\varepsilon} \in (0,1)$, there exists $\hat{\delta} = \hat{\delta}(\hat{\varepsilon}; n, k) > 0$ such that for any $0 < r \leq \hat{\delta}$ and $q_i \in M_i$ with*

$$d_{GH}((r^{-1}M_i, q_i), (\mathbb{R}^k, 0)) \leq \hat{\delta}$$

there are a family of subsets of good points $G_r(q_i) \subseteq B_r(q_i)$ with

$$\mathrm{vol}(G_r(q_i)) \geq (1-\hat{\varepsilon}) \cdot \mathrm{vol}(B_r(q_i))$$

and a sequence $\lambda_i \to \infty$ such that the following holds:

a) For every choice of base points $x_i \in G_r(q_i)$ and all sublimits $(Y, \cdot)$ of $(\lambda_i M_i, x_i)$ there exists a compact metric space K of dimension $l \leq n-k$ satisfying $\frac{1}{5} \leq \mathrm{diam}(K) \leq 1$ such that Y splits isometrically as a product

$$Y \cong \mathbb{R}^k \times K.$$

b) If $x_i^1, x_i^2 \in G_r(q_i)$ are base points such that, after passing to a subsequence,

$$(\lambda_i M_i, x_i^j) \to (\mathbb{R}^k \times K_j, \cdot)$$

for $1 \leq j \leq 2$ as before, then $\mathrm{dim}(K_1) = \mathrm{dim}(K_2)$.

The idea of the proof is to construct two families of sets $G_r^1(q_i)$ and $G_r^2(q_i)$, where r is sufficiently small, with the following properties: For any choice of points $x_i \in G_r^1(q_i)$ and for any rescaling sequence $\lambda_i \to \infty$, every (sub)limit of the sequence $(\lambda_i M_i, x_i)$ splits off an $\mathbb{R}^k$-factor. The second family of sets $G_r^2(q_i)$ is constructed together with a rescaling sequence $\lambda_i \to \infty$ such that for all large enough i and any point $x_i \in G_r^2(q_i)$ each single rescaled manifold $(\lambda_i M_i, x_i)$ is close to the product of $\mathbb{R}^k$ and a compact space, where this compact space depends on the choice of the regarded i and the base point x_i. After fixing this sequence $\lambda_i \to \infty$, the intersection of those two sets gives the result.

2.1 Construction of $G_r^1(q_i)$

This section deals with finding families of subsets $G_r^1(q_i) \subseteq B_r^1(q_i)$ such that all blow-ups of M_i with base points from $G_r^1(q_i)$ split off an $\mathbb{R}^k$-factor. Recall that a blow-up is the limit of the sequence of rescaled manifolds $\mu_i M_i$ for a sequence of scales $\mu_i \to \infty$. Thus, the natural question is under which condition such a splitting can be guaranteed.

By modifying certain distance functions, Cheeger and Colding obtained harmonic functions which were used to prove the following splitting theorem.

Theorem 2.2 ([CC96, Theorem 6.64])**.** *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a sequence of pointed n-dimensional Riemannian manifolds and let $R_i \to \infty$ and $\varepsilon_i \to 0$ be sequences of positive real numbers such that $B_{R_i}^{M_i}(p_i)$ has Ricci curvature at least $-(n-1)\cdot\varepsilon_i$. Assume $(B_{R_i}^{M_i}(p_i), p_i)$ to converge to some pointed metric space (Y, y) in the pointed Gromov-Hausdorff sense. If Y contains a line, then Y splits isometrically as $Y \cong \mathbb{R} \times X$.*

Assuming that the limit space already is the Euclidean space $\mathbb{R}^n$ (of the same dimension as the manifolds of the convergent sequence), Colding proved convergence of the volume of balls of radius 1 in the manifolds to the volume of the 1-ball in $\mathbb{R}^n$ by using n of such function (for every manifold), cf. [Col97, Lemma 2.1]. Using both the observations there and the proof of the above splitting theorem, Cheeger and Colding obtained the following statement which is stated as noted in [KW11, Theorem 1.3].

Theorem 2.3 ([CC00a, section 1]). *Let $(M_i, p_i) \to (\mathbb{R}^k, 0)$ be a sequence of pointed n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\operatorname{Ric}_{M_i} \geq -\frac{1}{i}$. Then there exist harmonic functions $b_1^i, \ldots, b_k^i : B_2(p_i) \to \mathbb{R}$ and a constant $C(n) \geq 0$ such that*

a) $|\nabla b_j^i| \leq C(n)$ *for all i and j and*

b) $\fint_{B_1(p_i)} \sum_{j,l=1}^k |\langle \nabla b_j^i, \nabla b_l^i \rangle - \delta_{jl}| + \sum_{j=1}^k \| \operatorname{Hess} b_j^i \|^2 \, dV_{M_i} \to 0$ *as $i \to \infty$.*

Moreover, the maps $\Phi^i = (b_1^i, \ldots, b_k^i) : B_2(p_i) \to \mathbb{R}^k$ provide ε_i-Gromov-Hausdorff approximations between $B_1(p_i)$ and $B_1(0)$ with $\varepsilon_i \to 0$.

Conversely, Kapovitch and Wilking proved the following in [KW11]: If there exist k functions with analogous properties on balls with radius $r_i \to \infty$, then the sublimit splits off an $\mathbb{R}^k$-factor.

Theorem 2.4 (Product Lemma, [KW11, Lemma 2.1]). *Let $(M_i, p_i)_{i \in \mathbb{N}}$ be a pointed sequence of n-dimensional manifolds with $\operatorname{Ric}_{M_i} > -\varepsilon_i$ for a sequence $\varepsilon_i \to 0$ and let $r_i \to \infty$ such that $\bar{B}_{r_i}(p_i)$ is compact for all $i \in \mathbb{N}$. Assume for every $i \in \mathbb{N}$ and $1 \leq j \leq k$ there are harmonic functions $b_j^i : B_{r_i}(p_i) \to \mathbb{R}$ which are L-Lipschitz and fulfil*

$$\fint_{B_R(p_i)} \sum_{j,l=1}^k |\langle \nabla b_j^i, \nabla b_l^i \rangle - \delta_{jl}| + \sum_{j=1}^k \| \operatorname{Hess} b_j^i \|^2 \, dV_{M_i} \to 0 \text{ for all } R > 0.$$

Then $(B_{r_i}(p_i), p_i)$ subconverges in the pointed Gromov-Hausdorff sense to a metric product $(\mathbb{R}^k \times X, p_\infty)$ for some metric space X. Moreover, $(b_1^i, \ldots, b_k^i)$ converges to the projection onto the Euclidean factor.

The above theorems will be generalised to the following statements: If all manifolds are sufficiently close to $\mathbb{R}^k$, then there exist harmonic functions similarly to Theorem 2.3 such that the average integral does not converge to zero but only is bounded, cf. Lemma 2.5. Consequently, an adaptation of the Product Lemma will be established: Under the following weaker hypothesis, the same conclusion holds, cf. Lemma 2.7: Only the average integral about the norms of the Hessian vanishes when passing to the limit whereas the average integrals about the scalar products of the gradients are bounded by a small constant.

First, maps similar to those in Theorem 2.3 will be constructed. A crucial step of the proof will be the rescaling of maps.

Lemma 2.5. *Given $n \in \mathbb{N}$, there exists $L = L(n) \geq 0$ such that the following holds: For arbitrary $\hat{\varepsilon} > 0$, $R > 0$, $k \leq n$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ with $\lim_{x\to 0} g(x) = 0$ there exists $\hat{\delta} = \hat{\delta}(\hat{\varepsilon}, g, R; n, k) \in (0,1)$ such that the following holds for every $\delta \leq \hat{\delta}$: For every pointed complete connected n-dimensional Riemannian manifold (M,p) with $\mathrm{Ric}_M \geq -(n-1)\cdot\delta^2$ and*

$$d_{GH}((M,p),(\mathbb{R}^k,0)) \leq g(\delta)$$

there exist harmonic functions $f_1,\ldots,f_k : B_R^M(p) \to \mathbb{R}$ such that $|\nabla f_j| \leq L$ and

$$\fint_{B_R^M(p)} \sum_{j,l=1}^k |\langle \nabla f_j, \nabla f_l\rangle - \delta_{jl}| + \sum_{j=1}^k \|\operatorname{Hess}(f_j)\|^2 \, dV_M < \hat{\varepsilon}.$$

Proof. Let $L := C(n)$ be the constant of Theorem 2.3. The proof is done by contradiction: Assume the statement is false and let $\hat{\varepsilon}$, R, k and g be contradicting. For every $i \in \mathbb{N}$, let $\hat{\delta}_i := \frac{1}{\sqrt{i(n-1)}} \in (0,1)$ and choose the contradicting $\delta_i \leq \hat{\delta}_i$ and (M_i,p_i) with

$$\mathrm{Ric}_{M_i} \geq -(n-1)\cdot\delta_i^2 \geq -\frac{1}{i} \quad \text{and} \quad d_{GH}((M_i,p_i),(\mathbb{R}^k,0)) \leq g(\delta_i)$$

such that for all harmonic Lipschitz maps $f_1^i,\ldots,f_k^i : B_1^{M_i}(p_i) \to \mathbb{R}$ with $|\nabla f_j^i| \leq L$,

$$\fint_{B_1^{M_i}(p_i)} \sum_{j,l=1}^k |\langle \nabla f_j^i, \nabla f_l^i\rangle - \delta_{jl}| + \sum_{j=1}^k \|\operatorname{Hess}(f_j^i)\|^2 \, dV_{M_i} \geq \hat{\varepsilon}.$$

Since $g(\delta_i) \to 0$ as $i \to \infty$, by Proposition 1.8 b), one has $(M_i,p_i) \to (\mathbb{R}^k,0)$, and so $(\frac{1}{R}M_i,p_i) \to (\mathbb{R}^k,0)$ as well. By Theorem 2.3, there exist harmonic functions

$$\tilde{f}_1^i,\ldots,\tilde{f}_k^i : B_1^{R^{-1}M_i}(p_i) \to \mathbb{R}$$

with $|\nabla \tilde{f}_j^i| \leq L$ satisfying

$$\fint_{B_1^{R^{-1}M_i}(p_i)} \sum_{j,l=1}^k |\langle \nabla \tilde{f}_j^i, \nabla \tilde{f}_l^i\rangle - \delta_{jl}| + \sum_{j=1}^k \|\operatorname{Hess}(\tilde{f}_j^i)\|^2 \, dV_{R^{-1}M_i} \to 0 \text{ as } i \to \infty.$$

In particular, for $f_j^i := R\cdot\tilde{f}_j^i : B_R^{M_i} \to \mathbb{R}$, one has $|\nabla f_j^i| \leq L$ and

$$\begin{aligned}
&\fint_{B_R^{M_i}(p_i)} \sum_{j,l=1}^k |\langle \nabla f_j^i, \nabla f_l^i\rangle - \delta_{jl}| + \sum_{j=1}^k \|\operatorname{Hess}(f_j^i)\|^2 \, dV_{M_i} \\
&= \fint_{B_1^{R^{-1}M_i}(p_i)} \sum_{j,l=1}^k |\langle \nabla \tilde{f}_j^i, \nabla \tilde{f}_l^i\rangle - \delta_{jl}| + \frac{1}{R^2}\cdot\sum_{j=1}^k \|\operatorname{Hess}(\tilde{f}_j^i)\|^2 \, dV_{M_i} \\
&\leq \left(1+\frac{1}{R^2}\right)\cdot\left(\fint_{B_1^{R^{-1}M_i}(p_i)} \sum_{j,l=1}^k |\langle \nabla \tilde{f}_j^i, \nabla \tilde{f}_l^i\rangle - \delta_{jl}| + \sum_{j=1}^k \|\operatorname{Hess}(\tilde{f}_j^i)\|^2 \, dV_{M_i}\right) \\
&\to 0 \text{ as } i \to \infty,
\end{aligned}$$

cf. Lemma 1.4. This is a contradiction. □

In order to generalise the Product Lemma, the following result of Cheeger and Colding is used. Again, the theorem is stated using the notation of [KW11, Theorem 1.5].

Theorem 2.6 (Segment Inequality, [CC96, Theorem 2.11])**.** *Given any dimension $n \in \mathbb{N}$ and radius $r_0 > 0$, there exists $\tau = \tau(n, r_0)$ such that the following holds: Let M be an n-dimensional Riemannian manifold which satisfies the lower Ricci curvature bound $\mathrm{Ric}_M \geq -(n-1)$ and $g : M \to \mathbb{R}^+$ be a non-negative function. Then for $r \leq r_0$,*

$$\fint_{B_r(p)\times B_r(p)} \int_0^{d(z_1,z_2)} g(\gamma_{z_1,z_2}(t))\, dt\, dV(z_1, z_2) \leq \tau \cdot r \cdot \fint_{B_{2r}(p)} g(q)\, dV(q),$$

where γ_{z_1,z_2} denotes a minimising geodesic from z_1 to z_2.

The following lemma is a generalisation of the Product Lemma where the average integral of scalar products of the gradients does not have to vanish, but only needs to be bounded.

Lemma 2.7. *Let $(M_i)_{i\in\mathbb{N}}$ be a sequence of connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)\cdot\varepsilon_i$ where $\varepsilon_i \to 0$. Let $r_i \to \infty$ and $q_i \in M_i$ be points such that the balls $\bar{B}_{r_i}(q_i)$ are compact. Furthermore, let $k \leq n$ and assume that for every $1 \leq j \leq k$ there is a harmonic L-Lipschitz map $b^i_j : B_{r_i}(q_i) \to \mathbb{R}$ satisfying*

$$\fint_{B_r(q_i)} \sum_{j=1}^{k} \|\operatorname{Hess}(b^i_j)\|^2\, dV \to 0 \quad \textit{and}$$

$$\fint_{B_r(q_i)} \sum_{j,l=1}^{k} |\langle \nabla b^i_j, \nabla b^i_l\rangle - \delta_{jl}|\, dV \leq 10^{-n^2}$$

for all $r \leq r_i$. Then each sublimit of $(B_{r_i}(q_i), q_i)$ is isometric to a product $(\mathbb{R}^k \times X, q_\infty)$ for some metric space X and some point $q_\infty \in \mathbb{R}^k \times X$.

Proof. Let (Y, y) be an arbitrary sublimit of $(B_{r_i}(q_i), q_i)$. Without loss of generality, assume convergence $(B_{r_i}(q_i), q_i) \to (Y, y)$. The concept of the proof is the following: For well chosen $\hat{q}_i$ and $c^i_{jl} := \langle \nabla b^i_j, \nabla b^i_l\rangle(\hat{q}_i)$, one gets $\fint_{B_1(q_i)} |\langle \nabla b^i_j, \nabla b^i_l\rangle - c^i_{jl}|\, dV \to 0$. In the second step, the corresponding statement for balls of arbitrary radius will be shown. Finally, after passing to a subsequence such that every $(c^i_{jl})_{i\in\mathbb{N}}$ converges to some limit c_{jl} and defining h_{jl} via the identity $\big((h_{jl})_{1\leq j,l\leq k}\big)^2 = \big((c_{jl})_{1\leq j,l\leq k}\big)^{-1}$, the linear combinations $d^i_j := \sum_{l=1}^{k} h_{jl} b^i_l$ satisfy the hypothesis of the Product Lemma, and thus, prove the claim.

a) Fix $1 \leq j, l \leq k$. This step provides $c^i_{jl} \in \mathbb{R}$ satisfying

$$\fint_{B_1(q_i)} |\langle \nabla b^i_j, \nabla b^i_l\rangle - c^i_{jl}|\, dV \to 0.$$

First, fix $R > 0$ and let $i_0 \in \mathbb{N}$ be large enough such that for all $i \geq i_0$, both $r_i \geq 2R$ and $\operatorname{Ric}_{M_i} \geq -(n-1) \cdot \varepsilon_i > -(n-1) \cdot 2\,\varepsilon_i \to 0$. Let $i \geq i_0$. Given $p, q \in B_R(q_i)$ such that q is not in the cut locus of p (recall that the set of these points has full measure), let $\gamma_{pq} : [0,1] \to M_i$ denote a minimising geodesic connecting p and q. Then the Segment Inequality provides $\tau = \tau(n, R)$ such that

$$\begin{aligned}
&\fint_{B_R(q_i)\times B_R(q_i)} \int_0^1 (\|\operatorname{Hess}(b^i_j)\| + \|\operatorname{Hess}(b^i_l)\|)(\gamma_{pq}(t))\,dt\,dV(q)\\
&\leq \tau \cdot R \cdot \fint_{B_{2R}(q_i)} (\|\operatorname{Hess}(b^i_j)\| + \|\operatorname{Hess}(b^i_l)\|)\,dV\\
&\to 0 \text{ as } i \to \infty
\end{aligned}$$

using that, by applying the hypothesis to $r = 2R \leq r_i$ for $i \geq i_0$,

$$\begin{aligned}
&\Big(\fint_{B_{2R}(q_i)} (\|\operatorname{Hess}(b^i_j)\| + \|\operatorname{Hess}(b^i_l)\|)\,dV\Big)^2\\
&\leq \fint_{B_{2R}(q_i)} (\|\operatorname{Hess}(b^i_j)\| + \|\operatorname{Hess}(b^i_l)\|)^2\,dV\\
&\leq 2 \cdot \fint_{B_{2R}(q_i)} \|\operatorname{Hess}(b^i_j)\|^2 + \|\operatorname{Hess}(b^i_l)\|^2\,dV\\
&\leq 4 \cdot \fint_{B_{2R}(q_i)} \sum_{j=1}^{k} \|\operatorname{Hess}(b^i_j)\|^2\,dV\\
&\to 0 \text{ as } i \to \infty.
\end{aligned}$$

Now prove the statement for radius 1: Suppose there exists $\varepsilon > 0$ such that for every $N \geq i_0$ there is $i \geq N$ with

$$\fint_{B_1(q_i)} \int_0^1 (\|\operatorname{Hess}(b^i_j)\| + \|\operatorname{Hess}(b^i_l)\|)(\gamma_{\hat{q}_i q}(t))\,dt\,dV(q) \geq \varepsilon$$

for all $\hat{q}_i \in B_{1/2}(q_i)$. For such an i, estimating the average value on the 1-ball by the one on the $\frac{1}{2}$-ball using $\operatorname{vol}(B_{1/2}(q_i)) \geq c_1(n) \cdot \operatorname{vol}(B_1(q_i))$ for $c_1(n) := \frac{1}{C_{BG}(n,-1,\frac{1}{2},1)}$ gives

$$\begin{aligned}
&\fint_{B_1(q_i)\times B_1(q_i)} \int_0^1 (\|\operatorname{Hess}(b^i_j)\| + \|\operatorname{Hess}(b^i_l)\|)(\gamma_{\hat{q}_i q}(t))\,dt\,dV(\hat{q}_i, q)\\
&\geq \frac{\operatorname{vol}(B_{1/2}(q_i))}{\operatorname{vol}(B_1(q_i))} \cdot \fint_{B_{1/2}(q_i)\times B_1(q_i)} \int_0^1 (\|\operatorname{Hess}(b^i_j)\| + \|\operatorname{Hess}(b^i_l)\|)(\gamma_{\hat{q}_i q}(t))\,dt\,dV(\hat{q}_i, q)
\end{aligned}$$

$$\geq c_1(n) \cdot \fint_{B_{1/2}(q_i)\times B_1(q_i)} \int_0^1 (\|\operatorname{Hess}(b_j^i)\| + \|\operatorname{Hess}(b_l^i)\|)(\gamma_{\hat{q}_i q}(t))\, dt\, dV(\hat{q}_i, q)$$
$$\geq c_1(n) \cdot \fint_{B_{1/2}(q_i)} \varepsilon\, dV(\hat{q}_i)$$
$$= c_1(n) \cdot \varepsilon > 0,$$

and this is a contradiction. Thus, for $i \geq i_0$, there exists $\hat{q}_i \in B_{1/2}(q_i)$ with

$$\fint_{B_1(q_i)} \int_0^1 (\|\operatorname{Hess}(b_j^i)\| + \|\operatorname{Hess}(b_l^i)\|)(\gamma_{\hat{q}_i q}(t))\, dt\, dV(q) \to 0.$$

Using $\|\dot{\gamma}_{\hat{q}_i,q}(\tau)\| = d(\hat{q}_i, q) \leq \frac{3}{2}$ due to $q \in B_1(q_i)$ and $\hat{q}_i \in B_{1/2}(q_i)$,

$$\Big|\langle \frac{d}{dt}_{|t=\tau} \nabla b_j^i(\gamma_{\hat{q}_i q}(t)), \nabla b_l^i(\gamma_{\hat{q}_i q}(\tau))\rangle\Big|$$
$$= |\langle \operatorname{Hess}(b_j^i) \cdot \dot{\gamma}_{\hat{q}_i q}(\tau), \nabla b_l^i\rangle(\gamma_{\hat{q}_i q}(\tau))|$$
$$\leq \|\operatorname{Hess}(b_j^i) \cdot \dot{\gamma}_{\hat{q}_i q}(\tau)\|(\gamma_{\hat{q}_i q}(\tau)) \cdot \|\nabla b_l^i\|(\gamma_{\hat{q}_i q}(\tau))$$
$$\leq \|\operatorname{Hess}(b_j^i)\|(\gamma_{\hat{q}_i q}(\tau)) \cdot \|\dot{\gamma}_{\hat{q}_i q}(\tau)\|(\gamma_{\hat{q}_i q}(\tau)) \cdot L$$
$$\leq \frac{3L}{2} \cdot \|\operatorname{Hess}(b_j^i)\|(\gamma_{\hat{q}_i q}(\tau))$$

for each $i \in \mathbb{N}$. Thus, for $c_{jl}^i := \langle \nabla b_j^i, \nabla b_l^i\rangle(\hat{q}_i)$,

$$\fint_{B_1(q_i)} |\langle \nabla b_j^i, \nabla b_l^i\rangle(q) - c_{jl}^i|\, dV(q)$$
$$= \fint_{B_1(q_i)} \Big| \int_0^1 \frac{d}{dt}_{|t=\tau} \langle \nabla b_j^i, \nabla b_l^i\rangle \circ \gamma_{\hat{q}_i q}(t)\, d\tau \Big|\, dV(q)$$
$$\leq \fint_{B_1(q_i)} \int_0^1 \Big| \frac{d}{dt}_{|t=\tau} \langle \nabla b_j^i(\gamma_{\hat{q}_i q}(t)), \nabla b_l^i(\gamma_{\hat{q}_i q}(t))\rangle \Big|\, d\tau\, dV(q)$$
$$= \fint_{B_1(q_i)} \int_0^1 \Big| \langle \frac{d}{dt}_{|t=\tau} \nabla b_j^i(\gamma_{\hat{q}_i q}(t)), \nabla b_l^i(\gamma_{\hat{q}_i q}(t))\rangle$$
$$+ \langle \nabla b_j^i(\gamma_{\hat{q}_i q}(t)), \frac{d}{dt}_{|t=\tau} \nabla b_l^i(\gamma_{\hat{q}_i q}(t))\rangle \Big|\, d\tau\, dV(q)$$
$$= \frac{3L}{2} \cdot \fint_{B_1(q_i)} \int_0^1 (\|\operatorname{Hess}(b_j^i)\| + \|\operatorname{Hess}(b_l^i)\|) \circ \gamma_{\hat{q}_i q}(\tau)\, d\tau\, dV(q)$$
$$\to 0 \text{ as } i \to \infty.$$

b) First, let $R > 0$ be arbitrary and prove

$$\fint_{B_R(q_i)} |\langle \nabla b_j^i, \nabla b_l^i\rangle - c_{jl}^i|\, dV \to 0.$$

For $R \geq 1$, $\frac{\mathrm{vol}(B_R(q_i))}{\mathrm{vol}(B_1(q_i))} \leq C_{BG}(n, -\varepsilon_i, 1, R) \leq C_{BG}(n, -1, 1, R) =: c_2(n, R)$y for i large enough. Thus, as in a),

$$\begin{aligned}
&\fint_{B_1(q_i)\times B_R(q_i)} \int_0^1 (\|\operatorname{Hess}(b_j^i)\| + \|\operatorname{Hess}(b_l^i)\|) \circ \gamma_{pq}(t)\, dt\, dV(p,q) \\
&\leq \frac{\mathrm{vol}(B_R(q_i))}{\mathrm{vol}(B_1(q_i))} \cdot \fint_{B_R(q_i)\times B_R(q_i)} \int_0^1 (\|\operatorname{Hess}(b_j^i)\| + \|\operatorname{Hess}(b_l^i)\|) \circ \gamma_{pq}(t)\, dt\, dV(p,q) \\
&\leq c_2(n,R) \cdot \fint_{B_R(q_i)\times B_R(q_i)} \int_0^1 (\|\operatorname{Hess}(b_j^i)\| + \|\operatorname{Hess}(b_l^i)\|) \circ \gamma_{pq}(t)\, dt\, dV(p,q) \\
&\to 0 \text{ as } i \to \infty.
\end{aligned}$$

If $R \leq 1$, using $\frac{\mathrm{vol}(B_1(q_i))}{\mathrm{vol}(B_R(q_i))} \leq C_{BG}(n, -1, R, 1) =: c_3(n, R)$ for sufficiently large i,

$$\begin{aligned}
&\fint_{B_1(q_i)\times B_R(q_i)} \int_0^1 (\|\operatorname{Hess}(b_j^i)\| + \|\operatorname{Hess}(b_l^i)\|) \circ \gamma_{pq}(t)\, dt\, dV(p,q) \\
&\leq c_3(n,R) \cdot \fint_{B_1(q_i)\times B_1(q_i)} \int_0^1 (\|\operatorname{Hess}(b_j^i)\| + \|\operatorname{Hess}(b_l^i)\|) \circ \gamma_{pq}(t)\, dt\, dV(p,q) \\
&\to 0 \text{ as } i \to \infty.
\end{aligned}$$

Similarly to a), a $\bar{q}_i \in B_1(q_i)$ is required satisfying both

$$\begin{aligned}
&\tilde{\varepsilon}_i := |\langle \nabla b_j^i, \nabla b_l^i\rangle(\bar{q}_i) - c_{jl}^i| \to 0 \text{ as } i \to \infty \quad \text{and} \\
&\fint_{B_R(q_i)} \int_0^1 (\|\operatorname{Hess}(b_j^i)\| + \|\operatorname{Hess}(b_l^i)\|)(\gamma_{\bar{q}_i q}(t))\, dt\, dV(q) \to 0 \text{ as } i \to \infty.
\end{aligned}$$

Assume no such $\bar{q}_i$ exists, i.e. there are $\varepsilon > 0$ and $\varepsilon' > 0$ such that for any $N \in \mathbb{N}$ there is an $i \geq N$ such that each $\bar{q}_i \in B_1(q_i)$ satisfies $|\langle \nabla b_j^i, \nabla b_l^i\rangle(\bar{q}_i) - c_{jl}^i| \geq \varepsilon$ or $\fint_{B_R(q_i)} \int_0^1 (\|\operatorname{Hess}(b_j^i)\| + \|\operatorname{Hess}(b_l^i)\|)(\gamma_{\bar{q}_i q}(t))\, dt\, dV(q) \geq \varepsilon'$. Assume the first condition is satisfied for infinitely many i. Then the corresponding subsequence is bounded from below by ε which is a contradiction to the sequence (of the average values of these terms) converging to 0. So the second condition must hold for infinitely many i. Then the corresponding subsequence is bounded from below by ε' in contradiction to the sequence converging to 0. This proves the existence of such a $\bar{q}_i$.

For this $\bar{q}_i$, as in the first step,

$$\begin{aligned}
&\fint_{B_R(q_i)} |\langle \nabla b_j^i, \nabla b_l^i\rangle(q) - c_{jl}^i|\, dV(q) \\
&\leq \fint_{B_R(q_i)} \int_0^1 \left| \frac{d}{dt}_{t=\tau} \langle \nabla b_j^i, \nabla b_l^i\rangle \circ \gamma_{\bar{q}_i q}(t) \right| + \left| \langle \nabla b_j^i, \nabla b_l^i\rangle(\bar{q}_i) - c_{jl}^i \right| d\tau\, dV(q)
\end{aligned}$$

$$\leq (1+R)\cdot L \cdot \fint_{B_R(q_i)} \int_0^1 (\| \operatorname{Hess}(b_j^i)\| + \| \operatorname{Hess}(b_l^i)\|) \circ \gamma_{\bar{q}_i q}(t)\, dt\, dV(q) + \tilde{\varepsilon}_i$$

$$\to 0 \text{ as } i \to \infty.$$

Thus, $\varepsilon_{jl}^i(q) := (\langle \nabla b_j^i, \nabla b_l^i\rangle - c_{jl}^i)(q)$ satisfies

$$\fint_{B_R(q_i)} | \varepsilon_{jl}^i |\, dV \to 0 \text{ as } i \to \infty.$$

c) Define a linear combination of the b_j^i that satisfies the hypothesis of the Product Lemma after passing to a subsequence:

As the b_j^i are L-Lipschitz, $c_{jl}^i := \langle \nabla b_j^i, \nabla b_l^i\rangle(\hat{q}_i) \in [-L^2, L^2]$ is a bounded sequence, and thus, has a convergent subsequence. Pass to the subsequence such that all $(c_{jl}^i)_{i\in\mathbb{N}}$ converge and denote the limits by $c_{jl} := \lim_{i\to\infty} c_{jl}^i \in [-L^2, L^2]$. Then

$$|c_{jl} - \delta_{jl}| = \lim_{i\to\infty} |c_{jl}^i - \delta_{jl}| \leq 10^{-n^2}$$

since

$$\begin{aligned} |c_{jl}^i - \delta_{jl}| &= \fint_{B_R(q_i)} |c_{jl}^i - \langle \nabla b_j^i, \nabla b_l^i\rangle + \langle \nabla b_j^i, \nabla b_l^i\rangle - \delta_{jl}|\, dV \\ &\leq \fint_{B_R(q_i)} |c_{jl}^i - \langle \nabla b_j^i, \nabla b_l^i\rangle|\, dV + \fint_{B_R(q_i)} |\langle \nabla b_j^i, \nabla b_l^i\rangle - \delta_{jl}|\, dV \end{aligned}$$

where the first summand converges to 0 and the second satisfies

$$\fint_{B_R(q_i)} |\langle \nabla b_j^i, \nabla b_l^i\rangle - \delta_{jl}|\, dV \leq \fint_{B_R(q_i)} \sum_{j,l=1}^k |\langle \nabla b_j^i, \nabla b_l^i\rangle - \delta_{jl}|\, dV \leq 10^{-n^2}.$$

Hence, the matrix $C := (c_{jl})_{1\leq j,l\leq k}$ is invertible, symmetric and positive definite. In particular, its inverse C^{-1} is diagonalisable with positive eigenvalues. Let C_D^{-1} denote the diagonal matrix and S the invertible matrix with $C^{-1} = S\cdot C_D^{-1}\cdot S^{-1}$ and define $C_D^{-1/2}$ as the diagonal matrix whose entries are the square roots of the diagonal entries of C_D^{-1}. Then $H := (h_{jl})_{jl} := S\cdot C_D^{-1/2}\cdot S^{-1}$ satisfies $H^2 = C^{-1}$. Now define $d_j^i := \sum_{l=1}^k h_{jl} b_l^i$. Obviously, these are Lipschitz and harmonic. Furthermore, they satisfy $\nabla d_j^i = \sum_{l=1}^k h_{jl} \nabla b_l^i$, and thus,

$$\begin{aligned} \langle \nabla d_{j_1}^i, \nabla d_{j_2}^i\rangle &= \sum_{l_1,l_2=1}^k h_{j_1 l_1} h_{j_2 l_2} \langle \nabla b_{l_1}^i, \nabla b_{l_2}^i\rangle = \sum_{l_1,l_2=1}^k h_{j_1 l_1} h_{j_2 l_2} (c_{l_1 l_2} + \varepsilon_{l_1 l_2}^i) \\ &= \delta_{j_1 j_2} + \sum_{l_1,l_2=1}^k h_{j_1 l_1} h_{j_2 l_2}\, \varepsilon_{l_1 l_2}^i \end{aligned}$$

due to

$$\sum_{l_1,l_2=1}^{k} h_{j_1l_1}h_{j_2l_2}c_{l_1l_2} = \sum_{l_1=1}^{k} h_{j_1l_1} \sum_{l_2=1}^{k} c_{l_1l_2}h_{l_2j_2} = \sum_{l_1=1}^{k} h_{j_1l_1}(C\cdot H)_{l_1j_2}$$
$$= (H\cdot C\cdot H)_{j_1j_2} = \delta_{j_1j_2}.$$

Thus,

$$\fint_{B_R(q_i)} \sum_{j_1,j_2=1}^{k} \left|\langle \nabla d^i_{j_1}, \nabla d^i_{j_2}\rangle - \delta_{j_1j_2}\right| dV$$
$$= \fint_{B_R(q_i)} \sum_{j_1,j_2=1}^{k} \left| \sum_{l_1,l_2=1}^{k} h_{j_1l_1}h_{j_2l_2}\, \varepsilon^i_{l_1l_2} \right| dV$$
$$\le \sum_{j_1,j_2,l_1,l_2=1}^{k} h_{j_1l_1}h_{j_2l_2} \fint_{B_R(q_i)} \left| \varepsilon^i_{j_1j_2} \right| dV$$
$$\to 0 \text{ as } i\to\infty.$$

Similarly,

$$\|\operatorname{Hess}(d^i_j)\|^2 = \Big\| \sum_{l=1}^{k} h_{jl}\cdot \operatorname{Hess}(b^i_l)\Big\|^2$$
$$= \sum_{l_1,l_2=1}^{k} h_{jl_1}h_{jl_2}\cdot \langle \operatorname{Hess}(b^i_{l_1}), \operatorname{Hess}(b^i_{l_2})\rangle$$
$$\le \sum_{l_1,l_2=1}^{k} h_{jl_1}h_{jl_2}\cdot \|\operatorname{Hess}(b^i_{l_1})\|\cdot\|\operatorname{Hess}(b^i_{l_2})\|$$
$$\le \frac{1}{2}\cdot \sum_{l_1,l_2=1}^{k} h_{jl_1}h_{jl_2}\cdot (\|\operatorname{Hess}(b^i_{l_1})\|^2 + \|\operatorname{Hess}(b^i_{l_2})\|^2)$$
$$\le \hat{h}\cdot \sum_{l=1}^{k} \|\operatorname{Hess}(b^i_l)\|^2$$

for $\hat{h} := \max\{h^2_{jl} \mid 1\le j,l\le k\}$. Finally,

$$\fint_{B_r(q_i)} \sum_{j=1}^{k} \|\operatorname{Hess}(d^i_j)\|^2\, dV$$
$$\le k\cdot\hat{h}\cdot \fint_{B_r(q_i)} \sum_{l=1}^{k} \|\operatorname{Hess}(b^i_l)\|^2\, dV$$
$$\to 0 \text{ as } i\to\infty.$$

Therefore, the hypothesis of the Product Lemma is satisfied, and, after passing to a subsequence, $(B_{r_i}(q_i), q_i)$ converges to $(\mathbb{R}^k \times X, (0, q_\infty))$ for some metric space X and $q_\infty \in X$. Since $(B_{r_i}(q_i), q_i)$ converges to (Y, y), this proves that Y is isometric to $\mathbb{R}^k \times X$. □

Applying the previous two lemmata proves that for sufficiently small balls there is a subset good base points of arbitrarily good volume such that all sublimits of sequences with respect to those base points split off an $\mathbb{R}^k$-factor. In order to verify this, the following statement, which in its first form was proven by Stein in [Ste93, p. 13], is needed in order to estimate the volume of a set where the so called ρ-maximum function is bounded from above. This statement will be useful later on as well. Again, the notation of [KW11, Lemma 1.4b)] is used.

Theorem 2.8 (Weak type 1-1 inequality)**.** *Let M be an n-dimensional Riemannian manifold satisfying the lower Ricci curvature bound $\mathrm{Ric}_M \geq -(n-1)$. For a non-negative function $f : M \to \mathbb{R}$ and $\rho > 0$, define the* ρ-maximum function of f *as*

$$\mathrm{Mx}_\rho\, f(p) := \sup_{r \leq \rho} \fint_{B_r(p)} f.$$

Especially, put $\mathrm{Mx}\, f(p) = \mathrm{Mx}_2\, f(p)$.

Then there is $C_{1\text{-}1}(n) > 0$ such that for any non-negative function $f \in L^1(M)$ and $c > 0$,

$$\mathrm{vol}(\{x \in M \mid \mathrm{Mx}_\rho\, f(x) > c\}) \leq \frac{C_{1\text{-}1}(n)}{c} \int_M f\, dV_M\,.$$

Using this, the first set needed for Proposition 2.1 can be constructed.

Lemma 2.9. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a collapsing sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$ and converge to a limit (X, p) of dimension $k < n$. For every $\hat{\varepsilon} \in (0, 1)$ there exists $\hat{\delta}_1 = \hat{\delta}_1(\hat{\varepsilon}; n, k) > 0$ such that for all $0 < r \leq \hat{\delta}_1$ and $q_i \in M_i$ with*

$$d_{GH}((r^{-1}M_i, q_i), (\mathbb{R}^k, 0)) \leq \hat{\delta}_1$$

there is a family of subsets of good points $G_r^1(q_i) \subseteq B_r(q_i)$ satisfying

$$\mathrm{vol}_{M_i}(G_r^1(q_i)) \geq (1 - \hat{\varepsilon}) \cdot \mathrm{vol}_{M_i}(B_r(q_i)) \quad \textit{and} \quad G_r^1(q_i) \subseteq \tilde{G}_i$$

where

$$\tilde{G}_i := \{q_i \in M_i \mid \textit{for all } \lambda_i \to \infty \textit{ and all sublimits } (Y, \cdot) \textit{ of } (\lambda_i M_i, q_i) \\ \textit{there exists } X \textit{ such that } Y \cong \mathbb{R}^k \times X \textit{ isometrically}\}.$$

Proof. This follows from Lemma 2.5 and Lemma 2.7 for well chosen $\hat{\delta} > 0$: For the constant $C_{1\text{-}1}(n)$ from Theorem 2.8, define $C(n) := C_{1\text{-}1}(n) \cdot 10^{n^2} \cdot C_{BG}(n, -1, 1, 2)$, and let $\hat{\delta}_1 = \hat{\delta}_1(\hat{\varepsilon}; n, k)$ be the constant $\hat{\delta}(\frac{\hat{\varepsilon}}{C(n)}, \mathrm{id}, 2; n, k)$ and $L(n)$ be as in Lemma 2.5. Let $0 < r \leq \hat{\delta}_1$ and $q_i \in M_i$ with

$$d_{GH}((r^{-1}M_i, q_i), (\mathbb{R}^k, 0)) \leq \hat{\delta}_1.$$

Then there exist harmonic and L-Lipschitz functions $f^i_j : B_2^{r^{-1}M_i} \to \mathbb{R}$ satisfying

$$\fint_{B_2^{r^{-1}M_i}(q_i)} \psi_\nabla(f^i) + \psi_H(f^i)\, dV_{r^{-1}M_i} \leq \frac{\hat{\varepsilon}}{C(n)}$$

where

$$\psi_\nabla(f^i) := \sum_{j,l=1}^{k} |\langle \nabla f^i_j, \nabla f^i_l \rangle - \delta_{jl}| \quad \text{and} \quad \psi_H(f^i) := \sum_{j=1}^{k} \| \operatorname{Hess}(f^i_j)\|^2.$$

Define

$$G_i := \{q_i \in B_1^{r^{-1}M_i}(q_i) \mid \mathrm{Mx}_1^{r^{-1}M_i}(\psi_\nabla(f^i) + \psi_H(f^i))(q_i) < 10^{-n^2}\}$$

where the 1-maximum function is taken with respect to the metric $d_{r^{-1}M_i} = \frac{1}{r} d_{M_i}$. Using Theorem 2.8, the volume of this set can be estimated by

$$\begin{aligned}
&\mathrm{vol}_{r^{-1}M_i}(B_1^{r^{-1}M_i}(q_i) \setminus G_i) \\
&\leq \frac{C_{1\text{-}1}(n)}{10^{-n^2}} \cdot \int_{B_1^{r^{-1}M_i}(q_i)} \psi_\nabla(f^i) + \psi_H(f^i)\, dV_{r^{-1}M_i} \\
&\leq \frac{C(n)}{C_{BG}(n,-1,1,2)} \cdot \int_{B_2^{r^{-1}M_i}(q_i)} \psi_\nabla(f^i) + \psi_H(f^i)\, dV_{r^{-1}M_i} \\
&\leq \frac{C(n)}{C_{BG}(n,-1,1,2)} \cdot \frac{\hat{\varepsilon}}{C(n)} \cdot \mathrm{vol}_{r^{-1}M_i}(B_2^{r^{-1}M_i}(q_i)) \\
&\leq \frac{\hat{\varepsilon}}{C_{BG}(n,-1,1,2)} \cdot C_{BG}(n,-1,1,2) \cdot \mathrm{vol}_{r^{-1}M_i}(B_1^{r^{-1}M_i}(q_i)) \\
&= \hat{\varepsilon} \cdot \mathrm{vol}_{r^{-1}M_i}(B_1^{r^{-1}M_i}(q_i)).
\end{aligned}$$

Hence, regarding $G^1_r(q_i) := G_i$ as a subset of M_i,

$$\frac{\mathrm{vol}_{M_i}(G^1_r(q_i))}{\mathrm{vol}_{M_i}(B_r^{M_i}(q_i))} = \frac{\mathrm{vol}_{r^{-1}M_i}(G_i)}{\mathrm{vol}_{r^{-1}M_i}(B_1^{r^{-1}M_i}(q_i))} \geq 1 - \hat{\varepsilon}.$$

It remains to prove $G^1_r(q_i) \subseteq \tilde{G}_i$: Let $x_i \in G^1_r(q_i)$ and $\lambda_i \to \infty$ be arbitrary. Define $r_i := \lambda_i \cdot r \to \infty$ and let $0 < \rho \leq r_i$. Since $B_{r_i}^{\lambda_i M_i}(x_i) = B_1^{r^{-1}M_i}(x_i) \subseteq B_2^{r^{-1}M_i}(q_i)$, the maps

$$\tilde{f}^i_j := r_i \cdot f^i_j : B_{r_i}^{\lambda_i M_i}(x_i) \to \mathbb{R}$$

are well defined, harmonic and L-Lipschitz and satisfy (cf. Lemma 1.4)

$$\begin{aligned}
&\fint_{B_\rho^{\lambda_i M_i}(x_i)} \psi_\nabla(\tilde{f}^i)\, dV_{\lambda_i M_i} \\
&= \fint_{B_{\rho/r_i}^{r^{-1}M_i}(x_i)} \psi_\nabla(f^i)\, dV_{r^{-1}M_i} \\
&\leq \mathrm{Mx}_1^{r^{-1}M_i}(\psi_\nabla(f^i) + \psi_H(f^i))(x_i) \\
&\leq 10^{-n^2}
\end{aligned}$$

and

$$\begin{aligned}
&\fint_{B_\rho^{\lambda_i M_i}(x_i)} \psi_H(\tilde{f}^i)\, dV_{\lambda_i M_i} \\
&= \frac{1}{r_i^2} \cdot \fint_{B_{\rho/r_i}^{r^{-1}M_i}(x_i)} \psi_H(f^i)\, dV_{r^{-1}M_i} \\
&\leq \frac{1}{r_i^2} \cdot \mathrm{Mx}_1^{r^{-1}M_i}(\psi_\nabla(f^i) + \psi_H(f^i))(x_i) \\
&\leq \frac{10^{-n^2}}{r_i^2} \to 0 \text{ as } i \to \infty.
\end{aligned}$$

By Lemma 2.7, any sublimit of $(\lambda_i M_i, x_i)$ has the form $(\mathbb{R}^k \times X, \cdot)$ for some metric space X. Thus, $G_r^1(q_i) \subseteq \tilde{G}_i$. □

2.2 Construction of $G_r^2(q_i)$ and λ_i

The aim of this section is to find a rescaling sequence $\lambda_i \to \infty$ and a family of subsets $G_r^2(q_i) \subseteq B_r(q_i)$ with the following two properties: On the one hand, every single rescaled manifold $\lambda_i M_i$ (with a base point from $G_r^2(q_i)$) is close to a product of $\mathbb{R}^k$ and a compact metric space. On the other hand, the sublimits of sequences $(\lambda_i M_i, x_i)$ with base points $x_i \in G_r^2(q_i)$ have the same dimension (depending not on the base points but only on the choice of the subsequence).

Before motivating the procedure, the notion of time-dependent vector fields needs to be introduced. A time-dependent vector field is a generalisation of a vector field on manifolds: In principle, if $X : I \times M \to TM$ is a time-dependent vector field and $t \in I$ a fixed time, then $X^t := X(t, \cdot) : M \to TM$ is a vector field in the usual sense. Since the notions of integral curves and flows require an additional time parameter, transferring these notion to time-dependent vector field is not completely trivial. Therefore, the following definition introduces all these concepts. Recall that, given a vector field in the usual sense, its flow is a 1-parameter family. The subsequent proposition states a corresponding property for time dependent flows. Both the definition and the proposition are essentially (up to different notation) taken from [Lee03, p. 451 f.].

Definition 2.10. Let (M, g) be a Riemannian manifold and $I \subseteq \mathbb{R}$ be an interval.

a) A continuous map $X : I \times M \to TM$ is called a *time-dependent vector field* if

$$X_p^t := X^t(p) := X(t, p) \in T_pM$$

for all $(t, p) \in I \times M$, i.e. X^t is a vector field for all $t \in I$.

b) A time-dependent vector field $X : I \times M \to TM$ is called *piecewise constant in time* if there exist disjoint sub-intervals $I = I_1 \amalg \ldots \amalg I_n$ such that for all $1 \leq i \leq n$ and all $s, t \in I_i$, $X^s = X^t$.

c) For arbitrary $s \in I$ and for $I - s := \{\tau - s \mid \tau \in I\}$, a curve $c : I - s \to M$ is called *s-integral curve of X* if

$$c'(t) = X_{c(t)}^{s+t}$$

for all $t \in I - s$. A 0-integral curve is also called *integral curve of X*.

d) There exists an open set

$$\Omega \subseteq \bigcup_{s \in I} \{s\} \times (I - s) \times M$$

and a map $\Phi : \Omega \to M$ such that for any $(s, p) \in I \times M$ the set

$$\Omega^{(s,p)} := \{t \in I - s \mid (s, t, p) \in \Omega\}$$

is an open interval which contains 0, and for any fixed $(s, p) \in I \times M$ the map $c : \Omega^{(s,p)} \to M$ defined by $c(t) := \Phi(s, t, p)$ is the unique maximal s-integral curve of X with starting point p. Using the notation $\varphi_t^s := \Phi(s, t, \cdot)$, this is equivalent to φ being a maximal solution of

$$\frac{d}{dt}_{|t=t_0} \varphi_t^s(p) = X_{\varphi_{t_0}^s(p)}^{s+t_0} \quad \text{and} \quad \varphi_0^s = \mathrm{id}\,.$$

Such a Φ is called *flow of X*.

e) A time-dependent vector field X has *compact support* if there exists a compact set $K \subseteq M$ such that for all $t \in I$ the vector fields X^t have support K. In this case, the flow Φ exists for all times.

Proposition 2.11. *For a time-dependent vector field $X : I \times M \to TM$ and its flow $\Phi : \Omega \to M$, denote $\varphi_t^s := \Phi(s, t, \cdot)$ as before and let $p \in M$ and s, t, u be times such that $(s, t, p) \in \Omega$ and $(s + t, u, \varphi_t^s(p)) \in \Omega$. Then $(s, t + u, p) \in \Omega$ and*

$$\varphi_u^{s+t} \circ \varphi_t^s(p) = \varphi_{t+u}^s(p).$$

In particular, if defined, φ_{-t}^{s+t} is the inverse of φ_t^s.

In order to prove that two blow-ups have the same dimension, the following will be established and used: Let $X_i : [0,1] \times M_i \to TM_i$ be time-dependent vector fields with $\int_0^1 (\mathrm{Mx}_{2r}(\|\nabla.X_i^t\|^{3/2})(c_i(t)))^{2/3}\, dt < \hat{\varepsilon}$ for all $i \in \mathbb{N}$ where $\hat{\varepsilon} > 0$ and the c_i are (0-)integral curves. Moreover, let the X_i be divergence free, i.e. the flows are measure preserving. Then any blow-ups coming from the sequences with base points $c_i(0)$ and $c_i(1)$, respectively, have the same dimension, i.e. if $\lambda_i \to \infty$ with

$$(\lambda_i M_i, c_i(0)) \to (Y_0, y_0) \quad \text{and} \quad (\lambda_i M_i, c_i(1)) \to (Y_1, y_1),$$

then $\dim(Y_0) = \dim(Y_1)$. This will be proven in section 2.3.

Since Gromov-Hausdorff convergence is preserved by shifting base points a little bit, the same statement is true if the base points $c_i(0)$ and $c_i(1)$, respectively, are replaced by points x_i and y_i, respectively, where $\lambda_i \cdot d(c_i(0), x_i) < C$ and $\lambda_i \cdot d(c_i(1), y_i) < C$ for some $C > 0$ (independent of i). This motivates the following definition.

Definition 2.12. Let M be a complete connected n-dimensional Riemannian manifold and $r, C, \hat{\varepsilon} > 0$. A point $q \in M$ has the $\mathcal{C}(M, r, C, \hat{\varepsilon})$-*property* if the following holds:

There is a subset $B_r(q)' \subseteq B_r(q)$ such that for all $x, y \in B_r(q)'$ there exists a time-dependent vector field $X : [0,1] \times M \to TM$ which is piecewise constant in time and has compact support and an integral curve $c : [0,1] \to M$ satisfying the following conditions:

a) The vector field X^t is divergence free on $B_{10r}(c(t))$ for all $0 \le t \le 1$,

b) $d(x, c(0)) < C$, $d(y, c(1)) < C$ and

c) $\int_0^1 (\mathrm{Mx}_{2r}(\|\nabla.X^t\|^{3/2})(c(t)))^{2/3}\, dt < \hat{\varepsilon}$.

In order to construct the subset $G_r^2(q_i)$, the following statement is used: There is a rescaling factor such that, if a manifold is sufficiently close to $\mathbb{R}^k$, the rescaled manifold is close to a product. This statement will be proven by contradiction using the following theorem of Kapovitch and Wilking where the first part is the first part of the original theorem and the second is taken from its proof.

Theorem 2.13 (Rescaling Theorem [KW11, Theorem 5.1]). *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a sequence of n-dimensional Riemannian manifolds and let $r_i \to \infty$ and $\mu_i \to 0$ be sequences of positive real numbers such that $B_{r_i}^{M_i}(p_i)$ has curvature larger than $-\mu_i$ and $\bar{B}_{r_i}^{M_i}(p_i)$ is compact. Suppose that (M_i, p_i) converges to $(\mathbb{R}^k, 0)$ for some $k < n$. After passing to a subsequence, there is a compact metric space K with* $\mathrm{diam}(K) = 10^{-n^2}$*, a family of subsets $G_1(p_i) \subseteq B_1(p_i)$ with $\frac{\mathrm{vol}(G_1(p_i))}{\mathrm{vol}(B_1(p_i))} \to 1$ and a sequence $\lambda_i \to \infty$ such that the following holds:*

a) For all $q_i \in G_1(p_i)$, the isometry type of the limit of any convergent subsequence of $(\lambda_i M_i, q_i)$ is given by the metric product $\mathbb{R}^k \times K$.

b) There exists a sequence $\hat{\varepsilon}_i \to 0$ such that p_i has the $\mathcal{C}(M_i, 1, C_i, \hat{\varepsilon}_i)$-property where $C_i := \frac{9^n}{\lambda_i}$.

Lemma 2.14. *For $\hat{\varepsilon} \in (0,1)$, $R > 0$, $\eta > 0$ and $k < n$ there is $\hat{\delta} = \hat{\delta}(\hat{\varepsilon}, R, \eta; n, k) > 0$ such that for all pointed n-dimensional Riemannian manifolds (M, p) with lower Ricci curvature bound $\mathrm{Ric}_M \geq -(n-1) \cdot \hat{\delta}^2$ satisfying that $\bar{B}_{1/\hat{\delta}}(p)$ is compact and*

$$d_{GH}((M,p), (\mathbb{R}^k, 0)) \leq \hat{\delta}$$

there is a factor $\lambda > 0$ such that the following holds:

a) *There are a subset of good points $G_1(p) \subseteq B_1(p)$ satisfying*

$$\mathrm{vol}_M(G_1(p)) \geq (1 - \hat{\varepsilon}) \cdot \mathrm{vol}_M(B_1(p))$$

and a compact metric space K of diameter 1 such that for all $q \in G_1(p)$ there is a point $\tilde{q} \in \mathbb{R}^k \times K$ with

$$d_{GH}(B_R^{\lambda M}(q), B_R^{\mathbb{R}^k \times K}(\tilde{q})) \leq \eta.$$

b) *The base point p has the $\mathcal{C}(M, 1, \frac{9^n \cdot 10^{n^2}}{\lambda}, \hat{\varepsilon})$-property.*

Proof. The proof is done by contradiction (using the Rescaling Theorem).

Assume that the statement is false and choose the contradicting $0 < \hat{\varepsilon} < 1$, $R > 0$, $\eta > 0$ and $k < n$. Thus, for $\hat{\delta} = \frac{1}{i}$, where $i \in \mathbb{N}$, there is (M_i, p_i) with lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -\frac{n-1}{i^2} > -\frac{2(n-1)}{i^2} \to 0$ satisfying that $\bar{B}_i(p_i)$ is compact and

$$d_{GH}((M_i, p_i), (\mathbb{R}^k, 0)) \leq \frac{1}{i}$$

such that for any $\lambda_i > 0$ statement a) or statement b) is not satisfied.

By Proposition 1.8 b), $(M_i, p_i) \to (\mathbb{R}^k, 0)$. Using the Rescaling Theorem and after passing to a subsequence, there exist a subset of good points $G_1(p_i) \subseteq B_1(p_i)$ such that

$$\frac{\mathrm{vol}(G_1(p_i))}{\mathrm{vol}(B_1(p_i))} \to 1 \text{ as } i \to \infty,$$

there exists $\tilde{\lambda}_i \to \infty$ and a compact metric space K with diameter 10^{-n^2} such that

a') for all $q_i \in G_1(p_i)$ and all sublimits $(Y, \cdot)$ of $(\tilde{\lambda}_i M_i, q_i)$ there exists $q' \in \tilde{K}$ such that $(Y, \cdot) \cong (\mathbb{R}^k \times \tilde{K}, (0, q'))$,

b') for all $x_i, y_i \in G_1(p_i)$ there is a time-dependent, piecewise constant in time vector field X_i with compact support and an integral curve c_i such that the vector field X_i^t is divergence free on $B_{10}(c_i(t))$ for all $0 \leq t \leq 1$, $d(x_i, c_i(0)) < \frac{9^n}{\tilde{\lambda}_i}$, $d(y_i, c_i(1)) < \frac{9^n}{\tilde{\lambda}_i}$ and $\int_0^1 (\mathrm{Mx}(\|\nabla . X_i^t\|^{3/2})(c_i(t)))^{2/3}\, dt \to 0$ as $i \to \infty$.

Choose these $G_1(p_i)$, let $\lambda_i := 10^{n^2} \cdot \tilde{\lambda}_i \to \infty$ and $K := 10^{n^2} \cdot \tilde{K}$. In particular, this satisfies $\mathrm{diam}(K) = 1$. Assume infinitely many i contradict the statement a). Without loss of generality, let i be large enough such that $\frac{\mathrm{vol}(G_1(p_i))}{\mathrm{vol}(B_1(p_i))} \geq 1 - \hat{\varepsilon}$. Pass to the subsequence of those contradicting i and choose points $q_i \in G_1(p_i)$ which contradict the statement a). After passing to a subsequence and using the Rescaling Theorem, $(\lambda_i M_i, q_i)$ is converging to $(\mathbb{R}^k \times K, \tilde{q})$ for some point $\tilde{q} \in \{0\} \times K$. In particular,

$$d_{GH}(B_R^{\lambda_i M_i}(q_i), B_R^{\mathbb{R}^k \times K}(\tilde{q})) \le \eta$$

for i large enough, and a) is satisfied by i. This is a contradiction. Hence, only finitely many i contradict statement a).

Without loss of generality, assume that all i satisfy a). Therefore, by assumption, b) is not satisfied by any i. Let $x_i, y_i \in G_1(p_i)$ be contradicting and choose X_i and c_i as in the Rescaling Theorem. In particular, X_i^t is divergence free on $B_{10}(c_i(t))$ for all $0 \le t \le 1$ and both $d(x_i, c_i(0)) < \frac{9^n \cdot 10^{n^2}}{\lambda_i}$ and $d(y_i, c_i(1)) < \frac{9^n \cdot 10^{n^2}}{\lambda_i}$. Moreover, for i large enough, $\int_0^1 (\mathrm{Mx}(\|\nabla . X_i^t\|^{3/2})(c_i(t)))^{2/3}\, dt < \hat{\varepsilon}$. This is a contradiction. □

Now rescaling the sequence M_i such that each element is close enough to $\mathbb{R}^k$ and applying the previous result, one obtains factors λ_i which basically are the sought-after rescaling sequence. However, the lemma does provide λ_i for every i, but does not give any hint about whether or not $\lambda_i \to \infty$ as $i \to \infty$. In order to prove $\lambda_i \to \infty$, the fact is needed that spaces of different dimensions are not close. This in turn follows from the fact that sequences of limit spaces do not increase dimension. For this, the following lemma is needed which states that, given a converging sequence of proper length spaces, there exists a rescaling such that the rescaled sequence converges to a tangent cone.

Lemma 2.15. *Let $(X_i, p_i) \to (X, p)$ be a converging sequence of proper length spaces. Then there exists $\mu_i \to \infty$ such that for all $\lambda_i \to \infty$ with $\lambda_i \le \mu_i$, $(\lambda_i X_i, p_i)$ subconverges to a tangent cone of (X, p).*

Proof. For $\varepsilon_i \to 0$ such that $d_{GH}((X_i, p_i), (X, p)) \le \varepsilon_i$, let $\mu_i := \varepsilon_i^{-1/2}$. For fixed $r > 0$, let i be large enough such that $r \le \varepsilon_i^{-3/2}$. Then $\frac{r}{\mu_i} \le \frac{1}{\varepsilon_i}$ and

$$d_{GH}(B_{r/\mu_i}^{X_i}(p_i), B_{r/\mu_i}^{X}(p)) \le 16 \cdot d_{GH}(B_{1/\varepsilon_i}^{X_i}(p_i), B_{1/\varepsilon_i}^{X}(p)) \le 16\, \varepsilon_i \to 0 \text{ as } i \to \infty$$

by Lemma 1.11 b). After passing to a subsequence, $(\mu_i X, p)$ converges to a tangent cone (Y, q). Then

$$d_{GH}(B_r^{\mu_i X_i}(p_i), B_r^Y(q)) \le \mu_i \cdot d_{GH}(B_{r/\mu_i}^{X_i}(p_i), B_{r/\mu_i}^{X}(p)) + d_{GH}(B_r^{\mu_i X}(p), B_r^Y(q)) \to 0,$$

and this proves that $(\mu_i X_i, p_i)$ subconverges to (Y, q).

Now let $\lambda_i \to \infty$ with $\lambda_i \le \mu_i$ and define $\alpha_i := \frac{\lambda_i}{\mu_i} \in [0, 1]$. After passing to a further subsequence, there is $\alpha \le 1$ such that $\alpha_i \to \alpha$. By Lemma 1.10 c), both $(\lambda_i X, p) \to (\alpha Y, q)$ and $(\lambda_i M_i, p_i) \to (\alpha Y, q)$. In particular, $(\lambda_i X_i, p_i)$ subconverges to a tangent cone of (X, p). □

Let $\mathcal{X}^n$ denote the class of all pointed metric spaces that can occur as Gromov-Hausdorff limit of a sequence of pointed complete connected n-dimensional Riemannian manifolds M_i with lower Ricci curvature bound $\mathrm{Ric}_{M_i} \ge -(n-1)$.

Lemma 2.16. *Let* $(X_i, p_i) \to (X, p)$ *be converging spaces in* $\mathcal{X}^n$ *with* $\dim(X_i) = k \le n$ *and* $\dim(X) = l \le n$. *Then* $l \le k$.

Proof. In order to estimate $\dim(X)$, take a generic point $x \in X_{\text{gen}}$ and construct a tangent cone $\mathbb{R}^{\dim(X)}$. The idea of this construction is to consider sequences of manifolds M_{ij} converging to X_i. For large i, these are sufficiently close to X, and applying Lemma 2.5 and Lemma 2.7 will give the claim. So, let $(M_{ij}, p_{ij}) \to (X_i, p_i)$ as $j \to \infty$.

Without loss of generality, let $p \in X$ be generic: Take any $x \in X_{\text{gen}}$. For $p_i^x \to x$, then $(X_i, p_i^x) \to (X, x)$ as well.

Choose a monotonically increasing sequence $\mu_i \to \infty$ such that $(\mu_i X_i, p_i^x) \to (\mathbb{R}^l, 0)$ as in Lemma 2.15. In particular, $\mathrm{Ric}_{\mu_i M_{ij}} \ge -(n-1) \cdot \mu_i^{-2}$. Since the rescaled spaces $\mu_i X_i$ are k-dimensional as well, without loss of generality, assume $(X, p) = (\mathbb{R}^l, 0)$ and $\mathrm{Ric}_{M_{ij}} \ge -(n-1) \cdot \delta_i$ for some monotonically decreasing sequence $\delta_i \to 0$.

Choose $\varepsilon_i \to 0$ as in Lemma 1.10 a) such that

$$d_{GH}(B^{X_i}_{1/\varepsilon_i}(p_i), B^{\mathbb{R}^l}_{1/\varepsilon_i}(0)) \le \frac{\varepsilon_i}{2}.$$

Without loss of generality,

$$d_{GH}(B^{M_{ij}}_{1/\varepsilon_i}(p_{ij}), B^{X_i}_{1/\varepsilon_i}(p_i)) \le \frac{\varepsilon_i}{2}$$

for al $j \in \mathbb{N}$. Hence,

$$d_{GH}(B^{M_{ij}}_{1/\varepsilon_i}(p_{ij}), B^{\mathbb{R}^l}_{1/\varepsilon_i}(0)) \le \varepsilon_i\,.$$

Define $g : \mathbb{R}^+ \to \mathbb{R}^+$ by

$$g(x) := \begin{cases} \varepsilon_i & \text{if } \delta_i \le x < \delta_{i-1}, \\ 1 & \text{if } x \ge \delta_1, \end{cases}$$

let $C_{1\text{-}1}(n)$ be the constant of Theorem 2.8, $c = c(n) := 2 \cdot C_{1\text{-}1}(n) \cdot C_{BG}(n, -1, \frac{1}{2}, 1)$ and choose $\hat{\delta} = \hat{\delta}(\frac{10^{-n^2}}{c}, g, 1; n, l)$ as in Lemma 2.5. Let i be large enough such that $\delta_i \le \hat{\delta}$ and let $j \ge J(i)$. Then $\mathrm{Ric}_{M_{ij}} \ge -(n-1) \cdot \delta_i$ and, since $g(\delta_i) = \varepsilon_i$,

$$d_{GH}(B^{M_{ij}}_{1/g(\delta_i)}(p_{ij}), B^{\mathbb{R}^l}_{1/g(\delta_i)}(0)) \le g(\delta_i).$$

Hence, there is a constant $L = L(n)$ and harmonic L-Lipschitz maps $f_h^{ij} : B_1^{M_{ij}}(p_{ij}) \to \mathbb{R}$, $1 \le h \le l$, such that

$$⨍_{B_1^{M_{ij}}(p_{ij})} \sum_{h_1,h_2=1}^{l} |\langle \nabla f^{ij}_{h_1}, \nabla f^{ij}_{h_2} \rangle - \delta_{h_1 h_2}| + \sum_{h=1}^{l} \| \mathrm{Hess}(f_h^{ij}) \|^2 \, dV_{M_{ij}} < \frac{10^{-n^2}}{c}.$$

In order to simplify notation, let

$$F^{ij} := \sum_{h_1,h_2=1}^{l} |\langle \nabla f^{ij}_{h_1}, \nabla f^{ij}_{h_2} \rangle - \delta_{h_1 h_2}| + \sum_{h=1}^{l} \| \mathrm{Hess}(f_h^{ij}) \|^2.$$

Since

$$\begin{aligned}
&\mathrm{vol}(\{p \in B_{1/2}^{M_{ij}}(p_{ij}) \mid \mathrm{Mx}_{1/2}(F^{ij}) > 10^{-n^2}\}) \\
&\leq \frac{C_{1\text{-}1}(n)}{10^{-n^2}} \cdot \int_{B_{1/2}^{M_{ij}}(p_{ij})} F^{ij}\, dV \\
&\leq \frac{C_{1\text{-}1}(n)}{10^{-n^2}} \cdot \int_{B_{1}^{M_{ij}}(p_{ij})} F^{ij}\, dV \\
&< \frac{C_{1\text{-}1}(n)}{10^{-n^2}} \cdot \frac{10^{-n^2}}{2 \cdot C_{1\text{-}1}(n) \cdot C_{BG}(n, -1, \frac{1}{2}, 1)} \cdot \mathrm{vol}(B_1^{M_{ij}}(p_{ij})) \\
&\leq \frac{1}{2} \cdot \mathrm{vol}(B_{1/2}^{M_{ij}}(p_{ij})),
\end{aligned}$$

the set

$$G_{ij} := \{p \in \bar{B}_{1/2}^{M_{ij}}(p_{ij}) \mid \mathrm{Mx}_{1/2}(F^{ij})(p) \leq 10^{-n^2}\}$$

is compact with $\mathrm{vol}(G_{ij}) \geq \frac{1}{2} \cdot \mathrm{vol}(B_{1/2}^{M_{ij}}(p_{ij}))$. Applying Proposition 1.17 a), the sequence $(G_{ij})_{j \in \mathbb{N}}$ subconverges to a set $G_i \subseteq \bar{B}_{1/2}^{X_i}(p_i)$ with positive volume, in particular, the intersection with $(X_i)_{\mathrm{gen}}$ is nonempty. Without loss of generality, assume that $(G_{ij})_{j \in \mathbb{N}}$ itself converges to G_i and choose $q_{ij} \in G_{ij}$ converging to a $q_i \in (X_i)_{\mathrm{gen}}$.

Since $(M_{ij}, q_{ij}) \to (X_i, q_i)$, there exists $\mu_j^i \to \infty$ (as $j \to \infty$) as in Lemma 2.15 such that $(\mu_j^i M_{ij}, q_{ij}) \to (\mathbb{R}^k, 0)$ as $j \to \infty$.

On the other hand, the maps

$$\tilde{f}^{ij} := \mu_j^i f^{ij} : B_{\mu_j^i}^{\mu_j^i M_{ij}}(q_{ij}) \to \mathbb{R}$$

are harmonic and L-Lipschitz. Furthermore, for arbitrary $r > 0$ and j large enough such that $2r < \mu_j^i$,

$$\begin{aligned}
&⨍_{B_r^{\mu_j^i M_{ij}}(q_{ij})} \sum_{h_1, h_2 = 1}^{l} |\langle \nabla \tilde{f}_{h_1}^{ij}, \nabla \tilde{f}_{h_2}^{ij} \rangle - \delta_{h_1 h_2}|\, dV_{\mu_j^i M_{ij}} \\
&= ⨍_{B_{r/\mu_j^i}^{M_{ij}}(q_{ij})} \sum_{h_1, h_2 = 1}^{l} |\langle \nabla f_{h_1}^{ij}, \nabla f_{h_2}^{ij} \rangle - \delta_{h_1 h_2}|\, dV_{M_{ij}} \leq 10^{-n^2}
\end{aligned}$$

and

$$\begin{aligned}
&⨍_{B_r^{\mu_j^i M_{ij}}(q_{ij})} \sum_{h=1}^{l} \| \mathrm{Hess}(\tilde{f}_h^{ij}) \|^2\, dV_{\mu_j^i M_{ij}} \\
&= \frac{1}{(\mu_j^i)^2} \cdot ⨍_{B_{r/\mu_j^i}^{M_{ij}}(q_{ij})} \sum_{h=1}^{l} \| \mathrm{Hess}(\tilde{f}_h^{ij}) \|^2\, dV_{M_{ij}} \to 0 \text{ as } j \to \infty.
\end{aligned}$$

By Lemma 2.7, there exists a metric space Z and a point $z \in \mathbb{R}^l \times Z$ such that

$$(\mu^i_j M_{ij}, q_{ij}) \to (\mathbb{R}^l \times Z, z) \text{ as } j \to \infty$$

In particular, $\mathbb{R}^k \cong \mathbb{R}^l \times Z$, and thus, $k \geq l$. □

Lemma 2.17. *For all $k < n$ there is $\varepsilon_0 = \varepsilon_0(n,k) \in (0, \frac{1}{100})$ such that the following is true: If $(X,p), (\mathbb{R}^k \times K, q) \in \mathcal{X}^n$ for a compact metric space K with* $\mathrm{diam}(K) = 1$ *and* $\dim(X) = k$*, then*

$$d_{GH}(B^X_{1/\varepsilon_0}(p), B^{\mathbb{R}^k \times K}_{1/\varepsilon_0}(q)) > \varepsilon_0\,.$$

Proof. The proof is done by contradiction to Lemma 2.16: Assume the statement is false and let $k < n$ be contradicting. For every $i \in \mathbb{N}$, $i > 100$, choose $(M_{ij}, p_{ij}) \to (X_i, p_i)$ and $(N_{ij}, q_{ij}) \to (\mathbb{R}^k \times K_i, q_i)$ as $j \to \infty$ with $\mathrm{diam}(K_i) = 1$, $\dim(X_i) = k$ and

$$d_{GH}((X_i, p_i), (\mathbb{R}^k \times K_i, q_i)) \leq \frac{1}{i}.$$

In particular, $(X_i, p_i), (\mathbb{R}^k \times K_i, q_i) \in \mathcal{X}^n$.

For every $i \in \mathbb{N}$ there is $J(i) \in \mathbb{N}$ such that

$$d_{GH}((M_{ij}, p_{ij}), (X_i, p_i)) \leq \frac{1}{i}$$

for all $j \geq J(i)$. Define inductively $j_1 := J(1)$ and $j_{i+1} := \max\{J(i), j_{i-1} + 1\}$. In particular, $j_i \to \infty$ as $i \to \infty$ and

$$d_{GH}((M_{ij_i}, p_{ij_i}), (X_i, p_i)) \leq \frac{1}{i}.$$

By Theorem 1.16, after passing to a subsequence, (M_{ij_i}, p_{ij_i}) converges to some (X, p) as $i \to \infty$. In particular, $(X, p) \in \mathcal{X}^n$. By Lemma 1.11 b), for arbitrary $r > 0$ and $i \geq r$

$$\begin{aligned}
&d_{GH}(B^{X_i}_r(p_i), B^{M_{ij_i}}_r(p_{ij_i})) \\
&\leq 16 \cdot d_{GH}(B^{X_i}_i(p_i), B^{M_{ij_i}}_i(p_{ij_i})) \\
&< \frac{16}{i} \to 0 \text{ as } i \to \infty.
\end{aligned}$$

This implies

$$\begin{aligned}
&d_{GH}(B^{X_i}_r(p_i), B^X_r(p)) \\
&\leq d_{GH}(B^{X_i}_r(p_i), B^{M_{ij_i}}_r(p_{ij_i})) + d_{GH}(B^{M_{ij_i}}_r(p_{ij_i}), B^X_r(p)) \to 0 \text{ as } i \to \infty.
\end{aligned}$$

Hence, $(X_i, p_i) \to (X, p)$, where, as seen before, $(X, p) \in \mathcal{X}^n$. With analogous argumentation, after passing to a further subsequence, $(\mathbb{R}^k \times K_i, q_i) \to (\mathbb{R}^k \times K, q)$ for some compact metric space K with $\mathrm{diam}(K) = 1$ and $(\mathbb{R}^k \times K, q) \in \mathcal{X}^n$.

On the other hand, for $r > 0$ and $i \geq r$,

$$\begin{aligned}
&d_{GH}(B_r^{\mathbb{R}^k \times K_i}(q_i), B_r^X(p)) \\
&\leq d_{GH}(B_r^{\mathbb{R}^k \times K_i}(q_i), B_r^{X_i}(p_i)) + d_{GH}(B_r^{X_i}(p_i), B_r^X(p)) \\
&\leq 16 \cdot d_{GH}(B_i^{\mathbb{R}^k \times K_i}(q_i), B_i^{X_i}(p_i)) + d_{GH}(B_r^{X_i}(p_i), B_r^X(p)) \to 0 \text{ as } i \to \infty.
\end{aligned}$$

Hence, $(\mathbb{R}^k \times K_i, q_i) \to (X, p)$. In particular, $X \cong \mathbb{R}^k \times K$ and $\dim(X) > k$. This is a contradiction to $\dim(X) \leq k$ by Lemma 2.16. □

Using this lemma, the sought-after rescaling sequence and family of sets can finally be constructed.

Lemma 2.18. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a collapsing sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$ and converge to a limit (X, p) of dimension $k < n$. For every $\hat{\varepsilon} \in (0, 1)$ there exists $\hat{\delta}_2 = \hat{\delta}_2(\hat{\varepsilon}; n, k) > 0$ such that for all $0 < r \leq \hat{\delta}_2$ and $q_i \in M_i$ with*

$$d_{GH}((r^{-1}M_i, q_i), (\mathbb{R}^k, 0)) \leq \hat{\delta}_2$$

there are a family of subsets of good points $G_r^2(q_i) \subseteq B_r(q_i)$ with

$$\mathrm{vol}(G_r^2(q_i)) \geq (1 - \hat{\varepsilon}) \cdot \mathrm{vol}(B_r(q_i))$$

and a sequence $\lambda_i \to \infty$ which satisfy:

a) For each $x_i \in G_r^2(q_i)$ there is a compact metric space K_i with diameter 1 and a point $\tilde{x}_i \in \{0\} \times K_i$ such that

$$d_{GH}(B_{1/\varepsilon_0}^{\lambda_i M_i}(x_i), B_{1/\varepsilon_0}^{\mathbb{R}^k \times K_i}(\tilde{x}_i)) \leq \frac{\varepsilon_0}{200}$$

for $\varepsilon_0 = \varepsilon_0(n, k)$ as in Lemma 2.17.

b) The points q_i have the $\mathcal{C}(M_i, r, \frac{9^n \cdot 10^{n^2}}{\lambda_i}, \hat{\varepsilon})$-property.

Proof. For carefully chosen $\hat{\delta}_2$, this is a direct consequence of rescaling the manifolds with a factor $\frac{1}{r}$ where $0 < r \leq \hat{\delta}_2$ and applying Lemma 2.14 to the rescaled manifolds: For $\hat{\varepsilon} \in (0, 1)$, let $\hat{\delta}_2 = \hat{\delta}_2(\hat{\varepsilon}; n, k)$ be the $\hat{\delta}(\hat{\varepsilon}, \frac{1}{\varepsilon_0}, \frac{\varepsilon_0}{200}; n, k)$ of Lemma 2.14.

Let $0 < r \leq \hat{\delta}_2$ and $q_i \in M_i$ with

$$d_{GH}((r^{-1}M_i, q_i), (\mathbb{R}^k, 0)) \leq \hat{\delta}_2.$$

Because of $\mathrm{Ric}_{M_i} \geq -(n-1)$,

$$\mathrm{Ric}_{r^{-1}M_i} \geq -(n-1) \cdot r^2 \geq -(n-1) \cdot \hat{\delta}_2^2.$$

Since M_i is complete, $\frac{1}{r}M_i$ is complete as well, and thus, $\bar{B}^{r^{-1}M_i}_{1/\hat{\delta}_2}(q_i)$ is compact. Due to Lemma 2.14, there exist $\tilde{\lambda}_i > 0$, a subset $\tilde{G}_1(q_i) \subseteq B^{r^{-1}M_i}_1(q_i)$ with

$$\mathrm{vol}(\tilde{G}_1(q_i)) \geq (1-\hat{\varepsilon}) \cdot \mathrm{vol}(B^{r^{-1}M_i}_1(q_i))$$

and a compact metric space K_i with $\mathrm{diam}(K_i) = 1$ such that for every $x_i \in \tilde{G}_1(q_i)$ there is $\tilde{x}_i \in \{0\} \times K_i$ with

$$d_{GH}(B^{\tilde{\lambda}_i r^{-1}M_i}_{1/\varepsilon_0}(x_i), B^{\mathbb{R}^k \times K_i}_{1/\varepsilon_0}(\tilde{x}_i)) \leq \frac{\varepsilon_0}{200}$$

and for all $x_i, y_i \in \tilde{G}_1(q_i)$ there are a time-dependent vector field X_i with compact support which is piecewise constant in time and an integral curve c_i such that both $d_{r^{-1}M_i}(x, c_i(0)) < \frac{9^n \cdot 10^{n^2}}{\tilde{\lambda}_i}$ and $d_{r^{-1}M_i}(y, c_i(1)) < \frac{9^n \cdot 10^{n^2}}{\tilde{\lambda}_i}$, the vector field X_i^t is divergence free on $B^{r^{-1}M_i}_{10}(c_i(t))$ for all $0 \leq t \leq 1$ as well as

$$\int_0^1 (\mathrm{Mx}_2^{r^{-1}M_i}(\|\nabla.X^t\|^{3/2})(c_i(t)))^{2/3}\, dt < \hat{\varepsilon}.$$

Define $\lambda_i := \frac{\tilde{\lambda}_i}{r}$ and regard $G^2_r(q_i) := \tilde{G}_1(q_i) \subseteq B^{r^{-1}M_i}_1(q_i) = B^{M_i}_r(q_i)$ as a subset of M_i. Then

$$\frac{\mathrm{vol}_{M_i}(G^2_r(q_i))}{\mathrm{vol}_{M_i}(B^{M_i}_r(q_i))} = \frac{\mathrm{vol}_{r^{-1}M_i}(\tilde{G}_1(q_i))}{\mathrm{vol}_{r^{-1}M_i}(B^{r^{-1}M_i}_1(q_i))} \geq 1 - \hat{\varepsilon}.$$

Moreover, given $x_i, y_i \in G^2_r(q_i)$, fix the corresponding X_i and c_i. Then X_i^t has compact support and is divergence free on $B^{M_i}_{10r}(c_i(t))$ for all $0 \leq t \leq 1$,

$$d_{M_i}(x, c_i(0)) < r \cdot \frac{9^n \cdot 10^{n^2}}{\tilde{\lambda}_i} = \frac{9^n \cdot 10^{n^2}}{\lambda_i} \quad \text{and} \quad d_{M_i}(y, c_i(1)) < \frac{9^n \cdot 10^{n^2}}{\lambda_i}$$

and

$$\int_0^1 (\mathrm{Mx}_{2r}(\|\nabla.X^t\|^{3/2})(c_i(t)))^{2/3}\, dt < \hat{\varepsilon}.$$

Assume that the sequence $(\lambda_i)_{i\in\mathbb{N}}$ is bounded. After passing to a subsequence, $\lambda_i \to \alpha$ and $(\lambda_i M_i, x_i) \to (\alpha X, q)$ for some $q \in X$. Since

$$d_{GH}(B^{\lambda_i M_i}_{1/\varepsilon_0}(x_i), B^{\mathbb{R}^k \times K_i}_{1/\varepsilon_0}(\tilde{x}_i)) \leq \frac{\varepsilon_0}{200},$$

one has

$$d_{GH}(B^{\alpha X}_{1/\varepsilon_0}(q), B^{\mathbb{R}^k \times K_i}_{1/\varepsilon_0}(\tilde{x}_i)) \leq \varepsilon_0$$

for all i large enough in contradiction to Lemma 2.17. Hence, $\lambda_i \to \infty$. □

This concludes the construction of $G^1_r(q_i)$, $G^2_r(q_i)$ and λ_i.

2.3 The $\mathcal{C}$-property and the dimension of blow-ups

In order to prove that the blow-ups with base points x_i and y_i, respectively, have the same dimension, a crucial argument is that the flow of a time-dependent vector field as in the definition of the $\mathcal{C}$-property is bi-Lipschitz on some small set. This result and the implication about dimensions are proven in this section.

For the proof it is important to know under which conditions large subsets of two balls intersect. The following lemmata deal with this question.

Lemma 2.19. *Let* (X, d, vol) *be a metric measure space and* $A' \subseteq A \subseteq X$*,* $B' \subseteq B \subseteq X$ *measurable with*

$$\begin{aligned} \mathrm{vol}(A') &\geq (1-\hat{\varepsilon}) \cdot \mathrm{vol}(A), \\ \mathrm{vol}(B') &\geq (1-\hat{\varepsilon}) \cdot \mathrm{vol}(B), \\ \mathrm{vol}(A \cap B) &> 2\hat{\varepsilon} \cdot \max\{\mathrm{vol}(A), \mathrm{vol}(B)\} \end{aligned}$$

for some $\hat{\varepsilon} > 0$*. Then* $\mathrm{vol}(A' \cap B') > 0$*, in particular,* $A' \cap B' \neq \emptyset$*.*

Proof. By hypothesis, $\mathrm{vol}((A \cap B) \setminus A') \leq \mathrm{vol}(A \setminus A') \leq \hat{\varepsilon} \cdot \mathrm{vol}(A) < \frac{1}{2} \cdot \mathrm{vol}(A \cap B)$. Analogously, $\mathrm{vol}((A \cap B) \setminus B') < \frac{1}{2} \cdot \mathrm{vol}(A \cap B)$. Thus,

$$\begin{aligned} \mathrm{vol}((A \cap B) \setminus (A' \cap B')) &\leq \mathrm{vol}((A \cap B) \setminus A') + \mathrm{vol}((A \cap B) \setminus B') \\ &< \mathrm{vol}(A \cap B). \end{aligned}$$

Therefore, $\mathrm{vol}(A' \cap B') = \mathrm{vol}(A \cap B) - \mathrm{vol}((A \cap B) \setminus (A' \cap B')) > 0$. □

Lemma 2.20. *Let* (M, g) *be a complete connected* n*-dimensional Riemannian manifold with lower Ricci curvature bound* $\mathrm{Ric}_M \geq -(n-1)$*. Given* $0 < \hat{\varepsilon} < \frac{1}{2}$ *and* $s > 0$*, there exist* $d_0(n, \hat{\varepsilon}, s), \delta_0(n, \hat{\varepsilon}, s) > 0$ *such that* $(\delta_0(n), \frac{1}{sd_0} - \frac{1}{2})$ *is non-empty and for* $\delta \in (\delta_0(n), \frac{1}{sd_0} - \frac{1}{2})$*, points* $p, q \in M$ *with distance* $d := d(p, q) < d_0$ *and* $R := \frac{d}{2} + \delta d < \frac{1}{s}$*,*

$$\mathrm{vol}(B_R(p) \cap B_R(q)) > 2\hat{\varepsilon} \cdot \max\{\mathrm{vol}(B_R(p)), \mathrm{vol}(B_R(q))\}.$$

Moreover, this δ_0 *can be chosen to be monotonically increasing in* d_0*.*

Proof. Let $p, q \in M$ be arbitrary, $\gamma : [0, d] \to M$ be a shortest geodesic connecting p and q and $m := \gamma(\frac{d}{2})$ be the midpoint of this geodesic, i.e. $d(p, m) = d(q, m) = \frac{d}{2}$.

First, let $r > 0$ be arbitrary. Observe $B_r(m) \subseteq B_{d/2+r}(p) \cap B_{d/2+r}(q)$. Moreover, $B_{d/2+r}(p) \subseteq B_{d/2+r+d(p,m)}(m) \subseteq B_{d+r}(m)$. Then, by the Bishop-Gromov Theorem,

$$\frac{\mathrm{vol}(B_{d/2+r}(p) \cap B_{d/2+r}(q))}{\mathrm{vol}(B_{d/2+r}(p))} \geq \frac{\mathrm{vol}(B_r(m))}{\mathrm{vol}(B_{d+r}(m))} \geq \frac{1}{C_{BG}(n, -1, r, d+r)}.$$

Now let $C_0 := \frac{1}{2\hat{\varepsilon}} > 1$ and $\hat{r} := \hat{r}(n, \hat{\varepsilon}, s) := \min\left\{\frac{\ln(C_0)}{2(n-1)}, \frac{1}{s}\right\}$. For $0 < d < 2\hat{r}$, define

$$\tilde{\delta}_0(n, \hat{\varepsilon}, d) := \inf\{\delta' > 0 \mid \forall \delta > \delta' : f_{n,\delta}(\delta d) < C_0\} \in [0, \infty]$$

where for $\delta > 0$ and $r > 0$,

$$f_{n,\delta}(r) := C_{BG}\left(n, -1, r, \left(1 + \frac{1}{\delta}\right) \cdot r\right).$$

In fact, this $\tilde{\delta}_0(n, \hat{\varepsilon}, d)$ is finite and monotonically increasing in d as will be proven next:

Assume $\tilde{\delta}_0(n, \hat{\varepsilon}, d) = \infty$, i.e. there exist $\delta_m \to \infty$ such that $f_{n,\delta_m}(\delta_m d) \geq C_0$. Then, applying Lemma 1.2,

$$\begin{aligned} f_{n,\delta_m}(\delta_m d) &= C_{BG}\left(n, -1, \delta_m d, \left(1 + \frac{1}{\delta_m}\right)\delta_m d\right) \\ &= C_{BG}(n, -1, \delta_m d, \delta_m d + d) \\ &\to e^{(n-1)d} \text{ as } m \to \infty, \end{aligned}$$

and this implies $e^{(n-1)d} \geq C_0$. On the other hand, $e^{(n-1)d} < e^{2(n-1)\hat{r}} \leq C_0$. This is a contradiction. Thus, $\tilde{\delta}_0(n, \hat{\varepsilon}, d) < \infty$.

Now let $d_1 < d_2$ and $\delta > \tilde{\delta}_0(n, \hat{\varepsilon}, d_2)$. Since $f_{n,\delta}$ is monotonically increasing in r,

$$C_0 > f_{n,\delta}(\delta d_2) \geq f_{n,\delta}(\delta d_1),$$

i.e. $\delta \geq \tilde{\delta}_0(n, \hat{\varepsilon}, d_1)$, and this proves the monotonicity of $\tilde{\delta}_0(n, \hat{\varepsilon}, \cdot)$.

Hence, $\tilde{\delta}_0(n, \hat{\varepsilon}, d)$ decreases for decreasing d whereas $\frac{1}{sd} - \frac{1}{2}$ increases. Therefore, there exists $d_0 = d_0(n, \hat{\varepsilon}, s) \leq 2\hat{r}$ such that $\tilde{\delta}_0(n, \hat{\varepsilon}, d) \leq \frac{1}{sd} - \frac{1}{2}$ for $d \leq d_0$. Let

$$\delta_0 = \delta_0(n, \hat{\varepsilon}, s) := \tilde{\delta}_0(n, \hat{\varepsilon}, d_0(n, \hat{\varepsilon}, s)) = \max\{\tilde{\delta}_0(n, \hat{\varepsilon}, d) \mid 0 < d \leq d_0\}$$

where the monotonicity of $\tilde{\delta}_0$ is used in the last equality. For $d \leq d_0$ and $\delta_0 < \delta < \frac{1}{sd_0} - \frac{1}{2}$, let

$$R := \frac{d}{2} + \delta d = \left(\frac{1}{2} + \delta\right) \cdot d < \left(\frac{1}{2} + \frac{1}{sd_0} - \frac{1}{2}\right) \cdot d_0 = \frac{1}{s}.$$

Then

$$\frac{\operatorname{vol}(B_R(p) \cap B_R(q))}{\operatorname{vol}(B_R(p))} \geq \frac{1}{C_{BG}(n, -1, \delta d, d + \delta d)} = \frac{1}{f_{n,\delta}(\delta d)} > \frac{1}{C_0} = 2\hat{\varepsilon}. \qquad \square$$

The next lemma will only be needed in chapter 3 but is already given here since its statement and the proof are similar to the previous one.

Lemma 2.21. *Let (M, g) be a complete connected n-dimensional Riemannian manifold with lower Ricci curvature bound $\operatorname{Ric}_M \geq -(n-1)$. For all $0 < \hat{\varepsilon} < \frac{1}{2}$ and $R > 0$ there is $d_0 = d_0(n, \hat{\varepsilon}, R) > 0$ such that for all $p, q \in M$ with $d(p, q) < d_0$,*

$$\operatorname{vol}(B_R(p) \cap B_R(q)) > 2\hat{\varepsilon} \cdot \max\{\operatorname{vol}(B_R(p)), \operatorname{vol}(B_R(q))\}.$$

Proof. Similarly to the proof of Lemma 2.20, for arbitrary points $p, q \in M$ with distance $d := d(p, q) < 2R$, observe

$$\frac{\mathrm{vol}(B_R(p) \cap B_R(q))}{\mathrm{vol}(B_R(p))} \geq \frac{1}{C_{BG}(n, -1, R - \frac{d}{2}, R + \frac{d}{2})}.$$

Since $C_{BG}(n, -1, R - \frac{d}{2}, R + \frac{d}{2}) \to 1$ as $d \to 0$, there is $d_0 = d_0(n, \hat{\varepsilon}, R) \in (0, 2R)$ such that $C_{BG}(n, -1, R - \frac{d}{2}, R + \frac{d}{2}) < \frac{1}{2\hat{\varepsilon}}$ for all $d \leq d_0$. In particular, for points $p, q \in M$ with $d(p, q) < d_0$,

$$\mathrm{vol}(B_R(p) \cap B_R(q)) > 2\hat{\varepsilon} \cdot \max\{\mathrm{vol}(B_R(p)), \mathrm{vol}(B_R(q))\}. \qquad \square$$

An important notion for investigating the $\mathcal{C}$-property is the *distortion* of a function which describes how much a function changes the distance of two points. In particular, it will be important to know how much a flow changes the distance of two points up to some fixed time.

Definition 2.22. For a map $f : M \to N$ between Riemannian manifolds the *distortion of f* is the function $dt^f : M \times M \to [0, \infty)$ defined by

$$dt^f(p, q) := |d_N(f(p), f(q)) - d_M(p, q)|.$$

If Φ is the flow of a time-dependent vector field on M, $t \in [0, 1]$ and $p, q \in M$, denote $\varphi_t := \Phi(0, t, \cdot)$ and let

$$dt(t)(p, q) := \max\{dt^{\varphi_\tau}(p, q) \mid 0 \leq \tau \leq t\},$$

and for $r \geq 0$, let

$$dt_r(t)(p, q) := \min\{r, dt(t)(p, q)\}.$$

The subsequent lemma generalises [KW11, Lemma 3.7] and can be proven completely analogously to it.

Lemma 2.23. *For $\tilde{\alpha} \in (1, 2)$ there exist $C = C(n, \tilde{\alpha})$ and $\hat{C} = \hat{C}(n, \tilde{\alpha})$ such that the following holds for any $0 < R \leq 1$: Let M be an n-dimensional Riemannian manifold with $\mathrm{Ric}_M \geq -(n-1)$ and $X : [0, 1] \times M \to TM$ be a time dependent, piecewise constant in time vector field with compact support and flow Φ, $\varphi_t^s := \Phi(s, t, \cdot)$ and $c : [0, 1] \to M$ be an integral curve of X such that X^t is divergence free on $B_{10R}(c(t))$ for all $t \in [0, 1]$.*

Let $\tilde{\varepsilon} := \int_0^1 (\mathrm{Mx}_R(\|\nabla . X^t\|)) \circ c(t)\, dt$. Then for any $r \leq \frac{R}{10}$,

$$\fint_{B_r(c(s)) \times B_r(c(0))} dt_r(1)(p, q)\, dV(p, q) \leq Cr \cdot \tilde{\varepsilon}.$$

Furthermore, there is a subset $B_r(c(0))' \subseteq B_r(c(0))$ with $c(0) \in B_r(c(0))'$ such that

$$\mathrm{vol}(B_r(c(0))') \geq (1 - C\tilde{\varepsilon}) \cdot \mathrm{vol}(B_r(c(0))).$$

Finally, for any $t \in [0,1]$,

$$\varphi_t^0(B_r(c(0))') \subseteq B_{\tilde{\alpha}r}(c(t))$$

and

$$\mathrm{vol}(B_r(c(t))) \leq \hat{C} \cdot \mathrm{vol}(B_r(c(0))).$$

Proof. The proof can be done completely analogously to the one of [KW11, Lemma 3.7] by replacing $\frac{r}{10}$ in the induction by $\frac{r}{m}$ where $m = 2 \cdot \frac{\tilde{\alpha}+1}{\tilde{\alpha}-1} > 0$. Again, the constants C and $\hat{C}$ can be made explicit in terms of the constant appearing in the Bishop-Gromov Theorem. □

The following lemma states that the flow of a time dependent vector field as in the definition of the $\mathcal{C}$-property is Lipschitz on certain small sets.

Lemma 2.24. *Given* $\alpha \in (1,2)$*, there exist* $C_0 = C_0(n,\alpha)$ *and* $C_0' = C_0'(n,\alpha)$ *such that for* $0 < \hat{\varepsilon} < \frac{1}{2C_0}$ *and* $0 < R \leq 1$ *there is* $\hat{r}_0 = \hat{r}_0(n,\hat{\varepsilon},\alpha,R) < \frac{R}{20\alpha}$ *satisfying the following:*

Let M *be an* n*-dimensional Riemannian manifold with lower Ricci curvature bound* $\mathrm{Ric}_M \geq -(n-1)$*,* $X : [0,1] \times M \to TM$ *a time dependent, piecewise constant in time vector field with compact support and flow* Φ*,* $\varphi_t^s := \Phi(s,t,\cdot)$*,* $\varphi_t := \varphi_t^0$*,* $c : [0,1] \to M$ *an integral curve of* X *such that* X^t *is divergence free on* $B_{10R}(c(t))$ *for all* $t \in [0,1]$ *and* $\int_0^1 (\mathrm{Mx}_{2R}(\|\nabla.X^t\|^{3/2})(c(t)))^{2/3}\,dt < \hat{\varepsilon}$*.*

Let $p := c(0)$ *and* $0 < r < \hat{r}_0$*. Then there exists a subset* $B_r(p)'' \subseteq B_r(p)$ *containing* p *with* $\mathrm{vol}(B_r(p)'') > (1 - C_0'\sqrt{\hat{\varepsilon}}) \cdot \mathrm{vol}(B_r(p))$ *such that* φ_t *is* α*-bi-Lipschitz on* $B_r(p)''$ *for any* $t \in [0,1]$*.*

Proof. Define $\tilde{\alpha} := \frac{\alpha+1}{2} \in (1, \frac{3}{2}) \subseteq (1,2)$ and fix the following constants:

- Let $C = C(n,\alpha)$ be the $C(n,\tilde{\alpha})$ and $\hat{C} = \hat{C}(n,\alpha)$ be the $\hat{C}(n,\tilde{\alpha})$ appearing in Lemma 2.23.
- Let $\tilde{C} = \tilde{C}(n) > 0$ be the constant of [KW11, formula (6)] satisfying
$$\mathrm{Mx}_\rho[\mathrm{Mx}_\rho(f)](x) \leq \tilde{C}(n) \cdot \left(\mathrm{Mx}_{2\rho}(f^{3/2})(x)\right)^{2/3}$$
for any $f \in L^{3/2}(M)$ and $0 < \rho \leq 1$.
- Let $C_0 = C_0(n,\alpha) := \tilde{C} \cdot C$.
- Let $C_0' = C_0'(n,\alpha) := \hat{C}^2 + \sqrt{\frac{C_0}{2}}$.

Fix $0 < \hat{\varepsilon} < \min\left\{\frac{1}{2C_0}, 1\right\}$ and $0 < R \leq 1$. First, observe

$$\begin{aligned}\tilde{\varepsilon} := \int_0^1 (\mathrm{Mx}_R(\|\nabla.X^t\|))(c(t))\,dt &\leq \int_0^1 \mathrm{Mx}_R(\mathrm{Mx}_R(\|\nabla.X^t\|))(c(t))\,dt \\ &\leq \tilde{C} \cdot \int_0^1 \mathrm{Mx}_{2R}(\|\nabla.X^t\|^{3/2})^{2/3}(c(t))\,dt \\ &< \tilde{C}\hat{\varepsilon}.\end{aligned}$$

In particular, $C\tilde{\varepsilon} < C_0\,\hat{\varepsilon} < \frac{1}{2}$. By Lemma 2.23, for all $r \le \frac{R}{10}$,

$$\fint_{B_r(p)\times B_r(p)} dt_r(1)\, dV \le Cr\cdot\tilde{\varepsilon} < C_0\,\hat{\varepsilon}\cdot r$$

and there is a subset $B_r(p)' \subseteq B_r(p)$ containing p with

$$\mathrm{vol}(B_r(p)') \ge (1 - C_0\,\hat{\varepsilon})\cdot \mathrm{vol}(B_r(p)) > (1 - C\tilde{\varepsilon})\cdot \mathrm{vol}(B_r(c(0)))$$

and $\varphi_t(B_r(p)') \subseteq B_{\alpha r}(c(t))$ for all $t \in [0,1]$. Furthermore, for $r \le \frac{R}{10}$,

$$\begin{aligned}
\frac{\mathrm{vol}(B_{\alpha r}(c(t)))}{\mathrm{vol}(B_r(p)')} &\le C_{BG}(n,-1,r,\alpha r)\cdot \frac{\mathrm{vol}(B_r(c(t)))}{\mathrm{vol}(B_r(p)')}\\
&\le C_{BG}\left(n,-1,\frac{1}{10},\frac{\alpha}{10}\right)\cdot \hat{C}\cdot \frac{\mathrm{vol}(B_r(c(0)))}{\mathrm{vol}(B_r(p)')}\\
&\le C_{BG}\left(n,-1,\frac{1}{10},\frac{\alpha}{10}\right)\cdot \frac{\hat{C}}{1-C\tilde{\varepsilon}}\\
&< 2\hat{C}\cdot C_{BG}\left(n,-1,\frac{1}{10},\frac{\alpha}{10}\right)\\
&= \hat{C}^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
&\fint_{B_r(p)'}\int_0^1 \mathrm{Mx}_R(\|\nabla.X^t\|)\circ\varphi_t(x)\,dt\,dV(x)\\
&= \int_0^1 \frac{1}{\mathrm{vol}(B_r(p)')}\cdot\int_{B_r(p)'} \mathrm{Mx}_R(\|\nabla.X^t\|)\circ\varphi_t(x)\,dV(x)\,dt\\
&= \int_0^1 \frac{1}{\mathrm{vol}(B_r(p)')}\cdot\int_{\varphi_t(B_r(p)')} \mathrm{Mx}_R(\|\nabla.X^t\|)(x)\,dV(x)\,dt\\
&\le \int_0^1 \frac{\mathrm{vol}(B_{\alpha r}(c(t)))}{\mathrm{vol}(B_r(p)')}\cdot\fint_{B_{\alpha r}(c(t))} \mathrm{Mx}_R(\|\nabla.X^t\|)(x)\,dV(x)\,dt\\
&\le \hat{C}^2\cdot\int_0^1 \max_{0\le\rho\le R}\fint_{B_\rho(c(t))} \mathrm{Mx}_R(\|\nabla.X^t\|)(x)\,dV(x)\,dt\\
&\le \hat{C}^2\cdot\tilde{C}\int_0^1 (\mathrm{Mx}_{2R}(\|\nabla.X^t\|^{3/2}))^{2/3}(c(t))\,dt\\
&< \hat{C}^2\cdot\tilde{C}\hat{\varepsilon}.
\end{aligned}$$

Define

$$B_r(p)'' := \{x \in B_r(p)' \mid \int_0^1 \mathrm{Mx}_R(\|\nabla.X^t\|)\circ\varphi_t(x)\,dt < \tilde{C}\cdot\sqrt{\hat{\varepsilon}}\}.$$

Observe $p \in B_r(p)''$ due to $\varphi_t(p) = c(t)$ and $\int_0^1 \mathrm{Mx}_R(\|\nabla.X^t\|) \circ c(t)\, dt < \tilde{C} \cdot \hat{\varepsilon} \leq \tilde{C} \cdot \sqrt{\hat{\varepsilon}}$. Furthermore,

$$\begin{aligned}
\mathrm{vol}(B_r(p)'') &\geq \left(1 - \frac{\fint_{B_r(p)'} \int_0^1 \mathrm{Mx}_R(\|\nabla.X^t\|) \circ \varphi_t(x)\, dt\, dV(x)}{\tilde{C} \cdot \sqrt{\hat{\varepsilon}}}\right) \cdot \mathrm{vol}(B_r(p)') \\
&> \left(1 - \frac{\hat{C}^2 \cdot \tilde{C}\hat{\varepsilon}}{\tilde{C} \cdot \sqrt{\hat{\varepsilon}}}\right) \cdot \mathrm{vol}(B_r(p)') \\
&\geq (1 - \hat{C}^2\sqrt{\hat{\varepsilon}}) \cdot (1 - C_0\hat{\varepsilon}) \cdot \mathrm{vol}(B_r(p)) \\
&\geq (1 - (\hat{C}^2\sqrt{\hat{\varepsilon}} + C_0\hat{\varepsilon})) \cdot \mathrm{vol}(B_r(p)) \\
&> (1 - C_0'\sqrt{\hat{\varepsilon}}) \cdot \mathrm{vol}(B_r(p))
\end{aligned}$$

using $C_0' = \hat{C}^2 + \sqrt{\frac{C_0}{2}} = \hat{C}^2 + C_0 \cdot \sqrt{\frac{1}{2C_0}} > \hat{C}^2 + C_0\sqrt{\hat{\varepsilon}}$.

Moreover, points in $B_r(p)''$ have the following property: Fix $t \in [0,1]$, $a \in B_r(p)''$ and let $\tilde{a} := \varphi_t(a)$. In particular, $\int_0^1 \mathrm{Mx}_R(\|\nabla.X^t\|) \circ \varphi_t(a)\, dt < \tilde{C} \cdot \sqrt{\hat{\varepsilon}}$ and, by Lemma 2.23, for any $\rho \leq \frac{R}{10}$ there are subsets $B_\rho(a)' \subseteq B_\rho(a)$ and $B_\rho(\tilde{a})' \subseteq B_\rho(\tilde{a})$ such that

$$\begin{aligned}
\mathrm{vol}(B_\rho(a)') &\geq (1 - C_0\sqrt{\hat{\varepsilon}}) \cdot \mathrm{vol}(B_\rho(a)) \quad \text{and} \quad \varphi_t(B_\rho(a)') \subseteq B_{\tilde{\alpha}\rho}(\tilde{a}), \\
\mathrm{vol}(B_\rho(\tilde{a})') &\geq (1 - C_0\sqrt{\hat{\varepsilon}}) \cdot \mathrm{vol}(B_\rho(\tilde{a})) \quad \text{and} \quad \varphi_{-t}^t(B_\rho(\tilde{a})') \subseteq B_{\tilde{\alpha}\rho}(a)
\end{aligned}$$

where $C_0\sqrt{\hat{\varepsilon}} < C_0 \cdot \hat{\varepsilon} < \frac{1}{2}$.

Let $d_0 = d_0(n, \hat{\varepsilon}, \alpha, R)$ and $\delta_0 = \delta_0(n, \hat{\varepsilon}, \alpha, R)$, respectively, denote the constants $d_0(n, C_0\sqrt{\hat{\varepsilon}}, \frac{10}{R})$ and $\delta_0(n, C_0\sqrt{\hat{\varepsilon}}, \frac{10}{R})$, respectively, of Lemma 2.20. This $d_0 < \frac{R}{10}$ can be chosen small enough such that $\delta_0 \leq \frac{1}{2} - \frac{1}{\alpha+1} = \frac{1}{2} - \frac{1}{2\tilde{\alpha}} < \frac{1}{2}$. Define

$$\hat{r}_0 = \hat{r}_0(n, \hat{\varepsilon}, \alpha, R) := \frac{d_0}{2\alpha} < \frac{R}{20\alpha}.$$

From now on, assume $r < \hat{r}_0$ and let $b \in B_r(p)''$ be another point. In particular, $d := d(a,b) < 2r < \frac{d_0}{\alpha} < d_0$. For arbitrary $\delta_0 < \delta < \frac{1}{2}$, let $\rho := (\frac{1}{2} + \delta)d < \frac{R}{10}$. By Lemma 2.19 and Lemma 2.20, there exists a point $z \in B_\rho(a)' \cap B_\rho(b)'$. Thus,

$$\begin{aligned}
d(\varphi_t(a), \varphi_t(b)) &\leq d(\varphi_t(a), \varphi_t(z)) + d(\varphi_t(z), \varphi_t(b)) \\
&< 2 \cdot \tilde{\alpha}\rho \\
&= \tilde{\alpha} \cdot (2\delta + 1)d.
\end{aligned}$$

Since $\delta > \delta_0$ was arbitrary and $(2\delta_0 + 1) \cdot \tilde{\alpha} \leq \alpha$, this proves

$$d(\varphi_t(a), \varphi_t(b)) \leq \alpha \cdot d(a,b).$$

For $\tilde{a} = \varphi_t(a)$ and $\tilde{b} = \varphi_t(b)$ as before, $d(\tilde{a}, \tilde{b}) = d(\varphi_t(a), \varphi_t(b)) \leq \alpha \cdot 2r < d_0$ and the same argumentation as before gives

$$d(a,b) = d(\varphi_{-t}^t(\tilde{a}), \varphi_{-t}^t(\tilde{b})) \leq \alpha \cdot d(\tilde{a}, \tilde{b}) = \alpha \cdot d(\varphi_t(a), \varphi_t(b)).$$

Thus, φ_t is α-bi-Lipschitz on $B_r(p)''$ for $r < \hat{r}_0$. □

If a sequence of manifolds satisfies the previous lemma and the rescaled manifolds endowed with the end points of the integral curve as base points converge, the limits have the same dimension.

Lemma 2.25. *Let $(M_i)_{i\in\mathbb{N}}$ be a sequence of n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$. For every $i \in \mathbb{N}$, let further $X_i : [0,1] \times M \to TM$ be a time dependent, piecewise constant in time vector field with compact support and flow Φ_i, $\varphi_i^t := \Phi_i(0,t,\cdot)$, $c_i : [0,1] \to M_i$ be an integral curve of X_i such that X_i^t is divergence free on $B_{10r}(c_i(t))$ for all $t \in [0,1]$ and $\int_0^1 (\mathrm{Mx}_{2r}(\|\nabla.X_i^t\|^{3/2})(c_i(t)))^{2/3}\, dt < \hat{\varepsilon}$ for some $0 < r \leq 1$ and $\hat{\varepsilon} > 0$.*

Assume $x_i' := c_i(0)$ and $y_i' := c_i(1)$ satisfy $d(x_i', y_i') \leq 2$ and let $\lambda_i \to \infty$ such that $(\lambda_i M_i, x_i') \to (X, x_\infty)$ and $(\lambda_i M_i, y_i') \to (Y, y_\infty)$ as $i \to \infty$. Then $\dim(X) = \dim(Y)$.

Proof. The proof consists of three steps: First, for arbitrary radius $r > 0$, construct a bi-Lipschitz map between subsets of $\bar{B}_r^X(x_\infty)$ and $\bar{B}_{\alpha r}^Y(y_\infty)$, cf. Figure 2.1. Next, observe that these subsets have positive volume. In particular, they intersect the set of generic points. Finally, repeating the argument for the limit spaces proves the claim.

Choose any $\alpha \in (1,2)$. Without loss of generality, let $i \in \mathbb{N}$ be large enough such that $r < \lambda_i \cdot \hat{r}_0$ where $\hat{r}_0 = \hat{r}_0(\alpha)$ is the constant from Lemma 2.24. Furthermore, let $B_{r/\lambda_i}^{M_i}(x_i')'' \subseteq B_{r/\lambda_i}^{M_i}(x_i')$ and $\varphi_i^1 : B_{r/\lambda_i}^{M_i}(x_i')'' \to B_{\alpha r/\lambda_i}^{M_i}(y_i')$ be as in Lemma 2.24. Since φ_i^1 is α-bi-Lipschitz, it can be extended to a map $\varphi_i^1 : \overline{B_{r/\lambda_i}^{M_i}(x_i')''} \to \bar{B}_{\alpha r/\lambda_i}^{M_i}(y_i')$ which is α-bi-Lipschitz as well.

In order to regard φ_i^1 as a map $\lambda_i M_i \to \lambda_i M_i$ instead of a map $M_i \to M_i$, let G_i denote this closure $\overline{B_{r/\lambda_i}^{M_i}(x_i')''}$ regarded as a subset of $\bar{B}_r^{\lambda_i M_i}(x_i') \subseteq \lambda_i M_i$. Correspondingly, define $f_i : G_i \to B_{\alpha r}^{\lambda_i M_i}(y_i')$ by $f_i(q) := \varphi_i^1(q)$, cf. Figure 2.1. By definition, this map is α-bi-Lipschitz and measure preserving.

By Lemma 1.12, there exists a compact set $S_r \subseteq \bar{B}_r^X(x_\infty)$ which is the sublimit of the G_i and an α-bi-Lipschitz homeomorphism $f_r : S_r \to f_r(S_r)$ such that $f_r(S_r)$ is a sublimit of the $f_i(G_i)$, cf. again Figure 2.1.

Now find a point $x_0 \in S_r$ such that both x_0 and $f_r(x_0)$ are generic: By Proposition 1.18,

$$\begin{array}{ccccc}
\bar{B}_{r/\lambda_i}^{M_i}(x_i') & \cong & \bar{B}_r^{\lambda_i M_i}(x_i') & \xrightarrow{i \to \infty} & \bar{B}_r^X(x_\infty) \\
\underline{\cup} & & \underline{\cup} & & \underline{\cup} \\
\overline{B_{r/\lambda_i}^{M_i}(x_i')''} & \cong & G_i & \xrightarrow{i \to \infty} & S_r \\
\varphi_i^1 \big\downarrow & & f_i \big\downarrow & & \big\downarrow f_r \\
\bar{B}_{\alpha r/\lambda_i}^{M_i}(y_i') & \cong & \bar{B}_{\alpha r}^{\lambda_i M_i}(y_i') & \xrightarrow{i \to \infty} & \bar{B}_{\alpha r}^Y(y_\infty)
\end{array}$$

Figure 2.1: Sets and maps used to construct $f_r : S_r \to \bar{B}_{\alpha r}(y_\infty)$.

there exists a constant $C > 0$ such that

$$\begin{aligned}&\operatorname{vol}_Y(f_r(S_r \cap X_{\text{gen}}) \cap Y_{\text{gen}})\\&= \operatorname{vol}_Y(f_r(S_r \cap X_{\text{gen}}))\\&= C \cdot \operatorname{vol}_X(S_r \cap X_{\text{gen}})\\&= C \cdot \operatorname{vol}_X(S_r) > 0.\end{aligned}$$

Hence, there exists $x_0 \in S_r \cap X_{\text{gen}}$ with image $f_r(x_0) \in Y_{\text{gen}}$. By similar arguments as before, the α-bi-Lipschitz maps $\lambda f_r : (\lambda S_r, x_0) \to (\lambda f_r(S_r), y_0)$ which are defined by $\lambda f_r(x) := f_r(x)$ (sub)converge to an α-bi-Lipschitz map $f : S_\infty \to S'_\infty$ as $\lambda \to \infty$. Since x_0 and $f(x_0)$ are generic, one has $S_\infty \subseteq \mathbb{R}^{\dim(X)}$ and $S'_\infty \subseteq \mathbb{R}^{\dim(Y)}$. Furthermore, $\operatorname{vol}(S_\infty) > 0$. This implies

$$\dim(X) = \dim(Y).$$ □

2.4 Proof of the main proposition

Now Proposition 2.1 can be proven: The idea is to intersect the sets constructed in Lemma 2.9 and Lemma 2.18. For fixed base points x_i in the intersection and the λ_i of Lemma 2.18, the $(\lambda_i M_i, x_i)$ are both close to products $(\mathbb{R}^k \times K_i, \cdot)$ and converging to a product $(\mathbb{R}^k \times Y, \cdot)$ where the K_i are compact with diameter 1 and Y is some metric space. The following (technical) lemmata show that this space Y in fact is compact. Subsequently, the main proposition can be proven.

A map $f : (X, d_X) \to (Y, d_Y)$ between two metric spaces is called *ε-isometry*, where $\varepsilon > 0$, if $|d_Y(f(p), f(q)) - d_X(p, q)| < \varepsilon$ for all $p, q \in X$.

Lemma 2.26. *Let $R > r \geq 0$, $\varepsilon > 0$, $k \in \mathbb{N}$, $f : S_R := S_R^{\mathbb{R}^k}(0) \to \bar{B}_R^{\mathbb{R}^k}(0) \setminus B_{R-r}^{\mathbb{R}^k}(0)$ be a continuous ε-isometry with $\varepsilon < 2 \cdot (R - r)$ and define $\operatorname{pr} : \bar{B}_R^{\mathbb{R}^k}(0) \setminus B_{R-r}^{\mathbb{R}^k}(0) \to S_R$ by $\operatorname{pr}(p) := \frac{R}{\|p\|} \cdot p$. Then $\operatorname{pr} \circ f : S_R \to S_R$ is surjective.*

Proof. Denote the distance function on $\mathbb{R}^k$ by d and distinguish the two cases of $r = 0$ and $r > 0$: First, let $r = 0$, i.e. $f(S_R) \subseteq S_R$ and $\operatorname{pr} = \operatorname{id}$. Assume that there exists a point $p \in S_R \setminus f(S_R)$ and define $j : S_R \setminus \{p\} \to \mathbb{R}^{k-1}$ as the stereographic projection. Then the composition $j \circ f : S_R \to \mathbb{R}^{k-1}$ is continuous and, by the theorem of Borsuk-Ulam, there exist $\pm q \in S_R$ such that $j \circ f(q) = j \circ f(-q)$. Since j is a homeomorphism, $f(q) = f(-q)$, and hence, $\varepsilon > |d(f(q), f(-q)) - d(q, -q)| = 2R$. This is a contradiction. Therefore, f is surjective.

Now let $r > 0$ be arbitrary. For any $p, q \in \bar{B}_R^{\mathbb{R}^k}(0) \setminus B_{R-r}^{\mathbb{R}^k}(0)$,

$$\begin{aligned}&|d(\operatorname{pr} \circ f(p), \operatorname{pr} \circ f(q)) - d(p, q)|\\&\leq |d(\operatorname{pr} \circ f(p), \operatorname{pr} \circ f(q)) - d(f(p), f(q))| + |d(f(p), f(q)) - d(p, q)|\\&\leq d(\operatorname{pr} \circ f(p), f(p)) + d(\operatorname{pr} \circ f(q), f(q)) + \varepsilon\\&\leq 2r + \varepsilon\,.\end{aligned}$$

Thus, $\operatorname{pr} \circ f$ is a continuous $2r + \varepsilon$-isometry and, by the first part, surjective. □

The following lemma states that, if two products $\mathbb{R}^k \times X$ and $\mathbb{R}^k \times Y$ are sufficiently close and X is compact, then Y is compact as well with similar diameter as X.

Lemma 2.27. *Let (X, d_X) and (Y, d_Y) be complete length spaces, X be compact, $x_0 \in X$, $y_0 \in Y$ and define*

$$\mathrm{rad}_Y(y_0) := \sup\{d_Y(y, y_0) \mid y \in Y\}.$$

Let $k \in \mathbb{N}$, $R > \mathrm{diam}(X)$ and $\varepsilon > 0$. Then the following is true:

a) If $\mathrm{diam}(X) + 4\,\varepsilon \leq \frac{2R}{3}$ and $d_{GH}(B_R^{\mathbb{R}^k \times X}((0, x_0)), B_R^{\mathbb{R}^k \times Y}((0, y_0))) < \frac{\varepsilon}{2}$, then

$$\min\{R, \mathrm{rad}_Y(y_0)\}^2 \leq \mathrm{diam}(X)^2 + 2\,\varepsilon \cdot \mathrm{diam}(X) + 4\,\varepsilon(R + \varepsilon).$$

b) If $\mathrm{diam}(X) = 0$ and $d_{GH}(B_R^{\mathbb{R}^k \times X}((0, x_0)), B_R^{\mathbb{R}^k \times Y}((0, y_0))) < \frac{R}{12}$, then Y is compact with $\mathrm{diam}(Y) < 2R$.

c) If $\mathrm{diam}(X) = 1$, $R \geq 100$ and $d_{GH}(B_R^{\mathbb{R}^k \times X}((0, x_0)), B_R^{\mathbb{R}^k \times Y}((0, y_0))) < \frac{1}{100R}$, then Y is compact with $c \leq \mathrm{diam}(Y) \leq 5c$ for some constant $c > 0$.

d) If $1000 \cdot \mathrm{diam}(X) \leq R < \frac{1}{2} \cdot \mathrm{diam}(Y)$, then

$$d_{GH}(B_R^{\mathbb{R}^k \times X}((0, x_0)), B_R^{\mathbb{R}^k \times Y}((0, y_0))) \geq 20 \cdot \mathrm{diam}(X).$$

Proof. a) The idea is to map both the set $S_R \times \{y_0\}$ and the point $(0, y_0)$ to $\mathbb{R}^k \times X$ via ε-approximations, to take the projection onto the Euclidean factor and to find an upper and lower estimate for the distance of the obtained set and point. Finally, comparison of this upper and lower bound gives the result.

As in Proposition 1.8, let (f, g) be ε-approximations between $\bar{B}_R^{\mathbb{R}^k \times X}((0, x_0))$ and $\bar{B}_R^{\mathbb{R}^k \times Y}((0, y_0))$ with $f((0, x_0)) = (0, y_0)$ and $g((0, y_0)) = (0, x_0)$. Let

$$d := \mathrm{diam}(X) \quad \text{and} \quad \delta := \min\{R, \mathrm{rad}_Y(y_0)\}.$$

For each $n \in \mathbb{N}$, $n \geq 1$, choose $y_n \in Y$ such that $\delta - \frac{1}{n} \leq \delta_n := d_Y(y_n, y_0) \leq \delta$. (If $\mathrm{rad}_Y(y_0) > R$, choose $y_n \in \partial\bar{B}_R(0)$ which is nonempty since Y is a length space; otherwise, by definition, there exists a sequence $\bar{y}_n$ satisfying $d_Y(\bar{y}_n, y_0) > \delta - \frac{1}{n}$.) In particular, δ_n is convergent with limit δ.

Let $S_R := \partial\bar{B}_R^{\mathbb{R}^k}(0) \subseteq \mathbb{R}^k$ and $S := g(S_R \times \{y_0\}) \subseteq \mathbb{R}^k \times X$, cf. Figure 2.2. Then

$$\begin{aligned}
&\sqrt{R^2 + \delta_n^2} - \varepsilon \\
&= d_{\mathbb{R}^k \times Y}((0, y_n), S_R \times \{y_0\}) - \varepsilon \\
&\leq d_{\mathbb{R}^k \times X}(g(0, y_n), S) \\
&= \min\left\{\sqrt{d_{\mathbb{R}^k}(\mathrm{pr}_{\mathbb{R}^k}(g(0, y_n)), \mathrm{pr}_{\mathbb{R}^k}(p))^2 + d_X(\mathrm{pr}_X(g(0, y_n)), \mathrm{pr}_X(p))^2} \mid p \in S\right\} \\
&\leq \sqrt{d_{\mathbb{R}^k}(\mathrm{pr}_{\mathbb{R}^k}(g(0, y_n)), \mathrm{pr}_{\mathbb{R}^k}(S))^2 + d^2}
\end{aligned}$$

$$\begin{array}{ccc} B_R^{\mathbb{R}^k \times Y}((0,y_0)) & \xrightarrow{\quad g \quad} & B_R^{\mathbb{R}^k \times X}((0,x_0)) \\ \cup\!| & & \cup\!| \\ \partial \bar{B}_R^{\mathbb{R}^k}(0) \times \{y_0\} = S_R \times \{y_0\} & \xrightarrow{\quad g \quad} & g(S_R \times \{y_0\}) = S \end{array}$$

Figure 2.2: Definition of S.

proves the lower bound

$$d_{\mathbb{R}^k}(\mathrm{pr}_{\mathbb{R}^k}(g(0,y_n)), \mathrm{pr}_{\mathbb{R}^k}(S)) \geq \sqrt{(\sqrt{R^2 + \delta_n^2} - \varepsilon)^2 - d^2}.$$

In order to find the upper bound, choose a number $m \in \mathbb{N}$, $m \geq 1$, with $\frac{2}{m} \leq \varepsilon$ and let Δ be a spherical triangulation of S_R such that the set of vertices $\tilde{\Gamma}$ of Δ is a finite $\frac{1}{m}$-net in S_R and each two vertices of a simplex have (spherical) distance at most $\frac{1}{m}$. (Notice that their Euclidean distance is at most $\frac{1}{m}$ as well.) Define $\Gamma := \tilde{\Gamma} \times \{y_0\}$ and $h := \mathrm{pr}_{\mathbb{R}^k} \circ g : \Gamma \to \mathbb{R}^k$ and extend h to a continuous map $H : S_R \times \{y_0\} \to \bar{B}_R^{\mathbb{R}^k}(0) \setminus B_{R-(d+3\,\varepsilon)}^{\mathbb{R}^k}(0)$ by mapping each (spherical) simplex of Δ with vertices γ_i continuously to the corresponding (Euclidean) simplex in $\mathbb{R}^k$ with vertices $h(\gamma_i)$, cf. Figure 2.3. Since Γ is finite, H is continuous.

Then $h(\Gamma)$ defines an $(\frac{1}{m} + \varepsilon)$-net in $H(S_R \times \{y_0\})$: Since each two vertices of a simplex in Δ have (Euclidean) distance at most $\frac{1}{m}$, their images have distance at most $\frac{1}{m} + \varepsilon$. Hence, each points $x \in H(S_R \times \{y_0\})$ is contained in a Euclidean simplex whose vertices have pairwise distance at most $\frac{1}{m} + \varepsilon$. Recall that, since the simplex is Euclidean, x has distance at most $\frac{1}{m} + \varepsilon$ to each of these vertices. Let $h(\gamma)$ denote one of those vertices. In particular, $x \in \bar{B}_{1/m+\varepsilon}(h(\gamma))$, and this proves $H(S_R \times \{y_0\}) \subseteq \bigcup\{\bar{B}_{1/m+\varepsilon}(\gamma') \mid \gamma' \in H(\Gamma)\}$.

Furthermore, H is a $(5\,\varepsilon + d)$-isometry: Let $p, q \in S_R$ be arbitrary. Choose points $\gamma_p, \gamma_q \in \tilde{\Gamma}$ such that $d_{\mathbb{R}^k}(p, \gamma_p) \leq \frac{1}{m}$ and $d_{\mathbb{R}^k}(q, \gamma_q) \leq \frac{1}{m}$. By construction,

$$d_{\mathbb{R}^k}(H(p, y_0), h(\gamma_p, y_0)) \leq \frac{1}{m} + \varepsilon,$$

and thus,

$$\begin{aligned} &|d_{\mathbb{R}^k}(H(p,y_0), H(q,y_0)) - d_{\mathbb{R}^k \times Y}((p,y_0),(q,y_0))| \\ &\leq |d_{\mathbb{R}^k}(H(p,y_0), H(q,y_0)) - d_{\mathbb{R}^k}(h(\gamma_p,y_0), h(\gamma_q,y_0))| \\ &\quad + |d_{\mathbb{R}^k}(h(\gamma_p,y_0), h(\gamma_q,y_0)) - d_{\mathbb{R}^k \times X}(g(\gamma_p,y_0), g(\gamma_q,y_0))| \\ &\quad + |d_{\mathbb{R}^k \times X}(g(\gamma_p,y_0), g(\gamma_q,y_0)) - d_{\mathbb{R}^k \times Y}((\gamma_p,y_0),(\gamma_q,y_0))| \\ &\quad + |d_{\mathbb{R}^k}(\gamma_p,\gamma_q) - d_{\mathbb{R}^k}(p,q)| \end{aligned}$$

$$
\begin{array}{ccc}
\Gamma = \tilde{\Gamma} \times \{y_0\} & \xrightarrow{h = \mathrm{pr}_{\mathbb{R}^k} \circ g} & h(\Gamma) \\
\cap\!\!\mid & & \cap\!\!\mid \\
S_R \times \{y_0\} & \xrightarrow{H} & H(S_R \times \{y_0\}) \subseteq \mathbb{R}^k \\
\cup\!\!\mid & & \cup\!\!\mid \\
\Delta(\gamma_1, \ldots, \gamma_m) & \longmapsto & \Delta(h(\gamma_1), \ldots, h(\gamma_m))
\end{array}
$$

Figure 2.3: Definition of H.

$$
\begin{aligned}
&\le d_{\mathbb{R}^k}(H(p, y_0), h(\gamma_p, y_0)) + d_{\mathbb{R}^k}(H(q, y_0), h(\gamma_q, y_0)) \\
&\quad + \big(d_{\mathbb{R}^k}(\mathrm{pr}_{\mathbb{R}^k} \circ g\,(\gamma_p, y_0), \mathrm{pr}_{\mathbb{R}^k} \circ g\,(\gamma_q, y_0))^2 \\
&\qquad\quad + d_X(\mathrm{pr}_X \circ g\,(\gamma_p, y_0), \mathrm{pr}_X \circ g\,(\gamma_q, y_0))^2\big)^{1/2} \\
&\qquad - d_{\mathbb{R}^k}(\mathrm{pr}_{\mathbb{R}^k} \circ g\,(\gamma_p, y_0), \mathrm{pr}_{\mathbb{R}^k} \circ g\,(\gamma_q, y_0)) \\
&\quad + \varepsilon \\
&\quad + d_{\mathbb{R}^k}(p, \gamma_p) + d_{\mathbb{R}^k}(q, \gamma_q) \\
&\le 2 \cdot \Big(\frac{1}{m} + \varepsilon\Big) + d_X(\mathrm{pr}_X \circ g\,(\gamma_p, y_0), \mathrm{pr}_X \circ g\,(\gamma_q, y_0)) + \varepsilon + 2 \cdot \frac{1}{m} \\
&\le \frac{4}{m} + 3\,\varepsilon + d \\
&\le 5\,\varepsilon + d.
\end{aligned}
$$

Finally, verify $H(S_R \times \{y_0\}) \subseteq \bar{B}_R(0) \setminus B_{R-(d+3\,\varepsilon)}(0)$: Let $p \in S_R$ be arbitrary and choose $\gamma_p \in \Gamma$ such that $d(p, \gamma_p) \le \frac{1}{m}$. Then

$$
\begin{aligned}
&d_{\mathbb{R}^k}(h(\gamma_p, y_0), \mathrm{pr}_{\mathbb{R}^k}(0, x_0)) \\
&= d_{\mathbb{R}^k}(\mathrm{pr}_{\mathbb{R}^k}(g(\gamma_p, y_0)), \mathrm{pr}_{\mathbb{R}^k}(g(0, y_0))) \\
&= \sqrt{d_{\mathbb{R}^k \times X}(g(\gamma_p, y_0), g(0, y_0))^2 - d_X(\mathrm{pr}_X \circ g\,(\gamma_p, y_0), \mathrm{pr}_X \circ g\,(0, y_0))^2} \\
&\ge \sqrt{(d_{\mathbb{R}^k \times Y}((\gamma_p, y_0), (0, y_0)) - \varepsilon)^2 - d^2} \\
&= \sqrt{(R - \varepsilon)^2 - d^2},
\end{aligned}
$$

and hence,

$$
\begin{aligned}
&d_{\mathbb{R}^k}(H(p, y_0), 0) \\
&\ge d_{\mathbb{R}^k}(h(\gamma_p, y_0), \mathrm{pr}_{\mathbb{R}^k}(0, x_0)) - d_{\mathbb{R}^k}(H(p, y_0), h(\gamma_p, y_0)) \\
&\ge \sqrt{(R - \varepsilon)^2 - d^2} - \Big(\frac{2}{m} + \varepsilon\Big) \\
&\ge ((R - \varepsilon) - d) - 2\,\varepsilon \\
&= R - (d + 3\,\varepsilon).
\end{aligned}
$$

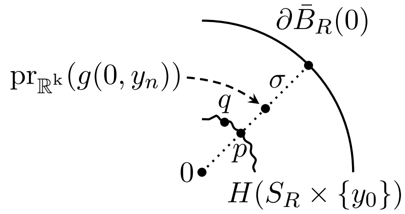

Figure 2.4: Points used to estimate $d_{\mathbb{R}^k}(\mathrm{pr}_{\mathbb{R}^k}(g(0,y_n)), \mathrm{pr}_{\mathbb{R}^k}(S))$.

Then the image of H intersects each segment σ from the origin to a point in $\partial\bar{B}_R(0)$: By Lemma 2.26, since $2(d+3\,\varepsilon)+(5\,\varepsilon+d)\leq 3(d+4\,\varepsilon)<2R$, the segment σ intersects $\partial\bar{B}_R(0)$ in a point contained in $\mathrm{pr}\circ H(S_R\times\{y_0\})$ where pr is the radial projection to the sphere defined as in Lemma 2.26. Since the projection is radial, σ intersects $H(S_R\times\{y_0\})$ as well.

Let p be this intersecting point for the segment through $\mathrm{pr}_{\mathbb{R}^k}(g(0,y_n))$, cf. Figure 2.4. Since $h(\Gamma\times\{y_0\})$ is a $(\frac{2}{m}+\varepsilon)$-net in $H(S_R)$, there exists a point $q\in h(\Gamma)$ such that $d_{\mathbb{R}^k}(p,q)\leq\frac{2}{m}+\varepsilon$. Thus, using $\mathrm{pr}_{\mathbb{R}^k}(S)=\mathrm{pr}_{\mathbb{R}^k}\circ g(S_R\times\{y_0\})\supseteq h(\Gamma)\ni q$ and that the segment from $\mathrm{pr}_{\mathbb{R}^k}(g(0,y_n))$ to p is part of a segment connecting the origin and the boundary of the R-ball,

$$\begin{aligned}
d_{\mathbb{R}^k}(\mathrm{pr}_{\mathbb{R}^k}(g(0,y_n)),\mathrm{pr}_{\mathbb{R}^k}(S)) &\leq d_{\mathbb{R}^k}(\mathrm{pr}_{\mathbb{R}^k}(g(0,y_n)),q)\\
&\leq d_{\mathbb{R}^k}(\mathrm{pr}_{\mathbb{R}^k}(g(0,y_n)),p)+d_{\mathbb{R}^k}(p,q)\\
&\leq R+\Big(\frac{2}{m}+\varepsilon\Big).
\end{aligned}$$

Now $m\to\infty$ proves

$$\sqrt{(\sqrt{R^2+\delta_n^2}-\varepsilon)^2-d^2}\leq d_{\mathbb{R}^k}(\mathrm{pr}_{\mathbb{R}^k}(g(0,y_n)),\mathrm{pr}_{\mathbb{R}^k}(S))\leq R+\varepsilon,$$

and thus,

$$\begin{aligned}
\delta_n &\leq \sqrt{(\sqrt{(R+\varepsilon)^2+d^2}+\varepsilon)^2-R^2}\\
&= \sqrt{2R\,\varepsilon+d^2+2\,\varepsilon^2+2\sqrt{(R\,\varepsilon+\varepsilon^2)^2+(\varepsilon\,d)^2}}\\
&\leq \sqrt{d^2+2\,\varepsilon\,d+4\,\varepsilon(R+\varepsilon)}.
\end{aligned}$$

Since this is true for all n and $\delta_n\to\delta$ as $n\to\infty$, this proves the claim.

b) Let $\varepsilon:=\frac{R}{6}$. Then $\mathrm{diam}(X)+4\,\varepsilon=\frac{2R}{3}$, and by a),

$$\min\{R,\mathrm{rad}_Y(y_0)\}^2\leq 4\,\varepsilon\cdot(R+\varepsilon)=\frac{24}{25}\cdot R^2<R^2.$$

Thus, $\mathrm{rad}_Y(y_0)<R$, and this implies $\mathrm{diam}(Y)\leq 2\cdot\mathrm{rad}_Y(y_0)<2R$.

c) Let $\varepsilon := \frac{1}{50R}$. Then $\operatorname{diam}(X) + 4\varepsilon \leq 1 + \frac{2}{25} < R \cdot \frac{\pi}{3}$ and a) can be applied. Since $R\varepsilon = \frac{1}{50} = 2 \cdot 10^{-2}$ and $\varepsilon \leq 2 \cdot 10^{-4}$,

$$\begin{aligned}\min\{R, \operatorname{rad}_Y(y_0)\}^2 &\leq \operatorname{diam}(X)^2 + 2\varepsilon \operatorname{diam}(X) + 4\varepsilon(R+\varepsilon) \\ &\leq \operatorname{diam}(X)^2 + 4 \cdot 10^{-4} \cdot \operatorname{diam}(X) + 8 \cdot 10^{-2} + 4 \cdot 10^{-8}.\end{aligned}$$

Using $\operatorname{diam}(X) = 1$,

$$\begin{aligned}\min\{R, \operatorname{rad}_Y(y_0)\}^2 &\leq 1 + 2 \cdot 10^{-2} + 10^{-4} + 10^{-8} \\ &< 1.05^2 \\ &< R^2.\end{aligned}$$

In particular, $\operatorname{diam}(Y) \leq 2 \cdot \operatorname{rad}_Y(y_0) < 2 \cdot 1.05 = \frac{21}{10}$.

On the other hand, by permuting X and Y,

$$\begin{aligned}\frac{1}{4} = \left(\frac{\operatorname{diam}(X)}{2}\right)^2 &\leq \operatorname{rad}_X(x_0)^2 \\ &= \min\{R, \operatorname{rad}_X(x_0)\}^2 \\ &\leq \operatorname{diam}(Y)^2 + 4 \cdot 10^{-4} \cdot \operatorname{diam}(Y) + 8 \cdot 10^{-2} + 16 \cdot 10^{-8},\end{aligned}$$

and this implies $\operatorname{diam}(Y) \geq \frac{21}{50} =: c$.

d) Assume $d_{GH}(B_R^{\mathbb{R}^k \times X}((0, x_0)), B_R^{\mathbb{R}^k \times Y}((0, y_0))) < 20d$ for $d := \operatorname{diam}(X)$ and define $\varepsilon := 40d$. By choice of R,

$$d \leq \frac{R}{1000} \quad \text{and} \quad \varepsilon \leq \frac{R}{25}.$$

In particular, $\operatorname{diam}(X) + 4\varepsilon = 161 \cdot d \leq \frac{161}{1000} \cdot R < \frac{2R}{3}$. Furthermore,

$$2 \operatorname{rad}_Y(y_0) \geq \operatorname{diam}(Y) > 2R.$$

By a),

$$\begin{aligned}R^2 &= \min\{R, \operatorname{rad}_Y(y_0)\}^2 \\ &\leq d^2 + 2\varepsilon d + 4\varepsilon(R+\varepsilon) \\ &\leq \frac{R^2}{10^6} + \frac{2R^2}{25 \cdot 10^3} + \frac{4R}{25} \cdot \frac{26R}{25} = \frac{166481}{10^6} \cdot R^2 < R^2.\end{aligned}$$

This is a contradiction. □

Using these results, the main proposition of this chapter finally can be proven.

Proposition 2.1. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a collapsing sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$ and converge to a limit (X, p) of dimension $k < n$. Given $\hat{\varepsilon} \in (0,1)$, there exists $\hat{\delta} = \hat{\delta}(\hat{\varepsilon}; n, k) > 0$ such that for any $0 < r \leq \hat{\delta}$ and $q_i \in M_i$ with*

$$d_{GH}((r^{-1}M_i, q_i), (\mathbb{R}^k, 0)) \leq \hat{\delta}$$

there are a family of subsets of good points $G_r(q_i) \subseteq B_r(q_i)$ with

$$\mathrm{vol}(G_r(q_i)) \geq (1-\hat{\varepsilon}) \cdot \mathrm{vol}(B_r(q_i))$$

and a sequence $\lambda_i \to \infty$ such that the following holds:

a) For every choice of base points $x_i \in G_r(q_i)$ and all sublimits $(Y, \cdot)$ of $(\lambda_i M_i, x_i)$ there exists a compact metric space K of dimension $l \leq n-k$ satisfying $\frac{1}{5} \leq \mathrm{diam}(K) \leq 1$ such that Y splits isometrically as a product

$$Y \cong \mathbb{R}^k \times K.$$

b) If $x_i^1, x_i^2 \in G_r(q_i)$ are base points such that, after passing to a subsequence,

$$(\lambda_i M_i, x_i^j) \to (\mathbb{R}^k \times K_j, \cdot)$$

for $1 \leq j \leq 2$ as before, then $\dim(K_1) = \dim(K_2)$.

Proof. Given $\hat{\varepsilon} \in (0,1)$, let

$$\hat{\delta}_1 = \hat{\delta}_1(\hat{\varepsilon}; n, k) > 0 \text{ be the } \hat{\delta}_1\Big(\frac{\hat{\varepsilon}}{2}; n, k\Big) \text{ of Lemma 2.9,}$$
$$\hat{\delta}_2 = \hat{\delta}_2(\hat{\varepsilon}; n, k) > 0 \text{ be the } \hat{\delta}_2\Big(\frac{\hat{\varepsilon}}{2}; n, k\Big) \text{ of Lemma 2.18}$$

and define

$$\hat{\delta} = \hat{\delta}(\hat{\varepsilon}; n, k) := \frac{1}{16} \cdot \min\{\hat{\delta}_1, \hat{\delta}_2\} > 0.$$

Furthermore, let $\varepsilon_0 = \varepsilon_0(n,k) \in (0, \frac{1}{100})$ be as in Lemma 2.17. Let $0 < r \leq \hat{\delta}$ and $q_i \in M_i$ with

$$d_{GH}((r^{-1}M_i, q_i), (\mathbb{R}^k, 0)) \leq \hat{\delta}.$$

In particular, by Lemma 1.11 b),

$$d_{GH}((r^{-1}M_i, q_i), (\mathbb{R}^k, 0)) \leq \hat{\delta}_1 \quad \text{and} \quad d_{GH}((r^{-1}M_i, q_i), (\mathbb{R}^k, 0)) \leq \hat{\delta}_2.$$

The remaining proof splits into several steps: First, define the family of subsets $G_r(q_i) \subseteq B_r(q_i)$ and the rescaling sequence $\lambda_i \to \infty$ and verify the volume estimate. Secondly, check that sublimits of the rescaled sequences split into a product of $\mathbb{R}^k$ and a compact factor with the claimed diameter bound. Finally, prove that each two sublimits coming from the same subsequence have the same dimension.

By Lemma 2.9, there is a family of subsets

$$G_r^1(q_i) \subseteq B_r(q_i)$$

with $\mathrm{vol}(G_r^1(q_i)) \geq (1 - \frac{\hat{\varepsilon}}{2}) \cdot B_r(q_i)$ such that for arbitrary $\mu_i \to \infty$ and $x_i \in G_r^1(q_i)$ each sublimit of $(\mu_i M_i, x_i)$ splits off an $\mathbb{R}^k$-factor. Furthermore, Lemma 2.18 gives a sequence $\lambda_i \to \infty$ and a family of subsets

$$G_r^2(q_i) \subseteq B_r(q_i)$$

satisfying $\mathrm{vol}(G_r^2(q_i)) \geq (1-\frac{\hat{\varepsilon}}{2}) \cdot \mathrm{vol}(B_r(q_i))$ such that for all $x_i \in G_r^2(q_i)$ there is a compact metric space K_i with $\mathrm{diam}(K_i) = 1$ and $\tilde{x}_i \in \{0\} \times K_i$ satisfying

$$d_{GH}(B_{1/\varepsilon_0}^{\lambda_i M_i}(x_i), B_{1/\varepsilon_0}^{\mathbb{R}^k \times K_i}(\tilde{x}_i)) \leq \frac{\varepsilon_0}{200}$$

and q_i has the $\mathcal{C}(M_i, r, \frac{9^n \cdot 10^{n^2}}{\lambda_i}, \frac{\hat{\varepsilon}}{2})$-property.

Fix this $\lambda_i \to \infty$ and define $G_r(q_i) := G_r^1(q_i) \cap G_r^2(q_i) \subseteq B_r(q_i)$. Clearly,

$$\mathrm{vol}(G_r(q_i)) \geq (1 - \hat{\varepsilon}) \cdot \mathrm{vol}(B_r(q_i)).$$

Let $x_i \in G_r(q_i)$ and (Y, y) be a sublimit of $(\lambda_i M_i, x_i)$. Using $x_i \in G_r^1(q_i)$, there are a metric space Y' and $y' \in \{0\} \times Y'$ such that $(Y, y) \cong (\mathbb{R}^k \times Y', y')$. On the other hand, since $x_i \in G_r^2(q_i)$,

$$d_{GH}(B_{1/\varepsilon_0}^{\lambda_i M_i}(x_i), B_{1/\varepsilon_0}^{\mathbb{R}^k \times K_i}(\tilde{x}_i)) \leq \frac{\varepsilon_0}{200}$$

for some compact metric space K_i with diameter 1 and $\tilde{x}_i \in \{0\} \times K_i$. Hence, by the triangle inequality and for i large enough,

$$d_{GH}(B_{1/\varepsilon_0}^{\mathbb{R}^k \times Y'}(y'), B_{1/\varepsilon_0}^{\mathbb{R}^k \times K_i}(\tilde{x}_i)) < \frac{\varepsilon_0}{100}.$$

By Lemma 2.27 c), there exists a constant $c > 0$ such that Y' is compact as well with $c \leq \mathrm{diam}(Y') \leq 5c$, and after rescaling with $\frac{1}{c}$ this finishes the first part of the claim.

So let $x_i, y_i \in G_r(q_i)$ and K_1 and K_2 be compact metric spaces such that, after passing to a subsequence,

$$(\lambda_i M_i, x_i) \to (\mathbb{R}^k \times K_1, x_\infty) \quad \text{and} \quad (\lambda_i M_i, y_i) \to (\mathbb{R}^k \times K_2, y_\infty).$$

Because of $x_i, y_i \in G_r^2(q_i)$, there is a time-dependent, piecewise constant in time vector field X_i with compact support and an integral curve c_i such that the vector field X_i^t is divergence free on $B_{10r}(c_i(t))$ for all $0 \leq t \leq 1$,

$$d(x_i, c_i(0)) < \frac{c(n)}{\lambda_i} \quad \text{and} \quad d(y_i, c_i(1)) < \frac{c(n)}{\lambda_i}$$

for $c(n) = 9^n \cdot 10^{n^2}$ and

$$\int_0^1 (\mathrm{Mx}_{2r}(\|\nabla.X_i^t\|^{3/2})(c_i(t)))^{2/3}\,dt < \frac{\hat{\varepsilon}}{2}.$$

Let $x_i' := c_i(0)$ and $y_i' := c_i(1)$. Since $d_{\lambda_i M_i}(x_i, x_i') \leq c(n)$ and $d_{\lambda_i M_i}(y_i, y_i') \leq c(n)$, there exists $x_\infty' \in \mathbb{R}^k \times K_1$ (cf. Lemma 1.10 b)), such that, after passing to a subsequence

$$(\lambda_i M_i, x_i') \to (\mathbb{R}^k \times K_1, x_\infty').$$

After passing to a further subsequence,

$$(\lambda_i M_i, y_i') \to (\mathbb{R}^k \times K_2, y_\infty')$$

for some $y_\infty' \in \mathbb{R}^k \times K_2$. Then Lemma 2.25 implies

$$\dim(K_1) = \dim(\mathbb{R}^k \times K_1) - k = \dim(\mathbb{R}^k \times K_2) - k = \dim(K_2). \qquad \square$$

Chapter 3

Global construction

Based on the 'local' version (Proposition 2.1) established in the last chapter, the proof of the following main result can now be given.

Theorem 3.1. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a collapsing sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound* $\mathrm{Ric}_{M_i} \geq -(n-1)$ *and converge to a limit (X, p) of dimension $k < n$. For $\varepsilon \in (0, 1)$ there exist a family of subsets of good points $G_1(p_i) \subseteq B_1(p_i)$ satisfying*

$$\mathrm{vol}(G_1(p_i)) \geq (1-\varepsilon) \cdot \mathrm{vol}(B_1(p_i)),$$

a sequence $\lambda_i \to \infty$ and a constant $D > 0$ such that for every choice of base points $q_i \in G_1(p_i)$ and every sublimit (Y, q) of $(\lambda_i M_i, q_i)$ there is a compact metric space K of dimension $l \leq n-k$ with $\frac{1}{D} \leq \mathrm{diam}(K) \leq D$ such that Y splits isometrically as a product

$$Y \cong \mathbb{R}^k \times K.$$

Moreover, for sequences $q_i^1, q_i^2 \in G_1(p_i)$ such that, after passing to a subsequence,

$$(\lambda_i M_i, q_i^j) \to (\mathbb{R}^k \times K_j, \cdot)$$

for $1 \leq j \leq 2$ as before, $\mathrm{dim}(K_1) = \mathrm{dim}(K_2)$.

The idea of the proof is to take (finitely many) sequences $(q_i)_{i\in\mathbb{N}}$ satisfying the hypothesis of Proposition 2.1 for some $r > 0$ and to define $G_1(p_i)$ as the union of the $G_r(q_i)$ obtained from Proposition 2.1. Instantly, the following question occurs:

(1) Why do sequences $(q_i)_{i\in\mathbb{N}}$ satisfying the hypothesis of Proposition 2.1 exist?

It will turn out that sequences $p_i^x \to x$, where $x \in X$ is a generic point, are candidates for these $(q_i)_{i\in\mathbb{N}}$: If $x \in X$ is generic, then $(\frac{1}{r}X, x)$ is close to $(\mathbb{R}^k, 0)$ for sufficiently small $r > 0$ and so is $(\frac{1}{r}M_i, p_i^x)$ for sufficiently large $i \in \mathbb{N}$. In fact, decreasing r only improves the situation.

Now assume that x, x' are two such generic points, let $r > 0$ be small enough and $\lambda_i \to \infty$ and $\lambda'_i \to \infty$, respectively, be the sequences given by Proposition 2.1. These sequences might be different, but Theorem 3.1 calls for one single rescaling sequence. This gives rise to the following question:

(2) Does Proposition 2.1 still hold for $(p_i^{x'})_{i\in\mathbb{N}}$ if $\lambda'_i \to \infty$ is replaced by $\lambda_i \to \infty$?

In order to answer this question, first consider the special case of $\lambda_i = 2\lambda'_i$: Obviously, if $q_i \in G_r(p_i^{x'})$ and $(\mathbb{R}^k \times K, \cdot)$ is a sublimit of $(\lambda'_i M_i, q_i)$, then $(\mathbb{R}^k \times 2K, \cdot)$ is a sublimit of $(\lambda_i M_i, q_i) = (2\lambda'_i M_i, q_i)$. Conversely, every sublimit of $(\lambda_i M_i, q_i)$ has the form $(\mathbb{R}^k \times 2K, \cdot)$ for a sublimit $(\mathbb{R}^k \times K, \cdot)$ of $(\lambda'_i M_i, q_i)$. It turns out that such a correspondence holds whenever the sequence $\left(\frac{\lambda'_i}{\lambda_i}\right)_{i\in\mathbb{N}}$ is bounded. In this way, λ'_i indeed can be replaced by λ_i if one allows weaker diameter bounds for the compact factors of the sublimits. Therefore, the question (2) can be reformulated in the following way:

(2') Under which condition is the quotient $\left(\frac{\lambda'_i}{\lambda_i}\right)_{i\in\mathbb{N}}$ of two such rescaling sequences bounded?

In fact, one can prove the following: If the subsets $G_r(p_i^x)$ and $G_r(p_i^{x'})$ have non-empty intersection, then $\left(\frac{\lambda'_i}{\lambda_i}\right)_{i\in\mathbb{N}}$ is bounded. An obvious approach for comparing points where these subsets do not intersect is to connect the points by a curve consisting of generic points only and to cover this curve by balls $B_{r_j}(y_j)$ such that for every two subsequent points y_j and y_{j+1} (and sufficiently large $i \in \mathbb{N}$) the subsets intersect. If this can be done using only finitely many y_j, an inductive argument proves the boundedness of $\left(\frac{\lambda'_i}{\lambda_i}\right)_{i\in\mathbb{N}}$. Usually, such covers are constructed by using a compactness argument: Let r^y denote the minimal radius such that all $r \leq r^y$ and p_i^y satisfy the hypothesis of Proposition 2.1. If this r^y is continuous in y, there exists $r > 0$ that can be used at each point of the (compact image of the) curve. Unfortunately, there is no reason for this r^y to depend continuously on y. It will turn out that a similar approach to compare λ_i and λ'_i can be performed if x and x' lie in the interior of a minimising geodesic such that all the points of this geodesic lying between x and x' are generic. The Hölder continuity result of Colding and Naber [CN12, Theorem 1.2] then allows a cover similar to the one described above. The subsequent question

(3) Does there exists a minimising geodesic such that x, x' lie in its interior?

can be answered affirmatively for a set of full measure (in $X \times X$) by applying further results of [CN12].

This chapter is subdivided into several sections answering the above questions: First, section 3.1 investigates generic points $x \in X$ and answers question (1) by applying Proposition 2.1 to the sequence $p_i^x \to x$. Both questions (2) and (2') are dealt with in section 3.2, which discusses the comparison of the different λ_i. Afterwards, section 3.3 treats question (3) by proving that the necessary conditions for performing the comparison are given on a set of full measure. Finally, section 3.4 deals with the proof of Theorem 3.1.

3.1 Application to generic points

A very important property of generic points is that, after rescaling, the manifolds with base points converging to a generic point are in some sense close to the Euclidean space.

Lemma 3.2. *Let (X,p) be the limit of a collapsing sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $-(n-1)$, $k=\dim(X)<n$, $x\in X_{\mathrm{gen}}$ and $p_i^x\to x$. For fixed $R>0$ and $\varepsilon>0$ there exists $\lambda_0=\lambda_0(x,R,\varepsilon)$ such that for all $\lambda\geq\lambda_0$,*

$$d_{GH}(B_R^{\lambda X}(x),B_R^{\mathbb{R}^k}(0))\leq\varepsilon\,.$$

Proof. Since x is a generic point, all tangent cones at x equal $\mathbb{R}^k$, i.e. $(\lambda X,x)\to(\mathbb{R}^k,0)$ for $\lambda\to\infty$. In particular, the R-balls converge and for sufficiently large λ the distance of these balls is bounded from above by ε. This proves that there exists

$$\lambda_0(x):=\min\{\lambda\geq1\mid\forall\mu\geq\lambda:d_{GH}(B_R^{\mu X}(x),B_R^{\mathbb{R}^k}(0))\leq\varepsilon\}<\infty.\qquad\square$$

Notation. From now on, for given $k<n$ and $\hat\varepsilon\in(0,1)$, let $\hat\delta=\hat\delta(\hat\varepsilon;n,k)$ be as in Proposition 2.1. For $r>0$, define

$$\mathcal{X}_r(\hat\varepsilon;n,k):=\Big\{x\in X_{\mathrm{gen}}\mid d_{GH}(B_{1/\hat\delta}^{r^{-1}X}(x),B_{1/\hat\delta}^{\mathbb{R}^k}(0))\leq\frac{\hat\delta}{2}\Big\}.$$

Lemma 3.3. *Let $(M_i,p_i)_{i\in\mathbb{N}}$ be a collapsing sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i}\geq-(n-1)$ and converge to a limit (X,p) of dimension $k<n$ and let $\hat\varepsilon\in(0,1)$.*

a) For $x\in X_{\mathrm{gen}}$ there is $0<r^x=r(\hat\varepsilon,x;n,k)\leq\hat\delta$ such that $x\in\mathcal{X}_r(\hat\varepsilon;n,k)$ for any $0<r\leq r^x$.

b) For $0<r\leq\hat\delta$, $x\in\mathcal{X}_r(\hat\varepsilon;n,k)$ and $p_i^x\to x$ there is $i_0\in\mathbb{N}$ such that for $i\geq i_0$ there are a subset of good points $G_r(p_i^x)\subseteq B_r(p_i^x)$ with

$$\mathrm{vol}(G_r(p_i^x))\geq(1-\hat\varepsilon)\cdot\mathrm{vol}(B_r(p_i^x))$$

and a sequence $\lambda_i\to\infty$ satisfying the following:

(i) For any choice of base points $x_i\in G_r(p_i^x)$ and all sublimits $(Y,\cdot)$ of (λ_iM_i,x_i) there exists a compact metric space K of dimension $l\leq n-k$ and diameter $\frac{1}{5}\leq\mathrm{diam}(K)\leq1$ such that Y splits isometrically as a product

$$Y\cong\mathbb{R}^k\times K.$$

(ii) If $x_i^1,x_i^2\in G_r(p_i^x)$ are base points such that, after passing to a subsequence,

$$(\lambda_iM_i,x_i^j)\to(\mathbb{R}^k\times K_j,\cdot)$$

for $1\leq j\leq2$ as before, then $\dim(K_1)=\dim(K_2)$.

Moreover, if ω is a fixed ultrafilter on $\mathbb{N}$, there exists $l \in \mathbb{N}$ such that the following holds: Given $q_i \in G_r(p_i^x)$, the ultralimit of $(\lambda_i M_i, q_i)$ is a product $\mathbb{R}^k \times K$ such that K is compact with

$$\frac{1}{5} \leq \operatorname{diam}(K) \leq 1 \quad \textit{and} \quad \dim(K) = l.$$

Proof. a) Let $\lambda_0 = \lambda_0(\hat{\varepsilon}; n, k)$ be the $\lambda_0(x, \frac{1}{\hat{\delta}}, \frac{\hat{\delta}}{2})$ appearing in Lemma 3.2 such that for all $\lambda \geq \lambda_0$,

$$d_{GH}(B_{1/\hat{\delta}}^{\lambda X}(x), B_{1/\hat{\delta}}^{\mathbb{R}^k}(0)) \leq \frac{\hat{\delta}}{2}.$$

Then $r^x := r(\hat{\varepsilon}, x; n, k) := \min\left\{\hat{\delta}, \frac{1}{\lambda_0}\right\} > 0$ proves the claim.

b) Let $x \in \mathcal{X}_r(\hat{\varepsilon}; n, k)$ be arbitrary, i.e.

$$d_{GH}(B_{1/\hat{\delta}}^{r^{-1}X}(x), B_{1/\hat{\delta}}^{\mathbb{R}^k}(0)) \leq \frac{\hat{\delta}}{2}.$$

Since $(\frac{1}{r} M_i, p_i^x) \to (\frac{1}{r} X, x)$, there is $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$,

$$d_{GH}(B_{1/\hat{\delta}}^{r^{-1}M_i}(p_i^x), B_{1/\hat{\delta}}^{r^{-1}X}(x)) \leq \frac{\hat{\delta}}{2}.$$

In particular, by the triangle inequality,

$$\begin{aligned} &d_{GH}(B_{1/\hat{\delta}}^{r^{-1}M_i}(p_i^x), B_{1/\hat{\delta}}^{\mathbb{R}^k}(0)) \\ &\leq d_{GH}(B_{1/\hat{\delta}}^{r^{-1}M_i}(p_i^x), B_{1/\hat{\delta}}^{r^{-1}X}(x)) + d_{GH}(B_{1/\hat{\delta}}^{r^{-1}X}(x), B_{1/\hat{\delta}}^{\mathbb{R}^k}(0)) \\ &\leq \frac{\hat{\delta}}{2} + \frac{\hat{\delta}}{2} = \hat{\delta}. \end{aligned}$$

Now Proposition 2.1 and Lemma 1.15 imply the claim. □

Notation. For $0 < r \leq \hat{\delta}$ and $x \in \mathcal{X}_r(\hat{\varepsilon}; n, k)$, let $\lambda_i^{\hat{\varepsilon},x}(r)$ and $G_r^{\hat{\varepsilon}}(p_i^x)$ be as in Lemma 3.3, i.e. for $q_i \in G_r^{\hat{\varepsilon}}(p_i^x)$ the sublimits of $(\lambda_i^{\hat{\varepsilon},x}(r)\, M_i, q_i)$ are isometric to products $(\mathbb{R}^k \times K, \cdot)$ where the K are compact metric spaces with $\operatorname{diam}(K) \in \left[\frac{1}{5}, 1\right]$. Moreover, for $x \in X_{\text{gen}}$, let $r^x(\hat{\varepsilon}; n, k)$ be as in Lemma 3.3, i.e. $x \in \mathcal{X}_r(\hat{\varepsilon}; n, k)$ for all $0 < r \leq r^x(\hat{\varepsilon}; n, k)$.

Furthermore, for a non-principal ultrafilter ω on $\mathbb{N}$, let $l_\omega^{\hat{\varepsilon},x}(r)$ be as in Lemma 3.3, i.e. for $q_i \in G_r^{\hat{\varepsilon}}(p_i^x)$, $\lim_\omega(\lambda_i^{\hat{\varepsilon},x}(r)\, M_i, q_i) = (\mathbb{R}^k \times K, \cdot)$ and $\dim(K) = l_\omega^{\hat{\varepsilon},x}(r)$ for some K as above.

All notations will be used throughout the remaining chapter without referring to Lemma 3.3 explicitly. Occasionally, if they are fixed, the dependences on n, k and $\hat{\varepsilon}$ will be suppressed.

3.2 Comparison of rescaling sequences

Throughout this section, let $k < n$ and $\hat{\varepsilon} \in (0, \frac{1}{2})$ be fixed and use the notation introduced in section 3.1.

Similar to chapter 2, investigate the set of all points such that all blow-ups split off an $\mathbb{R}^k$-factor. The following lemma states that for each two rescaling sequences whose limit spaces are some products containing a compact set, the quotient of these rescaling sequences is bounded. Especially, this holds in the situation of Lemma 3.3.

Lemma 3.4. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a collapsing sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$ and converge to a limit (X, p) of dimension $k < n$. As in Lemma 2.9, define*

$$\begin{aligned}\tilde{G}_i := \{q_i \in M_i \mid & \text{ for all } \lambda_i \to \infty \text{ and all sublimits } (Y, \cdot) \text{ of } (\lambda_i M_i, q_i) \\ & \text{there exists } X \text{ such that } Y \cong \mathbb{R}^k \times X \text{ isometrically}\}.\end{aligned}$$

a) Let $q_i \in \tilde{G}_i$ be arbitrary. For $1 \leq j \leq 2$, let $\lambda_i^j \to \infty$ and K_j be compact with $(\lambda_i^j M_i, q_i) \to (\mathbb{R}^k \times K_j, \cdot)$ as $i \to \infty$. Then the sequence $\big(\frac{\lambda_i^2}{\lambda_i^1}\big)_{i\in\mathbb{N}}$ is bounded.

b) For $1 \leq j \leq 2$, let $r_j > 0$, $x_j \in \mathcal{X}_{r_j}$, $p_i^j := p_i^{x_j}$ and $\lambda_i^j := \lambda_i^{\hat{\varepsilon}, x_j}(r_j)$. Moreover, assume $G_{r_1}(p_i^1) \cap G_{r_2}(p_i^2) \neq \emptyset$. Then the sequence $\big(\frac{\lambda_i^2}{\lambda_i^1}\big)_{i\in\mathbb{N}}$ is bounded.

Proof. a) The proof is done by contradiction: Without loss of generality, assume $\lambda_i^j > 0$ for $1 \leq j \leq 2$ and all $i \in \mathbb{N}$. Obviously, the sequence $\big(\frac{\lambda_i^2}{\lambda_i^1}\big)_{i\in\mathbb{N}}$ is bounded from below by 0. Assume the sequence is not bounded from above, i.e. $\frac{\lambda_i^2}{\lambda_i^1} \to \infty$ and, without loss of generality, $\lambda_i^1 < \lambda_i^2$ for all $i \in \mathbb{N}$.

There exists $\lambda_i \to \infty$ satisfying $\lambda_i^1 < \lambda_i < \lambda_i^2$ such that

$$(\lambda_i M_i, q_i) \to (\mathbb{R}^k \times N, y)$$

for some unbounded metric space N: Let $\mu_i \to \infty$ be as in Lemma 2.15 such that $(\mu_i' \lambda_i^1 M_i, q_i)$ subconverges to a tangent cone of $(\mathbb{R}^k \times K_1, x_1)$ for any $\mu_i' \to \infty$ with $\mu_i' \leq \mu_i$. Define

$$\mu_i' := \min\left\{\mu_i, \sqrt{\frac{\lambda_i^2}{\lambda_i^1}}\right\} \to \infty \quad \text{and} \quad \lambda_i := \mu_i' \cdot \lambda_i^1 \to \infty.$$

Without loss of generality, assume $\mu_i > 1$. Thus, $\lambda_i^1 < \lambda_i < \lambda_i^2$. Furthermore, using $q_i \in \tilde{G}_i$, there exists a metric space N and a point $q \in \mathbb{R}^k \times N$ such that $(\lambda_i M_i, q_i) \to (\mathbb{R}^k \times N, q)$ and this is a tangent cone of $(\mathbb{R}^k \times K_1, x_1)$. Hence, for some sequence $\alpha_i \to \infty$,

$$(\mathbb{R}^k \times \alpha_i K_1, x_1) \to (\mathbb{R}^k \times N, q).$$

Assume this N is compact and let $N' := \frac{1}{\operatorname{diam}(N)} \cdot N$ and $\beta_i := \frac{1}{\operatorname{diam}(N)} \cdot \alpha_i$. Then

$$(\mathbb{R}^k \times \beta_i K_1, x_1) \to (\mathbb{R}^k \times N', q),$$

and, for sufficiently large i,

$$d_{GH}(B_{100}^{\mathbb{R}^k \times \beta_i K_1}(x_1), B_{100}^{\mathbb{R}^k \times N'}(q)) \leq 10^{-4}.$$

By Lemma 2.27 c), the sequence $(\operatorname{diam}(\beta_i K_1))_{i \in \mathbb{N}}$ is bounded. This is a contradiction to $\beta_i \to \infty$. Hence, N is unbounded.

Now let $D := \operatorname{diam}(K_2)$ and fix $R > 1000D$. Since N is unbounded, $2R < \operatorname{diam}(N)$ and, by Lemma 2.27 d), $d_{GH}(B_R^{\mathbb{R}^k \times N}(q), B_R^{\mathbb{R}^k \times K_2}(x_2)) \geq 20D$. Thus, for i large enough,

$$\begin{aligned} d_{GH}(B_R^{\lambda_i M_i}(q_i), B_R^{\lambda_i^2 M_i}(q_i)) \geq\ & d_{GH}(B_R^{\mathbb{R}^k \times N}(q), B_R^{\mathbb{R}^k \times K_2}(x_2)) \\ & - d_{GH}(B_R^{\lambda_i M_i}(q_i), B_R^{\mathbb{R}^k \times N}(q)) \\ & - d_{GH}(B_R^{\lambda_i^2 M_i}(q_i), B_R^{\mathbb{R}^k \times K_2}(x_2)) \\ \geq\ & 10D. \end{aligned}$$

Since the maps $h_i : (0, \infty) \to (0, \infty)$ defined by

$$h_i(\mu_i) := d_{GH}(B_R^{\mu_i M_i}(q_i), B_R^{\lambda_i^2 M_i}(q_i))$$

are continuous with $h_i(\lambda_i^2) = 0$ and $h_i(\lambda_i) \geq 10D$, by the intermediate value theorem, there is a maximal $\lambda_i^1 < \lambda_i' \leq \lambda_i < \lambda_i^2$ such that $h_i(\lambda_i') = 5D$. Since $q_i \in \tilde{G}_i$, after passing to a subsequence,

$$(\lambda_i' M_i, q_i) \to (\mathbb{R}^k \times Y, y)$$

for some metric space Y. In particular,

$$d_{GH}(B_R^{\lambda_i' M_i}(q_i), B_R^{\lambda_i^2 M_i}(q_i)) \to d_{GH}(B_R^{\mathbb{R}^k \times Y}(y), B_R^{\mathbb{R}^k \times K_2}(x_2))$$

as $i \to \infty$. Hence, $d_{GH}(B_R^{\mathbb{R}^k \times Y}(y), B_R^{\mathbb{R}^k \times K_2}(x_2)) = 5D$. By Lemma 1.11 a),

$$d_{GH}(B_R^{\mathbb{R}^k \times K_2}(x_2), B_R^{\mathbb{R}^k}(0)) \leq \operatorname{diam}(K_2) = D.$$

Thus,

$$\begin{aligned} & d_{GH}(B_R^{\mathbb{R}^k \times Y}(y), B_R^{\mathbb{R}^k}(0)) \\ & \leq d_{GH}(B_R^{\mathbb{R}^k \times Y}(y), B_R^{\mathbb{R}^k \times K_2}(x_2)) + d_{GH}(B_R^{\mathbb{R}^k \times K_2}(x_2), B_R^{\mathbb{R}^k}(0)) \\ & \leq 6D < \frac{R}{12} \end{aligned}$$

and

$$\begin{aligned} & d_{GH}(B_R^{\mathbb{R}^k \times Y}(y), B_R^{\mathbb{R}^k}(0)) \\ & \geq d_{GH}(B_R^{\mathbb{R}^k \times Y}(y), B_R^{\mathbb{R}^k \times K_2}(x_2)) - d_{GH}(B_R^{\mathbb{R}^k \times K_2}(x_2), B_R^{\mathbb{R}^k}(0)) \\ & \geq 4D. \end{aligned}$$

By Lemma 2.27 b), Y is compact. Moreover,

$$\mathrm{diam}(Y) \geq d_{GH}(B_R^{\mathbb{R}^k \times Y}(y), B_R^{\mathbb{R}^k}(0)) \geq 4D > D,$$

in particular, Y is not a point.

Next, prove $\frac{\lambda_i^2}{\lambda_i'} \to \infty$: Assume the quotient is bounded. Hence, after passing to a subsequence, $\frac{\lambda_i^2}{\lambda_i'} \to \alpha$ and $\alpha \geq 1$ due to $\lambda_i' \leq \lambda_i^2$. Then, applying Lemma 1.10,

$$(\lambda_i^2 M_i, q_i) = \left(\frac{\lambda_i^2}{\lambda_i'} \cdot \lambda_i' M_i, q_i\right) \to (\mathbb{R}^k \times \alpha Y, y) \cong (\mathbb{R}^k \times K_2, x_2).$$

In particular, $\mathrm{diam}(Y) \leq \alpha \cdot \mathrm{diam}(Y) = \mathrm{diam}(K_2) = D$, and this is a contradiction. Thus, $\frac{\lambda_i^2}{\lambda_i'} \to \infty$. Analogously to the previous argumentation, there exists some maximal $\lambda_i' < \tilde{\lambda}_i < \lambda_i^2$ such that $h_i(\tilde{\lambda}_i) = 5D$. This is a contradiction to the maximal choice of λ_i'.

b) By construction, $G_{r_j}(p_i^j) \subseteq \tilde{G}_i$. Let $q_i \in G_{r_1}(p_i^1) \cap G_{r_2}(p_i^2) \subseteq \tilde{G}_i$ and $\alpha_i := \frac{\lambda_i^2}{\lambda_i^1}$. Assume $\alpha_i \to \infty$ and choose a subsequence $(i_j)_{j\in\mathbb{N}}$ such that $\alpha_{i_j} > j$ for all $j \in \mathbb{N}$. After passing to a further subsequence, there are compact metric spaces K_1 and K_2 such that $(\lambda_{i_j}^m M_{i_{j_l}}, q_{i_j}) \to (\mathbb{R}^k \times K_m, \cdot)$ for $1 \leq m \leq 2$. By a), the sequence $(\alpha_{i_j})_{j\in\mathbb{N}}$ is bounded. This is a contradiction to $\alpha_{i_j} \to \infty$. □

The following lemma gives a statement about the limit of such a bounded sequence of quotients.

Lemma 3.5. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a collapsing sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound* $\mathrm{Ric}_{M_i} \geq -(n-1)$ *and let $\lambda_i, \mu_i > 0$ such that $\left(\frac{\lambda_i}{\mu_i}\right)_{i\in\mathbb{N}}$ is bounded. If*

$$(\lambda_i M_i, p_i) \to (\mathbb{R}^k \times L, p_L) \quad \textit{and} \quad (\mu_i M_i, p_i) \to (\mathbb{R}^k \times M, p_M)$$

for some bounded metric spaces L and M, then

$$\frac{\lambda_i}{\mu_i} \to \frac{\mathrm{diam}(L)}{\mathrm{diam}(M)} \textit{ as } i \to \infty \quad \textit{and} \quad \mathrm{dim}(L) = \mathrm{dim}(M).$$

Proof. Let a be any accumulation point of $\left(\frac{\lambda_i}{\mu_i}\right)_{i\in\mathbb{N}}$ and $\left(\frac{\lambda_{i_j}}{\mu_{i_j}}\right)_{j\in\mathbb{N}}$ be the corresponding converging subsequence. By Lemma 1.10 c),

$$(\lambda_{i_j} M_{i_j}, p_{i_j}) = \left(\frac{\lambda_{i_j}}{\mu_{i_j}} \cdot \mu_{i_j} M_{i_j}, p_{i_j}\right) \to (a\,(\mathbb{R}^k \times M), p_M) = (\mathbb{R}^k \times (aM), p_M) \text{ as } j \to \infty.$$

Since this sequence converges to $(\mathbb{R}^k \times L, p_L)$ as well, there is an isometry

$$(\mathbb{R}^k \times (aM), p_M) \cong (\mathbb{R}^k \times L, p_L).$$

Thus, $\dim(L) = \dim(M)$ and $\operatorname{diam}(L) = a \cdot \operatorname{diam}(M)$. Hence, all accumulation points of the bounded sequence $(\frac{\lambda_i}{\mu_i})_{i\in\mathbb{N}}$ equal $\frac{\operatorname{diam}(L)}{\operatorname{diam}(M)}$, in particular, $\left(\frac{\lambda_i}{\mu_i}\right)_{i\in\mathbb{N}}$ is convergent. □

Next, the question will be answered under which condition the quotient of rescaling sequences belonging to two points in $\mathcal{X}_r$ is bounded. The first approach in order to prove this is the special case of their good subsets to intersect. In the general case, the idea is to connect the points by a curve which itself is contained in $\mathcal{X}_r$ and can be covered by finitely many balls such that subsequent subsets of good points intersect. In fact, this cannot be expected to be possible for the same $r > 0$. However, it turns out that the quotient of the rescaling sequences is bounded if the points are connected by a minimising geodesic contained in some $\mathcal{X}_{r'}$ of a possibly different r'. Making all of this precise is the subject of the following lemma.

Lemma 3.6. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a collapsing sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound* $\operatorname{Ric}_{M_i} \geq -(n-1)$ *and converge to a limit (X, p) of dimension $k < n$.*

a) Let $r_m > 0$, $x_m \in \mathcal{X}_{r_m}$ and $p_i^{x_m} \to x_m$ for $1 \leq m \leq 2$. If $G_{r_1}(p_i^{x_1}) \cap G_{r_2}(p_i^{x_2}) \neq \emptyset$ for all $i \in \mathbb{N}$, then

$$\frac{1}{5} \leq \frac{\lambda_i^{x_1}(r_1)}{\lambda_i^{x_2}(r_2)} \leq 5$$

for almost all $i \in \mathbb{N}$ and $l_\omega^{x_1}(r_1) = l_\omega^{x_2}(r_2)$.

b) Let $x \in X_{\text{gen}}$, $p_i^x \to x$ and $r^x \geq R > r > 0$. Then there is $m = m(n, \hat{\varepsilon}, r, R) \in \mathbb{N}$ such that

$$5^{-m} \leq \frac{\lambda_i^x(r)}{\lambda_i^x(R)} \leq 5^m$$

for almost all $i \in \mathbb{N}$ and $l_\omega^x(r) = l_\omega^x(R)$.

c) Let $\gamma : [0, l] \to X$ be a minimising geodesic with $\operatorname{im}(\gamma) \subseteq \mathcal{X}_r$ *for some $0 < r \leq \hat{\delta}$. Let $x = \gamma(0)$ and $y = \gamma(l)$. Then there is $m = m(n, \hat{\varepsilon}, l, r) \in \mathbb{N}$ such that*

$$5^{-m} \leq \frac{\lambda_i^x(r)}{\lambda_i^y(r)} \leq 5^m$$

for almost all $i \in \mathbb{N}$ and $l_\omega^x(r) = l_\omega^y(r)$.

d) Let $x, y \in \mathcal{X}_r$ and $\gamma : [0, l] \to X$ be a minimising geodesic with $\operatorname{im}(\gamma) \subseteq \mathcal{X}_{r'}$ for some $r' \leq r \leq \hat{\delta}$, $x = \gamma(0)$ and $y = \gamma(l)$. Then there is $m = m(n, \hat{\varepsilon}, l, r, r') \in \mathbb{N}$ such that

$$5^{-m} \leq \frac{\lambda_i^x(r)}{\lambda_i^y(r)} \leq 5^m$$

for almost all $i \in \mathbb{N}$ and $l_\omega^x(r) = l_\omega^y(r)$.

Proof. The proof of the first part will be straightforward. The second part will be proven by applying the first part to a sequence of radii r_j such that the subsets $G_{r_j}(p_i^j)$ and $G_{r_{j+1}}(p_i^{j+1})$ intersect. Similarly, the third part will be proven by subdividing the geodesics by points y_j such that the distance of two successive points is small enough to enforce that $G_r(p_i^{y_j})$ and $G_r(p_i^{y_{j+1}})$ intersect. The fourth part will turn out to be an immediate consequence of the other ones.

a) Without loss of generality, all $\lambda_i^{x_m}(r_m)$ are positive, where $1 \leq m \leq 2$. Define

$$a_i := \frac{\lambda_i^{x_1}(r_1)}{\lambda_i^{x_2}(r_2)} > 0$$

and let $q_i \in G_{r_1}(p_i^{x_1}) \cap G_{r_2}(p_i^{x_2})$ be arbitrary.

By Lemma 3.4, $(a_i)_{i \in \mathbb{N}}$ is bounded. Let a be an arbitrary accumulation point and $(a_{i_j})_{j \in \mathbb{N}}$ be the subsequence converging to a. Since $q_{i_j} \in G_{r_1}(p_{i_j}^{x_1})$, after passing to a subsequence,

$$(\lambda_{i_j}^{x_1}(r_1)\, M_{i_j}, q_{i_j}) \to (\mathbb{R}^k \times K_1, \cdot) \text{ as } j \to \infty$$

for some compact metric space K_1 with $\frac{1}{5} \leq \operatorname{diam}(K_1) \leq 1$. As $q_{i_j} \in G_{r_2}(p_{i_j}^{x_2})$ as well, after passing to a further subsequence,

$$(\lambda_{i_j}^{x_2}(r_2)\, M_{i_j}, q_{i_j}) \to (\mathbb{R}^k \times K_2, \cdot) \text{ as } j \to \infty$$

and K_2 satisfies $\frac{1}{5} \leq \operatorname{diam}(K_2) \leq 1$. By Lemma 3.5,

$$a = \lim_{j \to \infty} \frac{\lambda_{i_j}^{x_1}(r_1)}{\lambda_{i_j}^{x_2}(r_2)} = \frac{\operatorname{diam}(K_1)}{\operatorname{diam}(K_2)} \in \left[\frac{1}{5}, 5\right]$$

and $l_\omega^{x_1}(r_1) = l_\omega^{x_2}(r_2)$.

Since $(a_i)_{i \in \mathbb{N}}$ is a bounded sequence and all accumulation points are contained in $[\frac{1}{5}, 5]$, only finitely many a_i are not contained in $[\frac{1}{5}, 5]$.

b) Since $C_{BG}(n, -1, \beta R, R)$ is monotonically increasing for decreasing $\beta < 1$, there exists $\beta = \beta(n, \hat{\varepsilon}, R) < 1$ with $C_{BG}(n, -1, \beta R, R) < \frac{1}{\hat{\varepsilon}} - 1$. Fix this β.

Since $C_{BG}(n, -1, \beta\rho, \rho)$ is monotonically increasing for increasing ρ, all $\rho \leq R$ satisfy $C_{BG}(n, -1, \beta\rho, \rho) < \frac{1}{\hat{\varepsilon}} - 1$ as well.

Let $m = m(n, \hat{\varepsilon}, r, R) \in \mathbb{N}$ be maximal with $r \leq \beta^m \cdot R$ and define $r_j := \beta^j \cdot R$ for $0 \leq j < m$ and $r_m := r$. In particular,

$$r_m = r \leq \beta^m \cdot R = \beta \cdot r_{m-1}.$$

Then both

$$C_{BG}(n, -1, r_{j+1}, r_j) = C_{BG}(n, -1, \beta \cdot r_j, r_j) < \frac{1}{\hat{\varepsilon}} - 1$$

for $0 \leq j < m$ and

$$C_{BG}(n, -1, r_m, r_{m-1}) \leq C_{BG}(n, -1, \beta \cdot r_{m-1}, r_{m-1}) < \frac{1}{\hat{\varepsilon}} - 1.$$

Moreover, $x \in \mathcal{X}_{r_j}$ for all $0 \leq j \leq m$ due to $r_j \leq R \leq r^x$,
Assume $G_{r_j}(p_i^x) \cap G_{r_{j+1}}(p_i^x) = \emptyset$ for some $0 \leq k < m$ and $i \in \mathbb{N}$. This implies

$$G_{r_j}(p_i^x) \subseteq B_{r_{j+1}}(p_i^x) \setminus G_{r_{j+1}}(p_i^x),$$

in particular,

$$\begin{aligned}
(1 - \hat{\varepsilon}) \cdot \mathrm{vol}(B_{r_j}(p_i^x)) &\leq \mathrm{vol}(G_{r_j}(p_i^x)) \\
&\leq \mathrm{vol}(B_{r_{j+1}}(p_i^x) \setminus G_{r_{j+1}}(p_i^x)) \\
&\leq \hat{\varepsilon} \cdot \mathrm{vol}(B_{r_{j+1}}(p_i^x)) \\
&\leq \hat{\varepsilon} \cdot C_{BG}(n, -1, r_{j+1}, r_j) \cdot \mathrm{vol}(B_{r_j}(p_i^x)).
\end{aligned}$$

Hence, $1 - \hat{\varepsilon} \leq \hat{\varepsilon} \cdot C_{BG}(n, -1, r_{j+1}, r_j) < 1 - \hat{\varepsilon}$, and this is a contradiction.
Thus, $G_{r_j}(p_i^x) \cap G_{r_{j+1}}(p_i^x) \neq \emptyset$ for all $0 \leq j < m$ and $i \in \mathbb{N}$. By a),

$$\frac{1}{5} \leq \frac{\lambda_i^x(r_j)}{\lambda_i^x(r_{j+1})} \leq 5$$

for almost all i and $l_\omega^x(r_j) = l_\omega^x(r_{j+1})$. Inductively,

$$l_\omega^x(r) = l_\omega^x(r_m) = l_\omega^x(r_0) = l_\omega^x(R).$$

Then

$$\frac{\lambda_i^x(R)}{\lambda_i^x(r)} = \frac{\lambda_i^x(r_0)}{\lambda_i^x(r_m)} = \prod_{j=0}^{m-1} \frac{\lambda_i^x(r_j)}{\lambda_i^x(r_{j+1})}$$

proves the claim.

c) Let $d_0 = d_0(n, \hat{\varepsilon}, r)$ be as in Lemma 2.21 and $m_0 = m_0(n, \hat{\varepsilon}, l, r) \in \mathbb{N}$ be the minimal natural number with

$$l \leq m_0 \cdot d_0.$$

For $0 \leq j \leq m_0 - 1$, let $t_j := j \cdot d_0$ and $t_{m_0} := l > (m_0 - 1) \cdot d_0 = t_{m_0-1}$. In particular, for $0 \leq j < m_0 - 1$,

$$t_{j+1} - t_j = d_0$$

and

$$t_{m_0} - t_{m_0-1} = l - (m_0 - 1) \cdot d_0 \leq m_0 \cdot d_0 - (m_0 - 1) \cdot d_0 = d_0.$$

Hence, these t_j define a sequence $0 = t_0 < t_1 < \cdots < t_{m_0} = l$ with pairwise $t_{j+1} - t_j \leq d_0$.

For $0 \leq j \leq m_0$, define $y_j := \gamma(t_j) \in \mathcal{X}_r$. Now fix $0 \leq j < m_0$. By Lemma 2.21 and Lemma 2.19,

$$G_r(p_i^{y_j}) \cap G_r(p_i^{y_{j+1}}) \neq \emptyset.$$

Then a) implies $\frac{\lambda_i^{y_j}(r)}{\lambda_i^{y_{j+1}}(r)} \in [\frac{1}{5}, 5]$ for almost all $i \in \mathbb{N}$ and $l_\omega^{y_j}(r) = l_\omega^{y_{j+1}}(r)$. In particular, $l_\omega^x(r) = l_\omega^{y_0}(r) = l_\omega^{y_{m_0}}(r) = l_\omega^y(r)$ and

$$\frac{\lambda_i^x(r)}{\lambda_i^y(r)} = \prod_{j=0}^{m_0-1} \frac{\lambda_i^{y_j}(r)}{\lambda_i^{y_{j+1}}(r)} \in [5^{-m_0}, 5^{m_0}]$$

for almost all $i \in \mathbb{N}$.

d) Let $m_x := m_y := m(n, \hat{\varepsilon}, r', r)$ be as in b) and $m_0 = m_0(n, \hat{\varepsilon}, l, r')$ as in c). Then

$$\frac{\lambda_i^x(r)}{\lambda_i^x(r')} \in [5^{-m_x}, 5^{m_x}], \quad \frac{\lambda_i^y(r')}{\lambda_i^y(r)} \in [5^{-m_y}, 5^{m_y}] \quad \text{and} \quad \frac{\lambda_i^x(r')}{\lambda_i^y(r')} \in [5^{-m_0}, 5^{m_0}]$$

for almost all i and $l_\omega^x(r) = l_\omega^x(r') = l_\omega^y(r') = l_\omega^y(r)$. Finally, taking the product $m = m(n, \hat{\varepsilon}, l, r, r') := m_x \cdot m_0 \cdot m_y$ proves the claim. □

3.3 Generic points and geodesics

Throughout this section, fix a collapsing sequence $(M_i, p_i)_{i \in \mathbb{N}}$ of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$ and converge to a limit (X, p) of dimension $k < n$ and use the notation introduced in section 3.1. Moreover, minimising geodesics are assumed to be parametrised by arc length.

By Lemma 3.6, rescaling sequences corresponding to two different points can be compared if those points are connected by a geodesic lying in some $\mathcal{X}_r$. It remains to check for which points this is the case. It will turn out that, if the strict interior of a minimising geodesic (i.e. the interior bounded away from the endpoints) is generic, then it is already contained in $\mathcal{X}_r$ for sufficiently small $r > 0$. In fact, nearly all pairs of points lie in the interior of such a geodesic such that the part of the geodesic connecting these points is generic.

Notation. Define

$$\mathcal{G} := \{(x,y) \in X_{\text{gen}} \times X_{\text{gen}} \mid \exists \text{ minimising geod. } \gamma : [0,l] \to X, 0 < t_x < t_y < l : \\ x = \gamma(t_x), y = \gamma(t_y), \text{im}(\gamma_{|[t_x,t_y]}) \subseteq X_{\text{gen}}\},$$

and for $x \in X_{\text{gen}}$ denote the image under the projection to the second factor by

$$\mathcal{G}_x := \{y \in X_{\text{gen}} \mid (x,y) \in \mathcal{G}\}.$$

Finally, define

$$\mathcal{G}' := \{x \in X_{\text{gen}} \mid \mathcal{G}_x \text{ has full measure in } X\}.$$

Lemma 3.7. *The set $\mathcal{G}'$ has full measure in X.*

Proof. First, prove that $\mathcal{G}$ has full measure in $X \times X$. Let

$$S_1 := \{(x,y) \in X_{\text{gen}} \times X_{\text{gen}} \mid \exists \text{ minimising geodesic } c : [0,d] \to X : \\ x = c(0), y = c(d), \text{im}(c) \subseteq X_{\text{gen}}\}$$
$$S_2 := \{(x,y) \in X \times X \mid \exists \text{ minimising geodesic } \gamma : [0,l] \to X, 0 < t_x < t_y < l : \\ x = \gamma(t_x), y = \gamma(t_y)\}$$

and define $S := S_1 \cap S_2$. By [CN12, Theorem 1.20 (1)],

$$\text{vol}_X \times \text{vol}_X(X \times X \setminus S_1) = 0,$$

and by [CN12, Theorem A.4 (3)],

$$\text{vol}_X \times \text{vol}_X(X_{\text{gen}} \times X_{\text{gen}} \setminus S_2) = 0.$$

In particular, using that $\text{vol}_X(X \setminus X_{\text{gen}}) = 0$, cf. Theorem 1.19, this proves

$$\text{vol}_X \times \text{vol}_X(X \times X \setminus S) = 0.$$

Next, prove $S \subseteq \mathcal{G}$: Let $(x,y) \in S$, $c : [0,d] \to X_{\text{gen}}$ and $\gamma : [0,l] \to X$ be geodesics and $0 < t_x < t_y < l$ with $x = c(0) = \gamma(t_x)$ and $y = c(d) = \gamma(t_y)$. In particular, $d = d(x,y) = t_y - t_x \leq l$.

Define $\tilde{\gamma} : [0,l] \to X$ by

$$\tilde{\gamma}(\tau) := \begin{cases} \gamma(\tau) & \text{if } \tau \in [0,t_x] \cup [t_y,l], \\ c(\tau - t_x) & \text{if } \tau \in [t_x,t_y], \end{cases}$$

cf. Figure 3.1.

Obviously, this $\tilde{\gamma}$ is continuous. Moreover, it is a minimising geodesic: For arbitrary $0 \leq \tau_1 < \tau_2 \leq l$, $d_X(\tilde{\gamma}(\tau_1), \tilde{\gamma}(\tau_2)) = \tau_2 - \tau_1$ needs to be proven:

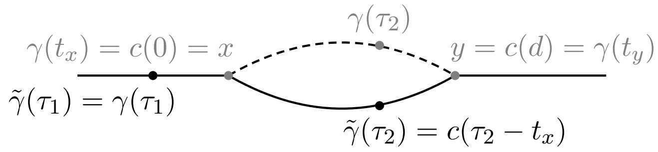

Figure 3.1: Construction of $\tilde{\gamma}$.

If τ_1, τ_2 are both contained in $[0, t_x] \cup [t_y, l]$ or both contained in $[t_x, t_y]$, this is true since γ and c are minimising. So let $0 \leq \tau_1 < t_x < \tau_2 \leq t_y$. Then

$$\begin{aligned} d_X(\tilde{\gamma}(\tau_1), \tilde{\gamma}(\tau_2)) &\leq d_X(\gamma(\tau_1), \gamma(t_x)) + d_X(c(0), c(\tau_2 - t_x)) \\ &\leq t_x - \tau_1 + \tau_2 - t_x = \tau_2 - \tau_1. \end{aligned}$$

Assume $d_X(\tilde{\gamma}(\tau_1), \tilde{\gamma}(\tau_2)) < \tau_2 - \tau_1$. In particular,

$$\begin{aligned} t_y - \tau_1 &= d_X(\gamma(\tau_1), \gamma(t_y)) \\ &\leq d_X(\gamma(\tau_1), \tilde{\gamma}(\tau_2)) + d_X(\tilde{\gamma}(\tau_2), \gamma(t_y)) \\ &= d_X(\tilde{\gamma}(\tau_1), \tilde{\gamma}(\tau_2)) + d_X(c(\tau_2 - t_x), c(t_y - t_x)) \\ &< \tau_2 - \tau_1 + (t_y - t_x - (\tau_2 - t_x)) \\ &= t_y - \tau_1, \end{aligned}$$

and this is a contradiction. The case $t_x \leq \tau_1 < t_y < \tau_2 \leq l$ can be done analogously. Then $\tilde{\gamma}$ verifies $(x, y) \in \mathcal{G}$, and this proves

$$\mathrm{vol}_X \times \mathrm{vol}_X(X \times X \setminus \mathcal{G}) = 0.$$

Using $X \times X \setminus \mathcal{G} = \bigcup_{x \in X} \{x\} \times (X \setminus \mathcal{G}_x)$,

$$0 = \mathrm{vol}_{X \times X}(X \times X \setminus \mathcal{G}) = \int_X \mathrm{vol}_X(X \setminus \mathcal{G}_x)\, dV(x) = \int_{X \setminus \mathcal{G}'} \mathrm{vol}_X(X \setminus \mathcal{G}_x)\, dV(x).$$

Since $\mathrm{vol}_X(X \setminus \mathcal{G}_x) > 0$ for all $x \in X \setminus \mathcal{G}'$, this proves that $X \setminus \mathcal{G}'$ has measure 0. □

So far it was seen that almost all points can be connected by a geodesic lying in X_{gen} which can be extended at both ends. By applying the following theorem of Colding and Naber, which describes the Hölder continuity of the geometry of small balls with the same radius, to this situation, one obtains that the interior of the regarded geodesics not only lies in X_{gen}, but in $\mathcal{X}_r$ for some $r > 0$.

Theorem 3.8 ([CN12, Theorem 1.1, Theorem 1.2]). *For $n \in \mathbb{N}$ there are $\alpha(n)$, $C(n)$ and $r_0(n)$ such that the following holds: Let M be a complete n-dimensional Riemannian manifold with $\mathrm{Ric}_M \geq -(n-1)$ or the limit space of a sequence of such manifolds, let*

$\gamma : [0,l] \to M$ a minimising geodesic (parametrised by arc-length) and fix $\beta \in (0,1)$. For $0 < r < r_0 \beta l$ and $\beta l < s < t < (1-\beta)l$,

$$d_{GH}(B_r^M(\gamma(s)), B_r^M(\gamma(t))) < \frac{C}{\beta l} \cdot r \cdot |s-t|^{\alpha(n)}.$$

Lemma 3.9. *Let $\hat{\varepsilon} \in (0, \frac{1}{2})$, $\gamma : [0,l] \to X$ be a minimising geodesic and $0 < s < t < l$ such that $\gamma_{|[s,t]}$ is contained in X_{gen}. Then there is $0 < r' = r'(\hat{\varepsilon}, l, s, t; n, k) \leq \hat{\delta}$ such that for all $0 < r \leq r'$,*

$$\operatorname{im}(\gamma_{|[s,t]}) \subseteq \mathcal{X}_r.$$

Proof. Define $\beta = \beta(l,s,t) := \frac{1}{2l} \cdot \min\{s, l-t\} > 0$. Then $t, s \in (\beta l, (1-\beta)l)$ due to

$$\beta l \leq \frac{s}{2} < s < t = l - (l-t) \leq (1-2\beta)l < (1-\beta)l.$$

Furthermore, let $\alpha(n), C(n), r_0(n)$ be as in Theorem 3.8 and define

$$d = d(\hat{\varepsilon}, l, s, t; n, k) := \sqrt[\alpha(n)]{\frac{\beta l \cdot \hat{\delta}^2}{2C(n)}}.$$

Let $m = m(s,t)$ be the natural number such that $(m-1)d \leq t - s < md$ and define

$$\tau_j := s + jd$$

for $0 \leq j \leq m$. By definition, $\tau_0 = s$ and $\tau_m = s + md > t$. Hence,

$$[s,t] \subseteq \bigcup_{j=0}^{m} (\tau_j - d, \tau_j + d).$$

For every $0 \leq j \leq m$, choose $\lambda_j = \lambda_j(\hat{\varepsilon}, l, s, t; n, k) > 1$ as in Lemma 3.2 such that

$$d_{GH}(B_{1/\hat{\delta}}^{\lambda X}(\gamma(\tau_j)), B_{1/\hat{\delta}}^{\mathbb{R}^k}(0)) \leq \frac{\hat{\delta}}{2}$$

for all $\lambda \geq \lambda_j$ and define $r' = r'(\hat{\varepsilon}, l, s, t; n, k) := \min\{\hat{\delta}, \frac{1}{\lambda_0}, \dots, \frac{1}{\lambda_m}, \hat{\delta} \cdot r_0(n) \cdot \beta l\}$. Let $0 < r \leq r'$ and $\tau \in [s,t]$ be arbitrary. Choose $0 \leq j \leq m$ with $|\tau - \tau_j| < d$. Recall that by definition of d,

$$|\tau - \tau_j|^{\alpha(n)} < d^{\alpha(n)} = \frac{\beta l \cdot \hat{\delta}^2}{2C(n)},$$

and so, using Theorem 3.8,

$$\begin{aligned}
&d_{GH}(B_{1/\hat{\delta}}^{r^{-1}X}(\gamma(\tau)), B_{1/\hat{\delta}}^{\mathbb{R}^k}(0)) \\
&\leq \frac{1}{r} \cdot d_{GH}(B_{r/\hat{\delta}}^{X}(\gamma(\tau)), B_{r/\hat{\delta}}^{X}(\gamma(\tau_j))) + d_{GH}(B_{1/\hat{\delta}}^{r^{-1}X}(\gamma(\tau_j)), B_{1/\hat{\delta}}^{\mathbb{R}^k}(0)) \\
&\leq \frac{1}{r} \cdot \frac{C(n)}{\beta l} \cdot \frac{r}{\hat{\delta}} \cdot |\tau_j - \tau|^{\alpha(n)} + \frac{\hat{\delta}}{2} \\
&< \hat{\delta}.
\end{aligned}$$

□

3.4 Proof of the main theorem

In order to prove Theorem 3.1, the following technical result is needed which gives an estimate for the number of balls a point can be contained in if the base points of these balls form an ε-net.

Lemma 3.10. *Let X be an n-dimensional Riemannian manifold with lower Ricci curvature bound* $\mathrm{Ric} \geq (n-1) \cdot \kappa$ *or the (pointed) Gromov-Hausdorff limit of a sequence of such manifolds. Then each point is contained in maximal $C_{BG}(n, \kappa, r, r+2R)$ balls with radii R whose base points have pairwise distance at least $2r$.*

Proof. This result is an immediate consequence of the Bishop-Gromov Theorem: Let $p_1, \ldots, p_m \in X$ be points with pairwise distance at least $2r$ and $q \in \bigcap_{i=1}^m B_R(p_i)$.

On the one hand, since $d(p_i, p_j) \geq 2r$, one has

$$B_r(p_i) \cap B_r(p_j) = \emptyset$$

for any $i \neq j$. On the other hand, for $\tilde{q} \in B_r(p_i)$,

$$d(\tilde{q}, q) \leq d(\tilde{q}, p_i) + d(p_i, q) < r + R,$$

and so $B_r(p_i) \subseteq B_{r+R}(q)$. Hence,

$$\coprod_{i=1}^{m} B_r(p_i) \subseteq B_{r+R}(q).$$

Furthermore, for any $1 \leq i \leq m$,

$$B_{r+R}(q) \subseteq B_{r+R+d(q,p_i)}(p_i) \subseteq B_{r+2R}(p_i).$$

Together, using the Bishop-Gromov Theorem,

$$\begin{aligned} 1 \geq \frac{\mathrm{vol}(\coprod_{i=1}^m B_r(p_i))}{\mathrm{vol}(B_{r+R}(q))} &= \sum_{i=1}^{m} \frac{\mathrm{vol}(B_r(p_i))}{\mathrm{vol}(B_{r+R}(q))} \\ &\geq \sum_{i=1}^{m} \frac{\mathrm{vol}(B_r(p_i))}{\mathrm{vol}(B_{r+2R}(p_i))} \\ &\geq \sum_{i=1}^{m} \frac{\mathrm{vol}(B_r(p_i))}{C_{BG}(n, \kappa, r, r+2R) \cdot \mathrm{vol}(B_r(p_i))} \\ &= \frac{m}{C_{BG}(n, \kappa, r, r+2R)}. \end{aligned}$$

Thus, $m \leq C_{BG}(n, \kappa, r, r+2R)$. □

It remains to prove the main theorem. Again, the notation introduced in section 3.1 and section 3.3 is used.

Proof of Theorem 3.1. The idea of the proof is the following: First, fix a bound $\hat{\varepsilon} \in (0, \frac{1}{2})$ and choose a radius R such that $\mathcal{X}_R(\hat{\varepsilon}; n, k)$ has sufficiently large volume. Inside of this set of points, choose a point x_0 and a finite R-net of points x_j such that $(x_0, x_j) \in \mathcal{G}$ and take the union of the subsets $G_R(p_i^{x_j})$. This has the required properties.

Let $\varepsilon \in (0, 1)$ be arbitrary and define

$$\hat{\varepsilon} = \hat{\varepsilon}(n, \varepsilon) := \frac{\varepsilon}{2 \cdot C_{BG}\left(n, -1, \frac{1}{8}, \frac{17}{8}\right)} \in \Big(0, \frac{1}{2}\Big).$$

For arbitrary $r > 0$, define

$$X'(r) := \{x \in B_{1-r}(p) \cap X_{\text{gen}} \mid r^x \geq r\}.$$

For $r_1 \leq r_2$, obviously $X'(r_2) \subseteq X'(r_1)$. Furthermore,

$$\bigcup_{r>0} X'(r) = B_1(p) \cap X_{\text{gen}}.$$

Thus, there exists a radius $0 < R = R(\varepsilon, X, p; n) \leq 1$ such that

$$\text{vol}_X(X'(r)) \geq \Big(1 - \frac{\varepsilon}{4}\Big) \cdot \text{vol}_X(B_1(p) \cap X_{\text{gen}}) = 1 - \frac{\varepsilon}{4}$$

for all $r \leq R$. Fix this $0 < R \leq 1$.

By Lemma 3.7,

$$\text{vol}_X(X'(R) \cap \mathcal{G}') = \text{vol}_X(X'(R)) \geq 1 - \frac{\varepsilon}{4},$$

so $X'(R) \cap \mathcal{G}'$ is non-empty. Fix an arbitrary point $x_0 \in X'(R) \cap \mathcal{G}'$, let $X' = X'(R) \cap \mathcal{G}_{x_0}$ and choose a maximal number of points $x_1, \ldots, x_l \in X'$ with pairwise distance at least R. By the maximality of the choice,

$$X' \subseteq \bigcup_{j=1}^{l} B_R(x_j).$$

Since, by definition of $\mathcal{G}'$, $\mathcal{G}_{x_0}$ has full measure,

$$\text{vol}_X\,\Big(\bigcup_{j=1}^{l} B_R(x_j)\Big) \geq \text{vol}_X(X') = \text{vol}_X(X'(R)) \geq 1 - \frac{\varepsilon}{4}.$$

On the other hand, by choice, $B_R(x_j) \subseteq B_{R+d(x_j,p)}(p) \subseteq B_1(p)$. Thus,

$$\text{vol}_X(B_1(p) \setminus \bigcup_{j=1}^{l} B_R(x_j)) \leq \frac{\varepsilon}{4}.$$

Let $p_i^{x_j} \to x_j$ and $i_0 \in \mathbb{N}$ be large enough such that for all $i \geq i_0$ and $1 \leq j < j' \leq l$,

$$d(p_i^{x_j}, p_i^{x_{j'}}) \geq \frac{1}{2} \cdot d(x_j, x_{j'}) \geq \frac{R}{2}$$

and

$$\frac{\mathrm{vol}_{M_i}(B_1(p_i) \setminus \bigcup_{j=1}^{l} B_R(p_i^{x_j}))}{\mathrm{vol}_{M_i}(B_1(p_i))} \leq 2 \cdot \frac{\mathrm{vol}_X(B_1(p) \setminus \bigcup_{j=1}^{l} B_R(x_j))}{\mathrm{vol}_X(B_1(p))}.$$

For the existence of this i_0, cf. Proposition 1.17 b). Fix $i \geq i_0$. Then

$$\begin{aligned} \mathrm{vol}_{M_i}(B_1(p_i) \setminus \bigcup_{j=1}^{l} B_R(p_i^{x_j})) &\leq 2 \cdot \frac{\mathrm{vol}_X(B_1(p) \setminus \bigcup_{j=1}^{l} B_R(x_j))}{\mathrm{vol}_X(B_1(p))} \cdot \mathrm{vol}_{M_i}(B_1(p_i)) \\ &\leq \frac{\varepsilon}{2} \cdot \mathrm{vol}_{M_i}(B_1(p_i)). \end{aligned}$$

By Lemma 3.10, every point of $\bigcup_{j=1}^{l} B_R(p_i^{x_j})$ is contained in at most M different $B_R(p_i^{x_j})$ where

$$M = M(\varepsilon; n, k) := C_{BG}\left(n, -1, \frac{R}{8}, \frac{17R}{8}\right) \leq C_{BG}\left(n, -1, \frac{1}{8}, \frac{17}{8}\right) = \frac{\varepsilon}{2\hat{\varepsilon}}.$$

Therefore,

$$\sum_{j=1}^{l} \mathrm{vol}_{M_i}(B_R(p_i^{x_j}) \leq M \cdot \mathrm{vol}_{M_i}\big(\bigcup_{j=1}^{l} B_R(p_i^{x_j})\big) \leq \frac{\varepsilon}{2\hat{\varepsilon}} \cdot \mathrm{vol}_{M_i}(B_1(p_i)).$$

Thus,

$$\begin{aligned} \mathrm{vol}_{M_i}\big(\bigcup_{j=1}^{l} B_R(p_i^{x_j}) \setminus \bigcup_{j=1}^{l} G_R(p_i^{x_j})\big) &\leq \mathrm{vol}_{M_i}\big(\bigcup_{j=1}^{l} (B_R(p_i^{x_j}) \setminus G_R(p_i^{x_j}))\big) \\ &\leq \sum_{j=1}^{l} \mathrm{vol}_{M_i}(B_R(p_i^{x_j}) \setminus G_R(p_i^{x_j})) \\ &\leq \sum_{j=1}^{l} \hat{\varepsilon} \cdot \mathrm{vol}_{M_i}(B_R(p_i^{x_j})) \\ &\leq \frac{\varepsilon}{2} \cdot \mathrm{vol}_{M_i}(B_1(p_i)). \end{aligned}$$

Hence,

$$\mathrm{vol}_{M_i}(B_1(p_i) \setminus \bigcup_{j=1}^{l} G_R(p_i^{x_j})) \leq \left(\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right) \cdot \mathrm{vol}_{M_i}(B_1(p_i)) = \varepsilon \cdot \mathrm{vol}_{M_i}(B_1(p_i)).$$

Now define

$$G_1(p_i) := \bigcup_{j=1}^{l} G_R(p_i^{x_j}) \quad \text{and} \quad \lambda_i := \lambda_i^{x_0}(R).$$

By construction,

$$\mathrm{vol}_{M_i}(G_1(p_i)) \geq (1-\varepsilon) \cdot \mathrm{vol}_{M_i}(B_1(p_i)).$$

From now on, let $\lambda_i^{x_j}$ denote $\lambda_i^{x_j}(R)$.

Fix $1 \leq j \leq l$. By construction, $(x_0, x_j) \in \mathcal{G}$ and $x_0, x_j \in \mathcal{X}_R$. Thus, there exists a minimising geodesic $\gamma_j : [0, l_j] \to X$ and $0 < s_j < t_j < l_j$ such that $\gamma_{j|[s_j,t_j]}$ is contained in X_{gen}, $\gamma_j(s_j) = x_0$ and $\gamma_j(t_j) = x_j$. By Lemma 3.9, there is $r'_j > 0$ such that for all $0 < r \leq r'_j$, $\gamma_{j|[s_j,t_j]}$ is contained in $\mathcal{X}_r$. Let $r_j := \min\{r'_j, R\}$. By Lemma 3.6 d), there is $m_j := m(n, \hat{\varepsilon}, d_X(x_0, x_j), r_j, R)$ satisfying

$$5^{-m_j} \leq \frac{\lambda_i^{x_0}}{\lambda_i^{x_j}} \leq 5^{m_j}$$

for almost all $i \in \mathbb{N}$ and $l_\omega^{x_0}(R) = l_\omega^{x_j}(R)$. From now on, let $i \in \mathbb{N}$ be large enough such that the above estimate holds for all $1 \leq j \leq l$.

Given $q_i \in G_1(p_i)$, let (Y, q) be an arbitrary sublimit of $(\lambda_i M_i, q_i)$, i.e. for a subsequence $(i_s)_{s\in\mathbb{N}}$,

$$(\lambda_{i_s} M_{i_s}, q_{i_s}) \to (Y, q) \text{ as } s \to \infty.$$

For a further subsequence $(i_{s_t})_{t\in\mathbb{N}}$ there is some $1 \leq j \leq l$ with $q_{i_{s_t}} \in G_R(p_{i_{s_t}}^{x_j})$ for all t and

$$(\lambda_{i_{s_t}}^{x_j} M_{i_{s_t}}, q_{i_{s_t}}) \to (\mathbb{R}^k \times \tilde{K}, \cdot) \text{ as } t \to \infty$$

for a compact metric space $\tilde{K}$ satisfying $\mathrm{diam}(\tilde{K}) \in [\frac{1}{5}, 1]$.

On the other hand,

$$\left(\frac{\lambda_{i_{s_t}}}{\lambda_{i_{s_t}}^{x_j}} \cdot \lambda_{i_{s_t}}^{x_j} M_{i_{s_t}}, q_{i_{s_t}} \right) = \left(\lambda_{i_{s_t}} M_{i_{s_t}}, q_{i_{s_t}} \right) \to (Y, q) \text{ as } t \to \infty,$$

and by Lemma 1.10 c), $\frac{\lambda_{i_{s_t}}}{\lambda_{i_{s_t}}^{x_j}}$ converges to some α and Y is isometric to the product $\mathbb{R}^k \times K$ for $K := \alpha \tilde{K}$. In particular, $5^{-m_j} \leq \alpha \leq 5^{m_j}$. Thus, for $D := 5^{\max\{m_j | 1 \leq j \leq l\}+1}$, $\mathrm{diam}(K) \in [\frac{1}{D}, D]$. Moreover, for any non-principal ultrafilter ω,

$$\dim(K) = \dim(\tilde{K}) = l_\omega^{x_j}(R) = l_\omega^{x_0}(R).$$

In particular, for any two sublimits $(\mathbb{R}^k \times K_1, \cdot)$ and $(\mathbb{R}^k \times K_2, \cdot)$ coming from the same subsequence of indices, let ω be a non-principal ultrafilter as in Lemma 1.15 such that these sublimits are ultralimits with respect to ω. Then $\dim(K_1) = \dim(K_2)$. $\square$

Appendix A

Gromov-Hausdorff convergence

Gromov-Hausdorff distance is an often used tool for measuring how far two compact metric spaces are from being isometric. This distance, which was introduced by Gromov in [Gro81], leads to the notion of Gromov-Hausdorff convergence which can be extended to non-compact metric spaces and allows to draw conclusions about the properties of the spaces 'near' to the limit space, if the limit space is well understood.

Many textbooks such as [BBI01, sections 7.3-7.5], [Pet06, section 10.1] and [BH99, p. 70ff.] give a (more or less) detailed introduction to the distance of compact metric spaces. Some even more detailed proofs can be found in [Ron10]. Since the literature on convergence of non-compact metric spaces usually is less comprehensive, this chapter treats the latter in detail. For the sake of completeness, it also contains a detailed introduction to the compact case, which is built on the literature cited above.

The first section deals with Gromov-Hausdorff distance of compact metric spaces. In addition, so called Gromov-Hausdorff approximations are introduced and the relation between those two terms is described. For both terms, a pointed and a non-pointed version is introduced, and it will be proven that these terms result in the same notion of convergence.

The second section deals with convergence of non-compact metric spaces, the major part of this chapter, and consists of four parts: First, for compact length spaces it will be proven that this notion of convergence coincides with the one for compact spaces. Secondly, several properties of pointed Gromov-Hausdorff convergence will be verified. After that, a convergence notion for points will be introduced and studied. Finally, convergence of (Lipschitz) maps will be investigated.

The third section deals with ultralimits, a more general tool to create 'limit spaces', and states some properties of those. In particular, a strong correspondence between ultralimits and subsequences converging in the pointed Gromov-Hausdorff sense will be established.

The fourth and final section reminds of the definition of measured Gromov-Hausdorff convergence as explained in [CC97].

A.1 The compact case

Given a metric space, an interesting question is whether it is possible to assign each two subsets a distance such that this distance in turn defines a metric. In [Hau65, Chapter VIII §6], Hausdorff answered this question by describing what nowadays is called the Hausdorff distance: For two subsets of a metric space, this is the minimal radius such that each subset is contained in the ball (with this radius) of the other subset. This was extended by Gromov in [Gro81, section 6] to describe how far two compact metric spaces are from being isometric by mapping two such spaces isometrically into a third one and measuring the Hausdorff distance of the images. This is the so called Gromov-Hausdorff distance.

Definition A.1. For bounded subsets A and B of a metric space (X, d), the *Hausdorff distance of A and B* is defined as

$$d_H^d(A, B) := \inf\{\varepsilon > 0 \mid A \subseteq B_\varepsilon^X(B) \text{ and } B \subseteq B_\varepsilon^X(A)\}$$

where $B_\varepsilon^X(B) := \{x \in X \mid \exists b \in B : d(x, b) < \varepsilon\}$. For two compact metric spaces (X, d_X) and (Y, d_Y), the *Gromov-Hausdorff distance of X and Y* is defined as

$$d_{GH}(X, Y) := \inf\{d_H^d(X, Y) \mid d \text{ admissible metric on } X \amalg Y\},$$

where a metric d on the disjoint union $X \amalg Y$ is called *admissible* if $d_{|X \times X} = d_X$ and $d_{|Y \times Y} = d_Y$.

On the space of (non-empty) compact subspaces of X, this d_H defines a metric, while d_{GH} defines a metric on the set of isometry classes of (non-empty) compact metric spaces. This will be proven below. From now on, all metric spaces are assumed to be non-empty. In order to compare two metric spaces with respect to some fixed base points, the pointed Gromov-Hausdorff distance is used.

Definition A.2. Let (X, d) be a metric space, $A, B \subseteq X$ bounded subsets and $a \in A$, $b \in B$ base points. The *pointed Hausdorff distance of (A, a) and (B, b)* is given by

$$d_H^d((A, a), (B, b)) := d_H^d(A, B) + d(a, b)$$

and the *pointed Gromov-Hausdorff distance between* two pointed compact metric spaces (X, x_0) and (Y, y_0) is defined as

$$d_{GH}((X, x_0), (Y, y_0)) := \inf\{d_H^d((X, x_0), (Y, y_0)) \mid d \text{ admissible metric on } X \amalg Y\}.$$

As in the non-pointed case, the pointed Gromov-Hausdorff distance defines a metric on the set of isometry classes of (non-empty) pointed compact metric spaces. In order to prove this, a notion strongly related to the one of Gromov-Hausdorff distance is used.

Definition A.3. Let $(X, d_X), (Y, d_Y)$ be metric spaces, $p \in X$, $q \in Y$ and $\varepsilon > 0$. A pair of (not necessarily continuous) maps $f : X \to Y$ and $g : Y \to X$ is called *(ε-)Gromov-Hausdorff approximations* or *ε-approximations* if for all $x, x_1, x_2 \in X$ and $y, y_1, y_2 \in Y$,

$$|d_X(x_1, x_2) - d_Y(f(x_1), f(x_2))| < \varepsilon, \qquad d_X(g \circ f(x), x) < \varepsilon,$$
$$|d_Y(y_1, y_2) - d_X(g(y_1), g(y_2))| < \varepsilon, \qquad d_Y(f \circ g(y), y) < \varepsilon\,.$$

The set of all such pairs is denoted by $Appr_\varepsilon(X, Y)$. In the pointed case, one restricts to pointed maps:

$$Appr_\varepsilon((X,p),(Y,q)) := \{(f,g) \in Appr_\varepsilon(X,Y) \mid f(p) = q \text{ and } g(q) = p\}.$$

Remark. In the literature, Gromov-Hausdorff approximations often are not defined as pairs of maps but as one map $f : X \to Y$ where f has distortion less than ε and $B_\varepsilon(f(X)) = Y$. Observe that $(f, g) \in Appr_\varepsilon(X, Y)$ already implies that f has these properties (for the same ε).

In the following it will be seen that Gromov-Hausdorff distance less than ε corresponds to ε-approximations (up to a factor). The next proposition shows that (up to another factor) the definition of Gromov-Hausdorff approximations used here can be replaced by the one described above.

Proposition A.4. *Let $\varepsilon > 0$ and $f : (X, d_X) \to (Y, d_Y)$ be a map between metric spaces such that*

$$|d_Y(f(x_1), f(x_2)) - d_X(x_1, x_2)| < \varepsilon$$

for all $x_1, x_2 \in X$. Then there exists $g : f(X) \to X$ such that $(f, g) \in Appr_\varepsilon(X, f(X))$. Moreover, if $Y = B_\varepsilon(f(X))$, then there is $h : Y \to X$ such that $(f, h) \in Appr_{3\varepsilon}(X, Y)$.

Proof. For each $y \in f(X)$ choose some $g(y) \in f^{-1}(y)$. In particular, $f \circ g = \mathrm{id}_{|f(X)}$. For $y_1, y_2 \in f(X)$,

$$|d_X(g(y_1), g(y_2)) - d_Y(y_1, y_2)| = |d_X(g(y_1), g(y_2)) - d_Y(f(g(y_1)), f(g(y_2)))| < \varepsilon,$$

and for $x \in X$,

$$d(x, g \circ f(x)) = |d(x, g(f(x))) - d(f(x), f(g \circ f(x)))| < \varepsilon\,.$$

Thus, $(f, g) \in Appr_\varepsilon(X, f(X))$.

Now assume $Y = B_\varepsilon(f(X))$. For $y \in f(X)$, define $h(y) := g(y)$, otherwise, choose $y' \in f(X)$ with $d_Y(y, y') < \varepsilon$ and define $h(y) := y'$. By construction, $h \circ f = g \circ f$, i.e. for all $x \in X$,

$$d_X(h \circ f(x), x) < \varepsilon\,.$$

For arbitrary $y \in Y$, using $f \circ g = \mathrm{id}_{|f(X)}$, $f \circ h(y) = f \circ g(y') = y'$ for $y' \in f(X) \cap B_\varepsilon(y)$ as in the definition of h. Hence,

$$d_Y(f \circ h(y), y) = d_Y(y', y) < \varepsilon\,.$$

Finally, for arbitrary $y_1, y_2 \in Y$,

$$
\begin{aligned}
&|d_X(h(y_1), h(y_2)) - d_Y(y_1, y_2)| \\
&\le |d_X(h(y_1), h(y_2)) - d_Y(f(h(y_1)), f(h(y_2)))| + |d_Y(f(h(y_1)), f(h(y_2))) - d_Y(y_1, y_2)| \\
&< \varepsilon + d_Y(f \circ h(y_1), y_1) + d_Y(f \circ h(y_2), y_2) \\
&< 3\,\varepsilon\,.
\end{aligned}
$$

□

Next, a strong connection between existence of Gromov-Hausdorff approximations and the Gromov-Hausdorff distance will be proven.

Proposition A.5. *Let X and Y be compact metric spaces with base points $p \in X$ and $q \in Y$, respectively, and $\varepsilon > 0$.*

a) If $d_{GH}(X, Y) < \varepsilon$, then $Appr_{2\varepsilon}(X, Y) \neq \emptyset$.

b) If $Appr_{\varepsilon}(X, Y) \neq \emptyset$, then $d_{GH}(X, Y) \le 2\,\varepsilon$.

c) If $d_{GH}((X, p), (Y, q)) < \varepsilon$, then $Appr_{2\varepsilon}((X, p), (Y, q)) \neq \emptyset$.

d) If $Appr_{\varepsilon}((X, p), (Y, q)) \neq \emptyset$, then $d_{GH}((X, p), (Y, q)) \le 2\,\varepsilon$.

Proof. As the proofs of a) and b), respectively, are very similar to, but slightly easier than those of c) and d), respectively, only the latter two are proven here.

c) Let $0 < \delta < \varepsilon - d_{GH}((X, p), (Y, q))$ and choose an admissible metric d with

$$d_H^d((X, p), (Y, q)) < d_{GH}((X, p), (Y, q)) + \delta < \varepsilon\,.$$

Then $d(p, q) < \varepsilon$ on the one hand and $d_H^d(X, Y) < \varepsilon$ on the other, i.e. for all $x \in X$ there exists $y_x \in Y$ that satisfies $d(x, y_x) < \varepsilon$. Analogously, for each $y \in Y$ there is $x_y \in X$ satisfying $d(y, x_y) < \varepsilon$. Define $f : X \to Y$ and $g : Y \to X$ by

$$f(x) := \begin{cases} q & \text{if } x = p, \\ y_x & \text{otherwise,} \end{cases} \qquad g(y) := \begin{cases} p & \text{if } y = q, \\ x_y & \text{otherwise.} \end{cases}$$

As seen above, $d(f(x), x) < \varepsilon$ for all $x \in X$. Thus, for all $x, x' \in X$,

$$|d_Y(f(x), f(x')) - d_X(x, x')| \le d(f(x), x)) + d(f(x'), x') < 2\,\varepsilon\,.$$

Analogously, $|d_X(g(y), g(y')) - d_Y(y, y')| < 2\,\varepsilon$ for all $y, y' \in Y$. Similarly, for $x \in X$,

$$
\begin{aligned}
d_X(g \circ f(x), x) &= d(g \circ f(x), x) \\
&\le d(g(f(x)), f(x)) + d(f(x), x) \\
&< 2\,\varepsilon,
\end{aligned}
$$

as well as $d_Y(f \circ g(y), y) < 2\,\varepsilon$ for all $y \in Y$. Thus,

$$(f, g) \in Appr_{2\varepsilon}((X, p), (Y, q)).$$

d) Fix an arbitrary pair $(f, g) \in Appr_\varepsilon((X, p), (Y, q))$. The definition of an admissible metric $d : (X \amalg Y) \times (X \amalg Y) \to \mathbb{R}$ requires $d_{|X\times X} := d_X$, $d_{|Y\times Y} := d_Y$ and $d(y, x) := d(x, y)$ for $x \in X$ and $y \in Y$. Hence, it suffices to define $d(x, y)$ for $x \in X$ and $y \in Y$. Then d is positive definite and symmetric by definition. Thus, in order to prove that d is a metric, it remains to check the triangle inequality. If done so, then d is in fact an admissible metric.

Define $d : (X \amalg Y) \times (X \amalg Y) \to \mathbb{R}$ via

$$d(x, y) := \frac{\varepsilon}{2} + \inf\{d_X(x, x') + d_Y(f(x'), y) \mid x' \in X\}$$

for $x \in X$ and $y \in Y$. It remains to check the triangle inequality. For $x_1, x_2 \in X$ and $y \in Y$,

$$\begin{aligned}
&d(x_1, x_2) + d(x_2, y)\\
&= d_X(x_1, x_2) + \frac{\varepsilon}{2} + \inf\{d_X(x_2, x') + d_Y(f(x'), y) \mid x' \in X\}\\
&= \frac{\varepsilon}{2} + \inf\{d_X(x_1, x_2) + d_X(x_2, x') + d_Y(f(x'), y) \mid x' \in X\}\\
&\geq \frac{\varepsilon}{2} + \inf\{d_X(x_1, x') + d_Y(f(x'), y) \mid x' \in X\}\\
&= d(x_1, y)
\end{aligned}$$

and

$$\begin{aligned}
&d(x_1, y) + d(y, x_2)\\
&= \varepsilon + \inf\{d_X(x_1, x') + d_Y(f(x'), y) + d_X(x_2, x'') + d_Y(f(x''), y) \mid x', x'' \in X\}\\
&\geq \varepsilon + \inf\{d_X(x_1, x') + d_Y(f(x'), f(x'')) + d_X(x_2, x'') \mid x', x'' \in X\}\\
&\geq \varepsilon + \inf\{d_X(x_1, x') + (d_X(x', x'') - \varepsilon) + d_X(x_2, x'') \mid x', x'' \in X\}\\
&\geq \inf\{d_X(x_1, x_2) \mid x', x'' \in X\}\\
&= d(x_1, x_2).
\end{aligned}$$

For $x \in X$ and $y_1, y_2 \in Y$, the triangle inequalities $d(x, y_1) + d(y_1, y_2) \geq d(x, y_2)$ and $d(y_1, x) + d(x, y_2) \geq d(y_1, y_2)$ can be proven analogously.

Using this metric d,

$$d(p, q) = \frac{\varepsilon}{2} + \inf\{d_X(p, x') + d_Y(f(x'), q) \mid x' \in X\} = \frac{\varepsilon}{2}$$

due to $0 \leq \inf\{d_X(p, x') + d_Y(f(x'), q) \mid x' \in X\} \leq d_X(p, p) + d_Y(f(p), q) = 0$. Furthermore, for $x \in X$,

$$d(x, f(x)) = \frac{\varepsilon}{2} + \inf\{d_X(x, x') + d_Y(f(x'), f(x)) \mid x' \in X\} = \frac{\varepsilon}{2}$$

using $x' = x$. For $y \in Y$, this implies

$$d(y, g(y)) \leq d(y, f \circ g(y)) + d(f \circ g(y), g(y)) < \varepsilon + \frac{\varepsilon}{2} = \frac{3\,\varepsilon}{2}.$$

Thus, $X \subseteq B^d_{\varepsilon/2}(f(X)) \subseteq B^d_{3\,\varepsilon/2}(Y)$ and $Y \subseteq B^d_{3\,\varepsilon/2}(X)$, i.e. $d^d_H(X, Y) \leq \frac{3\varepsilon}{2}$ and

$$d_{GH}((X, p), (Y, q)) \leq d^d_H((X, p), (Y, q)) = d^d_H(X, Y) + d(p, q) \leq 2\,\varepsilon\,. \qquad \square$$

Using these approximations, one can prove that the pointed Gromov-Hausdorff distance defines a metric. Two pointed metric spaces (X, p) and (Y, q) are called *isometric* if there exists an isometry $f : X \to Y$ with $f(p) = q$.

Proposition A.6. *On the space of isometry classes of (pointed) compact metric spaces, d_{GH} defines a metric.*

Proof. In order to prove that the Gromov-Hausdorff distance indeed defines a metric, one needs that the Hausdorff distance defines a metric. Therefore, this proof splits into several steps: First, the Hausdorff distance will be investigated. Then it will be proven that the Gromov-Hausdorff distance defines a pseudo-metric on the class of (pointed) compact metric spaces, i.e. it is not definite, but satisfies all the other properties of a metric. Finally, it will be proven that this already defines a metric up to isometry.

a) Let (X, d) be a metric space and $A, B, C \subseteq X$ be compact. First, prove that d_H is a metric in the non-pointed case:

   By definition, $d^d_H(B, A) = d^d_H(A, B)$, $d^d_H(A, B) \geq 0$ and $d^d_H(A, A) = 0$. In order to prove the triangle inequality, let $r_1 := d^d_H(A, B) \geq 0$, $r_2 := d^d_H(B, C) \geq 0$ and $\varepsilon > 0$ be arbitrary. For $a \in A$ there exists $b \in B$ with $d(a, b) < r_1 + \varepsilon$. Furthermore, there is $c \in C$ with $d(b, c) < r_2 + \varepsilon$. Hence, $d(a, c) < r_1 + r_2 + 2\,\varepsilon$ and this proves $A \subseteq B_{r_1+r_2+2\,\varepsilon}(C)$. An analogous argumentation proves $C \subseteq B_{r_1+r_2+2\,\varepsilon}(A)$, and therefore, $d^d_H(A, C) \leq r_1 + r_2 + 2\,\varepsilon$. Since $\varepsilon > 0$ was arbitrary, this proves $d^d_H(A, C) \leq r_1 + r_2 = d^d_H(A, B) + d^d_H(B, C)$.

   Assume that $A \neq B$ and $d^d_H(A, B) = 0$. Without loss of generality, assume there exists $a \in A$ with $a \notin B$. In particular, $d(a, b) > 0$ for all $b \in B$. Since B is compact, this proves $0 < \inf\{d(a, b) \mid b \in B\} \leq d^d_H(A, B)$, and this is a contradiction.

   Now fix $a \in A$, $b \in B$ and $c \in C$. Since d_H is a metric in the non-pointed case,

$$d^d_H((A, a), (B, b)) = d^d_H(A, B) + d(a, b) \geq 0$$

   and equality holds if and only if $A = B$ and $a = b$. Obviously, d_H is symmetric and

$$\begin{aligned} &d^d_H((A, a), (B, b)) + d^d_H((B, b), (C, c)) \\ &= d^d_H(A, B) + d^d_H(B, C) + d(a, b) + d(b, c) \\ &\geq d^d_H(A, C) + d(a, c) \\ &= d^d_H((A, a), (C, c)). \end{aligned}$$

   Thus, d_H defines a metric.

b) From now on, the proof restricts to the case of pointed metric spaces since the other one can be done completely analogously. Obviously, d_{GH} is non-negative and symmetric. It remains to prove the triangle inequality. Let (X, x_0), (Y, y_0) and (Z, z_0) be pointed compact metric spaces. For arbitrary $\varepsilon > 0$, choose admissible metrics d_{XY} on $X \amalg Y$ and d_{YZ} on $Y \amalg Z$ such that

$$\begin{aligned} d_H^{d_{XY}}((X, x_0), (Y, y_0)) &< d_{GH}((X, x_0), (Y, y_0)) + \varepsilon \quad \text{and} \\ d_H^{d_{YZ}}((Y, y_0), (Z, z_0)) &< d_{GH}((Y, y_0), (Z, z_0)) + \varepsilon\,. \end{aligned}$$

Define an admissible metric d_{XZ} on $X \amalg Z$ by

$$d_{XZ}(x, z) = \inf\{d_{XY}(x, y) + d_{YZ}(y, z) \mid y \in Y\}.$$

This actually defines a metric: Since everything else is obvious, only the triangle inequality needs to be checked. If all regarded points are contained in X or all in Z, there is nothing to prove. For $x_1, x_2 \in X$ and $z \in Z$,

$$\begin{aligned} &d_{XZ}(x_1, x_2) + d_{XZ}(x_2, z) \\ &= d_X(x_1, x_2) + \inf\{d_{XY}(x_2, y') + d_{YZ}(y', z) \mid y' \in Y\} \\ &= \inf\{d_{XY}(x_1, x_2) + d_{XY}(x_2, y') + d_{YZ}(y', z) \mid y' \in Y\} \\ &\geq \inf\{d_{XY}(x_1, y') + d_{YZ}(y', z) \mid y' \in Y\} \\ &= d_{XZ}(x_1, z) \end{aligned}$$

and

$$\begin{aligned} &d_{XZ}(x_1, z) + d_{XZ}(z, x_2) \\ &= \inf\{d_{XY}(x_1, y') + d_{YZ}(y', z) + d_{YZ}(z, y'') + d_{XY}(y'', x_2) \mid y', y'' \in Y\} \\ &\geq \inf\{d_{XY}(x_1, y') + d_Y(y', y'') + d_{XY}(y'', x_2) \mid y', y'' \in Y\} \\ &\geq \inf\{d_{XY}(x_1, y') + d_{XY}(y', x_2) \mid y' \in Y\} \\ &\geq d_X(x_1, x_2) \\ &= d_{XZ}(x_1, x_2). \end{aligned}$$

For $x \in X$ and $z_1, z_2 \in Z$, the inequalities $d_{XZ}(z_1, z_2) + d_{XZ}(z_2, x) \geq d_{XZ}(z_1, x)$ and $d(z_1, x) + d_{XZ}(x, z_2) \geq d_{XZ}(z_1, z_2)$ can be proven analogously. With similar arguments, one can prove that d_{XYZ} defines an admissible metric on $X \amalg Y \amalg Z$ where

$$d_{XYZ}(x, y) := \begin{cases} d_{XY}(x, y) & \text{if } x, y \in X \amalg Y, \\ d_{XZ}(x, y) & \text{if } x, y \in X \amalg Z, \\ d_{YZ}(x, y) & \text{if } x, y \in Y \amalg Z. \end{cases}$$

With those admissible metrics,

$$\begin{aligned} &d_{GH}((X,x_0),(Z,z_0)) \\ &\le d_H^{d_{XYZ}}(X,Z) + d_{XYZ}(x_0,z_0) \\ &\le d_H^{d_{XYZ}}(X,Y) + d_H^{d_{XYZ}}(Y,Z) + d_{XYZ}(x_0,y_0) + d_{XYZ}(y_0,z_0) \\ &\le d_H^{d_{XY}}(X,Y) + d_H^{d_{YZ}}(Y,Z) + d_{XY}(x_0,y_0) + d_{YZ}(y_0,z_0) \\ &< d_{GH}((X,x_0),(Y,y_0)) + d_{GH}((Y,y_0),(Z,z_0)) + 2\,\varepsilon, \end{aligned}$$

where in the second last inequality the fact is used that the inclusion $X \subseteq B_r^{d_{XY}}(Y)$ implies the inclusion $X \subseteq B_r^{d_{XYZ}}(Y)$, where $r > 0$ is arbitrary. Now letting $\varepsilon \to 0$ proves the triangle inequality for d_{GH}.

c) It is easy to see that the distance of isometric pointed compact spaces vanishes: Let (X,p) and (Y,q) be isometric via isometries f and g. For arbitrary $\varepsilon > 0$, then $(f,g) \in Appr_{\varepsilon/2}((X,p),(Y,q))$. By Proposition A.5, $d_{GH}((X,p),(Y,q)) \le \varepsilon$. Hence, $d_{GH}((X,p),(Y,q)) = 0$.

Conversely, let (X,p) and (Y,q) be two pointed compact metric spaces satisfying $d_{GH}((X,p),(Y,q)) = 0$. By definition, for each $n \ge 1$ there is an admissible metric d_n on $X \amalg Y$ with $d_H^{d_n}(X,Y) + d_n(p,q) < \frac{1}{n}$. Since X is compact and thus separable, there exists a countable dense subset $X' = \{x_i \mid i \in \mathbb{N}\} \subseteq X$ with $x_0 = p$.

Define $y_n^0 := q$. The constant sequence $(y_n^0)_{n\in\mathbb{N}}$ converges to q, and for each n, $d_n(x_0, y_n^0) = d_n(p,q) < \frac{1}{n}$.

Because of $d_H^{d_n}(X,Y) < \frac{1}{n}$, there exists some $y_n^1 \in Y$ such that $d_n(x_1, y_n^1) < \frac{1}{n}$. Since Y is compact, $(y_n^1)_n$ has a convergent subsequence $(y_{n_i}^1)_{i\in\mathbb{N}}$ with some limit $y_1 \in Y$. Then

$$d_{n_i}(x_1, y_1) \le d_{n_i}(x_1, y_{n_i}^1) + d_{n_i}(y_{n_i}^1, y_1) \to 0 \text{ as } i \to \infty.$$

The same argument for x_2 gives a subsequence $d_{n_{i_j}}$ of d_{n_i} and some $y_2 \in Y$ with $d_{n_{i_j}}(x_2, y_2) \to 0$ as $j \to \infty$. By a diagonal argument, there is a subsequence d_l of d_n and a sequence $(y_i)_{i\in\mathbb{N}}$ with $y_0 = q$ such that $d_l(x_i, y_i) \to 0$ as $l \to \infty$ for all i.

Define $f : X' \to Y$ by $f(x_i) := y_i$. Since the d_l are admissible metrics, for each l,

$$d_Y(f(x_i), f(x_j)) = d_l(f(x_i), f(x_j)) = d_l(y_i, y_j) \quad \text{and} \quad d_X(x_i, x_j) = d_l(x_i, x_j).$$

Therefore,

$$\begin{aligned} |d_Y(f(x_i), f(x_j)) - d_X(x_i, x_j)| &= |d_l(y_i, y_j) - d_l(x_i, x_j)| \\ &\le d_l(y_i, x_i) + d_l(x_j, y_j)) \\ &\to 0 \text{ as } l \to \infty. \end{aligned}$$

Hence, f is an isometry. Since X' is dense, f can be extended uniquely to an isometric embedding $f : X \to Y$ with $f(p) = q$. With a similar construction and using a subsequence of d_l, there is an isometric embedding $g : Y \to X$ with $g(q) = p$. After passing to this subsequence, for each x,

$$d_l(g \circ f(x), x) \leq d_l(g(f(x)), f(x)) + d_l(f(x), x) \to 0 \text{ as } l \to \infty.$$

Thus, f is an isometry with $f(p) = q$, i.e. (X, p) and (Y, q) are isometric. □

The definitions of pointed and non-pointed Gromov-Hausdorff distance essentially give the same notion of convergence. This will be proven next.

Proposition A.7. *Let X and Y be compact metric spaces.*

a) For each $x \in X$ and $y \in Y$, $d_{GH}(X, Y) \leq d_{GH}((X, x), (Y, y))$.

b) For any $x \in X$ there exists $y \in Y$ such that $d_{GH}((X, x), (Y, y)) \leq 2d_{GH}(X, Y)$.

Proof. Both statements follow easily from the definitions:

a) Let $x \in X$ and $y \in Y$ be arbitrary. Then

$$\begin{aligned} d_{GH}(X, Y) &= \inf\{d_H^d(X, Y) \mid d \text{ admissible metric on } X \amalg Y\} \\ &\leq \inf\{d_H^d(X, Y) + d(x, y) \mid d \text{ admissible metric on } X \amalg Y\} \\ &= \inf\{d_H^d((X, x), (Y, y)) \mid d \text{ admissible metric on } X \amalg Y\} \\ &= d_{GH}((X, x), (Y, y)). \end{aligned}$$

b) Let $r := d_{GH}(X, Y) \geq 0$. For arbitrary $n \in \mathbb{N}$, let d_n be an admissible metric on $X \amalg Y$ satisfying

$$d_H^{d_n}(X, Y) < d_{GH}(X, Y) + \frac{1}{n} = r + \frac{1}{n}.$$

Thus, $X \subseteq \bar{B}^{d_n}_{r+1/n}(Y)$, i.e. there exists $y_n \in Y$ such that $d_n(x, y_n) \leq r + \frac{1}{n}$. Since Y is compact, there exists a convergent subsequence $(y_{n_m})_{m \in \mathbb{N}}$ of $(y_n)_{n \in \mathbb{N}}$ with limit $y \in Y$. Then

$$\begin{aligned} d_H^{d_{n_m}}((X, x), (Y, y)) &= d_H^{d_{n_m}}(X, Y) + d_{n_m}(x, y) \\ &\leq r + \frac{1}{n_m} + d_{n_m}(x, y_{n_m}) + d_{n_m}(y_{n_m}, y) \\ &\leq 2r + \frac{2}{n_m} + d_Y(y_{n_m}, y) \end{aligned}$$

and

$$\begin{aligned} d_{GH}((X, x), (Y, y)) &= \inf\{d_H^d((X, x), (Y, y)) \mid d \text{ admissible metric on } X \amalg Y\} \\ &\leq \inf\{d_H^{d_{n_m}}((X, x), (Y, y)) \mid m \in \mathbb{N}\} \\ &\leq \inf\{2r + \frac{2}{n_m} + d_Y(y_{n_m}, y) \mid m \in \mathbb{N}\} \\ &= 2r. \end{aligned}$$

□

It is easy to give an example where the inequality in Proposition A.7 a) is strict.

Example A.8. Equip the interval $I := [-1, 1]$ with the induced metric from $\mathbb{R}$ and fix the points $0, 1 \in I$. Then

$$d_{GH}((I, 0), (I, 1)) \geq \frac{1}{2} > 0 = d_{GH}(I, I).$$

In order to see the first inequality, assume $d_{GH}((I, 0), (I, 1)) < \frac{1}{2}$. By Proposition A.5, there exists $(f, g) \in Appr_1((I, 0), (I, 1))$. In particular,

$$1 > |d_{\mathbb{R}}(g(1), g(-1)) - d_{\mathbb{R}}(1, -1)| = ||g(-1)| - 2| \geq 1$$

due to $0 \leq |g(-1)| \leq 1$. This is a contradiction.

Definition A.9. Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed compact metric spaces.

a) If $d_{GH}(X_i, X) \to 0$ as $i \to \infty$, then X_i *converges to* X.

b) If $d_{GH}((X_i, p_i), (X, p)) \to 0$ as $i \to \infty$, then (X_i, p_i) *converges to* (X, p).

If X_i converges to X, this is denoted by $X_i \to X$. If (X_i, p_i) converges to (X, p), this is denoted by $(X_i, p_i) \to (X, p)$.

Corollary A.10. *Let (X, d_X) and (X_i, d_{X_i}), $i \in \mathbb{N}$, be compact metric spaces.*

a) If $(X_i, x_i) \to (X, x)$ for some $x_i \in X_i$ and $x \in X$, then $X_i \to X$ as well.

b) If $X_i \to X$ and $x \in X$, then there exist $x_i \in X_i$ such that $(X_i, x_i) \to (X, x)$.

Recall that a metric space (X, d_X) is called *length space* if

$$d(x, y) = \inf\{L(c) \mid c \text{ continuous curve from } x \text{ to } y\}$$

for any $x, y \in X$, where $L(c)$ denotes the length of c.

Proposition A.11 ([BBI01, Theorem 7.5.1])**.** *A complete compact Gromov-Hausdorff limit of compact length spaces is a length space.*

In general, the Gromov-Hausdorff distance of two subsets of the same metric space, equipped with the induced metric, can be estimated by their Hausdorff distance. If this metric space is a length space and the subsets are balls, this estimate can be expressed by using the radii and the distance of the base points. This uses the property of length spaces that r-ball around a ball of radius s coincides with the $r + s$ ball (around the same base point).

Lemma A.12. *Let (X, d) be a length space, $p \in X$ and $r, s > 0$. Then*

$$B_r(B_s(p)) = B_{r+s}(p).$$

Proof. Let $q \in B_r(B_s(p))$, i.e. there exists $x \in B_s(p)$ with $d(x,q) < r$. Then

$$d(q,p) \leq d(q,x) + d(x,p) < r + s$$

proves $B_r(B_s(p)) \subseteq B_{r+s}(p)$. In fact, this inclusion holds in every metric space.

Conversely, let $q \in B_{r+s}(p)$. Since $B_s(p) \subseteq B_r(B_s(p))$, assume $q \in B_{r+s}(p) \setminus B_s(p)$. Let $l := d(p,q)$ denote the distance of p and q. In particular, $s \leq l < r + s$. Fix a shortest geodesic $\gamma : [0,l] \to X$ with $\gamma(0) = p$ and $\gamma(l) = q$. Define $\varepsilon := \frac{1}{2} \cdot \min\{s, r+s-l\} > 0$ and $t := s - \varepsilon \in (0,s) \subseteq [0,l]$. Then

$$d(\gamma(t),p) = t < s \quad \text{and} \quad d(\gamma(t),q) = l - t = l - s + \varepsilon < l - s + r + s - l = r.$$

Hence, $\gamma(t) \in B_s(p)$ and $q \in B_r(\gamma(t))$, and this verifies $B_{r+s}(p) \subseteq B_r(B_s(p))$. □

Lemma A.13. *Let (X,d) be a length space, $p,q \in X$, $r,s > 0$. Then*

$$d_H^d(\bar{B}_r(p), \bar{B}_s(q)) \leq d(p,q) + |r-s|.$$

Proof. Let $\varepsilon := d(p,q) + |r-s|$. If $\varepsilon = 0$, the claim holds due to $p = q$ and $r = s$. Hence, assume $\varepsilon > 0$. Then, applying Lemma A.12,

$$B_r(p) \subseteq B_{d(p,q)+r}(q) \subseteq B_{d(p,q)+|r-s|+s}(q) = B_{\varepsilon+s}(q) = B_\varepsilon(B_s(q)).$$

Analogously, $B_s(q) \subseteq B_\varepsilon(B_r(p))$. Therefore,

$$d_H^d(\bar{B}_r(p), \bar{B}_s(q)) = d_H^d(B_r(p), B_s(q)) \leq \varepsilon\,.$$ □

Corollary A.14. *Let (X,d) be a length space, $p,q \in X$, $r,s > 0$. Then*

a) $d_{GH}((\bar{B}_r^X(p),p),(\bar{B}_s^X(p),p)) \leq |r-s|$,

b) $d_{GH}((\bar{B}_r^X(p),p),(\bar{B}_r^X(q),q)) \leq 2d(p,q)$.

The diameters of metric spaces with small Gromov-Hausdorff distance are almost the same. In particular, for a convergent sequence of metric spaces, their diameters converge to the diameter of the limit space.

Proposition A.15. *For compact metric spaces (X,d_X) and (Y,d_Y),*

$$|\operatorname{diam}(X) - \operatorname{diam}(Y)| \leq 2d_{GH}(X,Y).$$

In particular, if $X_i \to X$ for compact metric spaces (X_i, d_{X_i}), $i \in \mathbb{N}$, then

$$\operatorname{diam}(X_i) \to \operatorname{diam}(X).$$

Proof. Let $\varepsilon := d_{GH}(X,Y)$, $\delta > 0$ and d be an admissible metric on $X \amalg Y$ such that

$$d_H^d(X,Y) < d_{GH}(X,Y) + \delta = \varepsilon + \delta.$$

This implies $Y \subseteq B^d_{\varepsilon+\delta}(X)$. Therefore, for any $y_1, y_2 \in Y$ there are $x_1, x_2 \in X$ with $d(x_i, y_i) < \varepsilon + \delta$ for $1 \leq i \leq 2$. Hence,

$$d_Y(y_1,y_2) \leq d(y_1,x_1) + d_X(x_1,x_2) + d(x_2,y_2) < 2\varepsilon + 2\delta + \operatorname{diam}(X).$$

Thus,

$$\operatorname{diam}(Y) = \sup\{d_Y(y_1,y_2) \mid y_1,y_2 \in Y\} \leq \operatorname{diam}(X) + 2\varepsilon + 2\delta.$$

Since $\delta > 0$ was arbitrary, $\operatorname{diam}(Y) \leq \operatorname{diam}(X) + 2\varepsilon$. The other inequality can be proven analogously. □

Corollary A.16. *If (X,d) is a compact metric space and $\{\mathrm{pt}\}$ the space consisting of only one point, then $d_{GH}(X,\{\mathrm{pt}\}) = \frac{1}{2} \cdot \operatorname{diam}(X)$.*

Proof. By Proposition A.15, $\operatorname{diam}(X) \leq 2 \cdot d_{GH}(X,\{\mathrm{pt}\})$. Thus, only the other inequality has to be proven.

Let $\delta = \frac{1}{2} \cdot \operatorname{diam}(X)$ and define an admissible metric d on the disjoint union $X \amalg \{\mathrm{pt}\}$ by $d(x,\mathrm{pt}) := \delta$. As usually, only the triangle inequality needs to be checked. For arbitrary $x_1, x_2 \in X$,

$$\begin{aligned} &d(x_1,x_2) + d(x_2,\mathrm{pt}) = d(x_1,x_2) + \delta \geq \delta = d(x_1,\mathrm{pt}) \quad \text{and} \\ &d(x_1,\mathrm{pt}) + d(\mathrm{pt},x_2) = 2\delta = \operatorname{diam}(X) \geq d(x_1,x_2). \end{aligned}$$

Using this metric,

$$d_{GH}(X,\{\mathrm{pt}\}) \leq d_H^d(X,\{\mathrm{pt}\}) = \delta.$$

□

For a metric space (X,d_X), let λX denote the metric space $(\lambda X, d_{\lambda X}) := (X, \lambda d_X)$. Rescaling of compact metric spaces behaves nicely under Gromov-Hausdorff distance. Observe $B_r^X(p) = \{q \in X \mid d_X(q,p) < r\} = \{q \in X \mid \lambda d_X(q,p) < \lambda r\} = B_{\lambda r}^{\lambda X}(p)$ for any $p \in X$ and $r > 0$.

Lemma A.17. *Let (X,d_X) and (Y,d_Y) be compact metric spaces.*

a) For the Hausdorff-distance, $d_H^{\lambda X} = \lambda \cdot d_H^X$ (both in the standard and in the pointed case).

b) For the Gromov-Hausdorff-distance, both $d_{GH}(\lambda X, \lambda Y) = \lambda \cdot d_{GH}(X,Y)$ and, for all $x \in X$ and $y \in Y$, $d_{GH}((\lambda X,x),(\lambda Y,y)) = \lambda \cdot d_{GH}((X,x),(Y,y))$.

Proof. a) Let $A, B \subseteq X$. Then

$$\begin{aligned} d_H^{\lambda X}(A,B) &= \inf\{\varepsilon > 0 \mid A \subseteq B_\varepsilon^{\lambda X}(B) \text{ and } B \subseteq B_\varepsilon^{\lambda X}(A)\} \\ &= \inf\{\lambda\tilde\varepsilon > 0 \mid A \subseteq B_{\tilde\varepsilon}^X(B) \text{ and } B \subseteq B_{\tilde\varepsilon}^X(A)\} \\ &= \lambda \cdot \inf\{\tilde\varepsilon > 0 \mid A \subseteq B_{\tilde\varepsilon}^X(B) \text{ and } B \subseteq B_{\tilde\varepsilon}^X(A)\} \\ &= \lambda \cdot d_H^X(A,B). \end{aligned}$$

Furthermore, for $a \in A$ and $b \in B$,

$$\begin{aligned} d_H^{\lambda X}((A,a),(B,b)) &= d_H^{\lambda X}(A,B) + d_{\lambda X}(a,b) \\ &= \lambda \cdot d_H^X(A,B) + \lambda \cdot d_X(a,b) \\ &= \lambda \cdot d_H^X((A,a),(B,b)). \end{aligned}$$

b) By definition, an admissible metric $\tilde{d}$ on $\lambda X \amalg \lambda Y$ is a metric on $X \amalg Y$ satisfying $\tilde{d}_{|X\times X} = d_{\lambda X} = \lambda \cdot d_X$ and $\tilde{d}_{|Y\times Y} = d_{\lambda Y} = \lambda \cdot d_Y$. Furthermore, $d := \frac{1}{\lambda} \cdot \tilde{d}$ is a metric if and only if $\tilde{d}$ is a metric. In addition, $d_{|X\times X} = \frac{1}{\lambda} \cdot \tilde{d}_{|X\times X} = d_X$ and $d_{|Y\times Y} = d_Y$. Thus, d is an admissible metric on $X \amalg Y$. On the other hand, using similar arguments, if d is an admissible metric on $X \amalg Y$, then $\tilde{d} := \lambda \cdot d$ is an admissible metric on $\lambda X \amalg \lambda Y$.

Hence,

$$\begin{aligned} d_{GH}(\lambda X, \lambda Y) &= \inf\{d_H^{\tilde{d}}(\lambda X, \lambda Y) \mid \tilde{d} \text{ admissible metric on } \lambda X \amalg \lambda Y\} \\ &= \inf\{d_H^{\lambda d}(\lambda X, \lambda Y) \mid \lambda \cdot d \text{ admissible metric on } \lambda X \amalg \lambda Y\} \\ &= \inf\{\lambda \cdot d_H^{d}(\lambda X, \lambda Y) \mid d \text{ admissible metric on } X \amalg Y\} \\ &= \lambda \cdot d_{GH}(X,Y). \end{aligned}$$

Analogously, $d_{GH}((\lambda X, x), (\lambda Y, y)) = \lambda \cdot d_{GH}((X,x),(Y,y))$. □

A.2 The non-compact case

For non-compact metric spaces, the above way of defining a metric (up to isometry) does not work: Using the Hausdorff distance as before on unbounded sets may give distance infinity. Thus, instead of defining a notion of distance for non-compact metric spaces, convergence is defined by using compact subspaces of these spaces only. On these, the previous definitions can be applied.

A metric space is called *proper* if all closed balls are compact. Throughout the remaining section, all metric spaces will assumed to be proper. Notice that proper metric spaces are complete.

For a metric space (X, d_X), $p \in X$ and $r > 0$, let $\bar{B}_r(p) := \{q \in X \mid d_X(p,q) \leq r\}$ denote the closed ball of radius r around p.

Definition A.18. Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed proper metric spaces. If

$$d_{GH}((\bar{B}_r^{X_i}(p_i), p_i), (\bar{B}_r^X(p), p)) \to 0 \text{ as } i \to \infty$$

for all $r > 0$, where the balls are equipped with the restricted metric, then (X_i, p_i) *converges to* (X,p) *(in the pointed Gromov-Hausdorff sense)*. If (X_i, p_i) converges to (X,p), this is denoted by $(X_i, p_i) \to (X,p)$ and (X,p) is called the *(pointed Gromov-Hausdorff) limit* of (X_i, p_i).

Frequently, a sequence (X_i, p_i) does not converge itself but has a converging subsequence. The limit of such a subsequence is called *sublimit* of (X_i, p_i), and (X_i, p_i) is said to *subconverge* to this limit.

Naturally, the question arises under which conditions a given sequence of metric spaces converges in the pointed Gromov-Hausdorff sense. For manifolds, the following theorem by Gromov states that in some cases at least a (Gromov-Hausdorff) sublimit exists. In section A.3, another, more general concept of creating and guaranteeing 'limits' will be introduced. It will turn out that these limits in fact are Gromov-Hausdorff sublimits as well.

Theorem A.19 (Gromov's Pre-compactness Theorem, [Pet06, Cor. 1.11])**.** *For $n \geq 2$, $\kappa \in \mathbb{R}$ and $D > 0$, the following classes are pre-compact, i.e. every sequence in the class has a convergent subsequence whose limit lies in the closure of this class:*

a) The collection of closed Riemannian manifolds with $\operatorname{Ric} \geq (n-1)\cdot\kappa$ *and* $\operatorname{diam} \leq D$.

b) The collection of pointed complete Riemannian manifolds with $\operatorname{Ric} \geq (n-1)\cdot\kappa$.

The section is structured as follows: In subsection A.2.1, the compability of the definition of pointed Gromov-Hausdorff convergence in Definition A.1 with the notion of convergence induced by the Gromov-Hausdorff distance of compact metric (length) spaces is verified. Subsequently, subsection A.2.2 deals with stating and verifying several properties of pointed Gromov-Hausdorff convergence. In this context, convergence of points and convergence of maps, respectively, are introduced in subsection A.2.3 and subsection A.2.4, respectively.

A.2.1 Comparison with the compact case

Applied to compact length spaces, the convergence in the pointed Gromov-Hausdorff sense coincides with the convergence of compact metric spaces in the pointed sense defined in the previous section. Conversely, given (non-pointed) convergence as defined for compact metric spaces and a fixed base point in the limit space, there exist base points such that the spaces converge in the pointed Gromov-Hausdorff sense.

In order to prove this, one uses the fact that approximations can be restricted to smaller balls. This is shown in the following lemma. Another statement of the lemma is that base points can be changed in a certain way. This will be useful later on as well.

Lemma A.20. *a) Let (X, d_X) and (Y, d_Y) be length spaces, $p, p' \in X$, $q, q' \in Y$ and $R \geq r > 0$ such that $\bar{B}^X_r(p') \subseteq \bar{B}^X_R(p)$ and $\bar{B}^Y_r(q') \subseteq \bar{B}^Y_R(q)$. Moreover, let $\varepsilon > 0$,*

$$(f, g) \in Appr_\varepsilon((\bar{B}^X_R(p), p), (\bar{B}^Y_R(q), q))$$

and $\delta := \max\{d(f(p'), q'), d(p', g(q'))\} \geq 0$. Then

$$Appr_{4\,\varepsilon+\delta}((\bar{B}^X_r(p'), p'), (\bar{B}^Y_r(q'), q')) \neq \emptyset$$

and

$$d_{GH}((\bar{B}_r^X(p'),p'),(\bar{B}_r^Y(q'),q')) \leq 8\,\varepsilon + 2\delta.$$

b) For pointed length spaces (X,d_X,p) *and* (Y,d_Y,q) *and* $R \geq r > 0$,

$$d_{GH}((\bar{B}_r^X(p),p),(\bar{B}_r^Y(q),q)) \leq 16 \cdot d_{GH}((\bar{B}_R^X(p),p),(\bar{B}_R^Y(q),q)).$$

Proof. a) For simplicity, let $\delta_f := d(f(p'),q')$, $\delta_g := d(p',g(q'))$, i.e. $\delta = \max\{\delta_f,\delta_g\}$, and $\tilde{\varepsilon} := 4\,\varepsilon + \delta$. As $\bar{B}_r^X(p') \subseteq \bar{B}_R^X(p)$, one can restrict f to $\bar{B}_r^X(p')$. For $x \in \bar{B}_r^X(p')$,

$$\begin{aligned} d_Y(f(x),q') &\leq d_Y(f(x),f(p')) + d_Y(f(p'),q') \\ &\leq (d_X(x,p') + \varepsilon) + \delta_f \\ &< r + \varepsilon + \delta_f. \end{aligned}$$

Hence, $f(\bar{B}_r^X(p')) \subseteq \bar{B}_{r+\varepsilon+\delta_f}^Y(q')$. Analogously, $g(\bar{B}_r^Y(q')) \subseteq \bar{B}_{r+\varepsilon+\delta_g}^X(p')$. Now modify f and g in order to obtain maps $\tilde{f}$ and $\tilde{g}$, respectively, whose images are contained in $\bar{B}_r^Y(q')$ and $\bar{B}_r^X(p')$, respectively, such that $(\tilde{f},\tilde{g})$ are $\tilde{\varepsilon}$-approximations: For $y \in \bar{B}_{r+\varepsilon+\delta_f}^Y(q') \setminus \bar{B}_r^Y(q')$ choose a shortest geodesic $c : [0,l] \to Y$ with $c(0) = q'$ and $c(1) = y$ where $r < l := d_Y(y,q') \leq r + \varepsilon + \delta_f$. Then $d_Y(c(r),q') = r$, in particular, $c(r) \in \bar{B}_r^Y(q')$, and for $\hat{y} := c(r)$,

$$\begin{aligned} d(y,\hat{y}) &= d_Y(y,q') - d_Y(\hat{y},q') \\ &< (r + \varepsilon + \delta_f) - r \\ &= \varepsilon + \delta_f. \end{aligned}$$

Using this, define $\tilde{f} : \bar{B}_r^X(p') \to \bar{B}_r^Y(q')$ by

$$\tilde{f}(x) := \begin{cases} q' & \text{if } x = p', \\ f(x) & \text{if } x \neq p' \text{ and } f(x) \in \bar{B}_r^Y(q'), \\ \widehat{f(x)} & \text{if } x \neq p' \text{ and } f(x) \notin \bar{B}_r^Y(q'). \end{cases}$$

Since $d_Y(\tilde{f}(p'),f(p')) = d_Y(q',f(p')) = \delta_f < \varepsilon + \delta_f$ and by construction,

$$d_Y(\tilde{f}(x),f(x)) < \varepsilon + \delta_f$$

for all $x \in \bar{B}_r^X(p')$. Similarly, define $\tilde{g} : \bar{B}_r^Y(q') \to \bar{B}_r^X(p')$. Using analogous arguments proves

$$d_X(\tilde{g}(y),g(y)) < \varepsilon + \delta_g$$

for all $y \in \bar{B}_r^Y(q')$.

By definition, $\tilde f(p') = q'$ and $\tilde g(q') = p'$, so it remains to prove that $(\tilde f, \tilde g)$ are $\tilde\varepsilon$-approximations. By construction,

$$\begin{aligned}
&|d_X(x_1,x_2) - d_Y(\tilde f(x_1), \tilde f(x_2))| \\
&\leq |d_X(x_1,x_2) - d_Y(f(x_1), f(x_2))| + |d_Y(f(x_1), f(x_2)) - d_Y(\tilde f(x_1), \tilde f(x_2))| \\
&< \varepsilon + (d_Y(f(x_1), \tilde f(x_1)) + d_Y(f(x_2), \tilde f(x_2))) \\
&< \varepsilon + 2(\varepsilon + \delta_f) \\
&< \tilde\varepsilon,
\end{aligned}$$

where $x_1, x_2 \in \bar B_r^X(p')$. Analogously, $|d_Y(y_1,y_2) - d_X(\tilde g(y_1), \tilde g(y_2))| < \tilde\varepsilon$ for arbitrary $y_1, y_2 \in \bar B_r^Y(q')$. Furthermore, for $x \in \bar B_r^X(p')$,

$$\begin{aligned}
&d_X(x, \tilde g \circ \tilde f(x)) \\
&\leq d_X(x, g \circ f(x)) + d_X(g \circ f(x), g \circ \tilde f(x)) + d_X(g \circ \tilde f(x), \tilde g \circ \tilde f(x)) \\
&< \varepsilon + (\varepsilon + d_Y(f(x), \tilde f(x))) + (\varepsilon + \delta_g) \\
&< 4\varepsilon + \delta_f + \delta_g \\
&= \tilde\varepsilon.
\end{aligned}$$

Analogously, $d_Y(y, \tilde f \circ \tilde g(y)) < \tilde\varepsilon$ for all $y \in \bar B_r^Y(q')$. Hence,

$$(\tilde f, \tilde g) \in Appr_{\tilde\varepsilon}((\bar B_r^X(p'), p'), (\bar B_r^Y(q'), q')),$$

and by Proposition A.5,

$$d_{GH}((\bar B_r^X(p'), p'), (\bar B_r^Y(q'), q')) \leq 2\tilde\varepsilon.$$

b) Let $\delta > 0$ be arbitrary and $\varepsilon := d_{GH}((\bar B_R^X(p), p), (\bar B_R^Y(q), q)) + \delta > 0$. By Proposition A.5,

$$Appr_{2\varepsilon}((\bar B_R^X(p), p), (\bar B_R^Y(q), q)) \neq \emptyset,$$

and by a),

$$d_{GH}((\bar B_r^X(p), p), (\bar B_r^Y(q), q)) \leq 16\,\varepsilon = 16 \cdot d_{GH}((\bar B_R^X(p), p), (\bar B_R^Y(q), q)) + 16\,\delta.$$

Since $\delta > 0$ was arbitrary, this implies the claim. □

To avoid confusion, for the next two statements, let $X_i \overset{GH}{\to} X$ and $(X_i, p_i) \overset{GH}{\to} (X, p)$, respectively, denote the convergence of compact metric spaces in the sense of Definition A.1 and Definition A.2, respectively. Further, denote by $(X_i, p_i) \overset{pGH}{\to} (X, p)$ the convergence in the pointed Gromov-Hausdorff sense of Definition A.18.

Proposition A.21. *Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed compact length spaces with $(X_i, p_i) \overset{pGH}{\to} (X, p)$. Then $X_i \overset{GH}{\to} X$, in particular,* $\operatorname{diam}(X_i) \to \operatorname{diam}(X)$.

Proof. Assume $\operatorname{diam}(X_i)$ is not bounded. Let $r > \operatorname{diam}(X)$. Without loss of generality, assume $\operatorname{diam}(X_i) > r$ for all $i \in \mathbb{N}$.

Let $0 < \varepsilon < r - \operatorname{diam}(X)$ and choose $x_i, y_i \in B_r^{X_i}(p_i)$ such that $d_{X_i}(x_i, y_i) \geq r - \frac{\varepsilon}{2}$. For $\varepsilon_i := 2 \cdot d_{GH}((X_i, p_i), (X, p))$ and $(f_i, g_i) \in Appr_{\varepsilon_i}((X_i, p_i), (X, p))$,

$$\operatorname{diam}(X) \geq d_X(f_i(x_i), f_i(y_i)) \geq r - \frac{\varepsilon}{2} - \varepsilon_i\,.$$

Since this holds for all $i \in \mathbb{N}$,

$$\operatorname{diam}(X) \geq r - \frac{\varepsilon}{2} > \operatorname{diam}(X) + \frac{\varepsilon}{2}.$$

This is a contradiction. Thus, there is an $R > \operatorname{diam}(X)$ such that $\operatorname{diam}(X_i) < R$ for all $i \in \mathbb{N}$. Then

$$d_{GH}(X_i, X) = d_{GH}(\bar{B}_R^{X_i}(p_i), \bar{B}_R^{X}(p)) \leq d_{GH}((\bar{B}_R^{X_i}(p_i), p_i), (\bar{B}_R^{X}(p), p)) \to 0 \text{ as } i \to \infty.$$

Hence, $X_i \to X$. Proposition A.15 implies the second part of the claim. □

Corollary A.22. *Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed compact length spaces. Then $(X_i, p_i) \overset{GH}{\to} (X, p)$ if and only if $(X_i, p_i) \overset{pGH}{\to} (X, p)$.*

Proof. The proof is done by proving both implications separately.

a) First, assume $(X_i, p_i) \overset{GH}{\to} (X, p)$ and let $r > 0$ be arbitrary.

   By Proposition A.15, $\operatorname{diam}(X_i) \to \operatorname{diam}(X)$, i.e. without loss of generality, assume a strict diameter bound D on all spaces X_i and X. In particular, for all $r \geq D$, $(\bar{B}_r^{X_i}(p_i), p_i) = (X_i, p_i)$ converges to $(X, p) = (\bar{B}_r^{X}(p), p)$.

   For $0 < r < D$,

$$\begin{aligned} d_{GH}((\bar{B}_r^{X_i}(p_i), p_i), (\bar{B}_r^{X}(p), p)) &\leq 16 \cdot d_{GH}((\bar{B}_D^{X_i}(p_i), p_i), (\bar{B}_D^{X}(p), p)) \\ &= 16 \cdot d_{GH}((X_i, p_i), (X, p)) \\ &\to 0 \end{aligned}$$

   by Lemma A.20. Hence, $(X_i, p_i) \overset{pGH}{\to} (X, p)$.

b) Now let $(X_i, p_i) \overset{pGH}{\to} (X, p)$. By Proposition A.21, $\operatorname{diam}(X_i) \to \operatorname{diam}(X)$. Without loss of generality, assume $\operatorname{diam}(X_i) \leq 2 \operatorname{diam}(X) =: r$. Thus,

$$d_{GH}((X_i, p_i), (X, p)) = d_{GH}((\bar{B}_r^{X_i}(p_i), p_i), (\bar{B}_r^{X}(p), p)) \to 0. \qquad □$$

In particular, if X_i, X are compact and $p \in X$, then, by Corollary A.10, there exist $p_i \in X_i$ such that $(X_i, p_i) \overset{GH}{\to} (X, p)$, and therefore, $(X_i, p_i) \overset{pGH}{\to} (X, p)$.

From now on, let $(X_i, p_i) \to (X, p)$ denote convergence in the pointed Gromov-Hausdorff sense.

A.2.2 Properties as in the compact case

This subsection deals with several properties which are familiar from the compact case. First of all, the Gromov-Hausdorff distance defines a metric on the set of the isometry classes of compact metric spaces. In general, if a sequence of pointed length spaces converges to a pointed space, it converges to its completion as well. Thus, those limit spaces can always be assumed to be complete. Under this assumption, the (complete) limit of pointed Gromov-Hausdorff convergence is unique up to isometry.

Proposition A.23. *Let (X, d_X, p), (Y, d_Y, q) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed length spaces. Assume X and Y to be complete, $(X_i, p_i) \to (X, p)$ and $(X_i, p_i) \to (Y, q)$. Then (X, p) and (Y, q) are isometric.*

Proof. For every $r > 0$, both $\bar{B}_r^X(p)$ and $\bar{B}_r^Y(q)$ are limits of $\bar{B}_r^{X_i}(p_i)$, and thus, there exists a (bijective) isometry $f_r : \bar{B}_r^X(p) \to \bar{B}_r^Y(q)$ with $f_r(p) = q$. Choose a countable dense subset $X' := \{x_0, x_1, x_2, \dots\}$ of X with $x_0 = p$ and let $y_i^n := f_n(x_i)$ for $n \in \mathbb{N}$.

For all $i \in \mathbb{N}$,

$$d_Y(y_i^n, q) = d_Y(f_n(x_i), f_n(p)) = d_X(x_i, p),$$

i.e. $(y_i^n)_{n\in\mathbb{N}}$ is a sequence in the compact subset $\bar{B}^Y_{d_X(x_i,p)}(q)$. By a diagonal argument, there exists a subsequence $(n_m)_{m\in\mathbb{N}}$ of the natural numbers such that for every $i \in \mathbb{N}$ the sequence $(y_i^{n_m})_{m\in\mathbb{N}}$ has a limit $y_i \in \bar{B}^Y_{d_X(x_i,p)}(q)$. In particular, $y_0^n = f_n(p) = q$ for all $n \in \mathbb{N}$ implies $y_0 = q$. For $i, j \in \mathbb{N}$, by construction,

$$d_Y(y_i, y_j) = \lim_{m\to\infty} d_Y(y_i^{n_m}, y_j^{n_m}) = \lim_{m\to\infty} d_Y(f_{n_m}(x_i), f_{n_m}(x_j)) = d_X(x_i, x_j),$$

i.e. the map $\tilde{f} : X' \to Y$ defined by $\tilde{f}(x_i) := y_i$ is an isometry with $\tilde{f}(p) = q$.

As Y is complete, there exists an extension of $\tilde{f}$ to an isometry $f : X \to Y$ with $f(p) = q$: Let $x \in X$ be arbitrary. Since X' was chosen to be dense, there exists a sequence $(x_{i_j})_{j\in\mathbb{N}}$ in X' converging to x. This is a Cauchy sequence, hence, $(\tilde{f}(x_{i_j}))_{j\in\mathbb{N}}$ is a Cauchy sequence as well and has a limit $y =: f(x)$.

This defines indeed an isometry $f : X \to Y$: Let $x, x' \in X$ be arbitrary and x_{i_j} and x_{i_l}, respectively, be sequences in X' converging to x and x', respectively. Then

$$d_Y(f(x), f(x')) = \lim_{j,l\to\infty} d_Y(\tilde{f}(x_{i_j}), \tilde{f}(x_{i_l})) = \lim_{j,l\to\infty} d_X(x_{i_j}, x_{i_l}) = d_X(x, x').$$

Thus, f is an isometry. It remains to prove that f is bijective:

Using a further subsequence n_{m_a} and the inverse maps $f^{-1}_{n_{m_a}}$, an isometry $g : Y \to X$ can be constructed analogously. For arbitrary $x \in X$, let $(y_{k_l})_{l\in\mathbb{N}}$ be the sequence in the dense subset $Y' \subseteq Y$ used in the construction of g converging to $f(x) \in Y$. Then

$$\begin{aligned} d_X(g \circ f(x), x) &= \lim_{a\to\infty} \lim_{l,j\to\infty} d_X(f^{-1}_{n_{m_a}}(y_{k_l}), x_{i_j}) \\ &= \lim_{a\to\infty} \lim_{l,j\to\infty} d_Y(y_{k_l}, f_{n_{m_a}}(x_{i_j})) \\ &= d_Y(f(x), f(x)) = 0. \end{aligned}$$

Analogously, $f \circ g = \mathrm{id}$. Thus, f is bijective. □

As in the compact case, Gromov-Hausdorff convergence preserves being a length space.

Proposition A.24. *Let (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed length spaces and (X, d_X, p) be a pointed complete metric space. If $(X_i, p_i) \to (X, p)$, then X is a length space.*

Proof. By [BBI01, Corollary 2.4.17], it suffices to prove that for arbitrary $x, y \in X$ and $\varepsilon > 0$ there are points $x = x_0, x_1, \ldots, x_m, x_{m+1} = y$ in X with

$$d_X(x_k, x_{k+1}) \leq \varepsilon \quad \text{and} \quad \sum_{k=0}^{m} d_X(x_k, x_{k+1}) < d_X(x, y) + \varepsilon\,.$$

Let $\varepsilon > 0$, $x, y \in X$ be arbitrary and choose $r > 0$ such that $x, y \in B_r^X(p)$. Since $\bar{B}_r^{X_i}(p_i) \to \bar{B}_r^X(p)$, there exist $\varepsilon_i \to 0$ and $(f_i, g_i) \in Appr_{\varepsilon_i}((\bar{B}_r^{X_i}(p_i), p_i), (\bar{B}_r^X(p), p))$, cf. Proposition A.5. Choose a shortest geodesic $c_i : [0, l_i] \to X_i$ with $c_i(0) = g_i(x)$ and $c_i(l_i) = g_i(y)$ where $l_i = d_{X_i}(g_i(x), g_i(y))$.

Let $m \in \mathbb{N}$ such that $m\,\varepsilon < l_i \leq (m+1)\,\varepsilon$. After passing to a subsequence, for all $1 \leq k \leq m$, the sequences

$$f_i(c_i(k\,\varepsilon)) \in \bar{B}_r^X(p)$$

converge to points x_k that have the required properties:

First of all, by construction, $d_X(f_i(c_i(0)), x) = d_X(f_i \circ g_i(x), x) < \varepsilon_i \to 0$ as $i \to \infty$. Hence, $f_i(c_i(0)) \to x$. Analogously, $f_i(c_i(l_i)) \to y$. Define $x_0 := x$ and $x_{m+1} := y$.

As the $f_i(c_i(\varepsilon))$ are contained in the compact set $\bar{B}_r^X(p)$, after passing to a subsequence, $f_i(c_i(\varepsilon))$ converges to some $x_1 \in \bar{B}_r^X(p)$. After passing to a further subsequence, $f_i(c_i(2\,\varepsilon))$ converges to a point $x_2 \in \bar{B}_r^X(p)$. Iterating this, there is a subsequence such that $f_i(c_i(0)) \to x_0$, $f_i(c_i(k\,\varepsilon)) \to x_k$ for all $1 \leq k \leq m$ and $f_i(c_i(l_i)) \to x_{m+1}$. Pass to this subsequence.

Since the c_i are shortest geodesics,

$$d_{X_i}(c_i(k\,\varepsilon), c_i((k+1)\,\varepsilon)) = L(c_{i|[k\,\varepsilon,(k+1)\,\varepsilon]}) = \varepsilon$$

for all $0 \leq k < m$ and

$$d_{X_i}(c_i(m\,\varepsilon), g_i(y)) = L(c_{i|[m\,\varepsilon, l_i]}) \leq \varepsilon\,.$$

Thus, the limits of these satisfy $d_X(x_k, x_{k+1}) \leq \varepsilon$. Furthermore, using $\varepsilon_i \to 0$,

$$\begin{aligned}
\sum_{k=0}^{m} d_X(x_k, x_{k+1}) &= \lim_{i\to\infty} \sum_{k=0}^{m-1} L(c_{i|[k\,\varepsilon,(k+1)\,\varepsilon]}) + L(c_{i|[m\,\varepsilon, l_i]}) \\
&= \lim_{i\to\infty} L(c_i) \\
&= \lim_{i\to\infty} d_{X_i}(g_i(x), g_i(y)) \\
&= d(x, y) \\
&< d(x, y) + \varepsilon\,.
\end{aligned}$$

□

As in the compact case, in the non-compact case there is a correspondence between (pointed) Gromov-Hausdorff convergence and approximations. In order to prove this, the following lemma is needed.

Lemma A.25. *For all $r > 0$, let $(\varepsilon_n^r)_{n\in\mathbb{N}}$ be a monotonically decreasing null sequence and $h : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ a function with $\lim_{x\to 0} h(x) = 0$. Then there exists a sequence $(r_n)_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} r_n = \infty$ and $\varepsilon_n^{r_n} \le h\big(\frac{1}{r_n}\big)$ for almost all $n \in \mathbb{N}$.*

Proof. Let $A := \{n \in \mathbb{N} \mid \forall r > 0 : \varepsilon_n^r > h\big(\frac{1}{r}\big)\}$ denote the set of all natural numbers n for which no such 'r_n' can exist. This set is finite: Fix $r > 0$. Then $\varepsilon_n^r > h\big(\frac{1}{r}\big)$ for all $n \in A$, but, since $(\varepsilon_n^r)_{n\in\mathbb{N}}$ is a null sequence, this inequality only holds for finitely many n. Hence, A is finite.

Without loss of generality, assume that for each n there is at least one $r > 0$ such that $\varepsilon_n^r \le h\big(\frac{1}{r}\big)$.

Let $R_n := \{r > 0 \mid \varepsilon_n^r \le h\big(\frac{1}{r}\big)\} \ne \emptyset$ denote the set of all radii which are possible candidates for 'r_n'. Then $(R_n)_{n\in\mathbb{N}}$ is an increasing sequence: Fix $r \in R_n$. Since $(\varepsilon_n^r)_{n\in\mathbb{N}}$ is monotonically decreasing, $\varepsilon_{n+1}^r \le \varepsilon_n^r \le h\big(\frac{1}{r}\big)$. Thus, $r \in R_{n+1}$.

Suppose that these sets are uniformly bounded, i.e. there exists $C > 0$ such that $\bigcup_{n\in\mathbb{N}} R_n \subseteq [0, C]$. Then $\varepsilon_n^r > h\big(\frac{1}{r}\big)$ for all n and all $r > C$. Consequently, for all $r > C$ the sequence $(\varepsilon_n^r)_{n\in\mathbb{N}}$ is bounded below by $h\big(\frac{1}{r}\big)$. This is a contradiction to $(\varepsilon_n^r)_{n\in\mathbb{N}}$ being a null sequence.

Therefore, $\bigcup_{n\in\mathbb{N}} R_n$ is unbounded, i.e. for all $C > 0$ there exists some $N \in \mathbb{N}$ such that $R_j \not\subseteq [0, C]$ for all $j \ge N$. In particular, for all $k \in \mathbb{N}$ there is a minimal $N_k \in \mathbb{N}$ such that for all $j \ge N_k$ there is some $r_j^k \in R_j$ with $r_j^k > k$. There are two cases:

1. Let $N_k \to \infty$. For every $n \in \mathbb{N}$, $n \ge N_0$, there is some $k \in \mathbb{N}$ with $N_k \le n < N_{k+1}$. Fix this k and define $r_n := r_n^k$ for some $r_n^k \in R_n$ satisfying $r_n^k > k$. Then, for arbitrary $k \in \mathbb{N}$ and all $n \ge N_k$, $r_n > k$. Thus, $r_n \to \infty$. Furthermore, by choice, $\varepsilon_n^{r_n} \le h\big(\frac{1}{r_n}\big)$.

2. Let $k_0 \in \mathbb{N}$ such that $N_k = N_{k_0}$ for all $k \ge k_0$. For $n < N_{k_0}$, define r_n as in the first case. For $n = N + m \ge N_{k_0} = N_{k_0+m}$, choose any $r_n := r_n^{k_0+m} \in R_n \cap (k_0 + m, \infty)$. Then $r_n \to \infty$ and $\varepsilon_n^{r_n} \le h\big(\frac{1}{r_n}\big)$. □

Proposition A.26. *Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be length spaces. Then the following statements are equivalent.*

a) $(X_i, p_i) \to (X, p)$.

b) For all functions $g : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ with $\lim_{x\to 0} g(x) = 0$ there exists $r_i \to \infty$ with $d_{GH}((\bar{B}_{r_i}^{X_i}(p_i), p_i), (\bar{B}_{r_i}^{X}(p), p)) \le g\big(\frac{1}{r_i}\big)$.

c) There exist $r_i \to \infty$ and $\varepsilon_i \to 0$ with $d_{GH}((\bar{B}_{r_i}^{X_i}(p_i), p_i), (\bar{B}_{r_i}^{X}(p), p)) \le \varepsilon_i$.

Proof. The proof is done by proving the implications a) ⇒ b), b) ⇒ c) and c) ⇒ a). First, let $(X_i, p_i) \to (X, p)$ and $g : \mathbb{R}^{>0} \to \mathbb{R}^{>0}$ with $\lim_{x\to 0} g(x) = 0$ be arbitrary. For fixed $r > 0$, define

$$\tilde{\varepsilon}_i^r := d_{GH}((\bar{B}_r^{X_i}(p_i), p_i), (\bar{B}_r^X(p), p)) \to 0 \text{ as } i \to \infty$$

and

$$\varepsilon_i^r := \sup\{\tilde{\varepsilon}_j^r \mid j \geq i\} \to 0 \text{ as } i \to \infty.$$

This sequence $(\varepsilon_i^r)_{i\in\mathbb{N}}$ is monotonically decreasing and satisfies $\varepsilon_i^r \geq \tilde{\varepsilon}_i^r$. By Lemma A.25, there exists $r_i \to \infty$ such that $\varepsilon_i^{r_i} \leq g\big(\frac{1}{r_i}\big)$ for all $i \in \mathbb{N}$. In particular,

$$d_{GH}((\bar{B}_{r_i}^{X_i}(p_i), p_i), (\bar{B}_{r_i}^X(p), p)) = \tilde{\varepsilon}_i^{r_i} \leq \varepsilon_i^{r_i} \leq g\Big(\frac{1}{r_i}\Big),$$

and this proves b). Obviously, b) implies c) via choosing $g := \mathrm{id}$ and $\varepsilon_i := \frac{1}{r_i}$.

Let $d_{GH}((\bar{B}_{r_i}^{X_i}(p_i), p_i), (\bar{B}_{r_i}^X(p), p)) \leq \varepsilon_i$ for some $r_i \to \infty$ and $\varepsilon_i \to 0$. Fix $r > 0$. Let $i \in \mathbb{N}$ be large enough such that $r < r_i$. By Lemma A.20,

$$d_{GH}((\bar{B}_r^{X_i}(p_i), p_i), (\bar{B}_r^X(p), p)) \leq 16\,\varepsilon_i,$$

and this implies the claim. □

Corollary A.27. *Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed length spaces. Then the following statements are equivalent.*

a) $(X_i, p_i) \to (X, p)$.

b) There is $\varepsilon_i \to 0$ such that $\mathrm{Appr}_{\varepsilon_i}((\bar{B}_{1/\varepsilon_i}^{X_i}(p_i), p_i), (\bar{B}_{1/\varepsilon_i}^X(p), p)) \neq \emptyset$ for all i.

c) There is $\varepsilon_i \to 0$ such that $d_{GH}((\bar{B}_{1/\varepsilon_i}^{X_i}(p_i), p_i), (\bar{B}_{1/\varepsilon_i}^X(p), p)) \leq \varepsilon_i$ for all i.

Proof. This is a direct consequence of Proposition A.5 and Proposition A.26. □

Similarly to the compact case, the Gromov-Hausdorff distance and convergence, respectively, is related to the diameters of the spaces: On the one hand, the distance of balls in X and $X \times Y$ are bounded from above by the diameter of Y. On the other hand, in the compact case it was proven that convergence of spaces implies convergence of the diameters. For length spaces, an analogous statement will be established.

Proposition A.28. *Let (X, d_X, x_0) and (Y, d_Y, y_0) be pointed proper metric spaces. If Y is compact, then $d_{GH}((\bar{B}_r^X(x_0), x_0), (\bar{B}_r^{X\times Y}((x_0, y_0)), (x_0, y_0))) \leq \mathrm{diam}(Y)$ for all $r > 0$.*

Proof. It suffices to define an admissible metric and to estimate the Hausdorff distance with respect to this metric.

Let $\delta > 0$ be arbitrary. Define an admissible metric d on $(X \times Y) \amalg X$ by

$$d((x,y),x') := \sqrt{d_X(x,x')^2 + d_Y(y,y_0)^2 + \delta^2}.$$

As usual, the only tricky part is to prove the triangle inequality: By the Minkowski inequality, for $x_1, x_1', x_2, x_2' \in X$ and $y_1, y_2 \in Y$,

$$\begin{aligned}
&d((x_1,y_1),x_1') + d(x_1',x_2') \\
&= \sqrt{d_X(x_1,x_1')^2 + d_Y(y_1,y_0)^2 + \delta^2} + d_X(x_1',x_2') \\
&\geq \sqrt{(d_X(x_1,x_1') + d_X(x_1',x_2')^2 + d_Y(y_1,y_0)^2 + \delta^2)} \\
&\geq \sqrt{(d_X(x_1,x_2')^2 + d_Y(y_1,y_0)^2 + \delta^2)} \\
&= d((x_1,y_1),x_2').
\end{aligned}$$

With completely analogous argumentation, one can prove the remaining inequalities

$$\begin{aligned}
d(x_1',(x_1,y_1)) + d((x_1,y_1),x_2') &\geq d(x_1',x_2'), \\
d((x_1,y_1),(x_2,y_2)) + d((x_2,y_2),x_2') &\geq d((x_1,y_1),x_2') \quad \text{and} \\
d((x_1,y_1),x_1') + d(x_1',(x_2,y_2)) &\geq d((x_1,y_1),(x_2,y_2)).
\end{aligned}$$

Fix $r > 0$. Let $(x,y) \in \bar{B}_r^{X\times Y}((x_0,y_0))$ be arbitrary, in particular, $x \in \bar{B}_r^X(x_0)$. Thus,

$$d((x,y),\bar{B}_r^X(x_0)) \leq d((x,y),x) = \sqrt{d_Y(y,y_0)^2 + \delta^2} \leq \sqrt{\operatorname{diam}(Y)^2 + \delta^2}.$$

Hence,

$$\bar{B}_r^{X\times Y}((x_0,y_0)) \subseteq \bar{B}^d_{\sqrt{\operatorname{diam}(Y)^2+\delta^2}}(\bar{B}_r^X(x_0)).$$

For arbitrary $x \in \bar{B}_r^X(x_0)$, one has $d((x,y_0),(x_0,y_0)) = d_X(x,x_0) < r$, and therefore, $(x,y_0) \in \bar{B}_r^{X\times Y}((x_0,y_0))$. Thus,

$$d(x,\bar{B}_r^{X\times Y}(x_0,y_0)) \leq d(x,(x,y_0)) = \delta$$

and

$$\bar{B}_r^X(x_0) \subseteq \bar{B}_\delta^d(\bar{B}_r^{X\times Y}(x_0,y_0)).$$

Hence,

$$\begin{aligned}
&d_{GH}((\bar{B}_r^X(x_0),x_0),(\bar{B}_r^{X\times Y}((x_0,y_0)),(x_0,y_0))) \\
&\leq d_H^d(\bar{B}_r^X(x_0),\bar{B}_r^{X\times Y}(x_0,y_0)) \\
&\leq \max\{\sqrt{\operatorname{diam}(Y)^2+\delta^2},\delta\} \\
&= \sqrt{\operatorname{diam}(Y)^2+\delta^2}.
\end{aligned}$$

Since δ was arbitrary, this proves the claim. □

In order to prove the convergence of diameters, one needs the following property of length spaces of infinite diameter: Any ball of radius r has diameter at least r. Though it is easy to see this, for the sake of completeness, the proof is given first.

Lemma A.29. *Let (X,d,p) be a pointed length space and $0 < r < \frac{1}{2}\cdot\operatorname{diam}(X)$. Then $\operatorname{diam}(\bar{B}_r^X(p)) \geq r$.*

Proof. Assume that $d(q,p) \leq r$ for all $q \in X$. Hence, $\bar{B}_r(p) = X$, and this implies $\operatorname{diam}(X) \leq 2r < \operatorname{diam}(X)$, which is a contradiction.

Hence, there is $q_r \in X$ such that $l_r := d(q_r,p) > r$. Fix a minimising geodesic $\gamma : [0,l_r] \to X$ with $\gamma(0) = p$ and $\gamma(l_r) = q_r$. Then $d(p,\gamma(r)) = r$, hence, $\gamma(r) \in \bar{B}_r(p)$. In particular, $\operatorname{diam}(\bar{B}_r(p)) \geq d(p,\gamma(r)) = r$. □

Proposition A.30. *Let (X,d_X,p) and (X_i,d_{X_i},p_i), $i\in\mathbb{N}$, be pointed length spaces. If $(X_i,p_i) \to (X,p)$, then $\operatorname{diam}(X_i) \to \operatorname{diam}(X)$. (Here, both $\operatorname{diam}(X_i)$ tending to infinity as well as the notion $\infty \to \infty$ are allowed.)*

Proof. Let $\varepsilon_i \to 0$ be as in Corollary A.27 with

$$d_{GH}((\bar{B}^{X_i}_{1/\varepsilon_i}(p_i),p_i),(\bar{B}^{X}_{1/\varepsilon_i}(p),p)) \leq \varepsilon_i\,.$$

By Proposition A.15, $|\operatorname{diam}(\bar{B}^{X_i}_{1/\varepsilon_i}(p_i)) - \operatorname{diam}(\bar{B}^{X}_{1/\varepsilon_i}(p))| \leq 2\,\varepsilon_i \to 0$. Distinguish the two cases of X being bounded and unbounded, respectively.

1. Let $\operatorname{diam}(X) < \infty$. Without loss of generality, assume $\operatorname{diam}(X) < \frac{1}{2\,\varepsilon_i}$ for all $i\in\mathbb{N}$. Then $X = B^X_{1/\varepsilon_i}(p)$ and

$$|\operatorname{diam}(\bar{B}^{X_i}_{1/\varepsilon_i}(p_i)) - \operatorname{diam}(X)| = |\operatorname{diam}(\bar{B}^{X_i}_{1/\varepsilon_i}(p_i)) - \operatorname{diam}(\bar{B}^{X}_{1/\varepsilon_i}(p))| \to 0,$$

   in particular, $\operatorname{diam}(\bar{B}^{X_i}_{1/\varepsilon_i}(p_i)) \to \operatorname{diam}(X)$ as $i\to\infty$. Without loss of generality, assume $\operatorname{diam}(\bar{B}^{X_i}_{1/\varepsilon_i}(p_i)) \leq 2\cdot\operatorname{diam}(X)$ for all $i\in\mathbb{N}$.

   Let $r_i := \min\{\frac{1}{\varepsilon_i}, \frac{1}{3}\cdot\operatorname{diam}(X_i)\} < \frac{1}{2}\cdot\operatorname{diam}(X_i)$. By Lemma A.29,

$$r_i \leq \operatorname{diam}(B^{X_i}_{r_i}(p_i)) \leq \operatorname{diam}(B^{X_i}_{1/\varepsilon_i}(p_i)) \leq 2\cdot\operatorname{diam}(X) < \frac{1}{\varepsilon_i}.$$

   Hence, $\operatorname{diam}(X_i) = 3r_i \leq 6\cdot\operatorname{diam}(X)$, the X_i are compact and Proposition A.15 implies the claim.

2. Let $\operatorname{diam}(X) = \infty$. Assume there is a subsequence $(i_j)_{j\in\mathbb{N}}$ and $C > 0$ such that $\operatorname{diam}(X_{i_j}) < C$ for all $j\in\mathbb{N}$. After passing to a further subsequence, $C < \frac{1}{\varepsilon_i}$ for all $i\in\mathbb{N}$. Then $X_i = B^{X_i}_{1/\varepsilon_i}(p_i)$ and $\operatorname{diam}(\bar{B}^{X_i}_{1/\varepsilon_i}(p_i)) = \operatorname{diam}(X_i) < C$. By Lemma A.29, $\operatorname{diam}(\bar{B}^{X}_{1/\varepsilon_i}(p)) \geq \frac{1}{\varepsilon_i}$ and

$$|\operatorname{diam}(\bar{B}^{X_i}_{1/\varepsilon_i}(p_i)) - \operatorname{diam}(\bar{B}^{X}_{1/\varepsilon_i}(p))| \geq \frac{1}{\varepsilon_i} - C \to \infty.$$

   This is a contradiction. Hence, $\operatorname{diam}(X_i) \to \infty$. □

Gromov-Hausdorff convergence is compatible with rescaling: Given a converging sequence of length spaces and a converging sequence of rescaling factors, the rescaled sequence converges and the limit space is the original one rescaled by the limit of the rescaling sequence. More generally, given a converging sequence of metric spaces and some bounded sequence of rescaling factors, the sublimits of the rescaled sequence correspond exactly to the sublimits of the rescaling sequence.

For a metric space (X,d), recall that αX denotes the rescaled metric space $(X,\alpha\, d)$.

Proposition A.31. *Let (X,d_X,p) and (X_i,d_{X_i},p_i), $i\in\mathbb{N}$, be pointed length spaces and $r_i,r,\alpha_i,\alpha>0$.*

a) If $(X_i,p_i)\to(X,p)$ and $r_i\to r$, then $(\bar{B}^{X_i}_{r_i}(p_i),p_i)\to(\bar{B}^X_r(p),p)$.

b) If $\alpha_i\to\alpha$, then $(\alpha_i X,p)\to(\alpha X,p)$.

c) If $(X_i,p_i)\to(X,p)$ and $\alpha_i\to\alpha$, then $(\alpha_i X_i,p_i)\to(\alpha X,p)$.

d) If $(X_i,p_i)\to(X,p)$ and $(\alpha_i X_i,p_i)\to(Y,q)$, then $\alpha_i\to\alpha$ and $(Y,q)\cong(\alpha X,p)$.

Proof. a) By Corollary A.14,

$$d_{GH}((\bar{B}^{X_i}_{r_i}(p_i),p_i),(\bar{B}^{X_i}_{r}(p_i),p_i))\le|r-r_i|\to 0,$$

and triangle inequality implies

$$d_{GH}(\bar{B}^{X_i}_{r_i}(p_i),\bar{B}^X_r(p))\le d_{GH}(\bar{B}^{X_i}_{r_i}(p_i),\bar{B}^{X_i}_r(p_i))+d_{GH}(\bar{B}^{X_i}_r(p_i),\bar{B}^X_r(p))\to 0.$$

b) Without loss of generality, let $\alpha=1$. There are two cases:

(i) Let X be compact. Define $f_i:X\to\alpha_i X$ and $g_i:\alpha_i X\to X$ by $f_i(x):=x$ and $g_i(x):=x$ for all $x\in X$ and let $0<\varepsilon_i:=2\cdot|\alpha_i-1|\cdot\operatorname{diam}(X)\to 0$. For any $x,x'\in X$,

$$|d_{\alpha_i X}(f_i(x),f_i(x'))-d_X(x,x')|=|\alpha_i-1|\cdot d_X(x,x')<\varepsilon_i\,.$$

Analogously,

$$|d_{\alpha_i X}(x,x')-d_X(g_i(x),g_i(x'))|<\varepsilon_i\,.$$

Furthermore, $d_X(x,g_i\circ f_i(x))=0<\varepsilon_i$ and $d_X(f_i\circ g_i(x),x)=0<\varepsilon_i$. Thus, $(f_i,g_i)\in \mathit{Appr}_{\varepsilon_i}((\alpha_i X,p),(X,p))$ and $(\alpha_i X,p)\to(X,p)$.

(ii) Let X be non-compact and $r>0$. Then, using a) and the compact case,

$$\begin{aligned}
&d_{GH}((\bar{B}^{\alpha_i X}_r(p),p),(\bar{B}^X_r(p),p))\\
&\le d_{GH}((\bar{B}^{\alpha_i X}_r(p),p),(\bar{B}^{\alpha_i X}_{\alpha_i r}(p),p))+d_{GH}((\bar{B}^{\alpha_i X}_{\alpha_i r}(p),p),(\bar{B}^X_r(p),p))\\
&=\alpha_i\cdot d_{GH}((\bar{B}^X_{r/\alpha_i}(p),p),(\bar{B}^X_r(p),p))+d_{GH}((\alpha_i\bar{B}^X_r(p),p),(\bar{B}^X_r(p),p))\\
&\to 0.
\end{aligned}$$

c) By the triangle inequality, for fixed $r > 0$,

$$\begin{aligned} &d_{GH}((\bar{B}_r^{\alpha_i X_i}(p_i), p_i), (\bar{B}_r^{\alpha X}(p), p)) \\ &\leq d_{GH}((\bar{B}_r^{\alpha_i X_i}(p_i), p_i), (\bar{B}_{\alpha_i r/\alpha}^{\alpha_i X}(p), p)) \\ &\quad + d_{GH}((\bar{B}_{\alpha_i r/\alpha}^{\alpha_i X}(p), p), (\bar{B}_r^{\alpha_i X}(p), p)) \\ &\quad + d_{GH}((\bar{B}_r^{\alpha_i X}(p), p), (\bar{B}_r^{\alpha X}(p), p)). \end{aligned}$$

By a),

$$d_{GH}((\bar{B}_r^{\alpha_i X_i}(p_i), p_i), (\bar{B}_{\alpha_i r/\alpha}^{\alpha_i X}(p), p)) = \alpha_i \cdot d_{GH}((\bar{B}_{r/\alpha_i}^{X_i}(p_i), p_i), (\bar{B}_{r/\alpha}^{X}(p), p)) \to 0,$$

by Corollary A.14,

$$d_{GH}((\bar{B}_{\alpha_i r/\alpha}^{\alpha_i X}(p), p), (\bar{B}_r^{\alpha_i X}(p), p)) \leq |r - \frac{\alpha_i}{\alpha} \cdot r| \to 0,$$

and by b),

$$d_{GH}((\bar{B}_r^{\alpha_i X}(p), p), (\bar{B}_r^{\alpha X}(p), p)) \to 0.$$

Hence, $(\bar{B}_r^{\alpha_i X_i}(p_i), p_i) \to (\bar{B}_r^{\alpha X}(p), p)$ for every $r > 0$.

d) Let α be an arbitrary accumulation point of $(\alpha_i)_{i\in\mathbb{N}}$. Hence, for a subsequence $(i_j)_{j\in\mathbb{N}}$, both $\alpha_{i_j} \to \alpha$, and by c), $(\alpha_{i_j} X_{i_j}, p_{i_j}) \to (\alpha X, p)$ as $j \to \infty$. On the other hand, $(\alpha_{i_j} X_{i_j}, p_{i_j}) \to (Y, q)$ as $j \to \infty$. Thus, (Y, q) and $(\alpha X, p)$ are isometric (cf. Proposition A.23). □

Corollary A.32. *Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed length spaces and $(\alpha_i)_{i\in\mathbb{N}}$ be a bounded sequence. If $(X_i, p_i) \to (X, p)$, then the sublimits of $(\alpha_i X_i, p_i)$ correspond to the $(\alpha X, p)$ for exactly the accumulation points α of $(\alpha_i)_{i\in\mathbb{N}}$.*

Proof. Let α be an accumulation point of $(\alpha_i)_{i\in\mathbb{N}}$ and $(\alpha_{i_j})_{j\in\mathbb{N}}$ be the subsequence converging to α. Then $(X_{i_j}, p_{i_j}) \to (X, p)$, and by Proposition A.31,

$$(\alpha_{i_j} X_{i_j}, p_{i_j}) \to (\alpha X, p).$$

Now let (Y, y) be a sublimit of $(\alpha_i X_i, p_i)$, i.e. $(\alpha_{i_j} X_{i_j}, p_{i_j}) \to (Y, y)$ for some subsequence $(i_j)_{j\in\mathbb{N}}$. Since $(\alpha_{i_j})_{j\in\mathbb{N}}$ is a bounded sequence, there exists a convergent subsequence $(\alpha_{i_{j_l}})_{l\in\mathbb{N}}$ with limit α. Thus, $(\alpha_{i_{j_l}} X_{i_{j_l}}, p_{i_{j_l}}) \to (Y, y)$, and the first part implies $(\alpha_{i_{j_l}} X_{i_{j_l}}, p_{i_{j_l}}) \to (\alpha X, p)$. Hence, (Y, y) is isometric to $(\alpha X, p)$ for an accumulation point α of $(\alpha_i)_{i\in\mathbb{N}}$. □

A.2.3 Convergence of points

In the previous section, convergent sequences of pointed metric (length) spaces were studied. Given such a sequence and using the corresponding approximations, a notion for convergence of points can be introduced.

Definition A.33. Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed length spaces. Assume $(X_i, p_i) \to (X, p)$ and let $\varepsilon_i \to 0$ and $(f_i, g_i) \in Appr_{\varepsilon_i}((\bar{B}^{X_i}_{1/\varepsilon_i}(p_i), p_i), (\bar{B}^{X}_{1/\varepsilon_i}(p), p))$ as in Corollary A.27. Let $q_i \in \bar{B}^{X_i}_{1/\varepsilon_i}(p_i)$ and $q \in X$. Then q_i *converges to* q, denoted by $q_i \to q$, if $f_i(q_i)$ converges to q (in X).

For $(X_i, p_i) \to (X, p)$ as in the definition, $p_i \to p$ due to $f_i(p_i) = p$. Moreover, for each $x \in X$ there exists such a sequence x_i satisfying $x_i \to x$, e.g. $x_i := g_i(x)$.

Convergence $q_i \to q$ depends on the choice of the underlying Gromov-Hausdorff approximations: Convergence with respect to one pair of approximations does not necessarily imply convergence for another, as the following example shows.

Example A.34. For $i \in \mathbb{N}$, let $X_i = X = \mathbb{S}^2$ be the 2-dimensional sphere, $p_i = p = N$ the north pole and $q_i = q$ some fixed point on the equator. Let φ denote the rotation of $\mathbb{S}^2$ by $\frac{\pi}{2}$ fixing p and define $f_i = g_i = f'_{2i} = g'_{2i} = \mathrm{id}_{\mathbb{S}^2}$, $f'_{2i+1} = \varphi$ and $g'_{2i+1} = \varphi^{-1}$.

Then both (f_i, g_i) and (f'_i, g'_i) are pointed isometries between (X_i, p_i) and (X, p) satisfying $f_i(q_i) = q$, but $f'_{2i}(q_{2i}) = q \neq \varphi(q) = f'_{2i+1}(q_{2i+1})$. Hence, $f_i(q_i)'$ is not convergent at all, but subconvergent with limits q and $\varphi(q)$.

In this example, after replacing the approximations, two sublimits occur: One sublimit is the limit corresponding to the original approximations, the other one is its image under an isometry of the limit space. Since Gromov-Hausdorff convergence distinguishes spaces only up to isometry, concretely $(X, p) \cong (h(X), h(p)) = (X, h(p))$ for any isometry h, this can be interpreted as follows: If q is a sublimit of q_i with respect to one Gromov-Hausdorff approximation, then it is a sublimit for all Gromov-Hausdorff approximations.

This is a general fact as the subsequent lemma shows. In order to prove this, the separability of a connected proper metric space is needed. Though it is easy to see that such a space is separable, for completeness, the proof is given first.

Lemma A.35. *A connected proper metric space is separable.*

Proof. Let (X, p) be a connected proper metric space and let $p \in X$ be arbitrary. Then

$$X = \bigcup_{q \in \mathbb{Q} \cap (0,\infty)} \bar{B}_q(p).$$

As a compact set, every $\bar{B}_q(p)$ is separable where $q \in \mathbb{Q}$ is positive. Therefore, there exists a countable dense subset $A_q \subseteq \bar{B}_q(p)$. Let $A := \bigcup_{q \in \mathbb{Q} \cap (0,\infty)} A_q$. This A is countable, and for arbitrary $x \in X$ there is a positive $q \in \mathbb{Q}$ such that $x \in \bar{B}_q(p)$, i.e. there exists a sequence $x_n \in A_q \subseteq A$ converging to x. Thus, A is dense in X, hence, X is separable. □

Lemma A.36. *Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed length spaces. Assume $(X_i, p_i) \to (X, p)$ and let $\varepsilon_i, \varepsilon'_i \to 0$, $r_i, r'_i \to \infty$ and*

$$(f_i, g_i) \in Appr_{\varepsilon_i}((\bar{B}^{X_i}_{r_i}(p_i), p_i), (\bar{B}^{X}_{r_i}(p), p)),$$
$$(f'_i, g'_i) \in Appr_{\varepsilon'_i}((\bar{B}^{X_i}_{r'_i}(p_i), p_i), (\bar{B}^{X}_{r'_i}(p), p)).$$

Let $q_i \in \bar{B}^{X_i}_{\min\{r_i,r_i'\}}(p_i)$ *and* $q \in X$*. If* $f_i(q_i) \to q$ *and* q' *is an accumulation point of* $f_i'(q_i)$*, then there exists an isometry* $h : X \to X$ *such that* $h(q) = q'$*.*

Proof. Without loss of generality, let $r_i = r_i'$: Otherwise, let $R_i := \min\{r_i, r_i'\}$ and, by Lemma A.20 and the construction in its proof, there are

$$(\tilde{f}_i, \tilde{g}_i) \in Appr_{\varepsilon_i}((\bar{B}^{X_i}_{R_i}(p_i), p_i), (\bar{B}^X_{R_i}(p), p))$$
$$(\tilde{f}_i', \tilde{g}_i') \in Appr_{\varepsilon_i'}((\bar{B}^{X_i}_{R_i}(p_i), p_i), (\bar{B}^X_{R_i}(p), p))$$

with

$$\tilde{f}_i(q_i) \to q \text{ if and only if } f_i(q_i) \to q,$$
$$\tilde{f}_i'(q_i) \to q \text{ if and only if } f_i'(q_i) \to q.$$

Define $h_i, \bar{h}_i : \bar{B}^X_{r_i}(p) \to \bar{B}^X_{r_i}(p)$ by

$$h_i := f_i' \circ g_i \quad \text{and} \quad \bar{h}_i := f_i \circ g_i'.$$

In particular, $h_i(p) = \bar{h}_i(p) = p$. For any $x, x' \in \bar{B}^X_{r_i}(p)$,

$$\begin{aligned} &|d_X(h_i(x), h_i(x')) - d_X(x, x')| \\ &\le |d_X(f_i'(g_i(x)), f_i'(g_i(x'))) - d_{X_i}(g_i(x), g_i(x'))| \\ &\quad + |d_{X_i}(g_i(x), g_i(x')) - d_X(x, x')| \\ &\le \varepsilon_i' + \varepsilon_i \to 0. \end{aligned}$$

Analogously, $|d_X(\bar{h}_i(x), \bar{h}_i(x')) - d_X(x, x')| \to 0$. Moreover,

$$\begin{aligned} &d_X(\bar{h}_i \circ h_i(x), x) \\ &= d_X(f_i \circ g_i' \circ f_i' \circ g_i(x), x) \\ &\le d_{X_i}(g_i \circ f_i \circ g_i' \circ f_i' \circ g_i(x), g_i(x)) + \varepsilon_i \\ &\le d_{X_i}(g_i' \circ f_i' \circ g_i(x), g_i(x)) + 2\,\varepsilon_i \\ &\le d_{X_i}(g_i(x), g_i(x)) + 2\,\varepsilon_i + \varepsilon_i' \to 0, \end{aligned}$$

and analogously, $d_X(h_i \circ \bar{h}_i(x), x) \to 0$. Hence, if the h_i and $\bar{h}_i$ (sub)converge (in some sense), their corresponding (sub)limits are isometries fixing p with $\bar{h} = h^{-1}$.

The idea for proving subconvergence is to choose a countable dense subset $A \subseteq X$, to define the sublimit of all $h_i(a)$ where $a \in A$ and to extend this limit to a continuous map on X. Doing the same simultaneously for $\bar{h}_i$ gives another sublimit that turns out to be the inverse of the first. In the end, identifying X with itself using this isometry proves the claim.

Choose a countable dense subset $A = \{a_n \mid n \in \mathbb{N}\} \subseteq X$ (cf. Lemma A.35) and, for i large enough such that $d_X(a_n, p) \le r_i$, define $z^i_n := h_i(a_n)$ and $\bar{z}^i_n := \bar{h}_i(a_n)$. Since

$$d_X(z^i_n, p) = d_X(h_i(a_n), h_i(p)) \to d_X(a_n, p),$$

the sequence $(d(z_n^i, p))_{i\in\mathbb{N}}$ is bounded from above by some $R > 0$. Hence, z_n^i is contained in $\bar{B}_R^X(p)$, and therefore, has a convergent subsequence. An analogous argument proves subconvergence for $(\bar{z}_n^i)_{i\in\mathbb{N}}$. Thus, using a diagonal argument, there is a subsequence $(i_j)_{j\in\mathbb{N}}$ such that for any $n \in \mathbb{N}$ the sequences $(z_n^{i_j})_{j\in\mathbb{N}}$ and $(\bar{z}_n^{i_j})_{j\in\mathbb{N}}$, respectively, converge to some $z_n \in X$ and $\bar{z}_n \in X$, respectively.

Define $h(a_n) := z_n$ and $\bar{h}(a_n) := \bar{z}_n$. In particular,

$$d_X(h(a_n), h(a_m)) = d_X(a_n, a_m) = d_X(\bar{h}(a_n), \bar{h}(a_m))$$

for all $n, m \in \mathbb{N}$. For arbitrary $x \in X$, choose a Cauchy sequence $(a_{n_k})_{k\in\mathbb{N}}$ in A converging to x and let

$$h(x) := \lim_{k\to\infty} h(a_{n_k}) \quad \text{and} \quad \bar{h}(x) := \lim_{k\to\infty} \bar{h}(a_{n_k}).$$

In fact, for any $k \in \mathbb{N}$,

$$\begin{aligned} d_X(h_{i_j}(x), h(x)) &\le d_X(h_{i_j}(x), h_{i_j}(a_{n_k})) + d_X(h_{i_j}(a_{n_k}), h(a_{n_k})) + d_X(h(a_{n_k}), h(x)) \\ &\le d_X(x, a_{n_k}) + \varepsilon_{i_j} + \varepsilon'_{i_j} + d_X(h_{i_j}(a_{n_k}), h(a_{n_k})) + d_X(h(a_{n_k}), h(x)) \\ &\to d_X(x, a_{n_k}) + d_X(h(a_{n_k}), h(x)) \text{ as } j \to \infty. \end{aligned}$$

Since this holds for every $k \in \mathbb{N}$ and $d_X(x, a_{n_k}) + d_X(h(a_{n_k}), h(x)) \to 0$ as $k \to \infty$,

$$h_{i_j}(x) \to h(x) \text{ as } j \to \infty.$$

Analogously, $\bar{h}_{i_j}(x) \to \bar{h}(x)$ as $j \to \infty$. In particular, $\bar{h}_{i_j} \circ h_{i_j} \to \bar{h} \circ h$ and vice versa. Thus, h is an isometry on X with inverse $\bar{h}$.

Now let $f_i(q_i) \to q$. Then

$$\begin{aligned} d_X(f'_{i_j}(q_{i_j}), h(q)) &\le d_{X_i}(g'_{i_j} \circ f'_{i_j}(q_{i_j}), g'_{i_j} \circ h(q)) + \varepsilon'_{i_j} \\ &\le d_{X_i}(q_{i_j}, g'_{i_j} \circ h(q)) + 2\,\varepsilon'_{i_j} \\ &\le d_X(f_{i_j}(q_{i_j}), f_{i_j} \circ g'_{i_j} \circ h(q)) + 2\,\varepsilon'_{i_j} + \varepsilon_{i_j} \\ &\le d_X(f_{i_j}(q_{i_j}), q) + d_X(q, \bar{h}'_{i_j} \circ h(q)) + 2\,\varepsilon'_{i_j} + \varepsilon_{i_j} \\ &\to 0 \text{ as } j \to \infty. \end{aligned}$$

This proves $f'_{i_j}(q_{i_j}) \to h(q)$ as $j \to \infty$. □

The following statements allow to change the base points of a given convergent sequence.

Proposition A.37. *Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed length spaces, and let $q_i \in X_i$ and $q \in X$. If $(X_i, p_i) \to (X, p)$ and $q_i \to q$, then $(X_i, q_i) \to (X, q)$.*

Proof. The proof is an immediate consequence of Lemma A.20 and Proposition A.26: Choose $\varepsilon_i \to 0$ and $(f_i, g_i) \in Appr_{\varepsilon_i}((\bar{B}^{X_i}_{1/\varepsilon_i}(p_i), p_i), (\bar{B}^X_{1/\varepsilon_i}(p), p))$ as in Definition A.33 with $f_i(q_i) \to q$. In particular,

$$d_{X_i}(q_i, g_i(q)) \le \varepsilon_i + d_X(f_i(q_i), f_i(g_i(q))) \le 2\,\varepsilon_i + d_X(f_i(q_i), q) \to 0.$$

Hence, $\delta_i := \max\{d_X(f_i(q_i), q), d_{X_i}(q_i, g_i(q))\} \to 0$.

Since $f_i(p_i) = p$,

$$d_{X_i}(p_i, q_i) \leq \varepsilon_i + d_X(p, q) + d_X(q, f_i(q_i)) \to d_X(p, q).$$

Let $r > 0$ be arbitrary. Fix i large enough such that $2(r + d_X(p, q)) \leq \frac{1}{\varepsilon_i}$ and such that $d_{X_i}(p_i, q_i) \leq 2d_X(p, q)$ or $d_{X_i}(p_i, q_i) \leq r$, respectively, if $p \neq q$ or $p = q$, respectively.

In particular,

$$\bar{B}_r^{X_i}(q_i) \subseteq \bar{B}^{X_i}_{r+d_{X_i}(p_i,q_i)}(p_i) \subseteq \bar{B}^{X_i}_{1/\varepsilon_i}(p_i), \quad \bar{B}_r^X(q) \subseteq \bar{B}^X_{r+d_X(p,q)}(p) \subseteq \bar{B}^X_{1/\varepsilon_i}(p)$$

and $Appr_{4\varepsilon_i+\delta_i}((\bar{B}_r^{X_i}(q_i), q_i), (\bar{B}_r^X(q), q)) \neq \emptyset$ by Lemma A.20. By Proposition A.5,

$$d_{GH}((\bar{B}_r^{X_i}(q_i), q_i), (\bar{B}_r^X(q), q)) \leq 8\varepsilon_i + 2\delta_i \to 0,$$

and Proposition A.26 implies the claim. □

Corollary A.38. *Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed length spaces. Let $q_i \in X_i$ with $d_{X_i}(p_i, q_i) \to 0$. If $(X_i, p_i) \to (X, p)$, then $(X_i, q_i) \to (X, p)$.*

Proof. Choose $\varepsilon_i \to 0$ and $(f_i, g_i) \in Appr_{\varepsilon_i}((\bar{B}^{X_i}_{1/\varepsilon_i}(p_i), p_i), (\bar{B}^X_{1/\varepsilon_i}(p), p))$ as in Corollary A.27. Then

$$d_X(f_i(q_i), p) = d_X(f_i(q_i), f_i(p)) \leq d_{X_i}(q_i, p_i) + \varepsilon_i \to 0.$$

Hence, $q_i \to p$, and Proposition A.37 implies the claim. □

Corollary A.39. *Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed length spaces. Let $q_i \in X_i$ with $d_{X_i}(p_i, q_i) \leq C$ for some $C > 0$. If $(X_i, p_i) \to (X, p)$, then there exists $q \in X$ such that (X_i, q_i) subconverges to (X, q).*

Proof. Let $\varepsilon_i \to 0$ and $(f_i, g_i) \in Appr_{\varepsilon_i}((\bar{B}^{X_i}_{1/\varepsilon_i}(p_i), p_i), (\bar{B}^X_{1/\varepsilon_i}(p), p))$ be as in Corollary A.27. For $R > C$ there is $i_0 > 0$ such that $C + \varepsilon_i \leq R$. Therefore, $f_i(q_i) \in \bar{B}_R(p)$ for all $i \geq i_0$. Since this ball is compact, there exists a convergent subsequence with limit $q \in \bar{B}_R(p)$. After passing to this subsequence, $q_i \to q$, and Proposition A.37 implies the claim. □

A.2.4 Convergence of maps

So far, statements about the convergence of metric spaces and points were made. But even statements about maps between those convergent space are possible: In fact, Lipschitz maps (sub)converge (in some sense) to Lipschitz maps. The proof of this seems to be rather technical, but in fact essentially only uses the same methods one can use to prove convergence of compact subsets (without bothering Gromov's Pre-compactness Theorem). Therefore, a proof of the latter is given in advance after establishing the following (technical) lemma.

Lemma A.40. *Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed length spaces. Assume $(X_i, p_i) \to (X, p)$ and let $\varepsilon_i \to 0$ and $(f_i, g_i) \in Appr_{\varepsilon_i}((\bar{B}^{X_i}_{1/\varepsilon_i}(p_i), p_i), (\bar{B}^{X}_{1/\varepsilon_i}(p), p))$ be as in Corollary A.27. Moreover, let $A_i \subseteq B^{X_i}_{1/\varepsilon_i}(p_i)$ and $A \subseteq X$ be compact and $f_i' : A_i \to A$, $g_i' : A \to A_i$ and $\delta_i \to 0$ satisfy*

$$d_X(f_i'(x_i), f_i(x_i)) \leq \delta_i \quad \text{and} \quad d_{X_i}(g_i'(x), g_i(x)) \leq \delta_i$$

for all $x_i \in A_i$ and $x \in A$. Then $A_i \to A$.

Proof. Prove $(f_i', g_i') \in Appr_{2(\varepsilon_i + \delta_i)}(A_i, A)$: For $x_i^1, x_i^2 \in A_i$,

$$\begin{aligned}
&|d_X(f_i'(x_i^1), f_i'(x_i^2)) - d_{X_i}(x_i^1, x_i^2)| \\
&\leq |d_X(f_i'(x_i^1), f_i'(x_i^2)) - d_X(f_i(x_i^1), f_i(x_i^2))| + |d_X(f_i(x_i^1), f_i(x_i^2)) - d_{X_i}(x_i^1, x_i^2)| \\
&< d_X(f_i'(x_i^1), f_i(x_i^1)) + d_X(f_i'(x_i^2), f_i(x_i^2)) + \varepsilon_i \\
&\leq \varepsilon_i + 2\delta_i.
\end{aligned}$$

Analogously, $|d_{X_i}(g_i'(x^1), g_i'(x^2)) - d_X(x^1, x^2)| < \varepsilon_i + 2\delta_i$ for all $x^1, x^2 \in A$. Moreover, for $x_i \in A_i$,

$$\begin{aligned}
&d_{X_i}(g_i' \circ f_i'(x_i), x_i)) \\
&\leq d_{X_i}(g_i' \circ f_i'(x_i), g_i \circ f_i'(x_i)) + d_{X_i}(g_i \circ f_i'(x_i), g_i \circ f_i(x_i)) + d_{X_i}(g_i \circ f_i(x_i), x_i) \\
&< \delta_i + (d_{X_i}(f_i'(x_i), f_i(x_i)) + \varepsilon_i) + \varepsilon_i \\
&\leq 2(\varepsilon_i + \delta_i),
\end{aligned}$$

and analogously, $d_X(f_i' \circ g_i'(x), x)) < 2(\varepsilon_i + \delta_i)$ for all $x \in A$. □

Proposition A.41. *Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be length spaces such that $(X_i, p_i) \to (X, p)$ and let $\varepsilon_i \to 0$ and*

$$(f_i, g_i) \in Appr_{\varepsilon_i}((\bar{B}^{X_i}_{1/\varepsilon_i}(p_i), p_i), (\bar{B}^{X}_{1/\varepsilon_i}(p), p))$$

be as in Corollary A.27. Let $K_i \in X_i$ are compact with $K_i \subseteq \bar{B}^{X_i}_R(p_i)$ for some $R > 0$. After passing to a subsequence, there exists $K \subseteq \bar{B}_r(p)$ such that K_i subconverges to K.

Proof. Without loss of generality, assume $R \leq \frac{1}{\varepsilon_i}$ and $\varepsilon_i \leq 1$ for all $i \in \mathbb{N}$.

Let $x_i \in K_i \subseteq \bar{B}^{X_i}_R(p_i)$ be arbitrary. Then $f_i(x_i) \in B^X_{R+\varepsilon_i}(p) \subseteq \bar{B}^X_{R+1}(p)$. Hence, the sequence $(f_i(x_i))_{i \in \mathbb{N}})$ is contained in a compact set, and therefore has a convergent subsequent. Unfortunately, for different choices of x_i different subsequences might converge. Therefore, a diagonal argument on countable dense subsets of the K_i will be used.

Let $A_i = \{a_i^n \mid n \in \mathbb{N}\} \subseteq K_i$ be a countable dense subset. As seen above, the sequence $(f_i(a_i^n))_{i \in \mathbb{N}}$, where $n \in \mathbb{N}$, has a convergent subsequence with limit $y_n \in \bar{B}^X_{R+1}(p)$. Moreover, this subsequence can be chosen such that, after passing to this subsequence, $d_X(f_i(a_i^n), y_n) < \frac{\varepsilon_i}{4}$. By a diagonal argument, there exists a common subsequence such that for every $n \in \mathbb{N}$ there is $y_n \in \bar{B}_{R+1}(p)$ with $d_X(f_i(a_i^n), y_n) < \frac{\varepsilon_i}{4}$ for all $i \in \mathbb{N}$. Pass to this subsequence.

Define $A := \{y_n \mid n \in \mathbb{N}\}$ as the set of all these limits and let $K := \bar{A}$ denote its closure. In particular, K is compact.

Define maps $f_i' : K_i \to K$ and $g_i' : K \to K_i$ in the following way: For $x_i \in A_i$, i.e. $x_i = a_i^n$ for some $n \in \mathbb{N}$, define $f_i'(x_i) := y_n \in A \subseteq K$. If $x_i \in K_i \setminus A_i$, choose $a_i^n \in A_i$ with $d_{X_i}(x_i, a_i^n) < \frac{\varepsilon_i}{4}$ and define $f_i'(x_i) := y_n \in A \subseteq K$. In particular,

$$\begin{aligned} d_X(f_i'(x_i), f_i(x_i)) &\leq d_X(y_n, f_i(a_i^n)) + d_X(f_i(a_i^n), f_i(x_i)) \\ &< \frac{\varepsilon_i}{4} + (\varepsilon_i + d_{X_i}(a_i^n, x_i)) \\ &< \frac{\varepsilon_i}{4} + \left(\varepsilon_i + \frac{\varepsilon_i}{4}\right) = \frac{3}{2}\,\varepsilon_i\,. \end{aligned}$$

For $x \in A$, i.e. $x = y_n$ for some $n \in \mathbb{N}$, let $g_i'(y_n) := a_i^n \in A_i \subseteq K_i$. For $x \in X \setminus A$, choose $y_n \in A$ with $d_X(x, y_n) < \frac{\varepsilon_i}{4}$ and define $g_i'(x) := a_i^n \in A_i \subseteq K_i$. Then

$$\begin{aligned} d_{X_i}(g_i'(x), g_i(x)) = d_{X_i}(a_i^n, g_i(x)) &< 2\,\varepsilon_i + d_X(f_i(a_i^n), x) \\ &\leq 2\,\varepsilon_i + d_X(f_i(a_i^n), y_n) + d_X(y_n, x) \\ &< \frac{5}{2}\,\varepsilon_i\,. \end{aligned}$$

Now Lemma A.40 implies the claim. ☐

Lemma A.42. *Let (X, d_X), (Y, d_Y), (X_i, d_{X_i}) and (Y_i, d_{Y_i}), $i \in \mathbb{N}$, be compact length spaces such that $X_i \to X$ and $Y_i \to Y$. Moreover, let $\alpha > 0$, $K_i \subseteq X_i$ be compact subsets and $f_i : K_i \to Y_i$ be α-bi-Lipschitz. After passing to a subsequence, the following holds:*

a) There exist compact subsets $K \subseteq X$ and $K' \subseteq Y$ which are Gromov-Hausdorff limits of K_i and $f_i(K_i)$, respectively, and an α-bi-Lipschitz map $f : K \to K'$ with $f(K) = K'$.

b) For any compact subset $L \subseteq K \subseteq X$ there exist compact subsets $L_i \subseteq K_i$ such that $L_i \to L$ and $f_i(L_i) \to f(L)$ in the Gromov-Hausdorff sense.

Proof. a) Pass to the subsequence of Proposition A.41. Then there are compact sets $K \subseteq X$ and $K' \subseteq Y$ such that $K_i \to K$ and $f_i(K_i) \to K'$. For these, fix $\varepsilon_i \to 0$, $(f_i^X, g_i^X) \in Appr_{\varepsilon_i}(K_i, K)$ and $(f_i^Y, g_i^Y) \in Appr_{\varepsilon_i}(f_i(K_i), K')$, cf. Figure A.1.

The idea is to define f as a limit of $h_i := f_i^Y \circ f_i \circ g_i^X : K \to K'$: For $x, x' \in K$,

$$\begin{aligned} d_Y(h_i(x), h_i(x')) &= d_Y(f_i^Y \circ f_i \circ g_i^X(x), f_i^Y \circ f_i \circ g_i^X(x')) \\ &\leq \varepsilon_i + d_{Y_i}(f_i \circ g_i^X(x), f_i \circ g_i^X(x')) \\ &\leq \varepsilon_i + (\alpha \cdot d_{X_i}(g_i^X(x), g_i^X(x'))) \\ &\leq \varepsilon_i + (\alpha \cdot (\varepsilon_i + d_X(x, x'))) \\ &= \alpha \cdot d_X(x, x') + (\alpha + 1) \cdot \varepsilon_i\,. \end{aligned}$$

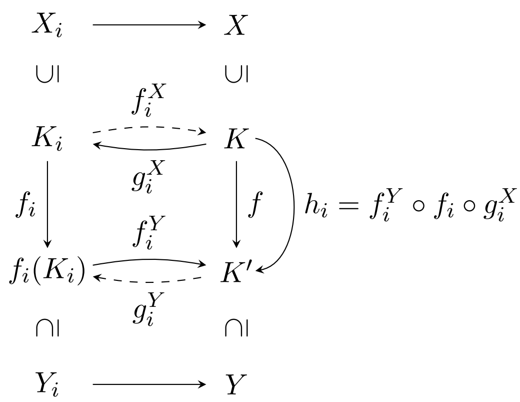

Figure A.1: Sets and maps used to construct $f : K \to K'$.

As in the proof of Proposition A.41, the $h_i(x)$ do not have to converge. Therefore, a diagonal argument on a dense subset of X will be used to construct a limit map which can be extended using the completeness of the limit space.

Let $A = \{x_j \mid j \in \mathbb{N}\}$ be a countable dense subset of K. Since $h_i(x_j) \in K'$ for all $i, j \in \mathbb{N}$ and K' is compact, by a diagonal argument, there is a subsequence $(i_n)_{n\in\mathbb{N}}$ such that $(h_{i_n}(x_j))_{n\in\mathbb{N}}$ converges for every $j \in \mathbb{N}$. Define $f : A \to K'$ by $f(x_j) = \lim_{n\to\infty} h_{i_n}(x_j)$. This map is α-bi-Lipschitz: For arbitrary $j, l \in \mathbb{N}$, with the above estimate,

$$\begin{aligned} d_Y(f(x_j), f(x_l)) &= \lim_{n\to\infty} d_Y(h_{i_n}(x_j), h_{i_n}(x_l)) \\ &\leq \lim_{n\to\infty} (\alpha + 1) \cdot \varepsilon_{i_n} + \alpha \cdot d_X(x_j, x_l) \\ &= \alpha \cdot d_X(x_j, x_l). \end{aligned}$$

Analogously, $d_Y(f(x_j), f(x_l)) \geq \frac{1}{\alpha} \cdot d_X(x_j, x_l)$.

Since A is a countable dense subset of K, f can be extended to an α-bi-Lipschitz map $f : K \to K'$ (cf. Lemma A.43) where $f(x) = \lim_{l\to\infty} f(x_{j_l})$ for $x \in K$ and $x_{j_l} \in A$ with $x_{j_l} \to x$. In particular, for $n \in \mathbb{N}$ and $l \in \mathbb{N}$,

$$\begin{aligned} &d_Y(f(x), h_{i_n}(x)) \\ &\leq d_Y(f(x), f(x_{j_l})) + d_Y(f(x_{j_l}), h_{i_n}(x_{j_l})) + d_Y(h_{i_n}(x_{j_l}), h_{i_n}(x)) \\ &\leq d_Y(f(x), f(x_{j_l})) + d_Y(f(x_{j_l}), h_{i_n}(x_{j_l})) + \alpha \cdot d_X(x_{j_l}, x) + (\alpha + 1) \cdot \varepsilon_{i_n} \\ &\to d_Y(f(x), f(x_{j_l})) + \alpha \cdot d_X(x_{j_l}, x) \text{ as } n \to \infty \\ &\to 0 \text{ as } l \to \infty. \end{aligned}$$

Hence, $f(x) = \lim_{n\to\infty} h_{i_n}(x)$.

Moreover, observe the following: Since f_i is α-bi-Lipschitz, it is injective. Therefore, the inverse f_i^{-1} of f_i exists on $f_i(K_i) \supseteq \mathrm{im}(g_i^X)$ and is α-bi-Lipschitz as well. Hence, for $x \in K$ and $y \in K'$,

$$\begin{aligned} d_Y(h_i(x), y) &= d_Y(f_i^Y \circ f_i \circ g_i^X(x), y) \\ &\leq 2\,\varepsilon_i + d_{Y_i}(f_i \circ g_i^X(x), g_i^Y(y)) \\ &\leq 2\,\varepsilon_i + \alpha \cdot d_{X_i}(g_i^X(x), f_i^{-1} \circ g_i^Y(y)) \\ &\leq 2\,\varepsilon_i + \alpha \cdot (2\,\varepsilon_i + d_{X_i}(x, f_i^X \circ f_i^{-1} \circ g_i^Y(y))) \\ &= 2(\alpha+1)\,\varepsilon_i + \alpha \cdot d_{X_i}(x, h_i'(y)) \end{aligned}$$

where $h_i' := f_i^X \circ f_i^{-1} \circ g_i^Y$. With analogous arguments and using a further subsequence $(i_{n_m})_{m\in\mathbb{N}}$ of $(i_n)_{n\in\mathbb{N}}$, there exists an α-bi-Lipschitz map $g : K' \to K$ with $g(y) = \lim_{m\to\infty} h'_{i_{n_m}}(y)$ for all $y \in K'$. In particular, for all $y \in K'$,

$$\begin{aligned} d_Y(f \circ g(y), y) &= \lim_{m\to\infty} d_Y(h_{i_{n_m}}(g(y)), y) \\ &\leq \lim_{m\to\infty} 2(\alpha+1)\,\varepsilon_{i_{n_m}} + \alpha \cdot d_X(g(y), h'_{i_{n_m}}(y)) \\ &= 0. \end{aligned}$$

Thus, $f \circ g = \mathrm{id}_{K'}$. Hence, $K' \subseteq \mathrm{im}(f)$ which proves $K' = f(K)$. In fact, with analogous argumentation, one can prove $g \circ f = \mathrm{id}_K$, i.e. g is the inverse of f.

b) This proof is based on the first part and is done with very similar methods.

   Let $(f_i^X, g_i^X) \in \mathit{Appr}_{\varepsilon_i}(K_i, K)$ and $(f_i^Y, g_i^Y) \in \mathit{Appr}_{\varepsilon_i}(f_i(K_i), K')$ be as before. Then $L_i := \overline{g_i^X(L)} \subseteq K_i$ is a compact subset of K_i. The proof of the subconvergences will be done in two steps. First, prove $L_i \to L$, then $f_i(L_i) \to f(L)$. For the maps defined below, cf. Figure A.2.

   (i) First, define $(\tilde{f}_i^X, \tilde{g}_i^X) \in \mathit{Appr}_{2\,\varepsilon_i}(L_i, L)$ as follows: For $x_i \in g_i^X(L)$, choose a point $y \in L$ with $x_i = g_i^X(y)$; for $x_i \in L_i \setminus g_i^X(L)$, choose $y \in L$ with $d_{X_i}(x_i, g_i^X(y)) < \frac{\varepsilon_i}{2}$. Then define $\tilde{f}_i^X(x_i) := y$. Finally, define $\tilde{g}_i^X := g_i^X$. By definition,

   $$d_{X_i}(\tilde{g}_i^X \circ \tilde{f}_i^X(x_i), x_i) = d_{X_i}(g_i^X \circ \tilde{f}_i^X(x_i), x_i) < \frac{\varepsilon_i}{2}$$

   for all $x_i \in L_i$. Conversely, for $x \in L$ and by applying this inequality,

   $$\begin{aligned} d_X(\tilde{f}_i^X \circ \tilde{g}_i^X(x), x) &= d_X(\tilde{f}_i^X \circ g_i^X(x), x) \\ &\leq d_{X_i}(g_i^X \circ \tilde{f}_i^X(g_i^X(x))), g_i^X(x)) + \varepsilon_i \\ &\leq \frac{3}{2} \cdot \varepsilon_i\,. \end{aligned}$$

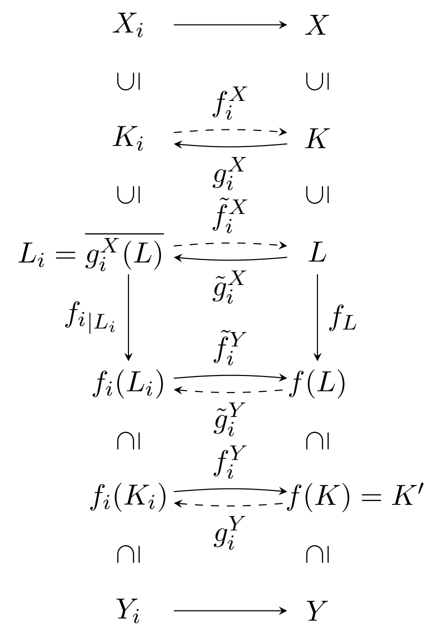

Figure A.2: Sets and maps used to construct $L_i \to L$.

Now let $x_i, x_i' \in L_i$ be arbitrary. Then

$$\begin{aligned}
&|d_X(\tilde{f}_i^X(x_i), \tilde{f}_i^X(x_i')) - d_{X_i}(x_i, x_i')| \\
&\leq |d_X(\tilde{f}_i^X(x_i), \tilde{f}_i^X(x_i')) - d_{X_i}(g_i^X(\tilde{f}_i^X(x_i)), g_i^X(\tilde{f}_i^X(x_i')))| \\
&\quad + |d_{X_i}(g_i^X(\tilde{f}_i^X(x_i)), g_i^X(\tilde{f}_i^X(x_i'))) - d_{X_i}(x_i, x_i')| \\
&< \varepsilon_i + d_{X_i}(g_i^X \circ \tilde{f}_i^X(x_i), x_i) + d_{X_i}(g_i^X \circ \tilde{f}_i^X(x_i'), x_i') \\
&< 2\,\varepsilon_i\,.
\end{aligned}$$

For $x, x' \in L$,

$$|d_{X_i}(\tilde{g}_i^X(x), \tilde{g}_i^X(x')) - d_X(x, x')| < \varepsilon_i < 2\,\varepsilon_i$$

by definition. This proves $(\tilde{f}_i^X, \tilde{g}_i^X) \in Appr_{2\,\varepsilon_i}(L_i, L)$.

(ii) Observe that the compactness of L_i and L, respectively, and the continuity of f_i and f, respectively, prove the compactness of $f_i(L_i)$ and $f(L)$, respectively. In order to prove the subconvergence of $f_i(L_i)$ to $f(L)$, let

$$\delta_i(x) := d_Y(h_i(x), f(x))$$

for $x \in L$ and

$$\delta_i := \sup_{x \in L} \delta_i(x).$$

For the subsequence $(i_n)_{n\in\mathbb{N}}$ of the first part, $\delta_{i_n}(x)$ converges to 0. Then δ_{i_n} converges to 0 as well: Assume this is not the case, i.e. there is $\epsilon > 0$ such that for all $l \in \mathbb{N}$ there exists $i_{n_l} \in \mathbb{N}$ and $x_{n_l} \in X$ with $\delta_{i_{n_l}}(x_{n_l}) \geq \varepsilon$. After passing to a subsequence, there is $x \in X$ such that $x_{n_l} \to x$ as $l \to \infty$. Then

$$\begin{aligned}
\varepsilon &\leq \delta_{i_{n_l}}(x_{n_l}) \\
&= d_Y(h_{i_{n_l}}(x_{n_l}), f(x_{n_l})) \\
&\leq d_Y(h_{i_{n_l}}(x_{n_l}), h_{i_{n_l}}(x)) + d_Y(h_{i_{n_l}}(x), f(x)) + d_Y(f(x), f(x_{n_l})) \\
&\leq (\alpha \cdot d_X(x_{n_l}, x) + (\alpha+1)\cdot \varepsilon_{i_{n_l}}) + \delta_{i_{n_l}}(x) + \alpha \cdot d_X(x, x_{n_l}) \\
&\to 0 \text{ as } l \to \infty.
\end{aligned}$$

This is a contradiction.
Construct $(\tilde{f}_i^Y, \tilde{g}_i^Y) \in Appr_{\tilde{\varepsilon}_i}(f_i(L_i), f(L))$ for $\tilde{\varepsilon}_i := (4\alpha+1)\,\varepsilon_i + 2\delta_i$ as follows: Define $\tilde{f}_i^Y := f \circ \tilde{f}_i^X \circ f_i^{-1}$ and $\tilde{g}_i^Y := f_i \circ g_i^X \circ f^{-1}$ (recall that f_i^{-1} exists on $f_i(L_i) \subseteq f_i(K_i)$ and that $f : K \to K'$ is bijective).
First, let $y_i \in L_i$ and $y \in L$ be arbitrary. Then

$$\begin{aligned}
d_{Y_i}(\tilde{g}_i^Y \circ \tilde{f}_i^Y(y_i), y_i) &= d_{Y_i}(f_i \circ g_i^X \circ \tilde{f}_i^X \circ f_i^{-1}(y_i), y_i) \\
&\leq \alpha \cdot d_{X_i}(g_i^X \circ \tilde{f}_i^X(f_i^{-1}(y_i)), f_i^{-1}(y_i)) \\
&< \alpha \cdot 2\,\varepsilon_i \leq \tilde{\varepsilon}_i,
\end{aligned}$$

and completely analogously,

$$d_Y(\tilde{f}_i^Y \circ \tilde{g}_i^Y(y), y) = d_Y(f \circ \tilde{f}_i^X \circ g_i^X \circ f^{-1}(y), y) < 2\alpha\,\varepsilon_i \leq \tilde{\varepsilon}_i.$$

For $y, y' \in L$,

$$\begin{aligned}
&|d_{Y_i}(\tilde{g}_i^Y(y), \tilde{g}_i^Y(y')) - d_Y(y, y')| \\
&\leq |d_{Y_i}(\tilde{g}_i^Y(y), \tilde{g}_i^Y(y')) - d_Y(f_i^Y \circ \tilde{g}_i^Y(y), f_i^Y \circ \tilde{g}_i^Y(y'))| \\
&\quad + |d_Y(f_i^Y \circ f_i \circ g_i^X \circ f^{-1}(y), f_i^Y \circ f_i \circ g_i^X \circ f^{-1}(y')) - d_Y(y, y')| \\
&< \varepsilon_i + d_Y(h_i \circ f^{-1}(y), f \circ f^{-1}(y)) + d_Y(h_i \circ f^{-1}(y'), f \circ f^{-1}(y')) \\
&\leq \varepsilon_i + 2\delta_i \leq \tilde{\varepsilon}_i.
\end{aligned}$$

Finally, let $y_i, y_i' \in Y_i$. Using the above estimates,

$$\begin{aligned}
&|d_Y(\tilde{f}_i^Y(y_i), \tilde{f}_i^Y(y_i')) - d_{Y_i}(y_i, y_i')| \\
&\leq |d_Y(\tilde{f}_i^Y(y_i), \tilde{f}_i^Y(y_i')) - d_{Y_i}(\tilde{g}_i^Y(\tilde{f}_i^Y(y_i)), \tilde{g}_i^Y(\tilde{f}_i^Y(y_i')))| \\
&\quad + |d_{Y_i}(\tilde{g}_i^Y(\tilde{f}_i^Y(y_i)), \tilde{g}_i^Y(\tilde{f}_i^Y(y_i'))) - d_{Y_i}(y_i, y_i')| \\
&< \varepsilon_i + 2\delta_i + d_{Y_i}(\tilde{g}_i^Y(\tilde{f}_i^Y(y_i)), y_i) + d_{Y_i}(\tilde{g}_i^Y(\tilde{f}_i^Y(y_i')), y_i') \\
&\leq \varepsilon_i + 2\delta_i + 2 \cdot 2\alpha\,\varepsilon_i = \tilde{\varepsilon}_i.
\end{aligned}$$

Thus, $(\tilde{f}_i^Y, \tilde{g}_i^Y) \in Appr_{\tilde{\varepsilon}_i}(f_i(L_i), f(L))$. Since $\tilde{\varepsilon}_{i_n} \to 0$ as $n \to \infty$, this proves $f_{i_n}(L_{i_n}) \to f(L)$ as $n \to \infty$. □

Lemma A.43. *Let (X, d_X) and (Y, d_Y) be metric spaces where Y is complete, let $A \subseteq X$ and $f : A \to Y$ be α-(bi)-Lipschitz for some $\alpha > 0$. Then f can be extended to an α-(bi)-Lipschitz map $\hat{f} : \bar{A} \to Y$.*

Proof. Let $a \in \bar{A} \setminus A$ be arbitrary. Then there exists a (Cauchy) sequence $(a_n)_{n\in\mathbb{N}}$ in A converging to a. By Lipschitz continuity of f, $(f(a_n))_{n\in\mathbb{N}}$ is a Cauchy sequence, and thus has a limit $\hat{a}$ in the complete metric space Y. If $(\tilde{a}_n)_{n\in\mathbb{N}}$ is another sequence with limit a, $d_Y(f(a_n), f(\tilde{a}_n)) \leq \alpha \cdot d_X(a_n, \tilde{a}_n) \to 0$, i.e. the limit $\hat{a}$ is independent of the choice of $(a_n)_{n\in\mathbb{N}}$. Now define $\hat{f}(a) := \hat{a}$ for $a \in \bar{A} \setminus A$ and $\hat{f}(a) := f(a)$ for $a \in A$. For arbitrary $a, b \in A$ and sequences $a_n \to a$, $b_n \to b$ in A,

$$d_Y(\hat{f}(a), \hat{f}(b)) = \lim_{n\to\infty} d_Y(f(a_n), f(b_n)) \leq \lim_{n\to\infty} \alpha \cdot d_X(a_n, b_n) = \alpha \cdot d(a, b).$$

Hence, $\hat{f}$ is α-Lipschitz. Analogously, if f is α-bi-Lipschitz, $\hat{f}$ is α-bi-Lipschitz. □

A.3 Ultralimits

Since sequences of proper spaces do not necessarily converge in the pointed Gromov-Hausdorff sense, a tool to enforce convergence can be useful. Such a tool are the so called ultralimits since they always exist and are sublimits in the pointed Gromov-Hausdorff sense. A basic reference from which the following definitions are taken is [BH99, section I.5]. Another, more set theoretical, reference is [Jec06, chapter 7]. In the following, ultralimits will be introduced and some properties will be investigated.

Definition A.44 ([BH99, Definition I.5.47])**.** A *non-principal ultrafilter on* $\mathbb{N}$ is a finitely additive probability measure ω on $\mathbb{N}$ such that all subsets $S \subseteq \mathbb{N}$ are ω-measurable with $\omega(S) \in \{0, 1\}$ and $\omega(S) = 0$ if S is finite.

Remark. If two sets have ω-measure 1, their intersection has ω-measure 1 as well: Let $\omega(A) = \omega(B) = 1$. Then $\omega(\mathbb{N} \setminus (A \cap B)) = \omega(\mathbb{N} \setminus A \cup \mathbb{N} \setminus B) \leq \omega(\mathbb{N} \setminus A) + \omega(\mathbb{N} \setminus B) = 0$, hence, $\omega(A \cap B) = 1$.

Using Zorn's Lemma, the existence of such a non-principal ultrafilter can be proven. But even more is true: Given any infinite set, there exists a non-principal ultrafilter such that the set has measure 1 with respect to this ultrafilter.

Lemma A.45. *Let $A \subseteq \mathbb{N}$ be an infinite set. Then there exists a non-principal ultrafilter ω on $\mathbb{N}$ such that $\omega(A) = 1$.*

Proof. Let

$$G := \{B \subseteq \mathbb{N} \mid B \supseteq A \text{ or } \mathbb{N} \setminus B \text{ is finite}\}.$$

For any $B_1, B_2 \in G$, the intersection $B_1 \cap B_2$ is non-empty: This is obviously correct if both $B_j \supseteq A$ or both $\mathbb{N} \setminus B_j$ are finite. Thus, let $B_1 \supseteq A$ and $\mathbb{N} \setminus B_2$ be finite: Then $A \setminus B_2$ is finite as well, hence, $B_1 \cap B_2 \supseteq A \cap B_2 = A \setminus (A \setminus B_2)$ is infinite since A is infinite. In particular, the intersection is non-empty.

Using that G contains all sets with finite complement, it follows from [Jec06, Lemma 7.2 (iii)], [Jec06, Theorem 7.5] and the subsequent remark therein that there exists a non-principal ultrafilter ω such that $\omega(X) = 1$ for all $X \in G$. In particular, $\omega(A) = 1$. □

Given a bounded sequence of real numbers, a non-principal ultrafilter provides a kind of 'limit'. In fact, these 'limits' are accumulation points and non-principal ultrafilters pick out convergent subsequences.

Lemma A.46 ([BH99, Lemma I.5.49])**.** *Let ω be a non-principal ultrafilter on $\mathbb{N}$. For every bounded sequence of real numbers $(a_i)_{i\in\mathbb{N}}$ there exists a unique real number $l \in \mathbb{R}$ such that*

$$\omega(\{i \in \mathbb{N} \mid |a_i - l| < \varepsilon\}) = 1$$

for every $\varepsilon > 0$. Denote this l by $\lim_\omega a_i$.

Lemma A.47. *If ω is a non-principal ultrafilter on $\mathbb{N}$ and $(a_i)_{i\in\mathbb{N}}$ a bounded sequence of real numbers, then $\lim_\omega a_i$ is an accumulation point of $(a_i)_{i\in\mathbb{N}}$. Moreover, there exists a subsequence $(a_{i_j})_{j\in\mathbb{N}}$ converging to $\lim_\omega a_i$ such that $\omega(\{i_j \mid j \in \mathbb{N}\}) = 1$.*

Conversely, if $(a_i)_{i\in\mathbb{N}}$ is a bounded sequence of real numbers and $a \in \mathbb{R}$ any accumulation point, then there exists a non-principal ultrafilter ω on $\mathbb{N}$ such that $a = \lim_\omega a_i$.

Proof. Let $(a_i)_{i\in\mathbb{N}}$ be any bounded sequence of real numbers.

First, fix a non-principal ultrafilter ω, let $a := \lim_\omega a_i$ and

$$A_\varepsilon := \{i \in \mathbb{N} \mid |a_i - a| < \varepsilon\}$$

for $\varepsilon > 0$. By definition, $\omega(A_\varepsilon) = 1$; in particular, A_ε has infinitely many elements. Thus, a is an accumulation point.

Next, prove that there exists $I \subseteq \mathbb{N}$ with $\omega(I) = 1$ such that the subsequence $(a_i)_{i\in I}$ converges to a. Assume this is not the case, i.e. every $I \subseteq \mathbb{N}$ satisfies $\omega(I) = 0$ or $(a_i)_{i\in I}$ does not converge to a. Since $\omega(\mathbb{N}) = 1$, $(a_i)_{i\in\mathbb{N}}$ does not converge to a. Hence, there exists $\varepsilon > 0$ such that $A_\varepsilon = \{i \in \mathbb{N} \mid |a_i - a| < \varepsilon\}$ is finite. In particular, $\omega(A_\varepsilon) = 0$ and this is a contradiction.

Now let $J \subseteq \mathbb{N}$ be a set of indices such that $\omega(J) = 1$ and the subsequence $(a_j)_{j\in J}$ converges to a. By Lemma A.45, there exists a non-principal ultrafilter ω such that $\omega(J) = 1$. By the first part, there exists a subsequence of indices $I \subseteq \mathbb{N}$ with $\omega(I) = 1$ and $a_j \to \lim_\omega a_i$ as $j \to \infty$ for $j \in I$. Now $\omega(I \cap J) = 1$ and both $a_j \to a$ and $a_j \to \lim_\omega a_i$ as $j \to \infty$ for $j \in I \cap J$. This proves $a = \lim_\omega a_i$. □

An immediate consequence of the above lemma is the following: Given two bounded sequences of real numbers, investigating sublimits coming from a common subsequence and investigating the 'limits' with respect to the same non-principal ultrafilter is the same.

Lemma A.48. *Let $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ be bounded sequences of real numbers.*

a) If ω is a non-principal ultrafilter on $\mathbb{N}$, then there exists a subsequence $(i_j)_{j\in\mathbb{N}}$ such that both $a_{i_j} \to \lim_\omega a_i$ and $b_{i_j} \to \lim_\omega b_i$ as $j \to \infty$.

b) *If there are $a, b \in \mathbb{R}$ and a subsequence $(i_j)_{j\in\mathbb{N}}$ such that both $a_{i_j} \to a$ and $b_{i_j} \to b$ as $j \to \infty$, then there exists a non-principal ultrafilter ω on $\mathbb{N}$ such that $a = \lim_\omega a_i$ and $b = \lim_\omega b_i$.*

Proof. a) By Lemma A.47, there are subsequences of indices $I, J \subseteq \mathbb{N}$ with measures $\omega(I) = \omega(J) = 1$ and

$$a_j \to \lim_\omega a_i \text{ as } j \to \infty \text{ for } j \in I \quad \text{and} \quad b_j \to \lim_\omega b_i \text{ as } j \to \infty \text{ for } j \in J.$$

In particular, $I \cap J$ has ω-measure 1. Hence, it is infinite and provides a common subsequence which satisfies the claim.

b) This follows directly from the second part of Lemma A.47 since the non-principal ultrafilter constructed there depends only on the indices of the convergent subsequence. □

Corollary A.49. *Let $(a_i)_{i\in\mathbb{N}}$ and $(b_i)_{i\in\mathbb{N}}$ be bounded sequences of real numbers.*

a) *If $a_i \leq b_i$ for all $i \in \mathbb{N}$, then $\lim_\omega a_i \leq \lim_\omega b_i$.*

b) *$\lim_\omega (a_i + b_i) = \lim_\omega a_i + \lim_\omega b_i$.*

Proof. Observe that Lemma A.48 holds not only for two but finitely many sequences for real numbers. Applying this and the corresponding statements for limits of sequences of real numbers implies the claim. □

An ultralimit is a 'limit space' assigned to a (pointed) sequence of metric spaces by using a non-principal ultrafilter. The construction of this ultralimit is related to Gromov-Hausdorff convergence in the sense that such a limit space is a sublimit in the pointed Gromov-Hausdorff sense. On the other hand, given any sublimit in the pointed Gromov-Hausdorff sense, there exists a non-principal ultrafilter such that the corresponding ultralimit is exactly this sublimit. This fact can be extended to a similar statement about finitely many different sequences and corresponding sublimits coming from a common subsequence.

Definition A.50 ([BH99, Definition I.5.50])**.** Let ω be a non-principal ultrafilter on $\mathbb{N}$, (X_i, d_i, p_i), $i \in \mathbb{N}$, be pointed metric spaces and

$$X_\omega := \{[(x_i)_{i\in\mathbb{N}}] \mid x_i \in X_i \text{ and } \sup_{i\in\mathbb{N}} d_i(x_i, p_i) < \infty\}$$

where

$$(x_i)_{i\in\mathbb{N}} \sim (y_i)_{i\in\mathbb{N}} \text{ if and only if } \lim_\omega d_i(x_i, y_i) = 0.$$

Furthermore, let $d_\omega([(x_i)_{i\in\mathbb{N}}], [(y_i)_{i\in\mathbb{N}}]) := \lim_\omega d_i(x_i, y_i)$. Then (X_ω, d_ω) is a metric space, called *ultralimit* of (X_i, d_i, p_i) and denoted by $\lim_\omega (X_i, d_i, p_i)$.

Remark. Let ω be a non-principal ultrafilter on $\mathbb{N}$, (X_i, d_i, p_i), $i \in \mathbb{N}$, be pointed metric spaces and $Y_i \subseteq X_i$. Then the limit $(Y_\omega, d_{Y_\omega}) := \lim_\omega(Y_i, d_i, p_i)$ is canonically a subset of $(X_\omega, d_{X_\omega}) := \lim_\omega(X_i, d_i, p_i)$: Obviously,

$$\{(y_i)_{i\in\mathbb{N}} \mid y_i \in Y_i \text{ and } \sup_i d_i(x_i, p_i) < \infty\}$$
$$\subseteq \{(x_i)_{i\in\mathbb{N}} \mid x_i \in X_i \text{ and } \sup_i d_i(x_i, p_i) < \infty\}.$$

Since the metric is the same on both X_i and Y_i and since the equivalence classes are only defined by using the ultrafilter and the metric, $Y_\omega \subseteq X_\omega$. With the same argumentation, the metric coincides: For $y_i, y_i' \in Y_i$,

$$d_{Y_\omega}([(y_i)_{i\in\mathbb{N}}]_{Y_\omega}, [(y_i')_{i\in\mathbb{N}}]_{Y_\omega}) = \lim_\omega d_i(y_i, y_i') = d_{X_\omega}([(y_i)_{i\in\mathbb{N}}]_{X_\omega}, [(y_i')_{i\in\mathbb{N}}]_{X_\omega}).$$

Lemma A.51 ([BH99, Lemma I.5.53])**.** *The ultralimit of a sequence of metric spaces is complete.*

In order to prove the correspondence of sublimits and ultralimits, first, compact metric spaces are investigated.

Proposition A.52. *Let ω be a non-principal ultrafilter on $\mathbb{N}$, (X_i, d_i, p_i), $i \in \mathbb{N}$, be pointed compact metric spaces with compact ultralimit (X_ω, d_ω) and $p_\omega := [(p_i)_{i\in\mathbb{N}}] \in X_\omega$. Then $\lim_\omega d_{GH}((X_i, p_i), (X_\omega, p_\omega)) = 0$.*

Proof. The statement will be proven by using ε-nets: First, finite ε-nets in X_i will be fixed and it will be proven that their ultralimit is a finite ε-net in X_ω. Then the Gromov-Hausdorff distance of these nets will be estimated. Finally, using the triangle inequality and $\varepsilon \to 0$ prove the claim.

Fix $\varepsilon > 0$. For every $i \in \mathbb{N}$, fix a finite ε-net $A_i^\varepsilon = \{a_i^1, \ldots, a_i^{n_i}\}$ in the compact space X_i with $a_i^1 = p_i$, i.e. $d(a_i^k, a_i^l) \geq \varepsilon$ for all $k \neq l$ and $X_i = \bigcup_{j=1}^{n_i} B_\varepsilon(a_i^j)$. Let A_ω^ε be the ultralimit of these A_i^ε, i.e.

$$A_\omega^\varepsilon = \{[(a_i)_{i\in\mathbb{N}}] \mid \forall i \in \mathbb{N}\ \exists 1 \leq j_i \leq n_i : a_i = a_i^{j_i}\} \subseteq X_\omega,$$

and let $p_\omega := [(p_i)_{i\in\mathbb{N}}] \in A_\omega^\varepsilon$. Then A_ω^ε is again a finite ε-net in X_ω:

Let $[(a_i^{k_i})_{i\in\mathbb{N}}], [(a_i^{l_i})_{i\in\mathbb{N}}] \in A_\omega^\varepsilon$. By definition,

$$[(a_i^{k_i})_{i\in\mathbb{N}}] = [(a_i^{l_i})_{i\in\mathbb{N}}] \text{ if and only if } \lim_\omega d_i(a_i^{k_i}, a_i^{l_i}) = 0.$$

Since $d_i(a_i^{k_i}, a_i^{l_i}) = 0$ exactly for those i with $k_i = l_i$ and $d_i(a_i^{k_i}, a_i^{l_i}) \geq \varepsilon$ otherwise, this implies

$$[(a_i^{k_i})_{i\in\mathbb{N}}] = [(a_i^{l_i})_{i\in\mathbb{N}}] \text{ if and only if } \omega(\{i \in \mathbb{N} \mid k_i = l_i\}) = 1.$$

In particular, for $[(a_i^{k_i})_{i\in\mathbb{N}}] \neq [(a_i^{l_i})_{i\in\mathbb{N}}]$, $d_{X_\omega}([(a_i^{k_i})_{i\in\mathbb{N}}], [(a_i^{l_i})_{i\in\mathbb{N}}]) = \lim_\omega d_i(a_i^{k_i}, a_i^{l_i}) \geq \varepsilon$. Furthermore, for arbitrary $[(x_i)_{i\in\mathbb{N}}]$ there are $a_i^{j_i}$ such that $x_i \in B_\varepsilon(a_i^{j_i})$. Thus,

$$d_\omega([(x_i)_{i\in\mathbb{N}}], [(a_i^{j_i})_{i\in\mathbb{N}}]) = \lim_\omega d_i(x_i, a_i^{j_i}) < \varepsilon\,.$$

This proves that A^ε_ω is an ε-net in X_ω. It remains to prove that A^ε_ω is finite: Assume it is not. Then $\bigcup_{p\in A^\varepsilon_\omega} B_\varepsilon(p)$ is an open cover of X_ω, and thus, has a finite subcover $X_\omega = \bigcup_{j=1}^k B_\varepsilon(q_j)$ with $q_j \in A^\varepsilon_\omega$. Hence, for any $q \in A^\varepsilon_\omega \setminus \{q_1, \dots, q_k\}$ there exists q_j such that $q \in B_\varepsilon(q_j)$. This is a contradiction to $d_\omega(q, q_j) \geq \varepsilon$.

Let $n_\omega < \infty$ denote the cardinality of A^ε_ω and $I := \{i \in \mathbb{N} \mid n_i = n_\omega\}$ be those indices such that A^ε_i and A^ε_ω have the same cardinality. Then $\omega(I) = 1$:

Let $A^\varepsilon_\omega = \{z_1, \dots, z_{n_\omega}\}$ and $z_k = [(a_i^{j_i^k})_{i\in\mathbb{N}}]$ with $1 \leq j_i^k \leq n_i$ for $1 \leq k \leq n_\omega$. For $k \neq l$, one has $1 = \omega(\{i \in \mathbb{N} \mid j_i^k \neq j_i^l\})$. Thus,

$$\begin{aligned} 1 &= \omega\big(\bigcap\nolimits_{1\leq k<l\leq n_\omega} \{i \in \mathbb{N} \mid j_i^k \neq j_i^l\}\big) \\ &= \omega(\{i \in \mathbb{N} \mid \forall 1 \leq k < l \leq n_\omega : j_i^k \neq j_i^l\}) \\ &\geq \omega(\{i \in \mathbb{N} \mid n_\omega \leq n_i\}) \\ &= \omega(I \cup J) \end{aligned}$$

where $J := \{i \in \mathbb{N} \mid n_i > n_\omega\}$. Assume $\omega(J) = 1$. For all $1 \leq j \leq n_\omega + 1$, let

$$q_i^j := \begin{cases} a_i^j & i \in J, \\ p_i & i \notin J \end{cases}$$

and $\tilde{z}_j := [(q_i^j)_{i\in\mathbb{N}}] \in A^\varepsilon_\omega$. By definition, $q_i^j = q_i^l$ if and only if $k \neq l$ or $i \in I$. Hence, if $k \neq l$, then $\omega(\{i \in \mathbb{N} \mid q_i^k = q_i^l\}) = \omega(\mathbb{N}\setminus J) = 1 - \omega(J) = 0$. Thus, $\tilde{z}_k \neq \tilde{z}_l$ and $\{\tilde{z}_1, \dots, \tilde{z}_{n_\omega+1}\} \subseteq A^\varepsilon_\omega$, hence, $n_\omega + 1 \leq n_\omega$. This is a contradiction. Therefore, $\omega(J) = 0$ and $\omega(I) = \omega(I \cup J) = 1$.

Similarly, for all $1 \leq j \leq n_\omega$, let

$$p_i^j := \begin{cases} a_i^j & i \in I, \\ p_i & i \notin I \end{cases}$$

and $y_j := [(q_i^j)_{i\in\mathbb{N}}] \in A^\varepsilon_\omega$. Analogously, $y_l = y_k$ if and only if $l = k$. This implies $A^\varepsilon_\omega = \{y_1, \dots, y_{n_\omega}\}$. In particular, $y_1 = p_\omega$.

For $1 \leq k < l \leq n_\omega$, define

$$\begin{aligned} I_\delta^{kl} :=& \{i \in I \mid |d_\omega(y_k, y_l) - d_i(a_i^k, a_i^l)| < \delta\} \\ =& \{i \in I \mid |d_\omega(y_k, y_l) - d_i(p_i^k, p_i^l)| < \delta\}. \end{aligned}$$

Since $d_\omega(y_k, y_l) = \lim_\omega d_i(p_i^k, p_i^l)$ by definition, $\omega(I_\delta^{kl}) = 1$ for any $\delta > 0$. Therefore, $\lim_\omega \delta_i^{kl} = 0$ for $\delta_i^{kl} := |d_\omega(y_k, y_l) - d_i(a_i^k, a_i^l)|$. Thus, for $\varepsilon_i := \max\{\delta_i^{kl} \mid 1 \leq k < l \leq n_\omega\}$, $\lim_\omega \varepsilon_i = 0$ as well.

Let $i \in I$ be fixed and define $f_i : A^\varepsilon_i \to A^\varepsilon_\omega$ and $g_i : A^\varepsilon_\omega \to A^\varepsilon_i$ by

$$f_i(a_i^j) := y_j \text{ and } g_i(y_j) = a_i^j$$

for $1 \leq j \leq n_\omega$. In particular, $f_i(p_i) = f_i(a_i^1) = y_1 = p_\omega$ and $g_i(p_\omega) = g_i(y_1) = a_i^1 = p_i$.

Obviously, $f_i \circ g_i = \mathrm{id}_{A_\omega^\varepsilon}$ and $g_i \circ f_i = \mathrm{id}_{A_i^\varepsilon}$. Furthermore, for $1 \leq k < l \leq n_\omega$,

$$|d_\omega(f_i(a_i^k), f_i(a_i^l)) - d_i(a_i^k, a_i^l)| = |d_\omega(y_k, y_l) - d_i(a_i^k, a_i^l)| = \delta_i^{kl} \leq \varepsilon_i,$$

and analogously,

$$|d_i(g_i(y_k), g_i(y_l)) - d_\omega(y_k, y_l)| \leq \varepsilon_i,$$

i.e. $(f_i, g_i) \in Appr_{\varepsilon_i}((A_i^\varepsilon, p_i), (A_\omega^\varepsilon, p_\omega))$. Therefore, $d_{GH}((A_i^\varepsilon, p_i), (A_\omega^\varepsilon, p_\omega)) \leq 2\,\varepsilon_i$ for any $i \in I$.

For any compact metric space (Z, d_Z) and ε-net $A \subseteq Z$,

$$d_H^{d_Z}(Z, A) = \inf\{r > 0 \mid B_r(A) \supseteq Z = B_\varepsilon(A)\} \leq \varepsilon\,.$$

Hence, for any $p \in A$, $d_{GH}((A,p),(Z,p)) \leq d_H(Z,A) + d_Z(p,p) \leq \varepsilon$.

Applying this general statement, for fixed $i \in I$ and $\varepsilon > 0$,

$$\begin{aligned}
&d_{GH}((X_i, p_i), (X_\omega, p_\omega)) \\
&\leq d_{GH}((X_i, p_i), (A_i^\varepsilon, p_i)) + d_{GH}((A_i^\varepsilon, p_i), (A_\omega^\varepsilon, p_\omega)) + d_{GH}((A_\omega^\varepsilon, p_\omega), (X_\omega, p_\omega)) \\
&\leq 2\,\varepsilon + 2\,\varepsilon_i\,.
\end{aligned}$$

In particular, $\lim_\omega d_{GH}((X_i, p_i), (X_\omega, p_\omega)) \leq 2\,\varepsilon$. Since this holds for all $\varepsilon > 0$,

$$\lim\nolimits_\omega d_{GH}((X_i, p_i), (X_\omega, p_\omega)) = 0. \qquad \square$$

Corollary A.53. *Let ω be a non-principal ultrafilter on $\mathbb{N}$. If the ultralimit of compact metric spaces is compact, it is a sublimit in the pointed Gromov-Hausdorff sense which comes from a subsequence with index set of ω-measure 1.*

Proof. Let (X_i, d_i, p_i), $i \in \mathbb{N}$, be pointed compact metric spaces, (X_ω, d_ω) their compact ultralimit and $p_\omega = [(p_i)_{i\in\mathbb{N}}]$. By the previous proposition,

$$\lim\nolimits_\omega d_{GH}((X_i, p_i), (X_\omega, p_\omega)) = 0,$$

and by Lemma A.47, there exists a subsequence $(i_j)_{j\in\mathbb{N}}$ of natural numbers satisfying $\omega(\{i_j \mid j \in \mathbb{N}\}) = 1$ such that

$$d_{GH}((X_{i_j}, p_{i_j}), (X_\omega, p_\omega)) \to 0 \text{ as } j \to \infty. \qquad \square$$

This result now gives a corresponding result for non-compact spaces.

Proposition A.54. *a) Let ω be a non-principal ultrafilter on $\mathbb{N}$. The ultralimit of a sequence of pointed proper length spaces is a sublimit in the pointed Gromov-Hausdorff sense (which comes from a subsequence with index set of ω-measure 1).*

b) The sublimit of a sequence of pointed proper length spaces in the pointed Gromov-Hausdorff sense is the ultralimit with respect to a non-principal ultrafilter.

Proof. a) Let (X_i, d_i, p_i), $i \in \mathbb{N}$, be pointed proper length spaces, (X_ω, d_ω) the corresponding ultralimit and $p_\omega := [(p_i)_{i\in\mathbb{N}}] \in X_\omega$. First it will be shown that an r-ball in the ultralimit is the ultralimit of r-balls. Then applying the corresponding statement for compact sets proves the claim.

(i) For $r > 0$, let $X_\omega^r \subseteq X_\omega$ denote the ultralimit of $(\bar{B}_r^{X_i}(p_i), d_i, p_i)$. This is a closed subset of X_ω: First, observe

$$X_\omega^r = \{[(q_i)_{i\in\mathbb{N}}] \mid q_i \in X_i \text{ and } d_i(q_i, p_i) \leq r\}.$$

Let $(z_n)_{n\in\mathbb{N}}$ be a sequence in X_ω^r which converges to a limit $z \in X_\omega$. Denote $z_n = [(q_i^n)_{i\in\mathbb{N}}]$ and $z = [(q_i)_{i\in\mathbb{N}}]$ where $q_i^n, q_i \in X_i$ with $d_i(q_i^n, p_i) \leq r$ for all $i, n \in \mathbb{N}$ and $\sup_{i\in\mathbb{N}} d_i(q_i, p_i) < \infty$. Moreover, $d_\omega(z_n, z) = \lim_\omega d_i(q_i^n, q_i) \to 0$ as $n \to \infty$. For all $n \in \mathbb{N}$, $d_\omega(z_n, p_\omega) = \lim_\omega d_i(q_i^n, p_i) \leq r$. Hence,

$$d_\omega(z, p_\omega) \leq \lim_{n\to\infty} d_\omega(z, z_n) + d_\omega(z_n, p_\omega) \leq r$$

and $z \in X_\omega^r$. This proves that X_ω^r is closed.
In fact, $X_\omega^r = \bar{B}_r^{X_\omega}(p_\omega)$: First, let $[(q_i)_{i\in\mathbb{N}}] \in X_\omega^r \subseteq X_\omega$ be arbitrary. Since

$$d_\omega([(q_i)_{i\in\mathbb{N}}], [(p_i)_{i\in\mathbb{N}}]) = \lim_\omega d_i(p_i, q_i) \leq r,$$

$[(q_i)_{i\in\mathbb{N}}] \in \bar{B}_r^{X_\omega}(p_\omega)$.
Now let $[(q_i)_{i\in\mathbb{N}}] \in B_r^{X_\omega}(p_\omega)$ and $I := \{i \in \mathbb{N} \mid d_i(p_i, q_i) < r\}$. Define

$$\tilde{q}_i := \begin{cases} q_i & i \in I, \\ p_i & i \notin I. \end{cases}$$

By definition, $[(\tilde{q}_i)_{i\in\mathbb{N}}] \in X_\omega^r$. Furthermore, $[(q_i)_{i\in\mathbb{N}}] = [(\tilde{q}_i)_{i\in\mathbb{N}}] \in X_\omega^r$: Since $[(q_i)_{i\in\mathbb{N}}] \in B_r^{X_\omega}(p_\omega)$, $0 \leq l := \lim_\omega d_i(q_i, p_i) < r$. For $\delta := r - l > 0$,

$$\begin{aligned} 1 &= \omega(\{i \in \mathbb{N} \mid |d_i(q_i, p_i) - l| < \delta\}) \\ &\leq \omega(\{i \in \mathbb{N} \mid d_i(q_i, p_i) < l + \delta = r\}) \\ &= \omega(I). \end{aligned}$$

Thus, for arbitrary $\varepsilon > 0$,

$$\begin{aligned} \omega(\{i \in \mathbb{N} \mid d_i(q_i, \tilde{q}_i) < \varepsilon\}) &\geq \omega(\{i \in \mathbb{N} \mid q_i = \tilde{q}_i\}) \\ &= \omega(I) = 1. \end{aligned}$$

Therefore, $\lim_\omega d_i(q_i, \tilde{q}_i) = 0$ and $[(q_i)_{i\in\mathbb{N}}] = [(\tilde{q}_i)_{i\in\mathbb{N}}] \in X_\omega^r$. Consequently, $B_r^{X_\omega}(p_\omega) \subseteq X_\omega^r$. Since X_ω^r is closed, this proves $\bar{B}_r^{X_\omega}(p_\omega) \subseteq X_\omega^r$, and hence, equality.

(ii) For any $r > 0$ and $\varepsilon_i^r := d_{GH}((\bar{B}_r^{X_i}(p_i), p_i), (\bar{B}_r^{X_\omega}(p_\omega), p_\omega))$, $\lim_\omega \varepsilon_i^r = 0$ by Proposition A.52. By Lemma A.55, there exists $r_i > 0$ with

$$\lim_\omega \frac{1}{r_i} = 0 \quad \text{and} \quad \omega\Big(\Big\{i \in \mathbb{N} \mid \varepsilon_i^{r_i} \leq \frac{1}{r_i}\Big\}\Big) = 1.$$

By Lemma A.47, there is $J = \{i_1 < i_2 < \dots\} \subseteq \mathbb{N}$ such that $\omega(J) = 1$ and $r_{i_j} \to \infty$. Let

$$I := J \cap \Big\{i \in \mathbb{N} \mid \varepsilon_i^{r_i} \leq \frac{1}{r_i}\Big\}.$$

Then $\omega(I) = 1$ and $I = \{i_{j_1} < i_{j_2} < \dots\} \subseteq J$. Thus, $r_{i_{j_l}} \to \infty$ and

$$d_{GH}((\bar{B}_{r_{i_{j_l}}}^{X_{i_{j_l}}}(p_{i_{j_l}}), p_{i_{j_l}}), (\bar{B}_{r_{i_{j_l}}}^{X_\omega}(p_\omega), p_\omega)) = \varepsilon_{i_{j_l}}^{r_{i_{j_l}}} \leq \frac{1}{r_{i_{j_l}}} \to 0$$

as $l \to \infty$. Now Corollary A.27 proves $(X_{i_{j_l}}, p_{i_{j_l}}) \to (X_\omega, p_\omega)$ in the pointed Gromov-Hausdorff sense where $\omega(\{i_{j_l} \mid l \in \mathbb{N}\}) = 1$.

b) This proof can be done completely analogously to the one of Lemma A.47. □

Lemma A.55. *Let ω be an ultrafilter on $\mathbb{N}$ and for every $r > 0$ let $(\varepsilon_i^r)_{i\in\mathbb{N}}$ be a sequence such that $\lim_\omega \varepsilon_i^r = 0$. Then there exists a sequence $(r_i)_{i\in\mathbb{N}}$ of positive real numbers such that $\lim_\omega \frac{1}{r_i} = 0$ and $\omega(\{i \in \mathbb{N} \mid \varepsilon_i^{r_i} \leq \frac{1}{r_i}\}) = 1$.*

Proof. For $i \in \mathbb{N}$, let $R_i := \{r > 0 \mid \varepsilon_i^r \leq \frac{1}{r}\}$. The idea of this proof, similar to the one of Lemma A.25, is to find a sequence $r_i \in R_i$ with $r_i > i$ for a set of indices of ω-measure 1. Since the R_i need to be non-empty, let $I := \{i \in \mathbb{N} \mid R_i \neq \emptyset\}$. Due to $\lim_\omega \varepsilon_i^1 = 0$,

$$\omega(I) = \omega\big(\big\{i \in \mathbb{N} \mid \exists\, r > 0 : \varepsilon_i^r \leq \frac{1}{r}\big\}\big) \geq \omega(\{i \in \mathbb{N} \mid \varepsilon_i^1 \leq 1\}) = 1,$$

i.e. $\omega(I) = 1$. Let $J := \{i \in \mathbb{N} \mid \neg\exists\, C > 0 : R_i \subseteq [0, C]\}$ be the indices of the unbounded sets. In particular, $J \subseteq I$. In the following, the cases of $\omega(J) = 0$ and $\omega(J) = 1$ will be distinguished.

In advance, observe that for sets of indices of ω-measure 1 the corresponding R_i cannot have a uniform upper bound: Let $A \subseteq \mathbb{N}$ be any subset such that there exists $C > 0$ with $\bigcup_{i\in A} R_i \subseteq [0, C]$ and let $r > C$. Then $i \in A$ implies $r \notin R_i$, i.e. $\varepsilon_i^r > \frac{1}{r}$. Thus, $\omega(A) \leq \omega(\{i \in \mathbb{N} \mid \varepsilon_i^r > \frac{1}{r}\}) = 0$.

First, let $\omega(J) = 1$. For $i \in J$, choose $r_i \in R_i \cap (i, \infty)$. For $i \in \mathbb{N}\setminus J$, let $r_i := 1$. Then

$$\omega\Big(\Big\{i \in \mathbb{N} \mid \varepsilon_i^{r_i} \leq \frac{1}{r_i}\Big\}\Big) \geq \omega(\{i \in \mathbb{N} \mid r_i \in R_i\}) \geq \omega(J) = 1.$$

For arbitrary $\varepsilon > 0$, choose $N \in \mathbb{N}$ with $\frac{1}{N} \leq \varepsilon$. For $i \in J$ with $i \geq N$,

$$\frac{1}{r_i} < \frac{1}{i} \leq \frac{1}{N} \leq \varepsilon$$

and

$$\omega\Big(\Big\{i \in \mathbb{N} \mid \frac{1}{r_i} \leq \varepsilon\Big\}\Big) \geq \omega(J \cap [N, \infty)) = 1.$$

Thus, $\lim_\omega \frac{1}{r_i} = 0$ and r_i has the desired properties.

Now let $\omega(J) = 0$. For $i \in I \cap J^c$, let $s_i := \sup R_i$ denote the least upper bound of R_i and choose $r_i \in [\frac{s_i}{2}, s_i] \cap R_i$. For $i \in I^c \cup J$, let $s_i := r_i := 1$. Then

$$\omega\Big(\Big\{i \in \mathbb{N} \mid \varepsilon_i^{r_i} \leq \frac{1}{r_i}\Big\}\Big) \geq \omega(\{i \in \mathbb{N} \mid r_i \in R_i\}) \geq \omega(I \cap J^c) = 1.$$

Let $\varepsilon > 0$ and $K_\varepsilon := \{i \in I \cap J^c \mid \frac{1}{s_i} > \varepsilon\}$. Then

$$\bigcup_{i \in K_\varepsilon} R_i \subseteq \bigcup_{i \in K_\varepsilon} [0, s_i] \subseteq \Big[0, \frac{1}{\varepsilon}\Big],$$

and thus, by the above argumentation, $\omega(K_\varepsilon) = 0$. Then, using $\omega(I \cap J^c) = 1$,

$$\begin{aligned}
\omega\big(\big\{i \in \mathbb{N} \mid \frac{1}{s_i} \leq \varepsilon\big\}\big) &= 1 - \omega\big(\big\{i \in \mathbb{N} \mid \frac{1}{s_i} > \varepsilon\big\}\big) \\
&= 1 - \omega\big(\big\{i \in I \cap J^c \mid \frac{1}{s_i} > \varepsilon\big\}\big) \\
&= 1 - \omega(K_\varepsilon) = 1.
\end{aligned}$$

Hence, $\lim_\omega \frac{1}{s_i} = 0$ and $\frac{1}{r_i} \leq \frac{2}{s_i}$ this proves the claim. □

As for bounded sequences of real numbers, investigating sublimits coming from the same subsequence is the same as investigating ultralimits.

Lemma A.56. *Let (X_i, d_{X_i}, p_i) and (Y_i, d_{Y_i}, q_i), $i \in \mathbb{N}$, be pointed proper length spaces.*

a) Let ω be a non-principal ultrafilter on $\mathbb{N}$. Then there exists a subsequence $(i_j)_{j\in\mathbb{N}}$ such that both

$$(X_{i_j}, p_{i_j}) \to \lim_\omega (X_i, d_{X_i}, p_i) \quad \text{and} \quad (Y_{i_j}, q_{i_j}) \to \lim_\omega (Y_i, d_{Y_i}, q_i) \quad \text{as } j \to \infty$$

in the pointed Gromov-Hausdorff sense.

b) Let (X, d_X, p) and (Y, d_Y, q) be pointed length spaces and $(i_j)_{j\in\mathbb{N}}$ be a subsequence such that both

$$(X_{i_j}, p_{i_j}) \to (X, p) \quad \text{and} \quad (Y_{i_j}, q_{i_j}) \to (Y, q) \quad \text{as } j \to \infty$$

in the pointed Gromov-Hausdorff sense. Then there exists a non-principal ultrafilter ω on $\mathbb{N}$ such that there are isometries

$$\lim_\omega (X_i, d_{X_i}, p_i) \cong (X, p) \quad \text{and} \quad \lim_\omega (Y_i, d_{Y_i}, q_i) \cong (Y, q).$$

Proof. Using Proposition A.54, the proof can be done completely analogously to the one of Lemma A.48. □

A.4 Measured Gromov-Hausdorff convergence

Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a sequence of pointed complete connected n-dimensional Riemannian manifolds with lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$. If $\mathrm{vol}_{M_i}(B_1(p_i)) \to 0$ as $i \to \infty$, this sequence is said to be *collapsing*. Observe the following: If (M_i, p_i) were converging to a limit (X, p) and given any (non-trivial) measure on X, the volume of the $B_1(p_i)$ would not converge to the volume of $B_1(p)$.

This phenomenon does not occur when using renormalised limit measures, cf. [CC97, section 1]: Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a collapsing sequence as above. Then (M_i, p_i) subconverges to a metric space (X, p) such that a 'renormalisation' of the measures vol_{M_i} converges to a limit measure vol_X. For a more detailed explanation, cf. again [CC97, section 1]. Essentially, this construction is done as follows:

For the renormalised measure

$$\mu_i(\cdot) := \frac{\mathrm{vol}_{M_i}(\cdot)}{\mathrm{vol}_{M_i}(B_1(p_i))},$$

after passing to a subsequence, (M_i, p_i) is converging to a metric space (X, p) and there exists a continuous map $V : X \times \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$V(q, R) = \lim_{i\to\infty} \mu_i(B_r(q_i))$$

for any $q_i \to q$ and all $R > 0$. For $\delta > 0$ and $A \subseteq X$, let

$$\nu_\delta(A) := \inf\Big\{ \sum_{j\in\mathbb{N}} V(z_j, r_j) \mid r_j \leq \delta \text{ and } A \subseteq \bigcup_{j\in\mathbb{N}} B_{r_j}(z_j) \Big\} \quad \text{and}$$
$$\nu(A) := \lim_{\delta\to 0} \nu_\delta(A).$$

This ν defines an outer measure and can be extended to a unique Radon measure vol_X on X. This can be summarized by the following theorem.

Theorem A.57 ([CC97, Theorem 1.6, Theorem 1.10]). *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a sequence of pointed complete connected n-dimensional Riemannian manifolds satisfying the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \geq -(n-1)$. Then (M_i, p_i) subconverges to a metric space (X, p) in the pointed Gromov-Hausdorff sense and there exists a Radon measure vol_X on X such that for all $x \in X$, $x_i \to x$ and $r > 0$,*

$$\frac{\mathrm{vol}_{M_i}(B_r^{M_i}(x_i))}{\mathrm{vol}_{M_i}(B_1^{M_i}(p_i))} \to \mathrm{vol}_X(B_r^X(x)) \text{ as } i \to \infty.$$

Moreover, for any $R \geq r > 0$ and $x \in X$,

$$\frac{\mathrm{vol}_X(B_R^X(x))}{\mathrm{vol}_X(B_r^X(x))} \leq \frac{V_{-1}^n(R)}{V_{-1}^n(r)}$$

where $V_{-1}^n(r)$ denotes the volume of a ball with radius $r > 0$ in the n-dimensional hyperbolic space.

Observe that the limit measure of a sequence (M_i, p_i) depends on the choice of the base points and the considered subsequence, cf. again [CC97, section 1].

The following lemma states a (technical) condition on compact subsets $A_i \subseteq M_i$ and $A \subseteq X$ which guarantees convergence of the renormalised measures $\mu_i(A_i) \to \mathrm{vol}_X(A)$.

Lemma A.58. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a sequence of pointed complete connected n-dimensional Riemannian manifolds converging to a metric space (X, p) in the measured Gromov-Hausdorff sense, and let $\varepsilon_i \to 0$ and*

$$(f_i, g_i) \in Appr_{\varepsilon_i}((\bar{B}^{M_i}_{1/\varepsilon_i}(p_i), p_i), (\bar{B}^X_{1/\varepsilon_i}(p), p))$$

be as in Corollary A.27. For $x \in X$, let p_i^x denote $g_i(x)$.

Assume $A_i \subseteq M_i$ and $A \subseteq X$ to be compact subsets with $A_i \subseteq B_R(p_i)$ and $A \subseteq B_R(p)$ for some $R > 0$. Moreover, assume that there is $\delta_i \to 0$ such that for all $r_j > 0$, $x_j \in X$ and $p_i^j \in B^{M_i}_{\varepsilon_i}(p_i^{x_j})$, where $j \in \mathbb{N}$, the following is true:

a) If $A_i \subseteq \bigcup_{j\in\mathbb{N}} B^{M_i}_{r_j}(p_i^j)$, then $A \subseteq \bigcup_{j\in\mathbb{N}} B^X_{r_j+\delta_i}(x_j)$.

b) If $A \subseteq \bigcup_{j\in\mathbb{N}} B^X_{r_j}(x_j)$, then $A_i \subseteq \bigcup_{j\in\mathbb{N}} B^{M_i}_{r_j+\delta_i}(p_i^j)$.

Then $\mu_i(A_i) \to \mathrm{vol}_X(A)$.

Proof. Define

$$\mu_i^\delta(A_i) := \inf\Big\{\sum_{j\in\mathbb{N}} \mu_i(B_{r_j}(p_i^j)) \mid r_j \le \delta, p_i^j \in M_i \text{ and } A_i \subseteq \bigcup_{j\in\mathbb{N}} B_{r_j}(p_i^j)\Big\}.$$

For measurable sets U_i, $\mu_i^\delta(U_i) = \mu_i(U_i)$ for any $\delta > 0$. The corresponding statement is true for ν_δ and vol_X. Since the A_i and A are assumed to be compact, they are measurable and these identities can be used in the following.

Fix $\delta > 0$. Without loss of generality, assume $R + \delta \le \frac{1}{\varepsilon_i}$ and $\delta_i \le \delta$ for all $i \in \mathbb{N}$. By definition,

$$\begin{aligned}\nu(A) &= \nu_\delta(A)\\ &= \inf\Big\{\sum_{j\in\mathbb{N}} \lim_{i\to\infty} \mu_i(B_{r_j}(p_i^{x_j})) \mid 0 < r_j \le \delta, x_j \in X \text{ and } A \subseteq \bigcup_{j\in\mathbb{N}} B_{r_j}(x_j)\Big\}.\end{aligned}$$

For such $r_j < \delta$ and $x_j \in X$, observe $A \cap B_{r_j}(x_j) \subseteq B_R(p) \cap B_\delta(x_j) = \emptyset$ if $d(p, x_j) \ge R + \delta$. Hence,

$$\nu(A) = \inf\Big\{\sum_{j\in\mathbb{N}} \lim_{i\to\infty} \mu_i(B_{r_j}(p_i^{x_j})) \mid 0 < r_j \le \delta, x_j \in B_{R+\delta}(p) \text{ and } A \subseteq \bigcup_{j\in\mathbb{N}} B_{r_j}(x_j)\Big\}.$$

Recall that $\lim_{i\to\infty}\mu_i(B_{r_j}(p_i^{x_j})) = \lim_{i\to\infty}\mu_i(B_{r_j}(p_i^j))$ if $p_i^i \to x_j$. In particular, this is true for all $p_i^j \in B_{\varepsilon_i}(p_i^{x_j})$. Therefore,

$$\nu(A) = \inf\big\{\sum_{j\in\mathbb{N}}\lim_{i\to\infty}\mu_i(B_{r_j}(p_i^j)) \mid 0 < r_j \le \delta, x_j \in B_{R+\delta}(p), p_i^j \in B_{\varepsilon_i}(p_i^{x_j}) \text{ and } A \subseteq \bigcup_{j\in\mathbb{N}} B_{r_j}(x_j)\big\}.$$

By the Bishop-Gromov Theorem, $1 \ge \frac{\mu_i(B_{r_j}(p_i^j))}{\mu_i(B_{r_j+\delta_i}(p_i^j))} \ge \frac{V_{-1}^n(r_j)}{V_{-1}^n(r_j+\delta_i)} \to 1$ as $i \to \infty$. Thus,

$$\begin{aligned}
\nu(A) &= \inf\big\{\sum_{j\in\mathbb{N}}\lim_{i\to\infty}\mu_i(B_{r_j+\delta_i}(p_i^j)) \mid 0 < r_j \le \delta, p_i^j \in B_{\varepsilon_i}(g_i(B_{R+\delta}(p))) \text{ and } A \subseteq \bigcup_{j\in\mathbb{N}} B_{r_j}(x_j)\big\}\\
&= \inf\big\{\sum_{j\in\mathbb{N}}\lim_{i\to\infty}\mu_i(B_{s_j}(p_i^j)) \mid 0 < s_j - \delta_i \le \delta, p_i^j \in B_{\varepsilon_i}(g_i(B_{R+\delta}(p))) \text{ and } A \subseteq \bigcup_{j\in\mathbb{N}} B_{s_j-\delta_i}(x_j)\big\}\\
&\ge \lim_{i\to\infty} \inf\big\{\sum_{j\in\mathbb{N}}\mu_i(B_{s_j}(p_i^j)) \mid 0 < s_j \le 2\delta, p_i^j \in M_i \text{ and } A_i \subseteq \bigcup_{j\in\mathbb{N}} B_{s_j}(p_i^j)\big\}\\
&\ge \lim_{i\to\infty} \mu_i^{2\delta}(A_i)\\
&= \lim_{i\to\infty} \mu_i(A_i).
\end{aligned}$$

Conversely,

$$\begin{aligned}
&\lim_{i\to\infty}\mu_i(A_i) = \lim_{i\to\infty}\mu_i^{\delta}(A_i)\\
&= \inf\big\{\sum_{j\in\mathbb{N}}\lim_{i\to\infty}\mu_i(B_{r_j}(p_i^j)) \mid 0 < r_j \le \delta, p_i^j \in M_i \text{ and } A_i \subseteq \bigcup_{j\in\mathbb{N}} B_{r_j}(p_i^j)\big\}\\
&= \inf\big\{\sum_{j\in\mathbb{N}}\lim_{i\to\infty}\mu_i(B_{r_j}(p_i^j)) \mid 0 < r_j \le \delta, p_i^j \in B_{R+\delta}(p_i) \text{ and } A_i \subseteq \bigcup_{j\in\mathbb{N}} B_{r_j}(p_i^j)\big\}.
\end{aligned}$$

For $p_i^j \in B_{R+\delta}(p_i)$ and $x_j = f_i(p_i^j) \in B_{R+\delta+\varepsilon_i}(p)$, note $p_i^j \in B_{\varepsilon_i}(p_i^{x_j})$. Thus, using again the Bishop-Gromov Theorem,

$$\begin{aligned}
&\lim_{i\to\infty}\mu_i(A_i)\\
&\ge \inf\big\{\sum_{j\in\mathbb{N}}\lim_{i\to\infty}\mu_i(B_{r_j+\delta_i}(p_i^j)) \mid 0 < r_j \le \delta, x_j \in B_{R+\delta+\varepsilon_i}(p), p_i^j \in B_{\varepsilon_i}(p_i^{x_j}) \text{ and } A_i \subseteq \bigcup_{j\in\mathbb{N}} B_{r_j}(p_i^j)\big\}
\end{aligned}$$

$$\geq \inf\Big\{\sum_{j\in\mathbb{N}} \lim_{i\to\infty} \mu_i(B_{s_j}(p_i^j)) \mid 0 < s_j \leq 2\delta, x_j \in X, p_i^j \in B_{\varepsilon_i}(p_i^{x_j}) \text{ and } A \subseteq \bigcup_{j\in\mathbb{N}} B_{s_j}(x_j)\Big\}$$
$$= \inf\Big\{\sum_{j\in\mathbb{N}} \lim_{i\to\infty} \mu_i(B_{s_j}(p_i^{x_j})) \mid 0 < s_j \leq 2\delta, x_j \in X \text{ and } A \subseteq \bigcup_{j\in\mathbb{N}} B_{s_j}(x_j)\Big\}$$
$$= \nu_{2\delta}(A)$$
$$= \mathrm{vol}_X(A).$$

This proves $\lim_{i\to\infty} \mu_i(A_i) = \mathrm{vol}_X(A)$. □

The next lemma provides a condition for sets such that these satisfy the hypothesis of Lemma A.58. This lemma will be used afterwards to prove volume convergence in several situations.

Lemma A.59. *Let (X, d_X, p) and (X_i, d_{X_i}, p_i), $i \in \mathbb{N}$, be pointed length spaces. Assume $(X_i, p_i) \to (X, p)$ and let $\varepsilon_i \to 0$ and $(f_i, g_i) \in Appr_{\varepsilon_i}((\bar{B}^{X_i}_{1/\varepsilon_i}(p_i), p_i), (\bar{B}^{X}_{1/\varepsilon_i}(p), p))$ be as in Corollary A.27. For $x \in X$, let p_i^x denote $g_i(x)$. Moreover, let $A_i \subseteq B^{X_i}_{1/\varepsilon_i}(p_i)$, $A \subseteq X$, $f'_i : A_i \to A$, $g'_i : A \to A_i$ and $\delta_i \to 0$ be such that*

$$d_X(f'_i(x_i), f_i(x_i)) \leq \delta_i \quad \text{and} \quad d_{X_i}(g'_i(x), g_i(x)) \leq \delta_i$$

for all $x_i \in A_i$ and $x \in A$. Let $r_j > 0$, $x_j \in X$ and $p_i^j \in B^{X_i}_{\varepsilon_i}(p_i^{x_j})$, where $j \in \mathbb{N}$.

a) If $A_i \subseteq \bigcup_{j\in\mathbb{N}} B^{X_i}_{r_j}(p_i^j)$, then $A \subseteq \bigcup_{j\in\mathbb{N}} B^{X}_{r_j+3\,\varepsilon_i+\delta_i}(x_j)$.

b) If $A \subseteq \bigcup_{j\in\mathbb{N}} B^{X}_{r_j}(x_j)$, then $A_i \subseteq \bigcup_{j\in\mathbb{N}} B^{X_i}_{r_j+3\,\varepsilon_i+\delta_i}(p_i^j)$.

Proof. First, assume $A_i \subseteq \bigcup_{j\in\mathbb{N}} B^{X_i}_{r_j}(p_i^j)$ and let $y \in A$ be arbitrary. Then $g'_i(y) \in A_i$. Therefore, there exists $j \in \mathbb{N}$ such that $g'_i(y) \in B^{X_i}_{r_j}(p_i^j) \subseteq B^{X_i}_{r_j+\varepsilon_i}(p_i^{x_j})$. Hence,

$$\begin{aligned} d_X(y, x_j) &< \varepsilon_i + d_{X_i}(g_i(y), g_i(x_j)) \\ &\leq \varepsilon_i + d_{X_i}(g_i(y), g'_i(y)) + d_{X_i}(g'_i(y), p_i^{x_j}) \\ &< \varepsilon_i + \delta_i + (r_j + \varepsilon_i), \end{aligned}$$

and this proves $A \subseteq \bigcup_{j\in\mathbb{N}} B^{X}_{r_j+2\,\varepsilon_i+\delta_i}(x_j) \subseteq \bigcup_{j\in\mathbb{N}} B^{X}_{r_j+3\,\varepsilon_i+\delta_i}(x_j)$. Conversely, assume $A \subseteq \bigcup_{j\in\mathbb{N}} B^{X}_{r_j}(x_j)$ and let $y_i \in A_i$ be arbitrary. Then $f'_i(y_i) \in B^{X}_{r_j}(x_j)$ for some $j \in \mathbb{N}$, and this implies

$$\begin{aligned} d_{X_i}(y_i, p_i^{x_j}) &< \varepsilon_i + d_X(f_i(y_i), f'_i(y_i)) + d_X(f'_i(y_i), x_j) + d_X(x_j, f_i(p_i^{x_j})) \\ &< \varepsilon_i + \delta_i + r_j + \varepsilon_i\,. \end{aligned}$$

Therefore, $A \subseteq \bigcup_{j\in\mathbb{N}} B^{X_i}_{r_j+2\,\varepsilon_i+\delta_i}(p_i^{x_j}) \subseteq \bigcup_{j\in\mathbb{N}} B^{X_i}_{r_j+3\,\varepsilon_i+\delta_i}(p_i^j)$. □

Proposition A.60. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a sequence of pointed complete connected n-dimensional Riemannian manifolds converging to a metric space (X, p) in the pointed Gromov-Hausdorff sense, and let $\varepsilon_i \to 0$ and*

$$(f_i, g_i) \in Appr_{\varepsilon_i}((\bar{B}^{M_i}_{1/\varepsilon_i}(p_i), p_i), (\bar{B}^X_{1/\varepsilon_i}(p), p))$$

be as in Corollary A.27. For $x \in X$, let p_i^x denote $g_i(x)$.

a) Let $A_i \subseteq M_i$ be compact with $A_i \subseteq \bar{B}_R(p_i)$ for some $R > 0$. After passing to a subsequence, there exists $A \subseteq X$ with $A_i \to A$ and $\mu_i(A_i) \to \mathrm{vol}_X(A)$.

b) Let $0 < R < 1$, $x_1, \dots, x_l \in B^X_{1-R}(p)$,

$$A_i := B_1^{M_i}(p_i) \setminus \bigcup_{j=1}^{l} B_R^{M_i}(p_i^{x_j}) \quad \text{and} \quad A := B_1^X(p) \setminus \bigcup_{j=1}^{l} B_R^X(x_j).$$

Then $\mu_i(A_i) \to \mathrm{vol}_X(A)$.

Proof. a) Pass to the subsequence of Proposition A.41. Then there exists $A \subseteq \bar{B}_R(p)$ such that $A_i \to A$. Moreover, by the proof of Proposition A.41, the hypothesis of Lemma A.59 is satisfied. Hence, Lemma A.58 implies the claim.

b) First, let $m \in \mathbb{N}$ as well as $r_k > 0$ and $y_k \in X$, where $1 \le k \le m$, be arbitrary and prove

$$\mu_i\Big(\bigcup_{k=1}^{m} B^{M_i}_{r_k}(p_i^{y_k})\Big) \to \mathrm{vol}_X\Big(\bigcup_{k=1}^{m} B^X_{r_k}(y_k)\Big).$$

This immediately implies the claim.

For every pair of points $y, z \in X$, fix a shortest geodesic c_{yz} (parametrised by arc length) connecting y with z. For $1 \le k \le m$, define $\mathrm{pr}^k : X \to \bar{B}^X_{r_k}(y_k)$ by

$$\mathrm{pr}^k(z) := \begin{cases} z & \text{if } d_X(z, y_k) \le r_k, \\ c_{y_k z}(r) & \text{if } d_X(z, y_k) > r_k. \end{cases}$$

Analogously, define $\mathrm{pr}_i^k : M_i \to \bar{B}^{M_i}_{r_k}(p_i^{y_k})$.

Let

$$K_i := \bigcup_{k=1}^{m} \bar{B}^{M_i}_{r_k}(p_i^{y_k}) \quad \text{and} \quad K := \bigcup_{k=1}^{m} \bar{B}^X_{r_k}(y_k)$$

and define maps $f_i' : K_i \to K$ and $g_i' : K \to K_i$ by

$$f_i'(z_i) := \mathrm{pr}^k \circ f_i(z_i) \text{ for the minimal } 1 \le k \le m \text{ with } z_i \in \bar{B}^{M_i}_{r_k}(p_i^{y_k}),$$
$$g_i'(z) := \mathrm{pr}_i^k \circ g_i(z) \text{ for the minimal } 1 \le k \le m \text{ with } z \in \bar{B}^X_{r_k}(y_k).$$

Let $z_i \in K_i$ be arbitrary and $1 \le k \le m$ be minimal with $z_i \in \bar{B}^{M_i}_{r_k}(p_i^{x_k})$. Then $f_i(z_i) \in \bar{B}^X_{r_k+\varepsilon_i}(f_i(p_i^{y_k})) \subseteq B^X_{r_k+2\,\varepsilon_i}(y_k)$ and

$$d_X(f_i'(z_i), f_i(z_i)) = d_X(\mathrm{pr}^k(f_i(z_i)), f_i(z_i)) \le 2\,\varepsilon_i\,.$$

Similarly,

$$d_{M_i}(g_i'(z), g_i(z)) \le \varepsilon_i$$

for all $z \in K$.

By Lemma A.59 and Lemma A.58,

$$\mu_i\big(\bigcup_{k=1}^m \bar{B}^{M_i}_{r_k}(p_i^{y_k})\big) \to \mathrm{vol}_X\,\big(\bigcup_{k=1}^m \bar{B}^X_{r_k}(y_k)\big)$$

Now

$$\begin{aligned}\mathrm{vol}_X\,\big(\bigcup_{k=1}^m \bar{B}^X_{r_k}(y_k) \setminus \bigcup_{k=1}^m B^X_{r_k}(y_k)\big) &\le \mathrm{vol}_X\,\big(\bigcup_{k=1}^m (\bar{B}^X_{r_k}(y_k) \setminus B^X_{r_k}(y_k))\big)\\ &\le \sum_{k=1}^m \mathrm{vol}_X(\bar{B}^X_{r_k}(y_k) \setminus B^X_{r_k}(y_k)) = 0\end{aligned}$$

implies

$$\mu_i\big(\bigcup_{k=1}^m B^{M_i}_{r_k}(p_i^{y_k})\big) \to \mathrm{vol}_X\,\big(\bigcup_{k=1}^m B^X_{r_k}(y_k)\big).$$

□

Proposition A.61. *Let $(M_i, p_i)_{i\in\mathbb{N}}$ be a sequence of pointed complete connected n-dimensional Riemannian manifolds which satisfy the uniform lower Ricci curvature bound $\mathrm{Ric}_{M_i} \ge -(n-1)$, let $x_i, y_i \in M_i$ with $d_{M_i}(x_i, y_i) \le 2$ and assume (M_i, x_i) and (M_i, y_i), respectively, to converge to metric spaces (X, x_∞) and (Y, y_∞), respectively. Moreover, let $r > 0$, $K_i \subseteq \bar{B}^{M_i}_r(x_i)$ be compact, $\alpha > 0$ and $f_i : K_i \to M_i$ be α-bi-Lipschitz and measure preserving with $f_i(x_i) = y_i$.*

After passing to a subsequence, there exist a compact subset $K \subseteq \bar{B}^X_r(x_\infty)$, an α-bi-Lipschitz map $f : K \to Y$ and a constant $C > 0$ such that $\mathrm{vol}_Y(f(A)) = C \cdot \mathrm{vol}_X(A)$ for any measurable subset $A \subseteq K$.

Proof. Pass to the subsequence of Lemma A.42. By Lemma A.42 a), there exist compact subsets $K \subseteq X$ and $K' \subseteq Y$ which are Gromov-Hausdorff limits of K_i and $f_i(K_i)$, respectively, and an α-bi-Lipschitz map $f : K \to K'$ with $f(K) = K'$.

By the Bishop-Gromov Theorem,

$$\frac{\mathrm{vol}_{M_i}(B^{M_i}_1(x_i))}{\mathrm{vol}_{M_i}(B^{M_i}_1(y_i))} \le \frac{\mathrm{vol}_{M_i}(B^{M_i}_3(y_i))}{\mathrm{vol}_{M_i}(B^{M_i}_1(y_i))} \le \frac{V^n_{-1}(3)}{V^n_{-1}(1)} =: c(n).$$

By an analogous argument,

$$\frac{1}{c(n)} \leq \frac{\mathrm{vol}_{M_i}(B_1^{M_i}(x_i))}{\mathrm{vol}_{M_i}(B_1^{M_i}(y_i))} \leq c(n)$$

for all $i \in \mathbb{N}$. After passing to a further subsequence, there exists $\frac{1}{c(n)} \leq C \leq c(n)$ with

$$\frac{\mathrm{vol}_{M_i}(B_1^{M_i}(x_i))}{\mathrm{vol}_{M_i}(B_1^{M_i}(y_i))} \to C.$$

In particular, $C > 0$.

Now let $L \subseteq K$ be compact, hence, measurable. By Lemma A.42 b) (and its proof), there exist compact subsets $L_i \subseteq K_i$ such that $L_i \to L$ and $f_i(L_i) \to f(L)$ satisfying the hypothesis of Lemma A.59. By Lemma A.58,

$$\begin{aligned} \mathrm{vol}_Y(f(L)) &= \lim_{i\to\infty} \frac{\mathrm{vol}_{M_i}(f_i(L_i))}{\mathrm{vol}_{M_i}(B_1^{M_i}(y_i))} \\ &= \lim_{i\to\infty} \frac{\mathrm{vol}_{M_i}(L_i)}{\mathrm{vol}_{M_i}(B_1^{M_i}(x_i))} \cdot \frac{\mathrm{vol}_{M_i}(B_1^{M_i}(x_i))}{\mathrm{vol}_{M_i}(B_1^{M_i}(y_i))} = C \cdot \mathrm{vol}_X(L). \end{aligned}$$

Finally, let $A \subseteq K$ be any measurable set. Since vol_X and vol_Y are Radon measures,

$$\begin{aligned} \mathrm{vol}_Y(f(A)) &= \sup\{\mathrm{vol}_Y(\tilde{L}) \mid \tilde{L} \subseteq f(A) \text{ is compact}\} \\ &= \sup\{\mathrm{vol}_Y(f(L) \mid L \subseteq A \text{ is compact}\} \\ &= \sup\{C \cdot \mathrm{vol}_X(L) \mid L \subseteq A \text{ is compact}\} \\ &= C \cdot \mathrm{vol}_X(A). \end{aligned}$$

□

Appendix B

Rescaling of metrics

Let (M, g) be an n-dimensional Riemannian manifold. For $\alpha > 0$, let αM denote the Riemannian manifold $(M, \alpha^2 g)$, i.e. all distances are rescaled by the factor α. This chapter states how notions connected to Riemannian geometry change when the metric is rescaled, starting with proving that distances change the claimed way.

From now on, let $\tilde{g} := \alpha^2 g$ denote the rescaled metric.

Definition B.1. For a Riemannian manifold M, a measurable subset $U \subseteq M$ and an integrable function $f : U \to \mathbb{R}$ let

$$\fint_U f \, dV_M := \frac{1}{\mathrm{vol}_M(U)} \cdot \int_U f \, dV_M$$

denote the *average integral*.

Lemma B.2. *Let $p \in M$ be arbitrary.*

a) For the distance function, $d_{\tilde{g}} = \alpha \cdot d_g$.

b) For the volume form, $dV_{\tilde{g}} = \alpha^n \cdot dV_g$.

c) For balls, $B_{\alpha r}^{\alpha M}(p) = B_r^M(p)$ and $\mathrm{vol}_{\alpha M}(B_{\alpha r}^{\alpha M}(p)) = \alpha^n \cdot \mathrm{vol}_M(B_r^M(p))$, where $r > 0$.

d) For the average integral, $\fint_{B_{\alpha r}^{\alpha M}(p)} f \, dV_{\alpha M} = \fint_{B_r^M(p)} f \, dV_M$ where $f : B_r^M(p) \to \mathbb{R}$ is integrable.

Proof. a) Let $p, q \in M$ and $c : [0, 1] \to M$ be a piecewise differentiable curve with starting point $c(0) = p$ and end point $c(1) = q$. Then, denoting by $L_g(c)$ the length of c with respect to the metric g,

$$\begin{aligned} L_{\tilde{g}}(c) &= \int_0^1 \sqrt{\tilde{g}_{c(t)}(\dot{c}(t), \dot{c}(t))} \, dt \\ &= \alpha \cdot \int_0^1 \sqrt{g_{c(t)}(\dot{c}(t), \dot{c}(t))} \, dt = \alpha \cdot L_g(c). \end{aligned}$$

Thus,

$$\begin{aligned} d_{\tilde{g}}(p,q) &= \inf\{L_{\tilde{g}}(c) \mid c \text{ piecewise differentiable curve from } p \text{ to } q\} \\ &= \alpha \cdot \inf\{L_g(c) \mid c \text{ piecewise differentiable curve from } p \text{ to } q\} \\ &= \alpha \cdot d_g(p,q). \end{aligned}$$

b) Recall that dV_g is the unique n-form on M which has value 1 for every oriented orthonormal basis. Observe the following correspondence between orthonormal basis with respect to the two Riemannian metrics g and $\tilde{g}$: Let $p \in M$ be arbitrary and $\{e_i\}_{1\le i\le n}$ be an orthonormal basis of $(T_pM, \tilde{g}_p)$. Then

$$\delta_{ij} = \tilde{g}(e_i, e_j) = g(\alpha e_i, \alpha e_j),$$

i.e. $\{\alpha e_i\}_{1\le i\le n}$ is an orthonormal basis with respect to g_p. Reversely, $\{\alpha^{-1} e_i\}_{1\le i\le n}$ is an orthonormal basis with respect to $\tilde{g}_p$ for any orthonormal basis $\{e_i\}_{1\le i\le n}$ of (T_pM, g_p).

Obviously, $\omega(\alpha e_1, \ldots, \alpha e_n) = \alpha^n \cdot \omega(e_1, \ldots, e_n)$ for any n-form ω. This proves the claim.

c) Using a), for $r > 0$,

$$\begin{aligned} B_r^M(p) &= \{q \in M \mid d_g(p,q) < r\} \\ &= \{q \in M \mid d_{\tilde{g}}(p,q) = \alpha \cdot d_g(p,q) < \alpha \cdot r\} \\ &= B_{\alpha r}^{\alpha M}(p). \end{aligned}$$

Hence, using b),

$$\mathrm{vol}_{\alpha M}(B_{\alpha r}^{\alpha M}(p)) = \alpha^n \cdot \mathrm{vol}_M(B_r^M(p)).$$

d) Let $f : B_r^M(p) \to \mathbb{R}$ be integrable. Then

$$\begin{aligned} ⨍_{B_{\alpha r}^{\alpha M}(p)} f \, dV_{\alpha M} &= \frac{1}{\mathrm{vol}_{\alpha M}(B_{\alpha r}^{\alpha M}(p))} \cdot \int_{B_{\alpha r}^{\alpha M}(p)} f \, dV_{\alpha M} \\ &= \frac{1}{\alpha^n \cdot \mathrm{vol}_M(B_r^M(p))} \cdot \alpha^n \cdot \int_{B_r^M(p)} f \, dV_M \\ &= ⨍_{B_r^M(p)} f \, dV_M \, . \end{aligned}$$

□

The next lemma states some facts about concepts related to curvature.

Lemma B.3. *a) For the Levi-Civita connection,* $\nabla^{\tilde{g}}_X Y = \nabla^g_X Y$.

b) For the Riemann curvature (3,1)-tensor, $R_{\alpha M} = R_M$.

c) For the sectional curvature, $K_{\alpha M} = \frac{1}{\alpha^2} \cdot K_M$.

d) For the Ricci curvature (2,0)-tensor, $\mathrm{Ric}_{\alpha M} = \mathrm{Ric}_M$. *Moreover,* $\mathrm{Ric}_{\alpha M} \geq \frac{\kappa}{\alpha^2}$ *if and only if* $\mathrm{Ric}_M \geq \kappa$.

Proof. a) The Levi-Civita connection is uniquely determined by the Koszul formula: For arbitrary vector fields X, Y and Z on M,

$$\begin{aligned}
2g(\alpha^2 \cdot \nabla^{\tilde{g}}_X Y, Z) &= 2\tilde{g}(\nabla^{\tilde{g}}_X Y, Z) \\
&= X\tilde{g}(Y, Z) + Y\tilde{g}(X, Z) - Z\tilde{g}(X, Y) \\
&\quad + \tilde{g}([X, Y], Z) - \tilde{g}([Y, Z], X) + \tilde{g}([Z, X], Y) \\
&= \alpha^2 \cdot \big(Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\
&\quad + g([X, Y], Z) - g([Y, Z], X) + g([Z, X], Y)\big) \\
&= \alpha^2 \cdot 2g(\nabla^g_X Y, Z) \\
&= 2g(\alpha^2 \cdot \nabla^g_X Y, Z).
\end{aligned}$$

b) Let X, Y and Z be vector fields on M. Then

$$\begin{aligned}
R_{\alpha M}(X, Y)Z &= \nabla^{\tilde{g}}_X \nabla^{\tilde{g}}_Y Z - \nabla^{\tilde{g}}_Y \nabla^{\tilde{g}}_X Z - \nabla^{\tilde{g}}_{[X,Y]} Z \\
&= \nabla^g_X \nabla^g_Y Z - \nabla^g_Y \nabla^g_X Z - \nabla^g_{[X,Y]} Z \\
&= R_M(X, Y)Z.
\end{aligned}$$

c) Let $p \in M$ and $v, w \in T_pM$. Then

$$\begin{aligned}
K_{\alpha M}(v, w) &= \frac{\tilde{g}(R_{\alpha M}(v, w)w, v)}{\tilde{g}(v, v) \cdot \tilde{g}(w, w) - \tilde{g}(v, w)^2} \\
&= \frac{\alpha^2 g(R(v, w)w, v)}{\alpha^2 g(v, v) \cdot \alpha^2 g(w, w) - (\alpha^2 g(v, w))^2} \\
&= \frac{1}{\alpha^2} \cdot K_M(v, w).
\end{aligned}$$

d) For arbitrary $p \in M$, $\{e_i\}_{1 \leq i \leq n}$ is an orthonormal basis of $(T_pM, \tilde{g}_p)$ if and only if $\{\alpha e_i\}_{1 \leq i \leq n}$ is an orthonormal basis with respect to g_p. Hence, for $v, w \in T_pM$,

$$\begin{aligned}
\mathrm{Ric}_{\alpha M}(v, w) &= \sum_{i=1}^n \tilde{g}(R_{\alpha M}(v, e_i)e_i, w) \\
&= \sum_{i=1}^n g(R(v, \alpha e_i)\alpha e_i, w) \\
&= \mathrm{Ric}_M(v, w).
\end{aligned}$$

Now let $\mathrm{Ric}_M \geq \kappa$, i.e. for any $v \in T_pM$,

$$\begin{aligned}\mathrm{Ric}_{\alpha M}(v,v) &= \mathrm{Ric}_M(v,v) \\ &\geq \kappa \cdot g(v,v) \\ &= \frac{\kappa}{\alpha^2} \cdot \tilde{g}(v,v).\end{aligned}$$ □

Recall that the operator norm of a linear operator $A_p : T_pM \to T_pM$ is defined by

$$\|A_p\|_{g_p} := \sup\{\|A_p \cdot w\|_{g_p} \mid w \in T_pM, \|w\|_{g_p} = 1\}.$$

This is independent under rescaling.

Lemma B.4. *For a linear operator $A_p : T_pM \to T_pM$, $\|A\|_{\tilde{g}} = \|A\|_g$.*

Proof. This is a direct computation:

$$\begin{aligned}\|A_p\|_{\tilde{g}_p} &= \sup\{\|A_p \cdot v\|_{\tilde{g}_p} \mid \|v\|_{\tilde{g}_p} = 1\} \\ &= \sup\{\|A_p \cdot \alpha v\|_{g_p} \mid \|\alpha v\|_{g_p} = 1\} \\ &= \sup\{\|A_p \cdot w\|_{g_p} \mid \|w\|_{g_p} = 1\} \\ &= \|A_p\|_{g_p}.\end{aligned}$$ □

The following lemma deals with the rescaling of smooth functions and their differentials.

Lemma B.5. *Let $f, f_i : M \to \mathbb{R}$ be smooth, $\tilde{f} := \alpha \cdot f$ and $\tilde{f}_i := \alpha \cdot f_i$, respectively.*

a) For the gradient, $\nabla^{\tilde{g}} \tilde{f} = \frac{1}{\alpha} \nabla^g f$ and $\tilde{g}(\nabla^{\tilde{g}} \tilde{f}_i, \nabla^{\tilde{g}} \tilde{f}_j) = g(\nabla^g f_i, \nabla^g f_j)$.

b) For the Hessian, $\mathrm{Hess}_{\tilde{g}}(\tilde{f}) = \frac{1}{\alpha} \cdot \mathrm{Hess}_g(f)$ and $\|\mathrm{Hess}_{\tilde{g}}(\tilde{f})\|_{\tilde{g}} = \frac{1}{\alpha} \cdot \|\mathrm{Hess}_g(f)\|_g$.

c) For the Laplacian, $\Delta_{\tilde{g}} \tilde{f} = \frac{1}{\alpha} \cdot \Delta_g f$. In particular, $\tilde{f}$ is harmonic if and only if f is harmonic.

d) Furthermore, $\tilde{f}$ is L-Lipschitz if and only if f is L-Lipschitz, where $L > 0$.

Proof. a) The gradient vector field $\nabla^g f$ is uniquely determined by

$$df_p(v) = g_p((\nabla^g f)_p, v) \text{ for all } p \in M \text{ and } v \in T_pM.$$

Thus, for $p \in M, v \in T_pM$,

$$\tilde{g}_p((\nabla^{\tilde{g}} \tilde{f})_p, v) = d\tilde{f}_p(v) = \alpha \cdot df_p(v) = \frac{1}{\alpha} \cdot \tilde{g}_p((\nabla^g f)_p, v).$$

So, $\nabla^{\tilde{g}} \tilde{f} = \frac{1}{\alpha} \nabla^g f$. Hence,

$$\tilde{g}(\nabla^{\tilde{g}} \tilde{f}_i, \nabla^{\tilde{g}} \tilde{f}_j) = \alpha^2 \cdot g\big(\frac{1}{\alpha} \nabla^g f_i, \frac{1}{\alpha} \nabla^g f_j\big) = g(\nabla^g f_i, \nabla^g f_j).$$

b) Since $\operatorname{Hess}_g(f)_p : T_pM \to T_pM$ is defined by $\operatorname{Hess}_g(f)_p(Y_p) = (\nabla^g_Y(\nabla^g f))(p)$,

$$(\nabla^{\tilde{g}}_Y(\nabla^{\tilde{g}}\tilde{f}))(p) = (\nabla^g_Y(\frac{1}{\alpha}\nabla^g f))(p) = \frac{1}{\alpha}(\nabla^g_Y(\nabla^g f))(p)$$

gives

$$\operatorname{Hess}_{\tilde{g}}(\tilde{f}) = \frac{1}{\alpha} \cdot \operatorname{Hess}_g(f).$$

Hence, using the previous lemma,

$$\| \operatorname{Hess}_{\tilde{g}}(\tilde{f})\|_{\tilde{g}} = \left\|\frac{1}{\alpha} \cdot \operatorname{Hess}_g(f)\right\|_g = \frac{1}{\alpha} \cdot \| \operatorname{Hess}_g(f)\|_g.$$

c) By definition of the Laplacian, for a $\tilde{g}_p$-orthonormal basis $(\tilde{e}_i)_{1\leq i\leq n}$ of T_pM, where $p \in M$, and the g_p-orthonormal basis given by $e_i := \alpha \cdot \tilde{e}_i$,

$$\begin{aligned}
(\Delta_{\tilde{g}}\tilde{f})_p &= \operatorname{tr}(\operatorname{Hess}_{\tilde{g}}(\tilde{f})_p) \\
&= \sum_{i=1}^{n} \tilde{g}_p(\operatorname{Hess}_{\tilde{g}}(\tilde{f})_p \cdot \tilde{e}_i, \tilde{e}_i) \\
&= \sum_{i=1}^{n} \alpha^2 \cdot g_p\Big(\frac{1}{\alpha}\operatorname{Hess}_g(f)_p \cdot \frac{1}{\alpha}e_i, \frac{1}{\alpha}e_i\Big) \\
&= \frac{1}{\alpha} \cdot (\Delta_g f)_p.
\end{aligned}$$

d) Let $L > 0$. Then f is L-Lipschitz with respect to d if and only if for any $p, q \in M$,

$$|f(p) - f(q)| \leq L \cdot d(p, q),$$

which is equivalent to

$$|\tilde{f}(p) - \tilde{f}(q)| = \alpha \cdot |f(p) - f(q)| \leq \alpha \cdot L \cdot d(p, q) = L \cdot \tilde{d}(p, q).$$

This again is true if and only if $\tilde{f}$ is L-Lipschitz with respect to $\tilde{d}$. □

Corollary B.6. *For smooth functions $f_i : M \to \mathbb{R}$, $1 \leq i \leq k$, let $\tilde{f}_i := \alpha \cdot f_i$. Define $f := (f_i)_{1\leq i\leq k}$ and $\tilde{f} = (\tilde{f}_i)_{1\leq i\leq k} = \alpha f$. Furthermore, define*

$$\psi^g_\nabla(f) := \sum_{i,j=1}^{k} |g(\nabla^g f_i, \nabla^g f_j) - \delta_{ij}| \quad \text{and} \quad \psi^g_H(f) := \sum_{i=1}^{k} \| \operatorname{Hess}_g(f_i)\|^2_g.$$

Then, using the analogous definitions for $\tilde{g}$,

$$\psi^{\tilde{g}}_\nabla(\tilde{f}) = \psi^g_\nabla(f) \quad \text{and} \quad \psi^{\tilde{g}}_H(\tilde{f}) = \frac{1}{\alpha^2} \cdot \psi^g_H(f).$$

Proof. These are direct computations:

$$\psi_{\nabla}^{\tilde{g}}(\tilde{f}) = \sum_{i,j=1}^{k} |\tilde{g}(\nabla^{\tilde{g}} \tilde{f}_i, \nabla^{\tilde{g}} \tilde{f}_j) - \delta_{ij}| = \sum_{i,j=1}^{k} |g(\nabla^{g} f_i, \nabla^{g} f_j) - \delta_{ij}| = \psi_{\nabla}^{g}(f)$$

and

$$\psi_{H}^{\tilde{g}}(\tilde{f}) = \sum_{i=1}^{k} \| \operatorname{Hess}_{\tilde{g}}(\tilde{f})\|_{\tilde{g}}^2 = \frac{1}{\alpha^2} \cdot \sum_{i=1}^{k} \| \operatorname{Hess}_{g}(f)\|_{g}^2 = \frac{1}{\alpha^2} \cdot \psi_{H}^{g}(f). \qquad \square$$

Appendix C

Bishop-Gromov volume comparison

Clearly, the volume of a ball at a given point with smaller radius has always as most the volume of a ball with larger radius at the same point. On manifolds satisfying a lower Ricci curvature bound, the following theorem allows an estimate in the other direction. This estimate is independent of the regarded manifold and depends only on the dimension and the lower Ricci curvature bound of the manifold and the radii.

Theorem C.1 (Bishop-Gromov Theorem, [Pet06, Chapter 9, Lemma 1.6])**.** *Let M be a complete n-dimensional Riemannian manifold with $\mathrm{Ric}_M \geq (n-1)\cdot\kappa$ for some $\kappa \in \mathbb{R}$ and let $p \in M$. Then the map*

$$r \mapsto \frac{\mathrm{vol}_M(B_r(p))}{V_\kappa^n(r)}$$

is monotonically decreasing with limit 1 as $r \to 0$, where $V_\kappa^n(r)$ is the volume of an r-ball in the n-dimensional space form of sectional curvature κ.

In particular, for $R \geq r > 0$,

$$\mathrm{vol}_M(B_R(p)) \leq \frac{V_\kappa^n(R)}{V_\kappa^n(r)} \cdot \mathrm{vol}_M(B_r(p)).$$

This factor is independent of M and denoted by

$$C_{BG}(n, \kappa, r, R) := \frac{V_\kappa^n(R)}{V_\kappa^n(r)}.$$

Let $R_0 > r_0 > 0$. By definition, the map

$$r \mapsto C_{BG}(n, \kappa, r, R_0)$$

is monotonically decreasing and the map

$$R \mapsto C_{BG}(n, \kappa, r_0, R)$$

is monotonically increasing. Further monotonicity properties are stated by the following lemma.

Lemma C.2. *Fix* $n \in \mathbb{N}$.

a) Let $R > r > 0$. *Then*

$$C_{BG}(n, \kappa, r, R) = \begin{cases} C_{BG}(n, 1, \sqrt{\kappa} r, \sqrt{\kappa} R) & \text{if } \kappa > 0, \\ C_{BG}(n, -1, \sqrt{-\kappa} r, \sqrt{-\kappa} R) & \text{if } \kappa < 0. \end{cases}$$

b) For any $\kappa \in \mathbb{R}$, $r > 0$ *and* $c \geq 1$, *the map* $r \mapsto C_{BG}(n, \kappa, r, c \cdot r)$ *is*

$$\begin{cases} \text{monotonically decreasing} & \text{if } \kappa > 0, \\ \text{constant} & \text{if } \kappa = 0, \\ \text{monotonically increasing} & \text{if } \kappa < 0 \end{cases}$$

with limit c^n *as* $r \to 0$.

c) For $\kappa \geq (\leq)\ 0$ *and* $R > r \geq 0$, $C_{BG}(n, \kappa, r, R) \leq (\geq) \left(\frac{R}{r}\right)^n$.

d) The map $\kappa \mapsto C_{BG}(n, \kappa, r, R)$, *where* $R \geq r > 0$, *is monotonically decreasing.*

Proof. a) First, $K_{\frac{1}{c} M^n_\kappa} = \left(\frac{1}{c}\right)^{-2} \cdot K_{M^n_\kappa} = c^2 \cdot \kappa$ implies $\frac{1}{c} \cdot M^n_\kappa = M^n_{c^2\kappa}$. Hence, $\mathrm{vol}_{M^n_\kappa}(B^{M^n_\kappa}_{cr}(\cdot)) = c^n \cdot \mathrm{vol}_{c^{-1} M^n_\kappa}(B^{c^{-1} M^n_\kappa}_r(\cdot))$ and $V^n_\kappa(c \cdot r) = c^n \cdot V^n_{c^2\kappa}(r)$. Thus,

$$V^n_1(\sqrt{\kappa} \cdot r) = \sqrt{\kappa}^n \cdot V^n_\kappa(r)$$

for $\kappa > 0$. Analogously,

$$V^n_{-1}(\sqrt{|\kappa|} \cdot r) = \sqrt{|\kappa|}^n \cdot V^n_\kappa(r)$$

for $\kappa > 0$. Now the definition of C_{BG} implies the claim.

b) Moreover, $V^n_\kappa(c \cdot r) = c^n \cdot V^n_{c^2\kappa}(r)$ implies

$$C_{BG}(n, \kappa, r, c \cdot r) = \frac{V^n_\kappa(c \cdot r)}{V^n_\kappa(r)} = c^n \cdot \frac{V^n_{c^2\kappa}(r)}{V^n_\kappa(r)}.$$

Let $c \geq 1$ be arbitrary and distinguish the different cases of κ: First, let $\kappa > 0$. Then

$$\mathrm{Ric}_{M^n_{c^2\kappa}} = (n-1) \cdot c^2\kappa \geq (n-1) \cdot \kappa,$$

i.e., by the Bishop-Gromov Theorem, the map

$$r \mapsto \frac{\mathrm{vol}_{M^n_{c^2\kappa}}\left(B^{M^n_{c^2\kappa}}_r(\cdot)\right)}{V^n_\kappa(r)} = \frac{1}{c^n} \cdot C_{BG}(n, \kappa, r, c \cdot r)$$

is monotonically decreasing with limit 1 as $r \to 0$. Hence, $C_{BG}(n, \kappa, r, c \cdot r) \to c^n$ as $r \to 0$.

The proof for $\kappa < 0$ can be done analogously using $\mathrm{Ric}_{M^n_\kappa} = (n-1)\cdot\kappa \geq (n-1)\cdot c^2\kappa$. Finally, for $\kappa = 0$,

$$C_{BG}(n, 0, r, c\cdot r) = c^n \cdot \frac{V_0^n(r)}{V_0^n(r)} = c^n.$$

c) Let $\kappa > 0$. Define $c := \frac{R}{r} \geq 1$ and $f(t) := C_{BG}(n,\kappa,t,ct)$. Since f is monotonically decreasing, for all $t > 0$,

$$f(t) \leq \lim_{t\searrow 0} f(t) = c^n.$$

In particular, $C_{BG}(n,\kappa,r,R) = f(r) \leq c^n = \frac{R^n}{r^n}$.

The cases $\kappa \leq 0$ can be done analogously.

d) Let $\kappa_0 \leq \kappa_1$. Then $\mathrm{Ric}_{M^n_{\kappa_1}} = (n-1)\cdot\kappa_1 \geq (n-1)\cdot\kappa_0$ and, by the Bishop-Gromov Theorem, the map

$$r \mapsto \frac{\mathrm{vol}(B_r^{M^n_{\kappa_1}}(\cdot))}{V^n_{\kappa_0}(r)} = \frac{V^n_{\kappa_1}(r)}{V^n_{\kappa_0}(r)}$$

is monotonically decreasing. Thus, for $r \leq R$,

$$\frac{V^n_{\kappa_1}(r)}{V^n_{\kappa_0}(r)} \geq \frac{V^n_{\kappa_1}(R)}{V^n_{\kappa_0}(R)},$$

or, equivalently,

$$C_{BG}(n,\kappa_0,r,R) = \frac{V^n_{\kappa_0}(R)}{V^n_{\kappa_0}(r)} \geq \frac{V^n_{\kappa_1}(R)}{V^n_{\kappa_1}(r)} = C_{BG}(n,\kappa_1,r,R).$$

This proves the claim. □

The following lemma provides formulas for the antiderivative of $\sinh^n$, where $n \in \mathbb{N}$. These are needed for volume computations in the hyperbolic space and will be used in the subsequent lemma.

Lemma C.3. *For $m \in \mathbb{N}$,*

$$\int \sinh^{2m-1}(x)\,dx = \frac{1}{2m-1}\cdot\cosh(x)\cdot\Big[\sum_{j=0}^{m-1}(-1)^{m-1-j}\cdot\Big(\prod_{k=j+1}^{m-1}\frac{2k}{2k-1}\Big)\cdot\sinh^{2j}(x)\Big]$$

and

$$\int \sinh^{2m}(x)\,dx = \frac{1}{2m}\cdot\cosh(x)\cdot\Big[\sum_{j=0}^{m-1}(-1)^{m-1-j}\cdot\Big(\prod_{k=j+1}^{m-1}\frac{2k+1}{2k}\Big)\cdot\sinh^{2j+1}(x)\Big] + \frac{1}{2m}\cdot(-1)^m\cdot\Big(\prod_{k=1}^{m-1}\frac{2k+1}{2k}\Big)\cdot x.$$

Proof. In order to prove the first part, let $a_k := \frac{2k}{2k-1}$ and $b_j := (-1)^{m-1-j} \cdot \prod_{k=j+1}^{m-1} a_k$. Then

$$b_{m-1} = 1 \quad \text{and} \quad b_{j-1} = (-1)^{m-1-(j-1)} \cdot \prod_{k=j}^{m-1} a_k = -b_j \cdot a_j = -\frac{2j}{2j-1} \cdot b_j.$$

Hence, $(2j-1) \cdot b_{j-1} = -2j \cdot b_j$, in particular, $b_0 = -2b_1$. Thus,

$$\begin{aligned}
&\frac{d}{dx}\Big(\cosh(x) \cdot \Big[\sum_{j=0}^{m-1} (-1)^{m-1-j} \cdot \Big(\prod_{k=j+1}^{m-1} \frac{2k}{2k-1}\Big) \cdot \sinh^{2j}(x)\Big]\Big) \\
&= \frac{d}{dx}\Big(\cosh(x) \cdot \Big[\sum_{j=0}^{m-1} b_j \cdot \sinh^{2j}(x)\Big]\Big) \\
&= \sum_{j=0}^{m-1} b_j \cdot \sinh^{2j+1}(x) + \sum_{j=1}^{m-1} b_j \cdot 2j \cdot \sinh^{2j-1}(x) \cdot \cosh^2(x) \\
&= b_0 \cdot \sinh(x) + \sum_{j=1}^{m-1} b_j \cdot \sinh^{2j-1}(x) \cdot (\sinh^2(x) + 2j \cdot \cosh^2(x)) \\
&= -2b_1 \cdot \sinh(x) + \sum_{j=1}^{m-1} b_j \cdot \sinh^{2j-1}(x) \cdot ((2j+1) \cdot \sinh^2(x) + 2j) \\
&= -2b_1 \cdot \sinh(x) + \sum_{j=1}^{m-1} (2j+1) \cdot b_j \cdot \sinh^{2j+1}(x) + \sum_{j=1}^{m-1} 2j \cdot b_j \cdot \sinh^{2j-1}(x) \\
&= -2b_1 \cdot \sinh(x) + \sum_{j=2}^{m} (2j-1) \cdot b_{j-1} \cdot \sinh^{2j-1}(x) \\
&\quad + 2b_1 \cdot \sinh(x) + \sum_{j=2}^{m-1} 2j \cdot b_j \cdot \sinh^{2j-1}(x) \\
&= (2m-1) \cdot b_{m-1} \cdot \sinh^{2m-1}(x) + \sum_{j=2}^{m-1} ((2j-1) \cdot b_{j-1} + 2j \cdot b_j) \cdot \sinh^{2j-1}(x) \\
&= (2m-1) \cdot \sinh^{2m-1}(x).
\end{aligned}$$

Similarly, let $a_k := \frac{2k+1}{2k}$ and $b_j := (-1)^{m-1-j} \cdot \big(\prod_{k=j+1}^{m-1} a_k\big)$ for the second statement. Then

$$\begin{aligned}
&\frac{d}{dx}\Big(\cosh(x) \cdot \Big[\sum_{j=0}^{m-1} b_j \cdot \sinh^{2j+1}(x)\Big] - b_0 \cdot x\Big) \\
&= \sum_{j=0}^{m-1} b_j \cdot \sinh^{2j}(x)(\sinh^2(x) + (2j+1) \cdot \cosh^2(x)) - b_0
\end{aligned}$$

$$= \sum_{j=0}^{m-1} (2j+2) \cdot b_j \cdot \sinh^{2j+2}(x) + \sum_{j=0}^{m-1} (2j+1) \cdot b_j \cdot \sinh^{2j}(x) - b_0$$
$$= (2m) \cdot b_{m-1} \cdot \sinh^{2m}(x) + \sum_{j=1}^{m-1} (2j \cdot b_{j-1} + (2j+1) \cdot b_j) \cdot \sinh^{2j}(x)$$
$$= 2m \cdot \sinh^{2m}(x)$$

since $b_{m-1} = 1$ and $b_{j-1} = -b_j \cdot a_j = -\frac{2j+1}{2j} \cdot b_j$ for $j \geq 1$. □

Essentially, volume growth in the hyperbolic space is exponential: Consider two balls of radii r and $r+c$, respectively. If $r \to \infty$, the quotient of the volumes of the balls converges to $e^{(n-1)c}$.

Lemma C.4. *For arbitrary $y > 0$, $C_{BG}(n, -1, x, x+y) \to e^{(n-1)y}$ as $x \to \infty$.*

Proof. For $x, y > 0$, observe

$$\frac{\cosh(x+y)}{\cosh(x)} = \cosh(y) + \sinh(y) \cdot \frac{\sinh(x)}{\cosh(x)} \to \cosh(y) + \sinh(y) = e^y$$

and

$$\frac{\sinh(x+y)}{\sinh(x)} = \cosh(y) + \sinh(y) \cdot \frac{\cosh(x)}{\sinh(x)} \to \cosh(y) + \sinh(y) = e^y$$

as well as

$$\frac{1}{\sinh^l(x)} \to 0 \text{ as } x \to \infty$$

where $l \in \mathbb{N}$ with $l \geq 1$. First, let $n = 2m$ for $m \geq 1$ and $x > 0$ be arbitrary. Then

$$C_{BG}(n, -1, x, x+y)$$
$$= \frac{V_{-1}^{2m}(x+y)}{V_{-1}^{2m}(x)} = \frac{\int_0^{x+y} \sinh^{2m-1}(t)\, dt}{\int_0^{x} \sinh^{2m-1}(t)\, dt}$$
$$= \frac{\cosh(x+y) \cdot \left[\sum_{j=0}^{m-1} (-1)^{m-1-j} \cdot \left(\prod_{k=j+1}^{m-1} \frac{2k}{2k-1}\right) \cdot \sinh^{2j}(x+y)\right]}{\cosh(x) \cdot \left[\sum_{j=0}^{m-1} (-1)^{m-1-j} \cdot \left(\prod_{k=j+1}^{m-1} \frac{2k}{2k-1}\right) \cdot \sinh^{2j}(x)\right]}$$
$$= \frac{\cosh(x+y)}{\cosh(x)} \cdot \left(\frac{\sinh(x+y)}{\sinh(x)}\right)^{2(m-1)}$$
$$\cdot \frac{1 + \sum_{j=0}^{m-2} (-1)^{m-1-j} \cdot \left(\prod_{k=j+1}^{m-1} \frac{2k}{2k-1}\right) \cdot \frac{1}{\sinh^{2(m-1-j)}(x+y)}}{1 + \sum_{j=0}^{m-2} (-1)^{m-1-j} \cdot \left(\prod_{k=j+1}^{m-1} \frac{2k}{2k-1}\right) \cdot \frac{1}{\sinh^{2(m-1-j)}(x)}}$$
$$\to e^y \cdot (e^y)^{(2m-2)} \cdot 1 = e^{(n-1)y}$$

as $x \to \infty$. The case $n = 2m+1$ can be proven analogously. □

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