Rigid-analytic L-Transforms

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RIGID-ANALYTIC L - TRANSFORMS

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In this talk I want to present a new method of defining p-adic L-functions for a certain class of elliptic curves. In the first section we shortly review the general philosophy of complex and p-adic L-functions and then explain the idea of the method which is based on the notion of a rigid-analytic automorphic form. The construction of a p-adic L-function associated with such an automorphic form is carried out in the second section.

I. THE STARTING POINT

Let $E_{/\mathbb{Q}}$ be an elliptic curve over the rationals. One of the most interesting invariants of E is its Hasse-Weil L-function

$$L(E,s) = \prod_{p \text{ good}} (1-t_p p^{-s}+p^{1-2s})^{-1} \cdot \prod_{p \text{ bad}} (1-t_p p^{-s})^{-1}$$

with $t_p := \begin{cases} p+1 - \#E(\mathbb{F}_p) & \text{if } E \text{ has good reduction at } p, \\ \pm 1 \text{ or } 0 & \text{otherwise,} \end{cases}$

It converges for $\mathrm{Re}(s) > 3/2$ and apparently collects arithmetic information about E. But in order to study its properties one needs analytic methods. We therefore now assume that E is a Weil curve, i.e., there exists a nonconstant $\mathbb{Q}$-morphism

$$X_0(N) \xrightarrow{\pi} E$$

such that $\pi(i\infty) = 0$ and $\pi^*\omega = c_\omega \cdot f$ for any holomorphic differential form ω on E, where c_ω is a constant and f is a normalized newform of weight 2 for $\Gamma_0(N)$.

Commentary:

1) $\pi^*\omega$ always is a cuspform of weight 2 for $\Gamma_0(N)$ which is an eigenform for all Hecke operators T_p, $p \nmid N$. The requirement "$\pi^*\omega$ newform" means that N is the minimal possible number for which such a π exists.

2) One of Weil's conjectures says that any $E_{/\mathbb{Q}}$ is a Weil curve.

The analytic properties of the Mellin transform

$$L(f,s) := \frac{(2\pi)^s}{\Gamma(s)} \cdot \int_0^\infty f(iy)y^{s-1}dy$$

of f are easy to obtain. But according to Eichler/Shimura, Igusa, and Deligne/Langlands we have

$$L(E,s) = L(f,s).$$

We therefore get analytic continuation and a functional equation also for $L(E,s)$.

On the other hand, at certain integer points $L(f,s)$ and its twists by Dirichlet characters have strong algebraicity and even integrality properties. Therefore there is a natural way to associate with $L(f,s)$ a p-adic analytic L-function $L_p(f,s)$ (p a prime number and s now a p-adic variable) such that the values of $L(f,s)$ and $L_p(f,s)$ at the "critical" integer points are closely related (Mazur/Swinnerton-Dyer, Manin, Amice-Velu, Visik). We emphasize that with this method $L_p(f,s)$ cannot be defined independently of $L(f,s)$. It also should be mentioned that there is a theory (Iwasawa,Mazur) how to define arithmetically a p-adic L-function $L_p(E,s)$; furthermore there is the "main conjecture" which relates $L_p(E,s)$ to $L_p(f,s)$.

Our idea to construct a p-adic L-function for E is to use directly Mumford's theory of p-adic uniformization. Let $\mathbb{C}_p$ denote the completion of an algebraic closure of $\mathbb{Q}_p$. The modular curve $X_0(N)_{/\mathbb{C}_p}$ itself is a Mumford curve if and only if $N = p$ (see [2]). Unfortunately, at present, no corresponding discrete group is known explicitly. But let us assume that N is square-free with an even number of prime divisors. Denote by D_N the quaternion algebra over $\mathbb{Q}$ which is ramified precisely at the prime divisors of N, and let Γ_N be the group of units of reduced norm 1 in a maximal order of D_N. If $S_{N/\mathbb{Q}}$ is the Shimura curve with $S_N(\mathbb{C}) = \Gamma_N \backslash \mathbb{H}$ then a result of Ribet ([8]) says that the Jacobian of S_N is $\mathbb{Q}$-isogenous to the new part of the Jacobian of $X_0(N)$:

$$J_0(N)^{new} \sim \operatorname{Jac} S_N.$$

We now fix a prime divisor p of N and denote by D_N' the quaternion algebra over $\mathbb{Q}$ which is ramified precisely at ∞ and at the prime divisors of N different from p. The image Γ_N' in $PGL_2(\mathbb{Q}_p)$ of the group of p-units (with respect to a maximal order) in D_N' is a discrete and finitely generated subgroup of $PGL_2(\mathbb{Q}_p)$. According to Čerednik ([1]) one has a rigid-analytic isomorphism

$$S_N(\mathbb{C}_p) \cong \Gamma_N' \backslash (\mathbb{C}_p \backslash \mathbb{Q}_p) \ .$$

Thus any Weil curve E with an analytic conductor N which fulfills the above assumptions (and consequently has multiplicative reduction at p) has a p-adic analytic uniformization

$$\Gamma_N' \backslash (\mathbb{C}_p \backslash \mathbb{Q}_p) \xrightarrow{\psi} E(\mathbb{C}_p)$$

which is "defined over $\mathbb{Q}$". Furthermore the rigid-analytic automorphic form $\psi^*\omega$ of weight 2 for Γ_N' up to a constant only depends on E.

In the next section we shall construct a p-adic analogue $L_p(g,s)$ of the classical Mellin transform for any rigid-analytic automorphic form g of arbitrary weight. In particular, we view $L_p(\psi^*\omega,s)$ as the p-adic L-function of E; of course, one first has to normalize the constant correctly (using Hecke operators). But we will not discuss this problem here, neither the question whether $L_p(\psi^*\omega,s)$ and $L_p(f,s)$ agree.

II. THE L-TRANSFORM

Let $K \subseteq \mathbb{C}_p$ be a finite extension field of $\mathbb{Q}_p$, let $\Gamma \subseteq SL_2(K)$ be a finitely generated discrete subgroup, and denote by $\mathcal{L} \subseteq K \cup \{\infty\}$ its set of limit points. Γ then acts discontinuously (via fractional linear transformations) on the analytic set

$$H := \mathbb{C}_p \cup \{\infty\} \backslash \mathcal{L}$$

and according to Mumford ([7] or [6]) $C := \Gamma \backslash H$ has a natural structure of a smooth projective curve over $\mathbb{C}_p$. We always make the following assumptions:

a) $\mathcal{L}$ is infinite (and therefore compact and perfect);

b) $\infty \in \mathcal{L}$.

DEFINITION:

A rigid-analytic function $f: H \to \mathbb{C}_p$ is called an automorphic form of weight $n \in \mathbb{Z}$ for Γ if

$$f(\gamma x) = (cx+d)^n f(x) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \text{ and } x \in H.$$

Furthermore $M_n(\Gamma)$ denotes the $\mathbb{C}_p$-vector space of all automorphic forms of weight n for Γ.

In a completely analogous way as in the classical case of a co-compact Fuchsian group one can compute the dimension of the vector space $M_n(\Gamma)$

for $n \neq 1$. We state the result only for a Schottky group Γ.

PROPOSITION:

Suppose that Γ *is free of rank* $r > 1$. *Then*

$$\dim_{\mathbb{C}_p} M_n(\Gamma) = \begin{cases} 0 & \text{for } n < 0, \\ 1 & \text{for } n = 0, \\ r & \text{for } n = 2, \\ (n-1)(r-1) & \text{for } n \geq 3. \end{cases}$$

Proof: We have $M_0(\Gamma) = \mathbb{C}_p$ since C is projective. On the other hand r is equal to the genus of C. $M_2(\Gamma)$ which is isomorphic to the vector space of holomorphic differentials on C therefore has the dimension r. The considerations in §4 of [5] imply the existence of a nonvanishing meromorphic function f_0 on H such that

$$f_0(\gamma x) = (cx+d) f_0(x) \quad \text{for all} \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \quad \text{and} \quad x \in H$$

and

$$\deg \operatorname{div}(f_0) = r - 1.$$

Consequently the map

$$\begin{aligned} \Gamma(C, \mathcal{O}(n \operatorname{div}(f_0))) &\longrightarrow M_n(\Gamma) \\ f &\longmapsto f \cdot f_0^n \end{aligned}$$

is an isomorphism. But the dimension on the left hand side for $n < 0$ or $n \geq 3$ is the required one by the Riemann-Roch theorem.

Γ not only acts on H but also on a certain tree T_Γ. Namely, let T_K be the Bruhat-Tits tree of $SL_2(K)$. The straight paths of T_K the ends of which correspond to the fixed points of a non-trivial hyperbolic element in Γ (i.e., the axes in T_K of the hyperbolic elements in Γ) form a subtree of T_K. The tree T_Γ is constructed from this subtree by neglecting all vertices P with the following two properties:

i. P has only two adjacent vertices P_1 and P_2;
ii. there is no nontrivial elliptic element in Γ which fixes P but not P_1 and P_2;

it only depends on Γ (not on the field K). The group Γ acts without inversion on T_Γ (use [9] II.1.3), and the quotient graph $S := \Gamma \backslash T_\Gamma$ is finite ([6] I.3.2.2). Furthermore, there is a canonical Γ-equivariant bijection

$$\mathcal{L} \longleftrightarrow \{\text{ends of } T_\Gamma\}$$
$$= \{\text{equivalence classes of halflines in } T_\Gamma\}$$

([6]I.2.5).

Notation: For any tree T we denote by Vert(T), resp. Edge(T), the set of vertices, resp. edges, of T. For any edge y of T, the vertices A(y) and E(y), resp. the edge $\bar{y}$, are defined to be the origin and the terminus, resp. the inverse edge, of y.

DEFINITION:

Let M be an abelian group. A harmonic cocycle on T_Γ with values in M is a map

$$c: \ \mathrm{Edge}(T_\Gamma) \longrightarrow M$$

with the properties

i. $c(\bar{y}) = -c(y)$ for all $y \in \mathrm{Edge}(T_\Gamma)$, and

ii. $\sum_{E(y)=P} c(y) = 0$ for all $P \in \mathrm{Vert}(T_\Gamma)$.

Let $C_{har}(T_\Gamma, M)$ denote the abelian group of all M-valued harmonic cocycles on T_Γ.

Our first basic observation will be that by "integration" one can construct a map from vector-valued holomorphic differential forms on H to vector-valued harmonic cocycles on T_Γ. By "integration" we mean the theory of residues which we shortly recall in the following. (I am grateful to F. Herrlich for some clarifying discussion about this point.) Let

$$F = \mathbb{C}_p \cup \{\infty\} \setminus (D_o \cup \cdots \cup D_m)$$

be a connected affinoid set where the D_i are pairwise disjoint open disks

$$D_o = \{x : |x-a_o|_p > |b_o|_p\} \quad \text{and}$$

$$D_i = \{x : |x-a_i|_p < |b_i|_p\} \quad \text{for } 1 \leq i \leq m;$$

for simplicity we only consider the case that $m \geq 1$ and $\infty \notin F$. Furthermore we can assume that $a_o \notin F$. Put

$$F_i := \mathbb{C}_p \cup \{\infty\} \setminus D_i$$

and

$$w_o(x) := \frac{x-a_0}{b_0}, \text{ resp. } w_i(x) := \frac{b_i}{x-a_i} \quad \text{for} \quad 1 \le i \le m .$$

These $w_i(x)$ obviously are invertible holomorphic functions on F. Any holomorphic differential form $\omega \in \Omega(F)$ on F has representations

$$\omega = f_i \, d \frac{1}{w_i} \qquad \text{with} \qquad f_i \in \mathcal{O}(F).$$

Let now

$$f_i = f_o^{(i)} + \ldots + f_m^{(i)} \qquad \text{with} \qquad f_j^{(i)} \in \mathcal{O}(F_j)$$

$$\text{and} \quad f_j^{(i)}(\infty) = 0 \qquad \text{for} \qquad 1 \le j \le m$$

be the Mittag-Leffler decomposition of f_i ([6] p. 41), which is uniquely determined and fulfills the following condition on the norms

$$(*) \qquad \| f_i \|_F = \max_{0 \le j \le m} \| f_j^{(i)} \|_{F_j} .$$

The differential form

$$\omega_i := f_i^{(i)} \, d \frac{1}{w_i}$$

then is meromorphic on F_i with at most one pole at $x = a_o$ in case $i = 0$, resp. $x = \infty$ in case $1 \le i \le m$. If

$$\omega_i = \sum_{\nu \in \mathbb{Z}} c_\nu^{(i)} w_i^\nu \, dw_i$$

denotes its development into a Laurent series we define

$$\text{res}_{D_i} \omega := c_{-1}^{(i)} .$$

This definition is independent of the particular representation of the disks D_i. Namely, for $1 \le i \le m$ already ω_i and therefore also $\text{res}_{D_i} \omega$ (see [4] p. 21) is independent; the case $i = 0$ then follows from the subsequent theorem of residues. Although this result is well known we will include a proof for the convenience of the reader.

<u>PROPOSITION</u>:

$\sum_{i=0}^{m} \text{res}_{D_i} \omega = 0$ *for any* $\omega \in \Omega(F)$.

<u>Proof</u>: If the assertion holds true for rational holomorphic differential forms on F then also for any holomorphic one by taking limits and using

(*). Let therefore

$$\omega = g(x)dx \in \Omega(F)$$

be a differential form where g(x) is a rational function without poles in F. If we view ω as a meromorphic differential form on $\mathbb{C}_p \cup \{\infty\}$ then we of course have

$$\sum_{a\in\mathbb{C}_p\cup\{\infty\}} \mathrm{res}_a \omega = 0$$

where $\mathrm{res}_a \omega$ is defined in the usual way by

$$\mathrm{res}_a \omega := \alpha_{-1},\ \omega = \begin{cases} \sum_{\nu\in\mathbb{Z}} \alpha_\nu \cdot (x-a)^\nu d(x-a) & \text{for } a \neq \infty, \\ \sum_{\nu\in\mathbb{Z}} \alpha_\nu \cdot (\frac{1}{x})^\nu \, d\,\frac{1}{x} & \text{for } a = \infty. \end{cases}$$

Since ω is holomorphic on F we get

$$\sum_{i=0}^{m} r_i(\omega) = 0 \qquad \text{with} \qquad r_i(\omega) := \sum_{a\in D_i} \mathrm{res}_a \omega .$$

The assertion then is proved if we show that one has

$$\mathrm{res}_{D_i} \omega = -r_i(\omega) .$$

Let us first consider the case $i \geq 1$. If

$$g = g_o + \ldots + g_m \quad \text{with rational} \quad g_j \in \mathcal{O}(F_j)$$
$$\text{and} \quad g_j(\infty) = 0 \text{ for } 1 \leq j \leq m$$

is the Mittag-Leffler decomposition of g, then $\left(\sum_{j\neq i} g_j(x)\right)dx$ is holomorphic on D_i which implies

$$r_i(\omega) = r_i(g_i(x)dx).$$

According to [4] p. 22 we have

$$r_i(\omega) = r_i(g_i(x)dx) = d_{-1}^{(i)}$$

where $g_i(x)dx = \sum_{\nu\in\mathbb{Z}} d_\nu^{(i)} (\frac{1}{w_i})^\nu d\,\frac{1}{w_i}$. On the other hand, from $\omega = g(x)dx = gb_i d\,\frac{1}{w_i}$ we derive $f_i = b_i g$ and therefore $\omega_i = b_i g_i d\,\frac{1}{w_i} = g_i(x)dx$. Together with $d\,\frac{1}{w_i} = -w_i^{-2} dw_i$ this implies

$$c_\nu^{(i)} = -d_{-\nu-2}^{(i)} \ .$$

In the case $i = 0$ the differential form $\left(\sum_{j=1}^{m} f_j^{(o)}\right) d\,\frac{1}{w_o}$ is holomorphic on D_o and we get

$$r_o(\omega) = r_o(f_o^{(o)}\, d\,\frac{1}{w_o}) \ .$$

But $f_o^{(o)}\, d\,\frac{1}{w_o}$ is holomorphic on $\{x: |x-a_o|_p = |b_o|_p\}$. We thus have

$$r_o(\omega) = r_o(f_o^{(o)}\, d\frac{1}{w_o}) = - \sum_{|a-a_o|_p<|b_o|_p} \operatorname{res}_a f_o^{(o)}\, d\frac{1}{w_o}$$

which according to [4] p. 22 is equal to $-c_{-1}^{(o)}$. Q.E.D.

We have to list some further useful properties of the residues the proof of which is an easy exercise.

<u>Remark</u>:

i. $\operatorname{res}_{D_i}(\omega_1+\omega_2) = \operatorname{res}_{D_i}\omega_1 + \operatorname{res}_{D_i}\omega_2$;

ii. let $F' = \mathbb{C}_p \cup \{\infty\} \setminus (D_o \cup \ldots \cup D_n \cup D'_{n+1} \cup \ldots) \supseteq F$ be an affinoid set containing F where the $D_o, \ldots, D_n, D'_{n+1}, \ldots$ $(1 \leq n < m)$ are pairwise disjoint open disks; for $0 \leq i \leq n$ and any $\omega \in \Omega(F')$ we then have

$$\operatorname{res}_{D_i}\omega = \operatorname{res}_{D_i}\omega|F \quad ;$$

iii. for any $\gamma \in PGL_2(\mathbb{C}_p)$ with $\infty \notin \gamma(F)$ we have

$$\operatorname{res}_{\gamma(D_i)}{}^{\gamma}\omega = \operatorname{res}_{D_i}\omega \quad \text{with} \quad {}^{\gamma}\omega := \omega\circ\gamma^{-1}$$

The second ingredient which we need for the construction of a map from the holomorphic differential forms $\Omega(H)$ on H to $C_{har}(T_\Gamma,\mathbb{C}_p)$ is a certain natural family of affinoid subsets of H. Its definition relies on ideas of Drinfeld ([3], see also [4] Chap. V). **We** first put

$U(y) := \{a \in \mathcal{L}$: a halfline in T_Γ corresponding to a passes through $y\}$

for any $y \in Edge(T_\Gamma)$. The $U(y)$ are compact and open in $\mathcal{L}$ and form a basis of the topology of $\mathcal{L}$.

Remark:

i. $\mathcal{L} = U(y) \cup U(\bar{y})$ and $\mathcal{L} = \bigcup_{E(y)=P} U(\bar{y})$ where the union is disjoint in each case;

ii. $U(\gamma(y)) = \gamma(U(y))$ for any $\gamma \in \Gamma$.

Let now

$$R: \mathbb{C}_p \cup \{\infty\} \longrightarrow \overline{\mathbb{F}}_p \cup \{\infty\}$$

$$a \longmapsto \begin{cases} a \bmod \mathfrak{m} & \text{if } |a|_p \leq 1, \\ \infty & \text{otherwise} \end{cases}$$

be the usual reduction map where $\mathfrak{m}$, resp. $\overline{\mathbb{F}}_p$, denotes the maximal ideal, resp. the residue class field, of $\mathbb{C}_p$; we set $R_\sigma := R \circ \sigma^{-1}$ for $\sigma \in PGL_2(K)$. Furthermore, we denote by P_o that vertex of T_K which is defined by the lattice $\mathfrak{o}_K \oplus \mathfrak{o}_K$ where $\mathfrak{o}_K$ is the ring of integers in K.

LEMMA:

For any $y \in Edge(T_\Gamma)$, *the set*

$$D_y := R_\sigma^{-1}(R_\sigma(U(\bar{y}))) \subseteq \mathbb{C}_p \cup \{\infty\}$$

where $\sigma \in PGL_2(K)$ *is such that* $E(y) = \sigma(P_o)$ *is an open disk and does not depend on the special choice of* σ.

Proof: The fibres of R_σ are open disks. So, it remains to show that $R_\sigma(U(\bar{y}))$ is a one-point set. We obviously can assume that $T_\Gamma = T_K$ and $\sigma = 1$ in which case that property is easily checked by explicit computation.

Thus, for any $P \in Vert(T_\Gamma)$,

$$F(P) := \mathbb{C}_p \cup \{\infty\} \setminus \bigcup_{E(y)=P} D_y$$

is a connected affinoid subset of H, and we have

$$F(\gamma(P)) = \gamma(F(P)) \qquad \text{for } \gamma \in \Gamma.$$

We now associate with a holomorphic differential form $\omega \in \Omega(H)$ the map

$$c_\omega: Edge(T_\Gamma) \longrightarrow \mathbb{C}_p$$

$$y \longmapsto res_{D_y}(\omega | F(E(y))) .$$

LEMMA:

c_ω *is a harmonic cocycle on* T_Γ.

Proof: The above proposition immediately implies $\sum_{E(y)=P} c_\omega(y) = 0$. Fix now an edge y of T_Γ and put $Q := A(y)$ and $P := E(y)$. The open disks D_z with $E(z) = P$, $z \neq y$ or $E(z) = Q$, $z \neq \bar{y}$ then are pairwise disjoint such that

$$F(y) := \mathbb{C}_p \cup \{\infty\} \setminus \bigcup_z D_z \supseteq F(Q) \cup F(P) \ ;$$

this follows from the general fact that, for any two edges y_1, y_2 of T_Γ with $E(y_2) = A(y_1)$ and $y_2 \neq \bar{y}_1$, we have

$$D_{y_2} \subseteq D_{y_1}$$

(reduce to the case $T_\Gamma = T_K$ and apply [6] I§2). Using again the above proposition we compute

$$c_\omega(y) = \operatorname{res}_{D_y}(\omega|F(P)) = -\sum_{\substack{E(z)=P\\ z\neq y}} \operatorname{res}_{D_z}(\omega|F(P))$$

$$= -\sum_{\substack{E(z)=P\\ z\neq y}} \operatorname{res}_{D_z}(\omega|F(y)) = \sum_{\substack{E(z)=Q\\ z\neq \bar{y}}} \operatorname{res}_{D_z}(\omega|F(y))$$

$$= \sum_{\substack{E(z)=Q\\ z\neq \bar{y}}} \operatorname{res}_{D_z}(\omega|F(Q)) = -\operatorname{res}_{D_{\bar{y}}}(\omega|F(Q))$$

$$= -c_\omega(\bar{y}) .$$

Q.E.D.

We therefore get the Γ-equivariant homomorphism

$$I: \ \Omega(H) \longrightarrow C_{har}(T_\Gamma, \mathbb{C}_p)$$
$$\omega \longmapsto I(\omega) := c_\omega \quad .$$

In order to derive from it maps from the automorphic forms to the harmonic cocycles we introduce the symmetric powers

$$W^n := \operatorname{Sym}^n W \qquad (n \geq 0)$$

of the natural representation of $\Gamma \subseteq SL_2(K)$ on the $\mathbb{C}_p$-vector space

$W = \mathbb{C}_p \oplus \mathbb{C}_p$. We then have the homomorphisms

$$I_n: \quad M_{n+2}(\Gamma) \longrightarrow H^o(\Gamma, \Omega(H) \otimes W^n) \longrightarrow H^o(\Gamma, C_{har}(T_\Gamma, W^n))$$

$$f \longmapsto \omega_f \longmapsto c_f := (I \otimes id_{W^n})(\omega_f)$$

where $\omega_f := \sum_{i=0}^{n} x^i f(x)dx \otimes (1,0)^i \cdot (0,1)^{n-i}$.

Remark:

There is a canonical map $\varepsilon_n: H^o(\Gamma, C_{har}(T_\Gamma, W^n)) \longrightarrow H^1(\Gamma, W^n)$ (see [9] II.2.8). We will show in another paper that

$$\varepsilon_n \circ I_n : M_{n+2}(\Gamma) \xrightarrow{\cong} H^1(\Gamma, W^n)$$

is an isomorphism (which can be viewed as an analogue of the Shimura isomorphism in the classical theory of automorphic forms).

The next basic observation is that harmonic cocycles on T_Γ are nothing else than certain distributions on the set of limit points $\mathcal{L}$.

DEFINITION:

For any abelian group M and any locally compact and totally disconnected space X let D(X,M) denote the abelian group of all M-valued finitely additive functions on the family of compact open subsets of X ("distributions on X"). In case X is compact put $D_o(X,M) := \{\mu \in D(X,M) : \mu(X) = 0\}$.

The following result due to Drinfeld ([3]) now is easy to prove.

LEMMA:

The map $D_o(\mathcal{L},M) \longrightarrow C_{har}(T_\Gamma,M)$ *is an isomorphism.*

$$\mu \longmapsto c_\mu(y) := \mu(U(y)).$$

Furthermore, if we set $\mathcal{L}_o := \mathcal{L} \setminus \{\infty\}$ then restriction of distributions induces an isomorphism $D_o(\mathcal{L},M) \xrightarrow{\cong} D(\mathcal{L}_o,M)$. Altogether we thus have constructed homomorphisms

$$M_{n+2}(\Gamma) \longrightarrow C_{har}(T_\Gamma, W^n) \cong D_o(\mathcal{L}, W^n) \cong D(\mathcal{L}_o, W^n)$$

$$f \longmapsto c_f \longmapsto \mu_f .$$

We consider μ_f as the p-adic L-transform of the automorphic form f. If f has weight 2 then μ_f even is a $\mathbb{C}_p$-valued measure (i.e., a bounded distribution) on $\mathcal{L}_o$. Namely, because of its Γ-invariance and the finiteness of the quotient graph S the cocycle c_f takes on only a finite number of different values. In general μ_f will not be a measure but we can describe its growth rather precisely. Let f always be an automorphic form of weight n+2 for Γ.

Notation: For any $\omega \in \Omega(H)$ and any $y \in \mathrm{Edge}(T_\Gamma)$ we put

$$\mathrm{res}_y\omega := \mathrm{res}_{D_y}(\omega|F(E(y))).$$

LEMMA:

For $0 \leq i \leq n$, $y \in \mathrm{Edge}(T_\Gamma)$, $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, *and* $e \in \mathbb{C}_p$ *such that* $\gamma(e) \neq \infty$ *we have*

$$\mathrm{res}_{\gamma(y)}(x-\gamma e)^i f(x)dx = (ce+d)^{n-2i} \cdot \sum_{j=0}^{n-i} \binom{n-i}{j} (e+\tfrac{d}{c})^{-j} \cdot \mathrm{res}_y (x-e)^{i+j} f(x)dx .$$

Proof: Using $(-cx+a) = (a-c\gamma(e)) - c(x-\gamma(e))$ and $(ce+d)(a-c\gamma(e)) = 1$ we compute

$$\mathrm{res}_y(x-e)^i f(x)dx = \mathrm{res}_{\gamma(y)}{}^{\gamma}((x-e)^i f(x)dx)$$

$$= \mathrm{res}_{\gamma(y)}\left(\frac{dx-b}{-cx+a} - e\right)^i (-cx+a)^{n+2} f(x)(-cx+a)^{-2}dx$$

$$= \mathrm{res}_{\gamma(y)}(ce+d)^i (x-\gamma e)^i (-cx+a)^{n-i} f(x)dx$$

$$= \sum_{j=0}^{n-i} \binom{n-i}{j} \mathrm{res}_{\gamma(y)}(ce+d)^i (x-\gamma e)^i (a-c\gamma(e))^j (-c)^{n-i-j} (x-\gamma e)^{n-i-j} f(x)dx$$

$$= \sum_{j=0}^{n-i} \binom{n-i}{j} (ce+d)^{i-j} (-c)^{n-i-j} \mathrm{res}_{\gamma(y)}(x-\gamma e)^{n-j} f(x)dx.$$

In particular, our assertion holds true if i = n. The general case then follows by an inductive argument using identities like

$$\sum_{j=i}^{m-1} (-1)^j \binom{m}{j}\binom{j}{i} = (-1)^{m+1}\binom{m}{i} \qquad \text{for } i < m. \qquad \text{Q.E.D.}$$

PROPOSITION:

There exists a constant $C > 0$ *such that we have*

$$\rho_y^{n/2-i} \cdot |\mathrm{res}_y (x-e)^i f(x)\,dx|_p < C$$

for all $0 \leq i \leq n$, $y \in \mathrm{Edge}(T_\Gamma)$ *with* $U(\bar{y}) \subseteq \mathcal{L}_o$, *and* $e \in U(\bar{y})$ *where*

$$\rho_y := \sup\{|u-v|_p : u,v \in D_y\} .$$

Proof: Since the quotient graph S is finite we can choose finitely many edges $y_1,\ldots,y_m$ of T_Γ such that $\infty \notin U(\bar{y}_1) \cup \ldots \cup U(\bar{y}_m)$ and such that any $y \in \mathrm{Edge}(T_\Gamma)$ with $\infty \notin U(\bar{y})$ is Γ-equivalent to one of the y_ν, say $y = \gamma(y_\nu)$ with $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. Using

$$|\gamma^{-1}(e) + \tfrac{d}{c}|_p = |\gamma^{-1}(e) - \gamma^{-1}(\infty)|_p \geq \rho_{y_\nu}$$

and

$$\rho_y = |c\gamma^{-1}(e)+d|_p^{-2} \cdot \rho_{y_\nu}$$

we derive from the above lemma

$$\rho_y^{n/2-i} \cdot |\mathrm{res}_y (x-e)^i f(x)\,dx|_p$$

$$\leq \rho_{y_\nu}^{n/2-i} \cdot \max_{0\leq j\leq n-i} |\gamma^{-1}(e)+\tfrac{d}{c}|_p^{-j} \cdot |\mathrm{res}_{y_\nu} (x-\gamma^{-1}(e))^{i+j} f(x)\,dx|_p$$

$$\leq \max_{0\leq j\leq n-i} \rho_{y_\nu}^{n/2-i-j} \cdot |\mathrm{res}_{y_\nu} (x-\gamma^{-1}(e))^{i+j} f(x)\,dx|_p .$$

But the last term obviously is bounded independently of $\gamma^{-1}(e) \in U(\bar{y}_\nu)$.

Q.E.D.

Let us define the $\mathbb{C}_p$-valued distributions $\mu_f^{(o)},\ldots,\mu_f^{(n)}$ on $\mathcal{L}_0$ by

$$\mu_f = \sum_{i=0}^{n} \mu_f^{(i)} \cdot (1,0)^i \cdot (0,1)^{n-i} .$$

Putting

$$\int_U x^i d\mu_f := \mu_f^{(i)}(U)$$

for $0 \leq i \leq n$ and any compact open subset $U \subseteq \mathcal{L}_o$ then induces a $\mathbb{C}_p$-linear map

$$\int \cdot\, d\mu_f : \mathcal{C}^n(\mathcal{L}_o) \longrightarrow \mathbb{C}_p$$

on the space $\mathcal{C}^n(\mathcal{L}_o)$ of all functions with compact support on $\mathcal{L}_o$ which are locally a polynomial in x of degree $\leq n$. The above proposition shows that this map satisfies a certain growth condition; we namely have

$$\int_{U(\bar{y})} (x-e)^i d\mu_f = - \operatorname{res}_y (x-e)^i f(x)dx$$

(under the appropriate assumptions). That property allows us to extend $\int . d\mu_f$ to a map on all functions with compact support on $\mathcal{L}_o$ which satisfy a certain condition of Lipschitz type. In order to be more specific let us make the following assumption which from an arithmetic point of view seems to be a natural one:

$$\Gamma \text{ is cocompact in } SL_2(\mathbb{Q}_p).$$

Then $T_\Gamma = T_{\mathbb{Q}_p}$ (use [9] II.1.5.5) and μ_f is a distribution on $\mathcal{L}_o = \mathbb{Q}_p$. In fact, the above proposition shows that $\int . d\mu_f$ induces an "admissible measure" on $\mathbb{Z}_p^x$ in the sense of Visik ([10]). The function

$$L_p(f,\chi) := \int_{\mathbb{Z}_p^x} \chi \, d\mu_f$$

therefore is well-defined and analytic in $\chi \in \operatorname{Hom}_{cont}(\mathbb{Z}_p^x, \mathbb{C}_p^x)$ (see [10]). In particular, if $\kappa\colon \mathbb{Z}_p^x \to 1+p\mathbb{Z}_p \subseteq \mathbb{C}_p^x$ denotes the canonical projection map then

$$L_p(f,s) := L_p(f,\kappa^{1-s})$$

is an analytic function on the open disk $\{s \in \mathbb{C}_p : |s|_p < qp^{-1/(p-1)}\}$ where $q = 4$ for $p = 2$ and $q = p$ otherwise.

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