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p-adic Weyl algebras

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Introduction

Let K be a complete non-archimedean field. We consider the d-th classical Weyl algebra A_d over K (see section 1.1 for a definition) and, for some $\varepsilon \in \mathbb{R}^{2d}_{>0}$, endow it with the non-archimedean K-vector space norm

$$|f|_\varepsilon = \max |a_{\alpha\beta}|\varepsilon^{(\alpha,\beta)}$$

if $f \in A_d$ is written in the form $f = \sum a_{\alpha\beta} X^\alpha Y^\beta$. If we require ε to be an element of $\mathbb{R}^{2d}_{>0}$ such that $|\gamma!|\varepsilon^{(-\gamma,-\gamma)}$ is bounded by some constant for all $\gamma \in \mathbb{N}^d$ then the norm is in fact an algebra norm on A_d (cf. lemma 1.2.1). The norm is multiplicative if and only if ε satisfies $\varepsilon_i \varepsilon_{d+i} \geq 1$ for all $1 \leq i \leq d$ (cf. lemma 1.2.4). We denote the completion of A_d with respect to this norm by $\mathcal{A}_{d,\varepsilon}$. The elements of $\mathcal{A}_{d,\varepsilon}$ can be written as formal power series in non-commuting variables such that the coefficients satisfy a certain convergence condition:

$$\mathcal{A}_{d,\varepsilon} = \{\sum a_{\alpha\beta} X^\alpha Y^\beta : |a_{\alpha\beta}|\varepsilon^{(\alpha,\beta)} \to 0 \text{ for } |\alpha| + |\beta| \to \infty\}.$$

We call the elements of $\mathcal{A}_{d,\varepsilon}$ restricted power series. Different versions of completed Weyl algebras appear in the literature. One can construct the algebra which is the union of all $\mathcal{A}_{d,\varepsilon}$ with $\varepsilon_1 = \ldots = \varepsilon_d = 1$ and $\varepsilon_i > 1$ for all $d+1 \leq i \leq 2d$ and the algebra which is the union of all $\mathcal{A}_{d,\varepsilon}$ with $\varepsilon_i > 1$ for all i. These algebras are considered in [Ber] and [MN]. The latter version is denoted by $\mathcal{A}_d^\dagger$ and is called the *Dwork-Monsky-Washnitzer-Weyl* algebra. The algebra $\mathcal{A}_{d,(1,\ldots,1)}$ appears in [Nar2]. We call it the *Tate-Weyl* algebra.

The fact that we can define a whole family of algebra norms on the classical Weyl algebra defined over a non-archimedean field is in sharp contrast to the fact that the classical Weyl algebra defined over the field of complex numbers has no algebra norm at all (see for example [Cun]).

The classical Weyl algebra A_d defined over an arbitrary field has been extensively studied during the last 50 years. The classical Weyl algebra A_d is a left and right Noetherian integral domain. The classical Weyl algebra A_d is simple if defined over a field of characteristic zero. In this case the Krull and the global dimension of A_d are d; the Krull dimension of A_d was first determined by Gabriel and Rentschler in [GR]. That the global dimension of A_d is d was proved by Rinehart [Rin] for $d = 1$ and in the general case by Roos [Roo]. The Krull and the global dimension of A_d are $2d$ if A_d is defined over a field of characteristic $p > 0$. The classical Weyl algebra A_d is an Auslander regular ring. Stafford proved that any left ideal of A_d has a set of 2 generators if A_d is defined over a field of characteristic zero [Sta]. The simple modules over the classical Weyl algebra A_1 were classified by Block [Blo]. For a long list of known and conjectured properties of the classical Weyl algebra see the introduction of [Bav].

In our thesis we are going to consider the question of which properties of the classical Weyl algebra over a complete non-archimedean field carry over to its various completions.

For almost all results we will assume that the components of ε lie in the value group $|K^\times|$. We take this as a general assumption for this introduction and consider only the case where the norm on $\mathcal{A}_{d,\varepsilon}$ is multiplicative.

In [Nar2] Narváez Macarro proves division theorems for the Tate- and Dwork-Monsky-Washnitzer-Weyl algebra under the assumption that the field K is discretely valued. We prove a division theorem for all Weyl algebras $\mathcal{A}_{d,\varepsilon}$ defined over an arbitrary complete non-archimedean field (cf. theorem 1.3.14). It was suggested to me by L. Narváez Macarro how to prove the division theorem for $\mathcal{A}_d^\dagger$ in the case of an arbitrary complete non-archimedean field K (cf. theorem 1.3.16). We use a technique similar to one used in [HM], [HN] and [NR]. In [Nar1] this technique is applied to the Dwork-Monsky-Washnitzer completion of the polynomial ring – a situation very similar to $\mathcal{A}_d^\dagger$.

The division theorems enable us to prove some of the basic properties of $\mathcal{A}_{d,\varepsilon}$ and $\mathcal{A}_d^\dagger$. The Weyl algebras $\mathcal{A}_{d,\varepsilon}$ and $\mathcal{A}_d^\dagger$ are Noetherian (cf. proposition 1.4.1). An element of $\mathcal{A}_{d,\varepsilon}$ or $\mathcal{A}_d^\dagger$ is a unit if and only if its exponent is zero (cf. proposition 1.4.3). We consider formal partial differentiation on elements of $\mathcal{A}_{d,\varepsilon}$ and $\mathcal{A}_d^\dagger$ and show that it respects two-sided ideals. As a consequence of this result, together with the characterization of units, we get that $\mathcal{A}_{d,\varepsilon}$ and

$\mathcal{A}_d^\dagger$ are simple rings if we assume the characteristic of K to be zero.

We prove that the Krull dimension and the global dimension of the completed Weyl algebra $\mathcal{A}_{d,\varepsilon}$ are bounded below by d (cf. propositions 3.1.2 and 3.1.3). The lower bound is given by $2d$ if we assume that the field K has characteristic zero (cf. propositions 3.1.5 and 3.1.6).

In the study of the classical Weyl algebra A_d it turns out to be very useful to consider the localizations of A_d with respect to the Ore sets $K[X_i]\backslash\{0\}$ resp. $K[Y_i]\backslash\{0\}$. One might expect that the multiplicative subsets $K\langle X_i\rangle_{\varepsilon_i}\backslash\{0\}$ and $K\langle Y_i\rangle_{\varepsilon_i}\backslash\{0\}$ of $\mathcal{A}_{d,\varepsilon}$, i.e. the sets of all non-zero restricted power series in X_i resp. Y_i, are Ore sets in $\mathcal{A}_{d,\varepsilon}$. However, this is not the case (cf. lemma 2.0.1) which is equivalent to the fact that the localizations of $\mathcal{A}_{d,\varepsilon}$ with respect to these sets do not exist.

Section 2 provides us with a construction of restricted skew power series rings. We use this construction to define ring extensions $\mathcal{B}_{d,\varepsilon}^{X_i}$ and $\mathcal{B}_{d,\varepsilon}^{Y_i}$ of $\mathcal{A}_{d,\varepsilon}$ (cf. section 3.2). These rings will to some extent play the role of the localizations in the case of the classical Weyl algebra. In fact, the rings $\mathcal{B}_{d,\varepsilon}^{X_i}$ resp. $\mathcal{B}_{d,\varepsilon}^{Y_i}$ are the microlocalizations of $\mathcal{A}_{d,\varepsilon}$ with respect to the sets $K\langle X_i\rangle_{\varepsilon_i}\backslash\{0\}$ resp. $K\langle Y_i\rangle_{\varepsilon_i}\backslash\{0\}$ (for the notion of microlocalizations see [LvO] or [Nag]). We set

$$\mathcal{B}_{d,\varepsilon} := \bigoplus_{i=1}^{d} \mathcal{B}_{d,\varepsilon}^{X_i} \oplus \bigoplus_{i=1}^{d} \mathcal{B}_{d,\varepsilon}^{Y_i}.$$

With the assumption that the characteristic of K is zero we prove the following lemma. For any maximal left ideal $I \subset \mathcal{A}_{d,\varepsilon}$ the left ideal $\mathcal{B}_{d,\varepsilon}I$ generated by I is not the unit ideal $\mathcal{B}_{d,\varepsilon}$ (cf. lemma 3.2.1). The proof involves both the division theorem for the Weyl algebra $\mathcal{A}_{d,\varepsilon}$ (cf. theorem 1.3.14) and the division theorems for $\mathcal{B}_{d,\varepsilon}^{X_i}$ and $\mathcal{B}_{d,\varepsilon}^{Y_i}$ (cf. theorem 2.2.4). This lemma will be an important ingredient to obtain upper bounds for the Krull dimension and the global dimension of $\mathcal{A}_{d,\varepsilon}$ in section 4.

In analogy to the fact that the localizations of the classical Weyl algebra A_1 mentioned above are simple principal left and right ideal domains, the rings $B_{1,\varepsilon}^{X}$ and $B_{1,\varepsilon}^{X}$ are simple principal left and right ideal domains, too (cf. propositions 3.2.4 and 2.2.8).

Under the additional assumption that the field K is discretely valued it is possible to define a complete and separated filtration on $\mathcal{A}_{d,\varepsilon}$ coming from the algebra norm. This allows us to apply the theory of filtered rings (for an introduction to the theory of filtered rings see [LvO]). We obtain the following. The completed Weyl algebra $\mathcal{A}_{d,\varepsilon}$ is Auslander regular (cf. proposition 4.3.3). We show that the Krull dimension and the global dimension of $\mathcal{A}_{d,\varepsilon}$ are bounded above by $2d$ (cf. proposition 4.3.6) which, when combined with the lower bounds computed in section 3, implies that the Krull dimension and the global dimension of $\mathcal{A}_{d,\varepsilon}$ are $2d$ if the characteristic of K is $p > 0$. We prove that the Krull dimension and the global dimension of $\mathcal{A}_{d,\varepsilon}$ are bounded above by $2d-1$ if the characteristic of K is zero (cf corollary 4.3.8). Hence the Krull dimension and the global dimension of $\mathcal{A}_{1,\varepsilon}$ are 1. For some special cases we also prove our conjecture that the Krull dimension and the global dimension of $\mathcal{A}_{d,\varepsilon}$ are d. For example, this is true for the Tate-Weyl algebra if the residue field k of K has characteristic zero (cf. remark 4.3.10). We prove an analog of Staffords theorem for $\mathcal{A}_{1,\varepsilon}$, i.e. any left ideal of $\mathcal{A}_{1,\varepsilon}$ has a set of 2 generators if the characteristic of K is zero (cf. corollary 4.3.9).

In section 5 we show that the so called saturation S_{sat} of the subset $K\langle X\rangle_{\varepsilon_1}\setminus\{0\}$ of $\mathcal{A}_{1,\varepsilon}$ is an Ore set in $\mathcal{A}_{1,d}$ (cf. proposition 5.1). The simple S_{sat}-torsionfree $\mathcal{A}_{1,d}$-modules are in bijection with the simple $(S_{\text{sat}})^{-1}\mathcal{A}_{1,d}$-modules (cf. corollary 5.3).

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1 Weyl algebras

Let K denote a field.

1.1 The classical Weyl algebra

The d-th Weyl algebra over K, here always called **classical Weyl algebra** and denoted by A_d, is the algebra with $2d$ generators $X_1, \dots, X_d, Y_1, \dots, Y_d$ and relations

$$Y_i X_j - X_j Y_i = \delta_{ij}$$

and

$$X_i X_j - X_j X_i = Y_i Y_j - Y_j Y_i = 0,$$

where δ_{ij} denotes the Kronecker delta (see [McCR] section 1.3). Note that if objects or properties have left and right versions we restrict to the left version as [McCR] always use right versions. The elements of A_d have a unique expression as finite sums

$$\sum_{\alpha,\beta \in \mathbb{N}^d} a_{\alpha\beta} X^\alpha Y^\beta$$

with coefficients $a_{\alpha\beta} \in K$ and the notation $X^\alpha = X_1^{\alpha_1} \cdots X_d^{\alpha_d}$ and $Y^\alpha = Y_1^{\alpha_1} \cdots Y_d^{\alpha_d}$. We always write elements in this form and get the following rules of multiplication.

Lemma 1.1.1. *We have*

$$Y^\beta X^\alpha = \sum_{\substack{\gamma \in \mathbb{N}^d \\ \gamma_i \le \alpha_i, \beta_i}} \gamma! \binom{\beta}{\gamma} \binom{\alpha}{\gamma} X^{\alpha-\gamma} Y^{\beta-\gamma}.$$

If $f = \sum a_{\alpha\beta} X^\alpha Y^\beta$, $g = \sum b_{\alpha\beta} X^\alpha Y^\beta$ *and* $fg = \sum c_{\alpha\beta} X^\alpha Y^\beta$ *then*

$$c_{\alpha\beta} = \sum_{\substack{\alpha', \beta', \alpha'', \beta'', \gamma \in \mathbb{N}^d \\ \alpha' + \alpha'' - \gamma = \alpha \\ \beta' + \beta'' - \gamma = \beta}} a_{\alpha'\beta'} b_{\alpha''\beta''} \gamma! \binom{\beta'}{\gamma} \binom{\alpha''}{\gamma}.$$

Proof. The first statement follows from [Dix] lemma 2.1. The second statement follows from the first. □

Here we used the notation $\gamma! := \gamma_1! \cdots \gamma_d!$ and $\binom{\alpha}{\gamma} := \binom{\alpha_1}{\gamma_1} \cdots \binom{\alpha_d}{\gamma_d}$. We write $|\alpha| := \alpha_1 + \ldots + \alpha_d$ for elements in $\mathbb{N}^d$. For $0 \neq f = \sum a_{\alpha\beta} X^\alpha Y^\beta \in A_d$ the **degree** is

$$\deg(f) := \max\{|\alpha| + |\beta| \in \mathbb{N} : a_{\alpha\beta} \neq 0\}$$

or $-\infty$ if $f = 0$ in agreement with the usual definition of degree of polynomials in several variables. As K-vector spaces the classical Weyl algebra A_d and the polynomial ring in $2d$ variables are isomorphic. To distinguish the two different algebra structures we write $\cdot$ for polynomial multiplication and $*$ for Weyl algebra multiplication if we want to emphasize in which ring we multiply.

Lemma 1.1.2. *Let $f, g \in A_d$. Then*

$$\deg(f * g) = \deg(f) + \deg(g)$$

and

$$\deg(f \cdot g - f * g) < \deg(f) + \deg(g).$$

Proof. The first statement follows from [Dix] lemma 2.4.(ii). The second statement follows from [Dix] lemma 2.4.(i). □

The classical Weyl algebra has the following basic algebraic properties.

Theorem 1.1.3. *The classical Weyl algebra A_d is a Noetherian integral domain. If the field K has characteristic zero then A_d is simple, i.e. has no two-sided ideals other than 0 and A_d.*

Proof. [McCR] theorem 1.3.5 and theorem 1.3.8.(i). □

1.2 Completions

Let $(K, |\ \ |)$ denote a complete non-archimedean field. On the d-th classical Weyl algebra A_d we have for any $\varepsilon \in \mathbb{R}^{2d}_{>0}$ the non-archimedean K-vector space norm $|\ \ |_\varepsilon$ defined by

$$|f|_\varepsilon := \max |a_{\alpha\beta}| \varepsilon^{(\alpha,\beta)}$$

for $f = \sum a_{\alpha\beta} X^\alpha Y^\beta \in A_d$, where $\varepsilon^{(\alpha,\beta)} = \varepsilon_1^{\alpha_1} \cdots \varepsilon_d^{\alpha_d} \varepsilon_{d+1}^{\beta_1} \cdots \varepsilon_{2d}^{\beta_d}$, i.e. we have

(i) $|f|_\varepsilon = 0$ iff $f = 0$,

(ii) $|af|_\varepsilon = |a||f|_\varepsilon$,

(iii) $|f+g|_\varepsilon \leq \max\{|f|_\varepsilon, |g|_\varepsilon\}$,

for all $f, g \in A_d$ and $a \in K$.

Lemma 1.2.1. *Let ε be an element of $\mathbb{R}^{2d}_{>0}$ such that $|\gamma!|\varepsilon^{(-\gamma,-\gamma)}$ is bounded by some constant $C > 0$ for all $\gamma \in \mathbb{N}^d$. Then*

$$|fg|_\varepsilon \leq C|f|_\varepsilon |g|_\varepsilon.$$

If $\varepsilon_i \varepsilon_{d+i} \geq 1$ for all i, we have $|fg|_\varepsilon \leq |f|_\varepsilon |g|_\varepsilon$.

Proof.

$$\begin{aligned}
|fg|_\varepsilon &= \max_{\alpha,\beta} \Big| \sum_{\substack{\alpha',\beta',\alpha'',\beta'',\gamma \in \mathbb{N}^d \\ \alpha'+\alpha''-\gamma=\alpha \\ \beta'+\beta''-\gamma=\beta}} a_{\alpha'\beta'} b_{\alpha''\beta''} \gamma! \binom{\beta'}{\gamma}\binom{\alpha''}{\gamma} \Big| \varepsilon^{(\alpha,\beta)} \\
&\leq \max_{\alpha',\beta',\alpha'',\beta'',\gamma \in \mathbb{N}^d} |a_{\alpha'\beta'}||b_{\alpha''\beta''}||\gamma!| \binom{\beta'}{\gamma}\binom{\alpha''}{\gamma} \Big| \varepsilon^{(\alpha'+\alpha''-\gamma,\beta'+\beta''-\gamma)} \\
&\leq \sup_{\gamma \in \mathbb{N}^d} |\gamma!|\varepsilon^{(-\gamma,-\gamma)} \max_{\alpha',\beta' \in \mathbb{N}^d} |a_{\alpha'\beta'}|\varepsilon^{(\alpha',\beta')} \max_{\alpha'',\beta'' \in \mathbb{N}^d} |b_{\alpha''\beta''}|\varepsilon^{(\alpha'',\beta'')} \\
&= C|f|_\varepsilon |g|_\varepsilon.
\end{aligned}$$

If ε satisfies $\varepsilon_i\varepsilon_{d+i} \geq 1$ for all $1 \leq i \leq d$ we have $|\gamma!|\varepsilon^{(-\gamma,-\gamma)} \leq 1$ for all γ, which gives the second part. □

Remark 1.2.2. *If for example $K = \mathbb{Q}_p$, we know that $|\gamma!|$ converges exponentially to zero as $|\gamma|$ goes to infinity. Hence we easily find an $\varepsilon \in \mathbb{R}^{2d}_{>0}$ with all $\varepsilon_i < 1$ such that $|\gamma!|\varepsilon^{(-\gamma,-\gamma)}$ is bounded by some constant C.*

Remark 1.2.3. *If $\varepsilon_i\varepsilon_{d+i} < 1$ for some i, then $|\ \ |_\varepsilon$ is not submultiplicative, for example*

$$1 = |X_iY_i + 1|_\varepsilon = |Y_iX_i|_\varepsilon \not\leq |Y_i|_\varepsilon |X_i|_\varepsilon = \varepsilon_{d+i}\varepsilon_i.$$

However instead of $|\ \ |_\varepsilon$ we can take the equivalent norm $|\ \ |'_\varepsilon$ defined by

$$|f|'_\varepsilon := \sup\{|fg|_\varepsilon |g|^{-1}_\varepsilon; 0 \neq g \in A_d\}.$$

This norm is submultiplicative (see [BGR] §1.2.1., prop. 2, and note that the proof is the same in the non-commutative case).

Lemma 1.2.4. *The norm $|\ \ |_\varepsilon$ on A_d is multiplicative if and only if $\varepsilon_i\varepsilon_{d+i} \geq 1$ for all $1 \leq i \leq d$.*

Proof. The "only if" is remark 1.2.3. Let $\prec$ be a total order on $\mathbb{N}^{2d}$ compatible with addition (see section 1.3). For $0 \neq f = \sum a_{\alpha\beta}X^\alpha Y^\beta \in A_d$ we define the ε-exponent to be

$$\varepsilon\text{-}\exp(f) := \max_{\prec}\{(\alpha,\beta) \in \mathbb{N}^{2d}; |a_{\alpha\beta}|\varepsilon^{(\alpha,\beta)} = |f|_\varepsilon\}.$$

We will define this exponent again in a slightly more general situation in section 1.3. For non-zero elements $f = \sum a_{\alpha\beta}X^\alpha Y^\beta$ and $g = \sum b_{\alpha\beta}X^\alpha Y^\beta$ in A_d put $(\alpha_1, \beta_1) = \varepsilon\text{-}\exp(f)$ and $(\alpha_2, \beta_2) = \varepsilon\text{-}\exp(g)$. We have

$$|fg|_\varepsilon \leq |f|_\varepsilon |g|_\varepsilon = |a_{\alpha_1\beta_1}b_{\alpha_2\beta_2}|\varepsilon^{(\alpha_1+\alpha_2,\beta_1+\beta_2)},$$

hence the desired equality follows if we show that the $(\alpha_1+\alpha_2, \beta_1+\beta_2)$-th coefficient of fg has absolute value equal to $|a_{\alpha_1\beta_1}b_{\alpha_2\beta_2}|$. Recall by lemma 1.1.1 the $(\alpha_1+\alpha_2, \beta_1+\beta_2)$-th coefficient of fg is given by the sum

$$\sum_{\substack{\alpha',\beta',\alpha'',\beta'',\gamma \in \mathbb{N}^d \\ \alpha'+\alpha''-\gamma = \alpha_1+\alpha_2 \\ \beta'+\beta''-\gamma = \beta_1+\beta_2}} a_{\alpha'\beta'}b_{\alpha''\beta''}\gamma!\binom{\beta'}{\gamma}\binom{\alpha''}{\gamma}.$$

We prove now the strict inequality

$$|a_{\alpha'\beta'}b_{\alpha''\beta''}\gamma!\binom{\beta'}{\gamma}\binom{\alpha''}{\gamma}| < |a_{\alpha_1\beta_1}b_{\alpha_2\beta_2}|$$

for all $\alpha', \beta', \alpha'', \beta'', \gamma \in \mathbb{N}^d$ with $(\alpha',\beta') + (\alpha'',\beta'') - (\gamma,\gamma) = (\alpha_1,\beta_1) + (\alpha_2,\beta_2)$ and $(\alpha',\beta',\alpha'',\beta'') \neq (\alpha_1,\beta_1,\alpha_2,\beta_2)$. Consider the following cases:

$\boldsymbol{(\alpha',\beta') \succ (\alpha_1,\beta_1)}$ implies $|a_{\alpha'\beta'}|\varepsilon^{(\alpha',\beta')} < |a_{\alpha_1\beta_1}|\varepsilon^{(\alpha_1,\beta_1)}$ ($(\alpha_1,\beta_1) = \varepsilon\text{-}\exp(f)$). Further we have $|b_{\alpha''\beta''}|\varepsilon^{(\alpha''-\gamma,\beta''-\gamma)} \leq |b_{\alpha''\beta''}|\varepsilon^{(\alpha'',\beta'')} \leq |b_{\alpha_2\beta_2}|\varepsilon^{(\alpha_2,\beta_2)}$. This gives

$$|a_{\alpha'\beta'}||b_{\alpha''\beta''}|\varepsilon^{(\alpha',\beta')+(\alpha''-\gamma,\beta''-\gamma)} < |a_{\alpha_1\beta_1}||b_{\alpha_2\beta_2}|\varepsilon^{(\alpha_1,\beta_1)+(\alpha_2,\beta_2)}$$

hence $|a_{\alpha'\beta'}b_{\alpha''\beta''}\gamma!\binom{\beta'}{\gamma}\binom{\alpha''}{\gamma}| \leq |a_{\alpha'\beta'}b_{\alpha''\beta''}| < |a_{\alpha_1\beta_1}b_{\alpha_2\beta_2}|$.

$\boldsymbol{(\alpha',\beta') \prec (\alpha_1,\beta_1)}$ implies $(\alpha'',\beta'') \succeq (\alpha'',\beta'') - (\gamma,\gamma) \succ (\alpha_2,\beta_2)$ and we proceed as in the first case.

$\boldsymbol{(\alpha', \beta') = (\alpha_1, \beta_1)}$ leads to the case $\boldsymbol{(\alpha'', \beta'') \succ (\alpha_2, \beta_2)}$ which is treated above. $\square$

Definition of completed Weyl algebras

We define the completion of A_d w.r.t. $|\ \ |_\varepsilon$ (for $\varepsilon > 0$) to be the K-Banach space of restricted non-commutative power series

$$\mathcal{A}_{d,\varepsilon} := \{\sum a_{\alpha\beta} X^\alpha Y^\beta ; |a_{\alpha\beta}|\varepsilon^{(\alpha,\beta)} \to 0 \text{ for } |\alpha + \beta| \to \infty\}$$

(for the commutative setting see e.g. [BGR] §6.1.5.). If we assume in addition that $|\gamma!|\varepsilon^{(-\gamma,-\gamma)}$ is bounded for varying $\gamma \in \mathbb{N}^d$, then this is a non-commutative K-Banach algebra, and whenever we write in future $\mathcal{A}_{d,\varepsilon}$ we mean this K-algebra, i.e. we always assume the above condition on ε. If furthermore $\varepsilon_i \varepsilon_{d+i} \geq 1$ for all i, then the norm is multiplicative. For $\varepsilon = (1, \ldots, 1)$ we write $\mathcal{A}_d = \mathcal{A}_{d,(1,\ldots,1)}$ and $|\ \ | = |\ \ |_{(1,\ldots,1)}$ and call this the **Tate-Weyl algebra**. Further we let $\mathcal{A}^\dagger = \bigcup_{\varepsilon > 1} \mathcal{A}_{d,\varepsilon}$ be endowed with the locally convex inductive limit topology, the **Dwork-Monsky-Washnitzer-Weyl algebra** (short: DMW-Weyl algebra) or **weak completion of the Weyl algebra**.

The multiplication formula of lemma 1.1.1 extends to any of the completions of A_d.

Lemma 1.2.5. *Let* $f = \sum a_{\alpha\beta} X^\alpha Y^\beta$ *and* $g = \sum b_{\alpha\beta} X^\alpha Y^\beta$ *be elements of* $\mathcal{A}_{d,\varepsilon}$*. If* $fg = \sum c_{\alpha\beta} X^\alpha Y^\beta$*, then*

$$c_{\alpha\beta} = \sum_{\substack{\alpha', \beta', \alpha'', \beta'', \gamma \in \mathbb{N}^d \\ \alpha' + \alpha'' - \gamma = \alpha \\ \beta' + \beta'' - \gamma = \beta}} a_{\alpha'\beta'} b_{\alpha''\beta''} \gamma! \binom{\beta'}{\gamma} \binom{\alpha''}{\gamma}.$$

Proof. Define $f_n = \sum\limits_{|\alpha|+|\beta| \leq n} a_{\alpha\beta} X^\alpha Y^\beta$ and equally g_n. Since $\lim f_n = f$ and $\lim g_n = g$ we have $\lim f_n g_n = fg$. However the coefficients of $f_n g_n$ are given by

$$c_{\alpha\beta} = \sum_{\substack{\alpha',\beta',\alpha'',\beta'',\gamma\in\mathbb{N}^d \\ \alpha'+\alpha''-\gamma=\alpha \\ \beta'+\beta''-\gamma=\beta \\ |\alpha'|+|\alpha''|\le n \\ |\beta'|+|\beta''|\le n}} a_{\alpha'\beta'} b_{\alpha''\beta''} \gamma! \binom{\beta'}{\gamma}\binom{\alpha''}{\gamma}.$$

Hence taking the limit gives the above formula. □

1.3 Division theorems

We say a total order $\prec$ on $\mathbb{N} \times \ldots \times \mathbb{N}$ is compatible with addition, if for $\alpha, \beta, \gamma \in \mathbb{N} \times \ldots \times \mathbb{N}$ we have
• if $\alpha \neq 0$, then $\beta \prec \beta + \alpha$ for all β and
• if $\alpha \prec \beta$, then $\alpha + \gamma \prec \beta + \gamma$ for all γ,
where by addition on $\mathbb{N} \times \ldots \times \mathbb{N}$ we mean component-wise addition.

Lemma 1.3.1. *For all subsets of $E \subseteq \mathbb{N}^d$ with $E + \mathbb{N}^d = E$ there exists a finite subset $F \subseteq E$ such that $E = F + \mathbb{N}^d$.*

Proof. We reproduce the proof of [Gal] lemma 1.1.8. Induction on d. For $d = 1$ this is clear. Suppose the assumption is true for all numbers $< d$.

Let $e = (e_1, \ldots, e_d) \in E$. For all $i = 1, \ldots, d$ and $j = 0, 1, 2, \ldots, e_i$ denote

$$E_{ij} = E \cap (\mathbb{N}^{i-1} \times \{j\} \times \mathbb{N}^{d-i}).$$

Hence we have $e + \mathbb{N}^d + \bigcup_{ij} E_{ij} = E$. If we write $\mathbb{N}^{d-1} = \mathbb{N}^{i-1} \times \{0\} \times \mathbb{N}^{d-i}$ we see that $E_{ij} + \mathbb{N}^{d-1} = E_{ij}$. By induction there is a finite set F_{ij} generating E_{ij}. The union of these sets $F = \bigcup_{ij} F_{ij} \cup \{e\}$ is a generating set for E. □

Lemma 1.3.2. *A total order $\prec$ on $\mathbb{N}^d$ which is compatible with addition is a well-ordering.*

Proof. Assume $\{\alpha_n\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence in $\mathbb{N}^d$, i.e.

$$\alpha_n \succ \alpha_{n+1} \quad \text{for all } n.$$

By the compatibility with addition on $\mathbb{N}^d$ we know that $\alpha_n + \gamma \neq \alpha_m$ for all $\gamma \in \mathbb{N}^d$ and all $n < m$. The set $E = \bigcup_n \alpha_n + \mathbb{N}^d$ satisfies $E + \mathbb{N}^d = E$. Hence by lemma 1.3.1 we find finitely many $\beta_m \in E$ say $\beta_m = \alpha_{n_m} + \gamma_m$ such that $\bigcup_n \beta_n + \mathbb{N}^d = E$. Now choose an n_0 such that $n_0 > n_m$ for the finitely many m. Then

$$\alpha_{n_0} = \beta_m + \gamma = \alpha_{n_m} + \gamma_m + \gamma,$$

a contradiction. $\square$

We consider now a total order on $\mathbb{N}^{1+d}$ which is compatible with addition and denote its restriction to $\{1, \ldots, m\} \times \mathbb{N}^d$ again by $\prec$. Such an order is called a **monomial order** on $\{1, \ldots, m\} \times \mathbb{N}^d$. Since $(i, \alpha) \prec (i, \beta)$ implies $(j, \alpha) \prec (j, \beta)$ for all $1 \leq j \leq m$, the order $\prec$ restricts to $\mathbb{N}^d$ if we define $\alpha \prec \beta$ if $(i, \alpha) \prec (i, \beta)$ for some $1 \leq i \leq m$.

Examples of monomial orders are the lexicographical order, the inverse lexicographical order, the diagonal order, and the Λ-order, where $\Lambda : \mathbb{R} \times \ldots \times \mathbb{R} \to \mathbb{R}$ is a certain linear form (see e.g. [Nar2]).

A further example is the following order on $\{1, \ldots, m\} \times \mathbb{N}^d$: An element $(i, \alpha_1, \ldots, \alpha_d)$ is said to be less than $(i', \alpha'_1, \ldots, \alpha'_d)$ if

$$\begin{array}{ll} & \sum \alpha_j < \sum \alpha'_j, \\ \text{or} & \sum \alpha_j = \sum \alpha'_j, \exists\, 1 \leq k \leq d : \alpha_k < \alpha'_k, \alpha_{k+1} = \alpha'_{k+1}\ , \ldots,\ \alpha_d = \alpha'_d, \\ \text{or} & \alpha_1 = \alpha'_1\ , \ldots,\ \alpha_d = \alpha'_d,\ i < i'. \end{array}$$

This order has the additional property that if an element $(i, \alpha_1, \ldots, \alpha_d)$ is less than $(i', \alpha'_1, \ldots, \alpha'_d)$ then $\sum \alpha_j \leq \sum \alpha'_j$. If a monomial order has this property we say it is compatible with the notion of degree.

Division in the polynomial algebra

Let $\prec$ be a monomial order on $\{1, \ldots, m\} \times \mathbb{N}^d$. Let $K[X]$ be the polynomial ring in d variables over K. Let $F = (f_1, \ldots, f_m)$ be an element of the free

$K[X]$-module $K[X]^m$ and write $f_i = \sum a_{i\alpha} X^\alpha$. Then

$$\operatorname{supp}(F) := \{(i, \alpha) \in \{1, \ldots, m\} \times \mathbb{N}^d; a_{i\alpha} \neq 0\}$$

is called the **support** of F, and if there is at least one non-zero polynomial f_i then

$$\exp_{K[X]^m}(F) := \max_{\prec}\{(i, \alpha) \in \operatorname{supp}(F)\}$$

is the **exponent** of F. For $F = 0$ we put $\exp(F) = -\infty$ with the convention that $-\infty < (i, \alpha)$ for all $(i, \alpha) \in \{1, \ldots, m\} \times \mathbb{N}^d$. If the notion of exponent is applied not to a vector but to an element $f = \sum a_\alpha X^\alpha \in K[X]$, we mean the analogue definition taking as total order the restriction of $\prec$ to $\mathbb{N}^d$ as explained above. For $\alpha \in \mathbb{N}^d$ and $(i, \beta) \in \{1, \ldots, m\} \times \mathbb{N}^d$ we define $\alpha + (i, \beta) := (i, \alpha + \beta)$.

Lemma 1.3.3. *If $f \in K[X]$ and $F, G \in K[X]^m$ then*
- $\exp(fF) = \exp(f) + \exp(F)$ *and*
- *if* $\exp(F) \neq \exp(G)$, *then* $\exp(F + G) = \max_{\prec}\{\exp(F), \exp(G)\}$,

with the usual conventions if $f = 0$ or $F = 0$.

Proof. This follows from lemma 1.3.7 if we view $K[X]$ as a subring of A_d. $\square$

For elements $F_1, \ldots, F_n \in K[X]^m$ with $F_j \neq 0$ for $1 \leq j \leq n$ we introduce the following notation

$$\begin{array}{rcl} \Delta_j & := & (\mathbb{N}^d + \exp(F_j)) \setminus \bigcup_{k=1}^{j-1} \Delta_k \subseteq \{1, \ldots, m\} \times \mathbb{N}^d \quad (1 \leq j \leq n), \\ \overline{\Delta} & := & \{1, \ldots, m\} \times \mathbb{N}^d \setminus \bigcup_{j=1}^{n} \Delta_j, \end{array}$$

where by the sum $\mathbb{N}^d + \exp(F_j)$ we mean the set $(i_j, \alpha_j + \mathbb{N}^d)$ if $\exp(F_j) = (i_j, \alpha_j)$. It is important to note that $\{1, \ldots, m\} \times \mathbb{N}^d$ is the disjoint union of the sets Δ_j. $1 \leq j \leq n$, and $\overline{\Delta}$.

Theorem 1.3.4. *Let $F_1, \ldots, F_n \in K[X]^m$ with $F_j \neq 0$ ($1 \leq j \leq n$). For all $G \in K[X]^m$ there exist unique polynomials $q_1, \ldots, q_n \in K[X]$ and a unique element $R \in K[X]^m$ such that*

(i) $G = \sum_j q_j F_j + R$,

(ii) $\operatorname{supp}(q_j) + \exp(F_j) \subseteq \Delta_j$ *for* $1 \leq j \leq n$,

(iii) $\operatorname{supp}(R) \subseteq \overline{\Delta}$.

Proof. We briefly recall the arguments of [Bay] prop. 2.2. Since $\prec$ is a well-ordering (cf. lemma 1.3.2) we proceed by induction. For $G = 0$ the result holds trivially. Let us assume the result holds for all $F \in K[X]^m$ with $\exp(F) \prec \exp(G)$. We use the notation $G = (g_1, \ldots, g_m)$ with $g_i = \sum b_{i\alpha} X^\alpha$ and $F_j = (f_1^{(j)}, \ldots, f_m^{(j)})$ with $f_i^{(j)} = \sum a_{i\alpha}^{(j)} X^\alpha$ and write $\exp(G) = (i_0, \alpha_0)$ and $\exp(F_j) = (i_j, \alpha_j)$. Now consider the two cases:

$\mathbf{exp(G) \in \overline{\Delta}}$. Let

$$G - (0, \ldots, b_{i_0\alpha_0} X^{\alpha_0}, \ldots, 0) = q_1 F_1 + \ldots + q_n F_n + R$$

be the expression for $G - (0, \ldots, g_{i_0\alpha_0} X^\alpha, \ldots, 0)$ by induction. Then we trivially have the following expression for G:

$$G = q_1 F_1 + \ldots + q_n F_n + R + (0, \ldots, g_{i_0\alpha_0} X^\alpha, \ldots, 0).$$

$\mathbf{exp(G) \in \Delta_j}$ **for some** $\boldsymbol{j}$**.** Let

$$G - \frac{b_{i_0\alpha_0}}{a_{i_j\alpha_j}^{(j)}} X^{\alpha_0 - \alpha_j} F_j = q_1 F_1 + \ldots + q_n F_n + R$$

be the unique expression of $G - \frac{b_{i_0\alpha_0}}{a_{i_j\alpha_j}^{(j)}} X^{\alpha_0 - \alpha_j} F_j$ by induction. Then

$$G = q_1 F_1 + \ldots + (q_j + \frac{b_{i_0\alpha_0}}{a_{i_j\alpha_j}^{(j)}} X^{\alpha_0 - \alpha_j}) F_j + \ldots + q_n F_n + R$$

is the desired decomposition of G. In both cases the uniqueness follows by induction, or by lemma 1.3.3 as in the proof of theorem 1.3.9. $\square$

For an element $F = (f_1, \ldots, f_m) \in K[X]^m$ with $f_i = \sum a_{i\alpha} X^\alpha$ we call

$$\deg(F) := \max\{\deg(f_1), \ldots, \deg(f_m)\}$$

the **degree** of F and

$$\sigma(F) := (\sum_{|\alpha|=\deg(F)} a_{1\alpha}X^{\alpha}, \ldots, \sum_{|\alpha|=\deg(F)} a_{m\alpha}X^{\alpha}).$$

the **symbol** of F. From now on assume that the monomial order $\prec$ is compatible with the notion of degree.

Remark 1.3.5. *We have* $\exp(F) = \exp(\sigma(F))$. *Indeed, the support of* $\sigma(F)$ *is a subset of the support of* F, *whence* $\exp(F) \succeq \exp(\sigma(F))$. *On the other hand* $\exp(F) = (i, \alpha)$ *implies* $a_{i\alpha} \neq 0$ *and* $|\alpha| = \deg(F)$, *since the order is compatible with the notion of degree; hence* $\exp(f) \preceq \exp(\sigma(F))$.

Corollary 1.3.6. *Let* $G \in K[X]^m$ *and* $F_j \in K[X]$, $F_j \neq 0$ *for* $1 \leq j \leq m$. *If* $G = \sum q_j F_j + R$ *is the unique decomposition for some* G *in the sense of the division theorem 1.3.4 we have*

$$\deg(G) = \max\{\deg(q_j F_j), \deg(R)\}.$$

Proof. By the division theorem we have

$$\exp(q_j F_j) \neq \exp(q_k F_k) \quad j \neq k \quad \text{and} \quad \exp(q_j F_j) \neq \exp(R).$$

Obviously $\deg(G) \leq \max\{\deg(q_j F_j), \deg(R)\}$. Suppose now

$$\deg(G) < \max\{\deg(q_j F_j), \deg(R)\},$$

hence there is a k with $\deg(G) < \deg(q_k F_k)$. Since we have $G = \sum q_j F_j + R$ we get $\sigma(q_k F_k) = \sigma(q_k F_k - G) = \sigma(-(\sum_{j \neq k} q_j F_j + R))$. Using remark 1.3.5 and the second part of lemma 1.3.3 we get the following contradiction

$$\exp(q_k F_k) = \exp(\sum_{j \neq k} q_j F_j + R) \in \{\exp(q_j F_j)(j \neq k), \exp(R)\}.$$

$\square$

Division in the classical Weyl algebra

Denote by A_d the d-th classical Weyl algebra over K. For an element $F \in A_d^m$ we have the same notion of exponent as in the case of the polynomial algebra. Note that now we work with $2d$ variables, i.e. we consider a monomial order $\prec$ on $\{1, \ldots, m\} \times \mathbb{N}^{2d}$ which is compatible with the notion of degree

Lemma 1.3.7. *For $f \in A_d$ and $F, G \in A_d^m$ we have*
- $\exp(fF) = \exp(f) + \exp(F)$, *and*
- *if* $\exp(F) \neq \exp(G)$ *then* $\exp(F+G) = \max_{\prec}\{\exp(F), \exp(G)\}$,

with the usual conventions if $f = 0$ or $F = 0$.

Proof. Let $f = \sum a_{\alpha\beta} X^\alpha Y^\beta$, $\exp(f) = (\alpha_1, \beta_1)$, $F = (f_1, \ldots, f_m)$ where $f_i = \sum b_{i\alpha\beta} X^\alpha Y^\beta$, $\exp(F) = (i_2, \alpha_2, \beta_2)$, and $ff_i = \sum c_{i\alpha\beta} X^\alpha Y^\beta$. Let us first prove that

$$c_{i\alpha\beta} = \sum_{\substack{\alpha',\beta',\alpha'',\beta'',\gamma \\ (\alpha',\beta')+(\alpha'',\beta'')-(\gamma,\gamma)=(\alpha,\beta)}} a_{\alpha'\beta'} b_{i\alpha''\beta''} \gamma! \binom{\beta'}{\gamma} \binom{\alpha''}{\gamma} = 0$$

for all $(i, \alpha, \beta) \succ (\alpha_1, \beta_1) + (i_2, \alpha_2, \beta_2)$. This certainly implies $\exp(fF) \preceq \exp(f) + \exp(F)$. We consider the following cases:

$\boldsymbol{(\alpha', \beta') \succ (\alpha_1, \beta_1)}$ implies $a_{\alpha'\beta'} = 0$.
$\boldsymbol{(\alpha', \beta') \preceq (\alpha_1, \beta_1)}$ together with

$$(\alpha', \beta') + (i, \alpha'', \beta'') - (\gamma, \gamma) = (i, \alpha, \beta) \succ (\alpha_1, \beta_1) + (i_2, \alpha_2, \beta_2)$$

implies $(i, \alpha'', \beta'') \succ (i_2, \alpha_2, \beta_2)$, hence $b_{i\alpha''\beta''} = 0$.

Now we show $c_{i_2,\alpha_1+\alpha_2,\beta_1+\beta_2} = a_{\alpha_1\beta_1} b_{i_2\alpha_2\beta_2}$, hence $\exp(fF) \succeq \exp(f) + \exp(F)$.

$\boldsymbol{(\alpha', \beta') \succ (\alpha_1, \beta_1)}$ implies $a_{\alpha'\beta'} = 0$.
$\boldsymbol{(\alpha', \beta') \prec (\alpha_1, \beta_1)}$ together with

$$(\alpha', \beta') + (i_2, \alpha'', \beta'') - (\gamma, \gamma) = (\alpha_1, \beta_1) + (i_2, \alpha_2, \beta_2)$$

implies $(i_2, \alpha'', \beta'') \succ (i_2, \alpha_2, \beta_2)$, hence $b_{i_2\alpha''\beta''} = 0$.
$\boldsymbol{(\alpha', \beta') = (\alpha_1, \beta_1)}$ implies $(i_2, \alpha'', \beta'') - (\gamma, \gamma) = (i_2, \alpha_2, \beta_2)$. If $\gamma \neq 0$ we have $|\alpha''| + |\beta''| > |\alpha_2| + |\beta_2|$. Hence $b_{i_2,\alpha'',\beta''} = 0$ and $c_{i_2,\alpha_1+\alpha_2,\beta_1+\beta_2} = a_{\alpha_1\beta_1} b_{i_2\alpha_2\beta_2}$.

The second statement is obvious. $\square$

Corollary 1.3.8. *Let $F_1, \ldots, F_n \in A_d^m$ such that $\exp(F_j) \neq \exp(F_k)$ for all $1 \leq j < k \leq n$, then $F_1 + \ldots + F_n \neq 0$, unless $n = 1$ and $F_1 = 0$; since $\exp(F_1 + \ldots + F_n) = \max\{\exp(F_1), \ldots, \exp(F_n)\}$.*

As in the case of a polynomial ring we set

$$\begin{array}{lll} \Delta_j & := & (\mathbb{N}^{2d} + \exp(F_j)) \setminus \bigcup_{k=1}^{j-1} \Delta_k \subseteq \{1, \ldots, m\} \times \mathbb{N}^{2d} \quad (1 \leq j \leq n), \\ \overline{\Delta} & := & \{1, \ldots, m\} \times \mathbb{N}^{2d} \setminus \bigcup_{j=1}^{n} \Delta_j. \end{array}$$

Theorem 1.3.9. *Let $F_1, \ldots, F_n \in A_d^m$ such that $F_j \neq 0$ $(1 \leq j \leq n)$. For all $G \in A_d^m$ there exist unique elements $q_1, \ldots, q_n \in A_d$ and a unique element $R \in A_d^m$ such that*

(i) $G = \sum_j q_j F_j + R$,

(ii) $\mathrm{supp}(q_j) + \exp(F_j) \subseteq \Delta_j$ *for* $1 \leq j \leq n$,

(iii) $\mathrm{supp}(R) \subseteq \overline{\Delta}$.

Remark 1.3.10. *All division theorems we consider also exist in a right version, i.e. with $G = \sum_j q_j F_j + R$ replaced by $G = \sum_j F_j q_j + R$.*

Proof. We recall the arguments of [Cas] theorem 2.1. *Uniqueness.* Suppose that

$$G = \sum q_j F_j + R = \sum q_j' F_j + R'$$

are different expressions. We consider now all non-zero differences $q_j - q_j'$, $R - R'$. For those differences we get from condition (ii) and (iii) $\exp((q_j - q_j')F_j) = \exp(q_j - q_j') + \exp(F_j) \in \Delta_j$, since $\exp(q_j - q_j') \in \mathrm{supp}(q_j) \cup \mathrm{supp}(q_j')$ and trivially $\exp(R - R') \in \overline{\Delta}$. Hence

$$\exp((q_j - q_j')F_j) \neq \exp((q_k - q_k')F_k) \text{ for } j \neq k$$

and

$$\exp((q_j - q_j')F_j) \neq \exp(R - R').$$

This implies, using corollary 1.3.8, that $\sum(q_j - q'_j)F_j + R - R' \neq 0$, a contradiction.

Existence. We proceed by induction on the degree of G (the degree is defined as in the polynomial case). The case $G = 0$ is trivial.

Suppose the assertion holds for all elements of degree strictly less than $\deg(G)$. Using theorem 1.3.4 one can write $G = \sum_j q_j \sigma(F_j) + R$ with multiplication in $K[X,Y]$. Since $\exp(\sigma(F_j)) = \exp(F_j)$ by remark 1.3.6 we have

$$\begin{aligned}
&\operatorname{supp}(q_j) + \exp(F_j) \in \Delta_j \text{ for all } 1 \leq j \leq n,\\
&\operatorname{supp}(R) \subseteq \overline{\Delta}.
\end{aligned}$$

Set

$$G' = G - \sum_j q_j F_j + R,$$

where here we multiply in A_d. Using for the moment the notation $\cdot$ for polynomial multiplication and $*$ for Weyl algebra multiplication we get

$$\begin{aligned}
\deg(G') &= \deg(G - (\sum q_j * F_j + R))\\
&= \deg(\sum q_j \cdot \sigma(F_j) + R - (\sum q_j * F_j + R))\\
&= \deg(\sum q_j \cdot \sigma(F_j) - \sum q_j * (\sigma(F_j) + \tilde{\sigma}(F_j)))\\
&= \deg(\sum (q_j \cdot \sigma(F_j) - q_j * \sigma(F_j)) - \sum q_j * \tilde{\sigma}(F_j))\\
&\leq \max\{\deg(q_j \cdot \sigma(F_j) - q_j * \sigma(F_j)), \deg(q_j * \tilde{\sigma}(F_j))\} \;<\; \deg(G)
\end{aligned}$$

if $\tilde{\sigma}(F_j) := F_j - \sigma(F_j)$. In the computation we used lemma 1.1.2 and the fact that $\deg(G) = \max\{\deg(q_j\sigma(F_j)), \deg(R)\}$ by corollary 1.3.6. Now since G' has degree strictly less than $\deg(G)$ we can decompose G' by induction. This gives us a decomposition for G which clearly satisfies the properties (ii) and (iii). $\square$

Division in the completed Weyl algebra $\mathcal{A}_{d,\varepsilon}$

On the free $\mathcal{A}_{d,\varepsilon}$-module $\mathcal{A}_{d,\varepsilon}^m$ we have the maximum norm, which will also be denoted by $|\ \ |_\varepsilon$. Let $F = (f_1, \ldots, f_m) \in \mathcal{A}_{d,\varepsilon}^m$ be an element where $f_i =$

$\sum a_{i\alpha\beta} X^\alpha Y^\beta$. The **$\varepsilon$-initial form** of F is defined to be

$$\varepsilon\text{-inform}(F) := \Big(\sum_{|a_{1\alpha\beta}|\varepsilon^{(\alpha,\beta)}=|F|_\varepsilon} a_{1\alpha\beta} X^\alpha Y^\beta, \ldots, \sum_{|a_{m\alpha\beta}|\varepsilon^{(\alpha,\beta)}=|F|_\varepsilon} a_{m\alpha\beta} X^\alpha Y^\beta \Big).$$

The **ε-exponent** of F in $\mathcal{A}^m_{d,\varepsilon}$ is

$$\varepsilon\text{-}\exp_{\mathcal{A}^m_{d,\varepsilon}}(F) := \exp_{A^m_d}(\varepsilon\text{-inform}(F))$$

where ε-inform(F) is viewed as an element of the free module A^m_d over the classical Weyl algebra. Again we have the exponent properties.

Lemma 1.3.11. *Let $\varepsilon \in \mathbb{R}^{2d}_{>0}$ with $\varepsilon_i \varepsilon_{d+i} \geq 1$, $f \in \mathcal{A}_{d,\varepsilon}$ and $F \in \mathcal{A}^m_{d,\varepsilon}$ then*
- *ε-$\exp(fF) = \varepsilon$-$\exp(f) + \varepsilon$-$\exp(F)$ and*
- *if ε-$\exp(F) \neq \varepsilon$-$\exp(G)$ then ε-$\exp(F+G) \in \{\varepsilon\text{-}\exp(F), \varepsilon\text{-}\exp(G)\}$ and ε-$\exp(F+G) \neq -\infty$, with the usual conventions if $f = 0$ or $F = 0$.*

Proof. Let $f = \sum a_{\alpha\beta} X^\alpha Y^\beta$, ε-$\exp(f) = (\alpha_1, \beta_1)$, $F = (f_1, \ldots, f_m)$ where $f_i = \sum b_{i\alpha\beta} X^\alpha Y^\beta$, ε-$\exp(F) = (i_2, \alpha_2, \beta_2)$, and $f f_i = \sum c_{i\alpha\beta} X^\alpha Y^\beta$.

We show $|c_{i\alpha\beta}|\varepsilon^{(\alpha,\beta)} < |fF|_\varepsilon$ for all $(i, \alpha, \beta) \succ (\alpha_1, \beta_1) + (i_2, \alpha_2, \beta_2)$, hence ε-$\exp(fF) \preceq \varepsilon$-$\exp(f) + \varepsilon$-$\exp(F)$. We have

$$\begin{aligned} |c_{i\alpha\beta}|\varepsilon^{(\alpha,\beta)} &= \Big| \sum_{\substack{\alpha',\beta',\alpha'',\beta'',\gamma \\ (\alpha',\beta')+(\alpha'',\beta'')-(\gamma,\gamma)=(\alpha,\beta)}} a_{\alpha'\beta'} b_{i\alpha''\beta''} \gamma! \binom{\beta'}{\gamma}\binom{\alpha''}{\gamma} \Big| \varepsilon^{(\alpha,\beta)} \\ &\leq \max_{(\alpha',\beta')+(\alpha'',\beta'')-(\gamma,\gamma)=(\alpha,\beta)} |a_{\alpha'\beta'}|\varepsilon^{(\alpha',\beta')} |b_{i\alpha''\beta''}|\varepsilon^{(\alpha'',\beta'')}. \end{aligned}$$

$\boldsymbol{(\alpha', \beta') \succ (\alpha_1, \beta_1)}$ implies $|a_{\alpha'\beta'}|\varepsilon^{(\alpha',\beta')} < |f|_\varepsilon$.
$\boldsymbol{(\alpha', \beta') \preceq (\alpha_1, \beta_1)}$ together with

$$(\alpha_1, \beta_1) + (i_2, \alpha_2, \beta_2) \prec (i, \alpha, \beta) = (\alpha', \beta') + (i, \alpha'', \beta'') - (\gamma, \gamma)$$

implies $(i_2, \alpha_2, \beta_2) \prec (i, \alpha'', \beta'')$, hence $|b_{i\alpha''\beta''}|\varepsilon^{(\alpha'',\beta'')} < |F|_\varepsilon$.

The assertion ε-$\exp(fF) \succeq \varepsilon$-$\exp(f) + \varepsilon$-$\exp(F)$ follows if we prove

$$|c_{i_2 \alpha_1+\alpha_2 \beta_1+\beta_2}|\varepsilon^{(\alpha_1+\alpha_2, \beta_1+\beta_2)} = |fF|_\varepsilon.$$

This is the case if and only if

$$|c_{i_2\alpha_1+\alpha_2\beta_1+\beta_2}| = |a_{\alpha_1\beta_1}||b_{i_2\alpha_2\beta_2}|,$$

however this was already shown in the proof of lemma 1.2.4.

Now we prove the second statement of the lemma. If $|F|_\varepsilon \neq |G|_\varepsilon$ we may assume $|F|_\varepsilon > |G|_\varepsilon$, hence the ε-exponent of $F+G$ is the ε-exponent of F.

If $|F|_\varepsilon = |G|_\varepsilon$ we may assume $\varepsilon\text{-exp}(F) \succ \varepsilon\text{-exp}(G)$. In this case the ε-exponent of $F+G$ is the ε-exponent of F. $\square$

Remark 1.3.12. *The proof of the second part of the lemma shows that we in fact have a stronger statement. If the ε-exponents of F and G are not equal, then the ε-exponent of the sum $F+G$ equals the ε-exponent of the element with the bigger ε-norm if the ε-norms are not equal and equals the maximum of the ε-exponents if the ε-norms are equal.*

Corollary 1.3.13. *Let $\varepsilon \in \mathbb{R}^{2d}_{>0}$ with $\varepsilon_i\varepsilon_{d+i} \geq 1$, $F_1, \dots, F_n \in A_d^m$ with $F_j \neq 0$, and $\varepsilon\text{-inform}(F_j) = F_j$ for all j. Apply theorem 1.3.9 to an element $G \in A_d^m$ and let $G = \sum q_j F_j + R$ be the unique decomposition. Then $|G|_\varepsilon = \max\{|q_jF_j|_\varepsilon, |R|_\varepsilon\}$.*

Proof. Note first that we have $\varepsilon\text{-exp}(q_jF_j) \neq \varepsilon\text{-exp}(q_kF_k)$ for $j \neq k$ and $\varepsilon\text{-exp}(q_jF_j) \neq \varepsilon\text{-exp}(R)$ for all j. This is because by lemma 1.3.11

$$\begin{aligned}
&\varepsilon\text{-exp}(q_jF_j) = \varepsilon\text{-exp}(q_j) + \varepsilon\text{-exp}(F_j) \in \operatorname{supp}(q_j) + \exp(F_j) \subseteq \Delta_j,\\
&\varepsilon\text{-exp}(R) \in \operatorname{supp}(R) \subseteq \overline{\Delta}.
\end{aligned}$$

Suppose that $|G|_\varepsilon < \max\{|q_jF_j|_\varepsilon, |R|_\varepsilon\}$. Then there is a k with $|G|_\varepsilon < |q_kF_k|_\varepsilon$. In combination with $G = \sum q_jF_j + R$ we get

$$\varepsilon\text{-exp}(q_kF_k) = \varepsilon\text{-exp}(\sum_{j\neq k} q_jF_j + R).$$

Now using the second part of lemma 1.3.11 we get

$$\varepsilon\text{-exp}(q_kF_k) = \varepsilon\text{-exp}(\sum_{j\neq k} q_jF_j + R) \in \{\varepsilon\text{-exp}(q_jF_j)(j \neq k), \varepsilon\text{-exp}(R)\},$$

which is a contradiction. $\square$

We use the Δ-notation of the classical Weyl algebra case, now of course since $F_j \in \mathcal{A}^m_{d,\varepsilon}$ we use the notion of ε-exponent defined above.

$$\begin{array}{rcl} \Delta_j & := & (\mathbb{N}^{2d} + \varepsilon\text{-}\exp(F_j)) \backslash \bigcup_{k=1}^{j-1} \Delta_k \subseteq \{1,\ldots,m\} \times \mathbb{N}^{2d} \quad (1 \leq j \leq n), \\ \overline{\Delta} & := & \{1,\ldots,m\} \times \mathbb{N}^{2d} \backslash \bigcup_{j=1}^{n} \Delta_j. \end{array}$$

Theorem 1.3.14. *Assume $\varepsilon \in |K^\times|^{2d}$ with $\varepsilon_i \varepsilon_{d+i} \geq 1$. Let $F_1, \ldots, F_n \in \mathcal{A}^m_{d,\varepsilon}$ such that $F_j \neq 0$ $(1 \leq j \leq n)$. For all $G \in \mathcal{A}^m_{d,\varepsilon}$ there exist unique elements $q_1, \ldots, q_n \in \mathcal{A}_{d,\varepsilon}$ and a unique element $R \in \mathcal{A}^m_{d,\varepsilon}$ such that*

(i) $G = \sum_j q_j F_j + R$,

(ii) $\operatorname{supp}(q_j) + \varepsilon\text{-}\exp(F_j) \subseteq \Delta_j$ *for* $1 \leq j \leq n$,

(iii) $\operatorname{supp}(R) \subseteq \overline{\Delta}$.

Moreover, we have $|G|_\varepsilon = \max\{|q_j F_j|_\varepsilon, |R|_\varepsilon\}$.

Proof. The uniqueness follows as in theorem 1.3.9 from the properties of the ε-exponent.

Existence. We may assume $|G|_\varepsilon = 1$ and $|F_j|_\varepsilon = 1$. Using the notation $F_j = (f_1^{(j)}, \ldots, f_m^{(j)})$ with $f_i^{(j)} = \sum a^{(j)}_{i\alpha\beta} X^\alpha Y^\beta$ we set

$$\delta := \max_{\substack{i,j,\alpha,\beta \\ |a^{(j)}_{i\alpha\beta}|\varepsilon^{(\alpha,\beta)} < 1}} |a^{(j)}_{i\alpha\beta}|\varepsilon^{(\alpha,\beta)} < 1$$

if the maximum is non-zero. Otherwise choose an arbitrary $0 < \delta < 1$. Now decompose $F_j = F_j^{>\delta} + F_j^{\leq\delta}$ where

$$\begin{array}{rcl} F_j^{>\delta} & := & \Big(\sum\limits_{|a^{(j)}_{1\alpha\beta}|\varepsilon^{(\alpha,\beta)} > \delta} a^{(j)}_{1\alpha\beta} X^\alpha Y^\beta, \ldots, \sum\limits_{|a^{(j)}_{m\alpha\beta}|\varepsilon^{(\alpha,\beta)} > \delta} a^{(j)}_{m\alpha\beta} X^\alpha Y^\beta \Big) \\ & = & \Big(\sum\limits_{|a^{(j)}_{1\alpha\beta}|\varepsilon^{(\alpha,\beta)} = 1} a^{(j)}_{1\alpha\beta} X^\alpha Y^\beta, \ldots, \sum\limits_{|a^{(j)}_{m\alpha\beta}|\varepsilon^{(\alpha,\beta)} = 1} a^{(j)}_{m\alpha\beta} X^\alpha Y^\beta \Big) \\ & = & \varepsilon\text{-inform}(F_j) \quad = \quad \varepsilon\text{-inform}(F_j^{>\delta}), \\ F_j^{\leq\delta} & := & \Big(\sum\limits_{|a^{(j)}_{1\alpha\beta}|\varepsilon^{(\alpha,\beta)} \leq \delta} a^{(j)}_{1\alpha\beta} X^\alpha Y^\beta, \ldots, \sum\limits_{|a^{(j)}_{m\alpha\beta}|\varepsilon^{(\alpha,\beta)} \leq \delta} a^{(j)}_{m\alpha\beta} X^\alpha Y^\beta \Big). \end{array}$$

Starting with $G = G_0$ we define a sequence G_k by the following procedure: as above we decompose $G_k = G_k^{>\delta} + G_k^{\leq\delta}$ and apply the division in the classical Weyl algebra to $G_k^{>\delta}$ and $F_j^{>\delta}$. Hence we have

$$G_k^{>\delta} = \sum_j q_{j,k} F_j^{>\delta} + R_k$$

with $|q_{j,k}|_\varepsilon \leq 1$, since $1 = |G_k^{>\delta}|_\varepsilon = \max\{|q_{jk}|_\varepsilon |F_j^{>\delta}|_\varepsilon, |R_k|_\varepsilon\}$ by corollary 1.3.13 and $|F_j^{>\delta}|_\varepsilon = 1$. We get

$$G_k = \sum_j q_{j,k} F_j + R_k + \tilde{G}_{k+1}$$

if

$$\tilde{G}_{k+1} := \sum_j q_{j,k} F_j^{>\delta} - \sum_j q_{j,k} F_j + G_k^{\leq\delta} = -\sum_j q_{j,k} F_j^{\leq\delta} + G_k^{\leq\delta}.$$

We have $|\tilde{G}_{k+1}|_\varepsilon \leq \max\{|q_{j,k}|_\varepsilon |F_j^{\leq\delta}|_\varepsilon, |G_k^{\leq\delta}|_\varepsilon\} \leq \delta$. If $\tilde{G}_{k+1} \neq 0$ choose an element $\pi_{k+1} \in K$ such that

$$G_{k+1} := \pi_{k+1}^{-1} \tilde{G}_{k+1}$$

has norm 1, hence $|\pi_{k+1}| \leq \delta$ ($\pi_0 = 1$). If $\tilde{G}_{k_0} = 0$ for some k_0 we put $G_k = q_{j,k} = R_k = 0$ for all j and all $k \geq k_0$. In either case this gives

$$G = \sum_j (\sum_{k=0}^{N} \prod_{l=0}^{k} \pi_l q_{j,k}) F_j + \sum_{k=0}^{N} \prod_{l=0}^{k} \pi_l R_k + \prod_{l=0}^{N+1} \pi_l G_{N+1}$$

for all $N \in \mathbb{N}$. Setting

$$q_j := \sum_{k=0}^{\infty} \prod_{l=0}^{k} \pi_l q_{j,k} \quad \text{and} \quad R := \sum_{k=0}^{\infty} \prod_{l=0}^{k} \pi_l R_k$$

we get the decomposition

$$G = \sum_j q_j F_j + R.$$

It is left to show that $\operatorname{supp}(q_j) + \varepsilon\text{-}\exp(F_j) \subseteq \Delta_j$ for $1 \leq j \leq n$ and $\operatorname{supp}(R) \subseteq \overline{\Delta}$. δ was chosen such that $F_j^{>\delta} = \varepsilon\text{-inform}(F_j)$, whence $\varepsilon\text{-}\exp_{\mathcal{A}_d^m}(F_j) =$

$\exp_{A_d^m}(F_j^{>\delta})$ and

$$\begin{aligned} \operatorname{supp}(q_j) + \varepsilon\text{-}\exp(F_j) &\subseteq \bigcup_k \operatorname{supp}(q_{j,k}) + \exp(F_j^{>\delta}) \subseteq \Delta_j, \text{ for } 1 \leq j \leq n \\ \operatorname{supp}(R) &\subseteq \bigcup_k \operatorname{supp}(R_k) \subseteq \overline{\Delta}. \end{aligned}$$

The final assertion is proved as in corollary 1.3.13. $\square$

Division in the DMW-Weyl algebra

Let $\tilde{L} : \mathbb{R}^{1+2d} \to \mathbb{R}$ be a linear form with non negative and $\mathbb{Z}$-linear independent coefficients $(\lambda_0, \lambda_1, \ldots, \lambda_{2d})$ and let $L : \mathbb{R}^{2d} \to \mathbb{R}$ be the linear form $L(\alpha, \beta) := \tilde{L}(0, \alpha, \beta)$ where $\alpha, \beta \in \mathbb{R}^d$. Denote by $\prec$ the total order (well order) on $\{1, \ldots, m\} \times \mathbb{N}^{2d}$ defined by $\tilde{L}$. In the case of the Tate-Weyl algebra $\mathcal{A}_d$, i.e. $\varepsilon = (1, \ldots, 1)$, we write exp instead of ε-exp. If f is an element of the DMW-Weyl algebra $\mathcal{A}_d^\dagger$ then we define its exponent to be the exponent of f as an element of $\mathcal{A}_d$ via the inclusion $\mathcal{A}_d^\dagger \subseteq \mathcal{A}_d$. For a real number $s > 1$ we write

$$\mathcal{A}_{d,s} := \mathcal{A}_{d,(s^{\lambda_1}, \ldots, s^{\lambda_{2d}})}.$$

Obviously $\mathcal{A}_d^\dagger = \bigcup_{s>1} \mathcal{A}_{d,s}$. Let $f = \sum a_{\alpha\beta} X^\alpha Y^\beta \in \mathcal{A}_{d,s}$, then by definition the norm on $\mathcal{A}_{d,s}$ is given by $\max |a_{\alpha\beta}| s^{L(\alpha,\beta)}$. We denote the norm by $| \ \ |_s$. Let $F = (f_1, \ldots, f_m) \in \mathcal{A}_{d,s}^m$ with $f_i = \sum a_{i\alpha\beta} X^\alpha Y^\beta$, then $\max_i |f_i|_s$ defines a Banach norm on $\mathcal{A}_{d,s}^m$, however in the following we will consider the equivalent norm given by

$$|F|_s := \max |a_{i\alpha\beta}| s^{\tilde{L}(i-1,\alpha,\beta)}.$$

Now let $F = (f_1, \ldots, f_m) \in \mathcal{A}_d^{\dagger m}$ where $f_i = \sum a_{i\alpha\beta} X^\alpha Y^\beta$ and let $(i_0, \alpha_0, \beta_0) = \exp(F)$. We define the **initial term** to be

$$\operatorname{in}(F) := (0, \ldots, a_{i_0\alpha_0\beta_0} X^{\alpha_0} Y^{\beta_0}, \ldots, 0)$$

where the monomial appears at the i_0-th place.

Lemma 1.3.15. *Let $0 \neq F \in \mathcal{A}_d^{\dagger m}$. Then there exists an $s_0 > 1$ such that for all $s \in]1, s_0]$*

$$|F - \operatorname{in}(F)|_s < \nu(s) |\operatorname{in}(F)|_s$$

with $\nu(s) < 1$.

Proof. This is a reproduction of lemma 3.5 of [Nar1]. Let $F = (f_1, \ldots, f_m) \in \mathcal{A}_d^{\dagger m}$ with $f_i = \sum a_{i\alpha\beta} X^\alpha Y^\beta$ and $\exp(F) = (i_0, \alpha_0, \beta_0)$. We may assume $|F| = \max |a_{i\alpha\beta}| = 1$. One can find an $s_1 > 1$ such that $|a_{i\alpha\beta}| s_1^{L(\alpha,\beta)} \to 0$ for $|\alpha| + |\beta| \to \infty$ and all $i = 1, \ldots, m$. Choose a constant C such that $L(\alpha, \beta) \leq C(|\alpha| + |\beta|)$ for all α, β and choose an integer N such that for $|\alpha| + |\beta| \geq N$ we have $|a_{i\alpha\beta}| s_1^{\tilde{L}(i-1,\alpha,\beta)} < \frac{1}{2}$ and $CN > \tilde{L}(i_0 - 1, \alpha_0, \beta_0)$.

We have to show (for suitable s_0 and $\nu(s)$) that for all $(i, \alpha, \beta) \neq (i_0, \alpha_0, \beta_0)$

$$|a_{i\alpha\beta}| s^{\tilde{L}(i-1,\alpha,\beta)} < \nu(s) s^{\tilde{L}(i_0-1,\alpha_0,\beta_0)}$$

for all $s \in]1, s_0]$.

If $(i,\alpha,\beta) \prec (i_0,\alpha_0,\beta_0)$ there is only a finite number of $a_{i\alpha\beta}$'s with this property, hence for all $s > 1$ there is a $\nu_1(s) < 1$ such that

$$|a_{i\alpha\beta}| s^{\tilde{L}(i-1,\alpha,\beta)} \leq s^{\tilde{L}(i-1,\alpha,\beta)} < \nu_1(s) s^{\tilde{L}(i_0-1,\alpha_0,\beta_0)}.$$

If $(i,\alpha,\beta) \succ (i_0,\alpha_0,\beta_0)$ and $|\alpha| + |\beta| < N$ we can find a constant $\nu' < 1$ with $|f_{i\alpha\beta}| < \nu'$ for all $(i, \alpha, \beta) \succ (i_0, \alpha_0, \beta_0)$. Choose a constant ν_2 such that $\nu' < \nu_2 < 1$. Then we have

$$|f_{i\alpha\beta}| s^{\tilde{L}(i-1,\alpha,\beta)} < \nu' s^{\tilde{L}(i-1,\alpha,\beta)} \leq \nu' s^{\lambda_0 m + C(|\alpha|+|\beta|)} < \nu' s^{\lambda_0 m + CN} < \nu_2 s^{\tilde{L}(i_0-1,\alpha_0,\beta_0)}$$

for all $1 < s < s_2 := (\frac{\nu_2}{\nu'})^{\frac{1}{\lambda_0 m + CN - \tilde{L}(i_0,\alpha_0,\beta_0)}}$.

If $(i,\alpha,\beta) \succ (i_0,\alpha_0,\beta_0)$ and $|\alpha| + |\beta| \geq N$ we have

$$\begin{aligned} |a_{i\alpha\beta}| s^{\tilde{L}(i-1,\alpha,\beta)} &= |a_{i\alpha\beta}| s_1^{\tilde{L}(i-1,\alpha,\beta)} \left(\frac{s}{s_1}\right)^{\tilde{L}(i-1,\alpha,\beta)} \\ &< \frac{1}{2}\left(\frac{s}{s_1}\right)^{\tilde{L}(i-1,\alpha,\beta)} < \frac{1}{2} < \frac{1}{2} s^{\tilde{L}(i_0-1,\alpha_0,\beta_0)} \end{aligned}$$

for all $1 < s < s_1$. We complete the lemma by taking $s_0 := \min\{s_1, s_2\}$ and $\nu(s) := \max\{\nu_1(s), \nu_2, \frac{1}{2}\}$. $\square$

As before we use the notation

$$\begin{aligned} \Delta_j &:= (\mathbb{N}^{2d} + \exp(F_j)) \setminus \bigcup_{k=1}^{j-1} \Delta_k \subseteq \{1, \ldots, m\} \times \mathbb{N}^{2d} \quad (1 \leq j \leq n), \\ \overline{\Delta} &:= \{1, \ldots, m\} \times \mathbb{N}^{2d} \setminus \bigcup_{j=1}^{n} \Delta_j. \end{aligned}$$

Theorem 1.3.16. *Let $F_1, \ldots, F_n \in \mathcal{A}_d^{\dagger m}$ such that $F_j \neq 0$ $(1 \leq j \leq n)$. For all $G \in \mathcal{A}_d^{\dagger m}$ there exist unique elements $q_1, \ldots, q_n \in \mathcal{A}_d^{\dagger}$ and a unique element $R \in \mathcal{A}_d^{\dagger m}$ such that*

(i) $G = \sum_j q_j F_j + R$,

(ii) $\mathrm{supp}(q_j) + \exp(F_j) \subseteq \Delta_j$ *for* $1 \leq j \leq n$,

(iii) $\mathrm{supp}(R) \subseteq \overline{\Delta}$.

Proof. We proceed as in the proof of theorem 3.6 in [Nar1], however the decomposition $u_s + v_s + w_s$ is taken from [HN]. We put

$$\nabla_s := \nabla_s(F_1, \ldots, F_n) := \{(q_1, \ldots, q_n) \in \mathcal{A}_s^n |\, \mathrm{supp}\, q_j + \exp(F_j) \subset \Delta_j\}$$

$$\Delta_s := \Delta_s(F_1, \ldots, F_n) = \{R \in \mathcal{A}_s^m |\, \mathrm{supp}(R) \subset \overline{\Delta}\}.$$

We endow the vector space $\nabla_s \oplus \Delta_s$ with the norm

$$|(q_1, \ldots, q_n, R)|_s := \max\{|q_j|_s \cdot |\, \mathrm{in}(F_j)|_s, |R|_s\}.$$

Now we use the following notation, for $q = \sum q_{\alpha\beta} X^\alpha Y^\beta$ denote by q^o the operator $\sum q_{\alpha\beta} X^\alpha T^\beta$ where Y is replaced by the **shift operator** T defined by $TX^\alpha Y^\beta := X^\alpha Y^{\beta+1}$, q' denotes the operator $q - q^o$.

We set $F_j' := F_j - \mathrm{in}(F_j)$ and consider the continuous linear maps u_s, v_s, w_s from $\nabla_s \oplus \Delta_s$ into the Banach space $\mathcal{A}_s^m$ defined by

$$\begin{aligned} u_s(q_1, \ldots, q_n, R) &:= q_1^o \,\mathrm{in}(F_1) + \ldots + q_n^o \,\mathrm{in}(F_n) + R \\ v_s(q_1, \ldots, q_n, R) &:= q_1' \,\mathrm{in}(F_1) + \ldots + q_n' \,\mathrm{in}(F_n) \\ w_s(q_1, \ldots, q_n, R) &:= q_1 F_1' + \ldots + q_n F_n' \end{aligned}$$

for all $s > 1$ sufficiently close to 1 (s.t. $F_j \in \mathcal{A}_s^m$ for $j = 1, \ldots, n$).

$\boldsymbol{u_s}$ **is a homeomorphism with** $\boldsymbol{\|u_s^{-1}\|_s = 1}$**.** This follows from the construction of $\nabla_s \oplus \Delta_s$ and from the choice of its norm.

v_s has norm < 1 for all $s > 1$. We use the notation $\exp(F_j) = (i_j, \alpha_j, \beta_j)$, $\mathrm{in}(F_j) = a_{i_j\alpha_j\beta_j} X^{\alpha_j} Y^{\beta_j}$, $q_j = \sum q^{(j)}_{\alpha\beta} X^\alpha Y^\beta$, and remember $q'_j = q_j - q^o_j$. For all j we have

$$\begin{aligned}
q'_j a_{i_j\alpha_j\beta_j} X^{\alpha_j} Y^{\beta_j} &= q_i a_{i_j\alpha_j\beta_j} X^{\alpha_j} Y^{\beta_j} - q^o_j a_{i_j\alpha_j\beta_j} X^{\alpha_j} Y^{\beta_j} \\
&= \sum q^{(j)}_{\alpha\beta} a_{i_j\alpha_j\beta_j} \gamma! \binom{\alpha_j}{\gamma}\binom{\beta}{\gamma} X^{\alpha+\alpha_j-\gamma} Y^{\beta+\beta_j-\gamma} \\
&\quad - \sum q^{(j)}_{\alpha\beta} a_{i_j\alpha_j\beta_j} X^{\alpha+\alpha_j} Y^{\beta+\beta_j} \\
&= \sum_{\gamma\neq 0} q^{(j)}_{\alpha\beta} f_{i_j\alpha_j\beta_j} \gamma! \binom{\alpha_j}{\gamma}\binom{\beta}{\gamma} X^{\alpha+\alpha_j-\gamma} Y^{\beta+\beta_j-\gamma}.
\end{aligned}$$

Hence

$$\begin{aligned}
|q'_j \,\mathrm{in}(F_j)|_s &= |(0, \ldots, \sum_{\gamma\neq 0} q^{(j)}_{\alpha\beta} a_{i_j\alpha_j\beta_j} \gamma! \binom{\alpha_j}{\gamma}\binom{\beta}{\gamma} X^{\alpha+\alpha_j-\gamma} Y^{\beta+\beta_j-\gamma}, \ldots, 0)|_s \\
&= \max_{\gamma\neq 0} |q^{(j)}_{\alpha\beta}||a_{i_j\alpha_j\beta_j}||\gamma! \binom{\alpha_j}{\gamma}\binom{\beta}{\gamma}| s^{\tilde{L}(i_j-1, \alpha+\alpha_j-\gamma, \beta+\beta_j-\gamma)} \\
&\leq \max |q^{(j)}_{\alpha\beta}| s^{L(\alpha,\beta)} \cdot |f_{i_j\alpha_j\beta_j}| s^{\tilde{L}(i_j-1,\alpha_j,\beta_j)} \cdot s^{-L(\gamma_0,\gamma_0)} \\
&= |q_j|_s \cdot |\,\mathrm{in}(F_j)|_s \cdot s^{-L(\gamma_0,\gamma_0)},
\end{aligned}$$

with $\gamma_0 \neq 0$. We can take $R = 0$, since v_s does not depend on R, and get

$$\begin{aligned}
\frac{|v_s(q_1, \ldots, q_n, 0)|_s}{|(q_1, \ldots, q_n, 0)|_s} &= \frac{|\sum q'_j \,\mathrm{in}(F_j)|_s}{\max\{|q_j|_s| \,\mathrm{in}(F_j)|_s\}} \\
&\leq \frac{\max |q'_j \,\mathrm{in}(F_j)|_s}{\max\{|q_j|_s| \,\mathrm{in}(F_j)|_s\}} \\
&\leq s^{-L(\gamma_0,\gamma_0)} \quad < \quad 1
\end{aligned}$$

for all $(q_1, \ldots, q_n, 0) \neq 0$, hence $\|v_s\|_s < 1$.

w_s has norm < 1 for all $s \in]1, s_0]$ and some $s_0 > 1$. Again we take $R = 0$, since w_s does not depend on R. We have

$$\begin{aligned}
\frac{|w_s(q_1, \ldots, q_n, 0)|_s}{|(q_1, \ldots, q_n, 0)|_s} &= \frac{|\sum q_j (F_j - \mathrm{in}(F_j))|_s}{\max\{|q_j|_s| \,\mathrm{in}(F_j)|_s\}} \\
&\leq \frac{\max\{|q_j|_s |(F_j - \mathrm{in}(F_j)|_s\}}{\max\{|q_j|_s| \,\mathrm{in}(F_j)|_s\}} \\
&< \nu(s) \quad < \quad 1
\end{aligned}$$

for all $(q_1, \dots, q_n, 0) \neq 0$ using lemma 1.3.15 and hence $\|w_s\|_s < 1$.

For all $s > 1$ sufficiently close to 1 we apply a standard argument (cf. [BGR] proposition 1.2.4/4) and obtain that

$$(q_1, \dots q_n, R) \mapsto (u_s + v_s + w_s)(q_1, \dots q_n, R) = \sum q_j F_j + R$$

is an isomorphism. □

1.4 First properties of completed Weyl algebras

From now on assume $\varepsilon \in |K^\times|^{2d}$ with $\varepsilon_i \varepsilon_{i+d} \geq 1$ for $1 \leq i \leq d$ or equivalently that $|\ |_\varepsilon$ on $\mathcal{A}_{d,\varepsilon}$ is multiplicative (cf. lemma 1.2.4). In this section we denote by $\mathcal{A}$ either the classical Weyl algebra A_d or one of its completions $\mathcal{A}_{d,\varepsilon}$ or $\mathcal{A}_d^\dagger$. By exp we denote the corresponding exponent.

Let I be a left ideal of $\mathcal{A}$. We define

$$\exp(I) = \{\exp(f) : 0 \neq f \in I\}.$$

Since $\exp(fg) = \exp(f) + \exp(g)$ we have $\exp(I) + \mathbb{N}^{2d} = \exp(I)$. The next proposition is a direct consequence of the division theorems in section 1.3.

Proposition 1.4.1. *All Weyl algebras $\mathcal{A}$ are Noetherian.*

Proof. The arguments of this proof can be found in [Gal] theorem 1.2.5. Let I be a left ideal of $\mathcal{A}$. By lemma 1.3.1 there is a finite set $f_1, \dots, f_n \in I$ such that the $\exp(f_i)$ generate $\exp(I)$, i.e. $\bigcup_i \exp(f_i) + \mathbb{N}^{2d} = \exp(I)$. Now to any $g \in I$ we can apply the division theorem using the above $f_1, \dots, f_n$ and get the decomposition

$$g = \sum q_i f_i + r.$$

Suppose $r \neq 0$. By the division theorem we have

$$\exp(r) \in \operatorname{supp}(r) \subseteq \overline{\Delta}.$$

With f_i and g the remainder r also lies in I, hence $\exp(r) \in \exp(I) = \bigcup_i \Delta_i$. This is a contradiction since the sets $\Delta_j, \overline{\Delta}$ are disjoint. □

Proposition 1.4.2. *Each left ideal $I \subseteq \mathcal{A}_{d,\varepsilon}$ is complete and hence closed in $I \subseteq \mathcal{A}_{d,\varepsilon}$.*

Proof. This is clear from [ST] proposition 2.1.(ii), however, we give an easy direct proof. Let $f_1, \dots, f_n$ be generators of I with $|f_j|_\varepsilon = 1$. Let $\sum_{i=0}^\infty g_i$ be convergent in $\mathcal{A}_{d,\varepsilon}$ with $g_i \in I$. We write $g_i = \sum_{j=1}^n q_{ij} f_j$. By the division theorem 1.3.14 we have $|g_i|_\varepsilon = \max\{|q_{ij} f_j|_\varepsilon\}$, hence $|q_{ij}|_\varepsilon \leq |g_i|_\varepsilon$ for all i and j. Therefore $\sum_{i=0}^\infty g_i = \sum_{j=1}^n (\sum_{i=0}^\infty q_{ij}) f_j \in I$. $\square$

Proposition 1.4.3. *An element f in $\mathcal{A}$ is invertible if and only if the exponent of f equals zero.*

Proof. Suppose $\exp(f) = 0$, applying the division theorem to 1 and dividing by f gives

$$1 = qf + r,$$

with $\mathrm{supp}(r) \in \emptyset$ (since $\exp(f) = 0$), hence $r = 0$. If we assume f to be invertible we get $0 = \exp(1) = \exp(f) + \exp(f^{-1})$, hence $\exp(f) = 0$. $\square$

Now we consider the usual operator of formal partial differentiation on the K-vector space of formal power series over K. Since this operator respects convergence, it extends to all completed Weyl algebras we consider. For the following lemma and corollary we do not need the assumptions $\varepsilon \in |K^\times|^{2d}$ and $\varepsilon_i \varepsilon_{d+i} \geq 1$.

Denote by ∂_{X_i} (resp. ∂_{Y_i}) the operator of formal differentiation with respect to the variable X_i (resp. Y_i), i.e. for $f = \sum a_{\alpha\beta} X^\alpha Y^\beta$ we define

$$\partial_{X_i} f = \sum_{\alpha,\beta \in \mathbb{N}^d} (\alpha_i + 1) a_{\alpha_1,\dots,\alpha_i+1,\dots,\alpha_d\,\beta} X^\alpha Y^\beta$$

$$\partial_{Y_i} f = \sum_{\alpha,\beta \in \mathbb{N}^d} (\beta_i + 1) a_{\alpha\,\beta_1,\dots,\beta_i+1,\dots,\beta_d} X^\alpha Y^\beta.$$

Lemma 1.4.4. *For $f \in \mathcal{A}$ we have*

$$f X_i - X_i f = \partial_{Y_i} f \quad \text{and} \quad Y_i f - f Y_i = \partial_{X_i} f.$$

Hence the operator respects two-sided ideals I of $\mathcal{A}$, i.e. we have $\partial_{Y_i} I \subset I$ and $\partial_{X_i} I \subset I$.

Proof. This follows from [Dix] lemma 2.2. □

A simple consequence of lemma 1.4.4 is

Corollary 1.4.5. *Let* $\operatorname{char} K = 0$*, then* K *is the center of* $\mathcal{A}$*.*

Proof. Suppose $f = \sum a_{mn} X^m Y^n$ lies in the center of $\mathcal{A}$. Then we have

$$X_i f - f X_i = \partial_{Y_i} f = 0 \quad \text{and} \quad f Y_i - Y_i f = \partial_{X_i} f = 0$$

for all $1 \leq i \leq d$. Hence all coefficients except the zeroth have to be zero, i.e. f lies in K. That elements of K lie in the center is clear. □

Proposition 1.4.6. *If* $\operatorname{char} K = 0$*, the algebra* $\mathcal{A}$ *is simple, i.e. has no proper two-sided ideals different from* 0*.*

Proof. Let I be a non-zero two-sided ideal of $\mathcal{A}_{d,\varepsilon}$ or $\mathcal{A}_d^\dagger$. We choose an element $0 \neq f = \sum a_{\alpha\beta} X^\alpha Y^\beta \in I$ and denote by (α_0, β_0) the exponent of f. We now use lemma 1.4.4, which says that the operators ∂_{X_i} and ∂_{Y_i} act on two-sided ideals. If $(\alpha_0, \beta_0) = (\alpha_{0,1}, \ldots, \alpha_{0,d}, \beta_{0,1} \ldots \beta_{0,d})$ and if we apply ∂_{X_i} $\alpha_{0,i}$ times to f and ∂_{Y_i} $\beta_{0,i}$ times for all i we get

$$\partial_X^{\alpha_0} \partial_Y^{\beta_0} f = \textstyle\sum_{\alpha,\beta \in \mathbb{N}^d} \frac{(\alpha+\alpha_0)!}{\alpha!} \frac{(\beta+\beta_0)!}{\beta!} a_{\alpha+\alpha_0\ \beta+\beta_0} X^\alpha Y^\beta \in I.$$

We have

$$|\alpha_0!\ \beta_0!\ a_{\alpha_0\beta_0}| > |\tfrac{(\alpha+\alpha_0)!}{\alpha!} \tfrac{(\beta+\beta_0)!}{\beta!} a_{\alpha+\alpha_0\ \beta+\beta_0}| \varepsilon^{(\alpha,\beta)}$$

if $(\alpha, \beta) \neq 0$. This is easily seen since on the one hand we know

$$|a_{\alpha_0\beta_0}| > |a_{\alpha+\alpha_0,\beta+\beta_0}| \varepsilon^{(\alpha,\beta)} \quad \text{if } (\alpha,\beta) \neq 0$$

by the choice of α_0 and β_0. And on the other hand we have

$$|\alpha_0!| \geq |\tfrac{(\alpha+\alpha_0)!}{\alpha!}| \text{ and equally } |\beta_0!| \geq |\tfrac{(\beta+\beta_0)!}{\beta!}|.$$

Hence the element we obtained above has exponent zero, which by proposition 1.4.3 means that it is a unit, i.e. $I = \mathcal{A}$. □

Tensor products of completed Weyl algebras

As before we denote by A_d the d-th classical Weyl algebra over K. We have a canonical K-algebra isomorphism

$$A_{d-1} \otimes_K A_1 \longrightarrow A_d$$

with $X_i \otimes 1 \mapsto X_i$, $Y_i \otimes 1 \mapsto Y_i$ for $1 \leq i \leq d-1$ and $1 \otimes X_d \mapsto X_d$, $1 \otimes Y_d \mapsto Y_d$, where A_{d-1} is the Weyl algebra in the $2(d-1)$ variables $X_1, \ldots, X_{d-1}, Y_1, \ldots, Y_{d-1}$ and A_1 is the Weyl algebra in the two variables X_d, Y_d. We write $\otimes$ instead of $\otimes_K$ in the following.

Let $\varepsilon \in \mathbb{R}^{2d}_{>0}$ with $\varepsilon_i \varepsilon_{d+i} \geq 1$. We define $\varepsilon' := (\varepsilon_1, \ldots, \varepsilon_{d-1}, \varepsilon_{d+1}, \ldots, \varepsilon_{2d-1})$ and $\varepsilon'' := (\varepsilon_d, \varepsilon_{2d})$. On A_{d-1} and A_1 we have the norms $| \ \ |_{\varepsilon'}$ and $| \ \ |_{\varepsilon''}$ respectively. This induces a norm on $A_{d-1} \otimes A_1$ if we set for an element $f \in A_{d-1} \otimes A_1$

$$|f|_{\varepsilon'\varepsilon''} := \inf\{\max_i |g_i|_{\varepsilon'} |h_i|_{\varepsilon''}\},$$

where the infimum runs through all representations $f = \sum g_i \otimes h_i$. We have of course also the norm $| \ \ |_{\varepsilon}$ on $A_{d-1} \otimes A_1$ coming from A_d via the above isomorphism. If we write

$$f = \sum a_{\alpha\beta} X_1^{\alpha_1} \cdots X_{d-1}^{\alpha_{d-1}} Y_1^{\beta_1} \cdots Y_{d-1}^{\beta_{d-1}} \otimes X_d^{\alpha_d} Y_d^{\beta_d} \in A_{d-1} \otimes A_1$$

in terms of the canonical K-basis $(X_1^{\alpha_1} \cdots X_{d-1}^{\alpha_{d-1}} Y_1^{\beta_1} \cdots Y_{d-1}^{\beta_{d-1}} \otimes X_d^{\alpha_d} Y_d^{\beta_d})_{\alpha,\beta \in \mathbb{N}^{2d}}$ of $A_{d-1} \otimes A_1$, this norm is simply $|f|_{\varepsilon} = \max |a_{\alpha\beta}| \varepsilon^{(\alpha,\beta)}$.

It is an easy exercise to prove that the norms $| \ \ |_{\varepsilon'\varepsilon''}$ and $| \ \ |_{\varepsilon}$ on $A_{d-1} \otimes A_1$ coincide. Now we consider the completed Weyl algebras $\mathcal{A}_{d-1,\varepsilon'}$ and $\mathcal{A}_{1,\varepsilon''}$. We take their algebraic tensor product and endow it with the norm $| \ \ |_{\varepsilon'\varepsilon''}$ as above. The canonical embedding

$$A_{d-1} \otimes A_1 \longrightarrow \mathcal{A}_{d-1,\varepsilon'} \otimes \mathcal{A}_{1,\varepsilon''}$$

is easily seen to be dense. Since the norm $| \ \ |_{\varepsilon'\varepsilon''}$ is multiplicative on $A_{d-1} \otimes A_1$ it is by density also multiplicative on $\mathcal{A}_{d-1,\varepsilon'} \otimes \mathcal{A}_{1,\varepsilon''}$. We denote its completion by $\mathcal{A}_{d-1,\varepsilon'} \widehat{\otimes} \mathcal{A}_{1,\varepsilon''}$.

Proposition 1.4.7. *The canonical map*

$$\mathcal{A}_{d-1,\varepsilon'} \widehat{\otimes} \mathcal{A}_{1,\varepsilon''} \longrightarrow \mathcal{A}_{d,\varepsilon}$$

with $X_i \otimes 1 \mapsto X_i$, $Y_i \otimes 1 \mapsto Y_i$ *for* $1 \leq i \leq d-1$ *and* $1 \otimes X_d \mapsto X_d$, $1 \otimes Y_d \mapsto Y_d$ *is an isometric isomorphism of* K*-Banach algebras.*

Proof. All maps in the following diagram are isometric and dense

$$\begin{tikzcd}
\mathcal{A}_{d-1,\varepsilon'} \widehat{\otimes} \mathcal{A}_{1,\varepsilon''} \arrow[rr] & & \mathcal{A}_{d,\varepsilon} \\
\mathcal{A}_{d-1,\varepsilon'} \otimes \mathcal{A}_{1,\varepsilon''} \arrow[u] & & \\
A_{d-1} \otimes A_1 \arrow[u] \arrow[rr] & & A_d . \arrow[uu]
\end{tikzcd}$$

□

2 Skew rings

In the study of the classical Weyl algebra A_d one is soon led to a certain extension ring namely the localization of A_d with respect to some variable.

The localization of A_d with respect to $K[X_i] \backslash \{0\}$ is $A_{d-1}(K(X_i))[Y_i, \partial_{X_i}]$, the skew polynomial ring (see section 2.1 or [McCR] paragraph 1.2 for a definition) over the $(d-1)$-th Weyl algebra over the field of fractions $K(X_i)$ of $K[X_i]$ with the usual derivation ∂_X on $K[X]$ extended to $A_{d-1}(K(X))$. The consideration of these localizations proves to be very useful in the investigation of the Weyl algebra.

If we consider the completed Weyl algebra $\mathcal{A}_{d,\varepsilon}$ the above remark motivates the following question: Does the localization of $\mathcal{A}_{d,\varepsilon}$ exist with respect to the multiplicative subset $K\langle X_i \rangle_{\varepsilon_i} \backslash \{0\}$, i.e. the subset of all non-zero elements in which only the variable X_i appears? The answer is negative.

Lemma 2.0.1. *Let* $\operatorname{char} K = 0$ *and* $\varepsilon \in |K^\times|^{2d}$. *Then the localization of* $\mathcal{A}_{d,\varepsilon}$ *with respect to* $K\langle X_i\rangle_{\varepsilon_i}\backslash\{0\}$ *does not exist. Equivalently,* $K\langle X_i\rangle_{\varepsilon_i}\backslash\{0\}$ *is not an Ore set in* $\mathcal{A}_{d,\varepsilon}$ *(see [McCR] 2.1.6 for a definition).*

Proof. By [McCR] 2.1.12 the two assertions of the lemma are equivalent, we are going to prove the second. If $K\langle X_i\rangle_{\varepsilon_i}\backslash\{0\}$ was an Ore set in $\mathcal{A}_{d,\varepsilon}$, then by the Weierstraß preparation theorem for Tate algebras (see e.g. [BGR] 5.2.2 theorem 1) $K[X_i]\backslash\{0\}$ would also be an Ore set in $\mathcal{A}_{d,\varepsilon}$. Hence it suffices to show that $K[X_i]\backslash\{0\}$ is not an Ore set in $\mathcal{A}_{d,\varepsilon}$.

Let $c \in K$ with $|c|\varepsilon_{d+i} < 1$. We claim that the element $f = \sum_\beta (cY_i)^\beta \in \mathcal{A}_{d,\varepsilon}$ has the property that $(\partial^\nu_{Y_i}(f))_{\nu\in\mathbb{N}}$ is a K-linearly independent family.

Recall that $\partial^\nu_{Y_i}(f) = \sum_\beta \frac{(\nu+\beta)!}{\beta!} c^{\nu+\beta} Y_i^\beta$. Let $a_\nu \in K$ with

$$0 = \sum_{\nu=0}^n a_\nu \partial^\nu_{Y_i}(f) = \sum_{\nu=0}^n a_\nu \sum_\beta \frac{(\nu+\beta)!}{\beta!} c^{\nu+\beta} Y_i^\beta = \sum_\beta (\sum_{\nu=0}^n a_\nu \frac{(\nu+\beta)!}{\beta!} c^{\nu+\beta}) Y_i^\beta.$$

Hence $\sum_{\nu=0}^n a_\nu \frac{(\nu+\beta)!}{\beta!} c^{\nu+\beta} = 0$ for all β. We denote by a the column vector $(a_0, \dots, a_n)$ and write the above equations for $0 \le \beta \le n$ in matrix form

$$((\frac{(\nu+\beta)!}{\beta!} c^{\nu+\beta})_{0\le\beta,\nu\le n})a = 0.$$

However, one can show that

$$\det((\frac{(\nu+\beta)!}{\beta!} c^{\nu+\beta})_{0\le\beta,\nu\le n}) = \prod_{k=0}^n k! c^{n(n+1)} \neq 0,$$

hence $a = 0$, whence the claim.

Finally, we prove that $K[X_i]\backslash\{0\}$ is not an Ore set in $\mathcal{A}_{d,\varepsilon}$.

We take $f = \sum (cY_i)^\beta \in \mathcal{A}_{d,\varepsilon}$ and $X_i \in K[X_i]\backslash\{0\}$ and assume the Ore condition holds, i.e. there exist $f' \in \mathcal{A}_{d,\varepsilon}$ and $s' \in K[X_i]\backslash\{0\}$ with $s'f = f'X_i$.

Let $s' = \sum_{\alpha=0}^n a_\alpha X_i^\alpha$. We use the formula $X_i^\alpha f = \sum_{\gamma=0}^\alpha (-1)^\gamma \binom{\alpha}{\gamma} \partial^\gamma_{Y_i}(f) X_i^{\alpha-\gamma}$, which is a consequence of lemma 1.4.4 and get

$$
\begin{aligned}
s'f &= \sum_{\alpha=0}^{n} a_\alpha \sum_{\gamma=0}^{\alpha} (-1)^\gamma \binom{\alpha}{\gamma} \partial_{Y_i}^{\gamma}(f) X_i^{\alpha-\gamma} \\
&= \sum_{\alpha=0}^{n} a_\alpha \sum_{\gamma=0}^{\alpha-1} (-1)^\gamma \binom{\alpha}{\gamma} \partial_{Y_i}^{\gamma}(f) X_i^{\alpha-\gamma} + \sum_{\alpha=0}^{n} a_\alpha (-1)^\alpha \partial_{Y_i}^{\alpha}(f) \\
&= \underbrace{(\sum_{\alpha=0}^{n} a_\alpha \sum_{\gamma=0}^{\alpha-1} (-1)^\gamma \binom{\alpha}{\gamma} \partial_{Y_i}^{\gamma}(f) X_i^{\alpha-1-\gamma})}_{=:g} X_i + \sum_{\alpha=0}^{n} a_\alpha (-1)^\alpha \partial_{Y_i}^{\alpha}(f).
\end{aligned}
$$

By the above claim $\sum_{\alpha=0}^{n} a_\alpha(-1)^\alpha \partial_{Y_i}^{\alpha}(f) \neq 0$, hence $f' \neq g$. The equality

$$
\varepsilon\text{-exp}((f'-g)X_i) = \varepsilon\text{-exp}(\sum_{\alpha=0}^{n} a_\alpha(-1)^\alpha \partial_{Y_i}^{\alpha}(f)),
$$

leads to a contradiction, since $\varepsilon\text{-exp}(\sum_{\alpha=0}^{n} a_\alpha(-1)^\alpha \partial_{Y_i}^{\alpha}(f)) = (0, \dots, *, \dots, 0)$ with some natural number at the $(d+i)$-th place, however, $\varepsilon\text{-exp}((f'-g)X_i) = \varepsilon\text{-exp}(f'-g) + \varepsilon\text{-exp}(X_i) = (*, \dots, *) + (0, \dots, 1, \dots, 0)$ where the 1 appears at the i-th place. $\square$

The subsequent will provide us with the construction of a ring that in the case of completed Weyl algebras plays to some extent the role of the localization. This ring is in fact a microlocalization, see [LvO] Chapter IV for the definition in the situation of filtered rings and [Nag] for the definition in the language of non-archimedean Banach algebras.

2.1 Skew polynomial rings

Let R be a unital associative ring, let $\sigma : R \longrightarrow R$ be a ring endomorphism, and $\delta : R \longrightarrow R$ a σ-derivation, i.e. a additive group endomorphism with

$$
\delta(ab) = \delta(a)b + \sigma(a)\delta(b) \quad \text{for all } a, b \in R.
$$

Let $R[X, \sigma, \delta]$ be the *skew polynomial ring* (see [McCR] section 1.2 for a definition). Note again that if objects or properties have left and right versions we restrict to the left version whereas [McCR] always use right versions. Every element of $R[X, \sigma, \delta]$ has a unique expression as a finite sum $\sum a_i X^i$ (cf. [McCR] 1.2.3).

We have the usual notion of degree of a polynomial in $R[X, \sigma, \delta]$ and if R has no zero divisors and σ is injective we have the following rule. Let $f = \sum_{i=0}^{m} a_i X^i$ and $g = \sum_{j=0}^{n} b_j X^j$ be in $R[X, \sigma, \delta]$ with a_m, b_n non-zero, then

$$\deg(fg) = \deg(f) + \deg(g).$$

This follows since fg has degree $\leq m + n$ and since the coefficient of X^{m+n} is $a_m \sigma^m(b_n)$ (see [McCR] proof of 1.2.9.(i)).

Theorem 2.1.1. (Division with remainder.) *We assume that σ is injective and that R is an integral domain. Let $f \in R[X, \sigma, \delta]$ with leading coefficient a unit. For all $g \in R[X, \sigma, \delta]$ there exist unique elements $q, r \in R[X, \sigma, \delta]$ with*

$$g = qf + r \qquad \textit{and} \qquad \deg(r) < \deg(f).$$

Proof. We proceed by induction on the degree of g. Let $f = \sum a_i X^i$ and $g = \sum b_i X^i$ with $\deg(f) = m$ and $\deg(g) = n$. If $n < m$ the theorem is clear. Hence assume $n \geq m$. Then we have by induction a unique expression

$$g - b_n(\sigma^{n-m}(a_m))^{-1} X^{n-m} f = qf + r.$$

The polynomial $b_n(\sigma^{n-m}(a_m))^{-1} X^{n-m} f$ is of degree n with leading coefficient $b_n(\sigma^{n-m}(a_m))^{-1} \sigma^{n-m}(a_m) = b_n$. □

Now we restrict to the case $\sigma = 1$ and we work over a complete non-archimedean ring R, by which we mean the following: A not necessarily commutative ring R with identity endowed with a map $|\ \ | : R \to \mathbb{R}_{\geq 0}$ satisfying the properties

(i) $|a| = 0 \iff a = 0$,

(ii) $|a + b| \leq \max\{|a|, |b|\}$,

(iii) $|ab| = |a||b|$,

is called *valued ring* (see [BGR] section 1.5.1 for the definition in the commutative setting). A valued ring which is complete with respect to the topology induced by $| \quad |$ is called a *complete non-archimedean ring.*

Let R be a complete non-archimedean ring. First of all the assumption $\sigma = 1$ implies that the multiplication is given by the formula

$$Xa = aX + \delta(a).$$

This implies

$$X^n a = \sum_{i=0}^{n} \binom{n}{i} \delta^{n-i}(a) X^i$$

(cf. [McCR] 1.2.8). Hence two elements of $R[X, \delta]$ are multiplied in the following way. For $(\sum a_i X^i)(\sum b_j X^j) = \sum c_k X^k$ we get

$$c_k = \sum_{j=0}^{k} \sum_{i=k-j}^{\infty} \binom{i}{k-j} a_i \delta^{i-k+j}(b_j). \tag{1}$$

This is of course a finite sum, however we write it in this form for later use.

Let $f = \sum a_i X^i \in R[X, \delta]$ be a skew polynomial and $\varepsilon \in \mathbb{R}_{>0}$. We define

$$|f|_\varepsilon := \max |a_i| \varepsilon^i.$$

Obviously $(R[X, \delta], | \quad |_\varepsilon)$ is a normed group (see [BGR] 1.1.3 for a definition). If $f \neq 0$ we call

$$\varepsilon\text{-exp}(f) := \max\{i : |a_i|\varepsilon^i = |f|_\varepsilon\}$$

the **$\boldsymbol{\varepsilon}$-exponent** of f. We define ε-exp$(f) := -\infty$ in case $f = 0$.

Proposition 2.1.2. *Let δ be norm decreasing with constant ε, i.e. $|\delta(a)| \leq \varepsilon |a|$ for all $a \in R$, then the norm $| \quad |_\varepsilon$ on $R[X, \delta]$ is multiplicative, hence $R[X, \delta]$ is a valued ring.*

Remark 2.1.3. *Note that if δ is norm decreasing with constant ε it is not necessarily norm decreasing in the ordinary sense.*

Proof. Let $f = \sum a_i X^i$ and $g = \sum b_j X^j$ be elements of $R[X, \delta]$. We write $fg = \sum c_k X^k$.

$$\begin{aligned}
|fg|_\varepsilon &= \max_k |c_k| \varepsilon^k \\
&= \max_k |\sum_{j=0}^{k} \sum_{i=k-j}^{\infty} \binom{i}{k-j} a_i \delta^{i-k+j}(b_j)| \varepsilon^k \\
&\leq \max_{\substack{i,j,k \\ k \leq i+j}} |a_i||\delta^{i-k+j}(b_j)| \varepsilon^k \\
&\leq \max_{\substack{i,j,k \\ k \leq i+j}} |a_i||b_j| \varepsilon^{i-k+j} \varepsilon^k \\
&= |f|_\varepsilon |g|_\varepsilon
\end{aligned}$$

Let $i_0 = \varepsilon\text{-exp}(f)$ and $j_0 = \varepsilon\text{-exp}(g)$. The assertion $|fg|_\varepsilon \geq |f|_\varepsilon |g|_\varepsilon$ follows if we show

$$|c_{i_0+j_0}| \varepsilon^{i_0+j_0} = |a_{i_0}| \varepsilon^{i_0} |b_{j_0}| \varepsilon^{j_0}.$$

This is the case if

$$|\binom{i}{i_0+j_0-j} a_i \delta^{i-i_0-j_0+j}(b_j)| \varepsilon^{i_0+j_0} < |a_{i_0}| \varepsilon^{i_0} |b_{j_0}| \varepsilon^{j_0}$$

for all $(i, j) \neq (i_0, j_0)$ with $i_0 + j_0 \leq i + j$. As above

$$|\binom{i}{i_0+j_0-j} a_i \delta^{i-i_0-j_0+j}(b_j)| \varepsilon^{i_0+j_0} \leq |a_i| \varepsilon^i |b_j| \varepsilon^j.$$

Hence if $i > i_0$ the strict inequality holds. If $i \leq i_0$ we get $j > j_0$ and again we have the strict inequality. ☐

2.2 Rings of restricted skew power series

Let $(R, |\ \ |)$ be a complete non-archimedean ring, δ a derivation, and $\varepsilon \in \mathbb{R}_{>0}$. We define $R\langle X, \delta\rangle_\varepsilon$ to be the completion of the normed group $(R[X, \delta], |\ \ |_\varepsilon)$. We can view elements of $R\langle X, \delta\rangle_\varepsilon$ as formal expressions $\sum_{i=0}^{\infty} a_i X^i$ with $a_i \in R$ and $|a_i|\varepsilon^i \to 0$ for $i \to \infty$. If δ is norm decreasing with respect to ε, i.e. if $|\delta(a)| \leq \varepsilon |a|$ for all $a \in R$, then $|\ \ |_\varepsilon$ is multiplicative on $R[X, \delta]$ (cf. Proposition 2.1.2) and $R\langle X, \delta\rangle_\varepsilon$ is a complete non-archimedean ring with coefficient wise addition, the multiplication given by formula (1) and multiplicative norm $|\ \ |_\varepsilon$. We call it the **ring of ε-restricted skew power series** over R. Whenever we write $R\langle X, \delta\rangle_\varepsilon$ in the following δ will tacitly be assumed to be norm decreasing with constant ε, so that $R\langle X, \delta\rangle_\varepsilon$ is a complete non-archimedean ring.

Alternative definition of complete Weyl algebras

Let $\varepsilon \in \mathbb{R}^{2d}_{>0}$ with $\varepsilon_i\varepsilon_{d+i} \geq 1$ for all $0 \leq i \leq d$. Let K be a complete non-archimedean field. We want to inductively redefine the d-th complete Weyl algebra $\mathcal{A}_{d,\varepsilon}$. We use the notation $\varepsilon(j) := (\varepsilon_1, \ldots, \varepsilon_j, \varepsilon_{d+1}, \ldots, \varepsilon_{d+j})$. We define $\mathcal{A}_0 := K$. The ring $\mathcal{A}_{j,\varepsilon(j)}$ is by induction hypothesis non-archimedean and complete. Hence taking $\delta = 0$ we can consider $\mathcal{A}_{j,\varepsilon(j)}\langle X_{j+1}\rangle_{\varepsilon_{j+1}}$, the ring of ε_{j+1}-restricted (skew) power series over $\mathcal{A}_{j,\varepsilon(j)}$. This is a complete ring with norm $|\ \ |_{\varepsilon_{j+1}}$ on which we have the operator ∂_{j+1} of formal differentiation with respect to X_{j+1}. For all $f = \sum a_i X_{j+1}^i \in \mathcal{A}_{j,\varepsilon(j)}\langle X_{j+1}\rangle_{\varepsilon_{j+1}}$ we have

$$\begin{aligned}
|\partial_{j+1}(f)|_{\varepsilon_{j+1}} &= \max_i |(i+1)a_{i+1}|\varepsilon_{j+1}^i \\
&\leq \max_i |a_i|\varepsilon_{j+1}^{i-1} \\
&\leq \max_i |a_i|\varepsilon_{j+1}^i\varepsilon_{d+j+1} \\
&= \varepsilon_{d+j+1}|f|_{\varepsilon_{j+1}}.
\end{aligned}$$

Hence the derivation ∂_{j+1} is norm decreasing with respect to ε_{d+j+1}. Therefore we can form

$$\mathcal{A}_{j+1,\varepsilon(j+1)} := (\mathcal{A}_{j,\varepsilon(j)}\langle X_{j+1}\rangle_{\varepsilon_{j+1}})\langle Y_{j+1}, \partial_{j+1}\rangle_{\varepsilon_{d+j+1}}$$

the ring of ε_{d+j+1}-restricted skew power series with norm $|\ \ |_{\varepsilon_{j+d+1}}$. The Weyl algebra $\mathcal{A}_{d,\varepsilon}$ we obtain by this procedure is the same as the Weyl algebra defined in section 1.2.

Remark 2.2.1. *Note that we can add the $2d$ variables in any order ($(2d)!$ choices) as long as we insure that we add the variable with respect to the corresponding ε_i and that we choose δ by the following rule. We take $\delta = 0$ if we add X_i (respectively Y_i) and Y_i (respectively X_i) has not jet been added. We* *take $\delta = \partial_{X_i}$ if we want to add Y_i and X_i has been added and we take $\delta = -\partial Y_i$ if we want to add X_i and Y_i has been added.*

Proof. For the algebra defined in this way we easily verify the relation $Y_iX_i = X_iY_i + 1$ for all i. Hence we can define a homomorphism from the completed Weyl algebra into this algebra by sending X_i to X_i and Y_i to Y_i. The map is injective, since writing elements as non-commutative power series is unique and it is surjective, since the convergence condition for the coefficients of elements is the same for both algebras. $\square$

Division in rings of skew power series

The notion of ε-exponent for elements in $R[X, \delta]$ extends to elements $f = \sum a_iX^i \in R\langle X, \delta\rangle_\varepsilon$.

Lemma 2.2.2. *Let $f, g \in R\langle X, \delta\rangle_\varepsilon$. Then*
- $\varepsilon\text{-exp}(fg) = \varepsilon\text{-exp}(f) + \varepsilon\text{-exp}(g)$ *and*
- *if* $\varepsilon\text{-exp}(f) \neq \varepsilon\text{-exp}(g)$ *then* $\varepsilon\text{-exp}(f+g) \in \{\varepsilon\text{-exp}(f), \varepsilon\text{-exp}(g)\}$ *and* $\varepsilon\text{-exp}(f+g) \neq -\infty$,

with the usual conventions if $f = 0$ or $g = 0$.

Proof. Let $f = \sum a_iX^i$, $g = \sum b_jX^j$ and $fg = \sum c_kX^k$. By the second part of the proof of proposition 2.1.2 we know

$$|c_{\varepsilon\text{-exp}(f)+\varepsilon\text{-exp}(g)}|\varepsilon^{\varepsilon\text{-exp}(f)+\varepsilon\text{-exp}(g)} = |fg|_\varepsilon,$$

hence $\varepsilon\text{-exp}(fg) \geq \varepsilon\text{-exp}(f) + \varepsilon\text{-exp}(g)$.

The inequality $\varepsilon\text{-exp}(fg) \leq \varepsilon\text{-exp}(f) + \varepsilon\text{-exp}(g)$ holds since there is no natural number $k > \varepsilon\text{-exp}(f) + \varepsilon\text{-exp}(g)$ with $|c_k|\varepsilon^k = |fg|_\varepsilon$. Indeed, let us assume $k > \varepsilon\text{-exp}(f) + \varepsilon\text{-exp}(g)$. We have

$$\begin{aligned} |c_k|\varepsilon^k &= |\sum_{j=0}^{k}\sum_{i=k-j}^{\infty}\binom{i}{k-j}a_i\delta^{i-k+j}(b_j)|\varepsilon^k \\ &\leq \max_{\substack{0\leq j\leq k \\ i+j\geq k}} |a_i||b_j|\varepsilon^{i+j}. \end{aligned}$$

If $i > \varepsilon\text{-exp}(f)$ then $|a_i|\varepsilon^i < |f|_\varepsilon$, hence $|a_i||b_j|\varepsilon^{i+j} < |fg|_\varepsilon$. The inequality $i < \varepsilon\text{-exp}(f)$ together with $i + j \geq k > \varepsilon\text{-exp}(f) + \varepsilon\text{-exp}(g)$ gives $j > \varepsilon\text{-exp}(g)$ and as above we obtain $|a_i||b_j|\varepsilon^{i+j} < |fg|_\varepsilon$, hence $|c_k|\varepsilon^k < |fg|_\varepsilon$.

The second statement follows as in the proof of lemma 1.3.11. □

Lemma 2.2.3. *Let $f, g, q, r \in R[X, \delta]$ with $g = qf + r$, $\deg(r) < \deg(f)$ and $\deg(f) = \varepsilon\text{-exp}(f)$. Then $|g|_\varepsilon = \max\{|qf|_\varepsilon, |r|_\varepsilon\}$.*

Proof. In proposition 2.1.2 we saw that the norm $|qf|_\varepsilon$ is given by the $\varepsilon\text{-exp}(q)+\varepsilon\text{-exp}(f)$-th coefficient. However, $\deg(r) < \deg(f)$, so this coefficient appears in g too, hence g has the same norm, whence the lemma. □

Definition. An element $0 \neq f = \sum a_i X^i \in R\langle X, \delta\rangle_\varepsilon$ is called **distinguished** if $a_{\varepsilon\text{-exp}(f)} \in R^\times$.

Theorem 2.2.4. *Let $\varepsilon \in |R|\backslash\{0\}$ and assume $|R|\backslash\{0\} = |R^\times|$. Let $f \in R\langle X, \delta\rangle_\varepsilon$ be distinguished. For all $g \in R\langle X, \delta\rangle_\varepsilon$ there is a unique element $q \in R\langle X, \delta\rangle_\varepsilon$ and a unique element $r \in R[X, \delta]$ with*

$$g = qf + r \qquad \textit{and} \qquad \deg(r) < \varepsilon\text{-exp}(f).$$

Moreover, we have $|g|_\varepsilon = \max\{|qf|_\varepsilon, |r|_\varepsilon\}$.

Proof. The proof is the same as the proof of theorem 1.3.14. However, we reproduce it for the sake of completeness. The uniqueness follows from lemma 2.2.2 (cf. the proof of theorem 1.3.9).

Existence. We may assume $|f|_\varepsilon = |g|_\varepsilon = 1$. Let $f = \sum a_i X^i$ and $g = \sum b_i X^i$. We put

$$\delta := \max_{\substack{i \\ |a_i|\varepsilon^i < 1}} |a_i|\varepsilon^i < 1$$

if the maximum is non-zero and choose an arbitrary $0 < \delta < 1$ otherwise. We denote by $f^{>\delta}$ the element $\sum_{|a_i|\varepsilon^i > \delta} a_i X^i$ and by $f^{\leq\delta}$ the element $\sum_{|a_i|\varepsilon^i \leq \delta} a_i X^i$. This gives a decomposition $f = f^{>\delta} + f^{\leq\delta}$.

Now starting with $g = g_0$ we define a sequence g_k by the following procedure. As above we decompose $g = g^{>\delta} + g^{\leq\delta}$ and apply theorem 2.1.1 to $g_k^{>\delta}$ and $f_k^{>\delta}$. Note that $f_k^{>\delta}$ has a unit as its leading coefficient! We get

$$g_k^{>\delta} = q_k f_k^{>\delta} + r_k$$

with $|q_k|_\varepsilon \leq 1$, since $1 = |g_k|_\varepsilon = \max\{|q_k f_k^{>\delta}|_\varepsilon, |r_k|_\varepsilon\}$ by lemma 2.2.3. Hence

$$g_k = q_k f_k + r_k + \tilde{g}_{k+1}$$

if

$$\tilde{g}_{k+1} := q_k f^{>\delta} - q_k f + g_k^{\leq\delta} = -q_k f_k^{\leq\delta} + g_k^{\leq\delta}.$$

We have $|\tilde{g}_{k+1}|_\varepsilon \leq \max\{|q_k f^{\leq\delta}|_\varepsilon, |g_k^{\leq\delta}|_\varepsilon\} \leq \delta$. If $\tilde{g}_{k+1} \neq 0$ choose $\pi_{k+1} \in R^\times$ such that

$$g_{k+1} := \pi_{k+1}^{-1}\tilde{g}_{k+1}$$

has norm 1, hence $|\pi_{k+1}| \leq \delta$ ($\pi_0 = 1$). If $\tilde{g}_{k_0} = 0$ for some k_0 we put $g_k = q_k = r_k = 0$ for all $k \geq k_0$. In either case this gives

$$g = \sum_{k=0}^{N}\prod_{l=0}^{k}\pi_l(q_k f + r_k) + \prod_{l=0}^{N+1}\pi_l g_{N+1}$$

for all $N \in \mathbb{N}$. Setting

$$q := \sum_{k=0}^{\infty}\prod_{l=0}^{k}\pi_l q_k \qquad \text{and} \qquad r := \sum_{k=0}^{\infty}\prod_{l=0}^{k}\pi_l r_k$$

we get the decomposition

$$g = qf + r.$$

Finally, we have to show ε-exp$(f) > \deg(r)$. By definition ε-exp$(f) = \deg(f^{>\delta})$. However, we have

$$\operatorname{supp}(r) \subseteq \bigcup_k \operatorname{supp}(r_k) \subseteq \{0, \ldots, \deg(f^{>\delta}) - 1\}.$$

The final assertion is clear since $|q|_\varepsilon \leq 1$. □

Proposition 2.2.5. *Assume $\varepsilon \in |R^\times| = |R|\backslash\{0\}$. An element $f \in R\langle X, \delta\rangle_\varepsilon$ is invertible if and only if it is distinguished with ε-exp$(f) = 0$.*

Proof. Suppose f is distinguished with ε-exp$(f) = 0$. Applying theorem 2.2.4 with $g = 1$ gives the inverse. If $f = \sum a_i X^i$ is invertible it is an immediate consequence of lemma 2.2.2 that ε-exp$(f) = 0$, i.e. $|a_0| > |a_i|\varepsilon^i$ for all $i > 0$. Let $f^{-1} = \sum b_i X^i$ be the inverse. Computing the constant coefficient of ff^{-1} using formula (1) gives $\sum_{i=0}^{\infty} a_i \delta^i(b_0) = 1$, hence $a_0 b_0 = 1 - \sum_{i=1}^{\infty} a_i \delta^i(b_0)$. The inequality

$$|a_0 b_0| > |a_i|\varepsilon^i |b_0| \geq |a_i \delta^i(b_0)| \quad \text{for all } i > 0$$

implies $|a| < 1$, where $a := \sum_{i=1}^{\infty} a_i \delta^i(b_0)$. Since R is complete $a_0 b_0$ is a unit with inverse $\sum_{i=0}^{\infty} a^i$. This proves that the element f is distinguished with ε-exp$(f) = 0$. $\square$

Theorem 2.2.4 is a generalization of the Weierstraß division theorem for Tate algebras (cf. [BGR] 5.2.1 theorem 2) in which $\delta = 0$.

As we saw above the completed Weyl algebras $\mathcal{A}_{d,\varepsilon}$ are rings of restricted skew power series. Hence the above division theorem applies to $\mathcal{A}_{d,\varepsilon}$. However since there are always elements which are not distinguished with respect to a given variable the theorem is not applicable to all elements of $\mathcal{A}_{d,\varepsilon}$. However in the case of the completed Weyl algebra $\mathcal{A}_\varepsilon = \mathcal{A}_{1,\varepsilon}$ we have the following lemma.

An element $f \in \mathcal{A}_\varepsilon$ is called Y-distinguished if under the identification with $K\langle X\rangle_{\varepsilon_1}\langle Y, \partial_X\rangle_{\varepsilon_2}$ it is distinguished. Of course, we have the similar notion of being X-distinguished using the identification with $K\langle Y\rangle_{\varepsilon_2}\langle X, -\partial_Y\rangle_{\varepsilon_1}$.

Lemma 2.2.6. *Let $\varepsilon = (\varepsilon_1, \varepsilon_2) \in |K^\times|^2$ with $\varepsilon_1\varepsilon_2 \geq 1$ and let $0 \neq f \in \mathcal{A}_\varepsilon$ be any element. There exists an isometric isomorphism $\sigma : \mathcal{A}_\varepsilon \longrightarrow \mathcal{A}_\varepsilon$ of K-Banach algebras such that $\sigma(f)$ is Y-distinguished (resp. X-distinguished).*

Proof. Let A denote the classical Weyl algebra in two variables X and Y. We get a well defined homomorphism of K-algebras

$$\sigma : A \longrightarrow \mathcal{A}_\varepsilon$$

if we put $\sigma(X) = X + cY^\mu$ and $\sigma(Y) = Y$ with $c \in K$ and $\mu \in \mathbb{N}$, since

$$Y(X + cY^\mu) = YX + cY^{\mu+1} = XY + 1 + cY^{\mu+1} = (X + cY^\mu)Y + 1.$$

We show now that σ is continuous with respect to the ε-norm on A and $\mathcal{A}_\varepsilon$ for all μ if we choose $c \in K$ such that $|c| = \varepsilon_1 \varepsilon_2^{-\mu}$. We have

$$\begin{aligned} |\sigma(f)|_\varepsilon &= |\sum_{\alpha\beta} a_{\alpha\beta}(X + cY^\mu)^\alpha Y^\beta|_\varepsilon \\ &\leq \max_{\alpha\beta} |a_{\alpha\beta}||(X + cY^\mu)|_\varepsilon^\alpha |Y|_\varepsilon^\beta \\ &= \max_{\alpha\beta} |a_{\alpha\beta}|\varepsilon^{(\alpha,\beta)} \\ &= |f|_\varepsilon. \end{aligned}$$

The classical Weyl algebra A is dense in $\mathcal{A}_\varepsilon$ and $\mathcal{A}_\varepsilon$ is complete, hence the continuous homomorphism σ extends uniquely to a continuous homomorphism $\sigma : \mathcal{A}_\varepsilon \to \mathcal{A}_\varepsilon$ with $|\sigma(f)|_\varepsilon \leq |f|_\varepsilon$. Taking $-c$ instead of c we again get a well defined continuous homomorphism $\tau : \mathcal{A}_\varepsilon \to \mathcal{A}_\varepsilon$ with $|\tau(f)|_\varepsilon \leq |f|_\varepsilon$. Obviously $\sigma \circ \tau = \tau \circ \sigma = \mathrm{id}$, i.e. σ is an isomorphism. Further we have

$$|f|_\varepsilon = |\tau(\sigma(f))|_\varepsilon \leq |\sigma(f)|_\varepsilon \leq |f|_\varepsilon,$$

hence $|\sigma(f)|_\varepsilon = |f|_\varepsilon$ for all $f \in \mathcal{A}_\varepsilon$.

We denote by $F^0\mathcal{A}_\varepsilon$ all elements of $\mathcal{A}_\varepsilon$ with $|\ \ |_\varepsilon \leq 1$. If we choose $a, b \in K$ with $|a|^{-1} = \varepsilon_1$ and $|b|^{-1} = \varepsilon_2$ we get a homomorphism of rings

$$\begin{aligned} \varphi :\quad F^0\mathcal{A}_\varepsilon \quad &\longrightarrow \quad A(k) \text{ or } k[X,Y], \\ \sum a_{\alpha\beta}X^\alpha Y^\beta \quad &\mapsto \quad \sum \overline{a_{\alpha\beta}a^{-\alpha}b^{-\beta}}X^\alpha Y^\beta \end{aligned}$$

mapping to the classical Weyl algebra in the two variables X and Y over the residue field k of K if $\varepsilon_1\varepsilon_2 = 1$ and mapping to the polynomial ring in the variables X and Y over k if $\varepsilon_1\varepsilon_2 > 1$ (cf. lemma 4.2.1).

An element $f \in F^0\mathcal{A}_\varepsilon$ is Y-distinguished if and only if $\varphi(f)$ written as a polynomial in Y with polynomials in X as coefficients has a constant as leading coefficient. Indeed, for $f = \sum a_{\alpha\beta}X^\alpha Y^\beta$ with $|f|_\varepsilon = 1$ the polynomial $\varphi(f)$ is of degree ε_2-$\exp(f)$ and $\sum a_{\alpha\,\varepsilon_2\text{-}\exp(f)}X^\alpha$ is a unit if and only if $|a_{0\,\varepsilon_2\text{-}\exp(f)}|_{\varepsilon_2} > |a_{\alpha\,\varepsilon_2\text{-}\exp(f)}|_{\varepsilon_2}\varepsilon_1^\alpha$ for all $\alpha > 0$ (cf. proposition 2.2.5). This in turn is the case if and only if $\sum \overline{a_{\alpha\,\varepsilon_2\text{-}\exp(f)}a^{-\alpha}b^{-\,\varepsilon_2\text{-}\exp(f)}}X^\alpha = \overline{a_{0\,\varepsilon_2\text{-}\exp(f)}b^{-\,\varepsilon_2\text{-}\exp(f)}}$.

Let us take $c := a^{-1}b^{\mu}$ to define σ. Then the homomorphism $\varphi \circ \sigma$ is given by

$$\begin{array}{ccc} F^0\mathcal{A}_\varepsilon & \longrightarrow & A(k) \text{ or } k[X,Y]. \\ \sum a_{\alpha\beta}X^\alpha Y^\beta & \mapsto & \sum \overline{a_{\alpha\beta}a^{-\alpha}b^{-\beta}}(X+Y^\mu)^\alpha Y^\beta \end{array}$$

To complete the proof assume without loss of generality that $f = \sum a_{\alpha\beta}X^\alpha Y^\beta \in \mathcal{A}_\varepsilon$ with $|f|_\varepsilon = 1$. Define $N := \{(\alpha,\beta) : \overline{a_{\alpha\beta}a^{-\alpha}b^{-\beta}} \neq 0\} \subset \mathbb{N}^2$ and let μ be in $\mathbb{N}$ such that $\mu > \alpha$ and $\mu > \beta$ for all $(\alpha,\beta) \in N$. Then there is a unique element $(\alpha_0,\beta_0) \in N$ such that $\mu\alpha_0 + \beta_0$ is maximal. Hence

$$\varphi(\sigma(f)) = \sum \overline{a_{\alpha\beta}a^{-\alpha}b^{-\beta}}(X+Y^\mu)^\alpha Y^\beta$$

is an element of degree $\mu\alpha_0 + \beta_0$ with a constant as leading coefficient. $\square$

Now we establish a Weierstraß preparation theorem for the completed Weyl algebra in two variables $\mathcal{A}_\varepsilon = K\langle X\rangle_{\varepsilon_1}\langle Y, \partial_X\rangle_{\varepsilon_2}$.

Theorem 2.2.7. *Assume $\varepsilon = (\varepsilon_1,\varepsilon_2) \in |K^\times|^2$ with $\varepsilon_1\varepsilon_2 \geq 1$. Let $f \in \mathcal{A}_\varepsilon$ be Y-distinguished. Then there exists a unique monic polynomial pol $\in K\langle X\rangle_{\varepsilon_1}[Y,\partial_X]$ of degree ε_2-$\exp(f)$ and a unique unit $u \in \mathcal{A}_\varepsilon$ such that $f = u \cdot pol$. Furthermore $|pol|_\varepsilon = \varepsilon_2^{\varepsilon_2\text{-}\exp(f)}$, hence pol is Y-distinguished with ε_2-$\exp(pol) = \deg_Y(pol)$.*

Proof. We apply the division theorem 2.2.4 to $Y^{\varepsilon_2\text{-}\exp(f)}$ and obtain elements $q \in \mathcal{A}_\varepsilon$ and $r \in K\langle X\rangle_{\varepsilon_1}[Y,\partial_X]$ with

$$Y^{\varepsilon_2\text{-}\exp(f)} = qf + r$$

and $\deg_Y(r) < \varepsilon_2$-$\exp(f)$. Moreover we have $\max\{|qf|_\varepsilon, |r|_\varepsilon\} = \varepsilon_2^{\varepsilon_2\text{-}\exp(f)}$. We set $pol := Y^{\varepsilon_2\text{-}\exp(f)} - r$. Hence $pol = qf$, $\deg_Y(pol) = \varepsilon_2$-$\exp(f)$ and $|pol|_\varepsilon = \varepsilon_2^{\varepsilon_2\text{-}\exp(f)}$. Now we show that q is a unit. We normalize the equation $pol = qf$ such that $|pol|_\varepsilon = |q|_\varepsilon = |f|_\varepsilon = 1$. As in the proof of lemma 2.2.6 we use the map

$$\varphi : F^0\mathcal{A}_\varepsilon \longrightarrow A(k) \text{ or } k[X,Y].$$

We get $\varphi(pol) = \varphi(q)\varphi(f)$ where $\varphi(pol)$ and $\varphi(f)$ are polynomials resp. skew polynomials in Y of the same degree with coefficients in $k[X]$ such that the leading coefficients are constant. This implies $\varphi(q) \in k$, hence q is Y-distinguished with ε_2-$\exp(q) = 0$, i.e. q is a unit (cf. proposition 2.2.5). Hence with $u := q$ we obtain a decomposition $pol = uf$ satisfying the properties of the proposition. Assume $pol' = u'f$ is another such decomposition. Then $pol - pol' = (u - u')f$ with $\deg_Y(pol - pol') < \varepsilon_2$-$\exp(f)$. If $pol \neq pol'$ this contradicts the uniqueness property of theorem 2.2.4. $\square$

Rings of restricted skew power series over a field

Now we work over a complete non-archimedean field K instead of a complete non-archimedean ring R.

Proposition 2.2.8. *Let $\varepsilon \in |K^\times|$ and let δ be a derivation on K which is norm decreasing with constant ε. Then all elements of $K\langle X, \delta\rangle_\varepsilon$ are distinguished and all left ideals are principal.*

Proof. Let I be a left ideal. Let ε-$\exp(I) := \{\varepsilon\text{-}\exp(f) : 0 \neq f \in I\}$. Choose an element $f \in I$ with ε-$\exp(f) = \min \varepsilon$-$\exp(I)$. By the division theorem 2.2.4, for any $g \in I$ we have a decomposition

$$g = qf + r$$

with $\deg(r) < \varepsilon$-$\exp(f)$. Assume $r \neq 0$. With g and qf in I we know $r \in I$. However ε-$\exp(r) \leq \deg(r) < \varepsilon$-$\exp(f)$, a contradiction. $\square$

We will compute the left Krull dimension $\mathcal{K}(K\langle X, \delta\rangle_\varepsilon)$. See [McCR] chapter 6 for the definition of the left Krull dimension of non-commutative rings. It is an open question if the left and right Krull dimension of left and right Noetherian rings coincide (cf. [McCR] 6.4.10 and 6.4.11). Here we restrict to the left Krull dimension, which we will call Krull dimension for simplicity. However, for the right Krull dimension we obtain the same results by symmetry.

Proposition 2.2.9. *Let $\varepsilon \in |K^\times|$ and let δ be a derivation on K which is norm decreasing with constant ε. The Krull dimension of $K\langle X, \delta\rangle_\varepsilon$ is 1.*

Proof. We directly verify the definition as it can be found in [McCR] 6.1.2. Consider the descending chain of left ideals generated by $X, X^2, \dots$. This chain never becomes constant (exponent lemma 2.2.2). Hence the Krull dimension is not zero. Let

$$K\langle X, \delta\rangle_\varepsilon \supseteq I_0 \supseteq I_1 \supseteq \dots$$

be any descending chain of left ideals. The Krull dimension of $K\langle X, \delta\rangle_\varepsilon$ is less than or equal to 1 if we show that for almost all i there are only finitely many left ideals between I_i and I_{i+1}. Since we have to show this for all but finitely many indices i we may assume $I_i \neq 0$ for all i. We have ε-$\exp(I_i) \supseteq \varepsilon$-$\exp(I_{i+1})$, both of which are subsets of $\mathbb{N}$. Recall that ε-$\exp(I_i) + \mathbb{N} = \mathbb{N}$ by the exponent lemma and hence that there are only finitely many different exponents between ε-$\exp(I_i)$ and ε-$\exp(I_{i+1})$. If $I \subseteq J$ are two left ideals with ε-$\exp(I) = \varepsilon$-$\exp(J)$ then $I = J$. To show this let f (resp. g) be a generating element of I (resp. J), f and g exist by proposition 2.2.8. Since ε-$\exp(I) = \varepsilon$-$\exp(J)$ we have ε-$\exp(f) = \varepsilon$-$\exp(g)$. However, there exists an element q with $f = qg$. By the exponent lemma ε-$\exp(q) = 0$ i.e. q is a unit (cf. proposition 2.2.5). Hence $I = J$. □

Chapter 7 of [McCR] introduces the notion of left and right global dimension for non-commutative rings. Although it is not true in general, we know that for a left and right Noetherian ring R the left and right global dimensions coincide (cf. [McCR] 7.1.11). In this case we simply speak of the global dimension of R denoted by $\mathrm{gld}(R)$.

Proposition 2.2.10. *Let $\varepsilon \in |K^\times|$ and let δ be a derivation on K which is norm decreasing with constant ε. The global dimension of $K\langle X, \delta\rangle_\varepsilon$ is 1.*

Proof. The ring $K\langle X, \delta\rangle_\varepsilon$ is not semisimple. Indeed, as shown above it has Krull dimension 1, whence it is not Artinian. By [McCR] 7.1.8 (a) it is therefore enough to show that all left ideals are projective. However, $K\langle X, \delta\rangle_\varepsilon$ is a principal left ideal domain (cf. proposition 2.2.8) and hence left ideals are module isomorphic to $K\langle X, \delta\rangle_\varepsilon$. □

3 Dimensions of Weyl algebras

The Krull dimension as well as the global dimension of the d-th classical Weyl algebra are equal to d if $\operatorname{char} K = 0$ and are equal to $2d$ if $\operatorname{char} K = p > 0$. We conjecture the same to be true for the d-th completed Weyl algebra $\mathcal{A}_{d,\varepsilon}$. We will prove that d serves as a lower bound for both the Krull and the global dimension. If $\operatorname{char} K = p > 0$ we show that $2d$ is a lower bound for both dimensions. That d is also the upper bound if $\operatorname{char} K = 0$ will only be proved for $d = 1$ under the additional assumption that K is discretely valued.

3.1 Lower bounds

Lemma 3.1.1. *Let $f \in \mathcal{A}_{d,\varepsilon}$ and let $k < d$. If fY_{k+1} is an element of the left ideal $\sum_{i=1}^{k} \mathcal{A}_{d,\varepsilon} Y_i$, then $f \in \sum_{i=1}^{k} \mathcal{A}_{d,\varepsilon} Y_i$.*

Proof. We use the convention that a coefficient is zero if it has negative indices. Let $f = \sum a_{\alpha\beta} X^\alpha Y^\beta$ and let $fY_{k+1} \in \sum_{i=1}^{k} \mathcal{A}_{d,\varepsilon} Y_i$. There exist $f_i = \sum a_{\alpha\beta}^{(i)} X^\alpha Y^\beta \in \mathcal{A}_{d,\varepsilon}$ with

$$\begin{aligned} fY_{k+1} &= \sum_{i=1}^{k} f_i Y_i \\ &= \sum_{i=1}^{k} \sum_{\alpha\beta} a_{\alpha\beta}^{(i)} X^\alpha Y^\beta Y_i \\ &= \sum_{\alpha\beta} (\sum_{i=1}^{k} a_{\alpha,\beta_1,\ldots,\beta_i-1,\ldots,\beta_d}^{(i)}) X^\alpha Y^\beta. \end{aligned}$$

On the other hand we have

$$fY_{k+1} = \sum_{\alpha\beta} a_{\alpha\beta} X^\alpha Y^\beta Y_{k+1} = \sum_{\alpha\beta} a_{\alpha,\beta_1,\ldots,\beta_{k+1}-1,\ldots,\beta_d} X^\alpha Y^\beta.$$

Together this gives $\sum_{i=1}^{k} a_{\alpha,\beta_1,\ldots,\beta_i-1,\ldots,\beta_d}^{(i)} = a_{\alpha,\beta_1,\ldots,\beta_{k+1}-1,\ldots,\beta_d}$ and hence

$$\sum_{i=1}^{k} a_{\alpha,\beta_1,\ldots,\beta_i-1,\ldots,\beta_d}^{(i)} = 0 \quad \text{if} \quad \beta_{k+1} = 0,$$

which will be important later.

Set $b^{(i)}_{\alpha\beta} := a^{(i)}_{\alpha,\beta_1,\ldots,\beta_{k+1}+1,\ldots,\beta_d}$ and $g_i := \sum b^{(i)}_{\alpha\beta} X^\alpha Y^\beta$. We have $g_i \in \mathcal{A}_{d,\varepsilon}$, since $|b^{(i)}_{\alpha\beta}|\varepsilon^{(\alpha,\beta)} \to 0$ for $|\alpha| + |\beta| \to \infty$. Finally,

$$\begin{aligned}
\sum_{i=1}^{k} g_i Y_i Y_{k+1} &= \sum_{i=1}^{k}\sum_{\alpha\beta} b^{(i)}_{\alpha\beta} X^\alpha Y^\beta Y_i Y_{k+1} \\
&= \sum_{i=1}^{k}\sum_{\alpha\beta} b^{(i)}_{\alpha,\beta_1,\ldots,\beta_i-1,\ldots,\beta_{k+1}-1,\ldots,\beta_d} X^\alpha Y^\beta \\
&= \sum_{\substack{\alpha\beta \\ \beta_{k+1}>0}} (\sum_{i=1}^{k} a^{(i)}_{\alpha,\beta_1,\ldots,\beta_i-1,\ldots,\beta_d}) X^\alpha Y^\beta \\
&= \sum_{\substack{\alpha\beta \\ \beta_{k+1}>0}} (\sum_{i=1}^{k} a^{(i)}_{\alpha,\beta_1,\ldots,\beta_i-1,\ldots,\beta_d}) X^\alpha Y^\beta \\
&\quad + \sum_{\substack{\alpha\beta \\ \beta_{k+1}=0}} (\sum_{i=1}^{k} a^{(i)}_{\alpha,\beta_1,\ldots,\beta_i-1,\ldots,\beta_d}) X^\alpha Y^\beta \\
&= \sum_{\alpha\beta}(\sum_{i=1}^{k} a^{(i)}_{\alpha,\beta_1,\ldots,\beta_i-1,\ldots,\beta_d}) X^\alpha Y^\beta \quad = \quad f Y_{k+1}
\end{aligned}$$

Hence $f = \sum_{i=1}^{k} g_i Y_i$, proving the lemma. □

Proposition 3.1.2. *The Krull dimension of $\mathcal{A}_{d,\varepsilon}$ is bounded below by d.*

Proof. Apply [McCR] proposition 6.5.9 to the left ideal generated by the elements $Y_1, \ldots, Y_d$. This is a proper ideal and the Y_i commute pairwise. Together with the property in lemma 3.1.1 the proposition follows. □

Proposition 3.1.3. *The global dimension of $\mathcal{A}_{d,\varepsilon}$ is bounded below by d.*

Proof. We consider the same ideal as in the proof of proposition 3.1.2. To apply [McCR] theorem 7.3.16 we need the additional property that $\sum Y_i \mathcal{A}_{d,\varepsilon}$ is a proper ideal, which is the case. The assertion follows. □

Remark 3.1.4. *To establish these lower bounds we did not need to assume K to be discretely valued nor ε to lie in $|K^\times|^{2d}$. Both these properties will be main ingredients to obtain upper bounds.*

If we assume $\operatorname{char} K = p > 0$ we immediately get as in the classical case the following strong result for the Krull dimension. Let K_{alg} denote an algebraic closure of K.

Proposition 3.1.5. *Assume* $\operatorname{char} K = p > 0$*, then the Krull dimension of* $\mathcal{A}_{d,\varepsilon}$ *is bounded below by* $2d$*. The Krull dimension of* $\mathcal{A}_{d,\varepsilon}$ *is* $2d$ *if* $\varepsilon \in |K_{alg}^\times|^{2d}$*.*

Proof. We proceed as in the proof in the classical case as it can be found in [McCR] 7.5.8. The elements X_i^p and Y_i^p in $\mathcal{A}_{d,\varepsilon}$ are central. Indeed, using the formula $fX_i^p = \sum_{j=0}^p \binom{p}{j} X_i^{p-j}\partial_{Y_i}^j$ (cf. lemma 1.4.4) we see that $fX_i^p = X_i^p f$, since $\binom{p}{j}$ is divisible by p for all $1 < j < p$ and all coefficients of $\partial_{Y_i}^p(f)$ are multiples of p. Hence the Tate algebra $T_{2d,\varepsilon^p} = K\langle X_1^p, \ldots, X_d^p, Y_1^p, \ldots, Y_d^p\rangle_{\varepsilon^p}$ in $2d$ variables is a subalgebra of $\mathcal{A}_{d,\varepsilon}$. In fact, $\mathcal{A}_{d,\varepsilon}$ is a free T_{2d,ε^p}-module of finite rank with basis $\{X^\alpha Y^\beta\}_{\alpha,\beta\in\{0,\ldots,p-1\}^d}$. Finally, using [McCR] corollary 6.5.3 we get

$$\mathcal{K}(\mathcal{A}_{d,\varepsilon}) = \mathcal{K}(T_{2d,\varepsilon^p}) \geq 2d$$

(cf. [BGR] remark 6.1.2 for the inequality). The final statement follows since we know $\mathcal{K}(T_{2d,\varepsilon^p}) = 2d$ if $\varepsilon \in |K_{\mathrm{alg}}^\times|$ (combine [BGR] remark 6.1.2 and the proof of [BGR] theorem 6.1.5/4). ☐

Proposition 3.1.6. *Assume* $\operatorname{char} K = p > 0$*, then the global dimension of* $\mathcal{A}_{d,\varepsilon}$ *is bounded below by* $2d$*.*

Proof. From the proof of proposition 3.1.5 above we know that $\mathcal{A}_{d,\varepsilon}$ is a free T_{2d,ε^p}-module of finite rank. Hence using [McCR] theorem 7.2.6 we get

$$\operatorname{gld}(\mathcal{A}_{d,\varepsilon}) \geq \operatorname{gld}(T_{2d,\varepsilon^p}) \geq 2d.$$

The fact that $\operatorname{gld}(T_{2d,\varepsilon^p}) \geq 2d$ follows with [McCR] theorem 7.3.16. ☐

3.2 Upper bounds ($d=1$)

Recall that by the alternative definition of Weyl algebras (remark 2.2.1) we have

$$\begin{aligned}\mathcal{A}_{d,\varepsilon} &\simeq (\mathcal{A}_{d-1,\hat{\varepsilon}^i}(K\langle X_i\rangle_{\varepsilon_i}))\langle Y_i, \partial_{X_i}\rangle_{\varepsilon_{d+i}} \\ &\simeq (\mathcal{A}_{d-1,\hat{\varepsilon}^i}(K\langle Y_i\rangle_{\varepsilon_{d+i}}))\langle X_i, -\partial_{Y_i}\rangle_{\varepsilon_i}\end{aligned}$$

with the notation $\hat{\varepsilon}^i = (\varepsilon_1, \ldots, \varepsilon_{i-1}, \varepsilon_{i+1}, \ldots, \varepsilon_{d+i-1}, \varepsilon_{d+i+1}, \ldots, \varepsilon_{2d})$ and where $\mathcal{A}_{d-1,\hat{\varepsilon}^i}$ is the Weyl algebra in the $2(d-1)$ variables $X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_d$ and $Y_1, \ldots, Y_{i-1}, Y_{i+1}, \ldots, Y_d$.

The derivation ∂_{X_i} (resp. ∂_{Y_i}) extends to a derivation on the Weyl algebra over the completion of the quotient field of $K\langle X_i\rangle_{\varepsilon_i}$ (resp. $K\langle Y_i\rangle_{\varepsilon_{d+i}}$) which is norm decreasing with constant ε_{d+i} (resp. ε_i). Hence we can form the ring of restricted skew power series

$$\begin{aligned} &\mathcal{B}_{d,\varepsilon}^{X_i} := (\mathcal{A}_{d-1,\hat{\varepsilon}^i}(\widehat{\mathrm{Quot}(K\langle X_i\rangle_{\varepsilon_i})}))\langle Y_i, \partial_{X_i}\rangle_{\varepsilon_{d+i}} \\ \text{resp.}\quad &\mathcal{B}_{d,\varepsilon}^{Y_i} := (\mathcal{A}_{d-1,\hat{\varepsilon}^i}(\widehat{\mathrm{Quot}(K\langle Y_i\rangle_{\varepsilon_{d+i}})}))\langle X_i, -\partial_{Y_i}\rangle_{\varepsilon_i}.\end{aligned}$$

We denote the norms of these K-Banach algebras by $|\ \ |_\varepsilon$. In the following we want to consider the canonical ring extension

$$\mathcal{A}_{d,\varepsilon} \longrightarrow \bigoplus_{i=1}^{d} \mathcal{B}_{d,\varepsilon}^{X_i} \oplus \bigoplus_{i=1}^{d} \mathcal{B}_{d,\varepsilon}^{Y_i} =: \mathcal{B}_{d,\varepsilon}.$$

The maximum norm on $\mathcal{B}_{d,\varepsilon}$ is again denoted by $|\ \ |_\varepsilon$.

Lemma 3.2.1. *Assume* $\mathrm{char}\, K = 0$ *and* $\varepsilon \in |K^\times|^{2d}$. *Let* $I \subset \mathcal{A}_{d,\varepsilon}$ *be a maximal left ideal. Then* $\mathcal{B}_{d,\varepsilon} I \subsetneq \mathcal{B}_{d,\varepsilon}$.

Proof. Assume the lemma to be false. Let $I \subset \mathcal{A}_{d,\varepsilon}$ be a maximal left ideal with $\mathcal{B}_{d,\varepsilon} I = \mathcal{B}_{d,\varepsilon}$. Then $1 \in \mathcal{B}_{d,\varepsilon}^{X_i} I$ and $1 \in \mathcal{B}_{d,\varepsilon}^{Y_i} I$ for $1 \le i \le d$.

Since $1 \in \mathcal{B}_{d,\varepsilon}^{X_i} I$, there is an element $f = \sum a_{\alpha\beta} X^\alpha Y^\beta \in I$, which is a unit in $\mathcal{B}_{d,\varepsilon}^{X_i}$, or equivalently, which is distinguished with ε_{d+i}- $\exp_{\mathcal{B}_{d,\varepsilon}^{X_i}}(f) = 0$ (cf.

proposition 2.2.5). By definition $\varepsilon_{d+i}\text{-}\exp_{\mathcal{B}^{X_i}_{d,\varepsilon}}(f) = 0$ implies

$$\begin{aligned} & \max_{\alpha,\beta_1,\ldots,\beta_{i-1},\beta_{i+1},\ldots,\beta_d} |a_{\alpha,\beta_1,\ldots,\beta_{i-1},0,\beta_{i+1},\ldots,\beta_d}| \varepsilon^{(\alpha,\beta_1,\ldots,\beta_{i-1},0,\beta_{i+1},\ldots,\beta_d)} \\ > & \max_{\alpha,\beta_1,\ldots,\beta_{i-1},\beta_{i+1},\ldots,\beta_d} |a_{\alpha\beta}|\varepsilon^{(\alpha,\beta)} \end{aligned}$$

for all $b_i > 0$. This together with the fact that f is distinguished combined with the fact that an element in $\mathcal{A}_{d-1,\hat{\varepsilon}^i}(\mathrm{Quot}\widehat{(K\langle X_i\rangle_{\varepsilon_i})})$ is a unit if and only if its $\hat{\varepsilon}^i$-exponent is zero (cf. proposition 1.4.3) gives

$$\begin{aligned} & \max_{\alpha_i} |a_{0,\ldots,\alpha_i,\ldots,0}|\varepsilon^{(0,\ldots,\alpha_i,\ldots,0)} \\ > & \max_{\alpha_i} |a_{\alpha,\beta_1,\ldots,\beta_{i-1},0,\beta_{i+1},\ldots,\beta_d}|\varepsilon^{(\alpha,\beta_1,\ldots,\beta_{i-1},0,\beta_{i+1},\ldots,\beta_d)} \end{aligned}$$

for all $(\alpha_1,\ldots,\alpha_{i-1},\alpha_{i+1},\ldots,\alpha_d,\beta_1,\ldots,\beta_{i-1},\beta_{i+1},\ldots,\beta_d) \neq 0$. Therefore, we can find an $\alpha_{f_i} \in \mathbb{N}$ with $\varepsilon\text{-}\exp_{\mathcal{A}_{d,\varepsilon}}(f) = (0,\ldots,\alpha_{f_i},\ldots,0)$ where α_{f_i} appears at the i-th place. Similarly, working in $\mathcal{B}^{Y_i}_{d,\varepsilon}$, we find an element $g \in I$ and an $\beta_{g_i} \in \mathbb{N}$ such that $\varepsilon\text{-}\exp_{\mathcal{A}_{d,\varepsilon}}(g) = (0,\ldots,\beta_{g_i},\ldots,0)$ where β_{g_i} appears at the $d+i$-th place.

Let $h_1,\ldots,h_n$ be elements in I such that $\varepsilon\text{-}\exp(I) = \bigcup_j \varepsilon\text{-}\exp(h_i) + \mathbb{N}^{2d}$ (cf. lemma 1.3.1). Hence $h_1,\ldots,h_n$ generate I (cf. proof of proposition 1.4.1). Applying theorem 1.3.14 we get an isomorphism of K-vector spaces

$$\begin{array}{ccc} \mathcal{A}_{d,\varepsilon}/I & \longrightarrow & \{f \in \mathcal{A}_{d,\varepsilon} : \mathrm{supp}(f) \subseteq \overline{\Delta}\}. \\ \sum q_i h_i + r & \mapsto & r \end{array}$$

We know that

$$\begin{aligned} \Delta &:= \textstyle\bigcup_{i=1}^d (0,\ldots,\alpha_{f_i},\ldots,0) + \mathbb{N}^{2d} \cup \bigcup_{i=1}^d (0,\ldots,\alpha_{g_i},\ldots,0) + \mathbb{N}^{2d} \\ &\subseteq \varepsilon\text{-}\exp(I) = \textstyle\bigcup_j \Delta_j. \end{aligned}$$

Hence $\overline{\Delta} \subseteq \mathbb{N}^{2d}\backslash\Delta = \{(\alpha,\beta) \in \mathbb{N}^{2d} : \alpha_i < \alpha_{f_i},\ \beta_i < \beta_{g_i}\}$, so that $\overline{\Delta}$ is finite and hence $\{f \in \mathcal{A}_{d,\varepsilon} : \mathrm{supp}(f) \subseteq \overline{\Delta}\}$ is finite dimensional over K. This is a contradiction to the fact that there are no simple finite dimensional $\mathcal{A}_{d,\varepsilon}$-modules which we are going to prove now.

The fact that there are no non-zero finite dimensional left $\mathcal{A}_{d,\varepsilon}$-modules is true for any simple infinite dimensional K-algebra, where K is any field. Indeed, let

M be a non-zero left $\mathcal{A}_{d,\varepsilon}$-module which is finite dimensional over K. Consider the ring $S := \mathrm{End}_{\mathcal{A}_{d,\varepsilon}}(M)$. M is a right S-module via $m\varphi := \varphi(m)$ for $\varphi \in S$, $m \in M$. Hence $\mathrm{End}_S(M)$ is finite dimensional over K, since $\mathrm{End}_S(M) \subseteq \mathrm{End}_K(M)$ is a K-subspace. Since $\mathcal{A}_{d,\varepsilon}$ is simple if $\mathrm{char}\, K = 0$ (cf. proposition 1.4.6) $\mathcal{A}_{d,\varepsilon} \to \mathrm{End}_S(M)$ with $f \mapsto (m \mapsto fm)$ is a ring extension, hence $\mathcal{A}_{d,\varepsilon}$ is finite dimensional over K, a contradiction. □

Now we restrict to the case $d = 1$. In this case we omit the subscript d, i.e. we write $\mathcal{A}_\varepsilon$ and $\mathcal{B}_\varepsilon = \mathcal{B}_\varepsilon^X \oplus \mathcal{B}_\varepsilon^Y$. To obtain the next results we assume the extension $\mathcal{A}_\varepsilon \subseteq \mathcal{B}_\varepsilon$ to be flat. We will show in section 4 proposition 4.3.4 that this is true at least if we further assume that K is discretely valued and that all components of ε lie in $|K^\times|$.

Proposition 3.2.2. *Assume $\mathcal{A}_\varepsilon \subseteq \mathcal{B}_\varepsilon$ to be a flat extension of rings and let $\varepsilon \in |K^\times|^{2d}$. If* $\mathrm{char}\, K = 0$*, then the Krull dimension of $\mathcal{A}_\varepsilon$ is* 1.

Proof. We already know that $\mathcal{K}(\mathcal{A}_\varepsilon) \geq 1$ (cf. lemma 3.1.2). The fact that $\mathcal{A}_\varepsilon \subseteq \mathcal{B}_\varepsilon$ is flat combined with lemma 3.2.1 implies that the extension is faithfully flat (cf. [McCR] proposition 7.2.3). Hence the map sending left ideals $I \subseteq \mathcal{A}_\varepsilon$ to the left ideals $\mathcal{B}_\varepsilon I \subseteq \mathcal{B}_\varepsilon$ preserves proper containments (cf. [Bou1] proposition I.3.5.9).

From proposition 2.2.9 we know that $\mathcal{K}(\mathcal{B}_\varepsilon^X) = \mathcal{K}(\mathcal{B}_\varepsilon^Y) = 1$. Hence $\mathcal{K}(\mathcal{B}_\varepsilon) = 1$. Indeed, the left ideals of $\mathcal{B}_\varepsilon$ are the direct sums of the left ideals of $\mathcal{B}_\varepsilon^X$ and $\mathcal{B}_\varepsilon^Y$. Hence with lemma 6.1.14 of [McCR] we have

$$\mathcal{K}(\mathcal{B}_\varepsilon) = \sup\{\mathcal{K}(\mathcal{B}_\varepsilon^X), \mathcal{K}(\mathcal{B}_\varepsilon^Y)\}.$$

Applying [McCR] 6.5.3.(i) we get $\mathcal{K}(\mathcal{A}_\varepsilon) \leq \mathcal{K}(\mathcal{B}_\varepsilon) = 1$. □

If the Weyl algebra is defined over a discretely valued field and the components of ε lie in $|K^\times|$ we will show in section 4 that the global dimension of $\mathcal{A}_{d,\varepsilon}$ is finite (cf. 4.3.6). To some extent this justifies the first assumption in the next proposition.

Proposition 3.2.3. *Assume the global dimension of $\mathcal{A}_\varepsilon$ to be finite and the extension $\mathcal{A}_\varepsilon \subseteq \mathcal{B}_\varepsilon$ to be flat. Let $\varepsilon \in |K^\times|^{2d}$. If* $\operatorname{char} K = 0$*, then the global dimension of $\mathcal{A}_\varepsilon$ equals* 1.

Proof. We know that 1 is a lower bound (cf. proposition 3.1.2). The extension $\mathcal{A}_\varepsilon \subseteq \mathcal{B}_\varepsilon$ is faithfully flat (cf. proof of proposition 3.2.2). Hence by [McCR] theorem 7.2.6 the global dimension of $\mathcal{A}_\varepsilon$ is bounded above by the global dimension of $\mathcal{B}_\varepsilon$. However, the global dimension of $\mathcal{B}_\varepsilon$ is 1 by proposition 2.2.10. $\square$

These results indicate that the ring extensions $\mathcal{B}_\varepsilon^X$ and $\mathcal{B}_\varepsilon^Y$, i.e. the microlocalizations, are the appropriate objects to consider. The localization of the classical Weyl algebra A in two variables X and Y with respect to the multiplicative subset $K[X]\backslash\{0\}$ is a principal left and right ideal domain with Krull and global dimension equal to 1. As we saw above the microlocalization $\mathcal{B}_\varepsilon^X$ of the completed Weyl algebra $\mathcal{A}_\varepsilon$ shares these properties. The analogy goes even further. Just as the localization in the classical case is a simple ring so is the microlocalization.

Proposition 3.2.4. *Assume $\varepsilon \in |K^\times|^{2d}$. The rings $\mathcal{B}_\varepsilon^X$ and $\mathcal{B}_\varepsilon^Y$ are simple, i.e. have no proper two-sided ideals different from zero.*

Proof. As in the case of the Weyl algebra we show first that formal differentiation of elements $f \in \mathcal{B}_\varepsilon^X$ with respect to Y is given by the formula

$$\partial_Y(f) = fX - Xf.$$

Let $f = \sum a_\beta Y^\beta$ and $X = \sum b_\beta Y^\beta$, i.e. $b_0 = X$ and $b_\beta = 0$ for all $\beta > 0$. Applying the multiplication formula (1) we get

$$\begin{aligned} fX &= \sum_k \Big(\sum_{j=0}^{k} \sum_{i=k-j}^{\infty} \binom{i}{k-j} a_i \partial_X^{i-k+j}(b_j)\Big) Y^k \\ &= \sum_k \Big(\sum_{i=k}^{\infty} \binom{i}{k} a_i \partial_X^{i-k}(X)\Big) Y^k \\ &= \sum_k (a_k X + (k+1) a_{k+1}) Y^k \end{aligned}$$

and

$$\begin{aligned} Xf &= \sum_k (\sum_{j=0}^{k} \sum_{i=k-j}^{\infty} \binom{i}{k-j} b_i \partial_X^{i-k+j}(a_j)) Y^k \\ &= \sum_k (a_k X) Y^k, \end{aligned}$$

proving the assertion. Let $f = \sum a_\beta Y^\beta$ be a non-zero element of a two-sided ideal of $\mathcal{B}_\varepsilon^X$. Set $\beta_0 := \varepsilon_2\text{-}exp(f)$. By the above assertion the element

$$\partial_Y^{\beta_0}(f) = \sum_\beta \tfrac{(\beta+\beta_0)!}{\beta!} a_{\beta+\beta_0} Y^\beta$$

is an element of the two-sided ideal. On the other hand we have

$$|\beta_0! a_{\beta_0}|_{\varepsilon_1} > |\tfrac{(\beta+\beta_0)!}{\beta!} a_{\beta+\beta_0}|_{\varepsilon_1} \varepsilon_2^\beta \quad \text{for all } \beta > 0$$

by the definition of the exponent and because $|\beta_0!| \geq |\frac{(\beta+\beta_0)!}{\beta!}|$ for all β. Hence $\partial_Y^{\beta_0}(f)$ is a unit (cf. proposition 2.2.5). □

4 Filtration

In this section we endow the completed Weyl algebra $\mathcal{A}_{d,\varepsilon}$, the microlocalizations $\mathcal{B}_{d,\varepsilon}^{X_i}$ and $\mathcal{B}_{d,\varepsilon}^{Y_i}$, and $\mathcal{B}_{d,\varepsilon}$, the sum of these microlocalizations, with a filtration. The associated graded rings turn out to be well known classical objects.

There is an extensive theory on how a filtered ring inherits properties from the associated graded ring (see for example [LvO]), however, for this process to work in our situation we have to assume that the complete non-archimedean field K is *discretely valued.*

4.1 Filtered rings

We briefly recall the definitions as they are used in [LvO]. Let R be an associative unital ring. We call R **filtered** if it is equipped with a family $\{F^n R\}_{n\in\mathbb{Z}}$

of additive subgroups $F^nR \subseteq R$ such that, for any $m, n \in \mathbb{Z}$,

$$\begin{array}{rl} \text{(i)} & F^mR \subseteq F^nR \text{ if } m \leq n, \\ \text{(ii)} & F^mR \cdot F^nR \subseteq F^{m+n}R, \\ \text{(iii)} & \bigcup_{n\in\mathbb{Z}} F^nR = R \text{ and } 1 \in F^0R. \end{array}$$

The **associated graded ring** is the ring

$$grR := \bigoplus_{n\in\mathbb{Z}} gr^nR$$

with the standard multiplication and where $gr^nR = F^nR/F^{n-1}R$. A filtration is said to be **complete** if it is Hausdorff ($\bigcap_{n\in\mathbb{Z}} F^nR = 0$) and the natural map

$$R \longrightarrow \varprojlim_n R/F^nR$$

is bijective (i.e. every Cauchy sequence converges).

Let K be a complete discretely valued field with uniformizing element $\pi \in K$ and residue field k. Let A be a K-Banach algebra (associative unital) with a non-archimedean norm $|\ \ |_A$ satisfying $|1|_A = 1$, $|ab|_A = |a|_A|b|_A$ and $|K| = |A|_A$. We can view A as a filtered ring if we define

$$F^nA := \{a \in A : |a|_A \leq |\pi|^{-n}\} \quad \text{for all } n \in \mathbb{Z}.$$

This filtration is complete. Note that grA is a graded k-algebra and that we have an isomorphism of graded k-algebras

$$grA \simeq k[\bar{\pi}, \bar{\pi}^{-1}] \otimes_k gr^0A,$$

where the graduation on the right hand side is given by the negative degree graduation of $k[\bar{\pi}, \bar{\pi}^{-1}]$. The subring F^0A of A is a filtered ring using the filtration of A and we have an isomorphism of graded k-algebras

$$grF^0A \simeq k[\bar{\pi}] \otimes_k gr^0A.$$

Lemma 4.1.1. *Assume that the Krull dimension of F^0A is bounded above by d resp. the global dimension of F^0A is bounded above by d and that F^0A is Noetherian. Then the Krull dimension resp. the global dimension of A is bounded above by $d-1$.*

Proof. We reproduce an argument of the proof of [ST] theorem 8.9. Let $\mathcal{O}_K$ be the valuation ring of K. As the elements of K commute with all elements of A the set $\mathcal{O}_K \backslash \{0\}$ is an Ore set in $F^0 A$. We have the ring isomorphisms

$$(\mathcal{O}_K \backslash \{0\})^{-1} F^0 A \simeq K \otimes_{\mathcal{O}_K} F^0 A \simeq A.$$

For all $f \in F^{0+}A$ the element $1-f$ has the inverse $1+f+f^2+\dots$ in $F^0 A$, hence $F^{0+}A$ lies in the Jacobson radical of $F^0 A$. This implies that any simple left $F^0 A$-module is $\mathcal{O}_K$-torsion (all $a \in \mathcal{O}_K \backslash \{0\}$ with $|a| < 1$ annihilate all elements of the module). The lemma follows if we apply [McCR] proposition 6.5.3 in the case of the Krull dimension and [McCR] corollary 7.4.3 and theorem 7.4.4 in the case of the global dimension. $\square$

4.2 Associated graded rings

From now on let K be a complete discretely valued non-archimedean field with uniformizing element $\pi \in K$ and residue field k. We consider the completed Weyl algebra $\mathcal{A}_{d,\varepsilon}$ defined over K. We endow $\mathcal{A}_{d,\varepsilon}$ with the filtration

$$F^n \mathcal{A}_{d,\varepsilon} := \{f \in \mathcal{A}_{d,\varepsilon} : |f|_\varepsilon \leq |\pi|^{-n}\} \quad \text{for } n \in \mathbb{Z}.$$

This filtration is complete. For $\varepsilon \in \mathbb{R}^{2d}_{>0}$ we will use the notation

$$d(\varepsilon) := \#\{j : \varepsilon_j \varepsilon_{d+j} = 1\}.$$

Lemma 4.2.1. *Let $\varepsilon \in |K^\times|^{2d}$ with $\varepsilon_j \varepsilon_{d+j} \geq 1$ for all $1 \leq j \leq d$. Then we have an isomorphism of graded k-algebras*

$$gr \mathcal{A}_{d,\varepsilon} \simeq k[\bar{\pi}, \bar{\pi}^{-1}] \otimes_k A_{d(\varepsilon)}(k) \otimes_k \mathrm{Pol}_{2(d-d(\varepsilon))}(k),$$

where $\mathrm{Pol}_{2(d-d(\varepsilon))}(k)$ denotes the polynomial ring in $2(d-d(\varepsilon))$ variables over k.

Proof. We may assume that $\varepsilon_j \varepsilon_{d+j} = 1$ for all $j = 1, \dots, d(\varepsilon)$ and that $\varepsilon_j \varepsilon_{d+j} > 1$ for all $j = d(\varepsilon)+1, \dots, d$. This is easily achieved if we consider for example the isomorphism

$$\mathcal{A}_{d,\varepsilon} \simeq \mathcal{A}_{1,\varepsilon_1,\varepsilon_{d+1}} \widehat{\otimes} \cdots \widehat{\otimes} \mathcal{A}_{1,\varepsilon_d,\varepsilon_{2d}}$$

of proposition 1.4.7. We choose $c_j \in K$ such that $|c_j|^{-1} = \varepsilon_j$ for $1 \leq j \leq 2d$ and such that $c_{d+j} = c_j^{-1}$ if $\varepsilon_j\varepsilon_{d+j} = 1$. We define a k-algebra homomorphism

$$\begin{array}{ccc} A_{d(\varepsilon)}(k) \otimes_k \mathrm{Pol}_{2(d-d(\varepsilon))}(k) & \longrightarrow & gr^0\mathcal{A}_{d,\varepsilon}, \\ X_j & \mapsto & \overline{c_jX_j} \\ Y_j & \mapsto & \overline{c_{d+j}Y_j} \end{array}$$

where the $2d(\varepsilon)$ variables of the classical Weyl algebra $A_{d(\varepsilon)}(k)$ are denoted by $X_1, \ldots, X_{d(\varepsilon)}, Y_1, \ldots, Y_{d(\varepsilon)}$ and the $2(d-d(\varepsilon))$ variables of the polynomial ring $\mathrm{Pol}_{2(d-d(\varepsilon))}(k)$ are denoted by $X_{d(\varepsilon)+1}, \ldots, X_d, Y_{d(\varepsilon)+1}, \ldots, Y_d$. This homomorphism is well defined since $\overline{c_j^{-1}Y_jc_jX_j} = \overline{c_jX_jc_j^{-1}Y_j} + 1$ for all $j = 1, \ldots, d(\varepsilon)$ and $\overline{c_{d+j}Y_jc_jX_j} = \overline{c_jX_jc_{d+j}Y_j}$ for all $j = d(\varepsilon)+1, \ldots, d$.

The homomorphism is bijective. Indeed, suppose $\sum \bar{a}_{\alpha\beta}X^\alpha Y^\beta \in A_{d(\varepsilon)}(k) \otimes_k \mathrm{Pol}_{2(d-d(\varepsilon))}(k)$ is send to zero. Then we have $\overline{\sum a_{\alpha\beta}c^{(\alpha,\beta)}X^\alpha Y^\beta} = 0 \iff |\sum a_{\alpha\beta}c^{(\alpha,\beta)}X^\alpha Y^\beta|_\varepsilon < 1 \iff \max|a_{\alpha\beta}| < 1 \iff \sum \bar{a}_{\alpha\beta}X^\alpha Y^\beta = 0$, where $c = (c_1, \ldots, c_{2d})$ and $c^{(\alpha,\beta)} = c_1^{\alpha_1} \cdots c_{2d}^{\beta_d}$. Hence the map is injective.

Let $\bar{f} = \overline{\sum a_{\alpha\beta}X^\alpha Y^\beta} \in gr^0\mathcal{A}_{d,\varepsilon}$ be any element. Then $\sum \overline{a_{\alpha\beta}c^{-(\alpha,\beta)}}X^\alpha Y^\beta$ is a preimage of $\bar{f}$. Hence the map is surjective.

Combining this with the isomorphism $gr\mathcal{A}_{d,\varepsilon} \simeq k[\bar{\pi}, \bar{\pi}^{-1}] \otimes_k gr^0\mathcal{A}_{d,\varepsilon}$ completes the proof. $\square$

We endow $\mathcal{B}_{d,\varepsilon}^{X_i}$ (resp. $\mathcal{B}_{d,\varepsilon}^{Y_i}$) (cf. section 3.2) defined over K with the complete filtration

$$F^n\mathcal{B}_{d,\varepsilon}^{X_i} := \{f \in \mathcal{B}_{d,\varepsilon}^{X_i} : |f|_\varepsilon \leq |\pi|^{-n}\} \quad \text{for } n \in \mathbb{Z}.$$

Lemma 4.2.2. *Let $\varepsilon \in |K^\times|^{2d}$ with $\varepsilon_j\varepsilon_{d+j} \geq 1$ for all $1 \leq j \leq d$. Then we have isomorphisms of graded k-algebras*

$$\begin{aligned} & gr\mathcal{B}_{d,\varepsilon}^{X_i} \simeq (k[X_i]\backslash\{0\})^{-1}(k[\bar{\pi}, \bar{\pi}^{-1}] \otimes_k A_{d(\varepsilon)}(k) \otimes_k \mathrm{Pol}_{2(d-d(\varepsilon))}(k)) \\ \text{and} \quad & gr\mathcal{B}_{d,\varepsilon}^{Y_i} \simeq (k[Y_i]\backslash\{0\})^{-1}(k[\bar{\pi}, \bar{\pi}^{-1}] \otimes_k A_{d(\varepsilon)}(k) \otimes_k \mathrm{Pol}_{2(d-d(\varepsilon))}(k)), \end{aligned}$$

where $\mathrm{Pol}_{2(d-d(\varepsilon))}(k)$ is the polynomial ring in $2(d-d(\varepsilon))$ variables over k.

Proof. We choose $c_j \in K$ such that $|c_j|^{-1} = \varepsilon_j$ for $1 \leq j \leq 2d$ and such that $c_{d+j} = c_j^{-1}$ if $\varepsilon_j \varepsilon_{d+j} = 1$. We have the following isomorphisms of k-algebras

$$\begin{array}{ccccc} k(X_i) & \longrightarrow & gr^0 \operatorname{Quot}(K\langle X_i\rangle_{\varepsilon_i}) & \longrightarrow & gr^0 \widehat{\operatorname{Quot}(K\langle X_i\rangle_{\varepsilon_i})} \\ \frac{\sum \bar{a}_\alpha X_i^\alpha}{\sum \bar{b}_\alpha X_i^\alpha} & \mapsto & \overline{\frac{\sum a_\alpha c_i^\alpha X_i^\alpha}{\sum b_\alpha c_i^\alpha X_i^\alpha}} & \mapsto & \overline{\frac{\sum a_\alpha c_i^\alpha X_i^\alpha}{\sum b_\alpha c_i^\alpha X_i^\alpha}} \end{array}$$

using the chosen $c_i \in K$. The second isomorphism is clear since the residue field of any non-archimedean valued filed is isomorphic to the residue field of its completion.

The first homomorphism is well defined by sending X_i to $\overline{c_i X_i}$ ($\overline{c_i X_i}$ being a transcendental element of the k-algebra $gr^0 \operatorname{Quot}(K\langle X_i\rangle_{\varepsilon_i})$). The homomorphism is injective since $\overline{\frac{\sum a_\alpha c_i^\alpha X_i^\alpha}{\sum b_\alpha c_i^\alpha X_i^\alpha}} = 0 \iff |\frac{\sum a_\alpha c_i^\alpha X_i^\alpha}{\sum b_\alpha c_i^\alpha X_i^\alpha}|_{\varepsilon_1} < 1 \iff \max |a_\alpha| < 1 \iff \frac{\sum \bar{a}_\alpha X_i^\alpha}{\sum \bar{b}_\alpha X_i^\alpha} = 0$. It is surjective since any element $\overline{\frac{\sum a_\alpha X_i^\alpha}{\sum b_\alpha X_i^\alpha}} \in gr^0 \operatorname{Quot}(K\langle X_i\rangle_{\varepsilon_i})$ has the preimage $\frac{\sum \overline{a_\alpha c_i^{-\alpha}} X_i^\alpha}{\sum \overline{b_\alpha c_i^{-\alpha}} X_i^\alpha}$.

We use the notation $\hat{\varepsilon}^i = (\varepsilon_1, \ldots, \varepsilon_{i-1}, \varepsilon_{i+1}, \ldots, \varepsilon_{d+i-1}, \varepsilon_{d+i+1}, \ldots, \varepsilon_{2d})$ as in section 3.2. Note that $d(\hat{\varepsilon}^i) = \#\{j : \varepsilon_j \varepsilon_{d+j} = 1, j \neq i\}$. By lemma 4.2.1 we have an isomorphism

$$\begin{array}{rcl} \varphi : A_{d(\hat{\varepsilon}^i)}(k(X_i)) \otimes_{k(X_i)} \operatorname{Pol}_{2(d-1-d(\hat{\varepsilon}^i))}(k(X_i)) & \rightarrow & gr^0 \mathcal{A}_{d-1,\hat{\varepsilon}^i}(\widehat{\operatorname{Quot}(K\langle X_i\rangle_{\varepsilon_i})}). \\ \sum\limits_{\alpha,\beta \in \mathbb{N}^{d-1}} \frac{\sum \bar{a}_{\alpha_i \alpha \beta} X_i^{\alpha_i}}{\sum \bar{b}_{\alpha_i \alpha\beta} X_i^{\alpha_i}} X^\alpha Y^\beta & \mapsto & \overline{\sum\limits_{\alpha,\beta} \frac{\sum a_{\alpha_i \alpha\beta} c_i^{\alpha_i} X_i^{\alpha_i}}{\sum b_{\alpha_i\alpha\beta} c_i^{\alpha_i} X_i^{\alpha_i}} \hat{c}^{i(\alpha,\beta)} X^\alpha Y^\beta} \end{array}$$

if we write $\alpha = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \ldots, \alpha_d)$ and $X = X_1 \cdots X_{i-1} X_{i+1} \cdots X_d$ and with the similar convention for β and Y. We use the notation $\hat{c}^{i(\alpha,\beta)}$ as in the proof of lemma 4.2.1 with $\hat{c}^i = (c_1, \ldots, c_{i-1}, c_{i+1}, \ldots, c_{d+i-1}, c_{d+i+1}, \ldots, c_{2d})$.

The k-algebra $gr^0 \mathcal{B}_{d,\varepsilon}^{X_i}$ contains the k-algebra $gr^0 \mathcal{A}_{d-1,\hat{\varepsilon}^i}(\widehat{\operatorname{Quot}(K\langle X_i\rangle_{\varepsilon_i})})$ since isometrically

$$\mathcal{A}_{d-1,\hat{\varepsilon}^i}(\widehat{\operatorname{Quot}(K\langle X_i\rangle_{\varepsilon_i})}) \subseteq \mathcal{A}_{d-1,\hat{\varepsilon}^i}(\widehat{\operatorname{Quot}(K\langle X_i\rangle_{\varepsilon_i})})\langle Y_i, \partial_{X_i}\rangle_{\varepsilon_{d+i}} = \mathcal{B}_{d,\varepsilon}^{X_i}.$$

The composition of φ with this inclusion is a homomorphism

$$A_{d(\hat{\varepsilon}^i)}(k(X_i)) \otimes_{k(X_i)} \operatorname{Pol}_{2(d-1-d(\hat{\varepsilon}^i))}(k(X_i)) \longrightarrow gr^0 \mathcal{B}_{d,\varepsilon}^{X_i}.$$

We have $c_{d+i}(Y_if) = c_{d+i}(fY_i + \partial_{X_i}(f))$ for all $f \in \mathcal{A}_{d-1,\hat{\varepsilon}^i}(\widehat{\mathrm{Quot}(K\langle X_i\rangle_{\varepsilon_i})})$. Hence if we assume $|f|_{\hat{\varepsilon}^i} \leq 1$ we have the following equality in $gr^0\mathcal{B}_{d,\varepsilon}^{X_i}$

$$\overline{c_{d+i}Y_i}\bar{f} = \bar{f}\,\overline{c_{d+i}Y_i} + \overline{c_{d+i}\partial_{X_i}(f)}. \tag{2}$$

It is easy to verify that

$$\varphi(\partial_{X_i}(\sum_{\alpha,\beta\in\mathbb{N}^{d-1}} \frac{\sum \bar{a}_{\alpha_i\alpha\beta}X_i^{\alpha_i}}{\sum \bar{b}_{\alpha_i\alpha\beta}X_i^{\alpha_i}} X^\alpha Y^\beta)) = \overline{c_i^{-1}\partial_{X_i}(\sum_{\alpha,\beta\in\mathbb{N}^{d-1}} \frac{\sum a_{\alpha_i\alpha\beta}c_i^{\alpha_i}X_i^{\alpha_i}}{\sum b_{\alpha_i\alpha\beta}c_i^{\alpha_i}X_i^{\alpha_i}} \hat{c}^{i(\alpha,\beta)}X^\alpha Y^\beta)}$$

for all $\sum_{\alpha,\beta\in\mathbb{N}^{d-1}} \frac{\sum \bar{a}_{\alpha_i\alpha\beta}X_i^{\alpha_i}}{\sum \bar{b}_{\alpha_i\alpha\beta}X_i^{\alpha_i}} X^\alpha Y^\beta \in A_{d(\hat{\varepsilon}^i)}(k(X_i)) \otimes_{k(X_i)} \mathrm{Pol}_{2(d-1-d(\hat{\varepsilon}^i))}(k(X_i))$. If we combine this in the case $\varepsilon_i\varepsilon_{d+i} = 1$ (recall that in this case $c_{d+i} = c_i^{-1}$) with equation (2) we get

$$\overline{c_{d+i}Y_i}\varphi(f) = \varphi(f)\overline{c_{d+i}Y_i} + \varphi(\partial_{X_i}(f))$$

for all $f \in A_{d(\hat{\varepsilon}^i)}(k(X_i)) \otimes_{k(X_i)} \mathrm{Pol}_{2(d-1-d(\hat{\varepsilon}^i))}(k(X_i))$. Hence by the universal property of skew polynomial rings we get a k-algebra homomorphism

$$\begin{array}{ccc} (A_{d(\hat{\varepsilon}^i)}(k(X_i)) \otimes_{k(X_i)} \mathrm{Pol}_{2(d-1-d(\hat{\varepsilon}^i))}(k(X_i)))[Y_i, \partial_{X_i}] & \longrightarrow & gr^0\mathcal{B}_{d,\varepsilon}^{X_i}. \\ \sum f_\beta Y_i^\beta & \mapsto & \sum \varphi(f_\beta)\overline{c_{d+i}Y_i}^\beta \end{array}$$

One can show that

$$|\partial_{X_i}(f)|_{\hat{\varepsilon}^i}\varepsilon_{d+i}^{-1} \leq |f|_{\hat{\varepsilon}^i}(\varepsilon_i\varepsilon_{d+i})^{-1}$$

for all $f \in \mathcal{A}_{d-1,\hat{\varepsilon}^i}(\widehat{\mathrm{Quot}(K\langle X_i\rangle_{\varepsilon_i})})$. If we combine this in the case $\varepsilon_i\varepsilon_{d+i} > 1$ with equation (2) we get

$$\overline{c_{d+i}Y_i}\varphi(f) = \varphi(f)\overline{c_{d+i}Y_i}$$

for all $f \in A_{d(\hat{\varepsilon}^i)}(k(X_i)) \otimes_{k(X_i)} \mathrm{Pol}_{2(d-1-d(\hat{\varepsilon}^i))}(k(X_i))$. Hence as above we get a well defined homomorphism of k-algebras

$$\begin{array}{ccc} (A_{d(\hat{\varepsilon}^i)}(k(X_i)) \otimes_{k(X_i)} \mathrm{Pol}_{2(d-1-d(\hat{\varepsilon}^i))}(k(X_i)))[Y_i] & \longrightarrow & gr^0\mathcal{B}_{d,\varepsilon}^{X_i}. \\ \sum f_\beta Y_i^\beta & \mapsto & \sum \varphi(f_\beta)\overline{c_{d+i}Y_i}^\beta \end{array}$$

We show now simultaneously that the two homomorphisms defined above are isomorphism. We suppose that $\sum(\sum \frac{\sum \bar{a}_{\alpha_i\beta_i\alpha\beta}X_i^{\alpha_i}}{\sum \bar{b}_{\alpha_i\beta_i\alpha\beta}X_i^{\alpha_i}} X^\alpha Y^\beta)Y_i^{\beta_i}$ is an element of

$(A_{d(\hat{\varepsilon}^i)}(k(X_i)) \otimes_{k(X_i)} \mathrm{Pol}_{2(d-1-d(\hat{\varepsilon}^i))}(k(X_i)))[Y_i, \partial_{X_i}]$ resp. $(A_{d(\hat{\varepsilon}^i)}(k(X_i)) \otimes_{k(X_i)} \mathrm{Pol}_{2(d-1-d(\hat{\varepsilon}^i))}(k(X_i)))[Y_i]$ which is zero under the corresponding homomorphism. Then

$$\begin{aligned} & \overline{\sum(\sum \frac{\sum a_{\alpha_i\beta_i\alpha\beta} c_i^{\alpha_i} X_i^{\alpha_i}}{\sum b_{\alpha_i\beta_i\alpha\beta} c_i^{\alpha_i} X_i^{\alpha_i}} \hat{c}^{i(\alpha,\beta)} X^\alpha Y^\beta) c_{d+i}^{\beta_i} Y_i^{\beta_i}} = 0 \\ \iff \quad & |\sum(\sum \frac{\sum a_{\alpha_i\beta_i\alpha\beta} c_i^{\alpha_i} X_i^{\alpha_i}}{\sum b_{\alpha_i\beta_i\alpha\beta} c_i^{\alpha_i} X_i^{\alpha_i}} \hat{c}^{i(\alpha,\beta)} X^\alpha Y^\beta) c_{d+i}^{\beta_i} Y_i^{\beta_i}|_\varepsilon < 1 \\ \iff \quad & \max_{\alpha_i\beta_i\alpha\beta} |a_{\alpha_i\beta_i\alpha\beta}| < 1. \end{aligned}$$

Hence the map is injective.

Let $\bar{f} = \overline{\sum f_\beta Y_i^\beta} \in gr^0 \mathcal{B}_{d,\varepsilon}^{X_i}$ be any element. Then $\sum \varphi^{-1}(\overline{f_\beta c_{d+i}^{-\beta}}) Y_i^\beta$ is a preimage of $\bar{f}$. Hence the map is surjective. □

Finally we endow $\mathcal{B}_{d,\varepsilon} = \bigoplus_{i=1}^d \mathcal{B}_{d,\varepsilon}^{X_i} \oplus \bigoplus_{i=1}^d \mathcal{B}_{d,\varepsilon}^{Y_i}$ (cf. section 3.2) with the complete filtration

$$F^n \mathcal{B}_{d,\varepsilon} := \{f \in \mathcal{B}_{d,\varepsilon} : |f|_\varepsilon \leq |\pi|^{-n}\} \quad \text{for } n \in \mathbb{Z}.$$

Corollary 4.2.3. *Let $\varepsilon \in |K^\times|^{2d}$ with $\varepsilon_j \varepsilon_{d+j} \geq 1$ for all $1 \leq j \leq d$. Let A_{gr} denote the k-algebra $k[\bar{\pi}, \bar{\pi}^{-1}] \otimes_k A_{d(\varepsilon)}(k) \otimes_k \mathrm{Pol}_{2(d-d(\varepsilon))}(k)$. Then we have an isomorphism of graded k-algebras*

$$gr\mathcal{B}_{d,\varepsilon} \simeq \bigoplus_{i=1}^d (K[X_i] \backslash \{0\})^{-1} A_{gr} \oplus \bigoplus_{i=1}^d (K[Y_i] \backslash \{0\})^{-1} A_{gr}.$$

Proof. The functor gr commutes with filtered direct sums, hence the corollary follows from lemma 4.2.2. □

4.3 Permanence properties

Proposition 4.3.1. *Let $\varepsilon \in |K^\times|^{2d}$ with $\varepsilon_j \varepsilon_{d+j} \geq 1$ for all $1 \leq j \leq d$. Then $\mathcal{A}_{d,\varepsilon}$, $\mathcal{B}_{d,\varepsilon}^{X_i}$, $\mathcal{B}_{d,\varepsilon}^{Y_i}$, and $\mathcal{B}_{d,\varepsilon}$ are Noetherian rings.*

Proof. With lemma 4.2.1 we have

$$gr\mathcal{A}_{d,\varepsilon} \simeq k[\bar{\pi},\bar{\pi}^{-1}] \otimes_k A_{d(\varepsilon)}(k) \otimes_k \mathrm{Pol}_{2(d-d(\varepsilon))}(k) \simeq \mathrm{Pol}_{2(d-d(\varepsilon))}(A_{d(\varepsilon)}(k)[\bar{\pi},\bar{\pi}^{-1}]).$$

$A_{d(\varepsilon)}(k)$ is Noetherian (cf. [McCR] 1.3.8), hence $A_{d(\varepsilon)}(k)[\bar{\pi},\bar{\pi}^{-1}]$ is Noetherian (cf. [McCR] 1.4.5), hence finally $\mathrm{Pol}_{2(d-d(\varepsilon))}(A_{d(\varepsilon)}(k)[\bar{\pi},\bar{\pi}^{-1}])$ is Noetherian (cf. [McCR] 1.2.9). As the localization resp. the sum of localizations of a Noetherian ring the rings $gr\mathcal{B}^{X_i}_{d,\varepsilon}$ and $gr\mathcal{B}_{d,\varepsilon}$ are Noetherian. Applying [LvO] proposition I.7.1.2 completes the proof. □

Remark 4.3.2. *We already know that $\mathcal{A}_{d,\varepsilon}$ defined over an arbitrary non-archimedean field K is Noetherian (cf. proposition 1.4.1). We also know that $\mathcal{B}^{X_i}_{d,\varepsilon}$ (resp. $\mathcal{B}^{Y_i}_{d,\varepsilon}$), and $\mathcal{B}_{d,\varepsilon}$ defined over any non-archimedean field K are Noetherian if $d = 1$ (cf. proposition 2.2.8).*

Now we show that the filtered rings considered above are Auslander regular rings. See [LvO] chapter III for an introduction to Auslander regular rings.

Proposition 4.3.3. *Let $\varepsilon \in |K^\times|^{2d}$ with $\varepsilon_j\varepsilon_{d+j} \geq 1$ for all $1 \leq j \leq d$. Then $\mathcal{A}_{d,\varepsilon}$, $\mathcal{B}^{X_i}_{d,\varepsilon}$, $\mathcal{B}^{Y_i}_{d,\varepsilon}$, and $\mathcal{B}_{d,\varepsilon}$ are Auslander regular rings.*

Proof. Since $\mathcal{A}_{d,\varepsilon}$, $\mathcal{B}^{X_i}_{d,\varepsilon}$, $\mathcal{B}^{Y_i}_{d,\varepsilon}$, and $\mathcal{B}_{d,\varepsilon}$ are complete filtered rings with Noetherian associated graded rings they are Zariski rings by [LvO] II.2.2.1. Hence by [LvO] proposition III.2.2.5 it suffices to show that the associated graded rings are Auslander regular.

The classical Weyl algebra over a field is Auslander regular (cf. [LvO] example III.2.4.4.(b)). Combining the proof of this result with corollary [LvO] III.2.3.6 implies that this is also true for the classical Weyl algebra defined over any Auslander regular ring.

By lemma 4.2.1 $gr\mathcal{A}_{d,\varepsilon}$ is isomorphic to the classical Weyl algebra over

$$k[\bar{\pi},\bar{\pi}^{-1},X_{d(\varepsilon)+1},\ldots,X_{2d},Y_{d(\varepsilon)+1},\ldots,Y_{2d}].$$

This ring is Noetherian and of finite global dimension (cf. [McCR] theorem 5.3), hence by [LvO] example III.2.4.3 Auslander regular.

To show that $gr\mathcal{B}_{d,\varepsilon}^{X_i}$ is Auslander regular is suffices by lemma 4.2.2 combined with the above results to check that $k(X)[Y, \partial_X]$ is Auslander regular. However, endowed with the degree filtration this ring has a commutative polynomial ring over a field as associated graded ring, hence $k(X)[Y, \partial_X]$ is Auslander regular.

Finally, $gr\mathcal{B}_{d,\varepsilon}$ is Auslander regular, since by lemma 4.2.3 it is the sum of Auslander regular rings. $\square$

Proposition 4.3.4. *Let $\varepsilon \in |K^\times|^{2d}$ with $\varepsilon_j\varepsilon_{d+j} \geq 1$ for all $1 \leq j \leq d$. Then the ring extension*

$$\mathcal{A}_{d,\varepsilon} \subseteq \mathcal{B}_{d,\varepsilon}$$

is flat. If $\operatorname{char} K = 0$*, then the extension is faithfully flat.*

Proof. By corollary 4.2.3 $gr\mathcal{B}_{d,\varepsilon}$ is the sum of localizations of $gr\mathcal{A}_{d,\varepsilon}$. Localizations are flat extensions, hence the extension $gr\mathcal{A}_{d,\varepsilon} \subseteq gr\mathcal{B}_{d,\varepsilon}$ is flat. This together with the fact that $gr\mathcal{B}_{d,\varepsilon}$ and $gr\mathcal{A}_{d,\varepsilon}$ are Noetherian (cf. proof of proposition 4.3.1) enables us to apply proposition 1.2 of [ST] which proves the flatness of the extension $\mathcal{A}_{d,\varepsilon} \subseteq \mathcal{B}_{d,\varepsilon}$.

If $\operatorname{char} K = 0$ we combine this with lemma 3.2.1 and deduce that the extension is faithfully flat (cf. [McCR] proposition 7.2.3). $\square$

Lemma 4.3.5. *Let $\varepsilon \in |K^\times|^{2d}$ with $\varepsilon_j\varepsilon_{d+j} \geq 1$ for all $1 \leq j \leq d$. Then the Krull dimension and the global dimension of the graded ring $gr\mathcal{A}_{d,\varepsilon}$ are $2d - d(\varepsilon) + 1$ if* $\operatorname{char} k = 0$ *and they are $2d + 1$ if* $\operatorname{char} k = p > 0$.

Proof. We have an isomorphism $gr\mathcal{A}_{d,\varepsilon} \simeq \operatorname{Pol}_{2(d-d(\varepsilon))}(A_{d(\varepsilon)}(k)[\overline{\pi}, \overline{\pi}^{-1}])$. The Krull dimension and the global dimension of $A_{d(\varepsilon)}(k)$ are $d(\varepsilon)$ if $\operatorname{char} k = 0$ and $2d(\varepsilon)$ if $\operatorname{char} k = p > 0$ (cf. [McCR] theorem 6.6.15 and proposition 6.6.14 resp. theorem 7.5.8 (iii) and (ii)). Forming the ring of Laurent polynomials over the Weyl algebra $A_{d(\varepsilon)}(k)$ increases the Krull dimension and the global dimension by one (cf. [McCR] proposition 6.5.4.(ii) resp. theorem 7.5.3.(iv)). Finally the polynomial ring in $2(d-d(\varepsilon))$ variables increases the Krull and the global dimension by $2(d-d(\varepsilon))$ (cf. [McCR] proposition 6.5.4.(i) resp. theorem 7.5.3.(iii)) which proves the lemma. $\square$

We can apply results of filtered ring theory which say that the Krull dimension resp. the global dimension of the graded ring serve as an upper bound for the Krull dimension resp. global dimension of the ground ring. Hence the lemma implies that the Krull dimension and the global dimension of $\mathcal{A}_{d,\varepsilon}$ are bounded above by $2d - d(\varepsilon) + 1$ if $\operatorname{char} k = 0$ and by $2d + 1$ if $\operatorname{char} k = p > 0$. However, we have a slightly stronger result.

Proposition 4.3.6. *Let $\varepsilon \in |K^\times|^{2d}$ with $\varepsilon_j \varepsilon_{d+j} \geq 1$ for all $1 \leq j \leq d$. Then the Krull dimension and the global dimension of $\mathcal{A}_{d,\varepsilon}$ are bounded above by $2d - d(\varepsilon)$ if $\operatorname{char} k = 0$ and they are bounded above by $2d$ if $\operatorname{char} k = p > 0$.*

Proof. Note first hat $grF^0\mathcal{A}_{d,\varepsilon} \simeq k[\bar{\pi}] \otimes_k gr^0\mathcal{A}_{d,\varepsilon}$. It follows from the proof of lemma 4.2.1 that

$$grF^0\mathcal{A}_{d,\varepsilon} \simeq k[\bar{\pi}] \otimes_k A_{d(\varepsilon)}(k) \otimes_k \mathrm{Pol}_{2(d-d(\varepsilon))}(k) \simeq \mathrm{Pol}_{2(d-d(\varepsilon))}(A_{d(\varepsilon)}(k)[\bar{\pi}]).$$

This ring has Krull and global dimension equal to $2d - d(\varepsilon) + 1$ if $\operatorname{char} k = 0$ and $2d+1$ if $\operatorname{char} k = p > 0$, which we show as in lemma 4.3.5. Applying [LvO] proposition I.7.1.2 and corollary I.7.2.2 we obtain that the Krull and the global dimension of $F^0\mathcal{A}_{d,\varepsilon}$ are bounded above by $2d - d(\varepsilon) + 1$ resp. $2d+1$ depending on the characteristic of k. Applying lemma 4.1.1 completes the proof. $\square$

Proposition 4.3.7. *Let $\varepsilon \in |K^\times|^{2d}$ with $\varepsilon_j \varepsilon_{d+j} \geq 1$ for all $1 \leq j \leq d$. Then the Krull and the global dimension of $\mathcal{B}^{X_i}_{d,\varepsilon}$, $\mathcal{B}^{Y_i}_{d,\varepsilon}$ and $\mathcal{B}_{d,\varepsilon}$ are bounded above by $2d - d(\varepsilon)$ if $\operatorname{char} k = 0$. If $\varepsilon_i \varepsilon_{d+i} > 1$ we have the stronger upper bound $2d - d(\varepsilon) - 1$ for $\mathcal{B}^{X_i}_{d,\varepsilon}$ and $\mathcal{B}^{Y_i}_{d,\varepsilon}$, and if $\varepsilon_j \varepsilon_{d+j} > 1$ for all $1 \leq j \leq d$ this is also an upper bound for $\mathcal{B}_{d,\varepsilon}$. If $\operatorname{char} k = p > 0$, then the Krull and the global dimension of $\mathcal{B}^{X_i}_{d,\varepsilon}$, $\mathcal{B}^{Y_i}_{d,\varepsilon}$ and $\mathcal{B}_{d,\varepsilon}$ are bounded above by $2d - 1$.*

Proof. As in the proof of proposition 4.3.6, the Krull dimension resp. the global dimension of $\mathcal{B}^{X_i}_{d,\varepsilon}$ is bounded above by the Krull dimension resp. the global dimension of $gr^0\mathcal{B}^{X_i}_{d,\varepsilon}$. In the proof of lemma 4.2.2 we showed that

$$gr^0\mathcal{B}^{X_i}_{d,\varepsilon} \simeq (A_{d(\hat{\varepsilon}^i)}(k(X_i)) \otimes_{k(X_i)} \mathrm{Pol}_{2(d-1-d(\hat{\varepsilon}^i))}(k(X_i)))[Y_i, \partial_{X_i}]$$

if $\varepsilon_i \varepsilon_{d+i} = 1$ and that

$$gr^0\mathcal{B}^{X_i}_{d,\varepsilon} \simeq (A_{d(\hat{\varepsilon}^i)}(k(X_i)) \otimes_{k(X_i)} \mathrm{Pol}_{2(d-1-d(\hat{\varepsilon}^i))}(k(X_i)))[Y_i]$$

if $\varepsilon_i\varepsilon_{d+i} > 1$. From the proof of lemma 4.3.5 we know that the Krull dimension and the global dimension of $A_{d(\hat{\varepsilon}^i)}(k(X_i)) \otimes_{k(X_i)} \mathrm{Pol}_{2(d-1-d(\hat{\varepsilon}^i))}(k(X_i))$ are equal to

$$2(d-1) - d(\hat{\varepsilon}^i)$$

if $\operatorname{char} k = 0$. Hence if $\varepsilon_i\varepsilon_{d+i} = 1$, i.e. if $d(\hat{\varepsilon}^i) = d(\varepsilon) - 1$ we know that the Krull dimension and the global dimension of $gr^0\mathcal{B}_{d,\varepsilon}$ are bounded above by

$$2(d-1) - d(\hat{\varepsilon}^i) + 1 = 2d - d(\varepsilon)$$

(cf. [McCR] proposition 6.5.4.(i) and theorem 7.5.3.(i)). If $\varepsilon_j\varepsilon_{d+j} > 1$, i.e. if $d(\hat{\varepsilon}^i) = d(\varepsilon)$ we know that the Krull dimension and the global dimension of $gr^0\mathcal{B}_{d,\varepsilon}$ are equal to

$$2(d-1) - d(\hat{\varepsilon}^i) + 1 = 2d - d(\varepsilon) - 1$$

(cf. [McCR] proposition 6.5.4.(i) and theorem 7.5.3.(iii)). Using the fact that

$$\mathcal{K}(\mathcal{B}_{d,\varepsilon}) = \sup_i\{\mathcal{K}(\mathcal{B}_{d,\varepsilon}^{X_i}), \mathcal{K}(\mathcal{B}_{d,\varepsilon}^{Y_i})\}$$

(cf. proof of proposition 3.2.2) completes the proof of the first part of the theorem. If $\operatorname{char} k = p > 0$ the Krull dimension and the global dimension of $A_{d(\hat{\varepsilon}^i)}(k(X_i)) \otimes_{k(X_i)} \mathrm{Pol}_{2(d-1-d(\hat{\varepsilon}^i))}(k(X_i))$ are $2(d-1)$ (cf. proof of lemma 4.3.5). Hence the Krull dimension and the global dimension of $gr^0\mathcal{B}_{d,\varepsilon}^{X_i}$ are at most $2(d-1) + 1 = 2d - 1$. $\square$

As a consequence of this proposition we get an improvement of results of proposition 4.3.6.

Corollary 4.3.8. *Let $\varepsilon \in |K^\times|^{2d}$ with $\varepsilon_j\varepsilon_{d+j} \geq 1$ for all $1 \leq j \leq d$ and assume that the characteristic of K is zero. Then the Krull dimension and the global dimension of $\mathcal{A}_{d,\varepsilon}$ are bounded above by $2d-1$, independent of the characteristic of the residue field k.*

Proof. We know that if $\operatorname{char} K = 0$ the map sending left ideals $I \subseteq \mathcal{A}_{d,\varepsilon}$ to left ideals $\mathcal{B}_{d,\varepsilon}I \subseteq \mathcal{B}_{d,\varepsilon}$ preserves proper containments (cf. proof of proposition 3.2.2). Hence $\mathcal{K}(\mathcal{A}_{d,\varepsilon}) \leq \mathcal{K}(\mathcal{B}_{d,\varepsilon}) \leq 2d-1$ where the first inequality follows with [McCR] lemma 6.5.3.(i) and the second inequality follows from proposition 4.3.7. The extension $\mathcal{A}_{d,\varepsilon} \subseteq \mathcal{B}_{d,\varepsilon}$ is faithfully flat (cf. proposition 4.3.4) and $\mathcal{A}_{d,\varepsilon}$ is Noetherian (cf. proposition 1.4.1) with finite global dimension (cf. proposition 4.3.6), hence applying [McCR] theorem 7.2.6 implies $\mathrm{gld}(\mathcal{A}_{d,\varepsilon}) \leq \mathrm{gld}(\mathcal{B}_{d,\varepsilon}) \leq 2d-1$. □

Corollary 4.3.9. *Let $\varepsilon \in |K^\times|^{2d}$ with $\varepsilon_j \varepsilon_{d+j} \geq 1$ for all $1 \leq j \leq d$ and assume that the characteristic of K is zero. Then every left ideal of $\mathcal{A}_{d,\varepsilon}$ has a set of $2d$ generators.*

Proof. $\mathcal{A}_{d,\varepsilon}$ is a left Noetherian simple ring (cf. proposition 1.4.1 and proposition 1.4.6). The Krull dimension is bounded above by $2d-1$ by corollary 4.3.8, hence the assertion follows with [McCR] corollary 6.7.8.(ii). □

Remark 4.3.10. *We believe that the Krull dimension and the global dimension of $\mathcal{A}_{d,\varepsilon}$ are equal to d if* $\mathrm{char}\, K = 0$. *We proved this for $d = 1$ in section 3 (cf. propositions 3.2.2 and 3.2.3). For $d = 1$ this is obviously also a consequence of corollary 4.3.8 combined with propositions 3.1.2 and 3.1.3 which establish the lower bounds. The conjectured statement follows from proposition 4.3.6 for arbitrary d in the special case where* $\mathrm{char}\, k = 0$ *and $d = d(\varepsilon)$.*

5 A note on simple modules

The simple left modules over the classical Weyl algebra A_1 in the two variables X and Y were classified by R. Block in [Blo]. The simple $K[X]$-torsion A_1-modules and the simple $K[X]$-torsionfree A_1-modules are considered separately. The simple $K[X]$-torsion A_1-modules are given by the maximal ideals of $K[X]$ (cf. [Blo] proposition 4.1) and the simple $K[X]$-torsionfree A_1-modules are in one-to-one correspondence with the simple modules over the localization of A_1 with respect to the Ore set $K[X]\backslash\{0\}$ (cf. [Blo] lemma 2.2.1 and corollary 2.2). Since the localization $(K[X]\backslash\{0\})^{-1}A_1$ is a principal left ideal domain the latter are given by similarity classes of irreducible elements (cf. introduction of [Blo]).

Recall that if we consider the completed Weyl algebra $\mathcal{A}_\varepsilon$ with $\varepsilon = (\varepsilon_1, \varepsilon_2) \in |K^\times|^2$ the subset $K\langle X\rangle_{\varepsilon_1}\backslash\{0\}$ is not an Ore set (cf. lemma 2.0.1). Hence it is not possible to imitate the above described strategy in the case of the completed Weyl algebra with respect to the multiplicative set $K\langle X\rangle_{\varepsilon_1}\backslash\{0\}$. However, if we replace $K\langle X\rangle_{\varepsilon_1}\backslash\{0\}$ by its saturation we obtain some initial results.

We briefly recall the notations as they are used in [LvO] chapter IV. Let R be a separated filtered ring and let S be a multiplicatively closed subset containing 1 but not 0. For $x \in F^nR\backslash F^{n-1}R$ we denote by $\sigma(x)$ the image of x in gr^nR. We put $\sigma(S) = \{\sigma(s) : s \in S\}$ and define the **saturation** of S in R to be the set

$$S_{\text{sat}} := \{r \in R : \sigma(r) \in \sigma(S)\}.$$

If we assume $\sigma(S)$ to be a multiplicatively closed subset of grR not containing 0 then S_{sat} is a multiplicatively closed subset of R.

As in section 4 we assume that K is a discretely valued non-archimedean field with residue field k. Let $\mathcal{A}_\varepsilon$ be the completed Weyl algebra in the two variables X and Y and let S be the multiplicatively closed subset $K\langle X\rangle_{\varepsilon_1}\backslash\{0\}$. We consider $\mathcal{A}_\varepsilon$ as a filtered ring with the filtration defined in section 4. The subset $\sigma(S)$ of $gr\mathcal{A}_\varepsilon$ is the multiplicatively closed set $(k[\bar{\pi}, \bar{\pi}^{-1}] \otimes_k k[X])\backslash\{0\}$. Recall that depending on ε the graded ring $gr\mathcal{A}_\varepsilon$ is either $k[\bar{\pi}, \bar{\pi}^{-1}] \otimes_k A_1(k)$ or $k[\bar{\pi}, \bar{\pi}^{-1}] \otimes_k k[X, Y]$ (cf. lemma 4.2.1). It is clear that $\sigma(S)$ is an Ore set in $gr\mathcal{A}_\varepsilon$, since $k[X]\backslash\{0\}$ is an Ore set in both $A_1(k)$ and $k[X, Y]$. It is also a consequence of lemma 4.2.1 that the saturation of S is the set

$$S_{\text{sat}} = \{f \in \mathcal{A}_\varepsilon\backslash\{0\} : \varepsilon\text{-inform}(f) \in K[X]\},$$

where ε-inform(f) denotes the ε-initial form as defined on page 21.

Proposition 5.1. *The set S_{sat} is an Ore set in $\mathcal{A}_\varepsilon$.*

Proof. Note that we use the conventions of [McCR] where an Ore set is a multiplicatively closed set satisfying the Ore condition (cf. [McCR] 2.1.6) this is called "second Ore condition" in [LvO]. By the proof of proposition 4.3.3 we know that $\mathcal{A}_\varepsilon$ a Zariski ring. Thus by definition its Rees ring is Noetherian. Hence the assertion follows with [LvO] proposition IV.1.19, since $\sigma(S_{\text{sat}}) = \sigma(S)$ is an Ore set in $gr\mathcal{A}_\varepsilon$, as was shown above. $\square$

Let A be a unital ring which has a localization $B = S^{-1}A$ and assume that B is not a field. Section 2.2 in [Blo] establishes a general relation between simple S-torsionfree A-modules and simple B-modules.

Proposition 5.2. *The map*

$$S^{-1} : \left\{ \begin{array}{c} \textit{isom. classes of} \\ \textit{simple } S\textit{-torsionfree} \\ \textit{left } A\textit{-modules} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{isom. classes} \\ \textit{of simple} \\ \textit{left } B\textit{-modules} \end{array} \right\}$$

defined by $M \mapsto B \otimes_A M =: S^{-1}M$ is injective. If we assume in addition that A has Krull dimension one, then the map S^{-1} is a bijection.

Proof. For the first assertion see [Blo] lemma 2.2.1. To prove the second assertion suppose N is a simple B-module. Choose $0 \neq n \in N$ then $\text{ann}_B(n) \neq 0$ (left annihilator) and hence the left ideal $I := \text{ann}_A(n)$ is non-zero. We have an isomorphism $An \simeq A/I$ of A-modules. Now choose $0 \neq a \in I$. We have the following inequalities of Krull dimensions $\mathcal{K}(A/I) \leq \mathcal{K}(A/Aa)) < \mathcal{K}(A) = 1$. The first inequality follows from [McCR] Lemma 6.2.4 and the second from [McCR] Lemma 6.3.9. Hence A/I is Artinian and the A-module N has a simple submodule. $\square$

This applies to our situation.

Corollary 5.3. *Let $\varepsilon = (\varepsilon_1, \varepsilon_2) \in |K^\times|^2$ and assume that $\mathcal{A}_\varepsilon$ is defined over a discretely valued non-archimedean field K with $\operatorname{char} K = 0$. Further, let $S_{\text{sat}} = \{f \in \mathcal{A}_\varepsilon : \varepsilon\text{-inform}(f) \in K[X]\}$ be as above. Then we have a bijection*

$$S_{\text{sat}}^{-1} : \left\{ \begin{array}{c} \textit{isom. classes of} \\ \textit{simple } S_{\text{sat}}\textit{-torsionfree} \\ \textit{left } \mathcal{A}_\varepsilon\textit{-modules} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \textit{isom. classes of} \\ \textit{simple left} \\ S_{\text{sat}}^{-1}\mathcal{A}_\varepsilon\textit{-modules} \end{array} \right\}.$$

Proof. By proposition 5.1 S_{sat} is an Ore set and by corollary 4.3.8 the Krull dimension of $\mathcal{A}_\varepsilon$ is 1. □

Remark 5.4. *Note that $S_{\mathrm{sat}}^{-1}\mathcal{A}_\varepsilon$ is a subring of the ring $\mathcal{B}_\varepsilon^X$ defined on page 51. Indeed, we have the inclusion $\mathcal{A}_\varepsilon \subset \mathcal{B}_\varepsilon^X$ and any element of S_{sat} is invertible in $\mathcal{B}_\varepsilon^X$ (cf. proposition 2.2.5), hence the assertion follows from the universal property of localizations.*

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