On Transfer Principles in Henselian Valued Fields

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Abstract

In this thesis, we study transfer principles in the context of certain Henselian valued fields, namely:

- Henselian valued fields of equicharacteristic 0,
- algebraically closed valued fields,
- algebraically maximal Kaplansky valued fields,
- and unramified mixed characteristic Henselian valued fields of perfect residue field.

First, we compute the burden of such a valued field in terms of the burden of its value group and its residue field. The burden is a cardinal related to the model theoretic complexity and a notion of dimension associated to NTP_2 theories. We showed for instance that the Hahn field $\mathbb{F}_p^{\mathrm{alg}}((\mathbb{Z}[1/p]))$ is inp-minimal (of burden 1), and that the ring of Witt vectors $W(\mathbb{F}_p^{\mathrm{alg}})$ over $\mathbb{F}_p^{\mathrm{alg}}$ is not strong (of burden ω). This result extends previous work due to Chernikov and Simon and realises an important step toward the classification of Henselian valued fields of finite burden.

Secondly, we show a transfer principle for the property that all types realised in a given elementary extension are definable. It can be written as follows: a valued field as above is stably embedded in an elementary extension if and only if its value group is stably embedded in the corresponding extension of value groups, its residue field is stably embedded in the corresponding extension of residue fields, and the extension of valued fields satisfies a certain algebraic condition. We show for instance that all types over the Hahn field $\mathbb{R}((\mathbb{Z}))$ are definable. Similarly, all types over the quotient field of $W(\mathbb{F}_p^{\mathrm{alg}})$ are definable. This extends a work of Cubides and Delon and of Cubides and Ye.

These distinct results use a common approach, which has been developed recently. It consists of establishing first a reduction to an intermediate structure called the leading term structure, or RV-sort, and then of reducing to the value group and residue field. This leads us to develop similar reduction principles in the context of pure short exact sequences of abelian groups.

À mes parents

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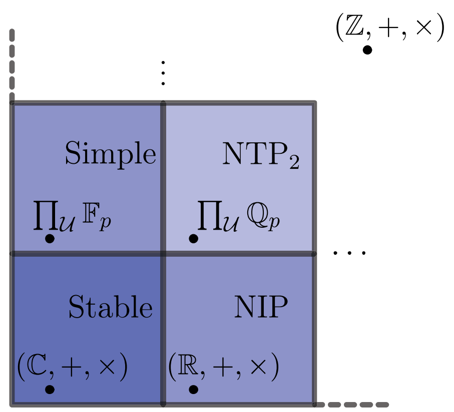
Contents

Introduction

This thesis treats some model theoretical aspects of valued fields. Model theory is a branch of mathematical logic with a focus on the study of algebraic structures and their definable sets. Valued fields are such algebraic structures, that are rich and interesting in many ways. The reader will find below an overview on this topic and a summary of all the main results of this text.

Classification theory.
Initiated by the work of Shelah in [66], an active area of model theory is the study and the classification of first order theories according to their complexity. The complexity of a theory is measured in terms of combinatorial configurations that it may encode. This is of course a natural idea, that one might find some origin in the famous theorems of Gödel, where the coding abilities of prime numbers are used - revealing at the same time the complexity of the theory of arithmetic. In general, the less a theory is able to encode complex configurations, the more it is considered as tame. A complex hierarchy of first order theories arises, expressing their relative tameness or complexity.

Modern classification of theories started with the study of the class of stable theories; a very tame and undoubtedly an important class of first or-



A classification of first order theories.

der theories. A theory is stable if no formula encodes "$i < j$" for $i, j \in \mathbb{N}$ (see e.g. [72] for a precise definition). This is a class to which belong the theory of infinite sets, algebraically closed fields and abelian groups. Many methods and tools have been specifically developed in order to analyse types and models, and these have structurally improved our abstract understanding of this class. Let us cite for instance Morley's Categoricity Theorem: if a first-order theory in a countable language is categorical in some uncountable cardinality, then it is categorical in all uncountable cardinalities. This framework has also established deep connections between the geometry of forking independence and properties of algebraic structures (groups and fields) definable in the theory.

However, many natural theories do not fit in this restrictive setting such as any ordered structure, or any 'random structure'. After some development, logicians have extended these tools to larger contexts, and have generalised results to some unstable theories. Already in [66], two generalisations capturing slightly wilder theories have been defined and studied, namely the class of simple theories (no formula encodes the tree property) – to which belong the theory of the random graph and the theory of pseudo finite fields – and the class of NIP theories (no formula encodes "$i \in J$", $i \in \mathbb{N}, J \subset \mathbb{N}$) – to which belong the theory of real closed fields, of dense linear order, of the field of p-adics and of algebraically closed valued fields. These two generalisations are 'orthogonal' in the sense that a theory which is both simple and NIP is in fact stable. The interest in these classes grew significantly after some tools and methods developed for stable theories found interesting extensions to more general contexts (e.g. see [69] and [50]). Ever since, model theory has pushed our knowledge to less and lesser tame theories by generalising these tools to wilder classes. In this conquest of the 'untame land' of first order theories, one of the latest developments concerns the class of NTP_2 theories (no formula encodes the tree property of the second kind). See [16] and [14]. It appeared to be a particularly interesting generalisation of the class of NIP and Simple theories. For instance, non-principal ultra-products of p-adics, the densely ordered random graph, and valued fields with a generic automorphism are NTP_2, but neither simple nor NIP. The main results of this thesis concerns specific NTP_2 theories of valued fields. Before going to details, let us mention instances of other developments in the framework of classification theory: on one hand, there is the classes of NSOP_n theories, $n \in \mathbb{N}^\star$, which form a proper hierarchy of generalisations of simple theories, and which is already meaningful in the study of fields. On the other hand, the classes of NIP_n theories (where no formula encodes "$i \in J$", $i \in \mathbb{N}^n, J \subset \mathbb{N}^n$) seem to be interesting generalisations of the class of NIP theories, highlighting the importance of combinatorial configurations of higher dimension.

The abstract study of these classes leads to a better understanding of algebraic structures, as some algebraic phenomena may or not occur, depending of the complexity of the theory (e.g. [48] and [17]). Of course, the more we study these classes, the more relevant it is to classify theories. It could be in general a difficult exercise, but many methods are known: one can show stability (resp. NIP) by counting types over small sets (resp. coheirs over models). Simple theories are exactly theories with some abstract independence relation ([50]). The situation is different when we look at a quantitative version of these notions of complexity.

Quantitative vs Qualitative notions of complexity.
Let us explain what we mean by 'quantitative', or at least, let us give a quick intuition. As we said, a theory is not stable, NIP, nor Simple if instances of a single formula realise a certain combinatorial configuration. For example, the formula $x|y$ in the theory of arithmetic, or the formula $x \in y$ in the theory of ZFC have the strong property to encode the tree property of the second kind, demonstrating that ZFC and the theory of $(\mathbb{N}, +, \times)$ have TP_2. Even if a theory cannot encode a given combinatorial property, it may however be able to do so partially, and to some extent. And so rises in a certain sense a notion of 'rank', or 'dimension'. There is probably no recipe which attaches an interesting quantitative notion of complexity to any class of the hierarchy. In our context, one can call 'dimension' of a theory the maximal size of sets of formulas $\phi_i(x, y_i)$, such that a set of y_i-instances, called a *pattern*, satisfies the combinatorial configuration. Applied to NIP, this gives us the notion of dp-rank, and applied to NTP_2, the notion of burden, which generalises the former. In particular, we no longer study only the intricacies of instances of one unique formula, but interaction of instances of several formulas. A major difference between quantitative and qualitative classifications seems to be the following: though one can use global reasoning in the second case (e.g. counting types), one needs however a rather good understanding of formulas in the first case. A good understanding of formulas could be a synonym for quantifier elimination, but in practice, one needs an even better description of one-dimensional definable sets than the one given by quantifier elimination. Hence, the quantitative classification of first order theories actually motivates the search for a good description of the definable sets. This is of course an important problem on its own, as such a description can be used for many purposes.

Computing the burden is indubitably a very active research area ([58], [47], [42]). In particular, finite burden or burden equal to 1 has some relevance ([1]). A theory of burden one is called *inp-minimal*, or if is NIP, *dp-minimal*. If it has 'non-infinite' burden, it is called *strong*, or if the theory is NIP, *strongly dependent*. In model theory of valued fields, it has been shown that the field of p-adics is dp-minimal ([28],[5]). Later in [19], Chernikov and Simon proved that any ultraproduct of the p-adics over an ultrafilter on the prime numbers is inp-minimal, improving Chernikov's earlier work in [14]). As we will see, this computation highlighted some constraint between value group and residue field, with some concrete applications.

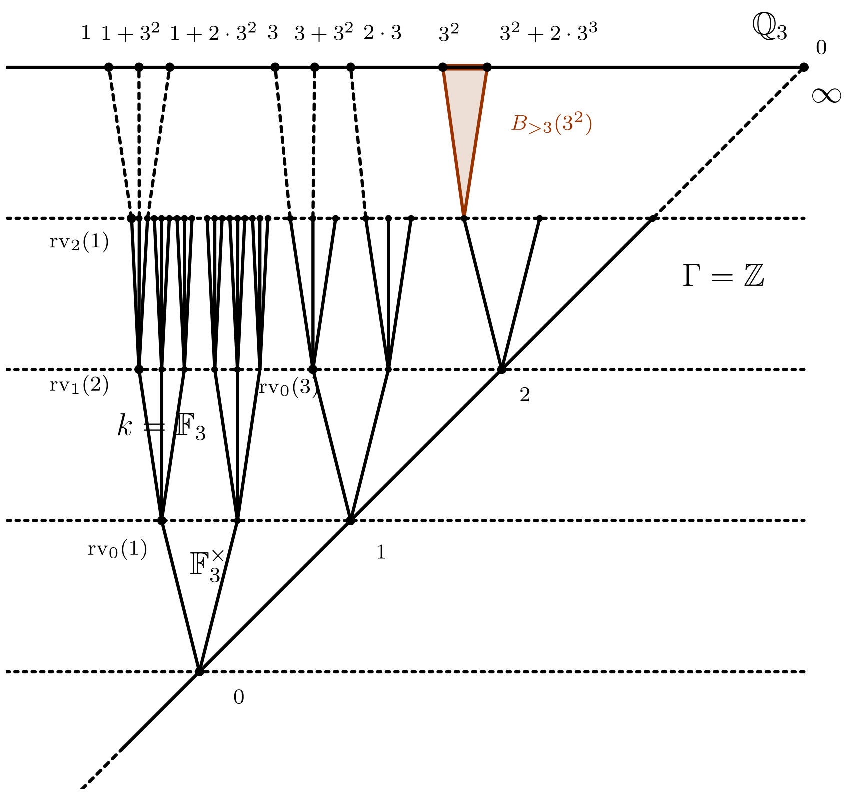
Stable embeddedness and definability of types

An other important notion present in this text is the notion of stable embeddedness. One can formulate two (equivalent) characterisations of stability: a theory is stable if and only if every complete type is definable if and only if every set is stably embedded. In model theory of unstable structures, the notion of stable embeddedness plays an important role. A subset A of an L-structure $\mathcal{M}$ is said to be stably embedded if any intersection of A and of a definable set is the intersection of A and of an A-definable set. By a duality between parameters/variables, it is equivalent to say that all types over A are definable over A. Understanding stably embedded definable subsets can be a crucial step in order to understand the whole structure. For instance, in the study of Henselian valued fields of equicharacteristic 0, the value group and residue field are stably embedded substructures that play a primordial role – we will discuss this fact longer in the introduction. The definition of stably embedded subsets is of course not limited to definable sets and other aspects concern indeed stably embedded submodels of the structure. Given a theory T, one can ask: when is a model $\mathcal{M}$ stably embedded in a given elementary extension $\mathcal{N}$? in all elementary extensions? One can also ask if this condition is first order: is the class of pairs of model $\mathcal{M} \leq \mathcal{N}$, such that $\mathcal{M}$ is stably embedded in $\mathcal{N}$, axiomatisable in the language of pairs? By an observation of van den Dries ([29]), being a stably embedded submodel of a real closed field is a first order property in the language of pairs. This was later generalised by Marker and Steinhorn to any o-minimal theories. They in fact prove that any o-minimal structure over which all 1-types are definable has all types definable. Few other structures satisfy analogous property, such as Presburger arithmetic and the theory of random graphs. Let us now introduce the structures that we mainly studied in this text, namely the valued fields.

Valued fields.

Many fields are naturally endowed with a valuation such as the field of formal power series $K((t))$ over a field K or the field of p-adics numbers $\mathbb{Q}_p$ for a prime p. Valued fields are rich algebraic structures and are very well studied in the framework of algebraic number theory, algebraic geometry and also model theory. In particular, model theorists found in the theory of valued fields a rich source of applications. Let us mention for instance the Denef rationality results for Poincaré series, the foundations of motivic integration and the Ax-Kochen-Eršhov principle – bringing the study of valued fields in the scope of model theory.

In the study of a valued field $\mathcal{K}$, the value group Γ and the residue field k play important roles, as we said. In the case that we will study, they will be always stably embedded substructures. It is often useful to consider a valued field as a leveled tree branching $|k|$-many times in $|\Gamma|$-many levels. It gives a picture for the valuation of an element or of the difference of two elements. The topology also becomes more intuitive. However, it fails to represent the field structure, in particular the multiplication. Nonetheless, it appears very useful in the context of this text. The reader will find some of these representations to support the intuition. We leave here a representation of the field $\mathbb{Q}_3$ of 3-adics, as an example.



The Ax-Kochen-Eršhov principle and 'benign' theories.
As usual in mathematics, one can prove relative results, and in the context of the classification, it can take the following form: understanding the complexity of a structure in term of that of a simpler substructure. The theory of valued fields is a fruitful playground for this 'philosophy' with the transfer principle of Ax-Kochen-Eršhov, discovered by Ax and Kochen ([6], [7], [8])

and independently by Eršhov ([32], [33],[34],[35],[36],[37],[38]). This principle states that the theory of Henselian valued fields of equicharacteristic 0 is completely determined by the theory of its residue field and that of its value group. One deduces from it the celebrated transfer principle between $\mathbb{F}_p((t))$ and $\mathbb{Q}_p$ for p large enough, and the approximate solution of Artin's conjecture. It has been shown that many theories are model complete relative to the residue field and the value group, such as

1. Henselian valued fields of characteristic $(0, 0)$,
2. algebraically closed valued fields,
3. algebraically maximal Kaplansky valued fields.

Notice that all these fields are Henselian - meaning that in these fields, a Newton process for extracting roots is possible. Although it is an important algebraic property – and moreover a first order one – we will rarely use explicitly its definition. However it will be implicitly used through the assumption of relative quantifier elimination. It is an important model theoretic property with a well known consequence that we discussed earlier: these valued fields enjoy the property that their residue fields (resp. value groups) are stably embedded and pure, in the sense that all sets definable in the residue field (resp. value group) may be defined in the language of fields (resp. ordered abelian group) with parameters in the residue field (resp. value group). In this text, we will refer to the theories listed above as *benign* theories of Henselian valued fields, as they behave nicely for the point of view of model theory. In Subsection 1.2.1, we will isolate few others important model theoretic properties that these theories of valued fields share and keep track of when they are used. The goal is to implicitly work axiomatically, and to emphasise model theoretic assumptions rather than algebraic ones.

One may be tempted to add to this list the theory of unramified Henselian valued fields of mixed characteristics with perfect residue field. We have indeed a result of Bélair in [12] who showed the model completeness relative to the value group and residue field. However the theory have some specificity – fortunately not too challenging. It will be treated in different paragraphs, as it needs different tools and observations. See the paragraph in Subsection 1.2.1 for mixed characteristic valued fields.

These 'benign' Henselian valued fields are opposed to wilder fields, such as $\mathbb{F}_p((t))$, which is Henselian but where an Ax-Kochen-Eršhov-like principle does not hold. In fact, a complete axiomatisation of $\mathbb{F}_p((t))$ is yet unknown: Kuhlmann showed that a natural set of axioms is not sufficient ([53]). This

It has been improved recently by Jahnke and Ascombe, who no longer assume the residue field to be perfect.

is an active problem, and recent progress has been made concerning its existential theory ([2]).

We will prove two distinct transfer principles for these benign valued fields and unramified mixed characteristic Henselian valued fields with perfect residue field. The first concerns its complexity with respect to the burden: What is the burden of such a valued field in terms of burden of its residue field and value group? The second concerns stably embedded elementary extensions of such fields: when is such a valued field stably embedded in an elementary extension? These two distinct questions are united by the fact that they can be approached by similar methods. We will discuss these methods after stating all the main results.

Transfer principles for burden.
One can expect, in principle, to understand the complexity of benign theories relatively to that of the value group and that of the residue field. A theorem of Delon [24] is the first instance of this approach in the context of the classification theory: a Henselian valued field of equicharacteristic 0 is NIP if and only if both its residue field and its value group are NIP. A more quantitative transfer in NIP Henselian valued fields was then showed by Shelah in [67]: a Henselian valued field of equicharacteristic 0 is strongly dependent if and only if both its residue field and its value group are strongly dependent. Both results were generalised to NTP_2 theories by Chernikov in [14]: a Henselian valued field of equicharacteristic 0 is NTP_2 (resp. strong, of finite burden) if and only if both of its residue field and its value group are NTP_2(resp. strong, of finite burden). Then, there is a finer result of Chernikov and Simon: consider a Henselian valued field of characteristic $(0, 0)$. Assume the residue field k satisfies

$$k^\star/(k^\star)^p \text{ is finite for every prime p.} \qquad (H_k)$$

Then the Henselian valued field of equicaracteristic 0 and residue field k is inp-minimal (*i.e.* of burden one) if and only if the residue field and the value group are both inp-minimal. They deduced the fact cited earlier: any ultraproduct $\Pi_{\mathcal{U}}\mathbb{Q}_p$ of the p-adics (over an ultrafilter $\mathcal{U}$ on the prime numbers) is inp-minimal. Interestingly, this result particularly echoes the original transfer principle of Ax-Kochen-Eršhov: the later indeed takes place in such an ultraproduct, and a good understanding of this structure is by consequence a long term goal. As many rank/dimension 1 version of complexity (strongly-minimal, SU-rank 1 etc.), this notion plays an important role. Let us cite

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It was later showed by Gurevich and Schmitt that any pure ordered abelian group is NIP.

(This theorem failed in general. It has been shown that $\mathbb{F}_p((t))$ has TP_2, while its value group and residue field are NTP_2. See [17]).

the classification of dp-minimal fields [46], and of dp-minimal valued fields [45]. It has been shown that dp-minimal ordered abelian groups $(\Gamma, +, 0, <)$ are exactly the *non-singular ones* ([45]), meaning that $\Gamma/p\Gamma$ is finite for all prime p. A natural question is then to consider the classification of valued fields of higher burden. This naturally requires generalisations of Chernikov and Simon's theorem. One can ask, as they did in [19, Problem 26], the following question: Given a Henselian valued field $\mathcal{K}$ of equicharacteristic 0, with residue field of burden n and value group of burden m. Can we compute burden of $\mathcal{K}$? We give a full answer to this question. Recall that, for us, a 'benign' Henselian valued field is either of equicharacteristic 0, algebraically closed or algebraically maximal Kaplansky.

Theorem 2.3.4. *Let $\mathcal{K} = (K, \Gamma, k)$ be a benign Henselian valued field, with value group Γ and residue field k. Then:*

$$\operatorname{bdn}(\mathcal{K}) = \max_{n \geq 0} \left(\operatorname{bdn}(k^\star/k^{\star n}) + \operatorname{bdn}(n\Gamma)\right).$$

It justifies *a postiori* the hypothesis in Chernikov-Simon's theorem, and shows that it is an optimal statement: an equicharacteristic 0 Henselian Non-trivially valued field with inp-minimal value group and inp-minimal residue field is inp-minimal if and only if the residue field satisfies (H_k). This answers at the same time [19, Problem 25].

The formula in itself as well as the proof shows a certain constrain between the value group and residue field within the valued field. In some sense, it behaves –in terms of complexity– like in a disjoint union structure $\Gamma \cup k$, rather than in a product structure $\Gamma \times k$. Hopefully, this should give some indication and intuition for future research in the classification of Henselian valued fields, where this distinction will be probably relevant.

Naturally, we can also consider a Henselian valued field with an angular component, *i.e.* a multiplicative map which associates to every element of the valued field a coefficient in the residue field and which coincides with the residue map on elements of value 0. We obtain:

Theorem 2.3.3. *Let $\mathcal{K}_{\mathrm{ac}} = (K, \Gamma, k, \mathrm{ac})$ be a benign Henselian valued field endowed with an* ac*-map* ac*, with value group Γ and residue field k. Then:* $\operatorname{bdn}(\mathcal{K}_{\mathrm{ac}}) = \operatorname{bdn}(k) + \operatorname{bdn}(\Gamma)$.

The study of valued fields without ac-map provide *a priori* finer results, but it is interesting to notice that the behavior changes – the complexity is now similar to that of the product structure $k \times \Gamma$ – and it is due to a simple reason that we going to make clear in this text. This difference yields for instance to the impossibility to define uniformly in p an angular component in the fields of p-adics (see Subsection 2.1.2), or in general, to the impossibility to define an ac-map in certain Henselian valued fields (see remark 2.3.6).

For unramified inequal characteristic Henselian valued fields with perfect residue field, we have the following:

Theorem 2.4.4. *Let $\mathcal{K} = (K, k, \Gamma)$ be an unramified mixed characteristic Henselian valued field with value group Γ and residue field k. We denote by $\mathcal{K}_{\mathrm{ac}_{<\omega}} = (K, k, \Gamma, \mathrm{ac}_n, n < \omega)$ the structure $\mathcal{K}$ endowed with compatible* ac*-maps. Assume that the residue field k is perfect. One has*

$$\mathrm{bdn}(\mathcal{K}) = \mathrm{bdn}(\mathcal{K}_{\mathrm{ac}_{<\omega}}) = \max(\aleph_0 \cdot \mathrm{bdn}(k), \mathrm{bdn}(\Gamma)).$$

Notice that this time, there is no difference between valued fields and valued fields endowed with ac-maps (maps with a plural, as it refers in this context to countably many maps. See 1.2.30). Notice that the multiplication sign between the two cardinals $\aleph_0 \cdot \mathrm{bdn}(k)$ cannot be replaced by an addition or a max, as we have to consider finite fields (of burden 0). It follows that the ring of Witt vectors $W(\mathbb{F}_p^{alg})$ – in fact any unramified inequal characteristic Henselian valued field with *infinite* perfect residue field – is not strong. This is perceived as a negative result.

We now describe our second type of transfer principles.

Transfer principle for definability of types.

As we said earlier, stably embedded elementary pairs of real closed fields form an elementary class in the language of pairs (([29])). Motivated by a similar question, Cubides and Delon have shown in [22] that an algebraically closed valued field $\mathcal{K}$ is stably embedded in an elementary extension $\mathcal{L}$ if and only if the valued fields extension $\mathcal{K} \preceq \mathcal{L}$ is *separated* (see Definition 3.1.1) and the small value group Γ_K is stably embedded in the larger value group Γ_L. Recently, Cubides and Ye proved in [23] a similar statement for p-adically closed valued fields and real closed valued fields. We give in this text a generalisation to benign Henselian valued fields. In fact, we give two statements:

Theorem 3.1.17. *Assume that T is a benign theory of Henselian valued fields. Let $\mathcal{K} \preceq \mathcal{L}$ be an elementary pair of models of T. The following are equivalent:*

1. *$\mathcal{K}$ is (uniformly) stably embedded in $\mathcal{L}$,*

Another reason is that k can for example be of burden '$\aleph_{\omega-}$'. See Subsection 1.1.2.

We distinguish in this text two natural notions of stable embeddedness/definability of types: the uniform and non-uniform one. See Definitions 1.1.42 and 1.1.46.

2. *The extension $\mathcal{L}/\mathcal{K}$ is separated, the residue field of $\mathcal{K}$ is (uniformly) stably embedded in the residue field of $\mathcal{L}$ and the value group of $\mathcal{K}$ is (uniformly) stably embedded in the value group of $\mathcal{L}$.*

Even if in practice, we will mainly consider elementary extensions of benign valued fields as the situation is simpler (see Subsection 1.1.3), we consider in the second statement non-elementary extensions of valued fields, in order to enlarge the scope of applications:

Theorem 3.1.16. *Assume that T is a benign theory of Henselian valued fields. Let $\mathcal{L}/\mathcal{K}$ be a separated extension of valued fields with $\mathcal{L} \models T$. Assume either*

- *that the value group of $\mathcal{K}$ is a pure subgroup of the value group of $\mathcal{L}$,*
- *or that the multiplicative group of the residue field k_L of $\mathcal{L}$ is divisible.*

The following are equivalent:

1. *K is stably embedded (resp. uniformly stably embedded) in $\mathcal{L}$,*
2. *the residue field of $\mathcal{K}$ is stably embedded (resp. uniformly stably embedded) in the residue field of $\mathcal{L}$ and the value group of $\mathcal{K}$ is stably embedded (resp. uniformly stably embedded) in the value group of $\mathcal{L}$.*

In particular, the field of p-adics $\mathbb{Q}_p$ is stably embedded in $\mathbb{C}_p$, the completion of its algebraic closure.

Again, with some additional observation, one can adapt these arguments and treat the case of unramified mixed characteristic Henselian valued fields with perfect residue field. We show indeed a similar statement for elementary pairs:

Theorem 3.2.3. *Let $\mathcal{K}$ be a unramified mixed characteristic Henselian valued field with perfect residue field and $\mathcal{L}$ be an elementary extension. The following are equivalent:*

1. *$\mathcal{K}$ is stably embedded (resp. uniformly stably embedded) in $\mathcal{L}$;*
2. *The extension $\mathcal{L}/\mathcal{K}$ is separated, the residue field of $\mathcal{K}$ is stably embedded (resp. uniformly stably embedded) in the residue field of $\mathcal{L}$ and the value group of $\mathcal{K}$ is stably embedded (resp. uniformly stably embedded) in the value group of $\mathcal{L}$.*

Also interesting is of course the method to prove these results. It is largely inspired by a method used in [19], which take source from well-known algebraic considerations. We will briefly summarise in these lines, but let us first discuss the limits of a direct approach.

Limit of the theorem of Pas.
In the setting of Henselian valued fields of characteristic $(0, 0)$, a very important tool is the theorem of Pas: Henselian valued fields of characteristic $(0, 0)$ endowed with an angular component eliminate field-sorted quantifiers. From this powerful result, one deduces easily the Ax-Kochen-Eršhov principle. However, it is insufficient in our context, as adding an ac-map to the language generates more structure and for instance more opportunities to partially encode the tree property of the second kind. This is indeed what happens: we show that the burden of a non-principal ultraproduct of p-adics endowed with an ac-map is equal to 2. Instead of using the theorem of Pas, Chernikov and Simon proceed with an intermediate step: before reducing to the residue field and value group, they consider another interpretable sort, the RV-sort. This is a rather unusual structure, highly relevant in the study of valued fields.

History of the RV-sort.
First traces of an intermediate structure between the valued field and the residue field and value group, can be found in a note of Krasner [51]. This idea has been brought to the scope of model theory by Basarab ([10]). After successive modifications ([52]), Flenner came up with the current definition of the RV-sort [40]. The RV-sort of a valued field K is the structure $K^\star/(1+\mathfrak{m})$ where $\mathfrak{m}$ is the maximal ideal of the valuation ring. We denote by rv the natural projection. The RV-sort offers an additional point of view: let us cite Hrushovski and Kazdan's work in motivic integration [44], where RV-sort are used, as opposed to Denef, Loeser and Cluckers' work [20], [21], [27] where ac-maps are used.

Benign theories of Henselian valued field also eliminate quantifiers relatively to the RV-sort. Similarly to the value group and residue field, one sees that the RV-sort is stably embedded and pure, and can be considered as an autonomous structure. Its full structure is induced by the exact sequence of abelian groups

$$1 \to k^\star \to \mathrm{RV} \to \Gamma \to 0$$

where $k^\star$ is the multiplicative group of the residue field, and Γ is the value (ordered abelian) group. It is also possible to consider it as a one-sorted structure $(\mathrm{RV}, \cdot, \oplus)$ where $\cdot$ is the group multiplication and $\oplus$ is some sort of non-associative addition, extending the addition of k (also defined in [40]).

One point of view, defended for instance in [40], would be to consider RV-sorts as stand alone structures, interesting on their own (as fields and groups are). Reduction principles to the RV-sort would be of equal importance, or could replace classical reduction principles to value group and residue field. A good understanding of some aspect of the RV-sort is indeed a crucial step in all the theorems above. In algebra of valued fields, traditional proofs require a distinction between three kind of extension: ramified (new value in the value group), residual (new elements in the residue field) and immediate extension (no new value and no new residue). Considering analogous statements for the RV-sort, one would have to only distinguish immediate and non-immediate extensions. But it is a complicated algebraic object, containing an ordered abelian group and a field. Adapting all classical statements of the algebra of valued fields from the point of the RV-sort requires some work, and it is not clear that algebraists will gain to do so. But towards this idea, the reader will find some results in this text. We produce in Appendix C an axiomatisation of RV and show that all such RV-structures are the RV-sort of a Henselian valued field. Overall, we chose a now common point of view – similar to that in [19]– which consists in seeing reduction to the RV-sort simply as an intermediate step. The strategy is indeed to reduce to the RV-sort and to continue the process of reduction later on, from the RV-sort to the value group and residue field. Let us mention our reduction principle down to the RV-sort.

Reduction to the RV-sort.

We show that the RV-sort of a benign valued field (see the list above) is as complex as the valued field itself with respect to the burden.

Theorem 2.1.2. *Let $\mathcal{K}$ be a benign valued field. Let M be a positive integer. Then $\mathcal{K}$ is of burden M if and only if the sort* RV *with the induced structure is also of burden M. In particular, $\mathcal{K}$ is inp-minimal if and only if* RV *is inp-minimal.*

For stably embedded substructure, we got the following theorem:

Theorem 3.1.8. *Let $\mathcal{L}/\mathcal{K}$ be a separated extension of valued fields, and assume that $Th(\mathcal{L})$ is a benign theory of Henselian valued field. The following are equivalent:*

1. *K is stably embedded (resp. uniformly stably embedded) in $\mathcal{L}$ (in* $\mathrm{Th}(\mathcal{L})$*),*

2. RV_K *is stably embedded (resp. uniformly stably embedded) in* RV_L.

We also proved similar statements for mixed characterisitic Henselian valued fields. See Theorems 2.1.1 and 3.2.1.

These reduction principles are possible due to the fact that benign valued fields eliminate quantifiers relatively to the RV-sort. In fact, our reduction of burden makes a decisive use of a slightly improved statement for one-dimensional definable set: every formula with one free variable x can be written in the following form:

$$\phi(\mathrm{rv}(x-a_0), \ldots, \mathrm{rv}(x-a_{n-1}), \beta)$$

where $\phi(x_0, \cdots, x_{n-1}, \beta)$ is an RV-formula. Notice that field-sorted terms are linear in x, and that a non-improved relative quantifier elimination result would only give polynomial field-sorted-terms). This result is due to Flenner in the case of characteristic 0, but we had to prove it in the general case of benign valued fields (see Theorem 1.2.19). This has some intuitive consequences, allowing us for instance to picture the valued field as a tree and to let aside algebraic consideration such as roots of polynomial. At the same time, this dependence of a description of one dimensional definable sets adds a layer of difficulty and an obstacle to potential generalisation, notably to valued fields with generic automorphism.

Resplendence of relative quantifier elimination.
Quantifier elimination relative to a sort S is said to be resplendent if it may be extended to any enrichment of the structure on S. This happens naturally and for syntactical reasons namely when the sort S is 'closed', meaning there is no function/predicate symbols from the sort S to the rest of the structure. This is a well known fact which has been formalised in [64]. All results of relative quantifier elimination in this text will be resplendent, and this notion will be intensively used through this text. In fact, all the main theorems cited above are resplendent – we mean that they still hold after enrichment of the value group and residue field or the RV-sort. This gives natural generalisations to richer theories and avoids unnecessary repetition ('one may prove similarly that ... '). Resplendency is for instance used to extend relative results on valued fields to valued fields with ac-map. One needs for that to notice that an ac-map is an enrichment of the RV-sort, and to treat the reduction from the RV-sort to the value group Γ and residue field k. The situation is then much simpler as the RV-sort may be identified with the product $k \times \Gamma$. Notice however that an ac-map is not an enrichment of neither the value group nor the residue field; a direct reduction to the value group and residue field would have not implied these statements. *A contrario*, working resplendently allows us to ignore 'superficial structure' and therefore helps to simplify proofs and notations.

Reduction to the residue field and value group.

As in [19], we see an RV-sort of a 'benign' valued field as an enriched short exact sequence

$$1 \to k^\star \to \mathrm{RV}^\star \to \Gamma \to 0$$

of abelian groups. Let us consider first 'benign' valued fields with an ac-map $\mathrm{ac} : K^\star \to k^\star$. The situation become simpler: one sees that adding an ac-map is equivalent to adding a spliting of the exact sequence above. By resplendence, it remains to reduce our problems from a product $k^\star \times \Gamma$ to Γ and k. We were naturally lead to the following proposition and lemma:

Proposition 1.1.28. *Let $\mathcal{K}$ and $\mathcal{H}$ be two structures, and consider the multisorted structure $\mathcal{G}$:*

$$\mathcal{G} = \{K \times H, \mathcal{K}, \mathcal{H}, \pi_K : K \times H \to K, \pi_H : K \times H \to H\},$$

called the direct product structure *(where π_K and π_H are the natural projections). Then $\mathcal{G}$ eliminates quantifiers relative to $\mathcal{K}$ and $\mathcal{H}$, and $\mathcal{K}$ and $\mathcal{H}$ are orthogonal and stably embedded within $\mathcal{G}$.*

We have

$$\mathrm{bdn}(\mathcal{G}) = \mathrm{bdn}(\mathcal{K}) + \mathrm{bdn}(\mathcal{H}).$$

Lemma 3.1.11. *Let $\mathcal{H}_1$ (resp. $\mathcal{K}_1$) be a substructure of a structure $\mathcal{H}_2$ (resp. $\mathcal{K}_2$). Then $\mathcal{H}_1 \times \mathcal{K}_1$ is a substructure of $\mathcal{H}_2 \times \mathcal{K}_2$ and we have:*

- *$\mathcal{H}_1 \times \mathcal{K}_1$ is stably embedded in $\mathcal{H}_2 \times \mathcal{K}_2$ if and only if $\mathcal{H}_1$ is stably embedded in $\mathcal{H}_2$ and $\mathcal{K}_1$ is stably embedded in $\mathcal{K}_2$.*
- *$\mathcal{H}_1 \times \mathcal{K}_1$ is uniformly stably embedded in $\mathcal{H}_2 \times \mathcal{K}_2$ if and only if $\mathcal{H}_1$ is uniformly stably embedded in $\mathcal{H}_2$ and $\mathcal{K}_1$ is uniformly stably embedded in $\mathcal{K}_2$.*

Naturally, the question of the reduction of benign valued fields without ac-map requires more work, as one has to really deal with the structure of a short exact sequence $1 \to k^\star \to \mathrm{RV}^\star \to \Gamma \to 0$. By resplendence, we can forget the additive structure on the residue field, and the ordered structure on the value group. As the following results are interesting in their own, we leave the traditional notation of valued fields, and name our sequences of abelian groups $0 \to A \to B \to C \to 0$.

Theorem 2.2.2. *Consider an $\{A\}$-$\{C\}$-enrichement of an exact sequence $\mathcal{M}$ of abelian groups*

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\nu} C \longrightarrow 0\,.$$

Assume that A is a pure subgroup of B: for all $a \in A$, $b \in B$ if $\iota(a) = nb$, then there is $a' \in A$ such that $a = na'$. We have

$$\operatorname{bdn} \mathcal{M} = \max_{n\in\mathbb{N}}(\operatorname{bdn}(A/nA) + \operatorname{bdn}(nC)).$$

In particular, if A/nA is finite for all $n \geq 1$, then

$$\operatorname{bdn} \mathcal{M} = \max(\operatorname{bdn}(A), \operatorname{bdn}(C)).$$

Corollary 3.1.14. Let $\mathcal{N} = \{0 \to A \to B \to C \to 0\}$ be a (possibly $\{A\}$-$\{C\}$-enrichement of a) pure short exact sequence of abelian groups, and let $\mathcal{M} \subseteq \mathcal{N}$ be a sub-short exact sequence. Assume either

- that $C(\mathcal{M})$ is a pure subgroup of $C(\mathcal{N})$,
- or that $\rho_n^{\mathcal{N}}(\mathcal{M})$ is finite for all $n \geq 1$.

Then, we have:

- $\mathcal{M}$ is stably embedded in $\mathcal{N}$ if and only if $A(\mathcal{M})$ is stably embedded in $A(\mathcal{N})$ and $C(\mathcal{M})$ is stably embedded in $C(\mathcal{N})$,
- $\mathcal{M}$ is uniformly stably embedded in $\mathcal{N}$ if and only if $A(\mathcal{M})$ is uniformly stably embedded in $A(\mathcal{N})$ and $C(\mathcal{M})$ is uniformly stably embedded in $C(\mathcal{N})$.

Is the study of the RV-sort necessary? It is at least very convenient and seems canonical in a certain way: the RV-sort seems to be the least substructure within benign valued fields containing the value group and residue field and where reduction principles apply. And as we saw, it also has the advantage to generalise in the context of valued fields with an ac-map.

Organisation of the text

A large part of this text is dedicated to preliminaries in Chapter 1. In Section 1.1, we define pure model theoretic notions such as burden and stable embeddedness. In Section 1.2, we describe essential model theoretic properties of algebraic structures: valued fields and short exact sequences of abelian groups. It also includes important lemmas and propositions.

We prove our transfer principles for burden in Chapter 2, which is split in four sections: in Section 2.1 we treat the reduction to the RV-sort, in Section

we will denote by $\rho_n^{\mathcal{N}}(\mathcal{M})$ the image of $\mathcal{M}$ in A/nA under a certain natural projection map. The condition is in particular satisfied if A/nA is itself finite

2.2 the reduction in pure short exact sequences of abelian groups, in Section 2.3 the reduction to the value group and residue field for benign valued fields, and in Section 2.4 the reduction to the value group and residue field for unramified mixed characteristic Henselian valued fields with perfect residue field.
We treat then our transfer principles for stably embedded pairs of valued fields in Chapter 3. For some technical reason, this chapter has a slightly different structure. It is split in three sections: Section 3.1 treats the reduction in the context of benign valued fields and Section 3.2 treats the reduction in the context of unramified mixed characteristic Henselian valued fields with perfect residue field. In both sections, we reduce first to the RV-sort. In the first, we provide a transfer principle for stably embedded pairs of pure short exact sequences. Then, in Section 3.3, we treat as a corollary the elementarity of the class of stably embedded elementary pairs of benign valued fields.

The reader will also find some sparse results. For instance, we took the opportunity to briefly study and construct non-uniformly stably embedded pairs of random graphs in Appendix A. In Appendix B, we produce a transfer principle for burden in Lexicographic products. We also gave an attempt in Subsection 1.1.2 to formalise and justify a common convention which consists of writing $\mathrm{bdn}(T) \leq \aleph_{0-}$ if a theory T is strong.

Reduction diagrams and heuristic

Let us conclude this introduction with some generalities about transfer principles. We summarise the strategy presented above by formalising the reduction in valued fields and pure short exact sequences of abelian groups. We introduce *reduction diagrams*. It is nothing else than a concise way to picture relative quantifier elimination and by extension, the strategy for proving reduction principles.

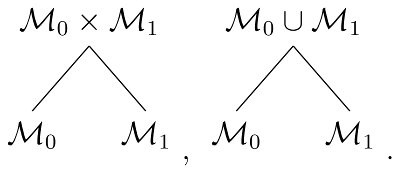
Heuristic 0.0.1. A *reduction diagram* of a structure $\mathcal{M}$ is a rooted tree such that:

- all nodes are pure sorts of $\mathcal{M}$ (in some $\emptyset$-interpretable language) endowed with their full structure;
- the root is $\mathcal{M}$;
- any node admits relative resplendent quantifier elimination (in some $\emptyset$-interpretable language) to the set of its children;
- any two sorts in two different branches are orthogonal.

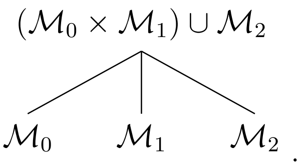
The idea is that one might be able to reduce certain questions on the structure $\mathcal{M}$ to the set of its leaves. Every node describes then an intermediate step. Reduction to a node would also have the advantage of being generalised to any enrichment of structure below the node.

In this text, we compute the burden (Definition 1.1.14) of the following examples in terms of the burden of the leaves. We also characterise stable embeddedness (Definition 1.1.42) of elementary pairs of models in terms of stable embeddedness of elementary pairs of structures in the leaves.

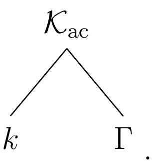
Example. 1. If $\mathcal{M}_0, \mathcal{M}_1$ are arbitrary structures, both the direct product $\mathcal{M}_0 \times \mathcal{M}_1$ and the disjoint union $\mathcal{M}_0 \cup \mathcal{M}_1$ reduce to $\mathcal{M}_0$ and $\mathcal{M}_1$ (Fact 1.1.37):



We can of course keep going:



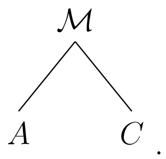
2. Let $\mathcal{K}_{ac} = \{K, \Gamma, k, \mathrm{ac} : K \to k\}$ be a Henselian valued field of equicharacteristic 0 of valued group Γ, residue field k, and angular component ac. It admits the following reduction diagrams (Theorem of Pas):



3. Let $\mathcal{M} = \{A, B, C, \iota, \nu\}$ be a short exact sequence of abelian groups

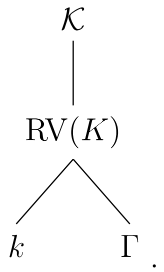
$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\nu} C \longrightarrow 0\,,$$

seen as a three-sorted structure. Assume A is a pure subgroup of B. It admits the following reduction diagram (Fact 1.2.39):

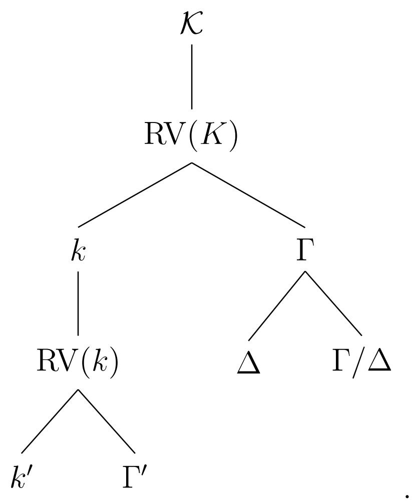


To get relative quantifier elimination, one has to consider interpretable maps from B to A/nA, $n \geq 0$. The sort A/nA are understood to be part of the induced structure on A.

4. Let $\mathcal{K} = \{K, \Gamma, k, \mathrm{RV}(K)\}$ be a Henselian valued field of equicharacteristic 0, value group Γ, residue field k and RV-sort $\mathrm{RV}(K)$ (definition in Subsection 1.2.1). It admits the following reduction diagram (Fact 1.2.27 and Fact 1.2.39):



5. If $\mathcal{K} = \{K, \Gamma, k\}$ is a Henselian valued field of equicharacteristic 0, where moreover the residue field k is endowed with a structure (k, Γ', k') of Henselian valued field of equicaracteristic 0, and Γ is endowed with a predicate for a convex subgroup Δ. Then by Corollary 1.2.40 (and resplendence), we have the following reduction diagram:



Chapter 1

Preliminaries

1.1 On pure model theory

Notations
We will assume the reader to be familiar with basic model theory concepts, and in particular with standard notations. One can refer to [72]. Symbols $x, y, z, \dots$ will usually refer to tuples of variables, $a, b, c, \dots$ to parameters. Capital letters $K, L, M, N, \dots$ will refer to sets, and calligraphic letters $\mathcal{K}, \mathcal{L}, \mathcal{M}, \mathcal{N}, \dots$ will refer to structures with respective base sets $K, L, M, N, \dots$. If there is no ambiguity, we may respectively name a very saturated elementary extension with blackboard bold letters $\mathbb{K}, \mathbb{L}, \mathbb{M}, \dots$ Languages will be denoted will a roman character $\mathrm{L}, \mathrm{L}', \mathrm{L}_{Rings}, \mathrm{L}_{\Gamma,k}$ etc.

In this chapter, we will consider any (possibly multi-sorted) first order language L, and an arbitrary L-structure $\mathcal{M}$.

1.1.1 Relative quantifier elimination and resplendence

In this text, we will use freely Rideau-Kikuchi's terminology about enrichment. We briefly recall it now. The reader can refer to [64, Annexe A] for a more detailed exposition. In particular, we will define the notion of resplendent relative quantifier elimination. The notion of resplendence will play a certain role in this text. It seems to unnecessary weigh down statements, but it is in fact a commodity than one should not avoid when it concerns transfer/reduction principles. It has the advantage to give effortless generalisations to richer structures, or to simplify the notation, by reducing the language to the strict necessity for producing transfer principles.

First, let us recall two notions of relative quantifier elimination.

Definition 1.1.1. Let $\mathcal{M}$ be a multisorted structure in a language L, and consider $\Pi \cup \Sigma$ a partition of the set of sorts. We denote by $L_{|\Sigma}$ the language of all function symbols and relation symbols in L involving only sorts in Σ. Then, we say that

- $\mathcal{M}$ eliminates Π-quantifiers if every formula $\phi(x)$ is equivalent to a formula without quantifier in a sort in Π.
- $\mathcal{M}$ eliminates quantifiers relatively to Σ if the theory of $\mathcal{M}^{\Sigma-\mathrm{Mor}}$ – obtained by naming all $L_{|\Sigma}$-definable sets (without parameters) with a new predicate– eliminates quantifiers.

As observed in [64, Annexe A], M eliminates quantifiers relatively to Σ, then it eliminates Π-quantifiers.

Definition 1.1.2. Let $\mathcal{M}$ be a multi-sorted structure in a language L, and let Σ be a set of sorts in L.

- a language L_e containing L is said to be a Σ-*enrichment* of L if all new function symbols and relation symbols only involve the sorts in Σ and the new sorts Σ_e in $L_e \setminus L$. An expansion $\mathcal{M}_e$ of $\mathcal{M}$ to L_e is called a Σ-*enrichment* of $\mathcal{M}$.
- Σ is said to be *closed* if any relation symbol involving a sort in Σ or any function symbol with a domain involving a sort in Σ only involves sorts in Σ.

Fact 1.1.3 ([64]). *Let $\mathcal{M}$ be a multisorted structure, and consider $\Pi \cup \Sigma$ a partition of the set of sorts. If Σ is a closed set of sorts, then $\mathcal{M}$ eliminates Π-quantifiers if and only if $\mathcal{M}$ eliminates quantifiers relatively to Σ.*

In the context of this text, these two notions of quantifier elimination will be often equivalent. Another consequence of closedness is the automatic resplendence of relative quantifier elimination:

Definition 1.1.4. Let $\mathcal{M}$ be a multi-sorted structure in a language L, and let Σ be a set of sorts in L. We say that $\mathcal{M}$ *eliminates quantifiers resplendently relatively* to Σ if for any Σ-enrichment $\mathcal{M}_e$ of $\mathcal{M}$, $\mathrm{Th}(\mathcal{M}_e)$ eliminates quantifiers relatively to $\Sigma \cup \Sigma_e$ (where Σ_e is the set of new sorts in M_e).

Fact 1.1.5 ([64, Proposition A.9]). *Let $\mathcal{M}$ be a multi-sorted structure in a language* L*, and assume that* $\mathrm{Th}(\mathcal{M})$ *eliminates quantifiers relative to a closed set of sorts Σ. Then* $\mathrm{Th}(\mathcal{M})$ *eliminates quantifiers resplendently relatively to Σ.*

Notice however that closedness does not characterise resplendence of relative quantifier elimination, as we will see later with pure short exact sequence of abelian groups. Let us introduce the notion of stable embedded definable sets and of pure sorts.

Definition 1.1.6.

- A definable subset D of $\mathcal{M}$ is called *stably embedded* if all definable subsets of D^n, $n \in \mathbb{N}$ can be defined with parameters in D.
- Two definable subsets D and D' of $\mathcal{M}$ are called *orthogonal* if for all formulas
$$\phi(x_0, \dots, x_{n-1}; x'_0, \dots, x'_{n-1}, a)$$
with parameters a in $\mathcal{M}$, there is finitely many formulas $\theta_i(x_0, \dots, x_{n-1}, a_i)$ and $\theta'_i(x'_0, \dots, x'_{n-1}, a'_i)$, with $i < k$ and parameters $a_0, \dots, a_{n-1}$, $a'_0, \dots, a'_{n-1}$ in $\mathcal{M}$, such that
$$\phi(D^n, D'^n, a) = \cup_{i<k} \theta_i(D^n, a_i) \times \theta'_i(D'^n, a_i).$$

If S is a sort, we use the following terminology in order to say that definable sets in S can be given by formulas with parameters in S and function/predicate symbols *contained* in S.

Definition 1.1.7. A sort S in an L-structure $\mathcal{M}$ is called *pure* or *unenriched* if definable subsets of S (with parameters) are given by $\mathrm{L}_{|S}(S)$-formulas where $\mathrm{L}_{|S}$ is the language restricted to function/predicated symbols which only involve S.

A pure sort S can be seen as an $\mathrm{L}_{|S}$-structure on its owns. In particular, it is stably embedded. Purity of a sort S is usually a simple corollary of quantifier elimination relative to S and closedness of S (see Fact 1.1.9).

Remark that the notion of closedness is syntactic, which is not ideal. One may indeed use another bi-interpretable language, where the sort is no longer closed, but where resplendent relative quantifier elimination still holds. Here is a sightly improved version of purity which can replace the notion of closedness. It is, in a certain sens, less dependent of the language. We will use it at some rare occasions.

This happens for instance with the residue field k of an equicharacteristic 0 Henselian valued field: it is closed in the traditional 3-sorted language of valued fields but it is also natural to interpret in addition a short exact sequences of abelian groups $1 \to k^\times \to \mathrm{RV}^\star \to \Gamma \to 0$. See Subsection 1.2.1

Definition 1.1.8. Consider $\mathcal{M}$ a structure. An imaginary sort $\mathcal{S} = (S, \ldots)$ endowed with an interpretable structure in a language L_S is called *pure with control of parameters* if every formula $\phi_i(x_S, b)$ where x_S is a tuple of S-variables and b is a tuple of parameters in M, is equivalent to a formula $\phi_S(x_S, \pi_S(t(b)))$ where π_S is the canonical projection onto S, ϕ_S is an L_S-formula and $t(x)$ is a tuple of L-terms.

The following is immediate:

Fact 1.1.9. *Consider $\mathcal{M}$ an* L*-structure. If S is a closed sort and $\mathcal{M}$ eliminates quantifiers relative to S, then S endowed with its induced structure in* $\mathrm{L}_{|S}$ *is pure with control of parameters. In particular, it is pure and stably embedded.*

Proposition 1.1.10. *Let $\mathcal{M}$ be an* L*-structure, and $\mathcal{S}$ an imaginary sort of arity n with some interpretable structure. Assume that $\mathcal{M}$ has quantifier elimination and that $\mathcal{S}$ is a pure imaginary structure with control of parameters. The (multisorted) structure $\{\mathcal{M}, \mathcal{S}, \pi_S : M^n \to S\}$ in the language* $\mathrm{L}_2 := \mathrm{L} \cup \mathrm{L}_S \cup \{\pi_S : M^n \to S\}$ *admits quantifier elimination relatively to $\mathcal{S}$.*

As $\mathcal{S}$ is by definition a closed sort in the language L_2, this is in fact a characterisation of purity with control of parameters:

Corollary 1.1.11. An interpretable structure $\mathcal{S}$ is pure with control of parameters if and only if $\{\mathcal{M}, \mathcal{S}, \pi_S : M^n \to S\}$ admits quantifier elimination relative to $\mathcal{S}$.

Proof. We use the usual back-and-forth argument. Let $\mathcal{N}$ be a $|M|$-saturated model of the theory of $\mathcal{M}$ in the language L_2. Let $f : (A, S_A) \to (B, S_B)$ be an isomorphism between a substructure (A, S_A) of $\mathcal{M}$ and a substructure (B, S_B) of $\mathcal{N}$. Assume that the restriction $f_{|S}$ to $\mathcal{S}$ is elementary. We want to extend f to an embedding of $\mathcal{M}$ into $\mathcal{N}$.
Step 0: We may assume that $S_A = S_M$.
Indeed, by elementarity of $f_{|S}$, there exists an isomophism $\tilde{f}_{|S} : S_M \to \tilde{f}_{|S}(S_M) \subset S_N$ extending $f_{|S}$. The union $f \cup \tilde{f}_{|S}$ is a partial isomorphism as the sort S is closed. Indeed, every quantifier-free formula $\phi(a, s)$ with parameters in (A, S_M) can be written of the form:

$$\bigvee \phi_{\mathrm{L}}(a) \wedge \phi_{\mathcal{S}}(s, \pi_S(t(a))),$$

where ϕ_{L} is an L-formula, $\phi_{\mathcal{S}}$ is an L_S-formula and t is a tuples of L-terms. As A is a structure, all terms $t(a)$ are elements of A. It follows that $f \cup \tilde{f}_{|S}$ preserves these formulas.

Step 1: We may assume that $A = M$ and thus conclude the proof.
Indeed, let $a \in M \setminus A$. We denote by $p(x)$ the quantifier free type of a over A. We want an appropriate answer for a, *i.e.* an element $\tilde{f}(a)$ of $\mathcal{N}$ satisfying the set of formulas:

$$\{\phi(x, f(b), f(s)) \mid \phi(x, b, s) \in p(x), b \in A,\ s \in S_M\}$$

By compactness, we need to show that it is finitely consistent. Consider a formula

$$\phi(x, b, s) \in p(x),$$

where $b \in A$ and $s \in S_M$. As S is pure with control of parameters, the formula

$$\exists x\ \phi(x, b, y_S)$$

is equivalent to an $\mathrm{L}_S(S_M)$-formula $\psi_S(t(b), y_S)$ (with a tuple of L-terms $t(y)$).

The formula

$$\theta(y) = \forall y_S\ \psi_S(t(y), y_S) \Leftrightarrow \exists x\ \phi(x, y, y_S)$$

is interpreted in the language L by a formula $\Sigma(y)$. As $\mathcal{M}$ has quantifier elimination in the language L, we may assume that $\Sigma(y)$ is quantifier-free. We have:

$$\mathcal{M} \models \Sigma(b),$$
$$\mathcal{M} \models \psi_S(t(b), s).$$

As f respects quantifier free-formula and $f|_{S_M}$ respects $\mathrm{L}_S(S_M)$-formula, we have

$$\mathcal{N} \models \Sigma(f(b)),$$
$$\mathcal{N} \models \psi_S(f(t(b)), f(s)).$$

Of course, $f(t(b)) = t(f(b))$. We get: $\mathcal{N} \models \exists x\ \phi(x, f(b), f(s))$. This concludes our proof. □

As an example, we treat the question of quantifier elimination in the field of p-adics in a two-sorted language of valued fields. This is a well known result, but we are not aware of a reference.

Example. Consider the theory T of the p-adics $\mathbb{Q}_p$ for some p. By Macintyre's theorem [54], it admits quantifier elimination in the language $\mathrm{L}_{Mac} := \mathrm{L}_{rings} \cup \{P_n\}_{n<\omega}$ where the predicate P_n interprets the n^{th}-powers.

- The value group Γ, simply considered as a set, is not pure. Indeed, the theory T in the language $\mathrm{L}_{Mac} \cup \{\Gamma\} \cup \{\mathrm{val}\}$ (no structure on Γ) does not eliminate the quantifiers in the formula encoding the addition:

$$\phi(x_\Gamma, y_\Gamma, z_\Gamma) \equiv \exists x, y \in K \ \ \mathrm{val}(x) = x_\Gamma \wedge \mathrm{val}(y) = y_\Gamma \wedge \mathrm{val}(xy) = z_\Gamma,$$

where $x_\Gamma, y_\Gamma, z_\Gamma$ are variables in Γ.

- By Bélair's theorem [12, Theorem 5.1], the structure $\{\mathbb{Q}_p, \mathcal{O}_n, \Gamma, \mathrm{val} : \mathbb{Q}_p \to \mathbb{Z}, \mathrm{ac}_n : \mathbb{Q}_p \to \mathcal{O}_n\}$ enriched with angular components (see Subsection 1.2.1) eliminates quantifiers in the sort for $\mathbb{Q}_p$. It results that the value group $(\Gamma, +, 0, <, \infty)$ –as an imaginary sort of $\mathbb{Q}_p$ in the ring language– is pure with control of parameters. Then, by the theorem, T eliminates quantifiers relatives to Γ in the language

$$\mathrm{L}_2 := \mathrm{L}_{Mac} \cup \{\Gamma, <, +, 0, \infty\} \cup \{\mathrm{val}\}$$

- To get full elimination of quantifiers, one only needs to eliminates quantifiers in $\{\Gamma, <, +, 0, \infty\}$. So the theory T eliminates quantifiers in the language $\mathrm{L}_{Mac} \cup \{\Gamma, <, +, P_{\Gamma,n}, 0, 1, \infty\} \cup \{\mathrm{val}\}$ where $P_{\Gamma,n}$ interprets the set of values divisible by n.

1.1.2 Classification theory

Introduction

In [66], Shelah defined the notion of burden as an invariant cardinal κ_{inp} and implicitly defined the *tree property of the second kind*. A theory which does not satisfy it is called NTP_2. Interest in the class of NTP_2 theories grew after the success of stability theory and with the necessity of extending methods to unstable contexts. In [16], Chernikov and Kaplan studied the forking relation in NTP_2 theories, establishing notably that types over models fork if and only if they divide. In [14], Chernikov continued the study of NTP_2 theories, establishing in particular a criterion with indiscernible sequences and the sub-multiplicativity of the burden.

We recall here a definition of burden, some of the results cited above and give some important lemmas required for the proof of Theorem 2.3.4. We will give a second definition (slightly different) of the burden in order to formalise a convention due to Adler([1]).

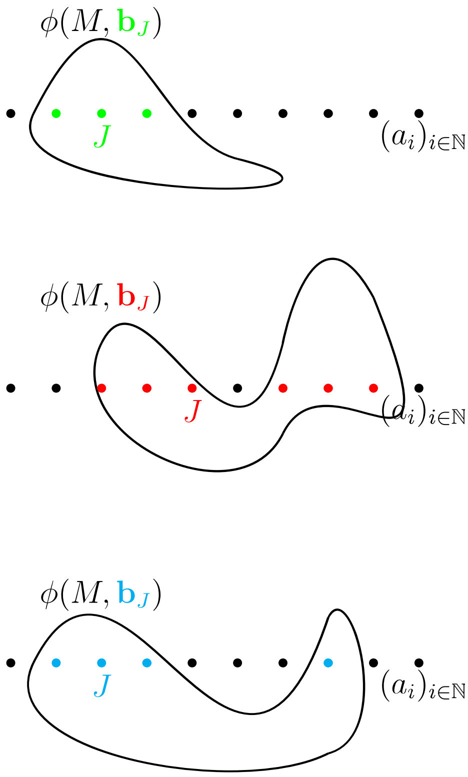
As an introduction, let us give first the definition of NIP theories. We also take the occasion to give a quick intuition on how this notion captures indeed a notion of complexity.

Let T be a complete first-order theory in a language L, $\mathcal{M}$ a model of T and let $\mathbb{M} \models T$ be a monster model. We denote by $\mathcal{S}$ the set of sorts in L.

Definition 1.1.12. A formula $\phi(x; y)$ has the Independence property (IP) if there is a sequence $(a_i)_{i\in\mathbb{N}}$ and $(b_J)_{J\subset\mathbb{N}}$ of elements in $\mathbb{M}$ such that for all $i \in \mathbb{N}$ and $J \subset \mathbb{N}$, we have:

$$\mathbb{M} \models \phi(a_i, b_J) \text{ if and only if } i \in J.$$

When $J \subset \mathbb{N}$ varies, it gives definable sets which must intersect or avoid each other in the points a_i. Then, one sees that an IP formula is a formula such that some instances interact in a specific way.



Definition 1.1.13. A theory T is NIP if no formula $\phi(x; y)$ has the independence property.

One may see then why NIP-ness is a notion of tameness: if such a configuration cannot be formed with instances of a single formula, it means that these instances behave rather nicely. We will not use any specific properties that NIP theories share, as our transfer principle for burden will apply to the more general class of NTP_2-theories. But it might be useful to recall already that the burden coincides with the dp-rank in the context of NIP-theories – a notion that the reader might be more familiar with. Let us define now the tree property of the second kind (TP_2) and the burden.

Burden of a theory

Definition 1.1.14. Let λ be a cardinal. For all $i < \lambda$, $\phi_i(x, y_i)$ is L-formula where x is a common tuple of free variables, $b_{i,j}$ are elements of $\mathbb{M}$ of size $|y_i|$ and k_i is a positive natural number. Finally, let $p(x)$ be a partial type. We say that $\{\phi_i(x, y_i), (b_{i,j})_{j\in\omega}, k_i\}_{i<\lambda}$ is *an inp-pattern of depth* λ in $p(x)$ if:

1. **for all $i < \lambda$, the i^{th} row is k_i-inconsistent**: any conjunction $\bigwedge_{l=1}^{k_i} \phi_i(x, b_{i,j_l})$ with $j_1 < \cdots < j_{k_i} < \omega$, is inconsistent.
2. **all (vertical) paths are consistent**: for every $f : \lambda \to \omega$, the set $\{\phi_i(x, b_{i,f(i)})\}_{i<\lambda} \cup p(x)$ is consistent.

Most of the time, we will not mention the k_i's and only say that the rows are finitely inconsistent. In all the definitions below, T and $\mathcal{M}$ are interchangeable.

Definition 1.1.15.

- Let $p(x)$ be a partial type. *The burden* of $p(x)$, denoted by $\mathrm{bdn}(p(x))$, is the cardinal defined as the supremum of the depths of inp-patterns in $p(x)$. If C is a small set of parameters, we write $\mathrm{bdn}(a/C)$ instead of $\mathrm{bdn}(\mathrm{tp}(a/C))$.
- The cardinal $\sup_{S\in\mathcal{S}} \mathrm{bdn}(\{x_S = x_S\})$ where x_S is a single variable from the sort S, is called the *burden* of the theory T, and it is denoted by $\kappa^1_{\mathrm{inp}}(T)$ or by $\mathrm{bdn}(T)$. The theory T is said to be *inp-minimal* if $\kappa^1_{\mathrm{inp}}(T) = 1$.
- More generally, for λ a cardinal, we denote by $\kappa^\lambda_{\mathrm{inp}}(T)$ the supremum of $\mathrm{bdn}(\{x = x\})$ where $|x| = \lambda$ and variables run in all sorts $S \in \mathcal{S}$. We always have $\kappa^\lambda_{\mathrm{inp}}(T) \geq \lambda \cdot \kappa^1_{inp}(T)$. In particular, if models of T are infinite, $\kappa^\lambda_{\mathrm{inp}}(T) \geq \lambda$.
- A formula $\phi(x, y)$ has TP_2 if there is an inp-pattern of the form $\{\phi(x, y), (b_{i,j})_{j<\omega}, k_i\}_{i<\omega}$. Otherwise, we say that $\phi(x, y)$ is NTP_2.
- The theory T is said NTP_2 if $\kappa^1_{\mathrm{inp}}(T) < \infty$. Equivalently, T is NTP_2 if and only if there is no TP_2 formula. (See [14, Remark 3.3])

In [14], Chernikov proves the following:

Fact 1.1.16 (Sub-multiplicativity). *Let $a_1, a_2 \in \mathbb{M}$. If there is an inp-pattern of depth $\kappa_1 \times \kappa_2$ in $\operatorname{tp}(a_1a_2/C)$, then either there is an inp-pattern of depth κ_1 in $\operatorname{tp}(a_1/C)$ or there is an inp-pattern of depth κ_2 in $\operatorname{tp}(a_2/a_1C)$.*

As a corollary, for $n < \omega$, we have $\kappa^n_{inp}(T) + 1 \leq (\kappa^1_{inp}(T) + 1)^n$ and then $\kappa^n_{inp}(T) = \kappa^1_{inp}(T) = \operatorname{bdn}(T)$ as soon as one of these cardinal is infinite.

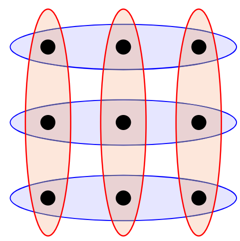
If the reader knows the notion of dp-rank of a theory T, usually denoted by dp-rank(T), let us say the following: it admits as well a similar definition in term of depth of *ict-patterns* and it has been showed that a theory T is NIP if and only if the depth of *ict*-pattern is bounded by some cardinal. In this paper, the reader only needs to know that the notions of dp-rank and burden coincide in NIP theories. If they are not familiar with the notion of dp-rank, they may take it as a definition.

Fact 1.1.17 ([1, Proposition 10]). *Let T be an NIP theory, and $p(x)$ a partial type. Then dp-rank$(p(x)) = \operatorname{bdn}(p(x))$.*

The previous fact is only stated with partial type $p(x) = \{x = x\}$, but the proof is the same.

Example.

- Any quasi-o-minimal theory is inp-minimal (see e.g. [69, Theorem A.16]). In particular, $\{\mathbb{Z}, 0, +, <\}$ is inp-minimal.
- Let $\mathrm{L} = \{R, B\}$ be the language with two binary predicates, and let $\mathcal{M}$ be a set with two cross-cutting equivalence relations with infinitely many infinite classes (for all a and b, there is infinitely many c such that aRc and bBc).

$\mathcal{M} :=$ 

One proves easily that $\operatorname{bdn}(\mathcal{M}) = 2$. For λ a cardinal, one can consider λ-many cross cutting equivalence relations, and shows that the structure is of burden λ.

The definition of burden of a theory, as many other notion of complexity, gives to unary sets an important role. But one has to notice that the notion of unary set is syntactic, and is not preserved under bi-interpretability:

Remark 1.1.18. *For $n \in \mathbb{N}$, one can consider the multisorted structure $(\mathcal{M}^n, \mathcal{M}, p_i, i < n)$ where $p_i : \mathcal{M}^n \to \mathcal{M}, (a_0, \dots, a_{n-1}) \mapsto a_i$ is the projection to the i^{th} coordinate. If we denote its theory by T^n, then we clearly have $\kappa^n_{inp}(T) = \kappa^1_{inp}(T^n)$.*

To clarify, let us introduce the following terminology:

Definition 1.1.19. Let $\mathcal{M}$ and $\mathcal{N}$ be two structures. We say that $\mathcal{N}$ is *interpretable on a unary set* in $\mathcal{M}$ if there is a bijection $f : N \to D/\sim$ where D is a unary definable set in $\mathcal{M}$, $\sim$ is a definable equivalence relation, and the pull-back in $\mathcal{M}$ of any graph of function and relation of $\mathcal{N}$ is definable. The structures $\mathcal{M}$ and $\mathcal{N}$ are said *bi-interpretable on unary sets* if $\mathcal{N}$ is interpretable on a unary set in $\mathcal{M}$ and $\mathcal{M}$ is interpretable on a unary set in $\mathcal{N}$.

We will work up to bi-interpretability on unary sets, meaning in particular that the main results of this text will only depend on the structure that we want to consider and not on the language.

Fact 1.1.20. *Let $\mathcal{M}$ and $\mathcal{N}$ be two structures, and assume that $\mathcal{N}$ is interpretable on a unary set in $\mathcal{M}$, then* $\mathrm{bdn}(\mathcal{N}) \leq \mathrm{bdn}(\mathcal{M})$. *In particular, if $\mathcal{M}$ and $\mathcal{N}$ are bi-interpretable on unary sets, then* $\mathrm{bdn}(\mathcal{M}) = \mathrm{bdn}(\mathcal{N})$.

For example, $\{\mathbb{Z}, 0, +, <\}$ does not interpret

$$\{\mathbb{Z} \times \mathbb{Z}, (\mathbb{Z}, 0, +, <), \pi_1 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}, \pi_2 : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}\}$$

on a unary set (the first being of burden 1, the second being of burden at least 2). However, if k is an imperfect field, we will see that k interprets $\{k \times k, (k, 0, 1, +, \cdot), \pi_1 : k \times k \to k, \pi_2 : k \times k \to k\}$ on a unary set.

Mutually-indiscernible arrays

Let λ be a cardinal. We have seen that an array $(b_{i,j})_{i<\lambda, j<\omega}$ occurs in the definition of an inp-pattern. A usual argument shows that we often may assume this array to be mutually-indiscernible. Since many demonstrations will use this fact, we give here a precise definition and a proof (Proposition 1.1.24). One can also consult [14], which is the reference for this paragraph. Let L be any first order language, $\mathcal{M}$ a L-structure of base set M.

Definition 1.1.21.
- A sequence $(b_j)_{j\in\lambda}$ of (tuples of) elements of M is indiscernible over a subset $A \subset M$ if for every $n \in \mathbb{N}$ and every formula $\phi(x_0, \dots, x_{n-1}, a)$ with parameters $a \in A$, we have

  $$\mathcal{M} \models \phi(b_0, \dots, b_{n-1}, a) \Leftrightarrow \phi(b_{j_0}, \dots, b_{j_{n-1}}, a)$$

  for every $j_0 < \dots < j_{n-1} \in \lambda$.

- An array $(b_{i,j})_{i\in\lambda,j\in\omega}$ is mutually indiscernible if every line $(b_{i,j})_{j\in\omega}$ is indiscernible over $\{b_{k,j}\}_{k\neq i,k\in\lambda,j<\omega}$.

We consider the set L_λ consisting of formulas in the language L with free variables $(x_{i,j})_{i<\lambda,j<\omega}$. Let $\Delta \subseteq L_\lambda$ be a subset.

Definition 1.1.22.
- An array $(b_{i,j})_{i\in\lambda,j\in\omega}$ is said to be Δ-*mutually indiscernible* if for any formula in Δ

$$\phi(x_{\alpha_1,1},\ldots,x_{\alpha_1,n_1},\ldots,x_{\alpha_k,1},\ldots,x_{\alpha_k,n_k})$$

  where $k\in\mathbb{N}$, $n_1,\ldots,n_k\in\mathbb{N}$ and $\alpha_1<\cdots<\alpha_k<\lambda$, we have

$$\begin{aligned}&\phi(b_{\alpha_1,i_{11}},\ldots,b_{\alpha_1,i_{1n_1}},\ldots,b_{\alpha_k,i_{k1}},\ldots,b_{\alpha_k,i_{kn_k}})\\ \Leftrightarrow\ &\phi(b_{\alpha_1,j_{11}},\ldots,b_{\alpha_1,j_{1n_1}},\ldots,b_{\alpha_k,j_{k1}},\ldots,b_{\alpha_k,j_{kn_k}}),\end{aligned}$$

  for all $i_{l1}<\cdots<i_{ln_l}<\omega,\ j_{l1}<\cdots<j_{ln_l}<\omega$ with $l=1,\ldots,k$.

- Let $(b_{i,j})_{i\in\lambda,j\in\omega}$ be an array. We denote by $\mathrm{EM}((b_{i,j}))$ the maximal set of formulas $\Delta\subset L_\lambda$ such that $(b_{i,j})$ is Δ-mutually indiscernible and $(b_{i,j})\models\Delta$. By this, we mean that, for any formula $\phi((x_{i,j})_{i\in\lambda,j\in\omega})$ in Δ, we have $\models\phi((b_{i,j})_{i\in\lambda,j\in\omega})$. Explicitly, $\mathrm{EM}((b_{i,j}))$ is equal to

$$\bigcup_{\substack{k,n_1,\ldots,n_k<\omega\\ \alpha_1,\ldots,\alpha_k<\lambda}}\left\{\phi(x_{\alpha_1,1}\ldots,x_{\alpha_1,n_1},\ldots,x_{\alpha_k,1},\ldots,x_{\alpha_k,n_k})\in L_\lambda\ \middle|\ \begin{array}{l}\phi(b_{\alpha_1,i_{11}},\ldots,b_{\alpha_1,i_{1n_1}},\ldots,b_{\alpha_k,i_{k1}}\ldots b_{\alpha_k,i_{kn_k}}),\\ \text{for all } i_{l1}<\cdots<i_{ln_l}<\omega,\ \text{with } l=1,\ldots,k\end{array}\right\}.$$

  Note in particular that if $(b_{i,j})$ is mutually indiscernible, $\mathrm{EM}((b_{i,j}))$ is a complete type.

- We also define the type $\mathrm{MI}((x_{\alpha,i})_{\alpha<\lambda,i<\omega})$ as the type saying that the array $(x_{\alpha,i})_{\alpha<\lambda,i<\omega}$ is mutually indiscernible:

$$\bigcup_{\substack{k,n<\omega\\ \alpha_1,\ldots,\alpha_k<\lambda}}\left\{\begin{array}{c}\phi(x_{\alpha_1,i_{11}},\ldots,x_{\alpha_1,i_{1n}},\ldots,x_{\alpha_k,i_{k1}}\ldots x_{\alpha_k,i_{kn}})\Leftrightarrow\\ \phi(x_{\alpha_1,j_{11}},\ldots,x_{\alpha_1,j_{1n}},\ldots,x_{\alpha_k,j_{k1}}\ldots x_{\alpha_k,j_{kn}})\end{array}\ \middle|\ \begin{array}{c}\phi((x_{\alpha,i})_{\alpha<\lambda,i<\omega})\in L_\lambda;\\ i_{l1}<\cdots<i_{ln}<\omega,\ j_{l1}<\cdots<j_{ln}<\omega\\ \text{with } l=1,\ldots,k\end{array}\right\}.$$

Using Ramsey, one can show the following lemma:

Lemma 1.1.23. *[15, Lemma 3.5] Let $(b_{\alpha,i})_{\alpha<n,i<\omega}$ be an array with $n < \aleph_0$, N an integer and Δ a* finite *subset of* L_n*. Then one can find a sub-array $(b_{\alpha,i_{\alpha 0}}, \ldots, b_{\alpha,i_{\alpha N-1}})_{\alpha<n} \subset (\bar{b}_\alpha)_{\alpha<n}$ which is Δ-mutually indiscernible.*

To be precise, our notion of Δ-indiscernibility (with $\Delta \subset \mathrm{L}_n$) is weaker than in [15], so the above Lemma is formally only a consequence of [15, Lemma 3.5].

Proposition 1.1.24.

- *Let $(b_{i,j})_{i<\lambda,j<\omega}$ be an array. There is a mutually indiscernible array $(\tilde{b}_{i,j})_{i<\lambda,j<\omega}$ such that* $\mathrm{EM}((b_{i,j})) \subseteq \mathrm{EM}((\tilde{b}_{i,j}))$.
- *If $p(x)$ is a partial type and if $\{\phi_i(x, y_i), (b_{i,j})_{j\in\omega}, k_i\}_{i<\lambda}$ is an inp-pattern in $p(x)$, there is an inp-pattern $\{\phi_i(x, y_i), (\tilde{b}_{i,j})_{j\in\omega}, k_i\}_{i<\lambda}$ in $p(x)$ with a mutually indiscernible array $(\tilde{b}_{i,j})_{i<\lambda,j<\omega}$.*

Proof. The first point follows by Compactness and Lemma 1.1.23, as the type

$$\mathrm{EM}((\bar{b}_i)_{i<\lambda})) \bigcup \mathrm{MI}((\bar{y}_i)_{i<\lambda}))$$

is finitely realised by sub-arrays of $(\bar{b}_i)_{i<\lambda}$. The second point follows easily from the first. ☐

Lemmas on inp-patterns

Let $\{\phi_i(x, y_i), (b_{i,j})_{j\in\omega}, k_i\}_{i<\lambda}$ be an inp-pattern. By Proposition 1.1.24, we may always assume that the array $(b_{i,j})_{i<\lambda,j\in\omega}$ is mutually indiscernible. We will now present some easy lemmas, which we will later use. They give us tools to 'transform' inp-patterns into simpler ones which are easier to analyse.

Lemma 1.1.25. *Let $\{\phi_i(x, y_i), (a_{i,j})_{j<\omega}, k_i\}_{i<\lambda}$ be an inp-pattern with $(a_{i,j})_{i<\lambda,j<\omega}$ mutually indiscernible. Assume for every $i < \lambda$, $\phi_i(x, a_{i,0})$ is equivalent to some formula $\psi_i(x, b_{i,0})$ with parameter $b_{i,0}$. Then we may extend $(b_{i,0})_{i<\lambda}$ to an mutually indiscernible array $(b_{i,j})_{i<\lambda,j<\omega}$ such that*

$$\{\psi_i(x, y_i), (b_{i,j})_{j<\omega}, k_i\}_{i<\lambda},$$

is an inp-pattern.

Proof. By 1-indiscernibility, we find $b_{i,j}$ such that $\phi_i(\mathbb{M}, a_{i,j}) = \psi_i(\mathbb{M}, b_{i,j})$. Then, the statement is clear. ☐

Remark 1.1.26. *Let D be a stably embedded definable set in $\mathbb{M}$, and $\{\phi_i(x, y_{i,j}), (a_{i,j})_{j<\omega}, k_i\}_{i<\lambda}$ an inp-pattern in D. This in particular implies that solutions of paths can be found in D but the parameters $(a_{i,j})$ may not belong to D. Using the previous lemma, we may actually assume that this is the case. It follows that D endowed with the induced structure is at least of burden λ.*

The next lemma shows that one can 'eliminate' disjunction symbols in inp-patterns. A direct consequence is that if the theory has quantifier elimination, then we may assume that formulas of inp-patterns are conjunctions of atomic and negation of atomic formulas.

Lemma 1.1.27. *Let $\{\phi_i(x, y_{i,j}), (a_{i,j})_{j<\omega}, k_i\}_{i<\lambda}$ be an inp-pattern with $(a_{i,j})_{i<\lambda, j<\omega}$ mutually indiscernible. Assume that $\phi_i(x, y_{i,j}) = \bigvee_{l \leq n_i} \psi_{l,i}(x, y_{i,j})$. Then there exists a sequence of natural numbers $(l_i)_{i<\lambda}$ such that $l_i \leq n_i$ and*

$$\{\psi_{l_i,i}(x, y_{i,j}), (a_{i,j})_{j<\omega}, k_i\}_{i<\lambda}$$

is an inp-pattern.

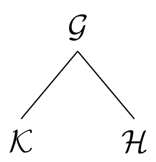
Proof. Let $d \models \{\phi_i(x, a_{i,0})\}_{j<\lambda}$. For every $i < \lambda$, let $l_i \leq n_i$ be such that $d \models \psi_{l_i,i}(x, a_{i,0})$. By the mutual-indiscernibility of $(a_{i,j})_{i<\lambda, j<\omega}$, every path of the pattern $\{\psi_{l_i,i}(x, y_{i,j}), (a_{i,j})_{j<\omega}, k_i\}_{i<\lambda}$ is consistent. The inconsistency of the rows follows immediately from the inconsistency of the rows of the initial pattern. □

'Elimination' of conjunction symbols may happen in more specific context. Notably:

Proposition 1.1.28. *Let $\mathcal{K}$ and $\mathcal{H}$ be two structures, and consider the multisorted structure $\mathcal{G}$:*

$$\mathcal{G} = \{K \times H, \mathcal{K}, \mathcal{H}, \pi_K : K \times H \to K, \pi_H : K \times H \to H\},$$

called the direct product structure *(where π_K and π_H are the natural projections). Then $\mathcal{G}$ eliminates quantifiers relative to $\mathcal{K}$ and $\mathcal{H}$, and $\mathcal{K}$ and $\mathcal{H}$ are orthogonal and stably embedded within $\mathcal{G}$.*



We have

$$\mathrm{bdn}(\mathcal{G}) = \mathrm{bdn}(\mathcal{K}) + \mathrm{bdn}(\mathcal{H}).$$

We prove an obvious generalisation for product of more than two structures in the next subsection.

Proof. Relative quantifier elimination, stable embeddedness and orthogonality are rather obvious. The inequality $\text{bdn}(\mathcal{G}) \geq \text{bdn}(\mathcal{K}) + \text{bdn}(\mathcal{H})$ is easy but we give a detailed proof. Let $\{\phi_i(x_K, y_i), (a_{i,j})_{j<\omega}\}_{i\in\lambda_1}$ be an inp-pattern in $\mathcal{K}$ and $\{\psi_i(x_H, y_i), (b_{i,j})_{j<\omega}\}_{i\in\lambda_2}$ an inp-pattern in $\mathcal{H}$. Then

$$\{\phi_i(\pi_K(x_K, x_H), y_i), (a_{i,j})_{j<\omega}\}_{i\in\lambda_1} \cup \{\psi_i(\pi_H(x_K, x_H), y_i), (b_{i,j})_{j<\omega}\}_{i\in\lambda_2}$$

is an inp-pattern in $\mathcal{G}$ of depth $\lambda_1 + \lambda_2$. Indeed, first notice that inconsistency of each rows is clear. Secondly, take a path $f : \lambda_1 \sqcup \lambda_2 \to \omega$. There is a element $d_K \in K$ satisfying $\{\phi_i(x_K, a_{i,f(i)})\}_{i\in\lambda_1}$ and an element $d_H \in H$ satisfying $\{\psi_i(x_H, b_{i,f(i)})\}_{i\in\lambda_2}$. Then, the element $d = (d_K, d_H)$ of $\mathcal{G}$ is a solution of the pattern along the path f.

For the other inequality, let $\{\theta_i(x, y_{i,j}), (c_{i,j})_{j<\omega}, k_i\}_{i<\lambda}$ be an inp-pattern in $\mathcal{G}$, with $(c_{i,j})_{i<\lambda,j<\omega}$ mutually indiscernible. We may assume $\theta_i(x, c_{i,j})$ is of the form $\phi_i(x_K, a_{i,j}) \wedge \psi_i(x_H, b_{i,j})$ where $x_K = \pi_K(x)$, $x_H = \pi(x_H)$, $c_{i,j} = a_{i,j} {}^\frown b_{i,j}$, $\phi_i(x_K, a_{i,j})$ is a $\mathcal{K}$-formula and $\psi_i(x_H, b_{i,j})$ is a $\mathcal{H}$-formula. Indeed, let $d \models \{\theta_i(x, c_{i,0})\}_{j<\lambda}$, by orthogonality, $\theta_i(x, c_{i,0})$ is equivalent to a formula of the form:

$$\bigvee_{k<n_i} \phi_{i,k}(x_K, a_{i,0}) \wedge \psi_{i,k}(x_H, b_{i,0}).$$

Then we conclude by using Lemmas 1.1.25 and 1.1.27. For every i, at least one of the sets $\{\phi_i(x_K, a_{i,j})\}_{j<\omega}$ and $\{\psi_i(x_H, b_{i,j})\}_{j<\omega}$ is k_i-inconsistent (by indiscernibility of $(c_{i,j})_j$). We may "eliminate" the conjunction as well and assume that every line is an L_K-formula or an L_H-formula. We conclude that $\lambda \leq \text{bdn}(\mathcal{K}) + \text{bdn}(\mathcal{H})$. □

Together with Fact 1.1.20, we get more generally:

Fact 1.1.29. *Let $M = (A, C, \dots)$ be a many-sorted structure. Assume that A and C are orthogonal and stably embedded in M. Then we have* $\text{bdn}(A\times C) = \text{bdn}(A) + \text{bdn}(C)$.

Let us finish this subsection with one more lemma:

Lemma 1.1.30. *Let D and D' two type-definable sets respectively given by the partial types $p(x)$ and $p'(x)$ and let $f : D \to D'$ be a surjective finite to one type-definable function. Then we have* $\text{bdn}(D) := \text{bdn}(p(x)) = \text{bdn}(p'(x)) =: \text{bdn}(D')$.

Proof. We may assume that D and D' are definable, the general case can be similarly deduced. Let $\{\phi'_i(x', y_i), (a_{i,j})_{j<\omega}, k_i\}_{i<\lambda}$ be an inp-pattern in D'. Clearly, $\{\phi'_i(f(x), y_i), (a_{i,j})_{j<\omega}, k_i\}_{i<\lambda}$ is an inp-pattern in D. Hence

$\mathrm{bdn}(D) \geq \lambda$. Conversely, let $\{\phi_i(x, y_i), (a_{i,j})_{j<\omega}, k_i\}_{i<\lambda}$ be an inp-pattern of depth λ in D. Consider the pattern

$$\{\phi'_i(x', y_{i,j}), (a_{i,j})_{j<\omega}\}_{i<\lambda},$$

where

$$\phi'_i(x', a_{i,j}) \quad \equiv \quad \exists x\ x \in D \wedge x' = f(x) \wedge \phi(x, a_{i,j}).$$

Clearly every path is consistent. Assume for some $i < \lambda$, the row $\{\phi'_i(x', a_{i,j})\}_{j<\omega}$ is consistent, witnessed by some h'. Note that h' is in D'. By the pigeonhole principle, there is $h \in D$ and an infinite subset J of ω such that $f(h) = h'$ and $h \models \{\phi_i(x, a_{i,j})\}_{j\in J}$, contradiction. It follows that $\{\phi_i(x', y_{i,j}), (a_{i,j})_{j<\omega}\}_{i<\lambda}$ is an inp-pattern in D'. We conclude that $\mathrm{bdn}(D') \geq \lambda$. □

More on burden and strength

We will formally introduce a well known convention with respect to the burden, which consists of writing $\mathrm{bdn}(\mathcal{M}) = \lambda_-$ for a limit cardinal λ if $\mathcal{M}$ admits inp-patterns of depth μ for all $\mu < \lambda$, but no inp-pattern of depth λ. It has been introduced in [1], and has the advantage to emphasising a relevant distinction. If the reader is not interested by such subtleties, they may move to the next subsection. Proposition 1.1.38 might be interesting on its own, as it corresponds to the 'baby case' for the difficulty that we will encounter for mixed-characteristic Henselian valued fields. One can refer to [1] for this paragraph.

Definition 1.1.31. We define the ordered class $(\mathrm{Card}^\star, <)$ as the linear order obtained from the ordered class of cardinals $(\mathrm{Card}, <)$ by adding for any limit cardinal λ a new element λ_- (called 'lambda minus'). This new element comes immediately before λ: $\lambda_- < \lambda$ and if $\mu \in \mathrm{Card}^\star$ with $\mu < \lambda$, then $\mu \leq \lambda_-$. In addition to the natural injection $\mathrm{Card} \hookrightarrow \mathrm{Card}^\star$, we define the *actualisation map* $\mathrm{act} : \mathrm{Card}^\star \to \mathrm{Card}$ as the map such that $\mathrm{act}(\lambda_-) = \lambda$ for every limit cardinal λ, and $\mathrm{act}(\kappa) = \kappa$ for any cardinal $\kappa \in \mathrm{Card}$. It will be convenient to also set $\kappa_- = \lambda$ when $\kappa = \lambda^+ \in \mathrm{Card}$ is a successor cardinal.

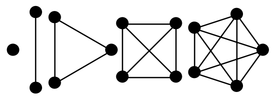
If λ is a limit cardinal, one should think λ as an 'actual' lambda and λ_- as a 'potential' lambda. We don't change our notion of cardinality of a set. As we will see, this definition of $\mathrm{Card}^\star$ is motivated by the burden, *i.e.* by a notion of dimension. It also motivates to (partially) extend the arithmetic operations of Card to $\mathrm{Card}^\star$. We will have to answer any question of the form: should the cardinal $\aleph_0 \cdot \aleph_{\omega-}$ be $\aleph_{\omega-}$ or $\aleph_\omega$? As the definitions themselves appear to be a bit technical, we prefer to first give intuition to the reader with a small digression on graphs.

Graphs and cliques

We consider symmetric graphs in the language $\mathrm{L} = \{R\}$. We denote by K_κ the complete graph on κ-many vertices, for κ a cardinal in Card. Given a graph $\mathcal{G}$, we denote by $C(\mathcal{G})$ the cardinal in $\mathrm{Card}^\star$:

$$C(\mathcal{G}) = \begin{cases} \kappa \in \mathrm{Card} \ \text{ if } K_\kappa \text{ embeds in } \mathcal{G} \text{ and } K_{\kappa^+} \text{ does not,} \\ \kappa_- \in \mathrm{Card}^\star \ \text{ if } K_\lambda \text{ embeds in } \mathcal{G} \text{ for all cardinal } \lambda < \kappa \text{ and } K_\kappa \text{ does not.} \end{cases}$$

Example. Let $\mathcal{G}$ be the disjoint union of graphs $\cup_{n<\aleph_0} K_n$:

$\mathcal{G} :=$ 

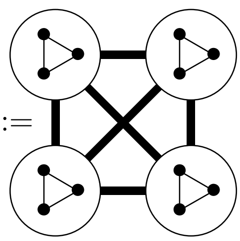
By definition, we have $C(\mathcal{G}) = \aleph_{0-}$.

In addition to the union of graphs, we want to consider another natural operation:

Definition 1.1.32. We define the lexicographic product of graphs $\mathcal{G}$ and $\mathcal{F}$ as the graph $\mathcal{G}[\mathcal{F}]$ with set of vertices $G \times F$ and a symmetric relation given by:

$$(g_0, f_0) R^{\mathcal{G}[\mathcal{F}]} (g_1, f_1) \ \Leftrightarrow \begin{cases} g_0 = g_1 \ \text{ and } f_0 R^{\mathcal{F}} f_1, \\ g_0 \neq g_1 \ \text{ and } g_0 R^{\mathcal{G}} g_1. \end{cases}$$

Example. Consider the lexicographic product of K_4 and K_3. We simply obtain K_{12}:

$K_4[K_3] :=$ 

If $C(\mathcal{G}), C(\mathcal{F}) \in \mathrm{Card}$ are cardinals greater or equal to 2, we have by pigeonhole principle that

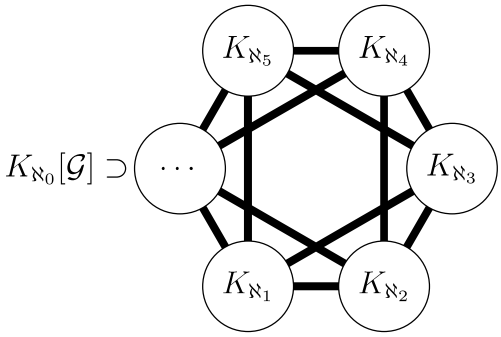
$$C(\mathcal{G}[\mathcal{F}]) = C(\mathcal{G}) \times C(\mathcal{F}).$$

This gives us the intuition of how one can define the product of cardinals in $\mathrm{Card}^\star$. Let us look at two examples:

Example. Consider $\mathcal{G} = \cup_{n<\omega} K_{\aleph_n}$ and $\mathcal{F} = \cup_{\alpha<\omega_1} K_{\aleph_\alpha}$. Then we have $C(\mathcal{G}) = \aleph_{\omega-}$ and $C(\mathcal{F}) = \aleph_{\omega_1-}$. If we consider the lexicographic product with $K_{\aleph_0}$, we obtain:

- $C(K_{\aleph_0}[\mathcal{G}]) = \aleph_\omega$,
- $C(K_{\aleph_0}[\mathcal{F}]) = \aleph_{\omega_1-}$.

We leave the proof to the reader, with the following picture for the intuition:



As a consequence, one might be tempted to write $\aleph_0 \cdot \aleph_{\omega-} = \aleph_\omega$ and $\aleph_0 \cdot \aleph_{\omega_1-} = \aleph_{\omega_1-}$. This is what we want to define now.

Arithmetic on Card*

We first define the cofinality of a cardinal λ in Card* as the cofinality of act(λ), denoted by cf(λ). Secondly, we define the following operations:

Definition 1.1.33. Let $\Lambda = (\lambda_i)_{i\in I}$ be a sequence in Card*. Let $\lambda = \sup_{i\in I}(\mathrm{act}(\lambda_i)) \in$ Card be the supremum in the usual sense, and $\mathrm{supp}(\Lambda) = \{i \in I \mid \lambda_i \neq 0\}$. We find a partition $I_1 \cup I_2 \cup I_3$ of I such that:

$$\Lambda = (\lambda_i)_{i\in I_1} \cup (\lambda_-)_{i\in I_2} \cup (\lambda)_{i\in I_3},$$

where $\lambda_i < \lambda_-$ for $i \in I_1$. We define $\sup^\star$ as follows:

- $\sup^\star_{i\in I}(\lambda_i) = \begin{cases} \lambda \text{ if } |I_3| \neq \emptyset. \\ \lambda_- \text{ otherwise.} \end{cases}$

If $|I|$ and λ are finite, the definition of the sum $\sum^\star$ in Card* is the sum in the usual sense:

$$\sum_{i\in I}^\star \lambda_i = \sum_{i\in I} \mathrm{act}(\lambda_i) = \sum_{i\in I} \lambda_i.$$

Otherwise, we set:

- $$\sum^{\star}_{i\in I}\lambda_i = \begin{cases} |\operatorname{supp}(\Lambda)| \text{ if } |\operatorname{supp}(\Lambda)| \geq \lambda, & \\ \begin{cases} \lambda \text{ if } I_3 \neq \emptyset, \\ \lambda \text{ if } |I_2| \geq \operatorname{cf}(\lambda), \\ \lambda \text{ if } \sup_{i\in I_1}(\operatorname{act}(\lambda_i)) = \lambda, \\ \lambda_- \text{ otherwise.} \end{cases} & \text{if } |\operatorname{supp}(\Lambda)| < \lambda. \end{cases}$$

For $\lambda, \mu \in \mathrm{Card}$, λ limit cardinal, we define the product $\cdot^\star : \mathrm{Card} \times \mathrm{Card}^\star \to \mathrm{Card}^\star$ in terms of sum:

- $$\sum^{\star}_{\mu}\lambda_- = \mu \cdot^\star \lambda_- = \begin{cases} \lambda_- \text{ if } \mu < \operatorname{cf}(\lambda), \\ \mu \cdot \lambda \text{ if } \mu \geq \operatorname{cf}(\lambda). \end{cases}$$

We see in particular that, under these definition, $\sup^\star$ and $\sum^\star$ do not necessary coincide anymore when there are infinite. However, it is clear that we recover the usual definition via the actualisation map:

$$\begin{array}{ccc} \mathrm{Card}^{\star|I|} & \xrightarrow{\sum^\star} & \mathrm{Card}^\star \\ \downarrow{\scriptstyle \mathrm{act}} & & \downarrow{\scriptstyle \mathrm{act}} \\ \mathrm{Card}^{|I|} & \xrightarrow{\sum} & \mathrm{Card} \end{array} \qquad \begin{array}{ccc} \mathrm{Card}^{\star|I|} & \xrightarrow{\sup^\star} & \mathrm{Card}^\star \\ \downarrow{\scriptstyle \mathrm{act}} & & \downarrow{\scriptstyle \mathrm{act}} \\ \mathrm{Card}^{|I|} & \xrightarrow{\sup} & \mathrm{Card} \end{array}$$

Here are the promised examples:

Examples.

- Consider the sequence $\Lambda_1 = \aleph_{\omega-}, 1, 2, 3, \ldots$. We have $\sup^\star \Lambda_1 = \sum^\star \Lambda_1 = \aleph_{\omega-}$.
- Consider the sequences $\Lambda_2 = (\aleph_{\omega-})_{i<\omega} = \aleph_{\omega-}, \aleph_{\omega-}, \ldots$ and $\Lambda_3 = (\aleph_i)_{i<\omega} = \aleph_0, \aleph_1, \ldots$. We have $\sup^\star \Lambda_2 = \sup^\star \Lambda_3 = \aleph_{\omega-}$ and $\sum^\star \Lambda_2 = \sum^\star \Lambda_3 = \aleph_\omega$.
- Consider $\Lambda_4 = (\aleph_i)_{i<\omega} \cup (\aleph_{2\omega-})$. Then $\sup^\star \Lambda_4 = \sum^\star \Lambda_4 = \aleph_{2\omega-}$.
- We have $\aleph_0 \cdot \aleph_{\omega-} = \aleph_\omega$, $\aleph_0 \cdot \aleph_{\omega_1-} = \aleph_{\omega_1-}$ and $\aleph_1 \cdot \aleph_{\omega_1-} = \aleph_{\omega_1}$.

Now, we go back to the burden.

Burden, strength and $\mathrm{Card}^\star$

In Definition 1.1.15, the burden of the complete theory T is the supremum (in $\mathrm{Card} \cup \{\infty\}$) of depth of inp-patterns in T. However this supremum is not necessarily attained by an actual inp-pattern. This distinction is in particular motivated by the following definition:

Definition 1.1.34 ([1])**.** A complete theory is called *strong* if there is no inp-pattern of infinite depth in T.

One sees that, paradoxically, some strong theories have burden $\aleph_0$ and some theories of burden $\aleph_0$ are not strong (see examples below). In other words, the definition of burden we gave failed to characterize strength. We will follow Adler's convention (see [1]) which gives a solution to this problem: burden will now take values in $\mathrm{Card}^\star \cup \{\infty\}$.

Definition 1.1.35. (second definition of burden) Let T be a complete theory. We denote by $\mathcal{S}$ the set of sorts.

- The burden $\mathrm{bdn}(\pi(x))$ of a partial type $\pi(x)$ is the supremum in $\mathrm{Card}^\star \cup \{\infty\}$ of the depths of inp-patterns in $p(x)$.
- The cardinal $\sup_{S\in\mathcal{S}} \mathrm{bdn}(\{x_S = x_S\})$ where x_S is a single variable from the sort S, is called the *burden* of the theory T, and it is denoted by $\kappa^1_{\mathrm{inp}}(T)$ or by $\mathrm{bdn}(T)$.

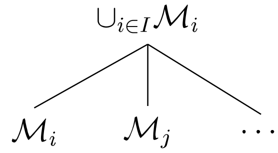
In other words, if the supremum $\lambda \in \mathrm{Card}$ of depth of inp-patterns is attained, the burden is equal to λ . Otherwise, it is equal to λ_-. In particular, strong theories are exactly theories of burden at most $\aleph_{0-}$. One can check that every lemma in the previous subsection -and its proof- still hold. Let us give a formal definition:

Definition 1.1.36. Let $\mathcal{M}_i = (M_i, \ldots)$ be a structure in a language L_i, for $i \in I$ a set of indices. We define the following multisorted structure:

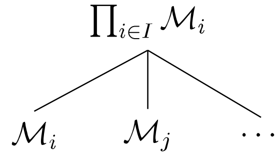
- The *disjoint union*
$$\bigcup_i \mathcal{M}_i = \{(M_i, \ldots)\}_{i\in I}.$$
with a sort for each $\mathcal{M}_i$'s.
- The *direct product*
$$\prod_{i\in I} \mathcal{M}_i = \{\prod_{i\in I} M_i, (M_i, \ldots)_{i\in I}, (\pi_i : \prod_{j\in I} M_j \to M_i)_{i\in I}\},$$
with a sort for each $\mathcal{M}_i$'s and a sort for the product and where $\pi_i : \prod_{j\in I} M_j \to M_i$ is the natural projection.

We have the following fact:

Fact 1.1.37.
- *The sorts $\mathcal{M}_i$ in the union $\cup_{i\in I}\mathcal{M}_i$ are stably embedded and pairwise orthogonal.*



- *The direct product* $\prod_{i\in I}\mathcal{M}_i$ *eliminates quantifiers relative to the sorts* $\mathcal{M}_i$. *In particular, the sorts* $\mathcal{M}_i$ *are stably embedded, and setwise orthogonal.*



Proof. The first point is easy and is solved by simple inspection on formulas. For the second point, we leave to the reader to prove quantifier elimination.

Stable embeddedness is clear by inspection: a formula $\phi(x_i)$ with variable $x_i \in \mathcal{M}_i$ without M-sorted quantifiers is a finite Boolean combination of formulas of the form

$$\phi_i(x_i, a_i, \pi_i(a)) \cup \bigcup_{j\in I\setminus\{i\}} \phi_j(a_j, \pi_j(a)) \cup \phi(a),$$

where ϕ_i is an L_i-formula, $\phi_j(a_j, \pi_j(a))$ are closed L_j-formula $j \in I \setminus \{i\}$ and $\phi(a)$ is a closed formula in the empty language. It is clearly equivalent to an L_i-formula with parameters in $\mathcal{M}_i$, as a closed formula is true or false and can be replaced either by $x_i = x_i$ or by $x_i \neq x_i$. Same argument for orthogonality.

□

Naturally, we have a generalisation of Proposition 1.1.28 for infinite products.

Proposition 1.1.38. *Let* $\mathcal{M}_i = (M_i, \dots)$ *be a structure in a language* L_i, *for* $i \in I$ *a set of indices. Assume they are not all finite. One has:*

- $\mathrm{bdn}(\bigcup_{i\in I}\mathcal{M}_i) = \sup^{\star}_{i\in I}\mathrm{bdn}(M_i)$,
- $\mathrm{bdn}(\prod_{i\in I}\mathcal{M}_i) = \sum^{\star}_{i\in I}\mathrm{bdn}(M_i)$.

Remark 1.1.39. *If all structures* $\mathcal{M}_i$ *are finite, there are two cases: either* $\#\{i \in I \mid |M_I| > 1\}$ *is infinite and* $\mathrm{bdn}(\prod_{i\in I}\mathcal{M}_i) = 1$, *or* $\#\{i \in I \mid |M_I| > 1\}$ *is finite and* $\mathrm{bdn}(\prod_{i\in I}\mathcal{M}_i) = 0$. *As the condition* $|M_i| > 1$ *cannot be seen in terms of burden, this case is told apart.*

Proof. The first point is clear: an inp-pattern $P(x)$ in $\bigcup_i \mathcal{M}_i$ has to 'choose' in which sort $\mathcal{M}_i$ its variable x lives. This sort, say $\mathcal{M}_{i_0}$, is stably embedded by Fact 1.1.37. By Remark 1.1.26, the depth of $P(x)$ is bounded by $\mathrm{bdn}(\mathcal{M}_{i_0})$. Going to the supremum, one sees that definitions match:

$$\mathrm{bdn}(\bigcup_{i\in I} \mathcal{M}_i) = \sup^{\star}_{i\in I} \mathrm{bdn}(M_i).$$

The second point is more subtle: if $Q(x)$ is an inp-pattern of depth μ in $\prod_{i\in I} \mathcal{M}_i$, with the variable x in the main sort, then the pattern refers to the sorts $\mathcal{M}_i$ simultaneously.

Claim 1. *Assume that $\prod_{i\in I} \mathcal{M}_i$ admits an inp-pattern of depth μ. Then there is an inp-pattern of depth μ in $\prod_{i\in I} \mathcal{M}_i$ of the following form:*

$$\{\phi_\alpha(\pi_{f(\alpha)}(x), y_{f(\alpha)}), (a_{f(\alpha),j})_{j<\omega}\}_{\alpha<\mu},$$

for some function $f : \mu \to I$ and where $\phi_\alpha(x_{f(\alpha)}, y_{f(\alpha)})$ is a $\mathcal{M}_{f(\alpha)}$-formula. In other word, we may assume that a line α "mentions" only one structure $\mathcal{M}_i$.

Proof. Let us assume that $\prod \mathcal{M}_i$ admits an inp-pattern $Q(x) = \{\psi_\alpha(x, \bar{y}_\alpha), (\bar{a}_{\alpha,j})_{j<\omega}, k_\alpha\}_{\alpha<\mu}$ of depth $\mu \geq 2$. We assume the array $(\bar{a}_{\alpha,j})_{\alpha<\mu, j<\omega}$ to be mutually indiscernible. To simplify the notation, a generic line of $Q(x)$ is denoted by $\{\psi(x, \bar{y}), (\bar{a}_j)_{j<\omega}, k\}$ (we drop the index α). By relative quantifier elimination and by Lemma 1.1.27, we may assume that formulas $\psi(x, \bar{y})$ in $Q(x)$ are of the form

$$\bigwedge_{n<N} x \neq y_n \wedge x = y \wedge \bigwedge \phi_i(\pi_i(x), y_i),$$

where $\phi_i(x, y_i)$ are L_i-formulas, $N \in \mathbb{N}$ and where $\bar{y} = (y_1, \ldots, y_N, y) \cup (y_i)_{i\in I}$ and $\bar{a}_j = (a_{1j}, \ldots, a_{Nj}, a_j) \cup (a_{ij})_{i\in I}$ for $j < \omega$. If the atomic formula $x = y$ does occur, for example in the first row, then consistency of paths contradicts k_2-inconsistency of the second row. Thus, we knows that formulas in $Q(x)$ are of the form

$$\bigwedge_{n<N} x \neq y_n \wedge \bigwedge \phi_i(\pi_i(x), y_i),$$

Now, the formula $\bigwedge_{n<N} x \neq y_n$ is co-finite. This implies that

$$\{\bigwedge \phi_i(\pi_i(x), a_{i,j})\}_{j<\omega}$$

is $k+1$-inconsistent. Indeed, otherwise, for one (equivalently for all) $k+1$-increasing tuple $j_0 < \cdots < j_k < \omega$, the set

$$\left\{\bigwedge \phi_i(\pi_i(x), a_{i,j_1}), \ldots, \bigwedge \phi_i(\pi_i(x), a_{i,j_k})\right\}$$

is satisfied by a_{n,j_l} for some $n < N$ and $l \leq k$. Without loss of generality, assume that $n = N-1$ and $l = k$. Then, by mutual indiscernibility, $(a_{N-1,j})_{j \geq k}$ are solutions of

$$\left\{\bigwedge \phi_i(\pi_i(x), a_{i,0}), \ldots, \bigwedge \phi_i(\pi_i(x), a_{i,k-1})\right\}$$

This contradicts the k-inconsistency of the line

$$\left\{\bigwedge_{n<N} x \neq a_{n,j} \wedge \bigwedge \phi_i(\pi_i(x), a_{i,j})\right\}_{j<\omega},$$

unless $(a_{N-1,j})_{j<\omega}$ is constant. In that case, this parameter can be ignore: replace the formula by

$$\bigwedge_{n<N-1} x \neq y_n \wedge \bigwedge \phi_i(\pi_i(x), y_i),$$

and we still have an inp-pattern. We get our contradiction by induction on N. Hence, we may assume that formulas $\psi(x, \bar{y})$ in $Q(x)$ are of the form

$$\bigwedge \phi_i(\pi_i(x), y_i).$$

We may now conclude using mutual indiscernibility that for at least one $i =: f(\alpha)$, the set

$$\{\phi_i(\pi_i(x), a_{i,j})\}_{j<\omega}$$

is k-inconsistant. We may replace the formula $\bigwedge \phi_i(\pi_i(x), y_i)$ by $\phi_{f(\alpha)}(\pi_{f(\alpha)}, y_{f(\alpha)})$. In other word, we may assume that only the index $i = f(\alpha)$ occurs in the formula of the line α. We found an inp-pattern of the desired form. $\square$

We denote $\mathrm{bdn}(\mathcal{M}_i)$ by λ_i and $\sup_{i \in I} \mathrm{act}(\lambda_i) \in \mathrm{Card}$ by λ. One immediate corollary is that

$$\mathrm{bdn}(\prod_{i\in I} \mathcal{M}_i) = \mathrm{bdn}(\prod_{\substack{i\in I\\ \lambda_i \neq 0}} \mathcal{M}_i) \geq 1,$$

(notice that we used that some $\mathcal{M}_i$ is infinite). We may assume that $I = \mathrm{supp}(\lambda_i)_{i\in I}$. Now, the proof is straight forward and is just a case study. We distinguish six cases:

First case: the cardinals $|I|$ and λ are finite. Then, this is immediate from the previous claim: $\mathrm{bdn}(\prod \mathcal{M}_i) = \sum \lambda_i = \sum^\star \lambda_i$.

Second case: we have $|I| \geq \lambda$ and $|I| \geq \aleph_0$. Then, let $(b_{i,j})_{j<\omega}$ be a sequence of pairwise distinct elements of $\mathcal{M}_i$. Let x be a variable in the main sort. Then, $\{\pi_i(x) = y_i, (b_{i,j})_{j<\omega}\}_{i\in I}$ is an inp-pattern of depth $|I|$. We have $\mathrm{bdn}(\prod M_i) \geq |I|$. Reciprocally, assume $\prod \mathcal{M}_i$ admits an inp-pattern $Q(x)$ of depth $\mu > |I|$. By the previous claim and pigeonhole principle, we find an inp-pattern of depth μ in some $\mathcal{M}_i$, which is a contradiction with $\lambda \leq |I| < \mu$. We get $\mathrm{bdn}(\prod \mathcal{M}_i) = \sum_{i\in I}^\star \lambda_i = |I|$.

Third case: we have $|I| < \lambda$ and $\lambda_i = \lambda \geq \aleph_0$ for some $i \in I$. Then clearly $\mathrm{bdn}(\prod \mathcal{M}_i) \geq \lambda$. Again, by pigeonhole principle, one gets $\mathrm{bdn}(\prod \mathcal{M}_i) \leq \lambda$.

Fourth case: we have $|I| < \lambda$ and $\mathrm{cf}(\lambda) \leq \#\{i \in I \mid \lambda_i = \lambda_-\}$. Then, choose any sequence of cardinals $(\mu_\alpha)_{\alpha<\mathrm{cf}(\lambda)}$ with supremum λ (in the usual sense) and $\mu_\alpha < \lambda$ for all α. We can assume that $I = \mathrm{cf}(\lambda)$ and that we have an inp-pattern $Q_i(x_i)$ in $\mathcal{M}_i$ of depth μ_i. The inp-pattern $Q(x) = \cup_{i\in I} Q_i(\pi_i(x))$ is of depth λ. We get $\mathrm{bdn}(\prod \mathcal{M}_i) = \sum_{i\in I}{}^\star \lambda_i = \lambda$.

Fifth case: we have $\sup\{\mathrm{act}(\lambda_i) \mid \lambda_i \notin \{\lambda_-, \lambda\}\} = \lambda$. We conclude as in the previous case that $\mathrm{bdn}(\prod \mathcal{M}_i) = \sum_{i\in I} \lambda_i = \lambda$.

Last case: we are not in the above cases. Then, by the previous claim, there is no inp-pattern of depth λ in $\prod \mathcal{M}_i$. We have then

$$\mathrm{bdn}(\prod \mathcal{M}_i) = \sum_{i\in I}{}^\star \lambda_i = \lambda_-. \qquad \square$$

In a supersimple theory, the burden of a complete type is always finite (see [1]). Hence, supersimple theories are examples of strong theories.

Example. The following structures have burden $\aleph_{0-}$:

- Any union structure $M = \bigcup_n M_n$, where for every $n \in \mathbb{N}$, M_n is a structure of burden n.
- Any model of ACFA, the model companion of the theory of algebraically closed fields with an automorphism.
- Any model of DCF_0, the theory of differentially closed fields.

The first example is clear by the previous discussion but could look artificial. It will naturally appear when we will discuss the burden of the

$\mathrm{RV}_{<\omega}$-sort in mixed characteristic (see Section 2.4). The last one is already given in [15]. The fact that the last two examples are of burden $\aleph_{0-}$ follows from the fact they are super-simple and from the next remark, once we notice that such fields are infinite dimensional vector spaces over respectively their fixed field and their constant field:

Remark 1.1.40. *[15, Remark 5.3] Let T be a simple theory and assume there is a n-dimensional type-definable vector space V over a type-definable infinite field F. Then there is a type in V of burden $\geq n$.*

Let us look at one natural example of a non-strong theory:

Remark 1.1.41. *Let k be an imperfect field of characteristic p, considered as a structure in the language of fields. Then* $\mathrm{bdn}(k) \geq \aleph_0$.

Proof. Let $e_0, e_1 \in k$ be two linearly independent elements over k^p. Then, we have the definable injective map

$$\begin{array}{cccc} f_2: & k \times k & \to & k \\ & (a,b) & \mapsto & a^p e_0 + b^p e_1 \end{array}$$

By induction, we define for $n \geq 2$:

$$\begin{array}{cccc} f_{n+1}: & k^{(n+1)} & \to & k \\ & (a_0, \dots, a_{n-1}) & \mapsto & f_n(a_0, \dots, a_{n-3}, f_2(a_{n-2}, a_{n-1})). \end{array}$$

Then, consider the formula for $n \geq 0$:

$$\phi_n(x, y_n) \equiv \exists y_0, \dots, y_{n-1}, y_{n+1}\ x = f_{n+2}(y_0, \dots, y_{n+1}),$$

and pairwise distinct parameters $b_{n,j} \in k$, for $j < \omega$. Then

$$\{\phi_n(x, y_n), (b_{n,j})_{j<\omega}\}_{n<\omega}$$

is an inp-pattern of depth $\aleph_0$. □

Similarly, one can show that if a model $\mathcal{M}$ is bi-interpretable on a unary set with the direct product $\mathcal{M} \times \mathcal{M}$, then $\mathrm{bdn}\,\mathcal{M} \geq \aleph_0$ (and it is never of the form λ_- where λ is a cardinal of cofinality $\aleph_0$).

1.1.3 Definition of stable embeddedness

Introduction

Stable embeddeness is usually seen as a property of definable sets: it means that any definable subset can be expressed with a formula with parameters in the set. As we mentioned in the introduction, in the context of benign Henselian valued fields (see definition 1.2.21), the value group and the residue field are both stably embedded. One can see that the definition also makes sense for any subset of the structure (not necessary definable). We have for instance the following well known equivalence: a theory is stable if and only if all sets are stably embedded. Then, it automatically comes to mind to consider submodels, which play an important role in the understanding of the structure. As Cubides, Delon and Ye, we will consider the question of stably embedded elementary pairs of certain valued fields and establish a transfer principle. As we do not consider definable subsets, it appears natural to distinguish two notions of stable embedddness. The purpose of this paragraph is to present these concepts. Fortunately, this distinction will not bring any difficulties: the same proofs for our transfer principle holds *mutatis mutandis* for the uniform and non-uniform case.

Definitions of stable embeddness

Let L be any first order language, and T be an L-theory. We defined in Subsection 1.1.1 the notion of stable embeddedness for definable sets and sorts. We define it now for arbitrary sets:

Definition 1.1.42 (stable embeddedness for arbitrary sets)**.** Let $\mathcal{M}$ be an L-structure. A subset $S \subseteq M$ is said to be *stably embedded in* $\mathcal{M}$ if for every formula $\phi(x, y)$ and for every tuple of parameters $a \in M^{|y|}$, there is another formula $\psi(x, z)$ and a tuple of parameters $b \in S^{|z|}$ such that $\phi(S^{|x|}, a) = \psi(S^{|x|}, b)$. In this case, we write $S \subseteq^{st} \mathcal{M}$.

Notice that $\psi(x, z)$ may depend on the parameter $a \in M$. For this reason, this definition is not the usual one. It results from this 'non-uniformity' the following:

Remark 1.1.43. *Let $\mathcal{M} \preceq \mathcal{M}'$ be two* L*-structures, and $D(x)$ be a* L*-formula with $|x| = 1$. Assume that $D(M')$ is stably embedded in $\mathcal{M}'$. Then $D(M)$ is stably embedded in $\mathcal{M}$. However, the converse does not always hold i.e., it is possible to have $D(M)$ stably embedded in $\mathcal{M}$, but $D(M')$ to fail to be stably embedded in $\mathcal{M}'$ (see example below). In other words, the property of* stable embeddedness *of a definable set is not in general preserved by the elementary extension relation.*

Example. We consider the atomic Boolean algebra $\mathcal{M} = \{\mathcal{P}_f(\omega) \cup \mathcal{P}_c(\omega), \cup, \cap, \cdot^c, 0, 1\}$, where $\mathcal{P}_f(\omega)$ is the set of finite subsets of ω and $\mathcal{P}_c(\omega)$ is the set of cofinite subsets of ω. We refer to the partial order of inclusion as *majoring*. By [61], one has quantifier elimination if we add unary predicates A_n, $n \in \mathbb{N}$, for the elements majoring exactly n atoms (here, a predicate for the subsets of ω of n elements). Then, one sees that the set of atoms $A_1^{\mathcal{M}}$, is stably embedded in $\mathcal{M}$. Indeed, all definable subsets of $A_1^{\mathcal{M}}$ are finite or cofinite and more generally, definable subsets of $(A_1^{\mathcal{M}})^n$ are finite Boolean combination of products of such sets and diagonals. However in an $\aleph_0$-saturated elementary extension $\mathcal{M}'$, we have an element f which is majoring infinitely many atoms and not majoring infinitely many other atoms. This allows us to define an infinite and co-infinite subset of $A_1^{\mathcal{M}'}$, which naturally cannot be defined using only parameters in $A_1^{\mathcal{M}'}$.

Definition 1.1.44. Let S be a subset of an L-structure $\mathcal{M}$, and $a \in M^{|x|}$ a tuple of elements. We say that the type $p(x) = \mathrm{tp}(a/S)$ is *definable* if for every formula $\phi(x, y)$, there is an L_S-formula $\psi(y, b)$ such that for all $s \in S$

$$p(x) \vdash \phi(x, s) \text{ if and only if } \mathcal{M} \models \psi(s, b).$$

Let us recall some notations from [22]. Notice that it has a slightly more general meaning, as we also consider non-elementary extensions $\mathcal{M} \subseteq \mathcal{N}$.

Notation. Let $\mathcal{N}$ be an L-structure and S be a subset. We write $T_n(S, \mathcal{N})$ if all n-types over S (for the theory of $\mathcal{N}$) realised in $\mathcal{N}$ are definable. If $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, we might write $T_n(\mathcal{M}, \mathcal{N})$ (both in curvy letters) instead of $T_n(M, \mathcal{N})$ to emphasise it. We write $T_n(\mathcal{M})$ if all n-types over $\mathcal{M}$ (in any elementary extension) are definable.

One sees immediately the following fact:

Fact 1.1.45. *A set S is stably embedded in a structure $\mathcal{M}$ if and only if $T_n(S, \mathcal{M})$ holds for all $n \in \mathbb{N}$.*

We give now a natural 'uniform' version of the notion of stable embeddedness and that of definable types (Definitions 1.1.46 and 1.1.49).

Definition 1.1.46. A subset S of a structure $\mathcal{M}$ is said to be *uniformly stably embedded in* $\mathcal{M}$ if for every formula $\phi(x, y)$, there is a formula $\psi(x, z)$ such that:

$$\text{for all } a \in M^{|y|}, \text{ there is a } b \in S^{|z|} \text{ such that } \phi(S^{|x|}, a) = \psi(S^{|x|}, b). \qquad (\star)$$

In that case, we write $S \subseteq^{ust} \mathcal{M}$.

When S is a definable subset of $\mathcal{M}$, ($\star$) is a first order property of the formula $\phi(x, y)$ and $\psi(x, z)$. So it is in particular preserved by elementary extension. It then makes sense to say that a formula is *uniformly stably embedded* in a given complete theory T. Following the usual convention, we will omit to specify 'uniform' in such a context. The reason is:

Remark 1.1.47. *Assume T is complete. Let $D(x)$ be an* L*-formula. The following statements are equivalent:*

1. *$D(M)$ is uniformly stably embedded in every model $\mathcal{M}$ of T,*
2. *$D(M)$ is uniformly stably embedded in some model $\mathcal{M}$ of T,*
3. *$D(M)$ is stably embedded in every model $\mathcal{M}$ of T,*
4. *$D(\mathbb{M})$ is stably embedded in an $|\mathrm{L}|$-saturated model $\mathbb{M}$ of T.*

Proof. (1) $\Rightarrow$ (2) and (1) $\Rightarrow$ (3) $\Rightarrow$ (4) are obvious. (2) $\Rightarrow$ (1) follows from the fact that ($\star$) is first order. It remains to prove (4) $\Rightarrow$ (2). It immediately follows from compactness, but we give few details here. We have to show that

$$D(\mathbb{M}) \subseteq^{st} \mathbb{M} \Rightarrow D(\mathbb{M}) \subseteq^{ust} \mathbb{M},$$

for an $|\mathrm{L}|$-saturated model $\mathbb{M}$ of T. Take such a model $\mathbb{M}$. If $|D(\mathbb{M})| < 2$, there is nothing to do. Assume that there is an L-formula $\phi(x, y)$ such that for all finite sets Δ of L-formulas $\psi(x, z)$, one has

$$\mathbb{M} \models \exists b_\Delta \bigwedge_{\psi(x,z)\in\Delta} \forall c \in D \; \exists a \in D \; \phi(a, b_\Delta) \not\Leftrightarrow \psi(a, c).$$

By compactness and $|\mathrm{L}|$-saturation, there is an element $b \in \mathbb{M}$ such that $\phi(D(\mathbb{M}), b)$ is not $\mathrm{L}(D(\mathbb{M}))$-definable. This a contradiction. We have shown that there is a finite set Δ of L-formulas such that

$$\mathbb{M} \models \forall b \bigvee_{\psi(x,z)\in\Delta} \exists c \in D \; \forall a \in D \; \phi(a, b) \Leftrightarrow \psi(a, c).$$

Since $|D(\mathbb{M})| \geq 2$, one can use new parameters to encode all formulas in Δ in a single one. In other words, we got an L-formula $\Psi(x, z')$ as wanted:

$$\mathbb{M} \models \forall b \; \exists c' \in D \; \forall a \in D \; \phi(a, b) \Leftrightarrow \Psi(a, c'). \qquad \square$$

Definition 1.1.48. We say that an L-formula $D(x)$ is *(uniformly) stably embedded* for T if $D(x)$ satisfies one (equivalently any) of the conditions in Remark 1.1.47 above.

Definition 1.1.49. Let S be a subset of an L-structure $\mathcal{M}$. We say that the family $(\text{tp}(a/S))_{a\in M^{|x|}}$ of all types over S realised in $\mathcal{M}$ is *uniformly definable* if for every formula $\phi(x,y)$, there is an L-formula $\psi(y,z)$ such that for every tuple $a \in M^{|x|}$, there is a tuple $b \in S^{|z|}$ such that for all $s \in S$:

$$\text{tp}(a/S) \vdash \phi(x,s) \text{ if and only if } \mathcal{M} \models \psi(s,b).$$

We use again the notations of [22], but adapted to this uniform definition.

Notation. Let $\mathcal{N}$ be an L-structure and let S be a subset. For $n \in \mathbb{N}$, we write $T_n^u(S,\mathcal{N})$ if the family of types $(\text{tp}^{\mathcal{N}}(a/S))_{a\in N^n}$ realised in $\mathcal{N}$ is uniformly definable. If $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, we might write $T_n^u(\mathcal{M},\mathcal{N})$ (both in curvy letters) instead of $T_n^u(M,\mathcal{N})$ to emphasise it. We write $T_n^u(\mathcal{M})$ if all n-types over $\mathcal{M}$ (in any elementary extension) are uniformly definable.

Similarly to the non-uniform case, one can also give a characterisation of uniform stable embeddedness in term of uniform definability of types:

Fact 1.1.50. *A set S is uniformly stably embedded in a structure $\mathcal{M}$ if and only if $T_n^u(S,\mathcal{M})$ holds for all $n \in \mathbb{N}$.*

Example. In every benign theory of Henselian valued fields (see the list in the introduction), the residue field k and the value group Γ are stably embedded. This is a well known corollary of relative quantifier elimination in the language augmented by angular components (see Subsection 1.2.1). In fact, they are said to be pure with control of parameters in the sense of the Definition 1.1.8.

We are interested in the following question: given a substructure $\mathcal{M}$ of an L-structure $\mathcal{N}$, when is M stably embedded (resp. uniformly stably embedded) in $\mathcal{N}$? The following (important) remark is immediately deduced from the stable embeddedness/ definability of types duality. It will be implicitly used in the remaining of this text.

Remark 1.1.51. *Let $\mathcal{N}$ be an* L*-structure, and M a subset of N and S an interpretable sort of $\mathcal{N}$. We denote by $S(M)$ the image of M under the projection of N to S.*

- *If $M \subseteq^{st} \mathcal{N}$ (resp. $M \subseteq^{ust} \mathcal{N}$) holds, then $S(M) \subseteq^{st} \mathcal{N}^{eq}$ (resp. $S(M) \subseteq^{ust} \mathcal{N}^{eq}$) holds.*
- *If $\mathcal{M}$ is an elementary submodel of $\mathcal{N}$, then $\mathcal{M} \subseteq^{st} \mathcal{N}$ (resp. $\mathcal{M} \subseteq^{ust} \mathcal{N}$) holds if and only if $\mathcal{M}^{eq} \subseteq^{st} \mathcal{N}^{eq}$ (resp. $\mathcal{M}^{eq} \subseteq^{ust} \mathcal{N}^{eq}$) holds.*

In other words, one can freely add some imaginary sorts to the language, or conversely remove them from the language. For instance, a theory of Henselian valued fields can either be described in the language L_{RV} or the language $L_{\Gamma,k}$ (see Subsection 1.2.1), and the question of stably embedded subsets will not be affected by this choice. For this reason, we will use different languages across the sections (notably in Subsections 3.1.2 and 3.1.4). However, adding new structure might change the notion of stably embedded substructures. The following is clear:

Example. The set of rationals $\mathbb{Q}$ is uniformly stably embedded in $(\mathbb{Q}^{rc}, 0, 1, +, \cdot, <)$, its real closure. However, it is not stably embedded in $(\mathbb{Q}^{rc}, 0, +, <)$ as an ordered abelian group (consider the cut in $\sqrt{2}$). Of course it is uniformly stably embedded again in $\mathbb{Q}^{rc}$, as a pure set.

In our study of Henselian valued fields, the question of stably embedded sub-valued fields can be asked in the language of valued fields enriched with angular components. We will be able to treat this question as well in Subsection 3.1.3.

One has to notice also that the set of stably embedded subsets is in general not closed under definable closure, as shown in the following example:

Example. The set of integers $\mathbb{Z}$ is uniformly stably embedded in the ordered abelian group $(\mathbb{R}, 0, +, <)$, but its definable closure $\mathbb{Q}$ is not.

More on definability of types

In order to apply Theorems 3.1.16 and 3.1.17, one has to understand stably embedded substructures in ordered abelian groups and fields. We will focus in this text on basic examples, namely on o-minimal theories and on Presburger arithmetic. The reader will also find in Appendix A a similar discussion on the random graph where we also construct a uniform stably embedded pair and a non-uniform stably embedded pair of random graphs.

Recall that L is any first order language. We saw that a substructure is stably embedded if and only if all realised types over this structure are definable. As we will see, in certain cases it is actually enough to show that 1-types are definable. We will call this property the '*Marker-Steinhorn criterion*', since it was first proved for elementary pairs of o-minimal structures by Marker and Steinhorn. The question of when definability of all 1-types implies definability of all n-types has been studied in the last two decades. In particular, counterexamples to natural generalisations of the Marker-Steinhorn criterion have been found (see, for example, the introduction to [22]).

O-minimal theories

Fact 1.1.52 (Marker-Steinhorn ([55])). *Let T be an o-minimal theory, and let $\mathcal{M} \preceq \mathcal{N}$ be two models. Then for all $n \in \mathbb{N}$, $T_1(\mathcal{M}, \mathcal{N}) \Rightarrow T_n(\mathcal{M}, \mathcal{N})$.*

From the proof of this fact follows a uniform version of it: if T is o-minimal, then

$$T_1^u(\mathcal{M}, \mathcal{N}) \Rightarrow T_n^u(\mathcal{M}, \mathcal{N}).$$

One can deduce this from the non-uniform theorem by a general argument: let us add a predicate P for the small model $\mathcal{M}$ to the language. Let $(\mathbb{N}, \mathbb{M})$ be an $|\mathrm{L}|$-saturated elementary extension of $(\mathcal{N}, \mathcal{M})$. As it is first order, we also have $T_1^u(\mathbb{M}, \mathbb{N})$ in the language L (however, it doesn't have to be true in the language $\mathrm{L}_P = \mathrm{L} \cup \{P\}$). In particular we have $T_1(\mathbb{M}, \mathbb{N})$ and then by Marker-Steinhorn, $T_n(\mathbb{M}, \mathbb{N})$ for all n. Using that $(\mathbb{N}, \mathbb{M})$ is $|\mathrm{L}|$-saturated and following the proof of Remark 1.1.47, we get that $T_n^u(\mathbb{M}, \mathbb{N})$ holds for all n. By elementarity, we have $T_n^u(\mathcal{M}, \mathcal{N})$ as wanted. As an immediate consequence, one deduces a previous result of Van den Dries: all types over an o-minimal expansion of $\mathbb{R}$ are definable, as the only possible cuts in $\mathbb{R}$ are of the form $a_-, a_+, +\infty$ or $-\infty$.

Presburger arithmetic

Let T be the theory of $(\mathbb{Z}, 0, 1, +, -, <, P_n)$ where $P_n(a)$ holds if and only if n divides a.

Remark 1.1.53. *Let $\mathcal{M}$ be a model of T, and let $\bar{a} = a_0, \ldots, a_{k-1}$ be a finite tuple of elements in an elementary extension $\mathcal{N}$ of $\mathcal{M}$. Then, by quantifier elimination:*

$$\mathrm{tp}(\bar{a}) \cup \bigcup_{z_0, \ldots, z_{k-1} \in \mathbb{Z}} \mathrm{tp}(\sum_{i<k} z_i a_i / M) \vdash \mathrm{tp}(\bar{a}/M).$$

It follows that for all $n \in \mathbb{N}$,

$$T_1(\mathcal{M}, \mathcal{N}) \Rightarrow T_n(\mathcal{M}, \mathcal{N}),$$

and

$$T_1^u(\mathcal{M}, \mathcal{N}) \Rightarrow T_n^u(\mathcal{M}, \mathcal{N}).$$

It is also clear that all types over $(\mathbb{Z}, 0, +, -, <)$ are uniformly definable. Indeed, any 1-type $\mathrm{tp}(a/\mathbb{Z})$ where a is an element of an elementary extension $\mathcal{Z}$ of $\mathbb{Z}$, is determined by the class modulo n of a (for every n) and by whether $a < \mathbb{Z}$ or $a > \mathbb{Z}$. To summarise, one has $\mathbb{Z} \preceq^{ust} \mathcal{Z}$ for every elementary extension $\mathcal{Z}$ of $(\mathbb{Z}, 0, +, <)$.

We conclude this small paragraph with a remark on ordered abelian groups:

Remark 1.1.54. *The only ordered abelian groups $\mathcal{Z}$ which satisfy $T_n(\mathcal{Z})$ for every n are the trivial group, $\mathbb{Z}$ and $\mathbb{R}$.*

Proof. Consider a non-trivial ordered abelian group $\mathcal{Z}$ which is stably embedded in all elementary extensions $\mathcal{Z}' \succeq \mathcal{Z}$. It needs in particular to be archimedian as otherwise an elementary extension will realise an irrational cut of the form $\mathcal{Z}_{<\omega\cdot a}/\mathcal{Z}_{>\omega\cdot a}$ where

$$\mathcal{Z}_{<\omega\cdot a} = \{z \in \mathcal{Z} \mid \text{there is } n < \omega \text{ such that } z < n \cdot a\},$$

and

$$\mathcal{Z}_{>\omega\cdot a} = \{z \in \mathcal{Z} \mid \text{for all } n < \omega,\ z > n \cdot a\}$$

where a is an element of $\mathcal{Z}$ such that the above sets are not empty. In other words, $\mathcal{Z}$ is isomorphic as an ordered abelian group to an additive subgroup of $\mathbb{R}$. We assume that it is a subgroup of $\mathbb{R}$. If $\mathcal{Z}$ is discrete, it is isomorphic to $\mathbb{Z}$. Assume it is not discrete, and so dense in $\mathbb{R}$. If there is $a \in \mathbb{R} \setminus \mathcal{Z}$, this element realises an irrational cut over $\mathcal{Z}$, and thus there would be an elementary extension $\mathcal{Z}'$ realising this irrational cut, which is a contradiction. This shows that $\mathcal{Z} = \mathbb{R}$. □

1.2 On model theory of algebraic structures

1.2.1 Valued fields

Introduction

We gather here some facts on valued fields. After some general statement on indiscernible sequences, we will introduce the RV-sort. Then, we will list the theories of valued fields that we will consider. We will assume the theory of Kaplansky known. We will need a few lemmas, such as a kind of transitivity of pseudo-limits, and a case study of indiscernible sequences. A valued field will be typically denoted by $\mathcal{K} = (K, \Gamma, k, \text{val})$ where K is the field (main sort), Γ the value group and k the residue field. The valuation is denoted by val, the maximal ideal $\mathfrak{m}$ and the valuation ring $\mathcal{O}$. We recall the two traditional languages of valued fields.

Notation and languages

We will work in different (many-sorted) languages. Let us define two of them:

- $\mathrm{L}_{\text{div}} = \{K, 0, 1, +, \cdot, |\}$, where $|$ is a binary relation symbol, interpreted by the division:

$$\text{for } a, b \in K, a \mid b \text{ if and only if } \text{val}(a) \leq \text{val}(b).$$

- $L_{\Gamma,k} = \{K, 0, 1, +, \cdot\} \cup \{k, 0, 1, +, \cdot\} \cup \{\Gamma, 0, \infty, +, <\} \cup \{\mathrm{val} : K \to \Gamma, \mathrm{Res} : K^2 \to k\}$.

where $\mathrm{Res} : K^2 \to k$ is the two-place residue map, interpreted as follows:

$$\mathrm{Res}(a,b) = \begin{cases} \mathrm{res}(a/b) \text{ if } \mathrm{val}(a) \geq \mathrm{val}(b) \neq \infty, \\ 0 \text{ otherwise.} \end{cases}$$

In the next paragraphs, we will also introduce the many-sorted languages L_{RV} and $L_{\mathrm{RV}_{<\omega}}$.

By bi-interpretability, a theory of valued fields can be expressed indifferently in either of these languages. Let $\mathcal{K}$ be a valued field. If the context is clear, we will often abusively denote by $K, \Gamma, k, \mathrm{RV}, ...$ the sorts in $\mathcal{K}$. In general, the sorts of a valued field $\mathcal{L}$ will be denoted by $L, \Gamma_L, k_L, \mathrm{RV}_L ...$ and of a valued field $\mathcal{K}'$ by $K', \Gamma', k', \mathrm{RV}', \dots$ etc.

Pseudo-Cauchy sequences

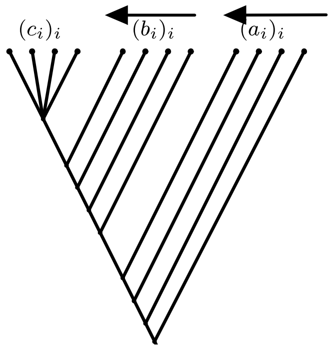
We will discuss here some simple facts about mutually indiscernible arrays in a valued field $\mathcal{K}$. We will denote by $\bar{\mathbb{Z}}$ the set of integers with extreme elements $\{-\infty, \infty\}$. We will assume that the reader is familiar with pseudo-Cauchy sequences. We recall however the basic definition:

Definition 1.2.1. Let $(I, <)$ be a totally ordered index set without greatest element. A sequence $(a_i)_{i \in I}$ of elements of K is *pseudo-Cauchy* if there is $i \in I$ such that for all indices $i < i_1 < i_2 < i_3$, $\mathrm{val}(a_{i_2} - a_{i_1}) < \mathrm{val}(a_{i_3} - a_{i_2})$. We say that $a \in K$ is a pseudo limit of the *pseudo-Cauchy* sequence $(a_i)_{i \in I}$ and we write $(a_i)_{i \in I} \Rightarrow a$ if there is $i \in I$ such that for all indices $i < i_1 < i_2$, we have $\mathrm{val}(a - a_{i_1}) = \mathrm{val}(a_{i_2} - a_{i_1})$.

The next two lemmas give some useful properties of indiscernible pseudo-Cauchy sequences.

Lemma 1.2.2.
1. *Assume $(a_i)_{i<\omega}$ is an indiscernible sequence and a is a pseudo limit of $(a_i)_{i<\omega}$. Then for any i, $\mathrm{val}(a_i - a) = \mathrm{val}(a_i - a_{i+1})$ depends only on i and not on the chosen limit a (for general pseudo-Cauchy sequence, this holds only for i big enough).*

2. *For three mutually indiscernible sequences $(a_i)_{i<\omega}$, $(b_i)_{i<\omega}$ and $(c_i)_{i<\omega}$, if $(a_i)_{i<\omega} \Rightarrow b_0$ and $(b_i)_{i<\omega} \Rightarrow c_0$, then we have $(a_i)_{i<\omega} \Rightarrow c_0$.*



3. *If $(a_i)_{i\in\mathbb{Z}}$ is an indiscernible sequence in K, then $(a_i)_{i\in\omega}\Rightarrow a_\infty$ or $(a_{-i})_{i\in\omega}\Rightarrow a_{-\infty}$ or for $i \neq j$, $\mathrm{val}(a_i - a_j)$ is constant (in this last case, $(a_i)_{i\in\mathbb{Z}}$ will be called a fan).*

Proof. 1. By definition of a pseudo-Cauchy sequence, $(\mathrm{val}(a_i - a_{i+1}))_i$ is eventually strictly increasing. By indiscernibility, it is strictly increasing. Let i_0 be such that $\mathrm{val}(a - a_i) = \mathrm{val}(a_{i+1} - a_i)$ for all $i > i_0$. Then $\mathrm{val}(a - a_{i_0}) = \min(\mathrm{val}(a - a_{i_0+1}), \mathrm{val}(a_{i_0+1} - a_{i_0})) = \min(\mathrm{val}(a_{i_0+2} - a_{i_0+1}), \mathrm{val}(a_{i_0+1} - a_{i_0})) = \mathrm{val}(a_{i_0+1} - a_{i_0})$. It holds also for $i = i_0$ and we can reiterate.

2. Notice that, by mutual indiscernibility and (1), $\mathrm{val}(a_i - b_0) = \mathrm{val}(a_i - a_{i+1}) = \mathrm{val}(a_i - b_j)$ for any $i, j < \omega$, *i.e.* $(a_i)_{i<\omega}\Rightarrow b_j$ for any j. Similarly, $(b_i)_{i<\omega}\Rightarrow c_j$ for any j. We have $\mathrm{val}(b_0 - b_1) \geq \mathrm{val}(b_0 - a_i) = \mathrm{val}(a_i - b_1)$. If $\mathrm{val}(b_0 - b_1) = \mathrm{val}(b_0 - a_i)$, we have by mutual indiscernibility that $(\mathrm{val}(b_0 - a_i))_{i<\omega}$ is constant, which is a contradiction with $(a_i)_{i<\omega}\Rightarrow b_0$. Then, we have $\mathrm{val}(b_0 - c_0) = \mathrm{val}(b_0 - b_1) > \mathrm{val}(b_0 - a_i)$. As $\mathrm{val}(a_i - c_0) \geq \min(\mathrm{val}(a_i - b_0), \mathrm{val}(b_0 - c_0))$, we deduce that $\mathrm{val}(a_i - c_0) = \mathrm{val}(a_i - b_0)$ for all i, *i.e.* $(a_i)\Rightarrow c_0$.

3. It is immediate by indiscernibility (consider for example $\mathrm{val}(a_0 - a_1)$ and $\mathrm{val}(a_1 - a_2)$).

□

Lemma 1.2.3. *Let $(a_j)_{j\in\mathbb{Z}}$ and $(b_l)_{l\in\mathbb{Z}}$ two mutually indiscernible sequences in K such that $(\mathrm{val}(a_j - b_l))_{j,l}$ is not constant. At least one of the following occurs:*

1. $(a_j)_{j<\omega}\Rightarrow b_0$,

2. $(b_l)_{l<\omega}\Rightarrow a_0$,

3. $(a_{-j})_{j<\omega}\Rightarrow b_0$,

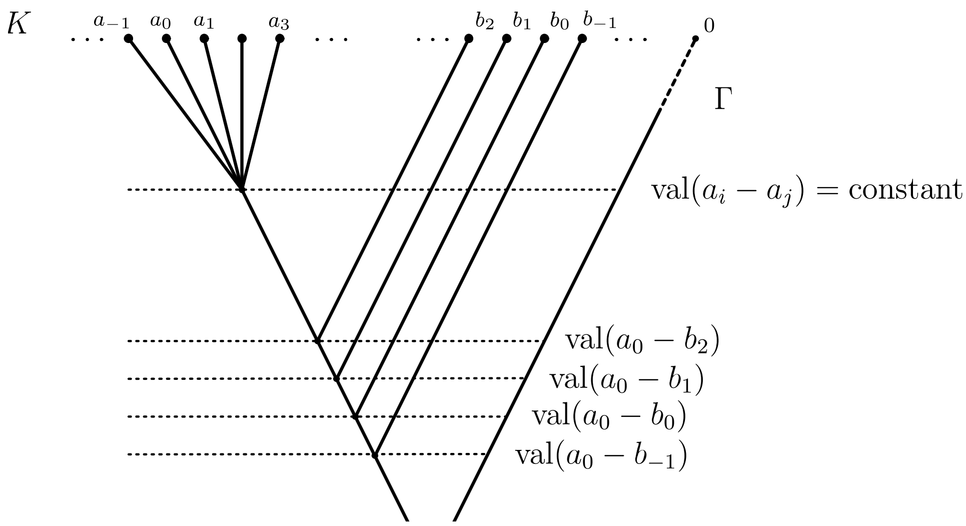
4. $(b_{-l})_{l<\omega} \Rrightarrow a_0$.

Note that if for example $(b_l)_{l<\omega} \Rrightarrow a_0$*, then by mutual indiscernibility,* $(b_l)_{l<\omega} \Rrightarrow a_j$ *for every* $j \in \mathbb{Z}$*.*

Proof. Since $\mathrm{val}(a_j - b_l)$ is not constant, using the mutual indiscernibility, one of the following occurs:

1. $\mathrm{val}(a_0 - b_0) < \mathrm{val}(a_1 - b_0)$,
2. $\mathrm{val}(a_0 - b_0) < \mathrm{val}(a_0 - b_1)$,
3. $\mathrm{val}(a_0 - b_0) < \mathrm{val}(a_{-1} - b_0)$,
4. $\mathrm{val}(a_0 - b_0) < \mathrm{val}(a_0 - b_{-1})$.

Indeed, if 1. and 3. do not hold, then the sequence $(\mathrm{val}(b_0 - a_j))_{j\in\mathbb{Z}}$ is constant. If 2. and 4. do not hold, then the sequence $(\mathrm{val}(b_l - a_0))_{l\in\mathbb{Z}}$ is constant. This cannot be true for both sequences as it would contradict the assumption. We conclude by indiscernibility.



□

RV-sorts

We will now define the RV-*sort* (or RV-*sorts*, as we may need to consider more than one sort) –an intermediate structure between the valued field and its value group and residue field. We also introduce corresponding languages L_{RV} and $\mathrm{L}_{\mathrm{RV}_{<\omega}}$. This paragraph is largely inspired by [40], which one can use as a reference. Let K be a Henselian valued field of characteristic $(0,p)$ with $p \geq 0$, of value group Γ and residue field k. If $\delta \in \Gamma_{\geq 0}$, we denote by $\mathfrak{m}_\delta$

the ideal of the valuation ring $\mathcal{O}$ defined by $\{x \in \mathcal{O} \mid v(x) > \delta\}$. The *leading term structure* of order δ is the quotient group

$$\mathrm{RV}_\delta^\star := K^\star/(1+\mathfrak{m}_\delta).$$

The quotient map is denoted by $\mathrm{rv}_\delta : K^\star \to \mathrm{RV}_\delta^\star$. The valuation $\mathrm{val} : K^\star \to \Gamma$ induces a group homomorphism $\mathrm{val}_{\mathrm{rv}_\delta} : \mathrm{RV}_\delta^\star \to \Gamma$. Since $\mathfrak{m} = \mathfrak{m}_0$ and $k^\star := (\mathcal{O}/\mathfrak{m}_0)^\star \simeq \mathcal{O}^\times/(1+\mathfrak{m}_0)$, we have the following short exact sequence:

$$1 \to k^\star \to \mathrm{RV}_0^\star \overset{\mathrm{val}_{\mathrm{rv}_0}}{\to} \Gamma \to 0.$$

In general, we denote by $\mathcal{O}_\delta$ the ring $\mathcal{O}/m_\delta$, called the *residue ring of order* δ . One has $\mathcal{O}_\delta^\times \simeq \mathcal{O}^\times/(1+\mathfrak{m}_\delta)$ and the following exact sequence:

$$1 \to \mathcal{O}_\delta^\times \to \mathrm{RV}_\delta^\star \overset{\mathrm{val}_{\mathrm{rv}_\delta}}{\to} \Gamma \to 0.$$

Furthermore, as $\mathfrak{m}_\gamma \subseteq \mathfrak{m}_\delta$ for any $\delta \leq \gamma$ in $\Gamma_{\geq 0}$, we have a projection map $\mathrm{RV}_\gamma^\star \to \mathrm{RV}_\delta^\star$ denoted by $\mathrm{rv}_{\gamma\to\delta}$ or simply by rv_δ. We add a new constant $\mathbf{0}$ to the sort $\mathrm{RV}_\delta^\star$ and we write $\mathrm{RV}_\delta := \mathrm{RV}_\delta^\star \cup \{\mathbf{0}\}$. We set the following properties:

- for all $\mathbf{x} \in \mathrm{RV}_\delta$, $\mathbf{0}\cdot\mathbf{x} = \mathbf{x}\cdot\mathbf{0} = \mathbf{0}$.
- $val_{\mathrm{rv}_\delta}(\mathbf{0}) = \infty, \quad \mathrm{rv}_\delta(0) = \mathbf{0}$.

Proposition 1.2.4. *For any $a, b \in K$ and $\delta \in \Gamma_{\geq 0}$, $\mathrm{rv}_\delta(a) = \mathrm{rv}_\delta(b)$ if and only if $\mathrm{val}(a-b) > \mathrm{val}(b) + \delta$ or $a = b = 0$.*

Proof. This follows easily from the definition: assume $\mathrm{rv}_\delta(a) = \mathrm{rv}_\delta(b)$ and $a \neq 0$. Then $a = b(1+\mu)$ for some $\mu \in m_\delta$ and $\mathrm{val}(a-b) = \mathrm{val}(b) + \mathrm{val}(\mu) > \mathrm{val}(b) + \delta$. Conversely, if $\mathrm{val}(a-b) > \mathrm{val}(b) + \delta$, one can write $a = b(1 + \frac{(a-b)}{b})$. □

As a group quotient, the sort RV_δ is endowed with a multiplication. As we will see, it also inherits from the field some kind of addition.

Notation. Let $0 \leq \delta_1, \delta_2, \delta_3$ be three elements of Γ and $\mathbf{x} \in \mathrm{RV}_{\delta_1}$, $\mathbf{y} \in \mathrm{RV}_{\delta_2}$, $\mathbf{z} \in \mathrm{RV}_{\delta_3}$ three variables. Then we define the following formulas:

$$\oplus_{\delta_1,\delta_2,\delta_3}(\mathbf{x},\mathbf{y},\mathbf{z}) \equiv \ \exists a, b \in K \ \ \mathrm{rv}_{\delta_1}(a) = \mathbf{x} \wedge \mathrm{rv}_{\delta_2}(b) = \mathbf{y} \wedge \mathrm{rv}_{\delta_3}(a+b) = \mathbf{z}$$

In our study of valued fields, we will consider the structures RV and $\mathrm{RV}_{<\omega}$, that we define now:

Definition 1.2.5. The RV-sort of a valued field $\mathcal{K}$ is the first order leading term structure

$$\mathrm{RV} = (\mathrm{RV}_0, \cdot, \oplus_{0,0,0}, \mathbf{0}, \mathbf{1})$$

endowed with its natural structure of abelian group and the ternary predicate described above. Following the usual convention, we drop the index 0, and write $\mathrm{RV}, \oplus$ and rv instead of $\mathrm{RV}_0, \oplus_{0,0,0}$ and rv_0.

Fact 1.2.6 (Flenner, [40, Proposition 2.8])**.** *The three-sorted structure* $\{(\mathrm{RV}, \mathbf{1}, \cdot, \mathbf{0}, \mathbf{1}), (k, 0, 1, +, \cdot), (\Gamma, 0, +, <), \iota, \mathrm{val}\}$ *and the one-sorted structure* $\{\mathrm{RV}, \mathbf{0}, \cdot, \oplus\}$ *are bi-interpretable on unary sets.*

This will mean, in the context of this paper, that these two points of view are equivalent, and we will swap between one to the other indifferently (see Fact 1.1.20 and Remark 1.1.51).

We defined the *leading term language* L_{RV} as the multisorted language with

- a sort for K and RV.
- the ring language for K,
- the (multiplicative) group language as well as the symbol 0 for RV.
- the ternary relation symbol $\oplus$ and the function symbols rv.

The structure $\mathcal{K} = (K, \mathrm{RV}, \mathrm{rv})$ becomes a structure in this language where all symbols are interpreted as before. This language is also bi-interpretable (without parameters) with the usual languages of valued fields, e.g. with $\mathrm{L}_{\mathrm{div}}$ (see [40, Proposition 2.8]).

Also, notice that the symbol $\oplus$ suggests a binary operation. Occasionally, we will indeed write $\mathrm{rv}(a) \oplus \mathrm{rv}(b)$ for $a, b \in K$ to denote the following element:

$$\mathrm{rv}(a) \oplus \mathrm{rv}(b) := \begin{cases} \mathrm{rv}(a+b) & \text{if } \mathrm{val}(a+b) = \min(\mathrm{val}(a), \mathrm{val}(b)), \\ \mathbf{0} & \text{otherwise.} \end{cases}$$

It is not hard to see that this is independent of the choice of representatives of $\mathrm{rv}(a)$ and $\mathrm{rv}(b)$. We will write $\bigoplus_{i \in I} \mathbf{a}_i$ for I a set of indices and $\mathbf{a}_i \in \mathrm{RV}$, when such a sum does not depend on any choices of parentheses. Notice that the law $\oplus$ is not an addition. If $a, b \in K$ with $\mathrm{val}(a) < \mathrm{val}(b)$, we have that $\mathrm{rv}(a) \oplus \mathrm{rv}(b) = \mathrm{rv}(a) = \mathrm{rv}(a+b)$. Also, notice that it is in general not true that $\mathrm{rv}(a+b) = \mathrm{rv}(a) \oplus \mathrm{rv}(b)$ (choose $a, b \in K$ such that $\mathrm{rv}(a) = -\mathrm{rv}(b)$ and $a \neq -b$). When we have indeed that $\mathrm{rv}(a+b) = \mathrm{rv}(a) \oplus \mathrm{rv}(b)$, we say that the sum $\mathrm{rv}(a) \oplus \mathrm{rv}(b)$ is well-defined.

In the specific context of mixed characteristic Henselian valued fields, we might have to consider a larger structure:

Definition 1.2.7. Assume that $\mathcal{K}$ is a valued field of characteristic 0 and residue characteristic $p \geq 0$. We reserve now the notation δ_n for $\delta_n = \mathrm{val}(p^n)$. We write $\mathrm{RV}_{<\omega}$ for the union of sorts leading term structure of finite order $\{(\mathrm{RV}_{\delta_n})_{n<\omega}, (\oplus_{\delta_l,\delta_m,\delta_n})_{n<l,m}, (\mathrm{rv}_{\delta_n\to\delta_m})_{m<n<\omega}\}$ endowed with ternary predicates $\oplus_{\delta_l,\delta_m,\delta_n}$ and a projective system of maps $(\mathrm{rv}_{\delta_n\to\delta_m})_{m<n<\omega}$. We also write $\mathrm{val}_{\mathrm{rv}_{<\omega}} : \mathrm{RV}_{<\omega} \to \Gamma \cup \{\infty\}$ for $\bigcup_{n<\omega}(\mathrm{val}_{\mathrm{rv}_{\delta_n}} : \mathrm{RV}_{\delta_n} \to \Gamma \cup \{\infty\})$, etc.

Remark 1.2.8. *In equicharacteristic* 0*, we have that* $\delta_n := \mathrm{val}(p^n) = 0$ *for all* $n < \omega$*. This leads to identifying* $\bigcup_{n<\omega} \mathrm{RV}_{\delta_n}$ *with* $\mathrm{RV} = \mathrm{RV}_0$.

In Section 2.4 and Section 3.2, we will have to use another language to describe the induced structure on $\mathrm{RV}_{<\omega}$:

Fact 1.2.9 (Flenner, [40, Proposition 2.8])**.** *The structure*

$$\{(\mathrm{RV}_{\delta_n})_{n<\omega}, (\oplus_{\delta_l,\delta_m,\delta_n})_{n<l,m}, (\mathrm{rv}_{\delta_n\to\delta_m})_{m<n<\omega}\}$$

and the structure

$$\big\{(\mathrm{RV}_{\delta_n})_{n<\omega}, (\mathcal{O}_{\delta_n}, \cdot, +, 0, 1)_{n<\omega}, (\Gamma, +, 0, <), (\mathrm{val}_{\mathrm{rv}_{\delta_n}})_{n<\omega},$$
$$(\mathcal{O}^{\times}_{\delta_n} \to \mathrm{RV}^{\times}_{\delta_n})_{n<\omega}, (\mathrm{rv}_{\delta_n\to\delta_m})_{m<n<\omega}\big\}$$

are bi-interpretable on unary sets.

As before, this is only to say that one can recover the valuation using the symbols $\oplus$ (see [40, Proposition 2.8]). And again, this fact means that we will be able to swap between one language to the other indifferently (see Fact 1.1.20 and Remark 1.1.51).

We defined the language $\mathrm{L}_{\mathrm{RV}_{<\omega}}$ as the multisorted language with

- sorts for K and RV_{δ_n} for $n < \omega$.
- the ring language for K,
- for all $n < \omega$, the (multiplicative) group language as well as the symbol 0 for RV_{δ_n}.
- relation symbols $\oplus_{\delta_l,\delta_m,\delta_n}$ for $n \leq l, m$ integers, function symbols $\mathrm{rv}_{\delta_n} : \mathcal{K} \to \mathrm{RV}_{\delta_n}$ and
  $\mathrm{rv}_{\delta_n\to\delta_m} : \mathrm{RV}_{\delta_n} \to \mathrm{RV}_{\delta_m}$ for $n > m$.

The structure $\mathcal{K} = (K, (\mathrm{RV}_{\delta_n})_{n<\omega}, (\oplus_{\delta_l,\delta_m,\delta_n})_{n<l,m}, (\mathrm{rv}_{\delta_n})_{n<\omega}, (\mathrm{rv}_{\delta_n\to\delta_m})_{m<n<\omega})$ becomes a structure in this language where all symbols are interpreted as before. This language is also bi-interpretable (without parameters) with the usual languages of valued fields, e.g. with $\mathrm{L}_{\mathrm{div}}$ (see [40, Proposition 2.8]).

Let $\mathcal{K}$ be any valued field. Let us state few lemmas.

Notation. Let $\delta_1, \delta_2, \delta_3 \in \Gamma$ be three values. We write:

$$\mathrm{WD}_{\delta_1,\delta_2,\delta_3}(\mathbf{x},\mathbf{y}) \equiv \qquad \exists! \mathbf{z} \in \mathrm{RV}_{\delta_3} \ \oplus_{\delta_1,\delta_2,\delta_3}(\mathbf{x},\mathbf{y},\mathbf{z}).$$

If the context is clear and in order to simplify notations, we will write:

- WD_{δ_3} instead of $\mathrm{WD}_{\delta_1,\delta_2,\delta_3}$,
- for any formula $\phi(\mathbf{z})$ with $\mathbf{z} \in \mathrm{RV}_{\delta_3}$, $\mathbf{x} \in \mathrm{RV}_{\delta_1}$ and $\mathbf{y} \in \mathrm{RV}_{\delta_2}$:

$$\phi(\mathrm{rv}_{\delta_3}(\mathbf{x}) + \mathrm{rv}_{\delta_3}(\mathbf{y}))$$

or

$$\phi(\mathrm{rv}_{\delta_3}(\mathbf{x}) + \mathrm{rv}_{\delta_3}(\mathbf{y})) \wedge \mathrm{WD}_{\delta_3}(\mathbf{x},\mathbf{y})$$

instead of

$$\exists \mathbf{z} \in \mathrm{RV}_{\delta_3} \ \oplus_{\delta_1,\delta_2,\delta_3}(\mathbf{x},\mathbf{y},\mathbf{z}) \wedge \phi(\mathbf{z}) \wedge \mathrm{WD}_{\delta_1,\delta_2,\delta_3}(\mathbf{x},\mathbf{y}).$$

Example. Take $K = \mathbb{R}((t))$ the field of power series over the reals endowed with the t-adic valuation. Consider $x = t^2+t^3+t^4+t^5$, $x' = t^2+t^3+t^4+2t^5 \in K$, $y = -t^2 - t^3 + t^4 - t^5 \in K$ and $z = 2t^4$, $z' = 2t^4 + t^5 \in K$.

Then, we have $\mathrm{rv}_2(x) = \mathrm{rv}_2(x')$ since $\mathrm{val}(x - x') = 5 > \mathrm{val}(x) + 2$ but $\mathrm{rv}_1(z) \neq \mathrm{rv}_1(z')$ since $\mathrm{val}(z - z') = 5 \not> \mathrm{val}(z) + 1$. We have:

$$\models \oplus_{2,2,1}(\mathrm{rv}_2(x), \mathrm{rv}_2(y), \mathrm{rv}_1(z)), \quad \models \oplus_{2,2,1}(\mathrm{rv}_2(x), \mathrm{rv}_2(y), \mathrm{rv}_1(z')),$$

Hence, the sum is not well-defined in RV_1:

$$\models \neg\, \mathrm{WD}_{2,2,1}(\mathrm{rv}_2(x), \mathrm{rv}_2(y))$$

We need to pass to RV_3 in order to get a well-defined sum in RV_1:

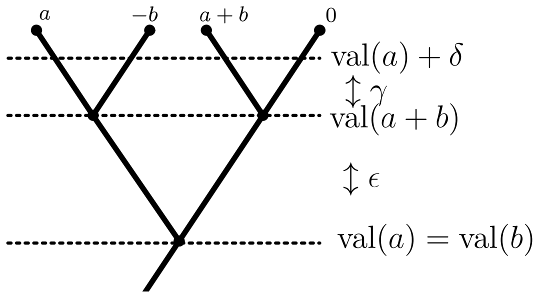
$$\models \oplus_{3,3,1}(\mathrm{rv}_3(x), \mathrm{rv}_3(y), \mathrm{rv}_1(z)), \quad \neg \models \oplus_{3,3,1}(\mathrm{rv}_3(x), \mathrm{rv}_3(y), \mathrm{rv}_1(z')),$$

$$\models \mathrm{WD}_{3,3,1}(\mathrm{rv}_3(x), \mathrm{rv}_3(y)).$$

More generally, we have the following proposition:

Proposition 1.2.10. *Let $0 \leq \gamma \leq \delta$ be two elements of $\Gamma_{\geq 0}$ and $\epsilon = \delta - \gamma \geq 0$. Then for every $a, b \in K^\star$:*

$$\mathrm{WD}_\gamma(\mathrm{rv}_\delta(a), \mathrm{rv}_\delta(b)) \quad \textit{if and only if} \quad \mathrm{val}(a+b) \leq \min\{\mathrm{val}(a), \mathrm{val}(b)\} + \epsilon.$$



Proof. Assume $\mathrm{val}(a+b) \leq \min\{\mathrm{val}(a), \mathrm{val}(b)\} + \epsilon$. Let $a' \in K$ such that $\mathrm{rv}_\delta(a') = \mathrm{rv}_\delta(a)$. This is equivalent to $\mathrm{val}(a - a') > \mathrm{val}(a) + \delta$, thus we have:

$$\mathrm{val}\left(a + b - (a' + b)\right) = \mathrm{val}(a - a') > \mathrm{val}(a) + \delta = \mathrm{val}(a) + \epsilon + \gamma \geq \mathrm{val}(a+b) + \gamma.$$

Hence, $\mathrm{rv}_\gamma(a' + b) = \mathrm{rv}_\gamma(a + b)$. We have proved the implication from right to left.

Conversely, assume that $\mathrm{val}(a + b) > \min\{\mathrm{val}(a), \mathrm{val}(b)\} + \epsilon$ and $\min\{\mathrm{val}(a), \mathrm{val}(b)\} = \mathrm{val}(a)$. Let $\eta = \mathrm{val}(a+b)+\gamma$ and take any $c \in K$ of valuation η. Then $\mathrm{rv}_\delta(a) = \mathrm{rv}_\delta(a + c)$ since $\mathrm{val}(a + c - a) = \eta > \mathrm{val}(a) + \delta$ and $\mathrm{rv}_\gamma(a + b) \neq \mathrm{rv}_\gamma(a + c + b)$ since $\mathrm{val}\left(a + c + b - (a + b)\right) = \eta = \mathrm{val}(a+b)+\gamma$. □

Remark 1.2.11. *To prove* $\mathrm{val}(a+b) \leq \min\{\mathrm{val}(a), \mathrm{val}(b)\}+\epsilon$ *with* $\epsilon \geq 0$*, it is enough to show that* $\mathrm{val}(a+b) \leq \mathrm{val}(a)+\epsilon$ *(or* $\mathrm{val}(a+b) \leq \mathrm{val}(b)+\epsilon$*). Indeed, if* $\mathrm{val}(a) = \mathrm{val}(b)$ *then this is clear. If* $\mathrm{val}(a) < \mathrm{val}(b)$ *or* $\mathrm{val}(b) < \mathrm{val}(a)$*, this is also clear since we have* $\mathrm{val}(a + b) = \mathrm{val}(a) \leq \mathrm{val}(a) + \epsilon$ *in the first case and* $\mathrm{val}(a + b) = \mathrm{val}(b) < \mathrm{val}(a) + \epsilon$ *in the second.*

The following lemma is immediate:

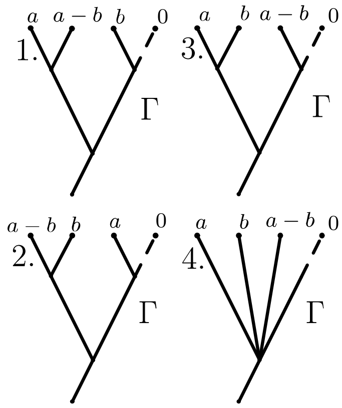
Lemma 1.2.12. *Let* a, b *and* $c = a - b$ *be elements of* K *and let* $\gamma \in \Gamma_{\geq 0}$*. At least one of the following holds:*

$$\models \mathrm{WD}_\gamma(\mathrm{rv}_\gamma(a), \mathrm{rv}_\gamma(b - a)) \tag{1.1}$$

$$\models \mathrm{WD}_\gamma(\mathrm{rv}_\gamma(b), \mathrm{rv}_\gamma(a - b)) \tag{1.2}$$

Proof. Notice that exactly one of the following occurs:

1. $\mathrm{val}\, a = \mathrm{val}\, c < \mathrm{val}\, b$,
2. $\mathrm{val}\, b = \mathrm{val}\, c < \mathrm{val}\, a$,
3. $\mathrm{val}\, a = \mathrm{val}\, b < \mathrm{val}\, c$,
4. $\mathrm{val}\, a = \mathrm{val}\, b = \mathrm{val}\, c$.



Let $\mathbf{a} = \mathrm{rv}_\gamma(a), \mathbf{b} = \mathrm{rv}_\gamma(b)$ and $\mathbf{c} = \mathrm{rv}_\gamma(c)$. In cases 2,3 and 4, the difference between $\mathbf{a}$ and $\mathbf{c}$ is well-defined.

$$\models \mathrm{WD}_\gamma(\mathbf{a}, -\mathbf{c}).$$

In cases 1,3 and 4, the sum of $\mathbf{b}$ and $\mathbf{c}$ is well-defined:

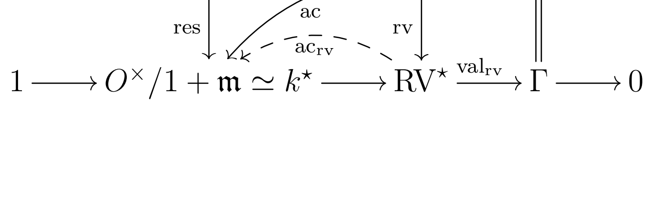
$$\models \mathrm{WD}_\gamma(\mathbf{b}, \mathbf{c}).$$

□

Benign theory of Henselian valued fields

Later in this text, we prove transfer principles for some rather nice Henselian valued fields, that we called here 'benign' (see Definition 1.2.21 below). The goal of this subsection is to discuss essential properties that these benign Henselian valued fields share. We will emphasise model theoretical properties, and briefly recall from which algebraic properties they can be deduced. The idea is to implicitly work axiomatically by listing required properties needed for proving Theorems 2.3.4 and 3.1.17.

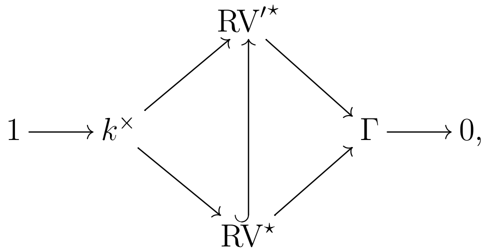
Let T be a (possibly incomplete) theory of Henselian valued fields. We need first to recall the definition of *an angular component* (or ac-*map*) . It is a group homomorphism usually denoted by $\mathrm{ac} : (K^\star, \cdot) \to (k^\star, \cdot)$ such that $ac_{|\mathcal{O}^\times} = \mathrm{res}_{\,|\mathcal{O}^\times}$. We also set $\mathrm{ac}(0) = 0$. We have the following diagram:



One remarks that an angular component gives a section $ac_{\mathrm{rv}} : \mathrm{RV} \to k^\star$ and, as a consequence, the sort RV becomes isomorphic as a group to the direct product $\Gamma \times k^\star$. Such a map always exists in an $\aleph_1$-saturated valued field $\mathcal{K}$: as $\mathcal{O}^\times$ is a pure subgroup of $K^\star$, there is a section $s : \Gamma \to K^\star$ of the valuation (see Fact 1.2.38). Then, the function $\mathrm{ac} : a \mapsto \mathrm{res}(a/s(v(a)))$ is an ac-map. Any theory T of Henselian valued fields in a language $\mathrm{L}_{\Gamma,k}$ admits a natural expansion – denoted by T_{ac} – in the language $\mathrm{L}_{\Gamma,k,\mathrm{ac}} = \mathrm{L}_{\Gamma,k} \cup \{\mathrm{ac} : K \to k\}$ by adding the axiom saying that ac is an angular component. We can now define some properties which T may or may not enjoy. Recall that an extension $\mathcal{K}' = (K', \mathrm{RV}', \Gamma', k')$ of $\mathcal{K} = (K, \mathrm{RV}, \Gamma, k)$ is said to be *immediate* if $\Gamma' = \Gamma$ and $k' = k$. We denote the following hypothesis:

$$\text{The set of models of } T \text{ is closed under maximal immediate extensions.} \tag{Im}$$

It is easy to see that extensions preserving the RV-sort are exactly immediate extensions, as the following commutative diagram:



implies $\mathrm{RV} = \mathrm{RV}'$.

If (Im) is satisfied, maximal immediate extensions are natural examples where one can apply the Ax-Kochen-Ershov principle. We name two such properties:

$$\text{for } \mathcal{K}, \mathcal{K}' \models T, \mathcal{K} \subseteq \mathcal{K}', \text{ we have } \mathcal{K} \preceq \mathcal{K}' \Leftrightarrow k \preceq k' \text{ and } \Gamma \preceq \Gamma'. \quad (\mathrm{AKE})_{\Gamma,k}$$

$$\text{for } \mathcal{K}, \mathcal{K}' \models T, \mathcal{K} \subseteq \mathcal{K}', \text{ we have } \mathcal{K} \preceq \mathcal{K}' \Leftrightarrow \mathrm{RV} \preceq \mathrm{RV}'. \quad (\mathrm{AKE})_{\mathrm{RV}}$$

These properties $(\mathrm{AKE})_{\mathrm{RV}}$ and $(\mathrm{AKE})_{\Gamma,k}$ are two different points of view, and often come together. We also denote the following properties:

$$\Gamma \text{ and } k \text{ are pure, stably embedded and orthogonal.} \quad (\mathrm{SE})_{\Gamma,k}$$

RV is pure and stably embedded. $(\mathrm{SE})_{\mathrm{RV}}$

As we will see, they are consequences of relative quantifier elimination:

T_{ac} has quantifier elimination (resplendently) relatively to Γ and k in the language $\mathrm{L}_{\Gamma,k,\mathrm{ac}}$. $((\mathrm{EQ})_{\Gamma,k,ac})$

T has quantifier elimination (resplendently) relatively to RV in the language L_{RV}. $((\mathrm{EQ})_{\mathrm{RV}})$

Notice that according to the terminology in Subsection 1.1.1, $\{\Gamma\}, \{k\}$ and $\{\mathrm{RV}\}$ are closed sets of sorts. Then resplendency automatically follows from relative quantifier elimination (Fact 1.1.5).

Here is a well known fact followed by an easy observation.

Fact 1.2.13. *$(EQ)_{\Gamma,k,\mathrm{ac}}$ implies $(AKE)_{\Gamma,k}$ and $(SE)_{\Gamma,k}$.*
$(EQ)_{\mathrm{RV}}$ implies $(AKE)_{\mathrm{RV}}$ and $(SE)_{\mathrm{RV}}$.

Observation 1.2.14. *1. $(AKE)_{\Gamma,k}$ implies $(AKE)_{\mathrm{RV}}$.*

2. $(EQ)_{\mathrm{RV}}$ implies $(EQ)_{\Gamma,k,ac}$.

We include a proof of this observation for completeness.

Proof. (1) This is immediate, as the value group and residue field are interpretable in the RV-structure: for any models $\mathcal{M}, \mathcal{N} \models T$, $\mathrm{RV}_M \preceq \mathrm{RV}_N$ implies that $k_M \preceq k_N$ and $\Gamma_M \preceq \Gamma_N$.
(2) We sketch a proof using the usual back-and-forth criterion. We assume $(\mathrm{EQ})_{\mathrm{RV}}$. Consider two models $\mathcal{M} = \{K_M, \Gamma_M, k_M\}$ and $\mathcal{N} = \{K_N, \Gamma_N, k_N\}$ of T in the language $\mathrm{L}_{\Gamma,k,\mathrm{ac}}$, and a partial automorphism $f = (f_K, f_\Gamma, f_k) : A = (K_A, \Gamma_A, k_A) \to B = (K_B, \Gamma_B, k_B)$ between a substructure $A \subseteq \mathcal{M}$ and a substructure $B \subseteq \mathcal{N}$. Moreover, we assume f_k and f_Γ to be elementary as morphisms respectively of fields and of ordered abelian groups. We want to extend f to an elementary embedding of M into N. By elementarity, we may extend f_Γ (resp. f_k) to an elementary embedding of ordered abelian groups $\tilde{f}_\Gamma : \Gamma_M \to \Gamma_N$ (resp. to an elementary embedding of fields $\tilde{f}_k : k_M \to k_N$). Then, by studying quantifier-free formulas, one sees that $\tilde{f} = f \cup \tilde{f}_\Gamma \cup \tilde{f}_k$ is a partial isomorphism of substructures. Without loss, assume that $\Gamma_A = \Gamma_M$ and $k_A = k_M$ and reset the notation. As the ac-map induces a splitting of the exact sequence

$$1 \to k^\star \xrightarrow{\iota} \mathrm{RV}^\star \xrightarrow{\mathrm{val_{rv}}} \Gamma \to 0,$$

we have the bijections $\mathrm{RV}_M^\star \simeq k_M^\star \times \Gamma_M$ and $\mathrm{RV}_N^\star \simeq k_N^\star \times \Gamma_N$. Hence, the partial isomorphism f induces an elementary embedding of RV-structure

$f_{\mathrm{RV}} : (\mathrm{RV}_M, \oplus, \cdot, \mathbf{1}, \mathbf{0}) \to (\mathrm{RV}_N, \oplus, \cdot, \mathbf{1}, \mathbf{0})$, and $f_K \cup f_{\mathrm{RV}}$ is a partial isomorphism of substructures in the language L_{RV}. By relative quantifier elimination down to RV, $f_K \cup f_{\mathrm{RV}}$ extends to an elementary embedding $\tilde{f} = (\tilde{f}_K, f_{\mathrm{RV}})$ of $\{M, \mathrm{RV}_M\}$ into $\{N, \mathrm{RV}_N\}$. One sees that $\tilde{f}_K \cup f_\Gamma \cup f_k : \mathcal{M} \to \mathcal{N}$ is an embedding extending the original partial isomorphism f. By back-and-forth, T satisfies $(\mathrm{EQ})_{\Gamma,k,\mathrm{ac}}$. □

More specifically, we will have to study 1-dimensional definable sets $D \subset K$. Flenner showed in [40] that in Henselian valued fields of characteristic 0, definable sets can be written with field-sorted linear terms (See Fact 1.2.28). This property will be of essential use. Let us give it also an abbreviation:

Definition 1.2.15. Let T be the theory of a Henselian valued field K in the language L_{RV}. We denote by $(\mathrm{Lin})_{\mathrm{RV}}$ the following property: any formula $\phi(x)$ with parameters in K and with $|x| = 1$ is equivalent to a formula of the form

$$\phi_{\mathrm{RV}}(\mathrm{rv}(x - a_1), \ldots, \mathrm{rv}(x - a_r), \alpha) \tag{1.3}$$

where $r \in \mathbb{N}$ and ϕ_{RV} is an RV-formula with a tuple of parameters $\alpha \in \mathrm{RV}(K)$ and $a_1, \ldots, a_r \in K^1$.

Notice that it is an improvement of a relative quantifier elimination down to RV for unary-definable sets: the term inside rv is linear in x where $(\mathrm{EQ})_{\mathrm{RV}}$ gives only a polynomial in x.

Its algebraic counterpart seems to be the following:

Definition 1.2.16. A valued field is called *algebraically maximal* if it admits no immediate algebraic extension.

In particular, Henselian valued fields of equicharacteristic 0 are algebraically maximal (by the fundamental equality) as well as algebraically closed valued fields. Delon proved that this is actually a first order property. For details, we refer to Delon's thesis [26] and a recent work of Halevi and Hasson in [41].

We show now that algebraically maximal valued fields with quantifier elimination relative to RV enjoy Property $(\mathrm{Lin})_{\mathrm{RV}}$. This fact was suggested by Yatir Halevi. Notice that a similar statement has been proved by Peter Sinclair for valued fields in the Denef-Pas language $\mathrm{L}_{\Gamma,k,ac}$ (see [70, Theorem 2.1.1.]). We thank both of them for their enlightenment.

Let $\mathcal{K} = \{(K, \cdot, +, 0, 1), (\mathrm{RV}, \mathbf{0}, \mathbf{1}, \cdot, \oplus), \mathrm{rv} : K^\star \to \mathrm{RV}^\star\}$ be a Henselian valued field viewed as a structure in the language L_{RV}. Let $\mathbb{K} \succ \mathcal{K}$ be a monster model. Let $x \in \mathbb{K} \setminus K$. We denote by $I_K(x)$ the set of values $\{\mathrm{val}(x - a) |\ a \in K\}$.

Fact 1.2.17 (Delon). *$I_K(x)$ has no maximum if and only if the extension $K(x)/K$ is immediate.*

Lemma 1.2.18. *Assume that $\mathcal{K}$ is algebraically maximal. Let $x \in \mathbb{K} \setminus K$.*

- *If $I_K(x)$ has no maximum, let $(\gamma_i = \operatorname{val}(x-a_i))_{i\in I}$ be a co-final sequence of values in $I_K(x)$. Then the quantifier-free type $\operatorname{qftp}(x/K)$ is implied by the type $\{\operatorname{val}(x-a_i) = \gamma_i\}_{i\in I}$.*
- *If $I_K(x)$ has a maximum, then there is $a \in K$ such that $\operatorname{tp}(\operatorname{rv}(x - a)/\operatorname{RV}(K)$ determined $\operatorname{qftp}(x/K)$. Moreover, $\operatorname{RV}(K(x))$ is generated by $\operatorname{RV}(K)$ and $\operatorname{rv}(x-a)$.*

Proof. Assume that $I_K(x)$ has no maximum, and let $(\gamma_i = \operatorname{val}(x - a_i))_{i\in I}$ be a co-final sequence of values in $I_K(x)$. Then, the sequence $(a_i)_{i\in I}$ is a pseudo-Cauchy sequence in K, with no pseudo-limit in K and which pseudo-converges to x. The extension is immediate and $\mathcal{K}$ is algebraically maximal. By [49], the pseudo-Cauchy sequence $(a_i)_{i\in I}$ is of transcendental type. Then, if $x' \in \mathbb{K}$ is another pseudo-limit of $(a_i)_{i\in I}$, the two extensions $K(x)$ and $K(x')$ are isomorphic over K. In other words, the quantifier-free type of x over K is uniquely determined by $\{\operatorname{val}(x - a_i) = \gamma_i\}_{i\in I}$.
Assume that $I_K(x)$ has a maximum $\gamma = \operatorname{val}(x - a)$. Then, we distinguish two cases.
Case 1: We have that $\operatorname{val}(x - a) \in \Gamma_K$.
Then, we have that $\operatorname{rv}(x - a) \notin \operatorname{RV}(K)$, as otherwise, it would exist $b \in K$ such that $\operatorname{val}(x-a-b) > \operatorname{val}(x-a)$, contradicting the maximality of $\operatorname{val}(x-a)$. Let $c \in K$ such that $\operatorname{val}(x - a) = \operatorname{val}(c)$. Then, since k_K is relatively algebraically closed in $k_{\mathbb{K}}$ (as $\mathbb{K}$ is an elementary extension of K) and since $\operatorname{rv}(\frac{x-a}{c})$ is in $k_{\mathbb{K}} \setminus k_K$, we have that $\operatorname{res}(\frac{x-a}{c})$ is transcendental over k_K. Without loss of generality, assume that $\frac{x-a}{c} = x$. The extension $K(x)$ is the Gauss extension, thus it is unique up to K-isomorphism (see e.g. [31]). In particular, the quantifier-free type of x over K is uniquely determined by $\operatorname{tp}(\operatorname{rv}(x)/\operatorname{RV}(K))$.
We show now that $\operatorname{RV}(K(x))$ is generated by $\operatorname{RV}(K)$ and $\operatorname{rv}(x)$ in the following sense: Consider $P(X) := \sum_{i<n} a_i X^i$ a non-trivial polynomial in K, and assume that $a_{i_0}, \dots, a_{i_{k-1}}$ are the coefficient of minimum value. Since $\operatorname{rv}(x^n) \in k^\star_{\mathbb{K}}$ for all n, and since $\operatorname{rv}(x)$ is transcendental over k_K, we have:

$$\operatorname{rv}\left(\frac{P(x)}{a_{i_0}}\right) = \sum_{j<k} \frac{\operatorname{rv}(a_{i_j})}{\operatorname{rv}(a_{i_0})} \operatorname{rv}(x^{i_j}) \in k^\star_{\mathbb{K}},$$

and so

$$\operatorname{rv}(P(x)) = \bigoplus_{j<k} \operatorname{rv}(a_{i_j}) \operatorname{rv}(x)^{i_j} = \bigoplus_{i<n} \operatorname{rv}(a_i) \operatorname{rv}(x)^i.$$

Case 2: We have that $\mathrm{val}(x-a) \notin \Gamma_K$. Then for all $n \in \mathbb{N}^\star$, $n \cdot \mathrm{val}(x-a) \notin \Gamma_K$, as $\mathbb{K}$ is an elementary extension of K. Then, for any polynomial $P(x) \in K(x)$, $\mathrm{val}(P(x-a))$ can be expressed in terms of $\mathrm{val}(x-a)$. The isomorphism type of $K(x)$ over K is uniquely determined by $\mathrm{rv}(x-a) = \alpha$, or in other words the quantifier-free type of x over K is uniquely determined by $\mathrm{tp}(\mathrm{rv}(x-a)/\mathrm{RV}(K))$. Without loss of generality, we may assume that $x = x-a$. One sees as well that $\mathrm{RV}(K(x))$ is generated by $\mathrm{RV}(K)$ and $\mathrm{rv}(x)$: consider $P(X) := \sum_{i<n} a_i X^i$ a non-trivial polynomial in K. As $n \cdot \mathrm{val}(x)$ is not in Γ_K for all n, we have that:

$$\mathrm{rv}(P(x)) = \bigoplus_{i<n} \mathrm{rv}(a_i)\,\mathrm{rv}(x^i).$$

□

Theorem 1.2.19. *Assume that $\mathcal{K}$ is algebraically maximal and admits quantifier elimination relative to* RV*. Then it also satisfies the property (Lin)$_{\mathrm{RV}}$.*

Proof. By compactness, it is enough to show that any (complete) 1-type $p(x) = \mathrm{tp}(b/K)$ over K is determined by formulas of the form 1.3.

- If $p(x) = \mathrm{tp}(b/K)$ is a realised type, i.e. $b \in K$, then the type is determined by $\{\mathrm{rv}(x-b) = \mathbf{0}\}$.
- If $K(b)/K$ is immediate, then by the previous lemma, $\mathrm{qftp}(b/K)$ is determined by the type $\{\mathrm{val}(x-a_i) = \gamma_i\}_{i \in I}$, where γ_i and a_i are given by the previous lemma. This can be written in the language L_{RV}: for $i \in I$ choose $c_i \in K$ of value γ_i. Then

  $$\mathrm{val}(x-a_i) = \gamma_i \Leftrightarrow \mathrm{rv}(x-a_i) \oplus \mathrm{rv}(c_i) \neq \mathrm{rv}(x-a_i) \wedge \mathrm{rv}(x-a_i) \oplus \mathrm{rv}(c_i) \neq \mathrm{rv}(c_i).$$

  As $\mathrm{RV}(K(b)) = \mathrm{RV}(K)$, and by quantifier elimination relative to the RV-sort, $\{\mathrm{val}(x-a_i) = \gamma_i\}_{i \in I} \cup \mathrm{tp}(\emptyset/\mathrm{RV}(K))$ determines $p(x) = \mathrm{tp}(b/K)$.
- If $K(b)/K$ is non-immediate, then by the previous lemma, there is an $a \in K$ such that $\mathrm{qftp}(b/K)$ is determined by $q(\mathrm{rv}(x-a))$ where $q = \mathrm{tp}(\mathrm{rv}(b-a)/\mathrm{RV})$. As $\mathrm{RV}(K(x))$ is generated by $\mathrm{RV}(K)$ and $\mathrm{rv}(b-a)$, and by quantifier elimination relative to RV, we got that $q(\mathrm{rv}(x-a))$ determines $p(x) = \mathrm{tp}(b/K)$.

□

Definition 1.2.20. A valued field of equicharacteristic $p > 0$ is said Kaplansky if the value group is p-divisible, the residue field is perfect and does not admit any finite separable extensions of degree divisible by p.

Definition 1.2.21. Any $\{\Gamma\}$-$\{k\}$-enrichment of one of the following theories of Henselian valued fields is called *benign*:

1. Henselian valued fields of characteristic $(0,0)$,
2. algebraically closed valued fields,
3. algebraically maximal Kaplansky Henselian valued fields.

A model of a benign theory will be called a *benign Henselian valued field.*

As promised, we have:

Fact 1.2.22. *Benign theories satisfy (Im), $(EQ)_{\mathrm{RV}}$ and $(Lin)_{\mathrm{RV}}$.*

By the discussion above, it implies $(\mathrm{EQ})_{\Gamma,k,ac}$, $(\mathrm{AKE})_{\Gamma,k}$, $(\mathrm{AKE})_{\mathrm{RV}}$,$(\mathrm{SE})_{\Gamma,k}$ and $(\mathrm{SE})_{\mathrm{RV}}$.

Proof. It is clear that the set of models of a benign theory is closed under maximal immediate extensions. Concerning the property $(\mathrm{EQ})_{\mathrm{RV}}$, we just give examples of references. We also leave here references for the property $(\mathrm{EQ})_{\Gamma,k,\mathrm{ac}}$. Notice that we might not refer to original proofs. The fact that Henselian valued fields of characteristic $(0,0)$ has property $(\mathrm{EQ})_{\Gamma,k,ac}$ is the classical theorem of Pas. The proof that it has $(\mathrm{EQ})_{\mathrm{RV}}$ is in [40]. Algebraically closed valued fields (in any characteristic) eliminate quantifiers by the theorem of Robinson. One deduces the property $(\mathrm{EQ})_{\Gamma,k,ac}$ from it. One can find a proof that algebraically closed valued fields of any characteristic have $(\mathrm{EQ})_{\mathrm{RV}}$ in [43]. Algebraically maximal Kaplansky valued fields have $(\mathrm{EQ})_{\Gamma,k,ac}$ and $(\mathrm{EQ})_{\mathrm{RV}}$ by [41]. As all these fields are algebraically maximal, they satisfy the condition of Theorem 1.2.19, and thus enjoy the property $(\mathrm{Lin})_{\mathrm{RV}}$.

Finally, all these properties hold for any $\{\Gamma\}$-$\{k\}$-enrichment, as it is a particular case of $\{\mathrm{RV}\}$-enrichment, and as the sorts Γ, k and RV are closed (Fact 1.1.5).

□

We will complete our study with some transfer principle for unramified mixed characteristic Henselian valued fields with perfect residue field. As it requires further techniques, it needs to be treated independently. We first need to introduce the Witt vector construction.

Witt vectors

In the theory of unramified mixed characteristic Henselian valued fields, we will understand RV_{δ_n}-structures thanks to the well known Witt vector construction. We start by briefly recalling the definition. For more details, one can see [65]. Let k be a field of characteristic p.

Definition 1.2.23 (Witt vectors)**.** Fix $X_0, X_1, X_2, \ldots$ and $Y_0, Y_1, Y_2, \ldots$ some indeterminates. We consider the polynomials $W_0, W_1, W_2, \ldots$ in $\mathbb{Z}[X_0, X_1, X_2, \ldots]$, called *Witt polynomials* and defined by:

$$W_i = \sum_{i=0}^{n} p^i X_i^{p^{n-i}}.$$

The ring of *Witt vectors over* k, denoted by $W(k)$, is a ring of base set k^ω. The sum of $\bar{x} = (x_0, x_1, \ldots)$ and $\bar{y} = (y_0, y_1, \ldots)$ is given by:

$$\bar{x} + \bar{y} = (S_n(x_0, \ldots, x_{n-1}, y_0, \ldots, y_{n-1}))_n$$

where $S_n(X_0, \ldots, X_{n-1})$ is the unique polynomial in $\mathbb{Z}[X_0, \ldots, X_{n-1}, Y_0, \ldots, Y_{n-1}]$ such that

$$W_n(X_0, \ldots, X_{n-1}) + W_n(Y_0, \ldots, Y_{n-1}) = W_n(S_0, \ldots, S_{n-1}).$$

The product $\bar{x} \cdot \bar{y}$ is defined similarly. These operations make $W(k)$ into a commutative ring.

The *residue map* π is simply the projection to the first coordinate. The natural section of the residue map, the so called *Teichmüller lift*, is defined as follows:

$$\begin{array}{rccl} \tau: & k & \longrightarrow & W(k) \\ & a & \longmapsto & [a] := (a, 0, 0, \ldots) \end{array}.$$

Finally, all the above definitions make sense if we restrict the base-set to k^n. One gets then the *truncated ring of Witt vectors of length* n denoted by $W_n(k)$, as well as its Teichmüller map $\tau_n : k \to W_n(k)$.

Observation 1.2.24. *$(W_n(k), +, \cdot, \pi)$ is interpretable in the field $(k, +, \cdot, 1, 0)$, with base set k^n. It is clear that* $\mathrm{bdn}(W_n(k)) := \kappa^1_{inp}(W_n(k)) \leq \kappa^n_{inp}(k)$ *(we will show that they are in fact equal).*

Recall that a p-ring is a complete local ring A of maximal ideal pA and perfect residue field A/pA. It is strict if $p^n \neq 0$ for every $n \in \mathbb{N}$. Here are some basic facts about Witt vectors:

Fact 1.2.25 (see e.g. [74, Chap. 6])**.** *Recall that the field k is perfect.*

1. *The ring of Witt vectors $W(k)$ is a strict local p-ring of residue field k, unique up to isomorphism with these properties.*

2. *The Teichmüller map is given by the following: let $a \in k$, then $\tau(a)$ is the limit (for the topology given by the maximal ideal $pW(k)$) of any sequence $(a_n^{p^n})_{n<\omega}$ such that $\pi(a_n)^{p^n} = a$ for all n.*

3. *In particular τ_n is definable in the structure $(W_n(k), +, \cdot, \pi)$. Indeed $\tau_n(a)$ is the (unique) element $a_{n-1}^{p^{n-1}} \in W_n$ such that $\pi(a_{n-1})^{p^{n-1}} = a$.*

4. *for $x = (x_n)_{n<\omega} \in W(k)$, one has $x = \sum_{n<\omega} [x_n^{p^{-n}}]p^n$.*

5. *In particular, the map $\chi_i : W(k) \to k,\ x = (x_0, x_1, \dots) \mapsto x_i$ is definable in the structure $(W(k), +, \cdot, \pi)$. One has indeed*

$$x_i = \pi \left(\frac{x - p^{i-1}[x_{i-1}^{p^{1-i}}] - \cdots - p[x_1^{p^{-1}}] - [x_0]}{p^i} \right)^{p^i}.$$

Similarly, for $0 \leq i \leq n-1$ the map $\chi_{n,i} : W_n(k) \to k,\ x = (x_0, x_1, \dots, x_{n-1}) \mapsto x_i$ is definable in $(W_n(k), +, \cdot, \pi)$.

We deduce the following:

Corollary 1.2.26.
- The structure $(W_n(k), +, \cdot, \pi : W_n(k) \to k)$ is bi-interpretable on unary sets with the structure $(k^n, k, +, \cdot, p_i, i < n)$, where $p_i : k^n \to k,\ (x_0, \dots, x_{n-1}) \mapsto x_i$ is the projection map. In other words, there is a bijection $W_n(k) \simeq k^n$ which leads to identify definable sets.

- Similarly, the structure $(W(k), +, \cdot, \pi : W(k) \to k)$ is bi-interpretable on unary sets with no parameters with the structure $(k^\omega, k, +, \cdot, p_i, i < \omega)$ where $p_i : k^\omega \to k,\ (x_0, x_1, \dots) \mapsto x_i$.

This corollary, coupled with Fact 1.1.20 and Remark 1.1.51, will be one of the main argument to treat our reduction principle in the context of unramified mixed characteristic valued fields with perfect residue field.

Unramified mixed characteristic Henselian valued fields

We give a short overview on unramified valued fields, by presenting the similarities with benign valued fields. The (partial) theory of Henselian valued fields of characteristic 0 does not satisfy either $(\mathrm{EQ})_{\Gamma,k,ac}$ or $(\mathrm{EQ})_{\mathrm{RV}}$. We indeed need to get 'information' modulo $\mathfrak{m}_{\delta_n}$ in a quantifier-free way. Let us recall the leading term language of finite order:

$$\mathrm{L}_{\mathrm{RV}_{<\omega}} = \{K, (\mathrm{RV}_{\delta_n})_{n<\omega}, (\oplus_{\delta_l,\delta_m,\delta_n})_{n<l,m}, (\mathrm{rv}_{\delta_n})_{n<\omega}, (\mathrm{rv}_{\delta_n\to\delta_m})_{m<n<\omega}\},$$

where $\oplus_n$ are ternary relation symbols and $\delta_n = \mathrm{val}(p^n)$. Let us just define all the analogous properties:

$$\mathcal{K}, \mathcal{K}' \models T, \mathcal{K} \subseteq \mathcal{K}', \text{ we have } \mathcal{K} \preceq \mathcal{K}' \Leftrightarrow \mathrm{RV}_{<\omega} \preceq \mathrm{RV}'_{<\omega}. \qquad (\mathrm{AKE})_{\mathrm{RV}_{<\omega}}$$

Let us cite two main results in [40]. First we have:

Fact 1.2.27. *[40, Proposition 4.3] Let T be the theory of characteristic 0 Henselian valued fields in the language $\mathrm{L}_{\mathrm{RV}_{<\omega}}$. Then T eliminates field-sorted quantifiers.*

$$\begin{array}{c}\mathcal{K}\\ \Big| \\ \mathrm{RV}_{<\omega}\end{array}$$

This result was already proved in [10]. Again, an important consequence is that the multisorted substructure $((\mathrm{RV}_{\delta_n})_{n<\omega}, (\oplus_{\delta_l,\delta_m,\delta_n})_{n<l,m}, (\mathrm{rv}_{\delta_n\to\delta_m})_{m<n<\omega})$ is stably embedded and pure. Secondly, we have its one-dimensional improved version:

Fact 1.2.28. *[40, Proposition 5.1] Let T be the theory of Henselian valued fields K of characteristic 0 in the language $\mathrm{L}_{\mathrm{RV}_{<\omega}}$. It has the following property denoted by $(Lin)_{\mathrm{RV}_{<\omega}}$: any formula $\phi(x)$ with parameters in K and with $|x| = 1$ is given by a formula of the form*

$$\phi_{\mathrm{RV}_{\delta_n}}(\mathrm{rv}(x-a_1), \ldots, \mathrm{rv}(x-a_r), \alpha) \tag{1.4}$$

where $\phi_{\mathrm{RV}_{\delta_n}}(\mathbf{x}_1, \ldots, \mathbf{x}_r, \mathbf{y})$ is an RV_{δ_n}-formula, with a tuple of parameters $\alpha \in \mathrm{RV}_{\delta_n}(K)$ and $a_1, \ldots, a_r \in K$ and $r \in \mathbb{N}$.

Again, notice that the improvement comes from the fact that the term inside rv_{δ_n} is linear in x where Fact 1.2.27 gives only a polynomial in x. These theorems also include the case of equicharacteristic 0, and it gives the same result as cited above. Indeed, in equicharacteristic 0, we may identify $\bigcup_{n<\omega} \mathrm{RV}_{\delta_n}$ with $\mathrm{RV} = \mathrm{RV}_0$ (Remark 1.2.8). We continue with a remark on enrichment.

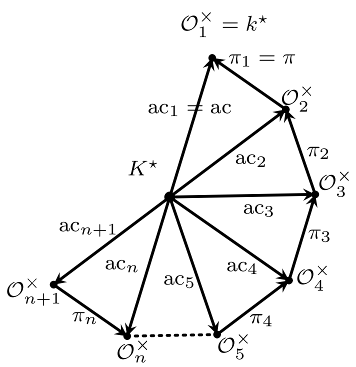
Remark 1.2.29. *By Fact 1.1.5, the first fact above holds in any* $\mathrm{RV}_{<\omega}$*-enrichment of* L_{RV}*. Indeed, first note that the* $\mathrm{RV}_{<\omega}$*-sort is closed in the language* $\mathrm{L}_{\mathrm{RV}_{<\omega}}$*, i.e. any relation symbol involving a sort* RV_{δ_n} *or any function symbol with a domain involving a sort* RV_{δ_n} *only involves such sorts. By Fact 1.1.5, the theory* T *of Henselian valued fields of characteristic* 0 *also eliminates quantifiers resplendently relative to* $\mathrm{RV}_{<\omega}$*. In other words, given an* $\mathrm{RV}_{<\omega}$*-enrichment* $\mathrm{L}_{\mathrm{RV}_{<\omega},e}$*, any complete* $\mathrm{L}_{\mathrm{RV}_{<\omega},e}$*-theory* $T_e \supset T$ *eliminates field-sorted quantifiers. A careful reading of Flenner's proof give us that Fact 1.2.28 holds resplendently as well.*

Now, let us discuss more specifically on the unramified mixed characteristic cases. We denote by T the theory of unramified mixed characteristic Henselian valued fields with perfect residue field. We assume now that $\mathcal{K}$ is such a valued field. There is by definition a smallest positive value $\mathrm{val}(p) = \delta_1$, that we denote by 1.

Notation: Notice that $\mathfrak{m} = p\mathcal{O}$ and in general that $\mathfrak{m}_{\delta_n} = \mathfrak{m}^{n+1} = p^{n+1}\mathcal{O}$ for all $n \geq 0$. We will write $\mathfrak{m}^{n+1}$ instead of $\mathfrak{m}_{\delta_n}$, $\mathcal{O}_{n+1}$ instead of $\mathcal{O}_{\delta_n}$ and RV_{n+1} instead of $\mathrm{RV}_{\delta_n} = K^\star/1 + p^{n+1}\mathcal{O}$. The projection map $\mathrm{res}_{\delta_n} : \mathcal{O} \to \mathcal{O}_{\delta_n}$ is written $\mathrm{res}_{n+1} : \mathcal{O} \to \mathcal{O}_{n+1}$ etc. The idea is to denote by RV_n the n^{th} RV-sort, as this makes sense in unramified (or finitely ramified) mixed characteristic valued fields. The purpose is also to fit with the usual notation, and it will help to simplify the notation, although this convention contradicts the previous one (RV_0 where 0 stands for the value $0 \in \Gamma$ is now RV_1, the first RV-sort).

In this context, let us define the angular component of degree n:

Definition 1.2.30. Let n be an integer greater than 0. *An angular component of order* n is a homomorphism $\mathrm{ac}_n : K^\star \to \mathcal{O}_n^\times$ such that for all $u \in \mathcal{O}^\times$, $\mathrm{ac}_n(u) = res_n(u)$. A system of angular component maps $(ac_n)_{n<\omega}$ is said to be *compatible* if for all n, $\pi_n \circ \mathrm{ac}_{n+1} = \mathrm{ac}_n$ where $\pi_n : \mathcal{O}_{n+1} \to \mathcal{O}_n \simeq \mathcal{O}_{n+1}/p^n\mathcal{O}_{n+1}$ is the natural projection.



The convention is to contract ac_1 to ac and π_1 to π. Then, let us complete the diagram given in Subsection 1.2.1:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}^\times & \longrightarrow & K^\star & \underset{\mathrm{val}}{\overset{s}{\rightleftarrows}} & \Gamma & \longrightarrow & 0 \\
 & & \downarrow \mathrm{res}_n & \swarrow \mathrm{ac}_n & \downarrow \mathrm{rv}_n & & \| & & \\
1 & \longrightarrow & \mathcal{O}^\times/(1+\mathfrak{m}^n) \simeq \mathcal{O}_n^\times & \longrightarrow & \mathrm{RV}_n^\star & \xrightarrow{\mathrm{val}_{\mathrm{rv}_n}} & \Gamma & \longrightarrow & 0
\end{array}$$

A section $s : \Gamma \to K^\star$ of the valuation gives immediately a compatible system of angular components (defined as $\mathrm{ac}_n := a \in K^\star \mapsto \mathrm{res}_n(a/s(v(a)))$). As $\mathcal{O}^\times$ is a pure subgroup of $K^\star$, such a section exists when $\mathcal{K}$ is $\aleph_1$-saturated (see Fact 1.2.38). As always, we assume that $\mathcal{K}$ is sufficiently saturated and we fix a compatible sequence $(\mathrm{ac}_n)_n$ of angular components.

We denote by $T_{ac_{<\omega}}$ the extension of T to the language $\mathrm{L}_{\Gamma,k,\mathrm{ac}_{<\omega}} := \mathrm{L}_{\Gamma,k} \cup \{\mathcal{O}_n, \mathrm{ac}_n : K \to \mathcal{O}_n,\ n \in \mathbb{N}\}$ where ac_n are interpreted as compatible angular components of degree n (see for instance [3]).

The following proposition is well known and has been used for example in [12, Corollary 5.2]. It states how the structure RV_n and the truncated Witt vectors W_n are related.

Proposition 1.2.31. *1. The residue ring $\mathcal{O}_n$ of order n is isomorphic to $W_n(k)$, the set of truncated Witt vectors of length n .*

2. The kernel of the valuation $\mathrm{val} : \mathrm{RV}_n^\star \to \Gamma$ *is given by $\mathcal{O}^\times/(1+\mathfrak{m}^n) \simeq (\mathcal{O}/\mathfrak{m}^n)^\times$. It is isomorphic to $W_n(k)^\times$, the set of invertible elements of $W_n(k)$.*

Proof. It is clear that (2) follows from (1) as $\mathcal{O}^\times/(1+\mathfrak{m}^n) \simeq (\mathcal{O}/\mathfrak{m}^n)^\times$.

Now, we prove (1) for any discrete value group Γ. Consider

$$W' := \varprojlim_{n<\omega} \mathcal{O}_n \subset \prod_{n<\omega} \mathcal{O}_n$$

the inverse limit of the $\mathcal{O}_n$'s. It is:

- strict, *i.e.* $p^n \neq 0$ in W' for every $n < \omega$, as $\pi_{n+1}(p^n) \neq 0$ in $\mathcal{O}_{n+1} = \mathcal{O}/p^{n+1}\mathcal{O}$,
- local, as a projective limit of the local rings $\mathcal{O}_n$,
- a p-ring. Its maximal ideal is pW', it is complete as projective limit, and its residue field is the perfect field k.

By uniqueness, W' is isomorphic to $W(k)$, the ring of Witt vectors over k. One just has to notice that $W'/p^nW' \simeq \mathcal{O}/p^n\mathcal{O}$ and it follows easily that $\mathcal{O}_n \simeq W_n(k)$ for every $n < \omega$. □

Note. In the above proof, one can also recover $W(k)$ by considering the coarsening $\dot{K}$ of K by the convex subgroup $\mathbb{Z} \cdot 1$. Indeed, if we denote by K° the residue field of the coarsening, as $\mathcal{K}$ is saturated enough, one has $\varprojlim_{n<\omega} \mathcal{O}_n \simeq \mathcal{O}(K^\circ)$ (see [10]).

Fact 1.2.32 (Bélair [12])**.** *The theory $T_{ac_{<\omega}}$ of Henselian mixed characteristic valued fields with perfect residue field and with angular components eliminates field-sorted quantifiers in the language $\mathrm{L}_{\Gamma,k,\mathrm{ac}_{<\omega}}$.*

Notice that in [12], Bélair doesn't assume that the residue field k is perfect, but it is indeed necessary in order to identify the ring $\mathcal{O}_n := \mathcal{O}/\mathfrak{m}^n$ with the truncated Witt vectors $W_n(k)$. This implies as well that the residue field k and the value group Γ are pure sorts, and are orthogonal. This can be seen by analysing field-sorted-quantifier-free formulas, and by noticing that $\mathcal{O}_n \simeq W_n(k)$ is interpretable in k (Corollary 1.2.26).

By analogy with the previous paragraph, we name the following properties:

$$T_{ac_{<\omega}} \text{ eliminates } K\text{-sorted quantifiers in the language } \mathrm{L}_{\mathrm{ac}_{<\omega}}. \tag{$\mathrm{EQ})_{\Gamma,k,ac_{<\omega}}$$

$$T \text{ has quantifier elimination (resplendently) relatively to } \mathrm{RV}_{<\omega}. \tag{$\mathrm{EQ})_{\mathrm{RV}_{<\omega}}$$

Again, notice that $\mathrm{RV}_{<\omega} = \bigcup_{n<\omega} \mathrm{RV}_n$ is a closed set of sorts. As before, we get:

Fact 1.2.33. *$(EQ)_{\mathrm{RV}_{<\omega}}$ implies $(AKE)_{\mathrm{RV}_{<\omega}}$.*

We also need to adapt the axiom (Im), as it is probably safer to look for a stronger property. Indeed, one can ask the following:

Question. When do we have that, for every n, $\mathrm{RV}_n = \mathrm{RV}'_n$ for all immediate extensions $\mathcal{K}'/\mathcal{K}$?

The Ax-Kochen-Ershov property and relative quantifier elimination for Henselian unramified mixed characteristic valued fields (with possibly imperfect residue field) has been proved in [3].

In general, an immediate extension of a mixed characteristic field can have a larger RV_n-sort. For instance, let us look at the field of rational functions $K = \mathbb{Q}(X)$ and the field of formal power series $K' = \mathbb{Q}((X))$. We consider the valuation ring $\mathcal{O}'$ on K' defined by

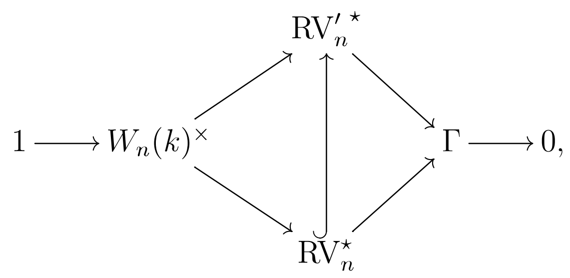
$$\mathcal{O}' := \{\sum_{i\geq n} a_i X^i) \mid n \in \mathbb{Z},\ \mathrm{val}_p(a_i) \geq 0 \text{ for all } i \text{ and } \mathrm{val}_p\, a_i = 0 \text{ only if } i \geq 0\}.$$

The value group can be identified with $(\mathbb{Z}\times\mathbb{Z}, (0,0), +, <_{lex})$ endowed with the lexicographic order $<_{lex}$ and where $\mathrm{val}(p) = (0,1) > \mathrm{val}(X) = (0,1)$. We consider the restriction of the valuation to K. Then the extension K'/K is immediate. One sees that $\mathrm{RV}_{(1,0)}(K')$ is uncountable (isomorphic as abelian group to $\mathbb{F}_p((X)) \times \Gamma$), and that $\mathrm{RV}_{(1,0)}(K)$ is countable (isomorphic as an abelian group to $\mathbb{F}_p(X) \times \Gamma$).

An extension $\mathcal{K}'/\mathcal{K}$ satisfying $\mathrm{RV}'_{<\omega} = \mathrm{RV}_{<\omega}$ will be called $\mathrm{RV}_{<\omega}$-*immediate*. Let us give a name to the condition saying that the previous question has a positive answer:

The set of models of T is closed under maximal immediate extensions and immediate extensions of models of T are $\mathrm{RV}_{<\omega}$-immediate. $(\mathrm{RV}_{<\omega}\text{-Im})$

This is satisfied by unramified mixed characteristic valued fields with perfect residue field. Indeed, the same argument as before proves that $\mathrm{RV}_n = \mathrm{RV}'_n$ for all $n \in \mathbb{N}$, as one has the commutative diagram:



where $W_n(k)$ is the ring of Witt vectors of order n over k.

To sum up, we have:

Fact 1.2.34. *The theory of unramified mixed characteristic Henselian valued fields with perfect residue field satisfies* $(\mathrm{RV}_{<\omega}\text{-}Im)$, $(EQ)_{\mathrm{RV}_{<\omega}}$, $(EQ)_{\Gamma,k,ac_{<\omega}}$, $(Lin)_{\mathrm{RV}_{<\omega}}$, $(AKE)_{\Gamma,k}$, $(SE)_{\Gamma,k}$ *and* $(SE)_{\mathrm{RV}_{<\omega}}$.

1.2.2 Abelian groups

We conclude these preliminaries with some facts on abelian groups. We are specifically interested in abelian groups for one main reason: we have to understand the structure of pure short exact sequences of abelian groups in order to produce our reduction principles for benign Henselian valued fields. As we obtain also reduction principles for such short exact sequences, we will take the occasion to apply it on explicit examples.

Burden in abelian groups

Let us gather here few facts on abelian groups. They will be used in Section 2.2, notably to provide examples to the Theorem 2.2.2. First, we have:

Fact 1.2.35 ([62, Theorem 2 $\mathbb{Z}$ 1])**.** *Let A be an abelian group. Let P_n be a predicate for n-divisibility in A. Then $\{A, +, -, 0, \{P_n\}_{n\in\mathbb{N}_{>1}}\}$ eliminates quantifiers.*

By the work of Szmielew in [71], and later by Eklof and Fisher [30], abelian groups have been classified up to elementary equivalence. Namely, for κ an uncountable cardinal, κ-saturated abelian groups are of the form

$$\mathbb{M} = \bigoplus_p (\bigoplus_n \mathbb{Z}(p^n)^{(\alpha_{p,n})} \oplus \mathbb{Z}_{(p)}^{(\beta_p)} \oplus \mathbb{Z}(p^\infty)^{(\gamma_p)}) \oplus \mathbb{Q}^{(\delta)},$$

where:

- $\mathbb{Z}(p^n)$ is the cyclic group of p^n elements,
- $\mathbb{Z}_{(p)}$ is the additive group of the integers localised in (p),
- $\mathbb{Z}(p^\infty)$ is the Prüfer p-group,
- and $\delta, \alpha_{p,n}, \beta_p$ and γ_p are some cardinals.

See [30, Proposition 1.11]. Furthermore, when they are finite, $\alpha_{p,n}, \beta_p$ and γ_p are invariants of the theory: there are preserved by elementary extensions.

The burden (or equivalently by stability, the dp-rank) of pure abelian groups has been computed in terms of these invariants, called *Szmielew invariants*, by Halevi and Palacín in [42]. We borrow from their work the following proposition, which says that a useful criterion to witness inp-patterns is a characterization in the case of one-based groups (and in particular, in the case of unenriched abelian groups):

Proposition 1.2.36 ([42, Proposition 3.4])**.** *A stable one-based group admits an inp-pattern of depth κ if and only if there exists* $\mathrm{acl}^{eq}(\emptyset)$*-definable subgroups* $(H_\alpha)_{\alpha<\kappa}$ *such that for any* $i_0 < \kappa$*, one has:*

$$\left[\bigcap_{\alpha\neq i_0} H_\alpha \ : \ \bigcap_{\alpha} H_\alpha\right] = \infty.$$

If $(b_{\alpha,j})_{j<\omega}$ *are representatives of pairwise distinct classes of* $\bigcap_{\alpha\neq i_0} H_\alpha$ *modulo* $\bigcap_\alpha H_\alpha$*, an inp-pattern of depth* κ *is given by* $\{x \in b_{\alpha,j}H_\alpha\}_{\alpha<\kappa, j<\omega}$*.*

We will use this criterion to provide examples to Theorem 2.2.2.

Quantifier elimination result in pure short exact sequences

Definition 1.2.37. Let B a group and A a subgroup. We say that A is a *pure subgroup* of B if for all a in A, $n \in \mathbb{N}$, a is n-divisible in B if and only if a is n-divisible in A.

We recall the following fundamental fact:

Fact 1.2.38. *Let* $\mathcal{M}$ *be an* $\aleph_1$*-saturated structure, and let* A, B *be two definable abelian groups, and assume that* A *is a pure subgroup of* B*. Then the exact sequence of abelian groups* $0 \to A \to B \to B/A \to 0$ *splits: there is a group homomorphism* $\alpha : B \to A$ *such that* $\alpha_{|A}$ *is the identity on* A*. In such case,* B *is isomorphic as a group to* $A \times B/A$*.*

More precisely, it is an immediate corollary of a more general statement on *pure-injectivity.* See [13, Theorem 20 p.171].

Assume that we have a pure short exact sequence of abelian groups

$$0 \longrightarrow A \xrightarrow{\ \iota\ } B \xrightarrow{\ \nu\ } C \longrightarrow 0\ .$$

(meaning that $\iota(A)$ is a pure subgroup of B). We treat it as a three-sorted structure (A, B, C, ι, ν), with a group structure for all sorts. In fact, in our main applications, we will consider such a sequence with more structure on A and C. Let us explicitly state all results resplendently, by working in an enriched language. So, let $\mathcal{M} = (A, B, C, \iota, \nu, \ldots)$ be an $\{A\}$-enrichment of a $\{C\}$-enrichment (for short: an $\{A\}$-$\{C\}$-enrichment) of the exact sequence in a language that we will denote by L, and we denote its theory by T. We will always assume that $\mathcal{M}$ is sufficiently saturated ($\aleph_1$ saturated will be enough).
Hypothesis of purity implies the exactness of the following sequences for $n \in \mathbb{N}$:

$$0 \longrightarrow A/nA \xrightarrow{\ \iota_n\ } B/nB \xrightarrow{\ \nu_n\ } C/nC \longrightarrow 0\ .$$

One has indeed that

$$\frac{A + nB}{nB} \simeq \frac{A}{A \cap nB} = \frac{A}{nA}.$$

We consider for $n \geq 0$ the following maps:

- the natural projections $\pi_n : A \to A/nA$,
- the map

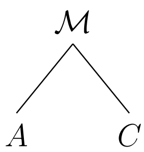
$$\begin{array}{rccc} \rho_n : & B & \to & A/nA \\ & b & \mapsto & \begin{cases} 0_{A/nA} & \text{if } b \notin \nu^{-1}(nC) \\ \iota_n^{-1}(b + nB) & \text{otherwise,} \end{cases} \end{array}$$

where $0_{A/nA}$ is the zero element of A/nA (often denoted by 0). Then let us consider the language

$$\mathrm{L}_q = \mathrm{L} \cup \{A/nA, \pi_n, \rho_n\}_{n \geq 0},$$

and let T_q be the natural extension of the theory T. By $< A >$, we denote the set of sorts containing A, A/nA and the new sorts possibly coming from the A-enrichment. Similarly, let $< C >$ be the set of sorts containing C and the new sorts possibly coming from the C-enrichment. By A-sort and C-sort, we will abusively refer to $< A >$ and $< C >$ respectively, and similarly for A-formulas and C-formulas. Aschenbrenner, Chernikov, Gehret and Ziegler prove the following result:

Fact 1.2.39 ([4]). *The theory T_q (resplendently) eliminates B-sorted quantifiers.*



More precisely, all L_q-formulas $\phi(x)$ with a tuple of variables $x \in B^{|x|}$ are equivalent to boolean combinations of formulas of the form:

1. *$\phi_C(\nu(t_0(x)), \dots, \nu(t_{s-1}(x)))$ where $t_i(x)$'s are terms in the group language, and ϕ_C is a C-formula,*

2. *$\phi_A(\rho_{n_0}(t_0(x)), \dots, \rho_{n_{s-1}}(t_{s-1}(x)))$ where the $t_i(x)$'s are terms in the group language, where $s, n_0, n_1, \dots, n_{s-1} \in \mathbb{N}$, and where ϕ_A is an A-formula.*

In particular there is no occurrence of the symbol ι.

In particular, notice that the formula $t(x) = 0$ is equivalent to $\nu(t(x)) = 0 \wedge \rho_0(t(x)) = 0$ and $\exists y\ ny = t(x)$ is equivalent to $\exists y_C\ ny_C = \nu(t(x)) \wedge \rho_n(t(x)) = 0$.

We have:

Corollary 1.2.40. In the theory T_q, $< A >$ and $< C >$ are stably embedded, pure (see Definition 1.1.7) and orthogonal to each other.

In fact, it can be easily deduced from the existence of a section. The following is more technical but highlights the fact that one does not need the function ι in order to express definable sets in $\bigcup_{n<\omega} A/nA$.

Proof. In this proof, A (resp. C) abusively refer to the union of the sorts $< A >$ (resp. $< C >$). The C-sort is pure and stably embedded by Fact 1.2.39 and closedness of C. It is also clear for the sort A, even if A is not a closed sort: one only needs to deal with the map $\iota : A \to B$. If D is a definable set in $A^{|x_A|}$, it is given by a disjunction of formulas of the form

$$\begin{aligned}\phi(x_A) =&\phi_A\left(\rho_{n_0}(k_0\iota(t_0(x_A)) + b_0), \dots, \rho_{n_{s-1}}(k_{s-1}\iota(t_{s-1}(x_A)) + b_{s-1}), a\right)\\ &\wedge \phi_C\left(\nu(\iota(t_0(x_A))), \dots, \nu(\iota(t_{s-1}(x_A))), c\right).\end{aligned}$$

where x_A is a tuple of A-variables, the $t_i(x_A)$'s are terms in the group language, $s, k_0, \dots, k_{s-1} \in \mathbb{N}$, $b_0, \dots, b_{s-1} \in B$, and $a \in A$ and $c \in C$ are tuples of parameters (notice that we also used that ι and ν are morphisms). We apply now the following transformation in order to get a new formula $\phi'(x_A)$:

- For $l < s$, if $\nu(b_l) \notin n_l C$, then replace $\rho_n(k_l\iota(t_l(x_A)) + b_l)$ by $0_{A/n_lA}$.
- For $l < s$, if $\nu(b_l) \in n_l C$, replace $\rho_{n_l}(k_l\iota(t_l(x_A)) + b_l)$ by $k_l\pi_{n_l}(t_l(x_A)) + \rho_{n_l}(b_l)$.
- For $l < s$, replace $\nu(\iota(t_l(x_A))$ by 0_C.

We obtain a pure A-formula $\phi'(x_A)$ such that $\phi'(A^{|x_A|}) = \phi(A^{|x_A|})$. Orthogonality can also be proved similarly. □

The proof shows that one does not need the function $\iota : A \to B$ in order to describe definable sets in A. In a certain sense, $< A >$ is a 'closure' of A, as it describes the induced structure on A, with no resort to any symbol from $\mathrm{L}_q \setminus \mathrm{L}_q|_{<A>}$.

Chapter 2

Burden in Henselian valued fields

In this chapter, we compute the burden of benign Henselian valued fields (Definition 1.2.21) and of unramified mixed characteristic Henselian valued fields with perfect residue field in terms of the burden of the value group and that of the residue field. The first section is common for both cases and treats the reduction from the valued field to the sort RV (resp. the sorts $\mathrm{RV}_{<\omega}$). For the reduction to the value group and residue field, we treat (separately) the case of benign Henselian valued fields in Section 2.3 and the case of unramified mixed characteristic henselian valued fields with perfect residue field in Section 2.4. They are both deduced from a computation of burden in short exact sequences of abelian groups, that we present in Subsection 2.2.

2.1 Reductions to RV and $\mathrm{RV}_{<\omega}$

We compute here the burden of Henselian valued fields of characteristc 0 in terms of burden of $\mathrm{RV}_{<\omega}$. As we explained in Section 1.2.1, mixed characteristic Henselian valued fields satisfy $(\mathrm{EQ})_{\mathrm{RV}_{<\omega}}$ (quantifier elimination relative to the union of sorts $\mathrm{RV}_{<\omega}$.), but do not satisfy in general $(\mathrm{EQ})_{\mathrm{RV}}$ (elimination of quantifiers relative to RV). Our result includes naturally the case of unramified mixed Henselian valued fields, and also equicharacteristic 0 Henselian valued fields. In the former case, we have a computation of the burden in term of the burden of RV, as in equicharacteristic 0 the structures RV and $\mathrm{RV}_{<\omega}$ can be identified (Remark 1.2.8). In fact, the proof that we are going to present can be adapted for all benign valued fields, and allows us to express their burden in term of the burden of their RV-sort.

2.1.1 Reductions

The aim of this subsection is to prove the following:

Theorem 2.1.1. *Let K be a Henselian valued field of characteristic* $(0, p)$, $p \geq 0$. *Let M be a positive integer and assume $\mathcal{K}$ is of burden M. Then, the sort* $\mathrm{RV}_{<\omega}$ *with the induced structure is also of burden M. In particular, $\mathcal{K}$ is inp-minimal if and only if* $\mathrm{RV}_{<\omega}$ *is inp-minimal.*

The demonstration (below) follows Chernikov and Simon's proof for the case of equicharacteristic 0 and burden 1 (see [19]). As we said, this statement also cover the case of equicharacteristic 0. One can also generalise the proof for infinite burden (see Corollary 2.1.3 for details). A careful reading of the proof shows that one only uses properties $(\mathrm{EQ})_{\mathrm{RV}_{<\omega}}$ and $(\mathrm{Lin})_{\mathrm{RV}_{<\omega}}$.

Of course, the proof can be written for equicaracteristic 0 fields only (it becomes simpler), and then it only uses Property $(\mathrm{EQ})_{\mathrm{RV}}$ and $(\mathrm{Lin})_{\mathrm{RV}}$. As algebraically maximal Kaplansky valued fields and algebraically closed valued fields satisfy these property, we obtain in fact:

Theorem 2.1.2. *Let $\mathcal{K}$ be a benign Henselian valued field. Let M be a positive integer and assume $\mathcal{K}$ is of burden M. Then, the sort* RV *with the induced structure is also of burden M. In particular, $\mathcal{K}$ is inp-minimal if and only if* RV *is inp-minimal.*

Proof of Theorem 2.1.1. We denote by $\bar{\mathbb{Z}}$ the set of natural numbers with extremal points $\mathbb{Z} \cup \{\pm\infty\}$. Let $\{\tilde{\phi}_i(x, y_i), (c_{i,j})_{j \in \bar{\mathbb{Z}}}, k_i\}_{i<M}$ be an inp-pattern in $\mathcal{K}$ of finite depth $M \geq 2$ with $|x| = 1$, where $c_{i,j} = a_{i,j}\mathbf{b}_{i,j} \in K^{k_1} \times \mathrm{RV}_{<\omega}^{k_2}$. Notice that the set of indices is $\bar{\mathbb{Z}}$, as we will make use of one of the extreme elements $\{a_{i,-\infty}, a_{i,+\infty}\})$ later. We have to find an inp-pattern of depth M in $\mathrm{RV}_{<\omega}$. Without loss of generality, we take $(c_{i,j})_{i,j}$ mutually indiscernible. By Fact 1.2.28 and mutual indiscernibility, we can assume the formulas $\tilde{\phi}_i$ are of the form

$$\tilde{\phi}_i(x, c_{i,j}) = \phi_i(\mathrm{rv}_{\delta_n}(x - a_{i,j;1}), \dots, \mathrm{rv}_{\delta_n}(x - a_{i,j;k_1}); \mathbf{b}_{i,j}),$$

for some integer n and where ϕ_i are $\mathrm{RV}_{<\omega}$-formulas. Also recall that δ_n denotes the value $\mathrm{val}(p^n)$. The arguments inside symbols rv_{δ_n} are linear terms in x. In some sense, difficulties coming from the field structure have been already treated and it only remains to deal with the structure coming from the valuation.

Let $d \models \{\phi_i(\mathrm{rv}_{\delta_n}(x - a_{i,0;1}), \dots, \mathrm{rv}_{\delta_n}(x - a_{i,0;k_1}); \mathbf{b}_{i,0})\}_{i<M}$ be a solution of the first column. Before we give a general idea of the proof, let us reduce to the case where only one term $\mathrm{rv}_{\delta_n}(x - a_{i,j})$ occurs in the formula $\tilde{\phi}_i$.

Claim 2. *We may assume that for all* $i < M$, $\tilde{\phi}_i(x, c_{i,j})$ *is of the form* $\phi_i(\mathrm{rv}_{\delta_n}(x - a_{i,j;1}); \mathbf{b}_{i,j})$, *i.e.* $|a_{i,j}| = k_1 = 1$.

Proof. We will first replace the formula $\tilde{\phi}_0(x, c_{0,j})$ by a new one with an extra parameter.

By Lemma 1.2.12, at least one of the following two cases occurs

1. $\mathrm{WD}_{\delta_n}(\mathrm{rv}_{\delta_n}(d - a_{0,0;1}), \mathrm{rv}_{\delta_n}(a_{0,0;1} - a_{0,0;2}))$ or
2. $\mathrm{WD}_{\delta_n}(\mathrm{rv}_{\delta_n}(d - a_{0,0;2}), \mathrm{rv}_{\delta_n}(a_{0,0;2} - a_{0,0;1}))$.

According to the case, we respectively define a new formula $\psi_0(x, c_{0,j} \frown \mathrm{rv}_{\delta_n}(a_{0,j;2} - a_{0,j;1}))$ by:

1.

$$\phi_0(\mathrm{rv}_{\delta_n}(x - a_{0,j;1}), \mathrm{rv}_{\delta_n}(x - a_{0,j;1}) + \mathrm{rv}_{\delta_n}(a_{0,j;1} - a_{0,j;2}), \mathrm{rv}_{\delta_n}(x - a_{0,j;3}), \dots, \mathrm{rv}_{\delta_n}(x - a_{0,j;k}); \mathbf{b}_{0,j}) \wedge \mathrm{WD}_{\delta_n}(\mathrm{rv}_{\delta_n}(x - a_{0,j;1}), \mathrm{rv}_{\delta_n}(a_{0,j;1} - a_{0,j;2})),$$

2.

$$\phi_0(\mathrm{rv}_{\delta_n}(x - a_{0,j;2}) + \mathrm{rv}_{\delta_n}(a_{0,j;2} - a_{0,j;1}), \mathrm{rv}_{\delta_n}(x - a_{0,j;2}), \dots, \mathrm{rv}_{\delta_n}(x - a_{0,j;k}); \mathbf{b}_{0,j}) \wedge \mathrm{WD}_{\delta_n}(\mathrm{rv}_{\delta_n}(x - a_{0,j;2}), \mathrm{rv}_{\delta_n}(a_{0,j;2} - a_{0,j;1})).$$

We will prove that the pattern where $\tilde{\phi}_0$ is replaced by ψ_0:

$\{\psi_0(x, y_0 \frown z), (c_{0,j} \frown \mathrm{rv}_{\delta_n}(a_{0,j;2} - a_{0,j;1}))_{j \in \bar{\mathbb{Z}}}, k_0\} \cup \{\tilde{\phi}_i(x, y_i), (c_{i,j})_{j \in \bar{\mathbb{Z}}}, k_i\}_{1 \le i < M}$

is also an inp-pattern. First note that we have added $\mathrm{rv}_{\delta_n}(a_{0,j;2} - a_{0,j;1})$ to the parameters $\mathbf{b}_{0,j}$, and it still forms a mutually indiscernible array. Clearly, d is still a realisation of the first column:

$$d \models \{\psi_0(x, c_{0,0} \frown \mathrm{rv}_{\delta_n}(a_{0,0;2} - a_{0,0;1}))\} \cup \{\tilde{\phi}_i(x, c_{i,0}) \mid 1 \le i < M\}.$$

By mutual indiscernibility of the parameters, every path is consistent. Since $\psi_0(\mathcal{K}) \subseteq \tilde{\phi}_0(\mathcal{K})$, inconsistency of the first row is also clear. By induction, it is clear that we may assume that ϕ_0 is of the desired form. We can do the same for all formulas ϕ_i, $0 < i < M$. □

If the array $(a_{i,j})_{i<M, j<\omega}$ is constant equal to some $a \in K$, then we obviously get an inp-pattern of depth M in $\mathrm{RV}_{<\omega}$: $\{\phi_i(\mathbf{x}, z_i), (\mathbf{b}_{i,j})_{j \in \bar{\mathbb{Z}}}, k_i\}_{i<M}$, where $\mathbf{x}$ is a variable in RV_{δ_n} (such a pattern is said to be *centralised*). Indeed, consistency of the path is clear. If a row is satisfied by some $\mathbf{d} \in \mathrm{RV}_{\delta_n}$

, any $d \in K$ such that $\mathrm{rv}_{\delta_n}(d-a) = \mathbf{d}$ will satisfy the corresponding row of the initial inp-pattern, which is absurd. Hence, the rows are inconsistent.

The idea of the proof is to reduce the general case (where the $a_{i,j}$'s are distinct) to this trivial case by the same method as above: removing the parameters $a_{i,j} \in K$ and adding new parameters from $\mathrm{RV}_{<\omega}$ to $\mathbf{b}_{i,j}$ and specifying the formula by adding a term of the form $\mathrm{WD}(\mathrm{rv}(x-a), \mathrm{rv}(a - a_{i,j}))$. The main challenge is to find a suitable $a \in K$ for a center.

Recall that $d \models \{\phi_i(\mathrm{rv}_{\delta_n}(x - a_{i,0}); b_{i,0})\}_{i<M}$ is a solution to the first column.

Claim 3. *For all $j < \omega$, and $i, k < M$ with $k \neq i$, we have* $\mathrm{val}(d - a_{i,j}) \leq \mathrm{val}(d - a_{k,0}) + \delta_n$.

Proof. Assume not: for some $j < \omega$, and $i, k < M$ with $k \neq i$:

$$\mathrm{val}(d - a_{i,j}) > \mathrm{val}(d - a_{k,0}) + \delta_n.$$

Then, $\mathrm{rv}_{\delta_n}(a_{i,j} - a_{k,0}) = \mathrm{rv}_{\delta_n}(d - a_{k,0})$. By mutual indiscernibility, we have

$$a_{i,j} \models \{\phi_k(\mathrm{rv}_{\delta_n}(x - a_{k,l}); \mathbf{b}_{k,l})\}_{l<\omega}.$$

This contradicts inconsistency of the row k. □

In particular, for all $i, k < M$, we have $|\mathrm{val}(d - a_{k,0}) - \mathrm{val}(d - a_{i,0})| \leq \delta_n$. For $i < M$, let us denote $\gamma_i := \mathrm{val}(d - a_{i,0})$ and let γ be the minimum of the γ_i's. By definition, we have the following for all $i, k < M$:

$$\mathrm{val}(a_{i,0} - a_{k,0}) \geq \min\{\mathrm{val}(d - a_{i,0}), \mathrm{val}(d - a_{k,0})\} \geq \gamma. \qquad (\star)$$

The following claim give us a correct centre a.

Claim 4. *We may assume that there is $i < M$ such that for all $k < M$, the following holds:*

$$\gamma_k = val(d - a_{k,0}) \quad \leq \quad \min\{\mathrm{val}(d - a_{i,\infty}), \mathrm{val}(a_{i,\infty} - a_{k,0})\} + \delta_n.$$

In particular, by Proposition 1.2.10, we have:

$$\mathrm{WD}_{\delta_n}(\mathrm{rv}_{2\delta_n}(d - a_{i,\infty}), \mathrm{rv}_{2\delta_n}(a_{i,\infty} - a_{k,0})).$$

Proof. By Remark 1.2.11, it is enough to find $i < M$ such that the following holds for all $k < M$:

$$\gamma_k \leq \mathrm{val}(d - a_{i,\infty}) + \delta_n \quad \text{or} \quad \gamma_k \leq \mathrm{val}(a_{i,\infty} - a_{k,0}) + \delta_n.$$

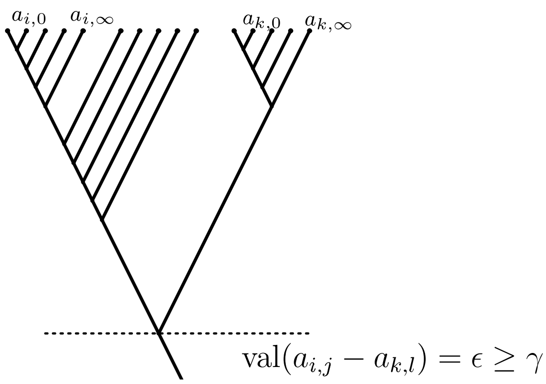
We will actually find i such that one of the following holds:

1. $\gamma_k \leq \mathrm{val}(d - a_{i,\infty}) + \delta_n$ for all $k < M$

2. $\gamma_k \leq \mathrm{val}(a_{i,\infty} - a_{k,0}) + \delta_n$ for all $k < M$

The first case will correspond to Case A, the second to Case B.

Case A : There are $0 \leq i, k < M$ with $i \neq k$ such that $\mathrm{val}(a_{i,j} - a_{k,l})$ is constant for all $j, l \in \omega$, equal to some ϵ. Note that $(\star)$ gives $\epsilon \geq \gamma$.



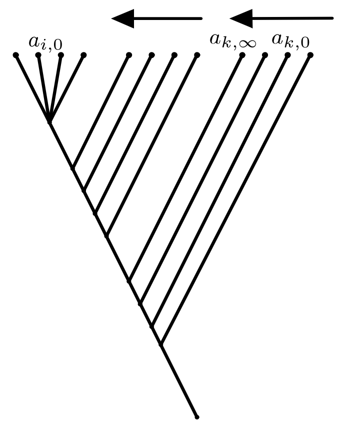
Then, we have:

$$\mathrm{val}(d - a_{i,\infty}) \geq \min\{\mathrm{val}(d - a_{i,0}), \mathrm{val}(a_{i,0} - a_{i,\infty})\} \geq \gamma$$

Indeed, $\mathrm{val}(a_{i,0} - a_{i,\infty}) \geq \min\{\mathrm{val}(a_{i,0} - a_{k,0}), \mathrm{val}(a_{k,0} - a_{i,\infty})\} = \epsilon \geq \gamma$. Hence, we have for every $0 \leq l < M$:

$$\mathrm{val}(d - a_{i,\infty}) + \delta_n \geq \gamma + \delta_n \geq \mathrm{val}(d - a_{l,0}) = \gamma_l.$$

Case B: For all $0 \leq i, k < M$ with $i \neq k$, $(\mathrm{val}(a_{i,j} - a_{k,l}))_{j,l}$ is not constant.
By Lemma 1.2.2 (2) and Lemma 1.2.3, there is $i < M$ such that for every $k < M$ and $k \neq i$, $(a_{k,l})_{l<\omega} \Rightarrow a_{i,0}$ or $(a_{k,-l})_{l<\omega} \Rightarrow a_{i,0}$. If needed, one can flip the indices and assume that for all $k \neq i$, $(a_{k,l})_{l<\omega} \Rightarrow a_{i,0}$. Note that only $(a_{i,j})_j$ could be a fan in this case.



Then we have

$$\mathrm{val}(a_{k,0} - a_{i,\infty}) = \mathrm{val}(a_{k,0} - a_{i,0}) \geq \gamma,$$

since $(a_{k,l}) \Rightarrow a_{i,\infty}$ as well. So

$$\mathrm{val}(a_{k,0} - a_{i,\infty}) + \delta_n \geq \gamma_k,$$

It remains to prove the inequality for $k = i$. Take $l \neq i$, $l < M$. We have:

$$\mathrm{val}(a_{i,0} - a_{i,\infty}) \geq \min\{\mathrm{val}(a_{i,0} - a_{l,0}), \mathrm{val}(a_{l,0} - a_{i,\infty})\} \geq \gamma.$$

Hence, $\mathrm{val}(a_{i,0} - a_{i,\infty}) + \delta_n \geq \gamma_i$.

□

Assume $i = 0$ satisfies the conclusion of the previous claim. For every $k < M$, we have

$$\mathrm{WD}_{\delta_n}(\mathrm{rv}_{2\delta_n}(d - a_{0,\infty}), \mathrm{rv}_{2\delta_n}(a_{0,\infty} - a_{k,0})).$$

Set $\tilde{b}_{i,j} := b_{i,j} \frown \mathrm{rv}_{2\delta_n}(a_{0,\infty} - a_{i,j})$ for $i < M, j < \omega$ and

$$\psi_i(\tilde{x}, \tilde{b}_{i,j}) := \phi_i\left(\mathrm{rv}_{\delta_n}(\tilde{x} + \mathrm{rv}_{2\delta_n}(a_{0,\infty} - a_{i,j})); b_{i,j}\right) \wedge \mathrm{WD}_{\delta_n}(\tilde{x}, \mathrm{rv}_{2\delta_n}(a_{0,\infty} - a_{i,j}))$$

where $\tilde{x}$ is a variable in $\mathrm{RV}_{2\delta_n}$.

This is an inp-pattern. Indeed, clearly, $\mathrm{rv}_{2\delta_n}(d - a_{0,\infty}) \models \{\psi_i(\tilde{x}, \tilde{b}_{i,0})\}_{i<M}$. By mutual indiscernibility of $(\tilde{b}_{i,j})_{i<M,j<\omega}$, every path is consistent. It remains to show that, for every $i < M$, $\{\psi_i(\tilde{x}, \tilde{b}_{i,j})\}_{j<\omega}$ is inconsistent. Assume there is $\alpha^\star \models \{\psi_i(\tilde{x}, \tilde{b}_{i,j})\}_{j<\omega}$ for some $i < M$, and let $d^\star$ be such that $\mathrm{rv}_{2\delta_n}(d^\star - a_{0,\infty}) = \alpha^\star$. Then, since $\mathrm{WD}_{\delta_n}(\alpha^\star, \mathrm{rv}_{2\delta_n}(a_{0,\infty} - a_{i,j}))$ holds for every $j < \omega$, $d^\star$ satisfies $\{\phi_i(\mathrm{rv}_{\delta_n}(x - a_{i,j}), b_{i,j})\}_{j<\omega}$, which is a contradiction. All rows are inconsistent, which concludes our proof.

□

With minor modifications, the proof goes through in the case of infinite burden λ. However, one must be careful regarding the precise statement of this generalisation. Assume we are in mixed characteristic $(0, p)$, and the burden λ is of cofinality $\mathrm{cf}(\lambda) = \omega$. Then the very first argument of the proof is no longer true: one cannot necessary assume that there are λ-many formulas $\tilde{\phi}_i(x, y_i)$ in the inp-pattern of the form

$$\tilde{\phi}_i(x, c_{i,j}) = \phi_i(\mathrm{rv}_{\delta_n}(x - a_{i,j;1}), \ldots, \mathrm{rv}_{\delta_n}(x - a_{i,j;k}); \mathbf{b}_{i,j}),$$

for a certain $n < \omega$. This depends of course of the cofinality of λ. Nonetheless, this is the only problem. One gets the following statement:

Corollary 2.1.3. Let λ be an infinite cardinal in $\mathrm{Card}^{\star}$.

- Let K be a mixed characteristic Henselian valued field. Assume that the union of sorts $\mathrm{RV}_{<\omega}$ with the induced structure is of burden λ. Then, the field K is of burden λ if $\mathrm{cf}(\lambda) > \omega$, and of burden λ or $\mathrm{act}(\lambda)$ if $\mathrm{cf}(\lambda) = \omega$.
- Let K be a benign Henselian valued field. Assume that the sort RV with the induced structure is of burden λ. Then the field K is of burden λ.

Proof. We treat the case of mixed characteristic Henselian valued field. We prove similarly the case of benign Henselian valued fields. Let $\kappa \geq \lambda$ be the burden of K, and let

$$\{\tilde{\phi}_i(x, y_i), (c_{i,j})_{j<\omega}\}_{i<\kappa}$$

be an inp-pattern of depth κ. If κ is of cofinality $\mathrm{cf}(\kappa) > \omega$, then there are κ-many formulas $\tilde{\phi}_i(x, y_i)$ in the inp-pattern of the form

$$\tilde{\phi}_i(x, c_{i,j}) = \phi_i(\mathrm{rv}_{\delta_n}(x - a_{i,j;1}), \ldots, \mathrm{rv}_{\delta_n}(x - a_{i,j;k}); \mathbf{b}_{i,j}),$$

for a certain $n < \omega$. We deduce an inp-pattern of depth κ in the $\mathrm{RV}_{<\omega}$-sort. Indeed, we follow the exact same proof with few changes in Claim 4:

- The minimum of $\{\gamma_k\}_{k<\lambda}$ may not exist, but one can pick γ in an extension of the monster model, realising the cut $\{\gamma \in \Gamma \mid \gamma < \gamma_k \text{ for all } k < \lambda\} \cup \{\gamma \in \Gamma \mid \gamma > \gamma_k \text{ for some } k < \lambda\}$. By Claim 3, we have for all $a \in K$, $\mathrm{val}(a) > \gamma$ implies $\mathrm{val}(a) + \delta_n > \gamma_k$ for all $k < \lambda$.
- Case A stays the same.

- Case B is slightly different, since an i such that for all k, $(a_{k,l})_{l<\omega} \Rightarrow a_{i,\infty}$ or $(a_{k,-l})_{l<\omega} \Rightarrow a_{i,\infty}$ does not necessarily exist either. We may distinguish three subcases:

  1. there is i such that for λ-many k, $(a_{k,l})_{l<\omega}$ or $(a_{k,-l})_{l<\omega}$ pseudo-converge to $a_{i,0}$. We conclude as in the proof.
  2. there is $i < \lambda$ such that $(a_{i,j})_{j<\omega}$ pseudo-converges to $a_{k,0}$ for λ-many k. For such a k, we have $\mathrm{val}(a_{k,0} - a_{i,\infty}) > \mathrm{val}(a_{k,0} - a_{i,0}) \geq \gamma$, and thus $\mathrm{val}(a_{k,0} - a_{i,\infty}) + \delta_n > \gamma_k$. We may conclude as well.
  3. there is $i < \lambda$ such that $(a_{i,-j})_{j<\omega}$ pseudo-converges to $a_{k,0}$ for λ-many k . This is an analogue to Subcase (2) just above, where $a_{i,-\infty}$ is taking the place of $a_{i,\infty}$.

Hence, we get $\lambda = \kappa$.
If κ is of cofinality ω, let $(\lambda_k)_{k\in\omega}$ be a sequence of successor cardinals cofinal in κ. By the previous discussion, we find an inp-pattern in $\mathrm{RV}_{<\omega}$ of depth λ_k for each λ_k. Hence, $\lambda_k \leq \lambda$ and $\kappa = \lambda$ or $\kappa = \mathrm{act}(\lambda)$. □

Remark 2.1.4.
- *Consider now an enriched Henselian valued field $\mathcal{K} = (K, \mathrm{RV}_{<\omega}, \ldots)$ of characteristic $(0, p)$, $p \geq 0$ in an $\mathrm{RV}_{<\omega}$-enrichment $\mathrm{L}_{\mathrm{RV}_{<\omega},e}$ of $\mathrm{L}_{\mathrm{RV}_{<\omega}}$. Then, the above proof still holds. The burden of K is equal (modulo the same subtleties when we consider the burden in $\mathrm{Card}^\star$) to the burden of $\mathrm{RV}_{<\omega} \cup \Sigma_e$ with the induced structure, where Σ_e is the set of new sorts in $\mathrm{L}_{\mathrm{RV}_{<\omega},e} \setminus \mathrm{L}_{\mathrm{RV}_{<\omega}}$.*
- *Similarly, an RV −enriched benign Henselian valued field has the same burden as $\mathrm{RV} \cup \Sigma_e$ where Σ_e is the set of new sorts in $\mathrm{L}_{\mathrm{RV},e} \setminus \mathrm{L}_{\mathrm{RV}}$.*

2.1.2 Applications to p-adic fields

In this subsection, p is a prime number. We will deduce from Theorem 2.1.2, as an application, that any finite extension of $\mathbb{Q}_p$ is dp-minimal. This is already known (in fact, all local fields of characteristic 0 are dp-minimal). One can refer to the classification on dp-minimal fields by Will Johnson [46]. The fact that $\mathbb{Q}_p$ is dp-minimal is due to Dolich, Goodrick and Lippel [28, Section 6] and Aschenbrenner, Dolich, Haskell, Macpherson and Starchenko in [5, Corollary 7.9.]. In Section 2.4, we will study more generally unramified mixed characteristic Henselian valued fields.

Theorem 2.1.5. *The theory of any finite extension of $\mathbb{Q}_p$ in the language of rings is dp-minimal.*

A characterisation of dp-minimality is the following: for any mutually indiscernible sequences $(a_i)_{i<\omega}$ and $(b_i)_{i<\omega}$ and any point c, one of these two sequences is indiscernible over c. As we already mention earlier, a theory is dp-minimal if and only if it is NIP and inp-minimal (see [68, Lemma 1.4]). Since finite extensions of $\mathbb{Q}_p$ are NIP, we have to prove that they are inp-minimal. Recall first that the valuation in a finite extension of $\mathbb{Q}_p$ is definable in the language of rings:

$$\mathrm{val}(x) \geq 0 \ \Leftrightarrow\ \exists y\ 1 + \pi x^q = y^q,$$

where π is an element of minimal positive valuation and q is a prime with $q \neq p$. We can safely consider $\mathbb{Q}_p$ in the two-sorted language of valued fields $\mathrm{L} = \mathrm{L}_{\mathrm{Mac}} \cup \mathrm{L}_{\mathrm{Pres}} \cup \{\mathrm{val}\}$, where $\mathrm{L}_{\mathrm{Mac}} = \mathrm{L}_{\mathrm{Rings}} \cup \{P_n\}_{n\geq 2}$ is the language of Macintyre with a predicate P_n for the subgroup of nth-power of $\mathbb{Q}_p$ and where $\mathrm{L}_{\mathrm{Pres}}$ is the language of Presburger arithmetic. We have the following well known result, that we already discussed in the example below Proposition 1.1.10:

Fact 2.1.6. *The theory* $\mathrm{Th}(\mathbb{Q}_p)$ *eliminates quantifiers. In particular, the value group is a pure sort.*

Let $\mathcal{K} = (K, \Gamma)$ be a finite extension of $\mathbb{Q}_p$ and let $\pi \in K$ be an element of minimal positive valuation. By interpretability, we obtain:

Remark 2.1.7. *The value group* Γ *is purely stably embedded in* $\mathcal{K}$*. Since* Γ *is a Z-group (as a finite extension of a Z-group), it is in particular inp-minimal.*

Fix some $n \in \mathbb{N}$. We have the following exact sequence

$$1 \longrightarrow \mathcal{O}^\times/(1+\mathfrak{m}_{\delta_n}) \longrightarrow \mathrm{RV}^\star_{\delta_n} \xrightarrow{\mathrm{val}_{\mathrm{rv}_{\delta_n}}} \Gamma \longrightarrow 0\ ,$$

where $\delta_n = \mathrm{val}(p^n)$ and $\mathfrak{m}_{\delta_n} = \{x \in K \mid \mathrm{val}(x) > \mathrm{val}(p^n)\}$. One sees that $(\mathcal{O}/\mathfrak{m}_{\delta_n})^\times \simeq \mathcal{O}^\times/(1+\mathfrak{m}_{\delta_n})$ is finite, or in other words, that the valuation map $\mathrm{val}_{\mathrm{rv}_{\delta_n}}$ is finite to one. It follows by Lemma 1.1.30 that RV_{δ_n} is also inp-minimal. Since this holds for arbitrary $n \in \mathbb{N}$, $\mathrm{RV} = \bigcup_n \mathrm{RV}_{\delta_n}$ is inp-minimal. We conclude by using Theorem 2.1.2.

The next application is a anticipation of the next paragraph. We provide a new proof of the non-uniform definability of an angular component. It can in fact already be deduced from [19]. Recall that an angular component is a group homomorphism $\mathrm{ac} : (K^\star, \cdot) \to (k^\star, \cdot)$ such that $ac_{|\mathcal{O}^\times} = \mathrm{res}_{\,|\mathcal{O}^\times}$.

$$\begin{array}{ccccccccc}
1 & \longrightarrow & O^\times & \longrightarrow & K^\star & \xrightarrow[\mathrm{val}]{} & \Gamma & \longrightarrow & 0 \\
 & & {\scriptstyle \mathrm{res}}\downarrow & \swarrow^{\mathrm{ac}}\ \ \swarrow^{\mathrm{ac}_{\mathrm{rv}}} & \downarrow{\scriptstyle \mathrm{rv}} & & \| & & \\
1 & \longrightarrow & O^\times/1+\mathfrak{m} \simeq k^\star & \longrightarrow & \mathrm{RV}^\star & \xrightarrow{\mathrm{val}_{\mathrm{rv}}} & \Gamma & \longrightarrow & 0
\end{array}$$

Consider any theory T_{ac} of a valued field endowed with an ac-map, and assume that both the value group Γ and residue field k are infinite. Then by Fact 1.1.29 and bi-interpretability on unary sets, on sees that the RV-sort is of burden at least 2. The set $\mathrm{RV}^\star$ is indeed in definable bijection with the direct product $\Gamma \times k^\star$.

In the field of p-adics $\mathbb{Q}_p$, an angular component ac is definable in the language of rings. We can show now easily that this definition cannot be uniform:

Corollary 2.1.8. There is no formula which gives a uniform definition of an ac-map in $\mathbb{Q}_p$ for every prime p.

Notice that this has already been observed by Pas in [59].

Proof. By Chernikov-Simon [19], we know that the ultraproduct of p-adic $\mathcal{F} = \prod_{\mathcal{U}} \mathbb{Q}_p$, where $\mathcal{U} \subset \mathcal{P}$ is an ultrafilter on the set of primes, is inp-minimal in the language of rings (recall that the p-adic valuation is uniformly definable in $\mathrm{L}_{\mathrm{Rings}}$). The residue field and the value group are infinite since they are respectively a pseudo-finite field and a $\mathbb{Z}$-group. By the above discussion, the ac-map cannot be defined in the language of rings, as it would contradict inp-minimality. □

2.2 Reduction in short exact sequences of abelian groups

We prove in this section that the burden of a pure short exact sequence of abelian groups

$$0 \to A \to B \to C \to 0$$

is given by the maximum of $\mathrm{bdn}(A/nA) + \mathrm{bdn}(nC)$ for $n \in \mathbb{N}$ (Theorem 2.2.2). As an RV-structure can be seen as an enrichment of such, we will deduce Theorem 2.3.4 from it. With more work, we will also be able to use it in order to prove Theorem 2.4.4.

2.2.1 Reduction

As in the paragraph 1.2.2, we consider a pure exact sequence $\mathcal{M}$ of abelian groups

$$0 \longrightarrow A \xrightarrow{\ \iota\ } B \xrightarrow{\ \nu\ } C \longrightarrow 0\ ,$$

in an $\{A\}$-$\{C\}$-enriched language L. In the following paragraph, we compute the burden of the structure $\mathcal{M}$ in terms of burden of A and that of C (in their induced structure). By bi-interpretability on unary sets, one can also consider it as a one-sorted structure $A \subset B$ where A is given by a predicate. It follows indeed from Fact1.1.20 that $\mathrm{bdn}(A \to B \to C) = \mathrm{bdn}(B, A)$. We often prefer the point of view of an exact sequence as it is more relevant for the computation of the burden. We write indifferently $\mathrm{bdn}(\mathcal{M})$, $\mathrm{bdn}(A \to B \to C)$ or $\mathrm{bdn}(B)$, as the sort B is understood as a sort of $\mathcal{M}$ with its full induced structure.

Notice that in the case where B/nB and $B_{[n]}$ are finite for all n and C is torsion free, a straight forward generalisation of [19, Proposition 4.1] gives that $\mathrm{bdn}(A \to B \to C) = \max(\mathrm{bdn}(A), \mathrm{bdn}(C))$. We will see that one can get rid of these hypothesis and obtain a more general result using Fact 1.2.39. We first show a trivial bound:

Fact 2.2.1 (Trivial bound). *Assume there is a section of the group morphism $\nu : B \to C$. Consider L_s the language L augmented by a symbol s, and interpret it by this section of ν.*

$$0 \longrightarrow A \xrightarrow{\iota} B \underset{\curvearrowleft s}{\xrightarrow{\nu}} C \longrightarrow 0 \,,$$

We have $\mathrm{bdn}_{\mathrm{L}}(C) = \mathrm{bdn}_{\mathrm{L}_s}(C)$ and $\mathrm{bdn}_{\mathrm{L}}(A) = \mathrm{bdn}_{\mathrm{L}_s}(A)$ as well as the following:

$$\max\{\mathrm{bdn}_{\mathrm{L}}(A), \mathrm{bdn}_{\mathrm{L}}(C)\} \leq \mathrm{bdn}_{\mathrm{L}}(B) \leq \mathrm{bdn}_{\mathrm{L}_s}(B) = \mathrm{bdn}_{\mathrm{L}}(A) + \mathrm{bdn}_{\mathrm{L}}(C).$$

Proof. The two first equalities are clear since A and C are stably embedded (and orthogonal) in both languages. The inequality $max\{\mathrm{bdn}_{\mathrm{L}}(A), \mathrm{bdn}_{\mathrm{L}}(C)\} \leq \mathrm{bdn}_{\mathrm{L}}(B)$ is obvious. As the burden only grows when we add structure, the inequality $\mathrm{bdn}_{\mathrm{L}}(B) \leq \mathrm{bdn}_{\mathrm{L}_s}(B)$ is also clear. The last equality come from the fact that in the language L_s, the structure $A \to B \to C$ and the structure $\{A \times C, A, C, \pi_A : A \times C \to A, \pi_C : A \times C \to C\}$ are bi-interpretable on unary sets. We conclude by Proposition 1.1.28 and Fact 1.1.20. □

Theorem 2.2.2. *Consider an $\{A\}$-$\{C\}$-enrichement of a pure exact sequence $\mathcal{M}$ of abelian groups*

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\nu} C \longrightarrow 0 \,,$$

in a language L. *We have* $\mathrm{bdn}\,\mathcal{M} = \max_{n \in \mathbb{N}}(\mathrm{bdn}(A/nA) + \mathrm{bdn}(nC))$.

In particular, if A/nA is finite for all $n \geq 1$, then $\mathrm{bdn}\,\mathcal{M} = \max(\mathrm{bdn}(A), \mathrm{bdn}(C))$.

To clarify, $\mathrm{bdn}(A/nA)$ is the burden computed in A/nA , i.e. the supremum of depth of patterns $P(x_n)$ with x_n an A/nA variable within the structure $\{(A, 0, +, \cdots), (A/nA, 0, +), \pi_n : A \to A/nA \mid n \geq 1\}$, where $\cdots$ denotes the enriched structure in A.

Remark 2.2.3.

- *If* $\mathrm{bdn}(A)$ *or* $\mathrm{bdn}(C)$ *is infinite (or equals* $\aleph_{0-}$*), then this is simply the trivial bound in Fact 2.2.1 (as then* $\max(\mathrm{bdn}(A), \mathrm{bdn}(C)) = \mathrm{bdn}(A) + \mathrm{bdn}(C)$*). Recall that the section exists as* A *is pure in* B *(Fact 1.2.38).*
- *The maximum is always attained by at least one* n*: if* $\mathrm{bdn}(A)$ *and* $\mathrm{bdn}(C)$ *are finite, this is trivial. If* $\mathrm{bdn}(A)$ *or* $\mathrm{bdn}(C)$ *is infinite, then* $n = 0$ *or* $n = 1$ *realises the maximum by the previous point.*
- *If* C *is torsion free, one has* $\mathrm{bdn}(nC) = \mathrm{bdn}(C)$ *for* $n > 0$ *(as the multiplication by* n *is a definable injection).*
- *If* $m|n$*, then* A/mA *can be seen as a quotient of* A/nA *and naturally, one has* $\mathrm{bdn}(A/mA) \leq \mathrm{bdn}(A/nA)$*.*

In the case that the sequence is unenriched, this gives us absolute results: assume that the induced structures on A/nA and nC are the structures of groups. Then, Proposition [42, Theorem 1.1.] together with Theorem 2.2.2 gives us a computation of $\mathrm{bdn}(A \to B \to C)$ in term of Szmielew invariants of A and C. We don't attempt to write a closed formula. Nonetheless, here are some examples:

Examples. We consider the following pairs of abelian groups $(A \subset B)$ with quotient C:

- $B = \mathbb{Z}_{(2)}^{(\omega)} \oplus \mathbb{Z}_{(3)}^{(\omega)}$ and $A = \mathbb{Z}_{(2)}^{(\omega)} \oplus \{0\}$. One has $\mathrm{bdn}(A/nA) + \mathrm{bdn}(nC) = 0 + 1 = 1$ for all $2 \nmid n$, and $\mathrm{bdn}(A/2nA) + \mathrm{bdn}(2nC) = 1 + 1 = 2$, which leads to $\mathrm{bdn}\,\mathcal{M} = 2$. This can already be deduced from Halevi and Palacín's work: the sort B, equipped only with its group structure, is already of burden 2. By the trivial bound, the structure $\mathcal{M}$ is also of burden 2.
- $B = \mathbb{Z}_{(2)}^{(\omega)} \oplus \mathbb{Z}_{(3)}^{(\omega)} \oplus \mathbb{Z}_{(2)}^{(\omega)} \oplus \mathbb{Z}_{(3)}^{(\omega)}$ and $A = \mathbb{Z}_{(2)}^{(\omega)} \oplus \mathbb{Z}_{(3)}^{(\omega)} \oplus \{0\} \oplus \{0\}$. Then $\mathrm{bdn}(\mathcal{M}) = 4$ (take $n = 6$ in Theorem 2.2.2). In term of subgroups, one can consider the subgroups $A + 4B$, $A + 9B$, $2B$ and $3B$. The intersection is
$$2\mathbb{Z}_{(2)}^{(\omega)} \oplus 3\mathbb{Z}_{(3)}^{(\omega)} \oplus 4\mathbb{Z}_{(2)}^{(\omega)} \oplus 9\mathbb{Z}_{(3)}^{(\omega)}.$$
One may see that these groups satisfy Proposition 1.2.36.

- $A = \mathbb{Z}(2^\infty)$, $C = \mathbb{Z}_{(2)}^{(\omega)} \oplus \mathbb{Z}_{(3)}^{(\omega)}$ and $B = A \times C$. One can see that $\mathrm{bdn}(C \to B \to A) = \mathrm{bdn}(C, B) = 3$. This equality is witnessed by the subgroups $2B, 3B$ and $C + B_{[2]}$. However, by Theorem 2.2.2, $\mathrm{bdn}\,(A \to B \to C) = \mathrm{bdn}(A, B) = 2$ as $A/nA = \{0\}$ for all $n \geq 1$.

Proof of Theorem 2.2.2. By Fact 1.1.20, we can work in the language L_q and use Fact 1.2.39. Recall that we abusively refer the union of sorts $\{A/nA\}_{n\in N}$ as the sort A. In particular, A-formula is a formula with potentially variables in A/nA for some integers n.

By Lemma 1.1.30, if A is finite, we get that $\mathrm{bdn}(B) = \mathrm{bdn}(C)$ and of course $\mathrm{bdn}(A/nA) = 0$ for all $n \in \mathbb{N}$. Assume that A is infinite. As A and C are orthogonal, so are in particular A/nA and nC. It follows by Fact 1.1.28 that $\mathrm{bdn}(A/nA \times nC) = \mathrm{bdn}(A/nA) + \mathrm{bdn}(nC)$. The definable (and surjective) map $\rho_n \times \nu : \nu^{-1}(nC) \to A/nA \times nC$ gives us that

$$\mathrm{bdn}(B) \geq \max_{n\in\mathbb{N}}(\mathrm{bdn}(A/nA) + \mathrm{bdn}(nC)).$$

It remains to show that $\mathrm{bdn}\, B \leq \max_{n\in\mathbb{N}}(\mathrm{bdn}(A/nA) + \mathrm{bdn}(nC))$. By Remark 2.2.3, we may assume that $\mathrm{bdn}(A)$ and $\mathrm{bdn}(C)$ are both finite. As A is infinite, $\mathrm{bdn}\, A \geq 1$. If $\mathrm{bdn}(\mathcal{M}) = 1$, the equality is clear. Assume that $\mathrm{bdn}(\mathcal{M}) > 1$ and let $P(x) = \{\phi_i(x, y_i), (a_{i,j})_{j<\omega}, k_i\}_{i<M}$ be an inp-pattern of finite depth $M \geq 2$, with $(a_{i,j})_{i,j}$ mutually indiscernible and $|x| = 1$. We need to show that $M \leq \mathrm{bdn}(A/nA) + \mathrm{bdn}(nC)$ for some $n \geq 0$. If x is a variable in the sort A (resp. in the sort C), $P(x)$ is an inp-pattern in A (resp. C) of depth bounded by $\mathrm{bdn}(A)$ (resp. $\mathrm{bdn}(C)$) by purity (Corollary 1.2.40). Then, the inequality holds for $n = 0$ (resp. $n = 1$).

Assume x is a variable in the sort B. Consider a line $\{\phi(x, y), (a_j)_{j<\omega}\}$ of $P(x)$ (we drop the index $i < M$ for the sake of clarity). By Fact 1.2.39, and by the fact that one can "eliminate" disjunctions in inp-patterns (see 1.1.27), we may assume that the formula $\phi(x, a_j)$ is of the form

$$\phi_A(\rho_{n_0}(t^0(x, \beta_j)), \ldots, \rho_{n_{s-1}}(t^{s-1}(x, \beta_j)), \alpha_j) \tag{2.1}$$

$$\wedge\phi_C(\nu(r^0(x, \beta_j)), \ldots, \nu(r^{k-1}(x, \beta_j)), \gamma_j), \tag{2.2}$$

where ϕ_A is an A-formula, ϕ_C a C-formula, and for $j < \omega$, $\alpha_j \in A$, $\beta_j \in B$, $\gamma_j \in C$ are parameters, $s, k, n_0, n_1, \ldots, n_{s-1} \in \mathbb{N}$ and the t^l's and r^l's are terms in the group language (one needs to keep in mind that s,k, n_l, t^l, r^l, β_j, α_j and γ_j depend on the line i). Also, notice that $\rho_{n_0}(t^0(x, \beta_j)) \neq 0 \in A/n_0A$ implies $\nu(t^0(x, \beta_j)) \in n_0C$ (a formula of the form (2.2)). By writing the following

$$\phi_A(\rho_{n_0}(t^0(x,\beta_j)),\dots,\rho_{n_{s-1}}(t^{s-1}(x,\beta_j)),\alpha_j) \simeq \\ \left(\phi_A(\rho_{n_0}(t^0(x,\beta_j)),\dots,\rho_{n_{s-1}}(t^{s-1}(x,\beta_j)),\alpha_j) \wedge \nu(t^0(x,\beta_j)) \in n_0C\right) \\ \bigvee \left(\phi_A(0,\rho_{n_1}(t^1(x,\beta_j)),\dots,\rho_{n_{s-1}}(t^{s-1}(x,\beta_j)),\alpha_j) \wedge \nu(t^0(x,\beta_j)) \notin n_0C\right),$$

and by eliminating once again the disjunction, one can assume that $\phi(x,a_j)$ (or more specifically, the formula $\phi_C(\nu(t^0(x,\beta_j),\dots,\nu(t^{s-1}(x,\beta_j)),\gamma_j)$) implies $t^0(x,\beta_j) \in \nu^{-1}(n_0C)$. We do the same for all terms $t^l(x,\beta_j)$, $l<s$. This means in particular that the list of terms $\{t^l\}_{l<s}$ is included in $\{r^l\}_{l<k}$.

Let $M' \geq 0$ be the number of rows such that

$$\left\{\phi_C(\nu(r^0(x,\beta_j),\dots,\nu(r^{k-1}(x,\beta_j)),\gamma_j)\right\}_{j<\omega}$$

is consistent. Without loss, they are the M' first rows of the pattern $P(x)$, and we denote by $P'(x)$ the sub-pattern consisting of these rows. At this point, one can prove that $M-M'$ is bounded by $\mathrm{bdn}(C)$, but we might need better (namely, $M-M' \leq \mathrm{bdn}(NC)$ for a certain N). So we will go back to this at the end of this proof. For now, we work with the sub-pattern $P'(x)$.

Terms $t(x,\beta_j)$ in the group language are of the form $kx+m\cdot\beta_j$, with $k\in\mathbb{N},\ m\in\mathbb{N}^{|\beta_j|}$.

Claim 5. *Assume that, in a line $\{\phi(x,y),(a_j)_{j<\omega}\}$ of $P'(x)$, a term $\rho_n(kx+m\cdot\beta_j)$ occurs. Then $\nu(m\cdot\beta_j) \mod nC$ is constant for all $j<\omega$.*

Proof. Assume not. By indiscernibility, $\nu(m\cdot\beta_j)$ are in distinct classes modulo nC. As

$$\phi_C(\nu(r^0(x,\beta_j),\dots,\nu(r^{s-1}(x,\beta_j)),\gamma_j) \vdash \nu(kx+m\cdot\beta_j) \in nC,$$

the ϕ_C part of the line

$$\left\{\phi_C(\nu(r^0(x,\beta_j),\dots,\nu(r^{s-1}(x,\beta_j)),\gamma_j)\right\}_{j<\omega}$$

is 2-inconsistent, contradicting the fact we chose one of the first M' lines. □

Claim 6 (Main claim)**.** *We may assume that in $P'(x)$, formulas are of the form*

$$\phi_A(\rho_N(x-d)),\alpha_j) \wedge \phi_C(\nu(x-d),\gamma_j) \tag{2.3}$$

*for a certain integer N and a certain parameter $d\in B$ **independent of the line**. In other words, in every lines, all terms $t^l(x,\beta_j)$ for $l<s$ and all terms $r^l(x,\beta_j)$ for $l<k$ are equal to $x-d$ (the pattern $P'(x)$ is said to be centralised), and $n_0=\dots=n_{s-1}=N$.*

Proof. Take any realisation d of the first column:

$$d \models \{\phi_i(x, a_{i,0})\}_{i<M}.$$

We fix an $i < M'$, and consider the i^{th} line $\{\phi(x, a_j)\}_{j<\omega}$, (again, we drop the index i for a simpler notation). For the following four steps, we fix a line i.

Step 1_i: We may assume that all terms $t^l(x, \beta_j) = k^l x + m^l \cdot \beta_j$, $l < s$ are of the form $k^l(x - d)$.
We change all terms one by one, starting with $t^0(x, \beta_j) = k^0 x + m^0 \cdot \beta_j$. We write

$$\rho_{n_0}(k^0 x + m^0 \cdot \beta_j) = \rho_{n_0}(k^0(x - d) + k^0 d + m^0 \cdot \beta_j).$$

Replace it by $\rho_{n_0}(k^0(x - d)) + \rho_{n_0}(k^0 d + m^0 \cdot \beta_j)$. This doesn't change the formula $\phi(x, a_j)$ as $\phi(x, a_j) \vdash \nu(k^0(x-d)) \in n_0 C$. Indeed, $\phi(x, a_j) \vdash \nu(k^0 x + m^0 \cdot \beta_j) \in n_0 C$ and $\nu(k^0 d + m^0 \cdot \beta_j) \in n_0 C$ since d is a solution of the first column and $\nu(m^0 \cdot \beta_j) \mod n_0 C$ is constant by the previous claim. Then, $\rho_{n_0}(k^0 d - m^0 \cdot \beta_j)$ is seen as a parameter in $A/n_0 A$, and it is added to α_j. We do the same for all terms $t^l(x, \beta_j)$, $l < s$.

Step 2_i: We may assume that all terms $r^l(x, \beta_j) = k^l x + m^l \cdot \beta_j$, $l < k$, are of the form $x - d$.
This is immediate, as ν is a morphism. Indeed, replace $\nu(r^l(x, \beta_j))$ by $k^l \nu(x - d) + \nu(k^l d + m^l \cdot \beta_j)$, where $\nu(k^l d + m^l \cdot \beta_j)$ is seen as a parameter in C, and is added to the parameters γ_j. In other words, we may assume that the formula in the ith line of the pattern $P'(x)$ is a conjunction of the form:

$$\phi_A(\rho_{n_0}(k^0(x - d), \ldots, \rho_{n_{s-1}}(k^{s-1}(x - d)), \alpha_j) \\ \wedge \phi_C(\nu(x - d), \gamma_j).$$

Step 3_i: We may assume that for $l < s$, k^l and n_l are co-prime.
We change the terms $k^l(x-d)$ one by one, starting with $k^0(x-d)$. Consider g the greatest common divisor of n_0 and k^0. Write $n_0' = \frac{n_0}{g}$ and $k^{0\prime} = \frac{k^0}{g}$. The morphism

$$\begin{aligned} A &\to g\tfrac{A}{nA} \\ a &\mapsto ga \mod nA \end{aligned}$$

induces the following isomorphism:

$$\frac{gA}{nA} \simeq \frac{A}{n_0' A + [A]_g} \simeq \frac{A/n_0' A}{\pi_{n_0'}([A]_g)}.$$

The latest quotient being interpretable with base set $A/n_0' A$, we see that our formula

$$\phi_A(\rho_{n_0}(k^0(x - d)), \ldots, \rho_{n_{s-1}}(k^{s-1}(x - d)), \alpha_j)$$

is equivalent to a formula of the form

$$\phi_A(\rho_{n'_0}(k^{0\prime}(x-d)), \rho_{n_1}(k^1(x-d)), \dots, \rho_{n_{s-1}}(k^{s-1}(x-d)), \alpha'_j).$$

By the same argument as before, we may assume that $\phi(x, a_j) \vdash \nu(k^{0\prime}(x - d)) \in n'_0C$. We proceed similarly for all remaining terms $k^l(x-d)$, $0 < l < s$.

Step 4_i**:** We may assume that $k^l = 1$ for all $l < s$.
As $\phi(x, a_j) \vdash \nu(k^l(x - d)) \in n_lC$ for all $l < s$, and as k^l and n_l are co-prime, we have that $\phi(x, a_j) \vdash \nu(x - d) \in n_lC$. Indeed, Bézout's identity give us some integer u, v such that:

$$k^lv(x-d) + un_l(x-d) = (x-d).$$

We can write $\rho_{n_l}(k^l(x-d)) = k^l\rho_{n_l}(x-d)$ for $l < s$, and the k^l's can now be "incorporated" in the formula ϕ_A.

So far, we may assume the formula $\phi(x, a_j)$ to be of the form:

$$\phi_A(\rho_{n_0}(x-d), \dots, \rho_{n_{s-1}}(x-d), \alpha_j)) \wedge \phi_C(\nu(x-d), \gamma_j). \tag{2.4}$$

Let us recall that d has been chosen independently of the row. We can apply these steps for all rows $i < M'$.

The last step is immediate and will conclude the proof of the main claim. Let N be the least common multiply of all n's which occur in **any** line of $P'(x)$.

Step 5: We may assume that N and only N occurs i.e. in any line, $n_0 = \dots = n_{s-1} = N$.
This follows from the fact that $A/n_{s-1}A$ is a quotient of A/NA (and so, is interpretable in A/NA with base set A/NA). In other words, the formulas $\phi_{i,A}(\rho_{n_{i,0}}(x-d), \dots, \rho_{n_{0,s_i-1}}(x-d), \alpha_j))$ can be seen in the sort A/NA. Notice that $\{\phi_i(x, a_{i,f(i)})\}_{i<M'}$ implies that $\nu(x-d) \in \bigcap_n nC = NC$ for any choice of function $f : M' \to \omega$. □

As now the pattern $P'(x)$ is centralised (i.e. only one K-term occurs), one can easily remark that for every line

$$\{\phi(x, a_j) = \phi_A(\rho_N(x-d)), \alpha_j) \wedge \phi_C(\nu(x-d), \gamma_j)\}_{j<\omega},$$

at most one of the following sets:

$$\{\phi_A(x_{A/NA}, y_A)), (\alpha_{i,j})_{j<\omega}\}$$

where $|y_{A/NA}| = |\alpha_j|$ and $x_{A/NA}$ is a variable in A/NA

or

$$\{\phi_C(x_C, y_C), (\gamma_j)_{j<\omega}\}$$

(where $|y_C| = |\gamma_j|$) is consistent. Indeed, this follows immediately from the fact that in the monster model, the sequence splits and $B \simeq A \times C$. Unfortunately, some ϕ_C-part have grown during the process of simplification (Step 3_i), and might no longer be consistent. It is not a problem: let M'' be the number of lines in $P'(x)$ such that the ϕ_A-part form an inconsistent line. One has $M'' \leq \mathrm{bdn}(A/NA)$ as it gives an inp-pattern in A/NA of depth M''. Without lost, they are the first M'' lines. Then, the ϕ_C-part of any of the $M - M''$ last lines of $P(x)$ is inconsistent by the remark above and the definition of $P'(x)$. One gets as well an inp-pattern of depth $M - M''$ in C. As any realisation r of $P'(x)$ (in particular, of $P(x)$) satisfies $\nu(r-d) \in NC$, one gets actually an inp-pattern in NC. It follows that $M - M'' \leq \mathrm{bdn}(NC)$. At the end, we get that $M \leq \mathrm{bdn}(A/NA) + \mathrm{bdn}(NC)$.

□

2.2.2 Applications

As main application of Theorem 2.2.2, we will deduce Theorem 2.3.4. This is the aim of the next section. For now, we want to emphasise the advantage of working resplendently by giving a straightforward generalisation of Theorem 2.2.2.

Corollary 2.2.4. Let $\mathcal{M}$ be an exact sequence of *ordered* abelian groups

$$0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\nu} C \longrightarrow 0\ ,$$

where $(A, <)$ is a convex subgroup of $(C, <)$. We consider it as a three sorted structure, with a structure of ordered abelian group for each sort, and function symbols for ι and ν. Then, we have:

$$\begin{aligned} \mathrm{bdn}\,\mathcal{M} &= \max_{n \in \mathbb{N}} (\mathrm{bdn}(A/nA) + \mathrm{bdn}(nC)) \\ &= \max\left(\mathrm{bdn}(A), \max_{n \in \mathbb{N}^\star}(\mathrm{bdn}(A/nA)) + \mathrm{bdn}(C)\right). \end{aligned}$$

Proof. As C is torsion free, $\iota(A)$ is pure in B. As for $b \in B$, $b > 0$ if and only if $\nu(b) > 0$ or $\nu(b) = 0$ and $\iota^{-1}(b) > 0$, $\mathcal{M}$ is an $\{A\}$-$\{C\}$-enrichment of a short exact sequence of abelian groups. It remains to apply Theorem 2.2.2. Notice that for all $n > 0$, we have $\mathrm{bdn}(nC) = \mathrm{bdn}(C)$ (as the multiplication by n in C is a definable injective morphism). □

2.3 Benign Henselian valued fields

Let $\mathcal{K} = (K, \Gamma, k)$ be a saturated enough benign Henselian valued field. We will compute the burden of $\mathrm{RV} := K^\star/1 + \mathfrak{m}$ in terms of the burden of k and

Γ. As the RV-sort is stably embedded, we will consider it as a structure on its own. By Fact 1.2.9, the induced structure is given by:

$$\{\mathrm{RV}, (k, \cdot, +, 0, 1), (\Gamma, +, 0, <), \mathrm{val}_{\mathrm{rv}} : \mathrm{RV} \to \Gamma, k^\star \to \mathrm{RV}\}.$$

Notice in particular that there is no need of the symbol $\oplus$ as we consider the sort k and Γ instead. The language is denoted by L. In other words the sort RV is no more than an enriched exact sequence of abelian groups:

$$1 \to k^\star \to \mathrm{RV}^\star \overset{\mathrm{val}_{\mathrm{rv}}}{\to} \Gamma \to 0,$$

where $k = k^\star \cup \{0\}$ is endowed with its field structure and Γ is endowed with its ordered abelian group structure. As Γ is torsion free, $k^\star$ is a pure subgroup of $\mathrm{RV}^\star$. The idea to consider RV as an enrichment of abelian groups is already present in [19] and has been developed in [4].

2.3.1 Reduction from RV to Γ and k

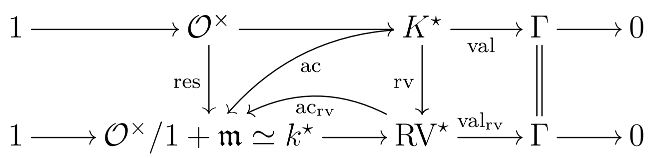
Let us recall a result of Chernikov and Simon:

Theorem 2.3.1 ([19, Theorem 1.4]). *Assume $\mathcal{K}$ is a Henselian valued field of equicharacteristic* 0. *Assume the residue field k satisfies*

$$k^\star/(k^\star)^p \text{ is finite for every prime } p. \qquad (H_k)$$

Then $\mathcal{K}$ is inp-minimal if and only if RV *with the induced structure is inp-minimal if and only if k and Γ are both inp-minimal.*

It will now be easy to extend this theorem. We have already seen the reduction to the RV-sort for any benign Henselian valued field, without the assumption (H_k). For the reduction to Γ and k, one can give first an easy bound, also independent of the assumption (H_k). Indeed, recall that in an $\aleph_1$-saturated model, any pure exact sequence of abelian groups splits (Fact 1.2.38). In particular, there exists a section $\mathrm{ac}_{\mathrm{rv}} : \mathrm{RV}^\star \to k^\star$ of the valuation $\mathrm{val}_{\mathrm{rv}}$ or equivalently, there exists an angular component $\mathrm{ac} : K^\star \to k^\star$ (as we already discussed in Subsection 2.1.2).



Recall that L_{ac} is the language L extended by a unary function $\mathrm{ac}_{\mathrm{rv}} : k \to \mathrm{RV}$. A direct translation of Fact 2.2.1 gives:

Fact 2.3.2 (Trivial bound)**.** *We have* $\mathrm{bdn}_{\mathrm{L}}(\Gamma) = \mathrm{bdn}_{\mathrm{L_{ac}}}(\Gamma)$ *and* $\mathrm{bdn}_{\mathrm{L}}(k) = \mathrm{bdn}_{\mathrm{L_{ac}}}(k)$ *as well as the following:*

$$\mathrm{bdn}_{\mathrm{L}}(\mathrm{RV}) \leq \mathrm{bdn}_{\mathrm{L_{ac}}}(\mathrm{RV}) = \mathrm{bdn}_{\mathrm{L}}(\Gamma) + \mathrm{bdn}_{\mathrm{L}}(k).$$

A valued field $\mathcal{K}_{\mathrm{ac}}$ together with an angular component ac can be considered as an RV-enrichment of $\mathcal{K}$. Using the enriched version of Theorem 2.1.2 (see Remark 2.1.4), we get:

Theorem 2.3.3. *Let* $\mathcal{K}_{\mathrm{ac}} = (\mathcal{K}, \Gamma, k, \mathrm{val}, \mathrm{ac})$ *be a benign Henselian valued field endowed with an* ac*-map. Then:* $\mathrm{bdn}(\mathcal{K}_{\mathrm{ac}}) = \mathrm{bdn}(k) + \mathrm{bdn}(\Gamma)$.

By Theorem 2.1.2 and Theorem 2.2.2, we can state one of our main theorem:

Theorem 2.3.4. *Let* $\mathcal{K}$ *be a benign Henselian valued field. Then:*

$$\mathrm{bdn}(\mathcal{K}) = \max_{n \geq 0} \left(\mathrm{bdn}(k^\star / k^{\star n}) + \mathrm{bdn}(n\Gamma) \right).$$

This gives a full answer to [19, Problem 4.3] and [19, Problem 4.4]:

Corollary 2.3.5. Let $\mathcal{K} = (K, \mathrm{RV}, k, \Gamma)$ be a benign Henselian valued field. Assume that:

$$k^\star / (k^\star)^p \text{ is finite for every prime p.} \qquad (H_k)$$

Then we have the equalities

$$\mathrm{bdn}(\mathcal{K}) = \mathrm{bdn}(\mathrm{RV}) = \max(\mathrm{bdn}(k), \mathrm{bdn}(\Gamma)).$$

Also, in the case that $\mathcal{K}$ is not trivially valued, the value group Γ is necessary of burden $\mathrm{bdn}(\Gamma) > 0$. It follows that a non-trivially valued benign Henselian field $\mathcal{K}$ is inp-minimal if and only if Γ, k are inp-minimal and k satisfies (H_k).

Similarly to the proof of non-existence of a uniform definition of the angular component of $\mathbb{Q}_p$, we can notice the following:

Remark 2.3.6. *Let* $\mathcal{K}$ *be a benign Henselian valued field of finite burden. Assume that the residue field is infinite and satisfies* (H_k)*. Then, no angular component is definable in the language of valued fields* L_{div}*.*

The reason is of course that in such a case, the two terms $\max(\mathrm{bdn}(k), \mathrm{bdn}(\Gamma))$ and $\mathrm{bdn}(k) + \mathrm{bdn}(\Gamma)$ are distinct.

All these results hold resplendently. In fact by definition, a benign Henselian valued field can have an enriched value group and residue field. Let us clarify by stating the previous theorem in an enriched language:

Remark 2.3.7. *If $\mathcal{K} = (K, \mathrm{RV}, k, \Gamma, \dots)$ is a $\{\Gamma\}$-$\{k\}$-enriched benign Henselian valued field in a $\{\Gamma\}$-$\{k\}$-enrichment $\mathrm{L}_{\Gamma,k,e}$ of $\mathrm{L}_{\Gamma,k}$, then*

$$\mathrm{bdn}(\mathcal{K}) = \mathrm{bdn}(\mathrm{RV} \cup \Sigma_e) = \max_n(\mathrm{bdn}(k^\star/k^{\star n}) + \mathrm{bdn}(n\Gamma), \mathrm{bdn}(\Sigma_e)),$$

where Σ_e is the set of new sorts in $\mathrm{L}_{\Gamma,k,e} \setminus \mathrm{L}_{\Gamma,k}$.

To conclude this short section, let us discuss more on the hypothesis (H_k) and bounded fields.

2.3.2 Bounded fields and applications

A *bounded field* is a field with finitely many extensions of degree n for every integer n. The absolute Galois group is called *small* if it contains finitely many open subgroups of index n. These two conditions are equivalent for perfect fields: a perfect field is bounded if and only if its absolute Galois group is small. Such a field K satisfies in particular the following:

$$K^\star/(K^\star)^p \text{ is finite for every prime p,} \qquad (H)$$

(see for example [39, Proposition 2.3]), and it's clear that (H) implies (H_k). It also implies:

$$\Gamma/p\Gamma \text{ is finite for every prime p,} \qquad (H_\Gamma)$$

The condition H_k might be restrictive but it allows various burdens for the residue field. However, the condition H_Γ implies inp-minimality for the value group. Indeed, an abelian group Γ satisfying (H_Γ) is called *non-singular*. In the pure structure of ordered abelian groups, non-singular ordered abelian groups are exactly the dp-minimal ones (see [45, Theorem 5.1]). We have the following examples :

Examples.

- The Hahn field $\mathbb{F}_p^{alg}((\mathbb{Z}[1/p]))$ is algebraically maximal Kaplansky Henselian. By Jahnke, Simon and Walsberg, the value group $\mathbb{Z}[1/p]$ is inp-minimal as it satisfies (H_Γ). The residue field $\mathbb{F}_p^{alg}$ satisfies (H_k) and is inp-minimal. By Theorem 2.3.4, this Hahn field is inp-minimal.
- In general, a bounded benign Henselian valued field $\mathcal{K}$ with residue field k has burden $\max(\mathrm{bdn}(k), 1)$.

Montenegro has computed the burden of some theories of bounded fields, namely bounded pseudo real closed fields (PRC fields) and pseudo p-adicaly closed fields (PpC fields). We recall here these theorems (see [58, Theorems 4.22 & 4.23]):

Theorem 2.3.8. *Let k be a bounded PRC field. Then* $\mathrm{Th}(k)$ *is* NTP_2*, strong and of burden the (finite) number of orders in k.*

Theorem 2.3.9. *Let k be an PpC field. Then* $\mathrm{Th}(k)$ *is* NTP_2 *if and only if* $\mathrm{Th}(k)$ *is strong if and only if k is bounded. In this case, the burden of* $\mathrm{Th}(k)$ *is the (finite) number of p-adic valuations in k.*

2.4 Unramified mixed characteristic Henselian valued fields

Let $\mathcal{K} = (K, \mathrm{RV}_{<\omega}, \Gamma, k)$ be an unramified Henselian valued field of characteristic $(0, p)$, $p \geq 2$ with perfect residue field k. We denote by 1 the valuation of p. The value group Γ contains $\mathbb{Z} \cdot 1$ as a convex subgroup. Recall that in this context, it is more convenient to denote the RV-sort of order n by $\mathrm{RV}_n^\star := K^\star/(1 + \mathfrak{m}^n)$ where $\mathfrak{m} = \{x \in K \mid \mathrm{val}(x) > 0\}$ is the maximal ideal of the valuation ring $\mathcal{O}$. Notice that $\mathfrak{m}^n = p^n\mathcal{O}$ for every integer n. Similarly to the previous section, we will compute the burden of $\mathrm{RV}_{<\omega} = \cup_{n\in\mathbb{N}} \mathrm{RV}_n$ in terms of the burden of k and Γ.

2.4.1 Reduction from $\mathrm{RV}_{<\omega}$ to Γ and k

Now we can look for the burden of RV_n. We start with a harmless observation:

Observation 2.4.1. *Let $m < n$ be integers. The element p^m is of valuation m. By [40, Proposition 2.8],* RV_m *is $\emptyset$-interpretable in* RV_n*, with base set* RV_n *quotiented by an equivalence relation. Hence the burden of* RV_n *can only grow with n: for $m < n$,* $\mathrm{bdn}(\mathrm{RV}_m) \leq \mathrm{bdn}(\mathrm{RV}_n)$.

Recall that in this context of unramified mixed characteristic Henselian valued fields with perfect residue field, the n^{th} residue ring $\mathcal{O}_n := \mathcal{O}/p^n\mathcal{O}$ is isomorphic to the n-truncated ring of Witt vectors (see Proposition 1.2.31). We work now in the following languages:

$$\begin{aligned}\mathrm{L} = \{&K, \Gamma, (\mathrm{RV}_n)_{n<\omega}, (W_n(k))_{n<\omega}, \mathrm{val} : K^\star \to \Gamma,\\ &(\mathrm{res}_n : \mathcal{O} \to W_n(k))_{n<\omega}, (\mathrm{rv}_n : K^\star \to \mathrm{RV}_n)_{n<\omega}\},\end{aligned}$$

which is a little variation of (and bi-interpretable with) the language $\mathrm{L}_{\mathrm{RV}_{<\omega}}$, where the structure of the RV_n's is described with exact sequences. We can also add the ac-maps to this language:

$$\mathrm{L}_{ac_{<\omega}} = \mathrm{L} \cup \{(\mathrm{ac}_n : K^\star \to W_n(k))_{n<\omega}\}.$$

Here is a consequence of Corollary 1.2.26, Remark 1.1.18 and Fact 1.1.20:

Corollary 2.4.2. We have:

- $\mathrm{bdn}(W_n(k)) = \kappa^1_{inp}(W_n(k)) = \kappa^n_{\mathrm{inp}}(k)$.
- $\mathrm{bdn}((W(k), +, \cdot, \pi : W(k) \to k)) = \kappa^{\aleph_0}_{inp}(k)$.

Recall that we have the following inequalities (see Subsection 1.1.2):

$$n \cdot \kappa^1_{\mathrm{inp}}(k) \leq \kappa^n_{\mathrm{inp}}(k) < (\kappa^1_{\mathrm{inp}}(k) + 1)^n.$$

In particular, if k is infinite then the burden of $(W_n(k), +, \cdot, \pi)$ is at least n.

In the language $\mathrm{L}_{\mathrm{ac}_{<\omega}}$, a consequence of Proposition 1.2.31 is that, for every $n < \omega$ the sort $W_n(k)$ is pure (in particular stably embedded) and orthogonal to Γ, as it is $\emptyset$-bi-interpretable with $(k^n, +, \cdot, p_i, i < n)$, which is a pure sort orthogonal to Γ. It follows that $W_n(k)$ doesn't have more structure in $\mathrm{L}_{\mathrm{ac}_{<\omega}}$ than in L. Similarly, the burden of Γ is the same in any of the above languages. Hence, we actually have the following equalities:

$$\mathrm{bdn}_{\mathrm{L}}(W_n(k)) = \mathrm{bdn}_{\mathrm{L}_{\mathrm{ac}_{<\omega}}}(W_n(k)), \tag{2.5}$$

$$\mathrm{bdn}_{\mathrm{L}}(\Gamma) = \mathrm{bdn}_{\mathrm{L}_{\mathrm{ac}_{<\omega}}}(\Gamma). \tag{2.6}$$

We are now able to give a relationship between $\mathrm{bdn}(\mathrm{RV}_n)$ and $\mathrm{bdn}(W_n(k))$.

Proposition 2.4.3. *[Trivial bound] We have*

$$\begin{aligned}\max(\mathrm{bdn}_{\mathrm{L}}(W_n(k)), \mathrm{bdn}_{\mathrm{L}}(\Gamma)) \leq \mathrm{bdn}_{\mathrm{L}}(\mathrm{RV}_n) &\leq \mathrm{bdn}_{\mathrm{L}_{\mathrm{ac}_{<\omega}}}(\mathrm{RV}_n) \\ &= \mathrm{bdn}_{\mathrm{L}}(W_n(k)) + \mathrm{bdn}_{\mathrm{L}}(\Gamma).\end{aligned}$$

Proof. By Proposition 1.2.31, we have the exact sequence of abelian groups:

$$1 \to W_n(k)^\times \to \mathrm{RV}^\star_n \to \Gamma \to 0.$$

The first inequality is clear if one shows that $\mathrm{bdn}_{\mathrm{L}}(W_n(k)) = \mathrm{bdn}_{\mathrm{L}}(W_n(k)^\times)$ where $W_n(k)^\times$ is endowed with the induced structure. The second inequality is also clear, as adding structure can only make the burden grow. Let $\{\phi_i(x, y_i), (a_{i,j})_{j<\omega}\}_{i\in\lambda}$ be an inp-pattern in $W_n(k)$, with $(a_{i,j})_{i<\lambda, j<\omega}$ mutually indiscernible. Let $d \models \{\phi(x, a_{i,0})\}_{i\in\lambda}$ be a realisation of the first column. In the case where $d \in W_n(k)^\times$, there is nothing to do.

Otherwise, $1+d \in W_n(k)^\times$ and $\{\phi_i(x-1, y_i), (a_{i,j})_{j<\omega}\}_{i\in\lambda}$ is an inp-pattern in $W_n(k)^\star$ of depth λ. This concludes the proof of the first inequality.

We work now in $\mathrm{L}_{\mathrm{ac}_{<\omega}}$, where we interpret $(ac_n)_n$ as a compatible sequence of angular components (it exists by $\aleph_1$-saturation). Recall that the burden may only increase. Then, the above exact sequences (definably) split in $\mathrm{L}_{\mathrm{ac}_{<\omega}}$, as we add a section. By the previous discussion, $W_n(k)^\times$ and Γ are orthogonal and stably embedded. We apply now Fact 1.1.29: the burden $\mathrm{bdn}_{\mathrm{L}_{\mathrm{ac}_{<\omega}}}(\mathrm{RV}_n^\star)$ is equal to $\mathrm{bdn}_{\mathrm{L}_{\mathrm{ac}_{<\omega}}}(W_n(k)^\times) + \mathrm{bdn}_{\mathrm{L}_{\mathrm{ac}_{<\omega}}}(\Gamma) = \mathrm{bdn}_{\mathrm{L}}(W_n(k)) + \mathrm{bdn}_{\mathrm{L}}(\Gamma)$.

□

Combining Corollary 2.1.3, Corollary 2.4.2 and Proposition 2.4.3, one gets:

Theorem 2.4.4. *Let $\mathcal{K} = (K, k, \Gamma)$ be an unramified mixed characteristic Henselian valued field. We denote by $\mathcal{K}_{\mathrm{ac}_{<\omega}} = (K, k, \Gamma, \mathrm{ac}_n, n < \omega)$ the structure $\mathcal{K}$ endowed with compatible* ac*-maps. Assume the residue field k is perfect. One has*

$$\mathrm{bdn}(\mathcal{K}) = \mathrm{bdn}(\mathcal{K}_{\mathrm{ac}_{<\omega}}) = \max(\aleph_0 \cdot \mathrm{bdn}(k), \mathrm{bdn}(\Gamma)).$$

And its enriched version:

Remark 2.4.5. *Let L_e be a $\{\Gamma\}$-$\{k\}$-enrichment of* L*. Let $\mathcal{K} = (K, k, \Gamma, \dots)$ be an enriched unramified mixed characteristic Henselian valued field in the language L_e. Assume the residue field k is perfect. We denote by $\mathcal{K}_{\mathrm{ac}_{<\omega}} = (K, k, \Gamma, \mathrm{ac}_n, n < \omega, \dots)$ the structure $\mathcal{K}$ endowed with compatible* ac*-maps. One has*

$$\mathrm{bdn}(\mathcal{K}) = \mathrm{bdn}(\mathcal{K}_{\mathrm{ac}_{<\omega}}) = \max(\aleph_0 \cdot \mathrm{bdn}(k), \mathrm{bdn}(\Gamma), \mathrm{bdn}(\Sigma_e)),$$

where Σ_e is the set of new sorts in $\mathrm{L}_e \setminus \mathrm{L}$.

This is a simple calculation, unless we want to consider burden in $\mathrm{Card}^\star$.

Remark 2.4.6. *Let $\mathcal{K}$ and $\mathcal{K}_{\mathrm{ac}_{<\omega}}$ as above. We consider the second definition of burden (Definition 1.1.33). We have*

$$\mathrm{bdn}(\mathcal{K}) = \mathrm{bdn}(\mathcal{K}_{\mathrm{ac}_{<\omega}}) = \max(\aleph_0 \cdot^\star \mathrm{bdn}(k), \mathrm{bdn}(\Gamma)).$$

This is what we will prove now. Remark that it implies that an unramified mixed characteristic valued field of infinite perfect residue field is never strong.

Proof. We use the same notation as before in this section. Unfortunately, due to the ambiguity in Corollary 2.1.3 concerning $\mathrm{bdn}(\mathcal{K}) \in \{\mathrm{bdn}(\mathrm{RV}_{<\omega}), \mathrm{act}(\mathrm{bdn}(\mathrm{RV}_{<\omega}))\}$ in the case that $\mathrm{cf}(\mathrm{bdn}(\mathrm{RV}_{<\omega})) = \omega$, we have to go back to the proof of Theorem 2.1.2.

We first show that $\mathrm{bdn}(\mathcal{K})$ is at least $\aleph_0 \cdot^\star \mathrm{bdn}(k)$. Recall that $W_n(k) \simeq \mathcal{O}_n := \mathcal{O}/\mathfrak{m}^n$ is interpretable (with one-dimensional base set $\mathcal{O} \subset K$), and so is the projective system $\{W_n(k), \pi_{n,m} : W_n(k) \to W_m(k),\ n > m\}$ and the projection maps $\chi_{n,n} : W_n(k) \to k, x = (x_1, \dots, x_n) \mapsto x_n$. If $\mathrm{cf}(\mathrm{bdn}(k)) > \aleph_0$, there is nothing to do as $\aleph_0 \cdot \mathrm{bdn}(k) = \aleph_0 \cdot^\star \mathrm{bdn}(k)$. Assume $\mathrm{cf}(\mathrm{bdn}(k)) \leq \aleph_0$. We write $\mathrm{bdn}(k) = \sup_{n<\omega} \lambda_n$ with $\lambda_n \in \mathrm{Card}$. Let $P_n(x_k)$ be an inp-pattern with $x_k \in k$, $|x_k| = 1$, of depth λ_n for every $n \in \omega$. Then, the pattern $P(x) = \cup_{n\in\omega} P_n(\chi_{n,n}(\pi_n(x)))$ is an inp-pattern in K of depth $\aleph_0 \cdot^\star \mathrm{bdn}(k)$. One gets:

$$\mathrm{bdn}(\mathcal{K}) \geq \max(\aleph_0 \cdot^\star \mathrm{bdn}(k), \mathrm{bdn}(\Gamma)).$$

We now prove that $\max(\aleph_0 \cdot^\star \mathrm{bdn}(k), \mathrm{bdn}(\Gamma))$ is an upper bound for $\mathrm{bdn}(\mathcal{K}_{\mathrm{ac}_{<\omega}})$.

Case 1: $\aleph_0 \cdot^\star \mathrm{bdn}(k) \geq \mathrm{bdn}(\Gamma)$.

Subcase 1.A: $\mathrm{cf}(\mathrm{bdn}(k)) > \aleph_0$. By Corollary 2.1.3, $\mathrm{bdn}(\mathcal{K}_{\mathrm{ac}_{<\omega}}) = \mathrm{bdn}(\mathrm{RV}_{<\omega}) = \sup_n(\kappa^n_{inp}(k), \mathrm{bdn}(\Gamma)) = \mathrm{bdn}(k) = \aleph_0 \cdot \mathrm{bdn}(k)$. We used the submultiplicativity of the burden, which gives here $\kappa^n_{inp}(k) = \kappa^1_{inp}(k) = \mathrm{bdn}(k)$ for all $n \in \mathbb{N}$.

Subcase 1.B: $\mathrm{cf}(\mathrm{bdn}(k)) \leq \aleph_0$. Then $\mathrm{act}(\mathrm{bdn}(\mathrm{RV}_{<\omega})) = \aleph_0 \cdot \mathrm{bdn}(k)$. By Corollary 2.1.3, we have $\mathrm{bdn}(\mathcal{K}_{\mathrm{ac}_{<\omega}}) \leq \aleph_0 \cdot \mathrm{bdn}(k)$.

Case 2: $\mathrm{bdn}(\Gamma) > \aleph_0 \cdot^\star \mathrm{bdn}(k)$. If $\mathrm{bdn}(\Gamma)$ is in Card, this is clear by Corollary 2.1.3. Assume $\mathrm{bdn}(\Gamma)$ is of the form λ_- for a limit cardinal $\lambda \in \mathrm{Card}$. Notice that this case occurs only if the sort Γ is enriched. We work in the corresponding enrichment of language $\mathrm{L}_{\mathrm{RV}_{<\omega}}$ together with ac_n-maps. We have to show that λ_- is an upper bound for $\mathrm{bdn}(\mathcal{K}_{\mathrm{ac}_{<\omega}})$. Let $P(x) = \{\theta_i(x, y_{i,j}), (c_{i,j})_{j\in\bar{\mathbb{Z}}}\}_{i\in\lambda}$ be an inp-pattern in K of depth λ with $|x| = 1$ and $(c_{i,j})_{i<\lambda, j\in\bar{\mathbb{Z}}}$ be a mutually indiscernible array. Then, by Fact 1.2.28, one can assume that each formula $\theta_i(x, c_{i,j})$ in $P(x)$ ($i \leq \lambda$, $j \in \bar{\mathbb{Z}}$) is of the form

$$\tilde{\theta}_i(\mathrm{rv}_{n_i}(x - \alpha^1_{i,j}), \dots, \mathrm{rv}_{n_i}(x - \alpha^m_{i,j}), \beta_{i,j}),$$

for some integers n_i and m, and where $\alpha^1_{i,j}, \dots, \alpha^m_{i,j} \in K$, $\beta_{i,j} \in \mathrm{RV}_{n_i}$ and $\tilde{\theta}_i$ is an RV_{n_i}-formula. As in the proof of Theorem 2.1.2, we may assume with no restriction that $m = 1$. As $\mathrm{RV}_{n_i} = W_{n_i}(k)^\times \times \Gamma$ is the direct product of the orthogonal and stably embedded sorts $W_{n_i}(k)^\times$ and Γ, we may assume $\theta_i(x, c_{i,j})$ is equivalent to a formula of the form

$$\phi_i(\mathrm{ac}_{n_i}(x - \alpha_{i,j}), a_{i,j}) \wedge \psi_i(\mathrm{val}(x - \alpha_{i,j}), b_{i,j})$$

where $\phi_i(x_{W_{n_i}}, a_{i,j})$ is a W_{n_i}-formula and $\psi_i(x_\Gamma, b_{i,j})$ is a Γ-formula. By Claim 4 in Theorem 2.1.2 (or more precisely, by a generalisation of Claim 4 to infinite depth $M = \lambda$), one may assume that there is $k < \lambda$ such that for all $i < \lambda$,

$$\mathrm{val}(d - \alpha_{i,0}) \leq \min\{\mathrm{val}(d - \alpha_{k,\infty}), \mathrm{val}(\alpha_{k,\infty} - \alpha_{i,0})\} + \max(n_i, n_k).$$

It follows that, if $\mathrm{val}(d - \alpha_{k,\infty}) = \mathrm{val}(\alpha_{k,\infty} - \alpha_{i,0})$, $\mathrm{val}(d - \alpha_{i,0})$ is equal to $\mathrm{val}(d - \alpha_{k,\infty}) + n'_i$ for some $0 \leq n'_i \leq \max(n_i, n_k)$. Otherwise, one has $\mathrm{val}(d - \alpha_{i,0}) = \min\{\mathrm{val}(d - \alpha_{k,\infty}), \mathrm{val}(\alpha_{k,\infty} - \alpha_{i,0})\}$. We can centralise $P(x)$ in $\alpha_{k,\infty}$, *i.e.* we can assume that each formula in $P(x)$ is of the form

$$\phi_i(\mathrm{ac}_{2n_i}(x - \alpha_{k,\infty}), a_{i,j}) \wedge \psi_i(\mathrm{val}(x - \alpha_{k,\infty}), b_{i,j})$$

(we add new parameters $\mathrm{val}(\alpha_{k,\infty} - \alpha_{i,j})$ and $\mathrm{ac}_{2n_i}(\alpha_{k,\infty} - \alpha_{i,j})$. Notice that once the difference of the valuation is known, $\mathrm{ac}_{n_i}(d - \alpha_{i,j})$ can be computed in terms of $\mathrm{ac}_{2n_i}(d - \alpha_{k,\infty})$ and $\mathrm{ac}_{2n_i}(\alpha_{i,j} - \alpha_{k,\infty})$). By indiscernibility, at least one of the following sets

$$\{\phi_i(x_{W_{2n_i}}, a_{i,j})\}_{j<\omega}$$

and

$$\{\psi_i(x_\Gamma, b_{i,j})\}_{j<\omega}$$

is inconsistent. Since $\lambda > \sup_n \mathrm{bdn}(W_n(k))$, we may assume that

$$\{\psi_i(x_\Gamma, y_i), (b_{i,j})_{j\in\mathbb{Z}})\}_{i<\lambda}$$

is an inp-pattern in Γ. This is a contradiction. Hence, we have $\mathrm{bdn}(\mathcal{K}) = \lambda_-$. $\square$

We end now with examples:

Examples.

1. Assume that k is an algebraically closed field of characteristic p, and Γ is a $\mathbb{Z}$-group. Then Γ is inp-minimal, *i.e.* of burden one (as it is quasi-o-minimal), and one has $\kappa^n_{inp}(k) = n$. By Theorem 2.4.4, any Henselian mixed characteristic valued field of value group Γ and residue field k has burden $\aleph_0$. In particular, the quotient field $Q(W(k))$ of the Witt vectors $W(k)$ over k is not strong.

2. Consider once again the field of p-adics $\mathbb{Q}_p$. We have $\kappa^n_{inp}(\mathbb{F}_p) = 0$ for all n, and $\mathrm{bdn}(\mathbb{Z}) = 1$. Then Theorem 2.4.4 gives $\mathrm{bdn}(\mathbb{Q}_p) = 1$.

Chapter 3

Stably embedded sub-valued fields

In this chapter we treat the question of definability of types over a submodel in benign Henselian valued fields (Definition 1.2.21) and unramified mixed characteristic Henselian valued fields of perfect residue field. We show for instance that all types realised in a given elementary extension are definable if and only if the value group and residue field satisfy an analogous property and the extension is separated (Definition 3.1.1). By Fact 1.1.45, this can be formulated this way: Such Henselian valued field is stably embedded (resp. uniformly stably embedded) in a given elementary extension if and only if the extension is separated, its value group is stably embedded in the corresponding extension of value group and its residue fields is stably embedded in its corresponding extension of residue field. In Section 3.1, we treat the case of benign valued fields. For that, we will have to prove a similar reduction for submodels of short exact sequences of abelian groups. In fact, we will also consider non-elementary extensions to enlarge the scope of applications. For example, we will show that the field of p-adics $\mathbb{Q}_p$ is stably embedded in $\mathbb{C}_p$, the algebraic closure of its completion (Corollary 3.1.19). In Section 3.2, we treat the case of elementary extension of unramified mixed characteristic Henselian valued fields with perfect residue field, using also the result on short exact sequences of abelian groups. In Section 3.3, we discuss the elementarity (in the language of pairs) of the class of elementary pairs $\mathcal{K} \prec \mathcal{L}$ where all types over $\mathcal{K}$ realised in $\mathcal{L}$ are definable (equivalently where $\mathcal{K}$ is stably embedded in $\mathcal{L}$).

3.1 Benign Henselian valued fields

Consider a benign theory T of Henselian valued fields. We want to discuss when a valued field $\mathcal{K}$ is stably embedded (resp. uniformly stably embedded) in an extension $\mathcal{L}$ which is a model of T. We need first to define the following:

Definition 3.1.1. An extension of valued fields $\mathcal{L}/\mathcal{K}$ is said separated if for any finite-dimensional K-vector subspace V of L, there is a K-basis $\{c_0, \dots, c_{n-1}\}$ of V such that for any $(a_0, \dots, a_{n-1}) \in K^n$,

$$v(\sum_{i<n} a_i c_i) = \min_{i<n}(v(a_i c_i)).$$

Equivalently, this means that for any $\{a_0, \dots, a_{n-1}\} \in K^n$,

$$\mathrm{rv}(\sum_{i<n} a_i c_i) = \bigoplus_{i<n} \mathrm{rv}(a_i)\,\mathrm{rv}(c_i).$$

Such a basis $\{c_0, \dots, c_{n-1}\}$ is called a *separating basis of V over $\mathcal{K}$*.

As we will see, it is a necessary condition for an elementary extension $\mathcal{L}/\mathcal{K}$ to be separated in order to be stably embedded. This property has been intensively studied (see [25] and [11] for more details).

Let us state the theorems of Cubides-Delon and Cubides-Ye:

Theorem 3.1.2 ([22, Theorem 1.9])**.** *Consider $\mathcal{K} \preceq \mathcal{L}$ be two algebraically closed valued fields. The following are equivalent:*

1. *$\mathcal{K}$ is stably embedded (resp. uniformly stably embedded) in $\mathcal{L}$,*
2. *the extension $\mathcal{L}/\mathcal{K}$ is separated and Γ_K is stably embedded (resp. uniformly stably embedded) in Γ_L.*

Theorem 3.1.3 ([23, Theorem 5.2.3.])**.** *Consider $\mathcal{K} \preceq \mathcal{L}$ be two real closed valued fields, or two p-adically closed valued fields. The following are equivalent:*

1. *$\mathcal{K}$ is stably embedded (resp. uniformly stably embedded) in $\mathcal{L}$,*
2. *the extension $\mathcal{L}/\mathcal{K}$ is separated, k_K is stably embedded (resp. uniformly stably embedded) in k_L and Γ_K is stably embedded (resp. uniformly stably embedded) in Γ_L.*

Only the non-uniform case is stated in [22, Theorem 1.9], but the proof goes through for the uniform case also. In [23], the uniform case can be deduced from the proof or by [23, Theorem 6.0.3]. We will generalise these theorems to extensions $\mathcal{L}/\mathcal{K}$ of benign theories T of Henselian valued fields (Subsection 1.2.1). In fact, we will not assume that $\mathcal{K}$ and $\mathcal{L}$ share the same complete theory. We proceed in two steps. For a separated extension of benign Henselian valued fields $\mathcal{L}/\mathcal{K}$, we characterise $K \subseteq^{st} \mathcal{L}$ (resp. $K \subseteq^{ust} \mathcal{L}$) by such property of the RV-sorts (Subsection 3.1.2), and later by such properties of the value groups and the residue fields (Subsection 3.1.4). First, we show that, in the case of elementary pairs, the notion of separatedness is indeed a necessary condition.

3.1.1 Separatedness as a necessary condition

We are going to prove that elementary submodels of benign valued fields are stably embedded only if the extension is separated (Proposition 3.1.4). This is a generalisation of (1 $\Rightarrow$ 2) in [22, Theorem 1.9]. Notice that our proof requires that the extension is elementary. In the next subsections, it will no longer be assumed.

Proposition 3.1.4. *Let $\mathcal{L}/\mathcal{K}$ be an elementary extension of valued fields, with* $\mathrm{Th}(\mathcal{L})$ *a completion of a benign theory of Henselian valued fields. If $\mathcal{K}$ is stably embedded in $\mathcal{L}$, then $\mathcal{L}/\mathcal{K}$ is a separated extension of valued fields.*

Remark. In fact, the proof below (more specifically the proof of Corollary 3.1.7) does not require relative quantifier elimination, but only the properties (Im) and $(\mathrm{AKE})_{\mathrm{RV}}$.

Proof. We start by defining the notion of valued vector spaces.

Definition 3.1.5. A valued $\mathcal{K}$-vector space $\mathcal{V}$ is a K-vector space V and a totally ordered set Γ_V together with:

- a group action $+ : \Gamma_K \times \Gamma_V \to \Gamma_V$ which is strictly increasing in both variables,
- a surjective map, called the *valuation*, $\mathrm{val} : V \setminus \{0\} \to \Gamma_V$ such that $\mathrm{val}(w + v) \geq \min(\mathrm{val}(w), \mathrm{val}(v))$ and $\mathrm{val}(\alpha \cdot v) = \mathrm{val}(\alpha) + \mathrm{val}(v)$ for all $v, w \in V$ and $\alpha \in K$. By convention, $\mathrm{val}(0) = \infty$.

Of course, the notion of separating basis and separated vector space extend naturally to this slightly more general setting. For $v \in V$ and $\gamma \in \Gamma_V$, we define the closed ball $B_{\geq\gamma}(v)$ by $\{v' \in V \mid \mathrm{val}(v - v') \geq \gamma\}$ and the open ball $B_{>\gamma}(v)$ by $\{v' \in V \mid \mathrm{val}(v - v') > \gamma\}$. The following lemma and proof are taken from a lecture course of Martin Hils.

Lemma 3.1.6. *Assume $\mathcal{K}' = (K', \mathrm{val})$ to be a maximal valued field. Let $\mathcal{V}' = (V', \mathrm{val})$ be a separated finite dimensional valued $\mathcal{K}'$-vector space. Then V' is spherically complete: if $(D_i)_{i\in I}$ is a family of nested balls, then the intersection $\bigcap_{i\in I} D_i$ is non-empty.*

Proof. We proceed by induction on $n \geq 1$. If V' is of dimension 1, we simply have that $(K', \mathrm{val}) \simeq (V', \mathrm{val})$ as $\mathcal{K}'$-valued vector spaces. Then, we only need to recall that $\mathcal{K}'$ is pseudo-complete as a maximal valued field. Assume the lemma to holds for all sub-$\mathcal{K}'$-vector spaces of dimension n and let $W = \{w_0, \ldots, w_n\}$ be a separating basis of V', a sub-$\mathcal{K}'$-vector space of dimension $n+1$. Let $(B_\alpha = B_{\geq\gamma_\alpha}(v_\alpha))_{\alpha<\lambda}$ be a decreasing sequence of closed balls in $\mathcal{V}'$, with λ a limit ordinal. For $\alpha < \lambda$, write $v_\alpha = \sum_{i\leq n} a_{\alpha,i} w_i$ with $a_{\alpha,i} \in K'$. Applying the definition of separating basis and by taking a subsequence, we may assume that $\gamma_\alpha \leq \mathrm{val}(v_{\alpha+1} - v_\alpha) = \mathrm{val}(a_{\alpha+1,n} - a_{\alpha,n}) + \mathrm{val}(w_n)$ for all $\alpha < \lambda$. It follows that the sequence $(a_{\alpha,n})_{\alpha<\lambda}$ is pseudo-Cauchy in $\mathcal{K}'$. As $\mathcal{K}'$ is pseudo-complete, one finds a pseudo-limit a_n in K'. We consider now the sequence $(v'_\alpha)_{\alpha<\lambda}$ where $v'_\alpha = \sum_{i<n} a_{\alpha,i} w_i + a_n w_n$. One has the following:

- $v'_\alpha \in B_\alpha$ i.e. $B_{\geq\gamma_\alpha}(v_\alpha) = B_{\geq\gamma_\alpha}(v'_\alpha)$,
- the set $B'_\alpha := \{\sum_{i<n} d_{\alpha,i} w_i \mid d_{\alpha,i} \in K', \sum_{i<n} d_{\alpha,i} w_i + a_n w_n \in B_\alpha\}$ is the (non-empty) closed ball
$$B_{\geq\gamma_\alpha}(\sum_{i<n} a_{\alpha,i} w_i)$$
in $V' =< w_0, \ldots, w_{n-1} >_{K'}$.
- $(B'_\alpha)_{\alpha<\lambda}$ is a decreasing sequence of closed balls in V' (of $\mathcal{K}'$-dimension n).

By the induction hypothesis, the sequence $(B'_\alpha)_{\alpha<\lambda}$ admits a non empty intersection. Let $v' \in \cap_{\alpha<\lambda} B'_\alpha$. Then, one has that $v = v' + a_n w_n \in \cap_{\alpha<\lambda} B_\alpha$. □

Corollary 3.1.7. Assume that $\mathcal{K}$ is a stably embedded elementary submodel of $\mathcal{L}$. Any finite dimensional separated sub-$\mathcal{K}$-vector space V of $\mathcal{L}$ is definably spherically complete: let $(D_i)_{i\in I}$ be a definable family of balls (D_i is a closed ball defined by a parameter i, and I is a definable set) with the finite intersection property (no finite intersection is empty). Then the intersection $\bigcap_{i\in I} D_i$ is non-empty.

Proof. Let $C = \{c_0, \ldots, c_{n-1}\}$ be a separating basis of V. We write $\gamma_i = \mathrm{val}(c_i) \in \Gamma_L$ for $i < n$. One can interpret V in K^n: elements are identified with their decomposition in the basis C, addition and scalar multiplication are defined as usual. Since K is stably embedded in $\mathcal{L}$, the type $\mathrm{tp}(c_0, \ldots, c_{n-1}/K)$ is definable. It follows that $v(\sum_{i<n} a_i c_i) > v(\sum_{i<n} b_i c_i)$ holds if and only if $(a_0, \ldots, a_{n-1})$ and $(b_0, \ldots, b_{n-1})$ satisfy a certain K-formula. So we also interpret the valuation. Let $(D_i)_{i \in I}$ be a uniformly definable family of nested balls. Consider K', the maximal immediate extension of K. By the previous lemma, the (definable) intersection $\bigcap_{i \in I'} D'_i$ has a point in K' (where I' is the definable set $I(K')$, $D'_i = D_i(K')$), and so $\bigcap_{i \in I} D_i$ is non-empty, as $K' \succeq K$ by the Ax-Kochen-Ershov principle. □

We prove by induction on n that any sub-$\mathcal{K}$-vector space of $\mathcal{L}$ of dimension n is separated. There is nothing to show for $n = 1$. Let V be a finite dimensional K-vector subspace of L, with a separating basis $C = \{c_0, \ldots, c_{n-1}\}$. Let a be any element of $L \setminus V$. Let us show that the K-vector space $\tilde{V} =< V, a >_K$ generated by V and a is also separated. It follows from Corollary 3.1.7 that $\{v(w - a) |\ w \in V\}$ has a maximum. Indeed, otherwise the family of balls $(B_{\geq \mathrm{val}(w-a)}(a))_{w \in V}$ will have an empty intersection. As $\mathrm{tp}(a, c_0, \ldots, c_{n-1}/\mathcal{K})$ is definable ($\mathcal{K}$ is stably embedded in $\mathcal{L}$), it is a definable family of balls, contradicting the fact that V is definably spherically complete by Corollary 3.1.7. Let $c_n = w - a$ realise this maximum. By a simple calculation, one sees that $\tilde{C} = \{c_0, \ldots, c_n\}$ is a separating basis of $\tilde{V} =< V, a >$. Indeed, consider any element $b \in \tilde{V} \setminus V$ and its decomposition $b = \sum_{i \leq n} b_i c_i$ in the basis $\tilde{C} = \{c_0, \ldots, c_n\}$. Notice that $\frac{\sum_{i<n} b_i c_i}{b_n}$ is an element of V. If $\mathrm{val}(\sum_{i<n} b_i c_i) \geq v(b_n c_n)$, then $\mathrm{val}(\frac{\sum_{i<n} b_i c_i}{b_n} + c_n) = v(c_n)$ by maximality of $\mathrm{val}(c_n)$, which gives $\mathrm{val}(\sum_{i<n} b_i c_i + b_n c_n) = \mathrm{val}(b_n c_n)$. If $\mathrm{val}(\sum_{i<n} b_i c_i) < v(b_n c_n)$, then $\mathrm{val}(\sum_{i<n} b_i c_i + b_n c_n) = \mathrm{val}(\sum_{i<n} b_i c_i)$. This proves that $\tilde{C} = \{c_0, \ldots, c_n\}$ is a separating $\mathcal{K}$-basis of $\tilde{V}$. □

3.1.2 Reduction to RV

In this paragraph, we reduce definability over sub-valued fields to the RV-sort for all benign theories of Henselian valued fields. The next two propositions consist simply of an RV-version of the theorem of Cubides-Delon. The proof is completely similar to that of $(2 \Rightarrow 1)$ in [22, Theorem 1.9]. Notice that the idea of adapting the proof of Cubides-Delon in the setting of an RV-sort can already be found in Rideau-Kikuchi's thesis [63]. Let T be a benign theory of Henselian valued fields. Recall that we can safely work in the language L_{RV} (or in an RV-enrichment of it), by Remark 1.1.51.

Theorem 3.1.8. *Let $\mathcal{L}/\mathcal{K}$ be a separated extension with $\mathcal{L} \models T$. The following are equivalent:*

1. *K is stably embedded (resp. uniformly stably embedded) in $\mathcal{L}$,*
2. *RV_K is stably embedded (resp. uniformly stably embedded) in RV_L.*

The proof below use the property $(\mathrm{EQ})_{\mathrm{RV}}$.

Proof. We prove the non-uniform case, the uniform case being similar.
$(2 \Rightarrow 1)$ Let $\phi(x, a)$ be a formula with parameters in L, x a tuple of field sorted variables. By relative quantifier elimination to RV, it is equivalent to a formula of the form

$$\psi(\mathrm{rv}(P_0(x)), \ldots, \mathrm{rv}(P_{k-1}(x)), \mathbf{b}),$$

where $\psi(\mathbf{x}_0, \ldots, \mathbf{x}_{k-1}, \mathbf{y})$ is an RV-formula, $P_i(x)$'s are polynomials with coefficients in L and $\mathbf{b} \in \mathrm{RV}_L$. For instance, notice that the formula $P(x) = 0$ for $P(X) \in L[X]$ is equivalent to $\mathrm{rv}(P(x)) = 0$. Consider V the finite dimensional K-vector-space generated by the coefficients of the P_i's, and let $c_0, \ldots, c_{n-1}$ be a separating basis of V. For $x \in K$, one has $P_i(x) = \sum_{j<n} P_i^j(x) c_j$ for some polynomials $P_i^j(x) \in K[x]$. By definition of separating basis, one has for $x \in K$:

$$\mathrm{rv}(P_i(x)) = \bigoplus_{j<n} \mathrm{rv}(P_i^j(x) c_j).$$

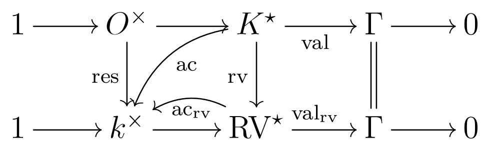
Hence, the trace in K of the formula $\phi(x, a)$ is given by $\theta((\mathrm{rv}(P_j^i(x)))_{i<k, j<n}, (\mathrm{rv}(c_j))_{j<n}, \mathbf{b})$, where θ is an RV-formula. Now, using that RV_K is stably embedded in RV_L, $\theta(\mathbf{x}, (\mathrm{rv}(c_j))_{j<n}, \mathbf{b})$ can be replaced by a formula $\xi(\mathbf{x}, \mathbf{d})$ with parameters $\mathbf{d}$ in RV_K, which induces the same set in RV_K. At the end, we get that $\phi(K, a)$ is definable with parameters in K.

$(1 \Rightarrow 2)$ By relative quantifier elimination to RV, if $\mathcal{K}$ is stably embedded in $\mathcal{L}$, so is RV_K in RV_L. Indeed, consider a formula $\phi(\mathbf{x}, \mathbf{a})$ with free variable $\mathbf{x} \in \mathrm{RV}$ and parameter $\mathbf{a} \in \mathrm{RV}_L$. As K is stably embedded in L, there is an $\mathrm{L}(K)$-formula $\psi(\mathbf{x}, b)$ with *a priori* field-sorted parameters $b \in K$ such that $\psi(\mathrm{RV}_K, b) = \phi(\mathrm{RV}_K, \mathbf{a})$. By relative quantifier elimination to RV, there is a field-sorted-quantifier-free formula $\theta(\mathbf{x}, \mathbf{y})$ and field-sorted terms $t(y)$ such that $\theta(\mathrm{RV}_K, \mathrm{rv}(t(b))) = \psi(\mathrm{RV}_K, b) = \phi(\mathrm{RV}_K, \mathrm{rv}(a))$. This concludes our proof. □

Remark 3.1.9. *The proof above holds for any RV-enrichment T^e of T in a language $\mathrm{L}^e_{\mathrm{RV}}$. Indeed T has quantifier elimination relative to RV only if T^e does (see Fact 1.1.5).*

3.1.3 Angular component

Let T be a benign theory of Henselian valued fields. We expand the language L_{RV} with the sort k, Γ, val, res and with a new function symbol $\mathrm{ac} : K \to k$ and consider the theory T_{ac} of the corresponding valued field with an ac-map. This can be considered as an RV-enrichment of T, as the map val and ac reduce to RV (see the diagram below).



As Theorem 3.1.8 holds in RV-enrichment, we deduce from it the following corollary:

Corollary 3.1.10. Let $\mathcal{L}$ be a model of T_{ac}. Assume that $\mathcal{L}/\mathcal{K}$ is a separated extension. The following are equivalent:

1. K is stably embedded (resp. uniformly stably embedded) in $\mathcal{L}$,
2. k_K is stably embedded (resp. uniformly stably embedded) in k_L and Γ_K is stably embedded (resp. uniformly stably embedded) in Γ_L

The proof is straightforward, once we prove Lemma 3.1.11. Again, it uses Property $(\mathrm{EQ})_{\mathrm{RV}}$ but Property (Im) is not required.

Given two structures $\mathcal{H}$ and $\mathcal{K}$ (possibly in a different language) with base set H and K, we already defined the product structure $\mathcal{H} \times \mathcal{K}$ as the three-sorted structure

$$\mathcal{H} \times \mathcal{K} = \{H \times K, \mathcal{H}, \mathcal{K}\} \cup \{\pi_H : H \times K \to H, \pi_K : H \times K \to K\}$$

where the function symbols π_H and π_K are interpreted by the canonical projections (for relative quantifier elimination, orthogonality and stable embeddedness, see Proposition 1.1.28). One sees that if H and K are two stably embedded and orthogonal definable sets in a structure $\mathcal{M}$, then the product $H \times K$ in M^2 with the full induced structure over parameters is isomorphic to $\mathcal{H} \times \mathcal{K}$, where $\mathcal{H}$ (resp. $\mathcal{K}$) is the set H (resp. K) endowed with its induced structure.

Lemma 3.1.11. *Let $\mathcal{H}_1$ (resp. $\mathcal{K}_1$) be a substructure of a structure $\mathcal{H}_2$ (resp. $\mathcal{K}_2$). Then $\mathcal{H}_1 \times \mathcal{K}_1$ is a substructure of $\mathcal{H}_2 \times \mathcal{K}_2$ and we have:*

- $\mathcal{H}_1 \times \mathcal{K}_1 \subseteq^{st} \mathcal{H}_2 \times \mathcal{K}_2$ *if and only if* $\mathcal{H}_1 \subseteq^{st} \mathcal{H}_2$ *and* $\mathcal{K}_1 \subseteq^{st} \mathcal{K}_2$.

- *$\mathcal{H}_1 \times \mathcal{K}_1 \subseteq^{ust} \mathcal{H}_2 \times \mathcal{K}_2$ if and only if $\mathcal{H}_1 \subseteq^{ust} \mathcal{H}_2$ and $\mathcal{K}_1 \subseteq^{ust} \mathcal{K}_2$.*

Proof. The fact that $\mathcal{H}_1 \times \mathcal{K}_1$ is a substructure of $\mathcal{H}_2 \times \mathcal{K}_2$ is obvious. We prove the non-uniform stable embeddedness transfer. The uniform case can be proved similarly, or can be deduced from the non-uniform case by considering a saturated enough extension in the language of pairs. First, we prove the left-to-right implication. This is almost a consequence of purity, but one needs control over the parameters, which is given by relative quantifier elimination. Assume $\mathcal{H}_1 \times \mathcal{K}_1 \subseteq^{st} \mathcal{H}_2 \times \mathcal{K}_2$. Let $\phi(x_H)$ be a H_2-formula, with x_H a tuple of H-sorted variables. We identify the formula $\phi(x_H)$ with the formula $\phi(\pi_H(x_{H,K}))$ where $x_{H,K}$ is a variable in $H \times K$(in other words, $\phi(H_2)$ is identified with $\phi(H_2) \times K_2$). By assumption, there is a formula $\psi(x_{H,K}, b)$ with parameters b in $H_1 \times K_1$ such that $\psi(H_1 \times K_1, b) = \phi(H_1) \times K_1$. By relative quantifier elimination, we may assume that $\psi(x_{H,K}, b)$ is a disjunction of formulas of the form:

$$\psi_H(\pi_H(x_{H,K}), \pi_H(b)) \wedge \psi_K(\pi_K(x_{H,K}), \pi_K(b))$$

where $\psi_H(x_H, y_H)$ is a H-formula and $\psi_K(x_K, y_K)$ is a K-formula. It follows that $\phi(H_1) \times K_1$ is a union of rectangles of the forms $\psi_H(H_1, \pi_H(b)) \times \psi_K(K_1, \pi_K(b))$. The union of the H-side gives a formula $\theta(x_H, \pi_H(b))$ with parameters in H_1 such that $\theta(H_1, \pi_H(b)) = \phi(H_1)$. This shows that $\mathcal{H}_1$ is stably embedded in $\mathcal{H}_2$ and similarly that $\mathcal{K}_1$ is stably embedded in $\mathcal{K}_2$.
Assume $\mathcal{H}_1 \subseteq^{st} \mathcal{H}_2$ and $\mathcal{K}_1 \subseteq^{st} \mathcal{K}_2$. Let $\phi(x_{H,K})$ be a $H_2 \times K_2$-formula. Again, by relative quantifier elimination, it is a finite union of rectangles, where the H-side has parameters in H_2 and the K-side has parameters in K_2. The trace in H_1 of each H-side is given by a formula with parameters in H_1. Similarly for the K-side. At the end, it gives a $H_1 \times K_1$-formula $\psi(x_{H,K})$ such that $\psi(H_1 \times K_1) = \phi(H_1 \times K_1)$. □

Proof of Corollary 3.1.10. By Theorem 3.1.8, or more precisely by Remark 3.1.9, K is stably embedded (resp. uniformly stably embedded) in $\mathcal{L}$ if and only if RV_K is stably embedded (resp. uniformly stably embedded) in the enriched RV-structure

$$\big((\mathrm{RV}^\star_L, \mathbf{1}, \cdot), (k_L, 0, 1, +, \cdot), (\Gamma_L, 0, +, <), \mathrm{ac}_{\mathrm{rv}} : \mathrm{RV}^\star_L \to k^\star_L, \\ \iota : k^\star_L \to \mathrm{RV}^\star_L, \mathrm{val}_{\mathrm{rv}} : \mathrm{RV}^\star_L \to \Gamma_L\big).$$

Notice that, in terms of structure, the injection $\iota : k^\star_L \to \mathrm{RV}^\star_L$ is superfluous, as is the multiplicative law in $\mathrm{RV}^\star_L$, since the graphs are respectively given by

$$\{(a, b) \in k^\star \times \mathrm{RV}^\star \mid \mathrm{ac}_{\mathrm{rv}}(b) = a \wedge \mathrm{val}_{\mathrm{rv}}(b) = 0\}$$

and

$$\{((b_1, b_2), b_3) \in \mathrm{RV}^2 \times \mathrm{RV} \mid \mathrm{ac}_{\mathrm{rv}}(b_1) \times \mathrm{ac}_{\mathrm{rv}}(b_2) = \mathrm{ac}_{\mathrm{rv}}(b_3) \wedge \mathrm{val}_{\mathrm{rv}}(b_1) + \mathrm{val}_{\mathrm{rv}}(b_2) = \mathrm{val}_{\mathrm{rv}}(b_3)\}.$$

In other word, RV_L is exactly the product structure $k_L \times \Gamma_L$:

$$(\mathrm{RV}_L, (k_L, 0, 1, +, \cdot), (\Gamma_L, 0, +, <), \mathrm{ac}_{\mathrm{rv}} : \mathrm{RV}_L^\star \to k_L^\star, \mathrm{val}_{\mathrm{rv}} : \mathrm{RV}_L^\star \to \Gamma_L).$$

Then the corollary is a direct consequence of Lemma 3.1.11. □

3.1.4 Reduction to Γ and k

We reduce definability of types over a submodel in RV to the corresponding conditions in the value group and residue field. We consider the multi-sorted structure $(\mathrm{RV}_K, \Gamma_K, k_K)$ and its (not necessary elementary) extension $(\mathrm{RV}_L, \Gamma_L, k_L)$. We are going to show that, under some reasonable conditions, RV_K is stably embedded (resp. uniformly stably embedded) in RV_L if and only if k_K is stably embedded (resp. uniformly stably embedded) in k_L and Γ_K is stably embedded (resp. uniformly stably embedded) in Γ_L. Recall once again that, if (RV, Γ, k) is an RV-structure, we have the following short exact sequence:

$$1 \longrightarrow k^\star \longrightarrow \mathrm{RV}^\star \longrightarrow \Gamma \longrightarrow 0 .$$

It will be seen as a sequence of enriched abelian groups. We will use the quantifier elimination result for short exact sequences of abelian groups due to Aschenbrenner, Chernikov, Gehret and Ziegler (see Subsection 1.2.2). Also, to fit with their notations, let us work in a more general context. Assume we have a (possibly $\{A\} - \{C\}$-enriched) short exact sequence of abelian groups

$$0 \longrightarrow A \xrightarrow{\ \iota\ } B \xrightarrow{\ \nu\ } C \longrightarrow 0 ,$$

where $\iota(A)$ is a pure subgroup of B. As in Subsection 1.2.2, we see it as an L-structure (resp. an L_q structure), and we denote by T (resp. T_q) its theory. Let us insist that A and C can be endowed with extra structure. By 'the sort A' and 'an A-formula', we abusively mean respectively the union of sort $\bigcup_{n<\omega} A/nA$ and a $\bigcup_{n<\omega} A/nA$-formula (so with potentially variables and parameters in A/nA).

We consider the question of stable embeddedness of a sub-short-exact-sequence $\mathcal{M}$ of a model $\mathcal{N} \models T$. We denote by $\bigcup_{n<\omega} \rho_n(\mathcal{M})$ the union of the images of $\mathcal{M}$ in $\bigcup_{n<\omega} A(\mathcal{N})/nA(\mathcal{N})$ under the maps $\rho_n^{\mathcal{N}}$'s. Notice that it should not be confused with $\bigcup_{n<\omega} A(\mathcal{M})/nA(\mathcal{M})$.

Proposition 3.1.12. *Let $\mathcal{N}$ be models of T and $\mathcal{M} \subseteq \mathcal{N}$ a sub-short exact sequence of groups. We have:*

- $\mathcal{M} \subseteq^{st} \mathcal{N}$ *if and only if* $\bigcup_{n<\omega} \rho_n(\mathcal{M}) \subseteq^{st} \bigcup_{n<\omega} A/nA(\mathcal{N})$ *and* $C(\mathcal{M}) \subseteq^{st} C(\mathcal{N})$ *and*
- $\mathcal{M} \subseteq^{ust} \mathcal{N}$ *if and only if* $\bigcup_{n<\omega} \rho_n(\mathcal{M}) \subseteq^{ust} \bigcup_{n<\omega} A/nA(\mathcal{N})$ *and* $C(\mathcal{M}) \subseteq^{ust} C(\mathcal{N})$.

Proof. We prove only the non-uniform case. We prove the right-to-left implication first.

Let $x = (x_0, \dots, x_{k-1})$ be a tuple of variables in the sort B. A term $t(x, b)$ with $b \in B(\mathcal{N})^l$ is of the form $n \cdot x + m \cdot b$ for $n \in \mathbb{N}^k, m \in \mathbb{N}^l$, where $n \cdot x = n_0 x_0 + \dots + n_{k-1} x_{k-1}$. By quantifier elimination, it is enough to check that the following formulas define on M some M-definable sets:

1. $\phi_C(\nu(t_0(x)), \dots, \nu(t_{s-1}(x)), c)$ where $t_i(x)$'s are terms with parameters in $B(\mathcal{N})$, $c \in C(\mathcal{N})$ and ϕ_C is a C-formula,
2. $\phi_A(\rho_{n_0}(t_0(x)), \dots, \rho_{n_{s-1}}(t_{s-1}(x)), a)$ where $t_i(x)$'s are terms with parameters in $B(\mathcal{N})$, $a \in \bigcup_{n<\omega} A/nA(\mathcal{N})$.

(1) We have $t_l(x) = n_l \cdot x + m_l \cdot b$ where b is some tuple of parameters in $B(\mathcal{N})$. Write $\nu(n \cdot x + m \cdot b)) = \nu(n \cdot x) + \nu(m \cdot b)$. Then $\nu(m \cdot b)$ is a parameter from $C(\mathcal{N})$ and one just needs to apply $C(\mathcal{M}) \subseteq^{st} C(\mathcal{N})$.
(2) Assume $t_0(x) = n \cdot x + m \cdot b$. If for all $g \in B(\mathcal{M})$, $\rho_{n_0}(n \cdot g + m \cdot b) = 0$, replace all occurrences of $t_0(x)$ by 0. Otherwise, for some $g \in B(\mathcal{M})$, we have $\nu(n \cdot g + m \cdot b) \in n_0 C$. If $\nu(n \cdot x + m \cdot b) \in n_0 C$, one can write $\rho_{n_0}(n \cdot x + m \cdot b) = \rho_{n_0}(n \cdot x + (-n) \cdot g) + \rho_{n_0}(n \cdot g + m \cdot b)$. The formula is equivalent to

$$\Big((\exists y_C\ \nu(n \cdot x + m \cdot b) = n_0 y_C) \ \wedge\ \phi_A(\rho_{n_0}(n \cdot x + (-n) \cdot g) + \rho_{n_0}(n \cdot g + m \cdot b), \dots, \rho_{n_{s-1}}(t_{s-1}(x)), a) \Big)$$
$$\vee \Big(\neg \exists y_C\ \nu(n \cdot x + m \cdot b) = n_0 y_C \wedge\ \phi_{A,n}(0, \rho_n(t_1(x)), \dots, \rho_n(t_{s-1}(x)), a) \Big)$$

Now $n \cdot x + (-n) \cdot g$ is a term with parameters in $B(\mathcal{M})$ and $\rho_n(n \cdot g + m \cdot b)$ is a parameter in $A/nA(\mathcal{N})$. We proceed similarly for all other terms $\rho_{n_i}(t_i(x))$, $0 < i < s$. As $\bigcup_{n<\omega} \rho_n(\mathcal{M}) \subseteq^{st} \bigcup_{n<\omega} A/nA(\mathcal{N})$, we conclude that the trace in M of the initial formula $\phi_A(\rho_{n_0}(t_0(x)), \dots, \rho_{n_{s-1}}(t_{s-1}(x)), a)$ is the trace in M of a formula with parameters in M.

It remains to prove the left-to-right implication. This is almost a consequence of purity of the sorts A and C, but one needs control over the set of parameters (we use implicitly Definition 1.1.8). Let $\phi_A(x_A, a)$ be an A-formula with parameters $a \in \bigcup_{n<\omega} A/nA(\mathcal{N})$. By stable embeddedness of M in $\mathcal{N}$, there is an L_q-formula $\psi(x_A, a, b, c)$ with parameters $a, b, c \in A(M)B(M)C(M)$ defining the same set on $A(M)$. As in the proof of Corollary 1.2.40, we may assume that $\psi(x_A, y_A, y_B, y_C)$ is of the form:

$$\psi_A(x_A, y_A, \rho_n(t(y_B)))$$

where ψ_A is an A-formula, $t(y_B)$ is a tuple of group terms (and with no occurrence of the variable y_C or of the function symbol ι). This proves that $\bigcup_{n<\omega} \rho_n(M)$ is stably embedded in $\bigcup_{n<\omega} A/nA(N)$. Similarly, we prove that $C(M)$ is stably embedded in $C(\mathcal{N})$. □

There is a particular case when the condition $\bigcup_{n<\omega} \rho_n(\mathcal{M}) \subseteq^{st} \bigcup_{n<\omega} A/nA(\mathcal{N})$ (resp. $\bigcup_{n<\omega} \rho_n(\mathcal{M}) \subseteq^{ust} \bigcup_{n<\omega} A/nA(\mathcal{N})$) is equivalent to $A(\mathcal{M}) \subseteq^{st} A(\mathcal{N})$ (resp. $A(\mathcal{M}) \subseteq^{ust} A(\mathcal{N})$), namely when

$$\rho_n(B(\mathcal{M})) = \pi_n(A(\mathcal{M})) \text{ for all } n \geq 1. \tag{3.1}$$

(notice that the equality $\rho_0(B(\mathcal{M})) = \pi_0(A(\mathcal{M})) = A(\mathcal{M})$ always holds). Indeed, by bi-interpretability and Remark 1.1.51, we have that $A(\mathcal{M}) \subseteq^{st} A(\mathcal{N})$ is equivalent to $\bigcup_{n<\omega} \rho_n(\mathcal{M}) \subseteq^{st} \bigcup_{n<\omega} A/nA(\mathcal{N})$.

Remark 3.1.13. *The following conditions implies the condition (3.1):*

- *$C(\mathcal{M})$ is pure in $C(\mathcal{N})$ (this holds in particular when $\mathcal{M}$ is an elementary submodel of $\mathcal{N}$),*
- *$A(\mathcal{N})/nA(\mathcal{N})$ is trivial for all $n \geq 1$.*

The second point is obvious. Let us show that the first one implies (3.1). If $b \in B$ is such that $\rho_n^{\mathcal{N}}(b) \neq 0$, then by definition $\nu(b) \in nC(\mathcal{N})$. By purity, $\nu(b) \in nC(\mathcal{M})$. Then there is $a \in A(\mathcal{M})$ such that $\iota(a) + nB(\mathcal{M}) = b + nB(\mathcal{M})$. In particular, $\iota(a) + nB(\mathcal{N}) = b + nB(\mathcal{N})$, which means that $\pi_n(a) = \rho_n(b)$. We have showed that $\rho_n(B(\mathcal{M})) = \pi_n(A(\mathcal{M}))$. We conclude by Remark 1.1.51 that $A(\mathcal{M}) \subseteq^{st} A(\mathcal{N})$ implies $\bigcup_{n<\omega} \rho_n(\mathcal{M}) \subseteq^{st} \bigcup_{n<\omega} A/nA(\mathcal{N})$.

We get the following:

Corollary 3.1.14. Let $\mathcal{N}$ be model of T and let $\mathcal{M} \subseteq \mathcal{N}$ be a sub-short-exact-sequence. Assume either

- that $C(\mathcal{M})$ is a pure subgroup of $C(\mathcal{N})$,
- or that $A(\mathcal{N})/nA(\mathcal{N})$ is trivial for all $n \geq 1$.

Then, we have:

- $\mathcal{M} \subseteq^{st} \mathcal{N}$ if and only if $A(\mathcal{M}) \subseteq^{st} A(\mathcal{N})$ and $C(\mathcal{M}) \subseteq^{st} C(\mathcal{N})$ and
- $\mathcal{M} \subseteq^{ust} \mathcal{N}$ if and only if $A(\mathcal{M}) \subseteq^{ust} A(\mathcal{N})$ and $C(\mathcal{M}) \subseteq^{ust} C(\mathcal{N})$.

If the reader is not looking for a perfect characterisation, they can easily slightly weaken the second condition in 3.1.14:

Corollary 3.1.15. Let $\mathcal{N}$ be models of T and let $\mathcal{M} \subseteq \mathcal{N}$ be a sub-short-exact-sequence. Assume that $A(\mathcal{N})/nA(\mathcal{N})$ is finite for all $n \geq 1$.

Then, we have:

- $A(\mathcal{M}) \subseteq^{st} A(\mathcal{N})$ and $C(\mathcal{M}) \subseteq^{st} C(\mathcal{N})$ imply $\mathcal{M} \subseteq^{st} \mathcal{N}$ and
- $A(\mathcal{M}) \subseteq^{ust} A(\mathcal{N})$ and $C(\mathcal{M}) \subseteq^{ust} C(\mathcal{N})$ imply $\mathcal{M} \subseteq^{ust} \mathcal{N}$.

Proof. We deduce from 3.1.12 and from the fact that if $A(\mathcal{N})/nA(\mathcal{N})$ is finite for all $n \geq 1$ that $A(\mathcal{M}) \subseteq^{st} A(\mathcal{N})$ implies $\bigcup_{n<\omega} \rho_n(\mathcal{M}) \subseteq^{st} \bigcup_{n<\omega} A/nA(\mathcal{N})$. Indeed, this is due to the fact that the union of a (uniformly) stably embedded set and a finite set is automatically (uniformly) stably embedded. □

Combining Theorem 3.1.8 and Proposition 3.1.12, we finally get the following theorem:

Theorem 3.1.16. *Assume T is a benign theory of Henselian valued fields. Let $\mathcal{L}/\mathcal{K}$ be a separated extension of valued fields with $\mathcal{L} \models T$. Assume either*

- *that Γ_K is a pure subgroup of Γ_L,*
- *or that $k_L^\star/(k_L^\star)^n$ is trivial for all $n \geq 1$.*

The following are equivalent:

1. *K is stably embedded (resp. uniformly stably embedded) in $\mathcal{L}$,*
2. *k_K is stably embedded (resp. uniformly stably embedded) in k_L, Γ_K is stably embedded (resp. uniformly stably embedded) in Γ_L.*

Recall that by Proposition 3.1.4, stably embedded elementary pairs are necessarily separated. As an elementary subgroup is automatically a pure subgroup, we get:

Theorem 3.1.17. *Assume T is a benign theory of Henselian valued fields. Let $\mathcal{K} \preceq \mathcal{L}$ be an elementary pair of models of T. The following are equivalent:*

1. *K is stably embedded (resp. uniformly stably embedded) in $\mathcal{L}$,*

2. *$\mathcal{L}/\mathcal{K}$ is separated, k_K is stably embedded (resp. uniformly stably embedded) in k_L and Γ_K is stably embedded (resp. uniformly stably embedded) in Γ_L.*

3.1.5 Applications

Let us apply Theorem 3.1.16 on some examples. We will need:

Fact 3.1.18 ([11])**.** *Any extension of a maximal valued field is separated.*

Hence, Hahn series $k((\Gamma))$ and Witt vectors $W(k)$ give us a very large branch of examples. We start with the field of p-adics:

Corollary 3.1.19. The field of p-adics $\mathbb{Q}_p$ is uniformly stably embedded in any algebraically closed valued field containing it. In particular, it is uniformly stably embedded in $\mathbb{C}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$.

Proof. By Theorem 3.1.16, it is enough to check that $\mathbb{Z}$ is uniformly stably embedded in any divisible ordered abelian groups containing it (the residue field of $\mathbb{Q}_p$ being finite, there is nothing to prove for the residue field). By an argument similar to that in Remark 1.1.53, we only have to prove that 1-types over $\mathbb{Z}$ in a divisible ordered abelian group are uniformly definable, which is immediate. □

With Theorem 3.1.17, we recover in particular the theorems of Cubides and Delon (Theorem 3.1.2), and of Cubides and Ye (Theorem 3.1.3) in the case of pairs of real closed fields. Here is a list of examples. Notice that some of them are new:

Examples. The Hahn series $\mathcal{K} = k((\Gamma))$ where

1. $k \models \mathrm{ACF}_0$, $k = (\mathbb{R}, 0, 1, +, \cdot)$ or $k = (\mathbb{Q}_p, 0, 1, +, \cdot)$;

2. $\Gamma = (\mathbb{R}, 0, +, <)$ or $\Gamma = (\mathbb{Z}, 0, +, <)$.

satisfies $\mathcal{K} \preceq^{ust} \mathcal{L}$ for every elementary extension $\mathcal{L}$ of $\mathcal{K}$.

Proof. This is a direct application of Theorem 3.1.17. Indeed:

- By maximality and Fact 3.1.18, all such extensions are separated.

- By Fact 1.1.52, $(\mathbb{R}, 0, +, <)$ and $(\mathbb{R}, 0, 1, +, \cdot)$ are uniformly stably embedded in any elementary extension.
- By Remark 1.1.53, $(\mathbb{Z}, 0, +, <)$ is uniformly stably embedded in any elementary extension.
- By the work of Cubides and Ye, $\mathbb{Q}_p$ is uniformly stably embedded in any elementary extension.

□

A natural example of an algebraically maximal Kaplansky valued field is the valued field

$$\mathcal{K} = \mathbb{F}_p^{alg}((\mathbb{Z}[1/p]))$$

where $\mathbb{Z}[1/p]$ is the additive group of all rational numbers with denominator a power of p. Unfortunately, one can find an elementary extension $\mathcal{L}$ so that $\mathcal{K}$ is not stably embedded in $\mathcal{L}$. In fact, it is the general case: very few valued fields have the property of being stably embedded in any elementary extension, as this property is rare for ordered abelian groups (see Remark 1.1.54)

3.2 Unramified mixed characteristic Henselian valued fields

A statement similar to Theorem 3.1.8 holds if $\mathcal{L}$ is a mixed characteristic Henselian valued field. As in the case of benign valued fields, separated pairs are stably embedded if and only if the corresponding pairs of sorts $\mathrm{RV}_{<\omega}$ are stably embedded:

Theorem 3.2.1. *Let $\mathcal{L}/\mathcal{K}$ be a separated extension of valued fields with $\mathcal{L}$ a mixed characteristic Henselian valued field. We see $\mathcal{L}$ as an $\mathrm{L}_{\mathrm{RV}_{<\omega}}$-structure. The following are equivalent:*

1. *K is stably embedded (resp. uniformly stably embedded) in $\mathcal{L}$,*
2. *$\mathrm{RV}_{<\omega}(K)$ is stably embedded (resp. uniformly stably embedded) in $\mathrm{RV}_{<\omega}(L)$.*

The proof of Theorem 3.1.8 also holds *mutatis mutandis* for this theorem, where the property $(\mathrm{EQ})_{\mathrm{RV}_{<\omega}}$ is used instead of $(\mathrm{EQ})_{\mathrm{RV}}$.

We treat the reduction to the value group and residue field in the more specific case of unramified mixed characteristic Henselian valued fields with perfect residue field. First, we can observe that separatedness is also a necessary condition for elementary pairs to be stably embedded:

Proposition 3.2.2. *Let $\mathcal{L}/\mathcal{K}$ be an elementary extension of unramified mixed characteristic Henselian valued fields with perfect residue field. If $\mathcal{K}$ is stably embedded in $\mathcal{L}$, then $\mathcal{L}/\mathcal{K}$ is a separated extension of valued fields.*

The proof of Proposition 3.1.4 also holds in the mixed characteristic case: the only difference is that one should use the properties $(\mathrm{AKE})_{\mathrm{RV}_{<\omega}}$ and $(\mathrm{Im} - \mathrm{RV}_{<\omega})$ instead of respectively the properties $(\mathrm{AKE})_{\mathrm{RV}}$ and (Im).

The reduction to the value group and residue field gives a generalisation of the theorem of Cubides and Ye in the case of p-adic fields (Theorem 3.1.3).

Theorem 3.2.3. *Let $\mathcal{K}$ be an unramified mixed characteristic Henselian valued field with perfect residue field, viewed as a structure in the three-sorted language $\mathrm{L}_{\Gamma,k}$ and let $\mathcal{L}$ be an elementary extension. The following are equivalent:*

1. *$\mathcal{K}$ is stably embedded (resp. uniformly stably embedded) in $\mathcal{L}$,*
2. *The extension $\mathcal{L}/\mathcal{K}$ is separated, k_K is stably embedded (resp. uniformly stably embedded) in k_L and Γ_K is stably embedded (resp. uniformly stably embedded) in Γ_L.*

Remark that this theorem only treats the case of elementary pairs. The statement like in Theorem 3.1.16 in the mixed characteristic case would in particular involve imperfect residue fields. Although quantifier elimination in such a context is known by the work of Anscombe and Jahnke [3], we chose to not develop this direction.

Proof. We treat only the non-uniform case. The uniform one is deduced similarly.
$(2 \Rightarrow 1)$. Assume that $k_K \subseteq^{st} k_L$ and $\Gamma_K \subseteq^{st} \Gamma_L$. Recall that we have the following diagrams for all $n \geq 0$:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}^{\times} & \longrightarrow & L^{\star} & \xrightarrow{\mathrm{val}} & \Gamma_L & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{res}_n} & & \downarrow{\scriptstyle \mathrm{rv}_n} & & \| & & \\
1 & \longrightarrow & W_n(k_L)^{\times} & \xrightarrow{\iota_n} & \mathrm{RV}_n^{\star}(L) & \xrightarrow{\mathrm{val}_{\mathrm{rv}_n}} & \Gamma_L & \longrightarrow & 0
\end{array}$$

where $W_n(k_L)$ is the truncated ring of Witt vectors of order n over k_L. The same diagram holds for $\mathcal{K}$. Consider an $\mathrm{RV}_{<\omega}$-formula $\phi(x, b)$ with a tuples of variables $x \in \mathrm{RV}_{<\omega}$ and a tuples of parameters $b \in \mathrm{RV}_{<\omega}(L)$. Without loss of generality, we may simplify the notation and assume that $\phi(x, b)$ only quantifies on the sorts RV_k for $k \leq n$, that all variables x are RV_n-variables and that parameters are all in $\mathrm{RV}_n(L)$. Define the structure $\mathrm{RV}_{\leq n}(L)$ obtained from $\mathrm{RV}_{<\omega}$ by restricting to the sort RV_k, $k \leq n$:

$$\mathrm{RV}_{\leq n}(L) := \{(\mathrm{RV}_k(L))_{k\leq n}, (W_k(k_L), \cdot, +, 0, 1)_{k\leq n}, (\Gamma_L, +, 0, <),\\ (\mathrm{val}_{\mathrm{rv}_k})_{k\leq n}, (\iota_k)_{k\leq n}, (\mathrm{rv}_{m\to k})_{k<m\leq n}\}$$

with the finite projective system of maps $(\mathrm{rv}_{m\to k} : \mathrm{RV}_m(L) \to \mathrm{RV}_k(L))_{k<m\leq n}$. We can also consider the finite projective system of maps $(\mathrm{res}_{m\to k} : W_m(L) \to W_k(L))_{k<m\leq n}$. As $\phi(x, b)$ only quantifies on $\mathrm{RV}_{\leq n}$, we have that

$$\text{For all } a \in \mathrm{RV}_n(L)^{|x|}, \quad \mathrm{RV}_{<\omega}(L) \models \phi(a, b) \ \Leftrightarrow\ \mathrm{RV}_{\leq n}(L) \models \phi(a, b)$$

(notice that $\mathrm{RV}_{\leq n}(L)$ is not endowed with the full induced structure inherited by $\mathrm{RV}_{<\omega}(L)$ as a formula in the induced structure may quantify over the sorts RV_N with $N > n$). We can see $\mathrm{RV}_{\leq n}(L)$ as a $\{W_n^\times(k_L)\}$-$\{\Gamma_L\}$-enriched exact sequence of abelian groups. Indeed, the kernel of the map $\mathrm{rv}_{m\to k}$ is given by $1 + p^m W_m(k_L)$, a subset of $W_m^\times(k_L)$. It follows that the diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & W_n^\times(k_L) & \xrightarrow{\iota_n} & \mathrm{RV}_n(L)^\star & \xrightarrow[\mathrm{val}_{\mathrm{rv}_n}]{} & \Gamma_L & \longrightarrow & 0\ , \\
 & & \Big\downarrow{\scriptstyle \mathrm{res}_{n\to n-1}} & & \Big\downarrow{\scriptstyle \mathrm{rv}_{n\to n-1}} & & \Big\| & & \\
1 & \longrightarrow & W_{n-1}^\times(k_L) & \xrightarrow{\iota_{n-1}} & \mathrm{RV}_{n-1}^\star(L) & \xrightarrow{\mathrm{val}_{\mathrm{rv}_{n-1}}} & \Gamma_L & \longrightarrow & 0 \\
 & & \vdots & & \vdots & & \Big\| & & \\
1 & \longrightarrow & W_1^\times(k_L) & \xrightarrow{\iota_1} & \mathrm{RV}_1^\star(L) & \xrightarrow{\mathrm{val}_{\mathrm{rv}_1}} & \Gamma_L & \longrightarrow & 0
\end{array}$$

is fully induced by the following diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & W_n^\times(k_L) & \xrightarrow{\iota_n} & \mathrm{RV}_n(L)^\star & \xrightarrow[\mathrm{val}_{\mathrm{rv}_n}]{} & \Gamma_L & \longrightarrow & 0\ . \\
 & & \Big\downarrow{\scriptstyle \mathrm{res}_{n\to n-1}} & & & & & & \\
 & & W_{n-1}(k_L)^\times & & & & & & \\
 & & \vdots & & & & & & \\
 & & W_1(k_L)^\times & & & & & &
\end{array}$$

This means that $\mathrm{RV}_{\leq n}$ is bi-interpretable with the multisorted structure

$$\{\mathrm{RV}_n(L), (W_k(k_L), \cdot, +, 0, 1)_{k\leq n}, (\Gamma_L, +, 0, <), \mathrm{val}_{\mathrm{rv}_n}, \iota_n, (\mathrm{res}_{m\to k})_{k<m\leq n}\}.$$

In order to conclude, recall that by Corollary 1.2.26, $W_n(k)$ is interpretable in k for any field k. In fact, we see as well that the structure

$$W_{\leq n}(k) := W_n(k) \to \cdots \to W_1(k)$$

is interpretable in k. Then, we have by Remark 1.1.51:

$$W_{\leq n}(k_K) \subseteq^{st} W_{\leq n}(k_L).$$

We can conclude using Corollary 3.1.14 that $\mathrm{RV}_{\leq n}(K) \subseteq^{st} \mathrm{RV}_{\leq n}(L)$. Them, there exists an $\mathrm{RV}_{\leq n}$-formula $\psi(x, c)$ with parameters $c \in \mathrm{RV}_{\leq n}(K)$ such that $\phi(\mathrm{RV}_n(K), b) = \psi(\mathrm{RV}_n(K), c)$. Then, we have

$$\begin{aligned} \text{For all } a \in \mathrm{RV}_n(K)^{|x|}, \quad \mathrm{RV}_{<\omega}(L) \models \phi(a, b) &\Leftrightarrow \mathrm{RV}_{\leq n}(L) \models \phi(a, b) \\ &\Leftrightarrow \mathrm{RV}_{\leq n}(L) \models \psi(a, c) \\ &\Leftrightarrow \mathrm{RV}_{<\omega}(L) \models \psi(a, c). \end{aligned}$$

This shows that $\mathrm{RV}_{<\omega}(K)$ is stably embedded in $\mathrm{RV}_{<\omega}(L)$. We can conclude using Theorem 3.2.1.

Let us prove ($1 \Rightarrow 2$). Assume that $\mathcal{K}$ is stably embedded in $\mathcal{L}$. By Theorem 3.2.1, we know that $\mathrm{RV}_{<\omega}(K)$ is stably embedded in $\mathrm{RV}_{<\omega}(L)$ and that the extension $\mathcal{L}/\mathcal{K}$ is separated. We show that k_K is stably embedded in k_L. Consider a formula $\phi(x, a)$ with free variables x in the residue sort, and a tuple of parameters $a \in k_L$. Again, we can see the residue field $k_L = W_1(k_L)$ as a sort in the structure

$$\begin{aligned} \mathrm{RV}_{\leq n}(L) := \{&(\mathrm{RV}_k(L))_{k\leq n}, (W_k(k_L), \cdot, +, 0, 1)_{k\leq n}, (\Gamma_L, +, 0, <), \\ &(\mathrm{val}_{\mathrm{rv}_k})_{k\leq n}, (\iota_k)_{k\leq n}, (\mathrm{rv}_{m\to k})_{k<m\leq n}\} \end{aligned}$$

and analogously for k_K. As $\mathrm{RV}_{<\omega}(K)$ is stably embedded in $\mathrm{RV}_{<\omega}(L)$, there is for some n an $\mathrm{RV}_{<\omega}$-formula $\psi(x, b)$ with some RV_n-sorted parameters $b \in K$ and quantifiers in RV_n such that $\psi(k_K^{|x|}, b) = \phi(k_K^{|x|}, a)$. As before, we observe that $\mathrm{RV}_{\leq n} := \cup_{m\leq n} \mathrm{RV}_m$ is a $\{W_n^\times(k)\}$-$\{\Gamma\}$-enrichment of the exact sequence of abelian groups:

$$1 \to W_n^\times(k) \to \mathrm{RV}_n \to \Gamma \to 0.$$

We may apply Fact 1.2.39: there is a formula $\theta(x, y)$ in the language

$$\begin{aligned} \{(W_k(k_L), \cdot, +, 0, 1)_{k\leq n}, (W_n^\times/(W_n^\times)^m)_{m>1}, (\pi_m : W_n^\star \to W_n^\star/(W_n^\star)^m)_{m>1}, \\ (\mathrm{res}_{l\to k})_{k<l\leq n}\} \end{aligned}$$

and group-sorted terms $t(y)$ such that

$$\theta(k_K^{|x|}, \rho_m(t(b))) = \psi(k_K^{|x|}, b) = \phi(k_K^{|x|}, a)$$

for some $m < \omega$, where ρ_m is defined as in Subsection 1.2.2. As the structure

$$W_{\leq n}(k) := W_n(k) \to W_{n-1}(k) \to \cdots \to W_1(k)$$

is interpretable in k for all fields k (Corollary 1.2.26), we have a formula $\xi(x,c)$ in the language of rings and with parameters c in k_K such that

$$\xi(k_K^{|x|},c) = \theta(k_K^{|x|},\rho_m(t(b))) = \psi(k_K^{|x|},b) = \phi(k_K^{|x|},a).$$

This shows that k_K is stably embedded in k_L. Similarly, we show that Γ_k is stably embedded in Γ_L. □

Example. Consider $\mathcal{K} = Q(W(\mathbb{F}_p^{alg}))$ the quotient field of the ring of Witt vectors over the algebraic closure of $\mathbb{F}_p$. Then $\mathcal{K}$ is uniformly stably embedded in any elementary extension.

Proof. By Fact 3.1.18, any elementary extension $\mathcal{L}/\mathcal{K}$ is separated. The value group $\mathbb{Z}$ of $\mathcal{K}$ is stably embedded in any elementary extension by Remark 1.1.54, and its residue field is even stable. We can conclude with Theorem 3.2.3. □

3.3 Axiomatisability of stably embedded pairs

Closely related to the question of definability of types is the question of elementarity of the class of stably embedded pairs. Consider T a theory in a language L and denote by T_P in the language $\mathrm{L}_P = \mathrm{L} \cup \{P\}$ the theory of elementary pairs $\mathcal{M} \preceq \mathcal{N}$ where P is a predicate for $\mathcal{M}$. We consider the following subclasses of models:

$$\mathcal{C}_T^{st} = \{(\mathcal{N},\mathcal{M}) \mid \mathcal{N} \succeq \mathcal{M} \models T,\ \mathcal{M} \text{ is stably embedded in } \mathcal{N}\}$$

and

$$\mathcal{C}_T^{ust} = \{(\mathcal{N},\mathcal{M}) \mid \mathcal{N} \succeq \mathcal{M} \models T,\ \mathcal{M} \text{ is uniformly stably embedded in } \mathcal{N}\}.$$

As Cubides and Ye in [23], we are asking whether these classes are first order. If that is the case, we denote respectively by T_P^{st} and T_P^{ust} their respective theory, and we say respectively that T_P^{st} and T_P^{ust} exist.

Remark 3.3.1. *If $\mathcal{M} \preceq^{st} \mathcal{N}$ is a stably embedded elementary pair of models which is not uniformly stably embedded, then any $|\mathrm{L}|$-saturated elementary extension of the pair is not stably embedded.*

Proof. By assumption, there is a formula $\phi(x,y)$ such that for all formulas $\psi(x,z)$, there is a $b \in N^{|y|}$ such that for any $c \in M^{|z|}$, $\phi(M^{|x|},b) \neq \psi(M^{|x|},c)$. By a usual coding trick, we have in fact that for any finite set of formulas Δ, there is a $b \in N^{|y|}$ such that for any $\psi(x, z) \in \Delta$ and $c \in M^{|z|}$, $\phi(M^{|x|}, b) \neq \psi(M^{|x|}, c)$. This is to say that the type in the language of pairs

$$p(y) = \{\forall c \in M^{|z|}, \phi(M^{|x|}, y) \neq \psi(M^{|x|}, c)\}_{\psi(x,z)\in \mathrm{L}}$$

is finitely satisfiable in $(\mathcal{N}, \mathcal{M})$. Thus, it is realised in any $|\mathrm{L}|$-saturated elementary extension of the pair. Such a pair will not be stably embedded. □

Corollary 3.3.2. If T_P^{st} exists, then T_P^{ust} exists and $T_P^{ust} = T_P^{st}$.

In a stable theory T, elementary submodels are always stably embedded, $\mathcal{C}_T^{st}$ and $\mathcal{C}_T^{ust}$ are simply the class of elementary pairs. In other words, we have $T_P^{st} = T_P^{ust} = T_P$. Let us analyse few more examples.

3.3.1 Examples

We quickly cover the case of o-minimal theories and the Presburger arithmetic. The reader will find in Appendix A the case of the theory of random graphs.

O-minimal theories

Consider T an o-minimal theory. By Marker-Steinhorn, an elementary pair (R_1, R_0) of models of T is stably embedded if and only if it is uniformly stably embedded, if and only if all elements in R_1 either realise a type at the infinity or a rational cut. So both T_P^{st} and T_P^{ust} exist, and $T_P^{st} = T_P^{ust}$. This theory is given by the theory T_P together with the axiom

$$\begin{aligned} \forall x \notin P\ (\forall b \in P\ x > b) \vee (\forall b \in P\ x < b) \\ \vee(\exists a \in P\ a < x\ \wedge\ \forall b \in P\ (a < b) \Rightarrow (x < b)) \\ \vee(\exists a \in P\ x < a\ \wedge\ \forall b \in P\ (b < a) \Rightarrow (b < x)). \end{aligned}$$

Presburger arithmetic

Consider the theory T of $(\mathbb{Z}, 0, 1, +, <)$. An elementary submodel $\mathcal{Z}$ of a model $\mathcal{Z}'$ is stably embedded if and only if no element in $\mathcal{Z}'$ realises a proper cut. So both T_P^{st} and T_P^{ust} exist and are equal. This theory is given by the theory T_P together with the axiom saying that P is convex.

3.3.2 Benign theories of Henselian valued fields

Let T be a completion of a benign theory of Henselian valued fields of equicharacteristic. We denote by T_Γ and T_k the corresponding theories of the value group and residue field. We require here T to be of equicharacteristic in order to get a canonical maximal Henselian valued field of given value group and given residue field, namely the Hahn series: if $\Gamma \models T_\Gamma$ and $k \models T_k$ then $k((\Gamma)) \models T$.

We have the following reduction:

Proposition 3.3.3.

- *T_P^{st} exists if and only if $(T_\Gamma)_P^{st}$ and $(T_k)_P^{st}$ exist,*
- *T_P^{ust} exists if and only if $(T_\Gamma)_P^{ust}$ and $(T_k)_P^{ust}$ exist.*

Notice that the case of algebraically closed valued fields (including ones of mixed characteristic) follows from the work of Cubides and Delon. The case of p-adically closed valued fields (which is not covered by this theorem) is treated together with the case of real closed valued fields in the work of Cubides and Ye. See [23, Theorem 4.2.4.]. Here is a list of examples (some of them are new):

Examples. Let T be the theory of the Hahn series $\mathcal{K} = k((\Gamma))$ where

1. $k \models \mathrm{ACF}_0$, $k = (\mathbb{R}, 0, 1, +, \cdot)$ or $k = (\mathbb{Q}_p, 0, 1, +, \cdot)$;
2. $\Gamma \models \mathrm{DOAG}$ or $\Gamma \models \mathrm{Th}(\mathbb{Z}, 0, +, <)$.

Then T_P^{st} and T_P^{ust} exist.

Proof. We prove the non-uniform case. The right-to-left implication is an easy consequence of Theorem 3.1.17. Indeed one has that the class $\mathcal{C}_T^{st}$ is axiomatised by:

$$((T_\Gamma)_P^{st} \cup (T_k)_P^{st} \cup \{K/P(K) \text{ is separated}\}.$$

Assume T_P^{st} exists. We prove that the class $\mathcal{C}_{T_k}^{st} = \{(k_1, k_2) \mid k_1 \preceq^{st} k_2 \models T_k, \}$ is axiomatisable. One can show that $\mathcal{C}_{T_\Gamma}^{st}$ is axiomatisable by the same argument. By Theorem 3.1.17, we have

$$\mathcal{C}_{T_k}^{st} = \{(k_1, k_2) \mid \text{there is } \Gamma \models T_\Gamma \text{ such that } k_1((\Gamma)) \preceq^{st} k_2((\Gamma)) \models T,\ k_1, k_2 \models T_k\}.$$

Indeed, T admits as models these Hahn series fields, and Hahn series are always maximal, so in particular every extension of such is separated (Fact 3.1.18). This class is closed under ultraproducts: let I be a set of indices and $(k_1^i, k_2^i) \in \mathcal{C}_k^{st}$ for all $i \in I$, with the corresponding $\Gamma^i \models T_\Gamma$. Let $\mathcal{U}$ be an ultrafilter on I. Then by Theorem 3.1.17, we have

$$\prod_{\mathcal{U}} (k_1^i((\Gamma^i))) \preceq^{st} \prod_{\mathcal{U}} (k_2^i((\Gamma^i))).$$

as $k_1^i((\Gamma^i)) \preceq^{st} k_2^i((\Gamma^i))$ for all i and as $\mathcal{C}^{st}_{\mathrm{Th}(K)}$ is closed by ultraproduct. Again by Theorem 3.1.17, we have:

$$\prod_{\mathcal{U}}(k_1^i) \preceq^{st} \prod_{\mathcal{U}}(k_2^i),$$

(the ultraproduct commutes with the residue map). Obviously, $\mathcal{C}^{st}_{\mathrm{Th}(k)}$ is stable under isomophism and if $(k_2, k_1) \preceq (k_2', k_1')$ with $k_1' \preceq^{st} k_2'$, then $k_1 \preceq^{st} k_2$ (Remark 1.1.43). This proves that $\mathcal{C}^{st}_{\mathrm{Th}(k)}$ is closed under elementary equivalence and, finally, that it is axiomatisable. □

3.3.3 Bounded formulas and elimination of unbounded quantifiers

Let T be a complete first order theory in a language L.

Definition 3.3.4. We say that an L_P-formula is bounded if it is of the form:

$$\mho_0 y_0 \in P \ \cdots \mho_{n-1} y_{n-1} \in P \ \phi(x, y_0, \ldots, y_{n-1}),$$

where $\phi(x, y_0, \ldots, y_{n-1})$ is an L-formula and $\mho_0, \ldots, \mho_{n-1} \in \{\forall, \exists\}$. We say that a theory extending the theory of pair T_P eliminates unbounded quantifiers when any formula is equivalent to a bounded formula.

Let us cite here examples of elimination of unbounded quantifiers.

Examples.

- Assume that T is an o-minimal theory extending the theory of ordered abelian groups with distinguished positive element 1. The theory of dense pairs of models of T eliminates unbounded quantifiers ([73, Theorem 2.5]).
- Assume that T is stable. The theory of belle pairs $T' \supseteq T_P$ eliminates unbounded quantifiers if and only if T does not have the finite cover property (see [60, Theorem 6]).
- Assume that T is NIP. Let $I \subseteq \mathcal{M}$ be an indiscernible sequence indexed by a dense complete linear order so that every type over I is realised in $\mathcal{M}$. Then $\mathrm{Th}(\mathcal{M}, \mathcal{I})$ is bounded ([9, Theorem 3.3]).

The reader will find an overview on elimination of unbounded quantifier in [18]. We give now another characterisation of stable embeddedness modulo elimination of unbounded quantifiers.

Proposition 3.3.5. *Let $\mathcal{M} \preceq \mathcal{N}$ be two models of T. Assume that the theory* $\mathrm{Th}(\mathcal{N}, \mathcal{M})$ *in the language of pairs* L_P *eliminates unbounded quantifiers. Then $\mathcal{M}$ is uniformly stably embedded in $\mathcal{N}$ if and only if P is a (uniformly) stably embedded predicate for* $\mathrm{Th}(\mathcal{N}, \mathcal{M})$.

Proof. Assume $\mathcal{M}$ to be uniformly stably embedded in $\mathcal{N}$, and let $(\mathbf{N}, \mathbf{M})$ be a monster model. So $\mathbf{M}$ is stably embedded in $\mathbf{N}$. Let $\phi(x, a)$ be an L_P-formula with parameter $a \in \mathbf{N}$. It is equivalent to a bounded formula

$$Q_0 y_0 \in P \ \cdots \ Q_{n-1} y_{n-1} \in P \ \psi(x, y_0, \dots, y_{n-1}, a),$$

with $\psi(x, y_0, \dots, y_{n-1}, z)$ an L-formula and $Q_0, \dots, Q_{n-1} \in \{\forall, \exists\}$. Then the definable set $\psi(\mathbf{M}^{n+1}, a)$ is given by some $\theta(\mathbf{M}^{n+1}, b)$ where $\theta(x, y_0, \dots, y_{n-1}, z)$ is an L-formula and $b \in \mathbf{M}^{|z|}$. Hence $P(x) \wedge \phi(x, a)$ is equivalent to

$$P(x) \wedge Q_0 y_0 \ \cdots Q_{n-1} y_{n-1} \theta(x, y_0, \dots, y_{n-1}, b).$$

This proves that $P(x)$ is a stably embedded predicate in $\mathrm{Th}(\mathcal{N}, \mathcal{M})$.

Assume that P is a (uniformly) stably embedded predicate. We want to show that $\mathcal{M}$ is uniformly stably embedded in $\mathcal{N}$. It is enough to show that if $(\mathcal{N}', \mathcal{M}') \succeq (\mathcal{N}, \mathcal{M})$, then $\mathcal{M}'$ is stably embedded in $\mathcal{N}'$ (see Remark 1.1.47). Let $(\mathcal{N}', \mathcal{M}')$ be an elementary extension of the pair and let $\phi(x, a)$ be an $\mathrm{L}(\mathcal{N}')$-formula. As the predicate P is stably embedded, the set $\phi(\mathcal{M}', a)$ is given by $\psi(\mathcal{M}', b)$ where $\psi(x, z)$ is an L_P-formula with parameters $b \in \mathcal{M}'$. By elimination of unbounded quantifier, we may assume that $\psi(x, z)$ is of the form

$$Q_0 y_0 \in P \ \cdots \ Q_{n-1} y_{n-1} \in P \ \theta(x, y_0, \dots, y_{n-1}, z).$$

Replace all bounded quantifier over P by the corresponding unbounded quantifier. We obtain an L-formula $\psi'(x, z)$ such that $\psi'(\mathcal{M}', b) = \psi(\mathcal{M}', b)$. This proves that $\mathcal{M}'$ is stably embedded in $\mathcal{N}'$. $\square$

Appendix A

On pairs of random graphs

We construct in this appendix two stably embedded pairs of random graphs: one is non-uniform and the other is uniform. This gives another illustration of the difference between these notions (Definitions 1.1.42 and 1.1.46). We then show that the class of uniformly (resp. non-uniformly) stably embedded pairs of random graphs is not elementary. We also take the occasion to talk about quantifier elimination in the Shelah expansion of a random graph.

Let T be the theory of the random graph in the language $\mathcal{L} = \{R\}$ and $\mathcal{G}$ a model of T. The Marker-Steinhorn criterion holds:

Remark A.0.1. *Let $\bar{a} = a_0, \dots a_{n-1}$ be a finite tuple of elements in an elementary extension $\mathcal{H}$ of $\mathcal{G}$. Then, by quantifier elimination, one has*

$$\mathrm{tp}(\bar{a}) \cup \bigcup_{i<n} \mathrm{tp}(a_i/G) \vdash \mathrm{tp}(\bar{a}/G).$$

It follows that for all $n \in \mathbb{N}$,

$$T_1(\mathcal{G}, \mathcal{H}) \Rightarrow T_n(\mathcal{G}, \mathcal{H}),$$

and

$$T_1^u(\mathcal{G}, \mathcal{H}) \Rightarrow T_n^u(\mathcal{G}, \mathcal{H}).$$

We will give two constructions of an elementary extension $\mathcal{H}$ of $\mathcal{G}$ such that all types over G realised in H are definable. The second one is simpler but 'careless' in the sense that types are actually not uniformly definable. We will give later a generalisation of the first construction.

Construction (H1). Assume $\lambda = |G|$. We are going to build an increasing sequence $(\mathcal{G}_j)_{j<\omega}$ of graphs containing $\mathcal{G}$ and of cardinality λ. We set $\mathcal{G}_0 := \mathcal{G}$. Assume that for some $j < \omega$, $\mathcal{G}_j$ has been constructed, and let

$(A_i, B_i)_{i<\lambda}$ be an enumeration of pairs of finite subsets of G_j such that for all $i < \lambda$, $A_i \cap B_i = \emptyset$. For each $i < \lambda$, pick any $m_i \in G$ such that m_i is related to $A_i \cap G$ and unrelated to $B_i \cap G$. Then, for each $i < \lambda$, we consider a new point δ_j^i. We set $G_{j+1} = G_j \cup \{\delta_j^i\}_{i<\lambda}$ and $R(\delta_j^i, g)$ if and only if ($g \in G$ and $R(m_i, g)$) or $g \in A_i$. Finally, we set $\mathcal{H}_1 = \bigcup_{j<\omega} \mathcal{G}_j$.

Remark A.0.2. *$\mathcal{H}_1$ is by construction a random graph containing $\mathcal{G}$. Also, every 1-type over G realised in $H_1 \setminus G$ is of the form:*

$$m_{\neq} :=< \{R(x, g) \mid g \in R(m, G)\} \cup \{x \neq g \mid g \in G\} >$$

where $m \in G$. Set of such 1-types is uniformly definable. Hence, one has $\mathcal{G} \preceq^{ust} \mathcal{H}_1$.

Construction (H2). Assume $\lambda = |G|$. We are going to create an increasing sequence $(\mathcal{G}_j)_{j<\omega}$ of graphs containing $\mathcal{G}$ and of cardinality λ. Set $\mathcal{G}_0 := \mathcal{G}$ and assume that $\mathcal{G}_j$ has been constructed. Let $(A_i)_{i\in\lambda}$ be an enumeration of finite subsets of G_j. For all $i < \lambda$, we consider a new point (δ_j^i). We set $G_{j+1} = G_j \cup \{\delta_j^i\}_{i<\lambda}$ and $R(\delta_j^i, g)$ if and only if $g \in A_i$. Finally, we set $\mathcal{H}_2 = \bigcup_{j<\omega} \mathcal{G}_j$.

Remark A.0.3. *$\mathcal{H}_2$ is by construction a random graph containing $\mathcal{G}$. Also, every 1-type over G realised in $H_2 \setminus G$ is of the form:*

$$p_A :=< \{R(x, g) \mid g \in A\} \cup \{\neg R(x, g) \mid g \notin A\} \cup \{x \neq g \mid g \in G\} >$$

where A is a finite subset of G. Such 1-types are definable. Hence one has $\mathcal{G} \preceq^{st} \mathcal{H}_2$.

One can easily see that G is not uniformly stably embedded in H_2. Indeed, by construction, for all $h \in H_2 \setminus G$, $R(h, G) = \{g \in G \mid R(h, g)\}$ is finite and conversely, any finite set is given by $R(h, G)$ for some h. If it were uniformly defined by another formula with parameters in G, this would contradict the fact that T eliminates $\exists^\infty$.

No axiomatisation of stably embedded pairs of random graphs

We show that the classes of (uniformly) stably embedded pairs of random graphs is not axiomatisable in the language of pairs $\mathrm{L}_P = \mathrm{L} \cup \{P\}$. Let us recall the notation of Section 3.3:

$$\mathcal{C}_T^{st} = \{(\mathcal{N}, \mathcal{M}) \mid \mathcal{N} \succeq \mathcal{M} \models T,\ \mathcal{M} \text{ is stably embedded in } \mathcal{N}\}$$

and

$$\mathcal{C}_T^{ust} = \{(\mathcal{N}, \mathcal{M}) \mid \mathcal{N} \succeq \mathcal{M} \models T,\ \mathcal{M} \text{ is uniformly stably embedded in } \mathcal{N}\}.$$

We show that $\mathcal{C}_T^{st}$ and $\mathcal{C}_T^{ust}$ are not axiomatisable.
As we have seen in Examples A.0.3, $\mathcal{C}_{RG}^{st}$ is not preserved by elementary equivalence: using the notation of Paragraph 1.1.3, we have that $\mathcal{G} \preceq^{st} \mathcal{H}_2$, but in any saturated enough extension $(\mathcal{G}', \mathcal{H}_2')$ of the pair $(\mathcal{G}, \mathcal{H}_2)$, $\mathcal{G}'$ is not stably embedded in $\mathcal{H}_2'$ (Remark 3.3.1).

However, $\mathcal{C}_T^{ust}$ is preserved by elementary extensions (as "for all y, there is a z such that $\phi(M^{|x|}, y) = \psi(M^{|x|}, z)$" is a first order property in the language of pairs of the formulas $\psi(x, z)$ and $\phi(x, y)$). Let us show that $\mathcal{C}_{RG}^{ust}$ is not stable by ultraproduct. We will need few facts about random graphs. We leave proofs (by induction) to the reader.

Let us denote by $\phi_{n,m}(x, y_0, \ldots, y_{m-1}, z_0, \ldots z_{n-1})$ the formula

$$\bigwedge_{i<n} R(x, y_i) \wedge \bigwedge_{j<m} \neg R(x, z_j),$$

for n, m positive integers, $n + m > 0$ and with all variables distinct. Let Φ be the set of such formulas.

Fact A.0.4. *Let $\mathcal{G}$ be a random graph and let n and m be positive integers, $n+m > 0$. For any disjoint finite subsets A and B of G, there is an instance of $\phi_{n,m}(x, g)$, with parameters $g = (g_0, \ldots, g_{m+n-1}) \in G^{n+m}$ all distinct such that $A \subseteq \phi_{n,m}(G, g) \subseteq B^{\complement}$.*

The following fact says that an instance of $\phi_{n,m}$ is 'bigger' than instances of $\phi_{n',m'}$ if $n + m < n' + m'$.

Fact A.0.5. *Let $\mathcal{G}$ be a random graph.*

- *Let $\phi(x, y) \in \Phi$, and let $a \in G^{|y|}$. Then $\phi(G, a)$ is infinite and co-infinite.*
- *Consider some distinct parameters $g = (g_0, \ldots, g_{n-1}, g_0', \ldots, g_{m-1}') \in G^{n+m}$ and some distinct parameters $h = (h_0, \ldots, h_{n'-1}, h_0', \ldots, h_{m'-1}') \in G^{n'+m'}$. Assume $n' + m' > n + m$. Then*

  $$\phi_{n,m}(G, g) \setminus \phi_{n',m'}(G, h)$$

  is infinite.

As we did previously with Construction $\mathcal{H}_1$, we construct an extension $\mathcal{H}_{n,m}$ such that the set of traces $\{R(G, h) \mid h \in H_{n,m}\}$ is contained in

$$\{\phi_{n,m}(G, g) \mid g = (g_0, \ldots, g_{n-1}, g_0', \ldots, g_{m-1}') \in G^{n+m} \text{ with all } g_i, g_j' \text{ distinct}\},$$

the set of instances of $\phi(x, y)$ with distinct parameters in G (we use Fact A.0.4). If $\mathcal{U}$ is a non-principal ultra-filter in $\mathbb{N} \times \mathbb{N} \setminus \{(0, 0)\}$, let us denote by $(\tilde{\mathcal{H}}, \tilde{\mathcal{G}})$ the ultraproduct $\prod_{\mathcal{U}}(\mathcal{H}_{n,m}, \mathcal{G})$. One sees that any set $R(\tilde{G}, a)$ for $a \notin \tilde{G}$ is infinite and co-infinite in $\tilde{G}$. But it cannot be given, even up to a finite set, by a (positive) boolean combination of instances of formulas in Φ with parameters in $\tilde{G}$. Indeed, consider such an instance $\phi_{n,m}(x, g)$. Then, by Fact A.0.5, the projection of $\phi_{n,m}(\tilde{G}, g) \setminus R(\tilde{G}, a)$ to $\mathcal{H}_{n',m'}$ is infinite if $n' + m' > n + m$. We conclude by Łoś's theorem (and the fact that Φ is closed under intersection). It follows that $\tilde{\mathcal{G}}$ is not stably embedded in $\tilde{\mathcal{H}}$.

More on expansions of random graphs: We briefly show that in any model $\mathcal{G}$ of the random graph, the Shelah expansion $\mathcal{G}^{\mathrm{sh}}$ (see e.g. [69, Definition 3.9]) does not eliminate quantifiers.

Claim 7. *There is a clique A of G consisting of (distinct) elements $a_{0,i}, a_{1,i}$ $i < \omega$ such that the subset $E = \{(a_{0,i}, a_{1,i})\}_{i<\omega}$ of G^2 is definable in $\mathcal{G}^{sh}$.*

Proof. First notice that any subset of G is externally definable. We construct by induction the subset $A = \{a_{0,i}, a_{1,i}\}_{i<\omega}$ and a subset $B = \{b_{0,i}, b_{1,i}\}_{i<\omega}$ of G such that:

- A is a clique: for all $i, j < \omega$, $\xi, \zeta \in \{0, 1\}$, $a_{\xi,i} R\, a_{\zeta,j}$
- for all $i, j < \omega$ and $\xi, \zeta \in \{0, 1\}$, $a_{\xi,i} R\, b_{\zeta,j}$ if and only if $i = j$ and $\xi \leq \zeta$.

The set B 'encodes' that the pairs $(a_{0,i}, a_{1,i})$ belong to E. Indeed, a definition of E will be given by:

$$(a_0, a_1) \in A^2 \wedge \exists b_0, b_1 \in B\ a_i R\, b_j \ \Leftrightarrow\ i \leq j.$$

For some $k < \omega$, assume that $(a_{0,i}, a_{1,i}, b_{0,i}, b_{1,i})_{i<k}$ has been constructed. Take $a_{0,k}, a_{1,k}$ any elements of G such that:

- $a_{0,k} R a_{1,k}$
- for all $i < k$ and $\xi, \zeta \in \{0, 1\}$, $a_{\xi,k} R\, a_{\zeta,i}$.
- for all $i < k$, and $\xi, \zeta \in \{0, 1\}$, $\neg a_{\xi,k} R b_{\zeta,i}$.

Then take $b_{0,k}$ and $b_{1,k}$ in G such that for all $i < k$ and $\xi, \zeta \in \{0, 1\}$, $a_{\xi,i} R\, b_{\zeta,k}$ if and only if $i = k$ and $\xi \leq \zeta$. This conclude our induction. □

It is easy to see that the set E is not externally definable. Indeed, as A is a clique, the only externally definable subsets of A^2 are –by quantifier elimination – Boolean combinations of rectangles $S \times S'$ (where $S, S' \subseteq A$)

and of the diagonal $\{(a,a) \mid a \in A\}$. Note also that the class of externally definable subsets is closed under Boolean combinations. The proposition follows:

Proposition A.0.6. *For any random graph $\mathcal{G}$, the Shelah expansion $\mathcal{G}^{sh}$ does not eliminate quantifiers.*

One can also ask if adding predicates for all subsets of G^n for a given n gives us quantifier elimination. For the same reason, this does not hold either: one can similarly show that their is a definable subset $E_n := \{(a_{0,i},\ldots,a_{n,i}) \mid i < \omega\}$ of $(G^{sh})^{n+1}$ where all a's are distinct and where $A = \{a_{0,i},\ldots,a_{n,i},\ i < \omega\}$ is a clique. Then, one sees that this set cannot be written as a finite Boolean combination of rectangles $S^{n_1} \times \cdots \times S^{n_k}$ with $2 \leq k \leq n+1$, $\sum_{i<k} n_i = n+1$ and $S^{n_i} \subseteq A^{n_i}$, and the diagonal $\{(a,\ldots,a) \in A^{n+1}\}$.

Appendix B

Reduction of burden in lexicographic products

Meir defined and studied the lexicographic product of relational structures in [56]. Using his quantifier elimination result, he notably proved a Stable and NIP transfer. We continue here to investigate the model theoretic complexity of such products with respect to the burden. In this last section, we show that the burden of the lexicographic product of pure relational structures is the maximum of the burden of these structures. This is a reduction principle for pure relational structures which appears to be similar to that of pure short exact sequences of abelian groups. However, the situation is here simpler since terms are trivial and inp-patterns are automatically centralised.

Consider a relational language L.

Definition B.0.1. Let $\mathcal{M}$ and $\mathcal{N}$ be two L-structures. We consider the language $\mathrm{L}_{\mathbb{U},s} = \mathrm{L} \cup \{R^{\mathbb{U}}\}_{R\in \mathrm{L}} \cup \{s\}$ where $R^{\mathbb{U}}$ are new unary predicates and s is a binary predicate. The lexicographic product $\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}$ of $\mathcal{M}$ and $\mathcal{N}$ is the $\mathrm{L}_{\mathbb{U},s}$-structure of base set $M \times N$ where the relations are interpreted as follows:

- $s^{\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}} := \{((a,b),(a,b')) \mid a \in M, b, b' \in N\}$
- if $R \in \mathrm{L}$ is an n-ary predicate,

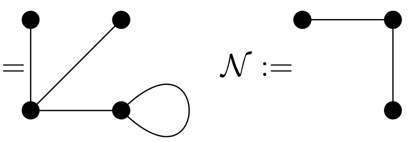
$$R^{\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}} := \{((a,b_1),\dots,(a,b_n)) \mid a \in M \text{ and } \mathcal{N} \models R(b_1,\dots,b_n)\} \cup \left\{((a_1,b_1),\dots,(a_n,b_n)) \mid \bigvee_{0\leq i\neq j\leq n} a_i \neq a_j \text{ and } \mathcal{M} \models R(a_1,\dots,a_n)\right\}.$$

- if $R \in \mathrm{L}$ is an n-ary predicate,

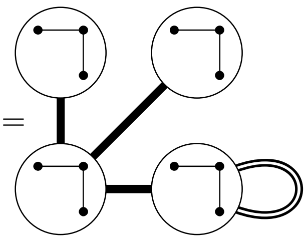
$$R^{\mathbb{U}\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}} := \Big\{ (a,b) \mid \mathcal{M} \models R(\underbrace{a,\ldots,a}_{n \text{ times}}) \Big\}.$$

We denote by $\mathcal{M}[\mathcal{N}^s]$ the restriction of $\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}$ to $\mathrm{L}_s := \mathrm{L} \cup \{s\}$.

Example. Consider the language of graphs $\mathrm{L} = \{R\}$, and let $\mathcal{M}$ and $\mathcal{N}$ be the graphs:

$\mathcal{M} :=$  $\mathcal{N} :=$

We obtain the following graph:

$\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}} :=$ 

where any point in a circle is related to any point from any linked circle. Notice that the loop is given by the predicate $R^{\mathbb{U}}$ (and not by R).

Within $\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}$, the structure $\mathcal{M}$ can be seen as an imaginary. It is indeed the quotient of $\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}$ modulo the equivalence relation $s(\mathbf{x}, \mathbf{y})$. We denote by $\pi_{\mathcal{M}} = \pi : \mathcal{M}[\mathcal{N}^s]^{\mathbb{U}} \to \mathcal{M}$ the projection. It is not an L-homomorphism. Indeed, if R is an n-ary predicate of L, and $D^n = \{(a,\ldots,a) \in M^n\}$ is the diagonal, one has:

$$\pi(R^{\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}}) = \begin{cases} R^{\mathcal{M}} \cup D^n & \text{if } R^{\mathcal{N}} \text{ is not empty,} \\ R^{\mathcal{M}} \setminus D^n & \text{if } R^{\mathcal{N}} \text{ is empty.} \end{cases}$$

Nonetheless, we recover the L-structure when we consider the additional symbol $R^{\mathbb{U}}$:

$$R^{\mathcal{M}} = \pi(R^{\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}}) \setminus D^n \cup \{(a,\ldots,a) \mid a \in \pi(R^{\mathbb{U}\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}})\}.$$

For $a \in M$, the equivalence class of a is a copy of N and it is denoted by N_a. Here, by definition

$$\begin{aligned} f_a : \ & \mathcal{N} \to \mathcal{M}[\mathcal{N}^s]^{\mathbb{U}} \\ & b \mapsto (a, b) \end{aligned}$$

is an L-isomorphism onto N_a. We will see that there is no additional structure on these sets (see Corollary B.0.3). Here is the result of quantifier elimination obtained by Meir:

Theorem B.0.2. *Let* L *be a relational language.* $\mathcal{M}$, $\mathcal{N}$ *be two* L*-structures admitting quantifier elimination. Then* $\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}$ *admits quantifier elimination.*

On can refer to [56, Theorem 2.6] in the case where $\mathcal{M}$ is transitive (for the action of the automorphism group). The theorem above can be found in Meir's thesis [57, Theorem 1.1.4]. We discuss here some immediate corollary. Let us denote by $\mathcal{M}|_{\mathbb{U}}$ the set $\pi(\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}})$ with the full induced structure, and for $a \in M$, let $\mathcal{N}_a$ be the subset $N_a \subset M[N]$ with the full induced structure.

Corollary B.0.3. The structures $\mathcal{M}|_{\mathbb{U}}$ and $\mathcal{N}_a$ for $a \in M$ are stably embedded and setwise orthogonal. The structures $\mathcal{M}|_{\mathbb{U}}$ and $\mathcal{M}$ on the one hand, and $\mathcal{N}_a$ and $\mathcal{N}$ on the other hand, have the same definable sets.

Proof. By quantifier eliminations, it is enough to look at atomic formulas. It is then a simple case study. □

To simplify the notation in our next proof, we will make $\mathcal{M}$ an $\mathrm{L}_{\mathbb{U}} := \mathrm{L} \cup \{R^{\mathbb{U}}\}_{R \in \mathrm{L}}$- structure, where the unary predicate $R^{\mathbb{U}}(x)$ is interpreted as $R(\underbrace{x, \ldots, x}_{n \text{ times}})$ if $R \in \mathrm{L}$ is an n-ary predicate.

Theorem B.0.4. $\mathrm{bdn}(\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}) = \max(\mathrm{bdn}(\mathcal{M}), \mathrm{bdn}(\mathcal{N}))$.

The theorem holds also if we consider the second definition of burden (Definition 1.1.35). The quantity $\max(\mathrm{bdn}(\mathcal{M}), \mathrm{bdn}(\mathcal{N}))$ is obviously a lower bound, as $\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}$ interprets on a unary set both $\mathcal{M}$ and $\mathcal{N}$.

Definition B.0.5. Let $\mathbf{x}, \mathbf{y}^0, \ldots, \mathbf{y}^{n-1}$ be single variables. An $\{s\}$-formula $\phi_s(\mathbf{x}, \mathbf{y}^0, \ldots, \mathbf{y}^{n-1})$ is a complete s-diagram in $\mathbf{x}$ over $\mathbf{y}^0, \ldots, \mathbf{y}^{n-1}$ if it is a maximally consistent conjunction of $\neg s(\mathbf{x}, \mathbf{y}^l)$ or $s(\mathbf{x}, \mathbf{y}^l)$ for $l < n$.

Proof. If $\mathcal{M}$ and $\mathcal{N}$ are finite, then so is $\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}$ and the theorem is trivially true. Assume $\mathcal{M}$ or $\mathcal{N}$ to be infinite. We may assume that $\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}$ is very saturated. Assume

$$\left\{\phi_i(\mathbf{x}, \mathbf{y}_i), (\mathbf{c}_{i,j} = (a_{i,j}, b_{i,j}))_{j<\omega}, k_i\right\}_{i<k}$$

is an inp-pattern in $\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}$ of depth $k > \max(\mathrm{bdn}(\mathcal{M}), \mathrm{bdn}(\mathcal{N})) \geq 1$ with $|\mathbf{x}| = 1$. By Ramsey and compactness, we can assume that the sequence of parameters $(a_{i,j}, b_{i,j})_{j<\omega}$ for $i < k$ are mutually indiscernible. By quantifier elimination and usual elimination of the disjunctions, we can also assume that for $i < k$ the formula $\phi_i(\mathbf{x}, \mathbf{y}_i)$ (where $\mathbf{y}_i = \mathbf{y}_i^1, \ldots, \mathbf{y}_i^{|\mathbf{y}_i|}$) is a conjunction of formulas of the form:

- $\phi_{i,s}(\mathbf{x}, \mathbf{y}_i) = \bigwedge_{l<|\mathbf{y}_i|}(\neg)s(\mathbf{x}, \mathbf{y}_i^l)$, a complete s-diagram in $\mathbf{x}$ over $\mathbf{y}_i^1, \ldots, \mathbf{y}_i^{|\mathbf{y}_i|}$.
- $\phi_{i,R}(\mathbf{x}, \mathbf{y}_i) = \bigwedge(\neg)R(\mathbf{x}, \mathbf{y}_i)$, for finitely many $R \in \mathrm{L}$,
- $\phi_{i,\mathbb{U}}(\mathbf{x}) = \bigwedge(\neg)R^{\mathbb{U}}(\mathbf{x})$, for finitely many $R \in \mathrm{L}$.

Then, the crucial step of the proof is to remark that no inp-pattern can have a line "talking about" $\mathcal{M}$ and another one "talking about" $\mathcal{N}$.

Case 1: Assume that the complete s-diagram in $\mathbf{x}$ of the first line implies $s(\mathbf{x}, \mathbf{y}_{0,0}^0)$.

Claim 8. *By consistency of paths, and inconsistency of the lines, the same holds for every line i: there is some $\mathbf{y}_i^{l_i}$, $l_i < |\mathbf{y}_i|$, such that $\phi_i(\mathbf{x}, \mathbf{y}) \vdash s(\mathbf{x}, \mathbf{y}_i^{l_i})$.*

Proof. Assume for example that line 1 implies $\neg s(\mathbf{x}, \mathbf{y}_1^l)$ for all $l < |\mathbf{y}_1|$. By mutual indiscernibility, $a_{0,0}^0 \neq a_{1,j}^l$ for all $j < \omega$ and $l < |\mathbf{y}_1|$. Take $\mathbf{d}_j \in \mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}$ satisfying

$$\phi_0(\mathbf{d}_j, \mathbf{c}_{0,0}) \wedge \phi_1(\mathbf{d}_j, \mathbf{c}_{1,j}),$$

(consistency of paths). As $s(\mathbf{d}_j, \mathbf{c}_{0,0}^0)$ and $a_{0,0}^0 \neq a_{1,j}^l$, this implies

$$\mathcal{M} \models \phi_{1,R}(a_{0,0}^0, a_{1,j}) \wedge \phi_{1,\mathbb{U}}(a_{0,0}^0)$$

and equivalently that

$$\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}} \models \phi_1(\mathbf{c}_{0,0}^0, \mathbf{c}_{1,j}).$$

Then, the line 1 would be realised by $\mathbf{c}_{0,0}^0$, contradiction. The same argument holds for any line $i > 0$. □

Without loss of generality, we may assume that for all i, $\phi_i(\mathbf{x}, \mathbf{y}) \vdash s(\mathbf{x}, \mathbf{y}_i^0)$ $(l_i = 0)$. It follows from the claim and by consistency of paths that:

Claim 9. *The parameters* $\{a_{i,n}^0\}_{i<k,n<\omega}$ *are all equal to some parameter* $a \in \mathcal{M}$.

Then by inconsistency of lines and by definition of $R^{\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}}}$, there is a subconjunction $\phi_i'(\mathbf{x}, \mathbf{y_i}')$ of $\phi_i(\mathbf{x}, \mathbf{y}_i)$, where $\mathbf{y}_i'$ is a subtuple of $\mathbf{y}_i$ containing $\mathbf{y}_i^0$, with a complete (positive) diagram $\bigwedge_{l<|\mathbf{y}_i'|} s(\mathbf{x}, \mathbf{y}_i^l)$ and which already forms an inconsistent line. This is simply to say that, if $\phi_i(\mathbf{x}, \mathbf{y_i}) \vdash \neg s(\mathbf{x}, \mathbf{y}_i^l)$ and $\phi_i(\mathbf{x}, \mathbf{y_i}) \vdash (\neg)R(\mathbf{x}, \mathbf{y}_i^l)$ for some $l < |y_i|$, then

$$\mathcal{M} \models \{(\neg)R(a, a_{i,j}^l)\}_{j<\omega}$$

or equivalently

$$\mathcal{M}[\mathcal{N}^s]^{\mathbb{U}} \models \{(\neg)R(\mathbf{c}, \mathbf{c}_{i,j}^l)\}_{j<\omega}.$$

whenever $\pi_{\mathcal{M}}(c) = a$.

So we may assume that the s-diagram in $\mathbf{x}$ in any lines is positive (with no occurrence of negation of $s(\mathbf{x}, \mathbf{y}_i^l)$). Thus, the inp-pattern translates to an inp-pattern of $\mathcal{N}$, of depth strictly bigger than $\mathrm{bdn}(\mathcal{N})$, namely:

$$\{\phi_{i,R}(x, y_i), (b_{i,j})_{j<\omega}\}_{i<k}$$

This is a contradiction.

Case 2: The complete s-diagram of any line is negative. Then we define the following pattern in $\mathcal{M}$:

$$P(x) := \left\{ \bigwedge_{l<|\mathbf{y}_i|} x \neq y_l \wedge \phi_{i,R}(x, y_i) \wedge \phi_{i,\mathbb{U}}(x), (a_{i,j})_{j<\omega} \right\}_{i<k}.$$

We can show – and this is a contradiction– that it is an inp-pattern of depth strictly greater than $\mathrm{bdn}(\mathcal{M})$. Indeed, lines are inconsistent: if $a \in \mathcal{M}$ realises a line

$$\left\{ \bigwedge_{l<|\mathbf{y}_i|} x \neq a_{i,j}^l \wedge \phi_{i,R}(x, a_{i,j}) \wedge \phi_{i,\mathbb{U}}(x) \right\}_{j<\omega},$$

then for any $b \in \mathcal{N}$, (a, b) satisfies the corresponding line of the original pattern

$$\{\phi_i(\mathbf{x}, \mathbf{c}_{i,j})\}_{j<\omega},$$

this is a contradiction. Paths are consistent: take $\mathbf{d} = (a, b)$ a realisation of the first column of our original inp-pattern:

$$\{\phi_i(\mathbf{x}, \mathbf{c}_{i,0})\}_{i<\kappa},$$

then a satisfies the first column of $P(x)$:

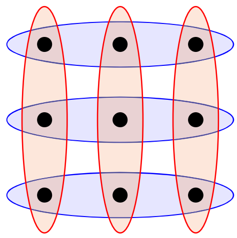
$$\left\{ \bigwedge_{l<|\mathbf{y}_i|} x \neq a^l_{i,0} \wedge \phi_{i,R}(x, a_{i,0}) \wedge \phi_{i,\mathbb{U}}(x) \right\}_{i<\kappa},$$

This concludes our proof.

□

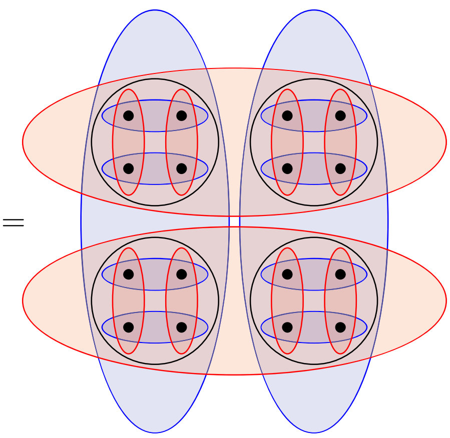
Example. Let $\mathrm{L} = \{R, B\}$ be the language with two binary predicates, and let $\mathcal{M}$ be a set with two cross-cutting equivalence relations with infinitely many infinite classes.

$\mathcal{M} :=$



We saw that $\mathrm{bdn}(\mathcal{M}) = 2$. We leave it to the reader to describe $\mathcal{M}[\mathcal{M}^s]^{\mathbb{U}}$ (where R and B are no longer equivalence relations). Then, one can verify that it is also of burden 2. We let the following picture for the intuition:

$\mathcal{M}[\mathcal{M}^s]^{\mathbb{U}} :=$



Appendix C

More on the RV-sort

We defined in Subsection 3.1.2 the leading term structure of order 0 $(\mathrm{RV}, \oplus, \cdot, \mathbf{0}, \mathbf{1})$ of a valued field $\mathcal{K} = (K, \Gamma, k)$ as the abelian group $K^\star/1+\mathfrak{m}$ endowed with some extra structure, where $\mathfrak{m}$ is the maximal ideal of the valuation ring. We saw that in the context of benign valued fields (Definition 1.2.21), and like the value group and residue field, the RV-sort appears to be a pure imaginary sort (even with control of parameters in the sense of Definition 1.1.8). This means that, like the value group and the residue field, it can be seen as a structure on its own. However, RV-sorts have not been extensively studied as algebraic structures. Let us cite Krasner, who defined in [51] a generalization of fields called 'corpoïde' and which includes in particular the RV-sorts. In this appendix, we give the definition of the RV-sort out of the context of valued fields – which we call an RV-structure. Then we give an axiomatisation in a language $\mathrm{L} := \{\oplus, \cdot, \mathbf{0}, \mathbf{1}\}$ with two constants and two binary functions, and show that every RV-structure is the RV-sort of a canonical Henselian valued field.

C.1 Axomatisation of RV-structures

Definition C.1.1. An RV-*structure* is an abelian group $(\mathrm{RV}^\star, \cdot)$ such that there are an ordered abelian group $(\Gamma, +, <, 0)$ and a field $(k, +, \cdot, 0, 1)$ and the following short exact sequence:

$$1 \to k^\star \overset{\iota}{\to} \mathrm{RV}^\star \overset{\mathrm{val}_{\mathrm{RV}}}{\to} \Gamma \to 0.$$

We abusively see $k^\star$ as a subset of $\mathrm{RV}^\star$. Outside of k, the addition is not defined, but one can extend it as follows. We call the group morphism $\mathrm{val}_{\mathrm{RV}} : \mathrm{RV}^\star \to \Gamma$ the *valuation map*.

Definition C.1.2. We first complete the embedding of $k^\star$ in $\mathrm{RV}^\star$ by adding a new element $\mathbf{0}$, absorbing for the multiplication (*i.e.* $\mathbf{0} \cdot \mathbf{a} = 0$ for all $\mathbf{a} \in \mathrm{RV}^\star \cup \{\mathbf{0}\}$). We denote by RV the set $\mathrm{RV}^\star \cup \{\mathbf{0}\}$. We also add a new element ∞ in Γ, with $\infty > \Gamma$, and extend the valuation map $\mathrm{val}_{\mathrm{RV}}$ in $\mathbf{0}$ by setting $\mathrm{val}_{\mathrm{RV}}(\mathbf{0}) = \infty$. Let $\mathbf{a}, \mathbf{b} \in \mathrm{RV}$. We denote by $\mathbf{a} \oplus \mathbf{b}$ the element:

$$\mathbf{a} \oplus \mathbf{b} = \begin{cases} \mathbf{0} & \text{if } \mathbf{a} = \mathbf{b} = 0, \\ \mathbf{a} & \text{if } \mathrm{val}_{\mathrm{RV}}(\mathbf{b}) > \mathrm{val}_{\mathrm{RV}}(\mathbf{a}), \\ \mathbf{b} & \text{if } \mathrm{val}_{\mathrm{RV}}(\mathbf{a}) > \mathrm{val}_{\mathrm{RV}}(\mathbf{b}), \\ (\mathbf{a}/\mathbf{b} + 1) \cdot \mathbf{b} & \text{otherwise.} \end{cases}$$

This new law $\oplus$ extends the addition in k. The phenomenon $\mathbf{a} \oplus \mathbf{b} = \mathbf{a}$ or $\mathbf{a} \oplus \mathbf{b} = \mathbf{b}$, corresponding to the first two cases, will be referred to as '*additive absorption*'. Notice that $\oplus$ is not associative. In particular the addition in K is not compatible with $\oplus$ through the valuation maps (e.g.: in $\mathbb{R}((T))$, $1 = \mathrm{val}_{\mathrm{RV}}(1 - 1 + t) \neq \mathrm{val}_{\mathrm{RV}}(\mathrm{rv}(1) \oplus \mathrm{rv}(-1 + t)) = +\infty$).

As the operation $\oplus$ is not (always) associative, we adopt a natural convention for the contraction $\bigoplus_{i<n} \mathbf{a}_i$:

Notation. Let $n \in \mathbb{N}$ and $\mathbf{a}_1, \ldots, \mathbf{a}_n \in \mathrm{RV}$. We will use the notation $\bigoplus_{i<n} \mathbf{a}_i$ for $((((\mathbf{a}_1 \oplus \mathbf{a}_2) \oplus \mathbf{a}_3) + \cdots) \oplus \mathbf{a}_n)$ only when the sum $((((\mathbf{a}_{\sigma(1)} \oplus \mathbf{a}_{\sigma(2)}) \oplus \mathbf{a}_{\sigma(3)}) + \cdots) \oplus \mathbf{a}_{\sigma(n)})$ is associative for any permutation σ (in other word, when this term does not depend on the choice of parenthesis). It happens in particular when all $\mathbf{a}_i$'s have the same valuation (dividing by $\mathbf{a}_1$, we get a sum in the field k) or *a contrario*, when they have pairwise distinct valuations (by additive absorption). Notice for instance that in the RV-sort of $\mathbb{R}((T))$, the sum $\mathbf{1} \oplus (-\mathbf{1} \oplus \mathbf{t})$ is not 'associative' whereas the sum $\mathbf{1} \oplus (\mathbf{t} \oplus -\mathbf{1})$ is.

Even if this operation $\oplus$ does not satisfy all the required properties, we refer to it as the 'addition' of RV. We attempt to describe its essential properties. Recall that we defined L as the one-sorted language with signature $\{\oplus, \cdot, \mathbf{0}, \mathbf{1}\}$. We will see that RV-structures are exactly L-structures satisfying the following list of axioms (1-7):

1. $(\mathrm{RV}^\star, \cdot, \mathbf{1})$ is an abelian group, where $\mathrm{RV}^\star = \mathrm{RV} \setminus \{\mathbf{0}\}$,
2. (neutral element for $\oplus$) $\forall \mathbf{a} \in \mathrm{RV},\ \mathbf{0} \oplus \mathbf{a} = \mathbf{a}$,
3. (semi or half-associativity) $[(\mathbf{a} \oplus \mathbf{b}) \oplus \mathbf{c} \neq \mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{c})] \Rightarrow \mathbf{a} \oplus \mathbf{b} = \mathbf{0}$ or $\mathbf{b} \oplus \mathbf{c} = \mathbf{0}$,
4. (commutativity for $\oplus$) $\forall \mathbf{a}, \mathbf{b} \in \mathrm{RV},\ \mathbf{a} \oplus \mathbf{b} = \mathbf{b} \oplus \mathbf{a}$,

5. (distributivity) $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathrm{RV},\ (\mathbf{a} \oplus \mathbf{b}) \cdot \mathbf{c} = \mathbf{ac} \oplus \mathbf{bc}$.

We define $k^\star$ as the set:

$$\{\mathbf{r} \in \mathrm{RV} \setminus \{\mathbf{0}\} \mid \mathbf{1} \oplus \mathbf{r} \neq \mathbf{1} \ \text{ and } \ \mathbf{1} \oplus \mathbf{r}^{-1} \neq \mathbf{1}\}.$$

We write $k := k^\star \cup \{0\}$ and we may denote its elements $a, b, c, \ldots \in k$ with the usual font and denote the restriction $\oplus_{|k}$ by the symbol $+$.

6. $(k := k^\star \cup \{0\}, \cdot, +, 0, 1)$ is a field,

7. (uniform additive absorption) $\forall \mathbf{a} \in \mathrm{RV},\ \forall r \in k^\star,\ (\mathbf{a} \oplus \mathbf{1} = \mathbf{1}) \Leftrightarrow (\mathbf{a} \oplus r = r)$.

The fact that all these properties are satisfied by any RV-structure is clear. From the axioms, one can show the following (8-10):

8. (multiplicative absorption) $\forall \mathbf{a} \in \mathrm{RV},\ \mathbf{0} \cdot \mathbf{a} = \mathbf{0}$.

Assume for some $\mathbf{a} \in \mathrm{RV}$, $\mathbf{0} \cdot \mathbf{a} \neq \mathbf{0}$. Then as $\mathbf{0}$ is a neutral element for $\oplus$ and by distributivity $\mathbf{0} \cdot \mathbf{a} = \mathbf{0} \cdot \mathbf{a} \oplus \mathbf{0} \cdot \mathbf{a}$. We may multiply by the multiplicative inverse of $\mathbf{0} \cdot \mathbf{a}$ and we get a contradiction in k.

9. (additive inverse) $\forall \mathbf{a} \in \mathrm{RV}, \exists! \mathbf{b}\ \mathbf{a} \oplus \mathbf{b} = \mathbf{0}$.

This inverse is given by $-\mathbf{a} := -\mathbf{1} \cdot \mathbf{a}$. Indeed we have $(\mathbf{a} \oplus -\mathbf{a}) = \mathbf{a} \cdot (\mathbf{1} + -\mathbf{1}) = \mathbf{0}$. Uniqueness is clear if $\mathbf{a} = \mathbf{0}$. Assume $\mathbf{a} \neq \mathbf{0}$, if $\mathbf{b} \in \mathrm{RV}^\star$ is such that $\mathbf{a} \oplus \mathbf{b} = \mathbf{0}$, in particular $\mathbf{b} \neq \mathbf{0}$. Then by distributivity and multiplicative absorption $\mathbf{b}/\mathbf{a} \in k^\star$ and from $\mathbf{b}/\mathbf{a} + \mathbf{1} = \mathbf{0}$ we get $\mathbf{b} = -\mathbf{a}$.

We recover the value group by setting $\Gamma := \mathrm{RV}^\star / k^\star$. For the order in Γ, one must define it as follows:

$$\forall [\mathbf{a}], [\mathbf{b}] \in \Gamma, [\mathbf{a}] < [\mathbf{b}] \ \Leftrightarrow \ \mathbf{1} \oplus \mathbf{b}/\mathbf{a} = \mathbf{1} \ \Leftrightarrow \ \mathbf{a} \oplus \mathbf{b} = \mathbf{a},$$

where $[\mathbf{a}]$ denote the classe of $\mathbf{a}$ modulo $k^\star$. By uniform additive absorption and distributivity, this definition does not depend of the representative $\mathbf{a}$ and $\mathbf{b}$ we have chosen. Indeed, if $r, r' \in k^\star$, one gets:

$$1 \oplus \mathbf{b}r/\mathbf{a}r' = \mathbf{1} \Leftrightarrow r'/r \oplus \mathbf{b}/\mathbf{a} = r'/r \overset{(7)}{\Leftrightarrow} \mathbf{1} \oplus \mathbf{b}/\mathbf{a} = \mathbf{1}.$$

10. $(\Gamma, <)$ is an ordered group.

Anti-symmetry of $<$ follows from the definitions of $k^\star$ and $<$, and transitivity is given by semi-associativity: Assume $\mathbf{a} \oplus \mathbf{b} = \mathbf{a}$ and $\mathbf{b} \oplus \mathbf{c} = \mathbf{b}$, then either $\mathbf{b} = \mathbf{0}$, or $\mathbf{a} \oplus \mathbf{b} \neq \mathbf{0}$ and $\mathbf{b} \oplus \mathbf{c} \neq \mathbf{0}$. In any case, $\mathbf{a} \oplus \mathbf{c} = (\mathbf{a} \oplus \mathbf{b}) \oplus \mathbf{c} = \mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{c}) = \mathbf{a} \oplus \mathbf{b} = \mathbf{a}$. It's a total order since for all $\mathbf{a}, \mathbf{b} \in \mathrm{RV}^\star$, either $\mathbf{a}/\mathbf{b} \in k^\star$, $\mathbf{a}/\mathbf{b} \oplus \mathbf{1} = \mathbf{1}$ or $\mathbf{b}/\mathbf{a} \oplus \mathbf{1} = \mathbf{1}$, which respectively gives $[\mathbf{a}] = [\mathbf{b}]$, $[\mathbf{a}] > [\mathbf{b}]$ or $[\mathbf{b}] > [\mathbf{a}]$. We complete the valuation map by setting $\mathrm{val}_{\mathrm{RV}}(\mathbf{0}) = \infty$ where $\infty > \Gamma$.

Note.

- To avoid the use of conventions for $\mathbf{0}$, one might define the quotient $\mathrm{RV}/k^\star$ as the set of orbits of RV under the action of $k^\star$. This action preserve the multiplication in RV. We get that $\mathbf{0}$ is the unique element in its orbit $[\mathbf{0}]$, which we denoted by ∞. the definition of $<$ gives then that $\infty > \Gamma$.

- In [51], structures satisfying Axioms (1) and (8) are called pseudo-group for the multiplication.

- One can replace (6) by:

$$(6')\ \forall r \in k,\ 1 + r \in k, 0 \neq 1, \text{ and } + \text{is associative in } k.$$

The fact that $(k, \cdot)$ is a multiplicative group can already be deduced from (7). Let $\mathbf{r}, \mathbf{s} \in \mathrm{RV}^\star$. If $\mathbf{s} \in k^\star$, one has:

$$\mathbf{1} \oplus \mathbf{r} \cdot \mathbf{s} = \mathbf{1} \overset{(5)}{\Leftrightarrow} \mathbf{1}/\mathbf{s} \oplus \mathbf{r} = \mathbf{1}/\mathbf{s} \overset{(7)}{\Leftrightarrow} \mathbf{1} + \mathbf{r} = \mathbf{r}$$

Hence, a product $\mathbf{r} \cdot \mathbf{s}$ is not in k if and only if $\mathbf{s}$ or $\mathbf{r}$ is not in k.

C.2 The Hahn field associated to RV

As we know, given any field k and any ordered abelian group Γ, there is always a Henselian valued field with residue field k and value group Γ: the Hahn field $k((\Gamma))$. We can ask the following question: is any RV-structure the RV-sort of a certain Henselian valued field? We define for that the Hahn field associated to RV.

Definition C.2.1 (of the Hahn field $\mathrm{RV}^{(\Gamma)}$)**.** The Hahn field associated to the RV-structure (RV, k, Γ) and denoted by $\mathrm{RV}^{(\Gamma)}$, is defined by the following set:

$$\{(\mathbf{a}_\gamma)_{\gamma\in\Gamma} \mid \forall \gamma \in \Gamma\ \mathbf{a}_\gamma \in \mathrm{RV}, \mathrm{val}_{\mathrm{RV}}(\mathbf{a}_\gamma) \in \{\gamma, \infty\} \text{ and } \mathrm{supp}(\mathbf{a}_\gamma)_\gamma \text{ is well-ordered}\}$$

where $\mathrm{supp}(\mathbf{a}_\gamma)_\gamma = \{\gamma \in \Gamma \mid \mathbf{a}_\gamma \neq \mathbf{0}\}$.
It is endowed with the following laws:

$$(\mathbf{a}_\gamma)_\gamma + (\mathbf{b}_\gamma)_\gamma = (\mathbf{a}_\gamma \oplus \mathbf{b}_\gamma)_\gamma$$

$$(\mathbf{a}_\gamma)_\gamma \cdot (\mathbf{b}_\gamma)_\gamma = (\bigoplus_{\delta+\epsilon=\gamma} \mathbf{a}_\delta \cdot \mathbf{b}_\epsilon)_\gamma$$

$$\mathrm{val}(\mathbf{a}_\gamma)_\gamma = \min \mathrm{supp}(\mathbf{a}_\gamma)_\gamma$$

An element $(\mathbf{a}_\gamma)_{\gamma\in\Gamma} \in \mathrm{RV}^{(\Gamma)}$ is written $\sum\limits_{\gamma\in\Gamma} \mathbf{a}_\gamma$ where the sign $\sum$ is purely formal.

Proposition C.2.2. *The Hahn field* $\mathrm{RV}^{(\Gamma)}$ *is a Henselian valued field of* RV-*sort* RV*.*

Proof. The proof is straightforward. As in the Hahn field $k((t^\Gamma))$, the difficult part is to show that every non-zero element of $\mathrm{RV}^{(\Gamma)}$ has a multiplicative inverse. We first show that it is a spherically complete ring. Then we will deduce that it is an actual field.

- (associativity for +): if $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathrm{RV}$ with $\mathrm{val}_{\mathrm{RV}}(\mathbf{a}) = \mathrm{val}_{\mathrm{RV}}(\mathbf{b}) = \mathrm{val}_{\mathrm{RV}}(\mathbf{c})$, then $(\mathbf{a} \oplus \mathbf{b}) \oplus \mathbf{c} = \mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{c})$. Then associativity for + in $\mathrm{RV}^{(\Gamma)}$ is clear as we sum componentwise.
- (commutativity for +): clear as $\oplus$ is commutative in RV.
- (neutral element for +): $0 := \sum_\gamma 0 \in \mathrm{RV}^{(\Gamma)}$ is a neutral element, as $0 \in \mathrm{RV}$ is a neutral element for $\oplus$.
- (inverse for +): if $a = \sum\limits_{\gamma\in\Gamma} \mathbf{a}_\gamma$, the inverse of a is given by $a = \sum\limits_{\gamma\in\Gamma} -\mathbf{a}_\gamma$. The support being the same, it is an element of $\mathrm{RV}^{(\Gamma)}$.

The multiplication in $\mathrm{RV}^{(\Gamma)}$ is well-defined: as the supports of a and b are well-ordered, the sum $\bigoplus\limits_{\delta+\epsilon=\gamma} \mathbf{a}_\delta \cdot \mathbf{b}_\epsilon$ is finite. As before, it is associative since all terms have same valuation. Then $\mathrm{supp}(a+b) \subset \mathrm{supp}(a) + \mathrm{supp}(b)$ and it is easy to see that is a well-ordered set of Γ. We have:

- (associativity for $\cdot$): Let $a = \sum\limits_{\delta\in\Gamma} \mathbf{a}_\delta, b = \sum\limits_{\epsilon\in\Gamma} \mathbf{b}_\epsilon, c = \sum\limits_{\zeta\in\Gamma} \mathbf{c}_\zeta$ be three elements of $\mathrm{RV}^{(\Gamma)}$, then a simple calculation gives $(a\cdot b)\cdot c = \bigoplus\limits_{\delta+\epsilon+\zeta=\gamma} (\mathbf{a}_\delta \cdot \mathbf{b}_\epsilon) \cdot \mathbf{c}_\zeta = \bigoplus\limits_{\delta+\epsilon+\zeta=\gamma} \mathbf{a}_\delta \cdot (\mathbf{b}_\epsilon \cdot \mathbf{c}_\zeta) = a \cdot (b \cdot c)$ (as $\cdot$ is associative in RV).

- (commutativity for $\cdot$): clear as $\cdot$ is commutative in RV.
- (neutral element for $\cdot$): Let $\mathbf{1} \in \mathrm{RV}^{(\Gamma)}$ be the element $\sum_\gamma \mathbf{a}_\gamma$ where $\mathbf{a}_\gamma = \begin{cases} \mathbf{0} \text{ if } \gamma \neq 0 \\ \mathbf{1} \text{ if } \gamma = 0 \end{cases}$. It is a neutral element for $\cdot$ as $\mathbf{1} \in \mathrm{RV}$ is a neutral element for the multiplication in RV.
- (distributivity): Let $a = \sum\limits_{\gamma\in\Gamma} \mathbf{a}_\gamma, b = \sum\limits_{\gamma\in\Gamma} \mathbf{b}_\gamma, c = \sum\limits_{\gamma\in\Gamma} \mathbf{c}_\gamma$ be three elements of $\mathrm{RV}^{(\Gamma)}$, then:

$$\begin{aligned}(a+b)\cdot c &= \sum_{\gamma\in\Gamma} \bigoplus_{\delta+\epsilon=\gamma} (\mathbf{a}_\delta \oplus \mathbf{b}_\delta)\cdot \mathbf{c}_\epsilon = \sum_{\gamma\in\Gamma} \bigoplus_{\delta+\epsilon=\gamma} (\mathbf{a}_\delta \cdot \mathbf{c}_\epsilon \oplus \mathbf{b}_\delta \cdot \mathbf{c}_\epsilon) \\ &= \sum_{\gamma\in\Gamma} \bigoplus_{\delta+\epsilon=\gamma} \mathbf{a}_\delta \cdot \mathbf{c}_\epsilon + \sum_{\gamma\in\Gamma} \bigoplus_{\delta+\epsilon=\gamma} \mathbf{b}_\delta \cdot \mathbf{c}_\epsilon \\ &= a\cdot c + b \cdot c.\end{aligned}$$

- (valuation): Homomorphism of groups is clear. The ultrametric inequality of the valuation is clear from the definition.
- (spherically complete): We give here a usual diagonal argument. Let $(a^i)_{i<\lambda}$ be a pseudo-Cauchy sequence in $\mathrm{RV}^{(\Gamma)}$, where λ is any limit ordinal. There is i_0 such that for all $i_0 < i < j < k$, $\mathrm{val}_{\mathrm{RV}}(a^i - a^j) < \mathrm{val}_{\mathrm{RV}}(a^j - a^k)$. For $i > i_0$, we denote by γ_i the value $\mathrm{val}_{\mathrm{RV}}(a^i - a^{i+1})$. We define
$$\mathbf{a}_\gamma := \begin{cases} \mathbf{a}^i_\gamma \text{ if } \gamma_i > \gamma \text{ for some } i < \lambda, \\ 0 \text{ otherwise.} \end{cases}$$
It is well defined by definition of the valuation (it does not depend on the choice of i). Let $a = \sum_\gamma \mathbf{a}_\gamma$, we get $\mathrm{val}_{\mathrm{RV}}(a - a^i) > \gamma_i$. We have proved that any pseudo-Cauchy sequence admits a pseudo-limit.
- (multiplicative inverse) Let $a = \sum_\gamma \mathbf{a}_\gamma \in \mathrm{RV}^{(\Gamma)}$. Assume it has no inverse and consider the set
$$\Delta := \{\mathrm{val}(1 - a\cdot b) \mid b \in \mathrm{RV}^{(\Gamma)}\}.$$
This set has no maximal element. Indeed, notice first if $\gamma = \mathrm{val}(1-a\cdot b)$, then $\gamma < \infty$ as a has no inverse. It follows that if $\mathbf{c}_\gamma \in \mathrm{RV}$ is the coefficient of value γ in $c = 1 - a\cdot b$, we have $\mathrm{val}(1 - a\cdot(b - \mathbf{c}_\gamma \cdot \mathbf{a}^{-1}_{\mathrm{val}(a)})) > \gamma$. So Δ has no maximal element. Let $(\gamma_\nu = \mathrm{val}(1-a\cdot b_\nu))_{\nu\in\lambda}$ be an co-final increasing sequence in Δ. Then, λ is a limit ordinal.

By definition, $(a \cdot b_\nu)_\nu$ is pseudo-Cauchy with pseudo-limit 1. Then, $(b_\nu)_{\nu\in\lambda}$ is a pseudo-Cauchy sequence (as multiplication by a preserve pseudo-Cauchy sequences). It converges to an element b *in* $\mathrm{RV}^{(\Gamma)}$. Then $\mathrm{val}(1 - a \cdot b) > \Delta$. Contradiction.

We have proved that $\mathrm{RV}^{(\Gamma)}$ is a spherically complete valued field, so in particular Henselian. Every element $\sum_\gamma \mathbf{a}_\gamma \in \mathrm{RV}^{(\Gamma)}$ can be written as $\mathbf{a}_\delta(1 + \sum_\gamma \mathbf{a}_\gamma/\mathbf{a}_\delta)$ where $\delta = \mathrm{val}(\sum_\gamma \mathbf{a}_\gamma)$. Clearly, we have that $\mathrm{RV}(\mathrm{RV}^{(\Gamma)}) = \mathrm{RV}$. □

Remarks C.2.3.

- *In the case where* $\mathrm{RV} = k \times \Gamma$, *one may easily show that* $\mathrm{RV}^{(\Gamma)}$ *is isomorphic to the Hahn field* $k((\Gamma))$.
- *The characteristic of* $\mathrm{RV}^{(\Gamma)}$ *is always equal to the characteristic of the residue field* k.

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List of Symbols

Adler's convention and more