Münster J. of Math. **8** (2015), 229–239
DOI 10.17879/65219672957
urn:nbn:de:hbz:6-65219673202

Münster Journal of Mathematics

On flow-equivalence of $\mathcal{R}$-graph shifts

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(Communicated by Joachim Cuntz)

Abstract. We show that Property (A) of subshifts and the semigroup that is associated to subshifts with Property (A) are invariants of flow equivalence. We show for certain $\mathcal{R}$-graphs that their isomorphism is implied by the flow equivalence of their $\mathcal{R}$-graph shifts.

1. Introduction

Let Σ be a finite alphabet, and let S_Σ be the shift on the shift space $\Sigma^{\mathbb{Z}}$,

$$S_\Sigma((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}, \quad (x_i)_{i\in\mathbb{Z}} \in \Sigma^{\mathbb{Z}}.$$

The S_Σ-invariant closed subsets X of $\Sigma^{\mathbb{Z}}$ (more precisely, with S_X denoting the restriction of S_Σ to X, the dynamical systems (X, S_X)) are called subshifts. These are the subject of symbolic dynamics. For an introduction to symbolic dynamics see [3] or [6].

A word is called admissible for a subshift $X \subset \Sigma^{\mathbb{Z}}$ if it appears in a point of X. We denote the set of admissible words of a subshift $X \subset \Sigma^{\mathbb{Z}}$ by $\mathcal{L}(X)$. The language $\mathcal{L}(X)$ is factorial and bi-extensible, and every factorial and bi-extensible language is the set of admissible words as a unique subshift.

Let $\bullet$ be a symbol that is not in Σ, and consider a subshift $X \subset \Sigma^{\mathbb{Z}}$. Denote by $\varphi^{(\sigma)}$ the mapping that assigns to a word $a \in \mathcal{L}(X)$ the word that is obtained from a by carrying out the substitution that replaces the symbol σ by the word $\sigma\bullet$. The set of subwords of the words in $\varphi^{(\sigma)}(\mathcal{L}(X))$ is a factorial and bi-extensible language, and we denote the subshift that it determines by $X^{(\sigma)}$. One says that the subshift $X^{(\sigma)}$ arises from the subshift X by symbol expansion. In Section 2 we describe some effects of symbol expansion.

Subshifts $X \subset \Sigma^{\mathbb{Z}}$ and $\widetilde{X} \subset \widetilde{\Sigma}^{\mathbb{Z}}$ are called flow equivalent if there exists a sequence Z_k, $1 \le k \le K$, $K \in \mathbb{N}$, of subshifts, such that $X = Z_1$ and $\widetilde{X} = Z_K$, and such that Z_k is topologically conjugate to Z_{k-1}, or Z_k is obtained from Z_{k-1} by symbol expansion, or Z_{k-1} is obtained from Z_k by symbol expansion, $1 < k \le K$. Flow equivalence was introduced by Parry and Sullivan in 1975 in [10]. Next to topological conjugacy it is one of the fundamental equivalence relations for subshifts.

The notions of $\mathcal{R}$-graph, $\mathcal{R}$-graph semigroup, and $\mathcal{R}$-graph shift were introduced in [5]. The class of $\mathcal{R}$-graph shifts contains the class of Markov–Dyck shifts [9]. In Section 5 we show for certain $\mathcal{R}$-graphs that the flow equivalence of their $\mathcal{R}$-graph shifts implies their isomorphism. This extends a result of Costa and Steinberg [1] for Markov–Dyck shifts. The proof uses Property (A) and the semigroup that is associated to subshifts with Property (A) (see [4]). In Section 3 we prove invariance under flow equivalence of Property (A) and in Section 4 we prove invariance under flow equivalence of the associated semigroup. For an extension of the theory beyond subshifts with Property (A) see Costa and Steinberg [1].

In Section 5 we consider $\mathcal{R}$-graph shifts. In [2] there was given a necessary and sufficient condition for an $\mathcal{R}$-graph to have an $\mathcal{R}$-graph shift with Property (A), whose associated semigroup is the $\mathcal{R}$-graph semigroup of the $\mathcal{R}$-graph. Under this condition we prove in Section 5 that the flow equivalence of $\mathcal{R}$-graph shifts implies the isomorphism of the underlying $\mathcal{R}$-graphs.

2. Symbol expansion

We introduce notation for subshifts $X \subset \Sigma^{\mathbb{Z}}$. We denote the S_X-orbit of a point $x \in X$ by $O_X(x)$, and for an S_X-invariant set $A \subset X$ we denote the set of S_X-orbits in A by $\Omega(A)$. The period of a periodic point $p \in X$ we denote by $\pi(p)$. For $x \in X$, $i, j \in \mathbb{Z}$, $i \le j$, we set

$$x_{[i,j]} = (x_k)_{i \le k \le j},$$

and

$$X_{[i,j]} = \{x_{[i,j]} \mid x \in X\}.$$

We use similar notation in the case that indices range in semi-infinite intervals. (The elements in $X_{[i,j]}, X_{[i,\infty)}, X_{(\infty,i)}$ can be identified with the words they carry. From the context it becomes clear if such an identification is made.) For $a \in X_{[i,j]}$, $i, j \in \mathbb{Z}$, $i \le j$, we set

$$\Gamma^+_X(a) = \{x^+ \in X_{(j,\infty)} \mid ax^+ \in X_{[i,\infty)}\}.$$

The notation Γ^- has the symmetric meaning. For $a \in X_{[i,j]}$, $i, j \in \mathbb{Z}$, $i \le j$, we also set

$$\omega^+_X(a) = \bigcap_{x^- \in \Gamma^-(a)} \{x^+ \in \Gamma^+(a) \mid x^- a x^+ \in X\}.$$

The notation ω^- has the symmetric meaning. And for $a \in X_{[i,j]}$, $i, j \in \mathbb{Z}$, $i \le j$, we set

$$\Gamma_X(a) = \{(x^-, x^+) \in \Gamma^-(a) \times \Gamma^+(a) \mid x^- a x^+ \in X\}.$$

Let $\sigma \in \Sigma$, let $\bullet$ be a symbol that is not in Σ, and consider for a subshift $X \subset \Sigma^{\mathbb{Z}}$ the subshift $X^{(\sigma)} \subset (\Sigma \cup \{\bullet\})^{\mathbb{Z}}$. We denote by $\varphi^{(\sigma)}_-$ (resp. $\varphi^{(\sigma)}_+$) the mapping that assigns to $x^- \in X_{(-\infty,0)}$ (resp. $x^+ \in X_{[0,\infty)}$) the point in $X^{(\sigma)}_{(-\infty,0)}$ (resp. $X^{(\sigma)}_{[0,\infty)}$) that is obtained from x^- (resp. x^+) by carrying out the substitution that replaces the symbol σ by the word $\sigma\bullet$. Also we denote by $\varphi^{(\sigma)}$ the mapping that assigns to a point $x \in X$ the point in $X^{(\sigma)}$ that is given by

$$\varphi^{(\sigma)}(x)_{(-\infty,0)} = \varphi_{-}^{(\sigma)}(x_{(-\infty,0)}),$$
$$\varphi^{(\sigma)}(x)_{[0,-\infty)} = \varphi_{+}^{(\sigma)}(x_{[0,-\infty)}).$$

One observes that

$$\varphi^{(\sigma)}(O_X(x)) \subset O_{X^{(\sigma)}}(\varphi^{(\sigma)}(x)), \quad x \in X.$$

With the notation $\ell^-(x,n)$ (resp. $\ell^+(x,n)$) for the length of $\varphi^{(\sigma)}(x_{[-n,0)})$ (resp. $\varphi^{(\sigma)}(x_{[0,n)})$), we note for precision that one has

$$\varphi^{(\sigma)}(S_X^n(x)) = S_{X^{(\sigma)}}^{-\ell^+(x,n)}(\varphi^{(\sigma)}(x)),$$
$$\varphi^{(\sigma)}(S_X^{-n}(x)) = S_{X^{(\sigma)}}^{\ell^-(x,n)}(\varphi^{(\sigma)}(x)), \quad n \in \mathbb{N}.$$

Also,

$$X^{(\sigma)} = \varphi^{(\sigma)}(X) \cup S_{X^{(\sigma)}}(\varphi^{(\sigma)}(X)).$$

We denote by ξ_σ the bijection of $\Omega(X)$ onto $\Omega(X^{(\sigma)})$ that assigns to the S_X-orbit of $x \in X$ the $S_{X^{(\sigma)}}$-orbit of $\varphi^{(\sigma)}(x)$.

Lemma 2.1. *For a subshift $X \subset \Sigma^{\mathbb{Z}}$ and for $\sigma \in \Sigma$, $a \in \mathcal{L}(X)$, one has*

$$\varphi_{+}^{(\sigma)}(\omega_X^+(a)) = \omega_{X^{(\sigma)}}^+(\varphi^{(\sigma)}(a)).$$

Proof. We prove that $\varphi^{(\sigma)}(\omega_X^+(a)) \subset \omega_{X^{(\sigma)}}^+(\varphi^{(\sigma)}(a))$. Let $x^+ \in \omega_X^+(a)$, and let

$$y^- \in \Gamma_{X^{(\sigma)}}^-(\varphi^{(\sigma)}(a)).$$

It follows from $\varphi^{(\sigma)}(a)_0 \neq \bullet$ that $y_{-1}^- \neq \sigma$, and one sees that y^- is in the image of $\varphi_{-}^{(\sigma)}$. Its inverse image x^- under $\varphi_{-}^{(\sigma)}$ is in $\Gamma_X^-(a)$. It follows that $x^-ax^+ \in X$, and therefore

$$\varphi^{(\sigma)}(x^-ax^+) = y^-\varphi^{(\sigma)}(a)\varphi_{+}^{(\sigma)}(x^+) \in X^{(\sigma)}.$$

This means that $\varphi_{+}^{(\sigma)}(x^+) \in \omega_{X^{(\sigma)}}^+(\varphi^{(\sigma)}(a))$.

For the converse one has a similar argument. $\square$

Lemma 2.2. *For a subshift $X \subset \Sigma^{\mathbb{Z}}$ and for $\sigma \in \Sigma$, $b, b' \in \mathcal{L}(X)$, one has*

$$\Gamma_X(b) = \Gamma_X(b')$$

if and only if

$$\Gamma_{X^{(\sigma)}}(\varphi^{(\sigma)}(b)) = \Gamma_{X^{(\sigma)}}(\varphi^{(\sigma)}(b')).$$

Proof. The lemma follows from

$$\Gamma_{X^{(\sigma)}}^+(\varphi^{(\sigma)}(a)) \subset \varphi_{+}^{(\sigma)}(\Gamma_X^+(a)),$$
$$\Gamma_{X^{(\sigma)}}^-(\varphi^{(\sigma)}(a)) \subset \varphi_{-}^{(\sigma)}(\Gamma_X^-(a)), \quad a \in \mathcal{L}(X). \qquad \square$$

3. Property (A)

Given a subshift $X \subset \Sigma^{\mathbb{Z}}$, we define for $n \in \mathbb{N}$ a subshift of finite type $A_n(X)$ by

$$A_n(X) = \bigcap_{i \in \mathbb{Z}} \big(\{x \in X \mid x_{[i,\infty)} \in \omega_X^+(x_{[i-n,i)})\} \\ \cap \{x \in X \mid x_{(-\infty,i]} \in \omega_X^-(x_{(i,i+n]})\}\big),$$

and we set

$$A(X) = \bigcup_{n \in \mathbb{N}} A_n(X).$$

Lemma 3.1. *For a subshift $X \subset \Sigma^{\mathbb{Z}}$ and for $\sigma \in \Sigma$, one has*

$$\xi_\sigma(\Omega(A_n(X))) \subset \Omega(A_{2n}(X^{(\sigma)})), \quad n \in \mathbb{N}, \tag{1}$$

$$\xi_\sigma^{-1}(\Omega(A_n(X^{(\sigma)}))) \subset \Omega(A_n(X)), \quad n \in \mathbb{N}. \tag{2}$$

Proof. We show (1). Let $n \in \mathbb{N}$, $x \in A_n(X)$, and $i \in \mathbb{Z}$. Let μ be the number of times that the symbol $\bullet$ appears in $\varphi^{(\sigma)}(x)_{[i,i+2n]}$. Assume that neither $x_i^{(\sigma)} = \bullet$, nor $x_{i+2n-1}^{(\sigma)} = \sigma$. Then

$$\varphi^{(\sigma)}(x_{[i,i+2n-\mu)}) = \varphi^{(\sigma)}(x)_{[i,i+2n]}.$$

From

$$x_{[i+2n-\mu,\infty)} \in \omega_X^+(x_{[i,i+2n-\mu)}),$$

it follows then by Lemma 2.1 that

$$\varphi^{(\sigma)}(x)_{[i+2n,\infty)} \in \omega_{X^{(\sigma)}}^+(\varphi^{(\sigma)}(x)_{[i,i+2n)}). \tag{3}$$

In the case that $x_i^{(\sigma)} = \bullet$ we have necessarily $x_{i-1}^{(\sigma)} = \sigma$, and in the case that $x_{i+2n-1}^{(\sigma)} = \sigma$ we have necessarily $x_{i+2n}^{(\sigma)} = \bullet$. In both cases it is seen that (3) also holds.

For (2) one has a similar argument. □

We recall from [4] the definition of Property (A). For $n \in \mathbb{N}$ a subshift $X \subset \Sigma^{\mathbb{Z}}$ has property (a, n, H), $H \in \mathbb{N}$, if for $h, \widetilde{h} \geq 3H$ and for $I_-, I_+, \widetilde{I}_-, \widetilde{I}_+ \in \mathbb{Z}$ such that

$$I_+ - I_-, \, \widetilde{I}_+ - \widetilde{I}_- \geq 3H,$$

and for $a \in A_n(X)_{(I_-,I_+]}$, $\widetilde{a} \in A_n(X)_{(\widetilde{I}_-,\widetilde{I}_+]}$ such that

$$a_{(I_-,I_-+H]} = \tilde{a}_{(\widetilde{I}_-,\widetilde{I}_++H]}, \quad a_{(I_+-H,I_+]} = \tilde{a}_{(\widetilde{I}_+-H,\widetilde{I}_+]},$$

one has

$$\Gamma_X(a) = \Gamma_X(\tilde{a}).$$

It is assumed that $A(X) \neq \varnothing$. The subshift $X \subset \Sigma^{\mathbb{Z}}$ has property (A) if there are H_n, $n \in \mathbb{N}$, such that X has the properties (a, n, H_n), $n \in \mathbb{N}$.

Theorem 3.2. *For a subshift $X \subset \Sigma^{\mathbb{Z}}$ and for $\sigma \in \Sigma$, one has that X has Property* (A) *if and only if $X^{(\sigma)}$ has Property* (A).

Proof. The theorem follows from Lemma 2.2 and Lemma 3.1. □

4. The associated semigroup

Consider a subshift $X \subset \Sigma^{\mathbb{Z}}$ with Property (A). We denote the set of periodic points in $A(X)$ by $P(A(X))$. We introduce a preorder relation $\gtrsim(X)$ into the set $P(A(X))$ where for $q, r \in P(A(X))$, $q \gtrsim(X)\ r$ means that there exists a point in $A(X)$ that is left asymptotic to the orbit of q and right asymptotic to the orbit of r. The equivalence relation on $P(A(X))$ that results from the preorder relation $\gtrsim(X)$ is denoted by $\approx(X)$. We denote the set of $\approx(X)$-equivalence classes by $\mathfrak{P}(X)$.

Lemma 4.1. *For a subshift $X \subset \Sigma^{\mathbb{Z}}$, for $\sigma \in \Sigma$, $q, r \in P(A(X))$, and for $\sigma \in \Sigma$, one has*

$$q \gtrsim(X)\ r$$

if and only if

$$\varphi^{(\sigma)}(q) \gtrsim(X^{(\sigma)})\ \varphi^{(\sigma)}(r).$$

Proof. This follows from Lemma 3.1. □

We recall the construction of the associated semigroup. For a Property (A) subshift $X \subset \Sigma^{\mathbb{Z}}$ we denote by $Y(X)$ the set of points in X that are left asymptotic to a point in $P(A(X))$ and also right-asymptotic to a point in $P(A(X))$. Let $y, \tilde{y} \in Y(X)$, let y be left asymptotic to $q \in P(A(X))$ and right asymptotic to $r \in P(A(X))$, and let $\tilde{y}$ be left asymptotic to $\tilde{q} \in P(A(X))$ and right asymptotic to $\tilde{r} \in P(A(X))$. Given that X has the properties (a, n, H_n), $n \in \mathbb{N}$, we say that y and $\tilde{y}$ are equivalent, $y \approx(X)\ \tilde{y}$, if $q \approx(X)\ \tilde{q}$ and $r \approx(X)\ \tilde{r}$, and if for $n \in \mathbb{N}$ such that $q, r, \tilde{q}, \tilde{r} \in P(A_n(X))$ and for $I, J, \tilde{I}, \tilde{J} \in \mathbb{Z}$, $I < J$, $\tilde{I} < \tilde{J}$, such that

$$y_{(-\infty, I]} = q_{(-\infty, 0]}, \quad y_{(J, \infty)} = r_{(0, \infty)},$$
$$\tilde{y}_{(-\infty, \tilde{I}]} = \tilde{q}_{(-\infty, 0]}, \quad \tilde{y}_{(\tilde{J}, \infty)} = \tilde{r}_{(0, \infty)},$$

and for $h \geq 3H_n$ and $a \in X_{(I-h, J+h]}$, $\tilde{a} \in X_{(\tilde{I}-h, \tilde{J}+h]}$ such that

$$\begin{aligned} a_{(I-H_n, J+H_n]} &= y_{(I-H_n, J+H_n]}, & \tilde{a}_{(\tilde{I}-H_n, \tilde{J}+H_n]} &= \tilde{y}_{(\tilde{I}-H_n, \tilde{J}+H_n]}, \\ a_{(I-h, I-h+H_n)} &= \tilde{a}_{(\tilde{I}-h, \tilde{I}-h+H_n)}, & a_{(J+h-H_n, J+h]} &= \tilde{a}_{(\tilde{J}+h-H_n, \tilde{J}+h]}, \\ a_{(I-h, I]} &\in A_n(X)_{(I-h, I]}, & \tilde{a}_{(\tilde{J}-h, \tilde{I}]} &\in A_n(X)_{(\tilde{J}-h, \tilde{I}]}, \\ a_{(J, J+h]} &\in A_n(X)_{(J, J+h]}, & \tilde{a}_{(\tilde{J}, \tilde{J}+h]} &\in A_n(X)_{(\tilde{J}, \tilde{J}+h]}, \end{aligned}$$

it holds that

$$\Gamma_X(a) = \Gamma_X(\tilde{a}).$$

To give $[Y(X)]_{\approx(X)}$ the structure of a semigroup, let $u, v \in Y(X)$, let u be right asymptotic to $q \in P(A(X))$ and let v be left asymptotic to $r \in P(A(X))$. If here $q \gtrsim(X)\ r$, then $[u]_{\approx(X)}[v]_{\approx(X)}$ is set equal to $[y]_{\approx(X)}$, where y is any point in Y such that there are $n \in \mathbb{N}$, $I, J, \hat{I}, \hat{J} \in \mathbb{Z}$, $I < J$, $\hat{I} < \hat{J}$, such that

$q, r \in A_n(X)$, and such that

$$u_{(I,\infty)} = q_{(I,\infty)}, \qquad v_{(-\infty,J]} = r_{(-\infty,J]},$$
$$y_{(-\infty,\hat{I}+H_n]} = u_{(-\infty,I+H_n]}, \quad y_{(\hat{J}-H_n,\infty)} = v_{(J-H_n,\infty)},$$

and

$$y_{(\hat{I},\hat{J}]} \in A_n(X)_{(\hat{I},\hat{J}]},$$

provided that such a point y exists. If such a point y does not exist, then $[u]_{\approx(X)}[v]_{\approx(X)}$ is equal to zero. Moreover, in the case that one does not have $q \gtrsim(X)\, r$, one sets $[u]_{\approx(X)}[v]_{\approx(X)}$ equal to zero.

Consider a subshift $X \subset \Sigma^{\mathbb{Z}}$ with Property (A). For $\mathfrak{p} \in \mathfrak{P}(X)$ we choose $d^{(\mathfrak{p})} \in \mathfrak{p}$ and set

$$\mathcal{D} = \{d^{(\mathfrak{p})} \mid \mathfrak{p} \in \mathfrak{P}(X)\}.$$

In order to facilitate the proof of its invariance under flow equivalence we give an alternate description of the semigroup that is associated to X in terms of the system $\mathcal{D} \subset Y_X$ of representatives of the equivalence relation $\approx(X)$. For $y \in O_X(d^{(\mathfrak{p})})$, $\mathfrak{p} \in \mathfrak{P}(X)$, we define $J(y, d^{(\mathfrak{q})}) \in \mathbb{Z}$ by

$$S_X^{-J(y,d^{(\mathfrak{p})})}(y) = d^{(\mathfrak{p})}, \quad 0 \leq \pi(d^{(\mathfrak{p})}) < \pi(d^{(\mathfrak{p})}).$$

For $\mathfrak{p} \in \mathfrak{P}(X)$ we set

$$H(d^{(\mathfrak{p})}) = \min\{H \in \mathbb{N} \mid \Gamma_X(\mathfrak{p}_{[0,H\pi(\mathfrak{p}))}) = \Gamma_X(\mathfrak{p}_{[0,(H+1)\pi(d^{(\mathfrak{p})}))})\}.$$

We denote by $Y_X^-(\mathcal{D})$ the set of points in Y_X that are left asymptotic to the orbit of a point in $\mathcal{D}$, and also right asymptotic to the orbit of a point in $\mathcal{D}$. More precisely, we denote by $Y_X^-(d^{(\mathfrak{p})})$ (resp. $Y_X^+(d^{(\mathfrak{p})})$) the set of points in Y_X that are left (right) asymptotic to the orbit of $d^{(\mathfrak{p})}$, $\mathfrak{p} \in \mathfrak{P}(X)$. For

$$y \in Y_X^-(d^{(\mathfrak{q})}q) \cap Y_X^+(d^{(\mathfrak{r})}), \quad \mathfrak{q}, \mathfrak{r} \in \mathfrak{P}(X),$$

we set

$$I^-(y) = \begin{cases} J(y, d^{(\mathfrak{q})}), & \text{if } y \in O_X(d^{(\mathfrak{q})}), \\ \max\{I \in \mathbb{Z} \mid y_{(-\infty,I)} = d^{(\mathfrak{q})}_{(-\infty,0)}\}, & \text{if } y \notin O_X(d^{(\mathfrak{q})}), \end{cases}$$

$$I^+(y) = \begin{cases} J(y, d^{(\mathfrak{r})}), & \text{if } y \in O_X(d^{(\mathfrak{r})}), \\ \min\{I \in \mathbb{Z} \mid y_{[I,\infty)} = d^{(\mathfrak{r})}_{[0,\infty)}\}, & \text{if } y \notin O_X(d^{(\mathfrak{r})}). \end{cases}$$

We say that $O, O' \in \Omega(Y_X^{(D)})$ are $\approx(D)$-equivalent if O and O' are left asymptotic to the same periodic obit, and also right asymptotic to the same periodic obit, and, with $\mathfrak{q} \in \mathfrak{P}$ such that y and y' are right asymptotic to the orbit of $d^{(\mathfrak{q})}$ and with $\mathfrak{r} \in \mathfrak{P}$ such that y and y' are left asymptotic to the orbit of $d^{(\mathfrak{r})}$, if there exist $y \in O$ and $y' \in O'$ such that

$$\begin{aligned}\Gamma_X(&d^{(\mathfrak{q})}_{[0,H(d^{(\mathfrak{q})})\pi(d^{(\mathfrak{q})}))} y_{[I^-(y),I^+(y))} d^{(\mathfrak{r})}_{[0,H(d^{(\mathfrak{r})})\pi(d^{(\mathfrak{r})}))}) \\ &= \Gamma_X(d^{(\mathfrak{q})}_{[0,H(d^{(\mathfrak{q})})\pi(d^{(\mathfrak{q})}))} y'_{[I^-(y'),I^+(y'))} d^{(\mathfrak{r})}_{[0,H(d^{(\mathfrak{r})})\pi(d^{(\mathfrak{r})}))}).\end{aligned}$$

To give $\Omega(Y_X^{(D)})$ the structure of a semigroup, let $\mathfrak{q}, \mathfrak{p}, \mathfrak{r} \in \mathfrak{P}$, and suppose that

$$u \in Y_X^-(\mathfrak{q}) \cap Y_X^+(\mathfrak{p}), \quad v \in Y_X^-(\mathfrak{p}) \cap Y_X^+(\mathfrak{r}).$$

In case that the word

$$\begin{aligned} &d^{(\mathfrak{q})}_{[0,H(\mathfrak{d}^{(\mathfrak{q})})\pi(d^{(\mathfrak{q})}))} y_{[I^-(u),I^+(u))} \\ &\quad * d^{(\mathfrak{p})}_{[0,H(d^{(\mathfrak{p})})\pi(d^{(\mathfrak{p})}))} y_{[I^-(v),I^+(v))} d^{(\mathfrak{r})}_{[0,H(d^{(\mathfrak{r})})\pi(d^{(\mathfrak{r})}))} \end{aligned} \tag{4}$$

is admissible for X, let a point $y[u,v] \in Y_X^-(\mathfrak{q}) \cap Y_X^+(\mathfrak{r})$ be given by

$$y[u,v]_{(-\infty,0)} = d^{(\mathfrak{q})}_{(-\infty,0)},$$

$$y[u,v]_{[0,\infty)} = d^{(\mathfrak{q})}_{[0,H(\mathfrak{q})\pi(\mathfrak{q}))} y_{[I^-(u),I^+(u))} d^{(\mathfrak{p})}_{[0,H(d^{(\mathfrak{p})})\pi(d^{(\mathfrak{p})}))} y_{[I^-(v),I^+(v))} d^{(d^{(\mathfrak{r})})}_{[0,\infty)}.$$

Then set

$$[O(u)]_{\approx(D)}[O(v)]_{\approx(D)} = [O(y[u,v])]_{\approx(D)}.$$

In case that the word (4) is not admissible for X, set

$$[O(u)]_{\approx(D)}[O(v)]_{\approx(D)} = 0.$$

Also, for $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, if

$$Y_X^-(\mathfrak{q}) \cap A(X) \cap Y_X^+(\mathfrak{r}) \neq \varnothing,$$

define a $\approx(D)$-equivalence class $\gamma(\mathfrak{q}, \mathfrak{r})$ by

$$\gamma(\mathfrak{q}, \mathfrak{r}) = [O(y)]_{\approx(D)}, \quad y \in Y_X^-(\mathfrak{q}) \cap A(X) \cap Y_X^+(\mathfrak{r}).$$

As a consequence of Property (A) of X the $\approx(D)$-equivalence class $\gamma(\mathfrak{q}, \mathfrak{r})$ is well defined. If

$$Y_X^-(\mathfrak{q}) \cap A(X) \cap Y_X^+(\mathfrak{r}) = \varnothing,$$

set

$$\gamma(\mathfrak{q}, \mathfrak{r}) = 0.$$

Identify $\mathfrak{p} \in \mathfrak{P}$ with $\gamma(\mathfrak{p}, \mathfrak{p})$. Finally, for $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$ and $u \in Y_X^+(\mathfrak{q})$, $v \in Y_X^-(\mathfrak{r})$, set

$$[O(u)]_{\approx(D)}[O(v)]_{\approx(D)} = [u]_{\approx(D)}\gamma(\mathfrak{q}, \mathfrak{r})[v]_{\approx(D)}.$$

An isomorphism $\eta_{\sigma,D}$ of $[Y_X]_{\approx(X)}$ onto $[\Omega(Y_X^{(D)})]_{\approx(D)}$ is obtained by choosing out of every $\approx(X)$-equivalence class α a point $\eta(\alpha) \in Y_X^{(D)}$, and by setting

$$\eta_X^{(D)}(\alpha) = [\eta(\alpha)]_{\approx(D)}.$$

Theorem 4.2. *For a subshift $X \subset \Sigma^{\mathbb{Z}}$ with Property* (A) *and for $\sigma \in \Sigma$ the semigroups that are associated to X and $X^{(\sigma)}$ are isomorphic.*

Proof. Set

$$d^{(\mathfrak{p}^{(\sigma)})} = \varphi^{(\sigma)}(d^{(\mathfrak{p})}), \quad \mathfrak{p} \in \mathfrak{P}.$$

One has

$$\pi(d^{(\mathfrak{p}^{(\sigma)})}) = \pi(d^{(\mathfrak{p})}), \quad \mathfrak{p} \in \mathfrak{P},$$

and, by Lemma 2.2,

$$H(d^{(\mathfrak{p}^{(\sigma)})}) = H(d^{(\mathfrak{p})}), \quad \mathfrak{p} \in \mathfrak{P}.$$

Setting

$$D^{(\sigma)} = \{d^{(\mathfrak{p}^{(\sigma)})} \mid \mathfrak{p} \in \mathfrak{P}\}$$

yields a system of representatives of the $\approx(X^{(\sigma)})$-equivalence classes in $\mathfrak{P}(X^{(\sigma)})$. By construction

$$\varphi^{(\sigma)}(y[u,v]) = y[\varphi^{(\sigma)}(u), \varphi^{(\sigma)}(v)], \quad u, v \in Y_X.$$

Also, by Lemma 3.1, for $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$, one has

$$Y_X^-(\mathfrak{q}) \cap A(X) \cap Y_X^-(\mathfrak{r}) \neq \varnothing$$

if and only if

$$Y_X^-(\mathfrak{q}^{(\sigma)}) \cap A(X^{(\sigma)}) \cap Y_X^-(\mathfrak{r}^{(\sigma)}) \neq \varnothing.$$

It follows that an isomorphism $\psi_{\sigma,D}$ of $[Y_X^{(D)}]_{\approx(D)}$ onto $[Y_{X^{(\sigma)}}^{(D^{(\sigma)})}]_{\approx(D^{(\sigma)})}$ is given by setting

$$\psi_{\sigma,D}([y]) = [\varphi^{(\sigma)}(y)], \quad y \in Y_X^{(D)},$$

and one obtains an isomorphism $\Xi^{(\sigma)}$ of $[Y_X]_{\approx(X)}$ onto $[Y_{X^{(\sigma)}}]_{\approx(X^{(\sigma)})}$ by setting

$$\Xi^{(\sigma)} = \eta_{\sigma,D}^{-1} \psi_{\sigma,D} \eta_{\sigma,D}. \qquad \square$$

See also [1, Thm. 9.20] for the invariance of the associated semigroup under flow equivalence, under the assumption that $A(X)$ is dense in X, or in the sofic case.

The semigroup $[Y_X^{(D)}]_{\approx(D)}$ is a set of equivalence classes of orbits. As originally done in [4], we have introduced the associated semigroup of a subshift with Property (A) in terms of equivalence classes of points, rather than equivalence classes of orbits. However, since points in Y_X that are in the same orbit are $\approx(X)$-equivalent, one can define the associated semigroup in the first place as a set of equivalence classes of orbits. The same remark applies to the set of idempotents $\mathfrak{P}$. When the associated semigroup is introduced as a set of equivalence classes of orbits, then the mapping ξ_σ is seen to induce the isomorphism of the associated semigroup of X onto the associated semigroup of $X^{(\sigma)}$.

5. $\mathcal{R}$-GRAPH SHIFTS

Given finite sets $\mathcal{E}^-$ and $\mathcal{E}^+$ and a relation $\mathcal{R} \subset \mathcal{E}^- \times \mathcal{E}^+$, we set

$$\mathcal{E}^-(\mathcal{R}) = \{e^- \in \mathcal{E}^- \mid \{e^-\} \times \mathcal{E}^+ \subset \mathcal{R}\},$$
$$\mathcal{E}^+(\mathcal{R}) = \{e^+ \in \mathcal{E}^+ \mid \mathcal{E}^- \times \{e^+\} \subset \mathcal{R}\},$$

and

$$\Omega_{\mathcal{R}}^+(e^-) = \{e^+ \in \mathcal{E}^+ \mid (e^-, e^+) \in \mathcal{R}\}, \quad e^- \in \mathcal{E}^-,$$
$$\Omega_{\mathcal{R}}^-(e^+) = \{e^- \in \mathcal{E}^- \mid (e^-, e^+) \in \mathcal{R}\}, \quad e^+ \in \mathcal{E}^+.$$

We recall from [5] the notion of an $\mathcal{R}$-graph. Let there be given a finite directed graph with vertex set $\mathfrak{P}$ and edge set $\mathcal{E}$. Furthermore, assume that a partition $\mathcal{E} = \mathcal{E}^- \cup \mathcal{E}^+$ is given. With s and t denoting the source and the target vertex of a directed edge we set

$$\mathcal{E}^-(\mathfrak{q},\mathfrak{r}) = \{e^- \in \mathcal{E}^- \mid s(e^-) = \mathfrak{q},\, t(e^-) = \mathfrak{r}\},$$
$$\mathcal{E}^+(\mathfrak{q},\mathfrak{r}) = \{e^- \in \mathcal{E}^+ \mid s(e^+) = \mathfrak{r},\, t(e^+) = \mathfrak{q}\}, \quad \mathfrak{q},\mathfrak{r} \in \mathfrak{P}.$$

We assume that $\mathcal{E}^-(\mathfrak{q},\mathfrak{r}) \neq \varnothing$ if and only if $\mathcal{E}^+(\mathfrak{q},\mathfrak{r}) \neq \varnothing$, $\mathfrak{q},\mathfrak{r} \in \mathfrak{P}$, and we assume that the directed graph $(\mathfrak{P},\mathcal{E}^-)$ is strongly connected, or equivalently that the directed graph $(\mathfrak{P},\mathcal{E}^+)$ is strongly connected. Furthermore, let there be given relations

$$\mathcal{R}(\mathfrak{q},\mathfrak{r}) \subset \mathcal{E}^-(\mathfrak{q},\mathfrak{r}) \times \mathcal{E}^+(\mathfrak{q},\mathfrak{r}), \quad \mathfrak{q},\mathfrak{r} \in \mathfrak{P}.$$

Set

$$\mathcal{R} = \bigcup_{\mathfrak{q},\mathfrak{r}\in\mathfrak{P}} \mathcal{R}(\mathfrak{q},\mathfrak{r}).$$

The resulting structure, for which we use the notation $\mathcal{G}_{\mathcal{R}}(\mathfrak{P},\mathcal{E}^-,\mathcal{E}^+)$, is called an $\mathcal{R}$-graph.

We also recall the construction of a semigroup (with zero) $\mathcal{S}_{\mathcal{R}}(\mathfrak{P},\mathcal{E}^-,\mathcal{E}^+)$ from an $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P},\mathcal{E}^-,\mathcal{E}^+)$ as given in [5]. The semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P},\mathcal{E}^-,\mathcal{E}^+)$ contains idempotents $\mathbf{1}_\mathfrak{p}$, $\mathfrak{p} \in \mathfrak{P}$, and has $\mathcal{E}$ as a generating set. Besides the relations $\mathbf{1}_\mathfrak{p}^2 = \mathbf{1}_\mathfrak{p}$, $\mathfrak{p} \in \mathfrak{P}$, one has for $\mathfrak{q},\mathfrak{r} \in \mathfrak{P}$, $\mathfrak{q} \neq \mathfrak{r}$, the relations $\mathbf{1}_\mathfrak{q}\mathbf{1}_\mathfrak{r} = 0$, and, with $\mathfrak{q},\mathfrak{r} \in \mathfrak{P}$, for the elements of the generating set $\mathcal{E}$ of $\mathcal{S}_{\mathcal{R}}(\mathfrak{P},\mathcal{E}^-,\mathcal{E}^+)$ the relations

$$f^-g^+ = \mathbf{1}_\mathfrak{q}, \quad f^- \in \mathcal{E}^-(\mathfrak{q},\mathfrak{r}),\, g^+ \in \mathcal{E}^+(\mathfrak{q},\mathfrak{r}),\, (f^-,g^+) \in \mathcal{R}(\mathfrak{q},\mathfrak{r}),$$
$$\mathbf{1}_\mathfrak{q} e^- = e^-\mathbf{1}_\mathfrak{r} = e^-, \quad e^- \in \mathcal{E}^-(\mathfrak{q},\mathfrak{r}),$$
$$\mathbf{1}_\mathfrak{r} e^+ = e^+\mathbf{1}_\mathfrak{q} = e^+, \quad e^+ \in \mathcal{E}^+(\mathfrak{q},\mathfrak{r}),$$
$$f^-g^+ = \begin{cases} \mathbf{1}_\mathfrak{q}, & \text{if } (f^-,g^+) \in \mathcal{R}(\mathfrak{q},\mathfrak{r}), \\ 0, & \text{if } (f^-,g^+) \notin \mathcal{R}(\mathfrak{q},\mathfrak{r}), \end{cases} \quad f^- \in \mathcal{E}^-(\mathfrak{q},\mathfrak{r}),\, g^+ \in \mathcal{E}^+(\mathfrak{q},\mathfrak{r}).$$

The semigroup $\mathcal{S}_{\mathcal{R}}(\mathfrak{P},\mathcal{E}^-,\mathcal{E}^+)$ is called an $\mathcal{R}$-graph semigroup.

The $\mathcal{R}$-graph shift $MD_{\mathcal{R}}(\mathfrak{P},\mathcal{E}^-,\mathcal{E}^+)$ of the $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P},\mathcal{E}^-,\mathcal{E}^+)$ is the subshift

$$MD_{\mathcal{R}}(\mathfrak{P},\mathcal{E}^-,\mathcal{E}^+) \subset \mathcal{E}^{\mathbb{Z}}$$

with the admissible words $(\sigma_i)_{1\leq i\leq I}$, $I \in \mathbb{N}$, of $MD_{\mathcal{R}}(\mathfrak{P},\mathcal{E}^-,\mathcal{E}^+)$ given by the condition

$$\prod_{1\leq i\leq I} \sigma_i \neq 0.$$

For an $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P},\mathcal{E}^-,\mathcal{E}^+)$ we denote by $\mathfrak{P}^{(1)}$ the set of vertices in $\mathfrak{P}$ that have a single predecessor vertex in $\mathcal{E}^-$, or equivalently that have a single successor vertex in $\mathcal{E}^+$. For $\mathfrak{p} \in \mathfrak{P}^{(1)}$ the predecessor vertex of $\mathfrak{p}$ in $\mathcal{E}^-$, which is identical to the successor vertex of $\mathfrak{p}$ in $\mathcal{E}^+$, is denoted by $\kappa(\mathfrak{p})$. We set

$$\mathcal{E}^-_{\mathcal{R}} = \bigcup_{\mathfrak{p}\in\mathfrak{P}^{(1)}} \mathcal{E}^-(\mathcal{R}(\kappa(\mathfrak{p}),\mathfrak{p})), \quad \mathcal{E}^+_{\mathcal{R}} = \bigcup_{\mathfrak{p}\in\mathfrak{P}^{(1)}} \mathcal{E}^+(\mathcal{R}(\kappa(\mathfrak{p}),\mathfrak{p})),$$

and

$$\mathfrak{P}^{(1)}_{\mathcal{R}} = \{\mathfrak{p} \in \mathfrak{P}^{(1)} \mid \mathcal{R}(\kappa(\mathfrak{p}), \mathfrak{p}) = \mathcal{E}^-(\kappa(\mathfrak{p}), \mathfrak{p}) \times \mathcal{E}^+(\kappa(\mathfrak{p}), \mathfrak{p})\}.$$

We formulate conditions (a), (b), (c) and (d) on an $\mathcal{R}$-graph $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ as follows:

(a−) $\Omega^+_{\mathcal{R}(\mathfrak{q},\mathfrak{r})}(e^-) \neq \Omega^+(\tilde{e}^-)$, $e^-, \tilde{e}^- \in \mathcal{E}^-(\mathfrak{q}, \mathfrak{r})$, $e^- \neq \tilde{e}^-$, $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$.

(a+) $\Omega^-_{\mathcal{R}(\mathfrak{q},\mathfrak{r})}(e^+) \neq \Omega^-(\tilde{e}^+)$, $e^+, \tilde{e}^+ \in \mathcal{E}^+(\mathfrak{r}, \mathfrak{q})$, $e^+ \neq \tilde{e}^+$, $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}$.

(b−) There is no nonempty cycle in $\mathcal{E}^-_{\mathcal{R}}$.

(b+) There is no nonempty cycle in $\mathcal{E}^+_{\mathcal{R}}$.

(c) For $\mathfrak{p} \in \mathfrak{P}^{(1)}$ such that $\kappa(\mathfrak{p}) \neq \mathfrak{p}$, $\mathcal{E}^-_{\mathcal{R}}(\mathfrak{p}) = \varnothing$ or $\mathcal{E}^-_{\mathcal{R}}(\mathfrak{p}) = \varnothing$.

(d) For $\mathfrak{q}, \mathfrak{r} \in \mathfrak{P}^{(1)}$, $\mathfrak{q} \neq \mathfrak{r}$, there do not simultaneously exist a path in $\mathcal{E}^-_{\mathcal{R}}$ from $\mathfrak{q}$ to $\mathfrak{r}$ and a path in $\mathcal{E}^+_{\mathcal{R}}$ from $\mathfrak{q}$ to $\mathfrak{r}$.

Theorem 5.1. *For $\mathcal{R}$-graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ that satisfy the conditions* (a), (b), (c) *and* (d) *the flow equivalence of the $\mathcal{R}$-graph shifts $D_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ implies the isomorphism of the $\mathcal{R}$-graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$.*

Proof. By [2, Thm. 2.3 and Thm. 6.1] the conditions imply that the $\mathcal{R}$-graph shift $D_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ has property (A), and that the semigroup that is associated to it is $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. By Theorem 4.2 the flow equivalence of the shifts $D_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$ implies the isomorphism of the $\mathcal{R}$-graph semigroups $\mathcal{S}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$, which in turn, by [5, Thm. 2.1], implies the isomorphism of the $\mathcal{R}$-graphs $\mathcal{G}_{\mathcal{R}}(\mathfrak{P}, \mathcal{E}^-, \mathcal{E}^+)$. □

Theorem 5.1 extends the result of Costa and Steinberg that the flow equivalence of Markov–Dyck shifts of finite irreducible directed graphs, in which every vertex has at least two incoming edges, implies the isomorphism of the graphs (see [1, Thm. 8.6]).

For $K > 1$, let B_K denote the full shift on K symbols, and let D_2 denote the Dyck shift on four symbols. The shifts $D_2 \times B_K$, $K > 1$, belong to the class of $\mathcal{R}$-graph shifts. They arise from the one-vertex $\mathcal{R}$-graphs $\mathcal{G}_{\mathcal{R}}(\{\mathfrak{p}\}, \mathcal{E}^-, \mathcal{E}^+)$, where

$$\begin{aligned}
\mathcal{E}^- &= \{e^-(m, \beta) \mid 1 \leq m \leq K,\ \beta = 0, 1\}, \\
\mathcal{E}^+ &= \{e^-(l, \beta) \mid 1 \leq l \leq K,\ \beta = 0, 1\},
\end{aligned}$$

and where

$$(e^-m, (\beta^-), e^-(l, \beta^+)) \in \mathcal{R}(\mathfrak{p}, \mathfrak{p})$$

if and only if

$$\beta^- = \beta^+, \quad 1 \leq m \leq K, \quad 1 \leq l \leq K.$$

These $\mathcal{R}$-graphs do not satisfy the conditions of Theorem 5.1, but the $\mathcal{R}$-graph shifts $D_2 \times B_K$, $K > 1$, have Property (A), and the flow equivalence of their $\mathcal{R}$-graph shifts $D_2 \times B_K$, $K > 1$, still implies the isomorphism of these $\mathcal{R}$-graphs. This can be seen from the invariance under flow equivalence of the K-groups of subshifts as shown by Matsumoto in [7], and from

$$K_0(D_2 \times B_K) = \mathbb{Z}[\tfrac{1}{n}]^\infty, \quad K > 1,$$

as also shown by Matsumoto [8, Section 8]. Note that the associated semigroup of $D_2 \times B_K$, $K > 1$, is the Dyck inverse monoid $\mathcal{D}_2$.

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Received July 22, 2014; accepted May 19, 2015

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