Münster J. of Math. **7** (2014), 489–514
DOI 10.17879/58269761901
urn:nbn:de:hbz:6-58269762205

Münster Journal of Mathematics
© Münster J. of Math. 2014

A class of C^*-algebras that are prime but not primitive

Gene Abrams and Mark Tomforde

(Communicated by Joachim Cuntz)

Abstract. We establish necessary and sufficient conditions on a (not necessarily countable) graph E for the graph C^*-algebra $C^*(E)$ to be primitive. Along with a known characterization of the graphs E for which $C^*(E)$ is prime, our main result provides us with a systematic method for easily producing large classes of (necessarily nonseparable) C^*-algebras that are prime but not primitive. We also compare and contrast our results with similar results for Leavitt path algebras.

1. Introduction

It is well known that any primitive C^*-algebra must be a prime C^*-algebra, and a partial converse was established by Dixmier in the late 1950's when he showed that every separable prime C^*-algebra is primitive (see [11, Corollaire 1] or [24, Thm. A.49] for a proof). For over 40 years after Dixmier's result, it was an open question as to whether every prime C^*-algebra is primitive. This was answered negatively in 2001 by Weaver, who produced the first example of a (necessarily nonseparable) C^*-algebra that is prime but not primitive [28]. Additional ad hoc examples of C^*-algebras that are prime but not primitive have been given in [10], [19, Prop. 31], and [21, Prop. 13.4], with this last example being constructed as a graph C^*-algebra. In this paper we identify necessary and sufficient conditions on the graph E for the C^*-algebra $C^*(E)$ to be primitive. Consequently, this will provide a systematic way for easily describing large classes of (necessarily nonseparable) C^*-algebras that are prime but not primitive. In particular, we obtain infinite classes of (nonseparable) AF-algebras, as well as infinite classes of non-AF, real rank zero C^*-algebras, that are prime but not primitive.

This work was supported by Collaboration Grants from the Simons Foundation to each author (Simons Foundation Grant #208941 to Gene Abrams and Simons Foundation Grant #210035 to Mark Tomforde).

Compellingly, but perhaps not surprisingly, the conditions on E for which $C^*(E)$ is primitive are identical to the conditions on E for which the Leavitt path algebra $L_K(E)$ is primitive for any field K [2, Thm. 5.7]. However, as is typical in this context, despite the similarity of the statements of the results, the proofs for graph C^*-algebras are dramatically different from the proofs for Leavitt path algebras, and neither result directly implies the other.

2. Preliminaries on graph C^*-algebras

In this section we establish notation and recall some standard definitions.

Definition 2.1. A *graph* (E^0, E^1, r, s) consists of a set E^0 of vertices, a set E^1 of edges, and maps $r : E^1 \to E^0$ and $s : E^1 \to E^0$ identifying the range and source of each edge.

Definition 2.2. Let $E := (E^0, E^1, r, s)$ be a graph. We say that a vertex $v \in E^0$ is a *sink* if $s^{-1}(v) = \varnothing$, and we say that a vertex $v \in E^0$ is an *infinite emitter* if $|s^{-1}(v)| = \infty$. A *singular vertex* is a vertex that is either a sink or an infinite emitter, and we denote the set of singular vertices by E^0_{sing}. We also let $E^0_{\text{reg}} := E^0 \setminus E^0_{\text{sing}}$, and refer to the elements of E^0_{reg} as *regular vertices*; i.e., a vertex $v \in E^0$ is a regular vertex if and only if $0 < |s^{-1}(v)| < \infty$. A graph is *row-finite* if it has no infinite emitters. A graph is *finite* if both sets E^0 and E^1 are finite. A graph is *countable* if both sets E^0 and E^1 are (at most) countable.

Definition 2.3. If E is a graph, a *path* is a finite sequence $\alpha := e_1 e_2 \ldots e_n$ of edges with $r(e_i) = s(e_{i+1})$ for $1 \leq i \leq n-1$. We say the path α has *length* $|\alpha| := n$, and we let E^n denote the set of paths of length n. We consider the vertices of E (i.e., the elements of E^0) to be paths of length zero. We also let $\operatorname{Path}(E) := \bigcup_{n \in \mathbb{N} \cup \{0\}} E^n$ denote the set of paths in E, and we extend the maps r and s to $\operatorname{Path}(E)$ as follows: for $\alpha = e_1 e_2 \ldots e_n \in E^n$ with $n \geq 1$, we set $r(\alpha) = r(e_n)$ and $s(\alpha) = s(e_1)$; for $\alpha = v \in E^0$, we set $r(v) = v = s(v)$. Also, for $\alpha = e_1 e_2 \cdots e_n \in \operatorname{Path}(E)$, we let α^0 denote the set of vertices that appear in α; that is,

$$\alpha^0 = \{s(e_1), r(e_1), \ldots, r(e_n)\}.$$

Definition 2.4. If E is a graph, the *graph C^*-algebra $C^*(E)$* is the universal C^*-algebra generated by mutually orthogonal projections $\{p_v \mid v \in E^0\}$ and partial isometries with mutually orthogonal ranges $\{s_e \mid e \in E^1\}$ satisfying

(1) $s_e^* s_e = p_{r(e)}$ for all $e \in E^1$,
(2) $s_e s_e^* \leq p_{s(e)}$ for all $e \in E^1$,
(3) $p_v = \sum_{\{e \in E^1 \mid s(e) = v\}} s_e s_e^*$ for all $v \in E^0_{\text{reg}}$.

Definition 2.5. We call Conditions (1)–(3) in Definition 2.4 the *Cuntz–Krieger relations*. Any collection $\{S_e, P_v \mid e \in E^1, v \in E^0\}$ of elements of a C^*-algebra A, where the P_v are mutually orthogonal projections, the S_e are partial isometries with mutually orthogonal ranges, and the Cuntz–Krieger relations are satisfied is called a *Cuntz–Krieger E-family* in A. The universal property of $C^*(E)$ says precisely that if $\{S_e, P_v \mid e \in E^1, v \in E^0\}$ is a Cuntz–Krieger E-family in a C^*-algebra A, the there exists a $*$-homomorphism $\phi : C^*(E) \to A$ with $\phi(p_v) = P_v$ for all $v \in E^0$ and $\phi(s_e) = S_e$ for all $e \in E^1$.

For a path $\alpha := e_1 \dots e_n$, we define $S_\alpha := S_{e_1} \cdots S_{e_n}$; and when $|\alpha| = 0$, we have $\alpha = v$ is a vertex and define $S_\alpha := P_v$.

Remark 2.6. Using the orthogonality of the projections $\{s_e s_e^* \mid e \in E^1\}$, we see that if $\alpha, \beta \in \mathrm{Path}(E)$, then $s_\alpha s_\alpha^* s_\beta s_\beta^*$ is nonzero if and only if $\alpha = \beta\gamma$ or $\beta = \alpha\delta$ for some $\gamma, \delta \in \mathrm{Path}(E)$; in the former case we get $s_\alpha s_\alpha^* s_\beta s_\beta^* = s_\alpha s_\alpha^*$, while in the latter we get $s_\alpha s_\alpha^* s_\beta s_\beta^* = s_\beta s_\beta^*$. Specifically, if $| \alpha |=| \beta |$, then $s_\alpha s_\alpha^* s_\beta s_\beta^*$ is nonzero precisely when $\alpha = \beta$, in which case the product yields $s_\alpha s_\alpha^*$.

Definition 2.7. A *cycle* is a path $\alpha = e_1 e_2 \dots e_n$ with length $|\alpha| \geq 1$ and $r(\alpha) = s(\alpha)$. If $\alpha = e_1 e_2 \dots e_n$ is a cycle, an *exit* for α is an edge $f \in E^1$ such that $s(f) = s(e_i)$ and $f \neq e_i$ for some i. We say that a graph satisfies *Condition (L)* if every cycle in the graph has an exit.

Definition 2.8. A *simple cycle* is a cycle $\alpha = e_1 e_2 \dots e_n$ with $r(e_i) \neq s(e_1)$ for all $1 \leq i \leq n-1$. We say that a graph satisfies *Condition (K)* if no vertex in the graph is the source of exactly one simple cycle. (In other words, a graph satisfies Condition (K) if and only if every vertex in the graph is the source of no simple cycles or the source of at least two simple cycles.)

Our main use of Condition (L) will be in applying the Cuntz–Krieger Uniqueness Theorem. The Cuntz–Krieger Uniqueness Theorem was proven for row-finite graphs in [6, Thm. 1], and for countably infinite graphs in [13, Cor. 2.12] and [23, Thm. 1.5]. The result for possibly uncountable graphs is a special case of the Cuntz–Krieger Uniqueness Theorem [20, Thm. 5.1] for topological graphs. Alternatively, one can obtain the result in the uncountable case by using the version for countable graphs and applying the direct limit techniques described in [23] and [14].

Theorem 2.9 (Cuntz–Krieger Uniqueness Theorem)**.** *If E is a graph that satisfies Condition (L) and $\phi : C^*(E) \to A$ is a $*$-homomorphism from $C^*(E)$ into a C^*-algebra A with the property that $\phi(p_v) \neq 0$ for all $v \in E^0$, then ϕ is injective.*

It is a consequence of the Cuntz–Krieger Uniqueness Theorem that if E is a graph satisfying Condition (L) and I is a nonzero ideal of $C^*(E)$, then there exists $v \in E^0$ such that $p_v \in I$. (To see this, consider the quotient map $q : C^*(E) \to C^*(E)/I$.)

Definition 2.10. If $v, w \in E^0$ we write $v \geq w$ to mean that there exists a path $\alpha \in \mathrm{Path}(E)$ with $s(\alpha) = v$ and $r(\alpha) = w$. (Note that the path α could be a single vertex, so that in particular we have $v \geq v$ for every $v \in E^0$.)

If E is a graph, a subset $H \subseteq E^0$ is *hereditary* if whenever $e \in E^1$ and $s(e) \in H$, then $r(e) \in H$. Note that a short induction argument shows that whenever H is hereditary, $v \in H$, and $v \geq w$, then $w \in H$.

For $v \in E^0$ we define

$$H(v) := \{w \in E^0 \mid v \geq w\}.$$

It is clear that $H(v)$ is hereditary for each $v \in E^0$. A hereditary subset H is called *saturated* if $\{v \in E^0_{\mathrm{reg}} \mid r(s^{-1}(v)) \subseteq H\} \subseteq H$. For any hereditary subset H, we let

$$\overline{H} := \bigcap \{K \mid K \subseteq H \text{ and } K \text{ is a saturated hereditary subset}\}$$

denote the smallest saturated hereditary subset containing H, and we call $\overline{H}$ the *saturation* of H. Note that if H is hereditary and we define $H_0 := H$ and $H_n := H_{n-1} \cup \{v \in E^0_{\mathrm{reg}} \mid r(s^{-1}(v)) \subseteq H_{n-1}\}$ for $n \in \mathbb{N}$, then $H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots$ and $\overline{H} = \bigcup_{n=0}^{\infty} H_n$. It is clear that both of the sets $\varnothing$ and E^0 are hereditary subsets of E^0.

Lemma 2.11. *Let $E = (E^0, E^1, r, s)$ be a graph. If $H \subseteq E^0$ and $K \subseteq E^0$ are hereditary subsets with $H \cap K = \varnothing$, then $\overline{H} \cap \overline{K} = \varnothing$.*

Proof. Since $H \cap K = \varnothing$, we have $H_0 \cap K_0 = \varnothing$. A straightforward inductive argument shows that $H_n \cap K_n = \varnothing$ for all $n \in \mathbb{N}$. Since $H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots$ and $K_0 \subseteq K_1 \subseteq K_2 \subseteq \ldots$, it follows that $\bigcup_{n=0}^{\infty} H_n \cap \bigcup_{n=0}^{\infty} K_n = \varnothing$. Hence $\overline{H} \cap \overline{K} = \varnothing$. □

If H is a hereditary subset, we let

$$I_H := \overline{\operatorname{span}} \left(\{s_\alpha s_\beta^* \mid \alpha, \beta \in \operatorname{Path}(E),\ r(\alpha) = r(\beta) \in H\}\right).$$

It is straightforward to verify that if H is hereditary, then I_H is a (closed, two-sided) ideal of $C^*(E)$ and $I_H = I_{\overline{H}}$. In addition, the map $H \mapsto I_H$ is an injective lattice homomorphism from the lattice of saturated hereditary subsets of E into the gauge-invariant ideals of $C^*(E)$. (In particular, if H and K are saturated hereditary subsets of E, then $I_H \cap I_K = I_{H \cap K}$.) When E is row-finite, the lattice homomorphism $H \mapsto I_H$ is surjective onto the gauge-invariant ideals of $C^*(E)$.

Definition 2.12. A graph E is *downward directed* if for all $u, v \in E^0$, there exists $w \in E^0$ such that $u \geq w$ and $v \geq w$.

Lemma 2.13. *Let $E = (E^0, E^1, r, s)$ be a graph, and let $v, w \in E^0$. If I is an ideal in $C^*(E)$, $p_v \in I$, and $v \geq w$, then $p_w \in I$.*

Proof. Let α be a path with $s(\alpha) = v$ and $r(\alpha) = w$. Since $p_v \in I$, we have

$$p_w = p_{r(\alpha)} = s_\alpha^* s_\alpha = s_\alpha p_{s(\alpha)} s_\alpha^* = s_\alpha p_v s_\alpha^* \in I. \qquad \square$$

The following graph-theoretic notion will play a central role in this paper.

Definition 2.14. Let $E = (E^0, E^1, r, s)$ be a graph. For $w \in E^0$, we define

$$U(w) := \{v \in E^0 \mid v \geq w\};$$

that is, $U(w)$ is the set of vertices v for which there exists a path from v to w. We say E satisfies the *Countable Separation Property* if there exists a countable set $X \subseteq E^0$ for which $E^0 = \bigcup_{x \in X} U(x)$.

Remark 2.15. It is useful to note that E does *not* satisfy the Countable Separation Property if and only if for every countable subset $X \subseteq E^0$ we have $E^0 \setminus \bigcup_{x \in X} U(x) \neq \varnothing$.

2.16. **The notions of "prime" and "primitive" for algebras, and for C^*-algebras.** When we are working with a ring R, an *ideal* in R shall always mean a two-sided ideal. When working with a C^*-algebra A, an *ideal* in A shall always mean a closed two-sided ideal. If we have a two-sided ideal in a C^*-algebra that is not closed, we shall refer to it as an *algebraic ideal.*

Many properties for rings are stated in terms of two-sided ideals. However, when working in the category of C^*-algebras it is natural to consider the corresponding C^*-algebraic properties stated in terms of closed two-sided ideals. Thus for C^*-algebras, one may ask whether a given ring-theoretic property coincides with the corresponding C^*-algebraic property. In the next few definitions we will consider the notions of prime and primitive, and explain how the ring versions of these properties coincide with the C^*-algebraic versions. In particular, a C^*-algebra is prime as a C^*-algebra if and only if it is prime as a ring, and a C^*-algebra is primitive as a C^*-algebra if and only if it is primitive as a ring. This will allow us to unambiguously refer to C^*-algebras as "prime" or "primitive".

If I, J are two-sided ideals of a ring R, then the product IJ is defined to be the two-sided ideal

$$IJ := \left\{ \sum_{\ell=1}^{n} i_\ell j_\ell \;\middle|\; n \in \mathbb{N}, i_\ell \in I, j_\ell \in J \right\}.$$

Definition 2.17. A ring R is *prime* if whenever I and J are two-sided ideals of R and $IJ = \{0\}$, then either $I = \{0\}$ or $J = \{0\}$.

If I, J are closed two-sided ideals of a C^*-algebra A, then the product IJ is defined to be the closed two-sided ideal

$$IJ := \overline{IJ} = \overline{\left\{ \sum_{\ell=1}^{n} i_\ell j_\ell \;\middle|\; n \in \mathbb{N}, i_\ell \in I, j_\ell \in J \right\}}.$$

Definition 2.18. A C^*-algebra A is *prime* if whenever I and J are closed two-sided ideals of A and $IJ = \{0\}$, then either $I = \{0\}$ or $J = \{0\}$.

Remark 2.19. We note that if a ring R admits a topology in which multiplication is continuous (e.g., if R is a C^*-algebra), then it is straightforward to show that R is prime if and only if R has the property that whenever I and J are *closed* two-sided ideals of R and $IJ = \{0\}$, then either $I = \{0\}$ or $J = \{0\}$.

Thus a C^*-algebra is prime as a ring (in the sense of Definition 2.17) if and only if it is prime as a C^*-algebra (in the sense of Definition 2.18). Moreover, since any C^*-algebra has an approximate identity, and any closed two-sided ideal of a C^*-algebra is again a C^*-algebra, we get that whenever I and J are closed two-sided ideals in a C^*-algebra, then $IJ = I \cap J$. Thus a C^*-algebra A is prime if and only if whenever I and J are closed two-sided ideals in A and $I \cap J = \{0\}$, then either $I = \{0\}$ or $J = \{0\}$. In addition, the existence of an approximate identity in a C^*-algebra implies that ideals of ideals are ideals. In other words, if A is a C^*-algebra, I is a closed two-sided ideal of A, and J is a closed two-sided ideal of I, then J is a closed two-sided ideal of A. Consequently, if I is a closed two-sided ideal in a prime C^*-algebra, then I is a prime C^*-algebra.

Definition 2.20. Recall that for a ring R a left R-module ${}_RM$ consists of an abelian group M and a ring homomorphism $\pi : R \to \mathrm{End}(M)$ for which $\pi(R)(M) = M$, giving the module action $r \cdot m := \pi(r)(m)$. We say that ${}_RM$ is faithful if the homomorphism π is injective, and we say ${}_RM$ is *simple* if $M \neq \{0\}$ and M has no nonzero proper R-submodules; i.e., there are no nonzero proper subgroups $N \subseteq M$ with $\pi(r)(n) \subseteq N$ for all $r \in R$ and $n \in N$. We make similar definitions for right R-modules M_R, faithful right R-modules, and simple right R-modules.

Definition 2.21. Let R be a ring. We say that R is *left primitive* if there exists a faithful simple left R-module. We say that R is *right primitive* if there exists a faithful simple right R-module.

There are rings that are primitive on one side but not on the other. The first example was constructed by Bergman [7, 8]. In his ring theory textbook [26, p. 159] Rowen also describes another example found by Jategaonkar that displays this distinction.

Definition 2.22. If A is a C^*-algebra, a *$*$-representation* is a $*$-homomorphism $\pi : A \to B(\mathcal{H})$ from the C^*-algebra A into the C^*-algebra $B(\mathcal{H})$ of bounded linear operators on some Hilbert space $\mathcal{H}$. We say that a $*$-representation $\pi : A \to B(\mathcal{H})$ is *faithful* if it is injective. For any subset $S \subseteq \mathcal{H}$, we define $\pi(A)S := \overline{\mathrm{span}}\{\pi(a)h \mid a \in A \text{ and } h \in S\}$. If $S = \{h\}$ is a singleton set, we often write $\pi(A)h$ in place of $\pi(A)\{h\}$. A subspace $\mathcal{K} \subseteq \mathcal{H}$ is called a *π-invariant subspace* (or just an *invariant subspace*) if $\pi(a)k \in \mathcal{K}$ for all $a \in A$ and for all $k \in \mathcal{K}$. Observe that a closed subspace $\mathcal{K} \subseteq \mathcal{H}$ is invariant if and only if $\pi(A)\mathcal{K} \subseteq \mathcal{K}$.

Definition 2.23. Let A be a C^*-algebra, and let $\pi : A \to B(\mathcal{H})$ be a $*$-representation.

(1) We say π is a *countably generated $*$-representation* if there exists a countable subset $S \subseteq \mathcal{H}$ such that $\pi(A)S = \mathcal{H}$.
(2) We say π is a *cyclic $*$-representation* if there exists $h \in H$ such that $\pi(A)h = H$. (Note that a cyclic $*$-representation could also be called a *singly generated $*$-representation.*)

(3) We say π is an *irreducible $*$-representation* if there are no closed invariant subspaces of $\mathcal{H}$ other than $\{0\}$ and $\mathcal{H}$. We note that π is irreducible if and only if $\pi(A)h = \mathcal{H}$ for all $h \in \mathcal{H} \setminus \{0\}$.

Remark 2.24. One can easily see that for any $*$-representation of a C^*-algebra, the following implications hold:

$$\text{irreducible} \implies \text{cyclic} \implies \text{countably generated.}$$

In addition, if $\mathcal{H}$ is a separable Hilbert space (and hence $\mathcal{H}$ has a countable basis), then π is automatically countably generated.

Definition 2.25. A C^*-algebra A is *primitive* if there exists a faithful irreducible $*$-representation $\pi : A \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

The following result is well known among C^*-algebraists.

Proposition 2.26. *If A is a C^*-algebra, then the following are equivalent.*

(1) *A is a left primitive ring (in the sense of Definition 2.21).*
(2) *A is a right primitive ring (in the sense of Definition 2.21).*
(3) *A is a primitive C^*-algebra (in the sense of Definition 2.25).*

Proof. If A is a C^*-algebra and A^{op} denotes the opposite ring of A, then we see that $a \mapsto a^*$ is a ring isomorphism from A onto A^{op}. Hence $A - \text{Mod}$ and $A^{op} - \text{Mod} \cong \text{Mod} - A$ are equivalent categories. (We mention that, in general, a C^*-algebra is not necessarily isomorphic as a C^*-algebra to its opposite C^*-algebra; see [22].)

The equivalence of (1) and (3) is proven in [12, Thm. 2.9.7, p. 57] and uses the results of [12, Cor. 2.9.6(i), p. 57] and [12, Cor. 2.8.4, p. 53], which show two things: (i) that any algebraically irreducible representation of A on a complex vector space is algebraically equivalent to a topologically irreducible $*$-representation of A on a Hilbert space, and (ii) that any two topologically irreducible $*$-representations of A on a Hilbert space are algebraically isomorphic if and only if they are unitarily equivalent. ☐

Remark 2.27. In light of Proposition 2.26, a C^*-algebra is *primitive* in the sense of Definition 2.25 if and only if it satisfies any (and hence all) of the three equivalent conditions stated in Proposition 2.26.

3. Prime and primitive graph C^*-algebras

In this section we establish graph conditions that characterize when the associated graph C^*-algebra is prime and when the associated graph C^*-algebra is primitive.

The following proposition was established in [19, Thm. 10.3] in the context of topological graphs, but for the convenience of the reader we provide a streamlined proof here for the context of graph C^*-algebras. We also mention that this same result was obtained for countable graphs in [5, Prop. 4.2].)

Proposition 3.1. *Let E be any graph. Then $C^*(E)$ is prime if and only if the following two properties hold:*

(i) *E satisfies Condition (L), and*
(ii) *E is downward directed.*

Proof. First, let us suppose that E satisfies properties (i) and (ii) from above. If I and J are nonzero ideals in $C^*(E)$, then it follows from property (i) and the Cuntz–Krieger Uniqueness Theorem for graph C^*-algebras that there exists $u, v \in E^0$ such that $p_u \in I$ and $p_v \in J$. By property (ii) there is a vertex $w \in E^0$ such that $u \geq w$ and $v \geq w$. It follows from Lemma 2.13 that $p_w \in I$ and $p_w \in J$. Hence $0 \neq p_w = p_w p_w \in IJ$, and so $C^*(E)$ is a prime C^*-algebra.

For the converse, let us suppose that $C^*(E)$ is prime and establish properties (i) and (ii). Suppose $C^*(E)$ is prime, and E does not satisfy Condition (L). Then there exists a cycle $\alpha = e_1 \ldots e_n$ in E that has no exits. Since α has no exits, the set $H := \alpha^0 = \{s(e_1), r(e_1), \ldots, r(e_{n-1})\}$ is a hereditary subset of E, and it follows from [5, Prop. 3.4] that the ideal $I_H = I_{\overline{H}}$ is Morita equivalent to the C^*-algebra of the graph $E_H := (H, s^{-1}(H), r|_H, s|_H)$. Since E_H is the graph consisting of a single cycle, $C^*(E_H) \cong M_n(C(\mathbb{T}))$ for some $n \in \mathbb{N}$ (see [16, Lemma 2.4]). Therefore, the ideal I_H is Morita equivalent to $C(\mathbb{T})$. However, since $C^*(E)$ is a prime C^*-algebra, it follows that the ideal I_H is a prime C^*-algebra. Because Morita equivalence preserves primality, and I_H is Morita equivalent to $C(\mathbb{T})$, it follows that $C(\mathbb{T})$ is a prime C^*-algebra. However, it is well known that $C(\mathbb{T})$ is not prime: If C and D are proper closed subsets of $\mathbb{T}$ for which $C \cup D = \mathbb{T}$, and we set $I_C := \{f \in C(\mathbb{T}) \mid f|_C \equiv 0\}$ and $I_D := \{f \in C(\mathbb{T}) \mid f|_D \equiv 0\}$, then I_C and I_D are ideals with $I_C \cap I_D = \{0\}$ but $I_C \neq \{0\}$ and $I_D \neq \{0\}$. This provides a contradiction, and we may conclude that E satisfies Condition (L) and that property (i) holds.

Next, suppose $C^*(E)$ is prime. Let $u, v \in E^0$, and consider $H(u) := \{x \in E^0 \mid u \geq x\}$ and $H(v) := \{x \in E^0 \mid v \geq x\}$. Then $H(u)$ and $H(v)$ are nonempty hereditary subsets, and the ideals $I_{H(u)} = I_{\overline{H(u)}}$ and $I_{H(v)} = I_{\overline{H(v)}}$ are each nonzero. Since $C^*(E)$ is a prime C^*-algebra, it follows that $I_{\overline{H(u)} \cap \overline{H(v)}} = I_{\overline{H(u)}} \cap I_{\overline{H(v)}} \neq \{0\}$. Thus $\overline{H(u)} \cap \overline{H(v)} \neq \varnothing$, and Lemma 2.11 implies that $H(u) \cap H(v) \neq \varnothing$. If we choose $w \in H(u) \cap H(v)$, then $u \geq w$ and $v \geq w$. Hence E is downward directed and property (ii) holds. □

The following is well known, but we provide a proof for completeness.

Lemma 3.2. *Any primitive C^*-algebra is prime.*

Proof. Let A be a primitive C^*-algebra. Then there exists a faithful irreducible $*$-representation $\pi : A \to B(\mathcal{H})$. Let I and J be ideals of A, and suppose that $IJ = \{0\}$. If $J \neq \{0\}$, then the faithfulness of π implies $\pi(J)\mathcal{H} \neq \{0\}$, and the fact that J is an ideal shows that $\pi(J)\mathcal{H}$ is a closed invariant subspace for π. Since π is irreducible, it follows that $\pi(J)\mathcal{H} = \mathcal{H}$. Using the fact that $IJ = \{0\}$, it follows that $\{0\} = \pi(IJ)\mathcal{H} = \pi(I)\pi(J)\mathcal{H} = \pi(I)\mathcal{H}$. Since π is faithful, this implies that $I = \{0\}$. Thus A is a prime C^*-algebra. □

The following two lemmas are elementary, but very useful.

Lemma 3.3. *If $X \subseteq (0,\infty)$ is an uncountable subset of positive real numbers, then the sum $\sum_{x\in X} x$ diverges to infinity.*

Proof. For each $n \in \mathbb{N}$, let $X_n := \{x \in X \mid x \geq \frac{1}{n}\}$. Then $X = \bigcup_{n=1}^{\infty} X_n$, and since X is uncountable, there exists $n_0 \in \mathbb{N}$ such that X_{n_0} is infinite. Thus $\sum_{x\in X} x \geq \sum_{x\in X_{n_0}} x \geq \sum_{i=1}^{\infty} \frac{1}{n} = \infty$. □

Lemma 3.4. *Let A be a C^*-algebra, and let $\{p_i \mid i \in I\}$ be a set of nonzero mutually orthogonal projections in A. If there exists a $*$-representation $\pi : A \to B(\mathcal{H})$ and a vector $\xi \in \mathcal{H}$ such that $\pi(p_i)\xi \neq 0$ for all $i \in I$, then I is at most countable.*

Proof. Let $P := \bigoplus_{i\in I} \pi(p_i)$ be the projection onto the direct sum of the images of the $\pi(p_i)$. Since the $\pi(p_i)$ are mutually orthogonal projections, the Pythagorean theorem shows $\sum_{i\in I} \|\pi(p_i)\xi\|^2 = \|P\xi\|^2 \leq \|\xi\|^2 < \infty$. Since each $\|\pi(p_i)\xi\|^2$ term is nonzero, and since any uncountable sum of positive real numbers diverges to infinity (see Lemma 3.3), it follows that the index set I is at most countable. □

The following proposition provides a necessary condition for a graph C^*-algebra to be primitive.

Proposition 3.5. *If E is a graph and $C^*(E)$ has a faithful countably generated $*$-representation, then E satisfies the Countable Separation Property.*

Proof. By hypothesis there is a faithful countably generated $*$-representation $\pi : C^*(E) \to B(\mathcal{H})$. Thus there exists a countable set of vectors $S := \{\xi_i\}_{i=1}^{\infty} \subseteq \mathcal{H}$ with $\pi(C^*(E))S = \mathcal{H}$. For every $n \in \mathbb{N} \cup \{0\}$ and for every $i \in \mathbb{N}$, define

$$\Gamma_{n,i} := \{\alpha \in \text{Path}(E) \mid |\alpha| = n \text{ and } \pi(s_\alpha s_\alpha^*)\xi_i \neq 0\}.$$

(Recall that we view vertices as paths of length zero, and in this case $s_v = p_v$.) By Remark 2.6, for any $n \in \mathbb{N} \cup \{0\}$ the set $\{s_\alpha s_\alpha^* \mid \alpha \in \text{Path}(E) \text{ and } |\alpha| = n\}$ consists of mutually orthogonal projections, and hence Lemma 3.4 implies that for any $n \in \mathbb{N} \cup \{0\}$ and for any $i \in \mathbb{N}$, the set $\Gamma_{n,i}$ is countable.

Define

$$\Gamma := \bigcup_{n=0}^{\infty} \bigcup_{i=1}^{\infty} \Gamma_{n,i},$$

which is countable since it is the countable union of countable sets. Also define

$$\Theta := \bigcup_{\alpha\in\Gamma} U(r(\alpha)).$$

Then $\Theta \subseteq E^0$ is a set of vertices, and we shall show that $\Theta = E^0$. We note that Θ consists precisely of the vertices v in E for which there is a path from v to w, where $w = r(\alpha)$ for a path α having the property that $\pi(S_\alpha S_\alpha^*)\xi_i$ is nonzero for some i.

Let $v \in E^0$, and let I denote the closed two-sided ideal of $C^*(E)$ generated by p_v. Since I is a nonzero ideal and π is faithful, it follows that $\pi(I)\mathcal{H} \neq \{0\}$. Thus

$$\pi(I)S = \pi(IC^*(E))S = \pi(I)\pi(C^*(E))S = \pi(I)\mathcal{H} \neq \{0\}$$

and hence there exists $a \in I$ and $\xi_i \in S$ such that $\pi(a)\xi_i \neq 0$. If $H(v) := \{w \in E^0 \mid v \geq w\}$ is the hereditary subset of E^0 generated by v (as given in Definition 2.10), then it follows from [5, §3] that

$$I = I_{H(v)} = \overline{\operatorname{span}}\{s_\alpha s_\beta^* \mid \alpha, \beta \in \operatorname{Path}(E) \text{ and } r(\alpha) = r(\beta) \in H(v)\}.$$

Since $a \in I$ and $\pi(a)\xi_i \neq 0$, it follows from the linearity and continuity of π that there exists $\alpha, \beta \in \operatorname{Path}(E)$ with $r(\alpha) = r(\beta) \in H(v)$ and $\pi(s_\alpha s_\beta^*)\xi_i \neq 0$. Hence

$$\pi(s_\alpha s_\beta^*)\pi(s_\beta s_\beta^*)\xi_i = \pi(s_\alpha s_\beta^* s_\beta s_\beta^*)\xi_i = \pi(s_\alpha s_\beta^*)\xi_i \neq 0,$$

and thus $\pi(s_\beta s_\beta^*)\xi_i \neq 0$. If we let $n := |\beta|$, then $\beta \in \Gamma_{n,i} \subseteq \Gamma$. Since $r(\beta) \in H(v)$, it follows that $v \geq r(\beta)$ and $v \in U(r(\beta)) \subseteq \bigcup_{\alpha \in \Gamma} U(r(\alpha)) = \Theta$. Thus $E^0 = \Theta := \bigcup_{\alpha \in \Gamma} U(r(\alpha))$, and since Γ is countable, E satisfies the Countable Separation Property. □

Corollary 3.6. *If E is a graph and $C^*(E)$ has a faithful cyclic $*$-representation, then E satisfies the Countable Separation Property.*

Corollary 3.7. *If E is a graph and $C^*(E)$ is primitive, then E satisfies the Countable Separation Property.*

Our main objective in this article is the following result, in which we provide necessary and sufficient conditions on a graph E for the C^*-algebra $C^*(E)$ to be primitive. In particular, we identify the precise additional condition on E that guarantees a prime graph C^*-algebra $C^*(E)$ is primitive. With the previously mentioned result of Dixmier [11, Corollaire 1] as context (i.e., that any prime separable C^*-algebra is primitive), we note that our additional condition is not a separability hypothesis on $C^*(E)$, but rather the Countable Separation Property of E.

Theorem 3.8. *Let E be any graph. Then $C^*(E)$ is primitive if and only if the following three properties hold:*

(i) *E satisfies Condition (L),*
(ii) *E is downward directed, and*
(iii) *E satisfies the Countable Separation Property.*

In other words, by Proposition 3.1, $C^*(E)$ is primitive if and only if $C^*(E)$ is prime and E satisfies the Countable Separation Property.

Proof. To prove this result we establish both the sufficiency and the necessity of properties (i), (ii), and (iii) for $C^*(E)$ to be primitive.

Proof of sufficiency. Since E satisfies the Countable Separation Property by (iii), there exists a countable set $X \subseteq E^0$ such that $E^0 = \bigcup_{x \in X} U(x)$. Since X is countable, we may list the elements of X as $X := \{v_1, v_2, \ldots\}$, where this list is either finite or countably infinite. For convenience of notation, let us write $X = \{v_i\}_{i\in I}$ where the indexing set I either has the form $I = \{1, \dots, n\}$ or $I = \mathbb{N}$.

We inductively define a sequence of paths $\{\lambda_i\}_{i\in I} \subseteq \operatorname{Path}(E)$ satisfying the following two properties:

(a) For each $i \in I$ we have $v_i \geq r(\lambda_i)$.
(b) For each $i \in I$ with $i \geq 2$ there exists $\mu_i \in \operatorname{Path}(E)$ such that $\lambda_i = \lambda_{i-1}\mu_i$.

To do so, define $\lambda_1 := v_1$ and note that for $i = 1$ Property (a) is satisfied trivially and Property (b) is satisfied vacuously. Next suppose that $\lambda_1, \dots \lambda_n$ have been defined so that Property (a) and Property (b) are satisfied for $1 \leq i \leq n$. By hypothesis (ii) E is downward directed, and hence there exists a vertex u_{n+1} in E such that $r(\lambda_n) \geq u_{n+1}$ and $v_{n+1} \geq u_{n+1}$. Let μ_{n+1} be a path from $r(\lambda_n)$ to u_{n+1}, and define $\lambda_{n+1} := \lambda_n \mu_{n+1}$. Then the paths $\lambda_1, \dots, \lambda_{n+1}$ satisfy Property (a) and Property (b) for all $1 \leq i \leq n$. Continuing in this manner, we either exhaust the elements of I or inductively create a sequence of paths $\{\lambda_i\}_{i\in I} \subseteq \operatorname{Path}(E)$ satisfying Property (a) and Property (b) for all $i \in I$.

Note that, in particular, for each $i < n$, we have $\lambda_n = \lambda_i \mu_{i+1} \dots \mu_n$, and hence by Remark 2.6 for $i \leq n$ we have

$$s_{\lambda_i} s_{\lambda_i}^* s_{\lambda_n} s_{\lambda_n}^* = s_{\lambda_n} s_{\lambda_n}^*.$$

We now establish for future use that every nonzero closed two-sided ideal J of $C^*(E)$ contains $s_{\lambda_n} s_{\lambda_n}^*$ for some $n \in I$. Using hypothesis (i), our graph satisfies Condition (L), and the Cuntz–Krieger Uniqueness Theorem (Theorem 2.9) implies that there exists $w \in E^0$ such that $p_w \in J$. By the Countable Separation Property there exists $v_n \in X$ such that $w \geq v_n$. In addition, by Property (a) above $v_n \geq r(\lambda_n)$, so that there is a path γ in E for which $s(\gamma) = w$ and $r(\gamma) = r(\lambda_n)$. Since $p_w \in J$, we have by Lemma 2.13 that $p_{r(\lambda_n)} \in J$, so that $s_{\lambda_n} s_{\lambda_n}^* = s_{\lambda_n} p_{r(\lambda_n)} s_{\lambda_n}^* \in J$ as desired.

Define

$$L := \left\{ \sum_{i=1}^{n} (x_i - x_i s_{\lambda_i} s_{\lambda_i}^*) \;\middle|\; n \in I \text{ and } x_1, \dots, x_n \in C^*(E) \right\}.$$

Recall that $\lambda_1 = v_1$, and by our convention (see Definition 2.5) $s_{\lambda_1} = p_{v_1}$ and $s_{\lambda_1} s_{\lambda_1}^* = p_{v_1}$. Clearly L is an algebraic (i.e., not necessarily closed) left ideal of $C^*(E)$. We claim that $p_{v_1} \notin L$. For otherwise there would exist $n \in I$ and $x_1, ..., x_n \in C^*(E)$ with

$$\sum_{i=1}^{n} (x_i - x_i s_{\lambda_i} s_{\lambda_i}^*) = p_{v_1}.$$

But then multiplying both sides of this equation by $s_{\lambda_n} s_{\lambda_n}^*$ on the right gives

$$\sum_{i=1}^{n} (x_i s_{\lambda_n} s_{\lambda_n}^* - x_i s_{\lambda_i} s_{\lambda_i}^* s_{\lambda_n} s_{\lambda_n}^*) = s_{\lambda_n} s_{\lambda_n}^*.$$

Using the previously displayed observation, the above equation becomes

$$\sum_{i=1}^{n}(x_i s_{\lambda_n} s^*_{\lambda_n} - x_i s_{\lambda_n} s^*_{\lambda_n}) = s_{\lambda_n} s^*_{\lambda_n},$$

which gives $0 = s_{\lambda_n} s^*_{\lambda_n}$, a contradiction. Hence we may conclude that $p_{v_1} \notin L$, and L is a proper left ideal of $C^*(E)$.

By the definition of L, we have $a - ap_{v_1} \in L$ for each $a \in C^*(E)$ and hence L is a modular left ideal of $C^*(E)$, a property which necessarily passes to any left ideal of $C^*(E)$ containing L. (See [9] for definitions of appropriate terms. Also, cp. [25, Chap. 2, Thm. 2.1.1].) Let $\mathcal{T}$ denote the set of (necessarily modular) left ideals of $C^*(E)$ that contain L but do not contain p_{v_1}. Since $L \in \mathcal{T}$, we have $\mathcal{T} \neq \varnothing$. By a Zorn's Lemma argument there exists a maximal element in $\mathcal{T}$, which we denote M. We claim that M is a maximal left ideal of $C^*(E)$. For suppose that M' is a left ideal of $C^*(E)$ having $M \subsetneqq M'$. Then by the maximality of M in $\mathcal{T}$ we have $p_{v_1} \in M'$, so that $xp_{v_1} \in M'$ for each $x \in C^*(E)$, which gives that $x = (x - xp_{v_1}) + xp_{v_1} \in L + M' = M'$ for each $x \in C^*(E)$, so that $M' = C^*(E)$. Thus M is a maximal left ideal, and since M is also modular, M is a maximal modular left ideal. Thus by [9, VII.2, Exer. 6] (cp. [25, Chap. 2, Cor. 2.1.4]) M is closed.

Since M is closed we may form the regular $*$-representation of $C^*(E)$ into $C^*(E)/M$; i.e., the homomorphism $\pi : C^*(E) \to \mathrm{End}(C^*(E)/M)$ given by $\pi(a)(b + M) := ab + M$. In this way $C^*(E)/M$ becomes a left $C^*(E)$-module. The submodules of $C^*(E)/M$ correspond to left ideals of $C^*(E)$ containing M, and by the maximality of M the only submodules of $C^*(E)/M$ are $\{0\}$ and M. Hence $C^*(E)/M$ is a simple module. We claim that $\ker \pi = \{0\}$. If $a \in \ker \pi$, then for all $b \in C^*(E)$ we have $ab + M = \pi(a)(b + M) = 0 + M$, so that $ab \in M$ for all $b \in C^*(E)$. If $\{e_\lambda\}_{\lambda \in \Lambda}$ is an approximate unit for $C^*(E)$, then the previous sentences combined with the fact that M is closed implies that $a = \lim_\lambda ae_\lambda \in M$. Hence we have established that $\ker \pi \subseteq M$. Furthermore, since $\ker \pi$ is a closed two-sided ideal of $C^*(E)$, it follows from above that if $\ker \pi$ is nonzero, then $s_{\lambda_n} s^*_{\lambda_n} \in \ker \pi$ for some $n \in I$. But then M contains $as_{\lambda_n} s^*_{\lambda_n}$ for every $a \in C^*(E)$. In addition, since $a - as_{\lambda_n} s^*_{\lambda_n} \in L \subseteq M$ for all $a \in C^*(E)$, it follows that $a \in M$ for all $a \in C^*(E)$, and $C^*(E) \subseteq M$, which is a contradiction. We conclude that $\ker \pi = \{0\}$. Hence $C^*(E)/M$ is a faithful left $C^*(E)$-module, which with the simplicity of $C^*(E)/M$ yields that $C^*(E)$ is primitive as a ring. It follows (see Remark 2.26) that $C^*(E)$ is primitive as a C^*-algebra.

Proof of necessity. If $C^*(E)$ is primitive, then $C^*(E)$ is necessarily prime by Lemma 3.2, so by Proposition 3.1 we get that E satisfies Condition (L) and is downward directed. In addition, Corollary 3.7 shows that if $C^*(E)$ is primitive, then E satisfies the Countable Separation Property, thus completing the proof. □

Corollary 3.9. *Let E be a graph. Then the following are equivalent:*

(i) *$C^*(E)$ is primitive.*
(ii) *$C^*(E)$ is prime and E satisfies the Countable Separation Property.*
(iii) *$C^*(E)$ is prime and $C^*(E)$ has a faithful cyclic $*$-representation.*
(iv) *$C^*(E)$ is prime and $C^*(E)$ has a faithful countably generated $*$-representation.*

Proof. The equivalence of (i) and (ii) follows from Theorem 3.8 and Proposition 3.1. The implications (i) $\implies$ (iii) $\implies$ (iv) are trivial. The implication (iv) $\implies$ (ii) follows from Corollary 3.7. □

Remark 3.10. In [21, Problem 13.6] Katsura asks whether a prime C^*-algebra is primitive if it has a faithful cyclic $*$-representation. Corollary 3.9 shows that the answer to Katsura's question is affirmative in the class of graph C^*-algebras. Moreover, Corollary 3.9 prompts us to ask the following more general question.

Question. If a C^*-algebra is prime and has a faithful countably generated representation, then is that C^*-algebra primitive?

Again, Corollary 3.9 provides us with an affirmative answer to this question for the class of graph C^*-algebras. In addition, an affirmative answer to this question in general implies an affirmative answer to Katsura's question in [21, Problem 13.6].

Remark 3.11. In the introduction of [28] Weaver notes that his example of a prime and not primitive C^*-algebra "... places competing demands on the set of partial isometries: it must be sufficiently abundant ... and [simultaneously] sufficiently sparse ..." Effectively, Theorem 3.8 identifies precisely how these two competing demands play out in the context of a graph C^*-algebra $C^*(E)$: primeness (abundance of partial isometries) corresponds to E satisfying Condition (L) and being downward directed, while nonprimitivity (sparseness of partial isometries) corresponds to E not satisfying the Countable Separation Property.

Remark 3.12. It is shown in [5, Prop. 4.2] that for a graph E having both E^0 and E^1 countable, $C^*(E)$ is primitive if and only if E is downward directed and satisfies Condition (L). When E^0 is countable, then E trivially satisfies the Countable Separation Property, with E^0 itself providing the requisite countable set of vertices. Thus Theorem 3.8 provides both a generalization of (since the countability of E^1 is not required), and an alternate proof for, the result in [5, Prop. 4.2].

4. Examples of prime but not primitive C^*-algebras

We now offer a number of examples of prime nonprimitive C^*-algebras that arise from the characterizations presented in Proposition 3.1 and Theorem 3.8. These examples are similar in flavor to the classes of examples that played an important role in [2].

Definition 4.1. Let X be any nonempty set, and let $\mathcal{F}(X)$ denote the set of finite nonempty subsets of X. We define three graphs $E_A(X)$, $E_L(X)$, and $E_K(X)$ as follows.

(1) The graph $E_A(X)$ is defined by

$$E_A(X)^0 = \mathcal{F}(X) \text{ and } E_A(X)^1 = \{e_{A,A'} \mid A, A' \in \mathcal{F}(X) \text{ and } A \subsetneqq A'\},$$

with $s(e_{A,A'}) = A$ and $r(e_{A,A'}) = A'$ for each $e_{A,A'} \in E_A(X)^1$.

(2) The graph $E_L(X)$ is defined by

$$E_L(X)^0 = \mathcal{F}(X) \text{ and } E_L(X)^1 = \{e_{A,A'} \mid A, A' \in \mathcal{F}(X) \text{ and } A \subseteqq A'\},$$

with $s(e_{A,A'}) = A$ and $r(e_{A,A'}) = A'$ for each $e_{A,A'} \in E_L(X)^1$.

(3) The graph $E_K(X)$ is defined by

$$E_K(X)^0 = \mathcal{F}(X) \text{ and}$$
$$E_K(X)^1 = \{e_{A,A'} \mid A, A' \in \mathcal{F}(X) \text{ and } A \subseteqq A'\} \cup \{f_A \mid A \in \mathcal{F}(X)\},$$

with $s(e_{A,A'}) = A$, and $r(e_{A,A'}) = A'$ for each $e_{A,A'} \in E_K(X)^1$, and with $s(f_A) = r(f_A) = A$ for all f_A.

Remark 4.2. Observe that for any nonempty set X, the graph $E_A(X)$ is acyclic, the graph $E_L(X)$ has as its only simple cycles the single loop at each vertex (so that $E_L(X)$ satisfies Condition (L), but not Condition (K)), and the graph $E_K(X)$ has as its only simple cycles the two loops at each vertex (so that $E_K(X)$ satisfies Condition (K)).

In this section all direct limits that we discuss will be direct limits in the category of C^*-algebras, so that the direct limit algebras discussed are C^*-algebras. An AF-algebra is typically defined to be a C^*-algebra that is the direct limit of a sequence of finite-dimensional algebras. Since it is a sequential direct limit, an AF-algebra is necessarily separable. Some authors, such as Katsura in [19], have considered arbitrary direct limits of finite-dimensional C^*-algebras. Following Katsura in [19], we shall define an *AF-algebra* to be a C^*-algebra that is the direct limit of finite-dimensional C^*-algebras. (Equivalently, an AF-algebra is a C^*-algebra A with a directed family of finite dimensional C^*-subalgebras whose union is dense in A.) An AF-algebra in our sense is a sequential direct limit (i.e., an AF-algebra in the traditional sense) if and only if it is separable.

Lemma 4.3. *Let $E_A(X), E_L(X)$, and $E_K(X)$ be the graphs presented in Definition 4.1.*

(1) *Each of the graphs $E_A(X), E_L(X)$, and $E_K(X)$ is downward directed.*
(2) *Each of the graphs $E_A(X), E_L(X)$, and $E_K(X)$ satisfies the Countable Separation Property if and only if X is countable.*

Proof. To establish (1), we observe that in each of the graphs $E_A(X), E_L(X)$, and $E_K(X)$, each pair of vertices A, A' corresponds to a pair of finite subsets of X, so that if B denotes the finite set $A \cup A'$ then there is an edge $e_{A,B}$ from

A to B and an edge $e_{A',B}$ from A' to B. Downward directedness of each of the three graphs follows.

For (2), we note that if X is countable then $\mathcal{F}(X)$ is countable, so that in this case each of the three graphs has countably many vertices, and thus trivially satisfies the Countable Separation Property. On the other hand, if X is uncountable, then any countable union of elements of $\mathcal{F}(X)$ includes only countably many elements of X, so that there exists some vertex (indeed, uncountably many vertices) which does not connect to the vertices represented by such a countable union. □

Proposition 4.4. *Let X be an infinite set, and let $E_A(X)$, $E_L(X)$, and $E_K(X)$ be the graphs presented in Definition 4.1. Then*

(1) *$C^*(E_A(X))$ is a prime AF-algebra for any set X. Furthermore, $C^*(E_A(X))$ is primitive if and only if X is countable. In addition, $C^*(E_A(X))$ is a separable AF-algebra if and only if X is countable.*
(2) *$C^*(E_L(X))$ is a prime C^*-algebra that is not AF for any set X. Furthermore, $C^*(E_L(X))$ is primitive if and only if X is countable. In addition, $C^*(E_L(X))$ contains an ideal that is not gauge invariant.*
(3) *$C^*(E_K(X))$ is a prime C^*-algebra of real rank zero that is not AF for any set X. Furthermore, $C^*(E_K(X))$ is primitive if and only if X is countable. In addition, every ideal of $C^*(E_K(X))$ is gauge invariant.*

Proof. The indicated primeness and primitivity properties of each of the three algebras $C^*(E_A(X))$, $C^*(E_L(X))$, and $C^*(E_K(X))$ follow directly from Proposition 3.1, Theorem 3.8, and Lemma 4.3. We now take up the discussion of the additional properties.

(1) Since $E_A(X)$ is a countable graph if and only if X is countable, it follows that $C^*(E_A(X))$ is separable if and only if X is countable.

(2) We see that $E_L(X)$ has exactly one loop at each vertex, and that these are the only simple cycles in $E_L(X)$. Thus every vertex of $E_L(X)$ is the base of exactly one simple cycle, so that $E_L(X)$ does not satisfy Condition (K). In addition, if α is the loop based at the vertex A, then since A is finite and X is infinite there exists an element $x \in X \setminus A$, and the edge from A to $A \cup \{x\}$ provides an exit for α. Hence every cycle in $E_L(X)$ has an exit and $E_L(X)$ satisfies Condition (L). Since $E_L(X)$ contains a cycle, $C^*(E_L(X))$ is not AF. Moreover, since $E_L(X)$ does not satisfy Condition (K), it follows that $C^*(E_L(X))$ contains an ideal that is not gauge invariant. (This is established for row-finite countable graphs in [27, Thm. 2.1.19], although the same argument works for non-row-finite or uncountable graphs. Alternatively, the result for uncountable graphs may also be obtained as a special case of [21, Thm. 7.6].)

(3) As $E_K(X)$ has two loops at each vertex, every vertex in $E_K(X)$ is the base point of two distinct simple cycles, so that $E_K(X)$ satisfies Condition (K). Since $E_K(X)$ contains a cycle, $C^*(E_K(X))$ is not AF. In addition, since $E_K(X)$ satisfies Condition (K), $C^*(E_K(X))$ has real rank zero. (This was established for C^*-algebras of locally-finite countable graphs in [18, Thm. 4.1] and for C^*-algebras of countable graphs in [15, Thm. 2.5], and can be extended to

uncountable graphs using the approximation methods of [23, Lemma 1.2].) Moreover, since $E_K(X)$ satisfies Condition (K), all ideals of $C^*(E_K(X))$ are gauge invariant. □

We can now produce infinite classes of C^*-algebras that are prime and not primitive. In fact, we are able to produce three such classes of C^*-algebras: one in which all the C^*-algebras are AF-algebras, one in which all the C^*-algebras are non-AF and contain ideals that are not gauge invariant, and one in which all the C^*-algebras are non-AF, have all of their ideals gauge invariant, and are real rank zero.

Proposition 4.5. *For a set X we let $|X|$ denote the cardinality of X.*

(1) *If $\mathcal{C} := \{X_i \mid i \in I\}$ is a collection of sets with $|X_i| > \aleph_0$ for all $i \in I$, and with $|X_i| \neq |X_j|$ for all $i, j \in I$ with $i \neq j$, then*
$$\{C^*(E_A(X_i)) \mid i \in I\}$$
is a collection of AF-algebras that are prime and not primitive. Moreover $C^*(E_A(X_i))$ is not Morita equivalent to $C^*(E_A(X_j))$ for all $i, j \in I$ with $i \neq j$.

(2) *If $\mathcal{C} := \{X_i \mid i \in I\}$ is a collection of sets with $|X_i| \geq 2^{\aleph_0}$ for all $i \in I$, and with $|X_i| \neq |X_j|$ for all $i, j \in I$ with $i \neq j$, then*
$$\{C^*(E_L(X_i)) \mid i \in I\}$$
is a collection of non-AF C^*-algebras each of which is prime and not primitive, and each of which has the property that it contains ideals that are not gauge invariant. Moreover, $C^*(E_L(X_i))$ is not Morita equivalent to $C^*(E_L(X_j))$ for all $i, j \in I$ with $i \neq j$.

(3) *If $\mathcal{C} := \{X_i \mid i \in I\}$ is a collection of sets with $|X_i| > \aleph_0$ for all $i \in I$, and with $|X_i| \neq |X_j|$ for all $i, j \in I$ with $i \neq j$, then*
$$\{C^*(E_K(X_i)) \mid i \in I\}$$
is a collection of non-AF C^*-algebras of real rank zero each of which is prime and not primitive, and each of which has the property that all of its ideals are gauge invariant. Moreover, $C^*(E_K(X_i))$ is not Morita equivalent to $C^*(E_K(X_j))$ for all $i, j \in I$ with $i \neq j$.

Proof. For (1), the fact that the $C^*(E_A(X_i))$ are AF-algebras that are prime and not primitive follows from Proposition 4.4(1). It remains to show the $C^*(E_A(X_i))$ are mutually non-Morita equivalent. For each $i \in I$, the graph $E_A(X_i)$ satisfies Condition (K), and hence the ideals in $C^*(E_A(X_i))$ are in one-to-one correspondence with admissible pairs (H, S), where H is a saturated hereditary subset of $E_A(X_i)^0$ and S is a subset of breaking vertices for H. For each $x \in X_i$ define $H_x := E_A(X_i)^0 \setminus \{x\}$. Because $\{x\}$ is a source and an infinite emitter in $E_A(X_i)$ that only emits edges into H_x, the set H_x is saturated and hereditary, and H_x has no breaking vertices. In addition, any saturated hereditary subset of $E_A(X_i)^0$ that contains H_x is either equal to H_x or equal to $E_A(X_i)^0$. Thus $I_{(H_x,\varnothing)}$ is a maximal ideal in $C^*(E_A(X_i))$. Conversely, any maximal ideal must have the form $I_{(H,S)}$ for an admissible pair (H,S), and in order to be a proper ideal there exists $x \in E_A(X_i)^0 \setminus H$. Because $H \subseteq H_x$, the maximality of $I_{(H,S)}$ implies $H = H_x$ and $S = \varnothing$. Thus we may conclude that the map $x \mapsto I_{(H_x,\varnothing)}$ is a bijection from X onto the set of maximal ideals of $C^*(E_A(X_i))$. Hence the cardinality of the set of maximal ideals of $C^*(E_A(X_i))$ is equal to $|X_i|$. Since any two Morita equivalent C^*-algebras have isomorphic lattices of ideals, any two Morita equivalent C^*-algebras have sets of maximal ideals with the same cardinality. Thus, when $i, j \in I$ with $i \neq j$, the fact that $|X_i| \neq |X_j|$ implies that $C^*(E_A(X_i))$ is not Morita equivalent to $C^*(E_A(X_j))$.

For (2), the fact that the C^*-algebras $C^*(E_L(X_i))$ are non-AF C^*-algebras that are prime and not primitive and that each contains ideals that are not gauge invariant follows from Proposition 4.4(2). It remains to show the $C^*(E_L(X_i))$ are mutually non-Morita equivalent. Fix $i \in I$, and let $J \triangleleft C^*(E)$ be a maximal ideal in $C^*(E_L(X_i))$. Then there exists exactly one $x \in X_i$ such that $I_{H_x} \subseteq J$, where $H_x := E_L(X_i)^0 \setminus \{x\}$. (If there did not exist such an x, then J would be all of $C^*(E_L(X_i))$, and if there existed more than one x, then J would not be maximal.) Since $I_{H_x} \subseteq J$, it follows that J corresponds to a maximal ideal of the quotient $C^*(E_L(X_j))/I_{H_x}$. Because the graph $E_L(X_i) \setminus H_x$ is a single vertex with a single loop, we see that $C^*(E_L(X_j))/I_{H_x} \cong C(\mathbb{T})$. For this maximal ideal there is a unique $z \in \mathcal{T}$ such that the ideal is equal to $\{f \in C(\mathbb{T}) \mid f(z) = 0\}$. This line of reasoning shows that the map $J \mapsto (x,z)$ is a bijection from the set of maximal ideals of $C^*(E_L(X_i))$ onto the set $X \times \mathbb{T}$. Since $|X_i| \geq 2^{\aleph_0}$ and $|\mathbb{T}| = 2^{\aleph_0}$, we may conclude that $|X_i \times \mathbb{T}| = |X_i|$. Thus the set of maximal ideals of $C^*(E_L(X_i))$ has cardinality equal to $|X_i|$. Since any two Morita equivalent C^*-algebras have isomorphic lattices of ideals, any two Morita equivalent C^*-algebras have sets of maximal ideals with the same cardinality. Thus, when $i, j \in I$ with $i \neq j$, the fact that $|X_i| \neq |X_j|$ implies that $C^*(E_L(X_i))$ is not Morita equivalent to $C^*(E_L(X_j))$.

For (3), the fact that the $C^*(E_K(X_i))$ are non-AF C^*-algebras of real rank zero that are prime and not primitive and that all ideals are gauge invariant follows from Proposition 4.4(3). It remains to show the $C^*(E_K(X_i))$ are mutually non-Morita equivalent. The proof follows much like the proof of part (1): For each $i \in I$, the graph $E_K(X_i)$ satisfies Condition (K), and hence the ideals in $C^*(E_K(X_i))$ are in one-to-one correspondence with admissible pairs (H,S), where H is a saturated hereditary subset $E_K(X_i)^0$ and S is a subset of breaking vertices for H. For each $x \in X_i$ define $H_x := E_K(X_i)^0 \setminus \{x\}$. Because $\{x\}$ is a source and an infinite emitter in $E_A(X_i)$, the set H_x is saturated and hereditary. In addition, because there is a loop at $\{x\}$, it is a (unique) breaking vertex for H_x. In addition, any saturated hereditary subset of $E_K(X_i)^0$ that contains H_x is either equal to H_x or equal to $E_A(X_i)^0$. Thus $I_{(H_x,\{x\})}$ is a maximal ideal in $C^*(E_K(X_i))$. Conversely, any maximal ideal must have the form $I_{(H,S)}$ for an admissible pair (H,S), and in order to be a proper ideal there exists $x \in E_K(X_i)^0 \setminus H$. Because $H \subseteq H_x$, the maximality of $I_{(H,S)}$ implies $H = H_x$ and $S = \{x\}$. Thus we may conclude that the map $x \mapsto I_{(H_x,\{x\})}$

is a bijection from X onto the set of maximal ideals of $C^*(E_K(X_i))$. Thus the cardinality of the set of maximal ideals of $C^*(E_K(X_i))$ is equal to $|X_i|$, and as argued in (1) this implies that when $i, j \in I$ with $i \neq j$, the C^*-algebra $C^*(E_K(X_i))$ is not Morita equivalent to C^*-algebra $C^*(E_K(X_j))$. □

Remark 4.6. Note that in each of parts (1)–(3) of Proposition 4.5 we are constructing a prime, nonprimitive C^*-algebra for each set in the collection $\mathcal{C}$. We mention that for any cardinal number κ, there exists a collection of κ sets of differing cardinalities all greater than or equal to $2^{\aleph_0}$. (This fact is well known; see e.g. [17, Lemma 7.7].) Hence in each of parts (1)–(3) of Proposition 4.5 one can choose the collection $\mathcal{C}$ to be of any desired cardinality κ.

Example 4.7. There are, of course, many examples of uncountable graphs whose associated C^*-algebras are primitive. For instance, let X be any uncountable set, let $\mathcal{P}(X)$ denote the set of *all* subsets of X, and let $E_{\mathcal{P}}(X)$ be the graph having

$$E^0_{\mathcal{P}(X)} = \mathcal{P}(X) \text{ and } E^1_{\mathcal{P}(X)} = \{e_{A,A'} \mid A, A' \in \mathcal{P}(X) \text{ and } A \subsetneqq A'\},$$

with $s(e_{A,A'}) = A$, and $r(e_{A,A'}) = A'$ for each $e_{A,A'} \in E^1_{\mathcal{P}(X)}$. Then $E_{\mathcal{P}(X)}$ is not a countable graph, and $C^*(E_{\mathcal{P}(X)})$ is not a separable C^*-algebra. However, $E_{\mathcal{P}(X)}$ satisfies the three conditions of Theorem 3.8, and hence $C^*(E_{\mathcal{P}(X)})$ is a primitive C^*-algebra. (In particular, we observe that any vertex in $E_{\mathcal{P}(X)}$ emits an edge pointing to $\{X\} \in E^0_{\mathcal{P}(X)}$, so $E_{\mathcal{P}(X)}$ trivially satisfies the Countable Separation Property.) The graph $E_{\mathcal{P}(X)}$ has no cycles, so that $C^*(E_{\mathcal{P}(X)})$ is an AF-algebra. In a like manner, we could construct additional examples of uncountable graphs having primitive graph C^*-algebras, and one could easily produce non-AF examples by adding one or two loops at every vertex of $E_{\mathcal{P}(X)}$.

The following definition provides a second graph-theoretic construction which produces examples of graphs whose corresponding graph C^*-algebras are prime but not primitive.

Definition 4.8. Let $\kappa > 0$ be any ordinal. We define the graph E_κ as follows:

$$E^0_\kappa = \{\alpha \mid \alpha < \kappa\}, \quad E^1_\kappa = \{e_{\alpha,\beta} \mid \alpha, \beta < \kappa, \text{ and } \alpha < \beta\},$$

$s(e_{\alpha,\beta}) = \alpha$, and $r(e_{\alpha,\beta}) = \beta$ for each $e_{\alpha,\beta} \in E^1_\kappa$. □

Recall that an ordinal κ is said to have *countable cofinality* in case κ is the limit of a countable sequence of ordinals strictly less than κ. For example, any countable ordinal has countable cofinality. The ordinal ω_1 does not have countable cofinality, while the ordinal ω_ω does have this property. With this definition in mind, it is clear that E^0_κ has the Countable Separation Property if and only if κ has countable cofinality. Thus by Theorem 3.8 we get

Proposition 4.9. *Let $\{\kappa_\alpha \mid \alpha \in \mathcal{A}\}$ denote a set of distinct ordinals, each without countable cofinality. Then the collection $\{C^*(E_{\kappa_\alpha}) \mid \alpha \in \mathcal{A}\}$ is a set of nonisomorphic AF-algebras, each of which is prime but not primitive.*

5. A comparison of the conditions for a graph C^*-algebra to be simple, to be primitive, and to be prime

In Proposition 3.1 and Theorem 3.8, we obtained conditions for a graph C^*-algebra to be prime and primitive, respectively. In this section we compare these conditions with the conditions for a graph C^*-algebra to be simple. Recall that an *infinite path p in E* is a nonterminating sequence $p = e_1e_2e_3\dots$ of edges in E, for which $r(e_i) = s(e_{i+1})$ for all $i \geq 1$. (Note that this notation, while standard, can be misleading; an infinite path in E is not an element of $\mathrm{Path}(E)$, as the elements of $\mathrm{Path}(E)$ are, by definition, finite sequences of edges in E.) We denote the set of infinite paths in E by E^∞.

Proposition 5.1. *Let E be a graph.*
*The graph C^*-algebra $C^*(E)$ is **simple** if and only if the following two conditions are satisfied*

(1) *E satisfies Condition (L), and*
(2) *E is cofinal (i.e., if $v \in E^0$ and $\alpha \in E^\infty \cup E^0_{\mathrm{sing}}$, then $v \geq \alpha^0$).*

*The graph C^*-algebra $C^*(E)$ is **primitive** if and only if the following three conditions are satisfied*

(1) *E satisfies Condition (L),*
(2) *E is downward directed (i.e., for all $v, w \in E^0$ there exists $x \in E^0$ such that $v \geq x$ and $w \geq x$), and*
(3) *E satisfies the Countable Separation Property (i.e., there exists a countable set $X \subseteq E^0$ such that $E^0 \geq X$).*

*The graph C^*-algebra $C^*(E)$ is **prime** if and only if the following two conditions are satisfied*

(1) *E satisfies Condition (L), and*
(2) *E is downward directed (i.e., for all $v, w \in E^0$ there exists $x \in E^0$ such that $v \geq x$ and $w \geq x$).*

Proof. Proposition 3.1 and Theorem 3.8 give the stated conditions for a graph C^*-algebra to be prime and primitive, respectively. The conditions for simplicity are established in [27, Thm. 2.12]. (Although all of [27] is done under the implicit assumption the graphs are countable, the proof of [27, Thm. 2.12] and the proofs of the results on which it relies all go through verbatim for uncountable graphs.) □

Every C^*-algebra has a nonzero irreducible representation. (This follows from the GNS construction, which shows that GNS-representations constructed from pure states are irreducible [24, Lemma A.12], and the Krein–Milman Theorem, which asserts that pure states exist for any C^*-algebra [24, Lemma A.13].) Thus any simple C^*-algebra has a faithful irreducible $*$-representation, and any simple C^*-algebra is necessarily primitive. Moreover, as was shown in Lemma 3.2, any primitive C^*-algebra is necessarily prime. Thus we have

$$C^*(E) \text{ is simple} \implies C^*(E) \text{ is primitive} \implies C^*(E) \text{ is prime.}$$

We observe that the form of each of the three results presented in Proposition 5.1, in which simplicity, primeness, and primitivity of $C^*(E)$ are given in graph-theoretic terms, may be described as "Condition (L) plus something extra". This having been said, our goal for this section is solely graph-theoretic: we show that these three "extra" conditions may be seen as arising in a common context, by considering subsets of E^0 of the form $\overline{H(v)}$.

Proposition 5.2. *Let E be a graph. Then the following equivalences hold.*
(1) *E is cofinal if and only if for all $v \in E^0$ one has $\overline{H(v)} = E^0$.*
(2) *E is downward directed if and only if for all $v, w \in E^0$ one has $\overline{H(v)} \cap \overline{H(w)} \neq \varnothing$.*
(3) *E satisfies the Countable Separation Property if and only if there exists a countable collection of subsets of vertices $\{S_i \mid i \in I\}$ (so, I is countable and $S_i \subseteq E^0$ for all $i \in I$) with $E^0 = \bigcup_{i \in I} S_i$ and with $\bigcap_{v \in S_i} \overline{H(v)} \neq \varnothing$ for all $i \in I$.*

Proof. For (1), suppose first that for all $v \in E^0$ we have $\overline{H(v)} = E^0$. Choose $v \in E^0$ and $\alpha \in E^\infty \cup E^0_{\text{sing}}$. If $\alpha \in E^0_{\text{sing}}$, then $\alpha \in \overline{H(v)} = E^0$. Since every element of $\overline{H(v)} \setminus H(v)$ is a regular vertex, it follows that $\alpha \in H(v)$ and $v \geq \alpha$. If instead $\alpha \in E^\infty$, then we may write $\alpha = e_1 e_2 e_3 \dots$ for $e_i \in E^1$ with $r(e_i) = s(e_{i+1})$ for all $i \in \mathbb{N}$. Since $s(e_1) \in \overline{H(v)} = E^0$, it follows that $s(e_i) \in H(v)_n$ for some $n \in \mathbb{N}$ (recall the notation of Lemma 2.11). Thus $s(e_2) = r(e_1) \in H(v)_{n-1}$, and continuing recursively we have $s(e_n) = r(e_{n-1}) \in H(v)_1$, and $s(e_{n+1}) = r(e_n) \in H(v)_0 = H(v)$. Hence $v \geq s(e_{n+1})$, and $v \geq \alpha^0$. Thus E is cofinal. Conversely, if there exists $v \in E^0$ with $\overline{H(v)} \neq E^0$, then there is a vertex $w \in E^0 \setminus \overline{H(v)}$. Since $\overline{H(v)}$ is saturated, either $w \in E^0_{\text{sing}}$ or there exists an edge $e_1 \in E^1$ with $s(e_1) = w$ and $r(e_1) \notin \overline{H(v)}$. Using $r(e_1)$ and continuing inductively, we either produce a singular vertex $z \in E^0 \setminus \overline{H(v)}$ or an infinite path $\alpha := e_1 e_2 e_3 \dots$ with $s(e_i) \in E^0 \setminus \overline{H(v)}$. Since $H(v) \subseteq \overline{H(v)}$, it follows that either there is a singular vertex z with $v \not\geq z$ or there is an infinite path α with $v \not\geq \alpha^0$. Hence E is not cofinal.

For (2), suppose first that E is downward directed. If $v, w \in E^0$, then the definition of downward directed implies that $H(v) \cap H(w) \neq \varnothing$. Since $H(v) \subseteq \overline{H(v)}$ and $H(w) \subseteq \overline{H(w)}$, it follows that $\overline{H(v)} \cap \overline{H(w)} \neq \varnothing$. Conversely, suppose that for all $v, w \in E^0$ one has $\overline{H(v)} \cap \overline{H(w)} \neq \varnothing$. Then for any $v, w \in E^0$, we may choose $x \in \overline{H(v)} \cap \overline{H(w)}$. Using the notation of Definition 2.10 write $\overline{H(v)} = \bigcup_{n=0}^\infty H(v)_n$ and $\overline{H(w)} = \bigcup_{n=0}^\infty H(w)_n$. Choose the smallest $n_1 \in \mathbb{N} \cup \{0\}$ such that $x \in H(v)_{n_1}$, and choose the smallest $n_2 \in \mathbb{N} \cup \{0\}$ such that $x \in H(w)_{n_2}$. If we let $n := \max\{n_1, n_2\}$, then $x \in H(v)_n \cap H(w)_n$. If $n = 0$, then $x \in H(v) \cap H(w)$ and we have that $v \geq x$ and $w \geq x$. If $n \geq 1$, then there exists an edge $e_1 \in E^1$ such that $s(e_1) = x$ and $r(e_1) \in H(v)_{n-1} \cap H(w)_{n-1}$. Using $r(e_1)$ next, and continuing recursively, we produce a finite path $\alpha := e_1 \dots e_n$ with $r(e_n) \in H(v) \cap H(w)$. Hence $v \geq r(e_n)$ and $w \geq r(e_n)$. Thus E is downward directed.

For (3), suppose first that E satisfies the Countable Separation Property. Then there is a countable nonempty set $X \subseteq E^0$ such that $E^0 = \bigcup_{x\in X} U(x)$. In addition, $x \in \bigcap_{v\in U(x)} \overline{H(v)}$ for all $x \in X$, so $\bigcap_{v\in U(x)} \overline{H(v)} \neq \varnothing$ for all $x \in X$. Thus the condition in (3) holds with $I := X$ and $S_i := U(i)$ for all $i \in I$. Conversely, suppose that there exists a countable collection of subsets of vertices $\{S_i \mid i \in I\}$ with $E^0 = \bigcup_{i\in I} S_i$ and with $\bigcap_{v\in S_i} \overline{H(v)} \neq \varnothing$ for all $i \in I$. For each $i \in I$, choose a vertex $v_i \in \bigcap_{v\in S_i} \overline{H(v)}$ and define

$$C_{v_i} := \{r(\alpha) \mid \alpha \in \text{Path}(E),\ s(\alpha) = v, \text{ and } s(\alpha_i) \in E^0_{\text{reg}} \text{ for all } 1 \leq i \leq |\alpha|\}.$$

(Note that if $v_i \in E^0_{\text{sing}}$, then $C_{v_i} := \{v_i\}$.) Since there are only a finite number of edges emitted from any regular vertex, we see that C_{v_i} is a countable set for all $i \in I$. We define $X := \bigcup_{i\in I} C_{v_i}$, and observe that since X is a countable union of countable sets, X is countable. If $w \in E^0$ is any vertex, then by the hypothesis that $E^0 = \bigcup_{i\in I} S_i$ there exists $i \in I$ such that $w \in S_i$. By the definition of v_i we then have that $v_i \in \overline{H(w)}$. Hence there exists a path $\alpha \in \text{Path}(E)$ such that $s(\alpha) = v_i$, $r(\alpha) \in H(w)$, and $s(\alpha_i) \in E^0_{\text{reg}}$ for all $1 \leq i \leq |\alpha|$. Thus $r(\alpha) \in C_{v_i}$, and $w \geq r(\alpha)$, so that $w \in \bigcup_{x\in X} U(x)$. We have therefore shown that $E^0 = \bigcup_{x\in X} U(x)$, and hence E satisfies the Countable Separation Property. □

It is clear that Property (1) of Proposition 5.2 implies both Property (2) and Property (3) of that Proposition. (Note that we may use the singleton set $S = E^0$ to establish Property (3) from Property (1).) Thus, as promised, using Proposition 5.1, we have established a natural connection between simplicity, primitivity, and primeness for graph C^*-algebras from a graph-theoretic point of view. We summarize this observation as the following result.

Corollary 5.3. *Let E be a graph.*
*The graph C^*-algebra $C^*(E)$ is **simple** if and only if the following two conditions are satisfied*

(1) *E satisfies Condition (L)*
(2) *If $v \in E^0$, then $\overline{H(v)} = E^0$.*

*The graph C^*-algebra $C^*(E)$ is **primitive** if and only if the following three conditions are satisfied*

(1) *E satisfies Condition (L)*
(2) *If $v, w \in E^0$, then $\overline{H(v)} \cap \overline{H(w)} \neq \varnothing$.*
(3) *There exists a countable collection of subsets of vertices $\{S_i \mid i \in I\}$ (so, I is countable and $S_i \subseteq E^0$ for all $i \in I$) such that $E^0 = \bigcup_{i\in I} S_i$ and $\bigcap_{v\in S_i} \overline{H(v)} \neq \varnothing$ for all $i \in I$.*

*The graph C^*-algebra $C^*(E)$ is **prime** if and only if the following two conditions are satisfied*

(1) *E satisfies Condition (L)*
(2) *If $v, w \in E^0$, then $\overline{H(v)} \cap \overline{H(w)} \neq \varnothing$.*

We conclude this graph-theoretic section with the following observation. Since the simplicity of $C^*(E)$ clearly implies its primeness, it is perhaps not surprising that there should be a direct graph-theoretic connection between the germane properties appearing in Proposition 5.1. Indeed, one can easily see that cofinal implies downward directed, as follows. If E is cofinal, and $v, w \in E^0$, then one may inductively create a sequence of edges $\alpha := e_1e_2e_3\dots$ with $s(e_1) = w$ and $s(e_i) = r(e_{i-1})$ for all $i \geq 2$, and such that this sequence either ends at a sink or goes on forever to produce an infinite path. Hence either $v \geq r(\alpha)$ (if α ends at a sink) or $v \geq \alpha^0$ (if α is an infinite path), and E is downward directed.

From this point of view, the difference between the notion of cofinal and the notion of downward directed can be viewed as follows: E is cofinal if and only if "for all $v, w \in E^0$ and for *all* $\alpha \in \text{Path}(E)$ with $s(\alpha) = w$ there exists $x \in E^0$ such that $v \geq x$ and $r(\alpha) \geq x$", while E is downward directed if and only if "for all $v, w \in E^0$ and for *some* $\alpha \in \text{Path}(E)$ with $s(\alpha) = w$ there exists $x \in E^0$ such that $v \geq x$ and $r(\alpha) \geq x$." Specifically, the cofinality property allows for the path from one of the vertices to start along any specified initial segment α, while the downward directedness property contains no such requirement.

6. Primality and primitivity of graph C^*-algebras compared with primality and primitivity of Leavitt path algebras

In this final section we compare the notions of primeness and primitivity for graph C^*-algebras $C^*(E)$ with primeness and primitivity for Leavitt path algebras. Briefly, for any graph E and any field K, one may define the K-algebra $L_K(E)$, the *Leavitt path algebra of E with coefficients in K*. When $K = \mathbb{C}$, then $L_{\mathbb{C}}(E)$ may be viewed as a dense $*$-subalgebra of $C^*(E)$. For reasons which remain not well understood, many structural properties are simultaneously shared by both $L_{\mathbb{C}}(E)$ and $C^*(E)$. We show in this section that the primitivity property may be added to this list. Additional information about Leavitt path algebras may be found in [1] or [2].

The map $\sum_{i=1}^n \lambda_i \alpha_i \beta_i^* \mapsto \sum_{i=1}^n \lambda_i \beta_i \alpha_i^*$ is a K-algebra isomorphism from $L_K(E)$ onto its opposite algebra $L_K(E)^{\text{op}}$. Hence there is a natural correspondence between left $L_K(E)$-modules and right $L_K(E)$-modules, which yields

Proposition 6.1 ([2, Prop. 2.2])**.** *If E is a graph and K is a field, then the algebra $L_K(E)$ is left primitive if and only if it is right primitive.*

Definition 6.2. In light of Proposition 6.1, we shall say a Leavitt path algebra $L_K(E)$ is *primitive* if it is left primitive (which occurs if and only if $L_K(E)$ is also right primitive).

Remark 6.3. When we say $L_K(E)$ is primitive, we mean that $L_K(E)$ is primitive as a ring. The astute reader may notice that it seems more natural to consider primitivity of $L_K(E)$ as an algebra; that is, to reformulate the definition of primitive as having a simple faithful left *K-algebra* module (not merely a simple faithful left *ring* module). However, since $L_K(E)$ has local units, any ring module also carries a natural structure as a K-algebra module. Hence any Leavitt path algebra is primitive as a ring if and only if it is primitive as a K-algebra. Likewise, again using that Leavitt path algebras have local units, any ring ideal of $L_K(E)$ is closed under scalar multiplication by K, and hence the ring ideals of $L_K(E)$ are precisely the K-algebra ideals of $L_K(E)$. Consequently, a Leavitt path algebra is prime as a ring if and only if it is prime as a K-algebra, and a Leavitt path algebra is simple as a ring if and only if it is simple as a K-algebra. Thus for Leavitt path algebras the ring-theoretic notions of primitive, prime, and simple coincide with the corresponding K-algebra-theoretic notions.

Theorem 6.4. *Let E be a graph. Then the following are equivalent.*

(i) *$C^*(E)$ is primitive.*
(ii) *$L_K(E)$ is primitive for some field K.*
(iii) *$L_K(E)$ is primitive for every field K.*
(iv) *E satisfies Condition (L), is downward directed, and satisfies the Countable Separation Property.*

Proof. The equivalence of (i) and (iv) is precisely Theorem 3.8. The equivalence of (ii), (iii), and (iv), is shown in [2, Thm. 5.7]. □

Theorem 6.4 provides yet another example of a situation in which the same ring-theoretic property holds for both of the algebras $C^*(E)$ and $L_{\mathbb{C}}(E)$ (indeed, $L_K(E)$ for any field K), but for which the proof that the pertinent property holds in each case is wildly different. In particular, no "direct" connection between $C^*(E)$ and $L_{\mathbb{C}}(E)$ is established. We note that the proof of the sufficiency direction of Theorem 3.8 looks, on the surface, nearly identical to the proof that $L_{\mathbb{C}}(E)$ is primitive whenever E satisfies Condition (L), is downward directed, and satisfies the Countable Separation Property [2, Thm. 3.5 with Prop. 4.8]. However, in the proof of the result herein we invoke the Cuntz–Krieger Uniqueness Theorem, whose justification is significantly different than that of the correspondingly invoked algebraic result [4, Cor. 3.3]. Furthermore, the proof of the necessity direction of Theorem 3.8 is significantly different than the proof of the analogous result for Leavitt path algebras [2, Prop. 5.6]. In this regard, it is worth noting that for Leavitt path algebras, in contrast to Lemma 3.4 for C^*-algebras, it is perfectly possible to have a graph E and left $L_K(E)$-module M containing an element m for which there exists an uncountable set of nonzero orthogonal projections in $L_K(E)$ which do not annihilate m. For example, let U be an uncountable set, and let E_U denote the graph having one vertex v, and uncountably many loops $\{e_i \mid i \in U\}$ at v. Let $R = L_K(E_U)$, and let $M = {}_RR$. Then for $m = 1_R \in M$, $\{e_ie_i^* \mid i \in U\}$ is such a set.

In contrast to the result presented in Theorem 6.4, the class of graphs which produce prime Leavitt path algebras is not the same as the class of graphs which produce prime graph C^*-algebras. For example, if we let E be the graph with one vertex and one edge

then for any field K, the Leavitt path algebra $L_K(E)$ is isomorphic to $K[x, x^{-1}]$, the algebra of Laurent polynomials with coefficients in K, which is prime. (Indeed, $K[x, x^{-1}]$ is a commutative integral domain.) However, the graph C^*-algebra $C^*(E)$ is isomorphic to $C(\mathbb{T})$, the C^*-algebra of continuous functions on the circle, which is not prime.

Thus "primeness" yields one of the relatively uncommon contexts in which an algebraic property of $L_K(E)$ does not coincide with the corresponding C^*-algebraic property of $C^*(E)$. Hence the conditions on E for $L_K(E)$ to be prime are different than the conditions on E for $C^*(E)$ to be prime.

Necessary and sufficient conditions for a Leavitt path algebra to be prime are given in [3, Cor. 3.10] (see also [2, Thm. 2.4]), which we state here.

Proposition 6.5. *Let E be a graph. Then the following are equivalent*

(i) *$L_K(E)$ is prime for some field K.*
(ii) *$L_K(E)$ is prime for every field K.*
(iii) *E is downward directed.*

We conclude this article with the following summary of comparisons of germane properties between Leavitt path algebras and graph C^*-algebras. A proof that the indicated conditions on E which yield the simplicity of $L_K(E)$ for any field K is given in [1]. The remaining comparisons follow from Proposition 6.5 with Proposition 3.1 and Theorem 6.4.

$L_K(E)$ is simple $\iff$ $C^*(E)$ is simple $\iff$ E is cofinal, and E satisfies Condition (L)

$L_K(E)$ prime $\iff$ E is downward directed

$C^*(E)$ prime $\iff$ E is downward directed, and E satisfies Condition (L)

$L_K(E)$ is primitive $\iff$ $C^*(E)$ is primitive $\iff$ E is downward directed, E satisfies Condition (L), and E has the Countable Separation Property

In particular, we note that $C^*(E)$ is prime if and only if $L_K(E)$ is prime and E satisfies Condition (L). Specifically, $C^*(E)$ prime implies $L_K(E)$ prime for every field K, but not conversely.

Acknowledgments. We thank Takeshi Katsura for providing us with useful comments and some helpful observations.

References

[1] G. Abrams, P. Ara, and M. Siles Molina, Leavitt path algebras. Lecture Notes in Mathematics, Springer Verlag, to appear.

[2] G. Abrams, J. P. Bell, and K. M. Rangaswamy, On prime nonprimitive von Neumann regular algebras, Trans. Amer. Math. Soc. **366** (2014), no. 5, 2375–2392. MR3165642

[3] G. Aranda Pino, E. Pardo and M. Siles Molina, Exchange Leavitt path algebras and stable rank, J. Algebra **305** (2006), no. 2, 912–936. MR2266860 (2007i:16009)

[4] G. Aranda Pino, D. Martín Barquero, C. Martín González, and M. Siles Molina, The socle of a Leavitt path algebra, J. Pure Appl. Algebra **212** (2008), no. 3, 500–509. MR2365329 (2008k:16022)

[5] T. Bates, J. Hong, I. Raeburn, and W. Szymański, The ideal structure of the C*-algebras of infinite graphs, Illinois J. Math. **46** (2002), no. 4, 1159–1176. MR1988256 (2004i:46105)

[6] T. Bates, D. Pask, I. Raeburn, and W. Szymański, The C*-algebras of row-finite graphs, New York J. Math. **6** (2000), 307–324 (electronic). MR1777234 (2001k:46084)

[7] G. M. Bergman, A ring primitive on the right but not on the left, Proc. Amer. Math. Soc. **15** (1964), 473–475. MR0167497 (29 #4770)

[8] G. M. Bergman, Errata: A ring primitive on the right but not on the left, Proc. Amer. Math. Soc. **15** (1964), 1000. MR0167497 (29 #4770)

[9] J. B. Conway, *A course in functional analysis*, Graduate Texts in Mathematics, 96, Springer, New York, 1985. MR0768926 (86h:46001)

[10] M. J. Crabb, A new prime C*-algebra that is not primitive, J. Funct. Anal. **236** (2006), no. 2, 630–633. MR2240178 (2007h:46068)

[11] J. Dixmier, Sur les C*-algèbres, Bull. Soc. Math. France **88** (1960), 95–112. MR0121674 (22 #12408)

[12] J. Dixmier, *Les C^*-algèbres et leurs représentations*, Deuxième édition. Cahiers Scientifiques, Fasc. XXIX, Gauthier-Villars Éditeur, Paris, 1969. MR0246136 (39 #7442)

[13] D. Drinen and M. Tomforde, The C*-algebras of arbitrary graphs, Rocky Mountain J. Math. **35** (2005), no. 1, 105–135. MR2117597 (2006h:46051)

[14] K. R. Goodearl, Leavitt path algebras and direct limits, in *Rings, modules and representations*, 165–187, Contemp. Math., 480, Amer. Math. Soc., Providence, RI. MR2508151 (2010h:16073)

[15] J. H. Hong and W. Szymański, Purely infinite Cuntz–Krieger algebras of directed graphs, Bull. London Math. Soc. **35** (2003), no. 5, 689–696. MR1989499 (2005c:46097)

[16] A. an Huef and I. Raeburn, The ideal structure of Cuntz–Krieger algebras, Ergodic Theory Dynam. Systems **17** (1997), no. 3, 611–624. MR1452183 (98k:46098)

[17] T. Jech, *Set theory*, The third millennium edition, revised and expanded. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. MR1940513 (2004g:03071)

[18] J. A. Jeong and G. H. Park, Graph C*-algebras with real rank zero, J. Funct. Anal. **188** (2002), no. 1, 216–226. MR1878636 (2002m:46085)

[19] T. Katsura, Non-separable AF-algebras, in *Operator Algebras: The Abel Symposium 2004*, 165–173, Abel Symp., 1, Springer, Berlin. MR2265049 (2007g:47117)

[20] T. Katsura, A class of C*-algebras generalizing both graph algebras and homeomorphism C*-algebras. I. Fundamental results, Trans. Amer. Math. Soc. **356** (2004), no. 11, 4287–4322 (electronic). MR2067120 (2005b:46119)

[21] T. Katsura, A class of C*-algebras generalizing both graph algebras and homeomorphism C*-algebras. III. Ideal structures, Ergodic Theory Dynam. Systems **26** (2006), no. 6, 1805–1854. MR2279267 (2007j:46090)

[22] N. C. Phillips and M. G. Viola, A simple separable exact C*-algebra not anti-isomorphic to itself, Math. Ann. **355** (2013), no. 2, 783–799. MR3010147

[23] I. Raeburn and W. Szymański, Cuntz–Krieger algebras of infinite graphs and matrices, Trans. Amer. Math. Soc. **356** (2004), no. 1, 39–59 (electronic). MR2020023 (2004i:46087)

[24] I. Raeburn and D. P. Williams, *Morita equivalence and continuous-trace C^*-algebras*, Mathematical Surveys and Monographs, 60, Amer. Math. Soc., Providence, RI, 1998. MR1634408 (2000c:46108)

[25] C. E. Rickart, *General theory of Banach algebras*, The University Series in Higher Mathematics, D. van Nostrand Co., Inc., Princeton, NJ, 1960. MR0115101 (22 #5903)

[26] L. H. Rowen, *Ring theory. Vol. I*, Pure and Applied Mathematics, 127, Academic Press, Boston, MA, 1988. MR0940245 (89h:16001)

[27] M. Tomforde, Structure of graph C*-algebras and their generalizations, Chapter in the book *Graph Algebras: Bridging the gap between analysis and algebra*, Eds. Gonzalo Aranda Pino, Francesc Perera Domènech, and Mercedes Siles Molina, Servicio de Publicaciones de la Universidad de Málaga, Málaga, Spain, 2006.

[28] N. Weaver, A prime C*-algebra that is not primitive, J. Funct. Anal. **203** (2003), no. 2, 356–361. MR2003352 (2004g:46075)

Received June 27, 2013; accepted August 3, 2013

Gene Abrams
Department of Mathematics, University of Colorado
Colorado Springs, CO 80918, USA
E-mail: `abrams@math.uccs.edu`
URL: `http://www.uccs.edu/~gabrams`

Mark Tomforde
Department of Mathematics, University of Houston
Houston, TX 77204-3008, USA
E-mail: `tomforde@math.uh.edu`
URL: `http://www.math.uh.edu/~tomforde`