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Homology and cohomology theories on manifolds

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Abstract. We study generalized homology and cohomology theories on the category of manifolds. Ordinary (co-)homology including the cup product is characterized axiomatically.

1. Introduction

In this paper, we study generalized homology and cohomology theories on the categories of smooth, PL, and topological manifolds. These are, by definition, absolute theories satisfying a Mayer-Vietoris property (see Section 4).

Recently, the first author constructed such a theory (of ordinary type) on smooth manifolds [4]; he expected that there is a uniqueness theorem telling that his theory is isomorphic to singular homology. Surprisingly, we could not find such a result in the literature. The incentive for writing this note was to provide a proof for this fact which we present in Section 5.

It turned out that some parts of our argument worked for generalized theories as well; Section 4 is written in this greater generality: We find a functor F which assigns to a generalized (co-)homology theory h_* on manifolds a theory $\hat{h}_*$ on the category $\mathcal{C}$ of countable CW complexes of finite dimension and we show that this functor is full and faithful, that is that F is a bijection on morphism sets. This reduces our original problem to a uniqueness result for ordinary (co-)homology on the category $\mathcal{C}$.

Sections 2 and 3 are preparatory; in Section 2 we collect some well-known facts about CW structures on manifolds and generalize these in Section 3 to triples.

In Section 6, we give, for certain coefficient rings, an axiomatic characterization of the cup product on the singular cohomology of manifolds.

2. Countable CW complexes of finite dimension

Let DIFF be the category consisting of all smooth manifolds without boundary and of the continuous maps between them. A manifold is of course assumed to be Hausdorff and to admit a countable basis for its topology; it is not assumed to be connected, but we require that all its components have the same (finite) dimension. The objects of DIFF are simply called differentiable manifolds or DIFF manifolds.

Similarly, there are the categories PL of PL manifolds and TOP of topological or TOP manifolds.

Fact 1. *A topological n-manifold has the homotopy type of a countable CW complex of dimension n.*

Proof. Let M be a topological n-manifold. As observed in [5], it follows from the work of Whitehead and Hanner that M has the homotopy type of a countable CW complex K. From Theorems E and C of [8] we conclude then that there is such a K of dimension n. Actually, Wall's Theorem E requires the hypothesis $n \geq 3$; but for $n \leq 2$ our assertion is anyway clear. □

Fact 2. *A countable CW complex X of dimension n is homotopy equivalent to a differentiable manifold of dimension $2n+1$.*

Proof. By Theorem 13 of [9], X has the homotopy type of a locally finite polyhedron P of dimension n. We can embed P in $\mathbb{R}^{2n+1}$. If U is the interior of a regular neighborhood of P in $\mathbb{R}^{2n+1}$ then U is a differentiable manifold which is homotopy equivalent to X. (Concerning regular neighborhoods of not necessarily compact polyhedra, cp. [1].) □

From these two facts, we conclude in particular:

Proposition 3. *Let* CAT $\in$ {DIFF, PL, TOP}. *A topological space X has the homotopy type of a CAT manifold if and only if X has the homotopy type of a countable CW complex of finite dimension.* □

3. Triples

Next, we have to generalize the considerations of the previous section from spaces to triples. For our purposes, a *triple* $(X; X_1, X_2)$ consists of a topological space X and two subspaces X_1 and X_2 with $X = X_1 \cup X_2$. Of course, a *map of triples* $f : (X; X_1, X_2) \to (Y; Y_1, Y_2)$ is a continuous map $f : X \to Y$ with $f(X_i) \subseteq Y_i$. We denote by $f_1 : X_1 \to Y_1$, $f_2 : X_2 \to Y_2$ and $f_3 : X_1 \cap X_2 \to Y_1 \cap Y_2$ the restrictions of f and put $f_0 := f$. We say that f is a *pseudo-equivalence of triples* if each f_i is a homotopy equivalence, $0 \leq i \leq 3$. And we call f a *homotopy equivalence of triples* if there is a map of triples $g : (Y; Y_1, Y_2) \to (X; X_1, X_2)$ such that g_i is a homotopy inverse of f_i for $0 \leq i \leq 3$.

A triple $(X; X_1, X_2)$ is called a *CW triple* if X is a CW complex and X_1 and X_2 are subcomplexes of X. A basic fact which we will use repeatedly and tacitly asserts that a pseudo-equivalence between CW triples is a homotopy equivalence ([7, Theorem (5.1)]).

We begin with the generalization of Fact 2, which is rather straightforward. Let us adopt the following conventions:

A *simplex* is a closed linear simplex in some $\mathbb{R}^N$. We write $\sigma \leq \tau$ if σ is a face of τ. By a *simplicial complex* K we understand a locally finite simplicial complex in some $\mathbb{R}^N$. The subset of $\mathbb{R}^N$ which is the union of the simplices of K is denoted by $|K|$.

Lemma 4. *Let K be a simplicial complex and let L_1, L_2 be subcomplexes of K. We write*

$$\begin{aligned} Z &:= |K| \\ X_i &:= |L_i|, \quad i = 1, 2, \\ X &:= |L_1 \cup L_2| \\ D &:= |L_1 \cap L_2| \end{aligned}$$

and assume that Z is a PL-manifold of dimension n. Then there are regular neighborhoods N_i of X_i in Z $(i = 1, 2)$ with the following properties:

(i) *$N_1 \cup N_2$ (resp. $N_1 \cap N_2$) is a regular neighborhood of X (resp. D) in Z.*

(ii) *Denoting by U_i the interior* Int N_i *of N_i in Z, we have*

$$U_1 \cup U_2 = \mathrm{Int}(N_1 \cup N_2), \quad U_1 \cap U_2 = \mathrm{Int}(N_1 \cap N_2).$$

(iii) *The inclusion $(X; X_1, X_2) \to (U_1 \cup U_2; U_1, U_2)$ is a homotopy equivalence.*

Proof. (i) For $S \subseteq Z$, let us write

$$N(S; K) := \{\sigma \in K \mid \text{ there is } \tau \in K \text{ with } \sigma \leq \tau \text{ and } \tau \cap S \neq \varnothing\}.$$

If K'' is the second barycentric subdivision of K, the sets

$$\begin{aligned} N_1 &:= |N(X_1, K'')| \\ N_2 &:= |N(X_2, K'')| \\ N &:= |N(X, K'')| \\ N_3 &:= |N(D, K'')| \end{aligned}$$

are regular neighborhoods of X_1, X_2, X and D in Z. It is obvious that $N_3 \subseteq N_1 \cap N_2$ and $N = N_1 \cup N_2$. So we have only to show that $N_1 \cap N_2 \subseteq N_3$.

Let $\sigma \in N(X_1, K'') \cap N(X_2, K'')$. Then there exist $\tau_i \in K$ with $\sigma \leq \tau_i$ and $\tau_i \cap X_i \neq \varnothing$ for $i = 1, 2$.

If $\tau_i \cap D \neq \varnothing$ for some i, we have $\sigma \in N(D, K'')$, hence $\sigma \subseteq N_3$. Therefore we may assume that

$$(\star) \quad \begin{cases} \sigma \in N(X_i \setminus D, K'') & \text{for } i = 1, 2, \\ \sigma \cap D = \varnothing. \end{cases}$$

According to the definition introduced in [1], the set

$$\tilde{N}_i := |N(X_i \setminus D, K'')|$$

is a regular neighborhood of X_i mod D in Z, and $\tilde{N} := \tilde{N}_1 \cup \tilde{N}_2$ is a regular neighborhood of X mod D in Z. By [1, Theorem 5.1], the inclusions

$$X \setminus D \hookrightarrow \tilde{N} \setminus D \text{ and } X_i \setminus D \hookrightarrow \tilde{N}_i \setminus D$$

are homotopy equivalences. In particular, they induce bijections of the sets of path components. By ($\star$), we have $\sigma \subseteq \tilde{N}_i \setminus D$ for $i = 1, 2$. This implies that σ belongs to different path components of $\tilde{N}$, which is absurd.

(ii) We have only to show that

$$\mathrm{Int}(N_1 \cup N_2) \subseteq U_1 \cup U_2.$$

Let $x \in \mathrm{Int}(N_1 \cup N_2)$, and assume that $x \notin U_1 \cup U_2$. Then we must have $x \in \partial N_1 \cap \partial N_2 \subseteq N_1 \cap N_2$. By (i), $N_1 \cap N_2$ is an n-manifold. Certainly, $x \notin \mathrm{Int}(N_1 \cap N_2)$; hence $x \in \partial(N_1 \cap N_2)$. Therefore, and since $x \in \mathrm{Int}(N_1 \cup N_2)$, there is an open n-ball B around x contained in $N_1 \cup N_2$ such that the set $S := B \setminus (N_1 \cap N_2)$ is connected and non-empty. Now S is the disjoint union of $S \cap N_1$ and $S \cap N_2$. These are closed subsets of S. So we arrive at a contradiction.

(iii) From (ii) and Theorem 5.1 of [1] it is obvious that

$$(X; X_1, X_2) \hookrightarrow (N_1 \cup N_2; N_1, N_2)$$

is a homotopy equivalence. Using collars, this implies that

$$(X; X_1, X_2) \hookrightarrow (U_1 \cup U_2; U_1, U_2)$$

is a homotopy equivalence. □

Let us say that a triple $(M; U, V)$ is a *CAT manifold triple* or simply a *CAT triple* if M is a CAT manifold and U, V are open in M.

Proposition 5. *Each CW triple $(X; X_1, X_2)$ with a countable CW complex X of dimension n is homotopy equivalent to a differentiable manifold triple of dimension $2n + 1$.*

Proof. An easy modification of the proof of Theorem 13 of [9] yields a homotopy equivalence between $(X; X_1, X_2)$ and a triple $(P; P_1, P_2)$ where P is a locally finite polyhedron of dimension n and P_1, P_2 are subpolyhedra. We embed P in $\mathbb{R}^{2n+1}$ and apply Lemma 4. □

It is not completely clear to us to what extent Fact 1 can be generalized. The following result is sufficient for our purposes:

Proposition 6. *Given a topological manifold triple $(M; U, V)$ of dimension n, there exist another topological manifold triple $(\tilde{M}; \tilde{U}, \tilde{V})$, a pseudo-equivalence from $(\tilde{M}; \tilde{U}, \tilde{V})$ to $(M; U, V)$ and a CW triple $(X; X_1, X_2)$ which is homotopy equivalent to $(\tilde{M}; \tilde{U}, \tilde{V})$ such that X is countable and*

$$\dim X = \max\{n + 1, 6\}.$$

Proof. Since we may replace M by $M \times \mathbb{R}^k$, we may assume that $n \geq 5$. Let

$$\begin{aligned} D &:= U \cap V, \\ \tilde{U} &:= (U \times]0,1[) \cup (D \times]0,3[), \\ \tilde{V} &:= (V \times]4,5[) \cup (D \times]2,5[), \\ \tilde{M} &:= \tilde{U} \cup \tilde{V}. \end{aligned}$$

The projection $(\tilde{M}; \tilde{U}, \tilde{V}) \to (M; U, V)$ is a pseudo-equivalence. Denote by A and B the closures of $\tilde{U}$ and $\tilde{V}$ in $\tilde{M}$. Then $(\tilde{M}; \tilde{U}, \tilde{V})$ is homotopy equivalent to the triple $(\tilde{M}; A, B)$. Since $\dim \tilde{M} \geq 6$ and since $A \cap B$ is a clean submanifold of $\tilde{M}$ in the sense of [3], Theorem 2.1 of Essay III of [3] yields that $\tilde{M}$ is a TOP handlebody on $A \cap B$ and that $A \cap B$ is itself a TOP handlebody. This implies our assertion. □

4. GENERALIZED HOMOLOGY AND COHOMOLOGY THEORIES

Let CAT be one of the categories DIFF, PL, or TOP. We consider the category $\mathcal{H}_{\mathrm{CAT}}$ of *generalized homology theories* on CAT. An object h_* of $\mathcal{H}_{\mathrm{CAT}}$ consists, by definition, of a sequence $(h_n)_{n \in \mathbb{Z}}$ of covariant homotopy functors on CAT with values in the category $\mathcal{A}$ of abelian groups and, for each CAT triple $(M; U, V)$, of natural homomorphisms

$$\Delta = \Delta_{U,V} : h_n(M) \to h_{n-1}(U \cap V)$$

which make the corresponding Mayer-Vietoris sequence exact.

Of course, a morphism α between two objects h_* and k_* of $\mathcal{H}_{\mathrm{CAT}}$ consists of natural transformations α_n between h_n and k_n such that

$$\alpha_{n-1} \Delta = \Delta \alpha_n$$

for each CAT manifold triple; we write $\alpha_M : h_n(M) \to k_n(M)$.

Similarly we have the category $\mathcal{H}^*_{\mathrm{CAT}}$ of *generalized cohomology theories* on CAT.

Let $\mathcal{C}$ be the category consisting of all countable CW complexes of finite dimension and the continuous maps between them. Let $\mathcal{H}_{\mathcal{C}}$ and $\mathcal{H}^*_{\mathcal{C}}$ be the categories of generalized (co-)homology theories on $\mathcal{C}$, defined exactly as $\mathcal{H}_{\mathrm{CAT}}$ and $\mathcal{H}^*_{\mathrm{CAT}}$ by using CW triples instead of CAT triples.

We wish to define functors

$$\begin{aligned} F: \quad \mathcal{H}_{\mathrm{CAT}} &\to \mathcal{H}_{\mathcal{C}}, & \mathcal{H}^*_{\mathrm{CAT}} &\to \mathcal{H}^*_{\mathcal{C}} \\ h_* &\mapsto \hat{h}_*, & h^* &\mapsto \hat{h}^*. \end{aligned}$$

We consider only homology; the case of cohomology theories is entirely similar.

For each object X of $\mathcal{C}$ we choose once and for all an object $\hat{X}$ of CAT, a homotopy equivalence $\varphi_X : X \to \hat{X}$, and a homotopy inverse $\psi_X : \hat{X} \to X$ of

φ_X. We put

$$\hat{h}_n(X) := h_n(\hat{X}).$$

For a morphism $f : Y \to X$ in $\mathcal{C}$, we define

$$f_* := (\varphi_X f \psi_Y)_* : \hat{h}_n(Y) \to \hat{h}_n(X).$$

This, so far, gives a homotopy functor $\hat{h}_n : \mathcal{C} \to \mathcal{A}$.

For a CW triple $(X; Y, Z)$ with $X \in \mathcal{C}$, we choose by Proposition 5 a homotopy equivalence f from $(X; Y, Z)$ to a CAT triple $(M; U, V)$ and define

$$\hat{\Delta} = \hat{\Delta}_{Y,Z} : \hat{h}_n(X) \to \hat{h}_{n-1}(Y \cap Z)$$

to be the composition

$$\hat{\Delta} := (f_3 \psi_{Y \cap Z})_*^{-1} \circ \Delta_{U,V} \circ (f \psi_X)_*.$$

It is easy to check that $\hat{\Delta}$ is well-defined, natural, and that the corresponding Mayer-Vietoris is exact.

All this means that we now have a generalized homology theory $\hat{h}_* \in \mathcal{H}_{\mathcal{C}}$. Finally, given a morphism $\alpha : h_* \to k_*$ in $\mathcal{H}_{\mathrm{CAT}}$, we obtain a morphism

$$\hat{\alpha} : \hat{h}_* \to \hat{k}_*$$

as follows: For $X \in \mathcal{C}$, define $\hat{\alpha} : \hat{h}_n(X) \to \hat{k}_n(X)$ to be the homomorphism $\alpha_{\hat{X}} : h_n(\hat{X}) \to k_n(\hat{X})$. The following is now obvious:

Proposition 7. *By $h_* \mapsto \hat{h}_*$ we obtain a functor F from $\mathcal{H}_{\mathrm{CAT}}$ to $\mathcal{H}_{\mathcal{C}}$. Similarly for cohomology.* □

Proposition 8. *The functor F is full and faithful, i.e. F is a bijection on morphism sets. In particular, two generalized homology theories h_* and k_* are isomorphic in $\mathcal{H}_{\mathrm{CAT}}$ if and only if $\hat{h}_*$ and $\hat{k}_*$ are isomorphic in $\mathcal{H}_{\mathcal{C}}$. Similarly for cohomology.*

Proof. Let $h_*, k_* \in \mathcal{H}_{\mathrm{CAT}}$ and let

$$\omega : \hat{h}_* \to \hat{k}_*$$

be a morphism in $\mathcal{H}_{\mathcal{C}}$. We wish to define a morphism

$$\check{\omega} : h_* \to k_*.$$

For a manifold M, choose $X \in \mathcal{C}$ and a homotopy equivalence $\rho : M \to X$. We define

$$\check{\omega}_M : h_n(M) \to k_n(M)$$

to be the composition

$$h_n(M) \xrightarrow{(\varphi_X \rho)_*} h_n(\hat{X}) \xrightarrow{=} \hat{h}_n(X) \xrightarrow{\omega_X} \hat{k}_n(X) \xrightarrow{=} k_n(\hat{X}) \xrightarrow{(\varphi_X \rho)_*^{-1}} k_n(M).$$

This does not depend on the choice of X and ρ. It is obvious that $\check{\omega}_M$ is natural with respect to continuous maps between manifolds. From Proposition 6, we conclude that $\check{\omega}\Delta = \Delta\check{\omega}$ for CAT triples. Hence, $\check{\omega}$ is a morphism from h_* to k_*. From the definition, it is immediately clear that $\omega \mapsto \check{\omega}$ is the inverse of the map $\alpha \mapsto \hat{\alpha}$ from $\mathrm{mor}(h_*, k_*)$ to $\mathrm{mor}(\hat{h}_*, \hat{k}_*)$. □

Remark. On $\mathcal{C}$, as on other categories of CW complexes, there is no essential difference between absolute homology theories in the sense of $\mathcal{H}_{\mathcal{C}}$ and relative homology theories. How to pass between these two aspects is generally known and will be used in the next section without further elaboration.

5. Axioms for ordinary (co-)homology on manifolds

A generalized homology theory $h_* \in \mathcal{H}_{\mathrm{CAT}}$ is called an *ordinary homology theory* on CAT if it satisfies the following three additional properties:

- $h_n = 0$ for $n < 0$.
- The Dimension Axiom [2] is satisfied, i.e. $h_n(*) = 0$ for all $n \neq 0$.
- h_* is *additive* [6] in the sense that for a manifold M of dimension 0, each group $h_n(M)$ is canonically isomorphic to $\bigoplus_{x \in M} h_n(\{x\})$.

We fix a reference point P_0 and call the group $h_0(P_0)$ the *coefficients* of the theory.

An *ordinary cohomology theory* on CAT is defined correspondingly; additivity means that for a manifold M of dimension 0, each group $h^n(M)$ is canonically isomorphic to $\prod_{x \in M} h^n(\{x\})$.

Lemma 9. *Let h_* be an ordinary homology theory on* CAT. *Suppose that the manifold M is a topological sum, $M = \coprod_i M_i$. Then the groups $h_n(M)$ are canonically isomorphic to $\bigoplus_i h_n(M_i)$.*

Proof. Step 1. Suppose that each M_i has the homotopy type of a sphere. In this special case, the assertion follows by induction on the dimension, using the standard Mayer-Vietoris argument.

Step 2. Now let M be an arbitrary manifold. In the case of TOP, we replace M by $M \times \mathbb{R}^k$ and assume that $\dim M \geq 6$. We use once more handlebody decompositions. Let us say that M is of type r if M can be obtained by attaching an open collar to a manifold M' (possibly with boundary, of course) such that M' has a decomposition with all handles of index $\leq r$. We proceed by a straightforward induction on the type of M, using Step 1. □

Proposition 10. *Let h_* be an ordinary homology theory on* CAT. *We denote singular homology theory on* CAT *with the same coefficients as h_* simply by H_*. For each isomorphism $\alpha_0 : H_0(P_0) \to h_0(P_0)$, there exists a unique isomorphism of homology theories $\alpha : H_* \xrightarrow{\cong} h_*$ (this means in particular that $\alpha\Delta = \Delta\alpha$) inducing α_0 on the coefficients. A corresponding result holds for cohomology theories.*

Proof. Again we consider only homology. Applying the functor F of Proposition 7, we obtain from h_* a homology theory $\hat{h}_* \in \mathcal{H}_{\mathcal{C}}$. Of course, we can also form $\hat{H}_*$; this is isomorphic to singular homology H_*, considered as an element of $\mathcal{H}_{\mathcal{C}}$. By Proposition 8, it suffices to show that there is a unique isomorphism $\omega : H_* \xrightarrow{\cong} \hat{h}_*$ in $\mathcal{H}_{\mathcal{C}}$ inducing α_0 on the coefficients.

To do this, we extend $\hat{h}_*$ to pairs (see the Remark at the end of Section 4) and obtain a homology theory satisfying all seven axioms of Eilenberg-Steenrod. Then we can apply the following lemma. □

Lemma 11. *Let us consider $\hat{h}_*$ and H_* as homology theories in the sense of Eilenberg-Steenrod on the admissible category $\hat{\mathcal{C}}$ whose objects are the pairs (X, Y) where X is an object of $\mathcal{C}$ and Y is a subcomplex of X. Given an isomorphism $\alpha_0 : H_0(P_0) \to \hat{h}_0(P_0)$, there exists a uniquely determined isomorphism $\omega : H_* \to \hat{h}_*$ (of homology theories on pairs) which equals α_0 on the coefficients.*

Proof. Let $\mathcal{C}'$ be the subcategory of $\hat{\mathcal{C}}$ consisting of pairs (X, Y) where X is a locally finite simplicial complex of finite dimension. As observed by Milnor [6] in a similar situation, the proof given in [2, pages 76 - 105], applies without essential change and provides us with the desired natural isomorphism on $\mathcal{C}'$. (This requires Lemma 9.) Then we use the fact that each object of $\hat{\mathcal{C}}$ is homotopy equivalent to an object of $\mathcal{C}'$. □

6. Multiplicative structures on ordinary cohomology

Having characterized axiomatically ordinary homology and cohomology on manifolds, we now turn to axioms for the ring structure on ordinary cohomology.

Let R be a commutative ring with 1. We consider singular cohomology $H^*(.\,; R)$ as an element of $\mathcal{H}^*_{\mathrm{CAT}}$. A *multiplicative structure* on $H^*(.\,; R)$ is, by definition, a natural multiplication $\bullet$ on the R-module $H^*(M; R)$ for all CAT manifolds M which gives $H^*(M; R)$ the structure of a graded R-algebra and which has the following property:

$$\lambda \bullet x = x \bullet \lambda = \lambda x$$

for $x \in H^*(M; R)$ and $\lambda \in H^0(M; R) = R$ if M is connected. (If M is not connected, it is obvious what we require instead.)

To obtain uniqueness, we need that a certain normalization axiom is satisfied. For a closed connected oriented n-manifold M, let

$$\iota_M \in H^n(M; R) = \mathrm{Hom}(H_n(M; \mathbb{Z}), R)$$

be the homomorphism sending $[M]$ to 1. We call a multiplicative structure $\bullet$ *normalized*, if for all $n, m \in \mathbb{N}$ we have

$$\mathrm{pr}_1^*(\iota_{S^n}) \bullet \mathrm{pr}_2^*(\iota_{S^m}) = \iota_{S^n \times S^m} \text{ in } H^*(S^n \times S^m; R).$$

Proposition 12. *Let R be one of the rings $\mathbb{Z}$ or $\mathbb{Z}/n\mathbb{Z}$ with $n \in \mathbb{N}$ and let $\bullet$ be a normalized multiplicative structure on the theory $H^*(.\,; R) \in \mathcal{H}^*_{\mathrm{CAT}}$. Then $\bullet$ equals the usual cup product.*

Proof. Let M be a manifold, $x \in H^n(M; R)$ and $y \in H^m(M; R)$. We have to show that $x \bullet y = x \cup y$. We may assume that $n, m > 0$. The cohomology classes x and y can be represented by maps into Eilenberg-MacLane spaces;

these can be replaced by finite skeleta which lie, up to homotopy, in CAT. So we are reduced to a "universal" situation. This, in turn, can be compared with the spheres S^n and S^m so that we can apply the normalization axiom. □

Remark. Note that we do not require that a normalized multiplicative structure is graded commutative. By the proposition, this is automatically the case for the coefficients in question.

References

[1] M. M. Cohen, A general theory of relative regular neighborhoods, Trans. Amer. Math. Soc. **136** (1969), 189–229. MR0248802 (40 #2052)

[2] S. Eilenberg and N. Steenrod, *Foundations of algebraic topology*, Princeton Univ. Press, Princeton, New Jersey, 1952. MR0050886 (14,398b)

[3] R. C. Kirby and L. C. Siebenmann, *Foundational essays on topological manifolds, smoothings, and triangulations*, Ann. of Math. Stud., 88, Princeton Univ. Press, Princeton, N.J., 1977. MR0645390 (58 #31082)

[4] M. Kreck, *Differential algebraic topology*, Graduate Studies in Mathematics, 110, Amer. Math. Soc., Providence, RI, 2010. MR2641092

[5] J. Milnor, On spaces having the homotopy type of a CW-complex, Trans. Amer. Math. Soc. **90** (1959), 272–280. MR0100267 (20 #6700)

[6] J. Milnor, On axiomatic homology theory, Pacific J. Math. **12** (1962), 337–341. MR0159327 (28 #2544)

[7] E. H. Spanier and J. H. C. Whitehead, The theory of carriers and S-theory. *Algebraic geometry and topology. A symposium in honor of S. Lefschetz*, 330–360, Princeton Univ. Press, Princeton, N.J. MR0084772 (18,919b)

[8] C. T. C. Wall, Finiteness conditions for CW-complexes, Ann. of Math. (2) **81** (1965), 56–69. MR0171284 (30 #1515)

[9] J. H. C. Whitehead, Combinatorial homotopy. I, Bull. Amer. Math. Soc. **55** (1949), 213–245. MR0030759 (11,48b)

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