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On Multivariate Stochastic Fixed Point Equations: The Smoothing Transform and Random Difference Equations

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On Multivariate Stochastic Fixed Point Equations: The Smoothing Transform and Random Difference Equations

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Summary

The thesis at hand is concerned with the study of random vectors $Y \in \mathbb{R}^d$, satisfying multivariate stochastic fixed point equations

$$Y \stackrel{d}{=} \sum_{i=1}^{N} \mathbf{T}_i Y_i + Q$$

($\stackrel{d}{=}$:same distribution). Here $N \geq 1$ fixed, Y_i are independent identically distributed (i.i.d.) copies of Y and independent of the random $d \times d$ matrices $(\mathbf{T}_i)_{i=1}^N$ (which can w.l.o.g. assumed to be identically distributed) and the random vector Q. The main interest is in the existence of fixed points (FPs) and in the asymptotic shape of their distribution, namely heavy tail behaviour. Both are encoded in a function m which is defined in terms of the distribution of $\mathbf{T}_1$. It is strictly log-convex with $m(0) = N$, hence there are at most two values where m equals 1, α and β, say.

It will be shown (in the setting of the multivariate smoothing transformation, $N \geq 2$, nonnegative matrices) that in addition to the known FPs with a finite moment of order α, there are also α-elementary FPs, i.e. FPs with tail index α. A full characterization of the set of α-elementary FPs is obtained and a one-to-one correspondence between FPs of the homogeneous ($Q \equiv 0$) and inhomogeneous equation similar to linear equations is proved, using the Markov renewal theory and a Choquet-Deny lemma in the setting of Kesten's renewal theorem. This is Part A of the thesis.

Part B studies the case $N = 1$, $Q \neq 0$, well-known as the random difference equation. Here $\alpha = 0$ and a unique FP exists whose tails are then governed by β. In the situation where $\mathbf{T}_1 \in GL(d, \mathbb{R})$ with spread-out distribution, this result is proved using regeneration techniques from the theory of Harris recurrent Markov chains. The question whether β is the precise tail index has been quite involved in previous studies, the regenerative structure now allows for a comparatively simple proof. Therefore, a bivariate minorzation condition which may be interesting in its own right is introduced and studied.

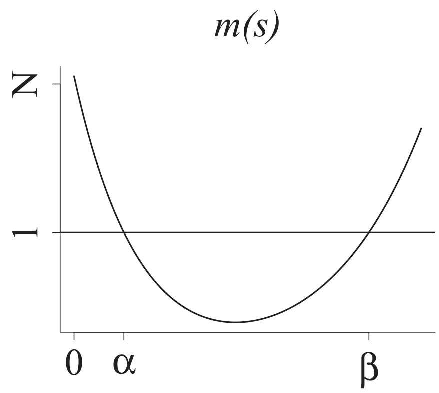

Figure 1.: A typical shape of m

Foreword

> The Road goes ever on and on
> Down from the door where it began.
> Now far ahead the Road has gone,
> And I must follow, if I can,
> Pursuing it with eager feet,
> Until it joins some larger way
> Where many paths and errands meet.
> And whither then? I cannot say.
> (John Ronald Reuel Tolkien [91])

For me, this poem perfectly reflects the process of writing a thesis and generally, thinking about maths. Solving one problem poses two new questions, many detours are made and possibly the final destination changes in due course. Nevertheless, I enjoyed travelling down this road and I appreciate that I had good guides, supporters and fellow travellers on my way to which I want to express my gratitude.

First of all, to my advisor Gerold Alsmeyer for introducing me into this subject, for his ongoing support and encouragement, for fruitful discussions providing me with new ideas but also giving me the chance to find and pursue my own interests. To Ewa Damek, for being my second advisor, for her interest in my research, for support and interesting and deep discussions. To her, Darek, Jacek and Mariusz, thank you for very good and motivating collaboration. To everybody at Wroclaw University, thank you for a good and long hospitality. I am grateful to John Nolan for sharing parts of his preprint about stable laws with me, as well as to Jeffrey Collamore, Yves Guivarc'h, Anja Janßen, Émile Le Page, Matthias Meiners and Matti Schneider for helpful discussions during the preparation of this thesis.I am very grateful to Christina Koch, Till Breuer and Matti Schneider, who have spent a lot of time for proofreading of this thesis.

To everybody from the "second floor", thank you for a very good time and atmosphere, in particular to Matti Schneider who shared the office and running tracks with me.

To the people who are the condicio sine qua non for any such travel: To my friends and family, thank you for your long patience during the final weeks and months and for friendship, advice and just being there, always. Here, a particular thank you to Nils Voelzke, who has been a fellow traveller from the very beginning and in particular through the ups and downs of writing a thesis. To my parents, thank you for always supporting and encouraging me - words are not enough. Christina, אני אוהב לך.

Contents

A. On Fixed Points of Multivariate Smoothing Transforms

1. Introduction and Basics

Let $N \geq 2$ be a fixed integer, $(\mathbf{T}_i)_{i=1}^N$ a random vector of $d \times d$ matrices with nonnegative entries. Consider the (homogeneous) multivariate smoothing transform $\mathcal{S}$ on the set $\mathcal{P}(\mathbb{R}^d_\geq)$ of probability measures on $\mathbb{R}^d_\geq = [0, \infty)^d$, defined by

$$\mathcal{S} \,:\, \nu \mapsto \mathcal{L}\left(\sum_{i=1}^N \mathbf{T}_i Y_i \right), \tag{ST}$$

where $(Y_i)_{i=1}^N$ are i.i.d. with distribution $\mathcal{L}(Y_1) = \nu$ and independent of $T = (\mathbf{T}_i)_{i=1}^N$.

This part of the thesis is concerned with the study of fixed points of $\mathcal{S}$, i.e. distributions $\eta \in \mathcal{P}(\mathbb{R}^d)$ satisfying $\mathcal{S}\eta = \eta$. In terms of random variables, if $\mathcal{L}(Y) = \eta$, then Y satisfies the stochastic fixed point equation

$$Y \stackrel{d}{=} \sum_{i=1}^N \mathbf{T}_i Y_i$$

with $(Y_i)_{i=1}^N$ being i.i.d. copies of Y, and independent of T and where $\stackrel{d}{=}$ means that both sides have the same distribution

A prominent example where stochastic fixed point equations of such a branching type appear is the study of stochastic processes on trees, see the review [1]. Multivariate equations appear e.g. if multiple types are considered. A survey of applications of the multivariate smoothing transform to the analysis of algorithms can be found in [77]. Moreover, multivariate equations with $d = 3$ arise very naturally when describing equilibrium distributions of particle speed in Maxwell gases, see [18]. There is also a close connection with multivariate α-stable distributions, which are the fixed points when $\mathbf{T}_1 = \ldots = \mathbf{T}_N = N^{-\frac{1}{\alpha}}\mathbf{Id}$, where $\mathbf{Id}$ is the $d \times d$ identity matrix and $\alpha \in (0, 1]$. Hence fixed points of $\mathcal{S}$ can be considered as generalizations of multivariate stable laws. Indeed, a very elegant characterization of the fixed points in terms of stable laws will be obtained.

1.1. Introduction

It is known that the structure of the set $\mathcal{F}$ of FPs η of $\mathcal{S}$ is governed by the function $\hat{m}(s) := \mathbb{E} \sum_{i=1}^N T_i^s$ in dimension $d = 1$. Considering only nontrivial solutions (i.e. $\eta \neq \delta_0$), there is a classical result of Durrett and Ligget for $d = 1$: There are

(I) no FPs if $\hat{m}(s) > 1$ for all $s \le 1$,

(II) FPs with finite expectation if $\hat{m}(1) = 1$ and $\hat{m}'(1) < 0$,

(III) FPs with infinite expectation if $\hat{m}(1) = 1$ and $\hat{m}'(1) = 0$ and

(IV) FPs with α-regularly varying tail if $\hat{m}(\alpha) = 1$ and $\hat{m}'(\alpha) < 0$ for some $0 < \alpha < 1$.

The multivariate extension of (II) was recently given by Buraczweski, Damek and Guivarc'h in [29]. Motivated by their results, the extension of the cases (III) and (IV) will be given in this thesis. The first step is to prove existence of nontrivial fixed points, which is more involved than in the one-dimensional case. The second step is to characterize the set of the fixed points of type (IV), the so-called *α-elementary fixed points* (see Iksanov [55]). A full description of this set in dimension $d \ge 2$ will be obtained.

Particular fixed points of the multivariate inhomogeneous smoothing transform

$$\mathcal{S}_Q \ : \ \nu \mapsto \mathcal{L}\left(\sum_{i=1}^N \mathbf{T}_i Y_i + Q\right),$$

where $(Y_i)_{i=1}^N$ i.i.d. with distribution ν and independent of the random element $((\mathbf{T}_i)_{i=1}^N, Q) \in M(d \times d, \mathbb{R}_\ge) \times \mathbb{R}^d_\ge$, have been studied by Mirek [75]. In this thesis, it will be shown that there are more fixed points, namely α-elementary ones, and that these fixed points are of the form "fixed point of the homogeneous smoothing transform + particular fixed point of the inhomogeneous smoothing transform", as it has been shown recently in the univariate setting by Alsmeyer and Meiners [6]. Two of the main results are stated at the end of this section, after some notation is introduced.

1.2. A Priori Definitions...

Most definitions and notations will be given adhoc when they occur for the first time, in order to make them present for the reader. The loss of quick reference is hopefully compensated by giving a list of symbols and abbreviations at the very end of this thesis. Nevertheless, there are some observations and definitions that are very basic or needed for the statement of the results, so it is convenient to give them now. This will make the rest of the presentation much more readable.

Unless otherwise noted, it is stipulated that all occuring random variables are defined on a common probability space with probability measure $\mathbb{P}$ and expectation symbol $\mathbb{E}$. The Laplace transform (LT) of a distribution η on $\mathbb{R}^d_\ge$ or a random variable Z (here with $\mathcal{L}(Z) = \eta$) is defined by

$$\phi_Z = \phi_\eta \ : \ \mathbb{R}^d_\ge \to \mathbb{R}_>, \quad x \mapsto \mathbb{E}\exp(-\langle x, Z\rangle) = \int_{\mathbb{R}^d} e^{-\langle x,z\rangle}\eta(dz).$$

Write $\mathfrak{L}$ for the associated mapping $\eta \mapsto \phi_\eta$. Consequently, $\mathcal{S}$ will be considered as a mapping on LTs by the natural definition $\mathcal{S}\phi_\eta = \phi_{\mathcal{S}\eta}$. Details will be given in Section 2. Abusing notation, a random variable Y is called fixed point of $\mathcal{S}$ if $\mathcal{S}\mathcal{L}(Y) = \mathcal{L}(Y)$. Uniqueness of fixed points is then always to be understood in terms of distributions. If it is not clear from the context whether the inhomogeneous or homogeneous case is adressed, the notation $\mathcal{S}_0$ will be used for the homogeneous smoothing transform as defined in (ST).

Write $\mathbb{N}_0$ for the natural numbers $\{0, 1, \dots\}$ and $\mathbb{N}$ for the positive integers $\{1, 2, \dots\}$. The nonnegative reals are denoted by $\mathbb{R}_\geq$, and the positive half-line by $\mathbb{R}_>$. The set of $d \times d$-matrices with entries from a given set E is denoted by $M(d \times d, E)$. Abbreviate $\mathcal{M}_+ = M(d \times d, \mathbb{R}_\geq)$ and $\breve{\mathcal{M}}_+ = M(d \times d, \mathbb{R}_>)$ for the set of matrices with positive entries. Write $\langle \cdot, \cdot \rangle$ for the euclidean scalar product $\langle x, y \rangle = \sum_{i=1}^N x_i y_i$ on $\mathbb{R}^d$, and $|\cdot|$ for the corresponding euclidean norm on $\mathbb{R}^d$, as well as for the absolute value on $\mathbb{R}$. Open balls of radius ε around x are denoted by $B_\varepsilon(x)$. The unit sphere in $\mathbb{R}^d$ is

$$\mathbb{S} := \{x \in \mathbb{R}^d \ : \ |x| = 1\}$$

and its intersection with the nonnegative cone is

$$\mathbb{S}_\geq = \mathbb{S} \cap \mathbb{R}^d_\geq.$$

The projection of a vector on the unit sphere is abbreviated by $\overline{x} := \frac{x}{|x|}$ and if $\mathbf{A}$ is a matrix, write

$$\mathbf{A} \cdot x := \overline{\mathbf{A}x} = \frac{\mathbf{A}x}{|\mathbf{A}x|}$$

for its action on the sphere. Denote by $\|\cdot\|$ the operator norm: For a mapping A between the normed spaces E and F with respective norms $|\cdot|_E$ and $|\cdot|_F$, then

$$\|A\| = \sup_{|x|_E = 1} |Ax|_F \, .$$

For a matrix $A \in \mathcal{M}_+$, define a corresponding lower bound by

$$\iota(\mathbf{A}) := \inf_{x \in \mathbb{S}_\geq} |\mathbf{A}x| \, .$$

For a metric space E, the set of continuous mappings $f : E \to \mathbb{R}$ is denoted by $\mathcal{C}(E)$. If E is compact, $\mathcal{C}(E)$ is equipped with the maximum norm $|\cdot|_\infty$,

$$|f|_\infty := \sup_{x \in E} |f(x)| \, ,$$

which yields the topology of uniform convergence. If E is locally compact, $\mathcal{C}(E)$ is equipped with the topology of uniform convergence on compact sets. The set of compactly supported continuous functions is denoted by $\mathcal{C}_c(E)$, bounded continuous functions by $\mathcal{C}_b(E)$ and continuous functions vanishing at infinity (i.e. $\forall \varepsilon > 0 \ \exists C \subset E$ compact with $|f(x)| < \varepsilon$ for all $x \notin C$) by $\mathcal{C}_0(E)$. The set of m-times continuously (Fréchet) differentiable mappings is denoted by $\mathcal{C}^m(E)$.

1.3. . . . and Observations

Note the important observation that since N is fixed, it may w.l.o.g be assumed that $(\mathbf{T}_i)_{i=1}^N$ are dependent, but identically distributed (see e.g. [31, A.1]). Hence the following **standing assumption** holds:

The weights $(\mathbf{T}_i)_{i=1}^N$ are dependent, but identically distributed with distribution μ.

Denote $\mu^* = \mathcal{L}\left(\mathbf{T}_1^\top\right)$ and write $(\mathbf{M}_n)_{n\in\mathbb{N}}$ for a sequence of i.i.d. random matrices with distribution μ^*.

Next is the multivariate analogue of the function $\hat{m}$. At this point, only its definition and some important properties are given. A motivation of this formula will be given in Subsections 4.1 and 7.2. Let $(\mathbf{T}_{(n)})_{n\in\mathbb{N}}$ be a sequence of i.i.d. random variables (r.v.s) with distribution μ. Set

$$\kappa(s) := \lim_{n\to\infty} \left(\mathbb{E}\left\|\mathbf{T}_{(1)}\cdots\mathbf{T}_{(n)}\right\|^s\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\mathbb{E}\left\|\mathbf{M}_1\cdots\mathbf{M}_n\right\|^s\right)^{\frac{1}{n}}, \tag{1.1}$$

$$m(s) := N\kappa(s). \tag{1.2}$$

The function $m(s)$ will be called *the spectral function*, for it will be seen (in Subsection 7.2) that it gives the spectral radius of a certain operator. Moreover, $m(s)$ is a strictly convex function, which is well defined on

$$I_\mu := \{s \geq 0 \ : \ \mathbb{E}\left\|\mathbf{T}_1\right\|^s < \infty\}$$

(this follows from the Hölder inequality resp. subadditivity of $\|\cdot\|$.) Write $s_\infty = \sup I_\mu$. Since $m(0) = N$, there are at most two values

$$0 < \alpha < \beta < s_\infty$$

with

$$m(\alpha) = m(\beta) = 1.$$

If both exists and if they are in the interior $\breve{I}_\mu$, then

$$m'(\alpha) < 0, \quad m'(\beta) > 0$$

by the strict convexity (which also implies the differentiability of m on $\breve{I}_\mu$.)

1.4. Statement of Results

One final piece, namely the main condition to be imposed on the distribution of the weight matrices, is needed before a first version of the main results can be stated.

Definition 1.1. A subsemigroup $\Gamma \subset \mathcal{M}_+$ is said to satisfy condition (C), if

1. no subspace $W \subset \mathbb{R}^d$ with $W \cap \mathbb{R}^d_\geq \neq \{0\}$ satisfies $\Gamma W \subset W$ and
2. $\Gamma \cap \breve{\mathcal{M}}_+ \neq \emptyset$.

Denote by $[\operatorname{supp}\mu]$ the smallest closed semigroup which contains $\operatorname{supp}\mu$, where as usual, the support is defined by

$$\operatorname{supp}\mu := \{x \in \mathcal{M}_+ \ : \ \mu(O) > 0 \,\forall \text{ open } O \text{ with } x \in O\}.$$

With the notation introduced above, a simplified version of the existence theorem for α-elementary fixed points of $\mathcal{S}_0$ can be stated as follows. See Subsection 9.3 for a more detailed statement and discussion.

Theorem 1.2 (Existence of Fixed Points). *Assume that the semigroup* $[\mathrm{supp}\ \mu]$ *satisfies condition* (C)*,*

$$\mathbb{E}(1+\|\mathbf{M}_1\|)\,(1+|\log\|\mathbf{M}_1\||+|\log\iota(\mathbf{M}_1)|)<\infty, \tag{M logM}$$

and that the spectral radius of $\mathbb{E}\mathbf{T}_1$ *is less than* N^{-1}*. Then there is* $\alpha\in(0,1)$ *with*

$$m(\alpha)=1,\quad m'(\alpha)<0.$$

For all $K>0$*,* $\mathcal{S}$ *possesses a nontrivial fixed point* Y_K*, and there is a continuous function* $e:\mathbb{S}_\geq\to\mathbb{R}_>$*, such that for all* $u\in\mathbb{S}_\geq$*,*

$$\lim_{t\to\infty} t^\alpha\mathbb{P}\left(\langle u,Y_K\rangle>t\right)=Ke(u)>0.$$

Concerning the characterisation of α-elementary fixed points of $\mathcal{S}_Q$, the main result 12.8 can be rephrased (in the spirit of [6, remark after Theorem 8.1]) as follows (where some technical details have been omitted):

Theorem 1.3 (Characterization Theorem). *Let* $(\mathbf{T}_i)_{i=1}^N$ *be i.i.d., let* $[\mathrm{supp}\,\mu]$ *satisfy* (C) *and let some natural moment assumptions hold. Assume there is* $\alpha\in(0,1)$ *with* $m(\alpha)=1$*,* $m'(\alpha)<0$*. Then* $\mathcal{S}_Q$ *possesses a one-parameter family of* α*-elementary fixed points* $(Y_K)_{K>0}$*, and their one-dimensional marginals satisfy*

$$\langle u,Y_K\rangle\stackrel{d}{=}\langle u,W^*\rangle+KW(u)^{1/\alpha}Z,$$

where $W^*\in\mathbb{R}^d_\geq$*,* $W(u)\in\mathbb{R}_\geq$ *are random variables which will be explicitly defined and independent of* Z*, which has a one-sided stable distribution with index* α*, i.e. with Laplace transform* $\mathbb{E}e^{-tZ}=e^{-t^\alpha}$*.*

The extra assumption that the $\mathbf{T}_i$ are independent is only needed for the existence of W^* which is not proved in this thesis, but cited from [75]. It is not necessary for the characterization itself as soon as W^* is given.

1.5. Further Organization

The further organization is as follows: At first, in Section 2 the weighted branching process (WBP) is introduced, which allows to study $\mathcal{S}$ in terms of random variables and Laplace transforms. Next is a section about different metrics and topologies on the set of probability measures which are used in Section 4 to derive some known results about existence of fixed points with a *finite* moment of order α, which are intended to explain the motivation of this work as well as the definition of m.

The methods which will be used subsequently in order to prove existence and characterization of α-elementary fixed points are inspired by Durrett and Liggett [41]: They analyze the action of $\mathcal{S}$ on LTs by means of an associated random walk and renewal theory, inter alia. The subsequent sections 5 - 8 introduce these tools in the multivariate setting, starting with a detailed review of LTs of multivariate stable distributions. Then a Markov random walk associated with the action of random matrices on $\mathbb{R}^d_\geq$ will be defined, corresponding transfer operators will be studied and a simple Markov renewal theorem, that complements Kesten's renewal theorem [60], will be proved.

Having all these tools at hand, the existence of nontrivial fixed points in the multivariate version of case (IV) will be proved and the definition of the Biggins martingale in the multivariate setting will be given in Section 9. Next, existence results in the boundary case (III) will be derived in Section 10. In order to describe the set $\mathcal{F}^\alpha$ of α-elementary fixed points, the question of uniqueness and the question whether the existence of an α-elementary fixed point readily implies $m(\alpha) = 1$, $m'(\alpha) < 0$ will be adressed. Using Krein-Milman theorem and a Choquet-Deny lemma due to Kesten [60], a positive answer will be given in Section 11. Finally, these results will be applied to fully characterize the set of α-elementary fixed points of $\mathcal{S}$ and $\mathcal{S}_Q$ in Section 12.

1.6. A Quick Readers Guide

An additional intention of this thesis is to give a comprehensive account to the theory needed for the study of multivariate stochastic fixed point equations, in particular to multivariate Laplace transforms of stable distributions, which are often neglected in the literature. Therefore, the introductory part of the thesis may seem unusually long. Though the full pleasure is only obtained by reading everything properly, these are the minimal prerequisites needed to understand the proofs of the main results in Sections 9 - 12: The reader should believe that the theory of one-dimensional Laplace transforms carries over to the multivariate case and that Laplace and Fourier transforms of stable distributions look quite similar. Additionally, read the short Section 2 and subsequently Propositions 4.3 & 4.4, Theorems 4.9 and 7.3, Subsection 8.1 and Proposition 8.9.

2. The Associated Weighted Branching Process and its Applications

In this section, a stochastic model associated with $\mathcal{S}$ is introduced which is suitable to describe iterations of the smoothing transform in terms of random variables. Additionally, it is used to describe the action of $\mathcal{S}$ on Laplace transforms. The exposition here is similar to the ones given in [4, Section 5.1] and [31, Section 3.2].

2.1. The Weighted Branching Process

Define the N-ary Ulam-Harris tree by

$$\mathfrak{T} := \bigcup_{n=0}^{\infty} \{1, \dots, N\}^n, \tag{2.1}$$

with the convention $\{1, \dots, N\}^0 = \emptyset$, the root. For a node $v = (i_1, \dots, i_k) \in \mathfrak{T}$, denote its level by $|v| = k$, its ancestor in the l-th level, $l \leq k$, by $v|l = (i_1, \dots, i_l)$ and its i-th child by $vi = (i_1, \dots, i_k, i)$.

Assign to each node v an independent copy

$$T(v) := (\mathbf{T}_1(v), \dots, \mathbf{T}_N(v), Q(v)) \tag{2.2}$$

of $T = (\mathbf{T}_1, \dots, \mathbf{T}_N, Q)$. Thus $\mathcal{T} := (T(v))_{v \in \mathfrak{T}}$ is a sequence of i.i.d. copies of T. The r.v. $T_i(v)$

can be understood as the weight of the vertex between v and vi. The product of the weights along the unique shortest path between the root $\emptyset$ and a node v is defined recursively by

$$\mathbf{L}(\emptyset) = \mathbf{Id}, \quad \mathbf{L}(vi) := \mathbf{L}(v)\mathbf{T}_i(v), \tag{2.3}$$

where $v \in \mathfrak{T}$, $1 \leq i \leq N$ and $\mathbf{Id}$ denotes the identity matrix. A natural filtration of the weight sequence is given by

$$\mathcal{T}_n := \sigma\left((T(v))_{|v|\leq n}\right).$$

When a random variable Y is given, assign to each node a copy $Y(v)$ of Y, such that again the sequence $\mathcal{Y} := (Y(v))_{v\in\mathfrak{T}}$ is i.i.d.and independent of $\mathcal{T}$.

Definition 2.1. The sequence

$$Y_n := \sum_{|v|=n} \mathbf{L}(v)Y(v) + \sum_{|w|<n} \mathbf{L}(w)Q(w), \tag{2.4}$$

$n \in \mathbb{N}_0$, is called the *weighted branching process* associated with $\mathcal{Y} \otimes \mathcal{T}$.

Lemma 2.2. *If $\mathbf{T}_1, \ldots, \mathbf{T}_N$ are identically distributed (which is the standing assumption), then for fixed $n \in \mathbb{N}$, the cumulative weights $(\mathbf{L}(v))_{|v|=n}$, are dependent, but identically distributed, and $\mathbf{L}(v)^\top \overset{d}{=} \mathbf{\Pi}_n$.*

The simple proof is omitted.

Furthermore, introduce the shift operator $[\cdot]_v$: If F is any function of $\mathcal{Y} \otimes \mathcal{T}$ and $v \in \mathfrak{T}$, set

$$[F(\mathcal{Y} \otimes \mathcal{T})]_v := F((Y(vw), T(vw))_{w\in\mathfrak{T}}).$$

The family $[\mathcal{Y} \otimes \mathcal{T}]_v$ corresponds to the subtree $[\mathfrak{T}]_v$, rooted in $v \in \mathfrak{T}$ and has the same distribution as the unshifted family $\mathcal{Y} \otimes \mathcal{T}$ and is independent of $(Y(w), T(w))_{|w|<|v|}$ as well as of all other subfamilies rooted at the same level. With this definition, it follows in particular

$$L(vw) = L(v)\,[L(w)]_v$$

for any $v, w \in \mathfrak{T}$. This allows inter alia to prove the following Lemma (see e.g. [4, Lemma 5.2]):

Lemma 2.3. *Let $\mathcal{L}(Y) = \eta$. Then the WBP $(Y_n)_{n\in\mathbb{N}_0}$ associated with $\mathcal{Y} \otimes \mathcal{T}$ satisfies*

$$\mathcal{L}(Y_n) = \mathcal{S}^n(\eta). \tag{2.5}$$

2.2. The Action of $\mathcal{S}$ on Laplace Transforms

The smoothing transform $\mathcal{S}$ acts on LTs of distributions on $\mathbb{R}^d_\geq$ by the canonical definition

$$\mathcal{S}\phi_\eta(x) = \phi_{\mathcal{S}\eta}(x) \tag{2.6}$$

for all $x \in \mathbb{R}^d_\geq$.

The following lemma corresponds to and is a consequence of Lemma 2.3.

Lemma 2.4. *Let ϕ_η be the LT of a distribution η on $\mathbb{R}^d_\geq$. Then for all $n \in \mathbb{N}$, $x \in \mathbb{R}^d_\geq$*

$$\mathcal{S}^n \phi_\eta(x) = \mathbb{E}\left(\exp\left(-\langle x, \sum_{|w|<n} \mathbf{L}(w)Q(w)\rangle\right) \prod_{|v|=n} \phi_\eta(\mathbf{L}(v)^\top x)\right). \tag{2.7}$$

Proof. Let $\mathcal{Y} = (Y(v))_{w\in\mathfrak{T}}$ and $\mathcal{T} = (T(w))_{w\in\mathfrak{T}}$ be i.i.d. random variables with distribution η resp. $\mathcal{L}(T)$ and such that $\mathcal{Y}$ and $\mathcal{T}$ are independent. Write $t = (\mathbf{t}(v))_{v\in\mathfrak{T}}$ for a deterministic sequence of weight matrices, and $\mathbf{l}(v)$ for the corresponding products along the paths. Referring to Lemma 2.3, $\mathcal{S}^n Y = \mathcal{L}(Y_n)$, where Y_n is the WBP associated with $\mathcal{Y} \otimes \mathcal{T}$. Considering (2.6),

$$\begin{aligned}
\mathcal{S}^n \phi_\eta(x) = \phi_{Y_n}(x) &= \mathbb{E}\left(\exp\left(-\langle x, \sum_{|v|=n} \mathbf{L}(v)Y(v) + \sum_{|w|<n} \mathbf{L}(w)Q(w)\rangle\right)\right)\\
&= \mathbb{E}\left(\exp\left(-\langle x, \sum_{|w|<n} \mathbf{L}(w)Q(w)\rangle\right) \mathbb{E}\left[\prod_{|v|=n} \exp\big(-\langle \mathbf{L}(v)^\top x, Y(v)\rangle\big) \middle| \mathcal{T}\right]\right)\\
&\overset{*}{=} \int \exp\left(-\langle x, \sum_{|w|<n} \mathbf{l}(w)q(w)\rangle\right) \mathbb{E}\left(\prod_{|v|=n} \exp\big(-\langle \mathbf{l}(v)^\top x, Y(v)\rangle\big)\right) \mathbb{P}(\mathcal{T} \in dt)\\
&= \int \exp\left(-\langle x, \sum_{|w|<n} \mathbf{l}(w)q(w)\rangle\right) \prod_{|v|=n} \Big(\mathbb{E}\exp\big(-\langle \mathbf{l}(v)^\top x, Y(v)\rangle\big)\Big) \mathbb{P}(\mathcal{T} \in dt)\\
&= \int \exp\left(-\langle x, \sum_{|w|<n} \mathbf{l}(w)q(w)\rangle\right) \prod_{|v|=n} \phi_Y(\mathbf{l}(v)^\top x) \mathbb{P}(T \in dt)\\
&= \mathbb{E}\left(\exp\left(-\langle x, \sum_{|w|<n} \mathbf{L}(w)Q(w)\rangle\right) \prod_{|v|=n} \phi_Y(\mathbf{L}(v)^\top x)\right).
\end{aligned}$$

The independence of $\mathcal{T}$ and $\mathcal{Y}$ allows to use the plug-in rule [28, Corollary 4.38] in *. □

Denote by $\mathcal{B}^1\left(\mathbb{R}^d_\geq\right)$ the set of bounded (Borel-) measurable functions from $\mathbb{R}^d_\geq$ to $\mathbb{R}$, uniformly bounded by 1. The smoothing transform induces a self-map of $\mathcal{B}^1\left(\mathbb{R}^d_\geq\right)$ by extension of equation (2.7), i.e.

$$\mathcal{S}f(x) := \mathbb{E}\left(\prod_{i=1}^N f(\mathbf{T}_i^\top x)\right) \tag{2.8}$$

for $f \in \mathcal{B}^1\left(\mathbb{R}^d_\geq\right)$. Then by a simple application of the theorem of bounded convergence, the following lemma results:

Lemma 2.5. *With the above definition, $\mathcal{S} : \mathcal{B}^1\left(\mathbb{R}^d_\geq\right) \to \mathcal{B}^1\left(\mathbb{R}^d_\geq\right)$ is a continuous mapping with respect to the pointwise convergence of functions.*

3. Convergence in Probability Spaces

Convergence of probability measures plays an important role in this thesis. In this section, several topologies and metrics on $\mathcal{P}(E)$ or its subspaces of $\mathcal{P}(E)$ will be introduced, where (E, d) is a priori any locally compact separable metric space equipped with the Borel σ-field $\mathfrak{E}$, in applications $E \in \{\mathbb{R}^d, \mathbb{R}^d_{\geq}, \mathbb{S}, \mathbb{S}_{\geq}, \mathbb{R}\}$. Some concepts are possibly well known, nevertheless it is convenient to mention them briefly (without proofs) to have all results at hand.

3.1. Weak and Vague Convergence

Information about weak and vague convergence on locally compact spaces can be found e.g. in [19, Chapter 2.4], the most important properties (for the present situation) are collected below. Write $\mathcal{M}^1(E)$ for the set of measures on E with total mass less or equal to 1.

Definition and Proposition 3.1. *A sequence $(\nu_n)_{n\in\mathbb{N}} \subset \mathcal{M}^1(E)$ is said to converge vaguely to $\nu \in \mathcal{M}^1(E)$, $\nu_n \overset{v}{\to} \nu$, if for all $f \in \mathcal{C}_c(E)$ or equivalently for all $f \in \mathcal{C}_0(E)$,*

$$\lim_{n\to\infty} \int f d\nu_n = \int f d\nu. \tag{3.1}$$

Another equivalent condition is that

$$\lim_{n\to\infty} \nu_n(B) = \nu(B) \tag{3.2}$$

for all relatively compact sets $B \subset \mathfrak{E}$ such that $\nu(\partial B) = 0$. Equipped with the topology of vague convergence, the set $\mathcal{M}^1(E)$ is compact.

Here ∂B denotes the topological boundary of B. Vague convergence commutes with the formation of product measures:

Lemma 3.2 ([19, Exercise 4.14])**.** *Let E, F be locally compact metric spaces. The mapping $G : \mathcal{M}^1(E) \times \mathcal{M}^1(F) \to \mathcal{M}^1(E \times F)$,*

$$G(\nu, \eta) = \nu \otimes \eta \tag{3.3}$$

is continuous w.r.t to the topology of vague convergence.

Definition and Proposition 3.3. *A sequence $(\nu_n)_{n\in\mathbb{N}} \subset \mathcal{M}^1(E)$ is said to converge weakly, $\nu_n \overset{d}{\to} \nu$, if (3.1) holds for all $f \in \mathcal{C}_b(E)$. Weak convergence $\nu_n \overset{d}{\to} \nu$ holds if and only if $\nu_n \overset{v}{\to} \nu$ and $\lim_{n\to\infty} \nu_n(E) = \nu(E)$.*

Weak convergence will also be denoted by $d{-}\lim\limits_{n\to\infty} \nu_n = \nu$. A weakly convergent sequence of probability measures converges towards a probability measure. Therefore, say that a sequence $(Y_n)_{n\in\mathbb{N}}$ converges in distribution, $Y_n \overset{d}{\to} Y$, iff $\mathcal{L}(Y_n) \overset{d}{\to} \mathcal{L}(Y)$. This convergence is the most important and will be implied by all of the subsequent types of convergence. Moreover, distributional convergence of random variables in $\mathbb{R}^d_{\geq}$ is equivalent to the convergence of their Laplace transforms (see below.)

3.2. The Prohorov Metric

The topology of weak convergence on $\mathcal{P}(E)$ is metrizable via the *Prohorov metric*. For $A \in \mathfrak{E}$ and $\varepsilon > 0$ define

$$A^\varepsilon := \{x \in E : d(x, A) < \varepsilon\}.$$

Definition and Proposition 3.4. *Let $\nu, \eta \in \mathcal{P}(E)$. The Prohorov distance $\varrho(\nu, \eta)$ is defined by*

$$\begin{aligned}\varrho(\nu,\eta) :=& \inf\{\varepsilon > 0 : \forall A \in \mathfrak{E},\ \nu(A) \leq \eta(A^\varepsilon) + \varepsilon\} \\ =& \inf\{\varepsilon > 0 : \forall A \in \mathfrak{E},\ \eta(A) \leq \nu(A^\varepsilon) + \varepsilon\}.\end{aligned} \tag{3.4}$$

This defines a metric on $\mathcal{P}(E)$ and for $\nu, (\nu_n)_{n\in\mathbb{N}} \in \mathcal{P}(E)$,

$$\nu_n \xrightarrow{d} \nu \quad \Leftrightarrow \quad \lim_{n\to\infty} \varrho(\nu_n, \nu) = 0.$$

More information can be found in [22, Section 6]. Note the following simple application of the above definiton:

Corollary 3.5. *Let $Z, (Z_n)_{n\in\mathbb{N}}$ be r.v.s in $\mathbb{R}$ with $Z_n \xrightarrow{d} Z$. Then for all $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$ such that for all $t \in \mathbb{R}$, $n \geq n_0$*

$$\mathbb{P}(Z_n > t) \leq \mathbb{P}(Z > t - \varepsilon) + \varepsilon \quad \textit{and} \quad \mathbb{P}(Z > t) \leq \mathbb{P}(Z_n > t - \varepsilon) + \varepsilon. \tag{3.5}$$

3.3. Total Variation

The strongest topology that will be introduced on $\mathcal{P}(E)$ is the topology of total variation. The total variation norm $\mathrm{tv}[\cdot]$ is defined on the vector space $\mathcal{M}^\pm(E) \supset \mathcal{P}(E)$ of regular bounded signed measures on E.

Definition and Proposition 3.6. *For a measure $\nu \in \mathcal{M}^\pm(E)$, its total variation norm is defined by*

$$\mathrm{tv}[\nu] := \sup\{\int_E f d\nu : f \in \mathcal{B}^1(E)\}.$$

If $\nu, (\nu_n)_{n\in\mathbb{N}} \in \mathcal{P}(E)$, then

$$\lim_{n\to\infty} \mathrm{tv}[\nu_n - \nu] = 0 \quad \Rightarrow \quad \nu_n \xrightarrow{d} \nu.$$

This can be found e.g. in [73, pp. 310 & 516].

3.4. Minimal L^s-Metric

The next two subsections consider metrics whose natural domain of definition are subspaces like

$$\mathcal{P}_s(E) := \{\nu \in \mathcal{P}(E) \ : \ \int |x|^s \nu(dx) < \infty\},$$

the set of probability measures on E with a finite moment of order s, or even subspaces of measures with a fixed first or second moment. On these subspaces, the metrics are complete and convergence in these metrics implies weak convergence.

This subsection considers the minimal L^s-distance l_s, which is a special case of the general concept of Wasserstein distances.

Definition and Proposition 3.7. *Let $s \in (0,1]$. For $\nu, \eta \in \mathcal{P}(E)$, set*

$$l_s(\nu,\eta) = \inf\{\mathbb{E}\,|Y - Z|^s \;:\; \mathcal{L}(Y) = \nu,\ \mathcal{L}(Z) = \eta\}.$$

This defines a metric on $\mathcal{P}(E)$.

A proof can be found in [83, Lemma 41]. For $\eta \in \mathcal{P}(E)$, define the subspace

$$\mathcal{P}_s(\eta) := \{\nu \in \mathcal{P}(E) \;:\; l_s(\nu,\eta) < \infty\}.$$

Observe that the measure η may have infinite moment of order s, as well as $\nu \in \mathcal{P}_s(\eta)$. Nevertheless, their l_s distance is well defined, this will turn out to be a important feature of the l_s-metric.

Proposition 3.8. *Let $s \in (0,1]$. For any $\eta \in \mathcal{P}(E)$, $(\mathcal{P}_s(\eta), l_s)$ is a complete metric space. If $\nu, (\nu_n)_{n\in\mathbb{N}} \in \mathcal{P}_s(\eta)$, then*

$$\lim_{n\to\infty} l_s(\nu_n,\nu) = 0 \quad \Rightarrow \quad \nu_n \xrightarrow{d} \nu.$$

This results from [38, Theorem 2].

3.5. The Zolotarev Metric

Last but not least, the Zolotarev metric ζ_s is introduced. It is particularly suitable for the study of (contraction properties of) the smoothing transform, as discussed in [87].

It was introduced in [96]. This subsection follows the recent exposition for probability measures on Hilbert spaces given in [39], where proofs can be found. Since it will be the only application, subsequently $E = \mathbb{R}^d$.

Denote by $D^k f$ the k-th Fréchet derivative of $f \in \mathcal{C}^k\left(\mathbb{R}^d\right)$ (see [37, Chapter VIII] for definitions). Remember that Df corresponds to the Jacobian matrix, while $D^2 f$ corresponds to the Hessian matrix.

For $s > 0$ and $k := \lceil s \rceil - 1$, define

$$\mathcal{D}_s := \left\{ f \in \mathcal{C}^k\left(\mathbb{R}^d\right) \;:\; \forall_{x,y\in\mathbb{R}^d}\ \left\| D^k f(x) - D^k f(y) \right\| \le |x-y|^{s-k} \right\} \tag{3.6}$$

If $s \le 1$, this is the set of s-Hölder functions on $\mathbb{R}^d$ with Hölder constant less or equal 1.

Definition 3.9. For $s > 0$, the Zolotarev distance ζ_s between r.v.s $Y, Z \in \mathbb{R}^d$ is defined by

$$\zeta_s(Y,Z) := \sup_{f\in\mathcal{D}_s} \left|\mathbb{E}\left(f(Y) - f(Z)\right)\right|. \tag{3.7}$$

Given $y \in \mathbb{R}^d$ and a symmetric positive definite matrix $\Sigma \in M(d \times d, \mathbb{R})$, define the subspaces

$$\mathcal{P}_s(\mathbb{R}^d) = \{\eta \in \mathcal{P}(\mathbb{R}^d) \ : \int |x|^s \, \eta(dx) < \infty\} \qquad s \in (0,1]$$

$$\mathcal{P}_{s,y}(\mathbb{R}^d) = \{\eta \in \mathcal{P}(\mathbb{R}^d) \ : \int |x|^s \, \eta(dx) < \infty, \int x \, \eta(dx) = y\} \qquad s \in (1,2]$$

$$\mathcal{P}_{s,y,\Sigma}(\mathbb{R}^d) = \{\eta \in \mathcal{P}(\mathbb{R}^d) \ : \int |x|^s \, \eta(dx) < \infty, \int x \, \eta(dx) = y, \mathbb{K}(\eta) = \Sigma\} \qquad s \in (2,3],$$

where $\mathbb{K}(\eta)$ denotes the covariance matrix of a generic random vector Z with $\mathcal{L}(Z) = \eta$. For brevity, when writing $\mathcal{P}_{s,*}(\mathbb{R}^d)$ the case distinction above as well as a particular choice of y and Σ, if necessary, will be stipulated.

A probability metric is called *simple*, if the distance between two random variables Y, Z depends only on their marginal distributions ν, η, say, and not on the particular coupling. This holds for the Zolotarev metric on particular subspaces of $\mathcal{P}(\mathbb{R}^d)$. Hence on these subspaces, $\zeta_s(\nu, \eta)$ is well defined.

Proposition 3.10. *Let $s \in (0,3]$. The Zolotarev metric ζ_s is simple on $\mathcal{P}_{s,*}$ and $(\mathcal{P}_{s,*}(\mathbb{R}^d), \zeta_s)$ is a complete metric space. If ν, $(\nu_n)_{n\in\mathbb{N}} \in \mathcal{P}_{s,*}(\mathbb{R}^d)$, then*

$$\lim_{n\to\infty} \zeta_s(\nu_n, \nu) = 0 \quad \Rightarrow \quad \nu_n \xrightarrow{d} \nu.$$

This is [39, Theorem 5.1].

4. Fixed Points of $\mathcal{S}$ with Finite α-Moment

In this section, it will be shown that $\mathcal{S}$ is a contraction with respect to (w.r.t.) ζ_s resp. l_s as soon as $m(s) < 1$. This motivates the definition of m. Considering Propositions 3.8 and 3.10, the usually approach via the Banach fixed point theorem yields existence and uniqueness of fixed points within particular subspaces of $\mathcal{P}(\mathbb{R}^d)$. Corresponding results will be given. Additionally, there are some very recent results about almost sure convergence towards these fixed points which will be cited.

It is remarkable that all fixed points obtained this way have a finite moment of order α which is due to the condition $m(s) < 1$ and the definition of $\mathcal{P}_{s,*}(\mathbb{R}^d)$. This leads to the question, whether there are also fixed points with an infinite moment of order α. Answering this question was a main motivation for this work and as it will turn out, indeed there are more fixed points.

4.1. Contraction Properties of $\mathcal{S}$ w.r.t ζ_s

The next Proposition, whose proof can be found in [76, Lemma 3.1], shows that $\mathcal{S}$ is Lipschitz on $(\mathcal{P}_{s,*}(\mathbb{R}^d), \zeta_s)$ for $s \in (0,3]$.

Proposition 4.1. *Let $s \in (0,3]$ and $\nu, \eta \in \mathcal{P}_{s,*}(\mathbb{R}^d)$. Then for any $n \in \mathbb{N}$,*

$$\zeta_s(\mathcal{S}^n\nu, \mathcal{S}^n\eta) \leq \left(\mathbb{E}\sum_{|v|=n} \|\mathbf{L}(v)\|^s\right)\zeta_s(\nu,\eta) = N^n\mathbb{E}\,\|\Pi_n\|^s\,\zeta_s(\nu,\eta). \tag{4.1}$$

This leads naturally to the definition of

$$m(s) = N \cdot \lim_{n\to\infty}\left(\mathbb{E}\,\|\Pi_n\|^s\right)^{1/n}, \tag{4.2}$$

for the Lipschitz factor in (4.1) is eventually smaller than 1 if and only if $m(s) < 1$.

Considering Proposition 3.10, conditions are needed that guarantee that $\mathcal{S}$ is a self-map of these spaces. They are given in the following Proposition.

Proposition 4.2. *Let $s \in (0,3]$ and*

$$\mathbb{E}\left(\|\mathbf{T}_1\|^s + |Q|^s\right) < \infty. \tag{s-moments}$$

- CASE $s \in (0,1]$*: Then $\mathcal{S}$ is a self-map of $\mathcal{P}_s(\mathbb{R}^d)$.*
- CASE $s \in (1,2]$*: Assume in addition, that*

$$y = N\mathbb{E}\mathbf{T}_1 y + \mathbb{E}Q \tag{eigenvector}$$

  holds. Then $\mathcal{S}$ is a self-map of $\mathcal{P}_{s,y}(\mathbb{R}^d)$.
- CASE $s \in (2,3]$*: Assume in addition, that $Q \equiv 0$ and the positiv definite matrix Σ satisfies*

$$\Sigma = \sum_{i=1}^N \mathbb{E}\left(\mathbf{T}_i\Sigma\mathbf{T}_i^\top\right) = N\mathbb{E}\left(\mathbf{T}\Sigma\mathbf{T}^\top\right). \tag{variance}$$

  Then $\mathcal{S}$ is a self-map of $\mathcal{P}_{s,0,\Sigma}(\mathbb{R}^d)$.

The naming (eigenvector) comes of course from the homogeneous situation $Q \equiv 0$, where y has to be an eigenvector of $\mathbb{E}\mathbf{T}_1$ with eigenvalue N^{-1}.

Proof. The eigenvalue condition follows by taking expectations in the fixed point equation $Y \overset{d}{=} \sum_{i=1}^N \mathbf{T}_i Y_i + Q$. The variance condition follows from a recursion formula for the covariance matrix of $\mathcal{S}^n Z$ given in [77, Lemma 4.5]. □

With the help of [77, Lemma 4.5], it is also possible to formulate (variance) in the case where $Q \neq 0$, but that formula is quite complicated and most applications are concerned with the centered case, e.g. [18]. This is why the inhomogeneous equation for $s \in (2,3]$ is not studied here.

4.2. Existence and Uniqueness of Fixed Points with a Finite α-Moment

For a square matrix $\mathbf{A}$, denote by $\mathrm{Eig}(\mathbf{A}, \lambda)$ the set of eigenvectors of $\mathbf{A}$ with eigenvalue λ, and

$$\mathrm{Eig}_0(\mathbf{A}, \lambda) = \mathrm{Eig}(\mathbf{A}, \lambda) \cup \{0\}.$$

Denote by $N(0, \Sigma)$ the multivariate Normal distribution with expectation 0 and covariance matrix Σ. Applying the Banach fixed point theorem, the following results (originally due to [84, 85]) concerning the subset

$$\mathcal{F}_s := \{\eta \ : \ \mathcal{S}\eta = \eta, \ \int |x|^s \, \eta(dx) < \infty\}$$

of fixed points with finite moment of order s can be obtained:

Proposition 4.3 (homogeneous case). *Let $T = (\mathbf{T}_1, \ldots, \mathbf{T}_N)$ be a random element of $M(d \times d, \mathbb{R})^N$, and $\mathcal{S}_0$ the homogeneous multivariate smoothing transform associated with T. Assume that there is $I_\mu \ni s > \alpha$ with $m(s) < 1$ and let* (s-moments) *hold.*

1. Case $\alpha < 1$*: Then $\mathcal{F}_s = \{\delta_0\}$.*
2. Case $\alpha \in [1, 2)$*: The following mapping is bijective:*

$$\begin{aligned} \mathrm{Eig}_0(\mathbb{E}\mathbf{T}_1, N^{-1}) &\to \mathcal{F}_s \\ y &\mapsto d\text{--}\lim_{n\to\infty} \mathcal{S}_0^n \delta_y \end{aligned}$$

3. Case $\alpha = 2$*. For every symmetric and positive definite matrix Σ satisfying* (variance)*, there is a unique fixed point $\eta \in \mathcal{P}_{s,0,\Sigma}(\mathbb{R}^d)$ and*

$$\eta = d\text{--}\lim_{n\to\infty} \mathcal{S}_0^n N(0, \Sigma).$$

Proposition 4.4 (inhomogeneous case). *Let $T = (\mathbf{T}_1, \ldots, \mathbf{T}_N, Q)$ be a random element of $M(d \times d, \mathbb{R})^N \times \mathbb{R}^d$ and $\mathcal{S}_Q$ the inhomogeneous multivariate smoothing transform associated with T. Assume there is $I_\mu \ni s > \alpha$ with $m(s) < 1$ and let* (s-moments) *hold.*

1. Case $\alpha < 1$*: Then $\#\mathcal{F}_s = 1$. If $\mathbb{P}(Q \neq 0) > 0$, then $\delta_0 \notin \mathcal{F}$, i.e. the fixed point is nontrivial.*
2. Case $1 \le \alpha < 2$, $\mathbb{E}Q = 0$*: The mapping*

$$\begin{aligned} \mathrm{Eig}_0(\mathbb{E}\mathbf{T}_1, N^{-1}) &\to \mathcal{F}_s \\ y &\mapsto d\text{--}\lim_{n\to\infty} \mathcal{S}_Q^n \delta_y \end{aligned}$$

   is bijective.
3. Case $1 \le \alpha < 2$, $\mathbb{E}Q \neq 0$*: Denote $O := \{y \in \mathbb{R}^d \ : \ y = N\mathbb{E}\mathbf{T}_1 y + \mathbb{E}Q\}$. Then the mapping*

$$\begin{aligned} O &\to \mathcal{F}_s \\ y &\mapsto d\text{--}\lim_{n\to\infty} \mathcal{S}_Q^n \delta_y \end{aligned}$$

   is bijective. If $O \neq \emptyset$, then $O \simeq \mathrm{Eig}_0(\mathbb{E}\mathbf{T}_1, N^{-1})$.

The last assertion is well known from the basic linear algebra. It is stated here to point out the relations between the homogeneous and inhomogeneous equation, which are very close to those of linear equations. These connection will be studied in more detail below in Subsection 4.4.

Remark 4.5. The restriction to the case $\alpha \leq 2$ should not be surprising, instead there are good reasons: In dimension $d = 1$, if $\hat{m}(s) < 1$ for some $s \in (2, 3]$ and Y is a fixed point of the homogeneous smoothing transform $\mathcal{S}$, then condition (variance) states that

$$Var(Y) = Var(Y) N \mathbb{E} T_1^2,$$

hence $Y \equiv c$ for some $c \in \mathbb{R}$ or $m(2) = 1$. In fact, it can be shown that if the one-dimensional smoothing transform with real-valued weights (homogeneous as well as inhomogeneous) has a non-constant fixed point Y with a finite moment of order $s > \alpha$, then necessarily $\alpha \leq 2$; see [5] for a detailed discussion.

Nevertheless, this is not true in dimension $d \geq 2$ without further assumptions, as it is shown by the following, even deterministic example (cf. [31]): Let

$$\mathbf{T}_1 = \cdots = \mathbf{T}_N = \begin{pmatrix} N^{-1/3} & 0 \\ 0 & N^{-1/2} \end{pmatrix}. \tag{4.3}$$

Then $m(s) = N \|\mathbf{T}_1\|^s = N(N^{-1/3})^s = N^{1-s/3}$, thus $\alpha = 3$. But if Y_2 has a standard normal distribution, the random vector $(0, Y_2)^\top$ is obviously a fixed point of the smoothing transform associated with (4.3). The point is that m is only concerned with the largest eigenvalue of $\mathbf{T}_1$, but there may be solutions concentrated on subspaces.

4.3. Almost Sure Convergence

Observe that all results above only yield convergence in distribution. Nevertheless, there are recent results for the multivariate smoothing transform that give almost sure convergence.

Buraczweski, Damek and Guivarch [29] proved the following result about fixed points of $\mathcal{S}_0$ for $\alpha = 1$: They show [29, Lemma 3.5] that if $[\operatorname{supp}\mu]$ satisfies (C) and $m(1) = 1$, then $\mathbb{E}\mathbf{T}_1$ has a unique eigenvector $y \in \mathbb{S}_\geq$ with eigenvalue N^{-1}.

Let W_n denote the WBP associated with w and $T = (\mathbf{T}_i)_{i=1}^N$, i.e.

$$W_n = \sum_{|v|=n} \mathbf{L}(v) y.$$

It is shown [29, p.2] that if $m(1) = 1$, then W_n is a nonnegative martingale w.r.t. to $\mathcal{T}_n$ (the Biggins martingale), hence it converges almost surely (a.s.) to a limit W. Since

$$W_n = \sum_{i=1}^N \mathbf{T}_i(\emptyset) \left[W_{n-1}\right]_i,$$

with $([W_{n-1}]_i)_{i=1}^N$ i.i.d. with the same distribution as W_{n-1} and independent of T, the limit W constitutes a fixed point of $\mathcal{S}_0$ – nevertheless, it may be trivial.

Theorem 4.6 (part of [29, Theorem 2.2])**.** *Let* $(\mathbf{T}_i)_{i=1}^N$ *be i.i.d. random matrices in* $M(d \times d, \mathbb{R})$*. Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C)*, that* $\mathbb{E}\,\|\mathbf{T}_1\| < \infty$*,* $m(1) = 1$*. Let* $y \in \mathbb{S}_{\geq}$ *be the unique normalized eigenvector of* $\mathbb{E}\mathbf{T}_1$ *with eigenvalue* N^{-1}*. Then the following are equivalent:*

1. *There is a fixed point* Y *of* $\mathcal{S}_0$ *with* $\mathbb{E}\,|Y| < \infty$ *and* $\mathbb{E}Y \neq 0$*.*
2. $\mathbb{E}W = y$*, in particular,* W *is nontrivial.*
3. $m'(1^-) < 0$*.*

Here and subsequently, $m'(s^-)$ denotes the left derivative of m in s.

In other words, for $\alpha = 1$ and $m'(\alpha^-) < 0$, W_n converges almost surely to the nontrivial fixed point of $\mathcal{S}$ with finite expectation y, described in Proposition 4.3.

For the inhomogeneous equation, Mirek [75] obtained the following a.s. convergence result:

Theorem 4.7 ([75, Theorem 1.7])**.** *Let* $(\mathbf{T}_i)_{i=1}^N$ *be i.i.d. random matrices in* $M(d \times d, \mathbb{R}_{\geq})$ *and* Q *a random vector in* $\mathbb{R}^d_{\geq}$ *with* $\mathbb{P}\,(Q \neq 0) > 0$*. Let* $[\operatorname{supp}\mu]$ *satisfy condition* (C)*. Assume that there are* $s_1 \in (0, 1/2]$*,* $s_2 > s_1$ *such that* $\mathbb{E}\,\|\mathbf{T}_1\|^{s_1} \leq \frac{1}{N}$*,* $\mathbb{E}\,\|\mathbf{T}_1\|^{s_2} \leq \frac{1}{N}$ *and* $\mathbb{E}\,|Q|^{s_2} < \infty$*. Then*

$$W_n^* := \sum_{k=0}^{n-1} \sum_{|v|=k} \mathbf{L}(v) Q(v)$$

converges almost surely to a r.v. W^* *which is a fixed point of* $\mathcal{S}_Q$*.*

Since $m(s) \leq N\mathbb{E}\,\|\mathbf{T}_1\|^s$, in this situation $\alpha \leq s_1 \leq \frac{1}{2}$. This restriction on α is due to technical reasons (see [75, proof of Lemma 3.12]). With some additional assumptions, the theorem gives the almost sure convergence of W_n^* to the unique fixed point with a finite moment of order α. It seems that if N is considered fixed (as in the present situation), then the assumption of independent weights is not necessary for the proofs in [29, 75].

4.4. A One-to-one Correspondence Between Fixed Points of $\mathcal{S}_0$ and $\mathcal{S}_Q$

In this subsection, the focus will be on the case $\alpha < 1$ with $m(s) < 1$ for some $s \in (\alpha, 1)$. Using the l_s-metric, a one-to-one correspondence between fixed points of the homogeneous and inhomogeneous smoothing transform will be derived. This approach is due to Rüschendorf [87, Section 3].

The first step is the following contraction lemma.

Lemma 4.8. *Let* $s \leq 1$*, let* (s-moments) *hold and let* $\eta \in \mathcal{P}(\mathbb{R}^d)$ *satisfy*

$$l_s(\eta, \mathcal{S}\eta) < \infty. \tag{4.4}$$

Then $\mathcal{S}$ *is a Lipschitz self map of* $\mathcal{P}_s(\eta)$ *and for* $n \in \mathbb{N}$

$$l_s(\mathcal{S}^n\eta, \mathcal{S}^n\nu) \leq N^n \mathbb{E}\,\|\Pi_n\|^s\, l_s(\nu, \eta).$$

This once more motivates the importance of the spectral function $m(s)$ – obviously, $\mathcal{S}$ is a contraction w.r.t. to l_s as soon as $m(s) < 1$. The proof of the univariate version, [87, Lemma 2.1] extends without efforts to the multivariate situation and is therefore omitted.

Using the Zolotarev metric, in Proposition 4.4 a numerical correspondence between fixed points of $\mathcal{S}_0$ and $\mathcal{S}_Q$ with a finite moment of order $s > \alpha$ was established. The use of the l_s-metric now gives the same result for general fixed points (with a possibly infinite moment of order α) in the case $\alpha < 1$. This result is due to Rüschendorf [87, Theorem 3.1].

Theorem 4.9. *Let $m(s) < 1$ for some $s \in (0, 1]$ and let* (s-moments) *hold. Then*

1. *For any fixed point η_0 of $\mathcal{S}_0$, there exists exactly one fixed point η_Q of $\mathcal{S}_Q$, such that $\eta_Q \in \mathcal{P}_s(\eta_0)$. It holds that*
$$d\text{–}\lim_{n\to\infty} \mathcal{S}_Q^n \eta_0 = \eta_Q.$$
2. *For any fixed point η_Q of $\mathcal{S}_Q$, there exists exactly one fixed point η_0 of $\mathcal{S}_0$, such that $\eta_0 \in \mathcal{P}_s(\eta_Q)$. It holds that*
$$d\text{–}\lim_{n\to\infty} \mathcal{S}_0^n \eta_Q = \eta_0.$$

Proof. Part 1 : By Lemma 4.8 and the Banach fixed point theorem, there is a unique fixed point of $\mathcal{S}_Q$ in $\mathcal{P}_s(\eta_0)$, as soon as
$$l_s(\eta_0, \mathcal{S}_Q\eta_0) < \infty.$$
Let $(Y_i)_{i=1}^N$ be i.i.d. copies of η_0, then, since η_0 is a fixed point of $\mathcal{S}_0$,
$$\mathcal{L}\left(\sum_{i=1}^N \mathbf{T}_i Y_i\right) = \eta_0$$
and thus $\left(\sum_{i=1}^N \mathbf{T}_i Y_i,\ \sum_{i=1}^N \mathbf{T}_i Y_i + Q\right)$ is a coupling of $\eta_0, \mathcal{S}_Q\eta_0$. Then
$$l_s(\eta_0, \mathcal{S}_Q\eta_0) \leq \mathbb{E}\left|\sum_{i=1}^N \mathbf{T}_i Y_i - (\sum_{i=1}^N \mathbf{T}_i Y_i + Q)\right|^s = \mathbb{E}\,|Q|^s < \infty$$
by (s-moments). Considering the Banach fixed point theorem, the convergence assertion in the l_s-metric can be obtained and by Proposition 3.8, this implies already the weak convergence of the measures.

The proof of part 2 is completely analogue. □

4.5. Resumé

This admittedly quite long review is primarily intended to motivate the upcoming results.

The reader should have observed that the existence of fixed points and the finiteness of their moments is closely connected with the spectral function $m(s)$ and that the fixed points (for $\alpha \in [1, 2)$) may be characterized by eigen*vectors* of $\mathbb{E}\mathbf{T}_1$. Moreover, only existence of fixed point with a finite moment of order α can be shown by the methods above. As said before, a main contribution of this thesis is the characterisation of fixed points with infinite moment of order α, more precisely, α-elementary fixed points. Their description will be in terms of eigen*functions* of an operator given by the action of $\mathbf{T}_1$ on $\mathbb{S}_\geq$, see Theorem 12.6.

The convergence and nontriviality of the Biggins martingale under the condition $\alpha = 1$, $m'(\alpha^-) < 0$ was mentioned. Subsequently, the definition of the Biggins martingale in the case $\alpha < 1$ as well as the proof that its limit is nontrivial will be given. In the last subsection, a one-to-one correspondence between fixed points of $\mathcal{S}_0$ and $\mathcal{S}_Q$ was derived. It will be used to prove the characterization of α-elementary fixed points of $\mathcal{S}_Q$.

5. On Multivariate Laplace Transforms

Most textbooks in probability theory only consider the one-dimensional LT on the positive half-line $\mathbb{R}_>$. Multivariate results are nevertheless known, but scattered around in literature. This is why a comprehensive account is given here including proofs or at least references where to find them. Moreover, the final part about the Hölder continuity seems to be new.

5.1. Basic Facts about Laplace Transforms on $\mathbb{R}^d_\geq$

This subsection contains the uniqueness and continuity theorem for multivariate LTs and a convergence result that will be useful later.

Theorem 5.1 (Uniqueness Theorem for Laplace Transforms)**.** *Let $\nu, \eta \in \mathcal{P}(\mathbb{R}^d_\geq)$. If $\phi_\nu(x) = \phi_\eta(x)$ for all $x \in \mathbb{R}^d_>$, then $\nu = \eta$.*

Theorem 5.2 (Continuity theorem for multivariate Laplace transforms)**.** *Let $(\nu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures on $\mathbb{R}^d_\geq$ with LTs $(\phi_n)_{n\in\mathbb{N}}$.*

1. *If $\nu_n \xrightarrow{v} \nu \in \mathcal{M}^1\left(\mathbb{R}^d_\geq\right)$ with LT ϕ, then $\phi_n(x) \to \phi(x)$ for all $x \in \mathbb{R}^d_>$.*
2. *If $\phi(x) := \lim_{n\to\infty} \phi_n(x)$ exists for all $x \in \mathbb{R}^d_>$, then ϕ can be continuously extended on $\mathbb{R}^d_\geq$, and then is the LT of a measure $\nu \in \mathcal{M}^1\left(\mathbb{R}^d_\geq\right)$, and $\nu_n \xrightarrow{v} \nu$. If $\lim_{n\to\infty} \phi_n(0) = \lim_{x\to 0} \phi(x)$, then ν is a probability measure, and $\nu_n \xrightarrow{d} \nu$.*

Using the results about weak and vague convergence of measures in $\mathcal{P}(\mathbb{R}^d)$ from Subsection 3.1, the proofs from the one-dimensional case carry over with the obvious modifactions. A slightly less detailed statement of these results can also be found in [89, Lemma 3].

At one point, a sequence of LTs will be considered which are evaluated at a convergent sequence of points. There the following Corollary of Lemma 3.2 will be helpful.

Corollary 5.3. *Let $\phi, (\phi_n)_{n\in\mathbb{N}}$ be Laplace transforms of measures $\eta, (\eta_n)_{n\in\mathbb{N}}$ with $\eta_n \xrightarrow{v} \eta$. Let $(a_n)_{n\in\mathbb{N}} \subset \mathbb{R}_>$ be a convergent sequence with the limit $a \in \mathbb{R}_>$. Then for all $x \in \mathbb{R}^d_>$,*

$$\lim_{n\to\infty} \phi_n(a_n x) = \phi(ax).$$

This result is well known if the measures converge weakly, since then the Laplace transforms converge uniformly on compact sets.

Proof. Consider the sequence of measures $(\delta_{a_n})_{n\in\mathbb{N}}$, which converges weakly (thus also vaguely) to δ_a. Regarding Lemma 3.2, $\eta_n \otimes \delta_{a_n}$ also converges vaguely to $\eta \otimes \delta_a$. For all $x \in \mathbb{R}^d_>$, the function $f_x(y,t) = \exp(-\langle tx, y\rangle)$ is in $\mathcal{C}_0\left(\mathbb{R}^d_\geq \times \mathbb{R}_\geq\right)$. Use Lemma 3.1 to conclude

$$\lim_{n\to\infty} \phi_n(a_n x) = \lim_{n\to\infty} \int_{\mathbb{R}^d_\geq \times \mathbb{R}_\geq} f_x(y,t)\, \eta \otimes \delta_a(dy,dt)) = \phi(ax). \qquad \square$$

5.2. Multivariate Stable Distributions and Their Laplace Transforms

In this subsection, multivariate stable distributions are introduced, their main properties are stated and then used to derive a formula for their Laplace transforms.

Definition 5.4. A r.v. Z in $\mathbb{R}^d$ is said to have a stable distribution if for all $n \geq 2$, there exists $a_n \in \mathbb{R}_>$ and $b_n \in \mathbb{R}^d$ such that

$$\sum_{i=1}^{n} Z_i \overset{d}{=} a_n Z + b_n, \tag{5.1}$$

where $(Z_i)_{i=1}^N$ are i.i.d. copies of Z.

It turns out (see e.g. [88, Theorem 2.1.2]) that necessarily $a_n = n^{1/\alpha}$ for some $\alpha \in (0,2]$. For $\alpha = 2$, the stable property uniquely identifies the Normal distributions. For $\alpha < 2$, Z is then said to be *α-stable* and Z is *strictly α-stable* if $b_n = 0$.

The characterisation of multivariate stable distributions via their Fourier transform (FT) is well known. Now the corresponding formula for Laplace transforms will be derived. The approach sketched here follows closely [97, I.6] and [78, Proposition 6.13] while properties of stable distributions are taken from [88].

Start with the classic representation theorem for one-dimensional stable distributions.

Proposition 5.5 ([88, Definition 1.1.6])**.** *A random variable Z is stable if and only if there are parameters $\alpha \in (0,2]$, $\sigma \in \mathbb{R}_\geq$, $\lambda \in [-1,1]$ and $b \in \mathbb{R}$ such that its FT has the following form:*

$$\mathbb{E}e^{itZ} = \begin{cases} \exp\left(-\sigma^\alpha |t|^\alpha \left[1 - i\lambda(\operatorname{sign} t)\tan\frac{\pi\alpha}{2}\right] + ibt\right) & \text{if } \alpha \neq 1, \\ \exp\left(-\sigma |t| \left[1 + i\lambda\frac{2}{\pi}(\operatorname{sign} t)\ln|t|\right] + ibt\right) & \text{if } \alpha = 1, \end{cases} \tag{5.2}$$

with the convention $\operatorname{sign}(0) = 0$.

Definition 5.6. If the FT of Z is the same as above, write $Z \overset{d}{=} S_\alpha(\sigma,\lambda,b)$.

The roles of the parameters are as follows (c.f. [88, Properties 1.2.2. & 1.2.3.]). Let as above $Z \overset{d}{=} S_\alpha(\sigma,\lambda,b)$.

- b is the *shift parameter*: Let $a \in \mathbb{R}$, then $Z + a \overset{d}{=} S_\alpha(\sigma,\lambda,b+\alpha)$.

- σ is the *scale parameter*: Let $a \in \mathbb{R} \setminus \{0\}$, then $aZ \stackrel{d}{=} S_\alpha(|a|\,\sigma, \operatorname{sign}(a)\lambda, ab)$ for $\alpha \neq 1$, and $aZ \stackrel{d}{=} S_\alpha(|a|\,\sigma, \operatorname{sign}(a)\lambda, ab - \frac{2}{\pi}a(\ln|a|)\sigma\lambda)$ for $\alpha = 1$.
- λ is the *skewness parameter*, it governs the ratio between left and right tails of Z, see [88, Property 1.2.15]. Moreover, if $\alpha \in (0,1)$, $\lambda = 1$ even implies that Z is supported on a half-line, as the following Proposition shows:

Proposition 5.7 ([88, Proposition 1.2.11])**.** *Let $\alpha \in (0,1)$, $Z \stackrel{d}{=} S_\alpha(\sigma, 1, b)$. Then $Z \geq b$ $\mathbb{P}$-a.s., for it can be written as a shifted limit of a random sum of Poisson random variables. The Laplace transform of Z exists and equals*

$$\mathbb{E}e^{-tZ} = \exp\left(-\left(\cos\frac{\pi\alpha}{2}\right)^{-1}(\sigma t)^\alpha - bt\right). \tag{5.3}$$

The next theorem, known as Lévy spectral representation theorem, gives the FT of multivariate stable distributions:

Theorem 5.8 ([88, Theorem 2.3.1])**.** *Let $\alpha \in (0,1)$ and Z a r.v. in $\mathbb{R}^d$. Then Z is α-stable if and only if there exists a probability measure ν on $\mathbb{S}$, $K \in \mathbb{R}_\geq$ and a vector $b \in \mathbb{R}^d$ such that*

$$\mathbb{E}e^{i\langle x,Z\rangle} = \exp\left(-K\int_{S^d}|\langle x,y\rangle|^\alpha\left(1 - i\operatorname{sign}(\langle x,y\rangle)\tan\frac{\pi\alpha}{2}\right)\nu(dy) + i\langle x,b\rangle\right) \tag{5.4}$$

for all $x \in \mathbb{R}^d$. The tupel (K,ν,b) is unique, hence write $\mathcal{L}(Z) = S_\alpha(K\nu, b)$.

Samorodnitsky and Taqqu use the notation $S_\alpha(\Gamma, b)$, where $\Gamma = K\nu$ is a finite measure on the unit sphere. The "decomposition" into K and a probability measure ν is used here, because it makes it more obvious that K is a scaling factor. The measure Γ resp. ν is called *spectral measure*.

One observes that all marginal distributions $\langle u, Z\rangle$, $u \in \mathbb{S}$ are one-dimensional α-stable (maybe degenerated), see [88, Theorem 2.1.2]. More precisely, $\langle u, Z\rangle \stackrel{d}{=} S_\alpha(\sigma_u, \lambda_u, b_u)$, with (see [88, Example 2.3.4]):

$$\sigma_u = \left(K\int_{\mathbb{S}}|\langle u,y\rangle|^\alpha\,\nu(dy)\right)^{1/\alpha}, \tag{5.5}$$

$$\lambda_u = \frac{\int_{\mathbb{S}}\operatorname{sign}\langle u,y\rangle\,|\langle u,y\rangle|^\alpha\,\nu(dy)}{\int_{\mathbb{S}}|\langle u,y\rangle|^\alpha\,\nu(dy)}, \tag{5.6}$$

$$b_u = \langle u, b\rangle \tag{5.7}$$

These identification now allows to derive the Laplace transform analogue of the spectral representation theorem. The proof of this final proposition is taken from [78].

Proposition 5.9 ([78, Proposition 6.13])**.** *Let $Z \stackrel{d}{=} S_\alpha(K\nu, 0)$, $\alpha \in (0,1)$. If the spectral measure ν is supported on $\mathbb{S}_\geq$, then* $\operatorname{supp} Z \subset \mathbb{R}^d_\geq$ *and the LT ϕ_Z of Z exists on $\mathbb{R}^d_\geq$. It holds that*

$$\phi_Z(x) = \exp\left(-K\left(\cos\frac{\pi\alpha}{2}\right)^{-1}\int_{\mathbb{S}_\geq}\langle x,y\rangle^\alpha\nu(dy)\right), \quad x \in \mathbb{R}^d_\geq. \tag{5.8}$$

Proof. If ν is supported on $\mathbb{S}_\geq$, then by (5.6) $\lambda_u = 1$ for all $u \in \mathbb{S}_\geq$. Referring to Proposition 5.7, the support of $\langle u, Z\rangle$ is contained in $\mathbb{R}_\geq$. Since this holds for any vector $u \in \mathbb{S}_\geq$, it follows that Z itself is supported in $\mathbb{R}^d_\geq$. This readily gives the existence of the Laplace transform on $\mathbb{R}^d_\geq$.

To verify the formula, write $x = ut$, $u \in \mathbb{S}_\geq$, $t \in \mathbb{R}_\geq$ and note that $\mathcal{L}\left(\langle x, Z\rangle\right) = S_\alpha(\sigma_u, 1, 0)$. Then by Proposition 5.7,

$$\mathbb{E}e^{-t\langle u,Z\rangle} = \exp\left(-\left(\cos\frac{\pi\alpha}{2}\right)^{-1} t^\alpha K \int_{\mathbb{S}} |\langle u, y\rangle|^\alpha \, \nu(dy)\right),$$

which gives the desired formula when taking into account that the domain of intergration is in fact supp $\nu = S_+$. □

Remark 5.10.

- For easing the presentation, it is convenient to get rid of the additional factor $(\cos \pi\alpha/2)^{-1}$: Write $Z \overset{d}{=} \tilde{S}_\alpha(K\nu, 0)$, if its LT ϕ_Z satisfies

$$\phi_Z(x) = \exp\left(-K \int_{\mathbb{S}_\geq} \langle x, y\rangle^\alpha \nu(dy)\right), \quad x \in \mathbb{R}^d_\geq. \tag{5.9}$$

- The formula (5.9) makes sense in the case $\alpha = 1$, too. Then ϕ_Z is the Laplace transform of the point mass at $v^* := K \int_{\mathbb{S}_\geq} y \, \nu(dy)$. In other words, $Z \equiv v^*$. This obviously is a 1-stable random variable.
- The existence of multivariate α-stable distributions, $\alpha \in (0, 1)$ and the formula for their Laplace transforms can also be shown by means of a multivariate version of the Bernstein theorem, which states that completely monotone functions define LTs of distributions. This approach does not make use of the Lévy spectral representation theorem and is very similar to the classical one-dimensional approach in Feller [45, XIII]. A multivariate version of the Bernstein theorem is stated in [19, Exercise 6.27], but dates at least back to Bochner [24, Theorem 4.2.1]. A proof of the multivariate Bernstein theorem can be found in [95].

5.3. Tail Behaviour

A special emphasis in this work will be put on heavy tail properties of distributions. Similar to the classical Tauberian theorem for Laplace transforms, the *asymptotics at zero* of $1 - \phi$ are in the multivariate case linked with the property of multivariate regular variation. This will be explained in this subsection.

As a first step, recall the classical Tauberian theorem. Write $\lim_{t\downarrow 0}$ for the right sided limit at zero.

Proposition 5.11 ([45, XIII.5, (5.22)]). *Let Z be a r.v. in $\mathbb{R}_\geq$ with LT ϕ. Then each of the relations*

$$\lim_{t\to\infty} L(t)t^\alpha \mathbb{P}\left(X > t\right) = \frac{c}{\Gamma(1-\alpha)} \quad \textit{and}$$
$$\lim_{t\downarrow 0} L(1/t)\frac{1-\phi(t)}{t^\alpha} = c$$

implies the other. Here $\alpha \in (0, 1)$, $c > 0$ and L is slowly varying at infinity.

Denote by $\mathcal{C}_c\left(\overline{\mathbb{R}^d_\geq}\setminus\{0\}\right)$ the set of functions $f\in\mathcal{C}_b\left(\mathbb{R}^d_\geq\right)$ with the additional property that

$$f(x)=0 \quad \forall x\in B_\delta(0)\cap\mathbb{R}^d_\geq \text{ for some } \delta>0,$$

i.e. f is supported away from the origin. Write λ^α for the α-homogeneous measure on the multiplicative group $\mathbb{R}_>$, i.e. $\lambda^\alpha(ds)=\frac{1}{s^{1+\alpha}}ds$.

Proposition 5.12. *Let Z be a r.v. in $\mathbb{R}^d_\geq$ with Laplace transform ϕ and let $\alpha\in(0,1)$. Then the following properties are equivalent:*

$$\lim_{t\to\infty} L(t)t^\alpha\mathbb{P}\left(\langle u,Z\rangle>t\right)=e(u) \qquad \forall\, u\in\mathbb{S}_\geq \tag{5.10}$$

$$\lim_{t\downarrow 0} L(1/t)\frac{1-\phi(ut)}{t^\alpha}=\Gamma(1-\alpha)e(u) \qquad \forall\, u\in\mathbb{S}_\geq \tag{5.11}$$

$$d\text{–}\lim_{t\to\infty}\frac{\mathbb{P}\left(|Z|>ts,\ Z/|Z|\in\cdot\right)}{\mathbb{P}\left(|Z|>t\right)}=s^{-\alpha}\varrho \qquad \forall\, s\in\mathbb{R}_> \tag{5.12}$$

$$\lim_{t\to\infty} t^\alpha L'(t)\mathbb{E}\left(f(t^{-1}Z)\right)=\int_0^\infty\int_{\mathbb{S}_\geq} f(sw)\,\varrho(dw)\lambda^\alpha(ds) \qquad \forall f\in\mathcal{C}_c\left(\overline{\mathbb{R}^d_\geq}\setminus\{0\}\right). \tag{5.13}$$

Here $e:\mathbb{S}_\geq\to(0,\infty)$ is a continuous function and ϱ a probability measure on $\mathbb{S}_\geq$. The functions L,L' are slowly varying at infinity and can be chosen as

$$L(t)=L'(t)=\left(t^\alpha\mathbb{P}\left(|Z|>t\right)\right)^{-1}.$$

With this choice, e and ϱ are uniquely determined and satisfy the following relation:

$$e(u)=\frac{1}{\alpha}\int_{\mathbb{S}_\geq}\langle u,w\rangle^\alpha\varrho(dw). \tag{5.14}$$

In this case, equivalence has to be understood in the following way. For instance, if (5.10) holds with a slowly varying function L and a continuous function e, then (5.11) holds with the same L and e; and there exist L' and a uniquely defined probability measure ϱ, such that (5.12) and (5.13) hold. Property (5.13) is called *multivariate regular variation*.

Proof. Step 1: The equivalence of properties (5.10) and (5.11) results from the classical Tauberian theorem above, while the equivalence of (5.10) and (5.12) was shown by Basrak, Davis and Mikosch [15, Theorem 1.1], see also Boman and Lindskog [25, Corollary 2]. Properties (5.12) and (5.13) are equivalent by function extension arguments (see [14, Theorem 2.1.4], also [25, Corollary 2]), when setting

$$L'(t)=\left(t^\alpha\mathbb{P}\left(|Z|>t\right)\right)^{-1}.$$

With this definition, L' is a slowly varying function by (5.12):

$$\lim_{t\to\infty}\frac{L(t)}{L(ts)}=\lim_{t\to\infty}\frac{s^\alpha t^\alpha\mathbb{P}\left(|Z|>ts\right)}{t^\alpha\mathbb{P}\left(|Z|>t\right)}=\varrho(\mathbb{S}_\geq)=1.$$

Step 2, proving (5.14): Observe that (5.13) implies the vague convergence

$$t^{\alpha}L'(t)\mathbb{P}\left(t^{-1}Z\in\cdot\right)\xrightarrow{v}\varrho\otimes\lambda^{\alpha}.$$

Fix $u\in\mathbb{S}_{\geq}$. The set $B=\{x\ :\ \langle u,x\rangle>1\}$ is compact in $\overline{\mathbb{R}^d_{\geq}}\setminus\{0\}$ with $\varrho\otimes\lambda^{\alpha}(B)=0$. Considering Proposition 3.1, it follows

$$\begin{aligned}\lim_{t\to\infty}t^{\alpha}L'(t)\mathbb{P}\left(\langle u,Z\rangle>t\right)&=\int_{\mathbb{S}_{\geq}}\int_0^{\infty}\mathbf{1}_{\{s\langle u,w\rangle>1\}}\lambda^{\alpha}(ds)\ \varrho(dw)\\&=\int_{\mathbb{S}_{\geq}}\int_{\langle u,w\rangle^{-1}}^{\infty}\frac{1}{s^{1+\alpha}}ds\ \varrho(dw)\\&=\int_{\mathbb{S}_{\geq}}\frac{1}{\alpha}\langle u,w\rangle^{\alpha}\varrho(dw).\end{aligned}$$

This gives that there is a constant $C>0$ such that for all $u\in\mathbb{S}_{\geq}$:

$$\frac{\alpha^{-1}\int_{\mathbb{S}_{\geq}}\langle u,w\rangle^{\alpha}\varrho(dw)}{e(u)}=C=\lim_{t\to\infty}\frac{L'(t)}{L(t)}.\tag{5.15}$$

W.l.o.g. $C=1$ by renorming $e\to Ce$, $L\to CL$. Then the asymptotics in (5.10) remain unchanged, when replacing L by L'. Thus, upon choosing $L=L'$, (5.15) yields

$$e(u)=\frac{1}{\alpha}\int_{\mathbb{S}_{\geq}}\langle u,w\rangle^{\alpha}\varrho(dw).$$

□

Remark 5.13.

- It is a classical result that if $\lim_{t\to\infty}\frac{1-\phi(tu)}{t}$ converges or $\mathbb{E}\langle u,Z\rangle<\infty$, then

$$\lim_{t\to\infty}\frac{1-\phi(ut)}{t}=\mathbb{E}\langle u,Z\rangle.$$

- It is easy to derive property (5.11) with $L(t)\equiv 1$ from the formula for the Laplace transform of $S_{\alpha}(K\nu,0)$ with

$$\frac{\Gamma(2-\alpha)}{1-\alpha}e(u)=\Gamma(1-\alpha)e(u)=K(\cos\pi\alpha/2)^{-1}\int_{\mathbb{S}_{\geq}}\langle u,y\rangle^{\alpha}\nu(dy).$$

  The limit obtained in (5.10) is then consistent with the result in [88, Property 1.2.15].

- If $Z\overset{d}{=}\tilde{S}_{\alpha}(K\nu,0)$, it would be convenient to conclude that $\varrho=\nu$, but the identity (5.14) is in general not sufficient to determine ϱ; the proof in [15] relies heavily on the convergence properties (5.10) resp. (5.12). Nevertheless, the following Proposition is a general result for multivariate α-stable laws that allows the conclusion $\varrho=\nu$.

Proposition 5.14 ([8, Corollary 3.6.20])**.** *Let $\mathcal{L}(Z)=S_{\alpha}(K\nu,\mu)$, $K>0$. Then*

1. *The function $t^{\alpha}\mathbb{P}\left(|Z|>t\right)$ is slowly varying.*

2. *The following weak convergence of measures on $\mathbb{S}_{\geq}$ holds:*

$$d\text{–}\lim_{t\to\infty} \frac{\mathbb{P}\left(|Z| > t, Z/\,|Z| \in \cdot\right)}{\mathbb{P}\left(|Z| > t\right)} = \nu.$$

Corollary 5.15. *Let ϕ be the LT of a r.v. Z in $\mathbb{R}^d_{\geq}$. If for a probability measure ν on $\mathbb{S}_{\geq}$ and some $\alpha \in (0,1)$*

$$\lim_{s\downarrow 0} \frac{1-\phi(su)}{s^\alpha} = K \int_{\mathbb{S}_{\geq}} \langle u, y\rangle^\alpha \nu(dy) \quad \forall u \in \mathbb{S}_{\geq}, \tag{5.16}$$

then Z is multivariate regular varying with index α. In particular,

$$t^\alpha \mathbb{P}\left(t^{-1} Z \in \cdot\right) \xrightarrow{v} C\nu \otimes \lambda^\alpha.$$

Proof. Consider the r.v. $Z_\alpha \stackrel{d}{=} S_\alpha(K,\nu,0)$. Its LT ϕ^α satisfies (5.16) with $L \equiv 1$. Thus by Proposition 5.12 also (5.12) holds and Proposition 5.14 yields $\eta = \nu$. But this conclusion is then true for any r.v. Z with LT ϕ satisfying (5.16), since the η in Proposition 5.12 is uniquely determined. Then from (5.13), the assertion results with

$$C = \lim_{t\to\infty} \frac{L'(t)}{L(t)} = \lim_{t\to\infty} L'(t) = \lim_{t\to\infty} \frac{1}{t^\alpha \mathbb{P}\left(|Z_\alpha| > t\right)}.$$

It has been shown in the proof of Proposition 5.12, that $\lim_{t\to\infty} L'(t)/L(t)$ exists and thus, since $L \equiv 1$, also the limit $\lim_{t\to\infty} L'(t)$ exists. ☐

5.4. Hölder Continuity

In this section a feature is shown, which only appears in the multivariate setting: Assuming that a r.v. in $\mathbb{R}^d_{\geq}$ is multivariate regular varying with index γ, the radial part of its LT ϕ is γ-Hölder if properly normalized.

Recall the definition of Hölder continuity:

Definition 5.16. Let (E,d) a metric space and $\gamma \in (0,1]$. A function $f : E \to \mathbb{R}$ is called γ-Hölder (continuous) with constant L, if

$$L := \sup_{x,y\in E} \frac{|f(x)-f(y)|}{d(x,y)^\gamma} < \infty. \tag{5.17}$$

Hölder continuity obviously implies continuity. A function is Lipschitz if and only if it is 1-Hölder. Denote by $H^\gamma(E) \subset \mathcal{C}\left(E\right)$ the set of γ-Hölder functions.

Abbreviate $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$ for $a,b \in \mathbb{R}$; and use the same notation for the componentwise minimum resp. maximum of vectors $a,b \in \mathbb{R}^d$, i.e. $(a\wedge b)_i = \min\{a_i,b_i\}$, $i = 1,\dots,d$. In the following two standard vectors in $\mathbb{R}^d_{\geq}$ resp. $\mathbb{S}_{\geq}$ will appear:

$$\vartheta_{\mathbf{d}} := (1,\dots,1)^\top \in \mathbb{R}^d, \quad \vartheta_{\mathbf{1}} = \frac{1}{\sqrt{d}}\vartheta_{\mathbf{d}}.$$

The proof of the following lemma makes an extensive use of inequalities for Laplace transforms, which are collected in the appendix.

Lemma 5.17. *Let ϕ be a Laplace transform of a r.v. Z on $\mathbb{R}^d_{\geq}$. If for some $\gamma \in (0,1]$ and $K \in \mathbb{R}_{>}$,*

$$\lim_{t\downarrow 0} \frac{(1-\phi(t\vartheta_{\mathbf{d}}))}{t^{\gamma}} = K, \tag{5.18}$$

then there is $A > 0$ such that for all $a \in [0, A]$, all $u, w \in \mathbb{S}_{\geq}$ and all $\chi \in (0, \gamma]$,

$$\left|\frac{1-\phi(au)-(1-\phi(aw))}{1-\phi(a\vartheta_{\mathbf{d}})}\right| \leq 8(|u-w| \wedge 1)^{\gamma} \leq 8(|u-w| \wedge 1)^{\chi} \tag{5.19}$$

and additionally, for all $b \geq 1$,

$$\left|\frac{1-\phi(bau)-(1-\phi(baw))}{1-\phi(ba\vartheta_{\mathbf{d}})}\right| \leq\ b \cdot 8(|u-w| \wedge 1)^{\gamma} \leq\ b \cdot 8(|u-w| \wedge 1)^{\chi}. \tag{5.20}$$

Proof. Step 1: Compute

$$\begin{aligned}
&\left|\frac{1-\phi(au)-(1-\phi(aw))}{1-\phi(a\vartheta_{\mathbf{d}})}\right| \\
&= \frac{1}{1-\phi(a\vartheta_{\mathbf{d}})}\left|\mathbb{E}\left(e^{-a\langle w,Z\rangle} - e^{-a\langle u,Z\rangle}\right)\right| \\
&\leq \frac{1}{1-\phi(a\vartheta_{\mathbf{d}})}\left(\left|\mathbb{E}\left(e^{-a\langle w\wedge u,Z\rangle} - e^{-a\langle u,Z\rangle}\right)\right| + \left|\mathbb{E}\left(e^{-a\langle w\wedge u,Z\rangle} - e^{-a\langle w,Z\rangle}\right)\right|\right) \\
&= \frac{1}{1-\phi(a\vartheta_{\mathbf{d}})}\left[\mathbb{E}\left(e^{-a\langle w\wedge u,Z\rangle}\left(1-e^{-a\langle u-w\wedge u,Z\rangle}\right)\right) + \mathbb{E}\left(e^{-a\langle w\wedge u,Z\rangle}\left(1-e^{-a\langle w-w\wedge u,Z\rangle}\right)\right)\right] \\
&\leq \frac{1}{1-\phi(a\vartheta_{\mathbf{d}})}\Big((1-\phi(a(u-w\wedge u))) + (1-\phi(a(w-w\wedge u)))\Big).
\end{aligned} \tag{5.21}$$

By assumption (5.18), the function $t \mapsto t^{-\gamma}(1-\phi(t\vartheta_{\mathbf{d}}))$ is continuous on $(0,\infty)$, hence there is $A \in (0,1]$ such that for all $a \in [0, A]$,

$$\frac{1-\phi(a\vartheta_{\mathbf{d}})}{a^{\gamma}} \in [\frac{K}{2}, 2K].$$

For all such a, $(1-\phi(a\vartheta_{\mathbf{d}}))^{-1} \leq 2K^{-1}a^{-\gamma}$. Considering the nominator, it follows, using (25.9), that

$$\begin{aligned}
&1-\phi(a(u-w\wedge u)) = 1-\phi(a\,|u-(w\wedge u)|\,\overline{u-w\wedge u}) \\
&\leq 1-\phi(a\,|u-w\wedge u|\,\vartheta_{\mathbf{d}}) \leq 2Ka^{\gamma}\,|u-w\wedge u|^{\gamma} \\
&\leq 2Ka^{\gamma}(|u-w| \wedge 1)^{\gamma}.
\end{aligned}$$

Consequently,

$$(1-\phi(a\vartheta_{\mathbf{d}}))^{-1}\,(1-\phi(a(u-w\wedge u)))\ \leq\ 2K^{-1}a^{-\gamma}2Ka^{\gamma}(|u-w| \wedge 1)^{\gamma}\ =\ 4(|u-w| \wedge 1)^{\gamma}.$$

The same calculation is valid for $(1-\phi(a\vartheta_{\mathbf{d}}))^{-1}(1-\phi(a(w-w\wedge u)))$ and thus putting both in (5.21), the first and main inequality in (5.19) is proved. For the second inequality in (5.19) use that the function $s \mapsto x^s = e^{-|\log x|s}$ is decreasing for $x \in [0,1]$, i.e.

$$(|u-w| \wedge 1)^\chi \geq (|u-w| \wedge 1)^\gamma$$

for $\chi \leq \gamma$.

Step 2: In order to prove (5.20), compute

$$\begin{aligned}
&\left|\frac{1-\phi(bau)-(1-\phi(baw))}{1-\phi(ba\vartheta_{\mathbf{d}})}\right| \overset{(25.7)}{\leq} \frac{|1-\phi(bau)-(1-\phi(baw))|}{1-\phi(a\vartheta_{\mathbf{d}})} \\
\leq\ & \frac{1}{1-\phi(a\vartheta_{\mathbf{d}})}\left((1-\phi(ba(u-w\wedge u)))+(1-\phi(ba(w-w\wedge u)))\right) \\
\overset{(25.8)}{\leq}\ & \frac{b}{1-\phi(a\vartheta_{\mathbf{d}})}\Big(1-\phi(a(u-w\wedge u))+(1-\phi(a(w-w\wedge u)))\Big).
\end{aligned}$$

From here, proceed as in the first step. □

6. The Simple Markov Renewal Theorem

Blackwell's renewal theorem for random walks (RWs) $(V_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ with a positive drift ([45, XI.9], [11]) gives asymptotics of the renewal measure

$$\mathbb{U}(I+t) := \mathbb{E}N(I+t) := \mathbb{E}\sum_{n=0}^{\infty} \mathbf{1}_I(V_n - t)$$

of an interval $I \subset \mathbb{R}$ for $t \to \infty$ as well as for $t \to -\infty$, the latter being zero.

The proof of the assertion for $t \to -\infty$ is (nowadays) quite simple – hence the result is called *simple renewal theorem*. One shows that $(\mathbb{N}(I+t))_{t\in\mathbb{R}}$ is uniformly integrable ([11, Step 1]), then it is a direct consequence of the strong law of large numbers (see [11, Remark 2]).

It is interesting to observe that this second part about asymptotics for $t \to -\infty$ does not appear in the statement of the Markov renewal theorem (MRT) neither by Kesten [60], nor by subsequent authors [2, 12, 63, 80]. This is why it will be given here, in the setting of Kesten [60, Theorem 2].

The basic idea of the subsequent proof is the same as in the classical non-Markov case, but the question of uniform integrability is more involved. Nevertheless, all necessary tools are already hidden in [60], the task is now to put them together properly.

6.1. The Setting and Prerequisites

In this subsection, notation related to Markov renewal theory is introduced and some special sets are defined for which the uniform integrability will hold. In the whole section, let (S,d) be a separable metric space equipped with the Borel σ-field.

Definition 6.1. Let $(X_n, U_n)_{n\in\mathbb{N}_0}$ be a temporally homogeneous Markov chain (MC) on $S \times \mathbb{R}$ such that

$$\mathbb{P}\left((X_{n+1}, U_{n+1}) \in A \times B \mid X_n, U_n)\right) = P(X_n, A \times B) \quad \text{a.s.} \tag{6.1}$$

for all $n \in \mathbb{N}_0$ and a transition kernel P. Then the associated sequence $(X_n, V_n)_{n\in\mathbb{N}_0}$ with $V_n = V_{n-1} + U_n$ for $n \in \mathbb{N}$ is also a MC and called Markov random walk (MRW) with driving chain $(X_n)_{n\in\mathbb{N}_0}$.

In this section, the convention $\mathbb{P}_x(X_0 = x, V_0 = 0) = 1$ is used. Denote the Markov renewal measure associated with the given MRW under $\mathbb{P}_x$ by $\mathbb{U}_x = \sum_{n=0}^{\infty} \mathbb{P}_x((X_n, V_n) \in \cdot)$.

Define subsets of S by

$$C_0 = \emptyset, \quad C_k = \left\{x \in S \,:\, \mathbb{P}_x\left(\frac{V_m}{m} \geq \frac{1}{k}\ \forall m \geq k\right) \geq \frac{1}{2}\right\} \tag{6.2}$$

for $k \geq 1$.

Investigating, e.g. C_1, it becomes obvious that after each visit of (X_n, V_n) to $C_1 \times [a, a+1]$, the MRW leaves this set forever after at most one more step (due to the transience of (V_n)) with probability at least $\frac{1}{2}$. In other words, the random number of renewals,

$$N(C_1 \times [t, t+1]) = \sum_{n=0}^{\infty} \mathbf{1}_{C_1\times[t,t+1]}(X_n, V_n)$$

of visits to $C_1 \times [t, t+1]$ is stochastically bounded by a r.v. N with geometric distribution, thus

$$\mathbb{U}(C_1 \times [t, t+1]) = \mathbb{E}N(C_1 \times [t, t+1]) \leq \mathbb{E}N < \infty$$

and this holds for any $a \in \mathbb{R}$. Hence there are at least some special sets with uniformly bounded Markov renewal measure. To be precise, the following Lemma holds.

Lemma 6.2. *Let $x \in S$, $t \in \mathbb{R}$, $a \in \mathbb{R}_>$, $k \in \mathbb{N}$. Then the family*

$$(N_t)_{t\in\mathbb{R}} := (N(C_k \times [t, t+a]))_{t\in\mathbb{R}} \tag{6.3}$$

is uniformly integrable w.r.t. to $\mathbb{P}_x$, and

$$\mathbb{U}_x(C_k \times [t, t+a]) = \mathbb{E}_x N(C_1 \times [t, t+a]) \leq 2(k+1+ka). \tag{6.4}$$

The proof is based upon the ideas in [60, Lemma 6] and can be found in the appendix, page 126.

Proposition 6.3. *Let $x \in S$ and assume that there is $l > 0$ such that*

$$\lim_{n\to\infty} \frac{V_n}{n} = l \quad \mathbb{P}_x\text{-a.s.}. \tag{6.5}$$

Then for all $k \in \mathbb{N}$, $a < b \in \mathbb{R}$

$$\lim_{t\to-\infty} \mathbb{U}_x(C_k \times [t+a, t+b]) = 0. \tag{6.6}$$

Proof. The convergence (6.5) assures that V_n is bounded from below for a.e. path, thus

$$\lim_{t\to-\infty} N(C_k \times [t+a, t+b]) = \lim_{t\to\infty} \sum_{n=0}^{\infty} \mathbf{1}_{C_k\times[a,b]}(X_n, V_n - t) = 0 \; \mathbb{P}_x\text{-a.s.}.$$

Considering Lemma 6.2, the family $(N(C_k \times [a+t, b+t]))$ is uniformly integrable, which gives convergence of the expectations:

$$\lim_{t\to-\infty} \mathbb{U}_x(C_k \times [t, t+a]) = \lim_{t\to\infty} \mathbb{E}_x N(C_k \times [t+a, t+b]) = 0. \qquad \square$$

6.2. Direct Riemann Integrability and the Simple Markov Renewal Theorem

This subsection is devoted to the formulation and the proof of the simple MRT.

Definition 6.4. A measurable function $g : S\times\mathbb{R} \to \mathbb{R}$ is called strongly directly Riemann integrable (sdRi) w.r.t. to $(\mathbb{P}_x)_{x\in S}$, if

$$g(u,\cdot) \text{ is Lebesgue-a.e. continuous, and}$$
$$\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}} (k+1) \sup\{|g(x,t)| \;:\; x \in C_{k+1} \setminus C_k, t \in [l, l+1]\} < \infty.$$

The dependence on $(\mathbb{P})_{x\in S}$ is via the sets C_k.

Theorem 6.5 (The Simple Markov Renewal Theorem)**.** *Assume that there is $l > 0$ such that*

$$\lim_{n\to\infty} \frac{V_n}{n} = l \quad \mathbb{P}_x\text{-a.s.} \quad \forall x \in S. \tag{6.7}$$

Then for every function g which is sdRi w.r.t. $(\mathbb{P}_x)_{x\in S}$ and all $x \in S$,

$$\lim_{t\to-\infty} g * \mathbb{U}_x(t) := \lim_{t\to-\infty} \mathbb{E}_x \left(\sum_{n=0}^{\infty} g(X_n, t - V_n) \right) = 0. \tag{6.8}$$

Proof. Fix $x \in S$. Referring to property (6.7), $S = \cup_{n=0}^{\infty} C_k$. Thus, it follows that for all $(y, s) \in S \times \mathbb{R}$

$$g(y,s) \le \hat{g}(y,s) := \sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}} \left(\sup_{C_{k+1}\setminus C_k\times[l,l+1]} |g| \right) \mathbf{1}_{C_{k+1}\setminus C_k\times[l,l+1]}(y,s). \tag{6.9}$$

Consequently, by an application of Lemma 6.2, for all $(x,t) \in S \times \mathbb{R}$,

$$\mathbb{E}_x \sum_{n=0}^{\infty} |g(X_n, t - V_n)| \le \mathbb{E}_x \sum_{n=0}^{\infty} \hat{g}(X_n, t - V_n)$$
$$= \sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}} \left(\sup_{C_{k+1}\setminus C_k\times[l,l+1]} |g| \right) \mathbb{E}_x \sum_{n=0}^{\infty} \mathbf{1}_{C_{k+1}\setminus C_k\times[l,l+1]}(X_n, t - V_n)$$

$$\leq \sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}}\left(\sup_{C_{k+1}\setminus C_k\times[l,l+1]}|g|\right)\mathbb{U}_x(C_{k+1}\times[\lceil t\rceil-l-2,\lceil t\rceil-l]) \qquad \left(=:\sum_{k,l}f_t(k,l)\right)$$
$$\leq \sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}}2(k+2+2k)\left(\sup_{C_{k+1}\setminus C_k\times[l,l+1]}|g|\right) \qquad \left(=:\sum_{k,l}f(k,l)\right)$$
$$\leq 6\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}}(k+1)\sup\{|g(x,t)|\ :\ x\in C_{k+1}\setminus C_k, t\in[l,l+1]\}<\infty.$$

This shows that the bounded convergence theorem may be applied to the sequence of functions

$$f_t(k,l)=\left(\sup_{C_{k+1}\setminus C_k\times[l,l+1]}|g|\right)\mathbb{U}_x(C_{k+1}\times[\lceil t\rceil-l-2,\lceil t\rceil-l]),$$

which converge to 0 pointwise by Proposition 6.3; this allows to conclude

$$\begin{aligned}
0 &\leq \limsup_{t\to-\infty}\left|\mathbb{E}_x\sum_{n=0}^{\infty}g(X_n,t-V_n)\right| \\
&\leq \limsup_{t\to-\infty}\mathbb{E}_x\sum_{n=0}^{\infty}|g(X_n,t-V_n)| \\
&\leq \limsup_{t\to-\infty}\sum_{k,l}f_t(k,l)=\lim_{t\to-\infty}\sum_{k,l}f_t(k,l)=\sum_{k,l}\lim_{t\to-\infty}f_t(k,l) \\
&=\sum_{k=0}^{\infty}\sum_{l\in\mathbb{Z}}\left(\sup_{C_{k+1}\setminus C_k\times[l,l+1]}|g|\right)\lim_{t\to-\infty}\mathbb{U}_x(C_{k+1}\times[\lceil t\rceil-l-2,\lceil t\rceil-l])=0.
\end{aligned}$$

This gives the asserted convergence. □

Corollary 6.6. *If g is sdRi w.r.t. $(\mathbb{P}_x)_{x\in S}$ and $S=\bigcup_{k=0}^{\infty}C_k$, then the family*

$$(|g|*\mathbb{U}_x(t))_{(x,t)\in S\times\mathbb{R}}=\left(\mathbb{E}_x\sum_{n=0}^{\infty}|g(X_n,t-V_n)|\right)_{(x,t)\in S\times\mathbb{R}} \tag{6.10}$$

is uniformly bounded.

7. Transfer Operators

This section considers operators in $\mathcal{C}(\mathbb{S}_{\geq})$, related to the action of the random matrix $\mathbf{T}_1$ on $\mathbb{S}_{\geq}$. Here the spectral function, resp. $\kappa(s)$ will reappear. The following operators in $\mathcal{C}(\mathbb{S}_{\geq})$ will be studied:

$$P^s f(x) := \mathbb{E}\left(|\mathbf{T}_1 x|^s f(\mathbf{T}_1\cdot x)\right), \tag{7.1}$$
$$P_*^s f(x) := \mathbb{E}\left(\left|\mathbf{T}_1^\top x\right|^s f(\mathbf{T}_1^\top\cdot x)\right)=\mathbb{E}\left(|\mathbf{M}_1 x|^s f(\mathbf{M}_1\cdot x)\right). \tag{7.2}$$

They are well defined for all $s \in I_\mu$ and define Markov transititon operators P, P_* if $s = 0$.

From the very beginning, properties of these transfer operators played an important role in the study of multivariate fixed point equations. They first appear in the proof of [59, Theorem 3] (there named $T_\varkappa$) and have been intensively studied by Guivarc'h and Le Page in [50, 52]. The results that will be presented in this section are mainly taken from [29]. The properties rely heavily on condition (C) and partially also on moment conditions on $\|\mathbf{T}_1\|$ and $\iota(\mathbf{T}_1)$. Before presenting the results, these assumptions will be discussed.

7.1. Condition (C)

In this subsection the conditions imposed on the distribution μ of the random matrix $\mathbf{T}_1$ are discussed.

This condition that $[\operatorname{supp} \mu]$ satisfies (C) is far from being restrictive as the following Lemma shows. It is stated without proof in [51], the idea of proof is due to Guivarc'h (private communication).

Lemma 7.1. *The set $\mathfrak{C}$ of measures μ, such that $[\operatorname{supp} \mu]$ satisfies condition (C), is dense in $\mathcal{P}(\mathcal{M}_+)$ with respect to the weak topology.*

Proof. The first reduction is that since $\mathcal{M}_+$ is separable, the Dirac measures form a dense subset of $\mathcal{P}(\mathcal{M}_+)$, hence it suffices to show that any Dirac $\delta_\mathbf{B}$ measure is a weak limit of measures in $\mathfrak{C}$.

Fix any $\mathbf{A} \in \breve{\mathcal{M}}_+$. Choose $\varrho > 0$ s.t. $B_\varepsilon(\mathbf{A}) \subset \breve{\mathcal{M}}_+$. Denote by λ^{d^2} the Lebesgue measure on $\mathcal{M}_+$, seen as a subset of $\mathbb{R}^{d^2}$. If W is a proper subspace of $\mathbb{R}^d$, then the orthogonal space $W^\perp \neq \emptyset$ and if $\mathbf{B}W \subset W$ for a matrix $\mathbf{B}$, then for all $x \in W, y \in W^\perp$,

$$\langle \mathbf{B}x, y \rangle = 0.$$

But the set of matrices (resp. matrix coefficients) that satisfy such an equation has the Lebesgue measure 0. Hence the normalized restriction of λ^{d^2} to an open ball, $\lambda^{d^2}(B_\varrho(\mathbf{A}))^{-1}\, \lambda^{d^2}\Big|_{B_\varrho(\mathbf{A})}$ is in $\mathfrak{C}$. Moreover, for any $\mathbf{B} \in \mathcal{M}_+$,

$$\delta_\mathbf{B} = \lim_{\varepsilon \to 0} (1-\varepsilon)\delta_\mathbf{B} + \varepsilon \lambda^{d^2}(B_\varrho(\mathbf{A}))^{-1}\, \lambda^{d^2}\Big|_{B_\varrho(\mathbf{A})}$$

in total variation, thus also in the weak topology. □

The second assumption is on the moments of $\mathbf{T}_1$ and in particular that $\mathbb{E}\,|\log \iota(\mathbf{T}_1)| < \infty$. This is on the one hand a lower bound for $|\mathbf{T}_1 x|$, on the other hand it guarantees that $\breve{\mathbb{S}}_\geq$ is invariant under the action of $\mathbf{T}_1$ – note that only $\breve{\mathcal{M}}_+ \breve{\mathbb{S}}_\geq \subset \breve{\mathbb{S}}_\geq$ holds a priori. The connection with the invariance of $\breve{\mathbb{S}}_\geq$ is stated in the following Lemma:

Lemma 7.2. *The following conditions are equivalent for $\mu \in \mathcal{P}(\mathcal{M}_+)$:*

(1) $\mu\{\mathbf{A} \,:\, \mathbf{A}$ *has no zero column* $\} = 1$.

(2) $\mu\{\mathbf{A} \,:\, \mathbf{A}^\top \cdot \breve{\mathbb{S}}_\geq \subset \breve{\mathbb{S}}_\geq\} = \mu^*\{\mathbf{M} \,:\, \mathbf{M} \cdot \breve{\mathbb{S}}_\geq \subset \breve{\mathbb{S}}_\geq\} = 1$.

(3) $\mu\{\mathbf{A} \,:\, \iota(\mathbf{A}) > 0\} = 1$

A sufficient condition is that $\mathbb{E}\left|\log \iota(\mathbf{T_1})\right| < \infty$.

Proof. (1) $\Leftrightarrow$ (2): The condition that $\mathbf{A}$ has no zero column can be stated as: for each i there is a j s.t. $\mathbf{A}_{ji} > 0$. Then the sets in (1) and (2) are equivalent by the formula

$$(\mathbf{A}^\top x)_i = \sum_{j=1}^{d} \mathbf{A}_{ji} x_j.$$

(1) $\Leftrightarrow$ (3): Reformulate property (1) again: It is equivalent to $a := \min_j \sum_{i=1}^{d} \mathbf{A}_{ij} > 0$. Then the equivalence of (1) and (3) follows by the inequalities in Subsection 25.2, valid for all $x \in \mathbb{S}_{\geq}$,

$$\sum_{i=1}^{d}\sum_{j=1}^{d} \mathbf{A}_{ij} x_j \;\geq\; |\mathbf{A}x| \;\geq\; d^{-\frac{1}{2}} \sum_{i=1}^{d}\sum_{j=1}^{d} \mathbf{A}_i j x_j \;\geq\; d^{-\frac{1}{2}} \sum_{j=1}^{d} a x_j \;\geq\; d^{-\frac{1}{2}} a\,|x|\,.$$

(1) follows by choosing particular x, e.g. the canonical basis of $\mathbb{R}^d$.

Writing the sufficient condition as $\int \left|\log \iota(\mathbf{A})\right| \mu(d\mathbf{A}) < \infty$, it becomes obvious that it implies (3). □

As additional properties of $\iota(\cdot)$ note that $\iota(\mathbf{A}) > 0$ for $\mathbf{A} \in \breve{\mathcal{M}}_+$. Also if $\mathbf{A} \in GL(d, \mathbb{R})$, then $\iota(\mathbf{A}) \geq \left\|A^{-1}\right\|^{-1} > 0$ and equality holds if the infimum is taken over $\mathbb{S}$ instead of $\mathbb{S}_{\geq}$, since

$$\inf_{x\in\mathbb{S}} |\mathbf{A}x| = \inf_{y\in\mathbb{S}} \left|\mathbf{A}(\mathbf{A}^{-1}\cdot y)\right| = \inf_{y\in\mathbb{S}} \frac{1}{|\mathbf{A}^{-1}y|} \left|\mathbf{A}\mathbf{A}^{-1}y\right| = \left(\sup_{y\in\mathbb{S}} \left|\mathbf{A}^{-1}y\right|\right)^{-1} = \left\|\mathbf{A}^{-1}\right\|^{-1} \quad (7.3)$$

7.2. Properties of Transfer Operators

The results of this subsection are taken from [29], which is based upon the fundamental paper [52]; see also [75]. The main properties of P^s, P^s_* (when assuming (C)) are contained in the following Theorem.

Notation will be abused for it will not be distinguished between an operator $Q : \mathcal{C}(\mathbb{S}_{\geq}) \to \mathcal{C}(\mathbb{S}_{\geq})$ and its adjoint (see [40, Chapter VI]) $Q' : \mathcal{C}(\mathbb{S}_{\geq})' \to \mathcal{C}(\mathbb{S}_{\geq})'$ in the set $\mathcal{C}(\mathbb{S}_{\geq})' = \mathcal{M}^{\pm}(\mathbb{S}_{\geq})$ of regular bounded signed measures on $\mathbb{S}_{\geq}$. Recall that Q' is defined by the identity

$$\int_{\mathbb{S}_{\geq}} f \, d(Q'\nu) = \int_{\mathbb{S}_{\geq}} (Qf)\, d\nu \quad \forall f \in \mathcal{C}(\mathbb{S}_{\geq}),\; \nu \in \mathcal{M}^{\pm}(\mathbb{S}_{\geq})\,.$$

So by writing $P^s\nu$, in fact the image measure $(P^s)'\nu$ is meant.

By the Perron-Frobenius theorem, any matrix $\mathbf{A} \in \breve{\mathcal{M}}_+$ has a unique dominant eigenvalue $\lambda_{\mathbf{A}}$ and the corresponding eigenvector (of unit length) $u_{\mathbf{A}}$ has strictly positive entries, i.e. $u_{\mathbf{A}} \in \breve{\mathbb{S}}_{\geq}$. Denote

$$\Lambda(\Gamma) = \overline{\{u_{\mathbf{A}} \;:\; \mathbf{A} \in \Gamma \cap \breve{\mathcal{M}}_+\}}.$$

Theorem 7.3 ([29, Theorem 3.3])**.** *Assume that* $[\operatorname{supp}\mu]$ *satisfies condition* (C) *and let* $s \in I_\mu$. *Then the following holds:*

1. *The spectral radius and the dominant eigenvalue of* P^s *are equal to* $\kappa(s)$.
2. *There is a unique strictly positive normalized function* e^s *(*$|e^s|_\infty = 1$*) and a unique probability measure* ν^s *such that*
$$P^s e^s = \kappa(s) e^s, \quad P^s \nu^s = \kappa(s)\nu^s. \tag{7.4}$$
3. *The function* e^s *is* $\min\{s,1\}$*-Hölder and* $\operatorname{supp}\nu^s = \Lambda([\operatorname{supp}\mu])$.
4. *For all* $f \in \mathcal{C}(\mathbb{S}_\geq)$,
$$\lim_{n\to\infty} \left| \frac{(P^s)^n f}{\kappa(s)^n} - \frac{\nu^s(f)}{\nu^s(e^s)} e^s \right|_\infty = 0. \tag{7.5}$$
5. *The function* $s \mapsto \kappa(s)$ *is strictly convex on* I_μ.
6. *In the same way*
$$P_*^s e_*^s = \kappa(s) e_*^s, \quad P_*^s \nu_*^s = \kappa(s)\nu_*^s,$$
for a unique probability measure ν_*^s *and a unique normalized strictly positive function* e_*^s *which satisfies the identity*
$$e_*^s(u) = \frac{\int \langle u, y\rangle^s \nu^s(du)}{\left|\int \langle \cdot, y\rangle^s \nu^s(dy)\right|_\infty}. \tag{7.6}$$

Defining $c_s := \left|\int \langle \cdot, y\rangle^s \nu^s(dy)\right|_\infty$, it follows that $c_s e_*^s(u) = \int_{\mathbb{S}_\geq} \langle u, y\rangle^s \nu^s(dy)$ and the quantity c_s measures, how “random” the distribution ν^s is – $c_s = 1$ for $s < 1$ if and only if ν^s is a point mass. Later, c_s will reappear as a norming constant.

Even if it is hard to calculate explicit values of $m(s)$ resp. $\kappa(s)$ due to the occurence of the limit, it is easy to give a sufficient condition for the existence of $\alpha \in (0,1]$. By the convexity of m resp. κ and the fact that $m(0) = N > 1$, α exists in $(0,1)$ if $m(1) < 1$ resp. $\kappa(1) < \frac{1}{N}$. In order to check this, the following lemma is very helpful.

Lemma 7.4 ([29, Lemma 3.5])**.** *Assume that* $[\operatorname{supp}\mu]$ *satisfies condition* (C), $\mathbb{E}\|\mathbf{T}_1\| < \infty$. *Let* $\mathbf{A} = \mathbb{E}\mathbf{T}_1$. *Then for some* $n \geq 1$, $\mathbf{A}^n \in \breve{\mathcal{M}}_+$. *If* $r(\mathbf{A})$ *denotes the spectral radius of* $\mathbf{A}$ *and* v_* *the Perron-Frobenius eigenvector of* $(\mathbf{A}^\top)$ *of unit length , then it holds that*

$$\kappa(1) = r(\mathbf{A}), \quad e^1(x) = \langle v_*, x\rangle, \quad \int_{\mathbb{S}_\geq} y\nu_*^1(dy) = v_* \tag{7.7}$$

Corollary 7.5. *Assume that* $[\operatorname{supp}\mu]$ *satisfies condition* (C), $\mathbb{E}\|\mathbf{T}_1\| < \infty$. *If the spectral radius of* $\mathbb{E}\mathbf{T}_1$ *is less than* $\frac{1}{N}$, *as a consequence there exists* $\alpha \in (0,1)$ *such that*

$$m(\alpha) = 1, \quad m'(\alpha) < 0.$$

Lemma 7.4 also allows to partly answer the question, whether there are fixed points for $\alpha > 1$:

Proposition 7.6. *Assume that* $[\operatorname{supp}\mu]$ *satisfies condition* (C), $\mathbb{E}\,\|\mathbf{T}_1\| < \infty$. *If there is a fixed point* $Y \in \mathbb{R}^d_{\geq}$ *of* $\mathcal{S}$ *with finite expectation, then* $m(1) = 1$, *in particular,* $\alpha \leq 1$.

Proof. Let Y be a fixed point of $\mathcal{S}$ with finite expectation, say $y = \mathbb{E}Y \in \mathbb{R}^d_{\geq} \setminus \{0\}$, let $(Y_i)_{i=1}^N$ be i.i.d. copies of Y, independent of T. Then

$$Y \stackrel{d}{=} \sum_{i=1}^{N} \mathbf{T}_i Y_i,$$

and, taking expectations, $y = N\mathbb{E}\mathbf{T}_1 y$. I.e., y is an eigenvector of $\mathbb{E}\mathbf{T}_1$ with eigenvalue $\frac{1}{N}$ and since $y \in \mathbb{R}^d_{\geq} \setminus \{0\}$, it is the Perron-Frobenius eigenvector of $\mathbb{E}\mathbf{T}_1$ Referring to Lemma 7.4, $\kappa(1) = \frac{1}{N}$, thus $m(1) = 1$. □

7.3. Simulation of $\kappa(s)$

Integrating (7.5) with respect to a probability measure $\eta \in \mathcal{P}(\mathbb{S}_{\geq})$, it follows that for all $f \in \mathcal{C}\left(\mathbb{S}_{\geq}\right)$

$$\lim_{n\to\infty} \frac{((P^s)^n \eta)(f)}{\kappa(s)^n} = \lim_{n\to\infty} \frac{\eta((P^s)^n f)}{\kappa(s)^n} = \nu^s(f)\frac{\eta(e^s)}{\nu^s(e^s)}.$$

Introducing the operator[1] $\tilde{P^s} \,:\, \mathcal{P}(\mathbb{S}_{\geq}) \to \mathcal{P}(\mathbb{S}_{\geq})$

$$\tilde{P^s}\eta := \frac{P^s\eta}{(P^s\eta)(\mathbf{1}_{\mathbb{S}_{\geq}})},$$

consequently

$$(\tilde{P^s})^n \eta \xrightarrow{d} \nu^s. \tag{7.8}$$

This property, together with the identity

$$\int_{\mathbb{S}_{\geq}} \int_{\mathcal{M}_+} |\mathbf{A}x|^s \, \mu(d\mathbf{A})\nu^s(dx) = (P^s\nu^s)(\mathbf{1}_{\mathbb{S}_{\geq}}) = \kappa(s)\nu^s(1) = \kappa(s) \tag{7.9}$$

which results from (7.4), is the basis for several simulation algorithms for $\kappa(s)$. The first algorithm was proposed by Basrak and Segers in [16], but it works only in special cases, see [17]. An alternative algorithm, which works and fits perfectly to the present situation, is introduced in the thesis of Janßen [56, Chapter 4].

The basic idea in [56, Section 4.4] is composed as follows: The convergence in (7.8) allows to approximate ν^s by $\nu_n := (\tilde{P^s})^n \nu$ via the Markov Chain Monte Carlo-methods. The convergence of ν_n can be checked by a Kolmogorov-Smirnov test. When a good candidate ν_n is obtained, draw samples from ν_n and μ and use identity (7.9) to compute an estimate for $\kappa(s)$ via Monte Carlo-integration.

[1] In fact, this operator already appears in the proof of Theorem 7.3 in [29].

The convergence result (7.8) also positively answers the question raised in [56, middle of p.69] whether the algorithm converges for any initial distribution.

There are several improvements concerning this algorithm, using different sample techniques and improving its speed. For 2×2-matrices, a good simulation of $\kappa(s)$ for 140 values of s, using one of the improved algorithms, runs in under 2 hours; which is a reasonably finite amount of time. This is work in progress by Holger Drees and Anja Janßen (private communication).

8. Markov Random Walks and Change of Measure

In this section, a MRW $(X_n, V_n)_{n \in \mathbb{N}_0}$ is introduced which is given by the action of $\mathbf{T}_1$. The results of Section 7 are used in order to define transformed probability measures for $(X_n, V_n)_{n \in \mathbb{N}_0}$ and to apply the simple MRT to $(X_n, V_n)_{n \in \mathbb{N}_0}$.

8.1. Change of Measure

In this subsection, transformed measures $\mathbb{Q}_x^s$ will be defined under which $(\mathbf{M}_n)_{n \in \mathbb{N}}$ are no longer i.i.d..

Let $\Omega := \mathbb{S}_\geq \times \mathcal{M}_+^{\mathbb{N}}$ and let $(X_0, (\mathbf{M})_{n \in \mathbb{N}})$ be the filtered identity mapping on Ω. For each $x \in \mathbb{S}$, define a probability measure $\mathbb{Q}_x$ on Ω by

$$\mathbb{Q}_x := \delta_x \otimes \bigotimes_{n=1}^{\infty} \mu^*. \tag{8.1}$$

I.e. $\mathbb{Q}_x(X_0 = x) = 1$ and under each $\mathbb{Q}_x$, $(\mathbf{M}_n)_{n \in \mathbb{N}}$ is a sequence of i.i.d. copies of $\mathbf{T}_1^\top$. Denote the associated expectation symbol by $\mathbb{E}_x$.

Write $\Pi_n := \mathbf{M}_n \dots \mathbf{M}_1$. Defining

$$X_n := \mathbf{M}_n \dots \mathbf{M}_1 \cdot X_0 = \Pi_n \cdot X_0, \quad V_n := \log |\mathbf{M}_n \dots \mathbf{M}_1 X_0| = \log |\Pi_n X_0|,$$

it follows that for all $n \in \mathbb{N}$

$$X_n = \mathbf{M}_n \cdot X_{n-1}, \quad V_n = V_{n-1} + U_n = V_{n-1} + \log |\mathbf{M}_n X_{n-1}|.$$

Hence $(X_n, V_n)_{n \in \mathbb{N}_0}$ is a MRW under each $\mathbb{Q}_x$.

Observe that the identity $(P_*^s)^n e_*^s = \kappa(s)^n e_*^s$, $n \in \mathbb{N}$, can be written as

$$e_*^s(x) = \frac{1}{\kappa(s)^n} \mathbb{E}_x \left(|\Pi_n X_0|^s \, e_*^s(\Pi_n \cdot X_0) \right) = \frac{1}{\kappa(s)^n} \mathbb{E}_x \left(e^{sV_n} e_*^s(X_n) \right). \tag{8.2}$$

Thus for $n \in \mathbb{N}$ new probability measures ${}_n\mathbb{Q}_x^s$ can be defined on $\mathbb{S}_\geq \times \mathcal{M}_+^n$ by

$${}_n\mathbb{Q}_x^s((X_0, \mathbf{M}_1, \dots, \mathbf{M}_n) \in A) := \frac{1}{e_*^s(x) \kappa^n(s)} \mathbb{E}_x \left(e^{sV_n} e_*^s(X_n) \mathbf{1}_A(X_0, \mathbf{M}_1, \dots, \mathbf{M}_n) \right) \tag{8.3}$$

for all Borel sets A. Referring to (8.2), the sequence $({}_n\mathbb{Q}_x^s)_n$ constitutes a projective system, thus by the Kolmogorov extension theorem [28, Corollary 2.19] it defines a probability measure $\mathbb{Q}_x^s$ on Ω. Denoting the corresponding expectation symbol by $\mathbb{E}_x^s$, the following identity holds:

$$\mathbb{E}_x^s\left(f(X_0, V_0, \dots X_n, V_n)\right) = \frac{1}{e_*^s(x)\kappa^n(s)}\mathbb{E}_x\left(e^{sV_n}e_*^s(X_n)f(X_0, V_0, \dots, X_n, V_n)\right). \tag{8.4}$$

It is valid for all bounded measurable functions f and all $n \in \mathbb{N}$.

Introduce the Markov transition kernel on $\mathcal{C}(\mathbb{S}_\geq)$,

Proposition 8.1 (contained in proof of [29, Theorem 3.3])**.** *Under each $\mathbb{Q}_x^s$, $(X_n)_{n\in\mathbb{N}_0}$ is a Markov Chain with transition kernel*

$$Q_*^s f(x) := \frac{1}{e_*^s(x)\kappa(s)} P_*^s(e_*^s f)(x). \tag{8.5}$$

and a unique stationary distribution π_*^s, which is given by

$$\pi_*^s(dx) = e_*^s(x)\nu_*^s(dx). \tag{8.6}$$

In the case when $\kappa(s) = 1$, the function $(x, t) \mapsto e_*^s(x)e^{st}$ is a harmonic function for the MC (X_n, V_n) under $\mathbb{Q}_x$. Hence the measure $\mathbb{Q}_x^s$ can also be obtained in terms of a harmonic transform. This approach is described in detail in [10].

Setting $\mathbb{Q}^s := \int \mathbb{Q}_x^s \pi_*^s(dx)$, it follows that $(X_n)_{n\in\mathbb{N}_0}$ is stationary under $\mathbb{Q}^s$. Concerning the drift of the random walk part $(V_n)_{n\in\mathbb{N}_0}$, the following theorem is useful:

Theorem 8.2 ([29, Theorem 3.7])**.** *Assume that $[\operatorname{supp}\mu]$ satisfies (C), $s \in I_\mu$ and*

$$\mathbb{E}\|\mathbf{M}\|^s\left(|\log\|\mathbf{M}\|| + |\log\iota(\mathbf{M})|\right) < \infty. \tag{8.7}$$

Then, for any $x \in \mathbb{S}_\geq$,

$$l(s) = \lim_{n\to\infty}\frac{V_n}{n} = \lim_{n\to\infty}\frac{1}{n}\log\|\mathbf{M}_n \dots \mathbf{M}_1\| \quad \mathbb{Q}_x^s\text{-a.s.}, \tag{8.8}$$

where

$$l(s) = \mathbb{E}_{\pi_*^s}^s V_1 = \frac{\kappa'(s^-)}{\kappa(s)}. \tag{8.9}$$

The number $l(0)$ is called the (upper) Lyapunov exponent associated with μ. See [26, Definition I.2.1] as well as [26, Section III.5] and Oseledec's multiplicative ergodic theorem [82] for more about Lyapunov exponents.

Here the quantity $\iota(\mathbf{M})$ reappears. Recall that by Lemma 7.2, the finiteness of $\mathbb{E}\,|\log\iota(\mathbf{M})|$ implies that for all $x \in \mathbb{S}_\geq$, $\mathbf{M}_1 x \neq 0$ a.s. and thus also $\Pi_n x \neq 0$ a.s.. This property accounts for the nondegeneracy of $\lim_{n\to\infty}\frac{V_n}{n}$ for any initial value $X_0 = x$, while the independence of $l(s)$ and X_0 is a consequence of condition (C), see [26, Chapter 3] and [52, Theorem 3.10] for details.

Next is a portmanteau moment condition, that will be assumed in all the main results. It incorporates both $\mathbb{E}\,\|\mathbf{M}_1\| < \infty$, which gives $[0,1] \in I_\mu$ as well as the validity of condition (8.7) for all $s \in [0,1]$:

$$\mathbb{E}(1 + \|\mathbf{M}_1\|)\,(1 + |\log\|\mathbf{M}_1\|| + |\log \iota(\mathbf{M}_1)|) < \infty. \tag{M logM}$$

Corollary 8.3. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C) (M logM) *holds, and* $\alpha \in (0,1]$. *Then for all* $x \in \mathbb{S}_\geq$,

$$\lim_{n\to\infty} \frac{V_n}{n} < 0 \quad \mathbb{Q}_x^\alpha\text{-a.s.} \tag{8.10}$$

and (8.4) *takes the form*

$$\mathbb{E}_x^\alpha\left(f(X_0, V_0, \ldots X_n, V_n)\right) = \frac{N^n}{e_*^\alpha(x)} \mathbb{E}_x\left(e^{\alpha V_n} e_*^\alpha(X_n) f(X_0, V_0, \ldots, X_n, V_n)\right). \tag{8.11}$$

8.2. The Simple Markov Renewal Theorem Revisited

Now the simple MRT and the property of direct Riemann integrability can be stated in the form that will be used in the proofs of the main theorems.

As a first step, a sufficient condition for strong direct Riemann integrability with respect to $(\mathbb{Q}_x^\alpha)_{x\in\mathbb{S}_\geq}$ can be derived.

Definition 8.4. Say that $g \in \mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$ is (multivariate) *directly Riemann integrable (dRi)*, if

$$\sum_{l\in\mathbb{Z}} \sup\left\{|g(u,t)|\ : u \in \mathbb{S}_\geq, t \in [l, l+1]\right\} < \infty. \tag{8.12}$$

Lemma 8.5. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C), (M logM) *holds and that there is* $\alpha \in (0,1]$ *with* $m(\alpha) = 1$ *and* $m'(\alpha) < 0$. *If* $g \in \mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$ *is (multivariate) dRi then* $\tilde{g}(y,s) := g(y,-s)$ *is sdRi w.r.t. to the measures* $(\mathbb{Q}_x^\alpha)_{x\in S}$.

Proof. Under the assumptions stated above, the MRW$(X_n, -V_n)_{n\in\mathbb{N}_0}$ satisfies the requirements of [29, Lemma 5.6], which yields that direct Riemann integrability of $\tilde{g}$ already implies that $\tilde{g}$ is sdRi w.r.t. $(\mathbb{Q}_x^\alpha)_{x\in\mathbb{S}_\geq}$. Since obviously, g is dRi iff $\tilde{g}$ is dRi, the assertion follows. □

Still, a more handy condition for the direct Riemann integrability can be derived. It stems from results for (univariate) dRi functions on the real line. See [9, V.4] for the definition of (univariate) direct Riemann integrability.

For a bounded measurable function $g : \mathbb{S}_\geq \times \mathbb{R} \to \mathbb{R}$ define

$$\hat{g} :\ t \mapsto \sup_{u\in\mathbb{S}_\geq} |g(u,t)|\,.$$

Lemma 8.6. *A function* $g \in \mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$ *is multivariate dRi if and only if* $\hat{g}$ *is univariate dRi.*

Proof. By [9, Proposition 4.1], a necessary and sufficient condition for the (univariate) direct Riemann integrability of $\hat{g}$ is that

1. $\hat{g}$ is bounded and a.e. continuous w.r.t Lebesgue measure on $\mathbb{R}$
2. $\sum_{l\in\mathbb{Z}} \sup_{t\in[l,l+1]} \hat{g}(t) < \infty$

Since $g \in \mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$ by assumption, (1) is always satisfied, and (2) is just a reformulation of (8.12). ☐

There are plenty of sufficient conditions for the (univariate) direct Riemann integrabibilty, see e.g. [9, Proposition 4.1]; here yet another is introduced. It comes from the following lemma:

Lemma 8.7. *If $f \geq 0$, $f \in L^1(\mathbb{R})$ and $f(t+\varepsilon) \leq h(\varepsilon)f(t)$ for all $\varepsilon > 0$ and $t \in \mathbb{R}$, where $h(\varepsilon) \to 1$ as $\varepsilon \downarrow 0$, then f is (univariate) dRi.*

Proof. Obviously, $f : \mathbb{R} \to \mathbb{R}_\geq$ is (univariate) dRi (in $L^1(\mathbb{R})$) if and only if $\overline{f}(t) := f(-t)$ is (univariate) dRi (in $L^1(\mathbb{R})$). Then the above is just [47, Lemma 9.1], applied to $\overline{f}$. ☐

This gives rise to the following sufficient condition for (multivariate) direct Riemann integrability.

Corollary 8.8. *Let $g \in \mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$. If $\hat{g} \in L^1(\mathbb{R})$ and there is $a > 0$ such that $t \mapsto e^{-at}\hat{g}(t)$ is decreasing, then g is (multivariate) dRi.*

Proof. The assumptions of Lemma 8.7 are satisfied, with $h(\varepsilon) = e^{a\varepsilon}$, $f(t) = \hat{g}(t)$. The (multivariate) direct Riemann integrability of g then follows by Lemma 8.6. ☐

At the time the reader has finished this section, he is allowed to forget everything from Markov renewal theory except for the sufficient condition (Cororllary 8.8) for the direct Riemann integrability and the following consequence of the simple MRT:

Proposition 8.9. *Assume that $[\operatorname{supp}\mu]$ satisfies (C), (M logM) holds, and there is $\alpha \in (0,1]$ with $m(\alpha) = 1$ and $m'(\alpha) < 0$. Let $g \in \mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$ be (multivariate) dRi. Then for all $x \in \mathbb{S}_\geq$,*

$$\lim_{t\to\infty} g * \mathbb{U}_x^\alpha(t) := \lim_{t\to\infty} \mathbb{E}_x^\alpha \left(\sum_{n=0}^{\infty} g(X_n, t - V_n) \right) = 0. \tag{8.13}$$

Proof. Apply the simple Markov Renewal Theorem 6.5 to the MRW $(X_n, -V_n)$ under $\mathbb{Q}_x^\alpha$ and the (by Lemma 8.5) sdRi function $\tilde{g}(y,s) := g(y,-s)$, to infer

$$\lim_{t\to-\infty} \mathbb{E}_x^\alpha \left(\sum_{n=0}^{\infty} \tilde{g}(X_n, t + V_n) \right) = 0.$$

But this is just a reformulation of the assertion. ☐

9. Existence of α-elementary Fixed Points

In this section, the announced existence result, Theorem 1.2, will be shown. It is contained with other results in subsection 9.3. In due course, the Biggins martingale will be defined in the case

$\alpha < 1$ and its nondegeneracy will be shown under the natural assumption $m'(\alpha) < 0$. In order to do so, the simple Markov renewal theorem will be a main tool.

9.1. A Convergent Sequence of Laplace Transforms

In this subsection, a pointwise increasing sequence of LTs $(\phi_n)_{n\geq 0}$ of distributions on $\mathbb{R}^d_\geq$ will be constructed, which converges to the LT of a fixed point of $\mathcal{S}$. Subsequently, write $\mathfrak{L}(F)$ for the set of Laplace transforms of a set of measures F.

The LT ϕ_0 defined in the following Corollary will be the starting point in order to find a convergent sequence $(\mathcal{S}^n\phi_0)_{n\in\mathbb{N}}$.

Corollary 9.1. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C) *and let* $m(\alpha) = 1$ *for some* $\alpha \in (0,1] \cap I_\mu$. *Then for each* $K \geq 0$, ϕ_0, *given by*

$$\phi_0(tu) := \exp\left(-\frac{K}{c_\alpha}\int_{\mathbb{S}_\geq}\langle tu, y\rangle^\alpha \nu^\alpha(dy)\right) \tag{9.1}$$

$$= \exp\left(-Kt^\alpha e^\alpha_*(u)\right) \tag{9.2}$$

with $u \in \mathbb{S}_\geq, t \in \mathbb{R}_\geq$, *is the LT of the multivariate stable distribution* $\tilde{S}_\alpha(c_s^{-1}K\nu^\alpha, 0)$ *on* $\mathbb{R}^d_\geq$.

It is often preferable to use formula (9.2). If a distribution has a LT given by (9.2) it will be denoted by $\tilde{S}_\alpha(Ke^\alpha_*, 0)$.

Proof. Since $\alpha \in I_\mu$, Theorem 7.3 yields the existence of ν^α, e^α_*. Referring to Theorem 5.9, formula (9.1) gives the Laplace transform of a multivariate stable law with index α. The second identity follows by another appeal to Theorem 7.3. □

The most important ingredient to the proof of existence of fixed points is the definition of the Biggins martingale ([20]) in the multivariate setting, which is given in the following Proposition:

Proposition 9.2. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C), *let* $m(\alpha) = 1$ *for some* $\alpha \in (0,1] \cap I_\mu$. *Then for all* $u \in \mathcal{S}$, *the process*

$$W_n(u) := c_\alpha^{-1}\sum_{|v|=n}\int_{\mathbb{S}_\geq}\langle \mathbf{L}(v)^\top u, y\rangle^\alpha \nu^\alpha(dy)$$

is a nonnegative martingale w.r.t. to the natural filtration $\mathcal{T}_n$ *and its limit* $W(u)$ *satisfies*

$$\mathbb{E}W(u) \leq c_\alpha^{-1}\int_{\mathbb{S}_\geq}\langle u, y\rangle^\alpha \nu^\alpha(dy) = e^\alpha_*(u).$$

Remark 9.3.
- If $\alpha = 1$, then $c_1 = 1$ by Lemma 7.4 and moreover, the formula simplifies to

$$W_n(u) = \sum_{|v|=n}\langle \mathbf{L}(v)^\top u, \int_{\mathbb{S}_\geq} y\nu^1(dy)\rangle = \langle u, \mathbf{L}(v)w\rangle = \langle u, W_n\rangle,$$

where W_n is the Biggins martingale in the special case $\alpha = 1$ as introduced in [29] (see Theorem 4.6). Thus the definition of the Biggins martingale above contains the previous one from [29] as a special case.

- It seems that this definition above first appeared in [21] for the study of multitype branching processes, there also a criterion [21, Theorem 2] for the nondegeneracy of W_n is given. The case of finite type space S was previously studied in [65]. A similar martingale, namely

$$V_n(u) = c_\alpha^{-1} N^n \int_{\mathbb{S}_\geq} \langle \Pi_n u, y\rangle^\alpha \nu^\alpha(dy)$$

was studied in [10], where also the change of measure is discussed.

- As the pointwise limit of measurable functions, the mapping $(u, \omega) \mapsto W_n(u)(\omega)$ is measurable – here ω denotes an element of the underlying probability space.

Proof. Each $W_n(u)$ is nonnegative, thus integrable. Then for all $n \in \mathbb{N}$,

$$\begin{aligned}
&\mathbb{E}\left[c_\alpha^{-1} \sum_{|v|=n+1} \int_{\mathbb{S}_\geq} \langle \mathbf{L}(v)^\top u, y\rangle^\alpha \nu^\alpha(dy) \,\middle|\, \mathcal{T}_n \right] \\
&= \mathbb{E}\left[c_\alpha^{-1} \sum_{|v|=n} \sum_{i=1}^N \int_{\mathbb{S}_\geq} \langle \mathbf{T}_i(v)^\top \mathbf{L}(v)^\top u, y\rangle^\alpha \nu^\alpha(dy) \,\middle|\, \mathcal{T}_n \right] \\
&= \sum_{|v|=n} c_\alpha^{-1} \sum_{i=1}^N \mathbb{E}\left[\int_{\mathbb{S}_\geq} \langle \mathbf{L}(v)^\top u, \mathbf{T}_i(v) y\rangle^\alpha \nu^\alpha(dy) \,\middle|\, \mathcal{T}_n \right] \\
&\overset{*}{=} c_\alpha^{-1} \sum_{|v|=n} \sum_{i=1}^N \int_{\mathbb{S}_\geq} \mathbb{E}\left[|\mathbf{T}_i(v) y|^\alpha \langle \mathbf{L}(v)^\top u, \mathbf{T}_i(v) \cdot y\rangle^\alpha \,\middle|\, \mathcal{T}_n \right] \nu^\alpha(dy) \\
&\overset{**}{=} c_\alpha^{-1} \sum_{|v|=n} N \int_{\mathbb{S}_\geq} P^\alpha \left(\langle \mathbf{L}(v)^\top u, \cdot \rangle^\alpha \right) (y) \nu^\alpha(dy) \\
&= c_\alpha^{-1} \sum_{|v|=n} \int_{\mathbb{S}_\geq} \langle \mathbf{L}(v)^\top u, y\rangle^\alpha \left(N P^\alpha \nu^\alpha \right)(dy) \\
&\overset{***}{=} c_\alpha^{-1} \sum_{|v|=n} \int_{\mathbb{S}_\geq} \langle \mathbf{L}(v)^\top u, y\rangle^\alpha \nu^\alpha(dy) \quad \mathbb{P}\text{-a.s.}
\end{aligned}$$

In *, Fubini's theorem was used and in ** it was used that for each v with $|v| = n$, $T(v) = (\mathbf{T}_i(v))_{i=1}^N$ are identically distributed and independent of $\mathcal{T}_n$. In ***, Theorem 7.3 was used together with $\kappa(\alpha) = \frac{1}{N} m(\alpha) = \frac{1}{N}$.

As a nonnegative martingale, $(W_n(u))_{n \in \mathbb{N}}$ converges almost surely, and its limit $W(u)$ satisfies by Fatou's lemma

$$\mathbb{E} W(u) = \mathbb{E} \liminf_{n\to\infty} W_n(u) \leq \liminf_{n\to\infty} \mathbb{E} W_n(u) = \mathbb{E} W_0(u) = c_\alpha^{-1} \int_{\mathbb{S}_\geq} \langle u, y\rangle^\alpha \nu^\alpha(dy). \qquad \square$$

At the end of this section, it will be shown that the limit $W(u)$ is indeed not degenerated and that $\mathbb{E}W(u) = e_*^\alpha(u)$ for all $u \in \mathbb{S}_\geq$. This is why the norming c_α^{-1} was introduced, this leads to $|\mathbb{E}W(u)|_\infty = 1$. These properties of the limit $W(u)$ will be proved by means of LTs of fixed points of the smoothing transform, the path will be laid out in the two subsequent propositions.

Proposition 9.4. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C)*, let* $m(\alpha) = 1$ *for some* $\alpha \in (0,1] \cap I_\mu$*. Let* ϕ_0 *be as in Corollary 9.1. Then for all* $n \in \mathbb{N}$, $(u,t) \in \mathbb{S}_\geq \times \mathbb{R}_\geq$,

$$\mathcal{S}^n\phi_0(tu) = \mathbb{E}\exp\left(-Kt^\alpha W_n(u)\right), \tag{9.3}$$

as well as

$$\mathcal{S}^{n+1}\phi_0(tu) > \mathcal{S}^n\phi_0(tu), \tag{9.4}$$

with strict inequality holding iff T *is not deterministic.*

Proof. By Lemma 2.4,

$$\begin{aligned}\mathcal{S}^n\phi_0(tu) &= \mathbb{E}\left(\prod_{|v|=n}\phi_0(t\mathbf{L}(v)^\top u)\right)\\ &= \mathbb{E}\exp\left(-\frac{K}{c_\alpha}\sum_{|v|=n}\int_{\mathbb{S}_\geq}\langle t\mathbf{L}(v)^\top u, y\rangle^\alpha \nu^\alpha(dy)\right)\\ &= \mathbb{E}\exp\left(-Kt^\alpha W_n(u)\right).\end{aligned}$$

As before, write $\mathcal{T}_n = (T(v))_{|v|\leq n}$ for the weights up to level n in $\mathfrak{T}$. Then (9.4) follows by an application of the conditional Jensen inequality ([28, Problem 4.16 b)]) and Proposition 9.2:

$$\begin{aligned}\mathcal{S}^{n+1}\phi_0(tu) &= \mathbb{E}\left(\mathbb{E}\left[\exp\left(-Kt^\alpha W_{n+1}(u)\right)\middle|\,\mathcal{T}_n\right]\right)\\ &\geq \mathbb{E}\left(\exp\left(\mathbb{E}\left[-Kt^\alpha W_{n+1}(u)\middle|\,\mathcal{T}_n\right]\right)\right)\\ &= \mathbb{E}\left(\exp\left(-Kt^\alpha W_n(u)\right)\right) = \mathcal{S}^n\phi_0(tu).\end{aligned}$$

□

Proposition 9.5. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C) *and let* $m(\alpha) = 1$ *for some* $\alpha \in (0,1] \cap I_\mu$*. The sequence* $\phi_n := \mathcal{S}^n\phi_0$ *converges pointwise to*

$$\psi(tu) = \mathbb{E}\exp\left(-Kt^\alpha W(u)\right)$$

for all $(u,t) \in \mathbb{S} \times \mathbb{R}_\geq$ *and* ψ *is the Laplace transfrom of a distribution on* $\mathbb{R}^d_\geq$*, with*

$$\mathcal{S}\psi = \psi.$$

Proof. The random variables $\exp\left(-Kt^\alpha W_n(u)\right)$ are uniformly bounded by 1, and converge by Proposition 9.2 a.s. to $\exp\left(-Kt^\alpha W_n(u)\right)$, thus referring to the bounded convergence theorem,

$$\lim_{n\to\infty}\phi_n = \psi$$

when taking expectations.

On the other hand, considering Lemma 2.4, $(\phi_n)_{n\in\mathbb{N}}$ is a sequence in $\mathfrak{L}(\mathcal{P}(\mathbb{R}^d_\geq))$. By Theorem 5.2, it converges to a LT ψ, and $\psi(0) = \mathbb{E}\exp(0) = 1$. Thus also $\psi \in \mathfrak{L}(\mathcal{P}(\mathbb{R}^d_\geq))$.

ψ is a fixed point of $\mathcal{S}$, since by Lemma 2.5, for all $x \in \mathbb{R}^d_\geq$,

$$\mathcal{S}\psi(x) = \mathcal{S}(\lim_{n\to\infty} \mathcal{S}^n\psi)(x) = \lim_{n\to\infty} \mathcal{S}^{n+1}\psi(x) = \psi(x). \qquad \square$$

Up to now, it is still possible that $\psi \equiv 1$, thus being the LT of the trivial fixed point δ_0 and this happens if and only if $W(u) \equiv 0$ for all $u \in \mathbb{S}_\geq$. In the next subsection, it will be shown that ψ is nontrivial except for the case $K = 0$. This will imply that also the martingale limit $W(u)$ is nontrivial.

Observe that up to now, the property $m'(\alpha) < 0$ was not used. It first appears in the proof of nontriviality in the next subsection, for the application of the simple MRT.

9.2. The Fixed Point is Nontrivial

It follows from formula (9.2), that if $K > 0$

$$\lim_{t\downarrow 0} \frac{1-\phi_0(tu)}{t^\alpha} = Ke_*^\alpha(u) > 0 \tag{9.5}$$

for all $u \in \mathbb{S}_\geq$. It will be shown by an adaption of the arguments in [41, Theorem 2.7], that also

$$\liminf_{t\downarrow 0} \frac{1-\psi(ut)}{t^\alpha} \geq Ke_*^\alpha(u) > 0.$$

This will particularly imply, that ψ is not degenerated.

Given a LT $\phi \in \mathfrak{L}(\mathcal{P}(\mathbb{R}^d_\geq))$, and $\chi \in (0,1] \cap I_\mu$, assume that $[\operatorname{supp}\mu]$ satisfies (C) in order to define the following functions on $\mathbb{S}_\geq \times \mathbb{R}$:

$$D_{\chi,\phi}(u,t) := \frac{e^{\chi t}}{e_*^\chi(u)}(1-\phi(e^{-t}u)) \tag{9.6}$$

$$G_{\chi,\phi}(u,t) := \frac{e^{\chi t}}{e_*^\chi(u)}\mathbb{E}\left(\prod_{i=1}^N \phi(e^{-t}\mathbf{T}_i^\top u) + \sum_{i=1}^N \left(1-\phi(e^{-t}\mathbf{T}_i^\top u)\right) - 1\right). \tag{9.7}$$

Substituting $s = e^{-t}$ and considering (9.2),

$$\lim_{t\to\infty} D_{\alpha,\phi_0}(u,t) = \lim_{s\to 0} \frac{1-\phi(su)}{e_*^\alpha(u)s^\alpha} = K.$$

The basic idea is to show that D_{α,ϕ_0} and $D_{\alpha,\psi}$ have the same limit for $t \to \infty$. The first Step in that direction is to link $D_{\alpha,\mathcal{S}\phi}$ and $D_{\alpha,\phi}$ by linearization of the product $\prod_{i=1}^N \phi(e^{-t}\mathbf{T}_i^\top u)$. The first ingredient needed therefore is the multivariate extension of [41, Lemma 2.3] given below.

Lemma 9.6. *Let* $[\operatorname{supp}\mu]$ *satisfy* (C), $\chi \in (0,1] \cap I_\mu$ *and* $\phi \in \mathfrak{L}(\mathcal{P}(\mathbb{R}^d_\geq))$. *It holds that*

$$D_{\chi,\mathcal{S}\phi}(u,t) = m(\chi)\mathbb{E}_u^\chi D_{\chi,\phi}(X_1, t - V_1) - G_{\chi,\phi}(u,t). \tag{9.8}$$

Proof. Recalling the properties of $\mathbb{E}_u^\chi$ from (8.4) to compute

$$\begin{aligned}
D_{\chi,\mathcal{S}\phi}(u,t) &= \frac{e^{\chi t}}{e_*^\chi(u)}\mathbb{E}\left(1 - \prod_{i=1}^N \phi(e^{-t}\mathbf{T}_i^\top u)\right) \\
&= \frac{e^{\chi t}}{e_*^\chi(u)}\mathbb{E}\left(\sum_{i=1}^N \left[1 - \phi(e^{-t}\mathbf{T}_i^\top u)\right]\right) - G_{\chi,\phi}(u,t) \\
&= \frac{e^{\chi t}}{e_*^\chi(u)}\sum_{i=1}^N \mathbb{E}\left(1 - \phi(e^{-t}e^{\log\left|\mathbf{T}_i^\top u\right|}\mathbf{T}_i^\top \cdot u)\right) - G_{\chi,\phi}(u,t) \\
&= \frac{e^{\chi t}}{e_*^\chi(u)}N\mathbb{E}_u\left(1 - \phi(e^{-t+V_1}X_1)\right) - G_{\chi,\phi}(u,t) \\
&= \frac{m(\chi)}{e_*^\chi(u)\kappa(\chi)}\mathbb{E}_u\left(e^{\chi V_1}e_*^\chi(X_1)\frac{e^{\chi(t-V_1)}}{e_*^\chi(X_1)}\left[1 - \phi(e^{-t+V_1}X_1)\right]\right) - G_{\chi,\phi}(u,t) \\
&= m(\chi)\mathbb{E}_u^\chi\left(\frac{e^{\chi(t-V_1)}}{e_*^\chi(X_1)}\left[1 - \phi(e^{-t+V_1}X_1)\right]\right) - G_{\chi,\phi}(u,t) \\
&= m(\chi)\mathbb{E}_u^\chi D_{\chi,\phi}(X_1, t - V_1) - G_{\chi,\phi}(u,t)
\end{aligned}$$

□

The following Lemma is a straightforward generalization of [41, Lemma 2.4]:

Lemma 9.7. *Let* $[\operatorname{supp}\mu]$ *satisfy* (C). *Let* $\phi, \varphi \in \mathfrak{L}(\mathcal{P}(\mathbb{R}^d_\geq))$ *and* $\chi \in (0,1] \cap I_\mu$. *Then*

1. $G_{\chi,\phi}(u,t) \geq 0$ *for all* $(u,t) \in \mathbb{S}_\geq \times \mathbb{R}$.
2. *For all* $u \in \mathbb{S}_\geq$, $t \mapsto e^{-\chi t}G_{\chi,\phi}(u,t)$ *is decreasing.*
3. *If* $\varphi(tu) \geq \phi(tu)$, *then*
$$G_{\chi,\varphi}(u,t) \leq G_{\chi,\phi}(u,t).$$
4. *The function*
$$t \mapsto e^{-\chi t}\left(\sup_{u \in \mathbb{S}_\geq} G_{\chi,\phi}(u,t)\right) \tag{9.9}$$
is decreasing.

Proof. The short proof, mainly taken from [90, Lemma 2.8] is included for completeness. Consider the function

$$h\ : [0,1]^N \to \mathbb{R}, \quad (s_1, \dots, s_N) \to \prod_{i=1}^N s_i + \sum_{i=1}^N (1 - s_i) - 1.$$

Then for all $1 \leq i \leq N$,

$$\frac{\partial}{\partial s_j} h(s_1, \dots, s_N) = -1 + \prod_{i \neq j} s_i \leq 0.$$

Thus if $r_i \leq s_i$ for all $1 \leq j \leq N$, then

$$h(r_1, \dots, r_N) \geq h(s_1, \dots, r_N). \tag{9.10}$$

Now the assertion follows by using that h is bounded, thus integrable and putting the following in (9.10):

1. $r_i = \phi(e^{-t}\mathbf{T}_i^\top u)$, $s_i = 1$.
2. $r_i = \phi(e^{-t_1}\mathbf{T}_i^\top u)$, $s_i = \phi(e^{-t_2}\mathbf{T}_i^\top u)$ for $t_1 > t_2$.
3. $r_i = \varphi(e^{-t}\mathbf{T}_i^\top u)$, $s_i = \phi(e^{-t}\mathbf{T}_i^\top u)$.

By (2), each of the functions $g_u(t) := e^{-\chi t} G_{\chi,\phi}(u,t)$ is decreasing in t. The pointwise supremum of decreasing functions is again decreasing, thus (4) follows. □

Abbreviate

$$D_{\alpha,n} := D_{\alpha,\mathcal{S}^n\phi_0} = D_{\alpha,\phi_n}, \quad G_{\alpha,n} = G_{\alpha,\mathcal{S}^n\phi_0}.$$

The next proposition gives the crucial identity that links $D_{\alpha,n}$ with $D_{\alpha,0}$:

Proposition 9.8. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C)*. Suppose further that* $m(\alpha) = 1$ *for some* $\alpha \in (0,1] \cap I_\mu$*. Let* $\phi_0 = \mathfrak{L}(\tilde{S}_\alpha(K\nu^\alpha, 0))$*. Then for all* $n \in \mathbb{N}$*,* $(u,t) \in \mathbb{S}_\geq \times \mathbb{R}$*,*

$$D_{\alpha,n}(u,t) \geq \mathbb{E}_u^\alpha D_{\alpha,0}(X_n, t - V_n) - \mathbb{E}_u^\alpha \sum_{k=0}^{n-1} G_{\alpha,0}(X_k, t - V_k) \tag{9.11}$$

as well as

$$D_{\alpha,n}(u,t) \leq \mathbb{E}_u^\alpha D_{\alpha,0}(X_n, t - V_n) \tag{9.12}$$

Proof. Referring to Lemma 9.4, $\phi_n \geq \phi_0$, hence by Lemma 9.7,(3), $G_{\alpha,\phi_n} \leq G_{\alpha,\phi_0}$. Use the identity from Lemma 9.6 with $m(\alpha) = 1$ to obtain

$$\begin{aligned} D_{\alpha,n}(u,t) &= D_{\alpha,\mathcal{S}\phi_{n-1}}(u,t) \\ &= \mathbb{E}_u^\alpha D_{\alpha,\phi_{n-1}}(X_1, t - V_1) - G_{\alpha,\phi_{n-1}}(u,t) \\ &\geq \mathbb{E}_u^\alpha D_{\alpha,\phi_{n-1}}(X_1, t - V_1) - G_{\alpha,0}(u,t). \end{aligned}$$

In other words, introducing the Markov kernel

$${}^\alpha P f(u,t) = \mathbb{E}_u^\alpha f(X_1, t - V_1),$$

it follows that

$$D_{\alpha,n}(u,t) \geq {}^\alpha P D_{\alpha,n-1}(u,t) - G_{\alpha,0}(u,t)$$

and thus by iteration

$$D_{\alpha,n}(u,t) \geq ({}^{\alpha}P)^n D_{\alpha,0}(u,t) - \sum_{k=0}^{n-1} ({}^{\alpha}P)^k G_{\alpha,0}(u,t),$$

which gives the first assertion. The proof of the second assertion goes along the same lines, using the estimate

$$D_{\alpha,n}(u,t) = \mathbb{E}_u^{\alpha} D_{\alpha,\phi_{n-1}}(X_1, t-V_1) - G_{\alpha,\phi_{n-1}}(u,t) \leq \mathbb{E}_u^{\alpha} D_{\alpha,\phi_{n-1}}(X_1, t-V_1),$$

which is valid since $G_{\alpha,\phi_{n-1}} \geq 0$ by Lemma 9.7. □

The next Step is to investigate the limit $n \to \infty$ in (9.11). The left-hand side (LHS) and the second member on the right-hand side (RHS) are easy to evaluate:

$$D_{\alpha,n}(u,t) \to D_{\alpha,\psi}(u,t)$$

by Proposition 9.5 and

$$\mathbb{E}_u^{\alpha} \sum_{k=0}^{n-1} G_{\alpha,0}(X_k, t-V_k) \to \mathbb{E}_u^{\alpha} \sum_{k=0}^{\infty} G_{\alpha,0}(X_k, t-V_k) = G_{\alpha,0} * \mathbb{U}_u^{\alpha}(t)$$

by monotone convergence. The simple MRT will be used to get rid of this term, but first consider the first member of the RHS in (9.11):

Lemma 9.9. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C). *Suppose further that* (M logM) *holds and that there is* $\alpha \in (0,1]$ *with* $m(\alpha) = 1$ *and* $m'(\alpha) < 0$. *Then for all* $(u,t) \in \mathbb{S}_{\geq} \times \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{E}_u^{\alpha} D_{\alpha,0}(X_n, t-V_n) = K.$$

Proof. Set $C' := \sup_{u \in \mathbb{S}_{\geq}} e_*^{\alpha}(u)$. Considering the inequality $1 - e^{-r} \geq r - \frac{1}{2}r^2$ from Lemma 25.7 in the appendix and the definition of ϕ_0 in Corollary 9.1, it holds that

$$\begin{aligned} D_{\alpha,0}(X_n, t-V_n) &= \frac{e^{\alpha(t-V_n)}}{e_*^{\alpha}(X_n)} \Big(1 - \exp(-K e^{\alpha(V_n-t)} e_*^{\alpha}(Y_n))\Big) \\ &\geq \frac{e^{\alpha(t-V_n)}}{e_*^{\alpha}(X_n)} \left(K e^{\alpha(V_n-t)} e_*^{\alpha}(X_n) - \frac{1}{2} K^2 e^{2\alpha(V_n-t)} e_*^{\alpha}(X_n)^2\right) \\ &\geq K - \frac{1}{2} K^2 C' e^{\alpha(V_n-t)}. \end{aligned}$$

Now by Theorem 8.2, $\lim_{n\to\infty} V_n = -\infty$ $\mathbb{Q}_u^{\alpha}$-a.s. Thus,

$$\liminf_{n\to\infty} D_{\alpha,0}(X_n, t-V_n) \geq K \quad \mathbb{Q}_u^{\alpha}\text{-a.s.}$$

It will be shown in Corollary 9.13 that $D_{\alpha,0}$ is bounded, thus the bounded convergence theorem can be applied to infer

$$\liminf_{n\to\infty} \mathbb{E}_u^\alpha D_{\alpha,0}(X_n, t - V_n) \geq K$$

for all $(u,t) \in \mathbb{S}_\geq \times \mathbb{R}$. In the same manner, use the inequality $1 - e^{-r} \leq r \leq e^{-(1-r)}$ (again from Lemma 25.7) to obtain

$$\limsup_{n\to\infty} \mathbb{E}_u^\alpha D_{\alpha,0}(X_n, t - V_n) \leq K.$$

Together this proves the assertion. □

Summarizing what has been proved up to now, the following result holds:

Corollary 9.10. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C). *Suppose further that* (M logM) *holds and that there is* $\alpha \in (0,1]$ *with* $m(\alpha) = 1$ *and* $m'(\alpha) < 0$. *Then for all* $(u,t) \in \mathbb{S}_\geq \times \mathbb{R}$,

$$K \geq D_{\alpha,\psi}(u,t) \geq K - \mathbb{E}_u^\alpha \left(\sum_{n=0}^{\infty} G_{\alpha,0}(X_n, t - V_n) \right). \tag{9.13}$$

The final Step is to show that the second term vanishes as t goes to infinity, this will be done by an application of the simple Markov Renewal Theorem, hence the task is to show that $G_{\alpha,0}$ is dRi, which will be the result of Proposition 9.14. Beforehand, several estimates will be proved, which are also useful for later purposes.

Lemma 9.11. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C). *Let* $\phi \in \mathfrak{L}(\mathcal{P}(\mathbb{R}^d_\geq))$ *and let* $\chi \in (0,1] \cap I_\mu$. *Introduce*

$$h(s) := e^{-(s\wedge N)} + (s \wedge N) - 1, \quad C(T) := \sum_{i=1}^{N} (\|\mathbf{T}_i\| \vee 1).$$

Then for all $u \in \mathbb{S}_\geq$, $t \in \mathbb{R}$,

$$G_{\chi,\phi}(u,t) \leq \frac{e^{\chi t}}{e_*^\chi(u)} \mathbb{E} h\Big(C(T)(1 - \phi(e^{-t}\vartheta_{\mathbf{d}})) \Big) \tag{9.14}$$

as well as, if $\mathbb{E}\|\mathbf{T}_1\| < \infty$

$$\lim_{t\to\infty} \sup_{u\in\mathbb{S}_\geq} \frac{G_{\chi,\phi}(u,t)}{e^{\chi t}(1 - \phi(e^{-t}\vartheta_{\mathbf{d}}))} = 0. \tag{9.15}$$

Proof. The proof is a generalization of the arguments given in Lemma [41, Lemma 2.6] to the multivariate situation. Properties of h are studied in Lemma 25.7 in the appendix.

Step 1: Compute, using the inequality $r \leq e^{-(1-r)}$ (see (25.14)) in the second and the inequality (25.11) in the last line:

$$\begin{aligned} G_{\chi,\phi}(u,t) &= \frac{e^{\chi t}}{e_*^\chi(u)} \mathbb{E} \left(\prod_{i=1}^{N} \phi(e^{-t}\mathbf{T}_i^\top u) + \sum_{i=1}^{N} \Big(1 - \phi(e^{-t}\mathbf{T}_i^\top u)\Big) - 1 \right) \\ &\leq \frac{e^{\chi t}}{e_*^\chi(u)} \mathbb{E} \left(\prod_{i=1}^{N} e^{-(1-\phi(e^{-t}\mathbf{T}_i^\top u))} + \sum_{i=1}^{N} \Big(1 - \phi(e^{-t}\mathbf{T}_i^\top u)\Big) - 1 \right) \end{aligned}$$

$$= \frac{e^{\chi t}}{e_*^\chi(u)} \mathbb{E}\left(e^{-\sum_{i=1}^N (1-\phi(e^{-t}\mathbf{T}_i^\top u))} + \sum_{i=1}^N \Big(1 - \phi(e^{-t}\mathbf{T}_i^\top u)\Big) - 1\right)$$
$$\leq \frac{e^{\chi t}}{e_*^\chi(u)} \mathbb{E}\left(h\Big(\sum_{i=1}^N (1-\phi(e^{-t}\mathbf{T}_i^\top u))\Big)\right)$$
$$\leq \frac{e^{\chi t}}{e_*^\chi(u)} \mathbb{E}\left(h\Big(\sum_{i=1}^N (\|\mathbf{T}_i\| \vee 1)(1-\phi(e^{-t}\vartheta_{\mathbf{d}}))\Big)\right).$$

This proves formula (9.14).

STEP 2: Use this bound and the substitution $r = (1 - \phi(e^{-t}\vartheta_{\mathbf{d}}))$ to deduce (9.15):

$$\frac{G_{\chi,\phi}(u,t)}{e^{\chi t}(1-\phi(e^{-t}\vartheta_{\mathbf{d}}))} \leq \frac{1}{e_*^\chi(u)(1-\phi(e^{-t}\vartheta_{\mathbf{d}}))} \mathbb{E}\, h\Big(C(T)(1-\phi(e^{-t}\vartheta_{\mathbf{d}}))\Big)$$
$$= \frac{\mathbb{E}\, h(C(T)r)}{e_*^\chi(u) r} = \frac{1}{e_*^\chi(u)} \mathbb{E}\, \frac{h(C(T)r)}{C(T)r} C(T).$$

Then taking the $\lim t \to \infty$ corresponds to $\lim r \to 0$. The function $s^{-1}h(s)$ is bounded and $\lim_{s\to 0} h(s)/s = 0$ by Lemma 25.7. Moreover,

$$\mathbb{E}\left(C(T)\right) \leq N \mathbb{E}\left(1 + \|\mathbf{T}_1\|\right) < \infty$$

as well as $C' := \sup_{u \in \mathbb{S}_\geq} e_*^\chi(u)^{-1} < \infty$.

Putting everything together and using the bounded convergence theorem, it results that

$$0 \leq \limsup_{t\to\infty} \frac{G_{\chi,\phi}(u,t)}{e^{\chi t}(1-\phi(e^{-t}\vartheta_{\mathbf{d}})} \leq \lim_{r\to 0} C' \mathbb{E}\, \frac{h(C(T)r)}{C(T)r} C(T) = 0. \qquad \square$$

Lemma 9.12. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C) *and let* $m(\alpha) = 1$ *for some* $\alpha \in (0,1] \cap I_\mu$*. Let* $\phi_0 = \mathfrak{L}(\tilde{S}_\alpha(K e_*^\alpha, 0))$ *and* $\psi = \lim_{n\to\infty} \mathcal{S}^n \phi_0$*. Then for all* $t \in \mathbb{R}$

$$1 - \phi_0(e^{-t}\vartheta_{\mathbf{d}}) \leq d^{\frac{\alpha}{2}} e^{-\alpha t}, \qquad 1 - \psi(e^{-t}\vartheta_{\mathbf{d}}) \leq d^{\frac{\alpha}{2}} e^{-\alpha t}. \tag{9.16}$$

Proof. In order to show the first estimate, use the definition of ϕ_0 , the inequality (25.14) and the convention that $|e_*^\alpha|_\infty = 1$ to infer that

$$1 - \phi_0(e^{-t}\vartheta_{\mathbf{d}}) = 1 - \exp\Big(-(e^{-t}\sqrt{d})^\alpha e_*^\alpha(\vartheta_{\mathbf{1}})\Big) \leq d^{\frac{\alpha}{2}} e^{-\alpha t} e_*^\alpha(\vartheta_{\mathbf{1}}) \leq d^{\frac{\alpha}{2}} e^{-\alpha t}.$$

The second estimate is then a direct consequence, since $1 - \mathcal{S}^n\phi_0$ is a decreasing sequence by Lemma 9.4. $\square$

Corollary 9.13. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C) *and let* $m(\alpha) = 1$ *for some* $\alpha \in (0,1] \cap I_\mu$*. The functions* D_{α,ϕ_0} *and* G_{α,ϕ_0} *are in* $\mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$*.*

Proof. The continuity of both functions is obvious from their very definition.

Abbreviate $C' := \sup_{y \in \mathbb{S}_\geq} e_*^\alpha(u)^{-1}$. To obtain the bound on D_{α,ϕ_0}, use the inequalities (25.9) and (9.16):

$$D_{\alpha,\phi_0}(u,t) = \frac{e^{\alpha t}}{e_*^\alpha(u)}(1-\phi_0(e^{-t}u)) \leq C' e^{\alpha t}(1-\phi_0(e^{-t}\vartheta_{\mathbf{d}})) \leq C' d^{\frac{\alpha}{2}}.$$

In order to bound $G_{\alpha,0}$, observe as a first step that it is bounded on the negative half-line by its very definition, in fact

$$\sup_{(u,t)\in \mathbb{S}_\geq \times \mathbb{R}_\leq} G_{\alpha,0}(u,t) \leq C' N.$$

Now considering the positive half-line, by (9.16)

$$\sup_{u\in\mathbb{S}_\geq} \frac{G_{\alpha,\phi_0}(u,t)}{e^{\alpha t}(1-\phi_0(e^{-t}\vartheta_{\mathbf{d}}))} \geq d^{-\frac{\alpha}{2}} \sup_{u\in\mathbb{S}_\geq} G_{\alpha,\phi_0}(u,t) \geq 0.$$

But considering (9.15), the LHS tends to 0 as $t\to\infty$, thus the same holds for $\sup_{u\in\mathbb{S}_\geq} G_{\alpha,\phi_0}(u,t)$ and consequently

$$\sup_{(u,t)\in \mathbb{S}_\geq \times \mathbb{R}_\geq} G_{\alpha,0}(u,t) < \infty.$$

□

Proposition 9.14. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C) *and let* $m(\alpha)=1$ *for some* $\alpha\in(0,1]\cap I_\mu$. *Let* $\mathbb{E}\,\|\mathbf{M}_1\|<\infty$. *Then* G_{α,ϕ_0} *is dRi.*

Proof. Introduce $\hat{g}: t\mapsto \sup_{u\in\mathbb{S}_\geq} G_{\alpha,\phi_0}(u,t)$. Referring to Corollary 9.13, $G_{\alpha,\phi_0}\in\mathcal{C}_b(\mathbb{S}_\geq\times\mathbb{R})$ and by Lemma 9.7, $t\mapsto e^{-\alpha t}\hat{g}(t)$ is decreasing. Thus using Corollary 8.8, it is sufficient to show that $\hat{g}\in L^1(\mathbb{R})$.

Setting again $C' := \sup_{u\in\mathbb{S}_\geq} \frac{1}{e_*^\chi(u)}$, it is a consequence of the estimates (9.14) and (9.16), that

$$\hat{g}(t) \leq C' e^{\alpha t}\mathbb{E} h\left(C(T) d^{\frac{\alpha}{2}} e^{-\alpha t}\right). \tag{9.17}$$

Now estimate

$$\begin{aligned}\int_{-\infty}^{\infty}\hat{g}(t)dt &\leq C'\int_{-\infty}^{\infty} e^{\alpha t}\mathbb{E} h\left(C(T)d^{\frac{\alpha}{2}}e^{-\alpha t}\right)dt \\ &= \mathbb{E}\, C'\int_{-\infty}^{\infty} e^{\alpha t} h\left(C(T)d^{\frac{\alpha}{2}}e^{-\alpha t}\right)dt = C'\mathbb{E}\int_0^\infty \frac{h(s)}{s^2} d^{\frac{\alpha}{2}}\frac{C(T)}{\alpha}ds \\ &= \frac{C'}{\alpha}\left(\int_0^\infty \frac{h(s)}{s^2}ds\right)\mathbb{E} C(T) \leq \frac{C'}{\alpha}\left(\int_0^\infty\frac{h(s)}{s^2}ds\right) N(1+\mathbb{E}\,\|\mathbf{M}_1\|)<\infty.\end{aligned}$$

Here the substitution

$$s = C(T)d^{\frac{\alpha}{2}}e^{-\alpha t},\quad dt = -\frac{\alpha}{d^{\frac{\alpha}{2}}C(T)}e^{\alpha t}ds = -\frac{\alpha}{d^{\frac{\alpha}{2}}C(T)}\, s\, ds,$$

Fubini's theorem and Lemma 25.7 were used for the final conclusion. □

9.3. Results and Discussion

In this subsection, the proof of Theorem 1.2 is finished, several remarks and extensions are given and it is discussed, why the approach to the existence of fixed points via the *stable transformation* is not applicable in the multivariate setting.

Summarizing what has been done in the previous subsections, the following Theorem can now be proved:

Theorem 9.15. *Assume that* $[\mathrm{supp}\,\mu]$ *satisfies* (C). *Suppose further that* (M logM) *holds and that there is* $\alpha \in (0,1]$ *with* $m(\alpha) = 1$ *and* $m'(\alpha) < 0$. *Let* ϕ_0 *be the LT of* $\tilde{S}_\alpha(Ke_*^\alpha, 0)$ *with* $K > 0$. *Then* $\psi := \lim_{n\to\infty} \mathcal{S}^n \phi_0$ *is (the LT of) a fixed point of* $\mathcal{S}$, *and for all* $u \in \mathbb{S}_\geq$

$$\lim_{s\downarrow 0} \frac{1-\psi(su)}{s^\alpha} = Ke_*^\alpha(u) > 0. \tag{9.18}$$

Proof. The convergence and fixed point property of ψ are contained in Proposition 9.5. In Corollary 9.10, the estimate

$$K \geq D_{\alpha,\psi}(u,t) = \frac{1-\psi(e^{-t}u)}{e_*^\alpha(u)(e^{-t})^\alpha} \geq K - \mathbb{E}_u^\alpha\left(\sum_{n=0}^\infty G_{\alpha,0}(X_n, t-V_n)\right) \tag{9.19}$$

was obtained. Referring to Proposition 9.14, $G_{\alpha,0}$ is directly Riemann integrable. Thus by the simple Markov Renewal Theorem applied to the present case (see Proposition 8.9) the last term tends to zero as $t \to \infty$. Consequently for all $u \in \mathbb{S}_\geq$,

$$\lim_{t\to\infty} \frac{1-\psi(e^{-t}u)}{e_*^\alpha(u)(e^{-t})^\alpha} = K.$$

Now replace $s = e^{-t}$ to obtain the assertion. □

Remark 9.16.
1. By Corollary 7.5, a sufficient condition for the existence of $\alpha \in (0,1)$ with $m(\alpha) = 1$, $m'(\alpha) < 0$ is that the spectral radius of $\mathbb{E}\mathbf{T}_1$ is less than N^{-1}.
2. In Subsection 7.3, it was explained how a numerical simulation of ν^α and m can be done.
3. Put $K < 0$ to obtain nontrivial fixed points supported on $\mathbb{R}^d_\leq$. The fixed point corresponding to $K = 0$ is the trivial one, $Y \equiv 0$.
4. By Corollary 5.15, if $\alpha < 1$ and Y has LT ψ, then Y is multivariate regular varying with index α, in particular, there is $C > 0$ such that

$$t^\alpha \mathbb{P}\left(t^{-1}Y \in \cdot\right) \xrightarrow{v} C\nu^\alpha \otimes \lambda^\alpha.$$

5. By Lemma 7.4, if $\alpha = 1$ and Y has LT ψ, then

$$Ke_*^\alpha(u) = \mathbb{E}\langle u, Y\rangle = K\langle u, y\rangle \quad \forall u \in \mathbb{S}_\geq,$$

where $y \in \mathbb{S}_\geq$ is the (essentially unique) eigenvector of $\mathbb{E}\mathbf{T}_1$ corresponding to the eigenvalue N^{-1}. In other words, for $\alpha = 1$ the fixed points described by Buraczweski, Damek and

Guivarc'h (see Theorem 4.6) were rediscovered.

Theorem 9.15 also gives the nontriviality of the limit of the Biggins martingale.

Theorem 9.17. *Under the assumptions of Theorem 9.15, it holds that* $\mathbb{E}W(u) = e_*^\alpha(u)$ *for all* $u \in \mathbb{S}_\geq$.

Proof. In Proposition 9.5, the formula $\psi(su) = \mathbb{E}e^{-Ks^\alpha W(u)}$ was obtained. The function

$$s \mapsto \frac{1 - e^{-Ks^\alpha W(u)}}{s^\alpha}$$

is decreasing, with limit $KW(u)$ for $s \to 0$, thus the monotone convergence theorem may be applied to deduce from formula (9.18), that

$$\begin{aligned} Ke_*^\alpha(u) &= \lim_{s \downarrow 0} \frac{1 - \mathbb{E}e^{-Ks^\alpha W(u)}}{s^\alpha} = \lim_{s \downarrow 0} \mathbb{E}\left(\frac{1 - e^{-Ks^\alpha W(u)}}{s^\alpha}\right) \\ &= \mathbb{E}\left(\lim_{s \downarrow 0} \frac{1 - e^{-Ks^\alpha W(u)}}{s^\alpha}\right) = \mathbb{E}\left(KW(u)\right). \end{aligned}$$ □

Finally, some words on what is different from the one-dimensional case:
In the one-dimensional case, it is sufficient to prove existence of fixed points in the case $\alpha = 1$, existence of fixed points for $\alpha < 1$ is then derived via the *stable transformation* (see [41, Section 3]). This method is recalled here to point out, why it breaks down in the multidimensional case:

Consider the one-dimensional smoothing transform $\mathcal{S}$ and assume $\alpha < 1$. Denote

$$\mathcal{S}_\alpha : \nu \mapsto \mathcal{L}\left(\sum_{i=1}^N T_i^\alpha Y_i\right)$$

with the usual conventions and let $\hat{m}_\alpha(s) = \sum_{i=1}^N \mathbb{E}\left(T_i^\alpha\right)^s$. Then $\hat{m}_\alpha(1) = 1$, $\hat{m}_\alpha'(1) = \alpha \hat{m}'(\alpha) < 0$, this is the situation "$\alpha = 1$". So suppose there is ϕ_α with $\mathcal{S}_\alpha \phi_\alpha = \phi_\alpha$. Then a fixed point of $\mathcal{S}$ is given by $\psi(t) := \phi_\alpha(t^\alpha)$, since this again defines a LT (see [41, Theorem 3.1] for details) and

$$\begin{aligned} \mathcal{S}\psi(t) &= \mathbb{E}\left(\prod_{i=1}^N \psi(T_i t)\right) = \mathbb{E}\left(\prod_{i=1}^N \phi_\alpha(T_i^\alpha t^\alpha)\right) \\ &= \mathcal{S}_\alpha \phi_\alpha(t^\alpha) = \phi_\alpha(t^\alpha) = \psi(t). \end{aligned}$$

The point is, that in the multivariate setting the weights $\mathbf{T}_i$ are matrices, so what is $\mathbf{T}_i^\alpha$? For a deterministic matrix $\mathbf{A}$, these powers can be defined via spectral calculus, see e.g. [40, Theorem VII.1.8]: But if $[\operatorname{supp} \mu]$ satisfies (C), as always assumed, then by part (1) of (C), the realisations of the random matrices do not have common eigenspaces, therefore it is impossible to write down a spectral decomposition of the random matrix $\mathbf{T}_1$. This is why the stable transformation is not applicable here.

10. Existence of Fixed Points in the Boundary Case

In this section, the case that $m'(\alpha^-) = 0$ is adressed, the so-called *boundary case*. The following theorem will be proved.

Theorem 10.1 (Existence of fixed points in the boundary case)**.** *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C), (M logM) *holds, and that there is* $\alpha \in (0,1]$ *with*

$$m(\alpha) = 1, \quad m'(\alpha^-) = 0.$$

Then $\mathcal{S}_0$ *possesses a nontrivial fixed point* Y*. If* $\alpha = 1$*, then* $\mathbb{E}\,|Y| = \infty$*.*

The strategy of proof, which will be given by several subsequent lemmata, is similar to [41, Theorem 3.5]: Consider small perturbations $\mathcal{S}_\chi$ of $\mathcal{S}$, which satisfy the assumptions of Theorem 9.15, thus possess nontrivial fixed points η_χ. Then it will be shown that there is a sequence η_{χ_k}, $\chi_k \to \alpha$, which converges weakly to a nontrivial fixed point of $\mathcal{S}$.

Lemma 10.2. *Let the assumptions of Theorem 10.1 be in force. Fix* $u_0 \in \mathbb{S}_\geq$*. Then for all* $\chi \in (0,\alpha)$*, the rescaled smoothing transform*

$$\mathcal{S}_\chi \;:\; \nu \mapsto \mathcal{L}\left(\sum_{i=1}^{N} \frac{1}{m(\chi)^{1/\chi}} \mathbf{T}_i X_i\right) \tag{10.1}$$

possesses a nontrivial fixed point η_χ *and its LT satisfies* $\psi_\chi(u_0) = 1/2$*.*

Proof. Step 1: Given the random variable $T = (\mathbf{T}_1, \ldots, \mathbf{T}_N)$ with $(\mathbf{T}_i)$ identically distributed with distribution μ, define for $\chi \in (0,\alpha)$ the rescaled weight vector

$$T_\chi = (\mathbf{T}_{\chi,1}, \ldots, \mathbf{T}_{\chi,N}) := m(\chi)^{-1/\chi}(\mathbf{T}_1, \ldots, \mathbf{T}_N). \tag{10.2}$$

Then $\mathcal{S}_\chi$ is the smoothing transform associated with T_χ and the $\mathbf{T}_{\chi,i}$ are identically distributed with law

$$\mu_\chi = \mathcal{L}\left(m(\chi)^{-1/\chi}\mathbf{T}_1\right).$$

Since $\chi \in (0,\alpha)$ it holds that $m(\chi) > m(\alpha) = 1$, thus the factor $m(\chi)^{-1/\chi} < 1$, it makes the matrices $\mathbf{T}_i$ "smaller". This is reflected in the corresponding spectral function m_χ which decays faster than m, as will be seen now. Let $(\mathbf{T}_{(n)})_{n\in\mathbb{N}}$ be a sequence of i.i.d. copies of $\mathbf{T}_1$ and compute

$$\begin{aligned} m_\chi(s) &= N \lim_{n\to\infty} \left(m(\chi)^{-1/\chi}\mathbb{E}\left\|\mathbf{T}_{(1)}\cdots\mathbf{T}_{(n)}\right\|^s\right)^{\frac{1}{n}} \\ &= N m(\chi)^{-s/\chi} \lim_{n\to\infty} \left(\mathbb{E}\left\|\mathbf{T}_{(1)}\cdots\mathbf{T}_{(n)}\right\|^s\right)^{\frac{1}{n}} \\ &= N m(\chi)^{-s/\chi}\kappa(s) = \frac{m(s)}{m(\chi)^{s/\chi}}. \end{aligned}$$

It follows that

$$m_\chi(0) = N, \qquad m_\chi(s) > 1 \text{ for } s \in (0,\chi), \qquad m_\chi(\chi) = 1.$$

In other words, $\alpha_\chi = \chi$. Since m_χ is again a spectral function, it is convex, thus it can be deduced that it is decreasing on $[0,\chi]$. But what is the derivative in χ?

By Proposition 7.3, κ and thus m are differentiable on $[0,1]$. Consequently, m_χ is differentiable and it follows that

$$\begin{aligned} m'_\chi(s) &= m'(s)m(\chi)^{-s/\chi} - \frac{\log m(\chi)}{\chi} m(s)m(\chi)^{-s/\chi} \\ &= \frac{m'(s)}{m(\chi)^{s/\chi}} - \frac{\log m(\chi)}{\chi} m_\chi(s). \end{aligned}$$

Thus $m'_\chi(\chi) = \frac{m'(\chi)}{m(\chi)} - \frac{\log m(\chi)}{\chi} < 0$, since $m(\chi) > m(\alpha) = 1$.

It is checked in the subsequent Lemma 10.3 that if $[\operatorname{supp}\mu]$ satisfies (C), then (C) remains valid for $[\operatorname{supp}\mu_\chi]$. Moreover, the moment condition (M logM) implies

$$\mathbb{E}(1 + \left\|\mathbf{T}_{\chi,1}^\top\right\|)\left(1 + \left|\log\left\|\mathbf{T}_{\chi,1}^\top\right\|\right| + \left|\log\iota(\mathbf{T}_{\chi,1}^\top)\right|\right) < \infty.$$

Thus Theorem 9.15 applied to $\mathcal{S}_\chi$ gives the existence of a nontrivial fixed point η_χ of $\mathcal{S}_\chi$ for any $\chi \in (0,\alpha)$.

Step 2: It still has to be shown that the fixed point Y_χ can be chosen such that $\psi_\chi(u_0) = 1/2$. By property (9.18), $\psi_\chi(su_0) < 1$ for some $s > 0$. This gives in particular that

$$t \mapsto \psi_\chi(tu_0),$$

which is the LT of the real-valued random variable $\langle u_0, Y_\chi\rangle$, is nontrivial, thus it is monotone decreasing from 1 to 0. In particular, there is $t_0 \in \mathbb{R}_>$ with $\psi_\chi(t_0u_0) = 1/2$. But then $\tilde{Y}_\chi := t_0 Y_\chi$ is also a fixed point of $\mathcal{S}_\chi$ and its LT $\tilde{\psi}_\chi$ satisfies

$$\tilde{\psi}_\chi(u_0) = \psi_\chi(t_0u_0) = 1/2. \qquad \square$$

Lemma 10.3. *With the definitions above, if* $[\operatorname{supp}\mu]$ *satisfies* (C)*, then* $[\operatorname{supp}\mu_\chi]$ *satisfies* (C).

Proof. Recall that $[A]$ denotes the smallest closed semigroup which contains A. That means, elements of $[\operatorname{supp}\mu]$ are either of the form

(A) $\mathbf{a}_1\cdots\mathbf{a}_n$ for some $n \in \mathbb{N}$, $\mathbf{a}_1,\dots,\mathbf{a}_n \in \operatorname{supp}\mu$, or

(B) $\lim_{n\to\infty}\mathbf{b}_n$, where $(\mathbf{b}_n)_{n\in\mathbb{N}}$ is a convergent sequence of elements of type (A).

Note that when taking (finite or infinite) products of elements of type (B), they are again of type (B) by diagonal selection methods. The proof will rely on the fact that geometrical properties of elements of type (A) of $[\operatorname{supp}\mu]$ and $[\operatorname{supp}\mu_\chi]$ are the same, since

$$\mathbf{a}_1,\dots,\mathbf{a}_n \in \operatorname{supp}\mu \quad\Leftrightarrow\quad m(\chi)^{-\frac{1}{\chi}}\mathbf{a}_1,\dots,m(\chi)^{-\frac{1}{\chi}}\mathbf{a}_n \in \operatorname{supp}\mu_\chi.$$

To be precise, have a look at the two properties of (C).

1. Take any subspace $\emptyset \neq W \subsetneq \mathbb{R}^d$ with $W \cap \mathbb{R}^d_{\geq} \neq \emptyset$. Then there is an element $\mathbf{c}$ of $[\operatorname{supp} \mu]$ that does not leave W invariant. If $\mathbf{c}$ is of type (A), then $\mathbf{c}_\chi := m(\chi)^{-\frac{n}{\chi}} \mathbf{c} \in [\operatorname{supp} \mu_\chi]$ for some $n \in \mathbb{N}$ and W is not invariant under $\mathbf{c}_\chi$ either. If it is of type (B), let $(\mathbf{b}_n)_{n\in\mathbb{N}}$ be a sequence of elements of type (A) that converges towards $\mathbf{c}$. Then there exist vectors $w \in W$, $w_\perp \in W^\perp$ such that
$$\lim_{n\to\infty} \langle \mathbf{b}_n w, w_\perp \rangle = \langle \mathbf{c} w, w_\perp \rangle > 0.$$
But then due to the convergence, there is $n_0 \in \mathbb{N}$ with
$$\langle \mathbf{b}_{n_0} w, w_\perp \rangle > 0,$$
i.e. $\mathbf{b}_{n_0}$ does not leave W invariant and $\mathbf{b_n0}$ is of type (A).
2. A similar argument applies here. If $\mathbf{c} \in [\operatorname{supp} \mu] \cap \breve{\mathcal{M}}_+$, then if it is of type (A), $\mathbf{c}_\chi$ as defined above is (for suitable n) in $[\operatorname{supp} \mu_\chi] \cap \breve{\mathcal{M}}_+$. If now $\mathbf{c} = \lim_{n\to\infty} \mathbf{b}_n$ is of type (B), convergence of matrices implies the convergence of all its entries, thus already $\mathbf{b}_{n_0} \in [\operatorname{supp} \mu] \cap \breve{\mathcal{M}}_+$ for some n_0 and $\mathbf{b}_{n_0}$ is of type (A).

□

Lemma 10.4. *Let the assumptions of Theorem 10.1 be in force. Let $(\eta_\chi)_{\chi\in(0,\alpha)}$ be given by Lemma 10.2. Then there exists a convergent sequence $(\eta_{\chi_n})_{n\in\mathbb{N}}$ and its weak limit η is a nontrivial fixed point of $\mathcal{S}$.*

Proof. Step 1: Take any sequence $(\eta_{\chi_k})_{k\in\mathbb{N}}$ with $\chi_k \to \alpha$. Use that the set $\mathcal{M}^1(\mathbb{R}^d_{\geq})$ is vaguely compact by Proposition 3.1. Thus there is a convergent subsequence $(\eta_{\chi_n})_{n\in\mathbb{N}}$ with vague limit $\eta \in \mathcal{M}^1(\mathbb{R}^d_{\geq})$. The continuity theorem 5.2 yields for the corresponding LTs $(\psi_{\chi_n})_{n\in\mathbb{N}}$ resp. ψ that
$$\lim_{n\to\infty} \psi_{\chi_n}(tu) = \psi(tu)$$
for all $(u,t) \in \mathbb{S}_{\geq} \times \mathbb{R}_{>}$. It remains to show that η is a nontrivial probability measure with $\mathcal{S}\psi = \psi$.

Step 2: Use $\lim_{n\to\infty} m(\chi_n)^{-\frac{1}{\chi_n}} = m(\alpha) = 1$ together with the vague convergence $\eta_{\chi_n} \xrightarrow{v} \eta$ to infer by an application of Corollary 5.3 that
$$\lim_{n\to\infty} \psi_{\chi_n}\left(m(\chi_n)^{-\frac{1}{\chi_n}} t\mathbf{T}_i^\top u\right) = \psi(t\mathbf{T}_i^\top u) \tag{10.3}$$
for all $(u,t) \in \mathbb{S}_{\geq} \times \mathbb{R}_{>}$. Since ψ_{χ_n} is a fixed point of $\mathcal{S}_{\chi_n}$,
$$\psi_{\chi_n}(tu) = \mathbb{E}\left(\prod_{i=1}^{N} \psi_{\chi_n}\left(m(\chi_n)^{-\frac{1}{\chi_n}} t\mathbf{T}_i^\top u\right)\right). \tag{10.4}$$
Now taking the limit $n \to \infty$ in (10.4) and using (10.3) together with the bounded convergence

theorem, it follows that for all $t \in \mathbb{R}_>$, $u \in \mathbb{S}_\geq$,

$$\psi(tu) = \mathbb{E}\left(\prod_{i=1}^{N} \psi(t\mathbf{T}_i^\top u)\right). \tag{10.5}$$

Step 3: As a Laplace transform of a finite measure, ψ can be continuously extended to 0 and the value $\psi(0)$ is the total mass of the measure. Since ψ is uniformly bounded by 1, it may be deduced, again using the bounded convergence theorem, that

$$\psi(0) = \lim_{|x|\downarrow 0} \psi(x) = \mathbb{E}\left(\prod_{i=1}^{N} \lim_{|x|\downarrow 0} \psi(\mathbf{T}_i^\top x)\right) = (\psi(0))^N .$$

Together with (10.5) it gives that η is a fixed point of $\mathcal{S}$ and moreover implies $\psi(0) \in \{0, 1\}$. But $\psi(u_0) = \lim_{n\to\infty} \psi_{\chi_n}(u_0) = 1/2$, thus (since LT are monotone in any direction) $\psi(0) = 1$. Consequently, η is a probability measure. Also $\eta \neq \delta_0$, for $\psi(u_0) = 1/2$. □

Remark 10.5. There is no statement about uniqueness. Eventually, if a different reference point u_0' is chosen, the limiting distribution η may be different, also when choosing a different subsequence. Moreover, the proof does not give that $\psi(u) < 1$ for all $u \in \partial\mathbb{S}_\geq$, so the distribution may also be concentrated on some subspace.

Lemma 10.4 proves Theorem 10.1 except for the last assertion that $\mathbb{E}\,|Y| = \infty$ if $\alpha = 1$. This results from [29, Theorem 2.2] (see Theorem 4.6) which states that if $\alpha = 1$, the existence of a nontrivial fixed point with finite expectation is equivalent to $m'(1) < 0$. Consequently, when $m'(1) = 0$, the nontrivial fixed point that was constructed above necessarily has an infinite expectation.

11. From ℵ-Elementary Fixed Points to the Existence of α

For $\aleph \in \mathbb{R}_>$, define the subset $\mathcal{F}^\aleph \subset \mathcal{F}$ of ℵ-*elementary fixed points* of $\mathcal{S}$ by

$$\mathfrak{L}\left(\mathcal{F}^\aleph\right) := \left\{\psi \in \mathfrak{L}(\mathcal{P}(\mathbb{R}^d_\geq)) \;:\; \mathcal{S}\psi = \psi, \lim_{t\downarrow 0} t^{-\aleph}(1 - \psi(t\vartheta_{\mathbf{1}})) \in (0, \infty)\right\}. \tag{11.1}$$

It has been shown in Section 9, that if $\aleph = \alpha$ with $m'(\alpha) < 0$, then $\mathcal{F}^\aleph \neq \emptyset$. This section is mainly devoted to the converse implication, namely that if $\mathcal{F}^\aleph \neq \emptyset$, then $\aleph = \alpha$, i.e. $\aleph \in (0, 1]$ with $m(\aleph) = 1$ and $m'(\aleph) \leq 0$. A further result will be that if ψ is (the LT of) an ℵ-elementary fixed point, then readily

$$\lim_{t\downarrow 0} t^{-\aleph}(1 - \psi(tu)) = K e_*^\aleph(u)$$

for some $K > 0$. This justifies the introduction of a reference point in the definition of $\mathcal{F}^\aleph$ and will be the basis for proofs of uniqueness in the subsequent section.

Note that the Greek letter α was developed from the Phoenician letter *Aleph* (see [71, Chapter 4]]) and that Aleph corresponds to the Hebrew letter ℵ. Figuratively speaking, the same evolution will happen in this section, from $\aleph$ to α.

As a first step, some a priori observations about properties of $\aleph$ will be made. Subsequently, asymptotics at zero of $t^{-\aleph}(1-\psi(tu))$ are studied in a general setting of *dilated Laplace transforms* (defined below). Convergence results are obtained by using compactness of special sets of functions, which contain these dilated Laplace transforms. Then by an application of the Krein-Milman theorem and Kesten's Choquet-Deny lemma, it will be deduced that the existence of $\aleph$-elementary fixed points implies $m(\aleph) = 1$, $m'(\aleph) \leq 0$.

11.1. $\aleph$-Elementary Fixed Points

The following observations are trivial and stated without proof for further reference.

Lemma 11.1. *The trivial fixed point δ_0 is not in $\mathcal{F}^{\aleph}$ for any $\aleph > 0$. It holds*

$$\lim_{t\downarrow 0} \frac{1-\psi(t\vartheta_{\mathbf{d}})}{t^{\aleph}} = K \quad \Leftrightarrow \quad \lim_{t\downarrow 0} \frac{1-\psi(t\vartheta_{\mathbf{1}})}{t^{\aleph}} = d^{\frac{\aleph}{2}} K \tag{11.2}$$

$$\textit{and both imply for all } a \in \mathbb{R}_>, \ \lim_{t\downarrow 0} \frac{(at)^{\aleph}}{1-\psi(t\vartheta_{\mathbf{d}})} = \frac{a^{\aleph}}{K}. \tag{11.3}$$

More detailed information is provided by the following lemma.

Lemma 11.2. *If $\mathcal{F}^{\aleph} \neq \emptyset$, then $\aleph \in (0,1]$. If $\aleph \in I_\mu$, then $m(s) \geq 1$ for all $s \in [0,\aleph)$.*

Proof. Let $\mathcal{L}(Y) \in \mathcal{F}^{\aleph}$ with LT ψ. Suppose $\aleph > 1$, then

$$\mathbb{E}\langle \vartheta_{\mathbf{d}}, Y\rangle = \lim_{t\downarrow 0} \frac{1-\psi(t\vartheta_{\mathbf{d}})}{t} = 0.$$

Since $Y \in \mathbb{R}^d_{\geq}$, this already implies $Y \equiv 0$ a.s.. But this cannot be the case by Lemma 11.1. Turning to the second assertion, observe first that

$$\mathbb{E}\,|Y|^s \leq \mathbb{E}\langle \vartheta_{\mathbf{d}}, Y\rangle^s = \int_0^\infty s t^{s-1} \mathbb{P}\left(\langle \vartheta_{\mathbf{d}}, Y\rangle > t\right) dt.$$

Referring to the Tauberian theorem for LTs, Proposition 5.11, the tails of $\langle \vartheta_{\mathbf{d}}, Y\rangle$ decay like $t^{-\aleph}$. Consequently, $\mathbb{E}\,|Y|^s < \infty$ for all $s < \aleph$. In other words, $Y \in \mathcal{F}_s$ and if now $m(s) < 1$ for some $s \in (0,\aleph) \subset (0,1)$, then Proposition 4.3 gives that $Y \stackrel{d}{=} \delta_0$, which is again a contradiction with Lemma 11.1. □

Corollary 11.3. *If $\mathcal{F}^{\aleph} \neq \emptyset$ then either $\aleph \leq \alpha$ or $m'(\alpha) = 0$ and $m(\aleph) > 1$.*

Proof. By Lemma 11.2, two cases are possible. Case 1: If $m(s) > 1$ for all $s \in [0,\aleph)$, then by definition $\aleph \leq \alpha$. Case 2: If there is $s \in [0,\aleph)$ with $m(s) = 1$, then $\alpha < \aleph$. Considering the strict convexity of m (see Theorem 7.3) and the fact that $m(s) \geq 1$ for all $s \in [0,\aleph)$ it follows that $m'(\alpha) = 0$ and $m(\aleph) > m(\alpha) = 1$. □

11.2. The Set J_χ

In this subsection, a compact subset of $\mathcal{C}(\mathbb{S}_\geq \times \mathbb{R})$ is introduced and it is shown that properly dilated LTs (to be defined below) of ℵ-elementary fixed points are in this set. The definition is stated with a general parameter $\chi \in (0, 1]$.

Definition 11.4. For $\chi \in (0, 1]$ let J_χ be the set of continuous functions

$$g : \mathbb{S}_\geq \times \mathbb{R} \to [0, \infty)$$

satisfying

(i) $\sup_{u \in S_+} g(u, 0) \cdot e_*^\chi(u) \leq 1$,

(ii) $t \mapsto g(u, t)e^{-\chi t}$ is decreasing for all $u \in \mathbb{S}_\geq$,

(iii) $t \mapsto g(u, t)e^{(1-\chi)t}$ is increasing for all $u \in \mathbb{S}_\geq$,

(iv) $u \mapsto g(u, t)e_*^\chi(u)e^{-\chi t}$ is χ-Hölder with constant (less or equal to) 8 for each $t \geq 0$ and

(v) $u \mapsto g(u, t)e_*^\chi(u)e^{(1-\chi)t}$ is χ-Hölder with constant (less or equal to) 8 for each $t < 0$.

The definition of this and the subsequent sets is in the spirit of [41, Lemma 2.11]. The multivariate setting necessitates the additional properties (iv) and (v). Since the Arzelà-Ascoli will be used to derive the compactness of J_χ, these conditions have to be uniform in g, this is why an explicit constant is given there and the explicit choice 8 was made because of the constant 8 in Lemma 5.17.

Proposition 11.5. *The set J_χ is a compact subset of $\mathcal{C}(\mathbb{S}_\geq \times \mathbb{R})$ w.r.t. the topology of uniform convergence on compact sets.*

The following bounds on $g \in J_\chi$ will be needed for several of the subsequent proofs. Therefore, they are noted in a seperate lemma:

Lemma 11.6. *For all $g \in J_\chi$ and $u \in \mathbb{S}_\geq$, the following uniform bounds hold:*

$$0 \leq g(u, t) \leq \begin{cases} e_*^\chi(u)^{-1}e^{\chi t} & t \geq 0 \\ e_*^\chi(u)^{-1}e^{-(1-\chi)t} & t \leq 0 \end{cases}. \tag{11.4}$$

Proof. Combining properties (ii) and (i), it follows that $g(u, t)e_*^\chi(u, t)e^{-\chi t} \leq 1$ for all g and $t \geq 0$, since this function is decreasing and for $t = 0$ bounded by 1 due to property (i). This implies the bound for $t \geq 0$. The bound for $t \leq 0$ follows similarly by combining properties (iii) and (i). The lower bound follows from the very definition of $g \in J_\chi$. □

The proof of Proposition 11.5 uses the Arzelà-Ascoli theorem. For the reader's convenience, the definition of equicontinuity is recalled:

Definition 11.7. Let (E, d_E), (G, d_G) be locally compact metric spaces. A family F of functions $(E, d_E) \to (G, d_G)$ is equicontinuous at $x \in E$, if

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall y \in B_\delta(x)\ \forall f \in F :\ f(y) \in B_\varepsilon(f(x)).$$

Proof of Proposition 11.5. $J_\chi \subset \mathcal{C}(S_+ \times \mathbb{R})$ by definition. To show that J_χ is compact, the general version of the Arzelà-Ascoli theorem for locally compact metric spaces [58, Theorem 7.18] will be applied. Therefore, it has to be checked that

- J_χ is closed in $\mathcal{C}(S_+ \times \mathbb{R})$,
- for all $(u,t) \in \mathbb{S}_\geq \times \mathbb{R}$, the closure of the orbit

$$\overline{J_\chi(u,t)} = \{g(u,t) \ : \ g \in J_\chi\}$$

  is compact in $\mathbb{R}$ and
- at each $(u,t) \in \mathbb{S}_\geq \times \mathbb{R}$, J_χ is equicontinuous.

Step 1, J_χ is closed: Let the sequence $(g_n)_{n\in\mathbb{N}} \subset J_\chi$ be convergent with limit $g \in \mathcal{C}(\mathbb{S}_\geq \times \mathbb{R})$. This convergence is uniform on compact sets and in particular implies pointwise convergence, which is sufficient to check validity of the properties (i) - (v) for the limit g.

Compactness of the orbits is a direct consequence of the uniform bounds on $(g(u,t))_{g\in J_\chi}$ given in Lemma 11.6.

Step 2, equicontinuity: Fix $(u_0,t_0) \in \mathbb{S}_\geq \times \mathbb{R}$ and $\varepsilon > 0$. In order to prove equicontinuity, first consider the variation in t. Let $\delta > 0$. Then for any $g \in J_\chi$, it follows from property (iii) that for all $u \in \mathbb{S}_\geq$ and $t \in [t_0-\delta, t_0+\delta]$,

$$g(u,t)e^{(1-\chi)(t_0+\delta)} \leq g(u,t)e^{(1-\chi)t} \leq g(u_0,t_0+\delta)e^{(1-\chi)(t_0+\delta)},$$

thus

$$g(u,t) \leq g(u,t_0+\delta)e^{2(1-\chi)\delta}.$$

Similarly, it is a consequence of property (ii) that $g(u,t) \geq g(u,t_0+\delta)e^{-2\chi\delta}$. Referring to Lemma 11.6, it holds that

$$M := \sup\{g(u,t) \ : \ g \in J_\chi, \ (u,t) \in \mathbb{S}_\geq \times [t_0-\delta, t_0+\delta]\} < \infty.$$

Combining the estimates above gives

$$\begin{aligned} |g(u,t) - g(u,t_0)| &\leq g(u,t_0+\delta)e^{2(1-\chi)\delta} - g(u,t_0+\delta)e^{-2\chi\delta} \\ &\leq M\left(e^{2(1-\chi)\delta} - e^{-2\chi\delta}\right). \end{aligned}$$

Hence there is $\delta_1 > 0$ such that

$$|g(u,t) - g(u,t_0)| < \frac{\varepsilon}{2} \tag{11.5}$$

for all $t \in B_{\delta_1}(t_0)$ and all $u \in \mathbb{S}_\geq$. In order to consider the variation in u, a case distinction has to be made.

Case $t_0 \geq 0$: Writing $h(u,t) = g(u,t)e_*^\chi(u)e^{-\chi t}$, it is a consequence of Lemma 11.6 that

$$L := \sup\{h(u,t) \ : \ g \in J_\chi, \ (u,t) \in \mathbb{S}_\geq \times [t_0-\delta_1, t_0+\delta_1]\} < \infty.$$

Considering property (iv), it follows that for all $u \in \mathbb{S}_{\geq}$,

$$\begin{aligned} |g(u,t_0) - g(u_0,t_0)| &= e^{\chi t_0} \left| \frac{h(u,t_0)}{e_*^{\chi}(u)} - \frac{h(u_0,t_0)}{e_*^{\chi}(u_0)} \right| \\ &\leq \frac{e^{\chi t_0}}{e_*^{\chi}(u_0)} |h(u,t_0) - h(u_0,t_0)| + e^{\chi t_0} \left| \frac{h(u_0,t_0)}{e_*^{\chi}(u)} - \frac{h(u_0,t_0)}{e_*^{\chi}(u_0)} \right| \\ &\leq 8 \frac{e^{\chi t_0}}{e_*^{\chi}(u_0)} |u - u_0|^{\chi} + e^{\chi t_0} L \left| \frac{1}{e_*^{\chi}(u)} - \frac{1}{e_*^{\chi}(u_0)} \right|. \end{aligned}$$

Hence there is $\delta_2 > 0$ such that

$$|g(u,t_0) - g(u_0,t_0)| \leq \varepsilon_2 \tag{11.6}$$

for all $u \in B_{\delta_2}(u_0)$. Combining (11.5) and (11.6), it holds that for all $(u,t) \in B_{\delta_2}(u_0) \times B_{\delta_1}(t_0)$,

$$|g(u,t) - g(u_0,t_0)| \leq |g(u,t) - g(u,t_0)| + |g(u,t_0) - g(u,t)| \leq \varepsilon.$$

This proves the equicontinuity in the case $t_0 \geq 0$. The Case $t_0 < 0$ can be treated completely similar, by using property (v) instead of property (iv). □

Let $\phi \in \mathfrak{L}(\mathcal{P}(\mathbb{R}^d_{\geq}))$. Define the *s-dilation of* ϕ, $s > 0$, by

$$h_s(u,t) := \frac{D_{\chi,\phi}(u, s+t)}{e^{\chi s}(1 - \phi(e^{-s}\vartheta_{\mathbf{d}}))} = \frac{e^{\chi t}}{e_*^{\chi}(u)} \frac{1 - \phi(e^{-(s+t)}u)}{1 - \phi(e^{-s}\vartheta_{\mathbf{d}})}.$$

Proposition 11.8. *Let* $\phi \in \mathfrak{L}(\mathcal{P}(\mathbb{R}^d_{\geq}))$, $\gamma \in (0,1]$ *and* $K \in \mathbb{R}_{>}$ *with*

$$\lim_{t \downarrow 0} \frac{1 - \phi(t\vartheta_{\mathbf{d}})}{t^{\gamma}} = K. \tag{11.7}$$

Then there is s_0 *such that* $(h_s)_{s \geq s_0} \in J_{\chi}$ *for any* $\chi \in (0,\gamma]$.

Moreover, if there is a function $f : \mathbb{S}_{\geq} \to [0,\infty)$ *with*

$$\lim_{t \downarrow 0} \frac{1 - \phi(tu)}{t^{\gamma}} = f(u) \quad \forall u \in \mathbb{S}_{\geq} \tag{11.8}$$

then this convergence is uniform on $\mathbb{S}_{\geq}$, *i.e.*

$$\lim_{t \downarrow 0} \left| \frac{1 - \phi(t\cdot)}{t^{\gamma}} - f \right|_{\infty} = 0. \tag{11.9}$$

Proof. Step 1: Let's check the properties of J_{χ} for h_s, $s > 0$.

(i) Using Inequality (25.9),

$$h_s(u,0) e_*^{\chi}(u) = \frac{1 - \phi(e^{-s}u)}{1 - \phi(e^{-s}\vartheta_{\mathbf{d}})} \leq 1.$$

(ii) By definition, $h_s(u,t)e^{-\chi t} = \left(e_*^\chi(u)(1-\phi(e^{-s}\vartheta_{\mathbf{d}}))\right)^{-1}(1-\phi(e^{-t}e^{-s}u))$. Note that s and u are fixed. As a function of t, it is decreasing since ϕ is a LT.

(iii) In this case

$$h_s(u,t)e^{(1-\chi)t} = \left(e_*^\chi(u)(1-\phi(e^{-s}\vartheta_{\mathbf{d}}))\right)^{-1}\frac{1-\phi(e^{-t}e^{-s}u)}{e^{-t}},$$

where s, u can be considered fixed. Recall that $r \mapsto (1-\phi(ru))/r$ is again a LT, thus decreasing in r (cf. Subsection 25.1). Now since $t \mapsto e^{-t}$ is as well decreasing,

$$t \mapsto \frac{1-\phi(e^{-t}e^{-s}u)}{e^{-t}}$$

is increasing.

(iv) This will result from an application of Lemma 5.17. Set $s_0 := -\log A$, Then for all $s \geq s_0$, $t \geq 0$ $a = e^{-(s+t)} \leq A$. Consequently, for all $u, w \in \mathbb{S}_\geq$

$$\begin{aligned}&\left|h_s(u,t)e_*^\chi(u)e^{-\chi t} - h_s(w,t)e_*^\chi(w)e^{-\chi t}\right| \\ &= \left|\frac{1-\phi(e^{-(s+t)}u)}{1-\phi(e^{-s}\vartheta_{\mathbf{d}})} - \frac{1-\phi(e^{-(s+t)}w)}{1-\phi(e^{-s}\vartheta_{\mathbf{d}})}\right| \overset{(5.19)}{\leq} 8(|u-w| \wedge 1)^\chi \\ &\leq 8\,|u-w|^\chi,\end{aligned}$$

which is the asserted Hölder continuity.

(v) Applying again Lemma 5.17 with the same s_0. Now $a = e^{-s} \leq A$ for all $s \geq s_0$ and $b = e^{-t} > 1$ for all $t < 0$. It follows that for all $u, w \in \mathbb{S}_\geq$,

$$\begin{aligned}&\left|h_s(u,t)e_*^\chi(u)e^{(1-\chi)t} - h_s(w,t)e_*^\chi(w)e^{(1-\chi)t}\right| \\ &= e^t\left|\frac{1-\phi(e^{-t}e^{-s}u)}{1-\phi(e^{-s}\vartheta_{\mathbf{d}})} - \frac{1-\phi(e^{-t}e^{-s}w)}{1-\phi(e^{-s}\vartheta_{\mathbf{d}})}\right| \\ &\overset{(5.20)}{\leq} e^t 8e^{-t}(|u-w| \wedge 1)^\chi \leq 8\,|u-w|^\chi,\end{aligned}$$

which is the asserted Hölder continuity.

Consequently, $h_s \in J_\chi$ for all $s \geq s_0 = -\log A$, where A is given by Lemma 5.17.

Step 2: Property (11.8) is equivalent to

$$\lim_{s\to\infty}\frac{1-\phi(e^{-(s+t)}u)}{e^{-s\gamma}} = e^{-\gamma t}f(u), \tag{11.10}$$

for all $(u,t) \in \mathbb{S}_\geq \times \mathbb{R}$ and, taking (11.7) into account, also equivalent to the *pointwise convergence*

$$\lim_{s\to\infty} h_s(u,t) = \lim_{s\to\infty}\frac{e^{\chi t}}{e_*^\chi(u)}\frac{1-\phi(e^{-(s+t)}u)}{1-\phi(e^{-s}\vartheta_{\mathbf{d}})} = \frac{e^{(\chi-\gamma)t}}{e_*^\chi(u)K}f(u) =: h(u,t). \tag{11.11}$$

Since the convergence in (11.7) is independent of u, it is sufficient to show that

$$\lim_{s\to\infty} |h_s(\cdot,0) - h(\cdot,0)|_\infty = 0$$

in order to prove (11.9).

Considering Step 1, there is $s_0 \in \mathbb{R}_>$ such that $h_s \in J_\chi$ for all $s \geq s_0$. J_χ is compact and hence any sequence h_{s_n} with $s_n \to \infty$ has a subsequence which is *uniformly convergent on compact sets*. But referring to (11.11), any subsequence has the same limit, namely h. Since $\mathcal{C}(\mathbb{S}_\geq \times \mathbb{R})$ is Hausdorff, then readily $h_s \to h$, now *uniformly on compact subsets* of $\mathbb{S}_\geq \times \mathbb{R}$. The assertion follows by considering the compact set $\mathbb{S}_\geq \times \{0\}$. □

11.3. The Set $H_{\chi,c}$

The set J_χ can be seen as a limit set for dilated Laplace transforms of arbitrary distributions on $\mathbb{R}^d_\geq$ with tail index < 1. In this subsection, a subset $H_{\chi,c} \subset J_\chi$ will be defined which will turn out to be the limit set for more specialized dilated LTs, namely those of ℵ-elementary fixed points.

Definition 11.9. Let $[\operatorname{supp}\mu]$ satisfy (C). For $\chi \in I_\mu \cap (0,1]$, $c \in (0,1]$ define the subset $H_{\chi,c} \subset J_\chi$ as follows: A function $g \in J_\chi$ is in $H_{\chi,c}$, if it satisfies the additional properties:

(i') $\sup_{u\in\mathbb{S}_\geq} g(u,0)e_*^\chi(u) = c$ and $g(u,0)e_*^\chi(u) \geq \min_i u_i$ for all $u \in \mathbb{S}_\geq$.

(vi) For all $(u,t) \in \mathbb{S}_\geq \times \mathbb{R}$, $g(u,t) = m(\chi)\mathbb{E}_u^\chi g(X_1, t - V_1)$.

(vii) Introducing

$$L_t : \mathbb{S}_\geq \times \mathbb{R} \to \mathbb{R}_>, \quad (u,r) \mapsto \frac{g(u,t+r)}{g(u,r)},$$

the following holds: For all $t \in \mathbb{R}$, all compact $C \subset \breve{\mathbb{S}}_\geq$, all $u, w \in C$:

$$\sup_{r\in\mathbb{R}} e^{-\chi t} |L_t(u,r) - L_t(w,r)| \leq 16\,(1 \vee e^{-t})(\min_{i=1,\dots,d; y\in C} y_i)^{-1} |u-w|^\chi.$$

Note that a priori, $g(u,t) = 0$ is possible for $t \neq 0$ and also $g(u,0) = 0$ for $u \in \partial\mathbb{S}_\geq$. The following lemma shows that this does not happen. Consequently, L_t and $H_{\chi,c}$ are well defined.

Lemma 11.10. *Let $[\operatorname{supp}\mu]$ satisfy (C). Let $g \in J_\chi$ for some $\chi \in (0,1] \cap I_\mu$ and let g satisfy properties (vi) and (i') for some $c \in (0,1]$. Then for all $(u,t) \in \mathbb{S}_\geq \times \mathbb{R}$, it holds that $g(u,t) > 0$. In particular, L_t is well defined and continuous on $\mathbb{S}_\geq \times \mathbb{R}$.*

Proof. Completely analogue to Lemma 11.6, for all $g \in H_\chi$ the lower bounds

$$g(u,t) \geq \begin{cases} g(u,0)e^{-\chi t} & t \leq 0 \\ g(u,0)e^{(1-\chi)t} & t \geq 0 \end{cases} \tag{11.12}$$

can be obtained. Thus as soon as $g(u,0) > 0$, readily $g(u,t) > 0$ for all $t \in \mathbb{R}$. Referring to property (i'), this already implies $g > 0$ on $\breve{\mathbb{S}}_\geq \times \mathbb{R}$.

Considering Theorem 7.3 (3) and (4), for every function f that is strictly positive on $\breve{\mathbb{S}}_{\geq}$, there is $n \in \mathbb{N}$ such that $(P_*^\chi)^n f(u) > 0$ for all $u \in \mathbb{S}_{\geq}$ i.e. $\mathbb{Q}_u^\chi\{f(X_n) > 0\} > 0$.

Applying this for $f = g(\cdot, 0)$, it follows that for some $n \in \mathbb{N}$, $\mathbb{Q}_u^\chi\{g(X_n) > 0\} > 0$. Since $\mathbb{Q}_u^\chi$ is a probability measure, there is even a compact set $C \subset \mathbb{S}_{\geq} \times \mathbb{R}$ such that

$$\mathbb{Q}_u^\chi\{g(X_n) > 0,\ (X_n, V_n) \in C\} > 0.$$

Write $r := \min_{(y,s)\in C} g(y) \left[e^{\chi s}\mathbf{1}_{[0,\infty)}(s) + e^{-(1-\chi)s}\mathbf{1}_{(-\infty,0)}(s)\right]$ and observe that this quantity is positive by the bounds obtained above. Now using property (6), for all $u \in \mathbb{S}_{\geq}$

$$\begin{aligned} g(u,0) &= \frac{1}{m(\chi)^n}\mathbb{E}_u^\chi g(X_n, -V_n) \\ &\geq \frac{1}{m(\chi)^n}\mathbb{E}_u^\chi g(X_n)\left[e^{\chi V_n}\mathbf{1}_{[0,\infty)}(V_n) + e^{-(1-\chi)V_n}\mathbf{1}_{(-\infty,0)}(V_n)\right] \\ &\geq \frac{1}{m(\chi)^n}\, r\, \mathbb{Q}_u^\chi\{g(X_n) > 0,\ (X_n, V_n) \in C\} \end{aligned}$$

where $r = \min_{(y,s\in C)} g(y) \left[e^{\chi s}\mathbf{1}_{[0,\infty)}(s) + e^{-(1-\chi)s}\mathbf{1}_{(-\infty,0)}(s)\right] > 0$. □

The statement of property (vii) looks quite awkward. There are several excuses for considering it: Firstly, it is satisfied by limits of s-dilated Laplace transforms of elemantary fixed points. Secondly, and more important, it is necessary in order to apply the Choquet-Deny lemma of Kesten. Thirdly, this particular formulation is compatible with the pointwise convergence of functions g. The latter will be used in the next result:

Proposition 11.11. *Let* $[\operatorname{supp}\mu]$ *satisfy* (C) *and* $\mathbb{E}\,\|\mathbf{M}_1\| < \infty$. *Then for each* $c \in (0,1]$ *and* $\chi \in (0,1]$, *the set* $H_{\chi,c}$ *is a compact subset of* $\mathcal{C}\,(S \times \mathbb{R})$ *with respect to the topology of uniform convergence on compact sets.*

Proof. The main part of the proof is already contained in Proposition 11.5. Since $H_{\chi,c} \subset J_\chi$ and J_χ is compact, it is sufficient to show that $H_{\chi,c}$ is closed, i.e. any uniform limit g of functions $g_n \in H_{\chi,c}$ is again an element of $H_{\chi,c}$. Uniform convergence on compact sets implies the pointwise convergence $g_n \to g$, hence it is even sufficient to show that the additional properties (i'), (vi) and (vii) are closed under pointwise convergence. For (i') and (vii), this is (more or less) obvious, it remains to consider (vi). Since $g_n \in H_{\chi,c}$, for all $(u,t) \in \mathbb{S}_{\geq} \times \mathbb{R}$

$$g_n(u,t) = m(\chi)\mathbb{E}_u^\chi g_n(X_1, t - V_1).$$

Then the same holds for g if it can be shown that the sequence of r.v.s $(g_n(X_1, t - V_1))_{n\in\mathbb{N}}$ is uniformly integrable w.r.t. $\mathbb{Q}_u^\chi$ because this would imply that

$$\mathbb{E}_u^\chi g_n(X_1, t - V_n) \to \mathbb{E}_u^\chi g(X_1, t - V_n).$$

The proof of uniform integrability is given in the subsequent lemma. □

Lemma 11.12. *Let* $[\operatorname{supp}\mu]$ *satisfy* (C) *and* $\mathbb{E}\,\|\mathbf{M}_1\| < \infty$. *Let* $\chi \in (0,1]$. *Then for all* $(u,t) \in \mathbb{S}_\geq \times \mathbb{R}$, *the family of r.v.s* $(g(X_1, t-V_1))_{g\in J_\chi}$ *is uniformly integrable w.r.t. to* $\mathbb{Q}_u^\chi$.

Proof. Let $r>0$, $g \in J_\chi$. Use the bounds (11.4) and the definition (8.4) of $\mathbb{Q}_u^\chi$ to compute

$$\begin{aligned}
&\int_{\{g(X_1,t-V_1)>r\}} g(X_1,t-V_1)\, d\mathbb{Q}_u^\chi \\
&\overset{(8.4)}{=} \frac{1}{e_*^\chi(u)\kappa(\chi)} \mathbb{E}_u \left(e^{\chi V_1} e_*^\chi(X_1) g(X_1,t-V_1) \mathbf{1}_{\{g(X_1,t-V_1)>r\}}\right) \\
&\overset{(11.4)}{\leq} \frac{1}{e_*^\chi(u)\kappa(\chi)} \mathbb{E}_u \Bigg(e^{\chi V_1} e_*^\chi(X_1) \Bigg[\frac{e^{\chi(t-V_1)}}{e_*^\chi(X_1)} \mathbf{1}_{\{t\geq V_1\}} \mathbf{1}_{\{\frac{e^{\chi(t-V_1)}}{e_*^\chi(X_1)}>r\}} \\
&\qquad + \frac{e^{-(1-\chi)(t-V_1)}}{e_*^\chi(X_1)} \mathbf{1}_{\{t<V_1\}} \mathbf{1}_{\{\frac{e^{-(1-\chi)(t-V_1)}}{e_*^\chi(X_1)}>r\}} \Bigg]\Bigg) \\
&= \frac{1}{e_*^\chi(u)\kappa(\chi)} \Bigg[e^{\chi t} \mathbb{P}_u \left(V_1 \leq t,\ e^{\chi V_1} < \frac{e^{\chi t} e_*^\chi(X_1)}{r} \right) \\
&\qquad + e^{-(1-\chi)t} \mathbb{E}_u \left(e^{V_1} \mathbf{1}_{\{V_1>t,\ e^{(1-\chi)V_1} > r e^{(1-\chi)t} e_*^\chi(X_1)\}} \right) \Bigg] \\
&\leq \frac{1}{e_*^\chi(u)\kappa(\chi)} \left[e^{\chi t} \mathbb{P}\left(|\mathbf{M}_1 u|^\chi < \frac{1}{r} C \right) + e^{-(1-\chi)t} \mathbb{E}\left(\|\mathbf{M}_1\| \mathbf{1}_{\{\|\mathbf{M}_1\|>rC'\}} \right) \right]
\end{aligned}$$

with

$$C = e^{\chi t} \sup_{y\in\mathbb{S}_\geq} e_*^\chi(y) < \infty \quad C' = e^{(1-\chi)t} \inf_{y\in\mathbb{S}_\geq} e_*^\chi(y) > 0$$

and independent of g. Since by assumption $\mathbb{E}\,\|\mathbf{M}_1\| < \infty$, the final expression converges to zero as $r \to \infty$. This gives the asserted uniform integrability. □

Proposition 11.13. *Let* $[\operatorname{supp}\mu]$ *satisfy* (C) *and* $\mathbb{E}\,\|\mathbf{M}_1\| < \infty$. *For* $\aleph \in (0,1]$, *let* $\psi \in \mathfrak{L}(\mathcal{F}^\aleph)$. *Choose* $\chi \leq \aleph$. *Then any sequence* $(h_{s_n})_{n\in\mathbb{N}}$ *of s-dilations of* ψ *with* $s_n \to \infty$ *has a convergent subsequence* $h_{s_{n_k}}$. *For the subsequence's limit* h, *there is* $c>0$ *such that* $h \in H_{\chi,c}$.

Proof. Step 1: By its very definition, the LT ψ of an ℵ-elementary fixed point satisfies

$$\lim_{t\to 0} \frac{1-\psi(t\vartheta_{\mathbf{d}})}{t^\aleph} = K \in \mathbb{R}_>.$$

Thus, for $\chi \leq \aleph$ Proposition 11.8 gives that for some $s_0 > 0$, $(h_s)_{s\geq s_0}$ is in the compact set J_χ. This implies the existence of a convergent subsequence and its limit h is in J_χ. The main burden is now to show that $h \in H_{\chi,c}$, i.e. it satisfies the additional properties (i'), (vi) and (vii).

Step 2, property (i'): On the one hand, $\sup_{u\in\mathbb{S}_\geq} h(u,0) e_*^\chi(u) \leq 1$ by property (i) of J_χ. On the other hand, for all $u \in \breve{\mathbb{S}}_\geq$ and all $s>0$, by inequality (25.12)

$$h_s(u,0) = \frac{1}{e_*^\chi(u)} \frac{1-\phi(e^{-s}u)}{1-\phi(e^{-s}\vartheta_{\mathbf{d}})} \geq \min_i u_i \frac{1-\phi(e^{-s}\vartheta_{\mathbf{d}})}{1-\phi(e^{-s}\vartheta_{\mathbf{d}})} = \min_i u_i > 0.$$

In particular, $h(\cdot,0)$ does not vanish, thus indeed $c := \sup_{u\in\mathbb{S}_\geq} h(u,0)e_*^\chi(u) \in (0,1]$, consequently, (i') holds for h.

Step 3, property (vi): Fix $(u,t)\in\mathbb{S}_\geq\times\mathbb{R}$. Since $\mathcal{S}\psi=\psi$, Lemma 9.6 gives

$$D_{\chi,\psi}(u,s+t) = m(\chi)\mathbb{E}_u^\chi D_{\chi,\psi}(X_1,s+t-V_1) - G_{\chi,\psi}(u,s+t).$$

Combining this with the defintion of h_s,

$$\begin{aligned} h_s(u,t) &= m(\chi)\frac{\mathbb{E}_u^\chi D_{\chi,\psi}(X_1,s+t-V_1)}{e^{\chi s}(1-\psi(e^{-s}\vartheta_{\mathbf{d}}))} - \frac{G_{\chi,\psi}(u,s+t)}{e^{\chi s}(1-\psi(e^{-s}\vartheta_{\mathbf{d}}))} \\ &= m(\chi)\mathbb{E}_u^\chi h_s(X_1,t-V_1) - \frac{G_{\chi,\psi}(u,s+t)}{e^{\chi(s+t)}(1-\psi(e^{-(s+t)}\vartheta_{\mathbf{d}}))}\frac{e^{\chi(s+t)}(1-\psi(e^{-(s+t)}\vartheta_{\mathbf{d}}))}{e^{\chi s}(1-\psi(e^{-s}\vartheta_{\mathbf{d}}))}. \end{aligned} \tag{11.13}$$

By Lemma 9.11, equation (9.15),

$$\lim_{s\to\infty}\frac{G_{\chi,\psi}(u,s+t)}{e^{\chi(s+t)}(1-\psi(e^{-(s+t)}\vartheta_{\mathbf{d}}))} = 0$$

and by inequality (25.5)

$$0 \leq \frac{e^{\chi(s+t)}(1-\psi(e^{-(s+t)}\vartheta_{\mathbf{d}}))}{e^{\chi s}(1-\psi(e^{-s}\vartheta_{\mathbf{d}}))} \leq e^{\chi t}\frac{1-\psi(e^{-s}\vartheta_{\mathbf{d}})}{1-\psi(e^{-s}\vartheta_{\mathbf{d}})} \leq e^{\chi t}, \tag{11.14}$$

thus the second term in (11.13) tends to zero as $s\to\infty$.

Now by the uniform integrability of functions $(h_{s_{n_k}}(X_1,t-V_1))_{n_k}\subset J_\chi$, which has been shown in Lemma 11.12,

$$h(u,t) = \lim_{k\to\infty} h_{s_{n_k}}(u,t) = \lim_{k\to\infty}\mathbb{E}_u^\chi h_{s_{n_k}}(X_1,t-V_1) = \mathbb{E}_u^\chi h(X_1,t-V_1).$$

Step 4, property (vii): Fix $t\in\mathbb{R}$, $C\subset\breve{\mathbb{S}}_\geq$ compact and compute for all $u,w\in C$

$$\begin{aligned} &e^{-\chi t}\left|\frac{h_s(u,t+r)}{h_s(u,r)} - \frac{h_s(w,t+r)}{h_s(w,r)}\right| \\ &= e^{-\chi t}\left|\frac{e^{\chi(t+r)}(1-\psi(e^{-(s+t+r)}u))}{e_*^\chi(u)(1-\psi(e^{-s}\vartheta_{\mathbf{d}}))}\cdot\frac{e_*^\chi(u)(1-\psi(e^{-s}\vartheta_{\mathbf{d}}))}{e^{\chi r}(1-\psi(e^{-(s+r)}u))}\right. \\ &\quad\left. - \frac{e^{\chi(t+r)}(1-\psi(e^{-(s+t+r)}w))}{e_*^\chi(w)(1-\psi(e^{-s}\vartheta_{\mathbf{d}}))}\cdot\frac{e_*^\chi(w)(1-\psi(e^{-s}\vartheta_{\mathbf{d}}))}{e^{\chi r}(1-\psi(e^{-(s+r)}w))}\right| \\ &= \left|\frac{1-\psi(e^{-(s+t+r)}u)}{1-\psi(e^{-(s+r)}u)} - \frac{1-\psi(e^{-(s+t+r)}w)}{1-\psi(e^{-(s+r)}w)}\right| \end{aligned}$$

Abbreviate the terms with a, b, c, d and continue

$$
\begin{aligned}
&= \left|\frac{a}{b} - \frac{c}{d}\right| = \left|\frac{a}{b}\cdot\frac{d-b}{d} + \frac{c}{d}\cdot\frac{a-c}{c}\right| \le \left|\frac{a}{b}\right|\left|\frac{d-b}{d}\right| + \left|\frac{c}{d}\right|\left|\frac{a-c}{c}\right| \\
&= \left(\frac{1-\psi(e^{-(s+t+r)}u)}{1-\psi(e^{-(s+r)}u)}\right)\left|\frac{d-b}{d}\right| + \left(\frac{1-\psi(e^{-(s+t+r)}w)}{1-\psi(e^{-(s+r)}w)}\right)\left|\frac{a-c}{c}\right| \\
&\le (1\vee e^{-t})\left|\frac{d-b}{d}\right| + (1\vee e^{-t})\left|\frac{a-c}{c}\right|
\end{aligned}
$$

In the last line, the inequalities (25.5) and (25.8) were used. Extending the fractions and using inequality (25.12) allows to continue by

$$
\begin{aligned}
&= (1\vee e^{-t})\frac{1-\psi(e^{-(s+r)}\vartheta_{\mathbf{d}})}{1-\psi(e^{-(s+r)}w)}\left|\frac{1-\psi(e^{-(s+r)}w) - \big(1-\psi(e^{-(s+r)}u)\big)}{1-\psi(e^{-(s+r)}\vartheta_{\mathbf{d}})}\right| \\
&\quad + (1\vee e^{-t})\frac{1-\psi(e^{-(s+t+r)}\vartheta_{\mathbf{d}})}{1-\psi(e^{-(s+t+r)}w)}\left|\frac{1-\psi(e^{-(s+t+r)}u) - \big(1-\psi(e^{-(s+t+r)}w)\big)}{1-\psi(e^{-(s+t+r)}\vartheta_{\mathbf{d}})}\right| \\
&\le (1\vee e^{-t})\left(\min_i w_i\right)^{-1}\left|\frac{1-\psi(e^{-(s+r)}w) - \big(1-\psi(e^{-(s+r)}u)\big)}{1-\psi(e^{-(s+r)}\vartheta_{\mathbf{d}})}\right| \\
&\quad + (1\vee e^{-t})\left(\min_i w_i\right)^{-1}\left|\frac{1-\psi(e^{-(s+t+r)}u) - \big(1-\psi(e^{-(s+t+r)}w)\big)}{1-\psi(e^{-(s+t+r)}\vartheta_{\mathbf{d}})}\right|
\end{aligned}
$$

By Lemma 5.17 there is $A \in \mathbb{R}_>$ such that as soon as

$$
e^{-(s+t+r)} \vee e^{-(s+r)} \le A, \tag{11.15}
$$

the following estimate is valid

$$
\begin{aligned}
\ldots &\le (1\vee e^{-t})\left(\min_{y\in C}\min_i y_i\right)^{-1}\cdot 2\cdot 8(|u-w|\wedge 1)^{\aleph} \\
&\le 16(1\vee e^{-t})\left(\min_{y\in C}\min_i y_i\right)^{-1}|u-w|^{\chi}
\end{aligned}
$$

The condition (11.15) holds for all $r \in \mathbb{R}$ in the limit $s \to \infty$ (recall that t is fixed). Luckily, the estimation just calculated remains valid unter pointwise convergence $h_{s_{n_k}} \to h$, so when taking the limit $s_{n_k} \to \infty$, it follows that the estimate is valid for all $r \in \mathbb{R}$ and consequently, h satisfies property (vii). □

Corollary 11.14. *Let* $[\operatorname{supp}\mu]$ *satisfy* (C) *and* $\mathbb{E}\,\|\mathbf{M}_1\| < \infty$*. If* $\mathcal{F}^{\aleph} \neq \emptyset$*, then for all* $\chi \in (0, \aleph]$*,* $H_{\chi,c} \neq \emptyset$ *for some* $c \in (0, 1]$.

11.4. The Set E_χ

It has been shown in Proposition 11.11 that $H_{\chi,c}$ is a compact subset of $\mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$. By the Krein-Milman theorem, it is contained in the convex hull of the set of its extremal points $E_{\chi,c}$, say. In this subsection, a description of $E_{\chi,c}$ will be given and in particular it will be proved that $E_{\chi,c} \neq \emptyset$ if and only if there is $\alpha \in (0,1]$ such that $m(\alpha) = 1$, $m'(\alpha) \leq 0$. This will lay out a path from $\mathcal{F}^\aleph \neq \emptyset$ to the conclusion that $\aleph = \alpha$.

Property (vi) states that – depending on the value of $m(\chi)$ – the functions in $H_{\chi,c}$ are sub-, super- or even harmonic for the MC $(X_n, V_n)_{n\in\mathbb{N}_0}$ under $\mathbb{Q}_u^\chi$. So in a sense, the description of $E_{\chi,c}$ will consist of an identification of extremal harmonic functions, which usually requires a result of Choquet-Deny type.

The Choquet-Deny lemma that will be used is due to Kesten [60, Lemma 1]. It is part of the proof of his MRT and is stated below in a version adapted to the present situation together with a proof that the reformulations are valid.

Lemma 11.15. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C) *and let*

$$\mathbb{E}\,|\log\|\mathbf{M}_1\|| + |\log\iota(\mathbf{M}_1)| + |\log\iota(\mathbf{T}_1)| < \infty. \tag{log-moments}$$

If $L \in \mathcal{C}_b(\mathbb{S}_\geq \times \mathbb{R})$ *satisfies*

(a) $L(u,s) = \mathbb{E}_u L(X_1, s - V_1)$ *for all* $(u,s) \in \breve{\mathbb{S}}_\geq \times \mathbb{R}$ *and*

(b) for each $u \in \breve{\mathbb{S}}_\geq$, $\lim_{v\to u}\sup_{s\in\mathbb{R}} |L(v,s) - L(u,s)| = 0$,

then L is constant.

The original statement contained in the proof of [60, Lemma 1, bottom of p. 362] can be rephrased as follows:

> **Kesten's Choquet-Deny lemma**
> Assume that conditions I.1 - I.3 are satisfied. Let L be a bounded function on $S\times\mathbb{R}$ that satisfies
>
> $$L(u,s) = \mathbb{E}_u L(X_1, s - V_1) \quad \forall\,(u,s) \in S \times \mathbb{R} \tag{(2.4)}$$
>
> and in addition, for all $h \in \mathcal{C}_c(\mathbb{R})$
>
> $$\limsup_{v\to u,\delta\downarrow 0}\ \sup_{|s'-s''|<\delta} |L_h(v,s') - L_h(u,s'')| = 0 \tag{(2.2)}$$
>
> where
>
> $$L_h(u,s) = \int_{-\infty}^{\infty} L(u, s+r)h(r)dr.$$
>
> Then L is a constant.

Here S is a separable metric space and the conditions I.1 - I.3, which will not be repeated here, can be found on [60, page 359].

Proof of Lemma 11.15. It is shown in [29, Proposition 5.5] that under the assumptions of the present lemma, Conditions I.1 - I.3 are satisfied for the MRW $(X_n, -V_n)_{n\in\mathbb{N}}$ w.r.t. $\mathbb{Q}_u$. The negative sign appears since in $I.2$ it is assumed that the random walk part has a positive drift, but by Theorem 8.2, $\lim_{n\to\infty} \frac{V_n}{n} < 0$ $\mathbb{Q}_u$-a.s. Nevertheless, considering $\tilde{L}(u,s) := L(u,-s)$ together with $(X_n, -V_n)_{n\in\mathbb{N}_0}$ leaves assumptions and assertions invariant. Hence without loss of generality (w.l.o.g.), the conditions I.1 - I.3 are satisfied for (X_n, V_n). Referring to Lemma 7.2, the assumption $\mathbb{E}\left|\log \iota(\mathbf{T}_1)\right|$ gives that $\breve{\mathbb{S}}_\geq \times \mathbb{R}$ is an invariant set for $(X_n, V_n)_{n\in\mathbb{N}_0}$. A close inspection of the proof of [29, Proposition 5.5] shows that conditions I.1 - I.3 remain valid for the restriction of the MRW to $\breve{\mathbb{S}}_\geq \times \mathbb{R}$.

Obviously, it is sufficient to show that L is constant on this $\breve{\mathbb{S}}_\geq \times \mathbb{R}$ since it is assumed that L is continuous on $\mathbb{S}_\geq \times \mathbb{R}$. Hence Kesten's Choquet-Deny Lemma will be applied with $S = \breve{\mathbb{S}}_\geq$ and L restricted to $\breve{\mathbb{S}}_\geq \times \mathbb{R}$.

It remains to check its assumptions ((2.2)) and ((2.4)). Condition ((2.4)) is just assumption (a). The boundedness of L and h implies

$$\lim_{\delta\downarrow 0} \sup_{u\in\breve{\mathbb{S}}_\geq} \sup_{|s'-s''|<\delta} \left|L_h(u,s') - L_h(u,s'')\right| = 0 \tag{11.16}$$

(this also appears in Kesten's proof as property [60, (2.5)]) by an appeal to the bounded convergence theorem . Combining this with assumption (b), it follows that for all $u \in \breve{\mathbb{S}}_\geq$

$$\begin{aligned} 0 &\leq \limsup_{v\to u,\delta\downarrow 0} \sup_{|s'-s''|<\delta} \left|L_h(v,s') - L_h(u,s'')\right| \\ &\leq \lim_{v\to u} \sup_{s'\in\mathbb{R}} \left|L_h(v,s') - L_h(u,s')\right| + \lim_{\delta\downarrow 0} \sup_{u\in\breve{\mathbb{S}}_\geq} \sup_{|s'-s''|<\delta} \left|L_h(u,s') - L_h(u,s'')\right| = 0. \end{aligned}$$

This is ((2.2)). Thus Kesten's Choquet-Deny lemma is applicable. □

Using this Choquet-Deny type result, the extremal functions in $H_{\chi,c}$ can be identified:

Lemma 11.16. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C)*, let* $\mathbb{E}\left\|\mathbf{M}_1\right\| < \infty$ *and let* (log-moments) *hold. For each* $\chi \in (0,1]$ *and* $c \in (0,1]$*, the extremal points of* $H_{\chi,c}$ *are contained in the set*

$$E_{\chi,c} := \left\{ (u,t) \mapsto c\frac{e_*^\gamma(u)}{e_*^\chi(u)} e^{(\chi-\gamma)t} \;:\; \gamma \in (0,1], m(\gamma) = 1 \right\}.$$

Proof. Let $g \in H_{\chi,c}$ be extremal.

Step 1: Use property (vi) to compute for $u \in \mathbb{S}_\geq$

$$\begin{aligned} g(u,t+s) &= m(\chi)\mathbb{E}_u^\chi g(X_1, t+s-V_1) \\ &= m(\chi) \int g(x,t+s-v)\mathbb{P}_u^\chi(X_1\in dx, V_1 \in dv) \qquad (11.17) \\ &= m(\chi) \int \frac{g(x,t+s-v)}{g(x,s-v)} g(u,s) \frac{g(x,s-v)}{g(u,s)} \mathbb{P}_u^\chi(X_1\in dx, V_1 \in dv) \qquad (11.18) \end{aligned}$$

Recall that by Lemma 11.10, $g > 0$, thus the denominators are positive. Using (11.17) with $t = 0$, it follows that

$$m(\chi) \int \frac{g(x,s-v)}{g(u,s)} \mathbb{P}_u^\chi(X_1 \in dx, V_1 \in dv) = 1.$$

Hence (11.18) is a convex combination of functions $g_{x,v}(u,s,t) = \frac{g(x,t+s-v)}{g(x,s-v)} g(u,s)$. Consequently, since g is extremal,

$$\frac{g(u,t+s)}{g(u,s)} = \frac{g(x,t+s-v)}{g(x,s-v)} \tag{11.19}$$

for all $u \in \mathbb{S}_{\geq}$, $t, s \in \mathbb{R}$ and all $(x,v) \in \operatorname{supp} \mathbb{P}_u^{\chi}((Y_1, V_1) \in \cdot) = \operatorname{supp} \mathbb{P}_u((Y_1, V_1) \in \cdot)$. This yields that $L_t(u,s) := \frac{g(u,t+s)}{g(u,s)}$ satisfies

$$L_t(u,s) = \mathbb{E}_u \left(L_t(X_1, s - V_1) \right). \tag{11.20}$$

Step 2: Lemma 11.15 will be applied in order to show that L_t is constant on $\mathbb{S}_{\geq} \times \mathbb{R}$, i.e. equation (11.19) holds for all $u, x \in \mathbb{S}_{\geq}$, $v, s, t \in \mathbb{R}$. Property (vii) yields condition (b) of the lemma, while (11.20) is its condition (a). It remains to show that L_t is bounded (for fixed t). If $t > 0$, by property (ii), $g(u,t+s)e^{-\chi(t+s)} \leq g(u,s)e^{-\chi s}$, thus

$$0 < L_t(u,s) \leq e^{\chi t} \frac{g(u,t+s)e^{-\chi(t+s)}}{g(u,s)e^{-\chi s}} \leq e^{\chi t}.$$

For $t \leq 0$, use property (iii) for an analogue argument.

Step 3: Validity of (11.19) for any $u, x \in \mathbb{S}_{\geq}$, $t, s, v \in \mathbb{R}$ implies that for some $\tilde{f} : \mathbb{S}_{\geq} \to (0, \infty)$, $a \in \mathbb{R}_{>}$, $b \in \mathbb{R}$,

$$g(u,t) = \tilde{f}(u) a e^{bt}.$$

Considering properties (ii) and (iii) it follows that $b \in [\chi - 1, \chi]$, i.e. $b = \chi - \gamma$ for some $\gamma \in [0,1]$. Rewriting $a\tilde{f}(u) =: e_*^{\chi}(u)^{-1} f(u)$, it follows that

$$g(u,t) = \frac{f(u)}{e_*^{\chi}(u)} e^{(\chi - \gamma)t}. \tag{11.21}$$

It remains to compute the possible values of f and γ. Therefore, use property (vi) which states $g = {}^{\chi}Pg$, hence

$$\begin{aligned}
f(u) &= e^{-(\chi-\gamma)t}\, e_*^{\chi}(u)\, m(\chi) \mathbb{E}_u^{\chi} \left(\frac{f(X_1)}{e_*^{\chi}(Y_1)} e^{(\chi-\gamma)(t-V_1)} \right) \\
&= e^{-(\chi-\gamma)t}\, e_*^{\chi}(u)\, m(\chi)\, \frac{1}{e_*^{\chi}(u)\kappa(\chi)} \mathbb{E}_u \left(e_*^{\chi}(X_1) \frac{f(X_1)}{e_*^{\chi}(X_1)} e^{\chi V_1} e^{(\chi-\gamma)(t-V_1)} \right) \\
&= N\, \mathbb{E}_u \left(f(X_1) e^{\gamma V_1} \right) = N\, \mathbb{E} \left(f(\mathbf{M}_1 \cdot u) \, |\mathbf{M}_1 u|^{\gamma} \right) \\
&= N P_*^{\gamma} f(u).
\end{aligned}$$

This means that f is an eigenfunction of P_*^{γ} with eigenvalue $\frac{1}{N}$. Referring to the definition of $H_{\chi,c}$, $f > 0$. By (7.5), scalar multiples of e_*^{γ} are the only strictly positive eigenfunctions of P_*^{γ}. Thus $f = ce_*^{\gamma}$ where c is given by property (i). The eigenvalue of P_*^{γ} corresponding to e_*^{γ} is $\kappa(\gamma)$. If now $\kappa(\gamma) = \frac{1}{N}$, then $m(\gamma) = N\kappa(\gamma) = 1$, which shows that all extremal points of $H_{\chi,c}$ are in $E_{\chi,c}$. □

It may happen, that the set $E_{\chi,c}$ is even too large, in the sense that not every element of $E_{\chi,c}$ is an extremal point of $H_{\chi,c}$, for it may be possible that not every element of $E_{\chi,c}$ is actually in $H_{\chi,c}$. In fact, if $\chi > \gamma$, by the methods used in the proof above it is not possible to show that g_γ satisfies the Hölder-continuity properties (iv) and (v). In the case $\chi \le \gamma$, the Hölder-continuity is a consequence of Theorem 7.3, (3). Nevertheless, the following Corollary holds true:

Corollary 11.17. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C)*, let* $\mathbb{E}\,\|\mathbf{M}_1\| < \infty$ *and let* (log-moments) *hold. For each* $\chi, c \in (0,1]$ *it holds that if* $H_{\chi,c} \neq \emptyset$*, then* $E_{\chi,c} \neq \emptyset$.

Proof. A non-void compact set in a locally convex linear topological space, e.g. $\mathcal{C}\left(\mathbb{S}_\ge \times \mathbb{R}\right)$, has extremal points (see [40, Lemma V.8.2]). Proposition 11.11 states that $H_{\chi,c}$ is compact in $\mathcal{C}\left(\mathbb{S}_\ge \times \mathbb{R}\right)$, thus if $H_{\chi,c}$ is non-void, the same holds for $E_{\chi,c}$, since by Lemma 11.16, it contains all extremal points of $H_{\chi,c}$. □

This allows to deduce the existence of α as soon as $\mathcal{F}^\aleph \neq \emptyset$:

Theorem 11.18. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C)*, let* $\mathbb{E}\,\|\mathbf{M}_1\| < \infty$ *and let* (log-moments) *hold. If* $\mathcal{F}^\aleph \neq \emptyset$ *for some* $\aleph \in (0,1]$*, then there is* $\alpha \in (0,1]$ *with* $m(\alpha) = 1$*,* $m'(\alpha) \le 0$.

Proof. The argumentation is now the same as in [41, Theorem 2.12]: If there is an ℵ-elementary fixed point, then by Corollary 11.14, there is $c \ge 0$, $\chi \le \aleph$ such that $H_{\chi,c} \neq \emptyset$. But then by Corollary 11.17 $E_{\chi,c}$ is not empty and thus there is $\gamma \in (0,1]$ with $m(\gamma) = 1$. Since $m(0) = N > 1$, the strict convexity implies that there are at most values where m equals 1, and the smaller one, α, satisfies $m'(\alpha) \le 0$. □

11.5. Dilated Laplace Transform of ℵ-Elementary Fixed Points

In this section, the final conclusion $\aleph = \alpha$ will be shown, and that the definition of ℵ-elementary fixed points is in fact independent of the reference point.

Proposition 11.19. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C)*, let* $\mathbb{E}\,\|\mathbf{M}_1\| < \infty$ *and let* (log-moments) *hold. Consider* $\chi, c \in (0,1]$*. Then every function in* $H_{\chi,c}$ *can be written as a convex combination*

$$\frac{c}{e_*^\chi(u)} \left(\lambda e_*^\alpha(u) e^{(\chi-\alpha)t} + (1-\lambda) e_*^\beta(u) e^{(\chi-\beta)t} \right)$$

for $\lambda \in [0,1]$*. In particular, if some* $g \in H_{\chi,c}$ *satisfies* $t \mapsto g(u,t) \equiv g(u,0)$ *for some* $u \in \mathbb{S}_\ge$*, then* $\lambda \in \{0,1\}$*,* $\chi \in \{\alpha,\beta\}$ *and thus* g *is already constant on* $\mathbb{R} \times \mathbb{S}_\ge$.

Proof. By the Krein-Milman theorem [40, Theorem V.8.4], $H_{\chi,c}$ as a compact subset of a locally compact vector space is contained in the closure of the convex envelope of the set of its extremal points and this set is in turn contained in $E_{\chi,c}$. This gives the representation of the functions.

Turning to the second assertion: If $\chi < \alpha$ or $\chi > \beta$, then both g_α and g_β are strictly decreasing resp. increasing in t, thus the same holds for any convex combination. If $\chi \in (\alpha,\beta)$, then $g_\beta(u,\cdot)$ is bounded by 1 on $\mathbb{R}_>$, while $\lim_{t\to\infty} g_\alpha(u,t) = \infty$ for each $u \in \mathbb{S}_\ge$, thus again any convex combination cannot be constant in t for some fixed u. Consequently $\chi \in \{\alpha,\beta\}$. Then exactly one of the functions g_β and g_α is constant everywhere, while the other is strictly monotone in t for all $u \in \mathbb{S}_\ge$. Hence $\lambda \in \{0,1\}$ and g is equal to the constant function. □

Theorem 11.20. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C)*, let* $\mathbb{E}\,\|\mathbf{M}_1\| < \infty$ *and let* (log-moments) *hold. Suppose* $\aleph \in (0,1]$ *and* $\psi \in \mathfrak{L}(\mathcal{F}^{\aleph})$*. Then* $\aleph = \alpha$ *and*

$$\lim_{t\downarrow 0}\left|\frac{1-\psi(t\cdot)}{t^{\alpha}} - Ke_*^{\alpha}\right|_{\infty} = 0 \tag{11.22}$$

for some K*>0.*

Proof. Step 1: On the one hand, combining (11.2) and (11.3) it follows that

$$\lim_{s\to\infty}\frac{1-\psi(e^{-(s+t)}\vartheta_{\mathbf{1}})}{1-\psi(e^{-s}\vartheta_{\mathbf{d}})} = \frac{e^{-\aleph t}}{\sqrt{d}} \quad \forall t \in \mathbb{R}. \tag{11.23}$$

On the other hand, by Proposition 11.13 applied for $\chi = \aleph$, any sequence $s_n \to \infty$ has a subsequence s_{n_k} such that for all $t \in \mathbb{R}$

$$\lim_{k\to\infty} h_{s_{n_k}}(\vartheta_{\mathbf{1}}, t) = \lim_{k\to\infty}\frac{e^{\aleph t}}{e_*^{\aleph}(\vartheta_{\mathbf{1}})}\frac{1-\psi(e^{-(s_{n_k}+t)}\vartheta_{\mathbf{1}})}{1-\psi(e^{-s_{n_k}}\vartheta_{\mathbf{d}})} = h(\vartheta_{\mathbf{1}}, t) \quad \forall t \in \mathbb{R} \tag{11.24}$$

for a function $h \in H_{\aleph,c}$. Comparing (11.23) with (11.24), it follows that

$$h(\vartheta_{\mathbf{1}}, t) = \frac{1}{\sqrt{d}^{\aleph}\, e_*^{\aleph}(\vartheta_{\mathbf{1}})} \quad \forall t \in \mathbb{R}. \tag{11.25}$$

Thus $t \mapsto h(\vartheta_{\mathbf{1}}, t)$ is constant. But then referring to Proposition 11.19, h is already constant on $\mathbb{S}_{\geq} \times \mathbb{R}$ and $\aleph \in \{\alpha, \beta\}$. Considering Corollary 11.3, $\aleph \leq \alpha$, thus $\aleph = \alpha$.

Step 2: Moreover, any subsequential limit is then necessarily equal to h, hence already

$$\lim_{s\to\infty} h_s = h$$

w.r.t. the topology of uniform convergence of compact sets. In particular, uniformly on the compact set $\mathbb{S}_{\geq} \times \{0\}$,

$$\lim_{s\to\infty}\frac{1-\psi(e^{-s}u)}{e^{-\alpha s}} = \lim_{s\to\infty} e_*^{\alpha}(u) h_s(u,0)\frac{1-\psi(e^{-s}\vartheta_{\mathbf{d}})}{e^{-\alpha s}} = Ke_*^{\alpha}(u)$$

for some $K > 0$. □

12. Characterization of α-Elementary Fixed Points

On the one hand, it follows from Theorem 11.20 that (LTs of) α-elementary fixed points have the same tails (asymptotics at zero) as $\tilde{S}_{\alpha}(Ke_*^{\alpha}, 0)$. On the other hand, it is a consequence of Theorem 9.15 that if $m'(\alpha) < 0$, then $\mathcal{S}^n\tilde{S}_{\alpha}(Ke_*^{\alpha}, 0)$ converges to an α-elementary fixed point. Up to now, there is no result about uniqueness (up to scaling) of these α-elementary fixed points.

In this section, a positive answer will be given under the condition $m'(\alpha) < 0$ by showing that for any random variable Z which has the same tails as $\tilde{S}_\alpha(Ke_*^\alpha, 0)$, $\mathcal{S}^n Z$ converges to the fixed point

$$Y := \lim_{n\to\infty} \mathcal{S}^n \tilde{S}_\alpha(Ke_*^\alpha, 0)$$

which was constructed in Section 9.

The tool (transience of the maximal position of a branching random walk with negative drift) will, together with the results from Subsection 4.4, also allow to obtain a full description of the set of α-elementary fixed points of the inhomogeneous smoothing transform. Consequently, in this section both the homogeneous $\mathcal{S}_0$ and inhomogeneous smoothing transform $\mathcal{S}_Q$ will be considered.

Beforehand, results about the asymptotics of fixed points of $\mathcal{S}_Q$ will be derived. Define the set $\mathcal{F}_Q^\alpha$ of α-elementary fixed points of $\mathcal{S}_Q$ by

$$\mathfrak{L}\left(\mathcal{F}_Q^\alpha\right) = \{\psi \in \mathfrak{L}(\mathbb{R}_\geq^d) \ : \ \mathcal{S}_Q\psi = \psi, \ \lim_{t\to 0} t^{-\alpha}(1-\psi(t\vartheta_{\mathbf{1}})) \in (0,\infty)\}$$

and similarly write $\mathcal{F}_0^\alpha$ for the set of α-elementary fixed points of $\mathcal{S}_0$.

In this section, the notations ψ_0 and ψ_Q will be used for the LTs of fixed points of $\mathcal{S}_0$ resp. $\mathcal{S}_Q$ (and not for the LTs of 0 resp. Q).

12.1. The Inhomogeneous Smoothing Transform

In Theorem 4.9, a one-to-one correspondence between $\mathcal{F}_Q$ and $\mathcal{F}_0$ was shown. The following Proposition shows that this correspondence respect the sets of α-elementary fixed points.

Proposition 12.1. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C)*, let* $\mathbb{E}\,\|\mathbf{M}_1\| < \infty$ *and let* (log-moments) *holds. Assume that* $m'(\alpha) < 0$ *and that there is* $s > \alpha$ *with* $m(s) < 1$*. Let* η_0 *and* η_Q *be corresponding fixed points of* $\mathcal{S}_0$ *resp.* $\mathcal{S}_Q$ *as given by Theorem 4.9, with LTs* ψ_0 *resp.* ψ_Q*. If one of them is* α*-elementary, then it holds that for all* $u \in \mathbb{S}_\geq$

$$\lim_{t\downarrow 0} \frac{1-\psi_Q(ut)}{1-\psi_0(ut)} = 1, \tag{12.1}$$

in particular, both fixed points are α*-elementary.*

Proof. Let (Y_0, Y_Q) be a coupling of η_0 and η_Q with $\mathbb{E}\,|Y_0 - Y_Q|^s < \infty$. A coupling with this property exists since $\eta_Q \in \mathcal{P}_s(\eta_0)$ by Theorem 4.9, i.e. $l_s(\eta_0, \eta_Q) < \infty$. Using the inequality $|a^s - b^s| \leq |a-b|^s$ which is valid for $s \in [0,1]$ and $a, b \in \mathbb{R}_\geq$, it follows that for all $u \in \mathbb{S}_\geq$

$$\mathbb{E}\,|\langle u, Y_Q\rangle^s - \langle u, Y_0\rangle^s| \leq \mathbb{E}\,|\langle u, Y_Q - Y_0\rangle|^s \leq \mathbb{E}\,|Y_Q - Y_0|^s < \infty.$$

Referring to the Goldie Lemma [47, Lemma 9.4] (see [4, Remark 4.4] for some corrections of this Lemma),

$$\int_0^\infty \frac{1}{t}\left(t^s\left|\mathbb{P}\left(\langle u, Y_Q\rangle > t\right) - \mathbb{P}\left(\langle u, Y_0\rangle > t\right)\right|\right) dt < \infty.$$

From the fact that $\int_1^\infty \frac{1}{t} dt$ diverges, it follows that necessarily

$$\limsup_{t\to\infty} t^s \left|\mathbb{P}\left(\langle u, Y_Q\rangle > t\right) - \mathbb{P}\left(\langle u, Y_0\rangle > t\right)\right| = 0.$$

Remember that $s > \alpha$, so the convergence is also true with t^α instead of t^s.

Case 1, Y_0 is α-elementary: Then it is a consequence of Theorem 11.20 combined with 5.12 that

$$\lim_{t\to\infty} t^\alpha \mathbb{P}\left(\langle u, Y_0\rangle > t\right) = K e_*^\alpha(u) > 0.$$

It follows that

$$\begin{aligned}
&\lim_{t\to\infty} \left| \frac{\mathbb{P}\left(\langle u, Y_Q\rangle > t\right)}{\mathbb{P}\left(\langle u, Y_0\rangle > t\right)} - 1 \right| \\
&= \lim_{t\to\infty} \left(t^\alpha \mathbb{P}\left(\langle u, Y_0\rangle > t\right)\right)^{-1} t^\alpha \left|\mathbb{P}\left(\langle u, Y_Q\rangle > t\right) - \mathbb{P}\left(\langle u, Y_0\rangle > t\right)\right| \\
&= \lim_{t\to\infty} \left(t^\alpha \mathbb{P}\left(\langle u, Y_0\rangle > t\right)\right)^{-1} \lim_{t\to\infty} t^\alpha \left|\mathbb{P}\left(\langle u, Y_Q\rangle > t\right) - \mathbb{P}\left(\langle u, Y_0\rangle > t\right)\right| \\
&= \left(K e_*^\alpha(u)\right)^{-1} \cdot 0 = 0.
\end{aligned} \tag{12.2}$$

First this gives

$$\lim_{t\to\infty} t^\alpha \mathbb{P}\left(\langle u, Y_Q\rangle > t\right) = K e_*^\alpha(u)$$

for all $u \in \mathbb{S}_\geq$, in other words, Y_0 and Y_Q have the same tail behaviour. From this, the asserted result (12.1) for the Laplace transforms follows by using Proposition 5.12.

Case 2, Y_Q is α-elementary: The calculations in (12.2) are valid for $u = \vartheta_{\mathbf{1}}$, with Y_Q and Y_0 interchanged. This allows to deduce

$$\lim_{t\to\infty} t^\alpha \mathbb{P}\left(\langle \vartheta_{\mathbf{1}}, Y_0\rangle > t\right) \in (0, \infty),$$

which is by Proposition 5.12 (and (11.2)) equivalent to

$$\lim_{t\to 0} t^{-\alpha}(1 - \psi_0(t\vartheta_{\mathbf{d}})) \in (0, \infty),$$

so Y_0 is α-elementary. Then Case 1 applies. □

Corollary 12.2. *Assume that* [supp μ] *satisfies* (C)*, let* $\mathbb{E}\left\|\mathbf{M}_1\right\| < \infty$ *and let* (log-moments) *hold. If* $\psi_Q \in \mathfrak{L}\left(\mathcal{F}_Q^\alpha\right)$*, then there is* $K > 0$ *such that*

$$\lim_{t\downarrow 0} \left| \frac{1 - \psi_Q(t\cdot)}{t^\alpha} - K e_*^\alpha \right|_\infty = 0. \tag{12.3}$$

Conversely, for each $K > 0$ *there is* $\psi_Q \in \mathfrak{L}\left(\mathcal{F}_Q^\alpha\right)$ *such that* (12.3) *holds.*

Proof. Step 1: If $\psi_Q \in \mathfrak{L}(\mathcal{F}_Q^\alpha)$ then by a first application of Proposition 12.1, the corresponding fixed point of $\mathcal{S}_0$, ψ_0 is also α-elementary, and its asymptotics at zero are given by $K e_*^\alpha$ due to

Theorem 11.20. Then using again Proposition 12.1, ψ_Q has the same asymptotics at zero. Finally, Proposition 11.8 gives the uniform convergence.

Step 2: In order to show the converse implication, combining Theorem 9.15 and Theorem 11.20, there is $\psi_Q \in \mathfrak{L}(\mathcal{F}_0^\alpha)$ such that (12.3) holds with ψ_Q replaced by ψ_0. Referring to Proposition 12.1, $\mathcal{S}_Q$ has an α-elementary fixed point ψ_Q with the same asymptotics at zero as ψ_0, i.e. $\lim_{t\downarrow 0} t^{-\alpha}(1-\psi_Q(tu)) = Ke_*^\alpha(u)$ for all $u \in \mathbb{S}_\geq$. In order to conclude that (12.3) holds, use again Proposition 11.8 to deduce the uniform convergence. □

12.2. Convergence of Fixed Points

In this subsection, it will be shown that for any $\phi \in \mathfrak{L}(\mathcal{P}(\mathbb{R}^d_\geq))$ which has the same asymptotics at zero as $\psi \in \mathfrak{L}(\mathcal{F}^\alpha)$, $\lim_{n\to\infty} \mathcal{S}^n\phi = \psi$. This is the final ingredient needed for the characterization of $\mathcal{F}^\alpha$ in the next subsection. Two more lemmata are needed beforehand. For a better stream of arguments, their proofs have been shortened or moved to the appendix.

Lemma 12.3. *Let $\alpha \in \breve{I}_\mu \cap (0,1)$ and $m'(\alpha) < 0$. If ϕ, $\varphi \in \mathfrak{L}(\mathcal{P}(\mathbb{R}^d_\geq))$ and there is $t_0 \in \mathbb{R}_>$ such that for all $(y,s) \in \mathbb{S}_\geq \times [0,t_0]$,*

$$\phi(sy) \leq \varphi(sy),$$

then for all $(u,t) \in \mathbb{S}_\geq \times \mathbb{R}_\geq$

$$\liminf_{n\to\infty} \mathcal{S}_Q^n\phi(tu) \leq \liminf_{n\to\infty} \mathcal{S}_Q^n\varphi(tu) \quad \textit{and} \quad \limsup_{n\to\infty} \mathcal{S}_Q^n\phi(tu) \leq \limsup_{n\to\infty} \mathcal{S}_Q^n\varphi(tu).$$

Proof. The proof can be found in the appendix on page 127. □

Lemma 12.4. *Let ϕ_1, $\phi_2 \in \mathfrak{L}(\mathcal{P}(\mathbb{R}^d_\geq))$ with*

$$\lim_{t\downarrow 0} \frac{1-\phi_1(tu)}{1-\phi_2(tu)} = 1 \quad \forall u \in \mathbb{S}_\geq \tag{12.4}$$

and

$$\lim_{t\downarrow 0} \frac{1-\phi_2(tu)}{t^\gamma} = e(u) > 0 \quad \forall u \in \mathbb{S}_\geq \tag{12.5}$$

for a strictly positive function e and $\gamma \in (0,1]$. Then the convergence in (12.4) *is uniform on $\mathbb{S}_\geq$.*

Proof. Write

$$\frac{1-\phi_1(tu)}{1-\phi_2(tu)} = \frac{1-\phi_1(tu)}{1-\phi_1(t\vartheta_{\mathbf{d}})} \frac{1-\phi_1(t\vartheta_{\mathbf{d}})}{1-\phi_2(t\vartheta_{\mathbf{d}})} \frac{1-\phi_2(t\vartheta_{\mathbf{d}})}{1-\phi_2(tu)}$$

and use Proposition 11.8 several times. □

The following theorem is the final step for characterizing the set of α-elementary fixed points. It is valid for both the homogeneous and the inhomogeneous smoothing transform.

Theorem 12.5. *Assume that* $[\operatorname{supp}\mu]$ *satisfies* (C)*, let* $\mathbb{E}\,\|\mathbf{M}_1\| < \infty$ *and let* (log-moments) *hold. Assume that* $\alpha \in (0,1)$ *and* $m'(\alpha) < 0$*. Let* $\psi \in \mathfrak{L}\,(\mathcal{F}^\alpha)$ *and let* $\phi \in \mathfrak{L}(\mathcal{P}(\mathbb{R}^d_\geq))$*, such that*

$$\lim_{t\downarrow 0} \frac{1-\phi(tu)}{1-\psi(tu)} = 1 \tag{12.6}$$

for all u *in* $\mathbb{S}_\geq$*. Then* $\lim_{n\to\infty} \mathcal{S}^n\phi = \psi$*.*

In particular, if ψ_1, ψ_2 are two α-elementary fixed points with the same asymptotics at zero, then readily $\psi_1 = \psi_2$.

Proof. The proof will be given for the inhomogeneous case; the homogeneous case is contained when setting $Q \equiv 0$. Referring to Corollary 12.2 (resp. Theorem 11.20 in the homogeneous case), there is $K > 0$ such that for all $p > 0$

$$\lim_{t\downarrow 0} \frac{1}{e^\alpha_*(u)} \frac{1-\psi(ptu)}{(pt)^\alpha K} = 1 = \lim_{t\downarrow 0} \frac{1}{e^\alpha_*(u)} \frac{1-\psi(tu)}{t^\alpha K}$$

uniformly in $u \in \mathbb{S}_\geq$. It follows that

$$\lim_{t\downarrow 0} \frac{1-\psi(ptu)}{1-\psi(tu)} = p^\alpha \tag{12.7}$$

and this convergence is uniform on $\mathbb{S}_\geq$. For $p > 1$ arbitrary but fixed, set

$$\underline{\psi}(tu) := \psi(ptu), \quad \text{and} \quad \overline{\psi}(tu) := \psi(p^{-1}tu).$$

Note that both $\underline{\psi}$, $\overline{\psi}$ are just scaled versions of the initial fixed point, consequently, they are fixed points themselves. Referring to Lemma 12.4, the convergence in (12.6) is uniform; and combining this with (12.7), it follows that

$$\lim_{t\downarrow 0} \frac{1-\phi(tu)}{1-\underline{\psi}(tu)} = p^\alpha > 1 \qquad \text{and} \qquad \lim_{t\downarrow 0} \frac{1-\phi(tu)}{1-\overline{\psi}(tu)} = p^{-\alpha} < 1 \tag{12.8}$$

uniformly in $u \in \mathbb{S}_\geq$. So there is $t_0 > 0$ such that for all $(y,s) \in \mathbb{S}_\geq \times [0,t_0]$

$$\underline{\psi}(sy) \leq \phi(sy) \leq \overline{\psi}(sy).$$

Considering Lemma 12.3,

$$\begin{aligned}\underline{\psi}(tu) &= \liminf_{n\to\infty} \mathcal{S}^n_Q\underline{\psi}(tu) \leq \liminf_{n\to\infty} \mathcal{S}^n_Q\phi(tu) \\ &\leq \limsup_{n\to\infty} \mathcal{S}^n_Q\phi(tu) \leq \limsup_{n\to\infty} \mathcal{S}^n_Q\overline{\psi}(tu) = \overline{\psi}(tu).\end{aligned}$$

Since p was arbitrary, $\overline{\psi}$ and $\underline{\psi}$ can be brought arbitrarily close to infer first the convergence of $\mathcal{S}^n\phi(tu)$ for any $t \in \mathbb{R}_\geq, u \in \mathbb{S}_\geq$ and next that $\lim_{n\to\infty} \mathcal{S}^n\phi(tu) = \psi(tu)$ for all $t \in \mathbb{R}_>, u \in \mathbb{S}_\geq$. □

12.3. Characterization of $\mathcal{F}^\alpha$

Denote by $\mathrm{Eig}^+(P_*^\alpha, N^{-1})$ the cone of positive eigenfunctions of P_*^α with eigenvalue N^{-1}. Then the characterization of the α-elementary fixed points of $\mathcal{S}$ is given by the following theorem, which is formulated in the spirit of Propositions 4.3 and 4.4.

Theorem 12.6. *Assume that* $[\mathrm{supp}\,\mu]$ *satisfies* (C)*, let* (M logM) *and* (log-moments) *hold. Let* $\mathbb{E}\,|Q| < \infty$*. Suppose that there is* $\alpha \in (0,1)$ *with* $m(\alpha) = 1$ *and* $m'(\alpha) < 0$*. Then the mappings*

$$\begin{aligned}\mathrm{Eig}^+(P_*^\alpha, N^{-1}) &\to \mathcal{F}_0^\alpha \\ e &\mapsto d\text{–}\lim_{n\to\infty} \mathcal{S}_0^n \tilde{S}_\alpha(e,0)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Eig}^+(P_*^\alpha, N^{-1}) &\to \mathcal{F}_Q^\alpha \\ e &\mapsto d\text{–}\lim_{n\to\infty} \mathcal{S}_Q^n \tilde{S}_\alpha(e,0)\end{aligned}$$

are well defined and bijective.

Proof. As the first step, consider $\mathcal{S}_0$. By Theorem 7.3,

$$\mathrm{Eig}^+(P_*^\alpha, N^{-1}) = \{K e_*^\alpha \;:\; K \in \mathbb{R}_>\}.$$

In Theorem 9.15, it was shown that $\mathcal{S}_0^n \tilde{S}_\alpha(K e_*^\alpha, 0)$ converges to an α-elementary fixed point of $\mathcal{S}_0$. The same theorem also gives injectivity since K is a scaling factor.

The mapping is surjective because by Theorem 11.20, the LT of each α-elementary fixed point ψ_0 of $\mathcal{S}$ has the same asymptotics at zero as $\mathfrak{L}(\tilde{S}_\alpha(K e_*^\alpha, 0))$ for some $K \in \mathbb{R}_>$. But then by Theorem 12.5, $d\text{–}\lim_{n\to\infty} \mathcal{S}_0^n \tilde{S}_\alpha(K e_*^\alpha, 0) = \psi_0$.

For the inhomogeneous smoothing transform, this follows by the same arguments, using Corollary 12.2 instead of Theorem 11.20 and 9.15. ☐

As mentioned before, fixed points of $\mathcal{S}_Q$ with a finite moment of order α were constructed in [75]. It was stated there (see [75, Remark 1.8]) that the constructed fixed point is unique within the set $\mathcal{P}_s(\mathbb{R}^d_\geq)$ for some $s > \alpha$. The theorem above now shows that there is no chance of finding a fixed point that is unique in $\mathcal{P}(\mathbb{R}^d_\geq)$, since there are more.

Recalling Proposition 9.5, the following description of $\mathfrak{L}(\mathcal{F}_0^\alpha)$ has been obtained:

Corollary 12.7 (Structure of $\mathcal{F}_0^\alpha$)**.** *Under the assumptions of Theorem 12.6,* $\mathfrak{L}(\mathcal{F}_0^\alpha)$ *equals the one-parameter family of Laplace transforms*

$$(u,t) \mapsto \mathbb{E}\exp\left(-K t^\alpha W(u)\right),$$

parametrised by $K \in \mathbb{R}_>$.

12.4. Structure of Fixed Points of $\mathcal{S}_Q$

The final result gives a very handy and simple description of $\mathcal{F}_Q^\alpha$ which is similar to [6, Theorem 8.1]. It shows that the "new" fixed points are somehow of the form *homogeneous solution + particular inhomogeneous solution*.

Theorem 12.8. *Let* $(\mathbf{T}_i)_{i=1}^N$ *be i.i.d. and assume that* $[\operatorname{supp}\mu]$ *satisfies* (C). *Let* $\mathbb{P}(Q \neq 0) > 0$, *let* (M logM) *and* (log-moments) *hold. Assume that there are* $s_1 \in (0, 1/2]$, $s_2 > s_1$ *such that* $\mathbb{E}\|\mathbf{M}_1\|^{s_1} \leq \frac{1}{N}$, $\mathbb{E}\|\mathbf{M}_1\|^{s_2} \leq \frac{1}{N}$ *and* $\mathbb{E}|Q|^{s_2} < \infty$. *Then* $\alpha \leq \frac{1}{2}$ *and the set* $\mathfrak{L}\left(\mathcal{F}_Q^\alpha\right)$ *equals the one-parameter family of Laplace transforms*

$$(u, t) \mapsto \mathbb{E}\exp\left(-t\langle u, W^*\rangle - Kt^\alpha W(u)\right) \tag{12.9}$$

parametrised by $K \in \mathbb{R}_>$.

For $K = 0$, the formula (12.9) is the LT of the fixed point constructed in [75, Theorem 1.7]. Note that the assumption of independent weights stems only from [75, Theorem 1.7] which is used here.

Proof. The moment assumptions are those of [75, Theorem 1.7] (which was restated in Theorem 4.7), the assumptions of Proposition 9.2 and Theorem 12.8 are given as well. The a.s. convergence of W_n^* to W^* and $W_n(u)$ to $W(u)$ for all $u \in \mathbb{S}_\geq$ holds by [75, Theorem 1.7] resp. Proposition 9.2. Referring to Theorem 12.6, the LT of each α-elementary fixed point of $\mathcal{S}_Q$ can be written as $d\text{-}\lim_{n\to\infty} \mathcal{S}_Q^n \phi_0$, where $\phi_0 = \mathfrak{L}(\tilde{S}_\alpha(Ke_*^\alpha, 0))$. This gives by an application of Lemma 2.4

$$\begin{aligned}
\psi_Q(u,t) = & \lim_{n\to\infty} \mathbb{E}\left(\exp\left(-t\langle u, \sum_{|w|<n} \mathbf{L}(w)Q(w)\rangle\right) \prod_{|v|=n} \phi_0(t\mathbf{L}(v)^\top u)\right) \\
= & \lim_{n\to\infty} \mathbb{E}\left(\exp\left(-t\langle u, W_n^*\rangle\right) \exp\left(-Kt^\alpha \sum_{|v|=n} \int_{\mathbb{S}_\geq} \langle \mathbf{L}(v)^\top u, y\rangle^\alpha \nu^\alpha(dy)\right)\right) \\
= & \lim_{n\to\infty} \mathbb{E}\left(\exp\left(-t\langle u, W_n^*\rangle\right) \exp\left(-Kt^\alpha W_n(u)\right)\right) \\
= & \mathbb{E}\left(\exp\left(-t\langle u, \lim_{n\to\infty} W_n^*\rangle\right) \exp\left(-Kt^\alpha \lim_{n\to\infty} W_n(u)\right)\right) \\
= & \mathbb{E}\exp\left(-t\langle u, W^*\rangle + Kt^\alpha W(u)\right)
\end{aligned}$$

for all $(u, t) \in \mathbb{S}_\geq \times \mathbb{R}_\geq$. In the final lines, the bounded convergence theorem and as the main ingredient, the a.s. convergence of W_n^* and $W_n(u)$ have been used. □

Corollary 12.9. *In the situation of Theorem 12.8, the one-dimensional marginals* $\langle u, Y_Q\rangle$ *can be written in the form*

$$\langle u, Y_Q\rangle \stackrel{d}{=} \langle u, W^*\rangle + KW(u)^{\frac{1}{\alpha}} Z,$$

where $K > 0$ *and* $Z \stackrel{d}{=} \mathcal{S}_\alpha(1, 1, 0)$ *and independent of* $W^*, W(u)$.

$W^*, W(u)$ can be interpreted as a random shift resp. random scaling where the randomness is inherited from the randomness of the weights. This becomes even more obvious, since both are functions of $\mathcal{T}$. Abusing the notation from Proposition 5.7, the above Corollary can be stated as

$$\langle u, Y_Q \rangle \stackrel{d}{=} \mathbb{E}\, S_\alpha \left(K' W(u), 1, W^* \right),$$

with $K' = \left(\cos\left(\frac{\pi\alpha}{2} \right) K \right)^{\frac{1}{\alpha}}$.

B. On Fixed Points of Multivariate Random Difference Equations

13. Introduction

The prototype of stochastic fixed point equations is the affine equation

$$Y \overset{d}{=} \mathbf{T}Y + Q, \tag{13.1}$$

where $(\mathbf{T}, Q)$ is a random element of $M(d \times d, \mathbb{R}) \times \mathbb{R}^d$ and independent of R.

It describes stationary solutions of *Random difference equations (RDEs)*, defined by

$$R_n = \mathbf{T}_{(n)} R_{n-1} + Q_n, \tag{RDE}$$

where $(\mathbf{T}_{(n)}, Q_n)_{n \in \mathbb{N}}$ is a sequence of i.i.d. copies of $(\mathbf{T}, Q)$. In turn, RDEs are a special and very important subclass of *Lipschitz recursions*, defined by

$$R_n = F_n(R_{n-1}), \tag{13.2}$$

where $(F_n)_{n \in \mathbb{N}}$ is an i.i.d. sequence of Lipschitz continuous mappings $F_n : \mathbb{R}^d \to \mathbb{R}^d$. The sequence R_n obviously constitutes a MC on $\mathbb{R}^d$. See the review by Diaconis and Freedman [36] for details.

13.1. Existence and Uniqueness of a Stationary Solution

Write $L(F)$ for the Lipschitz constant of a Lipschitz function F and define the (upper) Lyapunov exponent of the Lipschitz recursion (13.2) by

$$l := \lim_{n \to \infty} \frac{1}{n} \log L(F_n \circ \cdots \circ F_1) \quad \mathbb{P}\text{-a.s.}. \tag{13.3}$$

Note that there are some measure theoretic issues when defining the random variables $L(F_n)$, which will not be discussed here. If $\mathbb{E} \log^+ L(F_1) < \infty$, then l exists in $[-\infty, \infty)$ by Kingman's subadditive ergodic theorem [61] and equals

$$l = \lim_{n \to \infty} \frac{1}{n} \mathbb{E} \log L(F_n \circ \cdots \circ F_1).$$

In the case of RDEs, where $L(F_1) = \|\mathbf{T}_1\|$, this result was shown earlier and is known as the Furstenberg-Kesten theorem [46, Theorem 1 & 2].

If $\mathbb{E}\log^+ L(F_1) < \infty$, $l < 0$ and $\mathbb{E}\log^+ |x - F_1(x)| < \infty$ for some (and then for all) $x \in \mathbb{R}^d$, then the Lipschitz recursion (13.2) has a unique stationary distribution, which is given by the law of the then a.s. convergent series

$$Z_n := F_1 \circ \cdots \circ F_n(0),$$

the *backward process*. Observe that if $R_0 = 0$ then $Z_n \stackrel{d}{=} R_n$ for all $n \in \mathbb{N}$. Hence R_n converges in law to the stationary distribution. This result for general Lipschitz recursion was shown by Elton [42, Theorem 3].

Returning to the matrix recursion, the corresponding conditions for existence and uniqueness are

$$\mathbb{E}\log^+ \|\mathbf{T}\| + \log^+ |Q| < \infty \tag{logmom}$$

and negativity of the upper Lyapunov exponent

$$l = \mathbb{P}\text{-a.s.} - \lim_{n\to\infty} \frac{1}{n} \log \|\daleth_n\| < 0 \tag{l<0}$$

where

$$\daleth_n := \mathbf{T}_{(1)} \cdot \ldots \cdot \mathbf{T}_{(n)}.$$

The unique fixed point is then given by the law of the almost sure convergent series

$$R = \sum_{n=1}^{\infty} \daleth_{n-1} Q_n. \tag{13.4}$$

A classical reference for these existence and uniqueness results in dimension $d = 1$ is [93, Theorem 1.6]. The multivariate case is explicitly considered in [27, Theorem 1.1] with a special emphasis on the necessarity of condition (l<0).

13.2. A Moment for Motivation

From the explicit representation of R simple moment estimates can be derived.

Lemma 13.1. *Let $\beta > 0$ such that $\mathbb{E}\|\mathbf{T}\|^\beta + |Q|^\beta < \infty$ and for all $s < \beta$,*

$$m(s) := \lim_{n\to\infty} \left(\mathbb{E}\|\daleth_n\|^s\right)^{\frac{1}{n}} < 1.$$

Then $\mathbb{E}|R|^s < \infty$ for all $s < \beta$.

Proof. If $s \in (0, 1]$, use subadditivy to estimate

$$\mathbb{E}|R|^s \le \mathbb{E}\sum_{n=1}^{\infty} \|\daleth_{n-1}\|^s |Q_n|^s = \sum_{n=1}^{\infty} \mathbb{E}\|\daleth_{n-1}\|^s \,\mathbb{E}|Q|^s. \tag{13.5}$$

If $s > 1$, use the Minkowski inequality for

$$(\mathbb{E}\,|R|^s)^{\frac{1}{s}} \leq \sum_{n=0}^{\infty} (\mathbb{E}\,\|\urcorner_{n-1}\|^s)^{\frac{1}{s}}\,(\mathbb{E}\,|Q|^s)^{\frac{1}{s}}\,. \tag{13.6}$$

In both cases, the right hand side converges by domination with the geometric series if $m(s) <$ 1. □

This may serve as an heuristic argument that the spectral function $m(s)$ is closely connected with moments of R. Spitzer conjectured (for dimension d=1, cf. [59, bottom of p.208]) that $|R|$ is in the domain of attraction of a stable law with index β. In other words, Spitzer conjectured that R has the heavy tail property

$$\lim_{t\to\infty} t^{\beta}\mathbb{P}\left(|R| > t\right) = K > 0. \tag{13.7}$$

13.3. Previous Results and Aim of This Work

Spitzer's conjecture was proved in the multivariate setting by Kesten in his seminal paper [59] and independently by Grincevičjus [48, Theorem 2] in dimension $d = 1$. In dimension $d = 1$, Goldie [47] later gave a very much uni- and simplified approach, using so-called *implicit renewal theory*.

Kesten's main theorem [59, Theorem B] is formulated for $\mathbf{T} \in M(d \times d, \mathbb{R}_{\geq})$ under conditions similar to, but more restrictive than (C). At the end of his article, Kesten also stated without proof a theorem [59, Theorem 6] for the situation $(\mathbf{T}, Q) \in GL(d, \mathbb{R}) \times \mathbb{R}^d$. A proof was given by Le Page [66] and it dit not need Kesten's density assumption [59, Theorem 6 (iii)]. Over the years, Le Page's approach was further developed. A definite result was obtained by Guivarc'h and Le Page in [52]: Property (13.7) and even multivariate regular variation of R hold (under some additional moment assumptions) as soon as $[\operatorname{supp}\mu]$ satisfies condition $(i-p)$. Condition$(i-p)$ is the analogue of (C) in the setting of invertible matrices. The interested reader is referred to [51] for a shorteraccount of the main ideas of the proof in that fundamental paper.

The merit of the present work is to show how regeneration methods from the theory of Harris recurrent Markov chains can be used to provide a much shorter argument (particularly for the positivity of K) in the situation of [59, Theorem 6] when taking the density assumption into account. This calls for the development of bivariate regeneration schemes in the spirit of [13] and a detailed study of the action of $\mathbf{T}$ on $\mathbb{S}$ which are interesting in their own right.

Note that the present assumptions are stronger than those of [52], but cover many interesting situations for applications: E.g. if $\mu = \mathbb{P}\left(\mathbf{T} \in \cdot\right)$ has a component with a Lebesgue density on a ball centered at the identity matrix $\mathbf{Id}$, then these assumptions will be satisfied. Two further references should be mentioned: A situation similar to [59, Theorem 6] was considered by Klüppelberg and Pergamenchtchikov [64], but for a more specialized model and much closer along the lines of Kesten's proof. In the case where $\operatorname{supp}\mu$ is restricted to the group of similarities (products of a dilations and orthogonal matrices), related results were obtained by Buraczewski et al. [30].

Random Difference Equations appear in a broad variety of settings: Discretization of generalised Ornstein-Uhlenbeck processes [68], insurance ruin theory [81] or random walks in random environment on $\mathbb{Z}$ [44], to mention just a few recent articles.

13.4. A Moment for Notation

The same notation as in Chapter A will be used, mutatis mutandis: $\mathbb{S}_{\geq}$ is no longer a homogeneous space for the action of $GL(d,\mathbb{R})$, thus it has to be replaced by $\mathbb{S}$, the whole unit sphere in $\mathbb{R}^d$. The definition e.g. of the transfer operators has to be changed correspondingly. Write $\varrho = \mathcal{L}(\mathbf{T}, Q)$ and let $(\mathbf{T}_{(n)}, Q_n)_{n\in\mathbb{N}}$ be a sequence of i.i.d. copies of $(\mathbf{T}, Q)$ under $\mathbb{P}$. As before, $\mu^* = \mathcal{L}(\mathbf{T}^\top)$, and $(\mathbf{M}_n)_{n\in\mathbb{N}}$ will be a sequence of i.i.d. r.v.s with distribution μ^*.

Remark that the moment results above may also be obtained from Theorem 4.4 with $N = 1$ - in the setting of RDE, $m(s) = \kappa(s)$, $m(s) = 1$ and under the assumption (l<0), $\alpha = 0$. Figuratively speaking Chapter A was concerned with tail index α while Chapter B will be concerned with tail index β.

13.5. Statement of Results

The assumptions that will be imposed on μ are the following: First, assume that μ^* acts irreducibly on the unit sphere, i.e.

$$\forall_{x\in\mathbb{S}}\ \forall_{\text{open } U\subset\mathbb{S}} \qquad \max_{n\in\mathbb{N}} \mathbb{P}(\Pi_n \cdot x \in U) > 0. \tag{irred}$$

Secondly, assume that μ^* (resp. μ) is spread-out, i.e.

$$\exists_{\Gamma_0\in GL(d,\mathbb{R})}\ \exists_{c,p>0}\ \exists_{n_0\in\mathbb{N}} \qquad \mathbb{P}(\Pi_{n_0} \in \cdot) \geq p\mathbf{1}_{B_c(\Gamma_0)}\lambda^{d^2}. \tag{density}$$

Recall that λ^{d^2} denotes Lebesgue measure on $M(d\times d,\mathbb{R})$, seen as a subset of $\mathbb{R}^{d^2}$. For brevity, say that μ satisfies (i-d), if (irred) and (density) hold.

Observe that there will be no condition on the dependence structure of $(\mathbf{T}, Q)$ except for the necessary one,

$$\forall\, r\in\mathbb{R}^d \quad \mathbb{P}(\mathbf{T}r + Q = r) < 1, \tag{$R\neq r$}$$

which guarantees that the fixed point is not just a point mass.

Then the main result is as follows:

Theorem 13.2. *Let $(\mathbf{T}, Q)$ be a random element of $GL(d,\mathbb{R})\times\mathbb{R}^d$, let $\mu = \mathcal{L}(\mathbf{T})$ satisfy (i-d). Let $(R\neq r)$ hold and assume that there is $\beta > 0$ such that $m(\beta) = 1$, $m'(\beta^-) > 0$ and*

$$\mathbb{E}\left\|\mathbf{T}\right\|^\beta \left(\left|\log\left\|\mathbf{T}\right\|\right| + \left|\log\left\|\mathbf{T}^{-1}\right\|\right|\right) < \infty \tag{TlogT}$$

as well as

$$0 < \mathbb{E}\left|Q\right|^\beta < \infty. \tag{Q-beta}$$

Then the RDE (RDE) *has a unique stationary distribution R. R has unbounded support if and only if $(R\neq r)$ holds. In that case,*

$$\lim_{t\to\infty} t^\beta \mathbb{P}(\langle u, R\rangle > t) = K e_*^\beta(u) \tag{13.8}$$

for all $u\in\mathbb{S}$, where $e_*^\beta(u)$ is a strictly positive continuous function on $\mathbb{S}$.

As in Chapter A, the transfer operator P_*^s will be studied and it will be shown that it has the same properties under (i-d) as it has under (C). Thus e_*^β denotes the unique strictly positive eigenfunction of P_*^β with eigenvalue $\kappa(\beta) = 1$, and ν_*^β the corresponding eigenmeasure.

Remark 13.3.

- The constant K has the implicit representation

$$\frac{1}{\beta\, l(\beta)\, \nu_*^\beta(e_*^\beta)} \int_{\mathbb{S}} \mathbb{E}\left((\langle y, R\rangle^+)^\kappa - (\langle y, \mathbf{T}R\rangle^+)^\kappa \right) \nu_*^s(dy).$$

This is the multivariate version of the formula for K given by Goldie in [47, Theorem 4.1].

- Recall from (7.3) that if $\mathbf{A} \in GL(d, \mathbb{R})$ then $\iota(\mathbf{A}) = \left\| A^{-1} \right\|^{-1}$, so condition (TlogT) corresponds to (M logM) and is the precise adaption of (8.7) to the present situation of invertible matrices. Kesten does not impose a condition on $\left|\log \left\|T^{-1}\right\|\right|$ and indeed the theorem is valid without, but then the analysis of $l(\beta)$ becomes more involved. Since it is not a severe restriction (it is satisfied e.g. if the matrices are compactly supported) it is added here for easing the presentation.
- The statement of [59, Theorem 6] is slightly different at more points as well: In the theorem above, Kesten's condition (iv) is not needed. Moreover, instead of assuming the existence of β, Kesten imposes the condition

$$\exists_{s_0 > 0} \quad \mathbb{E} \inf_{u \in \mathbb{S}} \left| \mathbf{T}^\top x \right|^{s_0} \geq 1 \tag{$\exists\beta$}$$

which gives that $m(s_0) \geq 1$. Hence by convexity of m and the negativity of the Lyapunov exponent which is assumed in(l<0) the existence of β follows. The moment assumptions are replaced by similar ones formulated in terms of s_0. The condition $(\exists\beta)$ has caused some confusions, since it is not necessary for the existence of β, as pointed out in [64, Remark 2.8 (iii)].
- A simple situation in which (irred), (density) and $(R \neq r)$ are satisfied, is when (density) holds with $\Gamma_0 = \mathbf{Id}$ and Q is independent of $\mathbf{T}$.

In the spirit of Proposition 5.12, the following multivariate regular variation property holds.

Corollary 13.4. *Let $\beta \notin \mathbb{N}$. Then for all $f \in \mathcal{C}_c\left(\overline{\mathbb{R}^d} \setminus \{0\}\right)$,*

$$\lim_{t\to\infty} t^\beta \mathbb{E}\left(f(t^{-1}R)\right) = K \int_0^\infty \int_{\mathbb{S}} f(sw)\, \nu^\beta(dw) \frac{1}{s^{1+\beta}} ds \tag{13.9}$$

for some $K > 0$.

As before, the measure ν^β is an eigenmeasure of P^β with eigenvalue 1. See [52, Lemma 5.17] for a recent result covering all possible values of $\beta > 0$.

13.6. A Moment for Organization

The main tools of the proof are the theory of Harris recurrent chains, a newly developed bivariate minorization condition, a multivariate version of implicit renewal theory and a combination of regeneration techniques stemming from the bivariate minorization and a generalization of Lévy's symmetrization inequality.

Basic ideas from the theory of Harris chains are introduced first, in Section 14, to motivate the study of minorization properties in Section 18. These are refined in Section 16 where some useful results concerning a whole class of SFPE that are solved by R and obtained via the use of stopping times. In particular, geometric sampling will allow to simplify some assumptions in Theorem 13.2 before proving it. With this results at hand, the transfer operators P_*^s are studied in Section 17. This allows to reintroduce the harmonic change of measure in Section 18 where also another family of probability measures, useful for the proof of positivity of K is defined and an extension of the regeneration lemma of Athreya and Ney [13, Lemma 3.1] using bivariate minorization is given. In Section 19, the MRT for Harris chains is introduced and an extension is proved, using bivariate minorization. It is shown in the subsequent Section 20 that the intrinsic MRW $(X_n, V_n)_{n\geq 0}$ satisfies its assumption under the changed measure. By a first application of the MRT, it will be shown in Section 21 that $\lim_{t\to\infty} t^\beta \mathbb{P}\left(\sup_{n\in\mathbb{N}} |\Pi_n x| > t\right)$ exists and is positive, and that the same holds for $\liminf_{t\to\infty} t^\beta \mathbb{P}\left(\sup_{n\in\mathbb{N}} |\Pi_{\sigma_n - 1} x| > t\right)$, which is the restriction to regeneration times. This will also be needed to prove that K is positive. Finally, the proof of the main theorem is given in Section 22 where the convergence assertion is shown and in Section 23 which is concerned with the positivity of K. The Corollary about multivariate regular variation is proved in Section 24.

Parts of this results have been already published in [7], the result on multivariate regular variation has been published as part of the article [35].

14. Few Words on Harris Chains

As the theory of Harris recurrent Markov chains will be a main ingredient in the subsequent proofs and motivates some quite technical calculations, it is convenient to introduce its basic ideas at the outset.

A Markov chain $(X_n)_{n\in\mathbb{N}_0}$ on a separable metric space S with transition kernel P is called *strongly aperiodic Harris chain*, if there exists a measurable $\mathfrak{R} \subset S$, called *regeneration set*, such that

$$\mathbb{P}_x\left(X_n \in \mathfrak{R} \text{ infinitely often}\right) = 1$$

for all $x \in S$ (recurrence) and, furthermore,

$$\inf_{x\in\mathfrak{R}} P(x,\cdot) \geq \xi\, \Phi \tag{14.1}$$

for some $\xi > 0$ and a probability measure Φ with $\Phi(\mathfrak{R}) = 1$. Strong aperiodicity refers to the fact that (14.1) holds for P instead of just for P^m with $m \geq 2$. If S itself is regenerative then $(X_n)_{n\in\mathbb{N}_0}$ is called *Doeblin chain*. A Harris chain $(X_n)_{n\in\mathbb{N}_0}$ possesses a nice regenerative structure: One can redefine $(X_n)_{n\in\mathbb{N}_0}$ on a possibly enlarged probability space together with a filtration $(\mathcal{G}_n)_{n\in\mathbb{N}_0}$ and a sequence of stopping times $(\sigma_n)_{n\in\mathbb{N}_0}$, $\sigma_0 = 0$ w.r.t. $(\mathcal{G}_n)_{n\in\mathbb{N}_0}$ such that $(X_n)_{n\in\mathbb{N}_0}$ is still Markov adapted w.r.t. $(\mathcal{G}_n)_{n\in\mathbb{N}_0}$ and for each $k \in \mathbb{N}$, $x \in S$

$$\mathbb{P}_x\left((X_{\sigma_k+n})_{n\in\mathbb{N}_0} \in \cdot | \mathcal{G}_{\sigma_k - 1}\right) = \mathbb{P}_\Phi\left((X_n)_{n\in\mathbb{N}_0} \in \cdot\right). \tag{14.2}$$

This property allows to carry over many techniques of proof from the theory of discrete MCs. An important result that will be used later is the “ergodic theorem” for strongly aperiodic Doeblin chains:

Theorem 14.1. *Suppose there is $\Phi \in \mathcal{P}(S)$ and $\xi > 0$ such that for all $x \in S$,*

$$P(x, \cdot) \geq \xi\, \Phi,$$

i.e. P satisfies the strongly aperiodic Doeblin chain condition. Then P has a unique stationary distribution π, and is geometric ergodic:

$$\mathrm{tv}\,[P^n(x, \cdot) - \pi] \leq C(1-\xi)^n$$

for some $C > 0$ and all $x \in S$, $n \in \mathbb{N}$.

Proof. This is [73, Theorem 16.0.2], the assumption stated here corresponds to [73, Theorem 16.0.2 (v)] with $m = 1$ and $\nu_m = \Phi$. See [73, Section 5.2] for the definition of small sets. □

An extended minorization condition for bivariate Markov chains, e.g. MRWs, will be introduced in Section 18. There, also a proof of the existence of the regenerative structure will be given which contains the classical version as a special case. Thus the reader is refered to Section 18 for details, as well as to the introductory texts [9, Section VII.3] and [73, Section 5.1]. The theory of was developed around 1978 independently by Nummelin [79] and Athreya & Ney [13].

15. Minorization: Implications of (i-d)

In this section, a bivariate minorization condition for the sequence $(\Pi_n \cdot x, \Pi_n)_{n\in\mathbb{N}}$ will be shown. It is closely connected with condition (i-d)and is the basis for applying Harris chain theory later on.

First is a preparatory lemma.

Lemma 15.1. *Let $x \in \mathbb{S}$. For all $\varepsilon > 0$ there is $\eta > 0$ such that for all $u \in B_\eta(x)$ there is an orthogonal matrix $\mathbf{A}_u \in B_\varepsilon(\mathbf{Id})$ with $u = \mathbf{A}_u x$.*

Proof. If $u = x$, choose $\mathbf{A}_u = \mathbf{Id}$. If $u \neq x$, then choose an orthonormal basis $\hat{e}_1, \dots, \hat{e}_d$ with orthogonal transformation matrix $\mathbf{L}$ such that $x = \mathbf{L}e_1$, and $u = \mathbf{L}(\cos\theta e_1 + \sin\theta e_2)$ for some $\theta \in [0, 2\pi]$. Define

$$\hat{\mathbf{A}}_u = \begin{pmatrix} \mathbf{B} & 0 \\ 0 & \mathbf{C} \end{pmatrix} \text{ where } \mathbf{B} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

and $\mathbf{C}$ is the (d-2)-dimensional identity matrix. Set $\mathbf{A}_u := L\hat{\mathbf{A}}_u\mathbf{L}^{-1}$. Then $\mathbf{A}_u x = u$ and, since $\mathbf{L}$, $\mathbf{L}^{-1}$ are isometries,

$$\|\mathbf{A}_u - \mathbf{Id}\|^2 = \left\|\mathbf{L}\hat{\mathbf{A}}_u\mathbf{L}^{-1} - \mathbf{L}\,\mathbf{Id}\,\mathbf{L}^{-1}\right\|^2 = \left\|\hat{\mathbf{A}}_u - \mathbf{Id}\right\|^2$$

$$\leq \max_{x \in S^1} ((\cos\theta - 1)x_1 - \sin\theta x_2)^2 + ((\cos\theta - 1)x_2 + \sin\theta x_1)^2$$
$$\leq 4\left[(\cos\theta - 1)^2 + (\sin\theta)^2\right].$$

Thus $\mathbf{A}_u \to \mathbf{Id}$ if $u \to x$, this is the asserted continuity property. □

The formulation of the following proposition may look a bit scaring at first glance, but all properties will be used later and the sum will vanish in the next section, where geometric sampling will be introduced.

Proposition 15.2. *Let μ satisfy (i-d). Then for each $x \in \mathbb{S}$, there is $n \in \mathbb{N}$, $\xi, \delta > 0$, $C \subset GL(d, \mathbb{R})$ compact and a stochastic kernel Ψ from $\mathbb{S}$ to $\mathbb{S} \times GL(d, \mathbb{R})$ such that*

$$\sum_{k=1}^{n} 2^{-k} \mathbb{P}\left((\Pi_k \cdot y, \Pi_k) \in \cdot\right) \geq \xi \Psi(y, \cdot) \tag{MC1'}$$

and

$$\operatorname{supp} \Psi(y, \cdot) \subseteq B_\delta(x) \times C$$

for all $y \in \mathbb{S}$.

For the first marginal of Ψ the following holds: There is $\Phi \in \mathcal{P}(\mathbb{S})$, $\operatorname{supp}\Phi = B_\delta(x)$ such that for all $y \in \mathbb{S}$ and measurable $A \subset \mathbb{S}$

$$\Psi(y, A \times C) = \Phi(A) \tag{15.1}$$

and thus for all $y \in \mathbb{S}$

$$\sum_{k=1}^{n} 2^{-k} \mathbb{P}\left(\Pi_k \cdot y \in A, \Pi_k \in C\right) \geq \xi \Phi(A). \tag{MC2'}$$

For the second marginal of Ψ it holds that there is $L > 0$, such that for all $y \in \mathbb{S}$ and measurable $B \subset GL(d, \mathbb{R})$,

$$\Psi(y, B_\delta(x) \times B) = L \int_{B_\varsigma(\mathbf{Id})} \mathbf{1}_{B_\delta(x) \times B}(\mathbf{A} \cdot x, \mathbf{A}\mathbf{A}_y) \lambda^{d^2}(d\mathbf{A}), \tag{15.2}$$

where $\mathbf{A}_y$ is a deterministic matrix in $GL(d, \mathbb{R})$ for each fixed y.

Proof. Fix $x \in \mathbb{S}$. The first step is to show that (MC1') holds for all $y \in U$ for a specific open set U. Then the minorization is extended to all $y \in \mathbb{S}$ by using (irred).

The idea of proof for the first step is quite simple, but details are very technical. So let's give an heuristic how to prove (MC2'): Writing $x_0 = \Gamma_0^{-1} \cdot x$ and using (density),

$$\mathbb{P}(\Pi_n \cdot x_0 \in A) \geq \int_{B_c(\Gamma_0)} \mathbf{1}_A(\mathbf{A}\Gamma_0^{-1} \cdot x) \lambda^{d^2}(d\mathbf{A}).$$

Thus $\mathcal{L}(\Pi_n \cdot x_0)$ has an absolutely continuous component w.r.t. Lebesgue measure $\lambda_{\mathbb{S}}$ on $\mathbb{S}$, supported around x. If x_0 is replaced by y which is close to x_0, the integral on the right hand side changes smoothly and the idea is to find a common component for all $y \in B_\delta(x_0)$ with δ sufficiently small.

Step 1: Recall that p, c, Γ_0, n_0 are given by condition (density). It follows the existence of $c' > 0$ such that

$$B_{c'}(\mathbf{Id}) \subset B_c(\Gamma_0)\Gamma_0^{-1}.$$

Using the inequality

$$\|\mathbf{LA} - \mathbf{Id}\| \leq (1 + \|\mathbf{L} - \mathbf{Id}\|)\,\|\mathbf{A} - \mathbf{Id}\| + \|\mathbf{L} - \mathbf{Id}\|\,,$$

there is $\varepsilon > 0$ such that

$$B_{c'/2}(\mathbf{Id}) \subset B_{c'}(\mathbf{Id})\mathbf{A}$$

for all $\mathbf{A} \in B_\varepsilon(\mathbf{Id})$. Set $\varsigma := c'/2$.

Referring to Lemma 15.1, there is $\eta > 0$ such that for all $u \in U := \Gamma_0^{-1} \cdot B_\eta(x)$, there is a orthogonal matrix $\mathbf{A}_u \in B_\varepsilon(\mathbf{Id})$ with

$$u = \Gamma_0^{-1} \cdot \mathbf{A}_u x \quad \text{and} \quad B_\varsigma(\mathbf{Id}) \subset B_c(\Gamma_0)\Gamma_0^{-1}\mathbf{A}_u. \tag{15.3}$$

Next there is $\delta > 0$ such that

$$\int_{B_\varsigma(\mathbf{Id})} \mathbf{1}_A(\mathbf{A} \cdot x)\lambda^{d^2}(d\mathbf{A}) \tag{15.4}$$

defines a non-zero measure Θ with $B_\delta(x) \subset \operatorname{supp}(\Theta)$.

Now for fixed $u \in U$,

$$\begin{aligned}
\mathbb{P}\left((\Pi_{n_0} \cdot u, \Pi_{n_0}) \in B\right) &\geq \int_{B_c(\Gamma_0)} \mathbf{1}_B(\mathbf{A} \cdot u, \mathbf{A})\mathbb{P}\left(\Pi_{n_0} \in d\mathbf{A}\right) \\
&\geq p \int_{B_c(\Gamma_0)} \mathbf{1}_B(\mathbf{A} \cdot u, \mathbf{A})\lambda^{d^2}(d\mathbf{A}) \\
&= p \int_{B_c(\Gamma_0)} \mathbf{1}_B(\mathbf{A}\Gamma_0^{-1}\mathbf{A}_u \cdot x, \mathbf{A})\lambda^{d^2}(d\mathbf{A}) \\
&= p \int_{g^{-1}(B_c(\Gamma_0))} \mathbf{1}_B(g(\mathbf{A})\Gamma_0^{-1}\mathbf{A}_u \cdot x, g(\mathbf{A}))\,|\det Dg|\,\lambda^{d^2}(d\mathbf{A})
\end{aligned}$$

for a diffeomorphism g by the change-of-variables formula (see [86, Theorem 7.26]). With

$$g(\mathbf{A}) = \mathbf{A}\mathbf{A}_u^{-1}\Gamma_0,$$

and taking (15.3) into account,

$$g^{-1}(B_c(\Gamma_0)) = B_c(\Gamma_0)\Gamma_0^{-1}\mathbf{A}_u \supset B_\varsigma(\mathbf{Id}). \tag{15.5}$$

By [70, Equation (4.6)], the linear mapping $g : \mathbb{R}^{d^2} \to \mathbb{R}^{d^2}$ is given by the Kronecker product $(\mathbf{A}_u^{-1}\Gamma_0)^\top \otimes \mathbf{Id}$, and thus also its derivative Dg. By the determinant formula for the Kronecker product, [70, (ix)],

$$\det Dg = \det((\mathbf{A}_u\Gamma_0)^\top)^d \det(\mathbf{Id})^d = \det(\Gamma_0)^d.$$

Consequently,

$$\mathbb{P}\left((\Pi_{n_0}\cdot u, \Pi_{n_0}) \in B, \Pi_{n_0} \in \overline{B_c(\Gamma_0)}\right) \geq p\int_{B_\varsigma(\mathbf{Id})} \mathbf{1}_B(\mathbf{A}\cdot x, \mathbf{A}\mathbf{A}_u^{-1}\Gamma_0)\,|\det\Gamma_0|^d\,\lambda^{d^2}(d\mathbf{A}).$$

Step 2: Recall that by (irred), for any $y \in S$ there is $n_1(y) \in \mathbb{N}$ such that $\mathbb{P}\left(\Pi_{n_1(y)}\cdot y \in U\right) > 0$. It follows from a tightness argument that there is as well a compact subset $C(y) \subset GL(d,\mathbb{R})$ with $\mathbb{P}\left(\Pi_{n_1(y)}\cdot y \in U, \Pi_{n_1(y)} \in C(y)\right) > 0$. For fixed y, the mapping $z \mapsto \mathcal{L}\left(\Pi_{n_1(y)\cdot z}\right)$ is continuous w.r.t. the topology of weak convergence. Thus it is continuous w.r.t. the Prohorov metric. Hence there is $\varepsilon(y)$ such that

$$\inf_{z\in B_{\varepsilon(y)}(y)} \mathbb{P}\left(\Pi_{n_1(y)}\cdot z \in U, \Pi_{n_1(y)} \in C(y)\right) > 0.$$

By compactness, $\mathbb{S} = \bigcup_{i=1}^k B_{\varepsilon(y_i)}(y_i)$ for suitable $(y_i)_{i=1}^k$. Setting $C_1 = \bigcup_{i=1}^k C(y_i)$, this set is still compact as a finite union of compact sets. Let $n_1 = \max\{n_1(y_1),\dots,n_1(y_k)\}$, then

$$\xi' := \inf_{y\in S}\sum_{k=1}^{n_1} 2^{-(k)}\mathbb{P}\left(\Pi_n\cdot y \in U, \Pi_n \in C_1\right) > 0. \tag{15.6}$$

Set $n := n_0 + n_1$ and $C := C_1\overline{B_c(\Gamma_0)}$, which is a compact subset of $GL(d,\mathbb{R})$ as the continuous image under matrix multiplication of the compact $C_1 \times \overline{B_c(\Gamma_0)}$. Then for all $y \in \mathbb{S}$ and measurable $A \subset \mathbb{S}$, $B \subset GL(d,\mathbb{R})$

$$\begin{aligned}
&\sum_{k=1}^{n} 2^{-k}\mathbb{P}((\Pi_k\cdot y, \Pi_k) \in A\times B)\\
&\geq \sum_{k=1}^{n_1} 2^{-(k+n_0)}\mathbb{P}((\Pi_k\cdot y,\Pi_k) \in U\times C_1, (\Pi_{k+n_0}\cdot y, \Pi_{k+n_0}) \in A\times B, \Pi_{k+n_0}\in C)\\
&= \sum_{k=1}^{n_1} 2^{-(k+n_0)}\int \mathbb{P}\left((\Pi_{n_0}\mathbf{L}\cdot y, \Pi_{n_0}\mathbf{L}) \in A\times B, \Pi_{n_0} \in \overline{B_c(\Gamma_0)}\right)\mathbb{P}\left(\Pi_k \in d\mathbf{L}, \Pi_k\cdot y \in U, \Pi_k \in C_1\right)\\
&\geq \sum_{k=1}^{n_1} 2^{-(k+n_0)}\int_{C_1}\mathbf{1}_U(\mathbf{L}\cdot y)\left[p\int_{B_c(\Gamma_0)}\mathbf{1}_{A\times B}(\mathbf{A}\mathbf{L}\cdot y, \mathbf{A}\mathbf{L})\lambda^{d^2}(d\mathbf{A})\right]\mathbb{P}\left(\Pi_k\in d\mathbf{L}\right)\\
&\geq \sum_{k=1}^{n_1} 2^{-(k+n_0)}p\int_{C_1}\mathbf{1}_U(\mathbf{L}\cdot y)\left[\int_{B_\varsigma(\mathbf{Id})}\mathbf{1}_{A\times B}(\mathbf{A}\cdot x, \mathbf{A}\mathbf{A}_{\mathbf{L}\cdot y}^{-1}\Gamma_0)\,|\det\Gamma_0|^d\,\lambda^{d^2}(d\mathbf{A})\right]\mathbb{P}\left(\Pi_k\in d\mathbf{L}\right)\\
&\geq \xi\Psi(y, A\times B)
\end{aligned}$$

with $\xi = 2^{-n_0} p \xi'$ and

$$\Psi(y, A \times B) = L(y)^{-1} \sum_{k=1}^{n_1} 2^{-(k+n_0)} \int_{C_1} \mathbf{1}_U(\mathbf{L} \cdot y) p \times \int_{B_\varsigma(\mathbf{Id})} \mathbf{1}_{A \cap B_\delta(x) \times B}(\mathbf{A} \cdot x, \mathbf{A}\mathbf{A}_{\mathbf{L}\cdot y}^{-1}\Gamma_0) \left|\det \Gamma_0\right|^d \lambda^{d^2}(d\mathbf{A}) \mathbb{P}\left(\Pi_k \in d\mathbf{L}\right) \tag{15.7}$$

where

$$L(y) = |\det \Gamma_0|^d \int_{B_\varsigma(\mathbf{Id})} \mathbf{1}_{B_\delta(x)}(\mathbf{A} \cdot x) \lambda^{d^2}(d\mathbf{A}) \sum_{k=1}^{n_1} 2^{-(k+n_0)} \mathbb{P}\left(\Pi_k \cdot y \in U, \Pi_k \in C_1\right).$$

The assertion about $\Psi(y, B_\delta(x) \times \cdot)$ follows directly from this definition.

Now set

$$\tilde{\Phi}(A) := \int_{B_\varsigma(\mathbf{Id})} \mathbf{1}_{A \cap B_\delta(x)}(\mathbf{A} \cdot x) \left|\det \Gamma_0\right|^d \lambda^{d^2}(d\mathbf{A}).$$

By (15.4), $\tilde{\Phi}(A)$ is nonzero, has support $B_\delta(x)$, and its renormalization $\Phi := \tilde{\Phi}(B_\delta(x))^{-1}\tilde{\Phi}$ satisfies for all $y \in \mathbb{S}$

$$\Psi(y, \cdot \times C) = \Phi.$$

□

16. The Stopped RDE and Geometric Sampling

As already observed by Vervaat [93, Lemma 1.2], geometric sampling and, more generally, the use of stopping times for $(\mathbf{T}_{(n)}, Q_n)_{n\in\mathbb{N}}$ provides a useful technique the analysis of RDEs and is thus discussed in this section.

16.1. R is the Unique Solution of the Stopped Equation

Let $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ be a filtration such that $(\mathbf{T}_{(n)}, Q_n)_{n\in\mathbb{N}}$ is adapted to it and $(\mathbf{T}_{(k)}, Q_k)_{k>n}$ is independent of $\mathcal{F}_n$ for any $n \in \mathbb{N}_0$. Consider any a.s. finite stopping time τ w.r.t. $(\mathcal{F}_n)_{n\in\mathbb{N}_0}$ which, by suitable choice of the latter, includes the case that τ and $(\mathbf{T}_{(n)}, Q_n)_{n\in\mathbb{N}}$ are independent (pure randomization). Then it is readily checked that R defined in (13.4) satisfies

$$R = \daleth_\tau R^\tau + Q^\tau \tag{16.1}$$

where

$$Q^n := \sum_{k=1}^{n} \daleth_{k-1} Q_k \quad \text{and} \quad R^n := \sum_{k>n} \left(\prod_{j=n+1}^{k-1} \mathbf{T}_{(j)}\right) Q_k \tag{16.2}$$

for $n \in \mathbb{N}$. But since the sequence $(\mathbf{T}_{(\tau+n)}, Q_{\tau+n})_{n\in\mathbb{N}}$ is a copy of $(\mathbf{T}_{(n)}, Q_n)_{n\in\mathbb{N}}$ and independent of $(\mathbf{T}_{(n)}, Q_n)_{1\le n\le\tau}$ and τ, it follows that R^τ is independent of $(\daleth_\tau, Q^\tau)$ with

$$R^\tau \overset{d}{=} R. \tag{16.3}$$

In other words, (the law of) R also solves the stopped stochastic fixed point equation (SFPE)

$$Y \overset{d}{=} \daleth_\tau Y + Q^\tau \tag{16.4}$$

and provides a stationary distribution to the RDE

$$R_n = \mathbf{T}'_n R_{n-1} + Q'_n, \quad n \ge 1, \tag{16.5}$$

where $(\mathbf{T}'_n, Q'_n)_{n\in\mathbb{N}}$ is a sequence of i.i.d. copies of $(\daleth_\tau, Q^\tau)$. Uniqueness follows if (logmom) persists to hold for the "stopped pair" $(\daleth_\tau, Q^\tau)$ together with

$$\lim_{n\to\infty} \frac{1}{n} \log \|\daleth_{\sigma_n}\| < 0 \ \mathbb{P}\text{-a.s.}$$

where $(\sigma_n)_{n\ge0}$ denotes a zero-delayed renewal process such that $\sigma_1 = \tau$ and

$$\left(\sigma_n - \sigma_{n-1}, (\mathbf{T}_k, Q_k)_{\sigma_{n-1}<k\le\sigma_n}\right), \quad n \ge 1$$

are i.i.d.. For stopping times τ with finite mean this is indeed easily verified and the result is stated (without proof) in the following lemma.

Lemma 16.1. *The law of R forms the unique solution to the SFPE* (16.4) *whenever $\mathbb{E}\tau < \infty$.*

Study of R is now allowed within the framework of any stopped SFPE (16.4) with finite mean τ. The idea is to pick τ in such a way that $(\daleth_\tau, Q^\tau)$ has nice additional properties compared to $(\mathbf{T}, Q)$. Geometric sampling provides a typical example that will be used hereafter and therefore discussed next. Another use of this technique, in particular of identities (16.1) and (16.3) appears in Section 23.

16.2. Geometric Sampling

Suppose now that $(\sigma_n)_{n\ge0}$ is independent of $(\mathbf{T}_{(n)}, Q_n)_{n\ge1}$ with geometric(1/2) increments, that is $\mathbb{P}(\tau = n) = 1/2^n$ for each $n \in \mathbb{N}$. Then not only Lemma 16.1 holds true but also the following result:

Lemma 16.2. *If $(\mathbf{T}, Q)$ satisfies the assumption of Theorem 13.2, then so does $(\daleth_\tau, Q^\tau)$ with the same function m, in particular the same $\beta > 0$; and $n = n_0 = 1$ in* (irred), (density).

Proof. That (logmom) and $\lim_{n\to\infty} n^{-1} \log \|\Pi_{\sigma_n}\| < 0$ $\mathbb{P}$-a.s. persist to hold under any finite mean stopping time τ has already been pointed out before Lemma 16.1. As for (irred) and (density), just note that $\mathbb{P}(\Pi_\tau \in \cdot) = \sum_{k\ge1} 2^{-n} \mathbb{P}(\Pi_n \in \cdot)$. Assumption ($R \ne r$) ensures that the law of R is nondegenerate. But since R is also the unique solution to (16.4), $(R \neq r)$ must hold for $(\daleth_\tau, Q^\tau)$ as well.

For the assertion about m, use the subsequent Lemma 16.3 : The value $m(s)$ is given as the spectral radius of P_*^s (see below for a recap of the definition) while the value of $m_\tau(s)$, the spectral function associated with $\daleth_\tau$, is given as the spectral radius of $P_*^{s,\tau} = \sum_{k=1}^\infty 2^{-k}(P_*^s)^k$. Then it is a direct consequence of the spectral mapping theorem [40, VII.3.11], that $m(s) = m_\tau(s)$ for all $s \in I_\mu$. The remaining moment assertions (TlogT) and (Q-beta) are again easily verified by standard estimates. Further details are therefore omitted. □

Lemma 16.3. *Let μ satisfy (i-d). For each $s \in I_\mu$, the spectral radius $r(P_*^s)$ of P_*^s is given by*

$$r(P_*^s) = \kappa(s) = m(s).$$

Proof. Obviously,

$$r(P_*^s) = \lim_{n\to\infty} \sup_{x\in\mathbb{S}} (\mathbb{E}|\Pi_n x|^s)^{1/n} \leq \liminf_{n\to\infty} (\mathbb{E}\|\Pi_n\|^s)^{1/n}.$$

For the converse note that, by [26, Proposition III.3.2], $Z_{x_0} := \inf_{n\geq 0} \|\Pi_n\|^{-1} |\Pi_n x_0| > 0$ a.s. for any $x_0 \in \mathbb{S}$, whence

$$\sup_{x\in\mathbb{S}} \mathbb{E}|\Pi_n x|^s \;\geq\; \mathbb{E}\|\Pi_n\|^s \frac{\mathbb{E}|\Pi_n x_0|^s}{\mathbb{E}\|\Pi_n\|^s} \;\geq\; \mathbb{E}\|\Pi_n\|^s \frac{\mathbb{E}Z_{x_0} \|\Pi_n\|^s}{\mathbb{E}\|\Pi_n\|^s}$$

and therefore (using Jensen's inequality)

$$r(P_*^s) \;\geq\; \limsup_{n\to\infty} (\mathbb{E}\|\Pi_n\|^s)^{1/n} \lim_{n\to\infty} \frac{\mathbb{E}Z_{x_0}^{1/n} \|\Pi_n\|^s}{\mathbb{E}\|\Pi_n\|^s} \;=\; \limsup_{n\to\infty} (\mathbb{E}\|\Pi_n\|^s)^{1/n}$$

which completes the proof.

The assumption of [26, Proposition III.3.2] is that $T_\mu = [\operatorname{supp}\mu]$ (see [26, p. 43]]) is *strongly irreducible*: There is no finite union $\bigcup_{i=1}^k V_i$ of proper linear subspaces $\emptyset \neq V_1, \ldots, V_k \subsetneq \mathbb{R}^d$ that is invariant under $[\operatorname{supp}\mu]$, i.e.

$$\mathbf{A} \bigcup_{i=1}^k V_i \subset \bigcup_{i=1}^k V_i \tag{16.6}$$

for all $\mathbf{A} \in [\operatorname{supp}\mu]$ (see [26, Definition III.2.1]). For each choice of $(V_i)_{i=1}^k$ this gives a finite set of polynomial equations for the matrix coefficients, that have to be satisfied by all $\mathbf{A} \in [\operatorname{supp}\mu]$. But by (density), $B_c(\Gamma_0) \in [\operatorname{supp}\mu]$, so this cannot hold.

In [26, Propositon III.3.2], also the index of $[\operatorname{supp}\mu]$ (see [26, Definition III.1.3]) is addressed, but it is irrelevant to the part of the result that was used here. □

Considering Lemma 16.2, the following **standing assumption** holds:

If (irred),(density),(MC1') and (MC2') hold, they hold with $n_0 = n = 1$. (StA)

Corollary 16.4. *Let μ satisfy (i-d) and let* (StA) *be in force. Then the assertions of Proposition 15.2 hold with* (MC1'), (MC2') *replaced by*

$$\mathbb{P}\left((\Pi_1 \cdot y, \Pi_1) \in \cdot\right) \geq \xi \Psi(y, \cdot) \tag{MC1}$$

resp.

$$\mathbb{P}\left(\Pi_1 \cdot y \in \cdot, \Pi_1 \in C\right) \geq \xi \Phi. \tag{MC2}$$

17. Transfer Operators under Condition (i-d)

Recall the definition of the transfer operators, which are now considered as operators in $\mathcal{C}(\mathbb{S})$:

$$\begin{aligned} P^s f(x) &:= \mathbb{E}\left(|\mathbf{T}x|^s f(\mathbf{T} \cdot x)\right), \\ P^s_* f(x) &:= \mathbb{E}\left(\left|\mathbf{T}^\top x\right|^s f(\mathbf{T}^\top \cdot x)\right) = \mathbb{E}\left(|\mathbf{M}_1 x|^s f(\mathbf{M}_1 \cdot x)\right). \end{aligned}$$

This section is devoted to the proof of the following theorem, which is the analogue of Theorem 7.3 under condition (i-d). This time, proofs will be given. In the proofs, Harris chain theory and the minorization properties proved above will play an important role. For previous results, where μ has a density w.r.t. to the Haar measure on the group of unimodular matrices, see [92] as well as [26, Proposition V.2.6]

Theorem 17.1. *Let μ satisfy (i-d) and $s \in I_\mu$. Then the following holds:*

1. *The spectral radius and the dominant eigenvalue of P^s_* are equal to $\kappa(s)$.*
2. *There is a unique strictly positive normalized function $e^s_* \in \mathcal{C}(\mathbb{S})$ ($|e^s_*|_\infty = 1$) and a unique probability measure ν^s_* on $\mathbb{S}$ such that*
$$P^s_* e^s_* = \kappa(s) e^s_*, \quad P^s_* \nu^s_* = \kappa(s) \nu^s_*. \tag{17.1}$$
3. *The function e^s_* is symmetric, i.e. $e^s_*(x) = e^s_*(-x)$ for all $x \in \mathbb{S}$ and $\operatorname{supp}(\nu^s_*) = \mathbb{S}$.*
4. *For all $f \in \mathcal{C}(\mathbb{S})$,*
$$\lim_{n\to\infty} \left| \frac{(P^s_*)^n f}{\kappa(s)^n} - \frac{\nu^s_*(f)}{\nu^s_*(e^s_*)} e^s_* \right|_\infty = 0. \tag{17.2}$$
Moreover, for all $x \in \mathbb{S}$
$$\lim_{n\to\infty} \operatorname{tv}\left[\frac{(P^s_*)^n(x, \cdot)}{\kappa(s)^n} - \frac{\nu^s_*}{\nu^s_*(e^s_*)}\right] = 0. \tag{17.3}$$
5. *The function $s \mapsto \kappa(s)$ is convex on I_μ.*
6. *The mapping $s \mapsto e^s_*$ is continuous w.r.t. $|\cdot|_\infty$ and the mapping $s \mapsto \nu^s_*$ is continuous w.r.t. to the total variation norm.*

The proof will be given in a series of subsequent lemmata.

17.1. The Spectrum of P_*^s

The first assertion in 1. about the spectral radius $r(P_*^s)$ of P_*^s has already been shown in Lemma 16.3. The second assertion about the dominant eigenvalue is contained in the following Lemma. Its proof follows the same ideas as Kesten's original proof [59, Theorem 3, Step 1] and is valid under very general assumptions.

Lemma 17.2. *Assume that $s \in I_\mu$. Then P_*^s has an eigenvalue $\lambda_s \in \mathbb{C}$ with $|\lambda_s| = \kappa(s)$.*

Proof. Let's introduce some notation first. The conjugate space of $(\mathcal{C}(\mathbb{S}), |\cdot|_\infty)$ will be denoted by $\mathcal{C}(\mathbb{S})'$, this is the space $\mathcal{M}^\pm(\mathbb{S})$ of regular bounded signed measures on $\mathbb{S}$ equipped with the total variation norm $\mathrm{tv}[\cdot]$. The weak topology on $\mathcal{C}(\mathbb{S})'$ (which is also called the X topology of $X*$ in [40] sometimes) is the topology of weak convergence of measures.

STEP 1: Referring to [40, Definition IV.6.1 & Lemma VI.2.2]), the adjoint operator $(P_*^s)' : \mathcal{C}(\mathbb{S})' \to \mathcal{C}(\mathbb{S})'$ has the same operator norm as P_*^s and it is bounded since

$$\left\|(P_*^s)'\right\| = \|P_*^s\| \le \mathbb{E}\,\|T_1\|^s < \infty.$$

It follows that

$$(P_*^s)'D = \left\{(P_*^s)'\nu \ :\ \mathrm{tv}[\nu] \le 1\right\}$$

is bounded in $(\mathcal{C}(\mathbb{S})', \mathrm{tv}[\cdot])$. Hence the weak closure of $(P_*^s)'D$ is weakly compact, this is the assertion of [40, Corollary V.4.3]. This proves that $(P_*^s)'$ is weakly compact (see [40, Definition VI.4.1]).

STEP 2: Referring to [40, Theorem VI.4.8], P_*^s is then weakly compact as well. Considering [40, Corollary VI.7.5], it follows that $(P_*^s)^2$ is compact. With these properties, it is the assertion of [40, Theorems VII.4.5 & 6] that the spectrum of P^s is at most denumerable and has no point of accumulation except for possibly 0. Moreover, each non-zero number in the spectrum is an eigenvalue with a finite dimensional eigenspace. In particular, P_*^s has an eigenvalue λ_s which is in modulus equal to the spectral radius. □

17.2. The Eigenspaces of $\kappa(s)$

In order to deduce that in fact $\lambda_s = \kappa(s)$, condition (i-d) enters the stage. In particular property (irred) yields that the operators P_*^s are *strictly positive*: I.e. if $f \ge 0$, and $f(x) > 0$ for some $x \in \mathbb{S}$ then $P_*^s f(y) > 0$ for all $y \in \mathbb{S}$. Indeed, for any such f, the set $U_f = \{f > 0\}$ is nonempty and open by continuity. Now use (irred) with $n = 1$ to infer

$$P_*^s f(x) \ge \int |\mathbf{M}x|^s\, \mathbf{1}_{U_f}(\mathbf{M}\cdot x)\, f(\mathbf{M}\cdot x)\, \mathbb{P}\left(\mathbf{M}_1 \in d\mathbf{M}\right) > 0.$$

Then the final assertion in 1. as well as existence and uniqueness of normalized e_*^s result from the next lemma:

Lemma 17.3. *Let $s \in I_\mu$ and P_*^s be strictly positive. Then there is a unique strictly positive function*

e_*^s, $|e_*^s|_\infty = 1$, *such that* $P_*^s e_*^s = \kappa(s) e_*^s$. *Moreover,*

$$\mathrm{Eig}(P_*^s, \kappa(s)) = \mathbb{R} e_*^s.$$

Proof. The following argument goes back to Karlin [57, Section 5]. By Lemma 17.2, P_*^s has eigenvalue λ_s with $|\lambda_s| = \kappa(s)$. Let f be a corresponding eigenfunction. Hence,

$$\kappa(s)\,|f| = |\lambda_s f| = |P_*^s f| \le P_*^s\,|f|\,.$$

Suppose that $P_*^s\,|f| - \kappa(s)\,|f| \neq 0$. It is a consequence of the strict positivity of P_*^s that $P_*^s(P_*^s\,|f| - \kappa(s)\,|f|)$ is positive on $\mathbb{S}$ and thus bounded from below by some $\eta > 0$. Choose η small such that, furthermore, $\kappa(s) P_*^s\,|f| < 1/\eta$. It follows from these inequalities that

$$(P_*^s)^2\,|f| - P_*^s \kappa(s)\,|f| \;>\; \eta \;>\; \eta^2 \kappa(s)|f|$$

hence

$$(P_*^s)^2\,|f| > (1+\eta^2)\kappa(s) P_*^s\,|f|$$

and thereby

$$(P_*^s)^n P_*^s\,|f| > (1+\eta^2)^n \kappa(s)^n P_*^s\,|f|$$

for all $n \in \mathbb{N}$ upon iteration. Consequently, $\|(P_*^s)^n\| > (1+\eta^2)^n \kappa(s)^n$ for all $n \in \mathbb{N}$ and thus $r(P_*^s) > \kappa(s)$. This is a contradiction with Lemma 16.3 and leads to the conclusion that $P_*^s|f| = |f|$. Thus $e_*^s := |f|$ is a positive eigenfunction for the eigenvalue 1. It is positive everywhere due to the strict positivity of P_*^s.

Now, suppose there is another eigenfunction g, linearly independent of e_*^s and w.l.o.g. real-valued (for, if g is an eigenfunction, then so are its real and imaginary parts if nontrivial). Pick ε such that $h := e_*^s + \varepsilon g$ is nonnegative, but $h(x) = 0$ for some x. By linear independence, h does not vanish everywhere. Since it is again an eigenfunction, the strict positivity of P_*^s implies that it must be positive everywhere which is a contradiction. Hence e_*^s must be the unique eigenfunction modulo scalars. □

From this the corresponding properties of ν_*^s will be deduced. The idea of proof here is very similar to [52, Theorem 2.6] (see also [50]), but more straightforward due to the density assumption.

Lemma 17.4. *Let* μ *satisfy (i-d), let* $s \in I_\mu$ *and* (StA) *be in force. Then*

$$Q_*^s f(x) = \frac{1}{e_*^s(x)\kappa(s)} P_*^s(e_*^s f)(x) \tag{17.4}$$

defines a Markov transition operator on $\mathbb{S}$. *It has a unique stationary distribution* π_*^s *and is geometric ergodic: For all* $x \in \mathbb{S}$,

$$\mathrm{tv}\,[(Q_*^s)^n(x,\cdot) - \pi_*^s] \le (1-\xi)^n. \tag{17.5}$$

The probability measure

$$\nu_*^s(dx) := c e_*^s(x)^{-1} \pi_*^s(dx) \tag{17.6}$$

(with norming constant $c^{-1} = \int e_*^s(x)^{-1}\pi_*^s(dx)$*) satisfies*

$$P_*^s \nu_*^s = \kappa(s)\nu_*^s,$$

and $\operatorname{supp}(\nu_*^s) = \operatorname{supp}(\pi_*^s) = \mathbb{S}$.

Proof. Step 1: The operator Q_*^s maps positive functions onto positive functions since P_*^s and e_*^s are strictly positive. It is a Markov operator since

$$Q_*^s \mathbf{1}_{\mathbb{S}}(x) = \frac{1}{e_*^s(x)\kappa(s)} P_*^s(e_*^s)(x) = \frac{\kappa(s)e_*^s(x)}{\kappa(s)e_*^s(x)} = 1.$$

By canonical extension, for all measurable $A \subset \mathbb{S}$,

$$Q_*^s(x, A) = \frac{1}{e_*^s(x)\kappa(s)} \int \mathbf{1}_A(y) e_*^s(y) \left|\mathbf{M}x\right|^s \mathbb{P}\left(\Pi_1 \cdot x \in dy, \Pi_1 \in d\mathbf{M}\right).$$

Step 2: Choose arbitrary $x_0 \in \mathbb{S}$. By Corollary 16.4 resp. Proposition 15.2 there is $\xi > 0$, a compact subset $C \in GL(d, \mathbb{R})$ and a probability measure Φ such that $\mathbb{P}\left(\Pi_1 \cdot x \in dy, \Pi_1 \in C\right) \geq \xi\Phi(dy)$ for all $x \in \mathbb{S}$. Note that due to compactness of $\mathbb{S}$ and C

$$\xi_1 := \min_{z_1, z_2 \in \mathbb{S}} \frac{e_*^s(z_1)}{e_*^s(z_2)} > 0 \quad \text{and} \quad \xi_2 := \min_{z \in \mathbb{S}, \mathbf{A} \in C} |\mathbf{A}z|^s > 0.$$

Consequently, for each $x \in \mathbb{S}$ and measurable $A \subset \mathbb{S}$,

$$\begin{aligned} Q_*^s(x, A) &\geq \frac{\xi_1\xi_2}{\kappa(s)} \int \mathbf{1}_A(y) \mathbb{P}\left(\Pi_1 \cdot x \in dy, \Pi_1 \in C\right) \\ &\geq \frac{\xi\xi_1\xi_2}{\kappa(s)} \Phi(A). \end{aligned} \tag{17.7}$$

Thus Q_*^s satisfies the assumption of the ergodic theorem 14.1 which gives the geometric ergodicity and existence and uniqueness of π_*^s. The assertion about the support of π_*^s follows directly by an inspection of the minorization in (17.7): By Corollary 16.4, for any $x_0 \in \mathbb{S}$ there is $\xi, \delta > 0$ and Φ with $\operatorname{supp}(\Phi) = B_\delta(x)$. It follows that for all $\varepsilon > 0$

$$\pi_*^s(B_\varepsilon(x_0)) = \int_{\mathbb{S}} Q_*^s(x, B_\varepsilon(x_0)) \pi_*^s(dx) \geq \frac{\xi\xi_1\xi_2}{\kappa(s)} \Phi(B_\varepsilon(x_0)) > 0.$$

Step 3: Recall that $e_*^s > 0$ on $\mathbb{S}$, thus ν_*^s as defined in (17.6) is well defined and has the same support as π_*^s. In addition, for any $f \in \mathcal{C}(\mathbb{S})$

$$\begin{aligned} \int f(x) \nu_*^s(dx) &= \int \frac{f(x)}{e_*^s(x)} \pi_*^s(dx) \\ &= \int \frac{1}{e_*^s(x)\kappa(s)} P_*^s\left(\frac{f}{e_*^s} e_*^s\right)(x) \pi_*^s(dx) \end{aligned}$$

$$= \frac{1}{\kappa(s)} \int P_*^s f(x) \nu_*^s(dx)$$

which proves that $(P_*^s)' \nu_*^s = \kappa(s) \nu_*^s$. □

Lemma 17.5. *Let $s \in I_\mu$ and P_*^s be strictly positive. Then $\kappa(s)$ is an eigenvalue of $(P_*^s)'$ with one-dimensional eigenspace.*

Proof. By [40, Exercise VII.5.35], if $\kappa(s)$ is an eigenvalue of P_*^s and isolated in the spectrum with one-dimensional eigenspace, then the same holds true with P_*^s replaced by its adjoint $(P_*^s)'$. It was shown in the proof of Lemma 17.2, that each point (except maybe 0) of the spectrum of P_*^s is isolated, and Lemma 17.3 yields that the eigenspace of $\kappa(s)$ is one-dimensional. □

Thus, after renormalizing ν_*^s to a probability measure, it is unique, and assertions 2. is proved.

The convergence assertion in 4. is easily deduced from the geometric ergodicity: For any $f \in \mathcal{C}(\mathbb{S})$, $|f|_\infty \le \left|(e_*^s)^{-1}\right|_\infty$, (17.5) implies

$$|(Q_*^s)^n f(x) - \pi_*^s(f)| \le \left|(e_*^s)^{-1}\right|_\infty (1-\xi)^n,$$

thus this convergence is uniform in f and x. Now let $f \in \mathcal{C}(\mathbb{S})$, $|f|_\infty \le 1$. Let $g(x) = e_*^s(x)^{-1} f(x)$, then $|g|_\infty \le \left|(e_*^s)^{-1}\right|_\infty$. It follows

$$\left| \frac{(P_*^s)^n f(x)}{e_*^s(x)\kappa(s)^n} - \frac{\nu_*^s(f)}{\nu_*^s(e_*^s)} \right| = \left| (Q_*^s)^n(g) - \frac{\pi_*^s(g)}{\pi_*^s(1)} \right| \le \left|(e_*^s)^{-1}\right|_\infty (1-\xi)^n.$$

The assertion about the symmetry of e_*^s is as well a direct consequence: It can easily be seen that P_*^s maps symmetric functions onto symmetric functions. Referring to the convergence assertion above,

$$\frac{(P_*^s)^n \mathbf{1}(x)}{\kappa(s)^n} \to c e_*^s(x)$$

for some $c > 0$. Thus as a pointwise limit of symmetric functions, e_*^s is itself symmetric.

Lemma 17.6. *The function $s \mapsto \kappa(s)$ is convex on $\breve{I}_\mu$.*

Proof. Since $s \mapsto \|(P_*^s)^n f\|$ is log-convex on $\breve{I}_\mu$ for each $f \in \mathcal{C}(\mathbb{S})$ and $n \ge 1$ (use Hölder's inequality), the same holds true for $s \mapsto \|(P_*^s)^n\|$ as its pointwise supremum. Again as the pointwise limit of the log-convex functions $s \mapsto \|(P_*^s)^n\|^{1/n}$, $\kappa(s)$ is log-convex on $\breve{I}_\mu$. □

Turning finally to assertion 6., these convergence results can be proved by means of a perturbation theorem [54, Theorem III.8]. That theorem is applicable to the operators in $\mathcal{C}(\mathbb{S})$, defined for $\Re z \in I_\mu$ by

$$P_*^z f(x) := \mathbb{E}\left(|\mathbf{M}_1 x|^z f(\mathbf{M}_1 \cdot x)\right).$$

It is a consequence of that perturbation theorem, that for each $s_0 \in \breve{I}_\mu$, the mappings $s \mapsto \kappa(s)$, $s \mapsto e_*^s$ and $s \mapsto \nu_*^s$ are even holomorphic on $B_\varepsilon(s_0) \subset \mathbb{C}$ for some $\varepsilon > 0$. See [94, Section V.3] for the definition of holomorphic Banach space-valued functions. In particular, such functions are continuous w.r.t. to the norm topology by [94, Theorem V.3.1]. This gives the assertion.

18. Further Facts on Harris Chains

As announced in Section 14, in this section Markov chains satisfying an extended minorization condition will be studied and a regenerative structure for this chains will be developed.

First, the Markov Chains in question will be explicitly construct in the following subsection, which also contains the definition of the intrinsic MRW and the change of measure as in Section 8. Then a method, due to Athreya & Ney [13] and Nummelin [79] will be described that allows to construct a distributional copy of these Markov chains with an additional sequence of regeneration times.

18.1. Explicit Probability Spaces and Markov Chains

Recall that if not noted otherwise, it is assumed that all occuring random variables are defined on a common probability space, equipped with the measure $\mathbb{P}$. Subsequently, several random variables will be redefined on newly constructed probability spaces. It is far more convenient to write the same symbol for corresponding random variables on different probability spaces than to introduce new letters. But the reader should keep in mind, that in particular the identities (18.3) and (18.5) below are distributional identities.

The Measures $\mathbb{Q}_x^s$

For $n \in \mathbb{N}$, $s \in I_\mu$ define probability measures ${}_n\mathbb{Q}_x^s$ on $\mathbb{S} \times GL(d,\mathbb{R})^n$ by the property

$$ {}_n\mathbb{Q}_x^s(A) := \frac{1}{e_*^s(x)\kappa^n(s)} \mathbb{E}\left(e^{s \log|\Pi_n x|} e_*^s(\Pi_n \cdot x) \mathbf{1}_A(x, \mathbf{M}_1, \dots, \mathbf{M}_n) \right) \tag{18.1} $$

for all measureable sets A. The sequence $({}_n\mathbb{Q}_x^s)_n$ constitutes a projective system, thus referring to the Kolmogorov extension theorem [28, Corollary 2.19] it defines a probability measure $\mathbb{Q}_x^s$ on $\mathbb{S} \times GL(d,\mathbb{R})^{\mathbb{N}}$. Denote the corresponding expectation symbol by $\mathbb{E}_x^s$ and let $(X_0, (\mathbf{M}_n)_{n\in\mathbb{N}})$ be the fibered identity. Recalling the definition of $(X_n, V_n)_{n\in\mathbb{N}_0}$ in Subsection 6.1,

$$ X_n := \Pi_n \cdot X_0, \quad V_n := \log|\Pi_n X_0|, \quad U_n = V_n - V_{n-1}, $$

then (18.1) yields the identity

$$ \begin{aligned} &\mathbb{E}_x^s\left(f(X_0, V_0, \dots X_n, V_n) \right) \\ &= \frac{1}{e_*^s(x)\kappa^n(s)} \mathbb{E}\left(e^{s \log|\Pi_n x|} e_*^s(\Pi_n \cdot x) f(x, 0, \dots, \Pi_n \cdot x, \log|\Pi_n x|) \right) && (18.2) \\ &= \frac{1}{e_*^s(x)\kappa^n(s)} \mathbb{E}_x\left(e^{sV_n} e_*^s(X_n) f(X_0, V_0, \dots, X_n, V_n) \right) && (18.3) \end{aligned} $$

which is valid for all $n \in \mathbb{N}$ and all bounded measurable functions f. For the second identity, it was taken into account that $e_*^0 \equiv 1$. It is a consequence of Theorem 17.1 that the bivariate sequence

$(X_n, U_n)_{n\in\mathbb{N}_0}$ is a Markov chain under each $\mathbb{Q}_x^s$ with transition kernel

$$\hat{Q}_s((x,u), A\times B) = \frac{1}{e_*^s(x)\kappa(s)}\mathbb{E}\left(e_*^s(\mathbf{M}_1\cdot x)\,|\mathbf{M}_1 x|^s\,\mathbf{1}_A(\mathbf{M}_1\cdot x)\mathbf{1}_B(\log|\mathbf{M}_1 x|)\right). \tag{18.4}$$

Thus $(X_n, V_n)_{n\in\mathbb{N}_0}$ constitutes a Markov random walk under each $\mathbb{Q}_x^s$ and the associated Markov renewal measure will be denoted by $\mathbb{U}_x^s := \sum_{n=0}^\infty \mathbb{Q}_x^s((X_n, V_n)\in\cdot)$. Then along the same lines as Lemma 17.4, the following minorization result can be obtained.

Corollary 18.1. *For each $x_0\in\mathbb{S}$ there is $\xi_s, \delta>0$, $I\subset\mathbb{R}$ compact and a Markov kernel Υ from $\mathbb{S}$ to $\mathbb{S}\times\mathbb{R}$ with*

$$\operatorname{supp}\Upsilon(x,\cdot)\subseteq B_\delta(x_0)\times I, \quad \Upsilon(x,\cdot\times I)=\Phi$$

for all $x\in\mathbb{S}$. The Markov chain $(X_n, U_n)_{n\in\mathbb{N}_0}$ satisfies the bivariate minorization condition

$$\hat{Q}_s((x,u),\cdot)\geq \xi_s\Upsilon(x,\cdot)$$

for all $(x,u)\in\mathbb{S}\times\mathbb{R}$. In particular, $(X_n)_{n\in\mathbb{N}_0}$ is a strongly aperiodic Doeblin chain under each $\mathbb{Q}_x^s$.

Proof. This is a direct consequence of Corollary 16.4 and Proposition 15.2 with

$$\Upsilon(x, A\times B) = \int \mathbf{1}_B(\log|\mathbf{M}x|)\Psi(x, A\times d\mathbf{M})$$

and

$$I = [\min_{z\in\mathbb{S},\mathbf{A}\in C}\log|\mathbf{A}z|, \max_{\mathbf{A}\in C}\log\|\mathbf{A}\|]$$

with Ψ and C given by Proposition 15.2. □

The Measures $\mathbb{O}_x$

Define for each $x\in\mathbb{S}$ a probability measure on $\mathbb{S}\times(GL(d,\mathbb{R})\times\mathbb{R}^d)^{\mathbb{N}_0}$ by

$$\mathbb{O}_x := \delta(x)\otimes\delta(\mathbf{Id})\otimes\delta(0)\otimes\bigotimes_{n=1}^\infty \varrho$$

and denote the fibered identity by $(X_0, (\mathbf{T}_n, Q_n)_{n\in\mathbb{N}_0})$. Then for all $x\in\mathbb{S}$

$$\mathbb{O}_x([(X_0,\mathbf{T}_0,Q_0),(X_n,\mathbf{T}_n,Q_n)_{n\in\mathbb{N}}]\in\cdot) = \mathbb{P}\left([(x,\mathbf{Id},0),(\Pi_n\cdot x,\mathbf{T}_n,Q_n)_{n\in\mathbb{N}}]\in\cdot\right). \tag{18.5}$$

As before, write $\mathbf{M}_n = \mathbf{T}_n^\top$. The multivariate sequence $(X_n, \mathbf{M}_n, Q_n)_{n\in\mathbb{N}_0}$ is a Markov chain under each $\mathbb{O}_x$ with transition kernel

$$\hat{O}((x,\mathbf{A},q), A\times B\times C) = \mathbb{P}\left((\Pi_1\cdot x, \Pi_1, Q_1)\in A\times B\times C\right). \tag{18.6}$$

The following Corollary again results from Proposition 15.2 and Corollary 16.4:

Corollary 18.2. *For each $x_0 \in \mathbb{S}$ there is $\xi, \delta > 0$, $C \subset GL(d, \mathbb{R})$ compact and a Markov kernel Ψ from $\mathbb{S}$ to $\mathbb{S} \times GL(d, \mathbb{R})$ with*

$$\operatorname{supp} \Psi(x, \cdot) \subseteq B_\delta(x_0) \times C, \quad \Psi(x, \cdot \times C) = \Phi$$

for all $x \in \mathbb{S}$. The Markov chain $(X_n, \mathbf{M}_n, Q_n)_{n \in \mathbb{N}_0}$ satisfies the minorization condition

$$\begin{aligned} &\mathbb{O}_x \left((X_1, \mathbf{M}_1, Q_1) \in A \times B \times D \right) \\ &\geq \xi \int_B \mathbb{P}\left(Q \in D | \mathbf{T} = \mathbf{A}^\top \right) \Psi(x, A \times d\mathbf{A}) =: \xi \Xi(x, A \times B \times D), \end{aligned}$$

for all $x \in \mathbb{S}$. There are $L, \varsigma > 0$ such that

$$\Xi(x, B_\delta(x_0) \times B \times \mathbb{R}^d) = L \int_{B_\varsigma(\mathbf{Id})} \mathbf{1}_{B_\delta(x_0) \times B}(\mathbf{A} \cdot x, \mathbf{A}\mathbf{A}_x) \lambda^{d^2}(d\mathbf{A}) \tag{18.7}$$

for all $x \in \mathbb{S}$ where $\mathbf{A}_x$ is a deterministic matrix in $GL(d, \mathbb{R})$, only depending on x.

Remark 18.3. It will be important in the subsequent considerations that the image measures on the path spaces, $\mathbb{Q}_x((X_n, V_n)_{n \in \mathbb{N}_0} \in \cdot)$ and $\mathbb{O}_x((X_n, \mathbf{T}_n, Q_n)_{n \in \mathbb{N}_0} \in \cdot)$, may also be defined via the Markov transition kernels $\hat{Q}_s$ resp. $\hat{O}$ by means of the Ionescu-Tulcea theorem while the identities (18.3) resp. (18.5) still hold true.

18.2. A Multivariate Regeneration Lemma

The following Lemma extends the classical regeneration lemma [13, Lemma 3.1] to bivariate MCs satisfying bivariate minorization conditions as above. Note the initial distribution via the convention $\mathbb{P}_x(X_0 = x) = 1$.

Lemma 18.4. *Consider a MC $(X_n, Z_n)_{n \in \mathbb{N}_0}$ taking values in a separable metric space $S \times E$. Assume that it satisfies a bivariate minorization condition*

$$\mathbb{P}\left((X_1, Z_1) \in \cdot | X_0 = x, Z_0 = z \right) = P(x, \cdot) \geq \xi \Psi(x, \cdot) \qquad \forall x \in S$$

for Markov transition kernels P, Ψ with the additional property that for some probability measure $\Phi \in \mathcal{P}(S)$,

$$\Psi(x, \cdot \times E) = \Phi \qquad \forall x \in S.$$

Then the following holds: On a possibly enlarged probability space, one can redefine $(X_n, Z_n)_{n \in \mathbb{N}_0}$ together with an increasing sequence $(\sigma_n)_{n \in \mathbb{N}_0}$ of random epochs such that the following conditions are fulfilled under any $\mathbb{P}_{x,z}$, $x \in S \times E$:

(R1) There is a filtration $\mathcal{G} = (\mathcal{G}_n)_{n \geq 0}$ such that $(X_n, Z_n)_{n \in \mathbb{N}_0}$ is Markov adapted and each σ_n is a stopping time with respect to $\mathcal{G}$.

(R2) $(\sigma_n - \sigma_1)_{n \in \mathbb{N}}$ forms a zero-delayed renewal sequence with increment distribution $\mathbb{P}_\Phi(\sigma_1 \in \cdot)$ and is independent of σ_1.

(R3) For each $k \geq 1$, the sequence $(X_{\sigma_k+n})_{n\in\mathbb{N}_0}$ is independent of $(X_j)_{0\leq j\leq\sigma_k-1}$ with distribution $\mathbb{P}_\Phi\left((X_n)_{n\in\mathbb{N}_0}\in\cdot\right)$.

(R4) For each $k \geq 1$, $\mathbb{P}\left(X_{\sigma_k}, Z_{\sigma_k}\in\cdot|\mathcal{G}_{\sigma_k-1}\right) = \Psi(X_{\sigma_{k-1}},\cdot)$ $\mathbb{P}$-a.s..

The σ_n, called *regeneration epochs w.r.t.* Ψ, are obtained by the following coin-tossing procedure: At each step, a ξ-coin is tossed. If head comes up, then (X_{n+1}, Z_{n+1}) is generated according to $\Psi(X_n,\cdot)$, while it is generated according to $(1-\xi)^{-1}(P(X_n,\cdot)-\xi\Psi(X_n,\cdot))$ otherwise. Hence, the $\sigma_n - 1$ are those steps at which the coin toss produces a head. More formally, this is realized by introducing i.i.d. Bernoulli(ξ) (B(1,ξ)) variables $J_0, J_1, \ldots$ with the following properties:

(R5) For each $n \geq 0$, J_n is independent of $\sigma((X_k, Z_k)_{0\leq k\leq n})$.

(R6) $\sigma_0 := 0$ and $\sigma_n := \inf\{k > \sigma_{n-1} : \ J_{k-1} = 1\}$ for $n \geq 1$.

Then $(X_n, Z_n, J_n)_{n\geq 0}$ is defined as a Markov chain on $S \times E \times \{0,1\}$ with transition kernel $\hat{P}$ given by

$$\hat{P}((x,z,0), A\times B\times C) = (1-\xi)^{-1}\Big(P(x, A\times B) - \xi\Psi(x, A\times B)\Big)\mathrm{B}(1,\xi)(C)$$
$$\hat{P}((x,z,1), A\times B\times C) = \Psi(x, A\times B)\mathrm{B}(1,\xi)(C).$$

Denote by $\hat{\mathbb{P}}_{x,z}$ the measure induced by this transition kernel on the path space of (X_n, Z_n, J_n) with $(X_0, Z_0) = (x,z)$ and $\mathcal{L}(J_0) = \mathrm{B}(1,\xi)$. Introduce the canonical filtrations $\mathcal{G}_n$ and $\mathcal{F}_n$ for (X_n, Z_n, J_n) resp. (X_n, Z_n). Then use the “tower rule” ([28, Prop. 4.20 3)]) for conditional expectations to derive that for all $n \in \mathbb{N}$

$$\begin{aligned}
&\hat{\mathbb{P}}_{x,z}(X_n\in A, Z_n\in B|\mathcal{F}_{n-1})\\
=\ &\hat{\mathbb{E}}_{x,z}\left(\mathbf{1}_A(X_n)\mathbf{1}_B(Z_n)|\,\mathcal{F}_{n-1}\right)\\
=\ &\hat{\mathbb{E}}_{x,z}\left(\mathbb{E}\left(\mathbf{1}_A(X_n)\mathbf{1}_B(Z_n)|\mathcal{G}_{n-1}\right)|\,\mathcal{F}_{n-1}\right)\\
=\ &\hat{\mathbb{E}}_{x,z}\left(\hat{P}((X_{n-1}, Z_{n-1}, J_{n-1}), A\times B\times\{0,1\})\Big|\,\mathcal{F}_{n-1}\right)\\
=\ &\hat{\mathbb{E}}_{x,z}\Bigg(\ J_{n-1}\Psi(X_{n-1}, A\times B)\\
&+\ (1-J_{n-1})(1-\xi)^{-1}\Big(P(X_{n-1}, A\times B) - \xi\Psi(X_{n-1}, A\times B)\Big)\Big|\,\mathcal{F}_{n-1}\Bigg)\\
=\ &\xi J_{n-1}\Psi(X_{n-1}, A\times B) + (1-\xi)(1-\xi)^{-1}\Big(P(X_{n-1}, A\times B) - \xi\Psi(X_{n-1}, A\times B)\Big)\\
=\ &P(X_{n-1}, A\times B) \qquad \text{a.s..}
\end{aligned}$$

In the penultimate line, (R5) was used. It follows that the marginal sequence (X_n, Z_n) is a Markov chain with transition kernel P also on this enlarged space.

Remark 18.5.
- Any sequence $(\sigma_n)_{n\in\mathbb{N}_0}$ satisfying (R1)-(R4) with some kernel Ψ is called sequence of regeneration epochs for $(X_n, Z_n)_{n\in\mathbb{N}_0}$ w.r.t. Ψ.

- The classical regeneration lemma for a strongly aperiodic Harris recurrent MC gives assertions (R1)-(R3), (R1) of course just for $(X_n)_{n\in\mathbb{N}_0}$ instead of the bivariate chain. In this context, any sequence $(\sigma_n)_{n\in\mathbb{N}_0}$ satisfying (R1)-(R3) is called sequence of regeneration epochs for $(X_n)_{n\in\mathbb{N}_0}$.
- The (up to scalar multiplication) unique invariant measure of a strongly aperiodic Harris recurrent chain is finite if and only if $\mathbb{E}_\Phi\sigma_1 < \infty$ (see [13, Theorem 6.1]). Obviously, in the case of a strongly aperiodic Doeblin chain, σ_1 has a geometric(ξ)-distribution, thus σ_1 has finite expectation and $(X_n)_{n\in\mathbb{N}_0}$ has a unique stationary distribution.

18.3. Future Stopping Times

In view of Corollaries 18.1 and 18.2 and Remark 18.3, the measures induced by $\mathbb{Q}_x^s$ and $\mathbb{O}_x$ on the path spaces of $(X_n, V_n)_{n\in\mathbb{N}_0}$ resp. $(X_n, \mathbf{T}_n, Q_n)_{n\in\mathbb{N}_0}$ can be redefined to carry regeneration sequences, while the identities (18.3) and (18.5) still hold. Subsequently, the following stopping times will appear (for arbitrary but fixed $x_0 \in \mathbb{S}$):

- When considering measures $\mathbb{Q}_x^s$, the sequence $(\sigma_n)_{n\in\mathbb{N}_0}$ will always be a sequence of regeneration epochs w.r.t. to the bivariate minorization Υ, given by Corollary 18.1. In particular,

$$\mathbb{Q}_x\left((X_{\sigma_n}, U_{\sigma_n}) \in B_\delta(x_0) \times I\right) = 1$$

  for some $\delta > 0$, a compact interval $I \subset \mathbb{R}$ and all $x \in \mathbb{S}$, $n \in \mathbb{N}$.
- The subsequent hitting times of $(X_n)_{n\in\mathbb{N}_0}$ in $B_\delta(x_0)$ will be denoted by $(\tau_n)_{n\in\mathbb{N}}$, with the convention $\tau_0 = 0$.
- When considering measures $\mathbb{O}_x$, the sequence ϖ_n will be a sequence of regeneration epochs w.r.t. to the multivariate minorization Ξ given by Corollary 18.2 with the additional property that $X_{\varpi_{n-1}} \in B_\delta(x_0)$ for all $n \in \mathbb{N}$, with the $\delta > 0$ also given by Corollary 18.2. It will be introduced in Subsection 21.2. The particular formula (18.7) will be used.

19. The Markov Renewal Theorem for Strongly Aperiodic Doeblin Chains

In this section, first the MRT for MRWs with Harris recurrent driving chain [2, Theorem 1] will be formulated in the setting of strongly aperiodic Doeblin chains. Its convergence result a priori only holds for π-almost all $x \in S$, with π being the stationary distribution of the driving chain. Using a bivariate minorization property, it will be shown that the convergence assertion is valid for all $x \in S$ under an extra assumption on the random walk part. In this section, S denotes a separable metric space.

19.1. The Markov Renewal Theorem

Let $(X_n, V_n)_{n\in\mathbb{N}_0}$ be a MRW with strongly aperiodic Doeblin driving chain and stationary distribution π. The MRW $(X_n, V_n)_{n\in\mathbb{N}_0}$ is called *d-arithmetic*, if there exists a minimal $d > 0$ and a measurable function $f : S \to [0, d)$ such that

$$\mathbb{P}\left(U_1 - f(x) + f(y) \in d\mathbb{Z} | X_0 = x, X_1 = y\right) = 1$$

for $P_\pi((X_0, X_1) \in \cdot)$ almost all $(x, y) \in S^2$, and *nonarithmetic* otherwise.

Definition 19.1. A measurable function $g : S \times \mathbb{R} \to \mathbb{R}$ is called *π-directly Riemann integrable* if

$$g(x, \cdot) \text{ is Lebesgue-a.e. continuous for } \pi\text{-almost all } x \in S, \text{ and} \tag{19.1}$$

$$\int_S \sum_{n \in \mathbb{Z}} \sup_{t \in [n\delta, (n+1)\delta)} |g(x, t)| \, \pi(dx) < \infty \text{ for some } \delta > 0. \tag{19.2}$$

Defining the *first exit time* $N(t) := \inf\{n \geq 0 : V_n > t\}$ consider the *residual lifetime process* $R(t) := (V_{N(t)} - t)\mathbf{1}_{\{N(t)<\infty\}}$ and the *jump process* $Z(t) := X_{N(t)}\mathbf{1}_{\{N(t)<\infty\}}$. The following MRT is the main result of [2]:

Theorem 19.2. *Let $(X_n, V_n)_{n \in \mathbb{N}}$ be a nonarithmetic MRW with strongly aperiodic Doeblin driving chain $(X_n)_{n \in \mathbb{N}}$ with stationary distribution π. Let $l := \mathbb{E}_\pi V_1 > 0$. Then for every function g which is π-directly Riemann integrable,*

$$\lim_{t \to \infty} g * \mathbb{U}_x(t) = \lim_{t \to \infty} \mathbb{E}_x \left(\sum_{n \geq 0} g(X_n, t - V_n) \right) \to \frac{1}{l} \int_S \int_{\mathbb{R}} g(u, v) \, dv \, \pi(du). \tag{19.3}$$

for π-almost all $x \in S$. Moreover, if $f : S \times (0, \infty) \to (0, \infty)$ is bounded and continuous, then

$$\lim_{t \to \infty} \mathbb{E}_x \left(f(Z(t), R(t))\mathbf{1}_{\{N(t)<\infty\}} \right) = L(f) \tag{19.4}$$

for π-almost all $x \in S$ and some constant $L(f) > 0$.

Remark 19.3. The following extension of the above result follows directly upon inspection of the coupling proof given in [2, Section 7]: If Φ is any minorizing distribution for the transition kernel of the Harris driving chain $(X_n)_{n \geq 0}$, then $g * \mathbb{U}_\Phi(t)$ is a bounded function and converges to the limit given in (19.3). This fact will be used below.

19.2. The Convergence Holds Everywhere

In this subsection, it is shown that under some stronger assumptions, the convergence in 19.3 holds for all $x \in S$ instead of just for π-almost all.

Therefore, yet another definition of direct Riemann integrability is needed. It interpolates between the Definitions 8.4 and 19.1:

Definition 19.4. A function $g \in \mathcal{C}_b(\mathbb{S} \times \mathbb{R})$ is called *weakly directly Riemann integrable (wdRi)*, if

$$\sup_{u \in \mathbb{S}} \sum_{l \in \mathbb{Z}} \sup_{\in [l, l+1]} |g(u, t)| < \infty. \tag{19.5}$$

Obviously, this property is stronger than (19.2), but weaker than (8.12). Now for the announced bivariate minorization condition:

Definition 19.5. Let $(X_n, V_n)_{n\in\mathbb{N}_0}$ be a MRW with strongly aperiodic Doeblin driving chain. Say that the MRW has *bounded increments at regeneration epochs*, if there is $\xi > 0$, a finite interval $I \subset \mathbb{R}$ and a stochastic kernel Υ with

$$\operatorname{supp}(\Upsilon(x,\cdot)) \subset S \times I, \quad \Upsilon(x, \cdot \times I) = \Phi \tag{19.6}$$

for all $x \in S$ and some $\Phi \in \mathcal{P}(S)$, such that

$$\mathbb{P}_x(X_1, U_1 \in \cdot) \geq \xi\Upsilon(x,\cdot). \tag{19.7}$$

In other words, $(X_n, V_n)_{n\in\mathbb{N}_0}$ having bounded increments at regeneration epochs means

$$U_{\sigma_n} \in I \tag{19.8}$$

for any sequence $(\sigma_n)_{n\in\mathbb{N}_0}$ of regeneration epochs w.r.t. to Ψ and all $n \in \mathbb{N}$.

Proposition 19.6. *In the situation of Theorem 19.2, if the MRW has bounded increments at regeneration epochs and g is wdRi, then the convergence in* (19.3) *holds for all $x \in S$.*

Remark 19.7. The question, whether the convergence in (19.4) holds for all $x \in S$ as well is more delicate: It is derived by applying (19.3) to the MRW $(\ddot{X}_n, \ddot{V}_n)_{\in}\mathbb{N}_0) := (X_{\upsilon_n}, V_{\upsilon_n})_{n\in\mathbb{N}_0}$, where $\upsilon_0 = 0$ and υ_n, $n \in \mathbb{N}$ are the strictly ascending ladder epochs for the random walk part V_n and to the function

$$g(x,t) := \mathbb{E}_x\left(f(\ddot{X}_1, \ddot{V}_1 - t)\mathbf{1}_{\{\ddot{V}_1 > t\}}\right).$$

Properties of $(\ddot{X}_n, \ddot{V}_n)$ are studied in [3]; but it cannot be deduced from those results that $(\ddot{X}_n)_{n\in\mathbb{N}}$ satisfies the Doeblin condition, or at least, that there is a sequence of regeneration epochs $(\sigma_n)_{n\in\mathbb{N}}$ with $\sup_{x\in\mathbb{S}} \mathbb{E}_x\sigma_1 < \infty$ - this property is needed in Lemma 19.8. As a step in proving the main theorem, (19.4) will be applied, thus some of the interim results will hold only for π-almost all $x \in \mathbb{S}$. Nevertheless, the main result uses (19.3) whence it holds for all $x \in \mathbb{S}$.

Now for the proof of Proposition 19.6. Denote the RHS of (19.3) by K. Let Υ be a minorizing kernel for (X_n, U_n) and $I \subset \mathbb{R}$ a finite interval such that (19.6) holds. Let $(\sigma_n)_{n\geq 1}$ be an associated sequence of regeneration epochs and put $\sigma := \sigma_1$. The task is to show that $g * \mathbb{U}_x(t)$ converges to K for all $x \in S$. Begin by pointing out that

$$g * \mathbb{U}_x(t) = \mathbb{E}_x\left(\sum_{k\geq 0} g(X_k, t - V_k)\right) = G(x,t) + g * \mathbb{U}_{\varphi(x,\cdot)}(t) \tag{19.9}$$

where $\varphi(x,\cdot) := \mathbb{P}_x((X_\sigma, V_\sigma) \in \cdot)$ and

$$G(x,t) := \mathbb{E}_x\left(\sum_{k=0}^{\sigma-1} g(X_k, t - V_k)\right), \quad (x,t) \in S \times \mathbb{R}. \tag{19.10}$$

As for this last function, the following lemma holds:

Lemma 19.8. *The function G is bounded and satisfies $\lim_{t\to\infty} G(x,t) = 0$ for all $x \in S$.*

Proof. By (19.5), $C := \sup\{|g(x,t)| : x \in S,\ t \in \mathbb{R}\} < \infty$, and since $(X_n)_{n\geq 0}$ is a strongly aperiodic Doeblin chain, it follows

$$\sup_{x\in S, t\in\mathbb{R}} |G(x,t)| \leq C \sup_{x\in S} \mathbb{E}_x \sigma < \infty.$$

Just recall that a geometric number of coin tosses (the J_n) determines σ. Turning to the convergence assertion, observe that, again by property (19.5), $\lim_{t\to\infty} g(x,t) = 0$ for all $x \in S$, which implies the desired result by an appeal to the dominated convergence theorem. □

In view of (19.9), it remains to show that $g * \mathbb{U}_{\varphi(x,\cdot)}(t) \to K$. This requires one more lemma.

Lemma 19.9. *For each $x \in S$, the sequence $(X_\sigma, (X_n, U_n)_{n>\sigma})$ is independent of $(X_{\sigma-1}, V_{\sigma-1})$ under $\mathbb{P}_x$ with distribution given by $\mathbb{P}_\Phi\left((X_0, (X_n, U_n)_{n\geq 1}) \in \cdot\right)$.*

Proof. The first assertion follows directly when observing that, by regeneration, $(X_{\sigma+n})_{n\geq 0}$ and $(X_{\sigma-1}, V_{\sigma-1})$ are independent under $\mathbb{P}_x$, and the fact that the conditional distribution of U_k given $(X_n)_{n\geq 0}$ only depends on (X_k, X_{k-1}) (see Definition 6.1). The proof is completed by the observation that $\mathbb{P}_x\left((X_{\sigma+n})_{n\geq 0} \in \cdot\right) = \mathbb{P}_\Phi\left((X_n)_{n\geq 0}) \in \cdot\right)$. □

Define $V_{\sigma,n} := V_{\sigma+n} - V_\sigma$ for $n \geq 0$ and then

$$h(x,s,t) := \mathbb{E}_x\left(\sum_{k\geq 0} g(X_{\sigma+k}, t - s - V_{\sigma-1} - V_{\sigma,k})\right)$$

for $s, t \in \mathbb{R}$. Lemma 19.9 implies

$$h(x,s,t) = \int_{\mathbb{R}} g * \mathbb{U}_\Phi(t-s-r)\, {}^\kappa\mathbb{P}_x(V_{\sigma-1} \in dr).$$

As g satisfies (19.5), it follows from the MRT 19.2 and the subsequent remark that $g * \mathbb{U}_\Phi(t)$ is bounded and converges to K. By the dominated convergence theorem, the same limit holds for $\lim_{t\to\infty} h(x,s,t)$ for all s.

Finally, the connection between $h(x,s,t)$ and $g * \mathbb{U}_{\varphi(x,\cdot)}(t)$ becomes apparent after the following observations: By assumption, U_σ is taking its values in the finite interval I. Hence $g * \mathbb{U}_{\varphi(x,\cdot)}(t)$ can be estimated by

$$\inf_{s\in I} h(x,s,t) \leq g * \mathbb{U}_{\varphi(x,\cdot)}(t) \leq \sup_{s\in I} h(x,s,t).$$

Hence the sandwich theorem yields the desired conclusion that $\lim_{t\to\infty} g * \mathbb{U}_{\varphi(x,\cdot)}(t) = K$.

By the way, the subsequent Corollary has been proved:

Corollary 19.10. *If $(X_n, V_n)_{n\geq 0}$ is a MRW with strongly aperiodic Doeblin driving chain and bounded increments at regeneration epochs, then for every wdRi function g,*

$$\sup_{x\in\mathbb{S},t\in\mathbb{R}} |g| * \mathbb{U}_x(t) < \infty.$$

20. Study of a Markov Random Walk

In this section, properties of the MRW $(X_n, V_n)_{n\in\mathbb{N}_0}$ under the measures $\mathbb{Q}_x^s$ are studied in order to prove that the MRT 19.2 as well as its extension in Proposition 19.6 are applicable.

Lemma 20.1. *Under each $\mathbb{Q}_x^s$, the Markov chain $(X_n, U_{n+1})_{n\in\mathbb{N}_0}$ has a unique stationary distribution $\varphi = \mathbb{P}_{\pi_*^s}((X_0, U_1 \in \cdot)$.*

Proof. By the very definition of a MRW (Definition 6.1), the $(U_n)_{n\in\mathbb{N}}$ are conditionally independent, given $(X_n)_{n\in\mathbb{N}_0}$ and the distribution of U_{n+1} depends only on (X_n, X_{n+1}). Hence any stationary distribution π for the driving chain $(X_n)_{n\in\mathbb{N}_0}$ has a unique extension to a stationary distribution φ for $(X_n, U_{n+1})_{n\in\mathbb{N}_0}$, given by $\mathbb{P}_\pi((X_0, U_1) \in \cdot)$. Conversely, any stationary distribution of $(X_n, U_{n+1})_{n\in\mathbb{N}_0}$ reduces to a stationary distribution for $(X_n)_{n\in\mathbb{N}_0}$. Referring to Lemma 17.4, the driving chain has the unique stationary distribution π_*^s thus φ is the unique stationary distribution for $(X_n, U_{n+1})_{n\in\mathbb{N}_0}$. □

The following proposition corresponds to Theorem 8.2.

Proposition 20.2. *Let μ satisfy (i-d) and*

$$\mathbb{E}\,\|\mathbf{T}\|^s \left(|\log\|\mathbf{T}\|| + \left|\log\left\|\mathbf{T}^{-1}\right\|\right|\right) < \infty. \tag{20.1}$$

Then for all $x \in \mathbb{S}$,

$$\lim_{n\to\infty} \frac{V_n}{n} = \mathbb{E}^s_{\pi_*^s} V_1 \qquad \mathbb{Q}_x^s\text{-a.s.}. \tag{20.2}$$

Proof. The moment assumption 20.1 assures that

$$\begin{aligned}
\mathbb{E}^s_{\pi_*^s}|V_1| &= \int_{\mathbb{S}} \frac{1}{e_*^s(x)\kappa(s)} \mathbb{E}\left(|\mathbf{M}_1 x|^s\, e_*^s(\mathbf{M}_1\cdot x)\,|\log|\mathbf{M}_1 x||\right) \pi_*^s(dx) \\
&\leq \sup_{z_1,z_2\in\mathbb{S}} \frac{e_*^s(z_1)}{e_*^s(z_2)} \frac{1}{\kappa(s)} \mathbb{E}\,\|\mathbf{M}_1\|^s \left(|\log\|\mathbf{M}_1\|| + |\log\iota(\mathbf{M}_1)|\right) \\
&= \xi_1 \frac{1}{\kappa(s)} \mathbb{E}\,\|\mathbf{T}\|^s \left(|\log\|\mathbf{T}\|| + \left|\log\|(\|\,\mathbf{T}^{-1}\right|\right) < \infty,
\end{aligned}$$

i.e. $V_1 \in L^1(\mathbb{Q}^s_{\pi_*^s})$. Here (7.3) should be recalled.

Since φ is the unique stationary distribution for $(X_n, U_{n+1})_{n\in\mathbb{N}_0}$, the chain is indecomposable (see [28, Definition 7.13]). Referring to [28, Theorem 7.16], it is ergodic under $\mathbb{Q}^s_{\pi_*^s}$. Hence by Birkhoff's ergodic theorem [28, Theorem 6.28],

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}U_{k+1}=\lim_{n\to\infty}\frac{V_n}{n}=\mathbb{E}^s_{\pi^s_*}U_1=\mathbb{E}^s_{\pi^s_*}V_1 \qquad \mathbb{Q}^s_{\pi^s_*}\text{-a.s.}$$

Since $(X_n)_{n\in\mathbb{N}_0}$ is a strongly aperiodic Doeblin chain by Corollary 18.1, a coupling argument which is in spirit similar to the arguments given in the proof of Proposition 19.6, this convergence holds under each $\mathbb{Q}^s_x$ due to the Doeblin property of $(X_n)_{n\in\mathbb{N}_0}$.

□

The identification of the limit in (20.2) as $\frac{\kappa'(s^-)}{\kappa(s)}$ can be copied word by word from the proof of [29, Theorem 3.7]. In the present situation, the estimate

$$\mathbb{E}^\beta_{\pi^\beta_*}V_1>0$$

is sufficient. Since it can be obtained by a somewhat shorter argument, its proof will be given:

Lemma 20.3. *Let μ satisfy (i-d), let there be $\beta>0$ with $\kappa(\beta)=1$ and let*

$$\mathbb{E}\left\|\mathbf{T}\right\|^\beta\left(\left|\log\left\|\mathbf{T}\right\|\right|+\left|\log\left\|\mathbf{T}^{-1}\right\|\right|\right)<\infty. \tag{20.3}$$

Then

$$l(\beta)=\mathbb{E}^\beta_{\pi^\beta_*}V_1>0.$$

Proof. For $n\in\mathbb{N}$, consider the function

$$g_n:s\mapsto\int_{\mathbb{S}}\frac{1}{e^\beta_*(x)}\mathbb{E}\left|\Pi_n x\right|^s e^\beta_*(\Pi_n\cdot x)\pi^\beta_*(dx).$$

It is finite and convex on I_μ with $g_n(\beta)=1$. Under (20.3), $[0,\beta]\subset I_\mu$ and at least the left derivative in β exists, which can be expressed in two different ways. On the one hand, for all $s\in(0,\beta]$,

$$g_n'(s^-)=\int_{\mathbb{S}}\frac{1}{e^\beta_*(x)}\mathbb{E}\log\left|\Pi_n x\right|\left|\Pi_n x\right|^s e^\beta_*(\Pi_n\cdot x)\pi^\beta_*(dx),$$

in particular

$$g_n'(\beta^-)=\int_{\mathbb{S}}\mathbb{E}^\beta_x V_n\pi^\beta_*(dx)=\mathbb{E}^\beta_{\pi^\beta_*}V_n=n\,l(\beta)$$

by stationarity.

On the other hand,

$$g_n'(\beta^-)=\lim_{s\uparrow\beta}\frac{g_n(\beta)-g_n(s)}{\beta-s}.$$

Hence by convexity of g_n, $l(\beta)$ is positive as soon as there is $n\in\mathbb{N}$, $s\in(0,\beta)$ with $g_n(s)<1$.

But this results from the upper bound

$$g_n(s) = \int_{\mathbb{S}} \frac{1}{e_*^\beta}[(P_*^s)^n e_*^\beta](x)\pi_*^\beta(dx) \le \left(\max_{z\in\mathbb{S}} e_*^\beta(z)^{-1}\right) \|(P_*^s)^n\|\,,$$

valid for all $n \in \mathbb{N}$, $s \in I_\mu$. Since for $s \in (0,\beta)$, $r(P_*^s) = m(s) < 1$, the RHS tends to zero as n goes to infinity. Consequently, there is $n \in \mathbb{N}$, $s \in (0,\beta)$ with $g_n(s) < 1$. $\square$

One assumption of the MRT 19.2 remains to be checked: the lattice-type condition. As one would expect, it is mainly a consequence of (density).

Lemma 20.4. *Suppose that* (density) *and* (StA) *hold. Then for all $s \in I_\mu$, $(X_n, V_n)_{n\ge 0}$ is nonarithmetic under $(\mathbb{Q}_x^s)_{x\in\mathbb{S}}$, in fact*

$$\mathbb{E}_x^s \left|\mathbb{E}\left(e^{itV_1}|X_0, X_1\right)\right| < 1$$

for all $t \neq 0$ and π_*^s-almost all $x \in \mathbb{S}$.

Proof. Fix $s \in I_\mu$. If the assertion fails to hold, there exists a distribution ν on $\mathbb{S}$, absolutely continuous with respect to π_*^s, such that $\mathbb{E}_\nu^s \left|\mathbb{E}\left(e^{itV_1}|X_0, X_1\right)\right| = 1$ for some $t \neq 0$. In other words,

$$\mathbb{E}\left(e^{itV_1}|X_0, X_1\right) = e^{itf(X_0,X_1)} \ \mathbb{Q}_x^s\text{-a.s.}$$

for some measurable function f and ν-almost all $x \in \mathbb{S}$ or, equivalently,

$$\mathbb{Q}_\nu^s\left(V_1 \in f(X_0, X_1) + t^{-1}\mathbb{Z}\right) = 1. \tag{20.4}$$

W.l.o.g. suppose $t = 1$ hereafter. Due to (irred) and (StA) and referring to (18.3) & (18.4), a nonzero component of $\mathbb{Q}_x^s((X_1, V_1) \in \cdot)$ is given by

$$\Lambda_x(A \times B) := \xi_3 \int_{B_c(\Gamma_0)} \mathbf{1}_A(\mathbf{M}\cdot x)\mathbf{1}_B(\log|\mathbf{M}x|)\lambda^{d^2}(d\mathbf{M})$$

for measurable $A \subset S$, $B \subset \mathbb{R}$ and any $x \in \mathbb{S}$. Here

$$\xi_3 = p \min_{z_1,z_2} \frac{e_*^s(z_1)}{e_*^s(z_2)} \inf_{z\in\mathbb{S},\mathbf{M}\in B_c(\Gamma_0)} > 0.$$

The mapping $\mathbf{M} \mapsto \mathbf{M}x$ induces an absolutely continuous measure on $\mathbb{R}^d$ with some λ^d-density g, say. Switching to spherical coordinates, there are $\varepsilon_1, \varepsilon_2 > 0$ such that

$$\Lambda_x(A \times B) = \xi_3 \int_{|\Gamma_0 x|-\varepsilon_1}^{|\Gamma_0 x|+\varepsilon_1} \int_{B_{\varepsilon_2}(\Gamma_0\cdot x)\cap\mathbb{S}} \mathbf{1}_A(\omega)\mathbf{1}_B(s)g(s\omega)\sigma(d\omega)\frac{1}{s^{1+d}}\,ds$$

where σ is a measure on $\mathbb{S}$. Now, if (20.4) were true with $t = 1$, then

$$\Lambda_x(S \times \mathbb{R}) \ = \ \xi_3 \int_{B_{\varepsilon_2}(\Gamma_0\cdot x)\cap\mathbb{S}} \left(\int_{|\Gamma_0 x|-\varepsilon_1}^{|\Gamma_0 x|+\varepsilon_1} \mathbf{1}_{f(x,\omega)+\mathbb{Z}}(s)g(s\omega)\frac{1}{s^{1+d}}\,ds\right)\sigma(d\omega) \ > \ 0$$

for ν-almost all x which is impossible because the inner integral over a countable set is clearly zero for any fixed ω. □

Considering finally the assumptions of Proposition 19.6, these hold by Corollary 18.1, the minorizing kernel satisfies the definition of bounded increments, and the bivariate minorization condition contains the strongly aperiodic Doeblin chain property as a special case.

21. Renewal Theory for Products of Random Matrices

It has been shown in the previous Section that the MRW $(X_n, V_n)_{n\in\mathbb{N}_0}$ satisfies the assumptions of the MRT19.2 w.r.t. the measures $(\mathbb{Q}_x^\beta)_{x\in\mathbb{S}}$. Following Kesten [59, p.233f], the tail behaviour of $\sup_{n\geq 1} |\Pi_n x|$ as well as of $\sup_{n\geq 1} |\Pi_{\varpi_n - 1} x|$ (see Subsection 18.3) will be deduced.

21.1. Tail Behaviour of $\sup_{n\geq 1} |\Pi_n x|$

Proposition 21.1. *Let μ satisfy (i-d), assume there is $\beta > 0$ with $m(\beta) = 1$, $m'(\beta) > 0$ and let* (TlogT) *hold. Then*

$$\lim_{t\to\infty} t^\beta\, \mathbb{P}\left(\sup_{n\geq 1} |\Pi_n x| > t\right) = L\, e_*^\beta(x),$$

for π_*^s-almost all $x \in \mathbb{S}$ and some $L > 0$.

Proof. The function $f : \mathbb{S} \times \mathbb{R}_> \to \mathbb{R}_>,\ (y, s) \mapsto e^{-\beta s}/e_*^\beta(y)$ is bounded and continuous whence, by an application of the MRT 19.2,

$$L(f) := \lim_{t\to\infty} \mathbb{E}_x^\beta \left(f(Z(t), R(t))\mathbf{1}_{\{N(t)<\infty\}}\right)$$

exists for π_*^β-almost all $x \in \mathbb{S}$, is independent of x and positive. Now using (18.3),

$$\begin{aligned}
\mathbb{E}_x^\beta \left(f(Z(t), R(t))\mathbf{1}_{\{N(t)<\infty\}}\right) &= \sum_{n\geq 1} \mathbb{E}_x^\beta \left(f(X_n, V_n - t)\, \mathbf{1}_{\{N(t)=n\}}\right) \\
&= \sum_{n\geq 1} \mathbb{E}_x^\beta \left(\frac{1}{e_*^\beta(X_n)}\, e^{-\beta V_n + \beta t}\, \mathbf{1}_{\{N(t)=n\}}\right) \\
&= \frac{e^{\beta t}}{e_*^\beta(x)} \sum_{n\geq 1} \mathbb{E}_x \left(\frac{1}{e_*^\beta(X_n)}\, e^{-\beta V_n}\, e_*^\beta(X_n)\, e^{\beta V_n}\, \mathbf{1}_{\{N(t)=n\}}\right) \\
&= \frac{e^{\beta t}}{e_*^\beta(x)}\, \mathbb{Q}_x(N(t) < \infty) \\
&= \frac{e^{\beta t}}{e_*^\beta(x)}\, \mathbb{P}\left(\sup_{n\geq 1}\, \log |\Pi_n x| > \log e^t\right), \qquad (21.1)
\end{aligned}$$

which provides the asserted result upon substituting e^t by t. □

21.2. Tail Behaviour of $\sup_{n\geq 1}|x\Pi_{\sigma_n-1}|$

For any fixed $x_0 \in \mathbb{S}$, Corollary 18.2 yields the existence of $\delta > 0$ and a Markov kernel Ξ whose first marginal is a probability measure Φ which is supported on $B_\delta(x_0)$. Consider a sequence $(\varpi_n)_{n\in\mathbb{N}_0}$ of regeneration epochs under $\mathbb{O}_x$ w.r.t. to Ψ which satisfies the additional property that

$$X_{\varpi_n - 1} \in B_\delta(x_0)$$

for all $n \in \mathbb{N}$. Call such a sequence of regeneration epochs *feasible*. Writing $(\tau_n)_{n\in\mathbb{N}}$ for the subsequent hitting times of X_n in $B_\delta(x_0)$, it follows that $(\varpi_n)_{n\in\mathbb{N}}$ as well as $(\varpi_n - 1)_{n\in\mathbb{N}}$ are subsequences of $(\tau_n)_{n\in\mathbb{N}_0}$ (with the convention $\tau_0 = 0$.) In Section 23, it will needed and is therefore shown below that

$$\lim_{t\to\infty} t^\kappa\, \mathbb{O}_x\left(\sup_{n\geq 1}|\Pi_{\varpi_n-1}x| > t\right) > 0$$

for π_*^β-almost all $x \in B_\delta(x_0)$. The proof hinges on the following proposition similar to Proposition 21.1 above.

Proposition 21.2. *Let μ satisfy (i-d), assume there is $\beta > 0$ with $m(\beta) = 1$, $m'(\beta) > 0$ and let* (TlogT) *hold. Then there exists $L' > 0$ such that*

$$\lim_{t\to\infty} t^\kappa\, \mathbb{O}_x\left(\sup_{n\geq 1}|\Pi_{\tau_n}x| > t\right) = L'\, e_*^\beta(x),$$

for π_*^β-almost all $x \in B_\delta(x_0)$.

Proof. For $x \in B_\delta(x_0)$, write $\tau_n(x)$ for the subsequent hitting times of $\Pi_n \cdot x$ in $B_\delta(x_0)$, and $\tau_0(x) = 0$. Referring to (18.5), it has to be shown that

$$\lim_{t\to\infty} t^\kappa\, \mathbb{P}\left(\sup_{n\geq 1}\left|\Pi_{\tau_n(x)}x\right| > t\right) = L'\, e_*^\beta(x),$$

for π_*^β-almost all $x \in B_\delta(x_0)$. Since $V_{\tau_n} = \log\left|\Pi_{\tau_n(x)}x\right|$ a.s. under $\mathbb{Q}_x$ and $\mathbb{Q}_x^\beta$, one can proceed exactly as in the proof of Proposition 21.1, provided that the assumptions of the MRT 19.2 hold for the sequence $(X_{\tau_n}, V_{\tau_n})_{n\geq 0}$ under $(\mathbb{Q}_x^\beta)_{x\in B_\delta(x_0)}$, which is verified by the subsequent lemma. Note that (18.3) extends to

$$\begin{aligned}
&\mathbb{E}_x^\beta f(X_0, V_0, \ldots, X_{\tau_n}, V_{\tau_n}) \\
&\quad = \frac{1}{e_*^\beta(x)}\mathbb{E}\Big(e_*^\beta(\Pi_{\tau_n(x)}\cdot x)e^{\beta\log\left|\Pi_{\tau_n(x)}x\right|} f\left(x, 0, \ldots, \Pi_{\tau_n(x)}\cdot x, \log\left|\Pi_{\tau_n(x)}x\right|\right)\Big), \quad (21.2)
\end{aligned}$$

as one can easily see by applying (18.3) to $\mathbb{E}_x^\beta f(X_0, V_0, X_1, V_1, \ldots, X_k, V_k)\mathbf{1}_{\{\tau_n = k\}}$ for each $k \in \mathbb{N}$, which in turn is possible because the appearing indicator is a function of $(X_0, V_0, \ldots, X_k, V_k)$. $\square$

Lemma 21.3. *The hit chain $(X_{\tau_n})_{n\in\mathbb{N}_0}$ constitutes a strongly aperiodic Doeblin chain under each $\mathbb{Q}_x^\beta$, $x \in B_\delta(x_0)$ with stationary distribution $\pi = \pi_*^\beta(\cdot \cap B_\delta(x_0))/\pi_*^\beta(B_\delta(x_0))$. Moreover, the sequence $(X_{\tau_n}, V_{\tau_n})_{n\geq 0}$ is a nonarithmetic MRW under $(\mathbb{Q}_x^\beta)_{x\in\mathbb{S}}$ with positive drift.*

Proof. For the first statement, just note that $\{\sigma_n : n \geq 1\} \subset \{\tau_n : n \geq 1\}$ for any sequence of regeneration epochs w.r.t. Ψ. Next, due to Lemma 20.4 and the conditional independence of $U_1, U_2, \ldots$ given $(X_n)_{n\geq 0}$, it follows that for $t \neq 0$

$$\begin{aligned}\mathbb{E}_x^\beta \left|\mathbb{E}\left(e^{itV_{\tau_1}}|X_0, X_{\tau_1}\right)\right| \;&\leq\; \mathbb{E}_x^\beta \prod_{k=1}^{\tau_1} \left|\mathbb{E}\left(e^{itU_k}|X_{k-1}, X_k\right)\right| \\ &\leq\; \mathbb{E}_x^\beta \left|\mathbb{E}\left(e^{itV_1}|X_0, X_1\right)\right| \;<\; 1\end{aligned}$$

for π_*^β-almost all and thus π-almost all x. Consequently, $(X_{\tau_n}, V_{\tau_n})_{n\geq 0}$ is nonarithmetic under $(\mathbb{Q}_x^\beta)_{x\in\mathbb{S}}$. Finally, considering the drift,

$$\mathbb{E}_\pi^\beta V'_{\tau_1} \;=\; \lim_{n\to\infty} \frac{V_{\tau_n}}{n} \;\geq\; \lim_{n\to\infty} \frac{V_{\tau_n}}{\tau_n} \cdot \liminf_{n\to\infty} \frac{\tau_n}{n} \;\geq\; l(\beta)\cdot 1 \;>\; 0 \quad \mathbb{Q}_\pi^\beta\text{-a.s.}, \tag{21.3}$$

where the convergence of V_{τ_n}/n can be shown as in Proposition 20.2. □

Proposition 21.4. *Let $x_0 \in \mathbb{S}$ be fixed, let the assumptions of Proposition 21.2 be in force and $(\varpi_n)_{n\in\mathbb{N}_0}$ a sequence of feasible regeneration epochs. Then*

$$\lim_{t\to\infty} t^\beta\, \mathbb{O}_x\left(\sup_{n\geq 1} |\Pi_{\varpi_n - 1} x| > t\right) = L'' e_*^\beta(x) \tag{21.4}$$

for π_*^β-almost all $x \in B_\delta(x_0)$ and some $L'' > 0$.

Proof. Referring to Proposition 21.2,

$$\lim_{t\to\infty} t^\beta\, \mathbb{O}_x\left(\sup_{n\geq 1} \left|\Pi_{\tau_n(x)} x\right| > t\right) = L'\, e_*^\beta(x) > 0$$

for some $L' > 0$ and π_*^β-almost all $x \in \mathbb{S}$.
Fix any such x hereafter and put $\widehat{N}(t) := \inf\{n \geq 1 : |\Pi_{\tau_n} x| > t\}$, thus

$$\{\sup_{n\geq 1} |\Pi_{\tau_n} x| > t\} = \{\widehat{N}(t) < \infty\}.$$

Then it holds that

$$\begin{aligned}\mathbb{O}_x\left(\sup_{n\geq 1} |\Pi_{\varpi_n - 1} x| > t\right) &= \sum_{n\geq 0} \mathbb{O}_x\left(\hat{N}(t) = n, J_{\tau_n} = 1\right) \\ &\overset{*}{=} \xi \sum_{n\geq 0} \mathbb{O}_x\left(\hat{N}(t) = n\right) \\ &= \xi \mathbb{O}_x\left(\sup_{n\geq 1} |\Pi_{\tau_n} x| > t\right)\end{aligned}$$

where ξ comes from the minorization condition, and $*$ holds by property (R5). □

22. Implicit Markov Renewal Theory

In this section, the convergence assertion in the main result Theorem 13.2 will be proved, all assumptions of which will therefore be in force throughout, in fact in strengthened form given by (StA). Positivity of K is postponed to the next section.

Embarking on ideas by Goldie [47] and Le Page [66], the main tool is a comparison of the distribution functions of $\langle x, R\rangle$ and $\langle x, \mathbf{T}_1 R\rangle$ in order to make use of a Markov modulated version of Goldie's implicit renewal theory. This will prove that

$$K = \lim_{t\to\infty} \frac{t^\beta}{e_*^\beta(x)} \mathbb{P}\left(\langle x, R\rangle > t\right) \tag{22.1}$$

exists for all $x \in \mathbb{S}$, which proves the main assertion of Theorem 13.2. A formula for K, which is very similar to the one given in [47, Theorem 4.1] will be obtained as well.

Define

$$f(x,t) = \frac{e^{\beta t}}{e_*^\beta(x)} \mathbb{P}\left(\langle x, R\rangle > e^t\right).$$

The aim is to write the function f as a renewal function (potential) $g * \mathbb{U}_x^\beta(t)$ in order to apply Theorem 19.2 to prove that $\lim_{t\to\infty} f(x,t) = K'$. Then this obviously implies (22.1).

However the function f is not sufficiently smooth to satisfy all the hypotheses of the Markov renewal theorem, in particular the direct Riemann integrability. Therefore its smoothed version will be considered: For any function $g : \mathbb{S}\times\mathbb{R} \to \mathbb{R}$ define its exponential smoothing as convolution with a standard exponential distribution:

$$\bar{g}(y,t) = \int_{-\infty}^t e^{-(t-s)} g(y,s) ds.$$

By [47, Lemma 9.3], (or more generally, the monotone density theorem [23, Theorem 1.7.2]) if one of the functions $f(x,t)$ and $\bar{f}(x,t)$ converges for $t\to\infty$, then both of them converge to the same limit. So it is sufficient to consider the exponential smoothed version of f.

The better part of this section is devoted to the proof the following Proposition, from which the desired results will be derived by an application of the MRT 19.2. Recall that $\mathbb{U}_x^\beta$ denotes the Markov renewal measure associated with $\mathbb{Q}_x^\beta(X_1, V_1 \in \cdot)$

Proposition 22.1. *The function $\bar{f}$ satisfies $\bar{f}(x,t) = \bar{g} * \mathbb{U}_x^\beta(t)$, where*

$$g(x,t) = \frac{e^{\beta t}}{e_*^\beta(x)} \Big[\mathbb{P}\left(\langle x, R\rangle > e^t\right) - \mathbb{P}\left(\langle x, \mathbf{T}R\rangle > e^t\right) \Big], \tag{22.2}$$

and the function $\bar{g}$ is wdRi.

First, the wdRi of $\bar{g}$ will be shown (Subsection 22.1) for it will be used in the second step, the proof of the identity $\bar{f} = \bar{g} * \mathbb{U}^\beta$ (subsection 22.2).

22.1. Weak Direct Riemann Integrability

In order to show that $\bar{g}$ is wdRi the following Lemma due to Goldie, stated here without proof, will be useful:

Lemma 22.2 ([47, Lemma 9.2])**.** *Let $x \in \mathbb{S}$ such that $f(x,\cdot) \in L^1(\mathbb{R})$. Then*

$$\sum_{l\in\mathbb{Z}} \sup_{t\in[l,l+1]} \left|\bar{f}(x,t)\right| \;\leq\; e^2 \int |f(x,t)|\, dt < \infty.$$

Lemma 22.3. *Let* (TlogT) *and* (Q-beta) *hold. Then the function $\bar{g}$ as defined in 22.2, is wdRi.*

Proof. In view of the previous lemma, it suffices to show that $\int |g(x,t)|\, dt$ is uniformly bounded in $x \in \mathbb{S}$. Referring to [47, Lemma 9.4], the integral can be rewritten as an expectation as follows (the lemma is applied in *):

$$\begin{aligned}
\int_{\mathbb{R}} |g(x,t)|\, dt &= \int_{\mathbb{R}} \frac{e^{\beta t}}{e_*^{\beta}(x)} \left|\mathbb{P}\left(\langle x,R\rangle > e^t\right) - \mathbb{P}\left(\langle x,\mathbf{T}R\rangle > e^t\right)\right| \, dt \\
&= \int_{\mathbb{R}} \frac{e^{\beta t}}{e_*^{\beta}(x)} \left|\mathbb{P}\left(\langle x,\mathbf{T}R+Q\rangle > e^t\right) - \mathbb{P}\left(\langle x,\mathbf{T}R\rangle > e^t\right)\right| \, dt \\
&\overset{*}{=} \frac{1}{\beta e_*^{\beta}(x)} \mathbb{E}\Big| (\langle x,\mathbf{T}R+Q\rangle^+)^\beta - (\langle x,\mathbf{T}R\rangle^+)^\beta \Big|.
\end{aligned}$$

Thus it suffices to show that

$$\sup_{x\in\mathbb{S}} \mathbb{E}\Big| (\langle x,\mathbf{T}R+Q\rangle^+)^\beta - (\langle x,\mathbf{T}R\rangle^+)^\beta \Big| < \infty.$$

Case $\beta \leq 1$: Finiteness results directly from the inequality $|a^s - b^s| \leq |a-b|^s$, valid for all $a, b \in \mathbb{R}_{\geq}$, $s \in (0,1]$, since $\mathbb{E}\left|\langle x,Q\rangle^+\right|^\beta \leq \mathbb{E}\,|Q|^\beta$ and the latter is finite by assumption (Q-beta).

Case $\beta > 1$: The same a case-by-case analysis with respect to the signs of $\langle x,Q\rangle$ and $\langle x,\mathbf{T}R\rangle$ as in [47, Theorem 4.1] will be used. For shortness of notation, write $g_x(\beta) := (\langle x,\mathbf{T}R+Q\rangle^+)^\beta - (\langle x,\mathbf{T}R\rangle^+)^\beta$. Using the inequality $|a^s - b^s| \leq s\max\{a^s,b^s\}\,|a-b|^{s-1}$, valid for $a,b \in \mathbb{R}_>$, $s \geq 1$, one obtains

- $\mathbb{E}\left(|g_x(\beta)|\,\mathbf{1}_{\{\langle x,Q\rangle\geq 0,\ \langle x,\mathbf{T}R\rangle\geq 0\}}\right) \leq \mathbb{E}\left(\beta\max\left\{\langle x,\mathbf{T}R+Q\rangle^{\beta-1}, \langle x,Q\rangle^{\beta-1}\right\}\langle x,Q\rangle\mathbf{1}_{\{\dots\}}\right)$
  $\qquad\leq \beta\mathbb{E}\|\mathbf{T}\|^{\beta-1}\mathbb{E}\,|R|^{\beta-1}\,\mathbb{E}\,|Q| + \beta\mathbb{E}\,|Q|^\beta,$
- $\mathbb{E}\left(|g_x(\beta)|\,\mathbf{1}_{\{\langle x,Q\rangle> 0,\ \langle x,\mathbf{T}R\rangle< 0\}}\right) \leq \mathbb{E}\left(\langle x,Q\rangle^\beta\mathbf{1}_{\{\dots\}}\right) \leq \mathbb{E}\left(|Q|^\beta\right),$
- $\mathbb{E}\left(|g_x(\beta)|\,\mathbf{1}_{\{\langle x,\mathbf{T}R\rangle>-\langle x,Q\rangle> 0\}}\right) \leq \mathbb{E}\left(\beta\max\left\{\langle x,\mathbf{T}R+Q\rangle^{\beta-1}, \langle x,Q\rangle^{\beta-1}\right\}\langle x,Q\rangle\mathbf{1}_{\{\dots\}}\right)$
  $\qquad\leq \beta\mathbb{E}\,\|\mathbf{T}\|^{\beta-1}\,\mathbb{E}\,|R|^{\beta-1}\,\mathbb{E}\,|Q|,$
- $\mathbb{E}\left(|g_x(\beta)|\,\mathbf{1}_{\{-\langle x,Q\rangle>\langle x,\mathbf{T}R\rangle> 0\}}\right) = \mathbb{E}\left(\langle x,\mathbf{T}R\rangle^\beta\mathbf{1}_{\{\dots\}}\right) \leq \mathbb{E}\left((-\langle x,Q\rangle)^\beta\mathbf{1}_{\{\dots\}}\right) \leq \mathbb{E}\,|Q|^\beta.$

Recall that, by Lemma 13.1, $\mathbb{E}\,|R|^s < \infty$ for all $s < \beta$. Thus all bounds are finite and independent of $x \in \mathbb{S}$. □

22.2. Implicit Markov Renewal Theory

Lemma 22.4. *For all* $(x,t) \in \mathbb{S} \times \mathbb{R}$,

$$\bar{f}(x,t) = \sum_{k=0}^{n-1} \int \bar{g}(y, t-u)\, \mathbb{Q}_x^{\beta}\left(X_k \in dy, V_k \in du\right) \\ + \int_{-\infty}^{t} e^{-(t-s)} \frac{e^{\kappa s}}{e_*^{\beta}(x)} \mathbb{P}\left(\langle \Pi_n x, R\rangle > e^s\right) ds. \tag{22.3}$$

Proof. Recall the independence of R, $\mathbf{T}$, and $(\Pi_n)_{n\geq 1}$ under $\mathbb{P}$ and the definition of $\mathbb{Q}_x$, in particular the identity (18.3). For arbitrary $n \in \mathbb{N}$, $x \in \mathbb{S}$ and $s \in \mathbb{R}$, consider the following telescoping sum for $\mathbb{P}\left(\langle x, R\rangle > e^s\right)$

$$\begin{aligned}
&\sum_{k=1}^{n} \left[\mathbb{P}\left(\langle x, \neg_{k-1} R\rangle > e^s\right) - \mathbb{P}\left(\langle x, \neg_k R\rangle > e^s\right)\right] + \mathbb{P}\left(\langle x, \neg_n R\rangle > e^s\right) \\
&= \sum_{k=1}^{n} \Big[\mathbb{P}\left(\langle \Pi_{k-1} x, R\rangle > e^s\right) - \mathbb{P}\Big(\langle \Pi_{k-1} x, \mathbf{M}_k^{\top} R\rangle > e^s\Big)\Big] + \mathbb{P}\left(\langle \Pi_n x, R\rangle > e^s\right) \\
&= \sum_{k=1}^{n} \Big[\mathbb{P}\Big(e^{\log|\Pi_{k-1}x|}\langle \Pi_{k-1}\cdot x, R\rangle > e^s\Big) - \mathbb{P}\Big(e^{\log|\Pi_{k-1}x|}\langle \Pi_{k-1}\cdot x, \mathbf{T}R\rangle > e^s\Big)\Big] \\
&\qquad + \mathbb{P}\left(\langle \Pi_n x, R\rangle > e^s\right) \\
&= \sum_{k=0}^{n-1} \int \mathbb{P}\left(\langle y, R\rangle > e^{s-u}\right) - \mathbb{P}\left(\langle y, \mathbf{T}R\rangle > e^{s-u}\right)\, \mathbb{Q}_x\left(X_k \in dy, V_k \in du\right) \\
&\qquad + \mathbb{P}\left(\langle \Pi_n x, R\rangle > e^s\right)
\end{aligned}$$

Multiply by $e^{\beta s}/e_*^{\beta}(x) > 0$ and use again (18.3), this time with the transformed measure $\mathbb{Q}_x^{\beta}$ to obtain

$$\begin{aligned}
f(x,t) &= \sum_{k=0}^{n-1} \int \frac{e^{\beta(s-u)}}{e_*^{\beta}(x)} \left[\mathbb{P}\left(\langle y, R\rangle > e^{s-u}\right) - \mathbb{P}\left(\langle y, \mathbf{T}R\rangle > e^{s-u}\right)\right] \\
&\qquad \times \frac{e_*^{\beta}(y)}{e_*^{\beta}(y)} e^{\beta u}\, \mathbb{Q}_x(X_k \in dy, V_k \in du) + \frac{e^{\beta s}}{e_*^{\beta}(x)} \mathbb{P}\left(\langle \Pi_n x, R\rangle > e^s\right) \\
&= \sum_{k=0}^{n-1} \int g(y, s-u)\, \mathbb{Q}_x^{\beta}\left(X_k \in dy, V_k \in du\right) + \frac{e^{\beta s}}{e_*^{\beta}(x)} \mathbb{P}\left(\langle \Pi_n x, R\rangle > e^s\right).
\end{aligned}$$

Applying the exponential smoothing to both sides then gives the assertion. □

The proof of Proposition 22.1 is now finished by the subsequent lemma.

Lemma 22.5. *For all* $(x,t) \in \mathbb{S} \times \mathbb{R}$*:*

$$\lim_{n\to\infty} \int_{-\infty}^{t} e^{-(t-s)} \frac{e^{\kappa s}}{e_*^{\beta}(x)} \mathbb{P}\left(\langle \Pi_n x, R\rangle > e^s\right) ds = 0, \tag{22.4}$$

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \int \bar{g}(y, t-u)\, \mathbb{Q}_x^{\beta}\left(X_k \in dy, V_k \in du\right) = \bar{g} * \mathbb{U}_x(t). \tag{22.5}$$

Proof. Step 1: By the Cauchy-Schwarz inequality,

$$\mathbb{P}\left(\langle \Pi_n x, R\rangle > e^s\right) \le \mathbb{P}\left(|\Pi_n x|\,|R| > e^s\right) \le \mathbb{P}\left(|R| > e^{s-\log|\Pi_n x|}\right).$$

But the last term converges to 0 as $n \to \infty$ for any $s > 0$, since the upper Lyapunov exponent

$$l(0) = \lim_{n\to\infty} \frac{1}{n} \log|\Pi_n x| < 0 \quad \mathbb{P}\text{-a.s.}$$

is negative by assumption (l<0). Moreover, since t is fixed, the integrand is bounded. Thus assertion (22.4) follows by an appeal to the bounded convergence theorem.

Step 2: In Lemma 22.3, the wdRi of $\bar{g}$ was shown. By the results of Section 20 and referring to Corollary 19.10,

$$|\bar{g}| * \mathbb{U}_x(t) = \int \sum_{k=0}^{\infty} |\bar{g}(X_k, t-V_k)|\, d\mathbb{Q}_x^{\beta} < \infty.$$

Consequently, using the bounded convergence theorem,

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \int \bar{g}(y,t-u)\, \mathbb{Q}_x^{\beta}\left(X_k \in dy, V_k \in du\right) = \int \lim_{n\to\infty} \sum_{k=0}^{n-1} \bar{g}(X_k, t-V_k)\, d\mathbb{Q}_x^{\beta} = \bar{g} * \mathbb{U}_x(t).$$

and thus again the asserted convergence results from the bounded convergence theorem. □

22.3. Convergence Results

Summarizing what has been done so far, the main assertion can now be proved.

Theorem 22.6. *For all* $x \in \mathbb{S}$*,*

$$\lim_{t\to\infty} f(x,t) = \frac{1}{l(\beta)\beta} \int_{\mathbb{S}} \frac{1}{e_*^{\beta}(y)} \mathbb{E}\left((\langle y, R\rangle^+)^{\kappa} - (\langle y, \mathbf{T}R\rangle^+)^{\kappa} \right) \pi_*^s(dy) =: K < \infty.$$

Proof. By Proposition 22.1 and the monotone density theorem [23, Theorem 1.7.2]

$$\lim_{t\to\infty} f(x,t) = \lim_{t\to\infty} \bar{f}(x,t) = \lim_{t\to\infty} \bar{g} * \mathbb{U}_x(t).$$

It was shown in Section 20 that under $\mathbb{Q}_x^{\beta}$, the MRW $(X_n, V_n)_{n\in\mathbb{N}_0}$ satisfies the assumptions of the MRT 19.2 as well as the additional assumptions of Proposition 19.6. Referring again to Proposition

22.1, $\bar{g}$ is wdRi. Thus the limit $\lim_{t\to\infty} \bar{g} * \mathbb{U}_x(t)$ exists for all $x \in \mathbb{S}$, is finite and equals

$$\begin{aligned}
&\frac{1}{l(\beta)} \int_{\mathbb{S}} \int_{\mathbb{R}} \bar{g}(y,v)dv\, \pi_*^\beta(dy) = \frac{1}{l(\beta)} \int_{\mathbb{S}} \int_{\mathbb{R}} \int_{-\infty}^{v} e^{-(v-s)} \bar{g}(y,s)dsdv\, \pi_*^\beta(dy) \\
&\overset{*}{=} \frac{1}{l(\beta)} \int_{\mathbb{S}} \int_{\mathbb{R}} \left(\int_s^\infty e^{-(v-s)} dv \right) \bar{g}(y,s)ds\, \pi_*^\beta(dy) = \frac{1}{l(\beta)} \int_{\mathbb{S}} \int_{\mathbb{R}} g(y,s)\; ds\; \pi_*^\beta(dy) \\
&\overset{\#}{=} \frac{1}{l(\beta)} \int_{\mathbb{S}} \frac{1}{e_*^\beta(y)} \int_{\mathbb{R}} e^{\beta s} \left[\mathbb{P}\left(\langle y, R\rangle > e^s\right) - \mathbb{P}\left(\langle y, \mathbf{T}R\rangle > e^s\right)\right]\; ds\; \pi_*^\beta(dy) \\
&= \frac{1}{l(\beta)} \int_{\mathbb{S}} \frac{1}{e_*^\beta(y)} \int_0^\infty u^{\beta-1} \left[\mathbb{P}\left(\langle y, R\rangle > u\right) - \mathbb{P}\left(\langle y, \mathbf{T}R\rangle > u\right)\right]\; du\; \pi_*^\beta(dy) \\
&\overset{**}{=} \frac{1}{l(\beta)\beta} \int_{\mathbb{S}} \frac{1}{e_*^\beta(y)} \mathbb{E}\left((\langle y, R\rangle^+)^\beta - (\langle y, \mathbf{T}R\rangle^+)^\beta \right) \pi_*^\beta(dy).
\end{aligned}$$

In $*$, Fubini's theorem was used as well as the fact that the inner integral equals 1, for it is the density of a shifted standard exponential distribution. In $**$, again as in the proof of Lemma 22.3 the Goldie Lemma [47, Lemma 9.4] was used. □

Corollary 22.7. *For all* $x \in \mathbb{S}$*,*

$$\lim_{t\to\infty} t^\beta \mathbb{P}\left(|\langle x, R\rangle| > t\right) = \frac{1}{\beta\, l(\beta)\, \nu_*^\beta(e_*^\beta)} e_*^\beta(x) \int_{\mathbb{S}} \mathbb{E}\left(|\langle y, \mathbf{T}R + Q\rangle|^\beta - |\langle y, \mathbf{T}R\rangle|^\beta \right) \nu_*^\beta(dy).$$

Additionally,

$$e_*^\beta(x)K = \lim_{t\to\infty} t^\beta \mathbb{P}\left(\langle x, R\rangle > t\right) = \lim_{t\to\infty} t^\beta \mathbb{P}\left(\langle -x, R\rangle > t\right) = \frac{1}{2} \lim_{t\to\infty} t^\beta \mathbb{P}\left(|\langle x, t\rangle| > t\right) \quad (22.6)$$

Proof. In line # in the proof above, $\mathbb{P}\left(\langle y, R\rangle > e^s\right)$ may be replaced by $\mathbb{P}\left(\langle y, \mathbf{T}R + Q\rangle > e^s\right)$ in order to derive that

$$\lim_{t\to\infty} t^\beta \mathbb{P}\left(\langle x, R\rangle > t\right) = \frac{e_*^\beta(x)}{l(\beta)\beta} \int_{\mathbb{S}} \frac{1}{e_*^\beta(y)} \mathbb{E}\left((\langle y, \mathbf{T}R + Q\rangle^+)^\beta - (\langle y, \mathbf{T}R\rangle^+)^\beta \right) \pi_*^\beta(dy).$$

Substitute $-R$ for R to infer

$$\lim_{t\to\infty} t^\beta \mathbb{P}\left(\langle x, R\rangle < -t\right) = \frac{e_*^\beta(-x)}{l(\beta)\beta} \int_{\mathbb{S}} \frac{1}{e_*^\beta(y)} \mathbb{E}\left((\langle y, \mathbf{T}R + Q\rangle^-)^\beta - (\langle y, \mathbf{T}R\rangle^-)^\beta \right) \pi_*^\beta(dy).$$

Adding both, considering the symmetry of e_*^β (Theorem 17.1, 3.) and using that

$$\pi_*^\beta(dx) = \nu_*^\beta(e_*^\beta)^{-1} e_*^\beta(x) \nu_*^\beta(dx)$$

by Lemma 17.4 gives the first assertion.

For the second assertion, just note that e_*^β and hence also f are symmetric in x. □

When β is an even integer, this formula can sometimes be used to show that $K > 0$, e.g. if $\beta = 2$

and Q be independent of $\mathbf{T}$ with $\mathbb{E}Q = 0$: With $C = 2\beta l(\beta)\nu_*^\beta(e_*^\beta) > 0$,

$$\begin{aligned} CK &= \int_{\mathbb{S}} \mathbb{E}(\langle y, \mathbf{T}R\rangle + \langle y, Q\rangle)^2 - \langle y, \mathbf{T}R\rangle^2 \nu_*^s(dy) \\ &= \int_{\mathbb{S}} \mathbb{E}2\langle y, \mathbf{T}R\rangle\langle y, Q\rangle + \langle y, Q\rangle^2 \nu_*^s(dy) \\ &= \int_{\mathbb{S}} 2\langle y, \mathbb{E}Q\rangle\langle y, \mathbb{E}\mathbf{T}R\rangle + \mathbb{E}\langle y, Q\rangle^2 \nu_*^s(dy) = \mathbb{E}\langle Y, Q\rangle^2, \end{aligned}$$

for a r.v. Y independent of Q with $\mathcal{L}(Y) = \nu_*^\beta$. Since by Theorem 17.1, 3. the support of Y is the whole sphere $\mathbb{S}$, it follows that $CK = \mathbb{E}\langle Y, Q\rangle^2 > 0$. But in most cases, this expression is not suitable for checking whether $K > 0$ (and thus β is the precise tail index of R). This calls for a different argument, which will be given in the next section.

In the one-dimensional setting and even in the multivariate case if $\mathbf{T}$ is restricted to the group of similarities, a nice argument involving holomorphic extension of the expression for K (seen as a function in β) gives the positivity of K under some additional moment assumptions on Q and M, see [30] for details. That method was introduced by Guivarc'h in [49] and has been applied to more general stochastic fixed point equations as well, see [5, 31, 74].

In dimension $d = 1$, Enriquez et al. [43] and Collamore [34, Theorem 2.1] derived different representations for K, which give the positivity of K more easily. Nevertheless, up to now there is no multivariate version of this expressions though it was pointed out by Guivarc'h that these may be are closely related to the (multivariate) approach in [52].

23. The Constant K is Positive

To complete the proof of Theorem 13.2, it remains to show that K is positive, which is the topic of this final section.

23.1. On the Support of R

Taking Corollary 22.7 into account, it clearly suffices to show that $\liminf_{t\to\infty} t^\kappa \mathbb{P}(|\langle x, R\rangle| > t) > 0$ for some $x \in \mathbb{S}$. A necessary condition for this to hold is that $\operatorname{supp}\langle x, R\rangle$ is unbounded. The following lemma, originally due to Le Page [66, Lemma 3.11] proves the corresponding statement in the main theorem. It is this result where the nondegeneracy assumption ($R \neq r$), unused so far, enters in a crucial way.

Lemma 23.1. *Let μ satisfy (i-d). Assume that there is $\beta > 0$ such that $m(\beta) = 1$, $m'(\beta^-) > 0$ and let* (TlogT)*,* (Q-beta) *hold. Then exactly one of the following assertions hold:*

1. $\mathbb{P}(\mathbf{T}r + Q = r) = 1$ *for some* $r \in \mathbb{R}^d$
2. *For all $x \in \mathbb{S}$ and $t \in \mathbb{R}$,*

$$\mathbb{P}(\langle x, R\rangle \leq t) < 1. \tag{23.1}$$

If 1. holds, then this r is unique, and $R \equiv r$.

Proof. In this and the following proofs, inequality (6) from the appendix will be used several times. It states that for $x \in \mathbb{S}$, $\delta < 1$,

$$\inf_{z \in B_\delta(x)} \langle x, z \rangle \geq 1 - \delta.$$

STEP 1: If 1 holds for some $r \in \mathbb{R}^d$, then $R = r$ is a fixed point of the RDE. But under the assumptions stated, this fixed point is unique (as shown in the Introduction). This obviously yields the uniqueness of r, too.

STEP 2: Conversely, if $(R \neq r)$ holds then $\operatorname{supp} R$ is unbounded, as the following arguments show: Use (16.1) to infer for each $n \geq 1$,

$$\urcorner_n \operatorname{supp} R + Q^n \subseteq \operatorname{supp} R \quad \mathbb{P}\text{-a.s.}$$

Now assume, that $\operatorname{supp} R$ is bounded. By $(R \neq r)$, there exist at least two distinct $r_1, r_2 \in \operatorname{supp} R$. Defining $v := r_1 - r_2$, it then follows that for all $n \geq 1$ and some $C \in (0, \infty)$

$$|\urcorner_n v| \leq |\urcorner_n r_1 + Q^n| + |\urcorner_n r_2 + Q^n| \leq C \quad \mathbb{P}\text{-a.s.}$$

and thereupon for all $x \in \mathbb{S}$

$$|\Pi_n x| \, |\langle \Pi_n \cdot x, v \rangle| = |\langle x, \urcorner_n v \rangle| \leq |\urcorner_n v| \leq C \quad \mathbb{P}\text{-a.s..} \tag{23.2}$$

Consequently, the sequence $(|\Pi_n x| \, |\langle \Pi_n \cdot x, v \rangle|)_{n \in \mathbb{N}_0}$ is bounded by C $\mathbb{Q}_x^\beta$-a.s. as well, for the marginal distributions of Π_n under $\mathbb{Q}_x^\beta$ are $\mathbb{P}\left(\Pi_n \in \cdot\right)$-continuous by the very definition (18.1) of $\mathbb{Q}_x$. Equation (23.2) then reads

$$e^{V_n} \, |\langle X_n, v \rangle| \leq C \quad \mathbb{Q}_x^\beta\text{-a.s..}$$

Referring to Corollary 18.1, there is $\delta > 0$ such that $B_\delta(v)$ is a regenerative set for $(X_n)_{n \in \mathbb{N}_0}$ and the latter chain is a strongly aperiodic Doeblin chain w.r.t. to this minorization (see also Lemma 21.3). In particular, the hitting times $(\tau_n)_{n \in \mathbb{N}}$ of X_n in $B_\delta(v)$ are $\mathbb{Q}_x^\beta$-a.s. finite. By Inequality (6), $|\langle X_{\tau_n}, v \rangle| \geq 1 - \delta$. Together this yields

$$\limsup_{n \to \infty} e^{V_{\tau_n}} \leq \limsup_{n \to \infty} \frac{C}{|\langle X_{\tau_n}, v \rangle|} \leq \frac{C}{1 - \delta} \quad \mathbb{Q}_x^\beta\text{-a.s.}$$

for all $x \in \mathbb{S}$. Consequently,

$$\limsup_{n \to \infty} \frac{V_{\tau_n}}{\tau_n} \leq 0 \quad \mathbb{Q}_x^\beta\text{-a.s.}$$

for all $x \in \mathbb{S}$, which contradicts the fact that $\mathbb{V}_n$ has positive drift under $\mathbb{Q}_\pi^\beta$, see (21.3).

STEP 3: Having thus shown that $\operatorname{supp} R$ is not compact in $\mathbb{R}^d$, there exist sequences $(r_n)_{n \geq 1} \subset \operatorname{supp} R$ with $\lim_{n \to \infty} |r_n| = \infty$ whence, by compactness of $\mathbb{S}$, the following set is nonempty (recall the notation $\overline{x} = |x|^{-1} x$):

$$D := \left\{ y \in S \; : \; \exists \, (r_n)_{n \geq 1} \subset \operatorname{supp} R, \; \lim_{n \to \infty} |r_n| = \infty, \; \lim_{n \to \infty} \overline{r_n} = y \right\}.$$

Now suppose that $\mathbb{P}\left(\langle x_0, R\rangle \leq t_0\right) = 1$ for some $(x_0, t_0) \in \mathbb{S} \times \mathbb{R}$. This implies $\langle x_0, r\rangle \leq t_0$ for all $r \in \operatorname{supp} R$ by continuity of the scalar product. Then for any $y \in D$, there is a sequence $(r_n)_{n\geq 1} \subset \operatorname{supp} R$ such that $\overline{r_n} \to y$. Using that $\langle x_0, r_n\rangle \leq t_0$ for all n together with $|r_n| \to \infty$ implies that $\langle x_0, y\rangle \leq 0$ and this is true for all $y \in D$.

At the same time,

$$\mathbb{P}\left((\mathbf{T}_1 r_n + Q)_{n\in\mathbb{N}} \subset \operatorname{supp}(R)\right) = 1 \qquad \text{and} \qquad \mathbb{P}\left(\lim_{n\to\infty} |\mathbf{T}_1 r_n + Q| = \infty\right) = 1,$$

thus for any $y_0 \in D$,

$$\mathbb{P}\left(\lim_{n\to\infty} \overline{\mathbf{T}_1 r_n + Q} = \mathbf{T}_1 \cdot y_0 \in D\right) = 1.$$

Consequently,

$$\mathbb{P}\left(\langle x_0, \mathbf{T}_1 \cdot y_0\rangle \leq 0\right) = \mathbb{P}\left(\langle \mathbf{M}_1 \cdot x_0, y_0\rangle \leq 0\right) = 1,$$

in particular $\mathbb{P}(\mathbf{M}_1 \cdot x_0 \notin B_\delta(y_0)) = 0$ for sufficiently small $\delta > 0$ by Inequality (6). But this is a contradiction to (irred) (with $n = 1$). Thus (2) has been proved. □

Replacing x by $-x$ in (23.1) yields the additional inequality that for all $(x, t) \in \mathbb{S} \times R$,

$$\mathbb{P}\left(\langle x, R\rangle \geq t\right) < 1.$$

From the converse inequalities

$$\mathbb{P}\left(\langle x, R\rangle > t\right) > 0 \quad \text{and} \quad \mathbb{P}\left(\langle x, R\rangle < t\right) > 0, \tag{23.3}$$

valid for all $(x, t) \in \mathbb{S} \times \mathbb{R}$, the following Corollary can be deduced (assuming from now on, that all assumptions of the main theorem are in force, including $(R \neq r)$):

Corollary 23.2. *For each $x_0 \in \mathbb{S}$, $\xi \in \mathbb{R}_>$ and $\zeta \in (0, 1)$ there are $\eta > 0$ and $\delta \in (0, \zeta/2)$ such that $B_\delta(x_0)$ is a minorizing set in the sense of Corollary 18.2 and*

$$\mathbb{P}\left(\langle z, R\rangle > \xi\right) \geq \eta \quad \textit{and} \quad \mathbb{P}\left(\langle z, R\rangle < (1-\zeta)\xi\right) \geq \eta \tag{23.4}$$

for all $z \in B_\delta(x_0)$.

Proof. Referring to (23.3), there is $\eta > 0$ such that

$$\mathbb{P}\left(\langle x_0, R\rangle > \xi + 1\right) \geq 2\eta \quad \text{and} \quad \mathbb{P}\left(\langle x_0, R\rangle < (1-\zeta)\xi - 1\right) \geq 2\eta. \tag{23.5}$$

The mapping $z \mapsto \mathbb{P}\left(\langle z, R\rangle \in \cdot\right)$ is continuous w.r.t. weak convergence, thus by [22, Theorem 6.8] it is also continuous w.r.t. the Prohorov metric. Referring to (3.5), it follows that for all $\varepsilon \in (0, \eta)$ there is $\delta > 0$ such that for all $z \in B_\delta(x_0)$

$$\mathbb{P}\left(\langle z, R\rangle > \xi\right) \geq \mathbb{P}\left(\langle z, R\rangle > \xi + 1 - \varepsilon\right) \geq \mathbb{P}\left(\langle x_0, R\rangle > \xi + 1\right) - \varepsilon \geq \eta$$

as well as

$$\mathbb{P}\left(\langle z, R\rangle < (1-\zeta)\xi\right) \geq \mathbb{P}\left(\langle z, R\rangle < (1-\zeta)\xi - 1 + \varepsilon\right) \geq \mathbb{P}\left(\langle x_0, R\rangle < (1-\zeta)\xi - 1\right) - \varepsilon \geq \eta.$$

Possibly after making δ smaller, $B_\delta(x_0)$ is a minorizing set for $(X_n, \mathbf{T}_n, Q_n)_{n\in\mathbb{N}}$ by Corollary 18.2. □

23.2. A Generalized Symmetrization Inequality

Recall from Subsection 16.1 the definition of Q^n and R^n, the identity

$$R = \daleth_\tau R^\tau + Q^\tau \quad \mathbb{O}_x\text{-a.s}$$

as well as $\mathbb{O}_x\left(R^\tau \in \cdot\right) = \mathbb{P}\left(R \in \cdot\right)$ for any a.s. finite stopping time τ with respect to $(\mathcal{F}_n)_{n\geq 0}$, the natural filtration of $(X_n, \mathbf{T}_n, Q_n)_{n\geq 1}$ and all $x \in \mathbb{S}$. The next lemma is a generalization of Lévy's symmetrization inequality in the spirit of [47, Proposition 4.2]. The idea is to decompose

$$\langle x, R\rangle = \langle x, \daleth_n R^n + Q^n\rangle = |\Pi_n x| \langle \Pi_n \cdot x, R^n\rangle + \langle x, Q^n\rangle$$

w.r.t. the entrances τ_n of $\Pi_n \cdot x$ into $B_\delta(x_0)$ and to replace R^n by a deterministic vector ξy in order to compare the tail behaviour of $\langle x, R\rangle$ and $|\Pi_n x|$ in Proposition 23.4. In fact, instead of the hitting times τ_n, a feasible (see Subsection 21.2) sequence of regeneration epochs ϖ_n is considered in the subsequent lemma. This is due to the additional regeneration properties of $(X_{\varpi_n})_{n\in\mathbb{N}}$ which will be used in Propositon 23.4 and Lemma 23.5. The vector ξy should be interpreted as a generalized median - observe that if putting $\eta = 1/2$, $\zeta = 0$ in (23.4), then ξ is the median of $\langle x_0, R\rangle$.

Lemma 23.3. *Given any $x_0 \in \mathbb{S}$, $\xi \in \mathbb{R}_>$, there are $\delta, \eta > 0$ and a feasible sequence of regeneration epochs $(\varpi_n)_{n\in\mathbb{N}_0}$ w.r.t. $B_\delta(x_0)$, such that*

$$\mathbb{P}\left(|\langle x, R\rangle| > t\right) \;\geq\; \eta\, \mathbb{O}_x\left(\sup_{n\geq 1} |\langle x, Q^{\varpi_n}\rangle + \xi\langle \Pi_{\varpi_n} x, y\rangle| > t\right)$$

holds true for all $x \in \mathbb{S}$ and $y \in B_\delta(x_0)$.

Proof. Fix $x_0, x \in \mathbb{S}$ and $\zeta \in (0, 1)$. Then by Corollary 23.2 there are δ, η >0 such that (23.4) holds and $B_\delta(x_0)$ is a minorizing set in the sense of Corollary 18.2. Hence a feasible sequence $(\sigma_n)_{n\in\mathbb{N}_0}$ of regeneration epochs exists. In this proof, again the inequality (6) from the Appendix will be used, which gives that

$$\inf_{z,y\in B_\delta(x_0)} \langle z, y\rangle > 1 - 2\delta. \tag{23.6}$$

Note that this and (23.4) particularly hold for z=X_{ϖ_n}. As noted above, $\mathbb{O}_x\left(R^{\varpi_n} \in \cdot\right) = \mathbb{P}\left(R \in \cdot\right)$ for all $n \geq 1$. It follows that (23.4) holds for R^{ϖ_k} under $\mathbb{O}_x$ as well.

Step 1: Show that

$$\mathbb{P}\left(\langle x, R\rangle > t\right) \;\geq\; \eta\, \mathbb{O}_x\left(\sup_{n\geq 1} \langle x, Q^{\varpi_n}\rangle + \xi\left\langle \Pi_{\varpi_n} x, y\right\rangle > t\right).$$

In order to do so, define

$$C_k := \left\{\max_{1\leq j<k}\left(\langle x, Q^{\varpi_j}\rangle + \xi\left\langle x, \daleth_{\varpi_j} y\right\rangle\right) \leq t,\; \langle x, Q^{\varpi_k}\rangle + \xi\left\langle x, \daleth_{\varpi_k} y\right\rangle > t\right\}$$

$$\text{and} \quad D_k := \{\langle x, \daleth_{\varpi_k} R^{\varpi_k}\rangle > \xi \langle x, \daleth_{\varpi_k} y\rangle\} = \{\langle \Pi_{\varpi_k} x, R^{\varpi_k}\rangle > \xi \langle \Pi_{\varpi_k} x, y\rangle\}\,.$$

By (23.6), $0 < \langle \Pi_{\varpi_k} \cdot x, y\rangle \leq 1$ for all $y \in B_\delta(x_0)$, giving

$$D_k = \{\langle \Pi_{\varpi_k} \cdot x, R^{\varpi_k}\rangle > \xi \langle \Pi_{\varpi_k} \cdot x, y\rangle\} \supset \{\langle X_{\varpi_k}, R^{\varpi_k}\rangle > \xi\}$$

and thus $\mathbb{O}\left(D_k | \mathcal{F}_{\varpi_k}\right) \geq \eta\, \mathbb{O}_x$-a.s. In combination with $\bigcup_{k=1}^n (C_k \cap D_k) \subset \{\langle x, R\rangle > t\}$ and $C_k \in \mathcal{F}_{\varpi_k}$, this implies

$$\mathbb{P}(\langle x, R\rangle > t) = \mathbb{O}_x\left(\langle x, R\rangle > t\right) \;\geq\; \sum_{k=1}^n \int_{C_k} \mathbb{O}\left(D_k | \mathcal{F}_{\varpi_k}\right) d\mathbb{O}_x \;\geq\; \eta\, \mathbb{O}_x\left(\bigcup_{k=1}^n C_k\right),$$

and thus

$$\mathbb{P}(\langle x, R\rangle > t) \geq \eta\, \mathbb{O}_x\left(\sup_{n\geq 1}\left(\langle x, Q^{\varpi_n}\rangle + \xi\langle \Pi_{\varpi_n} x, y\rangle\right) > t\right)$$

by letting $n \to \infty$.

Step 2: Turning to the respective inequality for $\mathbb{P}\left(\langle x, R\rangle < -t\right)$, define

$$C_k' := \left\{\min_{1\leq j<k}\left(\langle x, Q^{\varpi_j}\rangle + \xi\,\langle x, \daleth_{\varpi_j} y\rangle\right) \geq -t,\; \langle x, Q^{\varpi_k}\rangle + \xi\,\langle x, \daleth_{\varpi_k} y\rangle < -t\right\}$$
$$\text{and} \quad D_k' := \{\langle X_{\varpi_k}, R^{\varpi_k}\rangle < \xi\langle X_{\varpi_k}, y\rangle\}\,.$$

Using again (23.6), $\langle X_{\varpi_k}, y\rangle \geq 1 - 2\delta > 1 - \zeta$ for all $y \in B_\delta(x_0)$, giving

$$D_k' \supset \{\langle X_{\varpi_k}, R^{\varpi_k}\rangle < (1-\zeta)\xi\}$$

and thus $\mathbb{O}\left(D_k' | \mathcal{F}_{\varpi_k}\right) \geq \eta\, \mathbb{O}_x$-a.s. Now reasoning as above,

$$\mathbb{P}(\langle x, R\rangle < -t) \;\geq\; \eta \lim_{n\to\infty} \mathbb{O}_x\left(\bigcup_{k=1}^n C_k'\right) \;=\; \eta\, \mathbb{O}_x\left(\inf_{n\geq 1}\left(\langle x, Q^{\varpi_n}\rangle + \xi\langle \Pi_{\varpi_n} x, y\rangle\right) < -t\right).$$

The desired result hence follows by a combination of this inequality with the one obtained for $\mathbb{P}\left(xR > t\right)$. ☐

23.3. K is Positive

Proposition 23.4. *There is $x \in \mathbb{S}$ such that $\liminf_{t\to\infty} t^\kappa\, \mathbb{P}(|xR| > t)$ is positive.*

Proof. Fix any $x_0 \in \mathbb{S}$ and $\xi \in \mathbb{R}_>$ and apply Lemma 23.3. Recall that Corollary 18.2 gives the existence of a probability measure Φ, supported on $\mathfrak{B}_\delta(x_0)$ and a compact set $C \subset GL(d, \mathbb{R})$ with $X_{\varpi_n} \stackrel{d}{=} \Phi$ and $\mathbf{M}_{\varpi_n} \in C$. The additional property (18.7) will be used in the subsequent Lemma 23.5, which will finish the present proof and thereby determine the choice of $y \in B_\delta(x_0)$ (see Lemma 23.3) which has at the moment to be seen as a parameter of the proof (as well as $\varepsilon > 0$). Recall furthermore, that by Proposition 21.4, $\lim_{t\to\infty} t^\beta \mathbb{O}_x\left(\sup_{n\geq 1} |x\Pi_{\varpi_n - 1}| > t\right)$ is positive for

π_*^β-almost all $x \in B_\delta(x_0)$. Fix any such x hereafter .

Define $\daleth_{j,k} := \mathbf{T}_j \cdot ... \cdot \mathbf{T}_k$, $Q^{j,n} := \sum_{k=j}^n \daleth_{j,k-1} Q_k$ and

$$\begin{aligned} T_n &:= \langle x, Q^{\varpi_n} \rangle + \xi \, \langle x, \daleth_{\varpi_n} y \rangle, \\ \Delta_n &:= Q^{\varpi_{n-1}+1, \varpi_n} - \xi \, (I - \daleth_{\varpi_{n-1}+1, \varpi_n}) \, y, \\ U_n &:= \langle x, \daleth_{\varpi_{n-1}} \Delta_n \rangle \end{aligned}$$

for $n \in \mathbb{N}$, with the convention $\varpi_0 := 0$. Then $T_n = T_{n-1} + U_n$ and $\{\sup_{n \geq 1} |T_n| > t\} \supset \{\sup_{n \geq 2} |U_n| > 2t\}$. Referring to Lemma 23.3,

$$\mathbb{P} \left(|\langle x, R \rangle| > t \right) = \mathbb{O}_x \left(\langle x, R \rangle > t \right) \geq \eta \, \mathbb{O}_x \left(\sup_{n \geq 1} |T_n| > t \right)$$

for some $\eta > 0$.

Since $\mathbf{M}_{\varpi_n} \in C$, $\inf_{z \in \mathbb{S}} |M_{\varpi_n} z| \geq c \, \mathbb{O}_x$-a.s. for all $n \in \mathbb{N}$ and a suitable $c > 0$. Set

$$A_k = \{ |\Pi_{\varpi_k - 1} x| \leq 2t/(c\varepsilon) \}$$

for $k \geq 1$ and some fixed $0 < \varepsilon < 1$ (to be chosen in Lemma 23.5 Hence, for all $t > 0$,

$$\begin{aligned} &\mathbb{P} \left(|\langle x, R \rangle| > t \right) \geq \eta \, \mathbb{O}_x \left(\sup_{n \geq 2} |U_n| \geq 2t \right) = \eta \mathbb{O}_x \left(\sup_{n \geq 1} |\langle \Pi_{\varpi_n} x, \Delta_{n+1} \rangle| \geq 2t \right) \\ &= \eta \, \mathbb{O}_x \left(\sup_{n \geq 1} |\Pi_{\varpi_n - 1} x| \, |\mathbf{M}_{\varpi_n} (\Pi_{\varpi_n - 1} \cdot x)| \, |\langle X_{\varpi_n}, \Delta_{n+1} \rangle| \geq 2t \right) \\ &\geq \eta \sum_{n \geq 1} \mathbb{O}_x \left(\bigcap_{k=1}^{n-1} A_k, \; |\Pi_{\varpi_n - 1} x| > \frac{2t}{c\varepsilon}, |\langle X_{\varpi_n}, \Delta_{n+1} \rangle| > \varepsilon \right) \\ &\overset{*}{\geq} \eta \sum_{n \geq 1} \mathbb{O}_x \left(\bigcap_{k=1}^{n-1} A_k, \; |\Pi_{\varpi_n - 1} x| > \frac{2t}{c\varepsilon} \right) \mathbb{O}_\Phi \left(|\langle X_0, \Delta_1 \rangle| > \varepsilon \right) \\ &\geq \eta \, \mathbb{O}_\Phi \left(|X_0 \Delta_1| > \varepsilon \right) \, \mathbb{O}_x \left(\sup_{n \geq 1} |x \Pi_{\varpi_n - 1}| > \frac{2t}{c\varepsilon} \right). \end{aligned}$$

In $*$, the regeneration property (R3) and the fact that $(X_n, \mathbf{T}_n, Q_n)_{n \in \mathbb{N}}$ can be defined as a Markov chain under $\mathbb{O}_x$ via (18.6) with the transition only depending on X_{n-1} have been used. The proof is finished by the subsequent lemma where the positivity of $\mathbb{O}_\Phi \left(|\langle X_0, \Delta_1 \rangle| > \varepsilon \right)$ will be shown. Together with (21.4) this clearly yields the desired conclusion. □

Lemma 23.5. *In the situation of Proposition 23.4, there exist $\varepsilon > 0$ and $y \in B_\delta(x_0)$ such that*

$$\mathbb{O}_\Phi \left(|\langle X_0, \Delta_1 \rangle| > \varepsilon \right) = \mathbb{O}_\Phi \Bigg(\; |\langle X_0, Q^{\varpi_1} - \xi \, (\mathbf{Id} - \Pi_{\varpi_1}) \, y \rangle| > \varepsilon \Bigg) > 0.$$

Proof. First, observe that it suffices to show the positivity of

$$\mathbb{O}_\Phi \left(|\langle X_0, \Delta_1 \rangle| > \varepsilon, \varpi_1 = 1 \right) = \mathbb{O}_\Phi \left(\left| \langle X_0, Q_{\varpi_1} - \xi \left(\mathbf{Id} - \mathbf{M}_{\varpi_1}^\top \right) y \rangle \right| > \varepsilon, \varpi_1 = 1 \right).$$

By Corollary 18.2 and the regeneration lemma 18.4, for all $x \in \operatorname{supp} \ \Phi = B_\delta(x_0)$,

$$\mathbb{O}_\Phi \left((X_{\varpi_1}, \mathbf{M}_{\varpi_1}, Q_{\varpi_1}) \in A \times B \times D | \, X_0 = x \right) = \Xi(x, A \times B \times D).$$

Recall from (18.7), that there are $L, \varsigma > 0$ such that

$$\Xi(x, B_\delta(x_0) \times B \times \mathbb{R}^d) = L \int_{B_\varsigma(\mathbf{Id})} \mathbf{1}_{B_\delta(x_0) \times B}(\mathbf{A} \cdot x, \mathbf{A}\mathbf{A}_x) \lambda^{d^2}(d\mathbf{A})$$

where $\mathbf{A}_x$ is a deterministic matrix in $GL(d, \mathbb{R})$, only depending on x. In other words, for all $x \in B_\delta(x_0)$, the conditional distribution of $\mathbf{M}_{\varpi_1}$ given $X_0 = x$ has a nonzero component which is absolutely continuous w.r.t. to the Lebesgue measure on $GL(d, \mathbb{R}) \subset \mathbb{R}^{d^2}$. Consequently,

$$\mathbb{O}_\Phi \left(\mathbf{Id} - \mathbf{M}_{\varpi_1}^\top \in GL(d, \mathbb{R}), \ \varpi_1 = 1 \right) =: p' > 0.$$

As said before, there is no information about the dependence structure between Q_{ϖ_1} and $\mathbf{M}_{\varpi_1}$, but nevertheless, the above yields that the affine mapping $Q_{\varpi_1} - \xi(\mathbf{Id} - \mathbf{M}_{\varpi_1}^\top)$ has full range $\mathbb{R}^d$ with probability at least $p' > 0$ under $\mathbb{O}_\Phi \left(\cdot, \varpi_1 = 1 \right)$.

Now suppose that

$$\langle X_0, Q_{\varpi_1} - \xi \left(I - \mathbf{M}_{\varpi_1}^\top \right) y \rangle = 0$$

$\mathbb{O}_\Phi \left(\cdot, \varpi_1 = 1 \right)$-a.s. for all $y \in B_\delta(x_0)$. Then the same holds true for all y in the convex hull of $B_\delta(x_0)$ in $\mathbb{R}^d$. But this convex hull contains a basis of $\mathbb{R}^d$ and thus the range of $Q_{\varpi_1} - \xi \left(\mathbf{Id} - \mathbf{M}_{\varpi_1} \right)$ and $\{tX_0 \, : \, t \in \mathbb{R}\}$ would be orthogonal $\mathbb{O}_\Phi \left(\cdot, \varpi_1 = 1 \right)$-a.s.. This is in contradiction with the above. □

24. On Multivariate Regular Variation

This section is also contained in the article [35].

As mentioned before in the proof of Proposition 5.12, Basrak et. al. [15] investigated conditions under which (13.8) already implies that R is multivariate regularly varying with index β. For non-integer β, this holds true, see [15, Theorem 1.1 (ii)] or [25, Corollary 2]. Writing $V := \mathbb{R}^d \setminus \{0\}$, it follows that for all $f \in \mathcal{C}_c \left(\overline{\mathbb{R}^d} \setminus \{0\} \right)$

$$\lim_{t \to \infty} t^\beta \mathbb{E} \left(f(t^{-1} R) \right) = K \int_0^\infty \int_{\mathbb{S}} f(sw) \, \nu(dw) \lambda^\beta(ds) \tag{24.1}$$

for some $\nu \in \mathcal{P}(\mathbb{S})$ and $\lambda^\beta(ds) = \frac{1}{s^{1+\beta}} ds$.

It remains to identify ν. The next proposition characterizes $\nu \otimes \lambda^\beta$ as a stationary measure of the Markov Chain on V, given by the action of μ on V.

Proposition 24.1. *The measure $\nu \otimes \lambda^\beta$ satisfies for all $f \in \mathcal{C}_c(V)$,*

$$\int_V f(x)\nu \otimes \lambda^\beta(dx) = \int_V \mathbb{E}\left(f(\mathbf{T}x)\right)\nu \otimes \lambda^\beta(dx). \tag{24.2}$$

Proof. Step 1: Proceed as in the proof of Lemma 2.19 in [30], to show that if $0 < \varepsilon < \min\{1, \beta\}$, then for all $f \in H^\varepsilon \cap \mathcal{C}_c(V)$, the space of ε-Hölder functions,

$$\lim_{t\to\infty} t^\beta \mathbb{E}\left(f(t^{-1}R) - f(t^{-1}\mathbf{T}R)\right) = 0.$$

Since $\lim_{t\to\infty} t^\beta \mathbb{E}\left(f(t^{-1}R)\right)$ exists, this yields the existence of $\lim_{t\to\infty} t^\beta \mathbb{E}\left(f(t^{-1}\mathbf{T}R)\right)$ and both are equal to $\int_V f(x)\nu \otimes \lambda^\beta(dx)$.

The assertion (24.2) follows if $\lim_{t\to\infty} t^\beta \mathbb{E}\left(f(t^{-1}\mathbf{T}R)\right)$ is also equal to $\int_V \mathbb{E}\left(f(\mathbf{T}x)\right)\nu \otimes \lambda^\beta(dx)$. Observe that $x \mapsto \mathbb{E}\left(f(\mathbf{T}x)\right)$ in general has unbounded support, i.e. it is not an element of $H^\varepsilon \cap \mathcal{C}_c(V)$, thus some more calculations are needed.

Step 2: Let $f \in H^\varepsilon \cap \mathcal{C}_c(V)$. Hence there is $\eta > 0$ such that $\operatorname{supp} f \subset \overline{B_\eta(0)}^c$ for some $\eta > 0$. Then for any $t \in \mathbb{R}_>$ and $\mathbf{A} \in GL(d, \mathbb{R})$,

$$\begin{aligned} t^\beta \mathbb{E}\left(f(t^{-1}\mathbf{A}R)\right) &\le t^\beta |f|_\infty \mathbb{E}\left(\mathbf{1}_{\{|t^{-1}\mathbf{A}R| > \eta\}}\right) \\ \le \|\mathbf{A}\|^\beta |f|_\infty &\left(t^\beta \|\mathbf{A}\|^\beta \mathbb{P}\left(t^{-1}\|\mathbf{A}\|\,|R| > \eta\right)\right) \\ \le \|\mathbf{A}\|^\beta |f|_\infty &\sup_{s>0} s^\beta \mathbb{P}\left(|R| > s\eta\right). \end{aligned}$$

It is a consequence of (13.8) that $C := \sup_{s>0} s^{-\beta}\mathbb{P}\left(s\,|R| > \eta\right) < \infty$. It follows that

$$\int \left(\|\mathbf{A}\|^\beta |f|_\infty \sup_{s>0} s^\beta \mathbb{P}\left(|R| > s\eta\right)\right)\mathbb{P}\left(\mathbf{T} \in d\mathbf{A}\right) = C\,|f|_\infty \mathbb{E}\|\mathbf{T}\|^\beta < \infty.$$

Using the bounded convergence theorem and that for fixed $\mathbf{A} \in GL(d, \mathbb{R})$, $x \mapsto f(\mathbf{A}x)$ is in $\mathcal{C}_c(V)$, it follows that

$$\begin{aligned} \lim_{t\to\infty} t^\beta \mathbb{E}\left(f(t^{-1}\mathbf{T}R)\right) &= \int \lim_{t\to\infty} t^\beta \mathbb{E}\left(f(t^{-1}\mathbf{A}R)\right)\mathbb{P}\left(\mathbf{T} \in d\mathbf{A}\right) \\ = \int \int_V f(\mathbf{A}x)\nu \otimes \lambda^\beta(dx)\mathbb{P}\left(\mathbf{T} \in d\mathbf{A}\right) &= \int_V \mathbb{E}\left(f(\mathbf{T}x)\right)\nu \otimes \lambda^\beta(dx). \end{aligned}$$

Finally observe that H^ε is dense in $\mathcal{C}_c(V)$ due to the Stone-Weierstrass theorem, hence the assertion holds for all $f \in \mathcal{C}_c(V)$. ☐

Remark 24.2. It is obvious from the definition of λ^β, that for all $\varepsilon > 0$ there exists $C > 0$ such that

$$\nu \otimes \lambda^\beta(S \times [C, \infty)) < \varepsilon.$$

This is the main ingredient for showing that (24.2) holds for the more general class of test functions

$$f \in \mathcal{C}_c\left(\overline{\mathbb{R}^d} \setminus \{0\}\right),$$

i.e. bounded continuous functions with limit at infinity, and supported away from 0. The proof goes along the same lines as the proof of Theorem 2.8 in [30] and is therefore omitted.

Lemma 24.3. *The measure ν satisfies $P^\beta \nu = \nu$.*

Proof. Due to Remark 24.2, the identity (24.2) also holds for bounded continuous functions f on V such that $\operatorname{supp} f \cap B_\eta(0) = \emptyset$ for some $\eta > 0$, in particular for functions $g_u(sv) = f(v)\mathbf{1}_{(u,\infty)}(s)$ where f is any continuous function on $\mathbb{S}$ and $u > 0$. Then

$$\begin{aligned}
&\int_u^\infty \int_{\mathbb{S}} f(v)\nu(dv)\lambda^\beta(ds) = \int_0^\infty \int_{\mathbb{S}} \mathbb{E}\left(f(\mathbf{T}\cdot v)\mathbf{1}_{(u,\infty)}(s\,|\mathbf{T}v|)\right)\nu(dv)\frac{1}{s^{\beta+1}}ds \\
&= \mathbb{E}\left(\int_{\mathbb{S}}\int_0^\infty f(\mathbf{T}v)\mathbf{1}_{(u,\infty)}(t)\frac{|\mathbf{T}v|^\beta}{t^{\beta+1}}dt\,\nu(dv)\right) \\
&= \int_u^\infty \int_{\mathbb{S}} \mathbb{E}\left(f(\mathbf{T}\cdot v)\,|\mathbf{T}v|^\beta\right)\nu(dv)\frac{1}{t^{\beta+1}}dt = \int_u^\infty \int_{\mathbb{S}} (P^\beta f)(v)\nu(dv)\frac{1}{t^{\beta+1}}dt.
\end{aligned}$$

Since u is arbitrary, it follows that $\int_S f(v)\nu(dv) = \int_S (P^\beta f)(v)\nu(dv)$ for all $f \in \mathcal{C}(\mathbb{S})$. Thus $P^\beta \nu = \nu$. □

Remark 24.4. Assuming additionally that

$$\forall_{x\in\mathbb{S}}\,\forall_{\text{open } U\subset\mathbb{S}} \qquad \max_{n\in\mathbb{N}} \mathbb{P}\left(\daleth_n \cdot x \in U\right) > 0, \tag{irred*}$$

it can be shown by the methods of Section 17 that ν is (up to scalar multiplication) the unique eigenmeasure of P^β with eigenvalue $m(\beta) = 1$.

Appendix

Inequalities

25.1. Inequalities for Laplace Transforms

This is a technical subsection, where a bunch of inequalities related to multivariate LTs is collected. If ϕ is the Laplace transform of a r.v. Z on $\mathbb{R}_{\geq}$, then integration by parts yields

$$\frac{1-\phi(t)}{t} = \int_0^{\infty} \exp(-tx)\mathbb{P}\left(Z > x\right) dx$$

(see [45, XIII.2 (2.7)]), thus $t^{-1}(1-\phi(t))$ is again a LT of a measure on $\mathbb{R}_{\geq}$. Consequently, it is decreasing and this yields the inequality, valid for all $u \in \mathbb{S}_{\geq}$, $t \in \mathbb{R}_{\geq}$, $0 < a < 1$:

$$\frac{1-\phi(at)}{at} \geq \frac{1-\phi(t)}{t} \quad \Rightarrow 1-\phi(at) \geq a(1-\phi(t)), \tag{25.3}$$

as well as, for $b \geq 1$,

$$1-\phi(bt) \leq b(1-\phi(t)). \tag{25.4}$$

For convenience, note this also in the multivariate setting:

Lemma 25.5. *For ϕ a Laplace transform of a distribution on $\mathbb{R}^d_{\geq}$, the following inequalities for $u \in \mathbb{S}_{\geq}$, $s \in \mathbb{R}_{>}$ hold:*

$$1-\phi(asu) \leq 1-\phi(su) \qquad \textit{for } a < 1, \tag{25.5}$$
$$1-\phi(asu) \geq a(1-\phi(su) \qquad \textit{for } a < 1, \tag{25.6}$$
$$1-\phi(bsu) \geq 1-\phi(su) \qquad \textit{for } b > 1, \tag{25.7}$$
$$1-\phi(bsu) \leq b(1-\phi(su)) \qquad \textit{for } b > 1. \tag{25.8}$$

Lemma 25.6. *Let ϕ be the Laplace transform of a distribution on $\mathbb{R}^d_{\geq}$, $u \in \mathbb{S}_{\geq}$, $t \in \mathbb{R}_{\geq}$ and $\mathbf{A} \in M(d \times d, \mathbb{R}_{\geq})$. Then*

$$1-\phi(tu) \leq 1-\phi(t\vartheta_{\mathbf{d}}) \tag{25.9}$$
$$1-\phi(t\mathbf{A}u) \leq 1-\phi(t\left|\mathbf{A}u\right|\vartheta_{\mathbf{d}}) \leq 1-\phi(t\left\|\mathbf{A}\right\|\vartheta_{\mathbf{d}}) \tag{25.10}$$
$$1-\phi(t\mathbf{A}u) \leq (\left\|\mathbf{A}\right\| \vee 1)\left(1-\phi(t\vartheta_{\mathbf{d}})\right) \tag{25.11}$$
$$1-\phi(tu) \geq 1-\phi(t(\min_i u_i)\vartheta_{\mathbf{d}}) \geq (\min_i u_i)(1-\phi(t\vartheta_{\mathbf{d}})) \tag{25.12}$$

Proof. Let Z be a r.v. with LT ϕ. For all $u \in \mathbb{S}_{\geq}$, $\langle u, Z\rangle \leq \langle \vartheta_{\mathbf{d}}, Z\rangle$. Thus

$$\begin{aligned}1-\phi(tu) &= \mathbb{E}\left(1-e^{-t\langle u,Z\rangle}\right) = \int_0^\infty te^{-tr}\mathbb{P}\left(\langle u, Z\rangle > t\right)dt \\ &\leq \int_0^\infty te^{-tr}\mathbb{P}\left(\langle \vartheta_{\mathbf{d}}, Z\rangle > t\right)dt = 1-\phi(t\vartheta_{\mathbf{d}}).\end{aligned}$$

From (25.9) and (25.5) now (25.10) follows:

$$\begin{aligned}1-\phi(t\mathbf{A}u) &= 1-\phi(t\left|\mathbf{A}u\right|\mathbf{A}\cdot u) \overset{(25.9)}{\leq} 1-\phi(t\left|\mathbf{A}u\right|\vartheta_{\mathbf{d}}) \\ &= 1-\phi(t\frac{\left|\mathbf{A}u\right|}{\left\|\mathbf{A}\right\|}\left\|\mathbf{A}\right\|\vartheta_{\mathbf{d}}) \overset{(25.5)}{\leq} 1-\phi(t\left\|\mathbf{A}\right\|\vartheta_{\mathbf{d}}).\end{aligned}$$

Then (25.11) follows by applying (25.5) resp. (25.8) in (25.10).

In order to prove (25.12), observe that

$$\langle u, Z\rangle = \sum_{i=1}^d u_i Z_i \geq \min_i u_i \sum_{i=1}^d Z_i = \min_i u_i \langle \vartheta_{\mathbf{d}}, Z\rangle.$$

Then the argument is the same as given for (25.9), with an additional use of (25.6). □

25.2. Norm Inequalities

Here several inequalities for vector and matrix norms, taken from [72] are listed.

(1) For $x, y \in \mathbb{R}^d \setminus \{0\}$ it holds that $|\overline{x}-\overline{y}| \leq \frac{2}{|x|}|x-y|$:

$$\begin{aligned}|\overline{x}-\overline{y}| &= \left|\frac{x}{|x|}-\frac{y}{|y|}\right| \\ &= \frac{1}{|x|\cdot|y|}\Big|\big|x\left|y\right|-y\left|x\right|\big|\Big| \\ &= \frac{1}{|x|\cdot|y|}\Big|\big|x\left|y\right|-y\left|y\right|+y\left|y\right|-y\left|x\right|\big|\Big| \\ &\leq \frac{1}{|x|\cdot|y|}\Big(|y|\cdot|x-y|+|y|\cdot\big||y|-|x|\big|\Big) \\ &\leq \frac{2}{|x|}\left|x-y\right|.\end{aligned}$$

(2) For $x \in \mathbb{S}_{\geq}$ it holds that $\sum_{i=1}^d x_i \geq |x|$:

Since all $x_i \geq 0$, $(\sum_{i=1}^d x_i)^2 \geq \sum_{i=1}^d x_i^2$. Now take the power $\frac{1}{2}$.

(3) Let $\mathbf{A} \in M(d\times d, \mathbb{R})$. Then $\left\|\mathbf{A}\right\| \leq \sqrt{\sum_{i,j=1}^d(\mathbf{A}(i,j))^2}$:

Let $x \in \mathbb{S}$. By Cauchy-Schwartz,

$$\sum_{i=1}^{d} x_i M(i,j))^2 = \langle x, \mathbf{A}(\cdot, j)^\top \rangle^2 \leq \|x\|^2 \, (\sum_{i=1}^{d} \mathbf{A}(i,j)^2$$

for all $1 \leq j \leq d$. This implies

$$\|\mathbf{A}\| = \sup_{x \in \mathbb{S}} \sqrt{\sum_{j=1}^{d} \left(\sum_{i=1}^{d} x_i \mathbf{A}(i,j) \right)^2} \leq \sqrt{\sum_{i,j=1}^{d} (\mathbf{A}(i,j))^2}.$$

(4) For $\mathbf{A} \in \mathcal{M}_+$, combining the above with (2) implies $\|\mathbf{A}\| \leq \sum_{i,j=1}^{d} M(i,j)$.

(5) For all $x \in \mathbb{R}^d$, $|x| \geq d^{-\frac{1}{2}} \sum_{i=1}^{d} x_i$:

Use the inequality of arithmetic and geometric means to obtain

$$\sqrt{\frac{\sum_{i=1}^{d} x_i^2}{d}} \geq \frac{\sum_{i=1}^{d} x_i}{d},$$

and multiply by $d^{\frac{1}{2}}$.

(6) Let $x \in \mathbb{S}$,$\delta < 1$. Then $\inf_{z \in B_\delta(x)} \langle x, z \rangle \geq 1 - \delta$: For all $z \in B_\delta$, $|z - x| < \delta$ and by an application of the Cauchy-Schwartz inequality

$$0 \leq 1 - \langle x, z \rangle = \langle x, x - z \rangle \leq |x| \, |x - z| \leq \delta$$

25.3. Some Calculus

Lemma 25.7. *Considering the function $f : \mathbb{R} \to \mathbb{R}$,*

$$f(s) = e^{-s} + s - 1, \tag{25.13}$$

it holds that

1. $f(s) \geq 0$ *for all* $s \in \mathbb{R}$.
2. $\lim_{s \to 0} s^{-1} f(s) = 0$.
3. $\lim_{s \to 0} s^{-2} f(s) = \frac{1}{2}$.
4. $\int_0^\infty \frac{f(s)}{s^2} ds < \infty$.

Moreover, the following inequalities hold:

$$1 - e^{-r} \leq r \leq e^{-(1-r)} \quad \forall r \in \mathbb{R}, \tag{25.14}$$

and

$$1 - e^{-r} \geq r - \frac{1}{2}r^2 \quad \forall r \in \mathbb{R}_{\geq}. \tag{25.15}$$

Proof. First, note

$$f'(s) = -e^{-s} + 1, \quad f''(s) = e^{-s}.$$

Thus f has a global minimum at $s = 0$ with $f(0) = 0$, thus (1) follows. This also gives the first inequality directly resp. by replacing $s = 1 - r$. The second inequality follows from considering $F(r) = 1 + \int_0^r f(s)ds$. (2) and (3) follow from L'Hôpital's rule. By (3), the function $g(s) := s^{-2}f(s)$ can be extended continuously in 0, and $\lim_{s\to\infty} g(s) = 0$. I.e. $g \in \mathcal{C}_0(\mathbb{R}_>)$, thus the integral in (4) converges. □

Additional Proofs

Lemma (Lemma 6.2)**.** *Let $x \in S$, $t \in \mathbb{R}$, $a \in \mathbb{R}_>$, $k \in \mathbb{N}$. Then the family*

$$(N_t)_{t\in\mathbb{R}} := (N(C_k \times [t, t+a]))_{t\in\mathbb{R}} \tag{25.16}$$

is uniformly integrable w.r.t. to $\mathbb{P}_x$, and

$$\mathbb{U}_x(C_k \times [t, t+a]) = \mathbb{E}_x N(C_1 \times [t, t+a]) \leq 2(k + 1 + ka). \tag{25.17}$$

Proof. This is the same proof as in [72]. A sufficient condition for the uniform integrability is to show that there is a r.v. $N \geq 0$ with $\mathbb{E}N < \infty$, such that

$$\sup_{t\in\mathbb{R}} \mathbb{P}_x(N_t \geq r) \leq \mathbb{P}_x(N \geq r) \quad \forall r > 0$$

(stochastic domination).

Introduce the stopping times

$$\begin{aligned}
\tau_0 &:= \inf\{n \geq 0 \ : \ X_n \in C_k,\ V_n \in [t, t+a]\},\\
\tau_{i+1} &:= \inf\{\tau_i + m \ : \ m > k, X_{\tau_i+m} \in C_k, V_{\tau_i+m} \in [t, t+a], V_{\tau_i+m} - V_{\tau_i} < mk^{-1}\}.
\end{aligned}$$

As soon as $m > ka$, the last condition is redundant, since $V_{\tau_i+m}, V_{\tau_i} \in [t, t+a]$ imply that

$$V_{\tau_i+m} - V_{\tau_i} \leq a < mk^{-1}.$$

Thus, taking also the first condition into account, it follows that

$$\begin{aligned}
N(C_k \times [t, t+a]) &= \sum_{n=0}^{\infty} \mathbf{1}_{C_k\times[t,t+a]}(X_n, V_n)\\
&= \sum_{i=0}^{\infty} \mathbf{1}_{\{\tau_i<\infty\}} \sum_{n=\tau_i}^{\tau_{i+1}-1} \mathbf{1}_{C_k\times[t,t+a]}(X_n, V_n)
\end{aligned}$$

$$\leq \sum_{i=0}^{\infty} \mathbf{1}_{\{\tau_i<\infty\}}(k+1+ka) \tag{25.18}$$

Let $\mathcal{F}_n$ be the canonical filtration of $(X_n, V_n)_{n\geq 0}$. Then for all $i \geq 0$

$$\begin{aligned}
&\mathbb{P}_x\left(\tau_{i+1}<\infty|\mathcal{F}_{\tau_i}\right)=\mathbb{P}_x\left(\tau_{i+1}<\infty, \tau_i<\infty, \ldots, \tau_0<\infty|\mathcal{F}_{\tau_i}\right)\\
&=\mathbb{P}_x\left(\tau_{i+1}<\infty|\mathcal{F}_{\tau_i}\right)\mathbf{1}_{\{\tau_i<\infty,\ldots,\tau_0<\infty\}}\\
&\leq \mathbb{P}_x\left(\exists m>k\ :\ V_{\tau_i+m}-V_{\tau_i}<mk^{-1}|\mathcal{F}_{\tau_i}\right)\mathbf{1}_{\{\tau_i<\infty,\ldots,\tau_0<\infty\}}\\
&\leq \left(1-\mathbb{P}_{X_{\tau_i}}\left(\forall m>k\ :\ V_m\geq mk^{-1}\right)\right)\mathbf{1}_{\{\tau_i<\infty,\ldots,\tau_0<\infty\}}\\
&\leq \frac{1}{2}\mathbf{1}_{\{\tau_i<\infty,\ldots,\tau_0<\infty\}},
\end{aligned}$$

where in the last line it was used that $X_{\tau_i} \in C_k$ on $\tau_i < \infty$. Thus by induction

$$\sup_{t\in\mathbb{R}} \mathbb{P}_x\left(\sum_{i=0}^{\infty}\mathbf{1}_{\{\tau_i<\infty\}}\geq m\right)=\sup_{t\in\mathbb{R}}\mathbb{P}_x\left(\tau_{m-1}<\infty\right)\leq\left(\frac{1}{2}\right)^m$$

for all $m \in \mathbb{N}$.

Thus, if N has a geometric distribution on the positive integers with parameter $\frac{1}{2}$, then for all $r > 0$

$$\sup_{t\in\mathbb{R}} \mathbb{P}_x\left(N(C_k\times[t,t+a])>r\right)\leq \mathbb{P}\left((k+1+ka)N>r\right).$$

This is the first assertion, the second follows directly:

$$\begin{aligned}
&\mathbb{E}_x\left(N(C_k\times[t,t+a])\right)=\int_0^\infty \mathbb{P}_x\left(N(C_k\times[t,t+a])>r\right)dr\\
&\leq \int_0^\infty \mathbb{P}\left((k+1+ka)N>r\right)dr=\mathbb{E}(k+1+ka)N=2(k+1+ka),
\end{aligned}$$

since $\mathbb{E}N = 2$. □

Lemma (Lemma 12.3)**.** *Let $\alpha \in \check{I}_\mu \cap (0,1)$ and $m'(\alpha) < 0$. If $\phi,\ \varphi \in \mathfrak{L}(\mathcal{P}(\mathbb{R}^d_\geq))$ and there is $t_0 \in \mathbb{R}_>$ such that for all $(y,s) \in \mathbb{S}_\geq \times [0,t_0]$,*

$$\phi(sy) \leq \varphi(sy),$$

then for all $(u,t) \in \mathbb{S}_\geq \times \mathbb{R}_\geq$

$$\liminf_{n\to\infty} \mathcal{S}_Q^n\phi(tu) \leq \liminf_{n\to\infty} \mathcal{S}_Q^n\varphi(tu) \quad \textit{and} \quad \limsup_{n\to\infty} \mathcal{S}_Q^n\phi(tu) \leq \limsup_{n\to\infty} \mathcal{S}_Q^n\varphi(tu).$$

The strategy of the proof is due to [69, Lemma 7.3] and adapted here to the inhomogeneous smoothing transform and the multivariate case which calls for some stronger assumptions. Beforehand, another Lemma is needed: Define

$$R_n := \max_{|v|=n} \|L(v)\| .$$

Lemma 25.8. *Let* $\alpha \in \breve{I}_\mu \cap (0,1)$ *and* $m'(\alpha) < 0$. *Then* $\lim_{n\to\infty} R_n = 0 \quad \mathbb{P}$*-a.s.*.

In dimension $d = 1$, this is a classical result (see e.g. [62]) which can be understood as a statement about the maximal position in a branching random walk: The logarithmic weights along the paths are additive,

$$\log L(vi) = \log L(v) + \log T_i(v)$$

and can be interpreted as displacements of particles which are generated by a Galton-Watson process. Then $\log R_n$ gives the maximal position of particles in the n-th generation which. With this interpretation it shows that the maximal position tends to $-\infty$ when $n \to \infty$. In the one-dimensional case, the condition $m'(\alpha^-) < 0$ is a sufficient assumption, for it holds as well that $N\mathbb{E}T_1^\alpha = m(\alpha) = 1$. But in the present multidimensional case $N\mathbb{E}\mathbf{T}_1^\alpha \geq m(\alpha) = 1$. This is why a slightly different proof under the assumption that $m(s) < 1$ for some $s \in (\alpha, 1)$ is given here. Note that the existence of such s is guaranteed by the assumption $\alpha \in \breve{I}_\mu$.

Proof. Step 1, reductions: $R_n \geq 0$ for all $n \in \mathbb{N}$, thus it suffices to show that $\limsup_{n\to\infty} R_n = 0$. Writing $R_{m,l} = \max_{|w|=ml} \|\mathbf{L}(w)\|$, it follows that

$$\limsup_{n\to\infty} R_n \leq \sum_{k=0}^{l-1} \sum_{|v|=k} \|\mathbf{L}(v)\| \limsup_{m\to\infty} \left[R_{m,l}\right]_v .$$

This is why it is enough to consider the maximum at each l-th generation for some $l \in \mathbb{N}$. By assumption, there is $s \in I_\mu$, such that $m(s) < 1$. Referring to the definition of $m(s)$, there is $l \in \mathbb{N}$ such that

$$\varrho(s) := \mathbb{E} \sum_{|v|=l} \|\mathbf{L}(v)\|^s = N^l \mathbb{E} \|\Pi_l\|^s < 1.$$

Fix this l. Define $Z_0 = 1$ and

$$Z_m = \sum_{|v|=l} \|\mathbf{L}(v)\|^s \left[Z_{m-1}\right]_v = \sum_{|v|=ml} \prod_{k=1}^{m} \left\| \left[\mathbf{L}(v|kl)\right]_{v|(k-1)l} \right\|^s$$

as the sum over the norms of the weights, taken in blocks of l generations. Hence $\mathbb{E}Z_1 = \varrho(s)$ and $(\left[R_{m,l}\right]_v)^s \leq \left[Z_m\right]_v$ for all $m \in \mathbb{N}$, $v \in \mathfrak{T}$.

Step 2: Considering the filtration $\mathcal{F}_m = \mathcal{T}_{ml} = \sigma((T(v))_{|v| \leq ml})$, it can be shown that $U_m := \varrho(s)^{-m} Z_m$ is a $\mathcal{F}_m$ martingale:

$$\begin{aligned} \mathbb{E}\left[U_{m+1} \middle| \mathcal{F}_m\right] &= \mathbb{E}\left[\varrho(s)^{-(m+1)} Z_{m+1} \middle| \mathcal{F}_m\right] \\ &= \mathbb{E}\left[\varrho(s)^{-(m+1)} \sum_{|v|=ml} \prod_{k=1}^{m} \left\| \left[\mathbf{L}(v|kl)\right]_{v|(k-1)l} \right\|^s \sum_{|w|=l} \left\| \left[\mathbf{L}(w)\right]_v \right\|^s \middle| \mathcal{F}_m\right] \end{aligned}$$

$$
\begin{aligned}
&= \varrho(s)^{-(m+1)} \sum_{|v|=ml} \prod_{k=1}^{m} \left\| [\mathbf{L}(v|kl)]_{v|(k-1)l} \right\|^{s} \mathbb{E} \sum_{|w|=l} \| [\mathbf{L}(w)]_{v} \|^{s} \\
&= \varrho(s)^{-(m+1)} \sum_{|v|=ml} \prod_{k=1}^{m} \left\| [\mathbf{L}(v|kl)]_{v|(k-1)l} \right\|^{s} \mathbb{E} \sum_{|w|=l} \| \mathbf{L}(w) \|^{s} \\
&= \varrho(s)^{-m} Z_m = U_m \quad \mathbb{P}\text{-a.s.}
\end{aligned}
$$

Moreover, U_m is a nonnegative martingale, thus it converges to a random variable U and by Fatou's lemma, $\mathbb{E}\, U \leq \mathbb{E}\, \varrho(s) Z_1 = 1$. In particular, U is almost sure finite, and this gives the final estimate

$$
\limsup_{m\to\infty} (R_{m,l})^s \leq \limsup_{m\to\infty} \varrho(s)^m U_m = 0 \quad \mathbb{P}\text{-a.s..} \qquad \square
$$

Proof of Lemma 12.3. Fix $(u,t) \in \mathbb{S}_{\geq} \times \mathbb{R}_{\geq}$. Let $\varepsilon > 0$. By Lemma 25.8, there is $n_0 \in \mathbb{N}$ such that $\mathbb{P}\,(tR_n > t_0) < \varepsilon$ for all $n \geq n_0$. On the set $tR_n \leq t_0$, by assumption $\phi(t\mathbf{L}(v)^\top u) \leq \varphi(t\mathbf{L}(v)^\top u)$ for all v with $|v| = n$ and the same holds true when multiplying both sides with $\exp(-t\langle u, \sum_{|w|<n} \mathbf{L}(w)Q(w)\rangle)$. Therefore, for all $n \geq n_0$

$$
\begin{aligned}
\mathcal{S}_Q^n \phi(tu) &= \mathbb{E} \exp(-t\langle u, \sum_{|w|<n} \mathbf{L}(w)Q(w)\rangle) \prod_{|v|=n} \phi(t\mathbf{L}(v)^\top u) \\
&= \mathbb{E}\mathbf{1}_{\{tR_n \leq t_0\}} \exp(-t\langle u, \sum_{|w|<n} \mathbf{L}(w)Q(w)\rangle) \prod_{|v|=n} \phi(t\mathbf{L}(v)^\top u) \\
&\quad + \mathbb{E}\mathbf{1}_{\{tR_n > t_0\}} \exp(-t\langle u, \sum_{|w|<n} \mathbf{L}(w)Q(w)\rangle) \prod_{|v|=n} \phi(t\mathbf{L}(v)^\top u) \\
&\leq \mathbb{E}\mathbf{1}_{\{tR_n \leq t_0\}} \exp(-t\langle u, \sum_{|w|<n} \mathbf{L}(w)Q(w)\rangle) \prod_{|v|=n} \varphi(t\mathbf{L}(v)^\top u) + \mathbb{P}\,(tR_n > t_0) \\
&\leq \mathbb{E} \exp(-t\langle u, \sum_{|w|<n} \mathbf{L}(w)Q(w)\rangle) \prod_{|v|=n} \varphi(t\mathbf{L}(v)^\top u) + \varepsilon = \mathcal{S}_Q^n \varphi(tu) + \varepsilon.
\end{aligned}
$$

Now take first the $\liminf_{n\to\infty}$ resp. $\limsup_{n\to\infty}$, and then let $\varepsilon \to 0$ to infer the assertion. $\square$

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Index

Acronyms

(i-d) irreducibility and density assumption.

a.s. almost surely.

dRi directly Riemann integrable.

FP fixed point.

FT Fourier transform.

i.i.d. independent identically distributed.

LHS left-hand side.

LT Laplace transform.

MC Markov chain.

MRT Markov renewal theorem.

MRW Markov random walk.

r.v. random variable.

RDE Random difference equation.

RHS right-hand side.

RW random walk.

sdRi strongly directly Riemann integrable.

SFPE stochastic fixed point equation.

w.l.o.g. without loss of generality.

w.r.t. with respect to.

WBP weighted branching process.

wdRi weakly directly Riemann integrable.

List of Symbols

The page numbers indicate where further information can be found. In Chapter A, $S = \mathbb{S}_{\geq}$ and in Chapter B, $S = \mathbb{S}$ as well as $\mathcal{M} = M(d \times d, \mathbb{R}_{\geq})$ resp. $\mathcal{M} = GL(d, \mathbb{R})$.

$(\exists\beta)$	$\exists_{s_0>0} \quad \mathbb{E}\inf_{u\in\mathbb{S}} \left\lvert \mathbf{T}^\top x\right\rvert^{s_0} \geq 1$	81
(density)	$\exists_{\Gamma_0\in GL(d,\mathbb{R})}\, \exists_{c,p>0}\, \exists_{n_0\in\mathbb{N}}\, \mathbb{P}\left(\Pi_{n_0} \in \cdot\right) \geq p\mathbf{1}_{B_c(\Gamma_0)}\lambda^{d^2}$	80
(eigenvalue)	$w = N\mathbb{E}\mathbf{T}_1 w + \mathbb{E}Q$	13
(irred)	$\forall_{x\in\mathbb{S}}\, \forall_{\text{open } U\subset\mathbb{S}}\ \max_{n\in\mathbb{N}} \mathbb{P}\left(\Pi_n \cdot x \in U\right) > 0$	80
(l<0)	$l = \mathbb{P}$-a.s. $-\lim_{n\to\infty} \frac{1}{n} \log \lVert \neg_n \rVert < 0$	78
(log-moments)	$\mathbb{E}\left\lvert\log \lVert\mathbf{M}_1\rVert\right\rvert + \lvert\log \iota(\mathbf{M}_1)\rvert + \lvert\log \iota(\mathbf{T}_1)\rvert < \infty$	64
(logmom)	$\mathbb{E}\log^+ \lVert\mathbf{T}\rVert + \log^+ \lvert Q\rvert < \infty$	78
(M logM)	$\mathbb{E}(1 + \lVert\mathbf{M}_1\rVert)\left(1 + \left\lvert\log\lVert\mathbf{M}_1\rVert\right\rvert + \lvert\log\iota(\mathbf{M}_1)\rvert\right) < \infty$	36
(s-moments)	$\mathbb{E}\left(\lVert\mathbf{T}_1\rVert^s + \lvert Q\rvert^s\right) < \infty$	13
(MC1)	$\mathbb{P}\left((\Pi_1 \cdot y, \Pi_1) \in \cdot\right) \geq \xi\Psi(y, \cdot)$	90
(MC2)	$\mathbb{P}\left(\Pi_1 \cdot y \in \cdot, \Pi_1 \in C\right) \geq \xi\Phi$	90
(Q-beta)	$0 < \mathbb{E}\lvert Q\rvert^\beta < \infty$	80
(R...)	properties of the regenerative structure, see Lemma 18.4	97
$(R \neq r)$	$\forall\, r \in \mathbb{R}^d \quad \mathbb{P}\left(\mathbf{T}r + Q = r\right) < 1$	80
(StA)	If (irred),(density),(MC1') and (MC2') hold, they hold with $n_0 = n = 1$	89
(TlogT)	$\mathbb{E}\lVert\mathbf{T}\rVert^\beta \left(\left\lvert\log\lVert\mathbf{T}\rVert\right\rvert + \left\lvert\log\left\lVert\mathbf{T}^{-1}\right\rVert\right\rvert\right) < \infty$	80
(variance)	$\Sigma = N\mathbb{E}\left(\mathbf{T}_1\Sigma\mathbf{T}_1^\top\right)$	13
$\perp$	For $W \subset \mathbb{R}^d$, $W^\perp = \{x \in \mathbb{R}^d \ :\ \langle x, y\rangle = 0\ \forall y \in W\}$	30
$\cdot$	action of matrices on the sphere, $\mathbf{A} \cdot x = \overline{\mathbf{A}x}$	3
$\vee$	$(a \vee b)_i = \max\{a_i, b_i\}$, $i = 1, \ldots, d$ for $a, b \in \mathbb{R}^d$	24
$\wedge$	$(a \wedge b)_i = \min\{a_i, b_i\}$, $i = 1, \ldots, d$ for $a, b \in \mathbb{R}^d$	24
$\bar{g}$	$\bar{g}(y, t) = \int_{-\infty}^t e^{-(t-s)} g(y, s) ds$	109
$\overline{x}$	$\overline{x} = \lvert x\rvert^{-1} x$ for $x \in \mathbb{R}^d$	3
$\lvert\cdot\rvert_\infty$	$\lvert f\rvert_\infty = \sup_{x\in E} \lvert f(x)\rvert$ for $f \in \mathcal{C}(E)$.	3
$\breve{}$	topological interior of the specified set	4
$\lVert\cdot\rVert$	operator norm: $\lVert A\rVert := \sup_{\lvert x\rvert=1} \lvert Ax\rvert$	3
$\langle\cdot, \cdot\rangle$	euclidean scalar product on $\mathbb{R}^d$	3
$[\cdot]_v$	tree shift operator	7
$\xrightarrow{v}$	vague convergence	9
$\xrightarrow{d}$	convergence in distribution, weak convergence	9
A^c	For $A \subset E$: $A^c = E \setminus A$	
α	$\alpha := \inf\{s \in I_\mu : m(s) \leq 1\}$. $\alpha \in \check{I}_\mu \Rightarrow m(\alpha) = 1, m'(\alpha) \leq 0$	4

$B_\varepsilon(x)$	$B_\varepsilon(x) := \{y \;:\; \lvert x-y\rvert < \varepsilon\}$	3
$\mathcal{B}^1(E)$	bounded Borel-measurable functions $E \to \mathbb{R}$ with $\lvert f\rvert_\infty \le 1$.	8
β	$\beta := \sup\{s \in I_\mu : m(s) \le 1\}$. $\beta \in \breve{I}_\mu \Rightarrow m(\beta) = 1, m'(\beta) \ge 0$	4
$\mathcal{C}(E)$	set of continuous mappings $f : E \to \mathbb{R}$	3
$\mathcal{C}^m(E)$	set of m-times continuously differentiable mappings $f : E \to \mathbb{R}$	3
$\mathcal{C}_0(E)$	set of continuous mappings $f : E \to \mathbb{R}$, that vanish at infinity	3
$\mathcal{C}_b(E)$	set of bounded continuous mappings $f : E \to \mathbb{R}$	3
$\mathcal{C}_c(E)$	set of compactly supported continuous mappings $f : E \to \mathbb{R}$	3
$\mathcal{C}_c\left(\overline{\mathbb{R}^d_\ge} \setminus \{0\}\right)$	functions $f \in \mathcal{C}_b\left(\mathbb{R}^d_\ge\right)$ which are supported away from the origin	22
$\mathcal{C}(E)'$	conjugate space of $(\mathcal{C}(E), \lvert\cdot\rvert_\infty)$: regular bounded signed measures on E, equipped with the total variation norm	31, 91
C_k	$C_k = \left\{x \in S \;:\; \mathbb{P}_x\left(\frac{V_m}{m} \ge \frac{1}{k}\ \forall m \ge k\right) \ge \frac{1}{2}\right\}$	27
c_s	$c_s = \left\lvert \int \langle\cdot, y\rangle^s \nu^s(dy)\right\rvert_\infty$	32
∂B	topological boundary of the set B	9
$d{-}\lim$	weak limit, limit in distribution	9
$D_{\alpha,n}$	$D_{\alpha,n} = D_{\alpha,\mathcal{S}^n\phi_0}$	43
δ	Dirac measure in the specified point	1
D^k	$D^k f$: k-th Fréchet derivative of f	11
$D_{\chi,\phi}$	$D_{\chi,\phi}(u,t) = \frac{e^{\chi t}}{e_*^s(u)}(1 - \phi(e^{-t}u))$	41
$\mathcal{D}_s$	$\left\{f \in \mathcal{C}^k\left(\mathbb{R}^d\right) \;:\; \forall_{x,y\in\mathbb{R}^d}\ \left\lVert D^k f(x) - D^k f(y)\right\rVert \le \lvert x-y\rvert^{s-k}\right\}$	11
$\mathrm{Eig}(\mathbf{A}, \lambda)$	eigenspace of $\mathbf{A}$ for eigenvalue λ	14
$\mathrm{Eig}_0(\mathbf{A}, \lambda)$	$\mathrm{Eig}(\mathbf{A}, \lambda) \cup \{0\}$	14
$\mathrm{Eig}^+(P_*^\alpha, \lambda)$	positive eigenfunctions of $\mathcal{P}_\alpha$ for eigenvalue λ	73
e^s	$P^s e^s = \kappa(s) e^s$, $\lvert e^s\rvert_\infty = 1$	32, 90
e_*^s	$P_*^s e_*^s = \kappa(s) e_*^s$, $\lvert e_*^s\rvert_\infty = 1$	32, 90
$\mathbb{E}_x$	expectation symbol of $\mathbb{Q}_x$	34
$\mathbb{E}_x^s$	expectation symbol of $\mathbb{Q}_x^s$, satisfying $\mathbb{E}_x^s\left(f((X_i,V_i)_{i=1}^n)\right) = \frac{1}{e_*^s(x)\kappa^n(s)}\mathbb{E}_x\left(e^{sV_n} e_*^s(X_n)((X_i,V_i)_{i=1}^n)\right)$ for all bounded measurable functions f and all $n \in \mathbb{N}$.	35, 95
$\mathcal{F}$	set of fixed points of $\mathcal{S}$	1
$\mathcal{F}^\aleph$	set of $\aleph$-elementary fixed points of $\mathcal{S}$	53
$\mathcal{F}_0^\alpha$	set of α-elementary fixed points of $\mathcal{S}_0$	69
$\mathcal{F}_Q^\alpha$	set of α-elementary fixed points of $\mathcal{S}_Q$	69
$\mathcal{F}_s$	$\mathcal{F} \cap \mathcal{P}_s(\mathbb{R}^d)$	14
$\hat{g}$	For $g : \mathbb{S}_\ge \times \mathbb{R} \to \mathbb{R}$, $\hat{g}(t) = \sup_{u\in S} \lvert g(u,t)\rvert$	36
Γ_0	invertible matrix, appearing in (density)	80
$G_{\alpha,n}$	$G_{\alpha,n} = G_{\alpha,\mathcal{S}^n\phi_0}$	43

$G_{\chi,\phi}(u,t)$	$= \frac{e^{\chi t}}{e_*^s(u)} \mathbb{E}\left(\prod_{i=1}^N \phi(e^{-t}\mathbf{T}_i^\top u) + \sum_{i=1}^N \left(1-\phi(e^{-t}\mathbf{T}_i^\top u)\right) - 1\right)$	41
$E_{\chi,c}$	extremal points of $H_{\chi,c}$, $E_{\chi,c} = \left\{(u,t) \mapsto c\frac{e_*^\gamma(u)}{e_*^\chi(u)} e^{(\chi-\gamma)t} \ : \ \gamma \in (0,1], m(\gamma)=1\right\}$.	65
$H_{\chi,c}$	compact subset of J_χ, see Def. 11.9	59
$H^\gamma(E)$	set of γ-Hölder functions on E: $H^\gamma = \{f \in \mathcal{C}(E) \ : \ \sup_{x,y\in E} \frac{\lvert f(x)-f(y)\rvert}{\lvert x-y\rvert^\gamma} < \infty\}$	24
h_s	$h_s(u,t) = \frac{D_{\chi,\phi}(u,s+t)}{e^{\chi s}(1-\phi(e^{-s}\vartheta_{\mathbf{d}}))} = \frac{e^{\chi t}}{e_*^\chi(u)} \frac{1-\phi(e^{-(s+t)}u)}{1-\phi(e^{-s}\vartheta_{\mathbf{d}})}$	57
$\mathbf{Id}$	identity matrix	7
I_μ	$I_\mu = \{s \geq 0 \ : \ \mathbb{E}\lVert\mathbf{T}_1\rVert^s < \infty\}$	4
$\iota(\mathbf{A})$	$\iota(\mathbf{A}) = \inf_{x\in\mathbb{S}_\geq} \lvert\mathbf{A}x\rvert$	3
J_χ	compact subset of $\mathcal{C}(\mathbb{S}_\geq \times \mathbb{R})$, see Def. 11.4	55
$(J_n)_{n\in\mathbb{N}_0}$	iid B(1,ξ) r.v.s, determining whether regeneration occurs, see Lemma 18.4	98
$\mathbb{K}$	covariance matrix	12
$\kappa(s)$	$\kappa(s) = \lim_{n\to\infty} \left(\mathbb{E}\lVert\daleth_n\rVert^s\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\mathbb{E}\lVert\Pi_n\rVert^s\right)^{\frac{1}{n}}$	4
$\mathfrak{L}$	Laplace transform mapping $\eta \mapsto \phi_\eta$	2
$\mathbf{L}(v)$	matricial path weights in the weighted branching tree, recursively defined by $\mathbf{L}(\emptyset) = \mathbf{Id}$, $\mathbf{L}(vi) = \mathbf{L}(v)\mathbf{T}_i(v)$	7
$l(s)$	$l(s) = \mathbb{E}_{\pi_*^s}^s V_1 = \frac{\kappa'(s^-)}{\kappa(s)}$	35
$l(0)$	$l(0) = \lim_{n\to\infty} \frac{1}{n} \log\lvert\Pi_n x\rvert < 0 \quad \mathbb{P}$-a.s., upper Lyapunov exponent	35
$\Lambda(\Gamma)$	$\Lambda(\Gamma) = \overline{\{u_{\mathbf{A}} \ : \ \mathbf{A} \in \Gamma \cap \check{\mathcal{M}}_+\}}$	32
$\lambda_{\mathbf{A}}$	Perron-Frobenius eigenvalue of $\mathbf{A}$	31
$\mathcal{L}$	law (distribution) of the specified random variable	1
λ^{d^2}	Lebesgue measure on $M(d\times d,\mathbb{R}) \subset \mathbb{R}^{d^2}$	30
λ^α	$\lambda^\alpha(ds) = \frac{1}{s^{1+\alpha}}ds$	22
$\lim_{t\downarrow 0}$	right sided limit in zero, $\lim_{t\downarrow 0} = \lim_{t\to 0,t>0}$	21
l_s	$l_s(\nu,\eta) := \inf\{\mathbb{E}\lvert Y-Z\rvert^s \ : \ \mathcal{L}(Y)=\nu, \mathcal{L}(Z)=\eta.\}$	11
L_t	$L_t(u,r) = \frac{g(u,t+r)}{g(u,r)}$ for $g \in H_{\chi,c}$	59
$(\mathbf{M}_n)_{n\in\mathbb{N}}$	sequence of i.i.d. random matrices with distribution μ^*	4
$m'(s^-)$	left derivative of m in s	16
$m(s)$	spectral function, Chapter A: $m(s) = N\kappa(s)$, Chapter B: $m(s) = \kappa(s)$	4
m_χ	spectral function of T_χ	50
$M(d\times d,E)$	set of $d\times d$ matrices with entries in E	3
$\mathcal{M}^\pm(E)$	(vector space) of regular bounded signed measures on E	10

$\mathcal{M}_+$	$\mathcal{M}_+ = M(d \times d, \mathbb{R}^d_{\geq})$	3
$[\operatorname{supp} \mu]$	smallest closed subsemigroup containing $\operatorname{supp} \mu$	4
supp	support of a measure μ: $\{x \; : \; \forall$ open O with $x \in O, \mu(O) > 0\}$	4
μ	$\mu = \mathcal{L}(\mathbf{T}_1) = \cdots = \mathcal{L}(\mathbf{T}_N)$	3
μ_χ	$\mu_\chi = \mathcal{L}\left(m(\chi)^{-\frac{1}{\chi}} \mathbf{T}_1\right)$	50
μ^*	$\mu^* = \mathcal{L}\left(\mathbf{T}_1^\top\right)$	4
$\mathbb{N}$	positive integers $\{1, 2, \dots\}$	3
$N(t)$	first exit time $N(t) = \inf\{n \geq 0 : V_n > t\}$	100
$\mathbb{N}_0$	natural numbers $\{0, 1, 2, \dots\}$	3
$N(0, \Sigma)$	multivariate Normal distribution with expectation 0 and covariance matrix Σ	14
$N(A)$	number of renewals (visits of the random walk) in the set A	27
ν^s	probability measure, $P^s \nu^s = \kappa(s) \nu^s$	32, 90
ν^s_*	probability measure, $P^s_* \nu^s_* = \kappa(s) \nu^s_*$	32, 90
$\hat{O}$	$\hat{O}((x, \mathbf{A}, q), A \times B \times C) = \mathbb{P}((\Pi_1 \cdot x, \Pi_1, Q_1) \in A \times B \times C)$, Markov transition operator of $(X_n, \mathbf{M}_n, Q_n)_{n \in \mathbb{N}_0}$ under $\mathbb{O}_x$	96
Ω	$\Omega = S \times \mathcal{M}^{\mathbb{N}}$	34
$\mathbb{O}_x$	$\mathbb{O}_x = \delta(x) \otimes \delta(\mathbf{Id}) \otimes \delta(0) \otimes \bigotimes_{n=1}^\infty \varrho$	96
Φ	minorizing measure: $\mathbb{P}(\Pi_1 \cdot y \in \cdot, \Pi_1 \in C) \geq \xi \Phi$ (under (StA)), $\Psi(y, \cdot \times C) = \Phi$ for all $y \in \mathbb{S}$	84
ϕ_Z	Laplace transform of Z, $\phi_Z(x) = \mathbb{E} \exp(-\langle x, Z \rangle)$	2
ϕ_0	(In Section 9:) $\phi_0(tu) = \exp(-K t^\alpha e^\alpha_*(u))$	38
$\mathbf{\Pi}_n$	$\mathbf{\Pi}_n = \mathbf{M}_n \dots \mathbf{M}_1$	34
$\urcorner_n$	$\urcorner_n = \mathbf{T}_{(1)} \cdot \ldots \cdot \mathbf{T}_{(n)}$	78
π^s_*	stationary distribution for Q^s_*, $\pi^s_*(dx) = e^s_*(x) \nu^s_*(dx)$	35
${}^\alpha P$	${}^s P f(u, t) = \mathbb{E}^s_u f(X_1, t - V_1)$	43
P^s	$P^s f(x) = \mathbb{E}\left(\lvert \mathbf{T}_1 x \rvert^s f(\mathbf{T}_1 \cdot x)\right)$	29, 90
$\mathcal{P}$	set of probability measures (on the specified measurable space)	1
$\mathcal{P}_s(E)$	set of probability measures on E with finite s-th moment	10
$\mathcal{P}_{w_d}(\eta)$	$\mathcal{P}_s(\eta) = \{\nu \in \mathcal{P}(\mathbb{R}^d) \; : \; l_s(\nu, \eta) < \infty\}$	11
$\mathcal{P}_{s,w}(E)$	set of probability measures on E with finite s-th moment and expectation w	12
$\mathcal{P}_{s,w,\Sigma}(E)$	set of probability measures on E with finite s-th moment, expectation w and covariance matrix Σ	12
Ψ	minorizing kernel: $\mathbb{P}((\Pi_1 \cdot y, \Pi_1) \in \cdot) \geq \xi \Psi(y, \cdot)$ (under (StA)), there is compact C with $\operatorname{supp} \Psi(y, \cdot) \subset B_\delta(x) \times C$ for all $y \in \mathbb{S}$	84
ψ	(In Section 9:) $\psi = \lim_{n \to \infty} \mathcal{S}^n \phi_0$	40
P^s_*	$P^s_* f(x) = \mathbb{E}\left(\left\lvert \mathbf{T}_1^\top x \right\rvert^s f(\mathbf{T}_1^\top \cdot x)\right) = \mathbb{E}\left(\lvert \mathbf{M}_1 x \rvert^s f(\mathbf{M}_1 \cdot x)\right)$	29, 90
$\mathbb{Q}^s$	$\mathbb{Q}^s = \int \mathbb{Q}^s_x \pi^s_*(dx)$, $(X_n)_{n \in \mathbb{N}_0}$ is stationary under $\mathbb{Q}^s$	35

$\hat{Q}_s$	$\hat{Q}_s((x,u), A \times B) = \frac{1}{e_*^s(x)\kappa(s)} \mathbb{E}\left(e_*^s(\mathbf{M}_1 \cdot x)\,\lvert\mathbf{M}_1 x\rvert^s\, \mathbf{1}_A(\mathbf{M}_1 \cdot x)\mathbf{1}_B(\log\lvert\mathbf{M}_1 x\rvert)\right)$. Markov transition operator of $(X_n, U_n)_{n\in\mathbb{N}_0}$ under $\mathbb{Q}_x^s$	96
Q_*^s	Markov transition kernel on $\mathcal{C}(S)$, $Q_*^s f(x) = \frac{1}{e_*^s(x)\kappa(s)} P_*^s(e_*^s f)(x)$	35
Q^n	$Q^n = \sum_{k=1}^n \urcorner_{k-1} Q_k$	87
${}_n\mathbb{Q}_x^s$	${}_n\mathbb{Q}_x^s((X_0, (\mathbf{M}_i)_{i=1}^n) \in A) := \frac{1}{e_*^s(x)\kappa^n(s)} \mathbb{E}_x\left(e^{sV_n} e_*^s(X_n) \mathbf{1}_A(X_0, (\mathbf{M}_i)_{i=1}^n)\right)$	34
$\mathbb{Q}_x^s$	probability measure on $(\Omega, \mathcal{A})$, projective limit of ${}_n\mathbb{Q}_x^s$	95
$\mathbb{Q}_x$	$\mathbb{Q}_x = \delta_x \otimes \bigotimes_{n=1}^\infty \mu^*$	34
$\mathbb{R}_\geq$	nonnegative real numbers $[0, \infty)$	1, 3
R	$R = \sum_{n=1}^\infty \urcorner_{n-1} Q_n$, $R \overset{d}{=} \mathbf{T}R + Q$	78
$r(Q)$	spectral radius of the operator **Q**	91
$R(t)$	residual lifetime process $R(t) = (V_{N(t)} - t)\mathbf{1}_{\{N(t)<\infty\}}$	100
ϱ	$\varrho = \mathcal{L}(\mathbf{T}, Q)$	80
R^n	$R^n = \sum_{k>n} \left(\prod_{j=n+1}^{k-1} \mathbf{T}_{(j)}\right) Q_k$	87
R_n	$R_n = \max_{\lvert v\rvert=n} \lVert L(v)\rVert$	128
$\mathbb{R}_>$	positive half-line $(0, \infty)$	3, 18
$S_\alpha(\sigma, \lambda, b)$	one-dimensional stable distribution with scale parameter σ, skewness parameter λ and shift parameter b	19
$\tilde{S}_\alpha(Ke_*^\alpha, 0)$	multivariate stable distribution with LT $\phi(tu) = \exp\left(-Kt^\alpha e_*^\alpha(u)\right)$	38
$S_\alpha(K\nu, b)$	multivariate stable distribution with scale parameter K, spectral measure ν and shift parameter b	20
$\tilde{S}_\alpha(K\nu, 0)$	multivariate stable distribution with LT $\exp\left(-K \int_{\mathbb{S}_\geq} \langle x, y\rangle^\alpha \nu(dy)\right)$, $x \in \mathbb{R}_\geq^d$	21
$\mathbb{S}$	unit sphere in $\mathbb{R}^d$	3
$(\sigma_n)_{n\in\mathbb{N}_0}$	regeneration epochs for $(X_n, U_n)_{n\in\mathbb{N}_0}$ under $\mathbb{Q}_x^\beta$ w.r.t. Υ.	99
s_∞	$s_\infty = \sup I_\mu$	4
$\mathbb{S}_\geq$	$\mathbb{S} \cap \mathbb{R}_\geq^d$	3
$\mathcal{S}$	smoothing transform	1
$\mathcal{S}_0$	homogeneous smoothing transform $\mathcal{S}_0 : \nu \mapsto \mathcal{L}\left(\sum_{i=1}^N \mathbf{T}_i Y_i\right)$	2
$\mathcal{S}_\chi$	$\mathcal{S}_\chi : \nu \mapsto \mathcal{L}\left(\sum_{i=1}^N \frac{1}{m(\chi)^{\frac{1}{\chi}}} \mathbf{T}_i X_i\right)$	50
$\mathcal{S}f(x)$	$\mathcal{S}f(x) = \mathbb{E}\left(\prod_{i=1}^N f(\mathbf{T}_i^\top x)\right)$	8
$\mathcal{S}_Q$	inhomogeneous smoothing transform $\mathcal{S}_Q : \nu \mapsto \mathcal{L}\left(\sum_{i=1}^N \mathbf{T}_i Y_i + Q\right)$	2
T	$T = (\mathbf{T}_i)_{i=1}^N$	1
$\mathfrak{T}$	N-ary Ulam-Harris tree, $\mathfrak{T} := \bigcup_{n=0}^\infty \{1, \ldots, N\}^n$	6
$(\tau_n)_{n\in\mathbb{N}}$	hitting times of $(X_n)_{n\geq 0}$ in $B_\delta(x_0)$	107

T_χ	$T_\chi = (\mathbf{T}_{\chi,1},\ldots,\mathbf{T}_{\chi,N}) = m(\chi)^{-\frac{1}{\chi}}(\mathbf{T}_1,\ldots,\mathbf{T}_N)$	50
$\vartheta_{\mathbf{1}}$	$\vartheta_{\mathbf{1}} := \sqrt{d}^{-1}(1,\ldots,1)^\top \in \mathbb{R}^d$	24
$\vartheta_{\mathbf{d}}$	$\vartheta_{\mathbf{d}} := (1,\ldots,1)^\top \in \mathbb{R}^d$	24
$(\mathbf{T}_i)_{i=1}^N$	random matrices in $M(d\times d,\mathbb{R}_\geq)$, w.l.o.g. identically distributed	3
$(\mathbf{T}_{(n)})_{n\in\mathbb{N}}$	sequence of i.i.d. random matrices with distribution μ	4
$\mathcal{T}$	$\mathcal{T} := (T(v))_{v\in\mathfrak{T}}$, sequence of i.i.d. copies of T	6
$\mathcal{T}_n$	filtration of $\mathcal{T}$, $\mathcal{T}_n = \sigma\left((T(v))_{\lvert v\rvert\leq n}\right)$	7
$\mathbb{U}$	renewal measure	26
$u_{\mathbf{A}}$	normalized Perron-Frobenius eigenvector of $\mathbf{A}$	31
Υ	minorizing kernel: $\hat{Q}_s((x,u),\cdot) \geq \xi_s\Upsilon(x,\cdot)$ There is compact $I\subset\mathbb{R}$ with supp $\Upsilon(x,\cdot)\subset B_\delta(x_0)\times I$ for all $x\in\mathbb{S}$	96
$\mathbb{U}_x$	Markov renewal measure, $\mathbb{U}_x = \sum_{n=0}^\infty \mathbb{P}_x\left((X_n,V_n)\in\cdot\right)$, $g*\mathbb{U}_x(t) = \mathbb{E}_x\left(\sum_{n=0}^\infty g(X_n,t-V_n)\right)$	27
$\mathbb{U}_x^s$	Markov renewal measure of $(X_n,V_n)_{n\in\mathbb{N}_0}$ under $\mathbb{Q}_x^s$	37
W_n	Biggins martingale for $\alpha=1$: $W_n = \sum_{\lvert v\rvert=n}\mathbf{L}(v)w$, $w = N\mathbb{E}\mathbf{T}_1 w$	15
$(\varpi_n)_{n\in\mathbb{N}_0}$	feasible sequence of regeneration epochs for $(X_n,\mathbf{M}_n,Q_n)_{n\in\mathbb{N}_0}$ under $\mathbb{O}_x$ w.r.t. Ξ, $X_{\varpi_n-1}\in B_\delta(X_0)$.	107
W_n^*	$W_n^* = \sum_{k=0}^{n-1}\sum_{\lvert v\rvert=k}\mathbf{L}(v)Q(v)$	16
$W_n(u)$	Biggins martingale for $\alpha<1$: $W_n(u) = \sum_{\lvert v\rvert=n}\int_{\mathbb{S}_\geq}\langle\mathbf{L}(v)^\top u,y\rangle^\alpha\nu^\alpha(dy)$	38
W^*	a.s. limit of W_n^*	16
$W(u)$	a.s. limit of $W_n(u)$	38
Ξ	minorizing kernel: $\hat{O}(x,\cdot)\geq\Xi(x,\cdot)$. There are $L,\varsigma>0$ such that $\Xi(x,B_\delta(x_0)\times B\times\mathbb{R}^d) = L\int_{B_\varsigma(\mathbf{Id})}\mathbf{1}_{B_\delta(x_0)\times B}(\mathbf{A}\cdot x,\mathbf{A}\mathbf{A}_x)\lambda^{d^2}(d\mathbf{A})$	97
$(X_n,V_n)_{n\in\mathbb{N}_0}$	Markov random walk; $X_n := \Pi_n\cdot X_0$, $V_n := \log\lvert\Pi_n X_0\rvert$	34, 64
Y_n	weighted branching process associated with $\mathcal{T}\otimes\mathcal{Y}$, $Y_n := \sum_{\lvert v\rvert=n}\mathbf{L}(v)Y(v) + \sum_{k=0}^{n-1}\sum_{\lvert v\rvert=k}\mathbf{L}(v)Q(v)$	7
$\mathcal{Y}$	$\mathcal{Y} := (Y(v))_{v\in\mathfrak{T}}$, sequence of i.i.d. copies of a r.v. Y	7
$Z(t)$	jump process $Z(t) = X_{N(t)}\mathbf{1}_{\{N(t)<\infty\}}$	100
ζ_s	Zolotarev metric	11